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COMPACTIFICATIONS OF A FIXED SET

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Abstract. If a Tychonoff space is fixed, then we may consider all possible Hausdorff compactifications of the space. If an infinite set is fixed, then we may vary Tychonoff topologies on the set and the compactifications may also be varied.Magills construction for compactifications of a fixed Tychonoff space through partitions is applied to derive compactifications of various Tychonoff spaces (X, τ), with a fixed set X and with a variation in Tychonoff topologies τ. The structure of required partitions is also analyzed. When topologies are varied, some possible extensions of mappings are obtained in this regard.

1. Introduction

Compactification of a space X is a compact space containing X as a dense subspace. If a Tychonoff space is fixed, then we may consider all possible Hausdorff compactifications of the space. If an infinite set is fixed, then we may vary Tychonoff topologies on the set and the compactifications may also be varied. Magill's [10] construction of compactifications through partitions is improved in the second section of this article, when topologies are also varied. The structure of required partitions is also analyzed in the second section. In a compact extension of a topological group, the inverse operation should be extendable homeomorphically from the base topological group (See:[1]). The third section of this article is to study such extensions of mappings, when topologies are also varied. The authors have also contributed a classical work for compactifications including order relations (See: [11], [13], [14], [15]). Recent works are also available in literature regarding compactifications and lattice structure of a collection of compactifications (See: [2], [3], [7]). The major application of Hausdorff compactifications is obtaining completeness under all uniformities inducing same topologies, apart from

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other applications (See: [4], [6], [8] and [9]). All definitions which are not defined here are followed from [12].

2. Set Fixation

Let us fix an infinite set X. We consider the collection of all (Hausdorff) Tychonoff topologies on X. If τ_1 and τ_2 are two Tychonoff topologies on X, then we write $\tau_1 \leq \tau_2$ if $\tau_1 \subseteq \tau_2$. The supremum of any collection of Tychonoff topologies does exist and it is also a Tychonoff space. Let (Y_1, τ_1') and (Y_2, τ_2') be two Hausdorff compactifications of (X, τ_1) and (X, τ_2) respectively, when X is fixed. Then we write $(Y_1, \tau_1') \geq (Y_2, \tau_2')$ if there is a continuous function f from (Y_1, τ_1') onto (Y_2, τ_2') such that $f(x) = x$, for all $x \in X$. In this case, $\{f^{-1}(y) : y \in Y_2\}$ form a partition for Y_1 by compact subsets of Y_1. Moreover, each $x \in X$ is in at most one partitioning set $f^{-1}(y)$. That is, $f^{-1}(y) \cap X$ is either an empty set or a singleton set, for every $y \in Y_2$.

On the other hand, let us consider a partition π for Y_1 by compact subsets of a compactification (Y_1, τ_1') of (X, τ_1) such that the following is true: To each $A \in \pi$, $A \cap X$ is either an empty set or a singleton set. Define $Y_2 = Y_1/\pi$, and let $f : Y_1 \to Y_1/\pi = Y_2$ be the natural quotient mapping. Endow Y_2 with the quotient topology τ_2' corresponding to the quotient mapping f. Then (Y_2, τ_2') is a compact space, which may not be Hausdorff. However we have the following Result 2.1 on Hausdorffness. A variation of the Theorem 2.1 may be found in [12, Problem 4Q]. Note that if $A \in \pi$ is such that $A \cap X$ is a singleton set $\{x\}$, say, then x is identified with $f(A)$ as an element of Y_2. In this way, X is considered as a dense subset of (Y_2, τ_2').

Theorem 2.1. *(Y_2, τ_2') is a Hausdorff compactification of (X, τ_2) (for some τ_2), if and only if for a given τ_1'-open subset U containing a given $A \in \pi$, there is a τ_1'-open set V, which is a union of members of π, such that $A \subseteq V \subseteq U$.*

Proof. Suppose (Y_2, τ_2') is Hausdorff. Let $A \in \pi$ and U be a τ_1'-open set containing A. Then $f(Y_1 \backslash U)$ is a τ_2'-compact subset of Y_2, because f is continuous. It is a τ_2'-closed set, because (Y_2, τ_2') is Hausdorff. Then $f(A) \in Y_2 \backslash (f(Y_1 \backslash U))$ or $A \subseteq f^{-1}(Y_2 \backslash (f(Y_1 \backslash U))) \subseteq U$, where $V = f^{-1}(Y_2 \backslash (f(Y_1 \backslash U)))$ is a τ_1'-open set, which is a union of members of π.

To prove the converse part, consider two distinct members $A, B \in \pi$. Since (Y_1, τ_1') is normal, there are disjoint τ_1'-open sets U_1 and V_1 such that $A \subseteq U_1$ and $B \subseteq V_1$. Then there are τ_1'-open sets U_2 and V_2, which are unions of members of π such that $A \subseteq U_2 \subseteq U_1$ and $B \subseteq V_2 \subseteq V_1$. Then $f(U_2)$ and $f(V_2)$ are two disjoint τ_2'-open sets of Y_2 such that $f(A) \in f(U_2)$ and $f(B) \in f(V_2)$. This proves the Hausdorffness of (Y_2, τ_2'). □

Let us now give a sufficient condition for a partition to obtain a Hausdorff compactification.

Theorem 2.2. *If the subfamily of all non singleton members of π is a locally finite family in (Y_1, τ_1'), then (Y_2, τ_2') is a Hausdorff compactification of (X, τ_2), for some Hausdorff topology τ_2 in X .*

Proof. To prove the Hausdorffness of (Y_2, τ_2'), consider two distinct elements y_1, y_2 in Y_2. Then there are $A, B \in \pi$ such that $A = f^{-1}(y_1)$ and $B = f^{-1}(y_2)$, respectively. For any $x \in A$, there is a τ_1'-open set U_x of x, which intersects only a finite number of non singleton members $C_1, C_2 \cdots C_n$ of π such that $\overline{U_x} \cap B = \phi$ and $C_i \neq A$, for every i. Define a τ_1'-open set $V_x = U_x \backslash (\overset{n}{\underset{i=1}{\cup}} C_i)$ containing x. Then $\{V_x : x \in A\}$ is an open cover of A and this cover has a finite subcover $\{V_{x_1}, V_{x_2}, \cdots V_{x_m}\}$, say. Let $U = \overset{m}{\underset{i=1}{\cup}} V_{x_i}$. Then U is a τ_1'-open set such that $A \subseteq U$; $\overline{U} \cap B = \phi$, and such that U is a union of members of π. Similarly, we can find a τ_1'-open set V such that $B \subseteq V$, $\overline{U} \cap \overline{V} = \phi$, and such that V is a union of members of π. Then $f(U)$ and $f(V)$ are disjoint τ_2'-open sets in Y_2 such that $f(A) \in f(U)$ and $f(B) \in f(V)$. This proves the Hausdorffness of (Y_2, τ_2'). $\square$

This Theorem 2.2 generalizes Lemma 2.1 in [10].

If we fix an infinite set X, vary Tychonoff topologies τ on X and vary (Hausdorff) compactifications (Y, τ') of (X, τ), then we obtain a complete upper semi-lattice $\mathcal{L}(X)$ under the relation " $\geq$ " defined above, that relates two compactifications. The largest element of this semi-lattice is the Stone-Čech compactification of X endowed with the discrete topology.

If a Tychonoff topology τ is fixed in X, then the collection $\mathcal{L}(X, \tau)$ of all compactifications of (X, τ) is a complete upper semi sublattice of $\mathcal{L}(X)$.

If $((X, \tau_i))_{i \in I}$ is a collection of Tychonoff topologies on an infinite set X, τ^* is the supremum of $(\tau_i)_{i \in I}$, and (Y_i, τ_i') is a compactification of (X, τ_i), for every $i \in I$, then the supremum of $(Y_i, \tau_i')_{i \in I}$ is of the form $(Y, {\tau^*}')$, where (X, τ^*) is a topological dense subspace of $(Y, {\tau^*}')$. Here $(Y, {\tau^*}')$ is the closure of the natural embedding of X into the Cartesian product $\prod_{i \in I} Y_i$, with the product topology. So, the mapping f from $\mathcal{L}(X)$ onto the complete upper semi-lattice of Tychonoff topologies on X, defined by $f((Y, \tau))$ = the subspace topology of τ on X, is an order preserving mapping and a join preserving mapping. This discussion leads to a convex structure of $\mathcal{L}(X, \tau)$ and a congruence relation through f (See: [5, p.17 and p.20]).

3. Self Extendable Mappings

Theorem 3.1. *Let (X, τ) be a locally compact Hausdorff space and (Y, τ') be its one point compactification, where $Y = X \cup \{\infty\}$, say. Let $h : (X, \tau) \to (X, \tau)$ be an onto homeomorphism. Then h has a unique homeomorphic extension $h' : (Y, \tau') \to (Y, \tau')$, and in this case $h'(\infty) = \infty$.*

Proof. Define $h'(\infty) = \infty$ and $h'(x) = h(x)$, for all $x \in X$. Fix a compact subset K of X. Then $h(K)$ and $h^{-1}(K)$ are compact subsets of X, and $h(X \backslash K)$ and $h^{-1}(X \backslash K)$ are open subsets of X. So h' and h'^{-1} are continuous at ∞. The continuity of h' and h'^{-1} at any point of X follows from the fact that X is open in (Y, τ'). This completes the proof. $\square$

Theorem 3.2. *Let $((X, \tau_i))_{i \in I}$ be a collection of Tychonoff spaces and $((Y_i, \tau_i'))_{i \in I}$ be a collection such that*

- (i) *Each (Y_i, τ_i') is a compactification of (X, τ_i).*
- (ii) *For any continuous mapping $h_i : (X, \tau_i) \to (X, \tau_i)$, there is a continuous extension $h_i' : (Y_i, \tau_i') \to (Y_i, \tau_i')$.*

Let $h : X \to X$ be a mapping such that $h : (X, \tau_i) \to (X, \tau_i)$ is continuous, for every $i \in I$. Then there is a continuous mapping $h' : (Y, {\tau^*}') \to (Y, {\tau^*}')$, that is an *extension of h, where* τ^* *is the supremum of* $(\tau_i)_{i\in I}$ *and* $(Y, \tau^{*'})$ *is the supremum of* $((Y_i, \tau_i'))_{i\in I}$.

Proof. Let $h_i' : (Y_i, \tau_i') \to (Y_i, \tau_i')$ be the continuous extension of $h : (X, \tau_i) \to (X, \tau_i)$. Define $H : \prod_{i\in I}(Y_i, \tau_i') \to \prod_{i\in I}(Y_i, \tau_i')$ by $H((y_i)_{i\in I}) = (h_i'(y_i))_{i\in I}$. Then H is continuous. Then the required $h' : (Y, \tau^{*'}) \to (Y, \tau^{*'})$ is the restriction of H to $(Y, \tau^{*'})$, where $(Y, \tau^{*'})$ is considered as a subspace of $\prod_{i\in I}(Y_i, \tau_i')$ as in Section 2. □

Remark. Suppose (ii) in Proposition 3.2 is replaced by

(ii)' For any surjective homeomorphism $h_i : (X, \tau_i) \to (X, \tau_i)$, there is a unique homeomorphic (or continuous) extension $h_i' : (Y_i, \tau_i') \to (Y_i, \tau_i')$.

Assume that $h : X \to X$ is a one to one and onto mapping such that $h_i : (X, \tau_i) \to (X, \tau_i)$ is an onto homeomorphism, for every $i \in I$. Then there is a homeomorphic (or continuous) mapping $h' : (Y, \tau^{*'}) \to (Y, \tau^{*'})$, that is an extension of h, for $(Y, \tau^{*'})$ given in Proposition 3.2.

Proof. If each h_i' is a homeomorphism, then H defined in the proof of the Proposition 3.2 is a homeomorphism. □

4. Conclusion

For a fixed infinite set, we may vary Tychonoff topologies on the set and the compactifications may also be varied. Magill's [10] construction of compactifications through partitions is improved and the structure of required partitions is also analyzed . In a compact extension of a topological group, the inverse operation should be extendable homeomorphically from the base topological group (See:[1]). Finally mappings are extended homeomorphically from topological space to its compact extension, when topologies are also varied.

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REDUCTION METHOD FOR FUNCTIONAL NONCONVEX DIFFERENTIAL INCLUSIONS

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Abstract. Our aim in this paper is to present a reduction method that solves first order functional differential inclusion in the nonconvex case. This approach is based on a discretization of the time interval, a construction of approximate solutions by reducing the problem to a problem without delay and an application of known results in this case. We generalize earlier results, the right hand side of the inclusion has nonconvex values and satisfies a linear growth condition instead to be integrably bounded. The lack of convexity is replaced by the topological properties of decomposable sets, that represents a good alternative in the absence of convexity.

1. Introduction

Let $\tau,\ T\ \geq 0$, be two non-negative real numbers, $\mathcal{C}_T := \mathcal{C}_{\mathbb{R}^n}([-\tau, T])$ is the Banach space of all continuous mappings from $[-\tau,\ 0]$ to $\mathbb{R}^n$ equipped with the norm of uniform convergence and $F : [0, T] \times \mathcal{C}_0 \rightharpoondown \mathbb{R}^n$ be a set-valued mapping with nonempty closed values. In this work, we study the existence of solutions for the following differential inclusion with delay

$$(\mathcal{DP}) \qquad \begin{cases} \dot{x}(t) \in F(t, \mathcal{T}(t)x) & \text{a.e. } t \in [0, T]; \\ x(t) = \varphi(t) & t \in [-\tau, 0]; \end{cases}$$

where $\varphi \in \mathcal{C}_0$ and $\mathcal{T}(t) : \mathcal{C}_T \longrightarrow \mathcal{C}_0$ defined by $\mathcal{T}(t)x(s) = x(t+s), \quad \forall s \in [-\tau,\ 0\ ]$. In [11], Fryszkowski proved an existence result for $(\mathcal{DP})$ when F is a set-valued mapping with nonconvex values, measurable, integrably bounded and lower semicontinuous in x. The proof of this theorem is based on the construction of a continuous selection for a class of nonconvex set-valued mapping. The existence of such selection is proved in [10]. In [12], Fryszkowski and Gorniewicz proved an

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existence result for differential inclusion of the form

$$(\mathcal{P}) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e.} \quad t \in [0, T] \\ x(0) = x_0, \end{cases}$$

where F is a set-valued mapping measurable in (t, x) and lower semicontinuous in x with nonconvex values, satisfying a linear growth condition. The main tool used in their proof is a continuous selection theorem for the set-valued mapping

$$K_F(x) = \{y \in L^1_{\mathbb{R}^n}([0, T]) : y(t) \in F(t, x(t)) \quad \text{a.e. on } [0, T]\},$$

which is well defined on $\mathcal{C}_{\mathbb{R}^n}([0, T])$ and is lower semicontinuous with decomposable values. Decomposable sets represent a good alternative in the absence of convexity. Our aim in this work, is to prove a general existence result for $(\mathcal{DP})$, where F satisfies a linear growth condition instead to be integrably bounded, that is

$$\|y\| \leq (\ 1 + \|\varphi\|\ )\, \rho(t), \text{ for every } y \in F(t, \varphi) \text{ and } (t, \varphi) \in [0, T] \times \mathcal{C}_0.$$

We extend also the existence result for the Cauchy problem without delay in the nonconvex case. Some applications have been obtained by considering such delayed set-valued mapping as perturbations (external forces applied) on systems governed by subdifferential operators, particularly in the case of the so-called Sweeping process, see for instance [7], [8]. We refer to [1]-[3] for recent results, [4], [5] and [13] for other approaches. The paper is organized as follows. In Section 2, we recall concepts and preliminaries needed in the paper. In Section 3, we provide the existence result for problem $(\mathcal{DP})$.

2. Preliminaries

Throughout the paper, we will use the following notations and definitions. Let $\mathbb{R}^n$ be the n dimensional Euclidean space and $\|\cdot\|$ its norm. $\mathcal{C}_T := \mathcal{C}_{\mathbb{R}^n}(\ [-\tau, T])$ is the Banach space of all continuous mappings from $[-\tau, T]$ to $\mathbb{R}^n$ endowed with the sup-norm, $L^1_{\mathbb{R}^n}([0, T])$ is the Banach space of all measurable mappings from $[0, T]$ to $\mathbb{R}^n$. Let $\mathcal{B}(\mathcal{C}_0)$ be the σ-algebra of Borel sets of $\mathcal{C}_0$ and $\mathcal{L}$ the σ-algebra of Lebesgue measurable subsets of $[0, T]$, $d(x, A)$ mean the usual distance from a point x to a set A, i.e., $d(x, A) := \inf_{y \in A} \|x - y\|$. A set-valued mapping $F : [0, T] \times \mathcal{C}_0 \rightharpoonup \mathbb{R}^n$ is integrably bounded if there exists an integrable function $\rho : [0, T] \to \mathbb{R}^+$ such that

$$\|F(t, \varphi)\| := sup\{\ \|y\|;\ y \in F(t, \varphi)\ \} \leq \rho(t),\ t \in\ [0, T],\ \varphi \in \mathcal{C}_0.$$

Definition 2.1. *([6]) Let X and Y be two topological spaces, $F : X \rightharpoonup Y$ a set-valued mapping with closes valued, is called lower semicontinuous (lsc for short) at a point $x_0 \in X$ if for any $y_0 \in F(x_0)$ and any neighborhood U of y_0 such that $F(x_0) \cap U \neq \emptyset$, there exists a neighborhood $V(x_0)$ of the point x_0 such that $F(x_0) \cap U \neq \emptyset$ for all $x \in V(x_0)$. A set-valued mapping F is said to be lower semicontinuous if it is so at every point $x_0 \in X$.*

If X and Y are metric spaces, it's equivalent to say: for each $x_0 \in [0, T]$ and $y_0 \in F(x_0)$ and any sequence $x_n \longrightarrow x_0$ there is $y_n \in F(x_n)$ such that $y_n \longrightarrow y_0$.

Lemma 2.1. *(Gronwall inequality) Let $u, v : [t_0, t_1] \longrightarrow \mathbb{R}^+$ two continuous functions such that, for any $C \geq 0$, we have*

$$u(t) \leq C + \int_{t_0}^{t} u(s)v(s)ds, \quad \forall t \in [t_0, t_1].$$

Then

$$u(t) \leq C \exp(\int_{t_0}^{t} v(s)ds), \quad \forall t \in [t_0, t_1].$$

3. Existence of solutions

In this section, we begin by the following result for the undelayed problem due to Fryszkowski and Gorniewicz (see [12]).

Theorem 3.1. *Let $G : [0,T] \times \mathbb{R}^n \rightharpoonup \mathbb{R}^n$ be a set-valued mapping with nonempty closed values satisfying*

(i) *G is $\mathcal{L} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable;*
(ii) *for every $t \in [0,T]$, $G(t,\cdot)$ is lsc in $\mathbb{R}^n$;*
(iii) *there exists an integrable function $\rho : [0,T] \longrightarrow \mathbb{R}^+$ such that*

$$\|y\| \leq (\ 1 + |x|\ )\, \rho(t), \text{ for every } y \in G(t,x) \text{ and } (t,x) \in [0,T] \times \mathbb{R}^n.$$

Then, $\forall\ x_0 \in \mathbb{R}^n$, the problem

$$\begin{cases} \dot{x}(t) \in G(t,x(t)) & a.e. \ on\ [0,T]; \\ x(0) = x_0; \end{cases} \tag{3.1}$$

admits at least one solution $x : [0,T] \to \mathbb{R}^n$ absolutely continuous on $[0,T]$.

The proof of this theorem is based on a selection theorem for decomposable sets stated in [11].

Now, we are able to give the existence result for the delayed problem.

Theorem 3.2. *Let $F : [0,T] \times \mathcal{C}_0 \rightharpoonup \mathbb{R}^n$ be a set-valued mapping with nonempty closed values such that*

(i) *F is $\mathcal{L} \otimes \mathcal{B}(\mathcal{C}_0)$ measurable;*
(ii) *for every $t \in [0,T]$, $F(t,\cdot)$ is lsc in $\mathcal{C}_0$;*
(iii) *for every $(t,\varphi) \in [0,T] \times \mathcal{C}_0$*

$$\|F(t,\varphi)\| \leq (1 + \|\varphi(0)\|)\rho(t).$$

Then, $\forall\ \varphi \in \mathcal{C}_0$, the problem $(\mathcal{DP})$ admits at least one continuous solution $x : [-\tau,T] \to \mathbb{R}^n$, absolutely continuous on $[0,T]$.

Proof. We will reduce our problem to a problem without delay and apply Theorem 3.1. For simplcity, we take $T = 1$ and consider for every $n \in \mathbb{N}$ a partition of $[0,T]$ defined by $t_i^n = i\mu_n T$, $\mu_n = 2^{-n}$, $i = 0,\ 1,\ \ldots\ldots,\ 2^n$.

Step 1 *Construction of approximate solutions.*

For every $(t,x) \in [-\tau, t_1^n] \times \mathbb{R}^n$, we define $f_0^n : [-\tau, t_1^n] \times \mathbb{R}^n$ by

$$f_0^n(t,x) = \begin{cases} \varphi(t) & \text{if } t \in [-\tau, 0]; \\ \varphi(0) + \frac{t}{\mu_n}(x - \varphi(0)) & \text{if } t \in]0, t_1^n]; \end{cases}$$

clearly $f_0^n(t_1^n, x) = x$, $\forall x \in \mathbb{R}^n$.

We define the set-valued mapping G_0^n on $[0, t_1^n] \times \mathbb{R}^n$ with closed values in $\mathbb{R}^n$ by

$$G_0^n(t,x) := F(t, \mathcal{T}(t_1^n) f_0^n(\cdot, x)) \ \ \forall (t,x) \in [0, t_1^n] \times \mathbb{R}^n.$$

Let show that G_0^n satisfies the conditions of Theorem 3.1. Remark first, that the function $x \longmapsto \mathcal{T}(t_1^n)f_0^n(\cdot,x)$ is Lipschitz. Indeed, for every $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}\|\mathcal{T}(t_1^n)f_0^n(\cdot,x)-\mathcal{T}(t_1^n)f_0^n(\cdot,y)\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau,0]}\|f_0^n(t_1^n+s,x)-f_0^n(t_1^n+s,y)\|\\ &= \sup_{s\in[-\mu_n,0]}\|f_0^n(t_1^n+s,x)-f_0^n(t_1^n+s,y)\|\\ &= \sup_{s\in[-\mu_n,0]}\|\tfrac{t_1^n+s}{\mu_n}(x-y)\|\\ &= \|x-y\|.\end{aligned}$$

The measurability and lower semicontinuity of G follows from that of F. Furthermore, by the condition $iii)$ of Theorem 3.2, we have, for every $t \in [0,t_1^n]$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\|G_0^n(t,x)\| = \|F(t,\mathcal{T}(t_1^n)f_0^n(\cdot,x))\| &\leq (1+\|\mathcal{T}(t_1^n)f_0^n(0,x)\|)\,\rho(t)\\ &= (1+\|f_0^n(t_1^n,x)\|)\,\rho(t)\\ &= (1+\|x\|)\,\rho(t).\end{aligned}$$

Hence G_0^n verifies the conditions of Theorem 3.1, this provides an absolutely continuous solution $v_0^n : [0,t_1^n] \longrightarrow \mathbb{R}^n$ to the problem

$$\left\{\begin{array}{lll} \dot{v}_0^n(t) & \in G_0^n(t,v_1^n(t)) & \text{a.e. on } \ [0,t_1^n];\\ v_0^n(t) & =\varphi(0)+\int_0^t \dot{v}_0^n(s)ds & \forall t \in]0,t_1^n];\\ v_0^n(0) & =\varphi(0). & \end{array}\right.$$

That is, v_0^n is a solution to

$$\left\{\begin{array}{lll} \dot{v}_0^n(t) & \in F(t,\mathcal{T}(t_1^n)f_0^n(\cdot,x)) & a.\ e.\ on\ \ [0,t_1^n];\\ v_0^n(0) & =\varphi(0). & \end{array}\right.$$

Put

$$x_n(t)=\left\{\begin{array}{ll} \varphi(t) & if\ t\in[-\tau,0];\\ v_0^n(t) & if\ t\in]0,t_1^n]. \end{array}\right.$$

As before, for every $(t,x) \in [-\tau,t_1^n]\times\mathbb{R}^n$, we define $f_1^n : [-\tau,t_2^n]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$f_1^n(t,x)=\left\{\begin{array}{ll} x_n(t) & \text{if } \ t\in[-\tau,t_1^n];\\ x_n(t_1^n)+\frac{t-t_1^n}{\mu_n}(x-x_n(t_1^n)) & \text{if } \ t\in]t_1^n,t_2^n]. \end{array}\right.$$

with $f_1^n(t_2^n,x)=x, \ \ \forall x\in\mathbb{R}^n$. Hence, we can define similarly the set-valued mapping G_1^n on $[t_1^n,t_2^n]\times\mathbb{R}^n$ with closed values of $\mathbb{R}^n$ by

$$G_1^n(t,x) := F(t,\mathcal{T}(t_1^n)f_1^n(\cdot,x)) \ \ \forall (t,x)\in[t_1^n,t_2^n]\times\mathbb{R}^n$$

satisfying for every $t\in[t_1^n,t_2^n]$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}\|G_1^n(t,x)\| = \|F(t,\mathcal{T}(t_2^n)f_1^n(\cdot,x))\| &\leq (1+\|\mathcal{T}(t_2^n)f_1^n(0,x)\|)\,\rho(t)\\ &= (\,1+\|f_1^n(t_2^n,x)\|)\,\rho(t)\\ &= (\,1+\|x\|)\,\rho(t).\end{aligned}$$

The function $x \longmapsto \mathcal{T}(t_2^n)f_1^n(\cdot,x)$ is Lipschitz since for all $x,y\in\mathbb{R}^n$ we have

$$\begin{aligned}
\|\mathcal{T}(t_2^n)f_1^n(\cdot,x)-\mathcal{T}(t_2^n)f_1^n(\cdot,y)\| &= \sup_{s\in[-\tau,0]}\|f_1^n(t_2^n+s,x)-f_1^n(t_2^n+s,y)\|\\
&= \sup_{s\in[-\mu_n,0]}\|f_1^n(t_2^n+s,x)-f_1^n(t_2^n+s,y)\|\\
&= \sup_{s\in[-\mu_n,0]}\|x_n(t_1^n)+\tfrac{t_2^n+s-t_1^n}{\mu_n}(x-x_n(t_1^n))\\
&- (x_n(t_1^n)+\tfrac{t_2^n+s-t_1^n}{\mu_n}(y-x_n(t_1^n)))\|\\
&= \sup_{s\in[-\mu_n,0]}\|\tfrac{t_2^n+s-t_1^n}{2^{-n}}(x-y)\|\\
&= \|\tfrac{t_2^n-t_1^n}{\mu_n}(x-y)\|\\
&= \|x-y\|.
\end{aligned}$$

Hence G_1^n verifies the conditions of Theorem 3.1, this provides an absolutely continuous solution $v_1^n:[t_1^n,t_2^n]\longrightarrow\mathbb{R}^n$ to the problem

$$\begin{cases}
\dot v_1^n(t) & \in G_1^n(t,v_1^n(t)) \qquad a.\ e.\ on\ [t_1^n,t_2^n];\\
v_1^n(t) & = x_n(t_1^n)+\int_{t_1^n}^t \dot v_1^n(s)ds \qquad \forall\ t\in]t_1^n,t_2^n];\\
v_1^n(t_1^n) & = x_n(t_1^n).
\end{cases}$$

So v_1^n is a solution of

$$\begin{cases}
\dot v_1^n(t) & \in F(t,\mathcal{T}(t_2^n)f_1^n(\cdot,x)) \qquad \text{a.e. on } [t_1^n,t_2^n];\\
v_1^n(t) & = x_n(t_1^n)+\int_{t_1^n}^t \dot v_1^n(s)ds \qquad \forall\ t\in]t_1^n,t_2^n];\\
v_1^n(0) & = \varphi(0).
\end{cases}$$

By induction, suppose that x_n is defined on $[-\tau,t_k^n]$, absolutely continuous on $[0,t_k^n]$, and satisfies

$$\begin{cases}
\dot x_n(t) & \in F(t,\mathcal{T}(t_{k-1}^n)f_{k-1}^n(\cdot,x)) \qquad \text{a.e. on } [t_{k-1}^n,t_k^n];\\
x_n(t) & = x_n(t_{k-1}^n)+\int_{t_{k-1}^n}^t \dot x_n(s)ds \qquad \forall\ t\in]t_{k-1}^n,t_k^n];
\end{cases}$$

and build a solution on $[t_k^n,t_{k+1}^n]$. For every $(t,x)\in[-\tau,t_1^n]\times\mathbb{R}^n$, we defined $f_k^n:[-\tau,t_{k+1}^n]\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ by

$$f_k^n(t,x)=\begin{cases}
x_n(t) & \text{if } t\in[-\tau,t_k^n];\\
x_n(t_k^n)+\frac{t-t_k^n}{\mu_n}(x-x_n(t_k^n)) & \text{if } t\in]t_k^n,t_{k+1}^n];
\end{cases}$$

with $f_k^n(t_{k+1}^n,x)=x$ and $f_k^n\in\mathcal{C}_{\mathbb{R}^n}([-\tau,t_{k+1}^n])$. The function $x\longmapsto\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,x)$ is Lipschitz. Indeed, for all $x,y\in\mathbb{R}^n$ we have

$$\begin{aligned}
&\|\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,x)-\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,y)\| =\\
&\sup_{s\in[-\tau,0]}\|f_k^n(t_{k+1}^n+s,x)-f_k^n(t_{k+1}^n+s,y)\|\\
&= \sup_{t\in[-\tau+t_{k+1}^n,t_{k+1}^n]}\|f_k^n(t,x)-f_k^n(t,y)\|.
\end{aligned}$$

We distinguish two cases

(1) if $-\tau+t_{k+1}^n\le t_k^n$, we have

$$\begin{aligned}
\sup_{t\in[-\tau+t_{k+1}^n,t_{k+1}^n]}\|f_k^n(t,x)-f_k^n(t,y)\| &= \sup_{t\in[t_k^n,t_{k+1}^n]}\|f_k^n(t,x)-f_k^n(t,y)\|\\
&= \sup_{t\in[t_k^n,t_{k+1}^n]}\|\tfrac{t-t_k^n}{\mu_n}(x-y)\|\\
&= \|x-y\|.
\end{aligned}$$

(2) if $t_k^n \le -\tau + t_{k+1}^n \le t_{k+1}^n$, we have

$$\begin{aligned}\sup_{t\in[-\tau+t_{k+1}^n,t_{k+1}^n]} \|f_k^n(t,x)-f_k^n(t,y)\| &\le \sup_{t\in[t_k^n,t_{k+1}^n]} \|f_k^n(t,x)-f_k^n(t,y)\| \\ &= \sup_{t\in[t_k^n,t_{k+1}^n]} \|\tfrac{t-t_k^n}{\mu_n}(x-y)\| \\ &= \|x-y\|.\end{aligned}$$

Similarly we can define G_k^n on $[t_k^n, t_{k+1}^n]\times\mathbb{R}^n$ with closed values of $\mathbb{R}^n$ by

$$G_k^n(t,x) := F(t,\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,x)) \ \ \forall (t,x)\in[t_k^n,t_{k+1}^n]\times\mathbb{R}^n$$

satisfying conditions of Theorem 3.1. Hence, there exists an absolutely continuous solution $v_k^n : [t_k, t_{k+1}] \longrightarrow \mathbb{R}^n$ to

$$\begin{cases} \dot v_k^n(t) & \in G_k^n(t, v_k^n(t)) & \text{a.e. on } [t_k^n, t_{k+1}^n];\\ v_k^n(t) & = x_n(t_k^n) + \int_{t_k^n}^t \dot v_k^n(s)ds & \forall\ t\in[t_k^n,t_{k+1}^n];\\ v_k^n(t_k^n) & = x_n(t_k^n). \end{cases}$$

So v_k^n is a solution of

$$\begin{cases} \dot v_k^n(t) & \in F(t,\mathcal{T}(t_{k+1}^n)f_k^n(\cdot,x)) & \text{a.e. on } [t_k^n, t_{k+1}^n];\\ v_k^n(t) & = x_n(t_k^n) + \int_{t_k^n}^t \dot v_k^n(s)ds & \forall\ t\in[t_k^n,t_{k+1}^n];\\ v_k^n(t_k^n) & = x_n(t_k^n). \end{cases}$$

Putting $x_n(t) = v_k^n(t)$ on $[t_k^n, t_{k+1}^n]$, we obtain

$$x_n(t) = \begin{cases} v_0^n(t) = \varphi(0) + \int_0^t \dot x_n(s)ds & if\ \ t\in[0,t_1^n];\\ v_1^n(t) = x_n(t_1^n) + \int_{t_1^n}^t \dot x_n(s)ds & if\ \ t\in[t_1^n,t_2^n];\\ \quad \cdot\ \cdot\ \cdot & \\ v_k^n(t) = x_n(t_k^n) + \int_{t_k^n}^t \dot x_n(s)ds & if\ \ t\in[t_k^n,t_{k+1}^n]. \end{cases}$$

For every $t\in[0,1]$, we set $\theta_n(t) = t_i^n$, $\delta_n(t) = t_{i+1}^n$, $\forall\ t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = 0$ and define $f_{\mu_n\delta_n(t)-1}^n \in \mathcal{C}_{\mathbb{R}^n([-\tau,\delta_n(t)])}$ by

$$f_{\mu_n\delta_n(t)-1}^n(t,x) = \begin{cases} x_n(t) & \text{if } \ t\in[-\tau,\theta_n(t)];\\ x_n(\theta_n(t)) + \frac{t-\theta_n(t)}{\mu_n}(x - x_n(\theta_n(t))) & \text{if } \ t\in]\theta_n(t),\delta_n(t)]. \end{cases}$$

Clearly x_n is continuous on $[-\tau,1]$, absolutely continuous on $[0,1]$ and satisfies

$$\begin{cases} \dot x_n(t) & \in F(t,\mathcal{T}(\delta_n(t))f_{\mu_n\delta_n(t)-1}^n(\cdot,x_n(t))) & a.\ e.\ on\ \ [0,1];\\ x_n(t) & = \varphi(0) + \int_0^t \dot x_n(s)ds & \forall\ t\in[0,1];\\ x_n(t) & = \varphi(t) & \forall\ t\in[-\tau,0]. \end{cases} \tag{3.2}$$

Step 2 *Uniform convergence.*
By the condition $iii)$ of Theorem 3.1 and (3.2), for almost $t\in[0,1]$, one has

$$\dot x_n(t) \in F(t,\mathcal{T}(\delta_n(t))f_{\mu_n\delta_n(t)-1}^n(\cdot,x_n(t))),$$

with $\mathcal{T}(\delta_n(t))f_{\mu_n\delta_n(t)-1}^n(\cdot,x_n(t))(0) = x_n(t)$ and

$$\|F(t,\mathcal{T}(\delta_n(t))f_{\mu_n\delta_n(t)-1}^n(\cdot,x_n(t)))\| \le (\ 1+\|x_n(t)\|\ )\,\rho(t).$$

Further, since x_n is absolutely continuous on $[0,1]$, we have

$$\begin{array}{rcl}\|x_n(t)-\varphi(0)\| & \leq & \int_0^t \|\dot{x}_n(s)\|\,ds \\ & \leq & \int_0^t (\,1+\|x_n(s)\|\,)\,\rho(s)ds \\ & \leq & \int_0^t (\,1+\|x_n(s)\|\rho(s)\,)\,ds \\ & = & \int_0^t \rho(s)ds+\int_0^t \rho(s)\|x_n(s)\|\;ds, \quad \forall t\in[0,1].\end{array}$$

Then, $\|x_n(t)\|\leq\|\varphi(0)\|+\int_0^t \rho(s)ds+\int_0^t \rho(s)\|x_n(s)\|\;ds$, $\forall t\in[0,1]$. Using Lemma 2.1, we obtain for all $t\in[0,1]$,

$$\|x_n(t)\|\leq(\|\varphi(0)\|+\int_0^t \rho(s)ds)\;\exp(\int_0^t \rho(s)ds).$$

Let $\alpha(t)=(\|\varphi(0)\|+\int_0^t \rho(s)ds)\;\exp(\int_0^t \rho(s)ds)$. Hence for almost every $t\in[0,1]$,

$$\|\dot{x}_n(t)\|\leq(\,1+\alpha(t))\,\rho(t). \tag{3.3}$$

By (3.3), $(\dot{x}_n(t))_n$ is relatively compact in $L^1_{\mathbb{R}^n}([0,1])$. By extracting a subsequence, we may assume that $(\dot{x}_n)_n$ converges $\sigma(L^1,L^\infty)$ to some $y\in L^1_{\mathbb{R}^n}([0,1])$. On the other hand, by (3.3) again, $(x_n)_n$ is equi-continuous, Ascoli's Theorem yields that $(x_n)_n$ converges uniformly in $[0,1]$ to x and

$$x(t)=\varphi(0)+\int_0^t y(s)ds,\;\forall t\in[0,1],$$

hence $\dot{x}(t)=y(t)$ almost everywhere. Now, let show that

$$\|\mathcal{T}(\delta_n(t))f^n_{\mu_n\delta_n(t)-1}(\cdot,x_n(t))-\mathcal{T}(t)x\|\longrightarrow 0,\;\text{when } n\longrightarrow\infty.$$

$$\begin{aligned}&\sup_{s\in[-\tau,0]}\|\mathcal{T}(\delta_n(t))f^n_{\mu_n\delta_n(t)-1}(s,x_n(t))-\mathcal{T}(t)x(s)\|_{\mathcal{C}_0}=\\ &\qquad\sup_{s\in[-\tau,0]}\|f^n_{\mu_n\delta_n(t)-1}(\delta_n(t)+s,x_n(t))-x(s+t)\|\\ &=\sup_{s\in[-\tau,0]}\|f^n_{\mu_n\delta_n(t)-1}(\delta_n(t)+s,x_n(t))-x(\delta_n(t)+s)+x(\delta_n(t)+s)-x(s+t)\|\\ &\leq\sup_{s\in[-\tau,0]}\|f^n_{\mu_n\delta_n(t)-1}(\delta_n(t)+s,x_n(t))-x(\delta_n(t)+s)\|+\\ &\qquad\sup_{s\in[-\tau,0]}\|x(\delta_n(t)+s)-x(s+t)\|.\end{aligned}$$

firstly,

$$\begin{aligned}&\sup_{s\in[-\tau,0]}\|f^n_{\mu_n\delta_n(t)-1}(\delta_n(t)+s,x_n(t))-x(\delta_n(t)+s)\|\\ &\leq\sup_{s\in[-\tau,-\mu_n]}\|f^n_{\mu_n\delta_n(t)-1}(\delta_n(t)+s,x_n(t))-x(\delta_n(t)+s)\|\\ &+\sup_{s\in[-\mu_n,0]}\|f^n_{\mu_n\delta_n(t)-1}(\delta_n(t)+s,x_n(t))-x(\delta_n(t)+s)\|\\ &=\sup_{s\in[-\tau,-\mu_n]}\|x_n(\delta_n(t)+s)-x(\delta_n(t)+s)\|+\\ &\sup_{s\in[-\mu_n,0]}\|x_n(\theta_n(t))+\frac{\delta_n(t)+s-\theta_n(t)}{\mu_n}(x_n(t)-x_n(\theta_n(t))-x(\delta_n(t)+s))\|\\ &=\sup_{s\in[-\tau,-\mu_n]}\|x_n(\delta_n(t)+s)-x(\delta_n(t)+s)\|\\ &+\sup_{s\in[-\mu_n,0]}\|\frac{s}{\mu_n}(x_n(t)-x_n(\theta_n(t)))+x_n(t)-x(\delta_n(t)+s)\|\\ &=\|x_n(\theta_n(t))-x(\theta_n(t))\|+\|x_n(t)-x_n(\delta_n(t))\|.\end{aligned}$$

On the other hand

$$\begin{aligned}\sup_{s\in[-\tau,0]}\|x(\delta_n(t)+s)-x(s+t)\| &\le \sup_{s\in[-\tau,-\mu_n]}\|x(\delta_n(t)+s)-x(s+t)\|\\ &\quad+\sup_{s\in[-\mu_n,0]}\|x(\delta_n(t)+s)-x(s+t)\|\\ &= \sup_{s\in[-\tau,-\mu_n]}\|x(\delta_n(t)+s)-x(s+t)\|\\ &\quad+\|x(\delta_n(t))-x(t)\|.\end{aligned}$$

Then

$$\sup_{s\in[-\tau,0]}\|\mathcal{T}(\delta_n(t))f^n_{\mu_n\delta_n(t)-1}(s,x_n(t))-\mathcal{T}(t)x(s)\|_{\mathcal{C}_0}\le$$
$$\|x_n(\theta_n(t))-x(\theta_n(t))\|+\|x_n(t)-x_n(\delta_n(t))\|+$$
$$\sup_{s\in[-\tau,-\mu_n]}\|x(\delta_n(t)+s)-x(s+t)\|+\|x(\delta_n(t))-x(t)\|.$$

As $|\theta_n(t)-t|\le\mu_n$ and $|\delta_n(t)-t|\le\mu_n$, $\forall t\in[0,1]$ then $\theta_n(t)\longrightarrow t$ and $\delta_n(t)\longrightarrow t$ for n large enough. Furthermore, $(x_n)_n$ converges uniformly to x, $\|x(\delta_n(t))-x(t)\|\longrightarrow 0, \|\ x_n(\delta_n(t))-x_n(t)\|\longrightarrow 0$ and $\|x_n(\theta_n(t))-x(\theta_n(t))\|\longrightarrow 0$. As x is uniformly continuous, there is $\lambda>0$ such that $|s-t|\le\lambda$ implies $\|x(s)-x(t)\|\le\epsilon$. But we have $|\delta_n(t)+s-(s+t)|\le\mu_n$ for all $s\in[-\tau,\mu_n]$. Hence

$$\sup_{s\in[-\tau,-\mu_n]}\|x(\delta_n(t)+s)-x(s+t)\|\le\epsilon \text{ for } \lambda\le\mu_n.$$

We can conclude that $\mathcal{T}(\delta_n(t))f^n_{\mu_n\delta_n(t)-1}(\cdot,x_n(t))\longrightarrow\mathcal{T}(t)x$ in $\mathcal{C}_0$.
Finally, since $\mathcal{T}(\delta_n(t))f^n_{\mu_n\delta_n(t)-1}(\cdot,x_n(t))\longrightarrow\mathcal{T}(t)x$ in $\mathcal{C}_0$, $(\dot{x}_n)_n$ converges $\sigma(L^1,L^\infty)$ to $\dot{x}\in L^1_{\mathbb{R}^n}([0,1])$ and the set-valued mapping $F(t,\cdot)$ is lsc with closed values on $\mathcal{C}_0$, then $\dot{x}(t)\in F(t,\mathcal{T}(t)x)$ (see [9]). So, x satisfies

$$\begin{cases}\dot{x}(t) & \in F(t,\mathcal{T}(t)x) \qquad \text{a.e. on } [0,T];\\ x(t) & =\varphi(0)+\int_0^t\dot{x}(s)ds \qquad \forall\, t\in[0,T];\\ x(t) & =\varphi(t) \qquad \forall\, t\in[-\tau,0].\end{cases}$$

The proof is then complete. □

4. Conclusion

In this paper, an existence result is obtained for first order functional differential inclusions with nonconvex right hand side. The approach used is an adaptation of a reduction method which consists of replacing the problem with delay with a problem without delay and applying the known results in this case.

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ON A MEAN METHOD OF SUMMABILITY

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Abstract. Let $p(x)$ be a nondecreasing real-valued continuous function on $R_+ := [0,\infty)$ such that $p(0)=0$ and $p(x)\to\infty$ as $x\to\infty$. Given a real or complex-valued integrable function f in Lebesgue's sense on every bounded interval $(0,x)$ for $x>0$, in symbol $f\in L^1_{loc}(R_+)$, we set

$$s(x)=\int_0^x f(u)du$$

and

$$\sigma_p(s(x))=\frac{1}{p(x)}\int_0^x s(u)dp(u),\quad x>0$$

provided that $p(x)>0$.

A function $s(x)$ is said to be summable to l by the weighted mean method determined by the function $p(x)$, in short, $(\overline{N},p)$ summable to l, if

$$\lim_{x\to\infty}\sigma_p(s(x))=l.$$

If the limit $\lim_{x\to\infty}s(x)=l$ exists, then $\lim_{x\to\infty}\sigma_p(s(x))=l$ also exists. However, the converse is not true in general. In this paper, we give an alternative proof a Tauberian theorem stating that convergence follows from summability by weighted mean method on $R_+ := [0,\infty)$ and a Tauberian condition of slowly decreasing type with respect to the weight function due to Karamata. These Tauberian conditions are one-sided or two-sided if $f(x)$ is a real or complex-valued function, respectively. Alternative proofs of some well-known Tauberian theorems given for several important summability methods can be obtained by choosing some particular weight functions.

1. introduction

Let $p(x)$ be a nondecreasing real-valued continuous function on $R_+ := [0,\infty)$. Throughout this paper, we assume that $p(0)=0$ and $p(x)\to\infty$ as $x\to\infty$. Given a real-valued integrable function f in Lebesgue's sense on every bounded interval $(0,x)$ for $x>0$, in symbol $f\in L^1_{loc}(R_+)$, we set

$$s(x)=\int_0^x f(u)du \tag{1.1}$$

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and

$$\sigma_p(s(x)) = \frac{1}{p(x)} \int_0^x s(u)dp(u), \quad x > 0$$

provided that $p(x) > 0$.

A function $s(x)$ is said to be summable to l by the weighted mean method determined by the function $p(x)$, in short, $(\overline{N}, p)$ summable to l, if

$$\lim_{x\to\infty} \sigma_p(s(x)) = l. \tag{1.2}$$

Clearly, if the ordinary limit

$$\lim_{x\to\infty} s(x) = l \tag{1.3}$$

exists, then (1.2) holds. However, the converse implication is not true in general. We may get the converse implication by adding some assumption(s) on $s(x)$, which is so-called Tauberian condition(s). Any theorem which states that convergence of (1.3) follows from (1.2) and a Tauberian condition is said to be a Tauberian theorem for summability by the weighted mean method.

A real-valued function $s(x)$ defined on R_+ is said to be slowly decreasing with respect to p if

$$\lim_{\lambda\to 1^+} \liminf_{t\to\infty} \min_{t\le x\le T} (s(x) - s(t)) \ge 0, \tag{1.4}$$

where

$$T := p^{-1}(\lambda p(t)), \;\; t > 0. \tag{1.5}$$

Note that the concept of slow decrease with respect to p is due to Karamata [4].

It is easy to see that a real-valued function $s(x)$ is slowly decreasing with respect to p if and only if for every $\epsilon > 0$ there exist $t_0 = t_0(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 1$ such that $s(x) - s(t) \ge -\epsilon$ whenever $t_0 \le t \le x \le T$.

An equivalent reformulation of (1.4) can be given as follows (see Fekete and Moricz [1]):

$$\lim_{\lambda\to 1^-} \liminf_{t\to\infty} \min_{T\le x\le t} (s(t) - s(x)) \ge 0, \tag{1.6}$$

where T is defined in (1.5). It is easy to see that a real valued function $s(x)$ is slowly decreasing with respect to p if and only if for every $\epsilon > 0$ there exist $t_1 = t_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon)$ with $0 < \lambda < 1$ such that $s(t) - s(x) \ge -\epsilon$ whenever $t_1 \le T \le x \le t$.

A real-valued function $s(x)$ defined on R_+ is said to be slowly decreasing if (1.4) holds, where $p(x) = x$ for all $x > 0$. Recall that the term "slow decrease" is introduced by Schmidt [7] for sequences of real numbers.

In [3], we obtained an alternative proof of Theorem 2.1 below when a Tauberian condition is of slowly decreasing type.

In this paper, we give an alternative proof a Tauberian Theorem stating that convergence follows from summability by weighted mean method over R_+ and a Tauberian condition of slowly decreasing type with respect to the weight function, due to Karamata [4].

Alternative proofs of some well-known Tauberian theorems given for several important summability methods can be obtained by choosing some particular weight functions.

2. MAIN RESULTS

By using proving techniques in [6], we give an alternative proof of the following Tauberian theorem [5] for the weighted mean summability of integrals of real-valued functions over R_+.

Theorem 2.1. *Let $p(x)$ be a nondecreasing real-valued continuous function on R_+ such that $p(0) = 0$ and $p(x) \to \infty$ as $x \to \infty$. If a real-valued function $f \in L^1_{loc}(R_+)$ is such that (1.2) holds and its integral function $s(x)$ is slowly decreasing with respect to p, then (1.3) holds.*

Proof. By the regularity of the summability method by the weighted mean, without loss of generalization, we assume that $l = 0$. Assume that $s(x)$ does not converge to 0 as $x \to \infty$.

Then, we have either $\limsup_{x\to\infty} s(x) > 0$ or $\liminf_{x\to\infty} s(x) < 0$.

First, we assume that $\limsup_{x\to\infty} s(x) > 0$. Then, there exist $\alpha > 0$ and a sequence (n_i) such that $s(n_i) \geq \alpha$ for all nonnegative integers i. Choosing $\epsilon = \frac{\alpha}{2}$ in the equivalent form of (1.4), we find $\lambda > 1$ and $t_0 \geq 0$ such that $s(x) \geq s(n_i) - \frac{\alpha}{2} \geq \frac{\alpha}{2}$ for $t_0 \leq n_i < x \leq m_i = p^{-1}(\lambda p(n_i))$.

Since

$$\begin{aligned}\sigma_p(s(m_i)) - \frac{p(n_i)}{p(m_i)}\sigma_p(s(n_i)) &= \sigma_p(s(m_i)) - \frac{1}{\lambda}\sigma_p(s(n_i)) \\ &= \frac{1}{p(m_i)}\int_{n_i}^{m_i} s(u)dp(u),\end{aligned}$$

we have

$$\begin{aligned}\sigma_p(s(m_i)) - \frac{p(n_i)}{p(m_i)}\sigma_p(s(n_i)) &\geq \frac{\alpha}{2p(m_i)}\int_{n_i}^{m_i} dp(u) \\ &= \frac{\alpha}{2}\left(1 - \frac{1}{\lambda}\right) \qquad (2.1)\end{aligned}$$

for $t_0 \leq n_i < x \leq m_i = p^{-1}(\lambda p(n_i))$. We conclude by (2.1) that $0 \geq \frac{\alpha}{2}\left(1 - \frac{1}{\lambda}\right)$. This contradicts our assumption that $\limsup_{x\to\infty} s(x) > 0$. Then, we have

$$\limsup_{x\to\infty} s(x) \leq 0. \qquad (2.2)$$

Next, we assume that $\liminf_{x\to\infty} s(x) < 0$. Then, there exist $\beta < 0$ and a sequence (n_i) such that $s(n_i) \leq \beta < 0$ for all nonnegative integers i. Choosing $\epsilon = -\frac{\beta}{2}$ in the equivalent form of (1.4), we find $0 < \lambda < 1$ and $t_1 = t_1(\epsilon)$ such that $s(x) \leq s(n_i) - \frac{\beta}{2} \leq \frac{\beta}{2}$ for $t_1 \leq m_i = p^{-1}(\lambda p(n_i)) < x \leq n_i$.

Since

$$\begin{aligned}\sigma_p(s(n_i)) - \frac{p(m_i)}{p(n_i)}\sigma_p(s(m_i)) &= \sigma_p(s(n_i)) - \lambda\sigma_p(s(m_i)) \\ &= \frac{1}{p(n_i)}\int_{m_i}^{n_i} s(u)dp(u),\end{aligned}$$

we have

$$\begin{aligned}\sigma_p(s(n_i)) - \frac{p(m_i)}{p(n_i)}\sigma_p(s(m_i)) &\leq \frac{\beta}{2p(n_i)}\int_{m_i}^{n_i} dp(u) \\ &= \frac{\beta}{2}(1 - \lambda) \qquad (2.3)\end{aligned}$$

for $t_1 \leq m_i = p^{-1}(\lambda p(n_i)) \leq x \leq n_i$. We conclude by (2.3) that $0 \leq \frac{\beta}{2}(1-\lambda)$. This contradicts our assumption that $\liminf_{x\to\infty} s(x) < 0$. Then, we have

$$\liminf_{x\to\infty} s(x) \geq 0. \tag{2.4}$$

Combining (2.2) and (2.4) gives convergence of $s(x)$ to 0 as $x \to \infty$. $\square$

A real-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly increasing with respect to p if $-s$ is slowly decreasing with respect to p.

Remark. Theorem 2.1 remains true if slow decrease of $s(x)$ with respect to p is replaced by slow increase of $s(x)$ with respect to p.

For a complex-valued integrable function f in Lebesgue's sense on every bounded interval $(0, x)$ for $0 < x < \infty$, we have the following Tauberian theorem.

Theorem 2.2. *Let $p(x)$ be a nondecreasing real-valued continuous function on R_+ such that $p(0) = 0$ and $p(x) \to \infty$ as $x \to \infty$. If a complex-valued function $f \in L^1_{loc}(R_+)$ is such that (1.2) holds and its integral function $s(x)$ is slowly oscillating with respect to p, then (1.3) holds.*

The proving technique in Theorem 2.1 is also valid for the proof of Theorem 2.2.

We remind the reader that a complex-valued function $s(x)$ defined on R_+ is said to be slowly oscillating with respect to p ([4]) if

$$\lim_{\lambda\to 1^+} \limsup_{t\to\infty} \max_{t\leq x\leq T} |s(x) - s(t)| = 0, \tag{2.5}$$

where T is defined as (1.5).

It is easy to see that a real-valued function $s(x)$ is slowly oscillating with respect to p if and only if for every $\epsilon > 0$ there exist $t_0 = t_0(\epsilon) > 0$ and $\lambda = \lambda(\epsilon) > 1$ such that $|s(x) - s(t)| \leq \epsilon$ whenever $t_0 \leq t \leq x \leq T$.

An equivalent reformulation of (2.5) can be given as follows (see Fekete and Moricz [1]):

$$\lim_{\lambda\to 1^-} \limsup_{t\to\infty} \max_{T\leq x\leq t} |s(t) - s(x)| = 0, \tag{2.6}$$

where T is defined in (1.5). It is easy to see that a real valued function $s(x)$ is slowly decreasing with respect to p if and only if for every $\epsilon > 0$ there exist $t_1 = t_1(\epsilon) > 0$ and $\lambda = \lambda(\epsilon)$ with $0 < \lambda < 1$ such that $|s(t) - s(x)| \leq \epsilon$ whenever $t_1 \leq T \leq x \leq t$.

A complex-valued function $s(x)$ defined on R_+ is said to be slowly oscillating if (2.5) holds, where $p(x) = x$ for all $x > 0$.

Recall that the concept of slow oscillation was introduced by Hardy [2] for sequences of real numbers.

3. Particular weights

Some particular choices of weight functions can lead to alternative proofs of some well-known Tauberian theorems given for several important summability methods. If $p(x) = x$ for all $x > 0$, then weighted mean method $(\overline{N}, p)$ reduces to the Cesàro summability method. If $p(x) = \ln x$ for all $x \in [1, \infty)$ and zero for all $x \in [0, 1)$, then then weighted mean method $(\overline{N}, p)$ reduces the harmonic mean method of first order. For other particular choices of the weight function p, we obtain the harmonic mean method of higher order. Our main Theorem 2.1 applies to all of these methods.

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ON (p, q)-ANALOG OF STANCU OPERATORS OF ROUGH λ-STATISTICALLY ρ-CAUCHY CONVERGENCE OF TRIPLE SEQUENCE SPACES

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Abstract. In this work, using the concept of natural density, we introduce the (p, q)-analogue of the Stancu-beta operators of rough λ-statistically ρ-Cauchy convergence on triple sequence spaces. We define the set of Bernstein Stancu beta opeators of rough statistical limit points of a triple sequence spaces and obtain to λ-statistical convergence criteria associated with this set. Also, we examine the relations between the set of Bernstein-Stancu beta operators of rough λ-statistically ρ-Cauchy convergence of triple sequences.

1. Introduction

We introduce the (p, q)-analogue of the Stancu-beta operators and study their approximation properties.

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$ and let us denote the set $K_{ij\ell} = \{(m, n, k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta_3 (K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

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First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Schurer operators etc.

Very recently, Khalid et al. have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p,q)-Bezier curves and surfaces based on (p,q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p,q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu beta operators based on (p,q)-integers.

Now we recall some basic definitions about (p,q)-integers. For any $u,v,w \in \mathbb{N}$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p-q} \text{ if } u,v,w \geq 1,$$

where $0 < q < p \leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]!_{p,q}[6]!_{p,q}...[uvw]!_{p,q} \text{ if } u,v,w \geq 1.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[uvw]!_{p,q}}{[mnk]!_{p,q}\,[(u-m)+(v-n)+(w-k)]!_{p,q}}$$

for all $u,v,w,m,n,k \in \mathbb{N}$ with $u \geq m, v \geq n, w \geq k$.

The formula for $(p,q)-$binomial expansion is as follows:

$$\begin{aligned}
&(ax+by)^{uvw}_{p,q}\\
&\quad = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{6}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}\\
&\quad \binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k},
\end{aligned}$$

$$\begin{aligned}
(x+y)^{uvw}_{p,q} &= (x+y)(px+qy)\left(p^6x+q^6y\right)\cdots\\
&\qquad \left(p^{(u-1)+(v-1)+(w-1)}x + q^{(u-1)+(v-1)+(w-1)}y\right),
\end{aligned}$$

$$\begin{aligned}
(1-x)^{uvw}_{p,q} &= (1-x)(p-qx)\left(p^6-q^6x\right)\cdots\\
&\qquad \left(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x\right), \text{ and}
\end{aligned}$$

$$(x)^{mnk}_{p,q} = x\,(px)\left(p^6x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}.$$

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} x^{m+n+k} \quad (1.1)$$

$$\cdot \prod_{u=0}^{(r-m-1)+(s-n-1)+(t-k-1)} (p^u - q^u x) f\left(\frac{[mnk]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}[rst]_{p,q}}\right), x \in [0,1].$$

Also, we have

$$(1-x)_{p,q}^{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} (-1)^{m+n+k} p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} \binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}.$$

(p,q)-Bernstein-Stancu operators are defined as follows:

$$S_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} x^{m+n+k}. \quad (1.2)$$

$$\prod_{u=0}^{(r-m-1)+(s-n-1)+(t-k-1)} (p^u - q^u x) f\left(\frac{p^{(r-m)+(s-n)+(t-k)}[mnk]_{p,q}+\eta}{[rst]_{p,q}+\mu}\right), x \in [0,1].$$

Note that for $\eta = \mu = 0$, (p,q)-Bernstein-Stancu operators given by (1.2) reduces into (p,q)-Bernstein operators. Also for $p = 1$, (p,q)-Bernstein-Stancu operators given by (1.1) turn out to be q-Bernstein-Stancu operators.

The definite integrals of a function f are defined by

$$\int_0^a\int_0^b\int_0^c f(xyz)\, d_{pq}x d_{pq}y d_{pq}z = (q-p)\, abc \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{p^{m+n+k}}{q^{(m+1)+(n+1)+(k+1)}} f\left(\frac{p^{m+n+k}}{q^{(m+1)+(n+1)+(k+1)}} abc\right),$$

$$\text{when } \left|\frac{p}{q}\right| < 1 \text{ and}$$

$$\int_0^a\int_0^b\int_0^c f(xyz)\, d_{pq}x d_{pq}y d_{pq}z = (p-q)\, abc \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{m+n+k}}{p^{(m+1)+(n+1)+(k+1)}} f\left(\frac{q^{m+n+k}}{p^{(m+1)+(n+1)+(k+1)}} abc\right),$$

$$\text{when } \left|\frac{q}{p}\right| < 1.$$

There are two (p,q)-analogues of the classical exponential function defined as follows:

$$e_{pq}(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{p^{\frac{u(u-1)+v(v-1)+w(w-1)}{2}}}{[uvw]_{pq}!} x^{u+v+w} \text{ and}$$
$$E_{pq}(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{\frac{u(u-1)+v(v-1)+w(w-1)}{2}}}{[uvw]_{pq}!} x^{u+v+w}.$$

It is easily seen that $e_{pq}(x)\ E_{pq}(-x) = 1$. For $m,n,k \in \mathbb{N}$, the (p,q)-beta and the (p,q)-Gamma functions are defined by

$$B_{pq}(m,n) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{x^{m-1}+x^{n-1}+x^{k-1}}{(1+x)^{m+n}} d_{pq}x$$

and

$$\Gamma_{pq}(u) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p^{\frac{u(u-1)+v(v-1)+w(w-1)}{2}} E_{pq}(-q(xyz))\, d_{pq}x,$$
$$\Gamma_{pq}(u+1,v+1,w+1) = [uvw]_{pq}!,$$

respectively. The functions are connected through

$$B_{pq}(m,n,k) = q^{\frac{6-[m(m-1)+n(n-1)+k(k-1)]}{2}} p^{\frac{-[m(m-1)+n(n-1)+k(k-1)]}{2}} \frac{\Gamma_{pq}(m)\,\Gamma_{pq}(n)\,\Gamma_{pq}(k)}{\Gamma_{pq}(m+n+k)}. \tag{1.3}$$

If $p = 1$ then the above notions of (p,q)-calculus reduce to the corresponding notations of q-calculus.

Let $0 < q < p < 1$ and $x \in [0,\infty)$. We introduce the (p,q)-Stancu-beta operators as follows:

$$S_{uvw,pq}(f,x) = \frac{1}{B_{pq}\left([uvw]_{pq}x, [uvw]_{pq}+3\right)}.$$
$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{(rst)^{[uvw]_{pq}(x-1)}}{(1+(rst))^{[uvw]_{pq}x+[uvw]_{pq}+3}}.$$
$$f\left(p^{[uvw]_{pq}x}, q^{[uvw]_{pq}x} rst\right) d_{pq}r\, d_{pq}s\, d_{pq}t.$$

Throughout the paper, $\mathbb{R}^3$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein-Stancu beta operators $(S_{uvw,p,q}(f,x))$ such that $(S_{uvw,p,q}(f,x)) \in \mathbb{R}$, $m,n,k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu-beta operators $(S_{uvw,p,q}(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st_3 - \lim S_{uvw,p,q}(f,x) = f(x)$, provided that the set

$$K_\epsilon := \left\{(m,n,k) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein-Stancu-beta operators. i.e., $\delta_3(K_\epsilon) = 0$. That is,

$$\lim_{u,v,w\to\infty} \frac{1}{uvw} \left|\{m \leq u, n \leq v, k \leq w : |S_{uvw,p,q}(f,x) - (f,x)| \geq \epsilon\}\right| = 0.$$

In this case, we write $\delta_3 - \lim S_{uvw,p,q}(f,x) = (f,x)$ or $S_{uvw,p,q}(f,x) \xrightarrow{st_3} (f,x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A-the characteristic function of $A \subset \mathbb{N}$. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d_3(A) = \lim_{i,j,\ell\to\infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \chi_A(K).$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [11], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [10] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

In this paper, we introduce the notion of Bernstein-Stancu beta operators of rough λ-statistically ρ-Cauchy sequences convergence. Defining the set of Bernstein-Stancu beta operators of rough λ-statistical limit points of a sequence, we obtain to λ-statistical convergence criteria associated with this set. Later, we prove that this set of rough λ-statistically ρ-Cauchy convergence of a triple sequence spaces.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}^3 \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Esi et al.* [2, 3, 4, 5], *Dutta et al.* [6], *Esi et al.* [7, 8], *Sahiner et al.* [12, 13], *Subramanian et al.* [14], *Debnath et al.* [9] and many others.

Throughout the paper let β be a nonnegative real number.

2. Definitions and Preliminaries

Definition 2.1. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,pq,}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers is said to be β-convergent to (f,x) denoted by $S_{uvw,p,q}(f,x) \to^{\beta} (f,x)$, provided that*

$$\forall \epsilon > 0 \; \exists (u_\epsilon, v_\epsilon, w_\epsilon) \in \mathbb{N}^3 : u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon \Rightarrow |S_{uvw,p,q}(f,x) - (f,x)| < \beta + \epsilon.$$

The set

$$\mathrm{LIM}^{\beta} x = \left\{ (f,x) \in \mathbb{R}^3 : S_{uvw,p,q}(f,x) \to^{\beta} (f,x) \right\}$$

is called the β-limit set of the triple sequences.

Definition 2.2. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu-beta operators of real numbers is said to be β-convergent if $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$. In this case, β is called the Bernstein-Stancu-beta operators of rough convergence degree of the triple sequence spaces. For $\beta = 0$, we get the ordinary convergence.*

Definition 2.3. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers is said to be β-statistically convergent to (f,x), denoted by $S_{uvw,p,q}(f,x) \to^{uvw} (f,x)$, provided that the set*

$$\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st - \limsup |S_{uvw,p,q}(f,x) - (f,x)| \leq \beta$$

is satisfied.

In addition, we can write $S_{uvw,p,q}(f,x) \to^{uvw} (f,x)$ if and only if the inequality

$$|S_{uvw,p,q}(f,x) - (f,x)| < \beta + \epsilon$$

holds for every $\epsilon > 0$ and almost all (u,v,w). Here β is called the Bernstein-Stancu beta operators of roughness of degree. If we take $\beta = 0$, then we obtain the ordinary statistical convergence.

In a similar fashion to the idea of classic Bernstein-Stancu beta operators of rough convergence, the idea of Bernstein-Stancu beta operators of rough statistical convergence of a triple sequence spaces can be interpreted as follows:

Assume that a Bernstein-Stancu beta operators of triple sequence space $(S_{uvw,p,q}(g,x))$ is statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or statistically approximated) triple sequence spaces $(S_{uvw,p,q}(f,x))$ satisfying $|S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \beta$ for all u,v,w (or for almost all u,v,w, i.e.,

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| > \beta\right\}\right) = 0.$$

Then the triple sequence spaces x is not statistically convergent any more, but as the inclusion

$$\begin{aligned} &\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (f,x)| \geq \epsilon\right\} \\ \supseteq &\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\} \end{aligned} \tag{2.1}$$

holds and we have

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (f,x)| \geq \epsilon\right\}\right) = 0,$$

i.e., we get

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right) = 0,$$

i.e., the triple sequence spaces x is β-statistically convergent in the sense of definition 2.3

In general, the Bernstein-Stancu beta operators of rough statistical limit may not unique for the Bernstein-Stancu beta operators of roughness degree $r > 0$. So we have to consider the so called Bernstein-Stancu beta operators of rough ness of β-statistical limit set is defined by

$$st - \mathrm{LIM}^{\beta}\, s_{uvw,p,q}(f,x) = \left\{(f,x) \in \mathbb{R}^3 : S_{uvw,p,q}(f,x) \to^{uvw} (f,x)\right\}.$$

The Bernstein-Stancu-beta operators of triple sequence space $S_{uvw,p,q}(f,x)$ is said to be Bernstein-Stancu beta operators of rough β-statistically convergent provided that $st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}(f,x) \neq \phi$. It is clear that if $st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}(f,x) \neq$

ϕ. We have

$$st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x) \\ =[st-\limsup S_{uvw,p,q}\,(f,x)-\beta,st-\liminf S_{uvw,p,q}\,(f,x)+\beta] \qquad (2.2)$$

We know that $\mathrm{LIM}^{\beta}=\phi$ for an unbounded triple sequence spaces might be rough statistically convergent. For instance, define

$$S_{uvw,p,q}\,(f,x)=\begin{cases} (-1)^{uvw} & , \text{ if} u\neq i^3, v\neq j^3, w\neq \ell^3 \ \ (i,j,\ell\in\mathbb{N}) \\ uvw & , \text{ otherwise}\end{cases}$$

in $\mathbb{R}$. Because the set $\{1,64,739,\ldots\}$ has natural density zero, we have

$$st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)=\begin{cases} \phi & , \text{ if } \beta<1 \\ [1-\beta,\beta-1] & , \text{ otherwise}\end{cases}$$

and $\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)=\phi$ for all $\beta\geq 0$.

As can be seen by the example above, the fact that $st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi$ does not imply $\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi$. Because a finite set of natural numbers has natural density zero, $\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi$ implies $st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi$. Therefore, we get $\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\subseteq st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)$. This obvious fact means $\left\{\beta\geq 0:\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi\right\}\subseteq\left\{\beta\geq 0: st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi\right\}$ in this language of sets and yields immediately

$$\inf\left\{\beta\geq 0:\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi\right\}\geq\inf\left\{\beta\geq 0: st-\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi\right\}.$$

Moreover, it also yields directly

$$\dim\left(\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\right)\leq\dim\left(st-\mathrm{LIM}^{\beta}\, s_{uvw,p,q}\,(f,x)\right).$$

Note. *The Bernstein-Stancu beta operators of rough statistical limit of a triple sequence spaces is unique for the roughness degree $\beta>0$.*

Definition 2.4. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}\,(f,x))$ be a triple sequence of Bernstein-Stancu-beta operators of real numbers is β-convergent, i.e., $\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)\neq\phi$. Take an arbitrary $L\in\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)$, for all $\epsilon>0\ \exists$ an $u_{\epsilon},v_{\epsilon},w_{\epsilon}\in\mathbb{N}^3$ such that $u\geq u_{\epsilon},v\geq v_{\epsilon},w\geq w_{\epsilon}$ implies*

$$|S_{uvw,p,q}\,(f,x)-(f,x)|\leq\beta+\frac{\epsilon}{2} \text{ and } |S_{uvw,p,q}\,(g,x)-(g,x)|\leq\beta+\frac{\epsilon}{2}$$

$$\Rightarrow |S_{uvw,p,q}(f,x)-S_{uvw,p,q}(g,x)|\leq|S_{uvw,p,q}(f,x)-(f,x)|+|S_{uvw,p,q}(g,x)-(g,x)| \\ \leq\beta+\frac{\epsilon}{2}+\frac{\epsilon}{2}\leq 2\beta+\epsilon.$$

Hence the Bernstein-Stancu beta operators of triple sequence spaces is a ρ-Cauchy sequence with $\rho=2\beta$. This Cauchy degree cannot be generally decreased. Indeed, let $z\in\mathbb{R}^3$ with $|z|=\beta$ and $S_{uvw,p,q}\,(f,x)=(-1)^{u+v+w}\,z$ then Bernstein-Stancu-beta operators of roughness is β-convergent with $0\in\mathrm{LIM}^{\beta}\, S_{uvw,p,q}\,(f,x)$, and $\rho=2\beta$ is its minimal Cauchy degree.

Conversely, let $\rho\geq 0$ be a Cauchy degree of some given Bernstein-Stancu-beta operators of triple sequence $(S_{uvw,p,q}\,(f,x))$ its convergence degree to equal $\frac{\rho}{2}$, i.e., $\mathrm{LIM}^{\frac{\rho}{2}}\, S_{uvw,p,q}\,(f,x)\neq\phi$. This condition always not true.

Definition 2.5. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers is said to be $\beta\lambda$-statistically convergent or $\beta\lambda$ st-convergent to (f,x), denoted by $S_{uvw,p,q}(f,x) \rightarrow^{\beta\lambda st} (f,x)$, provided that the set*

$$\lim_{u,v,w} \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right| = 0$$

Definition 2.6. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(s_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu-beta operators of real numbers and $\lambda = (\lambda_{uvw})$ be non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{(uvw)+1} \leq \lambda_{uvw}+1$, $\lambda_{111} = 1$. Hence the Bernstein-Stancu-beta operators of triple sequence is a $\rho-$ Cauchy sequence with $\rho = 2\beta$, i.e.,*

$$\lim_{uvw} \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}\right| = 0$$

If $\lambda = 1$ then it is called ordinary ρ-Cauchy sequences.

3. Main Results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers $\beta > 0$, a triple sequence $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st} (f,x) \Leftrightarrow (S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence.*

Proof. Assume that $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st} (f,x)$. Let $\epsilon > 0$. then we can write

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : \frac{1}{\lambda_{uvw}} |S_{uvw,p,q}(f,x) - (f,x)| \geq \epsilon\right\}\right) = 0,$$

we have $\delta(K_1) = 0$ and $\delta(K_2) = 0$, where

$$K_1 = \left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2}\right\}$$

and

$$K_2 = \left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2}\right\}.$$

Using the properties of natural density, we get

$$\frac{\delta\left(K_1^c \bigcap K_2^c\right)}{\lambda_{uvw}} = 1 \text{ as } u,v,w \to \infty.$$

Since the triple sequence is β convergent, $\mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x) \neq \phi$, take an arbitrary $(f,x) \in \mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x)$ for all $\epsilon > 0$ there exists an $u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}$ such that $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon$ and $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon$ implies $|S_{uvw,p,q}(f,x) - (f,x)| < \beta + \frac{\epsilon}{2}$ and $|S_{uvw,p,q}(g,x) - (g,x)| < \beta + \frac{\epsilon}{2}$,

$$\begin{aligned}
\Rightarrow \frac{1}{\lambda_{uvw}} &\left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \geq \rho + \epsilon\right\}\right| \\
&\leq \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right| + \\
&\qquad \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right| = 0.
\end{aligned}$$

Hence, $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence.

Conversely, suppose that the triple sequence spaces of Bernstein-Stancu beta operators of $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence. For every $\epsilon > 0$, we have

$$\Rightarrow \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right|$$
$$\leq \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right| +$$
$$\frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right|,$$

$$\Rightarrow \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right| \leq$$
$$\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2}\right\} +$$
$$\left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right|.$$
$$\Rightarrow (s_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st} (f,x).$$

□

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers $\beta > 0$,*

$$(S_{uvw,p,q}(f,x)) \rightarrow^{mnk} (f,x) \Longrightarrow st - \liminf\left(\frac{\lambda_{uvw}}{uvw}\right) \rightarrow^{\beta\lambda st} (f,x).$$

Proof. Omitted. □

Theorem 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers $\beta > 0$, if $(S_{uvw,p,q}(f,x)) \rightarrow^{mnk}$ ρ-Cauchy sequence and $\mathrm{LIM}^r S_{uvw,p,q}(f,x) - \liminf\left(\frac{\lambda_{uvw}}{uvw}\right) > 0$, then $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence.*

Proof. A Bernstein-Stancu beta operators of triple sequence $(S_{uvw,p,q}(f,x))$ be β-convergent, i.e., $\mathrm{LIM}^\beta S_{uvw,p,q}(f,x) \neq \phi$. Take an arbitrary $(f,x) \in \mathrm{LIM}^\beta S_{uvw,p,q}(f,x)$ for all $\epsilon > 0$ there exists an $u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}$ such that $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon \in \mathbb{N}$ and $\beta \geq u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}$ implies

$$\{u_\epsilon \leq u, v_\epsilon \leq v, w_\epsilon \leq w : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\}$$
$$\supset \left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}.$$

Therefore,

$$\frac{1}{uvw} |\{u_\epsilon \leq u, v_\epsilon \leq v, w_\epsilon \leq w : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\}|$$
$$\supset \frac{1}{uvw} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}\right|$$
$$\geq \frac{\lambda_{uvw}}{uvw} \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}\right|.$$

Taking limit as $u,v,w \to \infty$ and using $\mathrm{LIM}^\beta S_{uvw,p,q}(f,x) - \liminf\left(\frac{\lambda_{uvw}}{uvw}\right) > 0$, we get $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence. □

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STABILITY ANALYSIS OF A NOVEL ODE MODEL FOR HIV INFECTION

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Abstract. In this paper, we propose and investigate the stability of a novel 3-compartment ordinary differential equation (ODE) model of HIV infection of CD4$^+$ T-cells with a mass action term. Similar to various endemic models, the dynamics within the model is fully determined by the basic reproduction term R_0. If $R_0 < 1$, the disease-free (zero) equilibrium will be asymptotically stable. On the other hand, if $R_0 > 1$, there exists a positive equilibrium that is globally/orbitally asymptotically stable under certain conditions within the interior of a predefined region. Finally, numerical simulations are conducted to illustrate and verify the results.

1. Introduction

In the field of epidemiology, although our knowledge of viral dynamics and virus-specific immmune responses has not fully developed, numerous mathematical models have been developed an investigated to describe the immunological response to HIV infection (for example, [11, 2, 4, 18, 19, 12] and references therein). The models have been used to explain different phenomena within the host body, and by directly applying the models to real clinical data, they can also predict estimates of many measures, including the death rate of productively infected cells, the rate of viral clearance or the viral production rate.

These simple HIV models have played an essential role in providing a better understanding in the dynamics of this infectious diseases, while providing very important biological meanings for the (combined) drug therapies used against it. For more references and detailed meta mathematical analysis on these models in general, we can refer to survey papers written by Kirschner, 1996 [14] or Perelson and Nelson, 1999 [8]

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The simplest HIV model, only considering the dynamics of the virus concentration, is

$$\frac{dV}{dt} = P - cV \tag{1.1}$$

where

- P is an unknown function representing the rate of production of the virus,
- V is the virus concentration.

The dynamics of the population of target cells ($CD4^+$ T-cells for HIV or hepatic cells for HBV and HCV) is still not fully understood. Nevertheless, a reasonable, simple model for this population of cells, which can be extended further in various models, is

$$\frac{dT}{dt} = s - dT + aT\left(1 - \frac{T}{T_{\max}}\right) \tag{1.2}$$

with

- s representing the rate at which new T-cells are created from sources within the body, such as the thymus, or from the proliferation of existing T-cells,
- d being the death rate per T-cells,
- a is the maximum proliferation rate of target T-cells, when the proliferation is represented by a logistic function, and
- $T_{\max}$ is the population density of T-cells at which proliferation shuts off.

Human immunodeficiency virus, or HIV, is a virus belonging to the genus Lentivirus, part of the family Retroviridae [27]. It has an outer envelope of lipid and viral proteins, which encloses its core. The virion core contains two positive-sense single-stranded RNA and the enzyme reverse transcriptase, an RNA-dependent DNA polymerase.

HIV, like most viruses, cannot reproduce by itself. Therefore, they require a host cell and its materials to replicate. For HIV, it infects a variety of immune cells, including helper T cells, lymphocytes, monocytes, and dendritic cells by attaching to a specific receptor called the CD4 receptor contained in the cell membrane. Along with a chemokine coreceptor, the virus is granted entry into the cell. Inside the host cell, the viral RNA is transcribed into DNA by the enzyme reverse transcriptase. However, the enzyme has no proofreading capacity, so errors often occur during this process, giving rise to 1 to 3 mutations per newly synthesized virus particle. The DNA provirus is then transported into the nucleus and inserts itself into the host cell DNA with the aid of viral integrase. Thus, the viral genetic code becomes a stable part of the cell genome, which is then transcribed into a full-length mRNA by the host cell RNA polymerase. The full-length mRNA would be

(1) the genomes of progeny virus, which would be transported to the cytoplasm for assembly,
(2) translated to produce the viral proteins, including reverse transcriptase and integrase, and
(3) spliced, creating new translatable sequences

The nonstructural genes on the virus also encode regulatory proteins that have diverse effects on the host cell, including down-regulating host cell receptors like CD4 and major histocompatibility complex class I molecules, aiding in synthesizing full-length HIV RNAs and enabling transportation of the viral mRNAs out of the nucleus without being spliced by the host cell. Altogether, these effects enable viral mRNAs to be correctly translated into polypeptides and packaged into virions. These components are then transported to the plasma membrane and assembled into the mature virion, exiting the cell.

A person can contract the virus through one of four routes: sexual contact, either homo- or heterosexual; transfusions with whole blood, plasma, clotting factors and cellular fractions of blood; contaminated needles; perinatal transmission. The virus causes tissue destruction, immunodeficiency and can progress to acquired immunodeficiency syndrome (AIDS), completely breaking down the human body's defense mechanisms. These patients are now more susceptible to infections that should be harmless to a normal person, such as P.jiroveci pneumonia or tuberculosis, and the conditions are worse as well. So far, treatments for the disease mainly target reverse transcriptase, viral proteases, and viral integration and fusion, dealing with the virus infection before it progresses to AIDS. Currently, one treatment for HIV is highly active antiretroviral therapy (HAART), which includes a combination of drugs including nucleoside/nucleotide analog reverse transcriptase inhibitors, nonnucleoside reverse transcriptase inhibitors, protease inhibitors, fusion inhibitors, integrase inhibitors, and coreceptor blockers. These drugs are administered based on individualized criteria such as tolerability, drug-drug interactions, convenience/adherence, and possible baseline resistance. Although HAART can lower the viral load, the virus reemerges if the treatment is stopped. Therefore, HIV infection is currently both chronic and incurable. [28]

Whenever the population reaches $T_{\max}$, it will decrease, allowing us to impose an upper constrain $dT_{\max} < s$. With this constrain, the equation (1.2) has a unique equilibrium at

$$\hat{T} = \frac{T_{\max}}{2a}\left[a - d + \sqrt{(a-d)^2 + \frac{4as}{T_{\max}}}\right] \tag{1.3}$$

In 1989, Perelson [5] proposed a general model for the interaction between the human immune system and HIV; in the same paper, he also simplified that general model into a simpler model with four compartments, whose dynamics are described by a system of four ODEs:

- Concentration of cells that are uninfected (T),
- Concentration of cells that are latently infected (T^*),
- Concentration of cells that are actively infected (T^{**}), and
- Concentration of free infectious virus particles (v).

Later, he extended his own model in Perelson et al. (1993) [6] by proving various mathematical properties of the model, choosing parameter values from a restricted set that give rise to the long incubation period characteristic of HIV infection, and presenting some numerical solutions. He also observed that his model exhibits many clinical symptoms of AIDS, including:

- Long latency period,
- Low levels of free virus in the environment, and
- Depletion of $CD4^+$ cells.

The paper will be organized as follows: First, we will investigate a simplified ODE model from Perelson et al. (1993) [6] by considering three main components: the uninfected $CD4^+$ T-cells (T), the infected $CD4^+$ T-cells (I), and the free virus (V) with. This model is also assumed to have a saturation response of the infection rate. Next, the existence and stability of the infected steady state are considered through different theorems. Finally, numerical simulations are carried out, using `Julia`, to confirm the obtained results, before some remarks are included in the conclusion.

2. The proposal of the ODE model

Simplifying the model proposed in Perelson et al. (1993) [6] by reducing the number of dimensions and assuming that all of the infected cells have the ability of producing virus at an equal rate, we propose the following epidemic model of HIV infection of $CD4^+$ T-cells as follows:

$$\begin{aligned}\frac{dT}{dt} &= s - dT + aT\left(1 - \frac{T}{T_{\max}}\right) - \frac{\beta TV}{1+\alpha V} + \rho I \\ \frac{dI}{dt} &= \frac{\beta TV}{1+\alpha V} - (\delta + \rho)I \\ \frac{dV}{dt} &= qI - cV - k_1 VT\end{aligned} \tag{2.1}$$

where

- $T(t)$ is the concentration of healthy $CD4^+$ T-cells at time t (target cells),
- $I(t)$ is the concentration of infected $CD4^+$ T-cells at time t, and
- $V(t)$ is the viral load of the virions (concentration of free HIV at time t).

In infection modelling, it is very common to augment (2.1) with a "mass-action" term in which the rate of infection is given by βTV. This type of term is sensible, since the virus must interact with T-cells in order to infect and the probability of virus encountering a T-cell at a low concentration environment (where infected cells and viral load's motions are regarded as independent) can be assumed to be proportional to the product of the density, which is called linear infection rate. As a result, it follows that the classical models can assume that T-cells are infected at rate $-\beta TV$ and are generated at rate βTV.

With that simple mass-action infection term, the rates of change of uninfected cells, T, productively infected cells I, and free virus V, would be

$$\begin{aligned}\frac{dT}{dt} &= s - dT + aT\left(1 - \frac{T}{T_{\max}}\right) - \beta TV \\ \frac{dI}{dt} &= \beta TV - \delta I \\ \frac{dV}{dt} &= qI - cV\end{aligned} \tag{2.2}$$

Moreover, although the rate of infection in most HIV models is bilinear for the virus V and the uninfected target cells T, the actual incidence rates are probably not strictly linear for each variable in over the whole valid range. For example, a non-linear or less-than-linear response in V could occur due to the saturation at a high enough viral concentration, where the infectious fraction is significant for exposure to happen very likely. Thus, is it reasonable to assume that the infection rate of HIV modelling in saturated mass action is

$$\frac{\beta TV^x}{1+\alpha V^y}, \quad x, y, \alpha > 0 \tag{2.3}$$

In this paper, we will investigate the viral model with saturation response of the infection rate where $x = y = 1$, for the sake of simplicity. With that being said, we will proceed to explain the parameters within the model, with

- s is the rate at which new T-cells are created from source from precursors,
- d is the natural death rate of the $CD4^+$ T-cells,
- a is the maximum proliferation rate (growth rate) of T-cells (this means that $a > d$, in general),
- $T_{\max}$ is the T-cells population density at which proliferation shuts off (their carrying capacity),
- β is the rate constant of infection of T-cells with free virus,
- ρ is the "cure" rate, or the non-cytolytic loss of infected cells,
- δ is the death rate of the infected cells,
- q is the reproduction rate of the infected cells, and
- c is the clearance rate constant (loss rate) of the virions.

From the explanations above, we can say that

- $\delta+\rho$ is the total rate of disappearance of infected cells from the environment,
- $1/\delta$ is the average lifespan of a productively infected cell
- q/δ is the total number of virions produced by an actively infected cell during its lifespan, and
- q is the average rate of virus released by each cell.

Under the absence of virus (i.e, $I(t) = V(t) = 0 \quad \forall t > 0$), the T-cell population has a steady state value of

$$T_0 = \frac{T_{\max}}{2a}\left[(a-d) + \sqrt{(a-d)^2 + \frac{4a}{T_{\max}}}\right] \tag{2.4}$$

The system (2.1) needs to be initialized with the following initial conditions

$$T(0) > 0, \quad I(0) > 0, \quad V(0) > 0, \tag{2.5}$$

which lead us to denote that

$$R^3_+ = \{(T, I, V) \in \mathbb{R}^3 \| T \geq 0, I \geq 0, V \geq 0\}. \tag{2.6}$$

3. Equilibrium and stability of the proposed model

3.1. Equilibria and local stability. The system (2.1) has two steady states: the uninfected steady state $E_0 = (T_0, 0, 0)$ and the (positive) infected steady state $\bar{E} = \left(\bar{T}, \bar{I}, \bar{V}\right)$, where:

$$\begin{aligned}
\bar{T} &= \frac{T_{\max}}{2a}\left[a - d - \delta\frac{q\beta - (\delta+\rho)}{q\alpha(\delta+\rho)} + \sqrt{\left(a - d - \delta\frac{q\beta - (\delta+\rho)}{q\alpha(\delta+\rho)}\right)^2 - \frac{4a}{T_{\max}}\left(\frac{\delta c}{q\alpha} - s\right)}\right]\\
\bar{I} &= \frac{[q\beta - (\delta+\rho)k_1]\bar{T} - (\delta+\rho)c}{q\alpha(\delta+\rho)}\\
\bar{V} &= \frac{1}{\alpha}\left[\frac{q\beta\bar{T}}{\alpha(\delta+\rho)(c_1 + k_1 T} - 1\right].
\end{aligned} \tag{3.1}$$

Now, we will proceed to analyse the stability of the equilibria of system (2.1).

Since T_0 and $\bar{T}$ satisfy

$$\begin{aligned} s - dT_0 + aT_0\left(1 - \frac{T_0}{T_{\max}}\right) &= 0 \\ s - d\bar{T} + a\bar{T}\left(1 - \frac{\bar{T}}{T_{\max}}\right) &= \delta\bar{I} = \frac{\delta}{q\alpha(\delta+\rho)}\left[(q\beta - (\delta+\rho))T - (\delta+\rho)c\right] \end{aligned} \tag{3.2}$$

we get that

$$\bar{T} > \frac{c(\delta+\rho)}{q\beta - (\delta+\rho)k_1} \quad \Rightarrow \quad s - d\bar{T} + a\bar{T}\left(1 - \frac{\bar{T}}{T_{\max}}\right) > 0 \quad \Rightarrow \quad T_0 > \bar{T}, \tag{3.3}$$

and

$$\bar{T} < \frac{c(\delta+\rho)}{q\beta - (\delta+\rho)k_1} \quad \Rightarrow \quad s - d\bar{T} + a\bar{T}\left(1 - \frac{\bar{T}}{T_{\max}}\right) < 0 \quad \Rightarrow \quad T_0 < \bar{T}. \tag{3.4}$$

Hence,

- If $\bar{T} > \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)k_1}$, then $T_0 > \bar{T} > \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)k_1}$, which means that $E_0(T_0, 0, 0)$ is unstable, while the positive equilibrium $\bar{E}(\bar{T}, \bar{I}, \bar{V})$ exists.
- If $\bar{T} < \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)k_1}$, then $T_0 < \bar{T} < \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)k_1}$, which means that $E_0(T_0, 0, 0)$ is locally asymptotically stable, while the positive equilibrium $\bar{E}(\bar{T}, \bar{I}, \bar{V})$ is not feasible, as $\bar{I} < 0, \bar{V} < 0$.

Let

$$R_0 = \left(\frac{q\beta - (\delta+\rho)k_1}{c(\delta+\rho)}\right)\bar{T}. \tag{3.5}$$

We can see that R_0 is the bifurcation parameter. When $R_0 < 1$, the uninfected steady state E_0 is stable and the infected steady state $\bar{E}$ does not exist (unphysical). When $R_0 > 1$, E_0 becomes unstable and $\bar{E}$ exists.

For system (2.2), it is known that the basic reproductive ratio is given by:

$$R_{01} = \left(\frac{q\beta - (\delta+\rho)k_1}{c(\delta+\rho)}\right)T_0 \tag{3.6}$$

Once again, we emphasize the large difference of the basic reproduction ratio between the linear infection rate and the saturation infection rate.

- If $\alpha \to 0$, then $\bar{T} \to \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)}$, $\quad R_0 \to 1$;
- If $\alpha \to +\infty$, then $\bar{T} \to T_0, R_0 \to R_{01}$.

The `Jacobian matrix` of system (2.1) is:

$$\begin{pmatrix} (a-d) - \frac{2aT}{T_{\max}} - \frac{\beta V}{1+\alpha V} & \rho & -\frac{\beta T}{(1+\alpha V)^2} \\ \frac{\beta V}{1+\alpha V} & -(\delta+\rho) & \frac{\beta T}{(1+\alpha V)^2} \\ -k_1 V & q & -c - k_1 T \end{pmatrix}. \tag{3.7}$$

Let $E^*(T^*, I^*, V^*)$ be any arbitrary equilibrium. Then, the characteristic equation about E^* is:

$$\begin{vmatrix} \lambda + \left((d-a) + \frac{2aT^*}{T_{\max}} + \frac{\beta V^*}{1+\alpha V^*}\right) & -\rho & \frac{\beta T^*}{(1+\alpha V^*)^2} \\ -\frac{\beta V^*}{1+\alpha V^*} & \lambda + (\delta+\rho) & -\frac{\beta T^*}{(1+\alpha V^*)^2} \\ k_1 V^* & -q & \lambda + (c + k_1 T^*) \end{vmatrix} = 0. \tag{3.8}$$

For equilibrium $E_0 = (T_0, 0, 0)$, (3.8) reduces to

$$\left(\lambda - a + d + \frac{2aT_0}{T_{\max}}\right)\left[\lambda^2 + (c+\delta+\rho)\lambda + c(\delta+\rho) - q\beta T_0\right] = 0 \tag{3.9}$$

Hence, we can see that $E_0 = (T_0, 0, 0)$ is locally asymptotically stable if $R_0 < 1$, and it is a saddle point if $\dim W^s(E_0) = 2$, or if $\dim W^s(E_0) = 1$ while $R_0 > 1$. As a result, we have the following theorems.

Theorem 3.1. *If $R_0 < 1$, $E_0 = (T_0, 0, 0)$ is locally asymptotically stable; else, if $R_0 > 1$, $E_0 = E_0 = (T_0, 0, 0)$ is unstable.*

Theorem 3.2. *There exists $M > 0, M \in \mathbb{R}$ such that for any positive solution $(T(t), I(t), V(t))$ of system (2.1),*

$$T(t) \le M, I(t) \le M, V(t) \le M \tag{3.10}$$

for all large enough t.

Proof. Let $L(t) = T(t) + I(t)$ and assume that $L(0) = T(0) + I(0) = \text{const} = c$. Calculating the derivative of $L(t)$ using the equations in system (2.1), we have:

$$\begin{aligned} \frac{dL(t)}{dt} &= \frac{dT(t)}{dt} + \frac{dI(t)}{dt} \\ &= s - dT + aT\left(1 - \frac{T}{T_{\max}}\right) - \delta I \\ &= -dt - \delta I - \frac{a}{T_{\max}}\left(T - \frac{T_{\max}}{2a}\right)^2 + \frac{4s + aT_{\max}}{4} \\ &\le -(T+I)\min(d,\delta) - \frac{a}{T_{\max}}\left(T - \frac{T_{\max}}{2a}\right)^2 + \frac{4s + aT_{\max}}{4} \\ &= -hL(t) - M_0 \left(h = \min(d,\delta), M_0 = \frac{4s + aT_{\max}}{4}\right) \end{aligned} \tag{3.11}$$

Let $U(t) = L(t) - \frac{M_0}{h}$. This means that

$$\begin{aligned} U(0) = L(0) - \frac{M_0}{h} &= c - \frac{M_0}{h} \\ \frac{dU(t)}{dt} &= \frac{dL(t)}{dt} \end{aligned} \tag{3.12}$$

The inequality (3.11) can be rewritten as

$$\frac{dU(t)}{dt} \le (-h)U(t) \tag{3.13}$$

which yields, according to Gronwall's inequality,

$$\begin{aligned} U(t) &\leq U(0)\exp\left(\int_0^t (-h)ds\right) \\ &= \left(c - \frac{M_0}{h}\right)\exp\left([-hs]_0^t\right) \\ &= \left(c - \frac{M_0}{h}\right)\exp(-ht) \\ &\leq c - \frac{M_0}{h} \end{aligned} \tag{3.14}$$

or

$$T(t) + I(t) = L(t) = U(t) + \frac{M_0}{h} = c - \frac{M_0}{h} + \frac{M_0}{h} = c. \tag{3.15}$$

As $T(t) > 0, I(t) > 0\ \forall i \in \mathbb{Z}^+$, we can say that

$$V(t) \leq c, \quad I(t) \leq c. \tag{3.16}$$

Moreover, we also know that

$$\frac{dV}{dt} = qI - cV - k_1VT \leq qI - cV \leq qc - cV = -c(V - q). \tag{3.17}$$

Setting $V(0) = \text{const} = c_V$, using the exact same procedure with Gronwall's inequality, we obtain

$$V(t) \leq c_V \quad \forall t \in \mathbb{Z}^+. \tag{3.18}$$

With $M = \max(c, c_V)$, we would then conclude that

$$T(t) \leq M, \quad I(t) \leq M, \quad V(t) \leq M\ \forall t \in \mathbb{Z}^+. \tag{3.19}$$

We can easily see that this set is convex. As a consequence, the system (2.1) is dissipative.

The proof is complete. □

From this theorem, we define

$$D = \left\{(T, I, V) \in \mathbb{R}^3, 0 \leq T, I, V \leq M\right\}. \tag{3.20}$$

Denote

$$M = d - a + \frac{2a\bar{T}}{T_{\max}}, \quad N = \frac{\beta\bar{V}}{1 + \alpha\bar{V}}, \quad P = \frac{\beta\bar{T}}{(1 + \alpha\bar{V})^2}. \tag{3.21}$$

Then, the characteristic equation of the system around the equilibrium $\bar{E}(\bar{T}, \bar{I}, \bar{V})$ reduces to:

$$\lambda^3 + a_1\lambda^2 + (a_2 + a_4)\lambda + (a_3 + a_5) = 0, \tag{3.22}$$

where

$$\begin{aligned} a_1 &= M + (\delta + \rho + c_1 + k_1\bar{T}) \\ a_2 &= (\delta+\rho)(c_1 + k_1 T) + M(\delta+\rho+c_1+k_1\bar{T}) + (-k_1\bar{V}P) \\ a_3 &= \rho\left[-N(c_1+k_1\bar{T}) + Pk_1\bar{V}\right] + PNq \\ a_4 &= -NP \\ a_5 &= M(\delta+\rho)(c_1+k_1\bar{T}) - P(\delta+\rho)k_1\bar{V}. \end{aligned} \tag{3.23}$$

By the Routh-Hurwitz criterion [15], it follows that all eigenvalues of equation (3.22) have negative real parts if and only if

$$a_1 > 0, \quad a_3 + a_5 > 0, \quad a_1(a_2 + a_4) - (a_3 + a_5) > 0. \tag{3.24}$$

This leads us to the following theorems.

Theorem 3.3. *Suppose that*

(1) $R_0 > 1$,
(2) $a_1 > 0, \quad a_3 + a_5 > 0, \quad a_1(a_2 + a_4) - (a_3 + a_5) > 0.$

Then, the positive equilibrium $\bar{E}(\bar{T}, \bar{I}, \bar{V})$ *is asymptotically stable.*

Theorem 3.4. *If* $R_0 < 1$, *then* $E_0(T_0, 0, 0)$ *is globally asymptotically stable.*

Proof. First of all, as $R_0 < 1$, we would have

$$T_0 < \bar{T} < \frac{c(\delta+\rho)}{q\beta - (\delta+\rho)} \tag{3.25}$$

which means that

$$p < \frac{(c + k_1 T)(\delta+\rho)}{\beta T}. \tag{3.26}$$

From the system (2.1), we would have

$$\begin{aligned} \frac{dI}{dt} &\le \beta T V - (\delta+\rho)I, \\ \frac{dV}{dt} &= qI - cV - k_1 V T. \end{aligned} \tag{3.27}$$

Now, we would consider the following comparative system

$$\begin{aligned} \frac{dz_1}{dt} &= \beta T z_2 - (\delta+\rho)z_1 \\ \frac{dz_2}{dt} &= p z_1 - c z_2 - k_1 z_2 T. \end{aligned} \tag{3.28}$$

We will consider the following form of Lyapunov function:

$$L(\mathbf{X}) = V(z_1, z_2) = \frac{\delta+\rho}{(\beta T)^2} z_1^2 + \frac{1}{c + k_1 T} z_2^2. \tag{3.29}$$

The derivative of the function can be calculated as follows

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial z_1}\frac{dz_1}{dt} + \frac{\partial L}{\partial z_2}\frac{dz_2}{dt} \\
&= 2\frac{\delta+\rho}{(\beta T)^2} z_1 \left(\beta T z_2 - (\delta+\rho)z_1\right) + 2\frac{1}{c+k_1 T} z_2 \left(q z_1 - c z_2 - k_1 T z_2\right) \\
&= -2\left[\left(\frac{\delta+\rho}{\beta T} z_1\right)^2 + z_2^2 - \left(\frac{\delta+\rho}{\beta T} z_1 z_2 + \frac{q}{c+k_1 T}\right) z_1 z_2\right] \\
&\le -2\left[\left(\frac{\delta+\rho}{\beta T} z_1\right)^2 + z_2^2 - \left(\frac{\delta+\rho}{\beta T} + \frac{\beta+\rho}{\beta T}\right) z_1 z_2\right] \\
&= -2\left[\frac{\delta+\rho}{\beta T} z_1 - z_2\right]^2 \le 0 \quad \forall z_1, z_2.
\end{aligned}
\tag{3.30}
$$

We can see that the derivative is negative definite everywhere except at $(0,0)$. This means that $(z_1, z_2) = (0,0)$ is globally asymptotically stable.

As we can also see that

$$0 \le I(0) \le z_1(0), \quad 0 \le V(0) \le z_2(0) \tag{3.31}$$

which means that, if the system (3.28) admits the initial values $(z_1(0), z_2(0))$, we have that

$$I(t) \le z_1(t), \quad V(t) \le z_2(t) \quad \forall t > t_1 \tag{3.32}$$

or, in other words,

$$\lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} V(t) = 0. \tag{3.33}$$

From this, using the first equation of the system (2.1), for an $\epsilon\ in(0,1)$ infinitesimal,

$$s + (a - d - \delta\epsilon)T - \frac{aT^2}{T_{\max}} \le \frac{dT(t)}{dt} \le s + (a-d)T - \frac{aT^2}{T_{\max}} \quad \forall t > t_2 \tag{3.34}$$

which shows that

$$\lim_{t\to+\infty} T(t) = T_0. \tag{3.35}$$

From (3.33) and (3.35), we conclude that the system is globally asymptotically stable. The proof is complete. □

Theorem 3.5. *If $R_0 > 1$, then the system* (2.1) *is permanent.*

Proof. If $R_0 > 1$, we would have

$$(q\beta - (\delta+\rho)k_1)T_0 > (q\beta - (\delta+\rho)k_1)\bar{T} > c(\delta+\rho) \tag{3.36}$$

We will proceed to prove the weak permanence of this system using contradiction.

Assume that the system is not weakly permanent, from Theorem 3.4, there exists a positive orbit $(T(t), I(t), V(t))$ such that

$$\lim_{t\to+\infty} T(t) = T_0, \quad \lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} V(t) = 0. \tag{3.37}$$

Since $T_0 > \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)}$, combining with (3.37), we choose an arbitrary infinitesimal $\epsilon > 0$ such that there exists a $t_0 > 0$, for all $t > t_0$,

$$\begin{aligned} \frac{T_0-\epsilon}{1+\alpha\epsilon} &> \frac{c(\delta+\rho)}{q\beta-(\delta+\rho)} \\ T(t) &> T_0-\epsilon, \\ V(t) &< \epsilon. \end{aligned} \tag{3.38}$$

Under these conditions, the system (2.1) becomes

$$\begin{aligned} \frac{dI}{dt} &= \frac{\beta TV}{1+\alpha V} - (\delta+\rho)I \geq \frac{\beta(T_0-\epsilon)}{1+\alpha\epsilon}V - (\delta+\rho)I(t) \\ \frac{dV}{dt} &= qI - (c_1 + k_1 T) \approx qI - cV - k_1 T_0 \end{aligned} \tag{3.39}$$

Consider the following Jacobian matrix

$$J_\epsilon = \begin{pmatrix} -(\delta+\rho) & \frac{\beta(T_0-\epsilon)}{1+\alpha\epsilon} \\ q & -(c+k_1T_0). \end{pmatrix} \tag{3.40}$$

Since J_ϵ has positive off-diagonal element, according to the Perron - Frobenius theorem, for the maximum positive eigenvalue j_1 of J_ϵ, there is an associated positive eigenvector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Next, we consider a system associated with the Jacobian matrix J_ϵ

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{\beta(T_0-\epsilon)}{1+\alpha\epsilon}z_2 - (\delta+\rho)z_1 \\ \frac{dz_2}{dt} &= qz_1 - (c+k_1T_0)z_2. \end{aligned} \tag{3.41}$$

Let $z(t) = (z_1(t), z_2(t))$ be a solution of (3.41) through (lv_1, lv_2) at $t = t_0$, where $l > 0$ satisfies that

$$lv_1 < I(t_0), \quad lv_2 < V(t_0). \tag{3.42}$$

As we know that the semi-flow of (3.41) is monotone and $J_\epsilon v = v > 0$, $z_i(t)(t = 1, 2)$ is strictly increasing, meaning $\lim_{t\to+\infty} z_i(t) = +\infty$. This contradicts the Theorem 3.2, saying that the positive solution of (2.1) is bounded from above. This contradiction says that there exists no positive orbit of (2.1) tends to $(T_0, 0, 0)$ and $t \to +\infty$. Combining this and a result provided in [23], we conclude that the system (2.1) is permanent.

The proof is complete.

□

Theorem 3.6. *Assume that D is convex and bounded. Suppose that the system*

$$\frac{dX}{dt} = F(X), \quad X \in D \tag{3.43}$$

is competitive, permanent and has the property of stability of periodic orbits. If $\bar{X}_0$ is the only equilibrium point in intD and if it is locally asymptotically stable, then it is globally asymptotically stable in intD.

Proof. This matrix can easily be proven by considering the Jacobian matrix and choose the matrix H as

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.44}$$

By simple calculation, we obtain that

$$H\frac{\partial f}{\partial x}H = \begin{pmatrix} (a-d) - \frac{2aT}{T_{\max}} - \frac{\beta V}{1+\alpha V} & -\rho & -\frac{\beta T}{(1+\alpha V)^2} \\ -\frac{\beta V}{1+\alpha V} & -(\delta+\rho) & -\frac{\beta T}{(1+\alpha V)^2} \\ -k_1 V & -q & -c - k_1 T \end{pmatrix}. \tag{3.45}$$

This means that the system (2.1) is competitive in D, with respect to the partial order defined by the orthant

$$K = \left\{(T, I, V) \in \mathbb{R}^3 \| T \leq 0, I \geq 0, V \geq 0\right\}. \tag{3.46}$$

□

Remark. *As D is convex and the system* (2.1) *is competitive in D, we can say that the system* (2.1) *satisfies the Poincare - Bendixson property. This has been proven by Hirsch (1990)* [22]*, Zhu and Smith (1994)* [21] *and Smith and Thieme (1991)* [24] *that any three-dimensional competitive system that lie in convex sets would have the Poincaré - Bendixson property; in other words, any non-empty compact omega limit set that contains no equilibria must be a closed orbit.*

Theorem 3.7. *Let $c = I(0) + T(0)$ and suppose that*

(1) $R_0 > 1$,

(2) $a_1 > 0, a_3 + a_5 > 0, a_1(a_2 + a_4) - (a_3 + a_5) > 0.$

Then, the positive equilibrium $\bar{E}(\bar{T}, \bar{I}, \bar{V})$ of system (2.1) *is globally asymptotically stable provided that one of the following two assumptions hold*

(3) $T_{\max}\frac{a-d+k_1c}{2a} < m < T_0 < T_{\max}\frac{a-d+\delta+k_1c}{2a}$,

(4) $m > T_{\max}\frac{a-d+\delta+k_1c}{2a}$.

As we have already known that the system (2.1) is competitive and permanent (from Theorem 3.5 and Theorem 3.6), while $\bar{E}(\bar{T}, \bar{I}, \bar{V})$ is locally asymptotically stable if the two properties (i) and (ii) of Theorem 3.7 holds. As a result, in accordance with Theorem 3.6 (choosing $D = \Omega$), Theorem 3.7 if we can prove that the system (2.1) has the stability of periodic orbits. We will proceed to prove this under the following proposition.

Proposition 3.8. *Assume condition 3. or 4. of Theorem 3.7 hold true. Then, system* (2.1) *has the property of stability of periodic orbits.*

Proof. Let $P(t) = ((T(t), I(t), V(t))$ be a periodic solution whose orbit Γ is contained in intΩ. In accordance with the criterion given by Muldowney in [25], for the asymptotic orbital stability of a periodic orbit of a general autonomous system, it is sufficient to prove that the linear non-autonomous system

$$\frac{dW(t)}{dt} = \left(DF^{[2]}\left(P(t)\right)\right)W(t) \tag{3.47}$$

is asymptotically stable, where $DF^{[2]}$ is the second additive compound matrix of the Jacobian DF [1].

The Jacobian matrix of the system (2.1) is given by

$$J = \begin{pmatrix} (a-d) - \frac{2aT}{T_{\max}} - \frac{\beta V}{1+\alpha V} & \rho & -\frac{\beta T}{(1+\alpha V)^2} \\ \frac{\beta V}{1+\alpha V} & -(\delta+\rho) & \frac{\beta T}{(1+\alpha V)^2} \\ -k_1 V & q & -(c+k_1 T) \end{pmatrix}. \tag{3.48}$$

For the solution $P(t)$, the equation (3.47) becomes

$$\begin{aligned} \frac{dW_1}{dt} &= -\left(\delta + \rho - (a-d) + \frac{2aT}{T_{\max}} + \frac{\beta V}{1+\alpha V}\right) W_1 + \frac{\beta T}{(1+\alpha V)^2}(W_2 + W_3), \\ \frac{dW_2}{dt} &= qW_1 + \left(a - d - \frac{2aT}{T_{\max}} - \frac{\beta V}{1+\alpha V} - (c+k_1 T)\right) W_2 + \rho W_3, \\ \frac{dW_3}{dt} &= k_1 V W_1 + \frac{\beta V}{1+\alpha V} W_2 - (\delta + \rho + c + k_1 T) W_3. \end{aligned} \tag{3.49}$$

To prove that the system (3.49) is asymptotically stable, we shall use the following Lyapunov function, which is similar to the one found in [26] for the SEIR model:

$$L(W_1(t), W_2(t), W_3(t), T(t), I(t), V(t)) = \left\| \left(W_1(t), \frac{I(t)}{V(t)} W_2(t), \frac{I(t)}{V(t)} W_3(t) \right) \right\|, \tag{3.50}$$

where $\|\cdot\|$ is the norm in $\mathbb{R}^3$ defined by

$$\|(W_1, W_2, W_3)\| = \sup\{|W_1|, |W_2 + W_3|\}. \tag{3.51}$$

From Theorem 3.5, we obtain that the orbit of $P(t)$ remains at a positive distance from the boundary of Ω. Therefore,

$$I(t) \geq \eta, \quad V(t) \geq \eta, \quad \eta = \min\{\underline{\mathrm{I}}, \underline{\mathrm{V}}\} \quad \forall t \to +\infty. \tag{3.52}$$

Hence, the function $L(t)$ is well defined along $P(t)$ and

$$L(W_1, W_2, W_3; T, I, V) \geq \frac{\eta}{M} \left\|(W_1, W_2, W_3)\right\|. \tag{3.53}$$

Along a solution (W_1, W_2, W_3) of the system (3.49), $L(t)$ becomes

$$L(t) = \sup\left\{ |W_1(t)|, \frac{I(t)}{V(t)} \left(|W_2(t)| + |W_3(t)|\right) \right\}. \tag{3.54}$$

Then, we would have the following inequalities

$$\begin{aligned} D_+|W_1(t)| &\leq -\left(\delta + \rho - (a-d) + \frac{2aT}{T_{\max}} + \frac{\beta V}{1+\alpha V}\right)|W_1| + \frac{\beta T}{(1+\alpha V)^2}\left(|W_2(t)| + |W_3(t)|\right) \\ D_+|W_2(t)| &\leq q|W_1(t)| + \left(a - d - \frac{2aT}{T_{\max}} - \frac{\beta V}{1+\alpha V} - (c+k_1 T)\right)|W_2(t)| + \rho|W_3(t)| \\ D_+|W_3(t)| &\leq k_1 V|W_1(t)| + \frac{\beta V}{1+\alpha V}|W_2(t)| - (\delta+\rho+c+k_1 T)|W_3(t)|. \end{aligned} \tag{3.55}$$

From this, we get

$$
\begin{aligned}
D_+\frac{I}{V}(|W_2|+|W_3|) &= \left(\frac{dI/dt}{V}-\frac{IdV/dt}{V^2}\right)(|W_2|+|W_3|)+\frac{I}{V}D_+(|W_2|+|W_3|)\\
&\le \left(\frac{dI/dt}{I}-\frac{dV/dt}{V}\right)\frac{I}{V}(|W_2|+|W_3|)+\left(\frac{qI}{V}+k_1 I\right)|W_1|\\
&-\left(-a+d+\frac{2aT}{T_{\max}}+(c+k_1T)\right)\frac{I}{V}|W_2(t)|-(\delta+c+k_1T)\frac{I}{V}|W_3(t)|.
\end{aligned}
\tag{3.56}
$$

Thus, we can obtain

$$D_+L(t)\le \sup\{g_1(t),g_2(t)\}L(t), \tag{3.57}$$

where

$$
\begin{aligned}
g_1(t) &= -\delta-\rho+a-d-\frac{2aT}{T_{\max}}-\frac{\beta V}{1+\alpha V}+\frac{\beta TV}{I(1+\alpha V)^2}\\
g_2(t) &= \frac{qI}{V}+k_1I+\frac{dI/dt}{I}-\frac{dV/dt}{V}-G_1\\
G_1 &= \min\left\{-a+d+\frac{2aT}{T_{\max}}+(c+k_1T),\delta+c+k_1T\right\}.
\end{aligned}
\tag{3.58}
$$

From the second equation of the system (2.1), we obtain

$$
\begin{aligned}
g_1(t) &= -\delta-\rho+a-d-\frac{2aT}{T_{\max}}-\frac{\beta V}{1+\alpha V}+\frac{\beta TV}{I(1+\alpha V)^2}\\
&\le -\delta-\rho+a-d-\frac{2aT}{T_{\max}}-\frac{\beta V}{1+\alpha V}+\frac{\beta tV}{I(1+\alpha V)}\\
&= a-d-\frac{2aT}{T_{\max}}-\frac{\beta T}{1+\alpha V}+\frac{dI/dt}{I}.
\end{aligned}
\tag{3.59}
$$

Here, we consider two different cases.

- **Case 1:** If Point 3. of Theorem 3.7 holds, then

  $$-\delta<a-d-\frac{2aT}{T_{\max}}<0, \tag{3.60}$$

  that is

  $$G_1=-a+d+\frac{2aT}{T_max}+(c+k_1T). \tag{3.61}$$

  Then, we would obtain

  $$g_2(t)=a-d-\frac{2aT}{T_{\max}}+k_1I+\frac{dI/dt}{I}=g_1(t)+k_1I+\frac{\beta V}{1+\alpha V}>g_1(t). \tag{3.62}$$

  Hence,

  $$\sup\{g_1(t),g_2(t)\}\le a-d-\frac{2aT}{T_{\max}}+k_1I+\frac{dI/dt}{I}\le -\mu_1+\frac{dI/dt}{I}, \tag{3.63}$$

  where

  $$\mu_1>0,\quad a-d-\frac{2aT}{T_{\max}}+k_1I\le -\mu_1<0 \tag{3.64}$$

with the assumption that $k_1 I$ is negligible compare to the term $a - \frac{2aT}{T_{\max}}$. This assumption would be verified in the examples of the simulation part below.

- **Case 2:** If Point 4. of Theorem 3.7 holds, then

$$-a + d + \frac{2aT}{T_{\max}} \leq \delta, \tag{3.65}$$

which means that $G_1 = \delta + c + k_1 T$. Then, we obtain that

$$\mu_2 < 0, \quad g_1(t) < g_2(t) = k_1 T - \delta + \frac{dI/dt}{I} \leq -\mu_2 + \frac{dI/dt}{I} \tag{3.66}$$

with the same assumption that $k_1 T < \sigma$ in reasonably practical scenarios. Hence,

$$\sup\{g_1(t), g_2(t)\} \leq -\mu + \frac{dI/dt}{I}. \tag{3.67}$$

Let $\mu = \min\{\mu_1, \mu_2\}$. Then, form (3.63) and (3.66), we have

$$\sup\{g_1(t), g_2(t)\} \leq -\mu + \frac{dI/dt}{I}, \tag{3.68}$$

or

$$D_+ L(t) \leq \left(-\mu + \frac{dI/dt}{I}\right) L(t). \tag{3.69}$$

According to Gronwall's inequality, we would have

$$\begin{aligned} L(t) &\leq L(0) \exp\left(\int_0^t \left[-\mu + \frac{dI/dt}{I}\right] ds\right) \\ &= L(0) \exp\left([-\mu s + \ln(I(s))]_0^t\right) \\ &= L(0) \exp(-\mu t) \exp\left(\ln(I(t)) - \ln(I(0))\right) \\ &= L(0) \exp(-\mu t) \frac{I(t)}{I(0)} \\ &\leq \frac{M L(0)}{I(0)} \exp(-\mu t) \to 0 \quad \text{as} \quad t \to +\infty. \end{aligned} \tag{3.70}$$

From (3.53), we conclude that

$$(W_1(t), W_2(t), W_3(t)) \to 0 \quad \text{as} \quad t \to +\infty. \tag{3.71}$$

This implies that the linear system equation (3.49) is asymptotically stable, and, therefore, the periodic solution is asymptotically orbitally stable. This proves proposition 3.8.

□

Theorem 3.9. *Suppose that*

(1) $R_0 > 1$,
(2) $a_1 > 0, a_3 + a_5 > 0, a_1(a_2 + a_4) - (a_3 + a_5) > 0.$

Then, system (2.1) *has an orbitally asymptotically stable periodic solution.*

Proof. First, we perform a change of variables as follows:

$$z_1(t) = -T(t), \quad z_2(t) = I(t), \quad z_3(t) = -V(t) \tag{3.72}$$

Applying this transformation to the system (2.1), we obtain

$$\begin{aligned} \frac{dz_1(t)}{dt} &= -s - dz_1 + az_1\left(1 + \frac{z_1}{T_{\max}}\right) + \frac{\beta z_1 z_3}{1 - \alpha z_3} + \rho z_2 \\ \frac{dz_2(t)}{dt} &= \frac{\beta z_1 z_3}{1 - \alpha z_3} - (\delta + \rho) z_2 \\ \frac{dz_3(t)}{dt} &= -qz_2 - cz_3 + k_1 z_1 z_3. \end{aligned} \tag{3.73}$$

The Jacobian matrix of the system (3.73) is then given by

$$J(z) = \begin{pmatrix} a - d + \frac{2az_1}{T_{\max}} + \frac{\beta z_3}{1-\alpha z_3} & \rho & \frac{\beta z_1}{(1+\alpha z_3)^2} \\ \frac{\beta z_3}{1-\alpha z_3} & -(\delta+\rho) & \frac{\beta z_1}{(1+\alpha z_3)^2} \\ k_1 z_3 & -q & -c + k_1 z_1. \end{pmatrix} \tag{3.74}$$

Similar to the definition of the set D at 3.20, we define set E as:

$$E = \{(z_1, z_2, z_3) : z_1 \leq 0, z_2 \geq 0, z_3 \leq 0\}. \tag{3.75}$$

Since $J(z)$ has non-positive off diagonal elements at each point of E, (3.73) is competitive at E. Set $z^* = (-T^*, I^*, V^*)$. It is easy to see that z^* is unstable and $\det J(z^*) < 0$. Furthermore, it follows from Theorem 3.5 that there exists a compact set B in the interior of E such that for any $z_0 \in \mathrm{int}E$, there exists $T(z_0) > 0$ such that $z(t, z_0) \in B$ for all $t > T(z_0)$. Consequently, by Theorem 1.2 in Zhu and Smith (1994) [21] for the class of three-dimensional competitive systems, it has an orbitally asymptotically stable periodic solution.

The proof is complete. □

4. Numerical simulation

After providing all the analytical tools and qualitatively analysing the system for patterns on its dynamics, in this section, we will perform some numerical analysis on the model to verify the previous results.

4.1. **Simulation tools.**

The numerical simulation is conducted on the programming language `Julia` through the package `DifferentialEquation.jl`, A Performant and Feature-Rich Ecosystem for Solving Differential Equations in Julia by Rackauckas and Nie (2017) [29].

In order to avoid any stiffness in the ODE models, the algorithm for the Method of Steps in `Julia` is set to `Rosenbrock23`, which is the same as the classic ODE solver `ode23s` in `MATLAB`.

The simulation is conducted on a system with a 2.0 GHz dual core Intel core i5 with 16GB of RAM.

Parameters and Variables		Values
Dependent variables		
T	Uninfected CD4$^+$ T-cell population size	250 mm^{-3}
I	Infected CD4$^+$ T-cell density	50 mm^{-3}
V	Initial density of HIV RNA	160 mm^{-3}
Parameters and Constants		
s	Source term for uninfected CD4$^+$ T-cells	5 day^{-1} mm^{-3}
d	Natural death rate of CD4$^+$ T-cells	0.01 day^{-1}
a	Growth rate of CD4$^+$ T-cell population	0.8 day^{-1}
$T_{\max}$	Maximal population level of CD4$^+$ T-cells	1500 mm^{-3}
β	Rate CD4$^+$ T-cells became infected with virus	2.4×10^{-4} mm^{-3}
α	Saturated mass-action term	0.001
ρ	Rate of cure	0.01 day^{-1}
δ	Blanket death rate of infected CD4$^+$ T-cells	0.3 day^{-1}
q	Reproduction rate of the infected CD4$^+$ T-cells	500 mm^{-3} day^{-1}
c	Death rate of free virus	8 day^{-1}

Table 1. Preliminary values of variables and parameters for viral spread.

Parameters	Original scenario	Scenario #2	Scenario #3	Scenario #4
s	5	–	–	–
d	0.01	–	–	–
a	0.8	8	–	–
$T_{\max}$	1500	–	–	–
β	2.4×10^{-4}	–	0.0024	0.0024
α	0.001	0.0001	0.000001	0.000001
ρ	0.01	0.01	–	–
δ	0.3	5	–	–
q	500	–	2.5	2.5
c	8	1.3	3	1.3

Table 2. Values of parameters for viral spread in different scenarios.

4.2. **Simulation results.** Within the range of parameters that are proven to be realistic in medical research, we investigate the behavior of the model within 4 different scenarios.

- **The original scenario:** In this scenario, the conditions 1, 2 and 3 in Theorem 3.7 are satisfied. This means that, the positive equilibrium of the system (2.1) is globally asymptotically stable.
- **Scenario #2:** In this scenario, the conditions 1, 2 and 4 in Theorem 3.7 are satisfied. This means that, the positive equilibrium of the system (2.1) is also globally asymptotically stable.
- **Scenario #3:** In this scenario, the conditions 1 and 2 of Theorem 3.3 are satisfied. This means that, the positive equilibrium of the system (2.1) is locally asymptotically stable.
- **Scenario #4:** In this scenario, the conditions 1 and 2 of Theorem 3.9 are satisfied. This means that, the positive equilibrium of the system (2.1) is orbitally asymptotically stable.

Figure 1. The ODE model is locally asymptotically stable with parameters in the **original scenario**

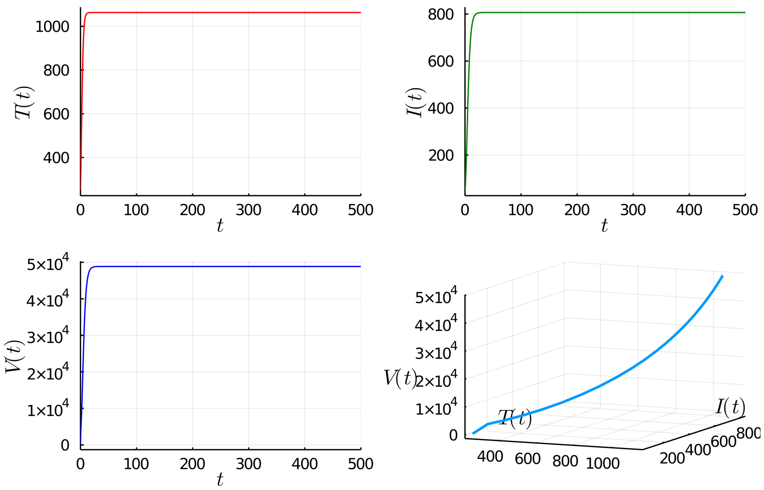

Figure 2. The ODE model is locally asymptotically stable with parameters in **Scenario #2**

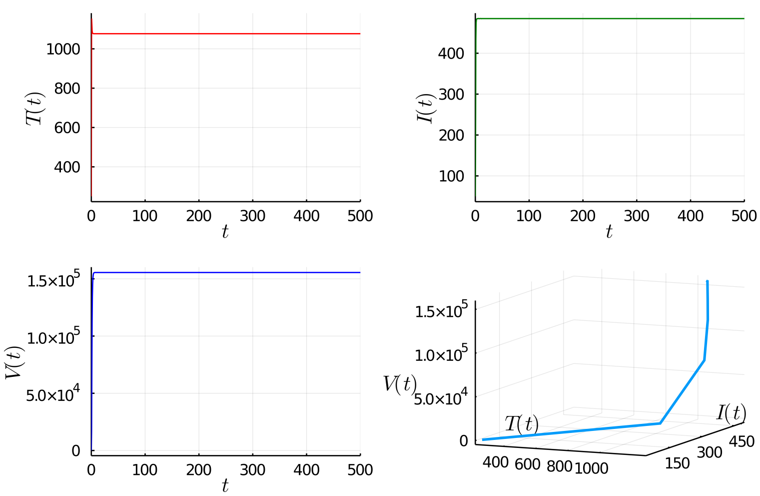

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FIGURE 3. The ODE model is locally asymptotically stable with parameters in **Scenario #3**

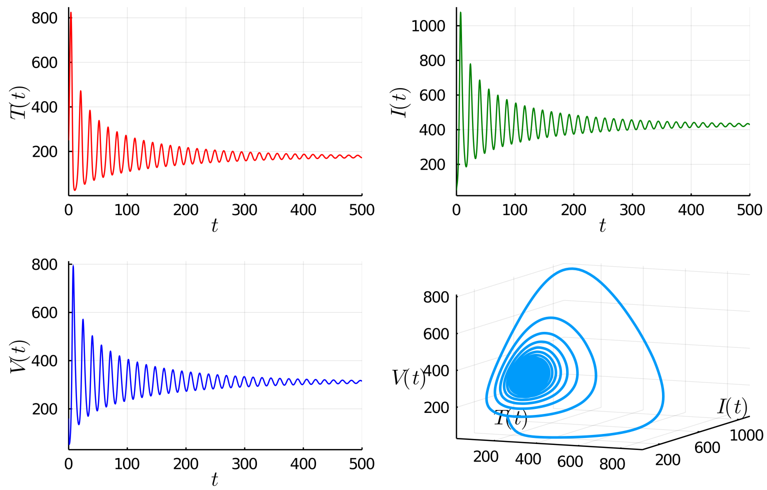
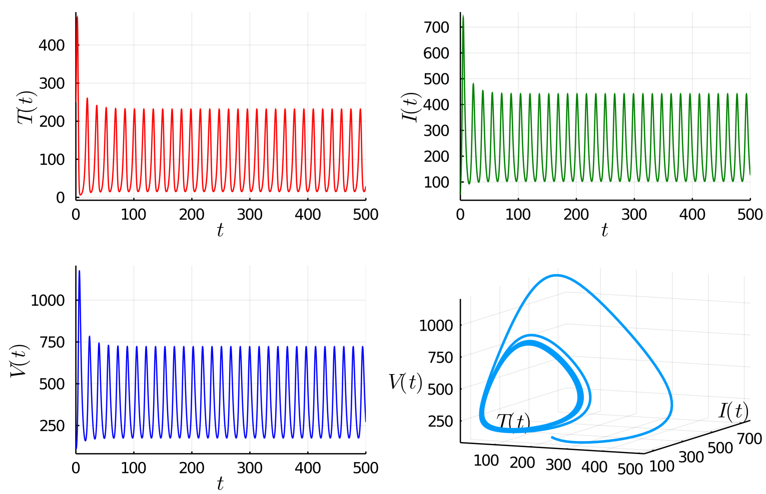


FIGURE 4. The ODE model is orbitally asymptotically stable with parameters in **Scenario #4**



APPENDIX A. DETAILED PROOF OF USED THEOREMS

Theorem A.1 (Gronwall, 1919)**.** *Let I denote an interval of the real line of the form $[a, \inf)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let β and u be real-valued continuous functions defined on I. If u is a differentiable function in the interior I^0 of I (the interval I without the end points a and possibly b) and satisfies the differential inequality*

$$u'(t) \leq \beta(t)u(t), t \in I^0 \tag{A.1}$$

then u is bounded by the solution of the corresponding differential equation $\nu'(t) = \beta(t)\nu(t)$:

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)ds\right). \tag{A.2}$$

Theorem A.2 (Lyapunov's stability)**.** *Let a function $V(\mathbf{X})$ be continuously differentiable in a neighbourhood U of the origin. The function $V(\mathbf{X})$ is called the Lyapunov function for an autonomous system*

$$\mathbf{X}' = f(\mathbf{X}) \tag{A.3}$$

if the following conditions are met:

(1) $V(\mathbf{X}) > 0$ for all $\mathbf{X} \in U \setminus \{0\}$;
(2) $V(0) = 0$;
(3) $\frac{dV}{dt} \leq 0$ for all $\mathbf{X} \in U$.

Then, if in a neighborhood U of the zero solution $\mathbf{X} = 0$ of an autonomous system there is a Lyapunov function $V(\mathbf{X})$ with a negative definite derivative $\frac{dV}{dt}$ for all $\mathbf{X} \in U \setminus \{0\}$, then the equilibrium point $\mathbf{X} = 0$ of the system is asymptotically stable.

Theorem A.3 (Perron - Frobenius Theorem)**.** [20] *Let A be an irreducible Metzler matrix (A Metzler matrix is a matrix whose all of its off-diagonal elements are non-negative). Then, λ_M, the eigenvalue of A of largest real part is real, and the elements of its associated eigenvector v_M are positive. Moreover, any eigenvector of A with non-negative elements belongs the the span of v_M.*

Theorem A.4 (Poincaré - Bendixson Theorem)**.** [3]

Given a differentiable real dynamical system defined on an open subset of the plane, every non-empty compact ω-limit set of an orbit, which contains only finitely many fixed points, is either

- *a fixed point,*
- *a periodic orbit, or*
- *a connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.*

Moreover, there is at most one orbit connecting different fixed points in the same direction. However, there could be countably many homoclinic orbits connecting one fixed point.

Next, we will give the definition of an additive compound matrix and consider the particular case when it's a square matrix [1]. A survey of properties of additive compound matrices, along with their connections to differential equations have been investigated in [25, 26].

We will start with the definition of the k-th exterior power (or multiplicative compound) of an $n \times m$ matrix.

Definition A.1 (Multiplicative compound of a matrix)**.** *Let A be an $n \times m$ matrix of real or complex numbers. Let $a_{i_1,i_2,...,i_k,j_1,j_2,...,j_k}$ be the minor of A determined by the rows $(i_1, ..., i_k)$ and the columns $(j_1, ..., j_k)$, $1 \leq i_1 < i_2 < ... < i_k \leq n, 1 \leq j_1 < j_2 < ... < j_k \leq m$. The k-th multiplicative compound matrix $A^{(k)}$ of A is the $\binom{n}{k} \times \binom{m}{k}$ matrix whose entries, written in lexicographic order, are $a_{i_1,...,i_k,j_1,...,j_k}$.*

In particular, when A is an $n \times k$ matrix with columns $a_1, a_2, ..., a_k$, $A^{(k)}$ is the exterior product $a_1 \vee a_2 \vee ... \vee a_k$.

In the case $m = n$, the additive compound matrices are defined as follows.

Definition A.2. *Let A be an $n \times n$ matrix. The k-th additive compound $A^{[k]}$ of A is the $\binom{n}{k} \times \binom{n}{k}$ matrix given by*

$$A[k] = D(I + hA)\|_{h=0}. \tag{A.4}$$

If $B = A^{[k]}$, the following formula for $b_{i,j}$ can be deduced from the equation (A.4)*. For any integer $i = 1, ..., \binom{n}{k}$, let $(i) = (i_1, i_2, ..., i_k)$ be the i-th member in the lexicographic ordering of all k-tuples of integers such that $1 \le i_1 < i_2 < ... < i_k \le n$. Then,*

$$b_{i,j} = \begin{cases} a_{i_1,i_1} + ... + a_{i_k,i_k} & \text{if } (i) = (j) \\ (-1)^{r+s} a_{i_s,j_r} & \text{if exactly one entry } i_s \text{ in } (i) \text{ does not occur in } (j) \\ & \text{and } j_r \text{ does not occur in } (i), \\ 0 & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries.} \end{cases} \tag{A.5}$$

In the extreme cases when $k = 1$ and $k = n$, we would have that $A^{[1]} = A$ and $A^{[n]} = \mathrm{tr}(A)$. For $n = 3$,which is the case that we are considering in this paper, we would have the matrices $A^{[k]}, k = 0, 1, 2$ as follows:

$$A^{[1]} = A, \quad A^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}, \quad A^{[3]} = a_{11} + a_{22} + a_{33}. \tag{A.6}$$

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