http://dergipark.gov.tr/cams

Communications in Advanced Mathematical Sciences

VOLUME IV
ISSUE I

ISSN 2651-4001

VOLUME IV ISSUE I
ISSN 2651-4001

March 2021
www.dergipark.org.tr/tr/pub/cams

COMMUNICATIONS IN ADVANCED MATHEMATICAL SCIENCES

Contents

Communications in Advanced Mathematical Sciences
Vol. IV, No. 1, 1-13, 2021
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.738324

Extended Newton-type Method for Generalized Equations with Hölderian Assumptions

M. Z. Khaton[1], M. H. Rashid[2]*

Abstract
In the present paper, we consider the generalized equation $0 \in f(x)+g(x)+\mathscr{F}(x)$, where $f:\mathscr{X}\to\mathscr{Y}$ is Fréchet differentiable on a neighborhood Ω of a point $\bar{x}$ in $\mathscr{X}$, $g:\mathscr{X}\to\mathscr{Y}$ is differentiable at point $\bar{x}$ and linear as well as $\mathscr{F}$ is a set-valued mapping with closed graph acting between two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$. We study the above generalized equation with the help of extended Newton-type method, introduced in [M. Z. Khaton, M. H. Rashid, M. I. Hossain, Journal of Mathematics Research, **10**(4) (2018), 1–18.], under the weaker conditions than that are used in Khaton *et al.* (2018). Indeed, semilocal and local convergence analysis are provided for this method under the conditions that the Fréchet derivative of f and the first-order divided difference of g are Hölder continuous on Ω. In particular, we show this method converges superlinearly and these results extend and improve the corresponding results in Argyros (2008) and Khaton *et al.* (2018).

Keywords: Divided difference, Extended Newton-type method, Generalized equations, Lipschitz-like mappings, Semilocal convergence.
2010 AMS: 49J53, 47H04, 65K10

[1] *Department of Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh and Officer on Special Duty, Directorate of Secondary and Higher Education, Dhaka, Bangladesh*
[2] *Department of Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh, ORCID: 0000-0001-8537-1596*
***Corresponding author**: harun_math@ru.ac.bd
Received: 16 May 2020, **Accepted:** 16 January 2021, **Available online:** 29 March 2021

1. Introduction

Robinson [27, 28] introduced generalized equation problems as an universal instrument for describing, analyzing and solving various type of problems in a framed way. This form of generalized equation problems have been discussed widely. Typical examples are systems of inequalities, systems of nonlinear equations, variational inequality problems, linear and nonlinear complementary problems and etc; see for examples [7, 19, 20]. Let Ω be a subset of $\mathscr{X}$. Let f be a Fréchet differentiable function from Ω to $\mathscr{Y}$ and ∇f be its Fréchet derivative, g be a differentiable at $\bar{x}$ but it may not be differentiable in a neighborhood Ω of $\bar{x}$ and linear function from Ω to $\mathscr{Y}$, $[x,y;g]$ denote the first-order divided difference at the points x and y and $\mathscr{F}$ be a set-valued mapping from $\mathscr{X}$ to $\mathscr{Y}$ with closed graph. To find a point x in Ω, we consider the generalized equation of the following form:

$$0 \in f(x)+g(x)+\mathscr{F}(x). \tag{1.1}$$

Pietrus and Alexis [1] associated the following Newton-like method for solving (1.1):

$$\begin{aligned} 0 \quad \in \quad & f(x_k)+g(x_k)+(\nabla f(x_k)+[2x_{k+1}-x_k,x_k;g])(x_{k+1}-x_k) \\ & +\mathscr{F}(x_{k+1}), \text{ for } k=0,1,\ldots \end{aligned} \tag{1.2}$$

and proved that the sequence generated by the process (1.2) converges superlinearlly. To solve the generalized equation (1.1), Rashid *et al*. [25] established the local convergence results using the weaker conditions than Alexis and Pietrus [1] for the method (1.2) and expanded the sequels by fixing a gap in the proof of [1, Theorem 1].

Furthermore, Hilout *et al*. [12] associated the following sequence for solving (1.1):

$$\begin{cases} x_0 \text{ and } x_1 \text{ are two starting points} \\ y_k = \alpha x_k + (1-\alpha)x_{k-1}; \quad \text{here } \alpha \in (0,1) \\ 0 \in f(x_k) + [y_k, x_k; f](x_{k+1} - x_k) + \mathscr{F}(x_{k+1}) \end{cases}$$

and they proved the superlinear convergence of the sequence generated by this method under the assumption that f is only differentiable and continuous at a solution x^*.

For approximating the solution of (1.1), Argyros and Hilout [4] considered the following Newton-like method :

$$0 \in f(x_k) + g(x_k) + \big(\nabla f(x_k) + [x_{k+1}, x_k; g]\big)(x_{k+1} - x_k) + \mathscr{F}(x_{k+1}), \text{for } k = 0, 1, \ldots, \tag{1.3}$$

and under Lipschitz continuity property of ∇f, they presented the quadratic convergence of the method (1.3).

Moreover, when $\mathscr{F} = \{0\}$, Argyros [2] investigated on local as well as semilocal convergence analysis for two-point Newton-like methods for solving (1.1) in a Banach space setting under very general Lipschitz type conditions. An extensive study on these issues has been investigated by Rashid [3, 19, 20, 21] and other researchers when $g = 0$. In the case when $\mathscr{F}$ is either zero mapping or nonzero mapping, a large number Newton-like iterative methods have been studied and we will not mention here all in detail.

Suppose that $x \in \mathscr{X}$ and $\mathscr{N}(x)$ is the subset of $\mathscr{X}$ which is defined as

$$\mathscr{N}(x) = \big\{d \in \mathscr{X} : 0 \in f(x) + g(x) + (\nabla f(x) + [x+d, x; g])d + \mathscr{F}(x+d)\big\}.$$

Under some suitable conditions, Khaton *et al*. [18] introduced and studied extended Newton-type method, when ∇f is continuous and Lipschitz continuous as well as g admits first-order divided difference satisfying Lipschitzian condition. Inspired by the work of Argyros in [4], Khaton *et al*. [18] considered the following "so called" extended extended Newton-type method (see Algorithm 1):

Algorithm 1 (Extended Newton-type Method)

Step 0. Pick $\eta \in [1, \infty)$, $x_0 \in \mathscr{X}$, and put $k := 0$.
Step 1. If $0 \in \mathscr{N}(x_k)$, then stop; otherwise, go to the next Step 2.
Step 2. If $0 \notin \mathscr{N}(x_k)$, choose $d_k \in \mathscr{N}(x_k)$ such that

$$\|d_k\| \le \eta \text{ dist } (0, \mathscr{N}(x_k)).$$

Step 3. Set $x_{k+1} := x_k + d_k$.
Step 4. Replace k by $k+1$ and go to Step 1.

In contrast Algorithm 1 with the known results, we have the following conclusions: When $F = 0$ and $g = 0$, it is obvious that Algorithm 1 is turned into the known Gauss-Newton method which is a famous iterative technique for solving nonlinear least squares (model fitting) problems and has been studied widely; see for example [8, 9, 13, 15, 29, 30]. Within the case when $g = 0$, several kind of methods for solving (1.1) were established by Rashid [22, 23, 24] and also obtained their local and semilocal convergence.

The objective of this article is to continue to study the semilocal and local convergence for the extended Newton-type method under the weaker conditions than [18], that is, ∇f is (L, q)-Hölder continuous and g admits the first-order divided difference satisfying q-Hölderian condition. The Lipschitz-like property of set-valued mappings which is the main tool of this study whose concepts can be found in Aubin [5] in the context of non smooth analysis and it has been studied by a huge number of mathematicians [1, 4, 10, 12, 17]. The main result of this study is semilocal analysis for the extended Newton-type method, that is, based on the information around the initial point, the main results are the convergence criteria, which provide few suitable conditions ensuring the convergence to a solution of any sequence generated by Algorithm 1. Consequently, the results of the local convergence for the extended Newton-type method are attained.

This article is organized as follows: Some necessary notations, notions, preliminary results and a fixed-point theorem are recalled in Section 2 that are used in the subsequent sections. In Section 3, we consider the extended Newton-type method defined by Algorithm 1 to approximate the solution of (1.1). Using the concept of Lipschitz-like property for the set-valued mapping, in this section we also establish the existence and superlinear convergence of the sequence generated by Algorithm 1 in both semilocal and local cases. At the end, we give a summary of the main results and present a comparison of this study with other known results.

2. Notations and Preliminaries

In this section, we evoke some notations and take out some results that will be helpful to verify our main results. Let $\mathscr{X}$ and $\mathscr{Y}$ be two complex or real Banach spaces. Also, let $p \in \mathscr{X}$ and $\mathbb{B}(p,\alpha) = \{u \in \mathscr{X} : \|u-p\| \leq \alpha\}$ denote the closed ball centered at p with radius $\alpha > 0$, and $\mathscr{F}$ be a set-valued mapping with closed graph. The domain of $\mathscr{F}$, can be stated as

$$\mathrm{dom}\mathscr{F} := \{p \in \mathscr{X} : \mathscr{F}(p) \neq \emptyset\}.$$

Let $q \in \mathscr{Y}$. Then the inverse of $\mathscr{F}$, denoted by $\mathscr{F}^{-1}$, is defined by

$$\mathscr{F}^{-1}(q) := \{p \in \mathscr{X} : q \in \mathscr{F}(p)\}.$$

The graph of $\mathscr{F}$, denoted by gph$\mathscr{F}$, is defined by

$$\mathrm{gph}\mathscr{F} := \{(p,q) \in \mathscr{X} \times \mathscr{Y} : q \in \mathscr{F}(p)\}.$$

Let M and N be two subsets of a non empty set $\mathscr{X}$ and p be a point in $\mathscr{X}$. The distance from a point p to a set M is defined by

$$\mathrm{dist}(p,M) := \inf\{\|p-m\| : m \in M\}.$$

In addition, the excess e from the set M to the set N is defined by

$$e(N,M) := \sup\{\mathrm{dist}(n,M) : n \in N\}.$$

The set $\mathscr{L}(\mathscr{X},\mathscr{Y})$ is the space of linear operators from $\mathscr{X}$ to $\mathscr{Y}$ and all the norms are denoted by $\|\cdot\|$.

Definition 2.1. *Suppose $f \in \mathscr{L}(\mathscr{X},\mathscr{Y})$. Then f is said to have the first order divided difference on the points x_1 and y_1 in $\mathscr{X}$ $(x_1 \neq y_1)$ if the following properties hold:*

(a) $[x_1,y_1;f](y_1-x_1) = g(y_1) - g(x_1)$ *for* $x_1 \neq y_1$*;*

(b) *if f is Fréchet differentiable at $x_1 \in \mathscr{X}$, then $[x_1,x_1;f] = \nabla f(x_1)$.*

Now we mention the notions of pseudo-Lipschitz and Lipchitz-like set-valued mappings, which was established by Aubin and have been studied widely. To see the more details about this topic, the reader could refer to [5, 6, 26].

Definition 2.2. *Let $\Psi : \mathscr{Y} \rightrightarrows \mathscr{X}$ be a set-valued mapping and $(\bar{q},\bar{p}) \in \mathrm{gph}\Psi$ with $\alpha_{\bar{p}}$, $\alpha_{\bar{q}}$ and ν are positive constants. Then Ψ is said to be*

(a) *Lipchitz-like on $\mathbb{B}(\bar{q},\alpha_{\bar{q}})$ relative to $\mathbb{B}(\bar{p},\alpha_{\bar{p}})$ with constant ν if the following inequality holds:*

$$e(\Psi(q_1) \cap \mathbb{B}(\bar{p},\alpha_{\bar{p}}), \Psi(q_2)) \leq \nu\|q_1 - q_2\| \quad \textit{for every } q_1,q_2 \in \mathbb{B}(\bar{q},\alpha_{\bar{q}}).$$

(b) *pseudo-Lipschitz around $(\bar{q},\bar{p})$ if there exist constants $\alpha'_{\bar{p}} > 0, \alpha'_{\bar{q}} > 0$ and $\nu' > 0$ such that Ψ is Lipchitz-like on $\mathbb{B}(\bar{q},\alpha'_{\bar{q}})$ relative to $\mathbb{B}(\bar{p},\alpha'_{\bar{p}})$ with constant ν'.*

The following lemma is due to Rashid *et al*. [26, Lemma 2.1], which is effective and the proof of this lemma is similar to that of [16, Theorem 1.49(i)].

Lemma 2.3. *Let $\Psi : \mathscr{Y} \rightrightarrows \mathscr{X}$ be a set-valued mapping and $(\bar{y},\bar{x}) \in$ gph Ψ. Also suppose that Ψ is Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ which is related to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant μ. Then*

$$\mathrm{dist}\,(x,\Psi(y)) \leq \nu\ \mathrm{dist}(y,\Psi^{-1}(x)),$$

for each $x \in \mathbb{B}(\bar{x},r_{\bar{x}})$ and $y \in \mathbb{B}(\bar{y},\frac{r_{\bar{y}}}{3})$ satisfying $\mathrm{dist}(y,\Psi^{-1}(x)) \leq \dfrac{r_{\bar{y}}}{3}$*, is hold.*

Dontchev and Hager [11] proved Banach fixed point theorem, which has been employing the standard iterative concept for contracting mapping. To prove the existence of the sequence generated by Algorithm 1, the following lemma will play an important rule in this study.

Lemma 2.4. *Let $\Phi : \mathscr{X} \rightrightarrows \mathscr{X}$ be a set-valued mapping. Let $x^* \in \mathscr{X}$, $0 < \lambda < 1$ and $r > 0$ be such that*

$$\mathrm{dist}(x^*, \Phi(x^*)) < r(1-\lambda) \tag{2.1}$$

and

$$e(\Phi(x_1) \cap \mathbb{B}(x^*, r), \Phi(x_2)) \leq \lambda \|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in \mathbb{B}(x^*, r). \tag{2.2}$$

Then Φ has a fixed point in $\mathbb{B}(x^*, r)$, that is, there exists $x \in \mathbb{B}(x^*, r)$ such that $x \in \Phi(x)$. Furthermore, if Φ is single-valued, then there exists a fixed point $x \in \mathbb{B}(x^*, r)$ such that $x = \Phi(x)$.

The preceding lemma is a generalization of a fixed point theorem and it has been taken from [14], where in the second assertion the excess e is updated by Hausdorff distance.

3. Convergence Analysis

Let $f : \Omega \subseteq \mathscr{X} \to \mathscr{Y}$ be a Fréchet differentiable function on a neighborhood Ω of $\bar{x}$ with its derivative denoted by ∇f, $g \colon \Omega \to \mathscr{Y}$ which is linear and differentiable at $\bar{x}$ and let $\mathscr{F} : \mathscr{X} \rightrightarrows \mathscr{Y}$ be a set-valued mapping with closed graph. This section is dedicated to prove the existence of a sequence generated by the extended Newton-type method, represented by Algorithm 1 and show the superlinear convergence of the sequence generated by this method.

Let $x \in \mathscr{X}$. Then for each $x \in \mathscr{X}$, we get

$$\begin{aligned} g(x) + [x+d, x; g]d &= g(x) - [x+d, x; g](x - (x+d)) \\ &= g(x) - (g(x) - g(x+d)) = g(x+d). \end{aligned} \tag{3.1}$$

Define a set-valued mapping $\mathscr{G}_x$ by

$$\mathscr{G}_x(\cdot) := f(x) + g(\cdot) + \nabla f(x)(\cdot - x) + \mathscr{F}(\cdot).$$

It holds, for the formation of $\mathscr{N}(x)$ and (3.1), that

$$\mathscr{N}(x) = \left\{ d \in \mathscr{X} : 0 \in \mathscr{G}_x(x+d) \right\}.$$

In addition, for any $z \in \mathscr{X}$ and $y \in \mathscr{Y}$, we get the following identity:

$$z \in \mathscr{G}_x^{-1}(y) \text{ if and only if } y \in f(x) + g(z) + \nabla f(x)(z - x) + \mathscr{F}(z). \tag{3.2}$$

Particularly, let $(\bar{x}, \bar{y}) \in \mathrm{gph}\mathscr{G}_{\bar{x}}$. Then, the definition of closed graphness of $\mathscr{G}_{\bar{x}}$ signifies that

$$\bar{x} \in \mathscr{G}_{\bar{x}}^{-1}(\bar{y}). \tag{3.3}$$

The following outcome constitutes the equivalence between $\mathscr{G}_{\bar{x}}^{-1}$ and $(f + g + \mathscr{F})^{-1}$. This result is due to [18].

Lemma 3.1. *Let $(\bar{x}, \bar{y}) \in$ gph $(f + g + \mathscr{F})$. Suppose that ∇f is continuous around $\bar{x}$. Assume that g admits first-order divided difference. Then the followings are equivalent:*

(i) *The mapping $(f + g + \mathscr{F})^{-1}$ is pseudo-Lipschitz at $(\bar{y}, \bar{x})$;*

(ii) *The mapping $\mathscr{G}_{\bar{x}}^{-1}$ is pseudo-Lipschitz at $(\bar{y}, \bar{x})$.*

For our suitability, let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $\mathbb{B}(\bar{x}, r_{\bar{x}}) \subseteq \Omega \cap \mathrm{dom}\mathscr{F}$. Suppose that ∇f is (L, q)-Hölder continuous on $\mathbb{B}(\bar{x}, r_{\bar{x}})$, that is , there exists $L > 0$ such that

$$\|\nabla f(x) - \nabla f(x')\| \leq L\|x - x'\|^q, q \in (0, 1], \quad \text{for any } x, x' \in \mathbb{B}(\bar{x}, r_{\bar{x}}), \tag{3.4}$$

g admits a first-order divided difference satisfying q-Hölder condition, that is, there exists $\nu > 0$ such that, for all $x, y, v, w \in \mathbb{B}(\bar{x}, r_{\bar{x}})$ $(x \neq y, v \neq w)$,

$$\|[x, y; g] - [v, w; g]\| \leq \nu(\|x - v\|^q + \|y - w\|^q), \tag{3.5}$$

and the mapping $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M, that is,

$$e(\mathscr{G}_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathscr{G}_{\bar{x}}^{-1}(y_2)) \leq M\|y_1 - y_2\| \quad \text{for any } y_1, y_2 \in \mathbb{B}(\bar{y}, r_{\bar{y}}). \tag{3.6}$$

Further, for $\bar{y}$, the closed graph property of $\mathscr{G}_{\bar{x}}$ implies that $f + g + \mathscr{F}$ is continuous at $\bar{x}$ i.e.

$$\lim_{x \to \bar{x}} \operatorname{dist}\big(\bar{y}, f(x) + g(x) + \mathscr{F}(x)\big) = 0 \tag{3.7}$$

is hold.
Let $\varepsilon_0 > 0$ and write

$$\bar{r} := \min\Big\{r_{\bar{y}} - 2\varepsilon_0 r_{\bar{x}}, \frac{r_{\bar{x}}(1 - M\varepsilon_0)}{4M}\Big\}. \tag{3.8}$$

Then

$$\bar{r} > 0 \text{ if and only if } \varepsilon_0 < \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}}, \ \frac{1}{M}\Big\}. \tag{3.9}$$

The following lemma is taken from [26, Lemma 3.1] and it plays a crucial role for convergence analysis of the extended Newton-type method.

Lemma 3.2. *Assume that $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M and that*

$$\sup_{x', x'' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})} \|\nabla f(x') - \nabla f(x'')\| \leq \varepsilon_0 < \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}}, \ \frac{1}{M}\Big\}. \tag{3.10}$$

Let $x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ and ε_0 be defined by (3.9). Suppose that ∇f is continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Let $\bar{r}$ be defined by (3.8) such that (3.10) is true. Then $\mathscr{G}_x^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \bar{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1 - M\varepsilon_0}$, that is,

$$e(\mathscr{G}_x^{-1}(y_1) \cap \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2}), \mathscr{G}_x^{-1}(y_2)) \leq \frac{M}{1 - M\varepsilon_0}\|y_1 - y_2\| \ \text{ for any } \ y_1, \ y_2 \in \mathbb{B}(\bar{y}, \bar{r}).$$

For our convenience, we would like to introduce some notations. First we define the mapping $J_x\colon \mathscr{X} \to \mathscr{Y}$, for each $x \in \mathscr{X}$, by

$$J_x(\cdot) := f(\bar{x}) + g(\cdot) + \nabla f(\bar{x})(\cdot - \bar{x}) - f(x) - g(x) - \big(\nabla f(x) + [\cdot, x; g]\big)(\cdot - x)$$

and the set-valued mapping $\Phi_x\colon \mathscr{X} \rightrightarrows \mathscr{X}$ by

$$\Phi_x(\cdot) := \mathscr{G}_{\bar{x}}^{-1}[J_x(\cdot)]. \tag{3.11}$$

Then for any $x', \ x'' \in \mathscr{X}$, we have

$$\begin{aligned} \|J_x(x') - J_x(x'')\| \quad &= \quad \|g(x') - g(x'') - [x', x; g](x' - x) + [x'', x; g](x'' - x) \\ &\qquad + (\nabla f(\bar{x}) - \nabla f(x))(x' - x'')\|. \end{aligned} \tag{3.12}$$

Furthermore, let $q \in (0, 1]$ and define

$$\hat{r} := \min\Big\{r_{\bar{y}} - 2Lr_{\bar{x}}^{q+1}, \frac{r_{\bar{x}}(1 - MLr_{\bar{x}}^q)}{4M}\Big\}. \tag{3.13}$$

Then

$$\hat{r} > 0 \Leftrightarrow L < \ \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}^{q+1}}, \frac{1}{Mr_{\bar{x}}^q}\Big\}. \tag{3.14}$$

3.1 Superlinear Convergence

In this section we will show that the sequence generated by Algorithm 1 converges superlinearly if ∇f is (L,q)-Hölderian and g admits first-order divided difference satisfying (ν,q)-Hölder condition. In fact, the following theorem provides some sufficient conditions ensuring the convergence of the extended Newton-type method with initial point x_0.

Theorem 3.3. *Let $\eta > 1$ and $q \in (0,1]$. Assume that $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M and that ∇f is (L,q)- Hölder continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ and g admits first-order divided difference that satisfies (3.5). Let $\hat{r}$ be defined by (3.13) so that (3.14) is satisfied. Let $\nu > 0$, $\delta > 0$ be such that*

(a) $\delta \le \min\left\{\frac{r_{\bar{x}}}{4}, (q+5)\hat{r}, 1, \left(\frac{3(q+1)r_{\bar{y}}}{[L(q+2)+2\nu(q+1)](6.2^q+1)}\right)^{\frac{1}{(q+1)}}\right\}$,

(b) $(2^q M+1)[L(q+2)+2\nu(q+1)]\Big(\eta(q+1)\delta^q + 4^{1-q}r_{\bar{x}}^q\Big) \le (q+1)$,

(c) $\|\bar{y}\| < \frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1}$.

Suppose that

$$\lim_{x\to\bar{x}} \operatorname{dist}(\bar{y}, f(x)+g(x)+\mathscr{F}(x)) = 0. \tag{3.15}$$

Then there exist some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 1 with initial point x_0 in $\mathbb{B}(\bar{x}, \hat{\delta})$ converges superlinearly to a solution x^* of (1.1).

Proof. According to the assumption (a) $4\delta \le r_{\bar{x}}$ and $\eta > 1$, by assumption (b) we can write the inequality as follows

$$\begin{aligned}(2^q M+1)(q+5)[L(q+2)+2\nu(q+1)]\delta^q &= (2^q M+1)[L(q+2)+2\nu(q+1)]\Big((q+1)\delta^q+4\delta^q\Big)\\ &\le (2^q M+1)[L(q+2)+2\nu(q+1)]\Big(\eta(q+1)\delta^q+4\delta^q\Big)\\ &\le (2^q M+1)[L(q+2)+2\nu(q+1)]\Big(\eta(q+1)\delta^q+4^{1-q}r_{\bar{x}}^q\Big)\\ &\le (q+1).\end{aligned} \tag{3.16}$$

Furthermore, using assumption (a) $4\delta \le r_{\bar{x}}$ and assumption(b) we can reduce the inequality as follows

$$\begin{aligned}\eta M[L(q+2)+2\nu(q+1)]\delta^q &< \eta 2^q M[L(q+2)+2\nu(q+1)](q+5)\delta^q\\ &\le (2^q M+1)[L(q+2)+2\nu(q+1)](\eta(q+1)\delta^q+4\delta^q) - 2^q ML4\delta^q\\ &\le (2^q M+1)[L(q+2)+2\nu(q+1)](\eta(q+1)\delta^q+4^{1-q}r_{\bar{x}}^q) - 2^q ML4^{1-q}r_{\bar{x}}^q\\ &\le (q+1) - 2^q ML4^{1-q}r_{\bar{x}}^q.\end{aligned}$$

Since $q \in (0,1]$ then, we get $2^q ML4^{1-q}r_{\bar{x}}^q \ge (q+1)MLr_{\bar{x}}^q$. Now using (3.16) in the above equation and it becomes

$$\eta M[L(q+2)+2\nu(q+1)]\delta^q \le (q+1) - (q+1)MLr_{\bar{x}}^q. \tag{3.17}$$

Putting

$$s := \frac{\eta M[L(q+2)+2\nu(q+1)]\delta^q}{(q+1)(1-MLr_{\bar{x}}^q)}.$$

Then, from (3.17) we have that

$$s \le 1. \tag{3.18}$$

Pick $0 < \hat{\delta} \le \delta$ such that, for each $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta})$,

$$\operatorname{dist}(0, f(x_0)+g(x_0)+F(x_0)) \le \frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1}. \tag{3.19}$$

Note that since (3.15) holds and assumption (c) is true, we assume that such $\hat{\delta}$ exists, which satisfies (3.19). Let $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta})$. By induction we will show that Algorithm 1 generates at least one sequence and such sequence $\{x_n\}$ generated by Algorithm 1 satisfies the following statements:

$$\|x_n - \bar{x}\| \leq 2\delta \tag{3.20}$$

$$\text{and} \qquad \|d_n\| \leq s\left(\frac{1}{3}\right)^{(q+1)^n} \delta, \tag{3.21}$$

hold for every $n = 0, 1, 2, \ldots$.

Define

$$r_x := \frac{(q+5)M}{4(q+1)}\Big([L(q+2)+2\nu(q+1)]\|x-\bar{x}\|^{(q+1)} + (q+1)\|\bar{y}\|\Big) \quad \text{for each } x \in X. \tag{3.22}$$

From (3.16) we get

$$2^q M[L(q+2)+2\nu(q+1)]\delta^q \leq \frac{q+1}{q+5}. \tag{3.23}$$

$$\text{and} \quad [L(q+2)+2\nu(q+1)]\delta^q \leq \frac{q+1}{q+5}. \tag{3.24}$$

Hence by the combination of $\delta \leq (q+5)\hat{r}$ in assumption (a) and inequality (3.24), we get

$$\begin{aligned} \|\bar{y}\| &< \frac{[L(q+2)+2\nu(q+1)]\delta^{q+1}}{3(q+1)} \\ &\leq \frac{(q+1)}{(q+1)\cdot(q+5)} \cdot \frac{(q+5)\hat{r}}{3} = \frac{\hat{r}}{3}. \end{aligned} \tag{3.25}$$

Utilizing (3.23) and assumption (c) together with (3.24), we get from (3.22) that

$$\begin{aligned} r_x &\leq \frac{(q+5)M}{4(q+1)}\Big([L(q+2)+2\nu(q+1)]\|\bar{x}-x_0\|^{q+1} + \frac{[L(q+2)+2\nu(q+1)]}{3}\delta^{q+1}\Big) \\ &< \frac{(q+5)M}{12(q+1)}\Big(3[L(q+2)+2\nu(q+1)](2\delta)^{q+1} + 2^q[L(q+2)+2\nu(q+1)]\delta^{q+1}\Big) \\ &= \frac{(q+5)M}{12(q+1)}[L(q+2)+2\nu(q+1)]\delta^{q+1}(3.2.2^q + 2^q) \\ &= \frac{(q+5)(6\cdot 2^q + 2^q)M}{12(q+1)}[L(q+2)+2\nu(q+1)]\delta^{q+1} \\ &= \frac{(q+5)7\cdot 2^q M}{12(q+1)}[L(q+2)+2\nu(q+1)]\delta^{p+1} \\ &= \frac{7(q+5)}{12(q+1)} \cdot \frac{(q+1)}{(q+5)}\delta < \frac{7}{12}\delta < 2\delta \quad \text{for each } x \in \mathbb{B}(\bar{x}, 2\delta). \end{aligned} \tag{3.26}$$

Observe that (3.20) is trivial for $n = 0$.

At first, we need to prove $\mathcal{N}(x_0) \neq \emptyset$ to show that (3.21) holds for $n = 0$. The nonemptyness of $\mathcal{N}(x_0)$ will ensure us to deduce the existence of the point x_1. We will apply Lemma 2.4 to the map Φ_{x_0} with $\eta_0 = \bar{x}$ for completing this. We have to show that Lemma 2.4 holds with $r := r_{x_0}$ and $\lambda := \dfrac{q+1}{q+5}$ satisfying both assertions (2.1) and (2.2). We get from (3.3) that $\bar{x} \in \mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta)$. According to the definition of the excess e and (3.11), defined as the mapping of Φ_{x_0}, we have that

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\leq e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(\bar{x})) \\ &\leq e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(\bar{x})) \\ &\leq e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(\bar{x})]). \end{aligned} \tag{3.27}$$

Since ∇f is (L,q)-Hölder continuous and g admits first-order divided difference satisfies Hölderian condition, for every $x \in \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, we have that

$$
\begin{aligned}
\|J_{x_0}(x) - \bar{y}\| &= \|f(\bar{x}) + g(x) + \nabla f(\bar{x})(x-\bar{x}) - f(x_0) - g(x_0) \\
&\quad -(\nabla f(x_0) + [x, x_0; g])(x - x_0) - \bar{y}\| \\
&\leq \|f(\bar{x}) - f(x_0) - \nabla f(x_0)(\bar{x} - x_0)\| + \|\big(\nabla f(x_0) - \nabla f(\bar{x})\big)(\bar{x} - x)\| \\
&\quad + \|g(x) - g(x_0) - [x, x_0; g](x - x_0)\| + \|\bar{y}\| \\
&\leq \frac{L}{q+1}\|\bar{x} - x_0\|^{q+1} + \|[x_0, x; g] - [x, x_0; g]\|\|x - x_0\| + \\
&\quad L\|x_0 - \bar{x}\|^q\|\bar{x} - x\| + \|\bar{y}\| \qquad (3.28) \\
&\leq \frac{L}{q+1}\|\bar{x} - x_0\|^{q+1} + \nu\big(\|x_0 - x\|^q + \|x - x_0\|^q\big)\|x - x_0\| + \\
&\quad L\|x_0 - \bar{x}\|^q\|\bar{x} - x\| + \|\bar{y}\| \\
&\leq \frac{L}{q+1}(2\delta)^{q+1} + L(2\delta)^q \cdot 2\delta + \nu\big((2\delta)^q + (2\delta)^q\big) \cdot 2\delta + \|\bar{y}\| \\
&\leq \frac{L(q+2) + 2\nu(q+1)}{q+1}\delta^{q+1} \cdot 2^{q+1} + \|\bar{y}\|. \qquad (3.29)
\end{aligned}
$$

Now through the assumptions (a) $\dfrac{[L(q+2)+2\nu(q+1)](6\cdot 2^q+1)}{3(q+1)}\delta^{q+1} \leq r_{\bar{y}}$ and (c), (3.28) gives that

$$
\begin{aligned}
\|J_{x_0}(x) - \bar{y}\| &\leq \frac{[L(q+2)+2\nu(q+1)]}{q+1}2^{q+1}\delta^{q+1} + \frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1} \\
&= \frac{[L(q+2)+2\nu(q+1)](3.2.2^q+1)}{3(q+1)}\delta^{q+1} \\
&< \frac{[L(q+2)+2\nu(q+1)](6\cdot 2^q+1)}{3(q+1)}\delta^{q+1} \\
&\leq r_{\bar{y}}. \qquad (3.30)
\end{aligned}
$$

This means that $J_{x_0}(x) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$. Moreover, let $x = \bar{x}$ in (3.28). Then it is easily proved that

$$J_{x_0}(\bar{x}) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$$

and

$$\|J_{x_0}(\bar{x}) - \bar{y}\| \leq \frac{[L+2\nu(q+1)]}{q+1}\|\bar{x} - x_0\|^{q+1} + \|\bar{y}\|. \qquad (3.31)$$

By using the Lipschitz-like property of $\mathscr{G}_{\bar{x}}^{-1}$ and (3.31) in (3.27), we obtain

$$
\begin{aligned}
\operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\leq M\|\bar{y} - J_{x_0}(\bar{x})\| \\
&\leq \frac{M[L(q+2)+2\nu(q+1)]}{q+1}\|\bar{x} - x_0\|^{q+1} + M\|\bar{y}\| \\
&\leq \frac{4}{q+5}r_{x_0} = \Big(1 - \frac{q+1}{q+5}\Big)r_{x_0} \\
&= (1-\lambda)r;
\end{aligned}
$$

i,e,. the statement (2.1) of Lemma 2.4 is hold.

Now, it is evident to show that statement (2.2) of Lemma 2.4 holds. Let $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0})$. Then we have that $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0}) \subseteq \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, r_{\bar{x}})$ by (3.26) and $J_{x_0}(x'), J_{x_0}(x'') \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ by (3.30). This together with Lipschitz-like property of $\mathscr{G}_{\bar{x}}^{-1}$ follows as

$$
\begin{aligned}
e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\leq e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(x'')) \\
&\leq e(\mathscr{G}_{\bar{x}}^{-1}[J_{x_0}(x')] \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathscr{G}_{\bar{x}}^{-1}[J_{x_0}(x'')]) \\
&\leq M\|J_{x_0}(x') - J_{x_0}(x'')\|. \qquad (3.32)
\end{aligned}
$$

Now, using the definition of first order divided difference of g in (3.12) we obtain

$$\begin{aligned}
\|J_{x_0}(x') - J_{x_0}(x'')\| &= \|g(x') - g(x'') - [x', x_0; g](x' - x_0) + [x'', x_0; g](x'' - x_0) \\
&\quad +(\nabla f(\bar{x}) - \nabla f(x_0))(x' - x'')\| \\
&\leq \|g(x') - g(x'') + [x', x_0; g](x_0 - x') - [x'', x_0; g](x_0 - x'')\| \\
&\quad +\|\nabla f(\bar{x}) - \nabla f(x_0)\|\|x' - x''\| \\
&\leq \|g(x') - g(x'') + g(x_0) - g(x') - g(x_0) + g(x'') \\
&\quad +\|\nabla f(\bar{x}) - \nabla f(x_0)\|\|x' - x''\| \\
&\leq \|\nabla f(\bar{x}) - \nabla f(x_0)\|\|x' - x''\| \leq L\|\bar{x} - x_0\|^q\|x' - x''\| \\
&\leq L.2^q\delta^q\|x' - x''\|. \qquad (3.33)
\end{aligned}$$

It follows from (3.32), that

$$e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) \leq ML.2^q\delta^q\|x' - x''\|.$$

Since $\nu, M, L > 0$ and $q \in (0, 1]$, then we can write $2^q ML\delta^q < 2^q M[L(q+2) + 2\nu(q+1)]\delta^p$ and hence the above inequality becomes

$$\begin{aligned}
e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\leq 2^q M[L(q+2) + 2\nu(q+1)]\delta^p\|x' - x''\| \\
&\leq \frac{q+1}{q+5}\|x' - x''\| \\
&= \lambda\|x' - x''\|.
\end{aligned}$$

Thus the statement (2.2) of Lemma 2.4 is also hold. Hence, both statements (2.1) and (2.2) of Lemma 2.4 are accomplished. Finally, it shows that Lemma 2.4 is adequate to presume the position of a point $\hat{x}_1 \in \mathbb{B}(\bar{x}, r_{x_0})$ such that $\hat{x}_1 \in \Phi_{x_0}(\hat{x}_1)$ which implies that $0 \in f(x_0) + g(x_0) + (\nabla f(x_0) + [\hat{x}_1, x_0; g])(\hat{x}_1 - x_0) + \mathscr{F}(\hat{x}_1)$ and hence $\mathscr{N}(x_0) \neq \emptyset$.

Next, it is sufficient to prove that (3.21) holds for $n = 0$. As ∇f is (L, q)- Hölder continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, we have for all $x', x'' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, that

$$Lr_{\bar{x}}^q \geq \sup_{x', x'' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})} \|\nabla f(x') - \nabla f(x'')\|. \qquad (3.34)$$

Observe the assumption (a) that $\hat{r} > 0$. Therefore, from (3.13) and (3.34)imply that Lemma 3.2 is satisfied with $\varepsilon_0 := Lr_{\bar{x}}^p$. According to our assumption $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$. Then, it implies from Lemma 3.2 that, $\mathscr{G}_{x_0}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \hat{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-MLr_{\bar{x}}^q}$ as $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta}) \subseteq \mathbb{B}(\bar{x}, \delta) \subseteq \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ by assumption (a) and the choice of $\hat{\delta}$. On the other hand, (3.19) follows as

$$\begin{aligned}
\text{dist}(0, \mathscr{G}_{x_0}(x_0)) &= \text{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)) \\
&\leq \frac{\hat{r}}{3}.
\end{aligned}$$

Inequality (3.25) shows that $0 \in \mathbb{B}(\bar{y}, \frac{\hat{r}}{3})$ and observe before that $x_0 \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Hence using Lemma 2.3, we get

$$\begin{aligned}
\text{dist}(x_0, \mathscr{G}_{x_0}^{-1}(0)) &\leq \frac{M}{1 - MLr_{\bar{x}}^q}\,\text{dist}(0, \mathscr{G}_{x_0}(x_0)) \\
&= \frac{M}{1 - MLr_{\bar{x}}^q}\,\text{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)).
\end{aligned}$$

This together with (3.1), gives

$$\begin{aligned}
\text{dist}(0, \mathscr{N}(x_0)) &= \text{dist}(x_0, \mathscr{G}_{x_0}^{-1}(0)) \\
&\leq \frac{M}{1 - MLr_{\bar{x}}^q}\,\text{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)). \qquad (3.35)
\end{aligned}$$

According to Algorithm 1 and using (3.35), (3.19) and then assumption (a), we have

$$\begin{aligned}
\|d_0\| &\leq \eta \ \mathrm{dist}(0, \mathscr{N}(x_0)) \\
&\leq \frac{\eta M}{(1 - M L r_{\bar{x}}^q)} \ \mathrm{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)) \\
&\leq \frac{\eta M [L(q+2) + 2\nu(q+1)]\delta^{q+1}}{3(q+1)(1 - M L r_{\bar{x}}^q)} = s\Big(\frac{1}{3}\Big)\delta.
\end{aligned}$$

This means that

$$\|x_1 - x_0\| = \|d_0\| \leq s\Big(\frac{1}{3}\Big)\delta,$$

and therefore, (3.21) is true for $n = 0$.

Suppose $x_1, x_2, \ldots, x_k$ are formed and (3.20), and (3.21) hold for $n = 0, 1, 2, \ldots, k-1$. We show that there exists x_{k+1} such that (3.20) and (3.21) also hold for $n = k$. Since (3.20) and (3.21) are true for each $n \leq k-1$, we have the following inequality:

$$\|x_k - \bar{x}\| \leq \sum_{i=0}^{k-1} \|d_i\| + \|x_0 - \bar{x}\| \leq s\delta \sum_{i=0}^{k-1} \Big(\frac{1}{3}\Big)^{(q+1)^i} + \delta \leq 2\delta.$$

This implies (3.20) holds for $n = k$. Now with all the same argument as we did for the case when $n = 0$, we can prove that $\mathscr{N}(x_k) \neq \emptyset$, that is, the point x_{k+1} exists and $\mathscr{G}_{x_k}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \hat{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1 - M L r_{\bar{x}}^q}$. Therefore, we have that

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|d_k\| \leq \eta \ \mathrm{dist}(0, \mathscr{N}(x_k)) \\
&\leq \eta \ \mathrm{dist}(x_k, \mathscr{G}_{x_k}^{-1}(0)) \\
&= \frac{\eta M}{1 - M L r_{\bar{x}}^q} \ \mathrm{dist}(0, f(x_k) + g(x_k) + \mathscr{F}(x_k)) \\
&\leq \frac{\eta M}{1 - M L r_{\bar{x}}^q} \|f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1}) \\
&\quad - \big(\nabla f(x_{k-1}) + [x_k, x_{k-1}; g]\big)(x_k - x_{k-1})\| \\
&\leq \frac{\eta M}{1 - M L r_{\bar{x}}^q} \big(\|f(x_k) - f(x_{k-1}) - \nabla f(x_{k-1})(x_k - x_{k-1})\| \\
&\quad + \|g(x_k) - g(x_{k-1}) - [x_k, x_{k-1}; g](x_k - x_{k-1})\|\big) \\
&\leq \frac{\eta M}{(q+1)(1 - M L r_{\bar{x}}^q)} \big(L\|x_k - x_{k-1}\|^{q+1} + \\
&\quad (q+1)\|[x_{k-1}, x_k; g] - [x_k, x_{k-1}; g]\| \|x_k - x_{k-1}\|\big) \\
&\leq \frac{\eta M}{(q+1)(1 - M L r_{\bar{x}}^q)} \big(L\|x_k - x_{k-1}\|^{q+1} + \\
&\quad (q+1)\nu(\|x_{k-1} - x_k\|^q + \|x_k - x_{k-1}\|^q)\|x_k - x_{k-1}\|\big) \\
&\leq \frac{\eta M [L + 2\nu(q+1)]}{(q+1)(1 - M L r_{\bar{x}}^q)} \|d_{k-1}\|^{q+1} \\
&\leq \frac{\eta M [L(q+2) + 2\nu(q+1)]}{(q+1)(1 - M L r_{\bar{x}}^q)} \|d_{k-1}\|^{q+1} \\
&\leq \frac{\eta M [L(q+2) + 2\nu(q+1)]}{(q+1)(1 - M L r_{\bar{x}}^q)} \Big(s\Big(\frac{1}{3}\Big)^{(q+1)^{k-1}} \delta\Big)^{q+1} \\
&\leq s\Big(\frac{1}{3}\Big)^{(q+1)^k} \delta.
\end{aligned}$$

This implies that (3.21) holds for $n = k$ and therefore the proof of the theorem is complete. □

Consider the special case when $\bar{x}$ is a solution of (1.1) (that is, $\bar{y} = 0$) in Theorem 3.3. We have the following corollary, which describes the local superlinear convergence result for the extended Newton-type method.

Corollary 3.4. *Suppose that $\bar{x}$ is a solution of (1.1). Let $q \in (0,1]$ and $\eta > 1$ and let $\mathscr{G}_{\bar{x}}^{-1}$ be pseudo-Lipschitz around $(0,\bar{x})$. Let $\tilde{r} > o$ and suppose that ∇f is (L,q)-Hölder continuous on $\mathbb{B}(\bar{x},\tilde{r})$ and g admits first-order divided difference satisfying Hölderian condition on $\mathbb{B}(\bar{x},\tilde{r})$. Assume that*

$$\lim_{x\to\bar{x}} \operatorname{dist}(0,\mathscr{G}_x(x)) = 0. \tag{3.36}$$

Then, with an initial point x_0, there exists some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 1 converges superlinearly to a solution x^* of (1.1).

Proof. Suppose that $\mathscr{G}_{\bar{x}}^{-1}$ is pseudo-Lipschitz around $(0,\bar{x})$. Then by definition of pseudo-Lipschitz continuty, there exist constants $M,\tilde{r}$ and r_0 such that $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_0)$ relative to $\mathbb{B}(\bar{x},\tilde{r})$ with constant M. Then, for each $0 < r_{\bar{x}} \le \tilde{r}$, we have that

$$e(\mathscr{G}_{\bar{x}}^{-1}(y_1)\cap\mathbb{B}(\bar{x},r_{\bar{x}}),\mathscr{G}_{\bar{x}}^{-1}(y_2) \le M\|y_1-y_2\| \ \text{ for any } \ y_1,\ y_2\in\mathbb{B}(0,r_0),$$

that is, $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_0)$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant M. Let $L \in (0,1]$, $q \in (0,1]$ and $\nu > 0$. By the (L,q)-Hölder continuity of ∇f we can select $r_{\bar{x}} \in (0,\tilde{r})$ such that $\frac{r_{\bar{x}}}{2} \le \tilde{r}$, $r_0 - 2Lr_{\bar{x}}^{q+1} > 0$, $MLr_{\bar{x}}^q < 1$ and

$$Lr_{\bar{x}}^q \ge \sup_{x',x''\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})} \|\nabla f(x') - \nabla f(x'')\|.$$

Then, define

$$\hat{r} := \min\left\{r_0 - 2Lr_{\bar{x}}^{q+1}, \frac{r_{\bar{x}}(1-MLr_{\bar{x}}^q)}{4M}\right\} > 0.$$

and

$$\min\left\{\frac{r_{\bar{x}}}{4}, (q+5)\hat{r}, \frac{3(q+1)r_0}{[L(q+2)+2\nu(q+1)](6.2^q+1)}\right\} > 0$$

Thus, we can choose $0 < \delta \le 1$ such that

$$\delta \le \min\left\{\frac{r_{\bar{x}}}{4}, (q+5)\hat{r}, \frac{3(q+1)r_0}{[L(q+2)+2\nu(q+1)](6.2^q+1)}\right\}$$

and

$$(2^qM+1)[L(q+2)+2\nu(q+1)]\left(\eta(q+1)\delta^q + 4^{1-q}r_{\bar{x}}^q\right) \le (q+1).$$

Now it is routine to check that conditions (a)-(c) of Theorem 3.3 are satisfied. Thus we can apply Theorem 3.3 to complete the proof. □

4. Conclusion

The semilocal and local convergence results are presented for the extended Newton-type method when $\eta > 1$, $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like, ∇f satisfies Hölderian condition and g admits first-order divided difference satisfying the Hölder condition defined by (3.5). In particular, we have presented semilocally superlinear convergence analysis for extended Newton-type method in Theorem 3.3 while the locally superlinear convergence analysis for extended Newton-type method is presented in Corollary 3.4. This result extends and improves the corresponding ones [4, 18].

Moreover, according to our main results, we have the following conclusions:

(i) If we set $q = 0$ in Theorem 3.3, it gives the semilocal linear convergence result for the extended Newton-type method and this result coincides with the result presented in [18, Theorem 3.1]. On the other hand, if we put $q = 0$ in Corollary 3.4, this result provides locally linear convergence result which is similar with the result presented in [18, Corollary 3.1].

(ii) If we put $q = 1$ in Theorem 3.3, it yields the semilocal quadratic convergence result for the extened Newton-type method and this result is analogous to the outcome presented in [18, Theorem 3.2]. Furthermore, if we give $q = 1$ in Corollary 3.4, it gives the local quadratic convergence result for this method which is resembling the work presented in [18, Corollary 3.2].

Acknowledgement

The research work of first author is supported by University Grant Commission, Bangladesh, grant No. 1.157 and 5340/2016 while the research work of second author is financed by Ministry of Science and Technology, Bangladesh, grant No. 39.00.0000.009.06.024.19/Phy's-543-559.

References

[1] C. J. Alexis, A. Pietrus, *On the convergence of some methods for variational inclusions*, Rev. R. Acad. Cien. serie A. Mat., **102**(2) (2008), 355–361.

[2] I. K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), 374–397.

[3] I. K. Argyros, *Computational theory of iterative methods*, Stud. Comp. Math. Elsevier, Amsterdam, 2007.

[4] I. K. Argyros, S. Hilout, *Local convergence of Newton-like methods for generalized equations*, Appl. Math. and Comp., **197** (2008), 507–514.

[5] J. P. Aubin, *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res., **9** (1984), 87–111.

[6] J. P. Aubin, H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990.

[7] M. C. Ferris, J.S. Pang, *Engineering and economic applications of complimentarity problems*, SIAM Rev., **39** (1997), 669–713.

[8] J. P. Dedieu, M. H. Kim, *Newton's method for analytic systems of equations with constant rank derivatives*, J. Complexity, **18** (2002), 187–209.

[9] J. P. Dedieu, M. Shub, *Newton's method for overdetermined systems of equations*, Math. Comp., **69** (2000), 1099–1115.

[10] A. L. Dontchev, *The Graves theorem revisited*, J. Convex Anal., **3** (1996), 45–53.

[11] A. L. Dontchev, W. W. Hager, *An inverse mapping theorem for set-valued maps*, Proc. Amer.Math. Soc., **121** (1994), 481–498.

[12] S. Hilout, C. J. Alexis, A. Pietrus, *A semilocal convergence of the secant-type method for solving a generalized equations*, Positivity, **10** (2006), 673–700.

[13] J. S. He, J. H. Wang, C. Li, *Newton's method for underdetemined systems of equations under the modified γ-condition*, Numer. Funct. Anal. Optim., **28** (2007), 663–679.

[14] A. D. Ioffe, V. M. Tikhomirov, *Theory of extremal problems*, Studies in Mathematics and its Applications, North-Holland, Amsterdam, 1979.

[15] C. Li, W. H. Zhang, X. Q. Jin, *Convergence and uniqueness properties of Gauss- Newton's method*, Comput. Math. Appl., **47** (2004), 1057–1067.

[16] B. S. Mordukhovich, *Variational Analysis and generalized differentiation I: Basic Theory*, Springer Berlin Heidelberg, New York, 2006.

[17] A. Pietrus, *Does Newton's method for set-valued maps converges uniformly in mild differentiability context?*, Rev. Colombiana Mat., **32** (2000), 49–56.

[18] M. Z. Khaton, M. H. Rashid, M. I. Hossain, *Convergence Properties of extended Newton-type Iteration Method for Generalized Equations*, Journal of Mathematics Research, **10**(4) (2018), 1–18.

[19] M. H. Rashid, *Extended Newton-type Method and Its Convergence Analysis for Implicit Functions and Their Solution Mappings*, WSEAS Transactions on Mathematics, **16** (2017), 133–142.

[20] M. H. Rashid, *Extended Newton-type Method and its Convergence Analysis for Nonsmooth Generalized Equations*, Journal of Fixed Point Theory and Applications, **19** (2017), 1295–1313.

[21] M. H. Rashid, *Convergence Analysis of a Variant of Newton-type Method for Generalized Equations*, International Journal of Computer Mathematics, **95**(3) (2018), 584–600.

[22] M. H. Rashid, *Convergence Analysis of extended Hummel-Seebeck-type Method for Solving Variational Inclusions*, Vietnam J. Math., **44**(1), (2015), 1–14.

[23] M. H. Rashid, A. Sardar, *Convergence of the Newton-type Method for Generalized Equations*, GANIT: J. Bangladesh Math. Soc., **35** (2015), 27–40.

[24] M. H. Rashid, *A Convergence Analysis of Gauss-Newton-type Method for Hölder Continuous Maps*, Indian J. Math., **57**(2) (2014), 181–198.

[25] M. H. Rashid, J. H. Wang, C. Li, *Convergence Analysis of a Method for Variational Inclusions*, Applicable Analysis, **91**(10) (2012), 1943–1956.

[26] M. H. Rashid, S. H. Yu, C. Li, S. Y. Wu, *Convergence analysis of the Gauss-Newton-type method for Lipschitz-like mappings*, J. Optim. Theory Appl., **158**(1) (2013), 216–233.

[27] S. M. Robinson, *Generalized equations and their solutions, Part I, basic theory*, Math. Program. Stud., **10** (1979), 128–141.

[28] S. M. Robinson, *Generalized equations and their solutions, part II: application to nonlinear programming*, Math. Program. Stud., **19** (1982), 200–221.

[29] X. B. Xu, C. Li, *Convergence of Newton's method for systems of equations with constant rank derivatives*, J. Comput. Math., **25** (2007), 705–718.

[30] X. B. Xu, C. Li, *Convergence criterion of Newton's method for singular systems with constant rank derivatives*, J. Math. Anal. Appl., **345** (2008), 689–701.

Communications in Advanced Mathematical Sciences
Vol. IV, No. 1, 14-25, 2021
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.845453

A study on Matrix Domain of Riesz-Euler Totient Matrix in the Space of p-Absolutely Summable Sequences

Merve İlkhan Kara[1]*, Mehmet Akif Bayrakdar[2]

Abstract
In this study, a special lower triangular matrix derived by combining Riesz matrix and Euler totient matrix is used to construct new Banach spaces. $\alpha-,\beta-,\gamma-$duals of the resulting spaces are obtained and some matrix operators are characterized. Finally by the aid of measure of non-compactness, the conditions for which a matrix operator on these spaces is compact are determined.

Keywords: Compact operators, Hausdorff measure of non-compactness, Matrix mappings, Sequence space, $\alpha-,\beta-,\gamma-$duals.
2010 AMS: 46B45, 47B07, 47H08

[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0002-0831-1474*
[2] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*
***Corresponding author**: merveilkhan@gmail.com
Received: 22 December 2020, **Accepted:** 12 March 2021, **Available online:** 29 March 2021

1. Introduction and background

A sequence space is a vector subspace of the space ω of all sequences with real entries. Well known classical sequence spaces are ℓ_p (the space of p-absolutely summable sequences, $1 \leq p < \infty$), ℓ_∞ (the space of bounded sequences), c_0 (the space of null sequences), c (the space of convergent sequences). On the other hand, bs, cs_0 and cs are the most frequently encountered spaces consisting of sequences generating bounded, null and convergent series, respectively. Further ψ is the space of all finite sequences. A Banach sequence space having continuous coordinates is called a BK space. Examples of BK spaces are c_0 and c endowed with the supremum norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, where $\mathbb{N} = \{1,2,3,...\}$.

By virtue of the fact that the matrix mappings between BK-spaces are continuous, the theory of matrix mappings plays an important role in the study of sequence spaces. Let X and Y be two sequence spaces, $\mathscr{A} = (a_{nk})$ be an infinite matrix with real entries and $\mathscr{A}_n$ indicate the n^{th} row of $\mathscr{A}$. If each term of the sequence $\mathscr{A}x = \{(\mathscr{A}x)_n\} = \{\sum_{k=1}^{\infty} a_{nk}x_k\}$ is convergent, this sequence is called $\mathscr{A}$-transform of $x = (x_n)$. Further, if $\mathscr{A}x \in \mathsf{Y}$ for every sequence $x \in \mathsf{X}$, then the matrix $\mathscr{A}$ defines a matrix mapping from X into Y. (X,Y) represents the collection of all matrices defined from X into Y. Additionally, $B(\mathsf{X},\mathsf{Y})$ is the set of all bounded (continuous) linear operators from X to Y. A matrix $\mathscr{A} = (a_{nk})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nk} = 0$ for $k > n$.

The matrix domain $\mathsf{X}_\mathscr{A}$ of the matrix $\mathscr{A}$ in the space X is defined by

$$\mathsf{X}_\mathscr{A} = \{x \in \omega : \mathscr{A}x \in \mathsf{X}\}.$$

Since this space is also a sequnce space, the matrix domain has a crucial role to construct new sequence spaces. Moreover given any triangle $\mathscr{A}$ and a BK-space X, the sequence space $\mathsf{X}_{\mathscr{A}}$ gives a new BK-space equipped with the norm $\|x\|_{\mathsf{X}_{\mathscr{A}}} = \|\mathscr{A}x\|_{\mathsf{X}}$. Several authors applied this technique to construct new Banach spaces with the help of special triangles. For relevant literature, the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] can be referred.

The spaces

$$\mathsf{X}^{\alpha} = \left\{ t = (t_k) \in \omega : \sum_{k=1}^{\infty} |t_k x_k| < \infty \text{ for all } x = (x_k) \in \mathsf{X} \right\},$$

$$\mathsf{X}^{\beta} = \left\{ t = (t_k) \in \omega : \sum_{k=1}^{\infty} t_k x_k \text{ converges for all } x = (x_k) \in \mathsf{X} \right\},$$

$$\mathsf{X}^{\gamma} = \left\{ t = (t_k) \in \omega : \sup_n \left| \sum_{k=1}^{n} t_k x_k \right| < \infty \text{ for all } x = (x_k) \in \mathsf{X} \right\},$$

are called the α-, β-, γ-duals of a sequence space X, respectively.

Let $(\mathsf{X}, \|.\|_{\mathsf{X}})$ be a normed space and $B_{\mathsf{X}} = \{x \in \omega : \|x\|_{\mathsf{X}} = 1\}$. Given any BK-space $\mathsf{X} \supset \psi$ and $t = (t_n) \in \omega$,

$$\|t\|^*_{\mathsf{X}} = \sup_{x \in B_{\mathsf{X}}} \left| \sum_k t_k x_k \right|$$

implies that $t \in \mathsf{X}^{\beta}$.

Lemma 1.1. *[16, Theorem 1.29] $\ell_1^{\beta} = \ell_\infty$ and $\ell_p^{\beta} = \ell_q$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The equality $\|t\|^*_{\ell_p} = \|t\|_{\ell_p^{\beta}}$ holds for all $t \in \ell_p^{\beta}$, where $1 \le p < \infty$.*

Lemma 1.2. *[16, Theorem 1.23 (a)] Given any BK-spaces X, Y and $\mathscr{A} \in (\mathsf{X}, \mathsf{Y})$, there exists a linear operator $\mathscr{L}_{\mathscr{A}} \in B(\mathsf{X}, \mathsf{Y})$ such that $\mathscr{L}_{\mathscr{A}}(x) = \mathscr{A}x$ for all $x \in \mathsf{X}$.*

Lemma 1.3. *[16] Let $\mathsf{X} \supset \psi$ be a BK-space and $\mathsf{Y} \in \{c_0, c, \ell_\infty\}$. If $\mathscr{A} \in (\mathsf{X}, \mathsf{Y})$, then*

$$\|\mathscr{L}_{\mathscr{A}}\| = \|\mathscr{A}\|_{(\mathsf{X},\mathsf{Y})} = \sup_{n \in \mathbb{N}} \|\mathscr{A}_n\|^*_{\mathsf{X}} < \infty.$$

Let $\mathscr{Q}$ be a bounded set in a metric space X and $B(x, \delta)$ be the open ball. The value

$$\chi(\mathscr{Q}) = \inf\{\varepsilon > 0 : \mathscr{Q} \subset \cup_{i=1}^{n} B(x_i, \delta_i), x_i \in \mathsf{X}, \delta_i < \varepsilon, n \in \mathbb{N}\}$$

is called the Hausdorff measure of noncompactness of $\mathscr{Q}$.

To compute the Hausdorff measure of noncompactness of a set in ℓ_p for $1 \le p < \infty$, the following result is essential.

Theorem 1.4. *[17] Let $\mathscr{Q}$ be a bounded subset in ℓ_p for $1 \le p < \infty$ and $P_r : \ell_p \to \ell_p$ be the operator defined by $P_r(x) = (x_0, x_1, x_2, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_p$ and each $r \in \mathbb{N}$. Then, we have*

$$\chi(\mathscr{Q}) = \lim_r \left(\sup_{x \in \mathscr{Q}} \|(I - P_r)(x)\|_{\ell_p} \right),$$

where I is the identity operator on ℓ_p.

A linear operator $\mathscr{L} : \mathsf{X} \to \mathsf{Y}$ is a compact operator if the domain of $\mathscr{L}$ is all of X and for every bounded sequence $x = (x_n)$ in X, the sequence $(\mathscr{L}(x_n))$ has a convergent subsequence in Y. The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of noncompactness. The Hausdorff measure of noncompactness of an operator $\mathscr{L} \in B(\mathsf{X}, \mathsf{Y})$, $\|\mathscr{L}\|_\chi = \chi(\mathscr{L}(B_{\mathsf{X}})) = 0$ if and only if $\mathscr{L}$ is compact.

In the theory of sequence spaces, the Hausdorff measure of noncompactness of a linear operator plays a role to characterize the compactness of an operator between BK spaces. For the relevant literature, see [18, 19, 20, 21, 22, 23, 24].

The Euler totient matrix $\Phi = (\phi_{nk})$ is defined as in [25]

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n, \end{cases}$$

where φ is the Euler totient function. In the recent time, by using this matrix, many new sequence and series spaces are defined and studied in the papers [26, 27, 28, 29, 30, 31, 32, 33].

For $p \in \mathbb{N}$ with $p \neq 1$, $\varphi(p)$ gives the number of positive integers less than p which are coprime with p and $\varphi(1) = 1$. Also, the equality

$$p = \sum_{k|p} \varphi(k)$$

holds for every $p \in \mathbb{N}$. For $p \in \mathbb{N}$ with $p \neq 1$, the Möbius function μ is defined as

$$\mu(p) = \begin{cases} (-1)^r & \text{if } p = p_1 p_2 ... p_r, \text{ where } p_1, p_2, ..., p_r \text{ are} \\ & \text{non-equivalent prime numbers} \\ 0 & \text{if } \tilde{p}^2 \mid p \text{ for some prime number } \tilde{p} \end{cases}$$

and $\mu(1) = 1$. The equality

$$\sum_{k|p} \mu(k) = 0 \tag{1.1}$$

holds except for $p = 1$.

The Riesz matrix $E = (e_{nk})$ is defined as

$$e_{nk} = \begin{cases} \frac{q_k}{Q_n} & , \quad \text{if } 0 \leq k \leq n \\ 0 & , \quad \text{if } k > n, \end{cases}$$

where (q_k) is a sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$. By using these matrix, the authors of [34] introduced the Riesz sequence spaces of non-absolute type.

The main purpose of this study is to construct new *BK* spaces $\ell_p(R_\Phi)$ for $1 \leq p < \infty$. The matrix R_Φ is obtained by combining Euler totient matrix and Riesz matrix. After studying certain properties of the resulting spaces, α-, β- and γ-duals are computed. Finally some matrix mappings from the spaces $\ell_p(R_\Phi)$ to the classical spaces are characterized and compact operators are studied.

2. The sequence space $\ell_p(R_\Phi)$

In the present section, we introduce the sequence space $\ell_p(R_\Phi)$ by using the matrix R_Φ, where $1 \leq p < \infty$. Also, we present some theorems which give inclusion relations concerning this space.

The matrix $R_\Phi = (r_{nk})$ is defined as

$$r_{nk} = \begin{cases} \frac{q_k \varphi(k)}{Q_n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n, \end{cases}$$

where $Q_n = q_1 + q_2 + ... + q_n$. We call this matrix as *Riesz Euler Totient matrix operator*.

The inverse $R_\Phi^{-1} = (r_{nk}^{-1})$ of the matrix R_Φ is computed as

$$r_{nk}^{-1} = \begin{cases} \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Now, we introduce the sequence space $\ell_p(R_\Phi)$ by

$$\ell_p(R_\Phi) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k \right|^p < \infty \right\} \quad (1 \leq p < \infty).$$

Unless otherwise stated, $y = (y_n)$ will be the R_Φ-transform of a sequence $x = (x_n)$, that is, $y_n = (R_\Phi x)_n = \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k$ for all $n \in \mathbb{N}$.

Theorem 2.1. *The space $\ell_p(R_\Phi)$ is a Banach space with the norm given by $\|x\|_{\ell_p(R_\Phi)} = \left(\sum_n \left| \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k \right|^p \right)^{1/p}$, where $1 \leq p < \infty$.*

Proof. We omit the proof which is straightforward. $\square$

Corollary 2.2. *The space $\ell_p(R_\Phi)$ is a BK-space, where $1 \leq p < \infty$.*

Theorem 2.3. *The space $\ell_p(R_\Phi)$ is linearly isomorphic to ℓ_p, where $1 \leq p < \infty$.*

Proof. Let f be a mapping defined from $\ell_p(R_\Phi)$ to ℓ_p such that $f(x) = R_\Phi x$ for all $x \in \ell_p(R_\Phi)$. It is clear that f is linear. Also it is injective since the kernel of f consists of only zero. To prove that f is surjective, consider the sequence $x = (x_n)$ whose terms are

$$x_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n} y_k$$

for all $n \in \mathbb{N}$, where $y = (y_k)$ is any sequence in ℓ_p. It follows from (1.1) that

$$\begin{aligned}(R_\Phi x)_n &= \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k = \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) \sum_{j|k} \frac{\mu(\frac{k}{j})}{\varphi(k)} \frac{Q_j}{q_k} y_j \\ &= \frac{1}{Q_n} \sum_{k|n} \sum_{j|k} \mu(\frac{k}{j}) Q_j y_j = \frac{1}{Q_n} \sum_{k|n} \left(\sum_{j|k} \mu(j) \right) Q_{\frac{n}{k}} y_{\frac{n}{k}} = \frac{1}{Q_n} \mu(1) Q_n y_n = y_n\end{aligned}$$

and so $x = (x_n) \in \ell_p(R_\Phi)$. f preserves norms since the equality $\|x\|_{\ell_p(R_\Phi)} = \|f(x)\|_{\ell_p}$ holds. $\square$

Remark 2.4. *The space $\ell_2(R_\Phi)$ is an inner product space with the inner product defined as $\langle x, \tilde{x} \rangle_{\ell_2(R_\Phi)} = \langle R_\Phi x, R_\Phi \tilde{x} \rangle_{\ell_2}$, where $\langle ., . \rangle_{\ell_2}$ is the inner product on ℓ_2 which induces $\|.\|_{\ell_2}$.*

Theorem 2.5. *The space $\ell_p(R_\Phi)$ is not an inner product space for $p \neq 2$.*

Proof. Consider the sequences $x = (x_n)$ and $\tilde{x} = (\tilde{x}_n)$, where

$$x_n = \begin{cases} \frac{\mu(n)}{\varphi(n)} \frac{Q_1}{q_n} + \frac{\mu(\frac{n}{2})}{\varphi(n)} \frac{Q_2}{q_n} & , \quad \text{if } n \text{ is even} \\ \frac{\mu(n)}{\varphi(n)} \frac{Q_1}{q_n} & , \quad \text{if } n \text{ is odd} \end{cases}$$

and

$$\tilde{x}_n = \begin{cases} \frac{\mu(n)}{\varphi(n)} \frac{Q_1}{q_n} - \frac{\mu(\frac{n}{2})}{\varphi(n)} \frac{Q_2}{q_n} & , \quad \text{if } n \text{ is even} \\ \frac{\mu(n)}{\varphi(n)} \frac{Q_1}{q_n} & , \quad \text{if } n \text{ is odd} \end{cases}$$

for all $n \in \mathbb{N}$. Then, we have $R_\Phi x = (1, 1, 0, ..., 0, ...) \in \ell_p$ and $R_\Phi \tilde{x} = (1, -1, 0, ..., 0, ...) \in \ell_p$. Hence, one can easily observe that

$$\|x + \tilde{x}\|_{\ell_p(R_\Phi)} + \|x - \tilde{x}\|_{\ell_p(R_\Phi)} \neq 2(\|x\|_{\ell_p(R_\Phi)} + \|\tilde{x}\|_{\ell_p(R_\Phi)}).$$

$\square$

Theorem 2.6. *The inclusion $\ell_p(R_\Phi) \subset \ell_q(R_\Phi)$ strictly holds for $1 \leq p < q < \infty$.*

Proof. It is clear that the inclusion $\ell_p(R_\Phi) \subset \ell_q(R_\Phi)$ holds since $\ell_p \subset \ell_q$ for $1 \leq p < q < \infty$. Also, $\ell_p \subset \ell_q$ is strict and so there exists a sequence $z = (z_n)$ in $\ell_q \backslash \ell_p$. By defining a sequence $x = (x_n)$ as

$$x_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n} z_k$$

for all $n \in \mathbb{N}$, we conclude that $x \in \ell_q(R_\Phi) \backslash \ell_p(R_\Phi)$. Hence, the desired inclusion is strict. $\square$

Before presenting the next result, we define the sequence space $\ell_\infty(R_\Phi)$ by

$$\ell_\infty(R_\Phi) = \{x \in \omega : R_\Phi x \in \ell_\infty\}.$$

Theorem 2.7. *The inclusion $\ell_p(R_\Phi) \subset \ell_\infty(R_\Phi)$ strictly holds for $1 \leq p < \infty$.*

Proof. The inclusion is obvious since $\ell_p \subset \ell_\infty$ holds for $1 \leq p < \infty$. Let $x = (x_n)$ be a sequence such that $x_n = \sum_{k|n} (-1)^k \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n}$ for all $n \in \mathbb{N}$. We obtain that $R_\Phi x = \left(\frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) \sum_{j|k} (-1)^j \frac{\mu(\frac{k}{j})}{\varphi(k)} \frac{Q_j}{q_k} \right) = ((-1)^n) \in \ell_\infty \backslash \ell_p$ which implies that $x \in \ell_\infty(R_\Phi) \backslash \ell_p(R_\Phi)$ for $1 \leq p < \infty$. $\square$

3. The α-, β- and γ-duals of the space $\ell_p(R_\Phi)$

In this section, we determine the α-, β- and γ-duals of the sequence space $\ell_p(R_\Phi)$, where $1 \leq p < \infty$. The following lemmas are required to prove our main results in this section. Here and in what follows $\mathscr{K}$ denotes the family of all finite subsets of $\mathbb{N}$.

Lemma 3.1. *[35] The following statements hold:*

$\mathscr{A} = (a_{nk}) \in (\ell_p, \ell_1)$ *if and only if*

$$\sup_{F \in \mathscr{K}} \sum_k \left| \sum_{n \in F} a_{nk} \right|^q < \infty \tag{3.1}$$

holds, where $1 < p < \infty$.

$\mathscr{A} = (a_{nk}) \in (\ell_\infty, \ell_1)$ *if and only if* (3.1) *holds with* $q = 1$.

$\mathscr{A} = (a_{nk}) \in (\ell_1, \ell_1)$ *if and only if*

$$\sup_k \sum_n |a_{nk}| < \infty \tag{3.2}$$

holds.

$\mathscr{A} = (a_{nk}) \in (\ell_p, c)$ *if and only if*

$$\lim_{n\to\infty} a_{nk} \text{ exists for each } k \in \mathbb{N} \tag{3.3}$$

and

$$\sup_n \sum_k |a_{nk}|^q < \infty \tag{3.4}$$

holds, where $1 < p < \infty$.

$\mathscr{A} = (a_{nk}) \in (\ell_\infty, c)$ *if and only if* (3.3) *and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n\to\infty} a_{nk} \right|$$

hold.

$\mathscr{A} = (a_{nk}) \in (\ell_1, c)$ *if and only if* (3.3) *and*

$$\sup_{n,k} |a_{nk}| < \infty \tag{3.5}$$

hold.

$\mathscr{A} = (a_{nk}) \in (\ell_p, c_0)$ *if and only if*

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N} \tag{3.6}$$

and (3.4) *holds, where* $1 < p < \infty$.

$\mathscr{A} = (a_{nk}) \in (\ell_\infty, c_0)$ *if and only if* (3.6) *and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0$$

hold.

$\mathscr{A} = (a_{nk}) \in (\ell_1, c_0)$ *if and only if* (3.5) *and* (3.6) *hold.*

$\mathscr{A} = (a_{nk}) \in (\ell_p, \ell_\infty)$ *if and only if* (3.4) *holds, where* $1 < p < \infty$.

$\mathscr{A} = (a_{nk}) \in (\ell_\infty, \ell_\infty)$ *if and only if* (3.4) *holds with* $q = 1$.

$\mathscr{A} = (a_{nk}) \in (\ell_1, \ell_\infty)$ *if and only if* (3.5) *holds.*

In the following theorem, we determine the α-duals of the spaces $\ell_p(R_\Phi)$ $(1 < p < \infty)$ and $\ell_1(R_\Phi)$.

Theorem 3.2. *The α-duals of the spaces $\ell_p(R_\Phi)$ $(1<p<\infty)$ and $\ell_1(R_\Phi)$ are as follows:*

$$(\ell_p(R_\Phi))^\alpha = \left\{ t=(t_n)\in\omega : \sup_{F\in\mathscr{K}} \sum_k \left| \sum_{n\in F, k|n} \frac{\mu(\frac{n}{k})}{\varphi(k)} \frac{Q_k}{q_n} t_n \right|^q < \infty \right\},$$

and

$$(\ell_1(R_\Phi))^\alpha = \left\{ t=(t_n)\in\omega : \sup_k \sum_{n\in\mathbb{N}, k|n} \left| \frac{\mu(\frac{n}{k})}{\varphi(k)} \frac{Q_k}{q_n} t_n \right| < \infty \right\}.$$

Proof. Consider the matrix $C=(c_{nk})$ defined by

$$c_{nk} = \begin{cases} \frac{\mu(\frac{n}{k})}{\varphi(k)} \frac{Q_k}{q_n} t_n & , \quad k \mid n \\ 0 & , \quad k \nmid n \end{cases}$$

for any sequence $t=(t_n)\in\omega$. Hence, given any $x=(x_n)\in\ell_p(R_\Phi)$ for $1\le p<\infty$, we have $t_n x_n=(Cy)_n$ for all $n\in\mathbb{N}$. This implies that $tx\in\ell_1$ with $x\in\ell_p(R_\Phi)$ if and only if $Cy\in\ell_1$ with $y\in\ell_p$. It follows that $t\in(\ell_p(R_\Phi))^\alpha$ if and only if $C\in(\ell_p,\ell_1)$ which completes the proof in view of Lemma 3.1. □

Theorem 3.3. *Let us define the following sets:*

$$A_1 = \left\{ t=(t_k)\in\omega : \lim_{n\to\infty} \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j \text{ exists for each } k\in\mathbb{N} \right\},$$

$$A_2 = \left\{ t=(t_k)\in\omega : \sup_n \sum_k \left| \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j \right|^q < \infty \right\},$$

and

$$A_3 = \left\{ t=(t_k)\in\omega : \sup_{n,k} \left| \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j \right| < \infty \right\}.$$

The β and γ-duals of the spaces $\ell_p(R_\Phi)$ $(1<p<\infty)$ and $\ell_1(R_\Phi)$ are as follows:
$(\ell_p(R_\Phi))^\beta = A_1\cap A_2$ *and* $(\ell_1(R_\Phi))^\beta = A_1\cap A_3$,
$(\ell_p(R_\Phi))^\gamma = A_2$ *and* $(\ell_1(R_\Phi))^\gamma = A_3$.

Proof. Let $t=(t_k)\in\omega$ and $B=(b_{nk})$ be an infinite matrix with terms

$$b_{nk} = \begin{cases} \sum_{j=k,k|j}^{n} t_j \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} & , \quad \text{if } 1\le k\le n \\ 0 & , \quad \text{if } k>n. \end{cases}$$

Hence it follows that

$$\sum_{k=1}^{n} t_k x_k = \sum_{k=1}^{n} t_k \left(\sum_{j|k} \frac{\mu(\frac{k}{j})}{\varphi(k)} \frac{Q_j}{q_k} y_j \right) = \sum_{k=1}^{n} \left(\sum_{j=k,k|j}^{n} t_j \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} \right) y_k = (By)_n$$

for any $x=(x_n)\in\ell_p(R_\Phi)$. This equality yields that $tx\in cs$ for $x\in\ell_p(R_\Phi)$ if and only if $By\in c$ for $y\in\ell_p$. That is, $t\in(\ell_p(R_\Phi))^\beta$ if and only if $B\in(\ell_p,c)$ for $1\le p<\infty$. Hence, by Lemma 3.1, it is concluded that $(\ell_p(R_\Phi))^\beta = A_1\cap A_2$ and $(\ell_1(R_\Phi))^\beta = A_1\cap A_3$.

This equality also yields that $tx\in bs$ for $x\in\ell_p(R_\Phi)$ if and only if $By\in\ell_\infty$ for $y\in\ell_p$. That is, $t\in(\ell_p(R_\Phi))^\gamma$ if and only if $B\in(\ell_p,\ell_\infty)$ for $1\le p<\infty$. Hence, by Lemma 3.1, it is concluded that $(\ell_p(R_\Phi))^\gamma = A_2$ and $(\ell_1(R_\Phi))^\gamma = A_3$.

□

4. Some matrix transformations related to the sequence space $\ell_p(R_\Phi)$

In this section, we give the characterization of the classes $(\ell_p(R_\Phi),\mathrm{Y})$, where $1\leq p<\infty$ and $\mathrm{Y}\in\{\ell_\infty,c,c_0,\ell_1\}$. Throughout this section, we write $d(n,k)=\sum_{j=0}^{n}d_{jk}$ for an infinite matrix $D=(d_{nk})$ and all $n,k\in\mathbb{N}$.

Theorem 4.1. *Let $1\leq p<\infty$ and Y be any sequence space. Then, we have $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),\mathrm{Y})$ if and only if*

$$D^{(n)}=\left(d_{mk}^{(n)}\right)\in(\ell_p,c)\ for\ each\ n\in\mathbb{N},$$

$$D=(d_{nk})\in(\ell_p,\mathrm{Y}),$$

where $d_{mk}^{(n)}=\begin{cases}0 & ,\quad k>m\\ \sum_{j=k,k|j}^{m}a_{nj}\frac{\mu(\frac{j}{k})}{\varphi(k)}\frac{Q_k}{q_j} & ,\quad 0\leq k\leq m\end{cases}$ *and* $d_{nk}=\sum_{j=k,k|j}^{\infty}a_{nj}\frac{\mu(\frac{j}{k})}{\varphi(k)}\frac{Q_k}{q_j}$ *for all $k,m,n\in\mathbb{N}$.*

Proof. We omit the proof since it follows with the same technique in [6, Theorem 4.1]. □

The following results are obtained by combining Theorem 4.1 with Lemma 3.1.

Theorem 4.2.
(a) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),\ell_\infty)$ if and only if

$$\lim_{m\to\infty}d_{mk}^{(n)}\ exists\ for\ each\ n,k\in\mathbb{N},\tag{4.1}$$

$$\sup_{m,k}\left|d_{mk}^{(n)}\right|<\infty\ for\ each\ n\in\mathbb{N}\tag{4.2}$$

and (3.5) holds with d_{nk} instead of a_{nk}.
(b) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),c)$ if and only if (4.1) and (4.2) hold, and (3.3) and (3.5) also hold with d_{nk} instead of a_{nk}.
(c) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),c_0)$ if and only if (4.1) and (4.2) hold, and (3.5) and (3.6) also hold with d_{nk} instead of a_{nk}.
(d) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),\ell_1)$ if and only if (4.1) and (4.2) hold, and (3.2) also holds with d_{nk} instead of a_{nk}.

Theorem 4.3. *Let $1<p<\infty$.*
(a) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),\ell_\infty)$ if and only if (4.1) and

$$\sup_m\sum_{k=0}^{m}\left|d_{mk}^{(n)}\right|^q<\infty\ for\ each\ n\in\mathbb{N}\tag{4.3}$$

hold, and (3.4) also holds with d_{nk} instead of a_{nk}.
(b) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),c)$ if and only if (4.1) and (4.3) hold, and (3.3) and (3.4) also hold with d_{nk} instead of a_{nk}.
(c) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),c_0)$ if and only if (4.1) and (4.3) hold, and (3.6) and (3.4) also hold with d_{nk} instead of a_{nk}.
(d) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),\ell_1)$ if and only if (4.1) and (4.3) hold, and (3.1) also holds with d_{nk} instead of a_{nk}.

The following results are derived by using Theorems 4.2-4.3.

Corollary 4.4. *The following statements hold:*
(a) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),bs)$ if and only if (4.1), (4.2) hold and (3.5) holds with $d(n,k)$ instead of a_{nk}.
(b) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),cs)$ if and only if (4.1), (4.2) hold and (3.3),(3.5) hold with $d(n,k)$ instead of a_{nk}.
(c) $\mathscr{A}=(a_{nk})\in(\ell_1(R_\Phi),cs_0)$ if and only if (4.1), (4.2) hold and (3.5),(3.6) hold with $d(n,k)$ instead of a_{nk}.

Corollary 4.5. *Let $1<p<\infty$. Then, the following statements hold:*
(a) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),bs)$ if and only if (4.1), (4.3) hold and (3.4) holds with $d(n,k)$ instead of a_{nk}.
(b) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),cs)$ if and only if (4.1), (4.3) hold and (3.3),(3.4) hold with $d(n,k)$ instead of a_{nk}.
(c) $\mathscr{A}=(a_{nk})\in(\ell_p(R_\Phi),cs_0)$ if and only if (4.1), (4.3) hold and (3.4),(3.6) hold with $d(n,k)$ instead of a_{nk}.

5. Compact operators on the space $\ell_p(R_\Phi)$

Let the matrix $\tilde{\mathscr{A}} = (\tilde{a}_{nk})$ defined by an infinite matrix $\mathscr{A} = (a_{nk})$ as

$$\tilde{a}_{nk} = \sum_{j=k,k|j}^{\infty} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} a_{nj}$$

for all $n,k \in \mathbb{N}$.

For a sequence $t = (t_k) \in \omega$, define a sequence $\tilde{t} = (\tilde{t}_k)$ as $\tilde{t}_k = \sum_{j=k,k|j}^{\infty} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j$ for all $k \in \mathbb{N}$.

Lemma 5.1. *Let $t = (t_k) \in (\ell_p(R_\Phi))^\beta$, where $1 \le p < \infty$. Then $\tilde{t} = (\tilde{t}_k) \in \ell_q$ and*

$$\sum_k t_k x_k = \sum_k \tilde{t}_k y_k$$

for all $x = (x_k) \in \ell_p(R_\Phi)$.

Lemma 5.2. *The following statements hold.*
(a) $\|t\|^*_{\ell_1(R_\Phi)} = \sup_k |\tilde{t}_k| < \infty$ for all $t = (t_k) \in (\ell_1(R_\Phi))^\beta$.
(b) $\|t\|^*_{\ell_p(R_\Phi)} = (\sum_k |\tilde{t}_k|^q)^{1/q} < \infty$ for all $t = (t_k) \in (\ell_p(R_\Phi))^\beta$, where $1 < p < \infty$.

Lemma 5.3. *Let X be any sequence space and $\mathscr{A} = (a_{nk})$ be an infinite matrix. If $\mathscr{A} \in (\ell_p(R_\Phi), \mathsf{X})$, then $\tilde{\mathscr{A}} \in (\ell_p, \mathsf{X})$ and $\mathscr{A}x = \tilde{\mathscr{A}}y$ for all $x \in \ell_p(R_\Phi)$, where $1 \le p < \infty$.*

Proof. It follows from Lemma 5.1. □

Lemma 5.4. *If $\mathscr{A} \in (\ell_1(R_\Phi), \ell_p)$, then we have*

$$\|\mathscr{L}_\mathscr{A}\| = \|\mathscr{A}\|_{(\ell_1(R_\Phi),\ell_p)} = \sup_k \left(\sum_n |\tilde{a}_{nk}|^p\right)^{1/p} < \infty,$$

where $1 \le p < \infty$.

Lemma 5.5. *[22, Theorem 3.7] Let $\mathsf{X} \supset \psi$ be a BK-space. Then, the following statements hold.*
(a) $\mathscr{A} \in (\mathsf{X}, \ell_\infty)$, then $0 \le \|\mathscr{L}_\mathscr{A}\|_\chi \le \limsup_n \|\mathscr{A}_n\|^*_\mathsf{X}$.
(b) $\mathscr{A} \in (\mathsf{X}, c_0)$, then $\|\mathscr{L}_\mathscr{A}\|_\chi = \limsup_n \|\mathscr{A}_n\|^*_\mathsf{X}$.
(c) If X has AK or $\mathsf{X} = \ell_\infty$ and $\mathscr{A} \in (\mathsf{X}, c)$, then

$$\frac{1}{2}\limsup_n \|\mathscr{A}_n - a\|^*_\mathsf{X} \le \|\mathscr{L}_\mathscr{A}\|_\chi \le \limsup_n \|\mathscr{A}_n - a\|^*_\mathsf{X},$$

where $a = (a_k)$ and $a_k = \lim_n a_{nk}$ for each $k \in \mathbb{N}$.

Lemma 5.6. *[22, Theorem 3.11] Let $\mathsf{X} \supset \psi$ be a BK-space. If $\mathscr{A} \in (\mathsf{X}, \ell_1)$, then*

$$\lim_r \left(\sup_{N \in \mathscr{K}_r} \left\|\sum_{n \in N} \mathscr{A}_n\right\|^*_\mathsf{X}\right) \le \|\mathscr{L}_\mathscr{A}\|_\chi \le 4\lim_r \left(\sup_{N \in \mathscr{K}_r} \left\|\sum_{n \in N} \mathscr{A}_n\right\|^*_\mathsf{X}\right)$$

and $\mathscr{L}_\mathscr{A}$ is compact if and only if $\lim_r \left(\sup_{N \in \mathscr{K}_r} \|\sum_{n \in N} \mathscr{A}_n\|^*_\mathsf{X}\right) = 0$, where $\mathscr{K}_r$ is the subcollection of $\mathscr{K}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

Theorem 5.7. *Let $1 < p < \infty$.*

1. *For $\mathscr{A} \in (\ell_p(R_\Phi), \ell_\infty)$,*

$$0 \le \|\mathscr{L}_\mathscr{A}\|_\chi \le \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}$$

holds.

2. *For $\mathscr{A} \in (\ell_p(R_\Phi), c)$,*
$$\frac{1}{2}\limsup_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q\right)^{1/q} \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q\right)^{1/q}$$
holds, where $\tilde{a} = (\tilde{a}_k)$ and $\tilde{a}_k = \lim_n \tilde{a}_{nk}$ for each $k \in \mathbb{N}$.

3. *For $\mathscr{A} \in (\ell_p(R_\Phi), c_0)$,*
$$\|\mathscr{L}_{\mathscr{A}}\|_\chi = \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}$$
holds.

4. *For $\mathscr{A} \in (\ell_p(R_\Phi), \ell_1)$,*
$$\lim_r \|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi),\ell_1)} \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq 4\lim_r \|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi),\ell_1)}$$
holds, where $\|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi),\ell_1)} = \sup_{N\in\mathscr{K}_r} \left(\sum_k |\sum_{n\in N} \tilde{a}_{nk}|^q\right)^{1/q}$ $(r \in \mathbb{N})$.

Proof.

1. Let $\mathscr{A} \in (\ell_p(R_\Phi), \ell_\infty)$. Since the series $\sum_{k=1}^\infty a_{nk}x_k$ converges for each $n \in \mathbb{N}$, we have $\mathscr{A}_n \in (\ell_p(R_\Phi))^\beta$. From Lemma 5.2 (b), we write $\|\mathscr{A}_n\|^*_{\ell_p(R_\Phi)} = \|\tilde{\mathscr{A}_n}\|^*_{\ell_p} = \|\tilde{\mathscr{A}_n}\|_{\ell_q} = (\sum_k |\tilde{a}_{nk}|^q)^{1/q}$ for each $n \in \mathbb{N}$. By using Lemma 5.5 (a), we conclude that
$$0 \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}.$$

2. Let $\mathscr{A} \in (\ell_p(R_\Phi), c)$. By Lemma 5.3, we have $\tilde{\mathscr{A}} \in (\ell_p, c)$. Hence, from Lemma 5.5 (c), we write
$$\frac{1}{2}\limsup_n \|\tilde{\mathscr{A}_n} - \tilde{a}\|^*_{\ell_p} \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \|\tilde{\mathscr{A}_n} - \tilde{a}\|^*_{\ell_p},$$
where $\tilde{a} = (\tilde{a}_k)$ and $\tilde{a}_k = \lim_n \tilde{a}_{nk}$ for each $k \in \mathbb{N}$. Moreover, Lemma 1.1 implies that $\|\tilde{\mathscr{A}_n} - \tilde{a}\|^*_{\ell_p} = \|\tilde{\mathscr{A}_n} - \tilde{a}\|_{\ell_q} = (\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q)^{1/q}$ for each $n \in \mathbb{N}$. This completes the proof.

3. Let $\mathscr{A} \in (\ell_p(R_\Phi), c_0)$. Since we have $\|\mathscr{A}_n\|^*_{\ell_p(R_\Phi)} = \|\tilde{\mathscr{A}_n}\|^*_{\ell_p} = \|\tilde{\mathscr{A}_n}\|_{\ell_q} = (\sum_k |\tilde{a}_{nk}|^q)^{1/q}$ for each $n \in \mathbb{N}$, we conclude from Lemma 5.5 (b) that
$$\|\mathscr{L}_{\mathscr{A}}\|_\chi = \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}.$$

4. Let $\mathscr{A} \in (\ell_p(R_\Phi), \ell_1)$. By Lemma 5.3, we have $\tilde{\mathscr{A}} \in (\ell_p, \ell_1)$. It follows from Lemma 5.6 that
$$\lim_r \left(\sup_{N\in\mathscr{K}_r} \left\|\sum_{n\in N} \tilde{\mathscr{A}_n}\right\|^*_{\ell_p}\right) \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq 4\lim_r \left(\sup_{N\in\mathscr{K}_r} \left\|\sum_{n\in N} \tilde{\mathscr{A}_n}\right\|^*_{\ell_p}\right).$$
Moreover, Lemma 1.1 implies that $\|\sum_{n\in N} \tilde{\mathscr{A}_n}\|^*_{\ell_p} = \|\sum_{n\in N} \tilde{\mathscr{A}_n}\|_{\ell_q} = (\sum_k |\sum_{n\in N} \tilde{a}_{nk}|^q)^{1/q}$ which completes the proof.

□

Corollary 5.8. *Let $1 < p < \infty$.*

1. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), \ell_\infty)$ if*

$$\lim_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q} = 0.$$

2. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), c)$ if and only if*

$$\lim_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q \right)^{1/q} = 0.$$

3. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), c_0)$ if and only if*

$$\lim_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q} = 0.$$

4. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), \ell_1)$ if and only if*

$$\lim_r \|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi), \ell_1)} = 0,$$

where $\|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi), \ell_1)} = \sup_{N \in \mathscr{K}_r} \left(\sum_k |\sum_{n \in N} \tilde{a}_{nk}|^q \right)^{1/q}$.

Theorem 5.9.

1. *For $\mathscr{A} \in (\ell_1(R_\Phi), \ell_\infty)$,*

$$0 \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \left(\sup_k |\tilde{a}_{nk}| \right)$$

holds.

2. *For $\mathscr{A} \in (\ell_1(R_\Phi), c)$,*

$$\frac{1}{2} \limsup_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right) \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right)$$

holds.

3. *For $\mathscr{A} \in (\ell_1(R_\Phi), c_0)$,*

$$\|\mathscr{L}_{\mathscr{A}}\|_\chi = \limsup_n \left(\sup_k |\tilde{a}_{nk}| \right)$$

holds.

4. *For $\mathscr{A} \in (\ell_1(R_\Phi), \ell_1)$,*

$$\|\mathscr{L}_{\mathscr{A}}\|_\chi = \lim_r \left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}| \right)$$

holds.

Proof. It follows with the same technique in Theorem 5.7. □

Corollary 5.10.

1. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), \ell_\infty)$ if*

$$\lim_n \left(\sup_k |\tilde{a}_{nk}| \right) = 0.$$

2. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), c)$ if and only if*

$$\lim_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right) = 0.$$

3. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), c_0)$ if and only if*

$$\lim_n \left(\sup_k |\tilde{a}_{nk}| \right) = 0.$$

4. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), \ell_1)$ if and only if*

$$\lim_r \left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}| \right) = 0.$$

References

[1] B. Altay, F. Başar, M. Mursaleen, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I*, Inform. Sci., **176**(10) (2006), 1450-1462.

[2] F. Başar, B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J., **55**(1) (2003), 136-147.

[3] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, İstanbul, 2012.

[4] M. İlkhan, *Certain geometric properties and matrix transformations on a newly introduced Banach space*, Fundam. J. Math. Appl., **3**(1) (2020), 45-51.

[5] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **2013**(38) (2013), 15 pages.

[6] M. Kirişçi, F. Başar, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl., **60** (2010), 1299-1309.

[7] M. Kirişçi, *Riesz type integrated and differentiated sequence spaces*, Bull. Math. Anal. Appl., **7**(2) (2015), 14-27.

[8] S.A. Mohiuddine, A. Alotaibi, *Weighted almost convergence and related infinite matrices*, J. Inequal. Appl., **2018**(15) (2018), 10 pages.

[9] M. Mursaleen, A.K. Noman, *On some new difference sequence spaces of non-absolute type*, Math. Comput. Modelling, **52**(3-4) (2010), 603-617.

[10] T. Yaying, B. Hazarika, *On sequence spaces defined by the domain of a regular Tribonacci matrix*, Math. Slovaca, **70**(3) (2020), 697-706.

[11] P. Zengin Alp, *A new paranormed sequence space defined by Catalan conservative matrix*, Math. Methods Appl. Sci., doi: https://doi.org/10.1002/mma.6530.

[12] T. Yaying, B. Hazarika, *On sequence spaces generated by binomial difference operator of fractional order*, Math. Slovaca, **69**(4) (2019), 901-918.

[13] T. Yaying, B. Hazarika, M. Mursaleen, *On sequence space derived by the domain of q-Cesaro matrix in ℓ_p space and the associated operator ideal*, J. Math. Anal. Appl., **493**(1) (2021), 124453.

[14] T. Yaying, B. Hazarika, S.A. Mohiuddine, M. Mursaleen, K.J. Ansari, *Sequence spaces derived by the triple band generalized Fibonacci difference operator*, Adv. Diff. Equ., 2020 (2020), 639.

[15] T. Yaying, *Paranormed Riesz difference sequence spaces of fractional order*, Kragujevac J. Math., **46**(2) (2022), 175-191.

[16] E. Malkowsky, V. Rakocevic, *An introduction into the theory of sequence spaces and measure of noncompactness*, Zb. Rad. (Beogr.), **17** (2000), 143-234.

[17] V. Rakocevic, *Measures of noncompactness and some applications*, Filomat, **12** (1998), 87-120.

[18] M. Başarır, E. E. Kara, *On some difference sequence spaces of weighted means and compact operators*, Ann. Funct. Anal. **2** (2011), 114-129.

[19] M. Başarır, E. E. Kara, *On the B-difference sequence space derived by generalized weighted mean and compact operators*, J. Math. Anal. Appl. **391** (2012), 67-81.

[20] M. Başarır, E. E. Kara, *On compact operators on the Riesz B^m-difference sequence spaces II*, Iran. J. Sci. Technol. Trans. A Sci., **36** (2012), 371-376.

[21] E. E. Kara, M. Başarır, *On compact operators and some Euler $B^{(m)}$-difference sequence spaces*, J. Math. Anal. Appl., **379**(2) (2011), 499-511.

[22] M. Mursaleen, A. K. Noman, *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal., **73**(8) (2010), 2541-2557.

[23] M. Mursaleen, A. K. Noman, *The Hausdorff measure of noncompactness of matrix operators on some BK spaces*, Oper. Matrices, **5** (2011), 473-486.

[24] T. Yaying, A. Das, B. Hazarika, P. Baliarsingh, *Compactness of binomial difference operator of fractional order and sequence spaces*, Rend. Circ. Mat. Palermo (II) Ser., **68** (2019), 459-476.

[25] M. İlkhan, E. E. Kara, *A new Banach space defined by Euler totient matrix operator*, Oper. Matrices, **13**(2) (2019), 527-544.

[26] S. Demiriz, S. Erdem, *Domain of Euler-totient matrix operator in the space $\mathscr{L}_p$*, Korean J. Math., **28**(2) (2020), 361-378.

[27] S. Demiriz, M. İlkhan, E. E. Kara, *Almost convergence and Euler totient matrix*, Ann. Funct. Anal., **11**(3) (2020), 604-616.

[28] S. Erdem, S. Demiriz, *4-dimensional Euler-totient matrix operator and some double sequence spaces*, Math. Sci. Appl. E-Notes, **8**(2) (2020), 110-122.

[29] G. C. Hazar Güleç, Merve İlkhan, *A new paranormed series space using Euler totient means and some matrix transformations*, Korean J. Math., **28**(2) (2020), 205-221.

[30] G. C. Hazar Güleç, Merve İlkhan, *A new characterization of absolute summability factors*, Commun. Optim. Theory, **2020** (2020), Article ID 15, 1-11.

[31] M. İlkhan, *Matrix domain of a regular matrix derived by Euler totient function in the spaces c_0 and c*, Mediterr. J. Math., **17** (2020), Article number 27.

[32] M. İlkhan, G. C. Hazar Güleç, *A study on absolute Euler totient series space and certain matrix transformations*, Mugla J. Sci. Technol., **6**(1) (2020), 112-119.

[33] M. İlkhan, S. Demiriz, E. E. Kara, *A new paranormed sequence space defined by Euler totient matrix*, Karaelmas Sci. Eng. J., **9**(2) (2019), 277-282.

[34] B. Altay, F. Başar, *Some paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math., **30**(4) (2006), 591-608.

[35] M. Stieglitz, H. Tietz, *Matrix transformationen von folgenraumen eine ergebnisübersicht*, Math. Z., **154** (1977), 1-16.

Communications in Advanced Mathematical Sciences
Vol. IV, No. 1, 26-38, 2021
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.837197

Global Analysis of a $(1,2)$-Type System of Non-Linear Difference Equations

Emine Yener[1], Mehmet Gümüş[2]*

Abstract
This paper deals with the study of global analysis of following $(1,2)-$type system of non-linear difference equations:

$$u_{n+1}=\frac{\alpha v_{n-1}}{\beta+\gamma v_n^p v_{n-2}^q},\qquad v_{n+1}=\frac{\alpha_1 u_{n-1}}{\beta_1+\gamma_1 u_n^p u_{n-2}^q},\qquad n=0,1,\dots$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_{1,}p,q$ and the initial conditions $u_i,v_i,\ i=-2,-1,0$ are non negative real numbers.

Keywords: $(1,2)$-type system, Difference equations, Equilibrium, Global stability, Rate of convergence.
2010 AMS: 39A10, 40A05

[1] *Zonguldak Bülent Ecevit University, Faculty of Science and Arts, Department of Mathematics, 67100, Zonguldak, Turkey*
[2] *Zonguldak Bülent Ecevit University, Faculty of Science and Arts, Department of Mathematics, 67100, Zonguldak, Turkey*
***Corresponding author**: m.gumus@beun.edu.tr
Received: 7 December 2020, **Accepted:** 28 December 2020, **Available online:**29 March 2021

1. Introduction

Difference equations (also called recursive sequences) appear in a lot of fields of pure and applied mathematics, both as discrete analogs of continuous behavior (analysis, numerical approximations) and as independent models for discrete behavior (population dynamics, economics, biology, ecology, etc.), [1]. In recent years, many models, especially in mathematical biology, are based on non-linear ones, [11]. Difference equation theory, especially nonlinear ones, is very fertile subject for scientists and is one of the important subjects of applied mathematics. So, many researchers have dealth with the qualitative behavior of nonlinear higher order rational difference equations and systems, see [1]-[33].

In [9], El-Owaidy *et al.* studied the global analysis of the following difference equation

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^p},\qquad n=0,1,\dots \tag{1.1}$$

with non-negative parameters and non-negative initial values.

In [5], Ahmed investigated the global asymptotic behavior and the perodic character for the rational difference equation

$$x_{n+1}=\frac{b x_{n-1}}{A+B x_n^p x_{n-2}^p},\qquad n=0,1,\dots \tag{1.2}$$

where the parameters $b,A,B,\ p,q$ are non-negative numbers and the initial values $x_{-2,}x_{-1,}x_0$ are arbitrary non-negative real numbers.

In [15], Gümüş and Soykan investigated the local asymptotic stability of equilibria, the periodic nature of solutions, the existence of unbounded solutions and the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-(k+1)}}{\beta + \gamma x_{n-k}^{p} x_{n-(k+2)}^{q}}, \qquad n = 0,1,... \tag{1.3}$$

where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers and the initial values $x_{-(k+2)}, x_{-(k+1)}, ..., x_{-1}, x_0 \in R^{+}$.

In [13], Gümüş and Soykan studied the dynamical behavior of positive solutions for a system of rational difference equation following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^{p}}, \qquad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^{p}}, \qquad n = 0,1,... \tag{1.4}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$ and the initial values u_{-i}, v_{-i} for $i = 0,1,2$ are positive real numbers.

In [14], Gümüş and Öcalan studied the dynamical behavior of positive solutions for a system of rational difference equations following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^{p} v_{n-2}^{q}}, \qquad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}, \qquad n = 0,1,... \tag{1.5}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p, q, p_1, q_1$ are positive real numbers and the initial values u_{-i}, v_{-i} are non-negative real numbers for $i = 0,1,2$.

In [28], Khan *et al.* investigated the asymptotic behavior of following anti-competitive system of rational difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha y_n}{\beta + \gamma x_n^{r}}, \\ y_{n+1} &= \frac{\alpha_1 x_n}{\beta_1 + \gamma_1 y_n^{t}}, \end{aligned}\right\}, n = 0,1,...$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, r \in (0,\infty)$ and $x_0, y_0 \in (0,\infty)$.

In [29], Qureshi and Din investigated the qualitative asymptotic behavior of positive solution for an anti-competitive system of third-order rational difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{y_{n-2}}{\beta + \gamma x_n x_{n-1} x_{n-2}}, \\ y_{n+1} &= \frac{\alpha_1 x_{n-2}}{\beta_1 + \gamma_1 y_n y_{n-1} y_{n-2}}, \end{aligned}\right\}, n = 0,1,...$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and $x_0, x_{-1}, x_{-2}, y_0, y_{-1}, y_{-2}$ are positive real numbers.

In [27], Qureshi and Khan studied the global dynamics of following $(1,2)-$type systems of difference equations

$$x_{n+1} = \frac{\eta y_{n-1}}{1 + \mu x_{n-2}^{p}}, \qquad y_{n+1} = \frac{\mu x_{n-1}}{1 + \eta y_{n-2}^{p}}, \qquad n = 0,1,... \tag{1.6}$$

$$x_{n+1} = \frac{\eta y_{n-1}}{1 + \mu y_{n-2}^{p}}, \qquad y_{n+1} = \frac{\mu x_{n-1}}{1 + \eta x_{n-2}^{p}}, \qquad n = 0,1,...$$

where η, μ, p and initial conditions $x_l, y_l, l = -2, -1, 0$ are non negative real numbers.

In the present paper, we will investigate of some properties, such as the local asymptotic stability, the global asymptotic stability, the existence of periodic solutions, the rate of converges etc., for $(1,2)-$type system of difference equations in the title, which has been investigated different versions of it in the known literarture. We first note down critical error for the results of the article [27]. Namely, to put it briefly, they can not obtain the equations they claim with their transformations. Using the transformations, they could get equations in this form;

$$\left.\begin{aligned} x_{n+1} &= \frac{\eta y_{n-1}}{1 + \mu y_{n-2}^{p}}, \\ y_{n+1} &= \frac{\eta_1 x_{n-1}}{1 + \mu_1 x_{n-2}^{p}}, \end{aligned}\right\}, n = 0,1,...$$

with

$$\eta = \frac{\alpha}{\beta}, \quad \eta_1 = \frac{\alpha_1}{\beta_1}$$

and

$$\mu_1 = \frac{\beta}{\gamma}, \ \ \mu = \frac{\beta_1}{\gamma_1}.$$

The same applies to the other equation. Let us also note that the theoretical results they obtained in their article are correct. However, an error was made only at the beginning.

The aim of this paper is to investigate the equilibrium points, the local asymptotic stability of these points, the global behavior of positive solutions, the existence of the prime two-periodic solutions and the rate of convergence of positive solutions of the following system

$$\left.\begin{aligned} u_{n+1} &= \frac{\alpha v_{n-1}}{\beta+\gamma v_n^p v_{n-2}^q}, \\ v_{n+1} &= \frac{\alpha_1 u_{n-1}}{\beta_1+\gamma_1 u_n^p u_{n-2}^q}, \end{aligned}\right\}, n=0,1,... \tag{1.7}$$

where the parameters $\alpha,\beta,\gamma,\alpha_1,\beta_1,\gamma_1,p,q$ are positive and initial condition $u_{-2},u_{-1},u_0,v_{-2},v_{-1},v_0 \in (0,\infty)$. Our results extend and complement some results in the literature.

If the initials conditions $u_i = v_i$ in the system (1.7) for $i \in \{-2,-1,0\}$ and $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \gamma_1$ then one obtain that $u_n = v_n$ for all $n \geqslant -2$, hence, the system (1.7) reduces to the difference equation

$$v_{n+1} = \frac{\alpha v_{n-1}}{\beta+\gamma v_n^p v_{n-2}^q}, \ n=0,1,...$$

which was studied by [4]. Therefore, to avoid degenerate situations, here we discuss the case $u_i \neq v_i$ for $i \in \{-2,-1,0\}$ and we investigate the system (1.7) basing on this condition.

It is clear that the system (1.7) can be reduced to the following system of difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{r y_{n-1}}{1+s_1 y_n^p y_{n-2}^q}, \\ y_{n+1} &= \frac{r_1 x_{n-1}}{1+s x_n^p x_{n-2}^q}, \end{aligned}\right\}, \ n=0,1,... \tag{1.8}$$

by the change of variables

$$u_n = \left(\frac{\beta\beta_1}{\gamma\gamma_1}\right)^{1/p+q} x_n$$

and

$$v_n = \left(\frac{\beta\beta_1}{\gamma\gamma_1}\right)^{1/p+q} y_n$$

with

$$r = \frac{\alpha}{\beta}, \quad r_1 = \frac{\alpha_1}{\beta_1}$$

and

$$s = \frac{\beta}{\gamma}, \quad s_1 = \frac{\beta_1}{\gamma_1}.$$

So, in order to study the system (1.7), we investigate the system (1.8).

2. Preliminaries

For the completenessin the paper, we find useful to remind some basic concepts of the difference equations theory as follows:

Let us introduce the six-dimensional discrete dynamical system

$$\begin{aligned} x_{n+1} &= f_1(x_n, x_{n-1}, x_{n-2}, y_{n,}, y_{n-1}, y_{n-2}), \\ y_{n+1} &= f_2(x_n, x_{n-1}, x_{n-2}, y_{n,}, y_{n-1}, y_{n-2}), \end{aligned} \tag{2.1}$$

$n \in \mathbb{N}$ where $f_1 : I_1^3 \times I_2^3 \to I_1$ and $f_2 : I_1^3 \times I_2^3 \to I_2$ are condinuously differentiable functions and I_1 ,I_2 are some invervals of real numbers. Then, for every initial conditions $(x_i, y_i) \in I_1 \times I_2$, for $i = -2, -1, 0$ the system (2.1) has a unique solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$.

Definition 2.1. *An equilibrium point of stsytem (2.1) is a point $(\bar{x}, \bar{y})$ that satisfies*

$$\begin{aligned} \bar{x} &= f_1(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}), \\ \bar{y} &= f_2(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}), \end{aligned}$$

Together with system (2.1), if we consider the associatedvector map

$$F = (f_1, x_n, x_{n-1}, x_{n-2}, f_2, y_n, y_{n-1}, y_{n-2}),$$

then the point $(\bar{x}, \bar{y})$ is also called o fixed point of the vector map F.

Definition 2.2. *If $(\bar{x}, \bar{y})$ be an equilibrium point of a map*

$$F = (f_1, x_n, x_{n-1}, x_{n-2}, f_2, y_n, y_{n-1}, y_{n-2})$$

where f_1 and f_2 are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system (2.1) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = BX_n$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and B is a Jacobian matrix of the system (2.1) about the equilibrium point $(\bar{x}, \bar{y})$.

Theorem 2.3. *For the system $X_{n+1} = F(X_n)$, $n = 0, 1, ...,$ of difference equations such that $\overline{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix B about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater then one, then $\overline{X}$ is unstable.*

3. Stability Character Of Equilibrium

In this section we will prove the stability nature of the zero equilibrium point. In the following theorem we will give the equilibrium points of system (1.8).

Theorem 3.1. *For all parameters r, r_1, s, s_1, system (1.8) have a unique zero equilibrium point.*

Proof. It is clear from the equilibrium definition. □

Before we give the following stability theorems about the local asymptotic stability of the zero equilibrium point, we build the corresponding linearized form of the system (1.8) and consider the following transformation;

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \to (f, f_1, f_2, g, g_1, g_2)$$

where

$$\begin{aligned}
f &= \frac{ry_{n-1}}{1+s_1 y_n^p y_{n-2}^q},\\
f_1 &= x_n,\\
f_2 &= x_{n-1},\\
g &= \frac{r_1 x_{n-1}}{1+s x_n^p x_{n-2}^q},\\
g_1 &= y_n,\\
g_2 &= y_{n-1}.
\end{aligned}$$

The Jacobian matrix about the fixed point $(\bar{x},\bar{y})$ under the above transformation is as follows:

$$B(\bar{x},\bar{y}) = \begin{pmatrix}
0 & 0 & 0 & -\frac{rs_1 p\bar{y}^{p+q}}{(1+s_1\bar{y}^{p+q})^2} & \frac{r}{1+s_1\bar{y}^{p+q}} & -\frac{rs_1 q\bar{y}^{p+q}}{(1+s_1\bar{y}^{p+q})^2}\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
-\frac{r_1 sp\bar{x}^{p+q}}{(1+s\bar{x}^{p+q})^2} & \frac{r_1}{1+s\bar{x}^{p+q}} & -\frac{r_1 sq\bar{x}^{p+q}}{(1+s\bar{x}^{p+q})^2} & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

where $r, s, r_1, s_1, p, q \in (0,\infty)$.

Theorem 3.2. *For system (1.8) the following properties hold:*
(i) The zero equilibrium point is locally asymptotically stable if $rr_1 < 1$.
(ii) The zero equilibrium point is locally unstable if $rr_1 > 1$.

Proof. (i) The linearized system of system (1.8) about the equilibrium point

$$(\bar{x}_0,\bar{y}_0) = (0,0)$$

is given by

$$X_{n+1} = B(\bar{x}_0,\bar{y}_0)X_n,$$

where

$$X_n = \begin{pmatrix} x_n\\ x_{n-1}\\ x_{n-2}\\ y_n\\ y_{n-1}\\ y_{n-2}\end{pmatrix}$$

and

$$B(\bar{x}_0,\bar{y}_0) = \begin{pmatrix}
0 & 0 & 0 & 0 & r & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & r_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

The characteristic equation of $B(\bar{x}_0,\bar{y}_0)$ is as follows:

$$P(\lambda) = \lambda^6 - (rr_1)\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned}\lambda_{1,2} &= 0,\\ \lambda_{3,4} &= \pm\sqrt[4]{rr_1},\\ \lambda_{5,6} &= \pm i\sqrt[4]{rr_1}.\end{aligned}$$

Since all eigenvalues of the Jacobian matrix B about

$$(\bar{x}_0,\bar{y}_0)=(0,0)$$

lie inside the open unit disk

$$|\lambda|<1,$$

the zero equilibrium point is locally asymptotically stable.
(ii) It is easy to see that if $rr_1>1$, then the zero equilibrium point of system (1.8) is unstable. □

Now, we will study the global asymptotic stability of system (1.8) about the zero equilibrium point.

Theorem 3.3. *The zero equilibrium point of system (1.8) is globally asymptotically stable when $r<1$ and $r_1<1$.*

Proof. In view of Theorem 3.2, it suffices to prove that

$$\lim_{n\to\infty}(x_n,y_n)=(0,0).$$

It is evident from (1.8) that

$$0\leqslant x_{n+1}=\frac{ry_{n-1}}{1+s_1y_n^py_{n-2}^q}<ry_{n-1}<y_{n-1}.$$

This implies that

$$x_{4n+1}<y_{4n-1}$$

and

$$x_{4n+5}<y_{4n+3}.$$

Besides this,

$$0\leqslant y_{n+1}=\frac{r_1x_{n-1}}{1+sx_n^px_{n-2}^q}<r_1x_{n-1}<x_{n-1}$$

This implies that

$$y_{4n+1}<x_{4n-1}$$

and

$$y_{4n+5}<x_{4n+3}.$$

So

$$x_{4n+5}<y_{4n+3}<x_{4n+2}$$

and

$$y_{4n+5}<x_{4n+3}<y_{4n+2}.$$

Hence, the subsequences

$$\{x_{4n+1}\},\{x_{4n+2}\},\{x_{4n+3}\},\{x_{4n+4}\}$$

and

$$\{y_{4n+1}\}, \{y_{4n+2}\}, \{y_{4n+3}\}, \{y_{4n+4}\}$$

are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are monotonic which are decreasing. Hence

$$\lim_{n\to\infty} x_n = 0$$

and

$$\lim_{n\to\infty} y_n = 0.$$

This completes the proof. □

4. Prime Periodic Two-Solutions 1.8

In this section we will investigate the periodic nature of system (1.8).

Theorem 4.1. *System (1.8) has no prime period two solutions.*

Proof. Assuming

$$\ldots,(a,b),(c,d),(a,b),(c,d),\ldots$$

is prime period two solutions of the system (1.8) such that

$$a,b,c,d \neq 0$$

and

$$a \neq c, b \neq d.$$

Then we have

$$a = \frac{rb}{1+s_1 d^{p+q}}, \; b = \frac{r_1 a}{1+sc^{p+q}} \tag{4.1}$$

and

$$c = \frac{rd}{1+s_1 b^{p+q}}, \; d = \frac{r_1 c}{1+sa^{p+q}} \tag{4.2}$$

After some tedious calculations from (4.1) and (4.2), we can obtain the following equilities;

$$(a+c)^2 - 4ac = 0,$$

and

$$(b+d)^2 - 4bd = 0.$$

But they are contrary to our assumption and therefore system (1.8) has no prime period-two solutions. This completes the proof. □

5. Rate of Convergence

In this section, we will give exact results about the rate of convergence of positive solutions that converge to the equilibrium point of the system (1.8), in the regions of parameters described in Theorem (3.3).

Consider the following system of difference equations

$$\left.\begin{aligned} x_{n+1} &= f_1(x_n, y_n), \; n = 0,1,\ldots \\ y_{n+1} &= f_2(x_n, y_n), \; n = 0,1,\ldots \end{aligned}\right\} \tag{5.1}$$

where f_1, f_2 are continuous functions that maps some set I into I. The set I is an interval of real numbers. System (5.1) is competitive if $f_1(x,y)$ is non-decreasing in x and non-increasing in y and $f_2(x,y)$ is non-increasing in x and non-decreasing in y. System (5.1) is called anti-competitive system, if the functions f_1 and f_2 have monotonic character opposite to the monotonic character in competitive system.

We state that the following theorems give precise information about the asymptotics of linear non-autonomous difference equations. Consider the scalar mth-order linear difference equation

$$y_{n+m} + p_1(n)y_{n+m-1} + p_m(n)y_n = 0 \tag{5.2}$$

where m is a positive integer and $p_i : \mathbb{Z}^+ \to \mathbb{C}$ for $i \in \{1,\ldots,m\}$. Suppose that

$$q_i = \lim_{n\to\infty} p_i(n), \text{ for } i = 1,2,\ldots,m, \tag{5.3}$$

exist in $\mathbb{C}$. For the following limitting equation of (5.2)

$$y_{n+m} + q_1 y_{n+m-1} + \ldots + q_m y_n = 0, \tag{5.4}$$

the asymptotics of solutions of (5.2) are given the following results. See [25].

Theorem 5.1. *(Poincaré's Theorem) Consider (5.2) based on the condition (5.3). Let λ_i for $i = 1,\ldots,m$ be the roots of the characteristic equation*

$$\lambda^m + q_1\lambda^{m-1} + \ldots + q_m = 0 \tag{5.5}$$

of the limiting equation (5.4) under the condition that $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$. If x_n is a positive solution of (5.2), then either $x_n = 0$ for all large n or there exists an index $j \in \{1,\ldots,m\}$ such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_j.$$

The releated results were obtained by Perron, and one of Perron's results was improved by Pituk, see [25].

Theorem 5.2. *Assume that (5.3) holds. If x_n is a positive solution of (5.2), then either eventually $x_n = 0$ or*

$$\lim_{n\to\infty} \sup(|x_{nj}|)^{1/n} = |\lambda_j|,$$

where $\lambda_1,\ldots,\lambda_m$ are the roots (not necessarily distinct) of the characteristic equation (5.5).

Consider

$$Y_{n+1} = [A + B(n)]Y_n \tag{5.6}$$

where Y_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix and

$$B : \mathbb{Z}^+ \to C^{m\times m}$$

is a matrix function satisfying

$$\|B(n)\| \to 0, \text{ when } n \to \infty, \tag{5.7}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm $\|.\|$. See [20].

Theorem 5.3. *(Pituk) Suppose that condition (5.7) holds for system (5.6). If Y_n is a solution of (5.6), then either*

$$Y_n = 0$$

for all large n or

$$\theta = \lim_{n\to\infty} \|Y_n\|^{1/n}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A.

Theorem 5.4. *(Pituk) Suppose that condition (5.7) holds for system (5.6). If Y_n is a solution of (5.6), then either*

$$Y_n = 0$$

for all large n or

$$\theta = \lim_{n\to\infty} \frac{\|Y_{n+1}\|}{\|Y_n\|}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A.

Using Theorem (5.3) and (5.4), we obtain the following rate of convergence result.

Theorem 5.5. *Suppose that $r < 1$ and $r_1 < 1$. Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be any positive solution of the system (1.8) such that*

$$\begin{aligned} \lim_{n\to\infty} x_n &= \bar{x}_1, \\ \lim_{n\to\infty} y_n &= \bar{x}_2 \end{aligned}$$

where $M = (\bar{x}_1, \bar{x}_2)$ and M is globally asymptotically stable. Then, the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}_{6\times 1} = \begin{pmatrix} x_n - \bar{x}_1 \\ x_{n-1} - \bar{x}_1 \\ x_{n-2} - \bar{x}_1 \\ y_n - \bar{x}_2 \\ y_{n-1} - \bar{x}_2 \\ y_{n-2} - \bar{x}_2 \end{pmatrix}_{6\times 1}$$

of every positive solution of the system (1.8) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n\to\infty} \|E_n\|^{1/n} &= |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, \ldots, 6 \\ \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} &= |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, \ldots, 6 \end{aligned}$$

where

$$|\lambda_i J_F(M)|$$

is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M.

Proof. Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be any positive solution of the system (1.8) such that

$$\lim_{n\to\infty} x_n = \bar{x}_1$$

and

$$\lim_{n\to\infty} y_n = \bar{x}_2.$$

To find the error terms, we have

$$\begin{aligned} x_{n+1} - \bar{x}_1 &= \sum_{i=0}^{2} A_i (x_{n-i} - \bar{x}_1) + \sum_{i=0}^{2} B_i (y_{n-i} - \bar{x}_2) \\ y_{n+1} - \bar{x}_2 &= \sum_{i=0}^{2} C_i (x_{n-i} - \bar{x}_1) + \sum_{i=0}^{2} D_i (y_{n-i} - \bar{x}_2). \end{aligned}$$

Set

$$\begin{aligned} e_n^1 &= x_n - \bar{x}_1, \\ e_n^2 &= y_n - \bar{x}_2; \end{aligned}$$

therefore, it follows that

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^{2} A_i e_{n-i}^1 + \sum_{i=0}^{2} B_i e_{n-i}^2 \\ e_{n+1}^2 &= \sum_{i=0}^{2} C_i e_{n-i}^1 + \sum_{i=0}^{2} D_i e_{n-i}^2 \end{aligned}$$

where

$$\begin{aligned}
A_0 &= 0,\ \ A_1 = 0,\ \ A_2 = 0,\\
B_0 &= -\frac{rs_1 p\overline{y}(y_n^p y_{n-2}{}^q - \overline{y}^{p+q})}{(1+s_1 y_n^p y_{n-2}{}^q)^2},\\
B_1 &= \frac{r}{1+s_1 y_n^p y_{n-2}{}^q},\\
B_2 &= -\frac{rs_1 q\overline{y}(y_n^p y_{n-2}{}^q - \overline{y}^{p+q})}{(1+s_1 y_n^p y_{n-2}{}^q)^2},\\
C_0 &= -\frac{r_1 sp\overline{x}(x_n^p x_{n-2}{}^q - \overline{x}^{p+q})}{(1+s x_n^p x_{n-2}{}^q)^2},\\
C_1 &= \frac{r_1}{1+s x_n^p x_{n-2}{}^q},\\
C_2 &= -\frac{r_1 sq\overline{x}(x_n^p x_{n-2}{}^q - \overline{x}^{p+q})}{(1+s x_n^p x_{n-2}{}^q)^2},\\
D_0 &= 0,\ \ D_1 = 0,\ \ D_2 = 0.
\end{aligned}$$

Taking the limits, it is clear that

$$\begin{aligned}
\lim_{n\to\infty} A_0 &= 0,\ \lim_{n\to\infty} A_1 = 0, \lim_{n\to\infty} A_2 = 0,\\
\lim_{n\to\infty} B_0 &= -\frac{rs_1 p\overline{y}(y_n^p y_{n-2}{}^q - \overline{y}^{p+q})}{(1+s_1 y_n^p y_{n-2}{}^q)^2},\\
\lim_{n\to\infty} B_1 &= \frac{r}{1+s_1 y_n^p y_{n-2}{}^q},\\
\lim_{n\to\infty} B_2 &= -\frac{rs_1 q\overline{y}(y_n^p y_{n-2}{}^q - \overline{y}^{p+q})}{(1+s_1 y_n^p y_{n-2}{}^q)^2},\\
\lim_{n\to\infty} C_0 &= -\frac{r_1 sp\overline{x}(x_n^p x_{n-2}{}^q - \overline{x}^{p+q})}{(1+s x_n^p x_{n-2}{}^q)^2},\\
\lim_{n\to\infty} C_1 &= \frac{r_1}{1+s x_n^p x_{n-2}{}^q},\\
\lim_{n\to\infty} C_2 &= -\frac{r_1 sq\overline{x}(x_n^p x_{n-2}{}^q - \overline{x}^{p+q})}{(1+s x_n^p x_{n-2}{}^q)^2},\\
\lim_{n\to\infty} D_0 &= 0,\ \lim_{n\to\infty} D_1 = 0,\ \lim_{n\to\infty} D_2 = 0.
\end{aligned}$$

That is

$$\begin{aligned}
B_0 &= -\frac{rs_1 p\overline{y}(y_1^p y_2{}^q - \overline{y}^{p+q})}{(1+s_1 \overline{y}_2^{p+q})^2} + \alpha_n,\\
B_1 &= \frac{r}{1+s_1 \overline{y}_2^{p+q}} + \beta_n,\\
B_2 &= -\frac{rs_1 q\overline{y}(y_1^p y_2{}^q - \overline{y}^{p+q})}{(1+s_1 \overline{y}_2^{p+q})^2} + \gamma_n,\\
\\
C_0 &= -\frac{r_1 sp\overline{x}(x_2^p x_1{}^q - \overline{x}^{p+q})}{(1+s \overline{x}_1^{p+q})^2} + \delta_n,\\
C_1 &= \frac{r_1}{1+s \overline{x}_1^{p+q}} + \eta_n,\\
C_2 &= --\frac{r_1 sq\overline{x}(x_2^p x_1{}^q - \overline{x}^{p+q})}{(1+s \overline{x}_1^{p+q})^2} + \theta_n,
\end{aligned}$$

where $\alpha_n \to 0$, $\beta_n \to 0$, $\gamma_n \to 0$, $\delta_n \to 0$, $\eta_n \to 0$, $\theta_n \to 0$ for $n \to \infty$.
Thus, the limitting system of error terms about the equilibrium M can be written as follows:

$$E_{n+1} = (C + D(n))E_n,$$

where $E_n = (e_n^1, e_{n-1}^1, e_{n-2}^1, e_n^2, e_{n-1}^2, e_{n-2}^2)^T$,

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & r & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{6\times 6},$$

$$D_n = \begin{pmatrix} 0 & 0 & 0 & \alpha_n & \beta_n & \gamma_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_n & \eta_n & \theta_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}$$

and $\|D(n)\| \to 0$, when $n \to \infty$. As desired. □

Corollary 5.6. *Assume that $rr_1 < 1$. Then, the error vector of every non-trivial solution of system 1.8 satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \|E_n\|^{1/n} = |\lambda_i J_F(M)|, \text{ for some } i = 1,2,3,4,5,6,$$

$$\lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_i J_F(M)|, \text{ for some } i = 1,2,3,4,5,6$$

where $|\lambda_i J_F(M)|$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M, i.e. $\{\lambda_{1,2} = 0, \lambda_{3,4} = \pm\sqrt[4]{rr_1}, \lambda_{5,6} = \pm i\sqrt[4]{rr_1}.\}$.

6. Conclusions

In the present paper, we described the qualitative behaviors of solutions of the system (1.8) of nonlinear difference equations. More precisely, we studied the equilibrium points, the local asymptotic stability, the global asymptotic stability of zero equilibrium, the existence of the prime two-periodic solutions and the rate of convergence of positive solutions of the aforementioned system. Also, we gave a correction about an article in the literature. Our system generalized the systems studied in [13, 14, 27].

The results in this paper can be extend to the following system of difference equations;

$$\left.\begin{aligned} u_{n+1} &= \frac{\alpha v_{n-1}}{\beta + \gamma \sum_{i=0}^{k} v_{n-2i}^{p_i}}, \\ v_{n+1} &= \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 \sum_{i=0}^{k} u_{n-2i}^{p_i}}, \end{aligned}\right\}, n = 0, 1, \ldots.$$

7. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this manuscript.

References

[1] R. Agarwal, *Difference equations and inequalities, theory, methods and applications*, Marcel Dekker Inc., New York 1992.

[2] R. Abo-Zeid and H. Kamal, *Global behavior of two rational third order difference equations,* Univers. J. Math. Appl., **2**(4) (2019), 212-217.

[3] S. Abualrub and M. Aloqeili, *Dynamics of the System of Difference Equations* $x_{n+1} = A + y_{n-k}/y_n$, $y_{n+1} = B + x_{n-k}/x_n$, Qual. Theory Dyn. Syst., **19**(2) (2020), 1-19.

[4] A. M. Ahmed, *On the Dynamics of Higher-Order Rational Difference Equation,* Discrete Dyn. Nat. Soc., **2011**, Article ID: 419789, 8 pages.

[5] A. M. Ahmed, *The Dynamics of the Recursive Sequences* $x_{n+1} = \frac{bx_{n-1}}{A+Bx_n^p x_{n-2}^p}$, J.Pure Appl. Math.: Adv. Applic., **1**(2) (2009), 215-223.

[6] M. M. Alzubaidi and E. M. Elsayed, *Analytical and Solutions of Fourth Order Difference Equations,* Commun. Adv. Math. Sci., **2**(1) (2019), 9-21.

[7] Q. Din, T. F. Ibrahim and A.Q. Khan, *Behavior of a competitive system of second order difference equations,* Sci. World J., Article ID: 283982.

[8] Q. Din, *Asymptotic behavior of an anti-competitive system of second-order difference equations,* J. Egyptian Math. Soc., 24(1) (2016), 37-43.

[9] H. M. El-Owaidy, A. M. Ahmed and A. M. Youssef, *The Dynamics of the Recursive Sequence* $x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^p}$, Appl. Math. Lett., **18** (2005), 1013-1018.

[10] M. M. El-Dessoky, E. M. Elsayed, E. M. Elabbasy and A. Asiri, *Expressions of the solutions of some systems of difference equations,* J. Comput. Anal. Appl., **27**(7) (2019), 1161-1172.

[11] S. Elaydi, *An Introduction to Difference Equations,* 3rd ed., Springer-Verlag, New York, 2005.

[12] M. Gocen and M. Guneysu, *The Global Attractivity of some ratinal difference equations*, J. Comp. Anal. Appl., 25 (7) (2018), 1233-1243.

[13] M. Gümüş and Y. Soykan**,** *Global Character of a Six-Dimensional Nonlinear System of Difference Equations,* Discrete Dyn. Nat. Soc., **2016**, Article ID 6842521.

[14] M. Gümüş and Ö. Öcalan, *The Qualitative Analysis of a Rational System of Difference Equations,* J. Fract. Calc. Appl., **9**(2) (2018), 113-126.

[15] M. Gümüş and Y. Soykan, *Dynamics of Positive Solutions of a Higher Order Fractional Difference Equation with Arbitrary Powers* , J. Appl. Math. Inf., **35**(3-4) (2017), 267-276.

[16] V. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications,* Kluwer Academic Publishers, Dordrecht, (1993).

[17] F. H. Gümüş, *Yüksek Mertebeden Fark Denklemlerinin Global Davranışları Üzerine,* Master Thesis, Afyon Kocatepe Üniversitesi Fen Bilimleri Enstitüsü Afyon (2015).

[18] A. Khelifa, Y. Halim and M. Berkal, *Solutions of a system of two higher-order difference equations in terms of lucas sequence,* Univers. J. Math. Appl., **2**(4) (2019), 202-211.

[19] M. A. Kerker, E. Hadidi, and A. Salmi, *Qualitative behavior of a higher-order nonautonomous rational difference equation,* J. Appl. Math. Comput., **64** (2020), 399–409.

[20] M. R. S. Kulenovic and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica.* Chapman and Hall/CRC, Boca Raton, London. (2002).

[21] M. R. S. Kulenović and G. Ladas, *Dynamics of second order rational difference equations*, Chapman and Hall/CRC, (2001).

[22] M. Kara, N. Touafek and Y. Yazlık, *Well-Defined Solutions of a Three-Dimensional System of Difference Equations,* Gazi Univ. J.Sci., **33**(3), 767-778.

[23] W. X. Ma, *Global Behavior of an Arbitrary-Order Nonlinear Difference Equation with a Nonnegative Function*, Mathematics, **8**(5) (2020), 825.

[24] M. Merdan and Ş. Şişman, *Investigation of linear difference equations with random effects*, Adv. Difference Equ., **2020** (1) (2020), 1-19.

[25] M. Pituk, *More on Poincares and Perrons Theorems for Difference Equations,* J. Difference Equ. Applic., **8**(3) (2002), 201-216.

[26] A. Sanbo, E. M. Elsayed and F. Alzahrani, *Dynamics of the nonlinear rational difference equation* $x_{n+1} = (Ax_{n-\alpha}x_{n-\beta} + Bx_{n-\gamma})/(Cx_{n-\alpha}x_{n-\beta} + Dx_{n-\gamma})$, Indian J. Pure Appl. Math., **50**(2) (2019), 385-401.

[27] M. N. Qureshi and A. Q. Khan, *Global dynamics of* $(1,2)$ *− type systems of difference equations,* Malaya J. Matematik, **6**(2), 408-416.

[28] M. N. Qureshi, Q. Din and A. Q. Khan, *Asymptotic behavior of an anti-competitive system of rational difference equations,* Life Sci. J., **11**(2014): 1-7.

[29] M. N. Qureshi and Q. Din, *Oualitative behavior of an anti-competitive system of third-order rational difference equations,* Comput. Ecology Software, **4**(2) (2014), 104-115.

[30] H. Sedaghat, *Nonlinear difference equations theory with applications to social science models,* 15. Springer Science and Business Media, (2003).

[31] D. T. Tollu and I. Yalçınkaya, *Global behavior of a three dimensional system of difference equations of order three,* Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**(1) (2019), 1-16.

[32] I. Yalçınkaya, *On the Global Asymptotic Stability of a Second-Order System of Difference Equations,* Discrete Dyn. Nat. Soc., **2008**, Article ID 860152.

[33] D. Zhang, W. Ji, L. Wang and X. Li, *On the symmetrical system of rational difference equation* $x_{n+1} = A + y_{n-k}/y_n$, $y_{n+1} = A + x_{n-k}/x_n$, Appl. Math., **4** (2013), 834-837.

Communications in Advanced Mathematical Sciences
Vol. IV, No. 1, 39-45, 2021
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.860254

Numerical Solution of a Quadratic Integral Equation through Classical Schauder Fixed Point Theorem

Merve Temizer Ersoy[1]*

Abstract
In this paper, we investigate the existence of at least one solution on the closed interval for quadratic integral equations with non-linear modification of the argument in Hölder spaces using the technique in the classical Schauder fixed point theorem.

Keywords: Fredholm integral equation, Hölder condition, Schauder fixed point theorem.
2010 AMS: Primary 45B05, 45G10, 47H10

[1] *Department of Mathematics, Faculty of Science and Arts, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, Turkey, ORCID: 0000-0003-4364-9144*

Received: 13 December 2021, **Accepted:** 10 March 2021, **Available online:** 29 March 2021

1. Introduction

Integral equations arise naturally in various applications in describing numerous real universe problems. As well, quadratic integral equations have numerous useful applications in describing uncountable events and problems of the real world. For instance, quadratic integral equations are often applicable in the traffic theory, in the theory of radiative transfer, in the theory of neutron transport and kinetic theory of gases. Several authors have comprehensively studied the integral equations and the solution of them in this references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Moreover, M. Benchohra and M. A. Darwish et al. [1] study the existence of the unique solution, defined on a semi-infinite interval $J:[0,\infty)$ for the following quadratic integral equations with a linear modification of the argument

$$x(t) = f(t) + (Ax)(t)\int_0^T u(t,s,x(s),x(\alpha s))ds, \;\; t \in J.$$

where $f : J \to \mathbb{R}$, $u : J \times J_T \times \mathbb{R}^2 \to \mathbb{R}$ are given functions, $0 < \alpha < 1$, $J_T = [0,T]$ and $A : C(J;\mathbb{R}) \to C(J;\mathbb{R})$ is an appropriate operator. Here $C(J;\mathbb{R})$ denotes the space of continuous functions $x : J \to \mathbb{R}$.

This article concerns the entity of solutions of the following a quadratic integral equation of Fredholm type,

$$x(t) = (T_1x)(t) + (T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau, \;\; t \in I = [0,1]. \tag{1.1}$$

where k is given function, T_1, T_2, T_3 are given operators satisfying conditions specified later and x is unknown function.

2. Preliminaries

Let $[a,b]$ be a closed interval in $\mathbb{R}$, by $C[a,b]$ we indicate the space of continuous functions defined on $[a,b]$ equipped with the supremum norm, i.e.,

$$\|x\|_\infty = \sup\{|x(t)| : t \in [a,b]\}$$

for $x \in C[a,b]$. For a fixed α with $0 < \alpha \leq 1$, by $H_\alpha[a,b]$ we will indicate the spaces of the real functions x defined on $[a,b]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that

$$|x(t) - x(s)| \leq H_x^\alpha |t-s|^\alpha \tag{2.1}$$

for all $t,s \in [a,b]$. It is well proved that $H_\alpha[a,b]$ is a linear subspaces of $C[a,b]$. Also, for $x \in H^\alpha[a,b]$, by H_x^α we will indicate the least possible stable for which inequality (2.1) is satisfied. Rather, we put

$$H_x^\alpha = \sup\left\{ \frac{|x(t) - x(s)|}{|t-s|^\alpha} : t,s \in [a,b] \text{ and } t \neq s \right\}. \tag{2.2}$$

The space $H_\alpha[a,b]$ with $0 < \alpha \leq 1$ may be equipped with the norm

$$\|x\|_\alpha = |x(a)| + H_x^\alpha$$

for $x \in H_\alpha[a,b]$. Here, H_x^α is defined by (2.2). In [2], the authors demonstrated that $(H_\alpha[a,b], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space.

Lemma 2.1. *For $0 < \alpha \leq 1$ and $x \in H_\alpha[a,b]$, we have:*

$$\|x\|_\infty \leq \max\left(1, (b-a)^\alpha\right) \|x\|_\alpha.$$

In particular, the inequality $\|x\|_\infty \leq \|x\|_\alpha$ is satisfied for $a = 0$ and $b = 1$, [2].

Lemma 2.2. *For $0 < \alpha < \beta \leq 1$, we have*

$$H_\beta[a,b] \subset H_\alpha[a,b] \subset C[a,b].$$

Furthermore, for $x \in H_\beta[a,b]$, we have:

$$\|x\|_\alpha \leq \max\left(1, (b-a)^{\beta-\alpha}\right) \|x\|_\beta.$$

Particularly, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\beta$ is satisfied for $a = 0$ and $b = 1$, [2].

Lemma 2.3. *Let's assume that $0 < \alpha < \beta \leq 1$ and E is a bounded subset in $H_\beta[a,b]$, then E is a relatively compact subset in $H_\alpha[a,b]$,* [3].

Lemma 2.4. *Assume that $0 < \alpha < \beta \leq 1$ and by B_r^β we indicate the ball centered at θ and radius r in the space $H_\beta[a,b]$, i.e., $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$. Then B_r^β is a closed subset of $H_\alpha[a,b]$,* [3].

Corollary 2.5. *Assume that $0 < \alpha < \beta \leq 1$ and $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$, then B_r^β is a compact subset in the space $H_\alpha[a,b]$,* [3].

Theorem 2.6 (Schauder's fixed point theorem)**.** *Let E be a nonempty, compact and convex subset of a Banach space $(X, \|\cdot\|)$, convex and let $T : E \to E$ be a continuity mapping. Then T has at least one fixed point in E,* [4].

3. Main Result

Theorem 3.1. *Assume that the following conditions $(i) - (iv)$ are satisfied:*

(i) *The operators $T_1, T_2 : H_\beta[0,1] \to H_\beta[0,1]$ are continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$. Also, T_1 and T_2 hold the inequalities*

$$\|T_1 x\|_\beta \leq f_1(\|x\|_\beta) \text{ and } \|T_2 x\|_\beta \leq f_2(\|x\|_\beta)$$

for any $x \in H_\beta[0,1]$, where α and β are the fixed constants satisfying $0 < \alpha < \beta \leq 1$ and the functions $f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing on $\mathbb{R}_+$.

(ii) *$k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function such that there exists a constant $k_\beta > 0$ satisfying*

$$|k(t,\tau) - k(s,\tau)| \leq k_\beta |t-s|^\beta,$$

for any $t,s,\tau \in [0,1]$.

(iii) *The operators $T_3 : H_\beta[0,1] \to C[0,1]$ is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$. Also, T_3 holds the inequality*

$$\|T_3x\|_\infty \leq f_3(\|x\|_\beta)$$

for any $x \in H_\beta[0,1]$, where α and β are the fixed constants satisfying $0 < \alpha < \beta \leq 1$ and the functions $f_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing on $\mathbb{R}_+$.

(iv) *There exists a positive solution r_0 of the inequality*

$$f_1(r) + (2K + k_\beta)f_2(r)f_3(r) \leq r,$$

where the constant K is defined by

$$\sup\left\{\int_0^1 |k(t,\tau)|d\tau : t \in [0,1]\right\} \leq K.$$

Then the equation (1.1) has at least one solution $x = x(t)$ belonging to space $H_\alpha[0,1]$.

Proof. We take for arbitrarily fixed $t, s \in [0,1], (t \neq s)$ and let us consider $x \in H_\beta[0,1]$ and the operator F defined on the space $H_\beta[0,1]$ by the formula:

$$(Fx)(t) = (T_1x)(t) + (T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau,$$

for $t \in [0,1]$. Then, in view of our assumptions we get

$$\begin{aligned}
(Fx)(t) - (Fx)(s) &= (T_1x)(t) + (T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau - (T_1x)(s) - (T_2x)(s)\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau \\
&= (T_1x)(t) - (T_1x)(s) + (T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau - (T_2x)(s)\int_0^1 k(s,\tau)(T_3x)(\tau)d\tau \\
&\quad + (T_2x)(s)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau - (T_2x)(s)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau \\
&= (T_1x)(t) - (T_1x)(s) + ((T_2x)(t) - (T_2x)(s))\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau \\
&\quad + (T_2x)(s)\int_0^1 (k(t,\tau) - k(s,\tau))(T_3x)(\tau)d\tau
\end{aligned}$$

and

$$\begin{aligned}
\frac{|(Fx)(t) - (Fx)(s)|}{|t-s|^\beta} &\leq \frac{|(T_1x)(t) - (T_1x)(s)|}{|t-s|^\beta} + \frac{|(T_2x)(t) - (T_2x)(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau)||(T_3x)(\tau)|d\tau \\
&\quad + \frac{|(T_2x)(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau) - k(s,\tau)||(T_3x)(\tau)|d\tau \leq H^\beta_{T_1x} + \|T_2x\|_\beta\|T_3x\|_\infty\int_0^1 |k(t,\tau)|d\tau \\
&\quad + \|T_2x\|_\infty\|T_3x\|_\infty\int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\beta}d\tau \\
&\leq H^\beta_{T_1x} + \|T_2x\|_\beta\|T_3x\|_\infty K + \|T_2x\|_\beta\|T_3x\|_\infty\int_0^1 k_\beta\frac{|t-s|^\beta}{|t-s|^\beta}d\tau \\
&\leq H^\beta_{T_1x} + f_2(\|x\|_\beta)f_3(\|x\|_\beta)K + f_2(\|x\|_\beta)f_3(\|x\|_\beta)k_\beta \\
&= H^\beta_{T_1x} + (K + k_\beta)f_2(\|x\|_\beta)f_3(\|x\|_\beta).
\end{aligned} \tag{3.1}$$

This demonstrates that the operator F maps $H_\beta[0,1]$ into itself. Besides, for any $x \in H_\beta[0,1]$, we get

$$\begin{aligned}
|(Fx)(0)| &\leq |(T_1x)(0)| + |(T_2x)(0)|\int_0^1 |k(0,\tau)|(T_3x)(\tau)|d\tau \\
&\leq |(T_1x)(0)| + \|T_2x\|_\infty\|T_3x\|_\infty K \\
&\leq |(T_1x)(0)| + \|T_2x\|_\beta\|T_3x\|_\infty K \\
&\leq |(T_1x)(0)| + f_2(\|x\|_\beta)f_3(\|x\|_\beta)K.
\end{aligned} \tag{3.2}$$

By the inequalities by (3.1) and (3.2), we derive that

$$\begin{aligned} \|Fx\|_\beta &\leq \|T_1x\|_\beta + (2K+k_\beta) f_2(\|x\|_\beta) f_3(\|x\|_\beta) \\ &\leq f_1(\|x\|_\beta) + (2K+k_\beta) f_2(\|x\|_\beta) f_3(\|x\|_\beta). \end{aligned} \tag{3.3}$$

Since positive number r_0 is the solution of the inequality given in hypothesis (iv), from (3.3), we conclude that the inequality

$$\|Fx\|_\beta \leq f_1(r_0) + (2K+k_\beta) f_2(r_0) f_3(r_0) \leq r_0 \tag{3.4}$$

holds. As a results, it follows that F transforms the ball

$$B_{r_0}^\beta = \{x \in H_\beta[0,1] : \|x\|_\beta \leq r_0\}$$

into itself. That is, $F : B_{r_0}^\beta \to B_{r_0}^\beta$. Thus, we have that the set $B_{r_0}^\beta$ is relatively compact in $H_\alpha[0,1]$ for any $0 < \alpha < \beta \leq 1$. Furthermore, $B_{r_0}^\beta$ is a compact subset in $H_\alpha[0,1]$.

We will show that the operator F is continuous on $B_{r_0}^\beta$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$. Let $y \in B_{r_0}^\beta$ be an arbitrary point in $B_{r_0}^\beta$. Then, we get

$$\begin{aligned} (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) = \; & (T_1x)(t) + (T_2x)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau \\ & - (T_1y)(t) - (T_2y)(t) \int_0^1 k(t,\tau)(T_3y)(\tau) d\tau \\ & - (T_1x)(s) - (T_2x)(s) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau \\ & + (T_1y)(s) + (T_2y)(s) \int_0^1 k(s,\tau)(T_3y)(\tau) d\tau \end{aligned} \tag{3.5}$$

for any $x \in B_{r_0}^\beta$ and $t,s \in [0,1]$. The equality (3.5) can be rewritten as:

$$\begin{aligned} (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) = \; & (T_1x)(t) - (T_1y)(t) - ((T_1x)(s) - (T_1y)(s)) \\ & + (T_2x)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau - (T_2y)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau \\ & + (T_2y)(t) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau - (T_2y)(t) \int_0^1 k(t,\tau)(T_3y)(\tau) d\tau \\ & - (T_2x)(s) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau + (T_2y)(s) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau \\ & - (T_2y)(s) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau + (T_2y)(s) \int_0^1 k(s,\tau)(T_3y)(\tau) d\tau. \end{aligned} \tag{3.6}$$

By (3.6), we have

$$\begin{aligned} (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) = \; & (T_1x)(t) - (T_1y)(t) - ((T_1x)(s) - (T_1y)(s)) \\ & + ((T_2x)(t) - (T_2y)(t)) \int_0^1 k(t,\tau)(T_3x)(\tau) d\tau \\ & + (T_2y)(t) \int_0^1 k(t,\tau)((T_3x)(\tau) - (T_3y)(\tau)) d\tau \\ & - ((T_2x)(s) - (T_2y)(s)) \int_0^1 k(s,\tau)(T_3x)(\tau) d\tau \\ & - (T_2y)(s) \int_0^1 k(s,\tau)((T_3x)(\tau) - (T_3y)(\tau)) d\tau. \end{aligned} \tag{3.7}$$

(3.7) yields the following equality:

$$\begin{aligned}((Fx)(t)-(Fy)(t))-((Fx)(s)-(Fy)(s)) &= (T_1x)(t)-(T_1y)(t)-((T_1x)(s)-(T_1y)(s))\\ &+[((T_2x)(t)-(T_2y)(t))-((T_2x)(s)-(T_2y)(s))]\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau\\ &+((T_2x)(s)-(T_2y)(s))\int_0^1 (k(t,\tau)-k(s,\tau))(T_3x)(\tau)d\tau\\ &+((T_2y)(t)-(T_2y)(s))\int_0^1 k(t,\tau)((T_3x)(\tau)-(T_3y)(\tau))d\tau\\ &+(T_2y)(s)\int_0^1 (k(t,\tau)-k(s,\tau))((T_3x)(\tau)-(T_3y)(\tau))d\tau. \end{aligned} \tag{3.8}$$

Since $|(T_3x)(\tau)|\le \|T_3x\|_\infty \le f_3(\|x\|_\beta)$ and $|(T_3x)(\tau)-(T_3y)(\tau)|\le \|T_3x-T_3y\|_\infty$ for all $x,y\in B_{r_0}^\beta$ and $\tau\in[0,1]$, taking into account (3.8) and hypotheses, we can write:

$$\begin{aligned}\frac{|(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s))|}{|t-s|^\alpha} &\le \frac{|(T_1x)(t)-(T_1y)(t)-((T_1x)(s)-(T_1y)(s))|}{|t-s|^\alpha}\\ &+\frac{|(T_2x)(t)-(T_2y)(t)-((T_2x)(s)-(T_2y)(s))|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)||(T_3x)(\tau)|d\tau\\ &+\frac{|(T_2x)(s)-(T_2y)(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)-k(s,\tau)||(T_3x)(\tau)|d\tau\\ &+\frac{|(T_2y)(t)-(T_2y)(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)||(T_3x)(\tau)-(T_3y)(\tau)|d\tau\\ &+\frac{|(T_2y)(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)-k(s,\tau)||(T_3x)(\tau)-(T_3y)(\tau)|d\tau\\ &\le \|T_1x-T_1y\|_\alpha+\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty K+\|T_2x-T_2y\|_\infty\|T_3x\|_\infty\int_0^1 k_\beta|t-s|^{\beta-\alpha}d\tau\\ &+\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty K+\|T_2y\|_\infty\|T_3x-T_3y\|_\infty\int_0^1 k_\beta|t-s|^{\beta-\alpha}d\tau\\ &\le \|T_1x-T_1y\|_\alpha+K\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+k_\beta\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty\\ &+K\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty+k_\beta\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty\\ &= \|T_1x-T_1y\|_\alpha+(K+k_\beta)\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+(K+k_\beta)\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty \end{aligned}\tag{3.9}$$

for all $t,s\in[0,1]$ with $t\ne s$. Besides, for $x,y\in B_{r_0}^\beta$, we obtain following equality:

$$\begin{aligned}(Fx)(0)-(Fy)(0) &= (T_1x)(0)+(T_2x)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau-(T_1y)(0)-(T_2y)(0)\int_0^1 k(0,\tau)(T_3y)(\tau)d\tau\\ &= (T_1x)(0)-(T_1y)(0)+(T_2x)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau\\ &\quad -(T_2y)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau+(T_2y)(0)\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau\\ &\quad -(T_2y)(0)\int_0^1 k(0,\tau)(T_3y)(\tau)d\tau\\ &= (T_1x)(0)-(T_1y)(0)+((T_2x)(0)-(T_2y)(0))\int_0^1 k(0,\tau)(T_3x)(\tau)d\tau\\ &\quad +(T_2y)(0)\int_0^1 k(0,\tau)((T_3x)(\tau)-(T_3y)(\tau))d\tau. \end{aligned}\tag{3.10}$$

By (3.10), we get that

$$\begin{aligned} |(Fx)(0)-(Fy)(0)| &= |(T_1x)(0)-(T_1y)(0)|+|(T_2x)(0)-(T_2y)(0)|K\int_0^1 |(T_3x)(\tau)|d\tau \\ &\quad +|(T_2y)(0)|K\int_0^1 |(T_3x)(\tau)-(T_3y)(\tau)|d\tau \\ &\le \|T_1x-T_1y\|_\infty+\|T_2x-T_2y\|_\infty K\|T_3x\|_\infty+\|T_2y\|_\infty K\|T_3x-T_3y\|_\infty \\ &\le \|T_1x-T_1y\|_\alpha+\|T_2x-T_2y\|_\alpha K\|T_3x\|_\infty+\|T_2y\|_\alpha K\|T_3x-T_3y\|_\infty. \end{aligned} \tag{3.11}$$

From (3.9) and (3.11), we have that

$$\begin{aligned} \|Fx-Fy\|_\alpha &= |(Fx-Fy)(0)|+H^\alpha_{Fx-Fy} \\ &= |(Fx)(0)-(Fy)(0)|+\sup\left\{\frac{|(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s))|}{|t-s|^\alpha} : t,s\in[0,1] \text{ and } t\neq s\right\} \\ &\le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+(2K+k_\beta)\|T_2y\|_\alpha\|T_3x-T_3y\|_\infty \\ &\le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)\|T_2x-T_2y\|_\alpha\|T_3x\|_\infty+(2K+k_\beta)\|T_2y\|_\beta\|T_3x-T_3y\|_\infty \\ &\le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)\|T_2x-T_2y\|_\alpha f_3(\|x\|_\beta)+(2K+k_\beta)f_2(\|y\|_\beta)\|T_3x-T_3y\|_\infty. \end{aligned} \tag{3.12}$$

Moreover, since $\|x\|_\beta\le r_0$ and $\|y\|_\beta\le r_0$, we derive from (3.12) that the following inequality holds:

$$\|Fx-Fy\|_\alpha \le 2\|T_1x-T_1y\|_\alpha+(2K+k_\beta)f_3(r_0)\|T_2x-T_2y\|_\alpha+(2K+k_\beta)f_2(r_0)\|T_3x-T_3y\|_\infty. \tag{3.13}$$

Since the operators $T_1,T_2 : H_\beta[0,1]\to H_\beta[0,1]$ and $T_3 : H_\beta[0,1]\to C[0,1]$ are continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, they are also continuous at the point $y\in B^\beta_{r_0}$. Let us take an arbitrary $\varepsilon>0$, then there exists the number $\delta=\delta(\varepsilon)>0$. The inequalities

$$\|T_1x-T_1y\|_\alpha<\frac{\varepsilon}{6}, \|T_2x-T_2y\|_\alpha<\frac{\varepsilon}{3(2K+k_\beta)f_3(r_0)}$$

and

$$\|T_3x-T_3y\|_\infty<\frac{\varepsilon}{3(2K+k_\beta)f_2(r_0)}$$

hold for all $x\in B^\beta_{r_0}$. Then, taking into account (3.13), we derive the following inequality:

$$\|Fx-Fy\|_\alpha<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

for all $x\in B^\beta_{r_0}$ with $\|x-y\|_\alpha<\delta$. Eventually, we infer that the operator F is continuous at the point $y\in B^\beta_{r_0}$. Since y was chosen arbitrarily, we conclude that F is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$. Because $B^\beta_{r_0}$ is compact in $H_\alpha[0,1]$, by the classical Schauder fixed point theorem, we get the desired consequence. □

4. Conclusion

This article concerns the entity of solutions of the following a quadratic integral equation of Fredholm type,

$$x(t)=(T_1x)(t)+(T_2x)(t)\int_0^1 k(t,\tau)(T_3x)(\tau)d\tau,\ \ t\in I=[0,1].$$

where k is given function, T_1,T_2,T_3 are given operators satisfying conditions specified later and x is unknown function.

References

[1] M. Benchohra, M. A. Darwish, *On unique Solvability of Quadratic Integral Equations with Linear Modification of the Argument*, Miskolc Math. Notes, **10** (2009), 3-10.

[2] J. Banaś, R. Nalepa, *On the space of functions with growths tempered by a modulus of continuity and its applications*, J. Funct. Space Appl. (2013), 13 pages, doi:http://dx.doi.org/10.1155/2013/820437.

[3] J. Caballero, M. Abdalla, K. Sadarangani, *Solvability of a quadratic integral equation of fredholm type in Hölder spaces*, Electron. J. Differ. Eq., **31** (2014), 1-10.

[4] J. Schauder, *Der Fixpunktsatz in Funktionalriiumen*, Studia Math., **2** (1930), 171-180.

[5] J. Banaś, A. Chlebowicz, *On an elementrary inequality and its application in theory of integral equations*, J. Math. Ineq., **11** (2) (2017), 595-605.

[6] J. Caballero, B. López, K. Sadarangani, *Existence of nondecreasing and continuous solutions of an integral equation with linear modification of the argument*, Acta Math. Sin. (Engl. Ser.), **23** (9) (2007), 1719-1728.

[7] M. A. Darwish, *On quadratic integral equation of fractional orders*, J. Math. Anal. Appl., **311**(1) (2005),112-119.

[8] S. Hu, M. Khavanin, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal., **34** (1989),261-266.

[9] C. A. Stuart, *Existence theorems for a class of non-linear integral equations*, Math. Z., **137** (1974) 49-66.

[10] J. Banaś, A. Chlebowicz, *On existence of integrable solutions of a functional integral equation under Carathéodory conditions*, Nonlinear Anal., **70** (2009), 3172-3179.

[11] J. Banaś, B. Rzepka, *On local attractivity and asymptotic stability of solutions of a quadratic Volterra integral equation*, Appl. Math. Comput., **213** (2009), 102-111.

[12] H. Deepmala, K. Pathak, *A study on some problems on existence of solutions for nonlinear functional-integral equations*, Acta Math. Scientia, **33** (2013), 1305-1313.

[13] H. Deepmala, K. Pathak, *Study on existence of solutions for some nonlinear functional-integral equations with applications*, Math. Commun., **18** (2013), 97-107.

[14] K. Maleknejad, K. Nouri, R. Mollapourasl, *Existence of solutions for some nonlinear integral equations*, Communications in Commun. Nonlinear Sci. Numer. Simul., **14** (2009), 2559-2564.

[15] K. Maleknejad, K. Nouri, R. Mollapourasl, *Investigation on the existence of solutions for some nonlinear functional-integral equations*, Nonlinear Anal., **71** (2009), 1575-1578.

[16] L. N. Mishra, R. P. Agarwal, M. Sen, *Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded interval*, Prog. Frac. Differ. Appli., **2** (3) (2016).

[17] L. N. Mishra, M. Sen, *On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order*, Appl. Math. Comput., (2016), http://dx.doi.org/10.1016/j.amc.2016.03.002.

[18] L. Liu, F. Guo, C. Wu, Y. Wu, *Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*, J. Math. Anal. Appl., **309** (2005), 638-649.

[19] J. Mallet-Paret, R.D. Nussbam, *Inequivalent measures of noncompactness and the radius of the essential spectrum*, Proc. Amer. Math. Soc., (2010), 917-930.

[20] G. Micula , G. Fairweather, *Direct numerical spline methods for first order Fredholm integro-differential equations*, Rev. Anal. Numer. Theory Approx., **22** (1) (1993), 59-66.

[21] M. Mursaleen , S.A. Mohiuddine, *Applications of noncompactness to the infinite system of differential equations in ℓ_p spaces*, Nonlinear Anal. Theory Methods Appl., **75** (4) (2012), 2111-2115.

[22] L. Olszowy, *Solvability of infinite systems of singular integral equations in Fréchet space of continuous functions*, Comp. Math. Appl. **59** (2010), 2794-2801.

[23] M. J. Caballero, R.Nalepa, K. Sadarangani, *Solvability of a quadratic integral equation of Fredholm type with Supremum in Hölder Spaces*, J. Funct. Space Appl., (2014), 7 pages, doi:http://dx.doi.org/10.1155/2014/856183.

Communications in Advanced Mathematical Sciences
Vol. IV, No. 1, 46-54, 2021
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.814296

On the Recursive Sequence $x_{n+1} = \dfrac{x_{n-29}}{1+x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24}}$

Burak Oğul[1]*, Dağıstan Şimşek[2]

Abstract
In this paper, we are going to analyze the following difference equation

$$x_{n+1} = \frac{x_{n-29}}{1+x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24}} \qquad n = 0,1,2,\ldots$$

where $x_{-29}, x_{-28}, x_{-27}, \ldots, x_{-2}, x_{-1}, x_0 \in (0,\infty)$.

Keywords: Difference equations, Recursive sequences, Period 30 solutions.
2010 AMS: 39A10

[1] *Department of Management Information Systems, School of Applied Science, Istanbul Aydin University, Istanbul, Turkey, ORCID: 0000-0002-3264-4340*
[2] *Department of Engineering Basic Sciences, Faculty of Engineering and Natural Sciences, Konya Technical University, Konya, Turkey, ORCID: 0000-0003-3003-807X*
***Corresponding author**: ogul.burak@hotmail.com
Received: 21 October 2020, **Accepted:** 27 March 2021, **Available online:** 29 March 2021

1. Introduction

Difference equation is a very diverse field that is effective in almost every branch of applied mathematics. Recently, researchers have shown great interest in studying the behavior of solutions of nonlinear difference equations. Difference equations are used in many fields such as population biology, economics, probability theory, genetics, psychology, mathematical modeling. There are many articles on difference equations, for example; [24]-[28]

Cinar, studied the following problem with positive initial values:

$$x_{n+1} = \frac{x_{n-1}}{-1+ax_nx_{n-1}},$$

for $n = 0,1,2,\ldots$ in [2] respectively.

Simsek et. al., studied the following problems with positive initial values,

$$x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}},$$

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$$

for $n = 0, 1, 2, ...$ in [5]-[7] respectively.

Elsayed studied the behavior of the solution of the following difference equation,

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n dx_{n-2}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-2}x_{-1}, x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. [15]

Devault et. al. studied the following problems

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$$

for $n = 0, 1, 2, ...$ in [23] and showed every positive solution of the equation where $A \in (0, \infty)$.

Stevic et. al. studied on a product-type system of difference equations of second order solvable in closed form in [28]. Shown that the following system of difference equations

$$z_{n+1} = \frac{z_n^a}{w_{n-1}^b}, w_{n+1} = \frac{w_n^c}{z_{n-1}^d}, n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$ is solvable in closed form.
In this work, the following non-linear difference equation was studied

$$x_{n+1} = \frac{x_{n-29}}{1 + x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24}} \tag{1.1}$$

where $x_{-29}, x_{-28}, ..., x_{-1}, x_0 \in (0, \infty)$.

2. Main Results

Let $\bar{x}$ be the unique positive equilibrium of the 1.1, then clearly,

$$\bar{x} = \frac{\bar{x}}{1 + \overline{xxxxx}} \Rightarrow \bar{x} + \bar{x}^6 = \bar{x} \Rightarrow \bar{x}^6 = 0 \Rightarrow \bar{x} = 0,$$

so $\bar{x} = 0$ can be obtained. For any $k \geq 0$ and $m > k$ notation $i = \overline{k, m}$ means $i = k, k+1, ..., m$

Theorem 2.1. *Consider the difference equation 1.1. Then the following statements are true.*

a) The sequences $x_{30n-29}, x_{30n-28}, ..., x_{30n-1}, x_{30n}$ are being decreased and

$$a_1, a_2, ..., a_{29}, a_{30} \geq 0$$

are existed in such that

$$\lim_{n\to\infty} x_{30n-29+k} = a_{1+k}, \quad k = \overline{0, 29}.$$

b)

$$\prod_{k=0}^{6} \lim_{n\to\infty} x_{35n-34-j+5k} = 0, \quad j = \overline{0, 4} \quad \text{or} \quad \prod_{k=0}^{6} a_{5k+i} = 0, \quad i = \overline{1, 5}.$$

c) $n_0 \in \mathbb{N}$ such that $x_{n+1} \leq x_{n-24}$ for all $n \geq n_0$, then

$$\lim_{n\to\infty} x_n = 0.$$

d) The following formulas below are hold:

$$x_{30n+1+k} = x_{-29+k}\left(1 - \frac{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}} \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1 + x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+6+k} = x_{-24+k}\left(1 - \frac{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+11+k} = x_{-19+k}\left(1 - \frac{x_{-4+k}x_{-9+k}x_{-14+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+16+k} = x_{-14+k}\left(1 - \frac{x_{-4+k}x_{-9+k}x_{-19+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+21+k} = x_{-9+k}\left(1 - \frac{x_{-4+k}x_{-14+k}x_{-19+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+26+k} = x_{-4+k}\left(1 - \frac{x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$k = \overline{0,4}$ *holds.*

e) *If* $x_{30n+1+k} \to a_{1+k} \neq 0$, $x_{30n+6+k} \to a_{6+k} \neq 0$, $x_{30n+11+k} \to a_{11+k} \neq 0$, $x_{30n+16+k} \to a_{16+k} \neq 0$, $x_{35n+21+k} \to a_{21+k} \neq 0$, *then* $x_{30n+26+k} \to a_{26+k} = 0$ *as* $n \to \infty$. $k = \overline{0,4}$.

Proof. a) Firstly, from the 1.1

$$x_{n+1} = \frac{x_{n-29}}{1 + x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24}}$$

is obtained. If $x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24} \in (0,+\infty)$, then $(1 + x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24}) \in ((1,+\infty)$. Since

$$x_{n+1} < x_{n-29},$$

$n \in \mathbb{N}$,

$$\lim_{n\to\infty} x_{30n-29+k} = a_{1+k}, \quad for \quad k = \overline{0,29}$$

existed formulas are obtained.

b) In view of the 1.1,

$$n = 30n \Rightarrow x_{30n+1} = \frac{x_{30n-29}}{1 + \prod_{k=0}^{5} x_{30n-29+5k}}$$

is obtained. If the limits are put on both sides of the above equality,

$$\prod_{k=0}^{6} \lim_{n\to\infty} x_{35n-34+5k} = 0 \quad \text{or} \quad \prod_{k=0}^{6} a_{5k+1} = 0$$

is obtained. Similarly for $n = 30n+1$, $n = 30n+2$, $n = 30n+3$ and $n = 30n+4$ we can obtain x_{30n+2}, x_{30n+3}, x_{30n+4} and x_{30n+5}.

c) If there exist $n_0 \in \mathbb{N}$ such that $x_{n+1} \leq x_{n-24}$ for all $n \geq n_0$, then, $a_1 \leq a_6 \leq a_{11} \leq a_{16} \leq a_{21} \leq a_{26} \leq a_1$, $a_2 \leq a_7 \leq a_{12} \leq a_{17} \leq a_{22} \leq a_{27} \leq a_2$, $a_3 \leq a_8 \leq a_{13} \leq a_{18} \leq a_{23} \leq a_{28} \leq a_3$, $a_4 \leq a_9 \leq a_{14} \leq a_{19} \leq a_{24} \leq a_{29} \leq a_4$, $a_5 \leq a_{10} \leq a_{15} \leq a_{20} \leq a_{25} \leq a_{30} \leq a_5$. Using (b) we get

$$\prod_{k=0}^{6} a_{5k+i} = 0, \quad i = \overline{1,5}.$$

Then we see that,

$$\lim_{n\to\infty} x_n = 0.$$

Hence the proof of (c) completed.

d) Subtracting x_{n-29} from the left and right-hand sides in 1.1

$$x_{n+1} - x_{n-29} = \frac{1}{1+x_{n-4}x_{n-9}x_{n-14}x_{n-19}x_{n-24}}(x_{n-4} - x_{n-34})$$

is obtained and the following formula is produced below, for $n \geq 5$

$$\begin{aligned}
x_{5n-24} - x_{5n-54} &= (x_1 - x_{-29}) \prod_{i=1}^{n-5} \frac{1}{1+x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}} \\
x_{5n-28} - x_{5n-53} &= (x_2 - x_{-28}) \prod_{i=1}^{n-5} \frac{1}{1+x_{5i-3}x_{5i-8}x_{5i-13}x_{5i-18}x_{5i-23}} \\
x_{5n-27} - x_{5n-52} &= (x_3 - x_{-27}) \prod_{i=1}^{n-5} \frac{1}{1+x_{5i-2}x_{5i-7}x_{5i-12}x_{5i-17}x_{5i-22}} \\
x_{5n-26} - x_{5n-51} &= (x_4 - x_{-26}) \prod_{i=1}^{n-5} \frac{1}{1+x_{5i-1}x_{5i-6}x_{5i-11}x_{5i-16}x_{5i-21}} \\
x_{5n-25} - x_{5n-50} &= (x_5 - x_{-25}) \prod_{i=1}^{n-5} \frac{1}{1+x_{5i}x_{5i-5}x_{5i-10}x_{5i-15}x_{5i-20}}.
\end{aligned} \tag{2.1}$$

$6j$ inserted in 2.1 by replacing n, $j = 0$ to $j = n$ is obtained by summing, for $k = \overline{0,4}$

$$x_{30n+1+k} - x_{-29+k} = (x_{1+k} - x_{-29+k}) \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}.$$

Also, $6j+1$ inserted in 2.1 by replacing n, $j = 0$ to $j = n$ is obtained by summing, for $k = \overline{0,4}$

$$x_{30n+6+k} - x_{-24+k} = (x_{6+k} - x_{-24+k}) \sum_{j=0}^{n} \prod_{i=1}^{6j+1} \frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}.$$

Also, $6j+2$ inserted in 2.1 by replacing n, $j = 0$ to $j = n$ is obtained by summing, for $k = \overline{0,4}$

$$x_{30n+11+k} - x_{-19+k} = (x_{11+k} - x_{-19+k}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}.$$

Also, $6j+3$ inserted in 2.1 by replacing n, $j = 0$ to $j = n$ is obtained by summing, for $k = \overline{0,4}$

$$x_{35n+16+k}-x_{-14+k}=(x_{16+k}-x_{-14+k})\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}.$$

Also, $6j+4$ inserted in 2.1 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,4}$

$$x_{30n+21+k}-x_{-9+k}=(x_{21+k}-x_{-9+k})\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}.$$

Also, $6j+5$ inserted in 2.1 by replacing n, $j=0$ to $j=n$ is obtained by summing, for $k=\overline{0,4}$

$$x_{30n+26+k}-x_{-4+k}=(x_{26+k}-x_{-4+k})\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}.$$

Now we obtained of the above formulas:

$$x_{30n+1+k}=x_{-29+k}\left(1-\frac{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+6+k}=x_{-24+k}\left(1-\frac{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+11+k}=x_{-19+k}\left(1-\frac{x_{-4+k}x_{-9+k}x_{-14+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+16+k}=x_{-14+k}\left(1-\frac{x_{-4+k}x_{-9+k}x_{-19+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+21+k}=x_{-9+k}\left(1-\frac{x_{-4+k}x_{-14+k}x_{-19+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$$x_{30n+26+k}=x_{-4+k}\left(1-\frac{x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}x_{-29+k}}{x_{-4+k}x_{-9+k}x_{-14+k}x_{-19+k}x_{-24+k}}\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{5i-4+k}x_{5i-9+k}x_{5i-14+k}x_{5i-19+k}x_{5i-24+k}}\right),$$

$k=\overline{0,4}$ holds.

e) Suppose that $a_1 = a_6 = a_{11} = a_{16} = a_{21} = a_{26} = 0$. By (d), the following formulas are produced below

$$\lim_{n\to\infty} x_{30n+1} = \lim_{n\to\infty} x_{-29}\left(1 - \frac{x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}} \sum_{j=0}^{n}\prod_{i=1}^{6j} \frac{1}{1+x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}\right)$$

$$a_1 = x_{-29}\left(1 - \frac{x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}} \sum_{j=0}^{\infty}\prod_{i=1}^{6j} \frac{1}{1+x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}x_{5i-29}}\right)$$

$$a_1 = 0 \Rightarrow \frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}. \tag{2.2}$$

Similarly,

$$a_6 = 0 \Rightarrow \frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-19}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}. \tag{2.3}$$

Similarly,

$$a_{11} = 0 \Rightarrow \frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}. \tag{2.4}$$

Similarly,

$$a_{16} = 0 \Rightarrow \frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}. \tag{2.5}$$

Similarly,

$$a_{21} = 0 \Rightarrow \frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-14}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}. \tag{2.6}$$

Similarly,

$$a_{26} = 0 \Rightarrow \frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-9}x_{-14}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+5} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}. \tag{2.7}$$

From 2.2 and 2.3

$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}} >$$
$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-19}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1} \frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}$$

thus, $x_{-29} > x_{-24}$. From 2.3 and 2.4

$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-19}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}} >$$
$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}$$

thus, $x_{-24} > x_{-19}$. From 2.4 and 2.5

$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-14}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}} >$$
$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}$$

thus, $x_{-19} > x_{-14}$. From 2.5 and 2.6

$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-9}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}} >$$
$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-14}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}$$

thus, $x_{-14} > x_{-9}$. From 2.6 and 2.7

$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-4}x_{-14}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}} >$$
$$\frac{1+x_{-4}x_{-9}x_{-14}x_{-19}x_{-24}}{x_{-9}x_{-14}x_{-19}x_{-24}x_{-29}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+5}\frac{1}{x_{5i-4}x_{5i-9}x_{5i-14}x_{5i-19}x_{5i-24}}$$

thus, $x_{-9} > x_{-4}$.
From here we obtain $x_{-29} > x_{-24} > x_{-19} > x_{-14} > x_{-9} > x_{-4}$. Similarly, we can obtain $x_{-28} > x_{-23} > x_{-18} > x_{-13} > x_{-8} > x_{-3}$, $x_{-27} > x_{-22} > x_{-17} > x_{-12} > x_{-7} > x_{-2}$, $x_{-26} > x_{-21} > x_{-16} > x_{-11} > x_{-6} > x_{-1}$ and $x_{-25} > x_{-20} > x_{-15} > x_{-10} > x_{-5} > x_0$. We arrive at a contradiction which completes the proof of theorem.

□

3. Conclusion

In this study, the theorem is given for the 1.1, and its solution and periodicity are investigated. By taking the coefficients of the 1.1, real numbers, sequence or function, new equations can be defined and their solutions can be examined.

References

[1] A.M. Amleh, G.A. Grove, G. Ladas, D.A. Georgiou, *On the recursive sequence* $y_{n+1} = \alpha + \frac{y_{n-1}}{y_n}$ J. of Math. Anal. App., **233**, (1999), 790-798.

[2] C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{x_{n-1}}{-1+\alpha x_n x_{n-1}}$, J. of App. Math. Comp., **158**(3), (2004), 793-797.

[3] C. Cinar, T. Mansour, I. Yalcinkaya, *On the difference equation of higher order*, Utilitas Mathematica, **92**, (2013), 161-166.

[4] C.H. Gibbons, M.R.S. Kulenovic, G. Ladas, *On the recursive sequence* $\frac{\alpha+\beta xn-1}{\chi+\beta xn-1}$, Math. Sci. Res. Hot-Line, **4**(2), (2000), 1-11.

[5] D. Simsek, C. Cinar, I. Yalcinkaya, *On the recursive sequence* $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, Int J. Contemp., **9**(12), (2006), 475-480.

[6] D. Simsek, C. Cinar, I. Yalcinkaya, *On the recursive sequence* $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}}$, Int J. Pure Appl. Math., **27**, (2006), 501-507.

[7] D. Simsek, C. Cinar, I. Yalcinkaya, *On the recursive sequence* $x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$, Int J. Pure Appl. Math., **28**, (2006), 117-124.

[8] D. Simşek, B. Ogul, C. Cinar, *Solution of the rational difference equation* $x_{n+1} = \frac{x_{n-17}}{1+x_{n-5}x_{n-11}}$, Filomat, **33**(5), (2019), 1353-1359.

[9] D. Simsek, B. Ogul, F. Abdullayev, *Solution of the Rational Difference Equation* $x_{n+1} = \frac{x_{n-13}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}}$, Applied Mathematics and Nonlinear Sciences, **5**(1), (2020), 485-494.

[10] E.M. Elsayed, *On the solution of some difference equation*, Europan Journal of Pure and Applied Mathematics, **4**(3), (2011), 287-303.

[11] E.M. Elsayed, *On the Dynamics of a higher order rational recursive sequence*, Communications in Mathematical Analysis, **12**(1), (2012), 117-133.

[12] E.M. Elsayed, *Solution of rational difference system of order two*, Mathematical and Computer Modelling, **5**, (2012), 378-384.

[13] E.M. Elsayed, *Behavior and expression of the solutions of some rational difference equations*, Journal of Computational Analysis Applications, **15**(1), (2013), 73-81.

[14] E.M. Elsayed, *Solution of rational difference system of order two*, Journal of Computational Analysis Applications, **33**(3), (2014), 751-765.

[15] E.M. Elsayed, *Solution and attractivity for a rational recursive sequence*, Discrete Dynamics in Nature and Society, **2011**, (2011), 1-17.

[16] E.M. Elabbasy, H. El-Metwally, E.M. Elsayed, *On the difference equation* $x_{n+1} = \frac{\alpha_x n_k}{\beta} + \gamma\pi = 0$, Journal of Concrete Applicable Mathematics, **5**(2), (2007),101-103.

[17] E.M. Erdogan, C. Cinar, I. Yalcinkaya, *On the dynamics of the recursive sequence*, Mathematical and Computer Modelling, **54**(5), (2011), 1481-1485.

[18] F. Belhannache, N. Touafek, R. Abo-Zeid, *On a higher-order rational difference equation* J. Appl. Math. Informatics, **34**(5-6), (2016), 369-382.

[19] M. Lui, Z. Guo, *Solvability of a higher-order nonlinear neutral delay difference equation*, Advances in Difference Equations, **2010**(1), (2010), 620-627.

[20] M. Ari, A. Gelisken, *Periodic and asymptotic behavior of a difference equation*, Asian-European Journal of Mathematics, **12**(6), (2019), 2040004.

[21] M.R.S. Kulenovic, G. Ladas, W.S. Sizer, *On the recursive sequence* $\frac{\alpha x_n+\beta x_{n-1}}{\chi x_n+\beta x_{n-1}}$, Math. Sci. Res.Hot-Line, **2**(5), (1998), 1-16.

[22] O. Moaaz, D. Chalishajar, O. Bazighifan, *Some qualitative behavior of solutions of general class of difference equations*. Mathematics, **7**(7), (2019), 585.

[23] R. DeVault, G. Ladas, S.W. Schultz, *On the recursive sequence* $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, Proc.Amer. Math. Soc., **126**(11), (1998), 3257-3261.

[24] R.P. Agarwal, E.M. Elsayed, *On the solution of fourth-order rational recursive sequence*, Advanced Studies in Contemporary Mathematics, **20**(4),(2010), 525-545.

[25] R. Karatas, C. Cinar, D. Simsek, *On Positive Solutions of the Difference Equation* $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci, **10**(1), (2006), 495-500.

[26] R. Karatas, *Global behavior of a higher order difference equation*, Int. J. Contemp. Math. Sci., **12**(3), (2017), 133-138.

[27] S.E. Das, M. Bayram, *On a system of rational difference equations*, World Applied Sciences Journal, **10**(11), (2010), 1306-1312.

[28] S. Stevic, B. Iricanin, Z. Smarda, *On a product-type system of difference equations of second order solvable in closed form*, Journal of Inequalities and Applications, **2015**(1), (2012), 327-334.

[29] T.F. Ibrahim, *Behavior of some higher order nonlinear rational partial difference equations*, Journal of the Egyptian Mathematical Society, **24**(4), (2016), 532-537.

[30] T.F. Ibrahim, *Bifurcation and periodically semicycles for fractional difference equation of fifth order* Journal of Nonlinear Sciences and Applications, **11**(3), (2018), 375-382.