FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

VOLUME IV ISSUE I

FUJMA

www.dergipark.org.tr/en/pub/fujma

ISSN 2645-8845

VOLUME 4 ISSUE 1
ISSN 2645-8845

March 2021
www.dergipark.org.tr/en/pub/fujma

FUNDAMENTAL JOURNAL OF MATHEMATICS AND APPLICATIONS

Editorial Secretariat

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 1-9
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.752758

Unrestricted Fibonacci and Lucas Quaternions

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Article Info

Keywords: *Binet's formula, Catalan's identity, Fibonacci quaternion, Generating function, Lucas quaternion*
2010 AMS: *11B39, 15B36, 11R52*
Received: *14 June 2020*
Accepted: *20 January 2021*
Available online: *29 January 2021*

Abstract

Many quaternion numbers associated with Fibonacci and Lucas numbers or even their generalizations have been defined and widely discussed so far. In all the studies, the coefficients of these quaternions have been selected from consecutive terms of these numbers. In this study, we define other generalizations for the usual Fibonacci and Lucas quaternions. We also present some properties, including the Binet's formulas and d'Ocagne's identities, for these types of quaternions.

1. Introduction

Since Sir William Rowan Hamilton introduced the quaternion algebra in 1843, their usage areas have developed rapidly. Due to the fact that the quaternions are encountered in many problems from elastodynamics, quantum mechanics, elasticity theory and many other fields of modern sciences, they have been studied widely.

A quaternion q can be regarded as a quadruple of real numbers and is formally defined by

$$q = q_0 + iq_1 + jq_2 + kq_3,$$

where q_0, q_1, q_2, and q_3 are any real numbers and the standard basis $\{1, i, j, k\}$ satisfies

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The conjugate of q is

$$q^* = q_0 - iq_1 - jq_2 - kq_3$$

and, the norm of q is

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$$N(q) = qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Fibonacci numbers are recursively defined as

$$F_n = F_{n-1} + F_{n-2},$$

with the initial terms $F_0 = 0$ and $F_1 = 1$. The Lucas numbers satisfy the same recurrence relation but with the initial terms $L_0 = 2$ and $L_1 = 1$. Binet's formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

respectively. Here, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the positive and negative roots of $x^2 - x - 1 = 0$, respectively.

Horadam [1] defined Fibonacci quaternions as

$$Q_n := F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}, \tag{1.1}$$

where F_n is the nth term of the Fibonacci sequence. Iyer [2] gave a similar definition for Lucas quaternions by the relation

$$T_n := L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}, \tag{1.2}$$

and gave their many properties, where L_n is the nth Lucas number.

Halici [3] gave Binet's formulas for the Fibonacci and Lucas quaternions as follows:

$$Q_n = \frac{\underline{\alpha}\alpha - \underline{\beta}\beta}{\alpha - \beta} \text{ and } T_n = \underline{\alpha}\alpha + \underline{\beta}\beta,$$

where $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$.

There exist many papers devoted to generalizations of the quaternion sequences in (1.1) and (1.2) today. For example, the references in [4]–[11] can be investigated. Note that all authors have used a known generalization of Fibonacci and Lucas numbers and have placed these numbers in coefficients of the basis vectors similar to the format given by Horadam and Iyer. In this study, we present an another perspective to the Fibonacci and Lucas quaternions. According to our approach, we define a new classes of quaternions whose the coefficients are arbitrarily selected from these splendid integers. The outline of this paper is as follows: In Section 2, we introduce the unrestricted Fibonacci and Lucas quaternions and give Binet's formulas and the generating functions for these quaternions. Furthermore, we present certain special identities such as d'Ocagne's identity and Catalan's identity; in Section 3, we display many fundamental properties and some sum formulas for these quaternion families.

2. Main results

Here, we present our definitions, some concepts and results. First of all, we give a definition in the following.

Definition 2.1. *Let p, r and s be arbitrary integers. Hence, nth unrestricted Fibonacci and Lucas quaternions are given by the relations*

$$\mathscr{F}_n^{(p,r,s)} := F_n + iF_{n+p} + jF_{n+r} + kF_{n+s} \tag{2.1}$$

and

$$\mathscr{L}_n^{(p,r,s)} := L_n + iL_{n+p} + jL_{n+r} + kL_{n+s}, \tag{2.2}$$

respectively.

According to our definitions, we have the following special cases:

- For $p=r=s=-n$, the usual Fibonacci numbers are obtained:

$$\mathscr{F}_n^{(-n,-n,-n)} = F_n$$

- For $p=1$ and $r=s=-n$, the Gaussian Fibonacci numbers are obtained:

$$\mathscr{F}_n^{(1,-n,-n)} = F_n + iF_{n+1}$$

- For $p=1$, $r=2$ and $s=3$, the well-known Fibonacci and Lucas quaternions are obtained:

$$\mathscr{F}_n^{(1,2,3)} = Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

$$\mathscr{L}_n^{(1,2,3)} = T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$$

Taking Eqs. (2.1) and (2.2) into account, we directly obtain

$$\mathscr{F}_n^{(p,r,s)} = \mathscr{F}_{n-1}^{(p,r,s)} + \mathscr{F}_{n-2}^{(p,r,s)} \tag{2.3}$$

and

$$\mathscr{L}_n^{(p,r,s)} = \mathscr{L}_{n-1}^{(p,r,s)} + \mathscr{L}_{n-2}^{(p,r,s)}, \tag{2.4}$$

by the recurrence relations of Fibonacci and Lucas numbers respectively.

To make it easier to present and prove the results of the rest of paper, we now present next theorem that states Binet's formulas for the unrestricted Fibonacci and Lucas quaternions.

Theorem 2.2. ***(Binet's Formula)*** *Let n be an integer. Then the Binet's formulas of the unrestricted Fibonacci and Lucas quaternions are*

$$\mathscr{F}_n^{(p,r,s)} = \frac{\breve{\alpha}\alpha^n - \breve{\beta}\beta^n}{\alpha - \beta} \tag{2.5}$$

and

$$\mathscr{L}_n^{(p,r,s)} = \breve{\alpha}\alpha^n + \breve{\beta}\beta^n, \tag{2.6}$$

where $\breve{\alpha} = 1 + i\alpha^p + j\alpha^r + k\alpha^s$ *and* $\breve{\beta} = 1 + i\beta^p + j\beta^r + k\beta^s$.

Proof. Applying Binet's formulas of the classic Fibonacci numbers to the definition of the unrestricted Fibonacci quaternions, we obtain

$$\begin{aligned}
\mathscr{F}_n^{(p,r,s)} &= F_n + iF_{n+p} + jF_{n+r} + kF_{n+s} \\
&= \frac{1}{\alpha-\beta}\Big(\alpha^n - \beta^n + i(\alpha^{n+p} - \beta^{n+p}) + j(\alpha^{n+r} - \beta^{n+r}) + k(\alpha^{n+s} - \beta^{n+s})\Big) \\
&= \frac{1}{\alpha-\beta}\Big(\alpha^n(1 + i\alpha^p + j\alpha^r + k\alpha^s) - \beta^n(1 + i\beta^p + j\beta^r + k\beta^s)\Big),
\end{aligned}$$

and the last equation gives Eq. (2.5). The other can be proved similarly. □

$\cdot$	$\breve{\alpha}$	$\breve{\beta}$	$\breve{\alpha}^*$	$\breve{\beta}^*$
$\breve{\alpha}$	$2\breve{\alpha}-N(\breve{\alpha})$	$B+\sqrt{5}C$	$N(\breve{\alpha})$	$A-\sqrt{5}C+2\gamma$
$\breve{\beta}$	$B-\sqrt{5}C$	$2\breve{\beta}-N\left(\breve{\beta}\right)$	$-A+\sqrt{5}C$	$N\left(\breve{\beta}\right)$
$\breve{\alpha}^*$	$N(\breve{\alpha})$	$-A-\sqrt{5}C$	$2\breve{\alpha}^*-N(\breve{\alpha})$	$-B+\sqrt{5}C-2\gamma+4$
$\breve{\beta}^*$	$A+\sqrt{5}C+2\gamma$	$N\left(\breve{\beta}\right)$	$-B-\sqrt{5}C-2\gamma+4$	$2\breve{\beta}^*-N\left(\breve{\beta}\right)$

Table 1: The multiplicative properties of $\breve{\alpha}$ and $\breve{\beta}$

Table 1 displays important features corresponding to the multiplication of $\breve{\alpha}$ and $\breve{\beta}$. They will undertake important tasks in the process of proofing the next theorems. Note that each of them can easily be proven by certain elementary operations and we omit the details. In Table 1, the following notations are used:

$$A=\sqrt{5}\mathscr{F}_0^{(p,r,s)}-\gamma,$$

$$B=\mathscr{L}_0^{(p,r,s)}-\gamma,$$

$$C=i(-1)^s F_{r-s}+j(-1)^p F_{s-p}+k(-1)^r F_{p-r}$$

and

$$\gamma=1+(-1)^p+(-1)^r+(-1)^s.$$

Note that $r-s$, $s-p$, and $p-r$ may be positive or negative.

The next theorem gives d'Ocagne's identities for the considered quaternions.

Theorem 2.3. *(**d'Ocagne's identity**) Let m and n be any integers. Hence, we have*

$$\mathscr{F}_m^{(p,r,s)}\mathscr{F}_{n+1}^{(p,r,s)}-\mathscr{F}_{m+1}^{(p,r,s)}\mathscr{F}_n^{(p,r,s)}=(-1)^n(BF_{m-n}+CL_{m-n})$$

and

$$\mathscr{L}_m^{(p,r,s)}\mathscr{L}_{n+1}^{(p,r,s)}-\mathscr{L}_{m+1}^{(p,r,s)}\mathscr{L}_n^{(p,r,s)}=-5(-1)^n(BF_{m-n}+CL_{m-n}).$$

Proof. From the Binet's formula in (2.5), we have

$$\begin{aligned}\mathscr{F}_m^{(p,r,s)}\mathscr{F}_{n+1}^{(p,r,s)}-\mathscr{F}_{m+1}^{(p,r,s)}\mathscr{F}_n^{(p,r,s)} &= \frac{1}{5}\left[\left(\breve{\alpha}\alpha^m-\breve{\beta}\beta^m\right)\left(\breve{\alpha}\alpha^{n+1}-\breve{\beta}\beta^{n+1}\right)-\left(\breve{\alpha}\alpha^{m+1}-\breve{\beta}\beta^{m+1}\right)\left(\breve{\alpha}\alpha^n-\breve{\beta}\beta^n\right)\right]\\ &= \frac{\sqrt{5}}{5}(-1)^n\left(\breve{\alpha}\breve{\beta}\alpha^{m-n}-\breve{\beta}\breve{\alpha}\beta^{m-n}\right)\end{aligned}$$

and by Table 1,

$$\begin{aligned}\mathscr{F}_m^{(p,r,s)}\mathscr{F}_{n+1}^{(p,r,s)}-\mathscr{F}_{m+1}^{(p,r,s)}\mathscr{F}_n^{(p,r,s)} &= \frac{\sqrt{5}}{5}(-1)^n\left[(B+C\sqrt{5})\alpha^{m-n}-(B-C\sqrt{5})\beta^{m-n}\right]\\ &= \frac{\sqrt{5}}{5}(-1)^n\left[B(\alpha^{m-n}-\beta^{m-n})+C\sqrt{5}(\alpha^{m-n}+\beta^{m-n})\right].\end{aligned}$$

We obtain the first identity from the last equation. Repeating the same procedure, the second identity can be obtained. □

Now we give Catalan's identities of the unrestricted Fibonacci and Lucas quaternions.

Theorem 2.4. *(**Catalan's identity**) For any integers m and n, we have*

$$\mathscr{F}_{m+n}^{(p,r,s)}\mathscr{F}_{m-n}^{(p,r,s)}-\left[\mathscr{F}_{m}^{(p,r,s)}\right]^2=(-1)^{m+n+1}F_n\left(BF_n+CL_n\right)$$

and

$$\mathscr{L}_{m+n}^{(p,r,s)}\mathscr{L}_{m-n}^{(p,r,s)}-\left[\mathscr{L}_{m}^{(p,r,s)}\right]^2=5(-1)^{m+n}F_n\left(BF_n+CL_n\right).$$

Proof. By Eq. (2.5) and Table 1, we can write

$$\begin{aligned}
\mathscr{F}_{m+n}^{(p,r,s)}\mathscr{F}_{m-n}^{(p,r,s)}-\left[\mathscr{F}_{m}^{(p,r,s)}\right]^2 &= \frac{1}{5}\left[(\breve{\alpha}\alpha^{m+n}-\breve{\beta}\beta^{m+n})(\breve{\alpha}\alpha^{m-n}-\breve{\beta}\beta^{m-n})-(\breve{\alpha}\alpha^{m}-\breve{\beta}\beta^{m})^2\right]\\
&= \frac{1}{5}\left[((-1)^{m-n+1}(\breve{\alpha}\breve{\beta}\alpha^{2n}+\breve{\beta}\breve{\alpha}\beta^{2n})+(-1)^m 2C\right]\\
&= \frac{1}{5}\left[(-1)^{m+n-1}\left((B+\sqrt{5}C)\alpha^{2n}+(B-\sqrt{5}C)\beta^{2n}\right)+(-1)^m B\right]\\
&= \frac{1}{5}\left[(-1)^{m+n-1}\left(B(\alpha^{2n}+\beta^{2n})+5C\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)\right)+(-1)^m B\right]\\
&= \frac{1}{5}\left[(-1)^{m+n-1}\left(BL_{2n}+5CF_{2n}\right)+(-1)^m B\right].
\end{aligned}$$

Substituting the identities $5F_n^2=L_{2n}-(-1)^n$ ([12, p.42]) and $F_{2n}=F_nL_n$ ([12, p.14]) give the desired result. Similarly, the second identity can be obtained. □

For the case $n=1$ in Theorem 2.4, we attain Cassini's identities, which are given in the following.

Corollary 2.5. *(**Cassini's identity**) For any integer m, we have*

$$\mathscr{F}_{m+1}^{(p,r,s)}\mathscr{F}_{m-1}^{(p,r,s)}-\left[\mathscr{F}_{m}^{(p,r,s)}\right]^2=(-1)^m(B+C)$$

and

$$\mathscr{L}_{m+1}^{(p,r,s)}\mathscr{L}_{m-1}^{(p,r,s)}-\left[\mathscr{L}_{m}^{(p,r,s)}\right]^2=-5(-1)^m(B+C).$$

Note that the above identities can be re-written for the usual forms of the Fibonacci and Lucas quaternions in the case $(p,r,s)=(1,2,3)$. In this case, we can summarize them as follows:

- D'Ocagne's identities are as follows:

$$Q_mQ_{n+1}-Q_{m+1}Q_n=(-1)^n\left[T_0F_{m-n}+(-Q_0+3k)L_{m-n}\right]$$

$$K_mK_{n+1}-K_{m+1}K_n=-5(-1)^n\left[T_0F_{m-n}+(-Q_0+3k)L_{m-n}\right]$$

- Catalan's identities are as follows:

$$Q_{m+n}Q_{m-n}-Q_m^2=(-1)^mF_{-n}\left[F_nT_0+(-Q_0+3k)L_n\right]$$

$$K_{m+n}K_{m-n}-K_m^2=-5(-1)^mF_{-n}\left[F_nT_0+(-Q_0+3k)L_n\right]$$

- Cassini's identities are as follows:

$$Q_{m+1}Q_{m-1} - Q_m^2 = (-1)^m(Q_{-1} + 3k)$$

$$K_{m+1}K_{m-1} - K_m^2 = -5(-1)^m(Q_{-1} + 3k)$$

The concept of Generating Function is so important research topic since that is helpful tool to solve linear homogeneous recurrence relations with constant coefficients. Here, we investigate both the ordinary generating functions and the exponential generating functions associated with our generalized quaternions. To do this, we introduce the following functions:

$$G_{\mathscr{F}}(x) = \sum_{n=0}^{\infty} \mathscr{F}_n^{(p,r,s)} x^n,$$

$$G_{\mathscr{L}}(x) = \sum_{n=0}^{\infty} \mathscr{L}_n^{(p,r,s)} x^n,$$

$$E_{\mathscr{F}}(x) = \sum_{n=0}^{\infty} \mathscr{F}_n^{(p,r,s)} \frac{x^n}{n!} \tag{2.7}$$

and

$$E_{\mathscr{L}}(x) = \sum_{n=0}^{\infty} \mathscr{L}_n^{(p,r,s)} \frac{x^n}{n!}.$$

Hence, we present another main results of the current paper.

Theorem 2.6. *The generating functions for the unrestricted Fibonacci and Lucas quaternions are*

$$G_{\mathscr{F}}(x) = \frac{\mathscr{F}_0^{(p,r,s)} + \mathscr{F}_{-1}^{(p,r,s)} x}{1 - x - x^2} \tag{2.8}$$

and

$$G_{\mathscr{L}}(x) = \frac{\mathscr{L}_0^{(p,r,s)} + \mathscr{L}_{-1}^{(p,r,s)} x}{1 - x - x^2}. \tag{2.9}$$

Proof. Substituting Eq. (2.3) into Eq. (2.8) and Eq. (2.4) into Eq. (2.9), the results are satisfied. So, this completes the proof. □

Theorem 2.7. *The exponential generating functions for the unrestricted Fibonacci and Lucas quaternions are*

$$E_{\mathscr{F}}(x) = \frac{\breve{\alpha} e^{\alpha x} - \breve{\beta} e^{\beta x}}{\alpha - \beta}$$

and

$$E_{\mathscr{L}}(x) = \breve{\alpha} e^{\alpha x} + \breve{\beta} e^{\beta x},$$

where e is the famous Euler's number.

Proof. Substituting Eq. (2.5) into Eq. (2.7) leads to

$$E_{\mathscr{F}}(x) = \sum_{n=0}^{\infty} \mathscr{F}_n^{(p,r,s)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{\breve{\alpha}\alpha^n - \breve{\beta}\beta^n}{\alpha - \beta} \frac{x^n}{n!} = \frac{1}{\alpha - \beta} \left(\breve{\alpha} \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} - \breve{\beta} \sum_{n=0}^{\infty} \frac{(\beta x)^n}{n!} \right).$$

Considering MacLaurin series of an exponential function, the result follows. The second identity is demonstrated similarly. □

3. More features

In this section, we present many properties for the unrestricted Fibonacci and Lucas quaternions and some sum formulas of them. The next theorem present these identities.

Theorem 3.1. *Let m,n and t be any integers. Then,*

$$\mathscr{F}_n^{(p+1,r+1,s+1)} = \mathscr{F}_n^{(p,r,s)} + \mathscr{F}_{n-1}^{(p,r,s)} - F_{n-1}, \tag{3.1}$$

$$\mathscr{L}_n^{(p+1,r+1,s+1)} = \mathscr{L}_n^{(p,r,s)} + \mathscr{L}_{n-1}^{(p,r,s)} - L_{n-1},$$

$$\mathscr{F}_{n+1}^{(p,r,s)} = \mathscr{F}_n^{(p+1,r+1,s+1)} + F_{n-1},$$

$$\mathscr{L}_{n+1}^{(p,r,s)} = \mathscr{L}_n^{(p+1,r+1,s+1)} + L_{n-1},$$

$$\mathscr{L}_n^{(p,r,s)} = \mathscr{F}_{n-1}^{(p,r,s)} + \mathscr{F}_{n+1}^{(p,r,s)},$$

$$\mathscr{L}_{m+n}^{(p,r,s)} \mathscr{F}_{m+t}^{(p,r,s)} - \mathscr{L}_{m+t}^{(p,r,s)} \mathscr{F}_{m+n}^{(p,r,s)} = 2(-1)^{m+n} B \mathscr{F}_{t-n}^{(p,r,s)},$$

$$\mathscr{F}_{m+n}^{(p,r,s)} + (-1)^n \mathscr{F}_{m-n}^{(p,r,s)} = \mathscr{F}_m^{(p,r,s)} L_n,$$

$$\mathscr{F}_m^{(p,r,s)} \mathscr{L}_n^{(p,r,s)} - \mathscr{L}_n^{(p,r,s)} \mathscr{F}_m^{(p,r,s)} = 2(-1)^m C L_{n-m},$$

$$\mathscr{F}_m^{(p,r,s)} \mathscr{L}_n^{(p,r,s)} - \mathscr{L}_m^{(p,r,s)} \mathscr{F}_n^{(p,r,s)} = 2(-1)^n (B F_{m-n} + C L_{m-n}),$$

$$\mathscr{F}_n^{(p,r,s)} \mathscr{F}_m^{(p,r,s)} - \mathscr{F}_m^{(p,r,s)} \mathscr{F}_n^{(p,r,s)} = 2(-1)^{m+1} C F_{n-m},$$

$$\mathscr{L}_n^{(p,r,s)} \mathscr{L}_m^{(p,r,s)} - \mathscr{L}_m^{(p,r,s)} \mathscr{L}_n^{(p,r,s)} = 10(-1)^m C F_{n-m},$$

$$\mathscr{F}_{m+n}^{(p,r,s)} \mathscr{F}_{m+n} - \mathscr{F}_{m-n}^{(p,r,s)} \mathscr{F}_{m-n} = \mathscr{F}_{2m}^{(p,r,s)} F_{2n},$$

$$\mathscr{L}_{m+n}^{(p,r,s)} L_{m+n} - \mathscr{L}_{m-n}^{(p,r,s)} L_{m-n} = 5 \mathscr{F}_{2m}^{(p,r,s)} F_{2n}, \tag{3.2}$$

$$\mathscr{F}_{m+n}^{(p,r,s)} L_{m+n} + \mathscr{F}_{m-n}^{(p,r,s)} L_{m-n} = \mathscr{F}_{2m}^{(p,r,s)} L_{2n} + 2(-1)^{m+n} \mathscr{F}_0^{(p,r,s)},$$

$$\mathscr{L}_{m+n}^{(p,r,s)} L_{m+n} + \mathscr{L}_{m-n}^{(p,r,s)} L_{m-n} = \mathscr{L}_{2m}^{(p,r,s)} L_{2n} + 2(-1)^{m+n} \mathscr{L}_0^{(p,r,s)},$$

$$5\left[\mathscr{F}_m^{(p,r,s)}\right]^2 - \left[\mathscr{L}_m^{(p,r,s)}\right]^2 = 4(-1)^{m+1} B,$$

$$\mathscr{F}_{m+n}^{(p,r,s)} + (-1)^n \mathscr{F}_{m-n}^{(p,r,s)} = \mathscr{F}_m^{(p,r,s)} L_n,$$

$$\mathscr{L}_{m+n}^{(p,r,s)} + (-1)^n \mathscr{L}_{m-n}^{(p,r,s)} = \mathscr{L}_m^{(p,r,s)} L_n$$

and

$$\mathscr{F}_{2m}^{(p,r,s)} = F_{m+1} \mathscr{F}_m^{(p,r,s)} + F_m \mathscr{F}_{m-1}^{(p,r,s)}.$$

Proof. To reduce the volume of the current paper, we give some proofs of the above identities. Since the proofs of the first six identities are done in the same way, we only prove Eq. (3.1). By the definition of unrestricted Fibonacci quaternions, we can write

$$\begin{aligned}
\mathscr{F}_n^{(p+1,r+1,s+1)} &= F_n + iF_{n+p+1} + jF_{n+r+1} + kF_{n+s+1} \\
&= F_n + i(F_{n+p} + F_{n+p-1}) + j(F_{n+r} + F_{n+r-1}) + k(F_{n+s} + F_{n+s-1}) \\
&= (F_n + iF_{n+p} + jF_{n+r} + kF_{n+s}) + (F_{n-1} + iF_{n+p-1} + jF_{n+r-1} + kF_{n+s-1}) - F_{n-1}.
\end{aligned}$$

Other proofs are based on Binet's formulas of the corresponding quaternions. As an example, we show that (3.2) holds. Hence, by employing Eq. (2.6) we obtain

$$\begin{aligned}
\mathscr{L}_{m+n}^{(p,r,s)} L_{m+n} - \mathscr{L}_{m-n}^{(p,r,s)} L_{m-n} &= \left(\breve{\alpha}\alpha^{m+n} + \breve{\beta}\beta^{m+n}\right)\left(\alpha^{m+n} + \beta^{m+n}\right) - \left(\breve{\alpha}\alpha^{m-n} + \breve{\beta}\beta^{m-n}\right)\left(\alpha^{m-n} + \beta^{m-n}\right) \\
&= \left(\breve{\alpha}\alpha^{2m+2n} + \breve{\beta}\beta^{2m+2n} - \breve{\alpha}\alpha^{2m-2n} - \breve{\beta}\beta^{2m-2n}\right) \\
&= \left(\breve{\alpha}\alpha^{2m+2n} + \breve{\beta}\beta^{2m+2n} - \breve{\alpha}\alpha^{2m}\beta^{2n} - \breve{\beta}\alpha^{2n}\beta^{2m}\right) \\
&= (\alpha-\beta)^2 \left(\frac{\breve{\alpha}\alpha^{2m} - \breve{\beta}\beta^{2m}}{\alpha-\beta}\right)\left(\frac{\alpha^{2n} - \beta^{2n}}{\alpha-\beta}\right) \\
&= 5\mathscr{F}_{2m}^{(p,r,s)} F_{2n}.
\end{aligned}$$

So, the proof is completed. □

Now, we list sum formulas for the considered quaternions in the following theorem.

Theorem 3.2. *The following summation formulas hold for any integer n.*

$$\sum_{t=0}^{n} \mathscr{F}_t^{(p,r,s)} = \mathscr{F}_{n+2}^{(p,r,s)} - \mathscr{F}_1^{(p,r,s)}, \tag{3.3}$$

$$\sum_{t=0}^{n} \mathscr{L}_t^{(p,r,s)} = \mathscr{L}_{n+2}^{(p,r,s)} - \mathscr{L}_1^{(p,r,s)},$$

$$\sum_{t=0}^{n} \mathscr{F}_{2t-1}^{(p,r,s)} = \mathscr{F}_{2n}^{(p,r,s)} - \mathscr{F}_0^{(p,r,s)},$$

$$\sum_{t=0}^{n} \mathscr{L}_{2t-1}^{(p,r,s)} = \mathscr{L}_{2n}^{(p,r,s)} - \mathscr{L}_0^{(p,r,s)},$$

$$\sum_{t=0}^{n} \mathscr{F}_{2t}^{(p,r,s)} = \mathscr{F}_{2n+1}^{(p,r,s)} - \mathscr{F}_{-1}^{(p,r,s)},$$

$$\sum_{t=0}^{n} \mathscr{L}_{2t}^{(p,r,s)} = \mathscr{L}_{2n+1}^{(p,r,s)} - \mathscr{L}_{-1}^{(p,r,s)},$$

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{F}_t^{(p,r,s)} = \mathscr{F}_{2n}^{(p,r,s)} \tag{3.4}$$

and

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{L}_t^{(p,r,s)} = \mathscr{L}_{2n}^{(p,r,s)}.$$

Proof. We will prove some of the above identities again. Let us consider $a_t = \mathscr{F}_{t+2}^{(p,r,s)} - \mathscr{F}_{1}^{(p,r,s)}$. Hence, by the definition of the unrestricted Fibonacci quaternions, we obtain

$$a_t - a_{t-1} = \mathscr{F}_t^{(p,r,s)}.$$

Applying the idea of creative telescoping [13] to Eq. (3.3), we conclude

$$\sum_{t=0}^{n} \mathscr{F}_t^{(p,r,s)} = \sum_{t=0}^{n} (a_t - a_{t-1}) = a_n - a_{-1},$$

and since $a_{-1} = 0$, Eq. (3.3) is attained. Proceeding as in the previous proof, we can obtain the other identities except last two identities.

Now, substituting Eq. (2.5) to Eq. (3.4) leads to

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{F}_t^{(p,r,s)} = \sum_{t=0}^{n} \binom{n}{t} \frac{\breve{\alpha}\alpha^n - \breve{\beta}\beta^n}{\alpha - \beta} = \frac{1}{\alpha - \beta} \left(\breve{\alpha} \sum_{t=0}^{n} \binom{n}{t} \alpha^n - \breve{\beta} \sum_{t=0}^{n} \binom{n}{t} \beta^n \right),$$

and considering the formal expression of the Binomial Theorem, we obtain

$$\sum_{t=0}^{n} \binom{n}{t} \mathscr{F}_t^{(p,r,s)} = \frac{\breve{\alpha}\,(\alpha+1)^n - \breve{\beta}\,(\beta+1)^n}{\alpha - \beta}.$$

Using $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, the proof of Eq. (3.4) is completed. The last equation can also be found in a similar way. □

4. Conclusions

In this study, we presented other generalizations for the usual Fibonacci and Lucas quaternions and gave many interesting properties of these definitions. In particular, the Binet's formulas, the generating function, some explicit formulas, and special identities such as d'Ocagne's Identities were obtained. Moreover, the examples regarding the reduced cases for our generalizations, from the quaternion forms to the usual integer sequence ones, were investigated. In addition to those, we considered sum formulas for our generalizations including the even and odd subscripts.

We think that the inferences of the paper can be used in several practical applications in applied sciences, e.g. control and system theory, and neural network, etc.

Acknowledgement

The authors would like to declare the financial support provided by the Research Fund of Kastamonu University under project number KÜBAP-01/2017-1.

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 10-16
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.811058

Fibonacci Elliptic Biquaternions[1]

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Article Info

Keywords: *Elliptic biquaternions, Fibonacci numbers, Matrix representations, Quaternions*
2010 AMS: *11B39, 11R52*
Received: *15 October 2020*
Accepted: *6 January 2021*
Available online: *29 January 2021*

Abstract

A. F. Horadam defined the complex Fibonacci numbers and Fibonacci quaternions in the middle of the 20th century. Half a century later, S. Halıcı introduced the complex Fibonacci quaternions by inspiring from these definitions and discussed some properties of them. Recently, the elliptic biquaternions, which are generalized form of the complex and real quaternions, have been presented. In this study, we introduce the set of Fibonacci elliptic biquaternions that includes the set of complex Fibonacci quaternions as a special case, and investigate some properties of Fibonacci elliptic biquaternions. Furthermore, we give the Binet formula and Cassini's identity in terms of Fibonacci elliptic biquaternions. Finally, we give elliptic and real matrix representations of Fibonacci elliptic biquaternions.

1. Introduction

Real quaternions and complex quaternions were introduced by Hamilton in 1843 [1] and 1853 [2], respectively. The set of real quaternions and the set of complex quaternions are represented as

$$H = \{w = w_0 + w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k} :\ w_0, w_1, w_2, w_3 \in \mathbb{R}\} \tag{1.1}$$

and

$$H_{\mathbb{C}} = \{W = W_0 + W_1\mathbf{i} + W_2\mathbf{j} + W_3\mathbf{k} :\ W_0, W_1, W_2, W_3 \in \mathbb{C}\}$$

respectively where the quaternionic units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\,,\ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\,,\ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}\,,\ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

As a consequence of the representations given above, a complex quaternion W can be written as

$$W = w + iw^*,\ \ i^2 = -1 \tag{1.2}$$

where w and w^* are real quaternions.

[1]This article is the completed version of the paper titled "Fibonacci Elliptic Biquaternions", which was presented as an virtual presentation at the 12th International Conference on Clifford Algebras and Their Applications in Mathematical Physics. The conference was held as an online conference between 3-7 August 2020.

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In 1963, $n-th$ Fibonacci quaternion

$$W_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}$$

was introduced by changing the components in equation (1.1) with the consecutive Fibonacci numbers by Horadam. Also, Horadam similarly defined the $n-th$ Lucas quaternion

$$T_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k}.$$

Here L_n and F_n are $n-th$ Lucas number and $n-th$ Fibonacci number, respectively [3].

On the other hand, the complex Fibonacci numbers were defined as

$$C_n = F_n + iF_{n+1}, \quad i^2 = -1$$

in [3]. Then, Halıcı expanded this definition to the complex quaternions with the name "$n-th$ complex Fibonacci quaternion". Halıcı gave this definition by changing the real quaternions w and w^* in the equation (1.2) with consecutive Fibonacci quaternions W_n and W_{n+1} as in the following [4]:

$$R_n = W_n + iW_{n+1}, \quad i^2 = -1.$$

There are many studies on Fibonacci quaternions and complex Fibonacci quaternions in the literature. The readers are referred to the studies [3]-[8] for these topics.

Recently, Özen and Tosun have expressed elliptic biquaternions comprising the complex and real quaternions [9]. The set of them is given as follows:

$$H\mathbb{C}_p = \left\{U = U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \,:\, U_0, U_1, U_2, U_3 \in \mathbb{C}_p\right\}$$

where $\mathbb{C}_p = \left\{u + Iv : u, v \in \mathbb{R}, I^2 = p, \, p \in \mathbb{R}^-\right\}$ indicates the set of elliptic numbers. The system of elliptic numbers is a one-parameter family of generalized complex number systems. The readers are referred to [10]-[14] for some interesting studies on the generalized complex numbers and elliptic numbers.

For any two elliptic biquaternions $U = U_0 + U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} \in H\mathbb{C}_p$ and $V = V_0 + V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k} \in H\mathbb{C}_p$, addition and scalar multiplication by $\lambda \in \mathbb{C}_p$ are given by

$$\begin{aligned} U + V &= (U_0 + V_0) + (U_1 + V_1)\,\mathbf{i} + (U_2 + V_2)\,\mathbf{j} + (U_3 + V_3)\,\mathbf{k} \\ \lambda U &= (\lambda U_0) + (\lambda U_1)\,\mathbf{i} + (\lambda U_2)\,\mathbf{j} + (\lambda U_3)\,\mathbf{k} \end{aligned}$$

and also, the multiplication of U and V is defined as in the following [9]:

$$\begin{aligned} UV &= [(U_0V_0) - (U_1V_1) - (U_2V_2) - (U_3V_3)] + [(U_0V_1) + (U_1V_0) + (U_2V_3) - (U_3V_2)]\,\mathbf{i} \\ &\quad + [(U_0V_2) - (U_1V_3) + (U_2V_0) + (U_3V_1)]\,\mathbf{j} + [(U_0V_3) + (U_1V_2) - (U_2V_1) + (U_3V_0)]\,\mathbf{k}. \end{aligned}$$

Moreover, the Hamiltonian, complex and total conjugates of U are as below:

$$\begin{aligned} \overline{U} &= U_0 - U_1\mathbf{i} - U_2\mathbf{j} - U_3\mathbf{k} \\ U^* &= U_0{}^* + U_1{}^*\mathbf{i} + U_2{}^*\mathbf{j} + U_3{}^*\mathbf{k} \\ U^\dagger &= U_0{}^* - U_1{}^*\mathbf{i} - U_2{}^*\mathbf{j} - U_3{}^*\mathbf{k} \end{aligned}$$

where superscript stars on U_0, U_1, U_2 and U_3 denote the usual complex conjugation. On the other hand, the semi-norm of U is defined as [9]:

$$N(U) = U_0{}^2 + U_1{}^2 + U_2{}^2 + U_3{}^2.$$

In the next section, we introduce Fibonacci elliptic biquaternions and give their some properties. In the last section, we give elliptic and real matrix representations of Fibonacci elliptic biquaternions.

2. Fibonacci elliptic biquaternions and their some properties

Thanks to [15], we know that $p-$complex Fibonacci numbers are given as in the following:

$$(C_p)_n = F_n + IF_{n+1}, \; I^2 = p \in \mathbb{R}$$

where F_n is the $n-th$ Fibonacci number. Here, we consider the case $p\in\mathbb{R}^-$ and call these numbers Fibonacci elliptic numbers. Thus, $n-th$ Fibonacci elliptic number is defined as

$$(C_p)_n = F_n + IF_{n+1},\ I^2 = p\in\mathbb{R}^-.$$

By following the method given in [4], we expand this definition to the elliptic biquaternions with the name "$n-th$ Fibonacci elliptic biquaternion". That is, $n-th$ Fibonacci elliptic biquaternion is given by

$$(U_p)_n = W_n + IW_{n+1},\quad I^2 = p\in\mathbb{R}^-$$

where

$$W_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}$$

and

$$W_{n+1} = F_{n+1} + F_{n+2}\mathbf{i} + F_{n+3}\mathbf{j} + F_{n+4}\mathbf{k}$$

are consecutive Fibonacci quaternions. Therefore, $(U_p)_n$ can be written as follows:

$$(U_p)_n = (F_n + IF_{n+1}) + (F_{n+1} + IF_{n+2})\mathbf{i} + (F_{n+2} + IF_{n+3})\mathbf{j} + (F_{n+3} + IF_{n+4})\mathbf{k},\quad I^2 = p\in\mathbb{R}^-. \tag{2.1}$$

As a consequence of the definition of Fibonacci elliptic numbers and the equation (2.1), $(U_p)_n$ can also be given in the following form

$$(U_p)_n = (C_p)_n + (C_p)_{n+1}\mathbf{i} + (C_p)_{n+2}\mathbf{j} + (C_p)_{n+3}\mathbf{k},\quad I^2 = p\in\mathbb{R}^-.$$

Thus, $(U_p)_n$ includes a scalar part

$$S\left((U_p)_n\right) = (C_p)_n$$

and a vectorial part

$$V\left((U_p)_n\right) = (C_p)_{n+1}\mathbf{i} + (C_p)_{n+2}\mathbf{j} + (C_p)_{n+3}\mathbf{k}.$$

The Hamiltonian, complex and total conjugates of $(U_p)_n$ can be found as

$$\begin{aligned}
\overline{(U_p)_n} &= (C_p)_n - (C_p)_{n+1}\mathbf{i} - (C_p)_{n+2}\mathbf{j} - (C_p)_{n+3}\mathbf{k}\\
(U_p)_n^* &= (C_p)_n^* + (C_p)_{n+1}^*\mathbf{i} + (C_p)_{n+2}^*\mathbf{j} + (C_p)_{n+3}^*\mathbf{k}\\
(U_p)_n^\dagger &= (C_p)_n^* - (C_p)_{n+1}^*\mathbf{i} - (C_p)_{n+2}^*\mathbf{j} - (C_p)_{n+3}^*\mathbf{k}
\end{aligned}$$

where $(C_p)_n^* = F_n - IF_{n+1}$. From here, it can be easily seen that the following identities hold:

$$\begin{aligned}
(U_p)_n + \overline{(U_p)_n} &= 2(C_p)_n\\
(U_p)_n + (U_p)_n^* &= 2W_n.
\end{aligned}$$

The semi norm of $(U_p)_n$ can be given as in the following:

$$N\left((U_p)_n\right) = (C_p)_n^2 + (C_p)_{n+1}^2 + (C_p)_{n+2}^2 + (C_p)_{n+3}^2.$$

If we use the identities

$$F_{n-1} + F_{n+1} = L_n,\ n\in\mathbb{Z}^+$$

and

$$F_n^2 + F_{n+1}^2 = F_{2n+1},\ n\in\mathbb{Z}$$

given in [16], we get

$$N\left((U_p)_n\right) = (F_{2n+1} + pF_{2n+3} + F_{2n+5} + pF_{2n+7}) + 2I\left(F_{n+1}L_{n+1} + F_{n+3}L_{n+3}\right).$$

Note that we will show the set of Fibonacci elliptic biquaternions with $FH\mathbb{C}_p$ throughout the paper. For negative indices, the Fibonacci elliptic biquaternions can be given as in the following lemma.

Lemma 2.1. *For $(U_p)_n \in FH\mathbb{C}_p$, the equality*

$$(U_p)_{-n} = (-1)^n \left[(-F_n + IF_{n-1}) + (F_{n-1} - IF_{n-2})\mathbf{i} + (-F_{n-2} + IF_{n-3})\mathbf{j} + (F_{n-3} - IF_{n-4})\mathbf{k}\right]$$

is satisfied.

Proof. By considering the definition of the Fibonacci elliptic biquaternions and the equality $F_{-n} = (-1)^{n+1} F_n$ given in [17], we can write the followings:

$$\begin{aligned}
(U_p)_{-n} &= (F_{-n} + IF_{-n+1}) + (F_{-n+1} + IF_{-n+2})\mathbf{i} + (F_{-n+2} + IF_{-n+3})\mathbf{j} + (F_{-n+3} + IF_{-n+4})\mathbf{k} \\
&= \left(F_{-n} + IF_{-(n-1)}\right) + \left(F_{-(n-1)} + IF_{-(n-2)}\right)\mathbf{i} + \left(F_{-(n-2)} + IF_{-(n-3)}\right)\mathbf{j} + \left(F_{-(n-3)} + IF_{-(n-4)}\right)\mathbf{k} \\
&= \left((-1)^{n+1} F_n + I(-1)^n F_{n-1}\right) + \left((-1)^n F_{n-1} + I(-1)^{n-1} F_{n-2}\right)\mathbf{i} \\
&\quad + \left((-1)^{n-1} F_{n-2} + I(-1)^{n-2} F_{n-3}\right)\mathbf{j} + \left((-1)^{n-2} F_{n-3} + I(-1)^{n-3} F_{n-4}\right)\mathbf{k} \\
&= (-1)^n \left[(-F_n + IF_{n-1}) + (F_{n-1} - IF_{n-2})\mathbf{i} + (-F_{n-2} + IF_{n-3})\mathbf{j} + (F_{n-3} - IF_{n-4})\mathbf{k}\right].
\end{aligned}$$

□

Now, we give the following theorem which reveals an essential relation between the Fibonacci numbers and Fibonacci elliptic biquaternions.

Theorem 2.2. *(Binet Formula) For $(U_p)_n \in FH\mathbb{C}_p$, Binet formula is given as*

$$(U_p)_n = (U_p)_1 F_n + (U_p)_0 F_{n-1}$$

where $n \geq 0$.

Proof. By direct calculation, the followings can be written easily:

$$\begin{aligned}
(U_p)_1 F_n + (U_p)_0 F_{n-1} &= \left[(F_1 + IF_2) + (F_2 + IF_3)\mathbf{i} + (F_3 + IF_4)\mathbf{j} + (F_4 + IF_5)\mathbf{k}\right] F_n \\
&\quad + \left[(F_0 + IF_1) + (F_1 + IF_2)\mathbf{i} + (F_2 + IF_3)\mathbf{j} + (F_3 + IF_4)\mathbf{k}\right] F_{n-1} \\
&= F_1F_n + F_2F_n\mathbf{i} + F_3F_n\mathbf{j} + F_4F_n\mathbf{k} + I\left(F_2F_n + F_3F_n\mathbf{i} + F_4F_n\mathbf{j} + F_5F_n\mathbf{k}\right) \\
&\quad + F_0F_{n-1} + F_1F_{n-1}\mathbf{i} + F_2F_{n-1}\mathbf{j} + F_3F_{n-1}\mathbf{k} + I\left(F_1F_{n-1} + F_2F_{n-1}\mathbf{i} + F_3F_{n-1}\mathbf{j} + F_4F_{n-1}\mathbf{k}\right) \\
&= \left[(F_0F_{n-1} + F_1F_n) + I\left(F_1F_{n-1} + F_2F_n\right)\right] + \left[(F_1F_{n-1} + F_2F_n) + I\left(F_2F_{n-1} + F_3F_n\right)\right]\mathbf{i} \\
&\quad + \left[(F_2F_{n-1} + F_3F_n) + I(F_3F_{n-1} + F_4F_n)\right]\mathbf{j} + \left[(F_3F_{n-1} + F_4F_n) + I(F_4F_{n-1} + F_5F_n)\right]\mathbf{k}.
\end{aligned}$$

Using the identity

$$F_nF_m + F_{n+1}F_{m+1} = F_{m+n+1}, \;\; m, n \in \mathbb{Z}^+$$

given in [5] and the equalities

$$F_0F_{n-1} + F_1F_n = 0F_{n-1} + 1F_n = F_n$$

and

$$F_1F_{n-1} + F_2F_n = 1F_{n-1} + 1F_n = F_{n-1} + F_n = F_{n+1},$$

the proof is completed. □

Theorem 2.3. *(Cassini's Identity) For $(U_p)_n \in FH\mathbb{C}_p$, Cassini's identity is given as*

$$(U_p)_{n-1}(U_p)_{n+1} - (U_p)_n^2 = (-1)^n \left(2W_1 - 3\mathbf{k} - pF_1^2T_0 - pF_2W_0 + 3pF_2\right) + I\left(W_{n-1}W_{n+2} - W_nW_{n+1}\right)$$

where $n \geq 1$.

Proof. By direct calculation, we get

$$\begin{aligned}
(U_p)_{n-1}(U_p)_{n+1} - (U_p)_n^2 &= (W_{n-1} + IW_n)(W_{n+1} + IW_{n+2}) - (W_n + IW_{n+1})^2 \\
&= W_{n-1}W_{n+1} + pW_nW_{n+2} + I\left(W_nW_{n+1} + W_{n-1}W_{n+2}\right) - \left(W_n^2 + pW_{n+1}^2 + 2IW_nW_{n+1}\right) \\
&= W_{n-1}W_{n+1} - W_n^2 + p\left(W_nW_{n+2} - W_{n+1}^2\right) + I\left(W_{n-1}W_{n+2} - W_nW_{n+1}\right).
\end{aligned}$$

Using the identities

$$W_{n-1}W_{n+1}-W_n^2=(-1)^n\left(2W_1-3\mathbf{k}\right)$$

and

$$W_{n+1-r}W_{n+1+r}-W_{n+1}^2=(-1)^{n-r}\left[F_r^2T_0+F_{2r}\left(W_0-3r\right)\right]\quad (for\ r=1)$$

given in [5], the proof is completed. □

3. Matrix representations of Fibonacci elliptic biquaternions

In this section, elliptic and real matrix representations of Fibonacci elliptic biquaternions are given by emphasizing the isomorphisms which determine these matrix representations.

3.1. 2x2 and 4x4 elliptic matrix representations of Fibonacci elliptic biquaternions

Thanks to Özen and Tosun [18], we know that there is a faithful relation between the elliptic biquaternions and 2x2 elliptic matrices. Every elliptic biquaternion $U=U_0+U_1\mathbf{i}+U_2\mathbf{j}+U_3\mathbf{k}\in H\mathbb{C}_p$ has a 2x2 elliptic matrix representation

$$\sigma(U)=\begin{bmatrix} U_0+\frac{1}{\sqrt{|p|}}IU_1 & -U_2-\frac{1}{\sqrt{|p|}}IU_3 \\ U_2-\frac{1}{\sqrt{|p|}}IU_3 & U_0-\frac{1}{\sqrt{|p|}}IU_1 \end{bmatrix}$$

which is determined by means of the following linear isomorphism [18]

$$\begin{aligned} \sigma: H\mathbb{C}_p &\to M_{2\times 2}(\mathbb{C}_p) \\ U &\to \sigma(U)=\begin{bmatrix} U_0+\frac{1}{\sqrt{|p|}}IU_1 & -U_2-\frac{1}{\sqrt{|p|}}IU_3 \\ U_2-\frac{1}{\sqrt{|p|}}IU_3 & U_0-\frac{1}{\sqrt{|p|}}IU_1 \end{bmatrix}. \end{aligned}$$

Let us consider the restriction of this isomorphism to the set of Fibonacci elliptic biquaternions. Then we get the following isomorphism:

$$\begin{aligned} \sigma^*: FH\mathbb{C}_p &\to \sigma(FH\mathbb{C}_p)\subset M_{2\times 2}(\mathbb{C}_p) \\ (U_p)_n &\to \sigma^*\left((U_p)_n\right)=\begin{bmatrix} (C_p)_n+\frac{1}{\sqrt{|p|}}I(C_p)_{n+1} & -(C_p)_{n+2}-\frac{1}{\sqrt{|p|}}I(C_p)_{n+3} \\ (C_p)_{n+2}-\frac{1}{\sqrt{|p|}}I(C_p)_{n+3} & (C_p)_n-\frac{1}{\sqrt{|p|}}I(C_p)_{n+1} \end{bmatrix}. \end{aligned}$$

Thus, we can give the following definition by using the equality $I^2=p=-\sqrt{|p|}\sqrt{|p|}$.

Definition 3.1. *The matrix*

$$\begin{bmatrix} \left(F_n-\sqrt{|p|}F_{n+2}\right)+I\left(1+\frac{1}{\sqrt{|p|}}\right)F_{n+1} & \left(-F_{n+2}+\sqrt{|p|}F_{n+4}\right)-I\left(1+\frac{1}{\sqrt{|p|}}\right)F_{n+3} \\ \left(F_{n+2}+\sqrt{|p|}F_{n+4}\right)+I\left(1-\frac{1}{\sqrt{|p|}}\right)F_{n+3} & \left(F_n+\sqrt{|p|}F_{n+2}\right)+I\left(1-\frac{1}{\sqrt{|p|}}\right)F_{n+1} \end{bmatrix}$$

derived from $\sigma^*\left((U_p)_n\right)$ is called 2x2 elliptic matrix representation of $(U_p)_n$.

On the other hand, there is an isomorphism between the matrix space $M=\left\{\begin{bmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0 \end{bmatrix}: X_0,X_1,X_2,X_3\in\mathbb{C}_p\right\}$ and the elliptic biquaternion space $H\mathbb{C}_p$ [19]:

$$\begin{aligned} \gamma: H\mathbb{C}_p &\to M \\ U &\to \gamma(U)=\begin{bmatrix} U_0 & -U_1 & -U_2 & -U_3 \\ U_1 & U_0 & -U_3 & U_2 \\ U_2 & U_3 & U_0 & -U_1 \\ U_3 & -U_2 & U_1 & U_0 \end{bmatrix}. \end{aligned}$$

Similarly above, if this isomorphism is restricted to the set of Fibonacci elliptic biquaternions, it is not difficult to see the following isomorphism

$$\begin{aligned}\gamma^* : FH\mathbb{C}_p &\rightarrow \gamma(FH\mathbb{C}_p) \subset M\\ (U_p)_n \rightarrow \gamma^*\left((U_p)_n\right) &= \begin{bmatrix} (C_p)_n & -(C_p)_{n+1} & -(C_p)_{n+2} & -(C_p)_{n+3}\\ (C_p)_{n+1} & (C_p)_n & -(C_p)_{n+3} & (C_p)_{n+2}\\ (C_p)_{n+2} & (C_p)_{n+3} & (C_p)_n & -(C_p)_{n+1}\\ (C_p)_{n+3} & -(C_p)_{n+2} & (C_p)_{n+1} & (C_p)_n \end{bmatrix}.\end{aligned}$$

Then, we can give the following according to definition of $n-th$ Fibonacci elliptic number.

Definition 3.2. *The matrix*

$$\begin{bmatrix} F_n+IF_{n+1} & -F_{n+1}-IF_{n+2} & -F_{n+2}-IF_{n+3} & -F_{n+3}-IF_{n+4}\\ F_{n+1}+IF_{n+2} & F_n+IF_{n+1} & -F_{n+3}-IF_{n+4} & F_{n+2}+IF_{n+3}\\ F_{n+2}+IF_{n+3} & F_{n+3}+IF_{n+4} & F_n+IF_{n+1} & -F_{n+1}-IF_{n+2}\\ F_{n+3}+IF_{n+4} & -F_{n+2}-IF_{n+3} & F_{n+1}+IF_{n+2} & F_n+IF_{n+1} \end{bmatrix}$$

derived from $\gamma^*\left((U_p)_n\right)$ *is called 4x4 elliptic matrix representation of* $(U_p)_n$.

3.2. 8x8 real matrix representations of Fibonacci elliptic biquaternions

In the study [20], Özen and Tosun obtained 8x8 real matrix representation of an arbitrary elliptic biquaternion in the space $H\mathbb{C}_p$. However, the isomorphism that determines this representation was not emphasized with its domain and range in this study. So, we are not able to apply the method of restriction as in the previous subsection. Here we give particular importance to get an isomorphism whose domain is $FH\mathbb{C}_p$ and whose range is a special 8x8 real matrix set. Because of this, we need some preparation.

From [21], it is known that there is an isomorphism between the elliptic matrix set $M_{4\times4}(\mathbb{C}_p)$ and real matrix set $M^{\Omega}_{8\times8}(\mathbb{R})$ as in the following:

$$\begin{aligned}\psi \,:\, M_{4\times4}(\mathbb{C}_p) &\rightarrow M^{\Omega}_{8\times8}(\mathbb{R})\\ A = A_1 + IA_2 &\rightarrow \psi(A) = \begin{bmatrix} A_1 & -\sqrt{|p|}A_2\\ \sqrt{|p|}A_2 & A_1 \end{bmatrix}\end{aligned}$$

where $M^{\Omega}_{8\times8}(\mathbb{R}) = \left\{ \begin{bmatrix} G & -\sqrt{|p|}H\\ \sqrt{|p|}H & G \end{bmatrix} : G,H \in M_{4\times4}(\mathbb{R}) \right\}$.

Since $\gamma(FH\mathbb{C}_p) \subset M \subset M_{4\times4}(\mathbb{C}_p)$, we can restrict the isomorphism ψ to the set $\gamma(FH\mathbb{C}_p)$. If we do this, we have the isomorphism $\psi^* : \gamma(FH\mathbb{C}_p) \rightarrow \psi(\gamma(FH\mathbb{C}_p)) \subset M^{\Omega}_{8\times8}(\mathbb{R})$. To obtain 8x8 real matrix representations of Fibonacci elliptic biquaternions is the aim of us. To do so, we write $\gamma^*\left((U_p)_n\right)$ as follows:

$$\gamma^*\left((U_p)_n\right) = B_1(n) + IB_2(n)$$

where

$$B_1(n) = \begin{bmatrix} F_n & -F_{n+1} & -F_{n+2} & -F_{n+3}\\ F_{n+1} & F_n & -F_{n+3} & F_{n+2}\\ F_{n+2} & F_{n+3} & F_n & -F_{n+1}\\ F_{n+3} & -F_{n+2} & F_{n+1} & F_n \end{bmatrix} \in M_{4\times4}(\mathbb{R})$$

and

$$B_2(n) = \begin{bmatrix} F_{n+1} & -F_{n+2} & -F_{n+3} & -F_{n+4}\\ F_{n+2} & F_{n+1} & -F_{n+4} & F_{n+3}\\ F_{n+3} & F_{n+4} & F_{n+1} & -F_{n+2}\\ F_{n+4} & -F_{n+3} & F_{n+2} & F_{n+1} \end{bmatrix} \in M_{4\times4}(\mathbb{R}).$$

Then, the compound function $\delta = \psi^* \circ \gamma^*$ yields the following isomorphism:

$$\begin{aligned}\delta :\; FH\mathbb{C}_p &\rightarrow \psi(\gamma(FH\mathbb{C}_p)) \subset M^{\Omega}_{8\times8}(\mathbb{R})\\ (U_p)_n \rightarrow \delta\left((U_p)_n\right) &= \begin{bmatrix} B_1(n) & -\sqrt{|p|}B_2(n)\\ \sqrt{|p|}B_2(n) & B_1(n) \end{bmatrix}.\end{aligned}$$

Consequently, we can give the following definition.

Definition 3.3. *The real matrix*

$$\begin{bmatrix} F_n & -F_{n+1} & -F_{n+2} & -F_{n+3} & -\sqrt{|p|}F_{n+1} & \sqrt{|p|}F_{n+3} & \sqrt{|p|}F_{n+3} & \sqrt{|p|}F_{n+4} \\ F_{n+1} & F_n & -F_{n+3} & F_{n+2} & -\sqrt{|p|}F_{n+2} & \sqrt{|p|}F_{n+4} & \sqrt{|p|}F_{n+4} & -\sqrt{|p|}F_{n+3} \\ F_{n+2} & F_{n+3} & F_n & -F_{n+1} & -\sqrt{|p|}F_{n+3} & -\sqrt{|p|}F_{n+1} & -\sqrt{|p|}F_{n+1} & \sqrt{|p|}F_{n+2} \\ F_{n+3} & -F_{n+2} & F_{n+1} & F_n & -\sqrt{|p|}F_{n+4} & -\sqrt{|p|}F_{n+2} & -\sqrt{|p|}F_{n+2} & -\sqrt{|p|}F_{n+1} \\ \sqrt{|p|}F_{n+1} & -\sqrt{|p|}F_{n+2} & -\sqrt{|p|}F_{n+3} & -\sqrt{|p|}F_{n+4} & F_n & -F_{n+1} & -F_{n+2} & -F_{n+3} \\ \sqrt{|p|}F_{n+2} & \sqrt{|p|}F_{n+1} & -\sqrt{|p|}F_{n+4} & \sqrt{|p|}F_{n+3} & F_{n+1} & F_n & -F_{n+3} & F_{n+2} \\ \sqrt{|p|}F_{n+3} & \sqrt{|p|}F_{n+4} & \sqrt{|p|}F_{n+1} & -\sqrt{|p|}F_{n+2} & F_{n+2} & F_{n+3} & F_n & -F_{n+1} \\ \sqrt{|p|}F_{n+4} & -\sqrt{|p|}F_{n+3} & \sqrt{|p|}F_{n+2} & \sqrt{|p|}F_{n+1} & F_{n+3} & -F_{n+2} & F_{n+1} & F_n \end{bmatrix}$$

derived from $\delta\left((U_p)_n\right)$ *is called 8x8 real matrix representation of* $(U_p)_n$.

4. Conclusion

In this study, Fibonacci elliptic biquaternions and their some properties are introduced. Also, Binet formula and Cassini's identity are given in terms of Fibonacci elliptic biquaternions. Moreover, real and elliptic matrix representations of Fibonacci elliptic biquaternions are derived.

When $p = -1$, the set of elliptic numbers matches up with the set of complex numbers. In that case, the set of elliptic biquaternions is reduced to the set of complex quaternions. Therefore, Fibonacci elliptic biquaternions can be seen as generalized form of complex Fibonacci quaternions that take an important place in the literature.

The use of matrix techniques gives us many advantages in many areas of science. In this respect, this study can be seen as the first step of the future studies which will be presented by using the matrix representations of Fibonacci elliptic biquaternions.

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 17-24
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.831447

Faber Polynomial Expansion for a New Subclass of Bi-univalent Functions Endowed with (p,q) Calculus Operators

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Article Info

Keywords: *Bi-univalent functions, Faber polynomial expansion, $(p,q)-$calculus*
2010 AMS: *30C45, 30C80, 05A30*
Received: *25 November 2020*
Accepted: *30 January 2021*
Available online: *31 January 2021*

Abstract

In this paper, we use the Faber polynomial expansion techniques to get the general Taylor-Maclaurin coefficient estimates for $|a_n|$, $(n \geq 4)$ of a generalized class of bi-univalent functions by means of $(p,q)-$calculus, which was introduced by Chakrabarti and Jagannathan. For functions in such a class, we get the initial coefficient estimates for $|a_2|$ and $|a_3|$. In particular, the results in this paper generalize or improve (in certain cases) the corresponding results obtained by recent researchers.

1. Introduction

Let $\mathscr{A}$ indicate the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0, f'(0) = 1$ for every $z \in \mathbb{D}$. Denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ containing of all univalent functions. Let Ω be the class of Schwarz functions ϕ, which are analytic in $\mathbb{D}$ satisfying the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. If f_1 and f_2 are analytic functions in $\mathbb{D}$, then we state f_1 is subordinate to f_2, denoted by $f_1 \prec f_2$, if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z))$ (see [1]).

According to the Koebe 1/4 Theorem [1], the range of $\mathbb{D}$ under every function f in the univalent function class $\mathscr{S}$ contains a disc $\{w : |w| < 1/4\}$ of radius $1/4$. Thus, every univalent function f has an inverse f^{-1} satisfying the conditions

$$f^{-1}(f(z)) = z, \ \ (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w, \ \ (|w| < r_0(f);\ r_0(f) \geq 1/4),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

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If both f and f^{-1} are univalent in $\mathbb{D}$, then a function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{D}$. The class of bi-univalent functions will be denoted by Σ in $\mathbb{D}$.

Not much is known about the bounds for $|a_n|$ of Faber polynomials in quantum calculus because the bi-univalency requirement makes the behaviour of the coefficients of the functions f and f^{-1} unpredictable. The quantum calculus has a great number of applications in the fields of special functions and other areas (see [2], [3]). There is a possibility to extend some of the results in quantum calculus to post quantum calculus in geometric function theory.

Let us first recall certain notations of the $(p,q)-$calculus. The $(p,q)-$twin-basic number $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \ \ (0 < q < p \leq 1, n = 0,1,2,...).$$

The $(p,q)-$derivative operator of a function f is given by

$$(D_{p,q}f)(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \ \ (z \neq 0) \tag{1.2}$$

and $(D_{p,q}f)(0) = f'(0)$ provided that the function f is differentiable at $z = 0$ (see [4]). For a function f given by (1.1), it can be easily concluded that

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}. \tag{1.3}$$

Note that, for $p = 1$, $(p,q)-$derivative operator reduces to the Jackson $q-$derivative ([5], [6]) given by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ \ (z \neq 0). \tag{1.4}$$

Also, for $p = 1$, $q-$bracket $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q}, \ \ (n = 0,1,2,...).$$

In 1903, G. Faber [7] in his thesis, introduced the polynomials which have since proved useful in analysis, and hence are known as Faber polynomials. By using the Faber polynomial expansion of functions $f \in \mathscr{A}$, researchers in [8] got the following useful results.

Lemma 1.1. *If f is of the form (1.1), then the coefficients of its inverse functions $g = f^{-1}$ are given by*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...) w^n := w + \sum_{n=2}^{\infty} b_n w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \tfrac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \tfrac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 + \tfrac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \tfrac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} [a_5 + (-n+2) a_3^2] \\ &+ \tfrac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] + \textstyle\sum_{l \geq 7} a_2^{n-l} V_l \end{aligned}$$

such that $V_l, (7 \leq l \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$. The first three terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3), \quad K_3^{-4} = -4(5a_2^3 - 5a_2 a_3 + a_4).$$

Making use of $(p,q)-$derivative operator defined in (1.2), we define the class $\mathscr{N}_{\Sigma}(p,q;\lambda,\delta,A,B)$ as below:

Definition 1.2. *Let A and B be real numbers such that $-1 \leq B < A \leq 1$. For $0 < q < p \leq 1, \lambda \geq 1, \delta \geq 0$, a bi-univalent function $f \in \Sigma$ is said to be in $\mathscr{N}_{\Sigma}(p,q;\lambda,\delta,A,B)$ if*

$$(1-\lambda)\frac{f(z)}{z} + \lambda (D_{p,q}f)(z) + \delta z D_{p,q}(D_{p,q}f)(z) \prec \frac{1+Az}{1+Bz}, \ \ (z \in \mathbb{D}) \tag{1.5}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda (D_{p,q}g)(w) + \delta w D_{p,q}(D_{p,q}g)(w) \prec \frac{1+Aw}{1+Bw}, \ \ (w \in \mathbb{D}) \tag{1.6}$$

where $g(w) = f^{-1}(w)$ for $w \in \mathbb{D}$.

By taking different values of the parameters p,q,λ,δ,A,B, we may obtain several new and known subclasses of the family $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$; for instance we have

(i) $\mathscr{G}_\Sigma(q;\lambda,\delta,A,B) \equiv \mathscr{N}_\Sigma(1,q;\lambda,\delta,A,B)$.
(ii) $\mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz}) \equiv \mathscr{N}_\Sigma(p,q;\lambda,0,A,B)$.
(iii) $\mathscr{R}_\Sigma(\lambda,\delta,\alpha) \equiv \mathscr{N}_\Sigma(1,1;\lambda,\delta,1-2\alpha,-1), (0 \le \alpha < 1)$, [9].
(iv) $\mathscr{T}_\Sigma(\lambda,\alpha) \equiv \mathscr{N}_\Sigma(1,1;\lambda,0,1-2\alpha,-1), (0 \le \alpha < 1)$, [10].
(v) $\mathscr{H}_\Sigma(\alpha) \equiv \mathscr{N}_\Sigma(1,1;1,0,1-2\alpha,-1), (0 \le \alpha < 1)$, [11].
(vi) $\mathscr{M}_\Sigma(\delta,\alpha) \equiv \mathscr{N}_\Sigma(1,1;1,\delta,1-2\alpha,-1), (0 \le \alpha < 1)$, [12].

Remark 1.3. *Note that the class $\mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$ in (i) is a new generalized class of bi-univalent functions defined by $D_q = \lim_{p\to 1} D_{p,q}$ given in (1.4).*

Remark 1.4. *The class $\mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz})$ in (ii) may be obtained by letting $\varphi = \frac{1+Az}{1+Bz}$ in the class $\mathscr{D}_\Sigma(p,q;\lambda,\varphi)$ which was studied in 2017 by Altınkaya and Yalçın [13]. The results in our paper improve the estimates of the corresponding bounds in [13]. Similarly, our results are also better than those determined in [11].*

In view of the relations witnessed in (i) to (vi) and Remarks 1.3 and 1.4, we conclude that the generalized class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$ unifies several subclasses of Σ.

2. Main results

We first give coefficient estimates of a function f in the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$ for all the coefficients except for the first initial coefficients a_2 and a_3.

Theorem 2.1. *For $0 < q < p \le 1$, $\delta \ge 0$, $\lambda \ge 1$, $-1 \le B < A \le 1$, let the function f given by (1.1) be in the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$. If $a_m = 0, (2 \le m \le n-1)$, then*

$$|a_n| \le \frac{A-B}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|}, \quad (n \ge 4). \tag{2.1}$$

Proof. If a function f given by (1.1) is in $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$, then by using (1.2) and (1.3), the left side of (1.5) gives

$$(1-\lambda)\frac{f(z)}{z} + \lambda(D_{p,q}f)(z) + \delta z D_{p,q}(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty}\left[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta\right]a_n z^{n-1}. \tag{2.2}$$

In view of (1.2), (1.3) and Lemma 1.1, the left side of (1.6) yields

$$\begin{aligned}(1-\lambda)\tfrac{g(w)}{w} &+ \lambda(D_{p,q}g)(w) + \delta w D_{p,q}(D_{p,q}g)(w)\\ &= 1+\textstyle\sum_{n=2}^{\infty}\left[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta\right]b_n w^{n-1}\\ &= 1+\textstyle\sum_{n=2}^{\infty}\left[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta\right]\times\tfrac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)w^{n-1},\end{aligned} \tag{2.3}$$

where $K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)$ are given in Lemma 1.1.

On the other hand, (1.5) and (1.6) imply the existence of two Schwarz functions $\phi(z) = \sum_{n=1}^{\infty} c_n z^n$, $(z \in \mathbb{D})$ and $\psi(w) = \sum_{n=1}^{\infty} d_n z^n, (w \in \mathbb{D})$ so that

$$(1-\lambda)\frac{f(z)}{z} + \lambda(D_{p,q}f)(z) + \delta z D_{p,q}(D_{p,q}f)(z) = \frac{1+A\phi(z)}{1+B\phi(z)} \tag{2.4}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda(D_{p,q}g)(w) + \delta w D_{p,q}(D_{p,q}g)(w) = \frac{1+A\psi(w)}{1+B\psi(w)}. \tag{2.5}$$

Moreover, by using the method given in [8] and [14], Jahangiri and Hamidi in [15] observed that

$$\frac{1+A\phi(z)}{1+B\phi(z)} = 1 - \sum_{n=1}^{\infty}(A-B)K_n^{-1}(c_1,c_2,\ldots,c_n,B)z^n, \tag{2.6}$$

and

$$\frac{1+A\psi(w)}{1+B\psi(w)} = 1 - \sum_{n=1}^{\infty}(A-B)K_n^{-1}(d_1,d_2,\ldots,d_n,B)w^n, \tag{2.7}$$

where $K_n^{-1}(k_1,k_2,...,k_n,B)$ are obtained by the general coefficients $K_n^j(k_1,k_2,...,k_n,B)$ for all $j\in\mathbb{Z}$ given by

$$\begin{aligned} K_n^j(k_1,k_2,...,k_n,B) &= \tfrac{j!}{(j-n)!(n)!}k_1^nB^{n-1}+\tfrac{j!}{(j-n+1)!(n-2)!}k_1^{n-2}k_2B^{n-2}\\ &+\tfrac{j!}{(j-n+2)!(n-3)!}k_1^{n-3}k_3B^{n-3}\\ &+\tfrac{j!}{(j-n+3)!(n-4)!}k_1^{n-4}[k_4B^{n-4}+\tfrac{j-n+3}{2}k_3^2B]\\ &+\tfrac{j!}{(j-n+4)!(n-5)!}k_1^{n-5}[k_5B^{n-5}+(j-n+4)k_3k_4B]+\textstyle\sum_{j\geq 6}k_1^{n-j}V_j, \end{aligned}$$

and where V_j is a homogeneous polynomial of degree j in the variables $k_2,k_3,...,k_n$; (see [8], [14], [15]).

In view of (2.2), (2.4) and (2.6), for every $n\geq 2$, we get

$$[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]a_n=-(A-B)K_{n-1}^{-1}(c_1,c_2,...,c_{n-1},B). \tag{2.8}$$

Similarly, because of (2.3), (2.5) and (2.7), for every $n\geq 2$, we have

$$[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]b_n=-(A-B)K_{n-1}^{-1}(d_1,d_2,...,d_{n-1},B). \tag{2.9}$$

Since $a_m=0$ for $2\leq m\leq n-1$, we have $b_n=-a_n$ and thus,

$$[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]a_n=-(A-B)c_{n-1},$$

$$-[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]a_n=-(A-B)d_{n-1}.$$

Recall that for the Schwarz functions ϕ and ψ, we have $|c_{n-1}|\leq 1$ and $|d_{n-1}|\leq 1$ (see [1]). Taking absolute values of the last two equalities and using $|c_{n-1}|\leq 1$ and $|d_{n-1}|\leq 1$, we obtain

$$|a_n|=\frac{(A-B)|c_{n-1}|}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|}=\frac{(A-B)|d_{n-1}|}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|},$$

thus we arrive at

$$|a_n|\leq\frac{A-B}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|}.$$

This completes the proof. □

Setting $p=1$ in (2.1) and using (i), we get the $q-$coefficient bounds of the Faber polynomials of the class $\mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$.

Corollary 2.2. *Let $q\in(0,1)$, $\delta\geq 0$, $\lambda\geq 1$ and $-1\leq B<A\leq 1$. If $f\in\mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$ and $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{A-B}{|1+([n]_q-1)\lambda+[n]_q[n-1]_q\delta|},\quad (n\geq 4).$$

Setting $\delta=0$ in (2.1) and in view of (ii) together with Remark 1.4, we get the following:

Corollary 2.3. *If $f\in\mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz})$ and $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{A-B}{|1+([n]_{p,q}-1)\lambda|},\quad (n\geq 4).$$

Remark 2.4. *In [13], the authors found that if $f\in\mathscr{D}_\Sigma(p,q;\lambda,\varphi)$ and $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{2}{|1+([n]_{p,q}-1)\lambda|},\quad (n\geq 4). \tag{2.10}$$

However, we find that the coefficient estimates in Corollary 2.3 further improve the estimates in (2.10) because

$$|a_n|\leq\frac{A-B}{|1+([n]_{p,q}-1)\lambda|}\leq\frac{2}{|1+([n]_{p,q}-1)\lambda|},\quad (n\geq 4)$$

for all $\lambda\geq 1,-1\leq B<A\leq 1$, (see [13]).

In view of (iii), Theorem 2.1 gives the next corollary:

Corollary 2.5. *[9] Let $\lambda \geq 1, \delta \geq 0, 0 \leq \alpha < 1$. If $f \in \mathscr{R}_\Sigma(\lambda,\delta,\alpha)$ and $a_m = 0, (2 \leq m \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\lambda+n(n-1)\delta}, \quad (n \geq 4).$$

Since $\mathscr{T}_\Sigma(\lambda,\alpha) \equiv \mathscr{N}_\Sigma(1,1;\lambda,0,1-2\alpha,-1)$ by (iv), Theorem 2.1 gives the next result:

Corollary 2.6. *[16] Let $\lambda \geq 1$, $0 \leq \alpha < 1$ and $a_m = 0, (2 \leq m \leq n-1)$. If $f \in \mathscr{T}_\Sigma(\lambda,\alpha)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\lambda}, \quad (n \geq 4).$$

Remark 2.7. *In view of (vi), if $f \in \mathscr{M}_\Sigma(\delta,\alpha)$, then we get corresponding result obtained in [12].*

For the next theorem, we need the following lemma.

Lemma 2.8. *[15] Let $\phi(z) = \sum_{n=1}^{\infty} c_n z^n$ be a Schwarz function satisfying $|\phi(z)| < 1$ for $|z| < 1$. If $\gamma \geq 0$, then*

$$|c_2 + \gamma c_1^2| \leq 1 + (\gamma - 1)|c_1|^2.$$

Theorem 2.9. *For $0 < q < p \leq 1$, $\delta \geq 0$, $\lambda \geq 1$, $-1 \leq B \leq A \leq 1$, let the function f given by (1.1) be in the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$. If*

$$\begin{aligned} t &= [3]_{p,q} = p^2 + pq + q^2 \\ \mu &= [2]_{p,q} = p + q, \end{aligned} \tag{2.11}$$

then

$$|a_2| \leq \min \begin{cases} \dfrac{A-B}{\sqrt{\left|(A-B)\left[1+(t-1)\lambda+t\mu\delta\right]+(1+B)\left[1+(\mu-1)\lambda+\mu\delta\right]^2\right|}}, & B \leq 0 \\[2ex] \dfrac{A-B}{|1+(\mu-1)\lambda+\mu\delta|}, & \end{cases} \tag{2.12}$$

$$|a_3| \leq \frac{(A-B)^2}{[1+(\mu-1)\lambda+\mu\delta]^2} + \frac{A-B}{|1+(t-1)\lambda+t\mu\delta|} \tag{2.13}$$

and

$$|a_3 - 2a_2^2| \leq \frac{(A-B)\left[1-(1+B)\dfrac{(1+(\mu-1)\lambda+\mu\delta)^2\,|a_2|^2}{(A-B)^2}\right]}{|1+(t-1)\lambda+t\mu\delta|} \quad (B \leq 0). \tag{2.14}$$

These results are sharp.

Proof. Upon setting 2 in place of n in (2.8), we obtain

$$[1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta]a_2 = (A-B)K_1^{-1}(c_1) = -(A-B)c_1. \tag{2.15}$$

Again, replacing $n = 3$ in (2.8), we have

$$[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]a_3 = (A-B)K_2^{-1}(c_2) = -(A-B)(Bc_1^2 - c_2). \tag{2.16}$$

Similarly, by substituting $n = 2$ and $n = 3$, respectively in (2.9), we observe

$$-[1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta]a_2 = -(A-B)d_1, \tag{2.17}$$

and

$$[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta](2a_2^2 - a_3) = -(A-B)(Bd_1^2 - d_2). \tag{2.18}$$

Using $|c_1| \leq 1$ and $|d_1| \leq 1$, it follows from (2.15) and (2.17) that $c_1 = -d_1$ and

$$|a_2| = \frac{(A-B)|c_1|}{|1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta|} = \frac{(A-B)|d_1|}{|1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta|},$$

then we get

$$|a_2| \leq \frac{A-B}{|1+(\mu-1)\lambda+\mu\delta|},$$

where μ is given by (2.11) and for $-1 \leq B \leq A \leq 1$.

Adding (2.16) to (2.18), and simple calculations gives

$$2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]a_2^2 = (A-B)(c_2+(-B)c_1^2+d_2+(-B)d_1^2).$$

Taking absolute values of both sides, we get

$$\left|2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]\right||a_2|^2 \leq (A-B)\left[\left|c_2+(-B)c_1^2\right|+\left|d_2+(-B)d_1^2\right|\right].$$

If $B \leq 0$, then by Lemma 2.8 we have

$$2\left|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta\right||a_2|^2 \leq (A-B)\left[2-(B+1)(|c_1|^2+|d_1|^2)\right].$$

Upon substituting c_1 and d_1 from (2.15) and (2.17), we obtain

$$2\left|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta\right||a_2|^2 \leq (A-B)\left[2-2(B+1)\frac{\left(1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta\right)^2|a_2|^2}{(A-B)^2}\right],$$

or equivalently

$$|a_2| \leq \frac{A-B}{\sqrt{\left|(A-B)\big(1+(t-1)\lambda+t\mu\delta\big)+(1+B)\big(1+(\mu-1)\lambda+\mu\delta\big)^2\right|}},$$

where t and μ are given by (2.11). This completes the proof of (2.12).

In order to obtain the coefficient estimates for $|a_3|$, we subtract (2.18) from (2.16), and we get

$$[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta](-2a_2^2+2a_3) = -(A-B)\left[(Bc_1^2-c_2)-(Bd_1^2-d_2)\right],$$

or

$$a_3 = a_2^2+\frac{(A-B)(c_2-d_2)}{2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]}. \tag{2.19}$$

Upon substituting the value of a_2^2 from (2.15) into (2.19), it follows that

$$a_3 = \frac{(A-B)^2c_1^2}{(1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta)^2}+\frac{(A-B)(c_2-d_2)}{2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]}.$$

Taking the absolute value and by using $|c_1| \leq 1$, $|c_2| \leq 1$ and $|d_2| \leq 1$, we get

$$|a_3| \leq \frac{(A-B)^2}{(1+(\mu-1)\lambda+\mu\delta)^2}+\frac{A-B}{|1+(t-1)\lambda+t\mu\delta|},$$

where t and μ are given by (2.11). This proves the inequality in (2.13).

Finally, (2.18) yields

$$2a_2^2-a_3 = \frac{(A-B)(d_2+(-B)d_1^2)}{1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta}.$$

By taking the absolute value of the above equation, we find

$$|a_3-2a_2^2| \leq \frac{(A-B)|d_2+(-B)d_1^2|}{|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta|}.$$

If $B \le 0$, then by Lemma 2.8 we have

$$|a_3 - 2a_2^2| \le \frac{(A-B)(1+(-B-1)|d_1|^2}{|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta|}.$$

Upon substituting the value of d_1 from (2.17), we get

$$|a_3 - 2a_2^2| \le \frac{(A-B)\left(1-(1+B)\frac{(1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta)^2|a_2|^2}{(A-B)^2}\right)}{|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta|}.$$

This proves the inequality given by (2.14). $\square$

In view of (i) and (ii), Theorem 2.9 leads to the following corollaries.

Corollary 2.10. *Let $q \in (0,1), \lambda \ge 1, \delta \ge 0$ and $-1 \le B < A \le 1$. If $f \in \mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$ and $a_m = 0, (2 \le m \le n-1)$, then*

$$|a_2| \le \min \begin{cases} \dfrac{A-B}{\sqrt{\left|(A-B)\left[1+(q+q^2)\lambda+(1+q+q^2)(1+q)\delta\right]+(1+B)\left[1+q\lambda+(1+q)\delta\right]^2\right|}}, & B \le 0 \\ \frac{A-B}{1+q\lambda+(1+q)\delta}, & \end{cases}$$

$$|a_3| \le \frac{(A-B)^2}{(1+q\lambda+(1+q)\delta)^2} + \frac{A-B}{1+(q+q^2)\lambda+(1+q+q^2)(1+q)\delta}$$

and

$$\left|a_3 - 2a_2^2\right| \le \frac{(A-B)\left[1-(1+B)\frac{\left[1+q\lambda+(1+q)\delta^2\right]|a_2|^2}{(A-B)^2}\right]}{1+(q+q^2)\lambda+(1+q+q^2)(1+q)\delta} \qquad (B \le 0).$$

Corollary 2.11. *Let $0 < q < p \le 1, \lambda \ge 1, -1 \le B < A \le 1$. If $f \in \mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz})$ and $a_m = 0, (2 \le m \le n-1)$, then*

$$|a_2| \le \min\left\{\frac{A-B}{|1+(p+q-1)\lambda|}, \frac{A-B}{\sqrt{\left|(A-B)\left[1+(p^2+pq+q^2-1)\lambda\right]+(1+B)\left[1+(p^2+pq+q^2-1)\lambda\right]^2\right|}}\right\},$$

$$|a_3| \le \frac{(A-B)^2}{(1+(p+q-1)\lambda)^2} + \frac{A-B}{|1+(p^2+pq+q^2-1)\lambda|}.$$

Remark 2.12. *Let $\lambda \ge 1, \delta \ge 0, 0 \le \alpha < 1$. If $f \in \mathscr{R}_\Sigma(\lambda,\delta,\alpha)$ and $a_m = 0, (2 \le m \le n-1)$, then Theorem 2.9 yields the corresponding results obtained in [9] for coefficients a_2, a_3 and $a_3 - 2a_2^2$.*

Remark 2.13. *Let $\lambda \ge 1, 0 \le \alpha < 1$. If $f \in \mathscr{T}_\Sigma(\lambda,\alpha)$ and $a_m = 0, (2 \le m \le n-1)$, then Theorem 2.9 satisfies the corresponding results obtained in [16] for coefficients a_2 and $a_3 - 2a_2^2$.*

Remark 2.14. *Setting $p = 1, q \to 1^-, A = 1-2\alpha, (0 \le \alpha < 1), B = -1$ and $\delta = 0$, Theorem 2.9 yields the corresponding results in [10] for coefficients a_2 and a_3.*

Remark 2.15. *Setting $p = 1, q \to 1^-, A = 1-2\alpha, (0 \le \alpha < 1), B = -1, \delta = 0$ and $\lambda = 1$, Theorem 2.9 yields the corresponding results in [11] for coefficient a_2.*

3. Conclusion

In this paper, we defined a new subclass of bi-univalent functions associated with (p,q)-derivative operator and investigated Faber polynomial coefficient estimates for this new class. We also concluded that the results are generalization of the corresponding results obtained by recent researchers.

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 25-32
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
DOI: https://dx.doi.org/10.33401/fujma.837479

Construction of Degenerate q-Daehee Polynomials with Weight α and its Applications

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Article Info

Keywords: *Weighted q-Bernoulli polynomials, Stirling numbers of first kind, Stirling numbers of second kind, Degenerate q-Daehee polynomials with weight α*
2010 AMS: *11B68, 11S80, 33C45*
Received: *8 December 2020*
Accepted: *31 January 2021*
Available online: *3 February 2021*

Abstract

The fundamental aim of the present paper is to deal with introducing a new family of Daehee polynomials which is called degenerate q-Daehee polynomials with weight α by using p-adic q-integral on $\mathbb{Z}_p$. From this definition, we obtain some new summation formulae and properties. We also introduce the degenerate q-Daehee polynomials of higher order with weight α and obtain some new interesting identities.

1. Introduction

The notion of p-adic numbers was firstly considered by Kurt Hensel (1861-1941). Motivated by this fruitful idea, many scientists begun to study new scientific tools using good and useful properties of them. Diverse effects of these new researches have emerged in mathematical physics in which they are used in the theory of ultrametric calculus, p-adic quantum mechanics, the p-adic mechanics, etc.

The one useful tool of p-adic analysis is Volkenborn integral (or so-called p-adic integral). Intense research activities in such an area as p-adic integral are principally motivated by their importance in special polynomials, especially the Bernoulli polynomials and their various generalizations. The other useful tool of p-adic analysis is q -analogue of p-adic invariant integral which is invented by Kim [10] . He showed that the Carlitz's q-Bernoulli polynomials and their different generalizations can be represented as a p-adic q-invariant integral which is called Witt's formula. Therefore, in recent years, p-adic integral and its various generalizations have been considered and extensively studied by many mathematicians, *cf.* [3], [4], [5], [7], [11], [12], [16], [20], [21], [22], [29].

We now begin with recalling some basic notations as follows.

Throughout this paper we use the following standard notations:

$$\mathbb{N} := \{1,2,3,\cdots\} \text{ and } \mathbb{N}_0 := \mathbb{N}\cup\{0\}.$$

The parameter p stands for the first letter of p-adic being a fixed prime number. The symbols denoted by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ mean p-adic integers field, p-adic rational numbers field and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The known

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p-adic norm denoted by $|\,.\,|_p$ is normalized by the equality $|p|_p = p^{-1}$. The $UD(\mathbb{Z}_p)$ means the space of $\mathbb{C}_p$-valued uniformly differentiable functions over $\mathbb{Z}_p$. The p-adic q-integral on $\mathbb{Z}_p$ of a function $f \in UD(\mathbb{Z}_p)$ is originally given by Kim [10] as follows:

$$I_q(f) := \int_{\mathbb{Z}_p} f(y)d\mu_q(y) = \lim_{N\to\infty} \sum_{k=0}^{p^N-1} f(k)\mu_q(k+p^N\mathbb{Z}_p), \tag{1.1}$$

where $\mu_q(k+p^N\mathbb{Z}_p) = \frac{q^k}{[p^N]_q}$ is Kim's q-Haar distribution. It follows from (1.1) that

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q} f'(0) + (q-1)f(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$.

In the year 2011, Kim [11] defined weighted q-Bernoulli polynomials (or known as q-Bernoulli polynomials with weight α) which can be represented by the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q}^{(\alpha)}(x)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} e^{z[x+y]_{q^\alpha}} d\mu_q(y), \tag{1.3}$$

or equivalently by

$$\beta_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx}[x]_{q^\alpha}^{n-l} \beta_{l,q}^{(\alpha)}, \quad (n \geq 0). \tag{1.4}$$

The pioneering of degenerate versions of Bernoulli and Euler polynomials was Carlitz who considered $(1+\lambda z)^{\frac{1}{\lambda}}$ instead of e^z in their generating functions. When $\lambda \to 0$, it returns to classical one. Actually, Carlitz [1], [2] gave the generating function of degenerate Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_n(x \mid \lambda)\frac{z^n}{n!} = \frac{z}{(1+\lambda z)^{\frac{1}{\lambda}} - 1}(1+\lambda z)^{\frac{x}{\lambda}}.$$

When $x = 0$, $\beta_n(0 \mid \lambda) := \beta_n(\lambda)$ are called degenerate Bernoulli numbers. It is noteworthy that

$$\lim_{\lambda\to 0} \beta_n(x \mid \lambda) = B_n(x),$$

where $B_n(x)$ are the Bernoulli polynomials, see [5], [30], [31], [32].

Kim also applied the idea of degenerate version to various special functions, polynomials and numbers, *cf.* [12], [13], [14], [15]. For example, Kim considered a new class of q-Bernoulli polynomials which is called degenerate q-Bernoulli polynomials given by

$$\int_{\mathbb{Z}_p} (1+\lambda z)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x|\lambda)\frac{z^n}{n!}, \tag{1.5}$$

where the parameters are assumed that $\lambda, z, q \in \mathbb{C}_p$ with $|\lambda z|_p < p^{-\frac{1}{p-1}}$ and $|1-q|_p < p^{-\frac{1}{p-1}}$, see [12].

Let $D_n(x)$ be Daehee polynomials given by

$$\sum_{n=0}^{\infty} D_n(x)\frac{z^n}{n!} = \frac{\log(1+z)}{z}(1+z)^x.$$

In the case when $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers, *cf.* [5], [7], [17], [18], [19], [21], [22], [24], [25], [26], [28], [29]. The degenerate version of Carlitz's type q-Daehee polynomials is considered by

$$\sum_{n=0}^{\infty} D_{n,q,\lambda}(x)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_q}{\lambda}} d\mu_q(y), \tag{1.6}$$

where the case $x = 0$, $D_{n,q,\lambda}(0) := D_{n,q,\lambda}$ stands for the degenerate of q-analogue Carlitz's type Daehee numbers, see [23]. Clearly that

$$D_{n,q,\lambda} \to D_{n,q} \text{ as } \lambda \to 0.$$

The Stirling numbers of first and second kinds are given, respectively, by

$$\sum_{n=k}^{\infty} S_1(n,k)\frac{z^n}{n!} = \frac{(\log(z+1))^k}{k!} \quad and \quad \sum_{n=k}^{\infty} S_2(n,k)\frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}, \tag{1.7}$$

satisfying

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \text{ see [8], [30], [32].}$$

Motivated by the works of [3], [4] and [12], we consider the degenerate q-Daehee polynomials with weight α as follows:

$$\int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!}.$$

From this definition, we obtain explicit identities and properties. We also introduce the degenerate q-Daehee polynomials of higher order with weight α.

2. The degenerate q-Daehee polynomials with weight α

We begin with the following definition.

Definition 2.1. *Let $\lambda, z, q \in \mathbb{C}_p$ with $|\lambda z|_p < p^{-\frac{1}{p-1}}$ and $|1-q|_p < p^{-\frac{1}{p-1}}$. The degenerate q-Daehee polynomials $D_{n,q;\alpha,\lambda}(x)$ are given by*

$$\sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y). \tag{2.1}$$

Remark 2.2. *Putting $\alpha = 1$ in Eq. (2.1) reduces to Eq. (1.6).*

Remark 2.3. *Traditionally, in the case $x = 0$, the polynomial reduces to its number. So, when $x = 0$ in (2.1), $D_{n,q;\alpha,\lambda}(0) := D_{n,q;\alpha,\lambda}$ will be called the degenerate q-Daehee numbers with weight α.*

It follows from Eq. (2.1) that

$$\begin{aligned}
\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{z^m}{m!} &= \int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y) \\
&= \int_{\mathbb{Z}_p} \sum_{j=0}^{\infty} \binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{j} \lambda^j (\log(1+z))^j d\mu_q(y) \\
&= \sum_{j=0}^{\infty} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} \sum_{m=j}^{\infty} S_1(m,j)\, d\mu_q(y)\frac{z^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} S_1(m,j)\, d\mu_q(y)\right)\frac{z^m}{m!},
\end{aligned}$$

where we have used

$$\left([\gamma+\zeta]_{q^\alpha}\right)_{j,\lambda} = [\gamma+\zeta]_{q^\alpha}\left([\gamma+\zeta]_{q^\alpha}-\lambda\right)\left([\gamma+\zeta]_{q^\alpha}-2\lambda\right)\cdots\left([\gamma+\zeta]_{q^\alpha}-(j-1)\lambda\right).$$

Thus we obtain the following theorem.

Theorem 2.4. *Let $m \in \mathbb{N}_0$. The degenerate q-Daehee polynomials with weight α satisfy*

$$D_{m,q;\alpha,\lambda}(x) = \sum_{j=0}^{m} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} S_1(m,j)\, d\mu_q(y).$$

Let $B_{k,q;\alpha,\lambda}(x)$ be degenerate q-Bernoulli polynomials with weight α which may be given by

$$\sum_{n=0}^{\infty} B_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda z)^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y). \tag{2.2}$$

Replacing z by e^z-1 in Eq. (2.1) gives

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{(e^z-1)^m}{m!} &= \int_{\mathbb{Z}_p}(1+\lambda z)^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_q(y), \\ &= \sum_{m=0}^{\infty} B_{m,q;\alpha,\lambda}(x)\frac{z^m}{m!}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{(e^z-1)^m}{m!} &= \sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{z^n}{n!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_{n,q;\alpha,\lambda}(x)S_2(m,n)\right)\frac{z^m}{m!}.\end{aligned} \tag{2.4}$$

Thus, from (2.3) and (2.4), we have the following theorem.

Theorem 2.5. *Let $m\in\mathbb{N}_0$. The following identity holds*

$$B_{m,q;\alpha,\lambda}(x)=\sum_{n=0}^{m} D_{n,q;\alpha,\lambda}(x)S_2(m,n).$$

Changing z to $\log(1+z)$ in Eq. (2.2) yields

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{z^n}{n!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_q(y) \\ &= \sum_{m=0}^{\infty} B_{m,q;\alpha,\lambda}(x)\frac{(\log(1+z))^m}{m!} \\ &= \sum_{m=0}^{\infty} B_{m,q;\alpha,\lambda}(x)\sum_{n=m}^{\infty}S_1(n,m)\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_{m,q;\alpha,\lambda}(x)S_1(n,m)\right)\frac{z^n}{n!}.\end{aligned}$$

By comparing coefficitents of $\frac{z^n}{n!}$ on the both sides of the above, we procure the following theorem.

Theorem 2.6. *Let $m\in\mathbb{N}_0$. The following summation formula satisfies*

$$D_{m,q;\alpha,\lambda}(x)=\sum_{m=0}^{n} B_{m,q;\alpha,\lambda}(x)S_1(n,m).$$

Since

$$q^x=e^{x\log q},$$

we have

$$\begin{aligned}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} &= e^{\frac{[x+y]_{q^\alpha}}{\lambda}\log(1+\lambda\log(1+z))} \\ &= \sum_{n=0}^{\infty}\left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)^n\frac{(\log(1+\lambda\log(1+z)))^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)^n\sum_{m=n}^{\infty}S_1(m,n)\lambda^m\frac{(\log(1+z))^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=0}^{j}[x+y]_{q^\alpha}^l\,\lambda^{j-l}S_1(j,l)S_1(n,j)\right)\frac{z^n}{n!}.\end{aligned} \tag{2.5}$$

Taking p-adic q-integral on $\mathbb{Z}_p$ on both sides of (2.5) becomes

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{z^m}{m!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}\,d\mu_q(y) \qquad (2.6)\\ &= \int_{\mathbb{Z}_p}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{l=0}^{j}[x+y]_{q^\alpha}^{l}\ \lambda^{j-l}S_1(j,l)S_1(m,j)\right)d\mu_q(y)\frac{z^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{l=0}^{j}\lambda^{j-l}S_1(j,l)S_1(m,j)\beta_{l,q}^{(\alpha)}(x)\right)\frac{z^m}{m!}. \qquad (2.7)\end{aligned}$$

By (2.6) and (2.7), we arrive at the following theorem.

Theorem 2.7. *Let $m\in\mathbb{N}_0$. The following relation holds*

$$D_{m,q;\alpha,\lambda}(x)=\sum_{j=0}^{m}\sum_{l=0}^{j}\lambda^{j-l}S_1(j,l)S_1(m,j)\beta_{l,q}^{(\alpha)}(x).$$

It is easy to check that

$$[x+y]_{q^\alpha}=\frac{1-q^{\alpha(x+y)}}{1-q^\alpha}=\frac{1-q^{\alpha x}}{1-q^\alpha}+\frac{q^{\alpha x}(1-q^{\alpha y})}{1-q^\alpha}=[x]_{q^\alpha}+q^{\alpha x}[y]_{q^\alpha}.$$

From here, we see that

$$\begin{aligned}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} &= (1+\lambda\log(1+z))^{\frac{[x]_{q^\alpha}+q^{\alpha x}[y]_{q^\alpha}}{\lambda}}\\ &= (1+\lambda\log(1+z))^{\frac{[x]_{q^\alpha}}{\lambda}}(1+\lambda\log(1+z))^{\frac{q^{\alpha x}[y]_{q^\alpha}}{\lambda}}\\ &= \left(\sum_{j=0}^{\infty}\binom{\frac{[x]_{q^\alpha}}{\lambda}}{j}\lambda^j(\log(1+z))^j\right)\left(\sum_{m=0}^{\infty}\frac{q^{m\alpha x}[y]_{q^\alpha}^m}{\lambda^m}\frac{(\log(1+\lambda\log(1+z)))^m}{m!}\right)\\ &= \left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left([x]_{q^\alpha}\right)_{j,\lambda}S_1(n,j)\right)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\lambda^{k-l}q^{\alpha l x}[y]_{q^\alpha}^{l}S_1(k,l)S_1(n,k)\right)\frac{z^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n}{j}\lambda^{k-l}\left([x]_{q^\alpha}\right)_{m,\lambda}q^{\alpha l x}[y]_{q^\alpha}^{l}S_1(k,l)S_1(j,k)S_1(n,m)\right)\frac{z^n}{n!}.\end{aligned}$$

Thus we have the following theorem.

Theorem 2.8. *Let n be nonnegative integer. The following implicit summation formula satisfies*

$$D_{n,q;\alpha,\lambda}(x)=\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n}{j}\lambda^{k-l}\left([x]_{q^\alpha}\right)_{m,\lambda}q^{\alpha l x}S_1(k,l)S_1(j,k)S_1(n,m)\beta_{l,q}^{(\alpha)}.$$

Now we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_q(y) \\
&= \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}q^y \\
&= \lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{y=0}^{dp^N-1}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}q^y \\
&= \lim_{N\to\infty}\frac{1}{[d]_q[p^N]_{q^d}}\sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1}q^a(1+\lambda\log(1+z))^{\frac{[x+a+dy]_{q^\alpha}}{\lambda}}q^{dy} \\
&= \frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+z))^{\frac{[d]_{q^\alpha}\left[\frac{x+a}{d}+y\right]_{q^{d\alpha}}}{\lambda}}q^{dy} \\
&= \frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\sum_{y=0}^{p^N-1}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{j=0}^{k}S_1(k,j)S_1(n,k)[d]_{q^\alpha}^j\left[\frac{a+x}{d}+y\right]_{q^\alpha}^j\lambda^{n-k}\right)q^{dy}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{[d]_q}\sum_{a=0}^{d-1}\sum_{k=0}^{n}\sum_{j=0}^{k}q^aS_1(k,j)S_1(n,k)[d]_{q^\alpha}^j\beta_{j,q^d}^{(\alpha)}\left(\frac{a+x}{d}\right)\lambda^{n-k}\right)\frac{z^n}{n!}.
\end{aligned}$$

Thus we get the following theorem.

Theorem 2.9. *Let n be nonnegative integer. The following distribution formula for degenerate q-Daehee polynomials with weight α holds*

$$D_{n,q;\alpha,\lambda}(x)=\frac{1}{[d]_q}\sum_{a=0}^{d-1}\sum_{k=0}^{n}\sum_{j=0}^{k}q^aS_1(k,j)S_1(n,k)[d]_{q^\alpha}^j\beta_{j,q^d}^{(\alpha)}\left(\frac{a+x}{d}\right)\lambda^{n-k}.$$

Recall from Eq. (1.2) that

$$qI_q(f_1)-I_q(f)=\frac{q-1}{\log q}f'(0)+(q-1)f(0).$$

Let us now consider the following function

$$f(y)=(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}},$$

then we find the following difference equation for degenerate q -Daehee polynomials with weight α as follows:

$$\begin{aligned}
qD_{n,q;\alpha,\lambda}(x+1)-D_{n,q;\alpha,\lambda}(x) &= (q-1)\sum_{k=0}^{n}\left([x]_{q^\alpha}\right)_{k,\lambda}S_1(n,k)+n\frac{\alpha}{[\alpha]_q}(q-1)^2q^{\alpha x} \\
&\times\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n}{j+1}(-1)^k\lambda^kk!D_{n-1-j,q;\alpha,\lambda}(x+1)S_1(j+1,k+1).
\end{aligned}$$

Now we introduce degenerate q-Daehee polynomials of higher order by using multivariate p-adic q-integral on $\mathbb{Z}_p$ defined by Kim in [16]:

$$\sum_{n=0}^{\infty}D_{n,q;\alpha,\lambda}^{(v)}(x)\frac{z^n}{n!}=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}}(1+\lambda\log(1+z))^{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}\overline{\mathbf{d}}\mu_q(y), \tag{2.8}$$

where

$$\mathbf{x}:=\sum_{i=1}^{v}x_i \text{ and } \overline{\mathbf{d}}\mu_q(y):=\prod_{i=1}^{v}d\mu_q(y_i).$$

It follows from the Eq. (2.8) that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}^{(v)}(x)\frac{z^n}{n!} &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} (1+\lambda\log(1+z))^{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}\,\overline{\mathbf{d}}\mu_q(y) \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} \sum_{l=0}^{\infty}\binom{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}{l}\lambda^l(\log(1+z))^l\,\overline{\mathbf{d}}\mu_q(y) \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} \sum_{l=0}^{\infty}\left([\overline{\mathbf{x}}+y]_{q^\alpha}\right)_{l,\lambda}\sum_{n=l}^{\infty}S_1(n,l)\,\overline{\mathbf{d}}\mu_q(y)\frac{z^n}{n!} \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\lambda^{l-j}S_1(l,j)S_1(n,l)[\overline{\mathbf{x}}+y]_{q^\alpha}^{j}\,\overline{\mathbf{d}}\mu_q(y)\right)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\lambda^{l-j}S_1(l,j)S_1(n,l)\beta_{j,q}^{(\alpha,v)}(x)\right)\frac{z^n}{n!}.
\end{aligned}$$

From those applications, we deduce the following theorem.

Theorem 2.10. *Let $n\in\mathbb{N}_0$. The following relation*

$$D_{n,q;\alpha,\lambda}^{(v)}(x)=\sum_{l=0}^{n}\sum_{j=0}^{l}\lambda^{l-j}S_1(l,j)S_1(n,k)\beta_{j,q}^{(\alpha,v)}(x),$$

holds true.

We finalize our paper replacing z by e^z-1 in Eq. (2.8):

$$\begin{aligned}
\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}^{(v)}(x)\frac{(e^z-1)^m}{m!} &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} (1+\lambda z)^{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}\,\overline{\mathbf{d}}\mu_q(y) \\
&= \sum_{n=0}^{\infty} B_{n,q;\alpha,\lambda}^{(v)}(x)\frac{z^n}{n!}
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}^{(v)}(x)\frac{(e^z-1)^m}{m!} &= \sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}^{(v)}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} D_{m,q;\alpha,\lambda}^{(v)}(x)S_2(n,m)\right)\frac{z^n}{n!}.
\end{aligned} \tag{2.10}$$

Thus, from (2.9) and (2.10), we have the following theorem.

Theorem 2.11. *Let $n\in\mathbb{N}_0$. The following identity holds*

$$B_{n,q;\alpha,\lambda}^{(v)}(x)=\sum_{m=0}^{n} D_{m,q;\alpha,\lambda}^{(v)}(x)S_2(n,m).$$

3. Conclusion

The pioneering of degenerate idea was Carlitz, see[1] and [2], who considered for Bernoulli and Euler polynomials. This idea was one of good advantages in order to introduce new families of special polynomials. As has been listed in the references, Kim and his research team have been working this fruitful idea for new special polynomials intensively.

In this paper, motivated by the works of Kim and his research team, we have dealt mainly with new family of polynomials which are called degenerate q-Daehee polynomials with weight α and degenerate q-Daehee polynomials with weight α of higher order. We have derived their explicit and summation formulae by using p-adic q-integral on $\mathbb{Z}_p$ and analytic methods.

Seemingly that these types of polynomials will be continued to be studied for a while due to their interesting reflections in the fields of mathematics.

Acknowledgement

We would like to thank the reviewers and editor for their valuable suggestions and comments which have improved the presentation of the paper substantially.

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 33-44
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.815819

Discrete Networked Dynamic Systems with Eigen-Spectrum Gap: Analysis and Performance

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Article Info

Keywords: *Discrete network dynamic systems (DNDS), Mode separation, Reduced-order modelling, State dependent networks*
2010 AMS: *15A39, 68R10, 93A15, 93B52, 93D05, 94C15*
Received: *24 October 2020*
Accepted: *26 February 2021*
Available online: *05 March 2021*

Abstract

This paper provides a detailed analysis and performance treatment of a class of discrete-time systems with an eigen-spectrum gap coupled over networks. We deploy tools from time-scale modeling (TSM) theory to develop rigorous reduced-order models to aid in the stability analysis of these multiple time-scale networked systems over fixed and undirected graph topology. We establish that the controller gain matrices can be determined by solving convex optimization problems in terms of finite linear matrix inequalities with prescribed $\mathbb{H}_\infty$ and $\mathbb{H}_2$ performance criteria. As demonstrated by simulation studies, the ensuing results provide designers with a network-centric approach to improve the performance and stability of such coupled systems.

1. Introduction

The usefulness of time-scale modeling (TSM) theory for the analysis and synthesis of dynamical control systems with slow and fast dynamics has been broadly recognized as a strong technique for over four decades [1, 2]. Different control methodologies have received great attention of various researchers in the theory of control systems that comprises time-scales [3]-[6]. An important feature of the existing results is that the control analysis and synthesis are accomplished in two stages, such that a suitable reduced-order dynamics is treated at each stage. Order reduction and control has been extended to discrete systems with two time scale [7]-[9] based on explicit invertible-transformations where quasi-steady-state is assumed [10]-[13]. It has been demonstrated that the discrete time dynamics can be reduced to (a) a slow sub-dynamics with large eigenvalues near the the unit disk and (b) a fast sub-dynamics with eigenvalues distributed near the origin of the disk. This decomposition can be satisfied if an inequality relating the norms of subsystem matrices holds. Therefore, this structure allows the user to implement feedback control using different gain matrices. Along with the enormous advancement of control theory, technological development of controlling a group of agents has been widely investigated and received increasing demands. A common structure for controlling a group of agents is the distributed cooperative and coordinated control techniques [14]-[19]. Recently, distributed coordination of multi-agent systems have received a tremendous interests in a wide range of practical applications, mainly including engineering, ecology, biology and sociology [20]-[30].
On another research avenue, discrete networked dynamic systems (DNDS) provides a high-level treatment of a general class of linear discrete-time dynamic systems interconnected over an information network processed in discrete-time environment, exchanging relative state measurement or output measurements. It seems encouraging that by exploiting the impact of the network properties, additional features of the dynamical systems can be revealed [31]. On a parallel development in view of the available results, it turns out that research avenues in multiagent systems offer great opportunities for further developments from theoretical, simulation and implementations standpoints [32, 33]. TSM theory [34, 35] is attractive for establishing these approximations as the obtained reduced-order dynamics guarantees the asymptotic behavior of the coupled-consensus

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dynamics and give a viable estimate of the performance of the network trajectories through a simpler set of equations, compared to the original complex structure. Reduced-order modeling and synchronization of a network of homogeneous linear agents that comprises two time-scale behavior over fixed and undirected graph topology are investigated in [36]. However, most of the existing results do not employ the time-scale separation that normally appears between the agent-layer and the network-layer. By adopting advantage of this framework, the technique addressed in this article is, therefore, flexible to be applied to a broad range of agent models and wide range network controllers. In addition, it gives important insights into the interplay between design parameters such as controller parameters and communication topology on the behavior of coupled-consensus dynamics. In this paper, tools from time-scale modeling (TSM) theory [37]–[42] are used to investigate reduced-order dynamics rigorously to help in the stability analysis of the multi-time-scale networked systems. Modeling and synchronizing reduced-order networks of a group of identical agents characterized by continuous singularly-perturbed dynamics over undirected graph topology have been addressed in [43].
The contributions of this paper are as follows:

A) We extend the preliminary findings of [7, 8] to networked formalism of discrete systems with eigen-spectrum gap, thereby exploring the relationship between the graph topology and the coupled system stability framework.
B) We develop a mode-separation methodology of expanding the stabilization control design to the synchronization problem. This is clarified by decomposing the overall network dynamics and designing the controls that synchronize the slow dynamics and the fast ones. By recomposing the slow and fast controllers to the network of two time-scale systems we obtain an approximation of the synchronization behavior imposed for each scale.
C) We established that the controller gain matrices can be determined by solving convex optimization problems in terms of finite linear matrix inequalities with prescribed $\mathbb{H}_\infty$ and $\mathbb{H}_2$ performance criteria.

Notations: Let Q^{-1}, Q^t and $||Q||$ denote the inverse, the transpose and induced-matrix-norm of square matrices Q, respectively. The notation $Q > 0$ is used to represent a symmetric positive-definite matrix Q and I_N represents the $N \times N$ identity matrix. If the dimension of any matrix is not not explicitly given, we assume it to have an appropriate dimension for algebraic operations. We use the notation $\bullet$ to denote an element that is induced by symmetry. Sometimes, the arguments of a function will be omitted when no confusion can arise.

2. Graph theory

In the sequel, we recall some definitions and properties of Graph theory, which will be used throughout the paper.
A weighted graph is a triple $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{W})$ consisting of a node (vertex) set $\mathbb{V} = \{1, \cdots n\}$ with cardinality $|\mathbb{V}| = n$, an edge set $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$ with cardinality $|\mathbb{E}| = m$, a positive weight set $\mathbb{W}$ with cardinality $|\mathbb{W}| = m$, a weighted adjacency matrix $\mathbb{A} = [a_{ij}]$ with non-negative adjacent elements a_{ij} and the corresponding vector of weights $\mathbf{w}$ with the order $\mathbf{w}_{ij}$ refers to the weight of the edge $\{i, j\}$ [33]. In what follows, we consider undirected graph such that $(i,\ j) \in \mathbb{E}$ is equivalent to $(j,\ i) \in \mathbb{E}$. In addition, we consider that the graph $\mathbb{G}$ contains no self-loop $(\forall i = 1, \ \cdots\ n)$, one has $(i,\ i) \notin \mathbb{E}$. The adjacency matrix associated with $\mathbb{G}$ is define as $\mathbb{A} = [a_{ij}] \in \Re^{n \times n}$ such that

$$\begin{cases} a_{ij} > 0 & if\ (i,\ j) \in \mathbb{E} \\ a_{ij} = 0 & otherwise \end{cases}$$

The (graph) Laplacian of $\mathbb{G}$ is a rank deficient and symmetric matrix defined by

$$\begin{aligned} \mathbb{L}(\mathbb{G}) &:= \mathbb{E}(\mathbb{G})\mathbb{E}(\mathbb{G})^t = \Delta(\mathbb{G}) - \mathbb{A}(\mathbb{G}) \\ &:= [\ell_{ij}],\ \ell_{ij} = -a_{ij},\ \ell_{ii} = \sum_{j=1}^{n} a_{ij}. \end{aligned}$$

Based on the definition of $\mathbb{L}$, the any of its rows sum is zero. Moreover, $\mathbb{L}(\mathbb{G})$ ha eigenvalues set $0 = \lambda_1(\mathbb{G}) \leq \cdots \leq \lambda_n(\mathbb{G}) = \lambda_{\max}(\mathbb{G})$ and associated with the set eigenvectors $v_1 := \frac{1}{n}\mathbf{1}, v_2, \ldots, v_n$. An attraction of these dynamics is that all subsystems converge to the consensus space defined as $\mathscr{X}_c = \{\mathbf{y} \in \Re^{np} | \mathbf{y}_1(k) = \cdots = \mathbf{y}_n(k)\}$ when $\mathbb{G}$ is a connected graph [33].

Definition 2.1. *In the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, a path of length p is defined as the union of edges as follows:*

$$\bigcup_{m=1}^{p} (i_m,\ j_m) \Rightarrow i_{m+1} = j_m,\ \forall m \in \{1, \ \cdots\ p-1\}$$

The agent j is said to be **connected** *with agent i if there one path exists joining i with j, i.e. $i_1 = i$ and $j_p = j$. If every two different agents has at least one path connecting them, the graph is said to be connected. Henceforth, we assume that the undirected graph $\mathbb{G}$ is connected.*

The following remark gives some important characteristic of the graph and its Laplacian matrix.

Remark 2.2. *Let $\lambda_1 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\mathbb{L}$. It follows from [32] that*

- $\lambda_1 = 0$ *is a unique zero eigenvalue of* $\mathbb{L}$ *corresponding to the eigenvector* $\mathbb{I} \triangleq [1,\ 1,\ \cdots,\ 1]^t$.
- λ_2 *is strictly positive if and only if the graph* $\mathbb{G}$ *is connected. This means that* $\mathbb{L} \geq 0$.
- $\lambda_1 = 0$ *is an eigenvalue with multiplicity r of the matrix* $\mathbb{L} \otimes I_r$. *Moreover,* $\mathbb{L} \otimes I_r$ *has r different normalized eigenvectors given by* $\mathbb{I} \otimes e_i, i = 1,\ldots,r$ *where* $e_i \in \Re^r$ *is the column vector whose* i^{th} *element is* 1 *and others are zeros.*
- *An orthonormal matrix exists:* $T \in \Re^{n\times n}$, $\mathbb{T}\mathbb{T}^t = \mathbb{T}^t\mathbb{T} = I_n$ *such that* $\mathbb{T}\mathbb{L}\mathbb{T}^t = \mathbb{D} \triangleq \boldsymbol{diag}(\lambda_1,\ \lambda_2, \cdots,\ \lambda_n)$

3. Mode-separation of discrete time dynamical systems

There are a wide class of linear discrete-time control systems with eigenvalue-separation. By reordering and/or rescaling of states, linear discrete system can be cast into the form

$$\begin{aligned} x(k+1) &= A_1x(k) + A_2z(k) + B_1u(k) + \Gamma_1\omega(k), \\ z(k+1) &= A_3x(k) + A_4z(k) + B_2u(k) + \Gamma_2\omega(k), \\ y(k) &= C_1x(k) + C_2z(k) \end{aligned} \tag{3.1}$$

where the disturbance weighting matrices are $\Gamma_1 \in \Re^{n_1\times s}$, $\Gamma_2 \in \Re^{q\times s}$. We seek to determine the conditions under which the modes of discrete systems can be separated. In (3.1), the state vector is formed by $x(k) \in \Re^{n_1}$ and $z(k) \in \Re^{n_2}$, and the control is $u(k) \in \Re^m$ and the disturbance vector $\omega(k) \in \Re^s$.

Assumption 1. *Let* $n = n_1 + n_2$. *System (3.1) is asymptotically Schur stable and its eigen-spectrum*

$$\begin{aligned} &1 > |\lambda_1| > \cdots > |\lambda_{n_1}| > |\lambda_{n_1+1}| > \cdots > |\lambda_n| \triangleq \lambda(A_s) \cup \lambda(A_f) \\ &\lambda(A_s) = \{\lambda_1,\ \cdots,\lambda_{n_1}\},\ \lambda(A_f) = \{\lambda_{n_1+1},\ \cdots,\lambda_n\} \end{aligned} \tag{3.2}$$

possesses a gap expressed by $\mu \triangleq |\lambda_{n_1+1}|/|\lambda_{n_1}| << 1$

A standard assumption in time-scale modeling theory, which ensures the well-posedness of (3.1) is that following.

Assumption 2. *The matrix* A_4 *is invertible.*

When *Assumption 1* is met, then system (3.1) is called a *two-time-scale system*. To this end $\lambda(A_s), \lambda(A_f)$ define, respectively, the eigenvalues of the slow (dominant) parts and are the eigenvalues of the fast (non-dominant) parts of system (3.1). A useful interpretation of (3.2) is that $[A_f]^k$ tends to zero much quicker that $[A_s]^k$. Recalling the facts for any square invertible matrix P that

$$|\lambda_{max}| \leq ||P||, \qquad 1/|\lambda_{min}| \leq ||P^{-1}||$$

An alternative expression of the eigen-spectrum property is

$$||A_s^{-1}||||A_f|| \ << \ 1 \tag{3.3}$$

which designates a matrix norm condition of mode separation in linear discrete systems.

Remark 3.1. *By looking at system (3.1) with property (3.2) or (3.3), it is significant that it enjoys the mode-separation implicitly through the recognition of a gap in the eigen-spectrum.*

3.1. Mode separation in networked systems

We consider a network of n identical linear discrete systems having an eigen-spectrum gap in the manner of (3.2). For any $i = 1,\cdots,n$ where the i^{th} system at discrete instant k, represented by the state $[x_i(k),\ z_i(k)] \in \Re^{n_1+n_2}$ and the input $u(k) \in \Re^m$, is given by

$$\begin{aligned} x_i(k+1) &= A_1x_i(k) + A_2z_i(k) + B_1u_i(k) + \Gamma_1\omega_i(k), \\ z_i(k+1) &= A_3x_i(k) + A_4z_i(k) + B_2u_i(k) + \Gamma_2\omega_i(k), \\ y_i(k) &= C_1x_i(k) + C_2z_i(k) \end{aligned} \tag{3.4}$$

$$\begin{aligned} &A_1 \in \Re^{n_1\times n_1},\ A_2 \in \Re^{n_1\times n_2},\ A_3 \in \Re^{n_2\times n_1}, \\ &A_4 \in \Re^{n_2\times n_2},\ \mathbf{rank}(B_1) \ = \ \mathbf{rank}(B_2) = m \end{aligned}$$

The consensus problem of n systems is first introduced:

Definition 3.2. *The n discrete systems with mode-separation defined by (3.4) achieve asymptotic synchronization using local information if there exists a state protocol of the form*

$$u_i(k) \quad = \quad \mathbb{K}_1 \sum_{j=1}^{n} a_{ij}[x_i(k) - x_j(k)] + \mathbb{K}_2 \sum_{j=1}^{n} a_{ij}[z_i(k) - z_j(k)] \tag{3.5}$$

where $\mathbb{K}_1 \in \Re^{m\times n_1}$, $\mathbb{K}_2 \in \Re^{m\times n_2}$ *such that*

$$\lim_{k\to\infty} ||x_i(k) - x_j(k)|| = 0, \; \lim_{k\to\infty} ||z_i(k) - z_j(k)|| = 0 \tag{3.6}$$

The prime objective hereafter is the characterization of the local controllers that use local information and asymptotically synchronize the two time-scale (TTS) discrete systems defined by (3.4). Toward our objective, we express the collective dynamics characterizing the performance of the collective dynamics of n feedback coupled-systems. In terms of

$$x(k) = [x_1^t(k), \ldots, x_n^t(k)]^t \in \Re^{nn_1}, \text{ and } z(k) = [z_1^t(k), \ldots, z_n^t(k)]^t \in \Re^{nn_2}$$

we note that the asymptotic synchronization (3.6) corresponds to

$$\lim_{k\to\infty} (\mathbb{L}\otimes I_{n_1})x(k) = 0, \; \lim_{k\to\infty} (\mathbb{L}\otimes I_{n_2})z(k) = 0 \tag{3.7}$$

Invoking the fact that $\mathbb{T}\mathbb{L} = \mathbb{D}\mathbb{T}$, it follows that (3.7) can be expressed as

$$\lim_{k\to\infty} (\mathbb{D}\otimes I_{n_1})(\mathbb{T}\otimes I_{n_1})x(k) = 0, \text{ and } \lim_{k\to\infty} (\mathbb{D}\otimes I_{n_2})(\mathbb{T}\otimes I_{n_2})z(k) = 0 \tag{3.8}$$

3.2. Closed-loop representation

On substituting protocol (3.5) in system (3.4), we obtain the closed-loop dynamics:

$$\begin{aligned} x(k+1) &= \hat{\mathbf{A}}_1 x(k) + \hat{\mathbf{A}}_2 z(k) + \hat{\Gamma}_1 \omega(k), \\ z(k+1) &= \hat{\mathbf{A}}_3 x(k) + \hat{\mathbf{A}}_4 z(k) + \hat{\Gamma}_2 \omega(k) \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \hat{\mathbf{A}}_1 &= (I_n\otimes A_1) - (I_n\otimes B_1\mathbb{K}_1)(\mathbb{L}\otimes I_{n_1}), \\ \hat{\mathbf{A}}_2 &= (I_n\otimes A_2) - (I_n\otimes B_1\mathbb{K}_2)(\mathbb{L}\otimes I_{n_1}), \\ \hat{\mathbf{A}}_3 &= (I_n\otimes A_3) - (I_n\otimes B_2\mathbb{K}_1)(\mathbb{L}\otimes I_{n_1}), \\ \hat{\mathbf{A}}_4 &= (I_n\otimes A_4) - (I_n\otimes B_2\mathbb{K}_2)(\mathbb{L}\otimes I_{n_1}), \\ \hat{\Gamma}_1 &= (I_n\otimes \Gamma_1)\, , \, \hat{\Gamma}_2 = (I_n\otimes \Gamma_2). \end{aligned}$$

It is significant to notice that unlike the invertibility of matrix $I_2 - A_4$, we can not guarantee that the matrix $I_2 - \hat{\mathbf{A}}_4$ is non-singular. Hence, the well-posedness of the closed-loop dynamics (3.9) has also to be guaranteed by the selection of the matrix gains. We now proceed by making another transformation of variables

$$\widehat{x}(k) = (\mathbb{T}\otimes I_{n_1})x(k)\, , \, \widehat{z}(k) = (\mathbb{T}\otimes I_{n_2})z(k),$$

converts the aggregate dynamics (3.9) into the form

$$\begin{bmatrix} \widehat{x}(k+1) \\ \widehat{z}(k+1) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{A}}_1 & \widehat{\mathbf{A}}_2 \\ \widehat{\mathbf{A}}_3 & \widehat{\mathbf{A}}_4 \end{bmatrix} \begin{bmatrix} \widehat{x}(k) \\ \widehat{z}(k) \end{bmatrix} + \begin{bmatrix} \widehat{\Gamma}_1 \\ \widehat{\Gamma}_2 \end{bmatrix} \omega(k) \tag{3.10}$$

where $\widehat{\Gamma}_1 = (\mathbb{T}\otimes I_{n_1})(I_n\otimes \Gamma_1)$, $\widehat{\Gamma}_2 = (\mathbb{T}\otimes I_{n_1})(I_n\otimes \Gamma_2)$, and

$$\begin{aligned} \widehat{\mathbf{A}}_1 &= (I_n\otimes A_1) - (I_n\otimes B_1\mathbb{K}_1)(\mathbb{D}\otimes I_{n_1}), \\ \widehat{\mathbf{A}}_2 &= (I_n\otimes A_2) - (I_n\otimes B_1\mathbb{K}_2)(\mathbb{D}\otimes I_{n_1}), \\ \widehat{\mathbf{A}}_3 &= (I_n\otimes A_3) - (I_n\otimes B_2\mathbb{K}_1)(\mathbb{D}\otimes I_{n_1}), \\ \widehat{\mathbf{A}}_4 &= (I_n\otimes A_4) - (I_n\otimes B_2\mathbb{K}_2)(\mathbb{D}\otimes I_{n_1}). \end{aligned} \tag{3.11}$$

The following results stand out:

Proposition 3.3. *The closed-loop system (3.10)-(3.11) can be decoupled in n independent TTS systems.*

Proof: Invoking the properties of Kronecker products [3], one uses the fact that for any matrices $\mathbb{M}$, $\mathbb{N}$ of appropriate dimensions we have

$$\begin{aligned}(I_n \otimes \mathbb{M}) - (I_n \otimes \mathbb{N})(\mathbb{D} \otimes I_m) &= (I_n \otimes \mathbb{M}) - (\mathbb{D} \otimes \mathbb{N}) \\ &= \mathbf{diag}[\mathbb{M}, \cdots, \mathbb{M}] - \mathbf{diag}[\lambda_1 \mathbb{N}, \cdots, \lambda_n \mathbb{N}] \\ &= \mathbf{diag}[\mathbb{M} - \lambda_1 \mathbb{N}, \cdots, \mathbb{M} - \lambda_n \mathbb{N}]\end{aligned}$$

which eventually results in

$$\begin{aligned}\widehat{\mathbf{A}}_1 &= \mathbf{diag}[A_1 - \lambda_1 B_1 \mathbb{K}_1, \cdots, A_1 - \lambda_n B_1 \mathbb{K}_1], \\ \widehat{\mathbf{A}}_2 &= \mathbf{diag}[A_2 - \lambda_1 B_1 \mathbb{K}_2, \cdots, A_2 - \lambda_n B_1 \mathbb{K}_2], \\ \widehat{\mathbf{A}}_3 &= \mathbf{diag}[A_3 - \lambda_1 B_2 \mathbb{K}_1, \cdots, A_3 - \lambda_n B_2 \mathbb{K}_1], \\ \widehat{\mathbf{A}}_4 &= \mathbf{diag}[A_4 - \lambda_1 B_2 \mathbb{K}_2, \cdots, A_4 - \lambda_n B_2 \mathbb{K}_2].\end{aligned}$$

This in turn casts the closed-loop system (3.10) into the form

$$\begin{bmatrix} \widehat{x}_i(k+1) \\ \widehat{z}_i(k+1) \end{bmatrix} = \begin{bmatrix} (A_1 - \lambda_i B_1 \mathbb{K}_1) & (A_2 - \lambda_i B_1 \mathbb{K}_2) \\ (A_3 - \lambda_i B_2 \mathbb{K}_1) & (A_4 - \lambda_i B_2 \mathbb{K}_2) \end{bmatrix} \begin{bmatrix} \widehat{x}_i(k) \\ \widehat{z}_i(k) \end{bmatrix} + \begin{bmatrix} \widehat{\Gamma}_1 \\ \widehat{\Gamma}_2 \end{bmatrix} \omega(k) \tag{3.12}$$

for $i = 1, \cdots\cdots, n$, which is the desired result. □

Proposition 3.4. *The asymptotic synchronization problem under consideration with local state information becomes a problem of feedback simultaneous stabilization of systems in (3.12) for $i = 2; \cdots; n$.*

Proof: Recall that (3.8) can be cast into

$$\lim_{k\to\infty} (\mathbb{D} \otimes I_{n_1})\widehat{x} = 0\ , \ \lim_{k\to\infty} (\mathbb{D} \otimes I_{n_2})(\mathbb{T} \otimes I_{n_1})\widehat{z} = 0.$$

In view of the fact that $\mathbb{D} = \mathbf{diag}[\lambda_1, \cdots, \lambda_n]$, $\lambda_1 \equiv 0$, it follows that the asymptotic synchronization condition reduces to

$$\lim_{k\to\infty} \widehat{x}_i = 0, \ \lim_{k\to\infty} \widehat{z}_i = 0, \ i = 2, \cdots, n$$

which completes the proof. □

Now, it follows from the definition of $\mathbb{T}$, that the following change of variables

$$\widehat{x}(k) = (\mathbb{T}^t \otimes I_{n_1})x(k)\ , \ \widehat{z}(k) = (\mathbb{T}^t \otimes I_{n_2})z(k)$$

also hold.

Proceeding further and following the time-scale design theory [11]-[40] with $\omega_i(k) \equiv 0$, the consensus manifold depends on the behavior of $[\widehat{x}(k); \widehat{z}(k)]$. Effectively, if the discrete-system

$$\begin{bmatrix} \widehat{x}_1(k+1) \\ \widehat{z}_1(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \widehat{x}_1(k) \\ \widehat{z}_1(k) \end{bmatrix} \tag{3.13}$$

has a stable equilibrium $[\widehat{x}^*(k); \widehat{z}^*(k)]$, then the original dynamics (3.1) reaches a finite synchronization asymptotically. If the system (3.13) has unstable equilibrium point then all the systems given in (3.4) achieves consensus on divergent paths.

Finally, the well-posedness of dynamics (3.4) is similar to the system (3.10) which in turn is guaranteed if all systems given in (3.12) are also well-posed. It must be emphasized that for $i = 1$, the system is well posed due to the non-singularity of $I_2 - A_4$. The rest of the systems in (3.10) are well-posed if $\mathbb{K}_2$ is selected such that $(A_4 - \lambda_i B_2 \mathbb{K}_2)$ invertible for $i = 2, \cdots, n$.

4. Control design

In this section, we aim to provide a control design method that gives completely decouple structure of the fast and slow modes that appear in the whole closed-loop system. Following the discrete quasi-steady state concept [7, 40], the fast dynamics associated with the small eigenvalues are crucial only within a short period of time. When that transient period finished, they become negligible and the trajectories behavior the original system can be characterized only by its slow dynamics.

Formally, setting $z_i(k+1) = z_i(k) \stackrel{\Delta}{=} z_{is}(k)$ in the dynamics (3.4) is the same as neglecting the effect of the fast dynamics. Under this condition, discrete quasi-steady state is given by

$$z_{is}(k) = [I_2 - A_4]^{-1}[A_3 x_{is}(k) + B_2 u_{is}(k) + \Gamma_2 \omega(k)]$$

As demonstrated in [11] the slow-mode control law $u_s(k) = \mathbb{K}_1 x_s(k)$ and the fast-mode control law $u_f(k) = \mathbb{K}_2 z_f(k)$ will eventually produce a composite control law $u_c(k)$ based on $u_c(k) = u_s(k) + u_f(k)$.

Note that the gains $\mathbb{K}_1$, $\mathbb{K}_2$ can be synthesized for slow and fast modes subject to specified performance objective. To this end and following a discrete-time quasi-steady-state technique [9]-[12], it can be readily investigated, the aggregate model (3.10)-(3.11) can be separated into a slow dynamics

$$\begin{aligned}
x_{is}(k+1) &= [\widehat{\mathbf{A}}_o - \lambda_i \widehat{\mathbf{B}}_o \mathbb{K}_1] x_{is}(k) + \Gamma_o \omega(k), \\
y_{is}(k) &= [\widehat{\mathbf{C}}_o + \widehat{\mathbf{D}}_o \mathbb{K}_1] x_{is}(k), \\
z_{is}(k) &= -(I - \widehat{\mathbf{A}}_4^{-1})[\widehat{\mathbf{A}}_3 - \lambda_i \widehat{\mathbf{B}}_2 \mathbb{K}_2] x_{is}(k) + \Gamma_2 w(k), \\
\widehat{\mathbf{A}}_o &= \widehat{\mathbf{A}}_1 + \widehat{\mathbf{A}}_2 (I - \widehat{\mathbf{A}}_4)^{-1} \widehat{\mathbf{A}}_3, \\
\widehat{\mathbf{B}}_o &= \widehat{\mathbf{A}}_1 + \widehat{\mathbf{A}}_2 (I - \widehat{\mathbf{A}}_4)^{-1} \widehat{\mathbf{B}}_2, \\
\widehat{\mathbf{C}}_o &= \widehat{\mathbf{C}}_1 + \widehat{\mathbf{C}}_2 (I - \widehat{\mathbf{A}}_4)^{-1} \widehat{\mathbf{A}}_3, \\
\widehat{\mathbf{D}}_o &= \widehat{\mathbf{C}}_2 (I - \widehat{\mathbf{A}}_4)^{-1} \widehat{\mathbf{B}}_2, \\
\Gamma_o &= \Gamma_1 + \widehat{\mathbf{A}}_2 (I - \widehat{\mathbf{A}}_4)^{-1} \Gamma_2,
\end{aligned} \tag{4.1}$$

of order n_1, and a fast dynamics:

$$\begin{aligned}
x_{if}(k+1) &= [\widehat{\mathbf{A}}_4 - \lambda_i \widehat{\mathbf{B}}_2 \mathbb{K}_2] x_{if}(k) + \Gamma_2 w(k), \\
y_{if}(k) &= \widehat{\mathbf{C}}_2 x_{if}(k)
\end{aligned} \tag{4.2}$$

of order n_2.
We are now in a position to establish the following result:

Theorem 4.1. *Let the gain matrices $\mathbb{K}_1$ and $\mathbb{K}_2$ be designed such that for $i = 2, \cdots, n$ the matrices*

$$[\widehat{\mathbf{A}}_o - \lambda_i \widehat{\mathbf{B}}_o \mathbb{K}_1] \ , \ [A_{i4} - \lambda_i B_2 \mathbb{K}_2]$$

are all Schur stable. Then, the composite controllers gain

$$\mathbb{K}_c = \Big[I_m - \mathbb{K}_2 (I_{n_2} - A_4)^{-1} B_2 \Big] \mathbb{K}_1 - \mathbb{K}_2 (I_{n_2} - A_4)^{-1} A_3$$

asymptotically synchronize systems (3.4) with local state information.

Proof: Following the results of [9] and selecting the gain matrices $\mathbb{K}_1$ and $\mathbb{K}_2$ to stabilize the slow and fast subsystems (4.1) and (4.2), respectively for $i = 2, \cdots, n$, guarantees that

$$\begin{aligned}
\widehat{x}_i(k) &= \widehat{x}_{is}(k) + \mathbf{O}(\mu), \\
\widehat{z}_i(k) &= (I_{n_2} - A_{i4})^{-1} [A_{i3} x_{is}(k) - \lambda_i B_2 \mathbb{K}_2 \widehat{x}_{is}(k)] + \widehat{x}_{if}(k) + \mathbf{O}(\mu)
\end{aligned}$$

hold for all sufficiently small $\mu > 0$ and all $k \in [0, \infty)$. Recalling that the asymptotic synchronization corresponds to

$$\lim_{k \to \infty} (\mathbb{L} \otimes I_{n_1}) x(k) = 0 \ , \ \lim_{k \to \infty} (\mathbb{L} \otimes I_{n_2}) z(k) = 0$$

which holds true in view of

$$\begin{aligned}
(\mathbb{L} \otimes I_{n_1}) x(k) &= (\mathbb{D} \otimes I_{n_1}) \widehat{x}(k), \\
&= [0, \lambda_2 \widehat{x}_2, \cdots, \lambda_n \widehat{x}_n]^t, \\
(\mathbb{L} \otimes I_{n_2}) z(k) &= (\mathbb{D} \otimes I_{n_2}) \widehat{z}(k), \\
&= [0, \lambda_2 \widehat{z}_2, \cdots, \lambda_n \widehat{z}_n]^t.
\end{aligned}$$

Therefore, the proof is completed. □

Remark 4.2. *Basically, Theorem 4.1 guarantees asymptotic synchronization of systems (3.1). In order to achieve that, both slow and fast dynamics should be separately synchronized by stabilizing the error between the different dynamics.*

Corollary 4.3. *Suppose that $\mathbb{K}_o$ is designed such that for $i = 2, \ldots, n$ that matrices $[A_o - \lambda_i B_o \mathbb{K}_1]$ are Schur stable. If $||A_4||_s < 1$, meaning that the matrix A_4 has a spectral radius less than 1, then a lower-order controller with $\mathbb{K}_1 = \mathbb{K}_c$ will asymptotically synchronize systems (3.1) as well.*

5. $\mathbb{H}_\infty$ and $\mathbb{H}_2$ control design

We next direct attention to the composite control design problem based on the $\mathbb{H}_\infty$ and $\mathbb{H}_2$ prescribed performance criteria. Our approach is to pursue a Lyapunov-design implementation of $\mathbb{H}_\infty$ and $\mathbb{H}_2$ controllers to guarantee stabilizing system (4.1)-(4.2), then we recompose them in the manner of *Theorem 4.1*. We start with the slow-control design.

5.1. Slow $\mathbb{H}_\infty$ controller

Consider a Lyapanov candidate function $V_s = x_s^t(k)\mathbb{P}_s x_s(k),\ \mathbb{P}_s > 0$ that is associated with the slow dynamics (4.1). Given a scalar $\gamma_s > 0,$ the objective of slow mode $\mathbb{H}_\infty$ control law is to determine the controller $u_s(k) = \mathbb{K}_1 x_s(k)$ that leads system (4.1) to stability and guarantees that $||y_s(k)||_2^2 < \gamma_s^2\ ||\omega(k)||_2^2$. The synthesis of the control problem is detailed by the following result:

Theorem 5.1. *: The dynamical system (4.1) is stabilized by the control law $u_s(k) = \mathbb{K}_1 x_s(k)$ and $||y_s(k)||_2^2 < \gamma_s^2\ ||\omega(k)||_2^2$ if there exist matrices $\mathbb{X}_s > 0,\ \mathbb{Y}_s$ and a scalar $\gamma_s > 0$ satisfying the following LMIs for $i = 2,\cdots,n$ are feasible*

$$\begin{bmatrix} -\mathbb{X}_s & 0 & \mathbb{X}_s\widehat{\mathbf{A}}_o^t + \mathbb{Y}_s^t\widehat{\mathbf{B}}_o^t & \mathbb{X}_s\widehat{\mathbf{C}}_o^t + \mathbb{Y}_s^t\widehat{\mathbf{D}}_o^t \\ \bullet & -\gamma_s^2 I & \Gamma_o^t & 0 \\ \bullet & \bullet & -\mathbb{X}_s & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{5.1}$$

The $\mathbb{H}_\infty$ slow gain matrix can be obtained as $\mathbb{K}_1 = \mathbb{Y}_s\mathbb{X}_s^{-1}$.

Proof: Based on robust control theory [44] that the solution of the slow-mode $\mathbb{H}_\infty$ control problem corresponds to obtaining the controller gain $\mathbb{K}_1$ that ensures the feasibility of the following inequality:

$$\Pi_s = \Delta V_s + y_s^t(k)y_s(k)\ -\ \gamma_s^2\ \omega^t(k)\omega(k) < 0. \tag{5.2}$$

Obtaining of difference of the Lyapunov function ΔV_s along the dynamics of (4.1) with the control law $u_s(k) = \mathbb{K}_1 x_s(k)$, we rewrite inequality (5.2) in its equivalent form:

$$\begin{aligned} \Pi_s &= \begin{bmatrix} x_s, \\ \omega_s \end{bmatrix}^t \Xi_s \begin{bmatrix} x_s, \\ \omega_s \end{bmatrix} < 0, \qquad (5.3)\\ \Xi_s &= \begin{bmatrix} \Xi s1 & \Xi_{s2}, \\ \bullet & -\Xi_{s3} \end{bmatrix}, \\ \Xi s1 &= -\mathbb{P}_s + (\widehat{\mathbf{A}}_o^t + \mathbb{K}_1^t\widehat{\mathbf{B}}_o^t)\mathbb{P}_s(\widehat{\mathbf{A}}_o + \widehat{\mathbf{B}}_o\mathbb{K}_1) + (\widehat{\mathbf{C}}_o^t + \mathbb{K}_1^t\widehat{\mathbf{D}}_o^t)(\widehat{\mathbf{C}}_o + \widehat{\mathbf{D}}_o\mathbb{K}_1), \\ \Xi_{s2} &= (\widehat{\mathbf{A}}_o^t + \mathbb{K}_1^t\widehat{\mathbf{B}}_o^t)\mathbb{P}_s\Gamma_o, \\ \Xi_{s3} &= \gamma_s^2 I - \Gamma_o^t\mathbb{P}_s\Gamma_o. \end{aligned}$$

Inequality (5.3) implies that $\Xi_s < 0$. Employing Schur complements to $\Xi_s < 0$ and using the following congruent transformation $\mathbb{X}_s,\ I,\ \mathbb{X}_s,\ I$ with $\mathbb{X}_s = \mathbb{P}_s^{-1},\ \mathbb{K}_1\mathbb{X}_s = \mathbb{Y}_s,$ we obtain the LMI (5.1). □

5.2. Fast $\mathbb{H}_\infty$-control

Using the same procedure of the slow-mode case, consider the Lyapunov function $V_f = x_f^t(k)\mathbb{P}_f x_f(k),\ \mathbb{P}_f > 0$ associated with the fast-dynamics (4.2). Given a scalar $\gamma_f > 0,$ the objective of fast-mode $\mathbb{H}_\infty$ control law is to obtain the controller $u_f(k) = \mathbb{K}_2 x_f(k)$ that stabilizes system (4.2) and guaranteeing that $||y_f(k)||_2^2 < \gamma_f^2\ ||\omega(k)||_2^2$. The synthesis of the control problem is detailed by the following result:

Theorem 5.2. *: System (4.2) is stabilizable by the controller $u_f(k) = \mathbb{K}_2 x_f(k)$ and $||y_f(k)||_2^2 < \gamma_f^2\ ||\omega(k)||_2^2$ if there exist matrices $\mathbb{X}_f > 0,\ \mathbb{Y}_f$ and a scalar $\gamma_f > 0$ such that such that the following LMIs for $i = 2,\cdots,n$ are feasible*

$$\begin{bmatrix} -\mathbb{X}_f & 0 & \mathbb{X}_f\widehat{\mathbf{A}}_4^t + \mathbb{Y}_f^t\widehat{\mathbf{B}}_2^t & \mathbb{X}_f\widehat{\mathbf{C}}_2^t \\ \bullet & -\gamma_f^2 I & \Gamma_2^t & 0 \\ \bullet & \bullet & -\mathbb{X}_f & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \tag{5.4}$$

The $\mathbb{H}_\infty$ fast gain matrix can be determined by $\mathbb{K}_2 = \mathbb{Y}_f\mathbb{X}_f^{-1}$.

Proof: The proof is similar to the proof of *Theorem 5.1*. □

We combine the results of *Theorems 4.1, 5.1 and 5.2*, such that the composite $\mathbb{H}_\infty$ control law is obtained by the following result:

Lemma 5.3. *Consider the dynamical system (4.1)-(4.2) and let $\mathbb{X}_s > 0,\ \mathbb{Y}_s$ and $\mathbb{X}_f > 0,\ \mathbb{Y}_f$ be the obtained solutions of the LMIs (5.1) and (5.4), respectively. Then, the $\mathbb{H}_\infty$ composite controller is obtained in the form*

$$u_c(k) \quad = \quad [(I-\mathbb{Y}_f\mathbb{X}_f^{-1}(I-A_4)^{-1}B_2)^{-1}\mathbb{Y}_s\mathbb{X}_s^{-1}-\mathbb{Y}_f\mathbb{X}_f^{-1}(I-A_4)^{-1}A_3]x(k)+\mathbb{Y}_f\mathbb{X}_f^{-1}z(k)$$

guarantees that $||y(k)||_2^2 < \gamma^2\ ||\omega(k)||_2^2$ with $\gamma \in [\gamma_s,\ \gamma_f]$. In addition, it yields an approximation of first-order to the states of the original dynamics (3.4).

If the fast-mode dynamics is asymptotically stable, we can derive a reduced-order $\mathscr{H}_\infty$ control based on the following lemma:

Lemma 5.4. *Consider the dynamics in (3.1) and assume $X_s > 0,\ Y_s$ are the solutions obtained by solving LMI (5.1). Then the reduced-order $\mathbb{H}_\infty$ control law is given as follows*

$$u_c(k) \quad = \quad \mathbb{Y}_s\mathbb{X}_s^{-1}x(k)$$

guarantees that $||y(k)||_2^2 < \gamma^2\ ||\omega(k)||_2^2$ with $\gamma \in [\gamma_s,\ \gamma_f]$. In addition, it yields an approximation of first-order to the states of the original dynamics (3.4).

Proof: The proof follows parallel details to the results in [7, 40]. □

Remark 5.5. *It is worth noting that that the results of Theorems 5.1 and 5.2 and Lemmas 5.3-5.4 are new in the field of two time-scale discrete-time dynamical systems. Morover, it also strengthen the idea that system (3.1) is represent a good model of discrete-time practical engineering dynamics with implicit representation of the mode-separation property.*

6. $\mathbb{H}_2$ Control design

Similarly, instead of synthesizing a full-order $\mathbb{H}_2$ control, we decompose it into separate $\mathbb{H}_2$ controllers for slow and fast modes. Moreover, we recompose the controllers similar ro the manner of *Theorem 4.1*.

6.1. Slow $\mathbb{H}_2$ controller

Consider a candidate Lyapunov function $V_s = x_s^t(k)\mathbb{P}_{s2}x_s(k),\ \mathbb{P}_{s2} > 0$ corresponding the slow dynamics (4.1). The objective of slow-mode $\mathbb{H}_2$ control law is to guarantee the stability of closed-loop slow mode and to maintain the $\mathbb{H}_2$-objective of the closed loop transfer function $H_{y_sw}(s)$ from the exogenous input ω to controlled output y_s as small as possible.
Substituting the slow-mode control law $u_s(k) = \mathbb{K}_{s1}x_s(k)$ into (4.1), the closed-loop slow subsystem becomes

$$\begin{aligned} x_s(k+1) &= \widehat{\mathbf{A}}_{co}x_s(k)+\Gamma_o\omega(k), \\ y_s(k) &= \widehat{\mathbf{C}}_{co}x_s(k), \\ \widehat{\mathbf{A}}_{co} &= \widehat{\mathbf{A}}_o+\widehat{\mathbf{B}}_o\mathbb{K}_{s1},\ \widehat{\mathbf{C}}_{co} = \widehat{\mathbf{C}}_o+\widehat{\mathbf{D}}_o\mathbb{K}_{s1}. \end{aligned} \tag{6.1}$$

Based on Lyapunov theory, given the control gain matrix $\mathbb{K}_{s1}$, the closed-loop dynamics (6.1) become asymptotically stable $\omega(k) \equiv 0$ if

$$\mathbb{P}_{s2}-\widehat{\mathbf{A}}_{co}^t\mathbb{P}_{s2}\widehat{\mathbf{A}}_{co} \ > \ 0.$$

Then, we can express the square of the $\mathbb{H}_2$-norm of the transfer function $H_{zw}(s)$ in terms of the solution of a Lyapunov equation (controllability Grammian). In addition, its minimization problem with respect to the gain matrix $\mathbb{K}_{s1}$ is characterized as

$$\min\left\{\mathbf{Tr}[\widehat{\mathbf{C}}_{co}\mathbb{P}_{s2}\widehat{\mathbf{C}}_{co}^t] : \mathbb{P}_{s2}-\widehat{\mathbf{A}}_{co}^t\mathbb{P}_{s2}\widehat{\mathbf{A}}_{co}+\Gamma_o\Gamma_o^t = 0\right\}$$

where $\mathbf{Tr}r[.]$ represents the trace of a matrix. Since $\mathbb{P}_{s2} < \mathscr{P}$ for any $\mathscr{P}$ satisfying

$$\mathscr{P}-\widehat{\mathbf{A}}_{co}^t\mathscr{P}\widehat{\mathbf{A}}_{co}+\Gamma_o\Gamma_o^t \ < \ 0 \tag{6.2}$$

it is readily verified that $||H_{zw}(s)||_2^2 = \mathbf{Tr}[\widehat{\mathbf{C}}_{co}\mathbb{P}_{s2}\widehat{\mathbf{C}}_{co}^t] < \nu$ if and only if there exists $\mathscr{P} > 0$ satisfying (6.2) and $\mathbf{Tr}[\widehat{\mathbf{C}}_{co}\mathbb{P}_{s2}\widehat{\mathbf{C}}_{co}^t] < \nu$. We introduce a new dummy variable $\mathscr{Z}$, to obtian the following synthesis result:

Theorem 6.1. *: The dynamical system (4.1) can bw stabilizable by the control law $u_s(k) = \mathbb{K}_{s1}x_s(k)$ and $||H_{zw}(s)||_2^2 < \nu$ for a prescribed ν if and only if there exist positive definite matrices $\mathscr{P}$, $\mathscr{Z}$, and a matrix $\mathscr{Q}$ with appropriate dimensions satisfying the following conditions:*

$$\begin{aligned} &\mathbf{Tr}(\mathscr{Z}) \ < \ \nu,\ \begin{bmatrix} \mathscr{Z} & \widehat{\mathbf{C}}_o\mathscr{P}+\widehat{\mathbf{D}}_o\mathscr{Q} \\ \bullet & \mathscr{P} \end{bmatrix} \ > \ 0, \\ &\begin{bmatrix} \mathscr{P} & \widehat{\mathbf{A}}_o\mathscr{P}+\widehat{\mathbf{B}}_o\mathscr{Q} & \Gamma_o \\ \bullet & \mathscr{P} & 0 \\ \bullet & \bullet & I \end{bmatrix} \ > \ 0 \end{aligned} \tag{6.3}$$

Moreover, the slow-mode gain matrix is obtained by $\mathbb{K}_{s1} = \mathscr{Q}\mathscr{P}^{-1}$

proof: It can be proved based on standard convex analysis similar to procedure presented in [45]. □

6.2. Fast $\mathbb{H}_2$ controller

Similarly, consider a Lyapunov function $V_f = x_f^t(k)\mathbb{P}_{f2}x_f(k)$, $\mathbb{P}_{f2} > 0$ associated with the fast dynamics (4.2). The objective of fast-mode $\mathbb{H}_2$ control law is to guarantee the stability of closed-loop fast-mode and to maintain a prescribed $\mathbb{H}_2$-performance the closed loop transfer function $H_{y_fw}(s)$ from ω to y_f as small as possible. The corresponding synthesis result is provided by the following result which follows a parallel development to *Theorem 6.1*:

Theorem 6.2. *: System (4.2) is stabilizable by the controller $u_f(k) = \mathbb{K}_{f2}x_f(k)$ and $||H_{y_fw}(s)||_2^2 < \nu$ for a prescribed ν if and only if there exist matrices $\mathscr{R} > 0$, $\mathscr{S}$, $\mathscr{W} > 0$ such that*

$$\boldsymbol{Tr}(\mathscr{W}) \; < \; \nu, \left[\begin{array}{cc} \mathscr{W} & \widehat{\mathbf{C}}_2\mathscr{R} \\ \bullet & \mathscr{R} \end{array}\right] > 0,$$

$$\left[\begin{array}{ccc} \mathscr{R} & \widehat{\mathbf{A}}_4\mathscr{R} + \widehat{\mathbf{B}}_2\mathscr{S} & \Gamma_2 \\ \bullet & \mathscr{R} & 0 \\ \bullet & \bullet & I \end{array}\right] > 0 \tag{6.4}$$

Moreover, the fast gain is given by $\mathbb{K}_{f2} = \mathscr{S}\mathscr{R}^{-1}$

Once again, by combining *Theorems 4.1 , 6.1 and 6.2*, the composite $\mathbb{H}_2$ controller is obtained by the following result:

Lemma 6.3. *Consider the dynamical system (3.1). Let $\mathscr{P} > 0$, $\mathscr{Q}$, $\mathscr{X} > 0$ and $\mathscr{R} > 0$, $\mathscr{S}$, $\mathscr{W} > 0$ be the given solutions of the conditions in (6.3) and (6.4). Then we obtain the $\mathscr{H}_2$ composite control as*

$$u_c(k) \;\; = \;\; [(I - \mathscr{S}\mathscr{R}^{-1}(I - A_4)^{-1}B_2)^{-1}\mathscr{Q}\mathscr{P}^{-1} - \mathscr{S}\mathscr{R}^{-1}(I - A_4)^{-1}A_3]x(k) + \mathscr{S}\mathscr{R}^{-1}z(k)$$

that guarantees the stability of closed-loop system and maintaining the $\mathbb{H}_2$-norm of the closed loop transfer function $H_{yw}(s)$ from ω to y_s as small as possible. In addition, it yields an approximation of first-order to the states of the original dynamics (3.1).

Remark 6.4. *In a similar way, the results of Theorems 6.1 and 6.2 and Lemmas 6.3 and 5.4 are contributions to the field of discrete systems with mode-separation. It is important to assert the relevance of the permutation and/or scaling in casting the discrete dynamics of the type (3.1) in the structure of two-time-scale discrete modes with the property of implicit characterization of the mode-separation.*

7. Simulation example

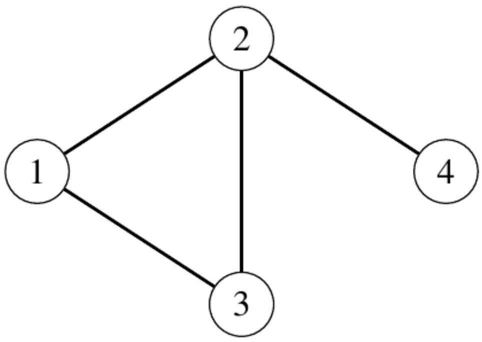

Figure 7.1: Connected topology of 4 agents.

Now, we apply the provided theoretical results to an engine model with dynamometer test. A linearizion is used to obtain the linear model as developed in [46].The state variables are selected as follows: the speed of the rotor, shaft-torque, speed of the engine and amplifier's current.The throttle-servo voltage and dynamometer current are input variables.Consider a group of 4 agent whose graph topology is shown in Fig 7.1. It can be easily shown that the model has a mode separation with two time scales: slow states ($n_1 = 2$) and three fast states ($n_2 = 3$). Using the data given in [46], the slow dynamics (4.1) is described by

$$A_o \;\; = \;\; \left[\begin{array}{cc} 0.762 & 0 \\ -0.029 & 0.689 \end{array}\right], B_o = \left[\begin{array}{cc} 0 & 1.049 \\ 0.090 & -0.018 \end{array}\right]$$

$$C_o \;\; = \;\; \left[\begin{array}{cc} 0 & 1 \\ -0.221 & 8.191 \end{array}\right], D_o = \left[\begin{array}{cc} 0 & 0 \\ 0.765 & -0.144 \end{array}\right]$$

whereas the fast model (4.2) is given by

$$A_4 = \begin{bmatrix} 0.160 & -0.002 & -0.258 \\ 0 & -0.038 & 0 \\ 0.231 & 0 & -0.381 \end{bmatrix}, B_2 = \begin{bmatrix} 0.702 & -0.083 \\ 0 & 22.400 \\ 0.142 & 0.026 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Based on *Theorems 5.1 and 5.2*, we obtain the $\mathbb{H}_\infty$ slow and fast matrix gains as follows

$$\begin{aligned} \mathbb{K}_1 &= \begin{bmatrix} 0.008 & -0.094 \\ 0.007 & 0.089 \end{bmatrix}, \gamma_s = 0.453, \\ \mathbb{K}_2 &= \begin{bmatrix} -0.286 & -0.001 & -0.079 \\ -0.277 & -0.011 & -0.084 \end{bmatrix}, \gamma_f = 0.629 \end{aligned}$$

This gives the $\mathbb{H}_\infty$ composite control law as

$$\begin{aligned} \mathbb{K}_c &= \begin{bmatrix} 0.054 & 0.030 & -0.288 & 0.012 & -0.078 \\ 0.051 & 0.114 & -0.269 & 0.078 & -0.103 \end{bmatrix}, \\ \gamma_c &\in [0.453,\ 0.629]. \end{aligned}$$

In addition, applying *Theorems 6.1 and 6.2* with $\nu = 1.245$ gives the following $\mathbb{H}_2$ slow and fast matrix gains as

$$\mathbb{K}_1 = \begin{bmatrix} 0.016 & -0.085 \\ 0.002 & 0.097 \end{bmatrix}, \mathbb{K}_2 = \begin{bmatrix} -0.305 & -0.013 & -0.044 \\ -0.225 & -0.001 & -0.103 \end{bmatrix}.$$

Based on the gain matrices $\mathbb{K}_1$ and $\mathbb{K}_1$, the $\mathbb{H}_2$ composite control law is obtained:

$$\mathbb{K}_c = \begin{bmatrix} 0.0784 & -0.248 & -50.87 & -0.0065 & -0.0771 \\ 0.095 & 0.0534 & -144.7 & -0.0047 & -0.246 \end{bmatrix}$$

According to *Lemmas 5.3-6.3*, the composite gains guarantee good approximation to the closed-loop state trajectories. Figure 7.2 shows the output response of the original system. The output disagreement of all agents are demonstrated in Figs 7.3, 7.4 and 7.5.

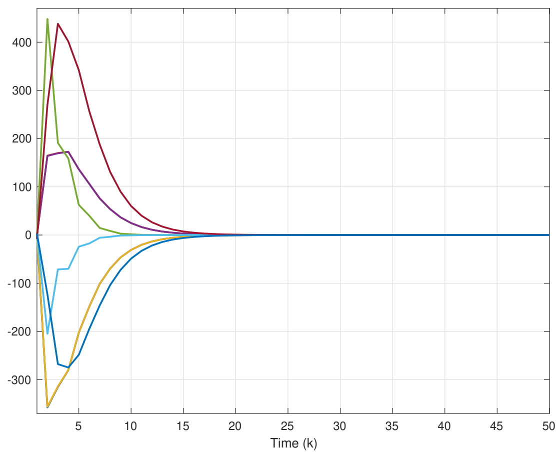

Figure 7.2: Output response of the original system.

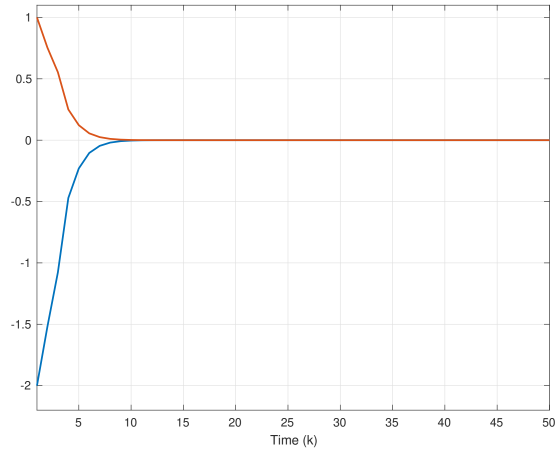

Figure 7.3: Disagreement among outputs $y_1(k) - y_2(k)$

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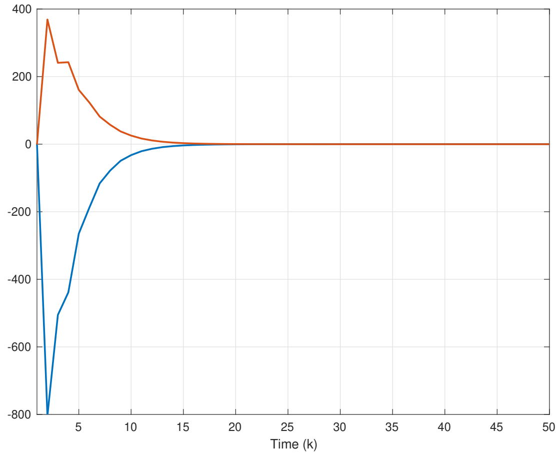

Figure 7.4: Disagreement among outputs $y_2(k) - y_3(k)$

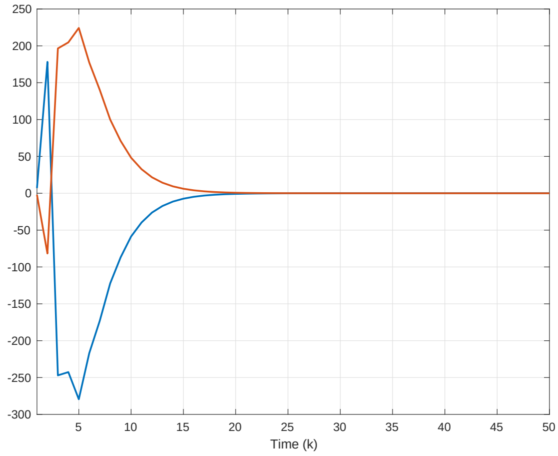

Figure 7.5: Disagreement among outputs $y_3(k) - y_4(k)$

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8. Conclusions

This article investigated feedback control synthesis problem of a broad range of discrete-time dynamics that possesses eigen-spectrum gap. The fast and slow modes are assumed to be observable and controllable. This assumption constitutes a very mild condition and less conservative than assuming observability and controllability of the original dynamical system. Adopting either the $\mathbb{H}_\infty$ or $\mathbb{H}_\infty$ optimization schemes, we have investigated two-stage design approach based on separate gain matrices for the slow and fast modes. We have constructed a composite controller to obtain first-order approximations to the behavior of the discrete-time dynamics. Moreover, the paper investigates the interactions between multiple time-scale-networked dynamics and gives guarantees on the stability of the disagreement among coupled systems. The addressed effectiveness of the presented methodologies have been demonstrated using a typical application model.

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Fundamental Journal of Mathematics and Applications, 4 (2) (2021) 45-48
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://doi.org/10.33401/fujma.842104

$\mathfrak{I}$-Limit and $\mathfrak{I}$-Cluster Points for Functions Defined on Amenable Semigroups

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Article Info

Keywords: *Amenable semigroups, Folner sequence, $\mathfrak{I}$-cluster points, $\mathfrak{I}$-convergence, $\mathfrak{I}$-limit points*
2010 AMS: *40A05, 40A35, 43A07*
Received: *18 December 2020*
Accepted: *25 February 2021*
Available online: *05 March 2021*

Abstract

In this paper firstly, for functions defined on discrete countable amenable semigroups (DCASG), the notions of $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points are introduced. Then, for the functions, the notions of $\mathfrak{I}$-limit superior and inferior are examined.

1. Preliminaries

The notion of $\mathfrak{I}$-convergence, based on the structure of the ideal $\mathfrak{I}$ of subset of the set of natural numbers $\mathbb{N}$, was introduced and studied by Kostyrko et al. [1, 2]. After than, regarding this notion, Demirci [3] examined the notions of $\mathfrak{I}$-limit superior and inferior.

One of first studies on amenable semigroups (ASG) is made by Day [4]. Then, Douglass [5] and Mah [6] studied the notions of summability in ASG. The notion of arithmetic mean was extended to ASG by Douglas [5] and Douglas obtained a characterization for the notion of almost convergence in ASG. Recently, Nuray and Rhoades [7] introduced the notions of convergence and statistical convergence in ASG.

The aim of this paper is to introduce some new notions for functions defined on DCASG and to examine some properties of them. Our new notions yield the notions in [2, 3] when the ASG is the additive positive integers.

Now, for better understanding our study, we recall the basic notations (see, [1, 2, 7, 8, 9]).

Let $\mathscr{G}$ be a DCASG with identity in which both left and right cancelation laws hold and $r(\mathscr{G})$ denote the space of real functions on $\mathscr{G}$.

If $\mathscr{G}$ is a countable amenable group, then there exists a sequence $\{\lambda_i\}$ of finite subsets of $\mathscr{G}$ such that

i. $\mathscr{G} = \bigcup_{i=1}^{\infty} \lambda_i$,

ii. $\lambda_i \subset \lambda_{i+1} \quad (i = 1, 2, \ldots)$,

iii. $\lim\limits_{i\to\infty} \frac{|\lambda_i \vartheta \cap \lambda_i|}{|\lambda_i|} = 1, \quad \lim\limits_{i\to\infty} \frac{|\vartheta \lambda_i \cap \lambda_i|}{|\lambda_i|} = 1, \quad$ for all $\vartheta \in \mathscr{G}$ (see, [9]).

If a sequence of finite subsets of $\mathscr{G}$ satisfy $(i) - (iii)$, then this sequence is called a Folner sequence of $\mathscr{G}$.

A familiar Folner sequence giving rise to the classical Cesàro method of summability is the sequence

$$\lambda_i = \{0, 1, 2, \ldots, i-1\}.$$

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Let $\mathscr{G}$ be a DCASG with identity in which both left and right cancelation laws hold. For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a function $h \in r(\mathscr{G})$ is called convergent to $l \in \mathbb{R}$ if every $\xi > 0$ there exists a $s_0 \in \mathbb{N}$ such that

$$|h(\vartheta) - l| < \xi,$$

for all $n > s_0$ and $\vartheta \in \mathscr{G} \setminus \lambda_n$.
Let $Y \neq \emptyset$. A family of sets $\mathfrak{I} \subseteq 2^Y$ (the power set of Y) is called an ideal if and only if

i. $\emptyset \in \mathfrak{I}$,

ii. $U \cup V \in \mathfrak{I}$ for $U, V \in \mathfrak{I}$,

iii. $V \in \mathfrak{I}$ for $U \in \mathfrak{I}$ and $V \subseteq U$.

An ideal $\mathfrak{I} \subseteq 2^Y$ is called non-trivial if $Y \notin \mathfrak{I}$. A non-trivial ideal $\mathfrak{I} \subseteq 2^Y$ is called admissible if

$$\mathfrak{I} \supset \{\{y\} : y \in Y\}.$$

All ideals in this paper are assumed to be admissible in $\mathbb{N}$.
Let $\mathscr{G}$ be a DCASG with identity in which both left and right cancelation laws hold. For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a function $h \in r(\mathscr{G})$ is called $\mathfrak{I}$-convergent to $l \in \mathbb{R}$ if every $\xi > 0$

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| \geq \xi\right\} \in \mathfrak{I}.$$

2. Main results

In this section firstly, for functions defined on DCASG, the notions of $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points are introduced.

Definition 2.1. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a number $l \in \mathbb{R}$ is called a $\mathfrak{I}$-limit point of a function $h \in r(\mathscr{G})$ if there exists a set $F \subset \mathscr{G}$ ($F \notin \mathfrak{I}$) such that*

$$\lim h(\vartheta) = l \quad (\vartheta \in F \setminus \lambda_i).$$

Definition 2.2. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a number $c \in \mathbb{R}$ is called an $\mathfrak{I}$-cluster point of a function $h \in r(\mathscr{G})$ if every $\xi > 0$*

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - c| < \xi\right\} \notin \mathfrak{I}.$$

For any function $h \in r(\mathscr{G})$, let $\mathfrak{I}^h_\Lambda(\mathscr{G})$ and $\mathfrak{I}^h_\Gamma(\mathscr{G})$ denote the set of all $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points of the function h, respectively.

Theorem 2.3. *For each function $h \in r(\mathscr{G})$,*

$$\mathfrak{I}^h_\Lambda(\mathscr{G}) \subseteq \mathfrak{I}^h_\Gamma(\mathscr{G}).$$

Proof. Let $l \in \mathfrak{I}^h_\Lambda(\mathscr{G})$. Then, there exists a set $F \notin \mathfrak{I}$ such that

$$\lim h(\vartheta) = l \quad (\vartheta \in F \setminus \lambda_i).$$

Hence, for every $\delta > 0$ there exists a $s_0 = s_0(\delta) \in \mathbb{N}$ such that for $\vartheta \in F \setminus \lambda_i$ we have

$$|h(\vartheta) - l| < \delta,$$

for all $i > s_0$. Therefore,

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| < \delta\right\} \supset F \setminus \lambda_i$$

and so

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| < \delta\right\} \notin \mathfrak{I},$$

which means that $l \in \mathfrak{I}^h_\Gamma(\mathscr{G})$. $\square$

Theorem 2.4. *For each function $h \in r(\mathscr{G})$, the set $\mathfrak{I}^h_\Lambda(G)$ is a closed set in $\mathbb{R}$.*

Proof. Let $l \in \overline{\mathfrak{I}_\Lambda^h(\mathscr{G})}$ and $\xi > 0$. Then, there exists

$$l_0 \in \mathfrak{I}_\Lambda^h(\mathscr{G}) \cap B(l,\xi).$$

Choose $\delta \geq 0$ such that

$$B(l_0,\delta) \subset B(l,\xi).$$

Obviously, we have

$$\left\{\vartheta \in \mathscr{G} : |l - h(\vartheta)| < \xi\right\} \supset \left\{\vartheta \in \mathscr{G} : |l_0 - h(\vartheta)| < \delta\right\}.$$

Therefore,

$$\left\{\vartheta \in \mathscr{G} : |l - h(\vartheta)| < \xi\right\} \notin \mathfrak{I}$$

and so $l \in \mathfrak{I}_\Lambda^h(\mathscr{G})$. □

Now secondly, for functions defined on DCASG, the notions of $\mathfrak{I}$-limit superior and inferior are examined.
For a function $h \in r(\mathscr{G})$, we define the following sets:

$$A_h := \left\{a \in \mathbb{R} : \{\vartheta \in \mathscr{G} : h(\vartheta) < a\} \notin \mathfrak{I}\right\},$$

similarly

$$B_h := \left\{b \in \mathbb{R} : \{\vartheta \in \mathscr{G} : h(\vartheta) > b\} \notin \mathfrak{I}\right\}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.

Definition 2.5. *For a function $h \in r(\mathscr{G})$, $\mathfrak{I}$-limit inferior is given by*

$$\mathfrak{I} - \liminf h = \begin{cases} \inf A_h & , \quad A_h \neq \emptyset \\ \infty & , \quad A_h = \emptyset \end{cases}$$

also, $\mathfrak{I}$-limit superior is given by

$$\mathfrak{I} - \limsup h = \begin{cases} \sup B_h & , \quad B_h \neq \emptyset \\ -\infty & , \quad B_h = \emptyset. \end{cases}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.

For any function $h \in r(\mathscr{G})$, it is easy to see that

$$\mathfrak{I} - \liminf h \leq \mathfrak{I} - \limsup h$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.

Definition 2.6. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a function $h \in r(\mathscr{G})$ is called $\mathfrak{I}$-bounded if there exists a M such that*

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta)| < M\right\} \in \mathfrak{I}.$$

Note that $\mathfrak{I}$-boundedness for a function $h \in r(\mathscr{G})$ implies that $\mathfrak{I} - \liminf h$ and $\mathfrak{I} - \limsup h$ are finite for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.
The following theorem can be proved by a simple least upper bound argument.

Theorem 2.7. *For any function $h \in r(\mathscr{G})$; if $\gamma = \mathfrak{I} - \liminf h$ is finite, then for every $\xi > 0$*

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) < \gamma + \xi\right\} \notin \mathfrak{I} \ \text{ and } \ \left\{\vartheta \in \mathscr{G} : h(\vartheta) < \gamma - \xi\right\} \in \mathfrak{I}, \tag{2.1}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.
Conversely if (2.1) holds for every $\xi > 0$, then

$$\mathfrak{I} - \liminf h = \gamma.$$

The dual statement for $\mathfrak{I} - \limsup h$ is as follows:

Theorem 2.8. *For any function $h \in r(\mathscr{G})$; if $\eta = \mathfrak{I} - \limsup h$ is finite, then for every $\xi > 0$*

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) > \eta - \xi\right\} \notin \mathfrak{I} \text{ and } \left\{\vartheta \in \mathscr{G} : h(\vartheta) > \eta + \xi\right\} \in \mathfrak{I}, \tag{2.2}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.
Conversely if (2.2) holds for every $\xi > 0$, then

$$\mathfrak{I} - \limsup h = \eta.$$

Theorem 2.9. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$; $\mathfrak{I} - \liminf h = \mathfrak{I} - \limsup h$ if and only if the $\mathfrak{I}$-bounded function h is $\mathfrak{I}$-convergent.*

Proof. For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, let

$$\gamma = \mathfrak{I} - \liminf h \text{ and } \eta = \mathfrak{I} - \limsup h.$$

Firstly, we assume that $\mathfrak{I} - \lim h = l$ and $\xi > 0$. Then,

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| \geq \xi\right\} \in \mathfrak{I}$$

and so

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) > l + \xi\right\} \in \mathfrak{I},$$

which implies that $\eta \leq l$. Also, we have

$$\left\{\vartheta : h(\vartheta) < l - \xi\right\} \in \mathfrak{I},$$

which implies that $l \leq \gamma$. Therefore $\eta \leq \gamma$, which we combine with the fact that

$$\mathfrak{I} - \liminf h \leq \mathfrak{I} - \limsup h,$$

to conclude that $\gamma = \eta$.
Now, secondly, we assume that for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$,

$$\mathfrak{I} - \liminf h = \mathfrak{I} - \limsup h.$$

If $\xi > 0$, then (2.1) and (2.2) imply

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) > l + \frac{\xi}{2}\right\} \in \mathfrak{I} \text{ and } \left\{\vartheta \in \mathscr{G} : h(\vartheta) < l - \frac{\xi}{2}\right\} \in \mathfrak{I}.$$

Hence, for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, we have

$$\mathfrak{I} - \lim h = l.$$

□

3. Conclusion

We investigated the notions of $\mathfrak{I}$-limit points and $\mathfrak{I}$-cluster points for functions defined on discrete countable amenable semigroups. These notions can also be studied for double sequences in the future.

Acknowledgement

This study is supported by TÜBİTAK (The Scientific and Technological Research Council of Turkey) with the project number 120F082.

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 49-58
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.809913

Circulant m-Diagonal Matrices Associated with Chebyshev Polynomials

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Article Info

Keywords: *Chebyshev polynomial, Circulant matrix, Eigenvalue, Eigenvector*
2010 AMS: *15Bxx, 15A18, 12E10*
Received: *13 October 2020*
Accepted: *14 March 2021*
Available online: *15 March 2021*

Abstract

In this study, we deal with an m banded circulant matrix, generally called circulant m-diagonal matrix. This special family of circulant matrices arise in many applications such as prediction, time series analysis, spline approximation, difference solution of partial differential equations, and so on. We firstly obtain the statements of eigenvalues and eigenvectors of circulant m-diagonal matrix based on the Chebyshev polynomials of the first and second kind. Then we present an efficient formula for the integer powers of this matrix family depending on the polynomials mentioned above. Finally, some illustrative examples are given by using maple software, one of computer algebra systems (CAS).

1. Introduction

Multiplying a vector by a circulant matrix is equivalent to a well-known operation called a circular convolution. Convolution operations, and so circulant matrices, arise in number of applications: digital signal processing, image compression, physics/engineering simulations, number theory, coding theory, cryptography, etc. Numerical solutions of certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [1]-[3].

A certain type of transformation of a set of numbers can be represented as the multiplication of a vector by a square matrix. Repetition of the operation is equivalent to multiplying the original vector by a power of the matrix. Solving some difference equations, differential and delay differential equations and boundary value problems, we need to compute the arbitrary integer powers of a square matrix [4, 5]. The powers of matrices are thus of considerable importance.

Computing the integer powers of circulant matrices depending on Chebyshev polynomials recently has been a very attractive problem [6]-[13]. For example, Rimas obtained a general expression for the entries of the r^{th} power $(r \in \mathbb{N})$ of the $n \times n$ real symmetric circulant $\operatorname{circ}_n(0,1,0,\dots,0,1)$ (see [6] or [7] for the odd case and [8] or [9] for the even case). In [10], Gutiérrez obtained a general expression for the entries of the positive integer powers of complex symmetric circulant matrix given by

$$\begin{array}{ll} \operatorname{circ}_n\left(b_0,b_1,\dots,b_{\frac{n-1}{2}},b_{\frac{n-1}{2}},\dots,b_1\right) & \text{if } n \text{ is odd,} \\ \operatorname{circ}_n\left(b_0,b_1,\dots,b_{\frac{n}{2}-1},b_{\frac{n}{2}},b_{\frac{n}{2}-1},\dots,b_1\right) & \text{if } n \text{ is even.} \end{array} \tag{1.1}$$

by generalizing the results derived by Rimas in [6]-[9].

In [11], Köken et al. obtained a general expression for the entries of the r^{th} power $(r \in \mathbb{N})$ of odd order circulant matrices of the type $\operatorname{circ}_n(0,a,0,\dots,0,b)$. In [12], we presented a single expression for the integer powers of the circulant matrix $\operatorname{circ}_n(a_0,a_1,0,\dots,0,a_{-1})$ of odd and even order by generalizing the results derived by Köken et al. in [11].

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In the current study, we consider an $n\times n$ circulant m-diagonal matrix A_n, that clearly is as,

$$\begin{aligned} A_n &= \operatorname{circ}_n\left(a_0,a_1,\ldots,a_m,0,\ldots,0,a_{-m},\ldots,a_{-1}\right) \\ &= \begin{bmatrix} a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 & a_{-m} & \cdots & a_{-1} \\ a_{-1} & a_0 & a_1 & \ddots & a_m & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & a_{-1} & a_0 & a_1 & \ddots & \ddots & \ddots & \ddots & \ddots & a_{-m} \\ a_{-m} & \ddots & a_{-1} & a_0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & a_{-m} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_m & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_m \\ a_m & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & a_{-m} & \cdots & a_{-1} & a_0 & a_1 \\ a_1 & \cdots & a_m & 0 & \cdots & 0 & a_{-m} & \cdots & a_{-1} & a_0 \end{bmatrix} \end{aligned} \tag{1.2}$$

for all $3\le n\in\mathbb{N}$, where $1\le m\le\left\lfloor\frac{n-1}{2}\right\rfloor$ and

$$a_{-i}=a_{n-i},\ \ i=1,\ldots,m. \tag{1.3}$$

We organize this paper as the following parts. In Section 2, we give some fundamental notations, definitions and important properties that we will need for the next sections. In Section 3, we introduce Lemma 3.1 and Theorem 3.3 that respectively give the statements of eigenvalues and eigenvectors of circulant m-diagonal matrix in (1.2) depending on the Chebyshev polynomials of the first and second kind. In Section 4, we obtain an efficient expression for the integer power of this matrix by means of the polynomials mentioned above. In Section 5, some illustrative examples are given. Finally, we will finish the paper with two Maple procedures.

2. Preliminaries

In this part, we present some fundamental notations, definitions and necessary properties for the next parts.
An $n\times n$ circulant matrix is defined in [14] as

$$C_n:=\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ddots & & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_0 \end{bmatrix}$$

where $c_{ij}=c_{(j-i)(mod\ n)}$. It can be clearly seen from above that each row of C_n is a cyclic shift of the previous row. Since C_n has at most n distinct elements in each row, it is often represented by

$$C_n:=\operatorname{circ}_n\left(c_0,c_1,\ldots,c_{n-1}\right).$$

Let $n\ge 1$ be a fixed integer and ω be the primitive nth root of unity; namely, $\omega=e^{\mathbf{i}\frac{2\pi}{n}}=\cos\left(\frac{2\pi}{n}\right)+\mathbf{i}\sin\left(\frac{2\pi}{n}\right)$, $\mathbf{i}=\sqrt{-1}$. The well-known eigenvalue decomposition of the matrix $C_n=\operatorname{circ}_n\left(c_0,c_1,\ldots,c_{n-1}\right)$ is that

$$C_n=F_n^*D_nF_n \tag{2.1}$$

where * denotes the conjugate transpose (i.e $F_n^*=\overline{F}_n^T$), F_n called $n\times n$ Fourier matrix that contains the eigenvectors of C_n such that

$$[F_n^*]_{u,v}=\frac{1}{\sqrt{n}}\omega^{(u-1)(v-1)},\ \ 1\le u,v\le n$$

and $D_n=\operatorname{diag}\left(\lambda_1,\lambda_2,\ldots,\lambda_n\right)$ with

$$\lambda_k=\sum_{r=1}^{n}c_{r-1}\omega^{(k-1)(r-1)},\ \ \ 1\le k\le n \tag{2.2}$$

are the eigenvalues of C_n [14].
It can be easily seen that the matrices F_n and F_n^* are symmetric:

$$F_n = F_n^T, \quad F_n^* = (F_n^*)^T = \overline{F_n} \tag{2.3}$$

where we can deduce $[F_n]_{u,v} = [F_n]_{v,u}$. It is also one of fundamental property that the matrix F_n is unitary: $F_n F_n^* = F_n^* F_n = I$ [14].
In [15], we have the eigenvector $f^{(k)}$ of C_n corresponding to the eigenvalue λ_k in (2.2) as the following

$$f^{(k)} = \frac{1}{\sqrt{n}} \left(1, \omega^{k-1}, \omega^{2(k-1)}, \ldots, \omega^{(n-1)(k-1)}\right)^T. \tag{2.4}$$

Since the product of two circulant matrices is again a circulant matrix, the r^{th} power ($r \in \mathbb{N}$) of C_n is also circulant and it is, from the well-known expression (2.1), obtained as

$$C_n^r = F_n^* D_n^r F_n = F_n^* \operatorname{diag}(\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r)\ F_n. \tag{2.5}$$

If C_n is nonsingular, then the expression (2.5) applies to negative integers.

Definition 2.1. *The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation*

$$T_n(x) = \cos n\theta \quad \text{when } x = \cos\theta.$$

Definition 2.2. *The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by*

$$U_n(x) = \sin(n+1)\theta / \sin\theta \quad \text{when } x = \cos\theta.$$

One can reach the following result about Chebyshev polynomials in [16]:
Let $T_k(x)$ and $U_k(x)$ $(k \in \mathbb{N} \cup \{0\})$ be the k^{th} degree Chebyshev polynomials of the first and second kind, respectively. Then

$$T_k(x) = \cos(k \arccos x) \text{ and } U_k(x) = \frac{\sin((k+1)\arccos x)}{\sin(\arccos x)} \tag{2.6}$$

for $-1 \le x \le 1$. Moreover, one can find more applications related this polynomials in [17]-[19].

3. Eigenvalues and eigenvectors of circulant m-diagonal matrix

In this part, we give the expressions of eigenvalues and eigenvectors of $A_n = \operatorname{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$ depending on Chebyshev polynomials of the first and second kind.

Lemma 3.1. *Consider $3 \le n \in \mathbb{N}$, $1 \le m \le \left\lfloor \frac{n-1}{2} \right\rfloor$ and $a_i \in \mathbb{R}$ $(i = 0, \pm 1, \ldots \pm m)$. Let $A_n = \operatorname{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$ be an $n \times n$ circulant matrix and $\alpha_k = \cos\frac{2\pi(k-1)}{n}$ for every $1 \le k \le n$. Then the eigenvalues of A_n are*

$$\lambda_k = a_0 + \sum_{l=1}^{m} \left((a_l + a_{-l})\, T_l(\alpha_k) + \mathbf{i}\,(a_l - a_{-l}) \operatorname{sgn}\left(\frac{n}{2} + 1 - k\right) \sqrt{1 - \alpha_k^2} \lim_{j \to k} U_{l-1}(\alpha_j) \right) \tag{3.1}$$

where λ_k is the k^{th} eigenvalue of A_n and sgn *denotes the signum function.*

Proof. Taking into account (2.2), (1.3) and $\omega^{(k-1)(n+2-r-1)} = \omega^{-(k-1)(r-1)}$ for all $2 \le r \le n$ (see [10]), we can write λ_k as

$$\begin{aligned} \lambda_k &= a_0 + a_1 \omega^{(k-1)1} + \ldots + a_m \omega^{(k-1)m} + a_{n-m} \omega^{(k-1)(n-m)} + \ldots + a_{n-1} \omega^{(k-1)(n-1)} \\ &= a_0 + a_1 \omega^{(k-1)} + \ldots + a_m \omega^{(k-1)m} + a_{-m} \omega^{-(k-1)m} + \ldots + a_{-1} \omega^{-(k-1)}. \end{aligned}$$

From the definition of ω, we get

$$\lambda_k = a_0 + \sum_{l=1}^{m} \left((a_l + a_{-l}) \cos\frac{2\pi(k-1)l}{n} + \mathbf{i}\,(a_l - a_{-l}) \sin\frac{2\pi(k-1)l}{n} \right). \tag{3.2}$$

Observe that from (2.6), we have

$$T_m\left(\cos\frac{2\pi(k-1)}{n}\right) = \cos\frac{2\pi(k-1)m}{n},$$

and

$$U_{m-1}\left(\cos\frac{2\pi(k-1)}{n}\right) = \frac{\sin\frac{2\pi(k-1)m}{n}}{\sin\frac{2\pi(k-1)}{n}}$$

where there exists indeterminate form $0/0$ for $k=1$ and $k=\frac{n}{2}+1$. Then we can construct the expression (3.2) as

$$\lambda_k = a_0 + \sum_{l=1}^{m}\left((a_l+a_{-l})\,T_l\left(\cos\frac{2\pi(k-1)}{n}\right) + \mathbf{i}\,(a_l-a_{-l})\sin\frac{2\pi(k-1)}{n}\lim_{j\to k}U_{l-1}\left(\cos\frac{2\pi(j-1)}{n}\right)\right).$$

Consequently, we reach the desired result by transforming $\cos\frac{2\pi(k-1)}{n}=\alpha_k$ and then

$$\sin\frac{2\pi(k-1)}{n} = \begin{cases} \sqrt{1-\alpha_k^2} & \text{if } \frac{n}{2}+1-k>0, \\ 0 & \text{if } \frac{n}{2}+1-k=0, \\ -\sqrt{1-\alpha_k^2} & \text{if } \frac{n}{2}+1-k<0. \end{cases} \tag{3.3}$$

□

With the help of Lemma 3.1, we reach a nice result for the eigenvalues of $A_n=\operatorname{circ}_n(a_0,a_1,\ldots a_m,0,\ldots,0,a_{-m},\ldots a_{-1})$. Since

$$\cos\frac{2\pi(n+2-k-1)}{n} = \cos\frac{2\pi(k-1)}{n}$$

and

$$\sin\frac{2\pi(n+2-k-1)}{n} = -\sin\frac{2\pi(k-1)}{n}.$$

We obtain that $\overline{\lambda_k}=\lambda_{n+2-k}$ $(2\le k\le n)$ from (3.2). Clearly, if we rewrite this eigenvalues in a diagonal matrix, then

$$\begin{aligned} D_n &= \operatorname{diag}\left(\lambda_1,\lambda_2,\ldots,\lambda_{\frac{n+1}{2}},\overline{\lambda_{\frac{n+1}{2}}},\ldots,\overline{\lambda_2}\right) && \text{if } n \text{ is odd}, \\ D_n &= \operatorname{diag}\left(\lambda_1,\lambda_2,\ldots,\lambda_{\frac{n}{2}},\lambda_{\frac{n}{2}+1},\overline{\lambda_{\frac{n}{2}}},\ldots,\overline{\lambda_2}\right) && \text{if } n \text{ is even}. \end{aligned} \tag{3.4}$$

If we take $n=8$ and $m=2$ for the matrix A_n in (1.2), then, from Lemma 3.1, we get

$$\begin{aligned} \lambda_3 &= a_0+(a_1+a_{-1})\cos\frac{\pi}{2}+\mathbf{i}\,(a_1-a_{-1})\sin\frac{\pi}{2} \\ &\quad +(a_2+a_{-2})\cos\pi+\mathbf{i}\,(a_2-a_{-2})\sin\pi \\ &= a_0-a_2-a_{-2}+\mathbf{i}\,(a_1-a_{-1}) \end{aligned}$$

and

$$\begin{aligned} \lambda_7 &= a_0+(a_1+a_{-1})\cos\frac{3\pi}{2}+\mathbf{i}\,(a_1-a_{-1})\sin\frac{3\pi}{2} \\ &\quad +(a_2+a_{-2})\cos 3\pi+\mathbf{i}\,(a_2-a_{-2})\sin 3\pi \\ &= a_0-a_2-a_{-2}-\mathbf{i}\,(a_1-a_{-1}). \end{aligned}$$

As can be seen above, $\lambda_7=\overline{\lambda_3}$.

Corollary 3.2. *Consider $3\le n\in\mathbb{N}$, $1\le m\le\left\lfloor\frac{n-1}{2}\right\rfloor$ and $a_i\in\mathbb{R}$ $(i=0,1,\ldots m)$. Let $B_n=\operatorname{circ}_n(a_0,a_1,\ldots,a_m,0,\ldots,0,a_m,\ldots,a_1)$ be an $n\times n$ symmetric circulant matrix and $\alpha_k=\cos\frac{2\pi(k-1)}{n}$ for every $1\le k\le n$. Then the eigenvalues of B_n are*

$$\mu_k = a_0+2\sum_{l=1}^{m}a_lT_l(\alpha_k) \tag{3.5}$$

where μ_k is the k^{th} eigenvalue of B_n.

Proof. The proof can be straightforwardly obtained from Lemma 3.1. □

Since $\cos\frac{2\pi(n+2-k-1)}{n}=\cos\frac{2\pi(k-1)}{n}$, we can easily see that $\mu_k=\mu_{n+2-k}$ $(2\le k\le n)$ from Corollary 3.2. Clearly, if we rewrite this eigenvalues in a diagonal matrix again, then

$$\begin{aligned} D_n &= \operatorname{diag}\left(\mu_1,\mu_2,\ldots,\mu_{\frac{n+1}{2}},\mu_{\frac{n+1}{2}},\ldots,\mu_2\right) && \text{if } n \text{ is odd}, \\ D_n &= \operatorname{diag}\left(\mu_1,\mu_2,\ldots,\mu_{\frac{n}{2}},\mu_{\frac{n}{2}+1},\mu_{\frac{n}{2}},\ldots,\mu_2\right) && \text{if } n \text{ is even}. \end{aligned}$$

If we take $n=8$ and $m=2$ for the matrix $B_n=\operatorname{circ}_n(a_0,a_1,\ldots a_m,0,\ldots,0,a_m,\ldots a_1)$, then, from Corollary 3.2, we get

$$\mu_3 = a_0+2\left(a_1\cos\frac{\pi}{2}+a_2\cos\pi\right) = a_0-2a_2$$

and

$$\mu_7 = a_0 + 2\left(a_1 \cos\frac{3\pi}{2} + a_2 \cos 3\pi\right) = a_0 - 2a_2.$$

As can be seen above, $\mu_7 = \mu_3$.
Now, from the expression (2.4), let us give the following result for the eigenvectors of $A_n = \mathrm{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$ depending on Chebyshev polynomials of the first and second kind

Theorem 3.3. *Let $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and $a_i \in \mathbb{R}$ $(i = 0, \pm 1, \ldots \pm m)$ and $\alpha_k = \cos\frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the eigenvector F_k of the matrix $A_n = \mathrm{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$ corresponding to the eigenvalue λ_k given by (3.1) is*

$$F_k = \begin{bmatrix} T_0(\alpha_k) + \mathbf{i}\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim\limits_{\mathbf{j}\to\mathbf{k}} U_{-1}(\alpha_j) \\ T_1(\alpha_k) + \mathbf{i}\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim\limits_{\mathbf{j}\to\mathbf{k}} U_0(\alpha_j) \\ \vdots \\ T_{n-1}(\alpha_k) + \mathbf{i}\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim\limits_{\mathbf{j}\to\mathbf{k}} U_{n-2}(\alpha_j) \end{bmatrix}. \tag{3.6}$$

Proof. From (2.4), (2.6) and (3.3), the result can be easily obtained. □

We must note that each one of all circulant matrices also have the eigenvectors generated by F_k given by (3.6)

4. Integer powers of circulant *m*-diagonal matrix

In this part, by using the symmetric relationship between the eigenvalues in (3.4), we give the efficient expression to compute the integer power of the circulant m-diagonal matrix $A_n = \mathrm{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$ based on Chebyshev polynomials of the first and second kind such that the method is faster than any of the classical methods which find the powers of A_n with an amount of computations.

Theorem 4.1. *Consider $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and $a_i \in \mathbb{R}$ $(i = 0, \pm 1, \ldots \pm m)$. Let $A_n = \mathrm{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$ be an $n \times n$ nonsingular circulant m-diagonal matrix and $\alpha_k = \cos\frac{2\pi(k-1)}{n}$ for every $1 \leq k \leq n$. Then the (u,v)th entry of A_n^r is that*

$$[A_n^r]_{u,v} = \frac{1}{n}(S_1 + S_2)$$

for all $r \in \mathbb{Z}$ and $1 \leq u, v \leq n$, where S_1 and S_2 are respectively such that

$$\begin{aligned} S_1 &= \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor + 1} \left(a_0 + \sum_{l=1}^{m}\left((a_l + a_{-l})T_l(\alpha_k) + \mathbf{i}(a_l - a_{-l})\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim_{\mathbf{j}\to\mathbf{k}} U_{l-1}(\alpha_j)\right)\right)^r \\ &\quad \times \left(T_{|u-v|}(\alpha_k) + \mathbf{i}\,\mathrm{sgn}(u-v)\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim_{\mathbf{j}\to\mathbf{k}} U_{|u-v|-1}(\alpha_j)\right) \end{aligned}$$

and

$$\begin{aligned} S_2 &= \sum_{k=2}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left(a_0 + \sum_{l=1}^{m}\left((a_l + a_{-l})T_l(\alpha_k) - \mathbf{i}(a_l - a_{-l})\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim_{\mathbf{j}\to\mathbf{k}} U_{l-1}(\alpha_j)\right)\right)^r \\ &\quad \times \left(T_{|u-v|}(\alpha_k) - \mathbf{i}\,\mathrm{sgn}(u-v)\,\mathrm{sgn}\left(\frac{n}{2}+1-k\right)\sqrt{1-\alpha_k^2}\lim_{\mathbf{j}\to\mathbf{k}} U_{|u-v|-1}(\alpha_j)\right). \end{aligned}$$

Here $\lfloor x \rfloor$ and sgn *denote the largest integer less than or equal to x and the signum function, respectively.*

Proof. By using (2.5) and (2.3), we get

$$\begin{aligned} [A_n^r]_{u,v} &= [F_n^* D_n^r F_n]_{u,v} = \sum_{k=1}^{n} [F_n^*]_{u,k} [D_n^r F_n]_{k,v} = \sum_{k=1}^{n} [F_n^*]_{u,k} \lambda_k^r [F_n]_{k,v}, \\ &= \sum_{k=1}^{n} \lambda_k^r [F_n^*]_{u,k} [F_n]_{v,k} = \frac{1}{n}\sum_{k=1}^{n} \lambda_k^r \omega^{(u-1)(k-1)} \omega^{-(v-1)(k-1)} \end{aligned}$$

and then

$$[A_n^r]_{u,v} = \frac{1}{n}\sum_{k=1}^{n} \lambda_k^r \omega^{(k-1)(u-v)} \tag{4.1}$$

where λ_k is already obtained as in (3.1). Since $\overline{\lambda_k} = \lambda_{n+2-k}$, $2 \le k \le n$ and $\omega^{(n+2-k-1)(u-v)} = \omega^{-(k-1)(v-u)}$, the second half of the sum in (4.1) can be written as

$$\sum_{k=\frac{n+1}{2}+1}^{n} \lambda_k^r \omega^{(k-1)(u-v)} = \sum_{k=2}^{\frac{n+1}{2}} \lambda_{n+2-k}^r \omega^{(n+2-k-1)(u-v)} = \sum_{k=2}^{\frac{n+1}{2}} \overline{\lambda_k}^r \omega^{-(k-1)(u-v)}$$

for the case n is odd. The same observations can be applied in the case n even and the result is that

$$[A_n^r]_{u,v} = \frac{1}{n}\left[\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor+1} \lambda_k^r \omega^{(k-1)(u-v)} + \sum_{k=2}^{\lfloor \frac{n+1}{2}\rfloor} \overline{\lambda_k}^r \omega^{-(k-1)(u-v)}\right] = \frac{1}{n}\left(S_1+S_2\right).$$

Thus, from the expression above, we can write

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor+1} \lambda_k^r \left(\cos\frac{2\pi(k-1)(u-v)}{n} + \mathbf{i}\sin\frac{2\pi(k-1)(u-v)}{n}\right)$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2}\rfloor} \overline{\lambda_k}^r \left(\cos\frac{2\pi(k-1)(u-v)}{n} - \mathbf{i}\sin\frac{2\pi(k-1)(u-v)}{n}\right).$$

Since, from (2.6),

$$T_{|u-v|}\left(\cos\frac{2\pi(k-1)}{n}\right) = \cos\frac{2\pi(k-1)(u-v)}{n}$$

and

$$U_{|u-v|-1}\left(\cos\frac{2\pi(k-1)}{n}\right) = \operatorname{sgn}(u-v)\frac{\sin\frac{2\pi(k-1)(u-v)}{n}}{\sin\frac{2\pi(k-1)}{n}}$$

with indeterminate form $0/0$ for $k=1$ and $k=\frac{n}{2}+1$, then

$$S_1 = \sum_{k=1}^{\lfloor \frac{n}{2}\rfloor+1} \lambda_k^r \left(T_{|u-v|}\left(\cos\frac{2\pi(k-1)}{n}\right) + \mathbf{i}\operatorname{sgn}(u-v)\sin\frac{2\pi(k-1)}{n}\lim_{\mathbf{j}\to\mathbf{k}} U_{|u-v|-1}\left(\cos\frac{2\pi(j-1)}{n}\right)\right),$$

and

$$S_2 = \sum_{k=2}^{\lfloor \frac{n+1}{2}\rfloor} \overline{\lambda_k}^r \left(T_{|u-v|}\left(\cos\frac{2\pi(k-1)}{n}\right) - \mathbf{i}\operatorname{sgn}(u-v)\sin\frac{2\pi(k-1)}{n}\lim_{\mathbf{j}\to\mathbf{k}} U_{|u-v|-1}\left(\cos\frac{2\pi(j-1)}{n}\right)\right).$$

The theorem follows by substituting λ_k in (3.1) and $\cos\frac{2\pi(k-1)}{n} = \alpha_k$ into the above expressions. □

From (2.5), we have the r^{th} power of any (symmetric) circulant matrix is also a (symmetric) circulant matrix.
Consider $a_i \in \mathbb{R}$ $(i = 0, \pm 1)$ and let $A_4 = \mathbf{circ}_4(a_0, a_1, 0, a_{-1})$ be circulant tridiagonal matrix. Then, by using Theorem 4.1, we get $A_4^r = \mathbf{circ}_4(\tau_0, \tau_1, \tau_2, \tau_3)$ with

$$\begin{aligned}
\tau_0 &= \frac{1}{4}\left[(a_0+a_1+a_{-1})^r + (a_0-(a_1+a_{-1}))^r + z^r + \overline{z}^r\right],\\
\tau_1 &= \frac{1}{4}\left[(a_0+a_1+a_{-1})^r - (a_0-(a_1+a_{-1}))^r - \mathbf{i}z^r + \mathbf{i}\overline{z}^r\right],\\
\tau_2 &= \frac{1}{4}\left[(a_0+a_1+a_{-1})^r + (a_0-(a_1+a_{-1}))^r - \mathbf{i}z^r - \mathbf{i}\overline{z}^r\right],\\
\tau_3 &= \frac{1}{4}\left[(a_0+a_1+a_{-1})^r - (a_0-(a_1+a_{-1}))^r + \mathbf{i}z^r - \mathbf{i}\overline{z}^r\right]
\end{aligned}$$

where $z = a_0 + \mathbf{i}(a_1 - a_{-1})$.
If we take $m = 1$ in Theorem 4.1, the expression given in [12, Theorem 2.1] can be easily seen.
Theorem 4.1 allows us to significantly reduce the computing process while finding the integer powers of the circulant m-diagonal matrix $A_n = \operatorname{circ}_n(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_{-m}, \ldots, a_{-1})$

Theorem 4.2. *Consider* $3 \leq n \in \mathbb{N}$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* $a_i \in \mathbb{R}$ $(i = 0, 1, \ldots m)$. *Let* $B_n = \mathrm{circ}_n\left(a_0, a_1, \ldots, a_m, 0, \ldots, 0, a_m, \ldots, a_1\right)$ *be an* $n \times n$ *nonsingular symmetric circulant m-diagonal matrix and* $\alpha_k = \cos\frac{2\pi(k-1)}{n}$ *for every* $1 \leq k \leq n$. *Then the (u,v)th entry of* B_n^r *is that*

$$[B_n^r]_{u,v} = \frac{1}{n}\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor + 1} l_{n-2k+2}\left(a_0 + 2\sum_{l=1}^{m} a_l T_l\left(\alpha_k\right)\right)^r T_{|u-v|}\left(\alpha_k\right)$$

where $\lfloor x \rfloor$ *denotes the largest integer less than or equal to x and*

$$l_s = \begin{cases} 1 & \text{if } s \in \{0, n\} \\ 2 & \text{in other cases.} \end{cases}$$

Proof. By using Theorem 4.1, we get

$$S_1 = \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor + 1}\left(a_0 + 2\sum_{l=1}^{m} a_l T_l\left(\alpha_k\right)\right)^r T_{|u-v|}\left(\alpha_k\right) + \mathbf{i}\,\mathrm{sgn}\left(u - v\right)\mathrm{sgn}\left(\frac{n}{2} + 1 - k\right)\sqrt{1 - \alpha_k^2}\,U_{|u-v|-1}\left(\alpha_k\right)$$

and

$$S_2 = \sum_{k=2}^{\left\lfloor \frac{n+1}{2} \right\rfloor}\left(a_0 + 2\sum_{l=1}^{m} a_l T_l\left(\alpha_k\right)\right)^r T_{|u-v|}\left(\alpha_k\right) - \mathbf{i}\,\mathrm{sgn}\left(u - v\right)\mathrm{sgn}\left(\frac{n}{2} + 1 - k\right)\sqrt{1 - \alpha_k^2}\,U_{|u-v|-1}\left(\alpha_k\right).$$

Since

$$\begin{cases} \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n+1}{2} \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n}{2} \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 = \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\begin{aligned} [B_n^r]_{u,v} &= \frac{1}{n}\left(S_1 + S_2\right) \\ &= \begin{cases} \frac{1}{n}\left[\left(a_0 + 2\sum\limits_{l=1}^{m} a_l\right)^r + 2\sum\limits_{k=2}^{\frac{n+1}{2}}\left(a_0 + 2\sum\limits_{l=1}^{m} a_l T_l\left(\alpha_k\right)\right)^r T_{|u-v|}\left(\alpha_k\right)\right] & \text{if } n \text{ is odd,} \\ \frac{1}{n}\left[\left(a_0 + 2\sum\limits_{l=1}^{m} a_l\right)^r + 2\sum\limits_{k=2}^{\frac{n}{2}}\left(a_0 + 2\sum\limits_{l=1}^{m} a_l T_l\left(\alpha_k\right)\right)^r T_{|u-v|}\left(\alpha_k\right) + \right. & \\ \left.\left(a_0 + 2\sum\limits_{l=1}^{m} a_l T_l\left(\alpha_{\frac{n}{2}+1}\right)\right)^r T_{|u-v|}\left(\alpha_{\frac{n}{2}+1}\right)\right] & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Therefore,

$$[B_n^r]_{u,v} = \frac{1}{n}\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor + 1} l_{n-2k+2}\left(a_0 + 2\sum_{l=1}^{m} a_l T_l\left(\alpha_k\right)\right)^r T_{|u-v|}\left(\alpha_k\right)$$

which is desired. □

Consider $a_i \in \mathbb{R}$ $(i = 0, 1, 2)$ and let $B_5 = \mathbf{circ}_5\left(a_0, a_1, a_2, a_2, a_1\right)$ be a symmetric circulant pentadiagonal matrix. Then, from Theorem 4.2, we get $B_5^r = \mathbf{circ}_4\left(\tau_0, \tau_1, \tau_2, \tau_2, \tau_1\right)$ with

$$\begin{aligned} \tau_0 &= \frac{1}{5}\left[\left(a_0 + 2a_1 + 2a_2\right)^r + 2\left(a_0 + \frac{1}{\phi}a_1 - \phi a_2\right)^r + 2\left(a_0 - \phi a_1 + \frac{1}{\phi}a_2\right)^r\right], \\ \tau_1 &= \frac{1}{5}\left[\left(a_0 + 2a_1 + 2a_2\right)^r + \frac{1}{\phi}\left(a_0 + \frac{1}{\phi}a_1 - \phi a_2\right)^r - \phi\left(a_0 - \phi a_1 + \frac{1}{\phi}a_2\right)^r\right], \\ \tau_2 &= \frac{1}{5}\left[\left(a_0 + 2a_1 + 2a_2\right)^r - \phi\left(a_0 + \frac{1}{\phi}a_1 - \phi a_2\right)^r + \frac{1}{\phi}\left(a_0 - \phi a_1 + \frac{1}{\phi}a_2\right)^r\right] \end{aligned}$$

where ϕ denotes the golden ratio.
Now, if we consider

$$\begin{cases} m = \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n-1}{2} & \text{for } n \text{ is odd,} \\ m = \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n}{2} & \text{for } n \text{ is even,} \end{cases}$$

in the symmetric circulant m-diagonal matrix $B_n = \mathrm{circ}_n\left(a_0, a_1, \ldots, a_m, 0, \cdots, 0, a_m, \cdots, a_1\right)$ and $a_{\frac{n}{2}} \neq 0$, then we get the symmetric circulant matrix in (1.1) discussed by Gutiérrez in [10]. And so, with the help of Theorem 4.2, we can straightforwardly reach the expression obtained by Gutiérrez in [10, Theorem 1] for positive integer powers of the matrix B_n in (1.1).

5. Illustrative examples

In this part, we give some illustrative examples. We will utilize Maple software in our calculations.

Example 5.1. *Let $A_6 = \operatorname{circ}_6(5,4,9,0,8,-2)$ be a circulant pentadiagonal matrix. we find the eigenvalues of A_6 by using (3.1) as*

$$\begin{array}{lcl} \lambda_1 & = & 24, \\ \lambda_2 & = & -2,5000+6,0621\mathbf{i}, \\ \lambda_3 & = & -4,5000+4,3301\mathbf{i}, \\ \lambda_4 & = & 20, \\ \lambda_5=\overline{\lambda_3} & = & -4,5000-4,3301\mathbf{i}, \\ \lambda_6=\overline{\lambda_2} & = & -2,5000-6,0621\mathbf{i} \end{array}$$

and from Theorem 4.1, the entries of A_6^3 as

$$A_6^3 = \operatorname{circ}_6(3778,1008,3483,938,3651,966).$$

Example 5.2. *Let $A_9 = \operatorname{circ}_9(-2,3,-4,9,0,0,6,5,-1)$ be a circulant heptadiagonal matrix. we find the eigenvalues of A_9 by using (3.1) as*

$$\begin{array}{lcl} \lambda_1 & = & 16, \\ \lambda_2 & = & -7,7942-3,6940\mathbf{i}, \\ \lambda_3 & = & -10,0923-1,73701\mathbf{i}, \\ \lambda_4 & = & 11,5000+11,2583\mathbf{i}, \\ \lambda_5 & = & -10,6133+9,7512\mathbf{i}, \\ \lambda_6=\overline{\lambda_5} & = & -10,6133-9,7512\mathbf{i}, \\ \lambda_7=\overline{\lambda_4} & = & 11,5000-11,2583\mathbf{i}, \\ \lambda_8=\overline{\lambda_3} & = & -10,0923+1,73701\mathbf{i}, \\ \lambda_9=\overline{\lambda_2} & = & -7,7942+3,6940\mathbf{i} \end{array}$$

and from Theorem 4.1, the entries of A_9^4 as

$$A_9^4 = \operatorname{circ}_9(-15410,26041,7866,-5401,16331,13209,-2024,3458,21466).$$

Example 5.3. *Let $B_7 = \operatorname{circ}_7(1,-3,2,0,0,2,-3)$ be a symmetric circulant pentadiagonal matrix. we find the eigenvalues of B_7 by using (3.5) as*

$$\begin{array}{lcl} \lambda_1 & = & -1, \\ \lambda_2 & = & -3,6310, \\ \lambda_3 & = & -1,2687, \\ \lambda_4 & = & 8,8987, \\ \lambda_5=\lambda_4 & = & 8,8987, \\ \lambda_6=\lambda_3 & = & -1,2687, \\ \lambda_7=\lambda_2 & = & -3,6310 \end{array}$$

and from Theorem 4.2, the entries of B_7^5 as

$$B_7^5 = \operatorname{circ}_7(15771,-14485,9987,-3388,-3388,9987,-14485).$$

Examples 5.1, 5.2 and 5.3 can be also confirmed by means of Maple procedures given by Appendix A and B.

Appendix A. Following Maple procedure firstly generates a $n\times n$ circulant heptadiagonal matrix $A_n = \operatorname{circ}_n(a_0,a_1,a_2,a_3,0,\ldots,0,a_{-3},a_{-2},a_{-1})$ and then compute eigenvalues and the k^{th} power ($r\in\mathbb{Z}$) of it.

```
restart:
with(LinearAlgebra):
m:='3':
n:='n':
r:='r':
a[0]:='a[0]':
a[1]:='a[1]':
a[-1]:='a[-1]':
a[2]:='a[2]':
a[-2]:='a[-2]':
```

```
a[3]:=’a[3]’:
a[-3]:=’a[-3]’:
f:=(i,j)→ piecewise(i=j,a[0],j>i and j-i<m+1,a[j-i],i>j and i-j<m+1,a[j-i],n-j+i<m+1,a[j-i-n],n-i+j<m+1,a[j-i+n]):
A[n]:=Matrix(n,n,f);
alpha:=k → evalf(cos(2*Pi*(k-1)/n)):
lambda:=k→ evalf((a[0]+sum((a[l]+a[-l])*ChebyshevT(l,alpha(k))+I*(a[l]-a[-l])*signum((n/2)+1-k)*sqrt(1-(alpha(k))^2)*limit
(ChebyshevU(l-1,alpha(j)),j=k),l=1..m))):
g:=(i,j)→ piecewise(i=j,lambda(i),0):
p:=(u,v)→evalf((1/n)*(sum((lambda(k)^r)*(ChebyshevT(abs(u-v),alpha(k))+I*signum(u-v)*signum((n/2)+1-k)*
sqrt(1-(alpha(k))^2)*limit(ChebyshevU(abs(u-v)-1,alpha(j)),j=k)),k=1..floor(n/2)+1)+sum(conjugate(lambda(k))^r*
(ChebyshevT(abs(u-v),alpha(k))-I*signum(u-v)*signum((n/2)+1-k)*sqrt(1-(alpha(k))^2)*ChebyshevU(abs(u-v)-1,alpha(k))),
k=2..floor((n+1)/2)))):
eigenvalues_of_A[n]:=Matrix(n,n,g);
the_rth_power_of_A[n]:=Matrix(n,n,p);
```

Appendix B. Following Maple procedure firstly generates a $n \times n$ symmetric circulant heptadiagonal matrix $B_n = \mathrm{circ}_n(a_0, a_1, a_2, a_3, 0, \ldots, 0, a_3, a_2, a_1)$ and then compute eigenvalues and the k^{th} power $(r \in \mathbb{Z})$ of it.

```
restart:
with(LinearAlgebra):
m:=’3’:
n:=’n’:
a[0]:=’a[0]’:
a[1]:=’a[1]’:
a[2]:=’a[2]’:
a[3]:=’a[3]’:
r:=’r’:
f:=(i,j)→ piecewise(i=j,a[0],i>j and i-j<m+1,a[i-j],i<j and j-i<m+1,
a[j-i],i<j and j-i>n-m-1,a[n-(j-i)],i>j and i-j>n-m-1,a[n-(i-j)],0):
B[n]:=Matrix(n,n,f);
alpha:=k→ evalf(cos(2*Pi*(k-1)/n)):
mu:=k → evalf(a[0]+2*sum(a[l]*ChebyshevT(l,alpha(k)),l=1..m)):
g:=(i,j)→ piecewise(i=j,μ(i),0):
l:=(s) → piecewise(s=0,1,s=n,1,2):
p:=(u,v)→ evalf((1/n)*((sum(((l(n-2*k+2)*(μ(k))^r)*(ChebyshevT(abs(u-v),alpha(k))),k=1..floor(n/2)+1))))):
eigenvalues_of_B[n]:=Matrix(n,n,g);
the_rt_power_of_B[n]:=Matrix(n,n,p);
```

6. Conclusion

There has been recently increasing research interest in circulant matrices in several areas, such as digital signal processing, image compression, physics/engineering simulations, number theory, coding theory, cryptography, and, naturally, linear algebra. This paper present eigenvalues, eigenvectors, powers of circulant m-diagonal matrix which is one type of circulant matrices by using some famous relations on chebyshev polynomials.

Acknowledgement

The author would like to thank the anonymous referees for their careful reading of the paper and very detailed proposals that helped improve the presentation of the paper.

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Fundamental Journal of Mathematics and Applications, 4 (1) (2021) 59-66
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.856647

Recurrence Relations for Knot Polynomials of Twist Knots

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Article Info

Keywords: *Knot polynomial, Oriented knot, Unoriented knot, Twist knot*
2010 AMS: *57M25, 57M27*
Received: *08 January 2021*
Accepted: *17 March 2021*
Available online: *20 March 2021*

Abstract

This paper gives HOMFLY polynomials and Kauffman polynomials L and F of twist knots as recurrence relations, respectively, and also provides some recursive properties of them.

1. Introduction

The knot polynomials are the most practical knot invariants for distinguishing knots from each other, where the coefficients of polynomials represent some properties of the knot. The first of the polynomial invariants is the Alexander polynomial [1] with one variable for oriented knots and links. There are generalizations of the Alexander polynomial and its Conway version [2], see [3]- [5]. Another important knot polynomial with one variable for oriented knots and links is the Jones polynomial [6]. Both the Jones polynomial was defined with new methods [7, 8] and studies were conducted on generalizations of the Jones polynomial [9]- [11]. One of the most important generalized polynomials is the HOMFLY polynomial [11]- [13] with two variable. The Alexander and Jones polynomials are special cases of the HOMFLY polynomial. For unoriented knots and links, there are the polynomials such as the BLM/Ho polynomial [14, 15] with one variable and the Kauffman polynomial F [16] with two variable whose primary version Kauffman polynomial L is an invariant of regular isotopy for unoriented knots and links. Both the Jones and the BLM/Ho polynomials are special cases of the Kauffman polynomial F.
The HOMFLY polynomial or HOMFLY-PT polynomial whose name is an acronym for its discoverers' last names is inspired by the Jones polynomial. The HOMFLY polynomial $P_K(a,z)$ is two variables Laurent polynomial for the oriented link diagram K. $P_K(a,z)$ is an ambient isotopy invariant of the link K determined by the following axioms:

$$
\begin{aligned}
a^{-1}P_{K_+}(a,z) - aP_{K_-}(a,z) &= zP_{K_0}(a,z) \qquad (1.1)\\
P_{\bigcirc}(a,z) &= 1, \qquad (1.2)
\end{aligned}
$$

where K_+, K_- and K_0 are skein diagrams drawn in Figure 1.1 and $\bigcirc$ is any diagram of the unknot.

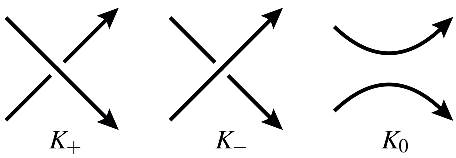

Figure 1.1: Skein Diagrams

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From the axioms (1.1) and (1.2), it is obtained that $a^{-1} - a = zP_{\bigcirc\bigcirc}$ or $\delta = (a^{-1} - a)z^{-1}$ with $\delta = P_{\bigcirc\bigcirc}$, where $\bigcirc\bigcirc$ is trivial link with two components. If $\bigcirc_\mu$ is a trivial μ-component link, then $P_{\bigcirc_\mu}(a,z) = \delta^{\mu-1}$. Also, $P_{K^*}(a,z) = P_K(a^{-1}, -z)$, where K^* is the mirror image of K.

In 1987, L. Kauffman [16, 17] discovered a new polynomial, denoted by L, which specializes to the bracket polynomial [7]. The Kauffman polynomial $L(a,x)$ is a two-variable Laurent polynomial for the unoriented link diagram K. $L(a,x)$ is a regular isotopy invariant of the link K satisfying the following axioms:

$$
\begin{aligned}
L_{K^+}(a,x) + L_{K^-}(a,x) &= x\left(L_{K^0}(a,x) + L_{K^\infty}(a,x)\right), && (1.3)\\
L_{\bigcirc}(a,x) &= 1, && (1.4)\\
L_{D^+}(a,x) &= aL_{D^0}(a,x), && (1.5)\\
L_{D^-}(a,x) &= a^{-1}L_{D^0}(a,x), && (1.6)
\end{aligned}
$$

where K^+, K^-, K^0 and K^∞ are unoriented diagrams drawn in Figure 1.2, $\bigcirc$ is any diagram of unknot and D^+, D^- and D^0 are unoriented diagrams drawn in Figure 1.3.

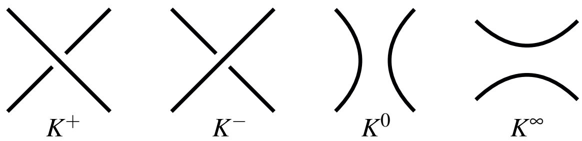

Figure 1.2: Crossings and splits

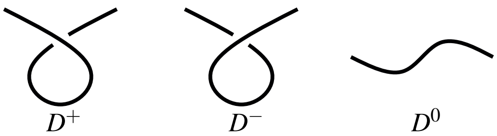

Figure 1.3: Diagrams related to Reidmeister moves of type I

The Kauffman polynomial F for oriented link diagram K by the formula [17]

$$F_K(a,x) = a^{-w(K)}L_K(a,x), \tag{1.7}$$

where L_K is defined on oriented link diagrams by forgetting the orientation and $w(K)$ denotes the writhe of oriented link diagram K ($w(K)$ is the sum of all crossing signs of K). Then the polynomial $F_K(a,x)$ is a Laurent polynomial invariant of ambient isotopy. From the axioms (1.3) and (1.5), it is obtained that $L_{\bigcirc\bigcirc}(a,x) = (a + a^{-1})x^{-1} - 1$ or $\delta = (a + a^{-1})x^{-1} - 1$ with $\delta = L_{\bigcirc\bigcirc}(a,x)$, where $\bigcirc\bigcirc$ is trivial link with two components. If $\bigcirc_\mu$ is a trivial μ-component link, then $L_{\bigcirc_\mu}(a,x) = F_{\bigcirc_\mu}(a,x) = \delta^{\mu-1}$. Also, $L_{K^*}(a,x) = L_K(a^{-1},x)$ and $F_{K^*}(a,x) = F_K(a^{-1},x)$, where K^* is the mirror image of link K.

The twist knots, which obtained by twisting a closed-loop repeatedly and then linking the ends together, are an essential class of knots. It could be found out lots of studies about their knot invariants (See [18]- [25] and others). Here, a twist knot is regarded with a clasp and right-handed n-half twists as drawn in Figure 2.1. Besides, the knot polynomials of some classes of knots and links were studied to give recursive formulas [26]- [31].

In this paper, it is aimed that deriving the recurrence relations for the HOMFLY polynomials of the oriented twist knots and the Kauffman polynomials L and F of the unoriented twist knots. While the HOMFLY polynomial and the Kauffman polynomials L and F of twist knots are defined as fourth-order recurrence relations, the $(2,n)$-torus link diagrams are encountered and their mentioned knot polynomials are utilized for some results. Also, some recursive properties of these relations are examined and it is provided the generating functions, the general solutions and the explicit forms.

2. Oriented and unoriented knot polynomials of twist knots

2.1. HOMFLY polynomials of twist knots

Suppose that $\mathscr{K}_n$ is an oriented digram of twist knot drawn in Figure 2.1, $\mathscr{K}_{(2,n)}$ is an oriented digram of $(2,n)$-torus link drawn in Figure 2.2 and P_n denotes the HOMFLY polynomial of $\mathscr{K}_n$ instead of $P_{\mathscr{K}_n}(a,z)$ for simplicity.

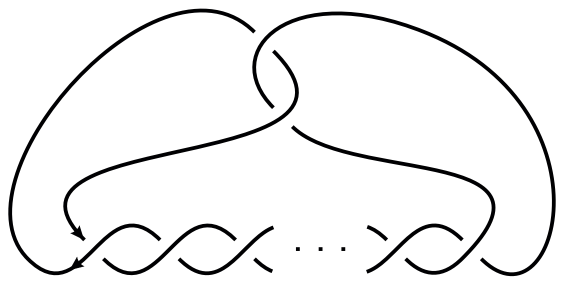

Figure 2.1: The twist knot with a clasp and right-handed n-half twists

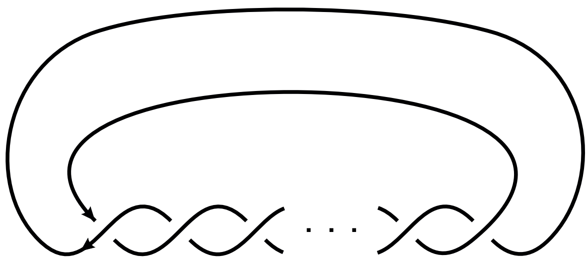

Figure 2.2: $(2,n)$-torus link

Theorem 2.1. *The HOMFLY polynomial of twist knot $\mathscr{K}_n$ satisfies the following relations:*

$$P_n = (a^2+1)P_{n-2} - a^2 P_{n-4}, \quad n \geq 4 \tag{2.1}$$

and

$$P_n = \begin{cases} azP_{\mathscr{K}_{(2,n+1)}} + a^2 & \text{if } n \text{ is odd}, \\ -a^{-1}zP_{\mathscr{K}_{(2,n)}} + a^{-2} & \text{if } n \text{ is even}. \end{cases} \tag{2.2}$$

Proof. Let the skein operations be applied to a designated half twist of the oriented diagram $\mathscr{K}_n$. If the crossing is switched, the resulting diagram is $\mathscr{K}_{n-2}$ twist knot by the second Reidemeister move. Then, if the crossing is smoothed, the resulting diagram is $(2,2)$-torus link, i.e. Hopf link, obtained by applying the first Reidemeister move $n-1$ times. Notice that, if n is odd, all crossings of the $(2,2)$-torus link are right-handed with counter-directed strands and if n is even, all crossings of the $(2,2)$-torus link are left-handed with same-directed strands. Hence, from the axiom (1.1), the following equations are obtained as

$$P_n = azP_{\mathscr{K}_{(2,2)}} + a^2 P_{n-2}$$

and

$$P_{n-2} = azP_{\mathscr{K}_{(2,2)}} + a^2 P_{n-4}.$$

Thus, the recurrence relation (2.1) is gotten from last two equations.
Let the skein operations be applied to a designated crossing of the clasp of the oriented diagram $\mathscr{K}_n$. The crossings of the clasp are right-handed and left-handed when n is odd and even, respectively. In case of n is odd, if the crossing is switched, the resulting diagram is an unknot by applying the second Reidemeister move and the first Reidemeister move n times. Then, if the crossing is smoothed, the resulting diagram is $\mathscr{K}_{(2,n+1)}$ torus link with counter-directed strands taking into consideration $n+1$ is even. Hence, from the axiom (1.1) and (1.2), the relation in (2.2) is obtained as

$$P_n = azP_{\mathscr{K}_{(2,n+1)}} + a^2.$$

In case of n is even, the relation in (2.2) is obtained similarly. □

Then, the recurrence relation (2.1) in Theorem 2.1 could be given with initial conditions as a fourth-order recurrence relation.

Definition 2.2. *The HOMFLY polynomials $\{P_n\}_{n=0}^{\infty}$ for the oriented diagrams of twist knots $\mathscr{K}_n$ is defined by the recurrence relation*

$$P_n = (a^2+1)P_{n-2} - a^2 P_{n-4}, \quad n \geq 4$$

with initial conditions

$$P_0 = 1, \quad P_1 = a^2z^2 - a^4 + 2a^2, \quad P_2 = a^2 - z^2 + a^{-2} - 1, \quad P_3 = a^4z^2 + a^2z^2 - a^6 + a^4 + a^2. \tag{2.3}$$

Also, since $P_{K^*}(a,z) = P_K(a^{-1},-z)$, where K^* is the mirror image of the diagram K, the following relation is obtained by using P_{-n} instead of $P_{\mathscr{K}_n^*}(a,z)$

$$P_{-n} = (a^{-2}+1)P_{-(n-2)} - a^{-2}P_{-(n-4)}.$$

The characteristic equation of (2.1) is a bi-quadratic equation as

$$\lambda^4 - (a^2+1)\lambda^2 + a^2 = 0$$

and the roots of this equation are

$$\lambda_1 = a, \quad \lambda_2 = -a, \quad \lambda_3 = 1, \quad \lambda_4 = -1. \tag{2.4}$$

Proposition 2.3. *The generating function of the sequence* $\{P_n\}$ *is*

$$g_P(\lambda) = \frac{-a^2\lambda^3 + (a^{-2} - z^2 - 2)\lambda^2 + (a^2z^2 - a^4 + 2a^2)\lambda + 1}{a^2\lambda^4 - (a^2+1)\lambda^2 + 1}. \tag{2.5}$$

Proof. The generating function of $\{P_n\}$ has the following form:

$$g_P(\lambda) = P_0 + P_1\lambda + P_2\lambda^2 + \dots$$

After the multiplications $(a^2+1)\lambda^2 g_P(\lambda)$ and $-a^2\lambda^4 g_P(\lambda)$, the following is provided by using (2.1)

$$\begin{aligned}(1-(a^2+1)\lambda^2 + a^2\lambda^4)g_P(\lambda) &= P_0 + P_1\lambda + (P_2 - (a^2+1)P_0)\lambda^2 + (P_3 - (a^2+1)P_1)\lambda^3 \\ &\quad + \sum_{n=4}^{\infty}\left(P_n - (a^2+1)P_{n-2} + a^2P_{n-4}\right)\lambda^n \\ &= P_0 + P_1\lambda + (P_2 - (a^2+1)P_0)\lambda^2 + (P_3 - (a^2+1)P_1)\lambda^3.\end{aligned}$$

Hence, the equality (2.5) is obtained from the below by using the equalities in (2.3).

$$g_P(\lambda) = \frac{P_0 + P_1\lambda + (P_2 - (a^2+1)P_0)\lambda^2 + (P_3 - (a^2+1)P_1)\lambda^3}{a^2\lambda^4 - (a^2+1)\lambda^2 + 1}.$$

□

Proposition 2.4. *The general solution of the recurrence relation* (2.1) *is*

$$P_n = Aa^n + B(-a)^n + C + D(-1)^n, \quad n \geq 0,$$

where

$$\begin{aligned} A &= -\frac{(a^2+a+1)(a^4 - a^2(z^2+2)+1)}{2a^2(a+1)}, & C &= \frac{a^4 - a^2z^2 + 1}{2a^2}, \\ B &= \frac{(a^2-a+1)(a^4 - a^2(z^2+2)+1)}{2a^2(a-1)}, & D &= -\frac{(a^2+1)(a^4 - a^2(z^2+2)+1)}{2a^2(a^2-1)}.\end{aligned}$$

Proof. The closed form of the sequence $\{P_n\}$ is given by

$$P_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n, \quad n \geq 0.$$

Then, the following linear equation system is provided from (2.3) and (2.4) as

$$\begin{aligned} P_0 &= A + B + C + D = 1, \\ P_1 &= A\lambda_1 + B\lambda_2 + C\lambda_3 + D\lambda_4 = a^2z^2 - a^4 + 2a^2, \\ P_2 &= A\lambda_1^2 + B\lambda_2^2 + C\lambda_3^2 + D\lambda_4^2 = a^2 - z^2 + a^{-2} - 1, \\ P_3 &= A\lambda_1^3 + B\lambda_2^3 + C\lambda_3^3 + D\lambda_4^3 = a^4z^2 + a^2z^2 - a^6 + a^4 + a^2.\end{aligned}$$

The values A, B, C and D is obtained by solving this system. Note that considering $x^n - y^n = (x-y)\sum_{i=0}^{n-1} x^k y^{n-1-k}$, the factors $(a-1)$ are simplified. □

Corollary 2.5. *For* $n \geq 2$*, the explicit formula for the HOMFLY polynomial of twist knot* $\mathscr{K}_n$ *is given by*

$$P_n = \begin{cases} \frac{z^2}{a+1}\left(\sum_{i=0}^{n} a^{i+2}\right) - a^{n+3} + a^{n+1} + a^2 & \text{if } n \text{ is odd}, \\ -\frac{z^2}{a+1}\left(\sum_{i=0}^{n-1} a^i\right) + a^n - a^{n-2} + a^{-2} & \text{if } n \text{ is even}. \end{cases} \tag{2.6}$$

Proof. From Corollary 1 in [28], the explicit formula for the HOMFLY polynomial of $(2,n)$-torus link $\mathscr{K}_{(2,n)}$ with counter-directed strands taking into consideration that n is even and the notations and diagrams mentioned in this paper is given by

$$P_{\mathscr{K}_{(2,n)}} = \left(\frac{a^n - 1}{a - a^{-1}}\right)z - a^n(a - a^{-1})z^{-1}. \tag{2.7}$$

Hence, the formulas in (2.6) are provided by using (2.2) and (2.7). □

Remark 2.6. *Since it is well known that the HOMFLY polynomial specializes to the Jones polynomial for* $a = t$ *and* $z = t^{1/2} - t^{-1/2}$*, the Alexander-Conway polynomial for* $a = 1$ *and the Alexander polynomial for* $a = 1$ *and* $z = t^{1/2} - t^{-1/2}$*, the recurrence relations for the mentioned knot polynomials of twist knot* $\mathscr{K}_n$ *could be easily obtained.*

2.2. Kauffman polynomials L and F of twist knots

Now, suppose that $\mathfrak{K}_n$ is an unoriented digram of twist knot drawn in Figure 2.1, $\mathfrak{K}_{(2,n)}$ is an unoriented digram of $(2,n)$-torus link drawn in Figure 2.2 and L_n denotes the Kauffman polynomial L of $\mathfrak{K}_n$ instead of $L_{\mathfrak{K}_n}(a,x)$ for simplicity.

Theorem 2.7. *The Kauffman polynomial L of twist knot $\mathfrak{K}_n$ satisfies the following relations:*

$$L_n = xL_{n-1} + (a^2-1)L_{n-2} - a^2xL_{n-3} + a^2L_{n-4}, \quad n \geq 4 \tag{2.8}$$

and

$$L_n = a^{-1}xL_{\mathfrak{K}_{(2,n)}} + xL_{\mathfrak{K}_{(2,n+1)}} - a^n. \tag{2.9}$$

Proof. Let the axiom (1.3) be applied to a designated half twist of the unoriented diagram $\mathfrak{K}_n$. If the crossing is switched, the resulting diagram is $\mathfrak{K}_{n-2}$ twist knot by the second Reidemeister move. If the crossing is split according to the K^0, the resulting diagram is $\mathfrak{K}_{n-1}$ twist knot. Then, if the crossing is split according to the K^∞, the resulting diagram is $(2,2)$-torus link, i.e. Hopf link, obtained by applying the first Reidemeister move $n-1$ times. Note that, if n is even, all crossings of the $(2,2)$-torus link are left-handed and if n is odd, all crossings of the $(2,2)$-torus link are right-handed. Hence, by using the axioms (1.4) and (1.5), the following equations are obtained as

$$L_n = xL_{n-1} + a^{n-1}xL_{\mathfrak{K}_{(2,2)}} - L_{n-2}$$

and

$$L_{n-2} = xL_{n-3} + a^{n-3}xL_{\mathfrak{K}_{(2,2)}} - L_{n-4}.$$

Thus, the recurrence relation (2.8) is gotten from last two equations.
Let the axiom (1.3) be applied to a designated crossing of the clasp of the unoriented diagram $\mathfrak{K}_n$. If the crossing is switched, the resulting diagram is an unknot obtained by applying the second Reidemeister move and the first Reidemeister move n times. If the crossing is split according to the K^0, the resulting diagram is torus link $\mathfrak{K}_{(2,n+1)}$. Then, if the crossing is split according to the K^∞, the resulting diagram is the image of torus link $\mathfrak{K}_{(2,n)}$ by applying the first Reidemeister move. Thus, the relation (2.9) is obtained by using the axioms (1.4), (1.5) and (1.6). □

Then, the recurrence relation (2.8) in Theorem 2.7 could be given with initial conditions as a fourth-order recurrence relation.

Definition 2.8. *The Kauffman polynomials $\{L_n\}_{n=0}^{\infty}$ for the unoriented diagrams of twist knots $\mathfrak{K}_n$ is defined by the recurrence relation*

$$L_n = xL_{n-1} + (a^2-1)L_{n-2} - a^2xL_{n-3} + a^2L_{n-4}, \quad n \geq 4$$

with initial conditions

$$\begin{gathered} L_0 = a^{-2}, \quad L_1 = (a+a^{-1})x^2 + (a^{-2}+1)x - 2a - a^{-1}, \\ L_2 = (a+a^{-1})x^3 + (a^2+a^{-2}+2)x^2 - (a+a^{-1})x - a^2 - a^{-2} - 1, \\ L_3 = (a+a^{-1})x^4 + (a^2+a^{-2}+2)x^3 + (a^3-a-2a^{-1})x^2 - (2a^{-2}+2)x - a^3 + a + a^{-1}. \end{gathered} \tag{2.10}$$

Since $L_{K^*}(a,x) = L_K(a^{-1},x)$, where K^* is the mirror image of the diagram K, the following relation is obtained by using L_{-n} instead of $L_{\mathscr{K}_n^*}(a,x)$

$$L_{-n} = xL_{-(n-1)} + (a^{-2}-1)L_{-(n-2)} - a^{-2}xL_{-(n-3)} + a^{-2}L_{-(n-4)}.$$

The characteristic equation of (2.8) is a quadratic equation as

$$\lambda^4 - x\lambda^3 - (a^2-1)\lambda^2 + a^2x\lambda - a^2 = 0$$

and the roots of this equation are

$$\lambda_1 = a, \quad \lambda_2 = -a, \quad \lambda_3 = \frac{1}{2}(x+\sqrt{x^2-4}), \quad \lambda_4 = \frac{1}{2}(x-\sqrt{x^2-4}). \tag{2.11}$$

Proposition 2.9. *The generating function of the sequence $\{L_n\}$ is*

$$g_L(\lambda) = \frac{a^3\lambda^3 + ((a^2+1)x^2 + ax - a^2 - 2)\lambda^2 + ((a+a^{-1})x^2 + x - 2a - a^{-1})\lambda + a^{-2}}{-a^2\lambda^4 + a^2x\lambda^3 - (a^2-1)\lambda^2 - x\lambda + 1}.$$

Proof. The generating function of $\{L_n\}$ has the following form:

$$g_L(\lambda) = L_0 + L_1\lambda + L_2\lambda^2 + \dots$$

After the multiplications $x\lambda g_L(\lambda)$, $(a^2-1)\lambda^2 g_L(\lambda)$, $-a^2x\lambda^3 g_L(\lambda)$ and $a^2\lambda^4 g_L(\lambda)$, the following is provided by using (2.8)

$$\begin{aligned}(1-x\lambda-(a^2-1)\lambda^2+a^2x\lambda^3-a^2\lambda^4)g_L(\lambda) &= L_0+(L_1-xL_0)\lambda+(L_2-xL_1-(a^2-1)L_0)\lambda^2+(L_3-xL_2-(a^2-1)L_1+a^2xL_0)\lambda^3 \\ &\quad+\sum_{n=4}^{\infty}\left(L_n-xL_{n-1}-(a^2-1)L_{n-2}+a^2xL_{n-3}-a^2L_{n-4}\right)\lambda^n \\ &= L_0+(L_1-xL_0)\lambda+(L_2-xL_1-(a^2-1)L_0)\lambda^2+(L_3-xL_2-(a^2-1)L_1+a^2xL_0)\lambda^3\end{aligned}$$

Hence, the equality (2.5) is obtained from the below by using the equalities in (2.3).

$$g_L(\lambda) = \frac{L_0+(L_1-xL_0)\lambda+(L_2-xL_1-(a^2-1)L_0)\lambda^2+(L_3-xL_2-(a^2-1)L_1+a^2xL_0)\lambda^3}{-a^2\lambda^4+a^2x\lambda^3-(a^2-1)\lambda^2-x\lambda+1}.$$

□

Proposition 2.10. *The general solution of the recurrence relation* (2.8) *is*

$$L_n = Aa^n + B(-a)^n + C\left(\frac{1}{2}(x+\sqrt{x^2-4})\right)^n + D\left(\frac{1}{2}(x-\sqrt{x^2-4})\right)^n, \quad n \geq 0, \tag{2.12}$$

where

$$\begin{aligned} A &= \frac{a^2\left(x^2-1\right)+ax+x^2-1}{a^2-ax+1}, \quad B=0, \\ C &= \frac{(a^2+1)\left(2a^3+a^2\left(x^3+x^2\sqrt{x^2-4}-\sqrt{x^2-4}-3x\right)+a\left(-x^2+x\sqrt{x^2-4}+2\right)-\sqrt{x^2-4}-x\right)}{2a^2\sqrt{x^2-4}\,(-a^2+ax-1)}, \\ D &= \frac{(a^2+1)\left(-2a^3+a^2\left(-x^3+x^2\sqrt{x^2-4}-\sqrt{x^2-4}+3x\right)+a\left(x^2+x\sqrt{x^2-4}-2\right)-\sqrt{x^2-4}+x\right)}{2a^2\sqrt{x^2-4}\,(-a^2+ax-1)}. \end{aligned}$$

Proof. The closed form of the sequence $\{L_n\}$ is given by

$$L_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + D\lambda_4^n, \quad n \geq 0.$$

Then, the following linear equation system is provided from (2.10) and (2.11) as

$$\begin{aligned} &L_0 = A+B+C+D = a^{-2}, \\ &L_1 = A\lambda_1+B\lambda_2+C\lambda_3+D\lambda_4 = (a+a^{-1})x^2+(a^{-2}+1)x-2a-a^{-1}, \\ &L_2 = A\lambda_1^2+B\lambda_2^2+C\lambda_3^2+D\lambda_4^2 = (a+a^{-1})x^3+(a^2+a^{-2}+2)x^2-(a+a^{-1})x-a^2-a^{-2}-1, \\ &L_3 = A\lambda_1^3+B\lambda_2^3+C\lambda_3^3+D\lambda_4^3 = (a+a^{-1})x^4+(a^2+a^{-2}+2)x^3+(a^3-a-2a^{-1})x^2-(2a^{-2}+2)x-a^3+a+a^{-1}. \end{aligned}$$

The values A, B, C and D is obtained by solving this system. □

Suppose that F_n denotes the Kauffman polynomial F of $\mathfrak{K}_n$ instead of $F_{\mathfrak{K}_n}(a,x)$ for simplicity.

Corollary 2.11. *The Kauffman polynomials $\{F_n\}_{n=0}^{\infty}$ for the unoriented diagrams of twist knots $\mathfrak{K}_n$ is defined by the recurrence relation*

$$F_n = \begin{cases} a^{-5}xF_{n-1}+(1-a^{-2})F_{n-2}-a^{-5}xF_{n-3}+a^{-2}F_{n-4} & \text{if } n \text{ is odd,} \\ a^3xF_{n-1}+(1-a^{-2})F_{n-2}-a^3xF_{n-3}+a^{-2}F_{n-4} & \text{if } n \text{ is even,} \end{cases} \quad n \geq 4 \tag{2.13}$$

with initial conditions

$$\begin{gathered} F_0 = 1, \quad F_1 = (a^{-2}+a^{-4})x^2+(a^{-3}+a^{-5})x-2a^{-2}-a^{-4}, \\ F_2 = (a+a^{-1})x^3+(a^2+a^{-2}+2)x^2-(a+a^{-1})x-a^2-a^{-2}-1, \\ F_3 = (a^{-4}+a^{-6})x^4+(a^{-3}+a^{-7}+2a^{-5})x^3+(a^{-2}+-a^{-4}-2a^{-6})x^2-(2a^{-7}+2a^{-5})x-a^{-2}+a^{-4}+a^{-6}. \end{gathered}$$

Also, the following relation is satisfied for F_n.

$$F_n = \begin{cases} a^{-3}xF_{\mathfrak{K}_{(2,n)}}+a^{-1}xF_{\mathfrak{K}_{(2,n+1)}}-a^{-2} & \text{if } n \text{ is odd,} \\ axF_{\mathfrak{K}_{(2,n)}}+a^3xF_{\mathfrak{K}_{(2,n+1)}}-a^2 & \text{if } n \text{ is even.} \end{cases} \tag{2.14}$$

Proof. It is proven by considering $w(\mathfrak{K}_n) = n+2$ and $w(\mathfrak{K}_n) = n-2$ if n is odd and even, respectively, and by using (1.7) in Definition 2.8 and the relation (2.9). □

In addition, since $F_{K^*}(a,x) = F_K(a^{-1},x)$, where K^* is the mirror image of the diagram K, the following relations are obtained by using F_{-n} instead of $F_{\mathscr{K}_n^*}(a,x)$

$$F_{-n} = \begin{cases} a^5 x F_{-(n-1)} + (1-a^2)F_{-(n-2)} - a^5 x F_{-(n-3)} + a^2 F_{-(n-4)} & \text{if } n \text{ is odd,} \\ a^{-3} x F_{-(n-1)} + (1-a^2)F_{-(n-2)} - a^{-3} x F_{-(n-3)} + a^2 F_{-(n-4)} & \text{if } n \text{ is even.} \end{cases}$$

By using same notation, it could be provided a relation from (2.14) for the mirror image of $\mathscr{K}_n^*$.

Corollary 2.12. *The general solution of the recurrence relation (2.13) is*

$$F_n = \begin{cases} a^{-n-2}\left(Aa^n + B(-a)^n + C(\frac{1}{2}(x+\sqrt{x^2-4}))^n + D(\frac{1}{2}(x-\sqrt{x^2-4}))^n\right) & \textit{if } n \textit{ is odd,} \\ a^{-n+2}\left(Aa^n + B(-a)^n + C(\frac{1}{2}(x+\sqrt{x^2-4}))^n + D(\frac{1}{2}(x-\sqrt{x^2-4}))^n\right) & \textit{if } n \textit{ is even,} \end{cases} \quad n \geq 0,$$

where

$$\begin{aligned} A &= \frac{a^2\left(x^2-1\right)+ax+x^2-1}{a^2-ax+1}, \quad B = 0, \\ C &= \frac{\left(a^2+1\right)\left(2a^3+a^2\left(x^3+x^2\sqrt{x^2-4}-\sqrt{x^2-4}-3x\right)+a\left(-x^2+x\sqrt{x^2-4}+2\right)-\sqrt{x^2-4}-x\right)}{2a^2\sqrt{x^2-4}\left(-a^2+ax-1\right)}, \\ D &= \frac{\left(a^2+1\right)\left(-2a^3+a^2\left(-x^3+x^2\sqrt{x^2-4}-\sqrt{x^2-4}+3x\right)+a\left(x^2+x\sqrt{x^2-4}-2\right)-\sqrt{x^2-4}+x\right)}{2a^2\sqrt{x^2-4}\left(-a^2+ax-1\right)}. \end{aligned}$$

Proof. The proof follows directly from (1.7) and (2.12) by considering $w(\mathfrak{K}_n) = n+2$ and $w(\mathfrak{K}_n) = n-2$ if n is odd and even, respectively. □

Remark 2.13. *Since it is well known that the Kauffman polynomial F specializes to the Jones polynomial for $a = -t^{3/4}$ and $x = t^{-1/4} + t^{1/4}$ and the BLM/Ho polynomial for $a = 1$, the recurrence relations for the mentioned knot polynomials of twist knot $\mathfrak{K}_n$ could be easily obtained.*

Corollary 2.14. *For $n \geq 1$, the explicit form of of $\{L_n\}$ and $\{F_n\}$ are*

$$L_n = a^{-1}xR_{n+2} + (ax^2 + a^{-2}x - a - a^{-1})R_{n+1} + (x^2 - ax + a^2 - a^{-2})R_n + (-x + a + a^{-1})R_{n-1} - a^n \tag{2.15}$$

and

$$F_n = \begin{cases} a^{-1}xS_{n+2} + (a^2x^2 + a^{-3}x - a^2 - 1)S_{n+1} + (x^2 - a^3x + a^4 + a^2 - a^{-2} - 1)S_n + (-ax + a^2 + 1)S_{n-1} - a^{-2} & \textit{if } n \textit{ is odd,} \\ a^3xS_{n+2} + (a^6x^2 + ax - a^6 - a^4)S_{n+1} + (a^4x^2 - a^7x + a^8 + a^6 - a^4 - a^2)S_n + (-a^5x + a^6 + a^4)S_{n-1} - a^2 & \textit{if } n \textit{ is even,} \end{cases} \tag{2.16}$$

where $\{R_n\}$ and $\{S_n\}$ are special cases of the following sequence $\{G_n\}$ with initial conditions $G_0 = G_1 = 0$, $G_2 = 1$ for $r = a + x$, $s = -(1 + ax)$, $t = a$ and $r = a^2 + ax$, $s = -(a^2 + a^3x)$, $t = a^4$, respectively.

$$G_n = \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n-2}{3} \rfloor} \binom{n-2-i-2j}{i+j}\binom{i+j}{j} r^{n-2-2i-3j} s^i (-1)^i t^j.$$

Proof. From Theorem 2.3 and 2.4 in [31], the explicit forms of the Kauffman polynomial L and F sequences of $\mathfrak{K}_{(2,n)}$ are given by

$$L_{\mathfrak{K}_{(2,n)}} = (a^{-1})R_{n+1} + \left(ax - (a+a^{-1})x^{-1}\right)R_n + \left((1+a^2)x^{-1} - a\right)R_{n-1}, \quad n \geq 1 \tag{2.17}$$

and

$$F_{\mathfrak{K}_{(2,n)}} = S_{n+1} + \left(a^3x - (a^3+a)x^{-1}\right)S_n + \left((a^5+a^3)x^{-1} - a^4\right)S_{n-1}, \quad n \geq 1. \tag{2.18}$$

Hence, the explicit form (2.15) is obtained by using (2.9) and (2.17). Then, the explicit form (2.16) is obtained by considering $w(\mathfrak{K}_n) = n+2$ and $w(\mathfrak{K}_n) = n-2$ if n is odd and even, respectively, and by using (2.14) and (2.18). □

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