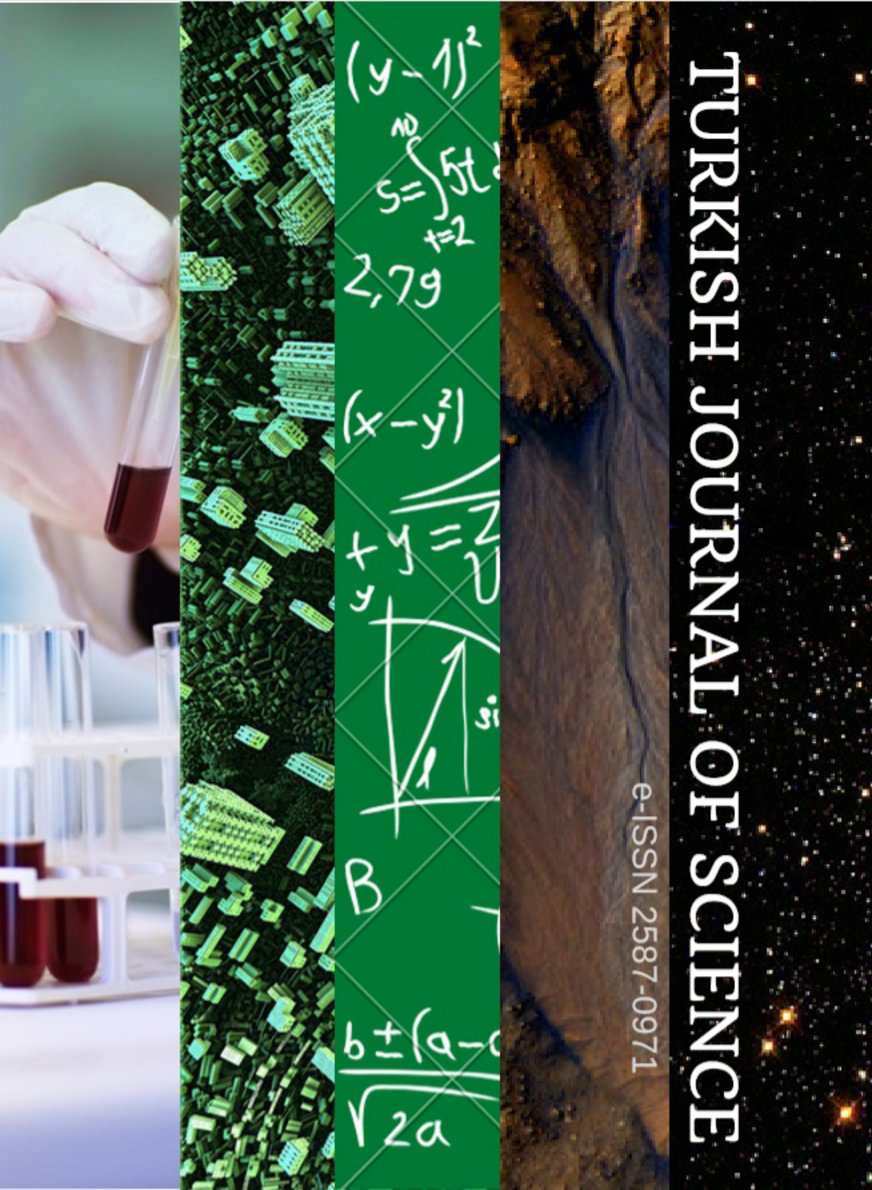
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New Refinements of Hadamard Integral Inequality via k-Fractional Integrals for P- Convex Function

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Abstract. In this study, we use k-fractional integrals to establish some new integral inequalities for p-convex function. These integral inequalities includes some new estimations for Hadamard inequality via k-fractional integrals.

1. Introduction

A function $\rho[\varepsilon, \epsilon'] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $u, v \in [\varepsilon, \epsilon']$ and $t \in [0, 1]$, the following inequality holds:

$$\rho(tu + (1-t)v) \leq t\rho(u) + (1-t)\rho(v).$$

We say that ρ is concave if $(-\rho)$ is convex. If ρ is both convex and concave, then ρ is to be said affin function. The affine functions are in the form $\varepsilon_1 u + \epsilon'_1$,for suitable constants $\varepsilon_1, \epsilon'_1$.

This definition has its origins in Jensen's results and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

The following double inequality is well known in the literature as Hadamard's inequality:

Let $\rho : [\varepsilon, \epsilon'] \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an subinterval of real numbers, $\eta, \eta' \in [\varepsilon, \epsilon']$ and $\eta < \eta'$, we have

$$\rho\left(\frac{\eta + \eta'}{2}\right) \leq \frac{1}{\eta' - \eta} \int_{\eta}^{\eta'} \rho(u)dx \leq \frac{\rho(\eta) + \rho(\eta')}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if ρ is concave. In [7], there are many inequalities associated with (1.1) for different function types.

The definition and basic elements about the subject are following.

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Definition 1.1. *[12] Let $\rho \in L_1[\varepsilon, \epsilon']$. The Riemann Liouville integrals $J^{\alpha}_{\varepsilon^+}f$ and $J^{\alpha}_{\epsilon'^-}f$ of order $\alpha > 0$ with $\varepsilon \geq 0$ are defined by*

$$J^{\alpha}_{\varepsilon^+}f\ (u) = \frac{1}{\Gamma(\alpha)} \int_{\varepsilon}^{u} (u-t)^{\alpha-1} f(t)\,dt\ , \qquad u > \varepsilon$$

and

$$J^{\alpha}_{\epsilon'^-}f\ (u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{\epsilon'} (t-u)^{\alpha-1} f(t)\,dt\ , \qquad u < \epsilon'$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here is $J^{0}_{\varepsilon^+}f(u) = J^{0}_{\epsilon'^-}f(u) = f(u)$.

In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

Definition 1.2. *[12] Let $\rho \in L_1[\varepsilon, \epsilon']$. The right and the left $k-$Riemann Liouville integrals $J^{\frac{\alpha}{k}-1}_{\varepsilon^+}\rho$ and $J^{\frac{\alpha}{k}-1}_{\epsilon'^-}\rho$ of order $\alpha > 0$, $k > 0$ with $\varepsilon > 0$ are defined by*

$$J^{\alpha,k}_{a^+}\rho(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{\varepsilon}^{u} (u-t)^{\frac{\alpha}{k}-1} \rho(t)\,dt\ , \qquad u > \varepsilon$$

and

$$J^{\alpha,k}_{\epsilon'^-}\rho(u) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{\epsilon'} (t-u)^{\frac{\alpha}{k}-1} \rho(t)\,dt\ , \qquad u < \epsilon'$$

Definition 1.3. *[7] We say that ρ:I $\to \mathbb{R}$ is a P-function, or that f belongs to the class P(I), if ρ is a non-negative function and for all u, v $\in$ I, $t \in [0,1]$, we have*

$$\rho(tu + (1-t)v) \leq \rho(u) + \rho(v).$$

P(I) contain all nonnegative monotone convex and quasi convex functions.

Definition 1.4. *[12] Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$: The function f is said to be quasi-convex on I if inequality*

$$\rho(tu + (1-t)v) \leq \sup\{\rho(u); \rho(v)\} \qquad (QC)$$

holds for all $u; v \in I$ and $t \in [0;1]$

In [14], M.Zeki Sarıkaya et.all proved the following inequality with connected (1.1) for fractional integrals using the definition of convexity:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^+}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

The aim of this paper is to rewrite inequality written in type (1.2) for the Riemann-Liouville $k-$fractional, using the P-convex function. In a way, it is a continuation of my previous works. see[12]

In [12] , we obtained the following lemma for $k-$Riemann Liouville fractional integrals.

Lemma 1.5. *Let ρ: $I \subset \mathbb{R} \to \mathbb{R}$ be a function on I ,where $\eta, \eta' \in I$ with $t \in [0,1]$. If $\rho \in L[\eta, \eta']$, then for all $\eta \leq u < v \leq \eta'$ and $\alpha > 0$ we have:*

$$\begin{aligned} &\frac{\rho(u)+\rho(v)}{v-u} + \frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I^{\alpha,k}_{u^+}\rho(v) + I^{\alpha,k}_{v^-}\rho(u)\right] \qquad (3) \\ = &\int_0^1 (1-t)^{\frac{\alpha}{k}} \rho'(tu + (1-t)v)\,dt \\ &+ \int_0^1 (1-t)^{\frac{\alpha}{k}} \rho'((1-t)u + tv)\,dt. \end{aligned}$$

for each $u, v \in [\eta, \eta']$.

Recently, Mathematicians have been published a huge amount papers for fractional order operators using the various convex functions, see [1–6, 8–11, 13, 15–18]

2. MAIN RESULTS

Theorem 2.1. *Let $\rho\colon S \subset \mathbb{R} \to \mathbb{R}$ be a function on I ,where $\eta, \eta' \in S$ with $t \in [0,1]$. If $\rho' \in L[\eta, \eta']$,*

for all $\eta \leq u < v \leq \eta'$ and $\alpha, k > 0$. If ρ' is $p-$convex on $[u, v]$. Then we have the inequality

$$\left|\frac{\rho(v)+\rho(u)}{v-u}+\frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v)+I_{v^-}^{\alpha,k}\rho(v)\right]\right| \leq 2\frac{k}{\alpha+k}\left[\left|\rho'(u)+\rho'(v)\right|\right]. \tag{4}$$

Proof. By using properties modulus and the identity in (1.3) with the p convexity of ρ'

$$\begin{aligned} &\left|\frac{\rho(v)+\rho(u)}{v-u}+\frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v)+I_{v^-}^{\alpha,k}\rho(v)\right]\right| \\ \leq\ & \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(tu+(1-t)v)\right|dt+\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'((1-t)u+tv)\right|dt. \end{aligned}$$

$$\begin{aligned} J_1 &= \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(tu+(1-t)v)\right|dt \\ &\leq \int_0^1 (1-t)^{\frac{\alpha}{k}}\left[\left|\rho'(u)\right|+\left|\rho'(v)\right|\right]dt=\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(u)\right|dt+\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(v)\right|dt \\ &= \left[\left|\rho'(u)\right|+\left|\rho'(v)\right|\right]\frac{k}{\alpha+k} \end{aligned}$$

□

and

$$\begin{aligned} J_2 &= \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'((1-t)u+tv)\right|dt \\ &\leq \int_0^1 (1-t)^{\frac{\alpha}{k}}\left[\left|\rho'(u)\right|+\left|\rho'(v)\right|\right]dt=\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(u)\right|dt+\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(v)\right|dt \\ &= \left[\left|\rho'(u)\right|+\left|\rho'(v)\right|\right]\frac{k}{\alpha+k} \end{aligned}$$

Then adding J_1 and J_2 we get the (2.1) inequality.

Corollary 2.2. *when $\alpha = k = 1$ in (2.1) we obtain the inequality*

$$\left|\frac{\rho(v)+\rho(u)}{2(v-u)}+\int_u^v f(t)\,dt\right| \leq \left[\left|\rho'(u)+\rho'(v)\right|\right].$$

Theorem 2.3. *Let $\rho\colon I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I ,where $\eta, \eta' \in I$ with $t \in [0,1]$. If $\rho' \in L[\eta, \eta']$, for all $\eta \leq u < v \leq \eta'$ and $\alpha, k > 0$. If $\left|\rho'\right|^q$ is $p-$convex on $[u, v]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ Then we have the* inequality

$$|J| \le 2\frac{k}{\alpha+k}\left[\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right]^{\frac{1}{q}}. \tag{5}$$

where

$$J=\frac{\rho(v)+\rho(u)}{v-u}+\frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v)+I_{v^-}^{\alpha,k}\rho(v)\right]$$

Proof. If we use the lemma (1.3) in view of the properties of modulus and Power Mean inequality with $p-$convex of $\left|\rho'\right|^q$ on $[u,v]$, we have

$$\begin{aligned}
|J_1| &= \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(tu+(1-t)v)\right|dt \\
&\le \left(\int_0^1 (1-t)^{\frac{\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\
&= \left(\int_0^1 (1-t)^{\frac{\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}\left(\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(u)\right|^q dt+\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(v)\right|^q dt\right)^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left[\left(\frac{\alpha}{\alpha+k}\right)\left|\rho'(u)\right|^q+\left(\frac{\alpha}{\alpha+k}\right)\left|\rho'(v)\right|^q\right]^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left[\left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right)\right]^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{q}}\left(\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right)^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

Similarly

$$\begin{aligned}
J_2 &= \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'((1-t)u+tv)\right|dt \\
&\le \left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

□

Now, then we obtain

$$|J| \le |J_1|+|J_2| = 2\left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right)^{\frac{1}{q}}$$

which proof the inequality (2.2).

Theorem 2.4. *Let ρ:$[\eta,\eta']\to\mathbb{R}$ be a differentiable mapping on $[\eta,\eta']$,where $\eta<\eta'$ such that $\rho'\in L[\eta,\eta']$ If $\left|\rho'\right|^q$ is $p-$convex on $[u,v]$ and $\eta\le u<v\le\eta'$ and . $p>1$,with $t\in[0,1]$. Then we have the*

$$\begin{aligned}
&\left|\frac{\rho(v)+\rho(u)}{v-u}+\frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v)+I_{v^-}^{\alpha,k}\rho(v)\right]\right| \qquad (6)\\
&\le 2\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left[\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1, \quad k>0, \ \alpha>1.$

Proof. By using the identity that is given in (1.3) with classic hölder inequaliy for each term and the definition $p-$convex of $\left|\rho'\right|$, we have

$$\begin{aligned} & \left|\frac{\rho(v)+\rho(u)}{v-u}+\frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v)+I_{v^-}^{\alpha,k}\rho(v)\right]\right| \\ \leq & \left(\int_0^1(1-t)^{\left(\frac{\alpha}{k}\right)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\rho'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\ & +\left(\int_0^1(1-t)^{\left(\frac{\alpha}{k}\right)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\rho'((1-t)u+tv)\right|^q dt\right)^{\frac{1}{q}} \\ \leq & 2\left(\frac{k}{k+\alpha p}\right)^{\frac{1}{p}}\left[\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right]^{\frac{1}{q}}. \end{aligned}$$

□

which proof the inequality (2.3) .

Corollary 2.5. *Under conditions of Theorem 3 we have*

$$\left|\frac{\rho(v)+\rho(u)}{2(v-u)}+\frac{\alpha\Gamma_k(\alpha)}{2(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v)+I_{v^-}^{\alpha,k}\rho(v)\right]\right| \leq \left[\left|\rho'(u)\right|^q+\left|\rho'(v)\right|^q\right]^{\frac{1}{q}}$$

Proof. Since $\lim\limits_{p\to\infty} 2\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}=2$ the result is clear. □

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The Category of Soft Topological Hyperrings

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Abstract. In this study, which focuses on the intersection of soft set theory and topological hyperrings, the concept of soft topological hyperrings is proposed and its relation with topological hyperrings is examined. Morever, some characterizations related to the family of soft topological hyperrings are obtained and the category of soft topological hyperrings is established. Finally, the concept of soft topological subhyperrings is described and several structural features are studied.

1. Introduction

Due to its compelling structure, the literature on ring theory has received less attention than group theory, but it has gained enough attention lately. [27] can be given as an example to the studies in this area. More interesting is the relationship between soft set theory and hyper structure of group/ring theory. The algebraic hyperstructures that emerged with the introduction of hypergroups by Marty are considered as a generalization of classical algebraic structures [1]. The concept of hyperrings plays important role in the the theory of algebraic hyperstructures. Hyperrings, defined by M. Krasner, have been studied by various researchers [19-21].In particular, the book "Hyperrings Theory and Applications" is a good review resource on this topic [18]. Although there are many algebraic studies on hyperrings, topological studies on them are very limited. By defining the concept of topological hyperrings, Nodehi et al. investigated some differences between the topological rings and topological hyperrings [20].

One of the fertile areas for many researchers working on theories modeling uncertainty is the soft set theory initiated by Molodstov, since it has many applications in economics, computer science, biology, engineering, environment, social science and medical science [2]. The easy applicability of this theory in other fields of mathematics, especially algebra and topology, has enabled many important studies. Firstly, Maji et al. developed the application of soft set theory in decision making problems and introduced some operations on soft sets [23]. Aktas and cagman presented the definition of soft groups and studied their fundamental operations [5]. Other algebraic works of soft set theory can be found [4, 6]. On the other hand, topological studies on soft sets were first put forth by Shabir and Naz [8]. By defining the notion of a soft topological space, they examined the separation axioms in a soft topological space. For more details, see [7, 9-11,13].

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Recently, some researchers have applied soft set theory to algebraic hyperstructures. Yamak, the first of these, developed the concepts of soft hypergroupoids and soft subhypergroupoids [12]. Later on, Wang et al. introduced the concepts of soft (normal) polygroups and soft (normal) subpolygroups [16]. Selvachandran and Salleh studied the concepts of soft hypergroups and soft subhypergroups as an expansion of soft hypergroupoids and soft subhypergroupoids [15]. Also, Selvachandran presented the definitions of Soft hyperrings and soft hyperring homomorphism [17]. In [14] Oguz developed the concept of soft topological polygroups by examining polygroups, an important subclass of hypergroups, with a soft topological approach.

The main purpose of this study is to present the concept of soft topological hyperrings by examining hyperrings which is one of the the algebraic hyperystructures with soft set theory from the topological point of view. Further, the relation between soft topological hyperrings and soft hyperrings is investigated and several theoretical results are given. By defining the concept of soft topological hyperring homomorphism, the category of soft topological hyperrings is establıshed. This study completed by giving the definition of soft topological subhyperrings and examining some relevant properties.

2. Preliminaries

In this section, some notions and results about soft sets, topological hyperrings and soft hyperrings to be used in the sequel will be presented.

Let X be an initial universe set and E be a set of parameters. Also, let $P(X)$ denotes the power set of X and $A \subset E$. The definition of a soft set introduced by Molodtsov is as follows:

Definition 2.1. [2] *A pair $(\mathcal{F}, A)$ is called a soft set over X, where $\mathcal{F}$ is a mapping defined by*

$$\mathcal{F} : A \longrightarrow P(X)$$

Note that a soft set over X is actually a parametrized family of subsets of the universe X.

Definition 2.2. [3] *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two soft sets over the common universe X. Then, $(\mathcal{F}, A)$ is called a soft subset of $(\mathcal{G}, B)$ (i.e., $(\mathcal{F}, A)\widetilde{\subset}(\mathcal{G}, B)$) if*
i. *$A \subseteq B$,*
ii. *$\mathcal{F}(\varepsilon)$ and $\mathcal{G}(\varepsilon)$ are identical approximations for all $\varepsilon \in A$.*

Definition 2.3. [12] *The support of a soft set (F, A) is defined as a set*

$$Supp(\mathcal{F}, A) = \{\varepsilon \in A : F(\varepsilon) \neq \emptyset\}$$

If $Supp(\mathcal{F}, A)$ is not equal to the empty set, then $(\mathcal{F}, A)$ is called non-null.

In the following, some generalizations are given for the nonempty family $\{(\mathcal{F}_\alpha, A_\alpha) | \alpha \in I\}$ of soft sets over the common universe X.

Definition 2.4. [24] *The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha)\ |\alpha \in I\}$ is defined by a soft set $(\mathcal{F}, A) = \widetilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha)$ such that $A = \bigcap_{\alpha\in I} A_i \neq \emptyset$ and $\mathcal{F}(\alpha) = \bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ for all $\varepsilon \in A_\alpha$.*

Definition 2.5. [24] *The extended intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha)|\alpha \in I\}$ is a soft set $(\mathcal{F}, A) = (\bigcap_{\mathcal{E}})_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha)$ such that $A = \bigcup_{\alpha\in I} A_\alpha$ and $\mathcal{F}(\varepsilon) = \bigcap_{\alpha\in I(\varepsilon)} \mathcal{F}_\alpha(\varepsilon)$, $I(\varepsilon) = \{\alpha \in I|\varepsilon \in A_\alpha\}$ for all $\varepsilon \in A_\alpha$.*

Definition 2.6. [24] *The $\wedge-$intersection of the family $\{(F_\alpha, A_\alpha)|\alpha \in I\}$ is defined by a soft set $(F, A) = \widetilde{\bigwedge}_{\alpha\in I}(F_\alpha, A_\alpha)$ such that $A = \Pi_{\alpha\in I}A_\alpha$ and $F((\varepsilon_\alpha)_{\alpha\in I}) = \bigcap_{\alpha\in I} F_\alpha(\varepsilon_\alpha)$ for all $(\varepsilon_\alpha)_{\alpha\in I} \in A_\alpha$.*

Definition 2.7. [26] *Let $\mathcal{R}$ be a non-empty set and $P^*(\mathcal{R})$ denote the family of non-empty subsets of $\mathcal{R}$. Then, the mapping $\cdot : \mathcal{R} \times \mathcal{R} \longrightarrow P^*(\mathcal{R})$ is called a hyperoperation and the pair $(\mathcal{R}, \cdot)$ is also called hypergroupoid.*

Definition 2.8. [26] *A hypergroup is a hypergroupoid* $(\mathcal{R}, \cdot)$ *which satisfies the following axioms:*

(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *for all* $x, y, z \in \mathcal{R}$

(ii) $x \cdot \mathcal{R} = \mathcal{R} \cdot x$ *for all* $x \in \mathcal{R}$.

A semi hypergroup is a hypergroupoid $(\mathcal{R}, \cdot)$ if for all $x, y, z \in \mathcal{R}$, we have $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Now, the definitions of topological hyperring, soft topological ring and soft hyperring will be recalled.

Definition 2.9. [18] *Let* $(\mathcal{F}, A)$ *be a non-null soft set on a commutative ring R endowed with the topology* τ*. Then, the triplet* $(\mathcal{F}, A, \tau)$ *is called a soft topological ring over R if the following conditions hold for all* $\varepsilon \in A$*:*
i. $\mathcal{F}(\varepsilon)$ *is a subring of G for all* $\varepsilon \in A$.
ii. *the mapping* $F(\varepsilon) \times F(\varepsilon) \longrightarrow F(\varepsilon)$ *defined by* $(x, y) \longmapsto x - y$ *is continuous.*
iii. *the mapping* $F(\varepsilon) \times F(\varepsilon) \longrightarrow F(\varepsilon)$ *defined by* $(x, y) \longmapsto x \cdot y$ *is continuous.*

Definition 2.10. [22] *A hyperring is an algebraic system* $(\mathcal{R}, +, \cdot)$ *which satisfies the following axioms:*
i. $(\mathcal{R}, +)$ *is a commutative hypergroup.*
ii. $(\mathcal{R}, \cdot)$ *is a semihypergroup.*
iii. *The hyperoperation "$\cdot$" is distributive with respect to the hyperoperation "+".*

Example 2.11. [22] *Let* $\mathcal{R} = \{0, 1\}$ *be a set with two hyperoperations defined as follows:*

$+$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

$\cdot$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0,1\}$

So it can be easily verified that $(\mathcal{R}, +, \cdot)$ *is a hyperring.*

Definition 2.12. [25] *A non-empty subset* $\mathcal{R}'$ *of a hyperring* $(\mathcal{R}, +, \cdot)$ *is said to be a subhyperring of* $\mathcal{R}$ *if* $(\mathcal{R}', +, \cdot)$ *itself is a hyperring.*

Definition 2.13. [20] *Let* $(\mathcal{R}, \tau)$ *be a topological space and* $P^*(\mathcal{R})$ *denote the family of non-empty subsets of* $\mathcal{R}$*. Then, the collection* $\mathcal{B}$ *consisting of all sets* $\mathcal{S}_V = \{U \in P^*(\mathcal{R}) : U \subseteq V, U \in \tau\}$ *is a base for a topology on* $P^*(\mathcal{R})$ *denoted by* τ^*.

Definition 2.14. [20] *Let* $(\mathcal{R}, +, \cdot)$ *be a hyperring and* $(\mathcal{R}, \tau)$ *be a topological space. Then, algebraic hyperstructure* $(\mathcal{R}, +, \cdot, \tau)$ *is called a topological hyperring if three hyperoperations " + ", " $\cdot$ " and "/" are continuous.*

Remark 2.15. [20] *Every topological ring is a topological hyperring by trivial hyperoperations.*

Definition 2.16. [17] *Let* $(\mathcal{F}, A)$ *be a non-null soft set over the hyperring R. Then the pair* (F, A) *is said to be a soft hyperring over R if* $\mathcal{F}(\varepsilon)$ *is a subhyperring of R for all* $\varepsilon \in Supp(\mathcal{F}, A)$.

Example 2.17. [25] *Consider a hyperring* $(\mathcal{R}, +, \cdot)$ *with the hyperoperations as follows:*

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	0	0	0

Define a soft set (F, A) *over* $\mathcal{R} = \{0, 1, 2, 3\}$*, where* $A = \mathcal{R}$*, by* $F(0) = \{0, 2\}$*,* $F(1) = \{0, 3\}$*,* $F(2) = \{0\}$ *and* $F(3) = \{0, 1\}$*. Then it is clear that* $F(0), F(1), F(2)$ *and* $F(3)$ *are subhyperrings of* $\mathcal{R}$*. Thus* (F, A) *is a soft hyperring over* $\mathcal{R}$*.*

3. Soft Topological Hyperrings

In this section, the concept of soft topological hyperrings will be introduced and some important characterizations of them will be established. By presenting the concept of soft topological subrings, the related structural properties will also be examined.

Definition 3.1. *Let τ be a topology on the hyperring $\mathcal{R}$. Let $\mathcal{F} : A \longrightarrow P(\mathcal{R})$ be a mapping, where $P(\mathcal{R})$ is the set of all subhyperrings of $\mathcal{R}$, and A is the set of parameters. The system $(\mathcal{F}, A, \tau)$ is called a soft topological hyperring over $\mathcal{R}$ if the following statements hold for all $\varepsilon \in Supp(\mathcal{F}, A)$:*
i. *$F(\varepsilon)$ is a subhyperring of $\mathcal{R}$;*
ii. *Three hyperoperations $+, \cdot, / : \mathcal{F}(\varepsilon) \times \mathcal{F}(\varepsilon) \longrightarrow P^*(\mathcal{F}(\varepsilon))$ are continuous with respect to the topologies induced by $\tau \times \tau$ and τ^*.*

A trivial verification shows that if $\mathcal{R}$ is a topological hyperring, it is sufficient to hold only the statement *i.* in the above definition in order to called the system $(\mathcal{F}, A, \tau)$ as a soft topological hyperring. Besides, the soft topological hyperring $(\mathcal{F}, A, \tau)$ can be considered as a parameterized family of subhyperrings of the topological hyperring $\mathcal{R}$.

Example 3.2. *Every soft topological ring is a soft topological hyperring.*

Example 3.3. *Consider the hyperring $\mathbb{R}$ of real numbers with its natural topology τ such that the hyperoperations $x + y = x \cdot y = \{x, y\}$ for all $x, y \in \mathbb{R}$. Suppose $A = \mathbb{N}$. Then for all $\varepsilon \in A$, the mapping F is defined as*

$$\begin{array}{rcl} F : \mathbb{N} & \longrightarrow & P^*(\mathbb{R}) \\ \varepsilon & \mapsto & F(\varepsilon) = \begin{cases} \{0, \varepsilon\} & \varepsilon \text{ tek} \\ \mathbb{Q} & \varepsilon \text{ çift} \end{cases} \end{array}$$

In either case, it can be clearly checked that $F(\varepsilon)$ is a subhyperring of the topological hyperring $\mathbb{R}$. Hence, the triplet $(\mathcal{F}, A, \tau)$ a soft topological hyperring over $\mathbb{R}$.

Definition 3.4. *Let $(\mathcal{F}, A, \tau)$ be a soft topological hyperring over $\mathcal{R}$. Then $(\mathcal{F}, A, \tau)$ is said to be*
i. *an identity soft topological hyperring if $F(\varepsilon) = \emptyset$ for all $\varepsilon \in A$.*
ii. *an absolute soft topological hyperring if $F(\varepsilon) = \mathcal{R}$ for all $\varepsilon \in A$.*

Example 3.5. *In the example above, assuming $A = \mathbb{R}$ and $F(\varepsilon) = \{\omega \in \mathbb{R} : \varepsilon + \omega = \{\varepsilon\}\}$ for all $\varepsilon \in A$, it is easily obtained that $(\mathcal{F}, A, \tau)$ is an identity soft topological hyperring over $\mathbb{R}$.*

In the following, we present the relationship between soft hyperrings and soft topological hyperrings.

Theorem 3.6. *Every soft hyperring on a topological hyperring $\mathcal{R}$ is a soft topological hyperring.*

Proof. Consider a soft hyperring $(\mathcal{F}, A)$ over the topological hyperring $\mathcal{R}$ with the topology τ. Since $\mathcal{F}(\varepsilon)$ is a subhyperring of $\mathcal{R}$ for all $\varepsilon \in A$, $\mathcal{F}(\varepsilon)$ is also a topological subhyperring of $\mathcal{R}$ with recpect to the topologies induced by τ and τ^* for all $\varepsilon \in A$. Thus, $(\mathcal{F}, A, \tau)$ is a soft topological hyperhyperring over $\mathcal{R}$. □

Remark 3.7. *Each soft hyperring $\mathcal{R}$ can be transformed into a soft topological hyperring by equipping both $\mathcal{R}$ and $P^*(\mathcal{R})$ with discrete or indiscrete topology. But the converse of this statement is not true, meaning that every soft hyperring over a hyperring is not a soft topological hyperring.*

Some generalizations for a nonempty family of soft topological hyperrings are introduced here:

Theorem 3.8. *Let $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ be a non-empty family of soft topological hyperrings over $\mathcal{R}$.*
i. *The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau)|\alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ is a soft topological hyperring over $\mathcal{R}$ if it is non-null.*
ii. *The extended intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha)|\alpha \in I\}$ is a soft topological hyperring over $\mathcal{R}$ if it is non-null.*
iii. *The $\wedge-$intersection $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ is a soft topological hypergroup over $\mathcal{R}$ if it is non-null*

Proof. **i**. The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ given by the soft set $\widetilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau) = (\mathcal{F}, A, \tau)$ such that $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ for all $\varepsilon \in A$ from Definition 2.4. Take $\varepsilon \in Supp(\mathcal{F}, A)$. By the assumption, $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ such that $\mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ for all $\alpha \in I$. Since $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ is a non-empty family of soft topological hyperrings over $\mathcal{R}$, this implies that $\mathcal{F}_\alpha(\varepsilon)$ is also a topological subhyperring of $\mathcal{R}$ for all $\alpha \in I$. It is then evident that $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ is a topological subhyperring of $\mathcal{R}$. Therefore, $(\mathcal{F}, A, \tau)$ is a soft topological hyperring over $\mathcal{R}$.
ii. The proof is similar to *i*.
iii. Choose $(\mathcal{F}, A, \tau) = \widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ for a non-empty family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ of soft topological hyperrings over $\mathcal{R}$. Let $\varepsilon \in Supp(\mathcal{F}, A)$. It follows from the hypothesis $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon_\alpha) \neq \emptyset$ that $\mathcal{F}_\alpha(\varepsilon_\alpha) \neq \emptyset$ for all $\alpha \in I$ and $(\varepsilon_\alpha)_{\alpha\in I} \in A_\alpha$. Thus, $\mathcal{F}_\alpha(\varepsilon_\alpha)$ is a topological subhyperring of $\mathcal{R}$ for all $\alpha \in I$ so that their intersection must be a topological subhyperring of $\mathcal{R}$ too. Clearly, $(\mathcal{F}, A, \tau)$ is a soft topological hyperring over $\mathcal{R}$. □

3.1. Soft Topological Hyperring Homomorphisms

Definition 3.9. *Let $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ be soft topological hyperrings over $\mathcal{R}$ and $\mathcal{S}$, respectively. Let $\phi : A \longrightarrow B$ and $\psi : \mathcal{R} \longrightarrow \mathcal{S}$ be two mappings. Then the pair (ψ, ϕ) is called a soft topological homomorphism if the following statements are satisfied:*
i. *ψ is a strong homomorphism;*
ii. *$\psi(\mathcal{F}(\varepsilon)) = \mathcal{K}(\phi(\varepsilon))$ for all $\varepsilon \in Supp(\mathcal{F}, A)$;*
iii. *$\psi_\varepsilon : (\mathcal{F}(\varepsilon), \tau_{\mathcal{F}(\varepsilon)}) \longrightarrow (\mathcal{K}(\phi(\varepsilon)), \tau'_{\mathcal{K}(\phi(\varepsilon))})$ continuous and open for all $\varepsilon \in Supp(\mathcal{F}, A)$.*

Namely, a soft topological homomorphism (ψ, ϕ) is a mapping of soft topological hyperrings. In this direction, we obtain a new category whose objects are soft topological hyperrings and whose arrows are soft topological homomorphisms.

Note that If ψ is a isomorphism, ϕ is bijective, then the pair (ψ, ϕ) is said to be a soft topological isomorphism, and $(\mathcal{F}, A, \tau)$ is soft topologically isomorphic to $(\mathcal{K}, B, \tau')$ denoted by $(\mathcal{F}, A, \tau) \simeq (\mathcal{K}, B, \tau')$.

Example 3.10. *Let $(\mathcal{K}, B, \tau)$ be a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. Then the pair $(\mathcal{I}, i)$ is a soft topological homomorphism from $(\mathcal{K}, B, \tau)$ to $(\mathcal{F}, A, \tau)$, where $i : B \longrightarrow A$ is an inclusion map and $\mathcal{I} : \mathcal{R} \longrightarrow \mathcal{R}$ is an identity map.*

Example 3.11. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be the two soft homomorphic hyperrings defined over $\mathcal{R}$ and $\mathcal{S}$, resp. Then, it is easy to obtain that $(\mathcal{F}, A, \tau)$ is soft topological homomorphic to $(\mathcal{K}, B, \tau)$ such that τ is discrete or anti-discrete topology. So, any soft homomorphic hyperrings can be reviewed as soft topological homomorphic hyperrings with the discrete or anti-discrete topology.*

At the moment, we can easily deduce that

Theorem 3.12. *Let the pair (ψ, ϕ) be a soft topological homomorphism between the soft topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ defined over $\mathcal{R}$ and $\mathcal{S}$, resp. Then if $\phi : A \longrightarrow B$ be an injective mapping, $(\psi(\mathcal{F}), B, \tau')$ is a soft topological hyperring over $\mathcal{S}$*

Proof. Consider two soft topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ over $\mathcal{R}$ and $\mathcal{S}$, respectively. Since $(\psi, \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{K}, B, \tau')$ is a soft topological homomorphism, it follows that $\phi(Supp(\mathcal{F}, A)) = Supp(\psi(\mathcal{F}), B)$. Consider $b \in Supp(\psi(\mathcal{F}), B)$. So there exist $\varepsilon \in Supp(\mathcal{F}, A)$ such that $\phi(\varepsilon) = b$ and hence $\mathcal{F}(\varepsilon) \neq \emptyset$. Also, it is evident that $\mathcal{F}(\varepsilon)$ is a topological subhyperring of $\mathcal{R}$ and is also a topological hyperring with respect to the topology induced by τ. Since ψ is a strong homomorphism, we obtain that $\psi(\mathcal{F}(\varepsilon))$ is a topological subhyperring of $\mathcal{H}'$ with respect to the topology induced by τ'. Consequently, $(\psi(\mathcal{F}), B, \tau')$ is a soft topological hypergroup over $\mathcal{S}$. □

Theorem 3.13. *Let the pair (ψ, ϕ) be a soft topological homomorphism between the soft topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ over $\mathcal{H}$ and $\mathcal{H}'$, resp. Then $(\psi^{-1}(\mathcal{K}), A, \tau)$ is a soft topological hyperring over $\mathcal{R}$ if it is non-null.*

Proof. Since the pair (ψ, ϕ) be a soft topological homomorphism, this implies

$$\phi(Supp(\psi^{-1}(\mathcal{K}), A)) = \phi^{-1}(Supp(\mathcal{K}, B))$$

for all $b \in Supp(\mathcal{K}, B)$. When $a \in Supp(\psi^{-1}(\mathcal{K}), A)$, we get $\phi(\varepsilon) \in Supp(\mathcal{K}, B)$. Thus, the nonempty set $\mathcal{K}(\phi(\varepsilon))$ is a topological subhyperring of $\mathcal{H}'$ and is also a topological hyperring with respect to the topology induced by τ'. Since ψ is a strong homomorphism, it follows that $\psi^{-1}(\mathcal{K}(\phi(\varepsilon))) = \psi^{-1}(\mathcal{K}(\varepsilon))$ is a topological subhyperring of $\mathcal{R}$ with respect to the topology induced by τ. This means that $(\psi^{-1}(\mathcal{K}), A, \tau)$ is a soft topological hyperring over $\mathcal{R}$. □

Theorem 3.14. *Let $(\mathcal{F}, A, \tau)$, $(\mathcal{K}, B, \tau')$ and $(\mathcal{N}, C, \tau'')$ be soft topological hyperrings over $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$, respectively. If $(\psi, \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{K}, B, \tau')$ and $(\psi', \phi') : (\mathcal{K}, B, \tau') \longrightarrow (\mathcal{N}, C, \tau'')$ are two soft topological homomorphisms, then the pair $(\psi' \circ \psi, \phi' \circ \phi)$ is a soft topological homomorphism.*

Proof. Consider two soft topological homomorphisms $(\psi, \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{K}, B, \tau')$ and $(\psi', \phi') : (\mathcal{K}, B, \tau') \longrightarrow (\mathcal{N}, C, \tau'')$. By Definition 3.9, it follows that $\psi : \mathcal{H} \longrightarrow \mathcal{H}'$ and $\psi' : \mathcal{H}' \longrightarrow \mathcal{H}''$ are two strong homomorphisms, and $\phi : A \longrightarrow B$ and $\phi' : B \longrightarrow C$ are two mappings such that the equalities $\psi(\mathcal{F}(\varepsilon)) = \mathcal{K}(\phi(\varepsilon))$ and $\psi'(\mathcal{K}(\epsilon)) = \mathcal{N}(\phi'(\epsilon))$ hold for all $\varepsilon \in Supp(\mathcal{F}, A)$, $\epsilon \in Supp(\mathcal{K}, B)$. So, We can easily deduce that $\psi' \circ \psi : \mathcal{H} \longrightarrow \mathcal{H}''$ is also strong homomorphism and $\phi' \circ \phi : A \longrightarrow C$ is a mapping so that the equality

$$(\psi' \circ \psi)(\mathcal{F}(\varepsilon)) = \psi'(\psi(\mathcal{F}(\varepsilon))) = \psi'(\mathcal{K}(\phi(\varepsilon))) = \mathcal{N}(\phi'(\phi(\varepsilon))) = \mathcal{N}((\phi' \circ \phi)(\varepsilon))$$

holds for all $\varepsilon \in Supp(\mathcal{F}, A)$. Also, $(\psi' \circ \psi)_\varepsilon : (\mathcal{F}(\varepsilon), \tau_{\mathcal{F}(\varepsilon)}) \longrightarrow (\mathcal{N}((\phi' \circ \phi)(\varepsilon)), \tau''_{\mathcal{N}((\phi' \circ \phi)(\varepsilon))})$ continuous and open for all $\varepsilon \in Supp(\mathcal{F}, A)$. Hence, it is concluded that $(\psi' \circ \psi, \phi' \circ \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{N}, C, \tau'')$ is a soft topological homomorphism. □

3.2. *Soft Topological Subhyperrings*

In this subsection, we define the notion of soft topological subhyperrings and establısh some its important characterizations.

Definition 3.15. *Let $(\mathcal{F}, A, \tau)$ be a soft topological hyperring over $\mathcal{R}$. Then, $(\mathcal{K}, B, \tau)$ is called a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if the following conditions are satisfied:*
i. *$B \subseteq A$;*
ii. *$\mathcal{K}(b)$ is a subhyperring of $\mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$;*
iii. *The hyperoperations $+, \cdot, / : \mathcal{F}(\varepsilon) \times \mathcal{F}(\varepsilon) \longrightarrow P^*(\mathcal{F}(\varepsilon))$ are continuous with respect to the topologies induced by $\tau \times \tau$ and τ^* for all $b \in Supp(\mathcal{K}, B)$.*

Example 3.16. *Consider a soft topological hyperring $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$ and $B \subseteq A$. Then, we can easily deduce that $(\mathcal{F}|_B, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$.*

Theorem 3.17. *If $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ and $(\mathcal{N}, C, \tau)$ is a soft topological subhyperring of $(\mathcal{K}, B, \tau)$, then $(\mathcal{N}, C, \tau)$ is the soft topological subhyperring of $(\mathcal{F}, A, \tau)$.*

Proof. Straightforward. □

Theorem 3.18. *Let $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ be two soft topological hyperrings over $\mathcal{R}$. Then $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$.*

Proof. Assume $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ are two soft topological hyperrings over $\mathcal{R}$. Clearly, if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$, it follows that $B \subseteq A$ and $\mathcal{K}(b) \subseteq \mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$. Thus, $\mathcal{K}(b)$ is a topological subhyperring of $\mathcal{F}(b)$ with respect to the topology induced by τ. Thus, $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$. □

After that, we discuss some generalized properties of soft topological subhyperrings.

Theorem 3.19. *Let $(\mathcal{F}, A, \tau)$ be a soft topological hyperring over $\mathcal{H}$ and $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ be a non-empty family of soft topological subhyperrings of $(\mathcal{F}, A, \tau)$.*
i. *The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if $\tilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau) \neq \emptyset$.*
ii. *The extended intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$.*
iii. *The extended union of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ with the topology τ if $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha, \beta \in I$, $\alpha \neq \beta$.*

Proof. We only prove **i**., and the proofs of **ii**. and **iii**. are similar. The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ defined by the soft set $\tilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau) = (\mathcal{F}, A, \tau)$ such that $\mathcal{F}(\varepsilon) = \bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ for all $\varepsilon \in A$. Let $\varepsilon \in Supp(\mathcal{F}, A)$. Assume $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ such that $\mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ for all $\alpha \in I$. Since $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a non-empty family of soft topological subhyperrings of $(\mathcal{F}, A, \tau)$, therefore $A_\alpha \subseteq A$ and $\mathcal{F}_\alpha(\varepsilon)$ is a topological subhyperring of $\mathcal{F}(\varepsilon)$ with respect to the topology induced by τ for all $\alpha \in I$. So $\bigcap_{\alpha\in I} A_\alpha \subseteq A$ and $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ is a topological subhyperring of $\mathcal{F}(\varepsilon)$. Consequently, the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$
□

Besides, we can obtain the following result:

Theorem 3.20. *Let $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ be a non-empty family of soft topological hyperrings over $\mathcal{H}$ and let $(\mathcal{K}_\alpha, B_\alpha, \tau)$ be a soft topological subhyperring of $(\mathcal{F}_\alpha, A_\alpha, \tau)$ for all $\alpha \in I$. Then, $\wedge-$intersection $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{K}_\alpha, B_\alpha, \tau)$ is a soft topological subhyperring of $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ if it is non-null.*

Proof. Suppose that $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a non-empty family of soft topological hyperrings over $\mathcal{R}$. By Theorem 3.5 (ii), it is clear that $\widetilde{\bigvee}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ is a soft topological hyperring over $\mathcal{R}$. Choose $\epsilon_\alpha \in Supp(\mathcal{K}_\alpha, B_\alpha)$. Then $\bigcap_{\alpha\in I} \mathcal{K}_\alpha(\epsilon_\alpha) \neq \emptyset$ which implies that $\mathcal{K}_\alpha(\epsilon_\alpha) \neq \emptyset$ for all $\alpha \in I$ and $(\epsilon_\alpha)_{\alpha\in I} \in B_i$. Further, $B_\alpha \subseteq A_\alpha$ and $\mathcal{K}_\alpha(\epsilon_\alpha)$ is a topological subhyperring of $\mathcal{F}_\alpha(\epsilon_\alpha)$ with respect to the topology induced by τ for all $\alpha \in I$ such that $\bigcap_{\alpha\in I} B_\alpha \subseteq \bigcap_{\alpha\in I} A_\alpha$ and $\bigvee_{\alpha\in I}(\mathcal{K}_\alpha(\epsilon_\alpha))$ is also a a topological subhyperring of $\bigvee_{\alpha\in I}(\mathcal{F}_\alpha(\epsilon_\alpha))$. Therefore, $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{K}_\alpha, B_\alpha, \tau)$ is a soft topological subhyperring of $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$.
□

Theorem 3.21. *Let $(\mathcal{K}, B, \tau)$ be a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. Then, the restricted intersection of $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if it is non-null.*

Proof. Suppose that $(\mathcal{K}, B, \tau)$ is a soft topological subhypergroup of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. If it is non-null, we have that $B \subseteq A$ and $\mathcal{K}(\epsilon)$ is a topological subhyperring of $\mathcal{F}(\epsilon)$ with respect to the topology induced by τ for all $\epsilon \in Supp(\mathcal{K}, B)$. Thus, we can obtain easily that $A \cap B \subseteq A$ and $\mathcal{K}(\epsilon) \cap \mathcal{F}(\epsilon)$ is a topological subhyperring of $\mathcal{F}(\epsilon)$ with respect to the topology induced by τ for all $\epsilon \in Supp(\mathcal{K}, B)$. Hence, the restricted intersection $(\mathcal{F}, A, \tau)\tilde{\cap}(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$.
□

Theorem 3.22. *Let $f : \mathcal{R} \longrightarrow \mathcal{R}'$ be a good homomorphism of the topological hyperrings $(\mathcal{F}, A, \tau')$ and $(\mathcal{K}, B, \tau')$ over $\mathcal{H}'$. Then $(f^{-1}(\mathcal{K}), B, \tau)$ is a soft topological subhyperring of $(f^{-1}(\mathcal{F}), A, \tau)$ if $(\mathcal{K}, B, \tau')$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau')$.*

Proof. Consider $(\mathcal{K}, B, \tau')$ as a soft topological subhyperring of $(\mathcal{F}, A, \tau')$ over $\mathcal{R}$. Let $\epsilon \in Supp(f^{-1}(\mathcal{K}), B)$. Because $(\mathcal{K}, B, \tau')$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau')$, we have that $B \subseteq A$ and $(\mathcal{K}(b))$ is a topological subhyperring of $(\mathcal{F}(\epsilon)$ with respect to the topology induced by τ' for all $\epsilon \in Supp(f^{-1}(\mathcal{K}), B)$. Further, since $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a good topological homomorphism, so $f^{-1}(\mathcal{F})(\epsilon) = f^{-1}(\mathcal{F}(\epsilon))$ is a topological subhyperring of $f^{-1}(\mathcal{K})(\epsilon) = f^{-1}(\mathcal{K}(\epsilon))$ with respect to the topology induced by τ for all $\epsilon \in Supp(f(\mathcal{K}), B)$. Therefore, $(f^{-1}(\mathcal{K}), B, \tau)$ is a soft topological subhyperring of $(f^{-1}(\mathcal{F}), A, , \tau)$. □

Theorem 3.23. *Let $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a good homomorphism of the topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ over $\mathcal{R}$. Then $(f(\mathcal{K}), B, \tau')$ is a soft topological subhyperring of $(f(\mathcal{F}), A, \tau')$ over $\mathcal{H}'$ if $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$.*

Proof. Assume $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. If $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$, this means that $B \subseteq A$ and $(\mathcal{K}(\epsilon))$ is a topological subhyperring of $(\mathcal{F}(\epsilon)$ with respect to the topology induced by τ for all $\epsilon \in Supp(\mathcal{K}, B)$. Also, because $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a good topological homomorphism, we have that $f(\mathcal{F})(\epsilon) = f(\mathcal{F}(\epsilon))$ is a topological subhyperring of $f(\mathcal{K})(\epsilon) = f(\mathcal{K}(\epsilon))$ with respect to the topology induced by τ' for all $\epsilon \in Supp(f(\mathcal{K}), B)$. Hence, $(f(\mathcal{K}), B, \tau')$ is a soft topological subhyperring of $(f(\mathcal{F}), A, \tau')$. □

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A nice copy of a degenerate Lorentz-Marcinkiewicz space that implies the failure of the fixed point property

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Abstract. Introducing the notion of asymptotically isometric copies inside Banach spaces, Dowling, Lennard and Turett made easier to detect failure of the fixed point property for nonexpansive mappings. Their tool was very usefull for indicating the failure. Since then, researchers have investigated alternative tools. Recently, Nezir introduced the notion of asymptotically isometric copies of $\ell^{1\boxplus 0}$. He noticed that a renorming of ℓ^1 turns out to be a degenerate Lorentz-Marcinkiewicz space and using its structure he introduced his notion which implies the failure of the fixed point property for nonexpansive mappings. In this study, we introduce another notion which is derived from the structure of another degenerate Lorentz-Marcinkiewicz space and we show that detecting our new tool in Banach spaces will indicate the failure of the fixed point property for nonexpansive mappings.

1. Intoduction and Preliminaries

While Dowling and Lennard initially wanted to prove that nonreflexive subspaces of $L^1[0,1]$ fail the fixed point property, they introduced the concept of a Banach space containing an asymptotically isometric copy of ℓ^1 and then used this notion to prove that every equivalent renorming of $\ell^1(\Gamma)$, for Γ uncountable, fails the fixed point property [4].

The notion of asymptotically isometric copies of the classical Banach spaces ℓ^1 has applications in metric fixed point theory because they arise naturally in many places. For example, every non-reflexive subspace of $(L_1[0,\ 1],\ \|.\|_1)$, every infinite dimensional subspace of $(\ell^1\ \|.\|_1)$, and every equivalent renorming of ℓ^∞ contains an asymptotically isometric copy of ℓ^1 and so all of these spaces fail the fixed point property [4, 6]. The concept of containing an asymptotically isometric copy ℓ^1 also arises in the isometric theory of Banach spaces in an intriguing way: a Banach space X contains an asymptotically isometric copy ℓ^1 if and only if X^* contains an isometric copy of $(L_1[0,\ 1],\ \|.\|_1)$ [6].

In 1996, Dowling, Lennard and Turett investigated Banach spaces containing asymptotically isometric copies of ℓ^1 deeply and they reached important results which has leaded researchers to test the failure of the fixed point property for nonexpansive mappings in Banach spaces they have studied. In fact, Lin was impressed with their work [5] which proved ℓ^1 with a norm does not contain any asymptotically isometric

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copy of ℓ^1 and then he later showed using a special version of the norm in [7] that ℓ^1 has the fixed point property.

Thus, importance of detecting nice copies of ℓ^1, after witnessing their applications especially, suggests the researchers to investigate alternative properties which will help explore the failure of the fixed point property for nonexpansive mappings. For example, recently, Álvaro, Cembranos and Mendoza [1] introduced another nice property, which they called N1, to detect failure of fixed point property for nonexpansive mappings. Their notion was more general than the concept of a Banach space containing an asymptotically isometric copy of c_0.

In 2019, the first author explored a new renorming of ℓ^1 and noticed that his renorming was actually yielding a degenerate Lorentz-Marcinkiewicz space. He investigated fixed point properties for the dual and predual of his renorming and obtained the results of their failure of the fixed point property for nonexpansive mappings [10]. Later, using the structure of these spaces, in [11], he introduced the notion of asymptotically isometric copies of $\ell^{1\boxplus 0}$ which implies failure of the fixed point property for nonexpansive mappings for nonexpansive mappings. One can say that detecting Nezir's construction in Banach spaces is a sign of detecting a nice copy of a degenerate Lorentz-Marcinkiewicz space.

In this study, we introduce another notion which is derived from the structure of another degenerate Lorentz-Marcinkiewicz space and we show that detecting our new tool in Banach spaces will indicate the failure of the fixed point property for nonexpansive mappings.

Now, we give some preliminaries for our study.

Definition 1.1. *[3] A Banach space $(X, \|\cdot\|)$ is said to contain an asymptotically isometric copy of ℓ^1 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n| \ ,$$

for all $(a_n)_n \in \ell^1$.

The usefulness of this notion can be found in the next result.

Theorem 1.2. *[4] If a Banach space X contains an asymptotically isometric copy of ℓ^1, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

Moreover, Dowling, Lennard and Turett provided the following theorem which shows an alternative way of detecting an asymptotically isometric copy of ℓ^1 in Banach spaces.

Theorem 1.3. *[3] A Banach space X contains an asymptotically isometric copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X such that there are constants $0<m\leq M<\infty$ so that for all $(t_n)_n \in \ell^1$,*

$$m\sum_{n=1}^{\infty}|t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq M\sum_{n=1}^{\infty}|t_n| \ ,$$

and $\lim_{n\to\infty}\|x_n\| = m$.

Now, let's recall the definition of Lorentz-Marcinkiewicz space and the degenerate one, the space Nezir introduced in [10].

First of all, we note that our reference for Lorentz spaces is [8, 9].

Now, we recall the construction of Lorentz-Marcinkiewicz spaces.

Let $w \in (c_0 \setminus \ell^1)^+$, $w_1 = 1$ and $(w_n)_{n\in\mathbb{N}}$ be decreasing; that is, consider a scalar sequence given by $w = (w_n)_{n\in\mathbb{N}}$, $w_n > 0, \forall n \in \mathbb{N}$ such that $1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_n \geq w_{n+1} \geq \ldots, \forall n \in \mathbb{N}$ with $w_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sum_{n=1}^{\infty} w_n = \infty$. This sequence is called a weight sequence. For example, $w_n = \frac{1}{n}, \forall n \in \mathbb{N}$.

Definition 1.4.

$$\ell_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{w,\infty} := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} x_j^{\star}}{\sum_{j=1}^{n} w_j} < \infty \right\}.$$

Here, $x^\star$ represents the decreasing rearrangement of the sequence x, which is the sequence of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in non-increasing order, followed by infinitely many zeros when $|x|$ has only finitely many non-zero terms.

This space is non-separable and an analogue of ℓ_∞ space.

Definition 1.5.

$$\ell^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \limsup_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} x_j^{\star}}{\sum_{j=1}^{n} w_j} = 0 \right\}.$$

This is a separable subspace of $\ell_{w,\infty}$ and an analogue of c_0 space.

Definition 1.6.

$$\ell_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{w,1} := \sum_{j=1}^{\infty} w_j\, x_j^{\star} < \infty \right\}.$$

This is a separable subspace of $\ell_{w,\infty}$ and an analogue of ℓ^1 space with the following facts: $(\ell^0_{w,\infty})^\star \cong \ell_{w,1}$ and $(\ell_{w,1})^\star \cong \ell_{w,\infty}$ where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic.

Now, we will introduce Nezir's construction.

For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define $|||x||| := \|x\|_1 + \|x\|_\infty = \sum\limits_{n=1}^{\infty} |x_n| + \sup\limits_{n\in\mathbb{N}} |x_n|$. Clearly $|||\cdot|||$ is an equivalent norm on ℓ^1 with $\|x\|_1 \le |||x||| \le 2\|x\|_1$, $\forall x \in \ell^1$.

We shall call $|||\cdot|||$ the $1 \boxplus \infty$-norm on ℓ^1.

Note that $\forall x \in \ell^1$, $|||x||| = 2x_1^* + x_2^* + x_3^* + x_4^* + \cdots$ where z^* is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}$, $\forall z \in c_0$.

Let $\delta_1 := 2, \delta_2 := 1, \delta_3 := 1, \cdots, \delta_n := 1$, $\forall n \ge 4$.

We see that $(\ell^1, |||\cdot|||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0 \backslash \ell^1$ (the usual Lorentz situation). This suggests that $\ell^0_{\delta,\infty} = (c_0, \|\cdot\|)$ is an isometric predual of $(\ell^1, |||\cdot|||)$ where for all $z \in c_0$, $\|z\| := \sup\limits_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} z_j^*}{\sum_{j=1}^{n} \delta_j}$.

In our study, we will consider the degenerate Lorentz-Marcinkiewicz space generated by the weight sequence $\delta = \left(1+1, 1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, 1+\frac{1}{16}, \cdots, 1+\frac{1}{2^n}, \cdots\right)$.

That is, we will consider the degenerate Lorentz-Marcinkiewicz space given by the following definition.

Definition 1.7.

$$\ell_{\delta,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{\ell,1} := \sum_{j=1}^{\infty} |x_j| + \sum_{j=1}^{\infty} \frac{x_j^{\star}}{2^{j-1}} < \infty \right\}.$$

Here, one can notice that for $x \in \ell_{\delta,1}$,

$$\begin{aligned} \|x\|_{\ell,1} &= \sum_{j=1}^{\infty} |x_j| + \sum_{j=1}^{\infty} \frac{x_j^{\star}}{2^{j-1}} \\ &= \sum_{j=1}^{\infty} x_j^{\star} + \sum_{j=1}^{\infty} \frac{x_j^{\star}}{2^{j-1}} \\ &= \sum_{j=1}^{\infty} \left(1 + \frac{1}{2^{j-1}}\right) x_j^{\star}. \end{aligned}$$

Inspired by the construction of degenerate Lorentz-Marcinkiewicz Nezir introduced in [10], in [11], Nezir introduced the structure of asymptotically isometric copies of $\ell^{1\boxplus 0}$. Then, he proved that if a Banach space contains an asymptotically isometric copies of $\ell^{1\boxplus 0}$, it fails the fixed point property for nonexpansive mappings. This was an alternative property to the concept of Banach spaces' containing an asymptotically isometric copies of ℓ^1. Now, we will recall this notion and the consequences it yields in fixed point theory.

Definition 1.8. *[11] A Banach space $(X, \|\cdot\|)$ is said to contain an asymptotically isometric copy of $\ell^{1\boxplus 0}$ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\frac{1}{2}\left[\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|a_n| + \sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n|\right] \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \frac{1}{2}\left[\sup_{n\in\mathbb{N}}|a_n| + \sum_{n=1}^{\infty}|a_n|\right],$$

for all $(a_n)_n \in \ell^1$.

Then, as we previously stated that he obtained the following result.

Theorem 1.9. *[11] If a Banach space X contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

He also showed that the above result could be given as the consequence of the following theorem.

Theorem 1.10. *[11] If a Banach space X contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then X contains an asymptotically isometric copy of ℓ^1 but the converse is not true.*

2. Main Results

In this section, we define two new properties that imply the failure of the fixed point property for nonexpansive mappings. That is, we show that if a Banach space has any of the properties we introduce then it fails to have the fixed point property for nonexpansive mappings. We find alternative ways of detecting our properties. Then, we show that a Banach space contains an asymptotically isometric copy of ℓ^1 if and only if it has any of the properties we introduce. Moreover, we show that the degenerate Lorentz-Marcinkiewicz space we introduce in the earlier section has any of the properties we introduce in this section but we show that a Banach space isomorphic to the degenerate Loretz-Marcinkiewicz space we introduce in the previous section does not contain any asymptotically isometric copy of ℓ^1 while oviously it has the properties we introduce in this section. Now, let's introduce those new properties and the results we have mentioned.

First of all, we give the definitions of our properties as follows:

Definition 2.1. *We will say that a Banach space $(X, \|\cdot\|)$ has property NM-1 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| + \sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)|a_n|}{2^{n-1}} \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}},$$

for all $(a_n)_n \in \ell^1$.

Definition 2.2. *We will say that a Banach space $(X, \|\cdot\|)$ has property NM-2 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2} \le \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \le \sqrt{\left[\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2},$$

for all $(a_n)_n \in \ell^1$.

First, we give an alternative ways of detecting our properties NM-1 and NM-2 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has one of the properties NM-1 and NM-2.

Theorem 2.3. *A Banach space $(X, \|\cdot\|)$ has property NM-1 if and only if there is a sequence $(x_n)_n$ in X such that*

1. *there exists $M \in [1, \infty)$ so that for any $(a_n)_n \in \ell^1$,*

$$\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq M \left[\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \right] \tag{1}$$

and

2.

$$\lim_{n\to\infty} \|x_n\| = 1. \tag{2}$$

Proof. Suppose that $(X, \|\cdot\|)$ has property NM-1. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sum_{n=1}^{\infty} (1-\varepsilon_n)|a_n| + \sum_{n=1}^{\infty} \frac{(1-\varepsilon_n)|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \, , \tag{3}$$

for all $(a_n)_n \in \ell^1$.

We may assume $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence since we may replace that with $\xi_j := \max_{k\geq n} \varepsilon_k$, for all $j \in \mathbb{N}$. Let $z_k = (1-\varepsilon_k)^{-1} x_k$ for each $k \in \mathbb{N}$. Then, for all $(a_k)_k \in \ell^1$,

$$\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n z_n \right\| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{1-\varepsilon_n} + \sum_{n=1}^{\infty} \frac{|a_n|}{(1-\varepsilon_n)2^{n-1}} \, .$$

Let $M = \frac{1}{1-\varepsilon_1}$. Then, condition (1) is achieved for the sequence $(z_n)_n$ in X. Also, it is clear to see the condition (2) is achieved for the sequence $(z_n)_n$ too since in inequality (3), taking $(a_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim_{n\to\infty} \|x_n\| = 1$ and so $\lim_{n\to\infty} \|z_n\| = 1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $M \in [1, \infty)$ so that for all $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq M \left[\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \right] \tag{4}$$

and $\lim_{n\to\infty} \|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. Since $\lim_{k\to\infty} \|x_k\| = 1$, and $\|x_k\| \geq 1$ for all $k \in \mathbb{N}$, by passing to subsequences, if necessary, we may suppose that $1 \leq \|x_k\| \leq 1 + \varepsilon_k$ for all $k \in \mathbb{N}$. Define $z_k = \frac{x_k}{1+\varepsilon_k}$ for every $k \in \mathbb{N}$. Then, since $\|z_k\| \leq 1$, we have

$$\left\| \sum_{n=1}^{\infty} a_n z_n \right\| \leq \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \quad \text{for every } (a_k)_k \in \ell^1 \, .$$

Also, from the left hand side inequality of (4), we have

$$\left\| \sum_{n=1}^{\infty} a_n z_n \right\| = \left\| \sum_{n=1}^{\infty} a_n \frac{x_n}{(1+\varepsilon_n)} \right\| \geq \sum_{n=1}^{\infty} \frac{|a_n|}{1+\varepsilon_n} + \sum_{n=1}^{\infty} \frac{|a_n|}{(1+\varepsilon_n)2^{n-1}} \geq \sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| + \sum_{n=1}^{\infty} (1-\varepsilon_n)\frac{|a_n|}{2^{n-1}} \, .$$

□

Now, we show the alternative way of detecting NM-2 property.

Theorem 2.4. *A Banach space $(X, \|\cdot\|)$ has property NM-1 if and only if there is a sequence $(x_n)_n$ in X such that*

1. *there exists $M \in [1,\infty)$ so that for any $(a_n)_n \in \ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \le \left\|\sum_{n=1}^{\infty}a_nx_n\right\| \le M\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \tag{5}$$

and

2.

$$\lim_{n\to\infty}\|x_n\| = 1. \tag{6}$$

Proof. Assume that $(X, \|\cdot\|)$ has property NM-2. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|\right]^2+\left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)\sum\limits_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2} \le \left\|\sum_{n=1}^{\infty}a_nx_n\right\| \le \sqrt{\left[\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}|a_j|\right]^2+\left[\sum_{n=1}^{\infty}\frac{\sum\limits_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2}, \tag{7}$$

for every $(a_n)_n \in \ell^1$.

Now for every $n\in\mathbb{N}$, define $z_n := x_n - x_{n-1}$ with $x_0 = 0$. So there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that for all $(a_n)_n \in \ell^1$

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n|\right]^2+\left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)|a_n|}{2^{n-1}}\right]^2} \le \left\|\sum_{n=1}^{\infty}a_nz_n\right\| \le \sqrt{\left[\sum_{n=1}^{\infty}|a_n|\right]^2+\left[\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right]^2}, \tag{8}$$

We may assume $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence since we may replace that with $\xi_j := \max\limits_{k\ge n}\varepsilon_k$, for all $j \in \mathbb{N}$. Let $y_k = (1-\varepsilon_k)^{-1}z_k$ for each $k \in \mathbb{N}$. Then, for all $(a_k)_k \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \le \left\|\sum_{n=1}^{\infty}a_ny_n\right\| \le \sqrt{\left(\sum_{n=1}^{\infty}\frac{|a_n|}{1-\varepsilon_n}\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{(1-\varepsilon_n)2^{n-1}}\right)^2}.$$

Let $M = \frac{1}{1-\varepsilon_1}$. Then, condition (5) is achieved for the sequence $(z_n)_n$ in X. Also, it is clear to see the condition (6) is achieved for the sequence $(z_n)_n$ too since in inequality (3), taking $(a_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim\limits_{n\to\infty}\|x_n\| = 1$ and so $\lim\limits_{n\to\infty}\|z_n\| = 1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $M \in [1,\infty)$ so that for all $(a_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \le \left\|\sum_{n=1}^{\infty}a_nx_n\right\| \le M\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \tag{9}$$

and $\lim\limits_{n\to\infty}\|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. Since $\lim\limits_{k\to\infty}\|x_k\| = 1$, and $\|x_k\| \ge 1$ for all $k \in \mathbb{N}$, by passing to subsequences, if necessary, we may suppose that $1 \le \|x_k\| \le 1+\varepsilon_k$ for all $k \in \mathbb{N}$. Define $z_k = \frac{x_k}{1+\varepsilon_k}$ for every $k \in \mathbb{N}$. Then, since $\|z_k\| \le 1$, we have

$$\left\|\sum_{n=1}^{\infty}a_nz_n\right\| \le \sum_{n=1}^{\infty}|a_n| \le \sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \quad \text{for every } (a_n)_n \in \ell^1 .$$

Also, from the left hand side inequality of (9), we have

$$\left\|\sum_{n=1}^{\infty} a_n z_n\right\| = \left\|\sum_{n=1}^{\infty} a_n \frac{x_n}{(1+\varepsilon_n)}\right\| \geq \sqrt{\left(\sum_{n=1}^{\infty} \frac{|a_n|}{1+\varepsilon_n}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{(1+\varepsilon_n)2^n}\right)^2}$$
$$\geq \sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n|\right)^2 + \left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|a_n|}{2^{n-1}}\right)^2}\ .$$

Now for each $n \in \mathbb{N}$, define $y_n := \sum_{j=1}^{n} z_j$. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that for all $(a_n)_n \in \ell^1$,

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2} \leq \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq \sqrt{\left[\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2}\ .$$

□

Now, we give important results for properties NM-1 and NM-2, one by one.

Theorem 2.5. *Let $(X, \|.\|)$ be a Banach space. Then, X has property NM-1 if and only if X contains an asymptotically isometric copy of ℓ^1.*

Proof. Suppose that X has property NM-1. Then, there is a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty}\|x_n\| = 1$ and there exists a constant $M \in [1,\infty)$ so that for any $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M\left[\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right].$$

Thus, letting $R := 2M$ we have

$$\sum_{n=1}^{\infty}|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq R\sum_{n=1}^{\infty}|a_n|\ .$$

Hence, by Theorem 1.3, X contains an asymptotically isometric copy of ℓ^1.

Conversely, suppose that a Banach space X contains an asymptotically isometric copy of ℓ^1. Then, by Theorem 1.3, there is a sequence $(x_n)_n$ in X with $\lim_n\|x_n\| = 1$ and there exists a constant $M \in [1,\infty)$ such that for all $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty}|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M\sum_{n=1}^{\infty}|a_n|\ .$$

Now, define $z_n =: \left(1 + \frac{1}{2^{n-1}}\right)x_n$ for each $n \in \mathbb{N}$ and let $K = 2M$, then we have

$$\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq M\left(\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right).$$

Hence,

$$\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq K\sum_{n=1}^{\infty}|a_n| \leq K\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}$$

and $\lim_n\|z_n\| = 1$.

Hence, by Theorem 2.4, X has property NM-2 and we are done. □

Therefore, we can give the following corollary using Theorem 1.2.

Corollary 2.6. *If a Banach space X has property NM-1, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

Theorem 2.7. *Let $(X, \|.\|)$ be a Banach space. Then, X has property NM-2 if and only if X contains an asymptotically isometric copy of ℓ^1.*

Proof. Suppose that X has property NM-2. Then, there is a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty} \|x_n\| = 1$ and there exists a constant $M \in [1, \infty)$ so that for any $(a_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2} \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M \sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2} .$$

Thus, letting $R := \sqrt{2}M$ we have

$$\sum_{n=1}^{\infty} |a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq R \sum_{n=1}^{\infty} |a_n| .$$

Hence, by Theorem 1.3, X contains an asymptotically isometric copy of ℓ^1.

Conversely, suppose that a Banach space X contains an asymptotically isometric copy of ℓ^1. Then, by Theorem 1.3, there is a sequence $(x_n)_n$ in X with $\lim_n \|x_n\| = 1$ and there exists a constant $M \in [1, \infty)$ such that for all $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M \sum_{n=1}^{\infty} |a_n| .$$

Now, define $z_n =: \left(1 + \frac{1}{2^{n-1}}\right) x_n$ for each $n \in \mathbb{N}$ and let $K = 2M$, then we have

$$\begin{aligned}\sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2} &\leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^{n-1}} \\ &\leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq M \left(\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{n-1}\right)^2} \leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| &\leq K \sum_{n=1}^{\infty} |a_n| \\ &\leq K \sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2}\end{aligned}$$

and $\lim_n \|z_n\| = 1$.

Hence, by Theorem 2.4, X has property NM-2 and we are done. □

Therefore, we can give the following corollary using Theorem 1.2.

Corollary 2.8. *If a Banach space X has property NM-2, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

3. Some Examples and Remarks

In this section, we will give examples that will show some utilization of our property in the fixed point theory.

As we mentioned in the introduction section, our construction appears in the structure of some degenerate Lorentz-Marcinkiewicz spaces. We have been impressed by the first author's solely works [10, 11] that introduce a degenerate Lorentz-Marcinkiewicz space and later give the definition of the concept of Banach spaces containing asymptotically isometric copies of $\ell^{1\boxplus 0}$. Now, let's consider the degenerate Lorentz-Marcinkiewicz space $\ell_{\delta,1}$ that we had talked about in the introduction section where its weight sequence δ is given by $\delta = \left(1 + \frac{1}{2^{n-1}}\right)_n$. In a recent, unpublished study by the authors of this paper, it was shown that $\ell_{\delta,1}$ contains an asymptotically isometric copy of ℓ_1 and so we can say by Theorem 2.5 that it has property NM-1 and so equivalently it has property NM-2. Then, this would prove that $\ell_{\delta,1}$ fails the fixed point property for nonexpansive mappings.

On the other hand, by Theorem 1.10, we know that if a Banach space contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then it contains an asymptotically isometric copy of ℓ_1. Thus, we can conclude by Theorem 2.5 and Theorem 2.7 that if a Banach space contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then it has properties NM-1 and NM-2. However, the following example shows that there exists a Banach space that has these properties but it does not contain any asymptotically isometric copy of $\ell^{1\boxplus 0}$. We have to note that the first author showed in [11] that there exists a Banach space that contains an asymptotically isometric copy of ℓ_1 but it does not contain any asymptotically isometric copy of $\ell^{1\boxplus 0}$. Thus, his example also verifies our remark but here we provide a different example.

Example 3.1. *One can easly see that Banach space ℓ^1 with its equivalent renorming given by for any $x = (x_n)_n \in \ell^1$, $|||x|||^\sim = \sum_{n=1}^{\infty} \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)|x_n|$ has the property NM-1 (so NM-2) but we can prove that it does not contain any asymptotically isometric copy of $\ell^{1\boxplus 0}$.*

Proof. We use the similiar ideas expressed in [5] and by contradiction, assume $\left(\ell^1, \ |||\cdot|||^\sim\right)$ does contain an asymptotically isometric copy of $\ell^{1\boxplus 0}$. That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and a sequence $(x_n)_n$ in ℓ^1 such that

$$\begin{aligned} \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)\,|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\varepsilon_n)\,|t_n| \ &\leq\ \left|\left|\left|\sum_{n=1}^{\infty} t_n\, x_n\right|\right|\right|^\sim \\ &\leq\ \frac{1}{2}\sup_{n\in\mathbb{N}}|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}|t_n|, \end{aligned} \tag{10}$$

for every $(t_n)_{n\in\mathbb{N}} \in \ell^1$.

Without loss of generality we suppose that $(x_n)_n$ is disjointly supported and that by passing to a subsequence, we can assume that (x_n) converges weak* (and so it is pointwise) to some $y \in \ell^1$.

Next, replacing x_n by the $|||\cdot|||^\sim$-normalization of $\left(\frac{x_{2n}-x_{2n-1}}{2}\right)_n$ satisfying (10), we can suppose that $y = 0$.

By the proof of the Bessaga-Pełczyński Theorem [2], we may pass to an essentially disjointly supported subsequence of x_n. Hence, when it is normalized and truncated this subsequence appropriately, we get a disjointly supported sequence satisfying (10). Also, by passing to subsequences if necessary, we may suppose that $\varepsilon_n < \frac{1}{3n}$ for all $n \in \mathbb{N}$.

Let $(m(k))_{k\in\mathbb{N}_0}$ with $m(0) = 0$ and $(\xi_k)_{k\in\mathbb{N}}$ a sequence of scalars such that for each $k \in \mathbb{N}$, $y_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j$.

Using the triangular inequality of the norm, for each $K \in \mathbb{N}$, we get

$$\begin{aligned}\frac{K - K\varepsilon_K}{2} + \frac{K + 1 - \varepsilon_1 - K\varepsilon_K}{2} &\leq |||x_1 + Kx_K|||^{\sim} \\ \leq \sum_{k=1}^{m(1)} \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)|\xi_k| + K \sum_{k=m(K-1)+1}^{m(K)} \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)|\xi_k| \\ \leq \frac{1}{2} \sum_{k=1}^{m(1)} |\xi_k| + K\left(\frac{1}{4} + \frac{1}{2^{m(K-1)+2}}\right) \sum_{k=m(K-1)+1}^{m(K)} |\xi_k| \,.\end{aligned}$$

Therefore, $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K \leq \frac{1}{2} + K\left(\frac{1}{4} + \frac{1}{2^{m(K-1)+2}}\right)$ for all $K \in \mathbb{N}$. But since $\varepsilon_1 < \frac{1}{3}$ and $K\varepsilon_K < \frac{1}{3}$, we have $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K > K$ and so

$$1 + \frac{1}{2K} - \frac{\varepsilon_1}{2K} - \varepsilon_K \leq \frac{3}{4K} + \left(\frac{1}{4} + \frac{1}{2^{m(K-1)+1}}\right), \text{ for all } K \in \mathbb{N}.$$

Thus, we get a contradiction by letting $K \to \infty$ since we would have $\frac{3}{4} \leq 0$. This completes the proof. □

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Analytic Functions Expressed with q-Poisson Distribution Series

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Abstract. Recently, the q- derivative operator has been used to investigate several subclasses of analytic functions in different ways with different perspectives by many researchers and their interesting results are too voluminous to discuss. The q- derivative operator are also used to construct some subclasses of analytic functions.

In this study, we introduce certain subclasses of analytic and univalent functions in the open unit disk defined by q-derivative. Here, we give some conditions for an analytic and univalent function to belonging to these classes. Also, in the study, we define two functions using q-derivative and we aim to find the conditions for this functions to belonging to defined above subclasses of analytic functions.

1. Intoduction

Let A be the class of analytic functions f in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0 = f'(0) - 1$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \; a_n \in \mathbb{C}. \tag{1}$$

Also, by Swe will denote the family of all functions in A which are univalent in U.

Let T denote the subclass of all functions f in A of the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \cdots - a_n z^n - \cdots = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0. \tag{2}$$

Some of the important and well-investigated subclasses of the univalent functions class S include the classes $S^*(\alpha)$ and $C(\alpha)$, respectively, starlike and convex functions of order $\alpha (\alpha \in [0, 1))$. By definition, we have (see for details, [2, 3], also [9])

$$S^*(\alpha) = \left\{ f \in A : \; \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \; z \in U \right\}, \; C(\alpha) = \left\{ f \in A : \; \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \; z \in U \right\}.$$

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For $\beta \in [0,1)$, interesting generalization of the classes $S^*(\alpha)$ and $C(\alpha)$ are the classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$ which, respectively, defined as follows

$$S^*(\alpha,\beta)=\left\{f\in S:\ \mathrm{Re}\left(\frac{zf'(z)}{\beta zf'(z)+(1-\beta)f(z)}\right)>\alpha,\ z\in U\right\},$$

$$C(\alpha,\beta)=\left\{f\in S:\ \mathrm{Re}\left(\frac{f'(z)+zf''(z)}{f'(z)+\beta zf''(z)}\right)>\alpha,\ z\in U\right\}.$$

The classes $TS^*(\alpha,\beta)$ and $TC(\alpha,\beta)$ were extensively studied by Altintaş and Owa [1] and certain conditions for hypergeometric functions and generalized Bessel functions for these classes were studied by Moustafa [5] and Porwal and Dixit [8].

For $\gamma \in [0,1]$, a generalization of the function classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$ is the class $S^*C(\alpha,\beta;\gamma)$ which is defined as follows:

$$S^*C(\alpha,\beta;\gamma)=\left\{f\in S: Re\left(\frac{zf'(z)+\gamma z^2f''(z)}{\gamma z(f'(z)+\beta zf''(z))+(1-\gamma)(\beta zf'(z)+(1-\beta)f(z))}\right)>\alpha\right\},z\in U.$$

In his fundamental paper [4], Jackson, for $q\in(0,1)$, introduced the q-derivative operator D_q of the an analytic function f as follows:

$$D_qf(z)=\begin{cases}\frac{f(z)-f(qz)}{(1-q)z}, & if\ \ z\neq 0,\\ f'(0) & ,\ if\ \ z=0.\end{cases}$$

The formulas for the q- derivative D_q of a product and a quotient of functions are

$$D_qz^n=[n]_q z^{n-1}, n\in \mathrm{N},$$

where

$$[n]_q=\frac{1-q^n}{1-q}=\sum_{k=1}^{n}q^{k-1}$$

is the q- analogue of the natural number n.

It is clear that $\lim\limits_{q\to 1^-}[n]_q=n$, $[0]_q=0$ and $\lim\limits_{q\to 1^-}D_qf(z)=f'(z)$ for the function $f\in A$.

For $q\in(0,1)$ and $\alpha\in[0,1)$, we define by $S_q^*(\alpha)$ and $C_q(\alpha)$ the subclass of A which we will call, respectively, q - starlike and q- convex functions of order α, as follows:

$$S_q^*(\alpha)=\left\{f\in S:\ Re\frac{zD_qf(z)}{f(z)}>\alpha, z\in U\right\}, C_q(\alpha)=\left\{f\in S:\ Re\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}>\alpha, z\in U\right\}.$$

Also, let's denote $TS_q^*(\alpha)=T\bigcap S_q^*(\alpha)$ and $TC_q(\alpha)=T\bigcap C_q(\alpha)$.

For $\beta\in[0,1)$, interesting generalization of the function classes $S_q^*(\alpha)$ and $C_q(\alpha)$ are the function classes $S_q^*(\alpha,\beta)$ and $C_q(\alpha,\beta)$, respectively, which we define as follows:

$$S_q^*(\alpha,\beta)=\left\{f\in A:\ \mathrm{Re}\left(\frac{zD_qf(z)}{\beta zD_qf(z)+(1-\beta)f(z)}\right)>\alpha,\ z\in U\right\}, C_q(\alpha,\beta)=\left\{f\in A:\ \mathrm{Re}\left(\frac{D_qf(z)+zD_q^2f(z)}{D_qf(z)+\beta zD_q^2f(z)}\right)>\alpha,\ z\in U\right\}.$$

Now let's define a generalization of the function classes $S_q^*(\alpha,\beta)$ and $C_q(\alpha,\beta)$ as follows:

Definition 1.1. *For $\alpha, \beta \in [0,1)$ and $\gamma \in [0,1]$ a function f given by* (1) *is said to be in the class $S_q^* C_q(\alpha, \beta; \gamma)$ if the following condition is satisfied*

$$Re\left(\frac{zD_q f(z) + \gamma z^2 D_q^2 f(z)}{\gamma z\left(D_q f(z) + \beta z D_q^2 f(z)\right) + (1-\gamma)\left(\beta z D_q f(z) + (1-\beta) f(z)\right)}\right) > \alpha, z \in U.$$

We will use $TS_q^* C_q(\alpha, \beta; \gamma) = T \cap S_q^* C_q(\alpha, \beta; \gamma)$.

It is clear that $S_q^* C_q(\alpha, \beta; 0) = S_q^*(\alpha, \beta)$, $S_q^* C_q(\alpha, \beta; 1) = C_q(\alpha, \beta)$, $\lim_{q \to 1^-} S_q^* C_q(\alpha, \beta; \gamma) = S^* C(\alpha, \beta; \gamma)$ and $\lim_{q \to 1^-} TS_q^* C_q(\alpha, \beta; \gamma) = TS^* C(\alpha, \beta; \gamma)$. So, function classes $S_q^* C_q(\alpha, \beta; \gamma)$ and $TS_q^* C_q(\alpha, \beta; \gamma)$ are generalization of the previously known function classes $S_q^*(\alpha, \beta)$, $C_q(\alpha, \beta)$, $S^* C(\alpha, \beta; \gamma)$ and $TS^* C(\alpha, \beta; \gamma)$ of analytic functions, respectively.

A variable x is said to have q- Poisson Distribution if it takes the values 0,1,2,3,... with probabilities e_q^{-p}, $\frac{p}{1!} e_q^{-p}$, $\frac{p^2}{2!} e_q^{-p}$, $\frac{p^3}{3!} e_q^{-p}$, ... , respectively, where p a parameter and

$$e_q^x = 1 + \frac{x}{[1]_q!} + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \cdots + \frac{x^n}{[n]_q!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{3}$$

is q- analogue of the exponential function e^x and

$$[n]_q! = [1]_q \cdot [2]_q \cdot [3]_q \cdots [n]_q$$

is the q- analogue of the factorial function $n! = 1 \cdot 2 \cdot 3 \cdots n$.

Thus, for q- Poisson Distribution, we have

$$P_q(x = n) = \frac{p^n}{n!} e_q^{-p}, \ n = 0, 1, 2, 3, ...$$

Now, we introduce a q- Poisson Distribution series as follows:

$$z + \sum_{n=2}^{\infty} \frac{p^{n-1} e_q^{-p}}{[n-1]_q!} z^n, \ z \in U. \tag{4}$$

We can easily show that series (4) is convergent and the radius of convergence is infinity.

Let us define functions $F_q : \mathrm{C} \to \mathrm{C}$ and $G_q : \mathrm{C} \to \mathrm{C}$ as

$$F_q(z) = z + \sum_{n=2}^{\infty} \frac{p^{n-1} e_q^{-p}}{[n-1]_q!} z^n, \ z \in U \tag{5}$$

and

$$G_q(z) = 2z - F_q(z) = z - \sum_{n=2}^{\infty} \frac{p^{n-1} e_q^{-p}}{[n-1]_q!} z^n, \ z \in U. \tag{6}$$

It is clear that $F_q \in A$ and $G_q \in T$, respectively.

In this study, using $q-$derivative we introduce certain subclasses of analytic and univalent functions in the open unit disk in the complex plane. Here, we give some conditions for an analytic and univalent function to belong to these classes. Applications of a q- Poisson Distribution series on the analytic functions are also given. In the study, we define two functions F_q and G_q by q- Poisson Distribution and we aim to find the conditions for this functions to belong to the classes of analytic functions which we define in the study.

2. Main Results

In this section, we will give sufficient condition for the function F_q defined by (5), belonging to the class $S_q^*C_q(\alpha,\beta;\gamma)$, and necessary and sufficient condition for the function G_q defined by (6), belonging to the class $TS_q^*C_q(\alpha,\beta;\gamma)$, respectively.

In order to prove our main results, we need the following theorems.

Theorem 2.1. *[6] Let $f \in A$. Then, $f \in S_q^*C_q(\alpha,\beta;\gamma)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}\left\{[n]_q\left[(1-\alpha\beta)\left(1+\gamma[n-1]_q\right)-(1-\beta)\alpha\gamma\right]-\alpha(1-\beta)(1-\gamma)\right\}|a_n| \leq 1-\alpha.$$

The result obtained here is sharp.

Theorem 2.2. *[6] Let $f \in T$. Then, $f \in TS_q^*C_q(\alpha,\beta;\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\left\{[n]_q\left[(1-\alpha\beta)\left(1+\gamma[n-1]_q\right)-(1-\beta)\alpha\gamma\right]-\alpha(1-\beta)(1-\gamma)\right\}|a_n| \leq 1-\alpha.$$

The result obtained here is sharp.

A sufficient condition for the function F_q defined by (5) to belonging to the class $S_q^*C_q(\alpha,\beta;\gamma)$ is given by the following theorem.

Theorem 2.3. *Let $p > 0$ and the following condition is provided*

$$\left\{\begin{array}{l}(1-\alpha\beta)\gamma p^2+\left\{(1-\alpha\beta)\left[1+(1+q)q\gamma e_q^{p(q-1)}\right]-(1-\beta)\alpha\gamma\right\}p \\ -[1-\alpha\beta-(1-\beta)\alpha\gamma]\left(1-e_q^{p(q-1)}\right)\end{array}\right\} e_q^p \leq 1-\alpha. \tag{7}$$

Then, the function F_q defined by (5) *belongs to the class$S_q^*C_q(\alpha,\beta;\gamma)$.*

Proof. Since $F_q \in A$ and

$$F_q(p,z)=z+\sum_{n=2}^{\infty}\frac{p^{n-1}}{[n-1]!}e_q^{-p}z^n,\ z\in U,$$

according to Theorem 2.1, we must show that

$$\sum_{n=2}^{\infty}\left\{[n]_q\left[(1-\alpha\beta)\left(1+\gamma[n-1]_q\right)-(1-\beta)\alpha\gamma\right]-\alpha(1-\beta)(1-\gamma)\right\}\frac{p^{n-1}}{[n-1]_q!}e_q^{-p} \leq 1-\alpha. \tag{8}$$

Let

$$L_q(\alpha,\beta,\gamma)=\sum_{n=2}^{\infty}\left\{[n]_q\left[(1-\alpha\beta)\left(1+\gamma[n-1]_q\right)-(1-\beta)\alpha\gamma\right]-\alpha(1-\beta)(1-\gamma)\right\}\frac{p^{n-1}}{[n-1]_q!}e_q^{-p}.$$

By setting

$$\begin{aligned}&[n]_q\left[(1-\alpha\beta)\left(1+\gamma[n-1]_q\right)-(1-\beta)\alpha\gamma\right]-\alpha(1-\beta)(1-\gamma)\\&=[n]_q[1-\alpha\beta-(1-\beta)\alpha\gamma]+[n]_q[n-1]_q(1-\alpha\beta)\gamma-\alpha(1-\beta)(1-\gamma)\end{aligned}$$

and using $[n]_q=[n-1]_q+q^{n-1}$, $[n]_q=[n-2]_q+q^{n-2}+q^{n-1}$, we write

$$\begin{aligned}&[n]_q\left[(1-\alpha\beta)\left(1+\gamma[n-1]_q\right)-(1-\beta)\alpha\gamma\right]-\alpha(1-\beta)(1-\gamma)\\&=(1-\alpha\beta)\gamma[n-2]_q[n-1]_q+\left[1-\alpha\beta-(1-\beta)\alpha\gamma+(1-\alpha\beta)\gamma\left(q^{n-2}+q^{n-1}\right)\right][n-1]_q\\&+q^{n-1}[1-\alpha\beta-(1-\beta)\alpha\gamma]-\alpha(1-\beta)(1-\gamma)\end{aligned} \tag{9}$$

Considering equality (9), by simple computation, we can write

$$L_q(\alpha,\beta,\gamma;p)=(1-\alpha\beta)\,\gamma e_q^{-p}\sum_{n=3}^{\infty}\frac{p^{n-1}}{[n-3]_q!}+\left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]e_q^{-p}\sum_{n=2}^{\infty}\frac{p^{n-1}}{[n-2]_q!}$$
$$+(1-\alpha\beta)(1+q)\,\gamma e_q^{-p}\sum_{n=2}^{\infty}\frac{(qp)^{n-1}}{[n-2]_q!}+\left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]e_q^{-p}\sum_{n=2}^{\infty}\frac{(qp)^{n-1}}{[n-1]_q!}$$
$$-\alpha(1-\beta)(1-\gamma)\,e_q^{-p}\sum_{n=2}^{\infty}\frac{p^{n-1}}{[n-1]_q!}.$$

Then, using the equality (3), we obtain

$$L_q(\alpha,\beta,\gamma;p)=(1-\alpha\beta)\,\gamma p^2+\left\{(1-\alpha\beta)\left[1+(1+q)\,q\gamma e_q^{p(q-1)}\right]-(1-\beta)\,\alpha\gamma\right\}p$$
$$-\left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right)+(1-\alpha)\left(1-e_q^{-p}\right).$$

Therefore, inequality (8) holds true if

$$(1-\alpha\beta)\,\gamma p^2+\left[1-\alpha\beta-(1-\beta)\,\alpha\gamma+(1-\alpha\beta)(1+q)\,q\gamma e_q^{p(q-1)}\right]p$$
$$-\left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right)+(1-\alpha)\left(1-e_q^{-p}\right)\le 1-\alpha$$

which is equivalent to (7).

Thus, the proof of Theorem 2.3 is completed.

□

From the Theorem 2.3, we can readily deduce the following results.

Corollary 2.4. *If $p>0$ and satisfied the following condition*

$$(1-\alpha\beta)\left(p-1+e_q^{p(q-1)}\right)e_q^{p}\le 1-\alpha,$$

then the function F_q defined by (5) *belongs to the class $S_q^*(\alpha,\beta)$.*

Corollary 2.5. *If $p>0$ and satisfied the following condition*

$$\left\{(1-\alpha\beta)\,p^2+\left[1-\alpha+(1-\alpha\beta)(1+q)\,qe_q^{p(q-1)}\right]p-(1-\alpha)\left(1-e_q^{p(q-1)}\right)\right\}e_q^{p}\le 1-\alpha,$$

then the function F_q defined by (5) *belongs to the class $C_q(\alpha,\beta)$.*

Now, we give necessary and sufficient condition for the function G_q defined by (6), to belonging to the class $TS_q^*C_q\left(\alpha,\beta;\gamma\right)$ with the following theorem.

Theorem 2.6. *If $p>0$, then the function G_q defined by* (6) *belongs to the class $TS_q^*C_q(\alpha,\beta;\gamma)$ if and only if satisfied the following condition*

$$\left\{\begin{array}{l}(1-\alpha\beta)\,\gamma p^2+\left\{(1-\alpha\beta)\left[1+(1+q)\,q\gamma e_q^{p(q-1)}\right]-(1-\beta)\,\alpha\gamma\right\}p\\ -\left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right)\end{array}\right\}e_q^{p}\le 1-\alpha. \tag{10}$$

Proof. Firstly, let us prove the sufficiency of the theorem.

First of all, let us state that we will use Theorem 2.2 to prove the theorem.

It is clear that $G_q\in T$. Let us show that the function G_q satisfies the sufficiency condition of Theorem 2.2

.

From the proof of Theorem 2.3, we write

$$\begin{aligned}
&\textstyle\sum_{n=2}^{\infty} \left\{[n]_q \left[(1-\alpha\beta)\left(1+\gamma\,[n-1]_q\right) - (1-\beta)\,\alpha\gamma\right] - \alpha\,(1-\beta)\,(1-\gamma)\right\} \frac{p^{n-1}}{[n-1]_q!} e_q^{-p} \\
&= (1-\alpha\beta)\,\gamma p^2 + \left\{(1-\alpha\beta)\left[1+(1+q)\,q\gamma e_q^{p(q-1)}\right] - (1-\beta)\,\alpha\gamma\right\} p \\
&- \left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right) + (1-\alpha)\left(1-e_q^{-p}\right)
\end{aligned} \tag{11}$$

Now, suppose that condition (10) is satisfied.
It follows that

$$\begin{aligned}
&(1-\alpha\beta)\,\gamma p^2 + \left\{(1-\alpha\beta)\left[1+(1+q)\,q\gamma e_q^{p(q-1)}\right] - (1-\beta)\,\alpha\gamma\right\} p - \left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right) \\
&\leq (1-\alpha)\,e_q^{-p},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&(1-\alpha\beta)\,\gamma p^2 + \left\{(1-\alpha\beta)\left[1+(1+q)\,q\gamma e_q^{p(q-1)}\right] - (1-\beta)\,\alpha\gamma\right\} p - \left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right) \\
&+ (1-\alpha)\left(1-e_q^{-p}\right) \leq 1-\alpha,
\end{aligned}$$

Considering equality (11), we write

$$\sum_{n=2}^{\infty} \left\{[n]_q \left[(1-\alpha\beta)\left(1+\gamma\,[n-1]_q\right) - (1-\beta)\,\alpha\gamma\right] - \alpha\,(1-\beta)\,(1-\gamma)\right\} \frac{p^{n-1}}{[n-1]_q!} e_q^{-p} \leq 1-\alpha. \tag{12}$$

Thus, the function G_q satisfies the sufficiency condition of Theorem 2.2. Hence, according to Theorem 2.2, the function G_q belongs to the class $TS_q^{*}C_q(\alpha,\beta;\gamma)$.
With this, the proof of the sufficiency of theorem is completed.
Now, let us we prove the necessity of theorem.
Assume that $G_q \in TS_q^{*}C_q(\alpha,\beta;\gamma)$. Then, from Theorem 2, we can say that condition (12) is satisfied.
It follows from (11) that

$$\begin{aligned}
&(1-\alpha\beta)\,\gamma p^2 + \left\{(1-\alpha\beta)\left[1+(1+q)\,q\gamma e_q^{p(q-1)}\right] - (1-\beta)\,\alpha\gamma\right\} p - \left[1-\alpha\beta-(1-\beta)\,\alpha\gamma\right]\left(1-e_q^{p(q-1)}\right) \\
&+ (1-\alpha)\left(1-e_q^{-p}\right) \leq 1-\alpha,
\end{aligned}$$

which is equivalent to (10).
This completes proof of the necessity of theorem.
Thus, the proof of Theorem 2.6 is completed.
□

From Theorem 2.6, we can readily deduce the following results.

Corollary 2.7. *If $p > 0$, then the function G_q defined by* (6) *belongs to the class $TS_q^{*}(\alpha,\beta)$ if and only if satisfied the following condition*

$$(1-\alpha\beta)\left(p-1+e_q^{p(q-1)}\right)e_q^{p} \leq 1-\alpha.$$

Corollary 2.8. *If $p > 0$, then the function G_q defined by* (6) *belongs to the class $TC_q(\alpha,\beta)$ if and only if satisfied the following condition*

$$\left\{(1-\alpha\beta)\,p^2 + \left[1-\alpha+(1-\alpha\beta)\,(1+q)\,qe_q^{p(q-1)}\right]p - (1-\alpha)\left(1-e_q^{p(q-1)}\right)\right\}e_q^{p} \leq 1-\alpha.$$

Remark 2.9. *The results obtained in Theorem 2.3, Theorem 2.6 and Corollaries 2.4, 2.5, 2.7, 2.8 are generalization of the results obtained in Theorem 3,4 and Corollary 5-8 in [7].*

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Repdigits as Product of Fibonacci and Pell numbers

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Abstract. In this paper, we find all repdigits which can be expressed as the product of a Fibonacci number and a Pell number. We use of a combined approach of lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method to prove our main result.

1. Introduction

Diophantine equations involving recurrence sequences have been studied for a long time. One of the most interesting of these equations is the equations involving repdigits.

A repdigit (short for "repeated digit") T is a natural number composed of repeated instances of the same digit in its decimal expansion. That is, T is of the form

$$x \cdot \left(\frac{10^t - 1}{9}\right)$$

for some positive integers x, t with $t \geq 1$ and $1 \leq x \leq 9$.

Some of the most recent papers related to the repdigits with well known recurrence sequences are [3, 5, 6, 8]. In this note, we use Fibonacci and Pell sequences in our main result.

Binet's formula for Fibonacci numbers is

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where $\varphi = \left(1 + \sqrt{5}\right)/2$ (the golden ratio) and $\psi = \left(1 - \sqrt{5}\right)/2$. From this formula, one can easiliy get

$$\varphi^{n-2} \leq F_n \leq \varphi^{n-1}. \tag{1}$$

Also, we can write

$$F_n = \frac{\varphi^n}{\sqrt{5}} + \theta \tag{2}$$

where $|\theta| \leq 1/\sqrt{5}$.

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Pell sequence, one of the most familiar binary recurrence sequence, is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$. Some of the terms of the Pell sequence are given by $0, 1, 2, 5, 12, 29, 70, \ldots$. Its characteristic polynomial is of the form $x^2 - 2x - 1 = 0$ whose roots are $\alpha = 1 + \sqrt{2}$ (the silver ratio) and $\beta = 1 - \sqrt{2}$. Binet's formula enables us to rewrite the Pell sequence by using the roots α and β as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}. \tag{3}$$

Also, it is known that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{4}$$

and

$$P_n = \frac{\alpha^n}{2\sqrt{2}} + \lambda \tag{5}$$

where $|\lambda| \leq 1/\left(2\sqrt{2}\right)$.

In this study, our main result is the following:

Theorem 1.1. *The only positive integer triples (n, t, x) with $1 \leq x \leq 9$ satisfying the Diophantine equation*

$$F_n P_n = x \cdot \left(\frac{10^t - 1}{9}\right) \tag{6}$$

as follows:

$$(n, t, x) \in \{(1, 1, 1), (2, 1, 2)\}.$$

2. Preliminaries

Before proceeding with the proof of our main result, let us give some necessary information for proof. We give the definition of the logarithmic height of an algebraic number and its some properties.

Definition 2.1. *Let z be an algebraic number of degree d with minimal polynomial*

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \cdot \prod_{i=1}^{d} (x - z_i)$$

where a_i's are relatively prime integers with $a_0 > 0$ and z_i's are conjugates of z. Then

$$h(z) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|z_i|, 1\}\right)\right)$$

is called the logarithmic height of z. *The following proposition gives some properties of logarithmic height that can be found in [9].*

Proposition 2.2. *Let $z, z_1, z_2, \ldots, z_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

1. $h(z_1 \cdots z_t) \leq \sum_{i=1}^{t} h(z_i)$
2. $h(z_1 + \cdots + z_t) \leq \log t + \sum_{i=1}^{t} h(z_i)$
3. $h(z^m) = |m| h(z)$.

We will use the following theorem (see [7] or Theorem 9.4 in [2]) and lemma (see [1] which is a variation of the result due to [4]) for proving our results.

Theorem 2.3. *Let $z_1, z_2, \ldots, z_s$ be nonzero elements of a real algebraic number field $\mathbb{F}$ of degree D, $b_1, b_2, \ldots, b_s$ rational integers. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := z_1^{b_1} \ldots z_s^{b_s} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log(sB)) \cdot A_1 \cdots A_s$$

where

$$A_i \geq \max\{D \cdot h(z_i), \left|\log z_i\right|, 0.16\}$$

for all $1 \leq i \leq s$. If $\mathbb{F} = \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

Lemma 2.4. *Let A, B, μ be some real numbers with $A > 0$ and $B > 1$ and let γ be an irrational number and M be a positive integer. Take p/q as a convergent of the continued fraction of γ such that $q > 6M$. Set $\varepsilon := \left\|\mu q\right\| - M\left\|\gamma q\right\| > 0$ where $\|\cdot\|$ denotes the distance from the nearest integer. Then there is no solution to the inequality*

$$0 < \left|u\gamma - v + \mu\right| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log \frac{Aq}{\varepsilon}}{\log B}.$$

3. The Proof of Theorem 1.1

Let us write Equations (2) and (5) in Equation (6). We get

$$\left(\frac{\varphi^n}{\sqrt{5}} + \theta\right)\left(\frac{\alpha^n}{2\sqrt{2}} + \lambda\right) = x \cdot \left(\frac{10^t - 1}{9}\right).$$

By using $|\theta| \leq 1/\sqrt{5}$ and $|\lambda| \leq 1/\left(2\sqrt{2}\right)$ we obtain

$$\left|\frac{(\varphi\alpha)^n}{\sqrt{5} \cdot 2\sqrt{2}} - \frac{x \cdot 10^t}{9}\right| < 0.8 \cdot \alpha^n.$$

To convert this inequality into form in Theorem 2.3, let us divide both sides by $(\varphi\alpha)^n / \left(\sqrt{5} \cdot 2\sqrt{2}\right)$. So, we have

$$\left|1 - 10^t \cdot (\varphi\alpha)^{-n} \cdot \left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right)\right| < 5.06 \cdot \varphi^{-n}. \tag{7}$$

Set

$$\Gamma := 10^t \cdot (\varphi\alpha)^{-n} \cdot \left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right) - 1.$$

We claim that $\Gamma \neq 0$. If $\Gamma = 0$, then one can easiliy see that $(\varphi)^{2n} \in \mathbb{Q}(\alpha)$. Since $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$ and φ is an quadratic algebraic number, the degree of $(\varphi)^{2n}$ is either 1 or 2. This means that $(\varphi)^{2n} \in \mathbb{Q}$ but from the Binomial theorem we know that $(\varphi)^{2n}$ is of the form $X_n + Y_n\sqrt{5}$ for some positive rational numbers X_n and Y_n which is a contradiction. Thus, we get $\Gamma \neq 0$.

Let us apply Theorem 2.3 to the inequality (7). Set

$$(z_1, z_2, z_3) = \left(10, \varphi\alpha, \left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right) \text{ and } (b_1, b_2, b_3) = (t, -n, 1).$$

Since $z_i \in \mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)$, we know that $D = 4$. So, we can take

$$\begin{aligned} 10 = A_1 &\geq 4 \cdot h(10) = 4 \cdot \log(10) \sim 9.21 \\ 3 = A_2 &\geq 4 \cdot h(\varphi\alpha) < 4 \cdot \log(2) \sim 2.77 \\ 25 = A_3 &\geq 4 \cdot h\left(x \cdot \sqrt{5}2\sqrt{2}/9\right) < 24.96. \end{aligned}$$

Now, let us try to estimate the value of B. From the inequalities (1) and (4), we can write

$$\varphi^{n-1} \cdot \alpha^{n-1} \geq F_n P_n = x \cdot \left(10^{t-1} - 1\right)/9 > 10^{t-1}$$

and this inequality implies that

$$1.68t - 1 < n. \tag{8}$$

Since $t < 1.68t - 1$ for $t > 1$ we can write $t < n$ from the inequality (8). Thus, we take

$$B := n.$$

So, due to the Theorem 2.3 we have

$$|\Gamma| > \exp\left(-C \cdot (1 + \log n) \cdot 10 \cdot 3 \cdot 25\right)$$

where $C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4^2 \cdot (1 + \log 4)$. From the inequality (7), we get

$$\frac{5.06}{\varphi^n} > \exp\left(-C \cdot (1 + \log n) \cdot 10 \cdot 3 \cdot 25\right).$$

Taking logarithm of both sides of the above inequality and considering $C < 5.5 \cdot 10^{12}$ and $1 + \log n < 2\log n$ for $n \geq 3$, we obtain

$$n < 7.1 \cdot 10^{17}. \tag{9}$$

By the inequality (8), we get

$$t < 4.3 \cdot 10^{17}. \tag{10}$$

Now, let us improve the bounds (9) and (10). Set

$$\Omega := t\log 10 - n\log(\varphi\alpha) + \log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right).$$

So, we can rewrite the Inequality (7) as

$$\left|1 - e^{\Omega}\right| < \frac{5.06}{\varphi^n}.$$

If $\Omega > 0$, then

$$\Omega < e^{\Omega} - 1 < 5.06 \cdot \varphi^{-n}.$$

Otherwise, i.e., $\Omega < 0$, then

$$1 - e^{-|\Omega|} = \left|e^{\Omega} - 1\right| < 5.06 \cdot \varphi^{-n}.$$

Thus,

$$|\Omega| < e^{|\Omega|} - 1 < \varphi^{-n} / \left(1 - \varphi^{-n}\right) < \varphi^{-n+1}.$$

From this inequality, we get

$$|\Omega| < 5.06 \cdot \varphi^{-n+1}. \tag{11}$$

Now, without loss of the generality, suppose $\Omega > 0$ (operations for the case $\Omega < 0$ are similar). From the Inequality (11), we obtain

$$\begin{aligned} 0 &< t\log 10 - n\log(\varphi\alpha) + \log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right) \\ &< 5.06 \cdot \varphi^{-(n-1)}. \end{aligned}$$

Dividing both sides of the above inequality by $\log(\varphi\alpha)$, we get

$$0 < t \cdot \frac{\log 10}{\log(\varphi\alpha)} - n + \frac{\log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right)}{\log(\varphi\alpha)} < 3.72 \cdot \varphi^{-(n-1)}.$$

In here, $\gamma := \log 10/\log(\varphi\alpha)$ is an irrational number. Hence, we can apply the Lemma 2.4 to the above inequality with the parameters

$$\mu := \frac{\log\left(\left(x \cdot \sqrt{5} \cdot 2\sqrt{2}\right)/9\right)}{\log(\varphi\alpha)},\ A := 3.72,\ B := \varphi \text{ and } w := n-1.$$

We can choose $M := 4.3 \cdot 10^{17}$ from the bound (10). So, 41th convergence of γ is satisfies the condition $q > 6M$. From this convergent, we get the smallest ε as 0.00207249. Thus, we have

$$\frac{\log(3.72 \cdot 2714452526429576634/0.00207249)}{\log\varphi} \sim 103.775 \leq n-1$$

and so, we get $n < 104$. Considering this bound on n, we obtain $t < 63$ from the inequality (8). Hence, in Mathematica, for the values $1 \leq n \leq 103$ and $1 \leq t \leq 62$ we get the solutions of the equality (6) as follows:

$$(n, t, x) \in \{(1, 1, 1), (2, 1, 2)\}.$$

This completes the proof.

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The Geometry of Ribbons

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Abstract. In this study, the representations of the ribbons in the 3-dimensional Euclidean space as the developable ruled surface are given. By calculating the average curvature of the ribbon surface, the results regarding the mean curvature according to the character of the centerline are obtained. In addition, examples supporting these results are given.

1. Introduction

In classical differential geometry, the ruled surfaces with additional property (constant mean curvature, constant Gauss curvature, minimal, etc.) are probably the simplest surface having the specified properties see for example [7],[8], [11] and references there in. These surfaces have many applications in surface modeling [12] and parametric design [13]. The structures formed are in the form of ribbon that increase in width and length in the [2] self-shaping process. The rulings don't correspond to the ribbon's central lines, but only to geometric lines that are constantly evolving during the deformation [5]. Helical ribbons are a significant class of 2-dimensional structures that often occurs in engineering and biology [14]. The predictive model for the mechanics and morphology of the stability of spiral bands is a new and important tool for research and design in various technologies such as biological sensing, nano-engineering coils for visual electronics [10].

This agreement aims to develop a common framework for discussing the above mathematical model of ribbons by classifying according to the centerline curves $\gamma(s)$. This article is structured as follows: Section 2 contains some notations and basic equations of the differential geometry of spatial curves in $\mathbb{E}^3$. Section 3 describes some geometrical properties of ruled surfaces at $\mathbb{E}^3$. Section 4 contains original results about ribbon surfaces in $\mathbb{E}^3$. These components also provide some of the key characteristics of ribbon surfaces and the construction of their curvatures. In section 5, we present some examples of ribbon surfaces. Finally, in section 6, we discuss our findings and decide details for future work.

2. Preliminaries

We will now analyze some notations of the differential distribution of spatial curves in 3-dimensional Euclidean space $\mathbb{E}^3$. Let $\gamma = \gamma(s) : I \to \mathbb{E}^3$ be a unit speed curve with $\gamma'(s) \neq 0$, where $\gamma'(s) = \frac{d\gamma}{ds}$. $T(s) = \gamma'(s)$

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is a unit tangent vector and is perpendicular to $T'(s) = \gamma''(s)$. If $\gamma''(s) \neq 0$, these vectors extend on the plane of oscillation γ in s. Specify the curvature of γ with $\kappa(s) = \left\| \gamma''(s) \right\|$. If $\kappa(s) \neq 0$, then the principal principal unit of the unit $N(s)$ of the curve γ in s is given by $T'(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector γ in s. Hence the Serret-Frenet formulas of γ are

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s) \end{aligned} \tag{1}$$

where $\tau(s)$ is the torsion of the curve γ at s [6].

A curve $\gamma = \gamma(s) : I \to \mathbb{E}^3$ with $\kappa(s) \neq 0$ is called a *conical geodesic (resp. cylindrical helix)* if the ratio $(\frac{\tau}{\kappa})'(s)$ (resp. $\frac{\tau}{\kappa}(s)$) is constant function [7]. If $\kappa(s) \neq 0$, $\tau(s)$ are both constant, then γ is known as *circular helix (W-curve)* [6].

In [4], B.Y. Chen defined a new type of curve in three-dimensional Euclidean space called a *rectifying curve*. According to his definition, a unit speed curve $\gamma : I \to \mathbb{E}^3$ is called the rectifying curve if γ satisfies the equation

$$\gamma(s) = m_1(s)T(s) + m_2(s)B(s) \tag{2}$$

for some real valued functions $m_1(s)$ and $m_2(s)$ [4]. By differentiating (2) and using the Frenet formulas one can obtain $m_1(s) = 1$, $m_2'(s) = 0$, $m_1\kappa - m_2\tau = 0$. As a result of these conditions, it is easy to show that the curve is a rectifying curve if and only if $\frac{\tau}{\kappa}(s) = as + b$ holds. Consequently, each rectifying curve is a kind of conical geodesic.

The Frenet motion formulas can be expounded as "if with the time variable s the motion point crosses the curve, the moving frame $\{T(s), N(s), B(s)\}$ moves based on (1). Consequently, the instantaneous rotational speed given by the Darboux vector

$$W(s) = \tau(s)T(s) + \kappa(s)B(s). \tag{3}$$

The Darboux vector is in its instantaneous axis direction of rotation and its length is

$$\omega(s) = \|W(s)\| = \sqrt{\kappa^2(s) + \tau^2(s)}.$$

The *modified Darboux* vector along the curve γ is defined by (see [7])

$$\widetilde{W}(s) = \frac{\tau(s)}{\kappa(s)}T(s) + B(s). \tag{4}$$

3. Material and Method

We now deal with the ruled surfaces in Euclidean space $\mathbb{E}^3$.

Definition 3.1 *With the ruled patch*

$$\varphi = \varphi_{(\gamma,\beta)}(s,u) = \gamma(s) + u\beta(s) \tag{5}$$

the surface is called a ruled surface, where, γ is the base curve and β is the director of the surface. The rulings of the surface are the lines $u \longmapsto \gamma(s) + u\beta(s)$ (see [1], [7],[8], [9] and [11]).

Let S be a ruled surface. In this case, the T_pS space is spanned by the following vectors;

$$\begin{aligned} \varphi_s(s,u) &= \frac{\partial \varphi_{(\gamma,\beta)}}{\partial s} = \gamma'(s) + u\beta'(s), \\ \varphi_u(s,u) &= \frac{\partial \varphi_{(\gamma,\beta)}}{\partial u} = \beta(s). \end{aligned}$$

The coefficients of the 1^{st} fundamental form are

$$\begin{aligned} g_{11} &= <\varphi_s(s,u), \varphi_s(s,u)>= \left\|\gamma'(s) + u\beta'(s)\right\|^2, \\ g_{12} &= <\varphi_s(s,u), \varphi_u(s,u)>= \langle\gamma'(s), \beta(s)\rangle + u\langle\beta'(s), \beta(s)\rangle, \\ g_{22} &= <\varphi_u(s,u), \varphi_u(s,u)>= \langle\beta(s), \beta(s)\rangle, \end{aligned} \tag{6}$$

where $\langle , \rangle$ is the inner product of $\mathbb{E}^3$. If the area element

$$\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\| = \sqrt{g_{11}g_{22} - g_{12}^2}, \tag{7}$$

does not vanish then $\varphi_{(\gamma,\beta)}$ is called *regular*. From now on we assume that $\varphi_{(\gamma,\beta)}$ is a regular patch. Then, the unit normal vector is

$$U(s,u) = \frac{\varphi_s(s,u) \times \varphi_u(s,u)}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|}. \tag{8}$$

Also, the partial derivatives of second order are:

$$\begin{aligned} \varphi_{ss}(s,u) &= \gamma''(s) + u\beta''(s), \\ \varphi_{su}(s,u) &= \beta'(s), \\ \varphi_{uu}(s,u) &= 0. \end{aligned} \tag{9}$$

Using (8) with (9) the coefficients of the 2^{nd} fundamental form become

$$\begin{aligned} L_{11} &= <\varphi_{ss}(s,u), U>= \frac{\det(\varphi_{ss}, \varphi_s, \varphi_u)}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|}, \\ L_{12} &= <\varphi_{su}(s,u), U>= \frac{\det(\varphi_{su}, \varphi_s, \varphi_u)}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|}, \\ L_{22} &= <\varphi_{uu}(s,u), U>= 0. \end{aligned} \tag{10}$$

Summing up these equations, one can write that the Gaussian curvature of S at point (s,u) is

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} = -\frac{\langle\beta'(s), \gamma'(s) \times \beta(s)\rangle^2}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|^2}, \tag{11}$$

and the mean curvature of S is

$$\begin{aligned} H &= \frac{L_{11}g_{22} + L_{22}g_{11} - 2g_{12}L_{12}}{2\left(g_{11}g_{22} - g_{12}^2\right)} \\ &= \frac{\langle\beta,\beta\rangle\det(\varphi_{ss}, \varphi_s, \varphi_u) - 2\left(\langle\gamma',\beta\rangle + u\langle\beta',\beta\rangle\right) + \det(\gamma', \varphi_s, \varphi_u)}{2\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|^3}. \end{aligned} \tag{12}$$

If K vanishes then the ruled surface is called *developable*. Further, if $\beta'(s) \times \beta(s) = 0$, then S is called cylindrical, otherwise it is non-cylindrical. The surface of $S \subset \mathbb{E}^3$ is minimal if and only if its mean curvature vanishes identically.

In [7], authors studied the *rectifying developable surfaces* given with the parametrization

$$\varphi_{(\gamma,\widetilde{W})}(s,u) = \gamma(s) + u\widetilde{W}(s) \tag{13}$$

where $\widetilde{W}(s)$ is the *modified Darboux* vector field defined by (4). They have proved that if the rectifying developable surface given with the parametrization (13) of is a cylindrical surface (resp. conical surface) then the base curve γ is a cylindrical helix (resp. conical helix).

4. Results

In [7], S. Izumiya and N. Takeuchi studied with ruled surface using the base curve $\widetilde{W}(s)$. They called them rectifying developable surfaces. In this section, we present an application of rectifying developable surface to ribbons in $\mathbb{E}^3$. However, a characterization of the mean curvature of the strip surfaces according to the character of the central line of the ribbon is given.

Definition 4.1 *A ribbon is a rectifying developable surface defined by the ruled patch*

$$\widetilde{\varphi} = \widetilde{\varphi}_{(\gamma,\widetilde{W})}(s,u) = \gamma(s) + u\widetilde{W}(s), \ s \in [0,L], \ u \in [-b,b] \tag{14}$$

where, $\widetilde{W}$ is the modified Darboux vector field defined by (4) (see, [3]).

Let R be a ribbon surface given with the ruled patch (14) then the tangent space of R is spanned by

$$\begin{aligned} \widetilde{\varphi}_s(s,u) &= (1 + u\rho'(s))\, T(s) \\ \widetilde{\varphi}_u(s,u) &= \rho(s)T(s) + B(s). \end{aligned} \tag{15}$$

where $\rho(s) = \frac{\tau(s)}{\kappa(s)}$ is the *harmonic curvature function of* γ. Then the coefficients of the 1^{st} fundamental form of R are found as

$$\begin{aligned} \widetilde{g}_{11} &= <\widetilde{\varphi}_s(s,u), \widetilde{\varphi}_s(s,u)>= (1 + u\rho'(s))^2 \\ \widetilde{g}_{12} &= <\widetilde{\varphi}_s(s,u), \widetilde{\varphi}_u(s,u)>= \rho(s)\,(1 + u\rho'(s)) \\ \widetilde{g}_{22} &= <\widetilde{\varphi}_u(s,u), \widetilde{\varphi}_u(s,u)>= 1 + (\rho(s))^2. \end{aligned} \tag{16}$$

Consequently, the area element of the ribbon becomes

$$\sqrt{\widetilde{g}_{11}\widetilde{g}_{22} - \widetilde{g}_{12}^2} = \left\|\widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u)\right\| = \left|1 + u\rho'(s)\right| \tag{17}$$

where

$$\begin{aligned} \widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u) &= (1 + u\rho'(s))\, T(s) \times (\rho(s)T(s) + B(s)) \\ &= -\,(1 + u\rho'(s))\, N(s) \end{aligned} \tag{18}$$

is the surface normal. So, the unit normal vector field of R becomes

$$\widetilde{U}(s,u) = \frac{\widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u)}{\left\|\widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u)\right\|} = -N(s). \tag{19}$$

The second partial derivatives of $\widetilde{\varphi}(u,v)$ are expressed as follows

$$\begin{aligned} \widetilde{\varphi}_{ss}(s,u) &= u\rho''(s)T(s) + \kappa(s)\,(1 + u\rho'(s))\, N(s), \\ \widetilde{\varphi}_{su}(s,u) &= \rho'(s)T(s), \\ \widetilde{\varphi}_{uu}(s,u) &= 0. \end{aligned} \tag{20}$$

Using (19) with (20) the coefficients of the 2^{nd} fundamental form become

$$\begin{aligned} \widetilde{L}_{11} &= <\widetilde{\varphi}_{ss}(s,u), U>= -\kappa(s)\,(1 + u\rho'(s))\,, \\ \widetilde{L}_{12} &= <\widetilde{\varphi}_{su}(s,u), U>= 0, \\ \widetilde{L}_{22} &= <\widetilde{\varphi}_{uu}(s,u), U>= 0. \end{aligned} \tag{21}$$

From the equations (21) and (11) it can be easily seen that the ribbon R is a flat surface. Furthermore, summing up (16)-(20) and using (12) we obtain the following results;

Theorem 4.2 *Let R be a ribbon surface given by (14), then the mean curvature vector of R becomes*

$$\widetilde{H}(s,u) = -\frac{\kappa(s)\left(1+(\rho(s))^2\right)}{2\left|1+u\rho'(s)\right|} \tag{22}$$

where $\rho(s) = \frac{\tau(s)}{\kappa(s)}$ *is the harmonic curvature (function) of the central line* γ.

With the equation (22) we have the following results;

Corollary 4.3 *The ribbon surface R given by the parametrization (14) can not be minimal.*

Corollary 4.4 *Let R be a ribbon surface given by (14). If the center line of R is a circular helix then the mean curvature of the ribbon is constant i.e., the ribbon is of constant mean curvature.*

Corollary 4.5 *Let R be a ribbon surface given by (14). If the center line of R is a cylindrical helix then the mean curvature of the ribbon turns into* $\widetilde{H}(s,u) = \delta\kappa(s)$, *where* $\delta = -\frac{1+\rho}{2}$ *is a constant function.*

The following result contains a characterization of the Serret-Frenet curvatures of the spherical (constant slope) helix curves.

Lemma 4.6 *[6] Let* $\gamma = \gamma(s)$ *be a unit-speed curve in* $\mathbb{E}^3$ *that has constant slope* $cot\theta = \frac{\tau}{\kappa}$ *with respect to a unit vector* $\overrightarrow{u} \in \mathbb{E}^3$, *where* $0 < \theta < \frac{\pi}{2}$. *Assume also that* γ *lies on a sphere of radius* $r > 0$ *then the curvature and torsion of* γ *are given by*

$$\kappa^2(s) = \frac{1}{r^2 - s^2\tan^2\theta} = \frac{1}{r^2 - s^2\cot^2\theta}, \tau^2(s) = \frac{1}{r^2\tan^2\theta - s^2}. \tag{23}$$

Proposition 4.7 *Let R be a ribbon surface whose center line is a spherical helix given with the Serret-Frenet curvatures* κ *and* τ. *Then, the mean curvature H of the ribbon R is given by* $\widetilde{H} = \lambda\kappa(s)$,*where* λ *is a constant function defined by* $\lambda = -\frac{1+\cot^2\theta}{2}$

Proof. Assume that the centerline of the strip is a spherical slope curve (helix) then the ratio of curvatures must be constant. Thus, with the help of equations (23) and (22) we get the result. □

Corollary 4.8 *Let R be a ribbon surface whose center line is a conical geodesic, i.e.* $\rho''(s) = 0$. *Then, the mean curvature* $\widetilde{H}$ *of the ribbon R is the multiple of the curvature* $\kappa(s)$,*with a smooth function*

$$\mu(s,u) = \frac{1+(as+b)^2}{2\left|1+au\right|},\ a,b \in \mathbb{R}. \tag{24}$$

The *geodesic curvature*, the *normal curvature* and the *geodesic torsion* of the surface associated the curve $\gamma(s)$ are defined as follows;

$$\kappa_g = \langle U \times T, T' \rangle, \kappa_n = \langle U, \gamma'' \rangle, \tau_g = \langle U \times U', T' \rangle \tag{25}$$

where U is the unit normal of the surface. From this consideration, a curve $\gamma(s)$ is an *asymptotic line* (resp. *geodesic line* or *principal line*) if and only if normal curvature κ_n (resp. geodesic curvature κ_g or geodesic torsion τ_g) vanishes identically [6].

Proposition 4.9 *The center line* γ *of the ribbon R is geodesically principal and the normal curvature of R corresponds to the curvature* κ *of* γ.

Proof. By the use of Serret-Frenet frame (1) with (25) the geodesic curvature, the normal curvature and the geodesic torsion of R become

$$\begin{aligned} \kappa_g &= \langle U \times T, T' \rangle = \kappa \langle B, N \rangle = 0, \\ \tau_g &= \langle U \times U', T' \rangle = \kappa \langle D, N \rangle = 0, \\ \kappa_n &= \langle U, \gamma'' \rangle = -\kappa \langle N, N \rangle = -\kappa, \end{aligned} \tag{26}$$

respectively. □

Definition 4.10 *A unit speed planar curve $\gamma : I \longrightarrow \mathbb{E}^2$ whose curvature is a given piecewise-continuous function $\kappa : I \longrightarrow \mathbb{R}^+$ is parametrized by*

$$\gamma(s) = \left(\int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b \right);\ \theta(s) = \int \kappa(s)ds + c \tag{27}$$

where, a, b, c are constant of integration [6].

If the center line of the ribbon is a regular curve $\gamma(s) = (x(s), y(s)\,, 0)$ then the the resultant ribbon R becomes a cylindrical ruled surface with the parametrization

$$\widetilde{\varphi}_{(\gamma,\widetilde{W})}(s,u) = \left(\int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b, u \right), s \in [0, L], u \in [-b, b]. \tag{28}$$

Thus, we have the following result;

Corollary 4.11 *Let R be a ribbon surface given by the parametrization (28). Then the mean curvature of R is a multiple of the curvature κ of the form $\widetilde{H} = -\frac{\kappa(s)}{2}$.*

5. Visualization

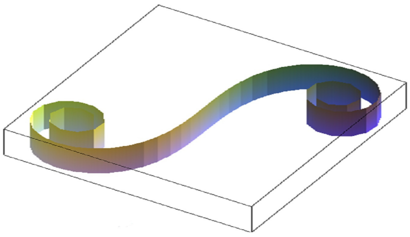
Geometric models of curves and surfaces have an important place in computer-aided geometric design. Therefore, in the present section, geometric visualization of some ribbon models is given with the help of maple.

Example 5.1 *In this example we construct three kind of ribbon using the plane curve given with the parametrization (21);*

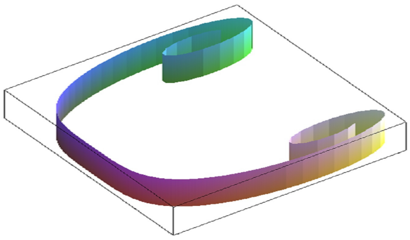
(a) For $\kappa(s) = as + b$ the center line is a Cornu spiral and the graph of resultant ribbon is cylindrical (Figure 1-(a)).

(b) For $\kappa(s) = as^2 + bs + c$ the center line is a generalized Cornu spiral and the graph of resultant ribbon is cylindrical (Figure 1-(b)).

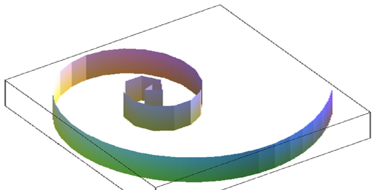
(c) For $\kappa(s) = \frac{a}{s+b}$ the center line is a logarithmic spiral and the graph of resultant ribbon is cylindrical (Figure 1-(c)).



(a) Cornu spiral



(b) Generalized cornu spiral



(c) Logarithmic spiral

Figure 1: Ribbon Surfaces in $\mathbb{E}^3$

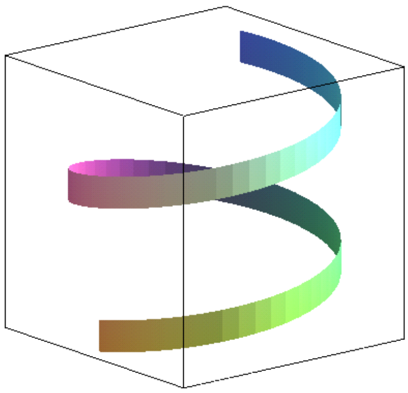
Example 5.2 *We take the center line curve γ as a right circular helix*

$$\gamma(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin(\frac{s}{c}), \frac{bs}{c}\right),\ a^2 + b^2 = c^2.$$

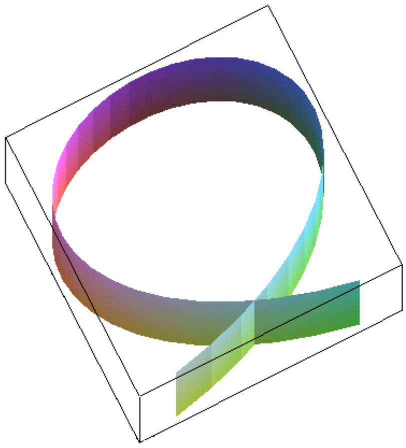
The Serret-Frenet curvatures of γ are constant functions $\kappa(s) = \frac{a}{c^2}, \tau(s) = \frac{b}{c^2}$ and $\cot\theta(s) = \frac{b}{a}$. A simple calculation shows that the ribbon R has constant mean curvature $H = -\frac{1}{2a}$. In Figure 2-(a) we pictured the ribbon taking the values $a = 3$, $b = 4$ and $c = 5$.

Example 5.3 *Consider the planar curve $\gamma(s) = (s\cos(s), s\sin s, 0)$ with curvature $\kappa(s) = -\frac{2+s^2}{(1+s^2)^{3/2}}$. The resultant surface becomes a braid ribbon which has self intersection (Figure 2-(b)). Further, the mean curvature of the ribbon becomes*

$$\widetilde{H}(s,u) = -\frac{2+s^2}{2\,(1+s^2)^{3/2}}.$$



(a) Cylindrical ribbon



(b) The braid ribbon

Figure 2: Ribbon Surfaces in $\mathbb{E}^3$

6. Conclusions

In conclusion, the paper presents a simple method for constructing developable surface patches bounded by space curves. These surfaces have many applications in surface modeling and parametric design. The most relevant feature of this construction is that the parametrization of the resultant ruled surface gives a ribbon in 3-dimensional Euclidean space. The method is founded on finding a special type of ruled surface taking the director curve as Darboux vector of the base curve. It has been shown that the mean curvature of the helical ribbon surfaces are related with the Serret-Frenet curvature of the center line of the ribbon. Nowadays, helical ribbons getting popular in nanoengineering. The detailed exploration of this analogy is the subject of future work. Especially, We would also like to extend our calculations to a parallel transport frame of ribbon configurations.

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An Application of Interior and Closure in General Topology: A Key Agreement Protocol

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Abstract. In this paper, a new key agreement scheme is constructed using the notions of the interior of a set, the closure of a set, open function, closed function, continuous function in topological spaces. An implementation of this scheme is presented between the two parties and it is shown that they generate the common secret key.

1. Introduction

Key agreement, a protocol that enables two or more parties to create a secret key together over an unprotected channel, does not require an active role of the trusted authority, unlike most other key distribution techniques. Key agreement schemes can be divided into two categories based on private keys and based on public keys. Consider a n-user network. In a secret key-based key agreement scheme, it is a requirement that each user stores n-1 secret keys. On the other hand, this requirement is reduced to only one pair of public and private keys in a key agreement scheme based public key. This indicates that public key-based key agreement is more useful. (see [1] for details).

The first work on the key agreement scheme was done by Merkle [2] in 1978. However, the article published by Diffie, Merkle's doctoral advisor, and Helmman [3] in 1976 is the first article published on this subject in the literature. This is because the study Merkle submitted in 1975 was in a lengthy evaluation process.

The Diffie-Hellman key agreement scheme uses the commutativity property provided by cyclic groups. In this scheme, the associativity property of group axioms is used in the generation of a common secret key, and the cyclicity of the group is used in making it difficult to find this secret key by an adversary.

In 2017, Partala [4] published a study based on the computation of homomorphic images, which included an algebraically generalized Diffie-Hellman key-agreement scheme. The security of this scheme lies in the difficulty of solving the homomorphic image problem. This problem is the problem of computing the image of a given group element under an indefinite homomorphism.

Çağman et al. [5] introduced a key agreement scheme based on a group action of special orthogonal group on the complex projective line whose elements are 2×2 matrices with real entries.

In this paper, a new key agreement scheme is constructed using the notions of interior and closure in topological spaces.

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2. Preliminaries

Let's give some information about the general topology from [6–8].

Definition 2.1. *Let (X, τ) be a topological space and $A \subseteq X$.*

1. A point $x \in A$ is called *an interior point of* A if there exists an open set G such that $x \in G \subseteq X$. The set of all interior point of A, denoted by $\mathrm{Int}(A)$, is called *the interior of* A.
2. A point $x \in X$ is called *an closure point of* A if every open set containing x contains at least one point of A. The set of all closure point of A, denoted by $\mathrm{Cl}(A)$, is called *the closure of* A.

Proposition 2.2. *Let (X, τ) be a topological space. For every pair of subsets A, B of X, the followings hold:*

1. $\mathrm{Int}(A \cap B) = \mathrm{Int}\, A \cap \mathrm{Int}\, B$,
2. $\mathrm{Cl}(A \cup B) = \mathrm{Cl}\, A \cup \mathrm{Cl}\, B$.

Definition 2.3. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, $f\colon X \to Y$ a function and let $x \in X$.*

1. f is called *a continuous function at the point* x if, for every τ_Y-open set V containing $f(x)$, $f^{-1}(V)$ is a τ_X-open set. f is called *a continuous function* if f is continous at every point of X.
2. f is called *an open function at the point* x if, for every τ_X-open set U containing x, $f(U)$ is a τ_Y-open set. f is called *an open function* if f is open at every point of X.
3. f is called *an closed function at the point* x if, for every τ_X-closed set U containing x, $f(U)$ is a τ_Y-closed set. f is called *a closed function* if f is closed at every point of X.

Proposition 2.4. *Let (X, τ) be a topological space, $f\colon X \to Y$ a function. The followings hold.*

1. If f is continuous, then $f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A))$ for every subset A of X.
2. If f is injective and continuous, then $\mathrm{Int}(f(A)) \subseteq f(\mathrm{Int}(A))$ for every subset A of X.
3. If f is open, then $f(\mathrm{Int}(A)) \subseteq \mathrm{Int}(f(A))$ for every subset A of X.
4. If f is closed, then $\mathrm{Cl}(f(A)) \subseteq f(\mathrm{Cl}(A))$ for every subset A of X.

3. Key Agreement Scheme

Let's assume that Alice and Bob must have a common secret key as shown in Figure 3 to communicate securely with each other.

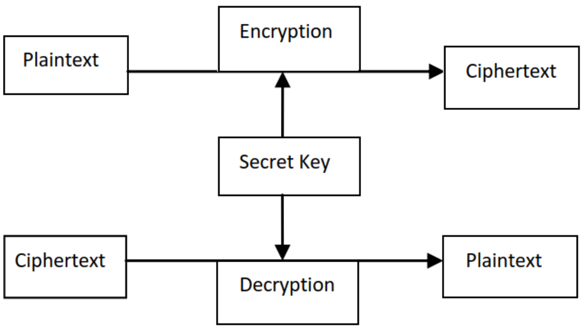

Figure 1: A communication diagram for Alice and Bob

In Table 1, they follow their steps to agree on a common secret key.

Step	Alice	Bob
1	They specify arbitrary positive integers n, m as public.	
2	Choose an arbitrary topological space (X, τ_X) publicly such that $\|X\| = n$.	Choose an arbitrary topological space (Y, τ_Y) publicly such that $\|Y\| = n$.
3	They choose an arbitrary pair of m-tuples $A = (A_1, A_2, \dots, A_m)$ and $B = (B_1, B_2, \dots, B_m)$ whose components are subsets of X as public.	
4	Set f and g secretly as m-tuples $(f_k)_{k \leq m}$ and $(g_k)_{k \leq m}$, respectiveley, where each f_k is an injective, open and continuous functions from X to Y, and each g_k is a closed and continuous functions from X to Y.	Let G and F be m-tuples whose the general terms are $G_k = \mathrm{Int}_{\tau_Y}(A_k \cap B_k)$ and $F_k = \mathrm{Cl}_{\tau_Y}(A_k \cup B_k)$, respectively.
5	Set the m-tuples $f[A], g[A], f[B], g[B]$ whose the general terms are $f[A]_k = f_k(A_k)$, $f[B]_k = f_k(B_k)$, $g[A]_k = g_k(A_k)$, $g[B]_k = g_k(B_k)$, respectively, and send them to Bob as public.	Send m-tuples G and F to Alice.
6	By using m-tuples G and F, generate the secret key as the pair $K = (K^{\mathrm{Int}}, K^{\mathrm{Cl}})$ where K^{Int} and K^{Cl} are the m-tuples whose the general terms are $K^{\mathrm{Int}}_k = f_k(G_k)$ and $K^{\mathrm{Cl}}_k = g_k(F_k)$, respectively.	By using m-tuples $f[A]$, $g[A]$, $f[B]$ and $g[B]$, generate the secret key as the pair $K = (K^{\mathrm{Int}}, K^{\mathrm{Cl}})$ where K^{Int} and K^{Cl} are the m-tuples whose the general terms are $K^{\mathrm{Int}}_k = \mathrm{Int}_{\tau_Y}(f[A]_k \cap f[B]_k)$ and $K^{\mathrm{Cl}}_k = \mathrm{Cl}_{\tau_Y}(g[A]_k \cup g[B]_k)$, respectively.

Table 1: Key agreement scheme

In the first step, the sizes of the topologies and the tuples to be created are fixed, say n and m, respectively. In the next step, Alice chooses an arbitrary topology on an X set with n elements as public, while Bob chooses an arbitrary topology on a Y set with the same number of elements as public. In the third step, They together arbitrarily choose a pair of m-tuples whose components are subsets of X.

In the next two steps, Alice secretly and arbitrarily chooses an m-tuple f whose components are injective, open and continuous functions from X to Y, and an m-tuple g whose components are closed and continuous from X to Y, and then send m-tuples $f[A], g[A], f[B], g[B]$ to Bob whose the general terms are

$$f[A]_k = f_k(A_k), f[B]_k = f_k(B_k), g[A]_k = g_k(A_k), g[B]_k = g_k(B_k)$$

while Bob set G and F as the m-tuples whose general terms are

$$G_k = \mathrm{Int}_{\tau_Y}(A_k \cap B_k) \text{ and } F_k = \mathrm{Cl}_{\tau_Y}(A_k \cup B_k)$$

, respectively, and sends them to Alice.

The last step shows how to generate the common secret key K by each of parties. Alice computes two components of K as m-tuples with the general terms $f_k(G_k)$ and $g_k(F_k)$, respectively while Bob computes two components of K as m-tuples with the general terms

$$\mathrm{Int}_{\tau_Y}(f[A]_k \cap f[B]_k) \text{ and } \mathrm{Cl}_{\tau_Y}(g[A]_k \cup g[B]_k)$$

, respectively.

Let's examine the steps in the key agreement scheme on an example.

Example 3.1. *Alice and Bob set $n = 4$ and $m = 3$. Then, Alice set $X = \{a, b, c, d\}$ while Bob $Y = \{1, 2, 3, 4\}$. Alice and Bob set*

$$\begin{aligned}\tau_X = \{&\emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\},\\ &\{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}\end{aligned}$$

and

$$\tau_Y = \{\emptyset, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, Y\},$$

respectively. Then, They set two 3-tuples $A = (A_1, A_2, A_3)$ *and* $B = (B_1, B_2, B_3)$ *where*

$$A_1 = \emptyset, \quad A_2 = \{c\}, \quad A_3 = \{a,d\},$$
$$B_1 = \{b,c\}, \quad B_2 = \{b,c\}, \quad B_3 = \{a,c,d\}.$$

Alice sets $f = (f_1, f_2, f_3)$*, each component of which is an injective, open and continuous function from* X *to* Y *where*

$$a \overset{f_1}{\mapsto} 1, b \overset{f_1}{\mapsto} 3, c \overset{f_1}{\mapsto} 4, d \overset{f_1}{\mapsto} 2,$$
$$a \overset{f_2}{\mapsto} 1, b \overset{f_2}{\mapsto} 3, c \overset{f_2}{\mapsto} 4, d \overset{f_2}{\mapsto} 2,$$
$$a \overset{f_3}{\mapsto} 1, b \overset{f_3}{\mapsto} 3, c \overset{f_3}{\mapsto} 4, d \overset{f_3}{\mapsto} 2,$$

and $g = (g_1, g_2, g_3)$*, each component of which is a closed and continuous function from* X *to* Y *where*

$$a \overset{g_1}{\mapsto} 1, b \overset{g_1}{\mapsto} 1, c \overset{g_1}{\mapsto} 1, d \overset{g_1}{\mapsto} 1,$$
$$a \overset{g_2}{\mapsto} 2, b \overset{g_2}{\mapsto} 2, c \overset{g_2}{\mapsto} 2, d \overset{g_2}{\mapsto} 3,$$
$$a \overset{g_3}{\mapsto} 3, b \overset{g_3}{\mapsto} 1, c \overset{g_3}{\mapsto} 1, d \overset{g_3}{\mapsto} 3.$$

Then, Alice computes m-tuples $f[A], g[A], f[B], g[B]$ *whose the general terms are* $f[A]_k = f_k(A_k)$*,* $f[B]_k = f_k(B_k)$*,* $g[A]_k = g_k(A_k)$*,* $g[B]_k = g_k(B_k)$*, respectively, as*

$$f[A] = (\emptyset, \{4\}, \{1,2\}),$$
$$g[A] = (\emptyset, \{2\}, \{3\}),$$
$$f[B] = (\{3,4\}, \{3,4\}, \{1,2,4\}),$$
$$g[B] = (\{1\}, \{2\}, \{1,3\}),$$

and send them to Bob while Bob computes m-tuples G *and* F *whose the general terms are* $G_k = \mathrm{Int}_{\tau_Y}(A_k \cap B_k)$ *and* $F_k = \mathrm{Cl}_{\tau_Y}(A_k \cup B_k)$*, respectively, as*

$$G = (\emptyset, \{c\}, \{a,d\}),$$
$$F = (\{b,c\}, \{b,c\}, X),$$

and send them to Alice.

Using m-tuples G *and* F*, Alice computes the secret common key* $K = (K^{\mathrm{Int}}, K^{\mathrm{Cl}})$ *whose general terms of components are* $K_k^{\mathrm{Int}} = f_k(G_k)$ *and* $K_k^{\mathrm{Cl}} = g_k(F_k)$*, respectively, as*

$$K = ((\emptyset, \{4\}, \{1,2\}), (\{1\}, \{2\}, \{1,3\})).$$

On the other hand, by using m-tuples $f[A]$*,* $g[A]$*,* $f[B]$ *and* $g[B]$*, Bob computes the secret common key* $K = (K^{\mathrm{Int}}, K^{\mathrm{Cl}})$ *whose the general terms of components are* $K_k^{\mathrm{Int}} = \mathrm{Int}_{\tau_Y}(f[A]_k \cap f[B]_k)$ *and* $K_k^{\mathrm{Cl}} = \mathrm{Cl}_{\tau_Y}(g[A]_k \cup g[B]_k)$*, respectively, as*

$$K = ((\emptyset, \{4\}, \{1,2\}), (\{1\}, \{2\}, \{1,3\})).$$

Thus Alice and Bob have produced the same public key K *to use in communication.*

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