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On $\mathscr{I}_\theta$-convergence in Neutrosophic Normed Spaces

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Abstract

The purpose of this article is to investigate lacunary ideal convergence of sequences in neutrosophic normed space (NNS). Also, an original notion, named lacunary convergence of sequence in NNS, is defined. Also, lacunary $\mathscr{I}$-limit points and lacunary $\mathscr{I}$-cluster points of sequences in NNS have been examined. Furthermore, lacunary Cauchy and lacunary $\mathscr{I}$-Cauchy sequences in NNS are introduced and some properties of these notions are studied.

1. Introduction and background

Theory of fuzzy sets (FSs) was firstly given by Zadeh [1]. The publication of the paper affected deeply all the scientific fields. This notion is significant for real-life conditions, but has not adequate solution to some problems and so these problems lead to original quests.
Intuitionistic fuzzy sets (IFSs) for such cases were initiated by Atanassov [2]. Atanassov et al. [3] used this concept in decision-making problems. Kramosil and Michalek [4] investigated fuzzy metric space (FMS) utilizing the concepts fuzzy and probabilistic metric space. The FMS as a distance between two points to be a non-negative fuzzy number was examined by Kaleva and Seikkala [5]. George and Veeramani [6] gave some qualifications of FMS. Some basic features of FMS were given and significant theorems were proved in [7]. Moreover, FMS has used by practical researches as for example decision-making, fixed point theory, medical imaging. Park [8] generalized FMSs and defined IF metric space (IFMS). Park utilized George and Veeramani's [6] opinion of using t-norm and t-conorm to the FMS meantime describing IFMS and investigating its fundamental properties. Saadati and Park [9] initially examined properties of intuitionistic fuzzy normed space (IFNS).
The statistical convergence initially introduced by [10]. Statistical convergence in IFNS was given by Karakuş et al [11]. Notable results on this topic can be found in [12]-[17].
By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be indicated by $I_r = (k_{r-1}, k_r]$. Using lacunary sequence, Fridy and Orhan [18] examined the concept of lacunary statistical convergence. The publication of the paper affected deeply all the scientific fields. Some works in lacunary statistical convergence can be found in [19]-[23].
The concept neutrosophy implies impartial knowledge of thought and then neutral describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The neutrosophic set (NS) was investigated by F. Smarandache [24] who defined the degree of indeterminacy (i) as indepedent component. In [25], neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth (T), falsity (F), and indeterminacy (I). A Neutrosophic set (NS) is determined as a set where every component of the universe has a degree of T, F and I.
In IFSs the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. FSs can be thought as a remarkable case of an IFS where the 'degree of non-belongingness' of an element is absolutely equal to '1- degree of belongingness'. Uncertainty is based on the belongingness degree in IFSs, whereas the uncertainty in NS is considered independently from T and F values. Since no any limitations among the degree of T, F, I, NSs are actually more general than IFS.
Neutrosophic soft linear spaces (NSLSs) were considered by Bera and Mahapatra [26]. Subsequently, in [27], the concept neutrosophic soft normed linear (NSNLS) was defined and the features of (NSNLS) were examined. Significant results on this topic can be found in [28]-[32].

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Kirişçi and Şimşek [33] defined new concept known as neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. Some notable features of NMS have been examined.
Neutrosophic normed space (NNS) and statistical convergence in NNS has been investigated by Kirişci and Şimşek [34]. Neutrosophic set and neutrosophic logic has used by applied sciences and theoretical science such as decision making, robotics, summability theory. Some noteworthy results on this topic can be examined in [35]-[39].
In [39], lacunary statistical convergence of sequences in NNS was examined. Also, lacunary statistically Cauchy sequence in NNS was given and lacunary statistically completeness in connection with a neutrosophic normed space was presented.
Firstly, we recall some definitions used throughout the paper.
For $K \subset \mathbb{N}$ and $j \in \mathbb{N}$, if

$$\delta_j(K) = \frac{|K \cap \{1,2,...,j\}|}{j},$$

then $\delta_j(K)$ is named jth partial density of K. If

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|, \left(\text{i.e., } \delta(K) = \lim_{j\to\infty} \delta_j(K)\right)$$

exists, it is named the natural density of K. $\Psi = \{K \subset \mathbb{N} : \delta(K) = 0\}$ is denoted the zero density set.
A sequence (x_n) is said to be statistically convergent to ξ if for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |x_k - \xi| \ge \varepsilon\}) = 0,$$

i.e., $\{n \in \mathbb{N} : |x_k - \xi| \ge \varepsilon\} \in \Psi$. We demonstrate $st - \lim x_n = \xi$ or $x_n \xrightarrow{st} \xi$, $(n \to \infty)$.
In the wake of the study of ideal convergence defined by Kostyrko et al. [40], there has been comprehensive research to discover applications and summability studies of the classical theories. Ideal convergence became a notable topic in summability theory after the researches of [41]-[52].
Let $\emptyset \ne S$ be a set, and then a non empty class $\mathscr{I} \subseteq P(S)$ is said to be an *ideal* on S iff (i) $\emptyset \in \mathscr{I}$, (ii) $\mathscr{I}$ is additive under union, (iii) for each $A \in \mathscr{I}$ and each $B \subseteq A$ we find $B \in \mathscr{I}$. An ideal $\mathscr{I}$ is called *non-trivial* if $\mathscr{I} \ne \emptyset$ and $S \notin \mathscr{I}$. A non-empty family of sets $\mathscr{F}$ is called *filter* on S iff (i) $\emptyset \notin \mathscr{F}$, (ii) for each $A, B \in \mathscr{F}$ we get $A \cap B \in \mathscr{F}$, (iii) for every $A \in \mathscr{F}$ and each $B \supseteq A$, we obtain $B \in \mathscr{F}$. Relationship between ideal and filter is given as follows:

$$\mathscr{F}(\mathscr{I}) = \{K \subset S : K^c \in \mathscr{I}\},$$

where $K^c = S - K$.
A non-trivial ideal $\mathscr{I}$ is (i) an *admissible ideal* on S iff it contains all singletons.
A sequence (x_n) is named to be ideal convergent to ξ if for every $\varepsilon > 0$, i.e.

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \ge \varepsilon\} \in \mathscr{I}.$$

We take $\mathscr{I}$ as admissible ideal throughout the paper.
Triangular norms (t-norms) (TN) were given by Menger [53]. TNs are used to generalize with the probability distribution of triangle inequality in metric space terms. Triangular conorms (t-conorms) (TC) known as dual operations of TNs. TNs and TCs are important for fuzzy operations (intersections and unions).

Definition 1.1. *([53]) Let $* : [0,1] \times [0,1] \to [0,1]$ be an operation. When $\circ$ satisfies following situations, it is called continuous TN. Take $p,q,r,s \in [0,1]$,*

(a) $p * 1 = p$,
(b) *If* $p \le r$ *and* $q \le s$, *then* $p * q \le r * s$,
(c) $*$ *is continuous,*
(d) $*$ *associative and commutative.*

Definition 1.2. *([53]) Let $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ be an operation. When $\diamondsuit$ satisfies following situations, it is said to be continuous TC.*

(a) $p \diamondsuit 0 = p$,
(b) *If* $p \le r$ *and* $q \le s$, *then* $p \diamondsuit q \le r \diamondsuit s$,
(c) $\diamondsuit$ *is continuous,*
(d) $\diamondsuit$ *associative and commutative.*

Definition 1.3. *([34]) Let F be a vector space, $\mathscr{N} = \{\langle u, \mathscr{G}(u), \mathscr{B}(u), \mathscr{Y}(u)\rangle : u \in F\}$ be a normed space (NS) such that $\mathscr{N} : F \times \mathbb{R}^+ \to [0,1]$. While following conditions hold, $V = (F, \mathscr{N}, *, \diamondsuit)$ is called to be NNS. For each $u, v \in F$ and $\lambda, \mu > 0$ and for all $\sigma \ne 0$,*

(a) $0 \le \mathscr{G}(u,\lambda) \le 1$, $0 \le \mathscr{B}(u,\lambda) \le 1$, $0 \le \mathscr{Y}(u,\lambda) \le 1$ $\forall \lambda \in \mathbb{R}^+$,
(b) $\mathscr{G}(u,\lambda) + \mathscr{B}(u,\lambda) + \mathscr{Y}(u,\lambda) \le 3$ *(for* $\lambda \in \mathbb{R}^+$*),*
(c) $\mathscr{G}(u,\lambda) = 1$ *(for* $\lambda > 0$*) iff* $u = 0$,
(d) $\mathscr{G}(\sigma u,\lambda) = \mathscr{G}\left(u, \frac{\lambda}{|\sigma|}\right)$,
(e) $\mathscr{G}(u,\mu) * \mathscr{G}(v,\lambda) \le \mathscr{G}(u+v, \mu+\lambda)$,
(f) $\mathscr{G}(u,.)$ *is non-decreasing continuous function,*
(g) $\lim_{\lambda\to\infty} \mathscr{G}(u,\lambda) = 1$,
(h) $\mathscr{B}(u,\lambda) = 0$ *(for* $\lambda > 0$*) iff* $u = 0$,
(i) $\mathscr{B}(\sigma u,\lambda) = \mathscr{B}\left(u, \frac{\lambda}{|\sigma|}\right)$,

(j) $\mathscr{B}(u,\mu)\lozenge\mathscr{B}(v,\lambda)\geq\mathscr{B}(u+v,\mu+\lambda)$,
(k) $\mathscr{B}(u,.)$ *is non-decreasing continuous function,*
(l) $\lim_{\lambda\to\infty}\mathscr{B}(u,\lambda)=0$,
(m) $\mathscr{Y}(u,\lambda)=0$ *(for* $\lambda>0$*) iff* $u=0$,
(n) $\mathscr{Y}(\sigma u,\lambda)=\mathscr{Y}\left(u,\frac{\lambda}{|\sigma|}\right)$,
(o) $\mathscr{Y}(u,\mu)\lozenge\mathscr{Y}(v,\lambda)\geq\mathscr{Y}(u+v,\mu+\lambda)$,
(p) $\mathscr{Y}(u,.)$ *is non-decreasing continuous function,*
(r) $\lim_{\lambda\to\infty}\mathscr{Y}(u,\lambda)=0$,
(s) *If* $\lambda\leq 0$*, then* $\mathscr{G}(u,\lambda)=0,\mathscr{B}(u,\lambda)=1$ *and* $\mathscr{Y}(u,\lambda)=1$.
Then $\mathscr{N}=(\mathscr{G},\mathscr{B},\mathscr{Y})$ *is called Neutrosophic norm (NN).*

Definition 1.4. *([34]) Let V be an NNS, the sequence (x_k) in V, $\varepsilon\in(0,1)$ and $\lambda>0$. Then, the sequence (x_k) is converges to ξ iff there is $N\in\mathbb{N}$ such that $\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon$, $\mathscr{B}(x_k-\xi,\lambda)<\varepsilon$, $\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$. That is, $\lim_{n\to\infty}\mathscr{G}(x_k-\xi,\lambda)=1$, $\lim_{n\to\infty}\mathscr{B}(x_k-\xi,\lambda)=0$ and $\lim_{n\to\infty}\mathscr{Y}(x_k-\xi,\lambda)=0$ as $\lambda>0$. In that case, the sequence (x_k) is named a convergent sequence in V. The convergent in NNS is indicated by $\mathscr{N}-\lim x_k=\xi$.*

Definition 1.5. *([34]) A sequence (x_k) in V, $\varepsilon\in(0,1)$ and $\lambda>0$. Then, the sequence (x_k) is Cauchy in NNS V if there is a $N\in N$ such that $\mathscr{G}(x_k-x_m,\lambda)>1-\varepsilon$, $\mathscr{B}(x_k-x_m,\lambda)<\varepsilon$, $\mathscr{Y}(x_k-x_m,\lambda)<\varepsilon$ for $k,m\geq N$.*

Definition 1.6. *([34]) A sequence (x_m) is said to be statistically convergent to $\xi\in F$ with regards to NN (SC-NN), if, for each $\lambda>0$ and $\varepsilon>0$ the set*

$$P_\varepsilon:=\{m\leq n:\mathscr{G}(x_m-\xi,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(x_m-\xi,\lambda)\geq\varepsilon,\ \mathscr{Y}(x_m-\xi,\lambda)\geq\varepsilon\}$$

or equivalently

$$P_\varepsilon:=\{m\leq n:\mathscr{G}(x_m-\xi,\lambda)> 1-\varepsilon \text{ or } \mathscr{B}(x_m-\xi,\lambda)<\varepsilon,\ \mathscr{Y}(x_m-\xi,\lambda)<\varepsilon\}.$$

has ND zero. That is $d(P_\varepsilon)=0$ or

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m\leq n:\mathscr{G}(x_m-\xi,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(x_m-\xi,\lambda)\geq\varepsilon,\ \mathscr{Y}(x_m-\xi,\lambda)\geq\varepsilon\}\right|=0.$$

It is denoted by $S_{\mathscr{N}}$-$\lim x_m=\xi$ or $x_k\to\xi\,(S_{\mathscr{N}})$. The set of SC-NN will be denoted by $S_{\mathscr{N}}$.

Definition 1.7. *([34]) The sequence (x_k) is called statistical Cauchy with regards to NN $\mathscr{N}$ (SCa-NN) in NNS V, if there exists $N=N(\varepsilon)$, for every $\varepsilon>0$ and $\lambda>0$ such that*

$$C_\varepsilon:=\{m\leq n:\mathscr{G}(x_m-x_N,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(x_m-x_N,\lambda)\geq\varepsilon,\ \mathscr{Y}(x_m-x_N,\lambda)\geq\varepsilon\}$$

has ND zero. That is, $d(C_\varepsilon)=0$.

Definition 1.8. *([34]) Let V be an NNS. For $\lambda>0$, $w\in F$ and $\varepsilon\in(0,1)$,*

$$B(w,\varepsilon,\lambda)=\{u\in F:\mathscr{G}(w-u,\lambda)>1-\varepsilon,\ \mathscr{B}(w-u,\lambda)<\varepsilon,\ \mathscr{Y}(w-u,\lambda)<\varepsilon\}$$

is called open ball with center w, radius ε.

2. Main results

Definition 2.1. *Take an NNS V. For a lacunary sequence θ, a sequence $x=(x_k)$ is named to be lacunary convergent to $\xi\in F$ with regards to NN (LC-NN), if for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that*

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$$

for all $r\geq r_0$. We indicate $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi$.

Theorem 2.2. *Let V be an NNS. If x is lacunary convergent with regards to NN, then $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x$ is unique.*

Proof. Presume that $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi_1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi_2$. Given $\varepsilon>0$, select $\rho\in(0,1)$ such that $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\lozenge\rho<\varepsilon$. For each $\lambda>0$, there is $r_1\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi_1,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi_1,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi_1,\lambda)<\varepsilon$$

for all $r\geq r_1$. Also, there is $r_2\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi_2,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi_2,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi_2,\lambda)<\varepsilon$$

for all $r \geq r_2$. Think $r_0 = \max\{r_1, r_2\}$. Then, for $r \geq r_0$, we take a $m \in \mathbb{N}$ such that

$$\mathscr{G}\left(x_m - \xi_1, \tfrac{\lambda}{2}\right) > \tfrac{1}{h_r} \sum_{k \in I_r} \mathscr{G}\left(x_k - \xi_1, \tfrac{\lambda}{2}\right) > 1 - \rho,$$

$$\mathscr{G}\left(x_m - \xi_2, \tfrac{\lambda}{2}\right) > \tfrac{1}{h_r} \sum_{k \in I_r} \mathscr{G}\left(x_k - \xi_2, \tfrac{\lambda}{2}\right) > 1 - \rho.$$

Then, we obtain

$$\begin{aligned} \mathscr{G}(\xi_1 - \xi_2, \lambda) &\geq \mathscr{G}\left(x_m - \xi_1, \tfrac{\lambda}{2}\right) * \mathscr{G}\left(x_m - \xi_2, \tfrac{\lambda}{2}\right) \\ &> (1-\rho) * (1-\rho) > 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is abritrary, we get $\mathscr{G}(\xi_1 - \xi_2, \lambda) = 1$ for all $\lambda > 0$, which gives that $\xi_1 = \xi_2$. □

Definition 2.3. *Let $\theta = (k_r)$ be a lacunary sequence, $\mathscr{I} \subset 2^{\mathbb{N}}$ and let V be an NNS. A sequence $x = (x_k)$ is said to be lacunary $\mathscr{I}$-convergent to $\xi \in F$ with regards to NN ($\mathscr{I}_\theta C$-NN), if, for every $\varepsilon \in (0,1)$ and $\lambda > 0$, the set*

$$\left\{ \begin{array}{c} r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) \leq 1 - \varepsilon \\ \text{or } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) \geq \varepsilon,\ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) \geq \varepsilon \end{array} \right\} \in \mathscr{I}.$$

ξ is called the lacunary $\mathscr{I}$-limit of the sequence of (x_k), and we demonstrate $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x = \xi$.

Now, we prepare an example to denote the sequence $\mathscr{I}_\theta$-convergent in an NNS.

Example 2.4. *Let $(F, \|.\|)$ be a NNS, $\mathscr{I}$ be a non-trivial admissible ideal. For all $u, v \in [0,1]$, take the TN $u * v = uv$ and the TC $u \Diamond v = \min\{u+v, 1\}$. For all $x \in F$ and every $\lambda > 0$, we contemplate $\mathscr{G}(x,\lambda) = \frac{\lambda}{\lambda + \|x\|}$, $\mathscr{B}(x,\lambda) = \frac{\|x\|}{\lambda + \|x\|}$ and $\mathscr{Y}(x,\lambda) = \frac{\|x\|}{\lambda}$. Then, V is an NNS. We define a sequence (x_k) by*

$$x_k = \begin{cases} 1, & \text{if } k = t^2 \ (t \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any $\lambda > 0$ and for all $\varepsilon \in (0,1)$, the following set

$$\begin{aligned} A(\varepsilon,\lambda) &= \left\{ k \in \mathbb{N} : \tfrac{\lambda}{\lambda + \|x_k\|} \leq 1 - \varepsilon \text{ or } \tfrac{\|x_k\|}{\lambda + \|x_k\|} \geq \varepsilon,\ \tfrac{\|x_k\|}{\lambda} \geq \varepsilon \right\} \\ &= \left\{ k \in \mathbb{N} : \|x_k\| \geq \tfrac{\lambda\varepsilon}{1-\varepsilon},\ \text{or } \|x_k\| \geq \lambda\varepsilon \right\} \\ &= \{ k \in \mathbb{N} : \|x_k\| = 1 \} = \{ k \in \mathbb{N} : k = t^2 \ (t \in \mathbb{N}) \} \end{aligned}$$

i.e.,

$$A(\varepsilon,\lambda) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k,\lambda) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k,\lambda) \geq \varepsilon,\ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k,\lambda) \geq \varepsilon \right\}$$

will be a finite set. So, $\delta(A(\varepsilon,\lambda)) = 0$, and as a result $A(\varepsilon,\lambda) \in \mathscr{I}$. We show that $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x = 0$.

Lemma 2.5. *For every $\varepsilon > 0$ and $\lambda > 0$, the following situations are equivalent.*

(a) $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x = \xi$,

(b) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) \leq 1 - \varepsilon \right\} \in \mathscr{I}$ *and*

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) \geq \varepsilon \right\} \in \mathscr{I}$,

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) \geq \varepsilon \right\} \in \mathscr{I}$,

(c) $\left\{ \begin{array}{c} r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1 - \varepsilon \\ \text{and } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon \\ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \end{array} \right\} \in \mathscr{F}(\mathscr{I})$,

(d) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1 - \varepsilon \right\} \in \mathscr{F}(\mathscr{I})$ *and*

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon \right\} \in \mathscr{F}(\mathscr{I})$,

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \right\} \in \mathscr{F}(\mathscr{I})$ *and*

(e) $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim \mathscr{G}(x_k - \xi, \lambda) = 1$ *and* $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim \mathscr{B}(x_k - \xi, \lambda) = 0$, $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim \mathscr{Y}(x_k - \xi, \lambda) = 0$.

Theorem 2.6. *If a sequence $x=(x_k)$ is lacunary $\mathscr{I}$-convergent with regards to the NN, then $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x$ is unique.*

Proof. Presume that $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_1$ and $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_2$. Select $\varepsilon\in(0,1)$. Then, for a given $\rho\in(0,1)$, $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\Diamond\rho<\varepsilon$. For any $\lambda>0$, let's denote the following sets:

$$\begin{aligned}
K_{\mathscr{G}1}(\rho,\lambda)&=\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_1,\tfrac{\lambda}{2}\right)\le 1-\rho\right\},\\
K_{\mathscr{G}2}(\rho,\lambda)&=\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_2,\tfrac{\lambda}{2}\right)\le 1-\rho\right\},\\
K_{\mathscr{B}1}(\rho,\lambda)&=\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-\xi_1,\tfrac{\lambda}{2}\right)\ge \rho\right\},\\
K_{\mathscr{B}2}(\rho,\lambda)&=\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-\xi_2,\tfrac{\lambda}{2}\right)\ge \rho\right\},\\
K_{\mathscr{Y}1}(\rho,\lambda)&=\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-\xi_1,\tfrac{\lambda}{2}\right)\ge \rho\right\},\\
K_{\mathscr{Y}2}(\rho,\lambda)&=\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-\xi_2,\tfrac{\lambda}{2}\right)\ge \rho\right\}.
\end{aligned}$$

Since $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_1$, using Lemma 2.5, we obtain $K_{\mathscr{G}1}(\rho,\lambda)$, $K_{\mathscr{B}1}(\rho,\lambda)$, $K_{\mathscr{Y}1}(\rho,\lambda)\in\mathscr{I}$. Utilizing $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_2$, we get $K_{\mathscr{G}2}(\rho,\lambda)$, $K_{\mathscr{B}2}(\rho,\lambda)$, $K_{\mathscr{Y}2}(\rho,\lambda)\in\mathscr{I}$.
Let

$$\begin{aligned}
K_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)\quad&:=(K_{\mathscr{G}1}(\rho,\lambda)\cup K_{\mathscr{G}2}(\rho,\lambda))\cap(K_{\mathscr{B}1}(\rho,\lambda)\cup K_{\mathscr{B}2}(\rho,\lambda))\\
&\cap(K_{\mathscr{Y}1}(\rho,\lambda)\cup K_{\mathscr{Y}2}(\rho,\lambda)).
\end{aligned}$$

Then, $K_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)\in\mathscr{I}$, which implies that $\emptyset\ne K^c_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)\in\mathscr{F}(\mathscr{I})$. If $r\in K^c_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)$, then we have three possible cases. That is, $r\in\left(K^c_{\mathscr{G}1}(\rho,\lambda)\cap K^c_{\mathscr{G}2}(\rho,\lambda)\right)$, $r\in\left(K^c_{\mathscr{B}1}(\rho,\lambda)\cap K^c_{\mathscr{B}2}(\rho,\lambda)\right)$ or $r\in\left(K^c_{\mathscr{Y}1}(\rho,\lambda)\cap K^c_{\mathscr{Y}2}(\rho,\lambda)\right)$. First, think that $r\in\left(K^c_{\mathscr{G}1}(\rho,\lambda)\cap K^c_{\mathscr{G}2}(\rho,\lambda)\right)$. Then, we obtain

$$\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_1,\tfrac{\lambda}{2}\right)>1-\rho \text{ and } \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_2,\tfrac{\lambda}{2}\right)>1-\rho.$$

Now, obviously, we get a $m\in\mathbb{N}$ such that

$$\begin{aligned}
\mathscr{G}\left(x_m-\xi_1,\tfrac{\lambda}{2}\right)&>\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_1,\tfrac{\lambda}{2}\right)>1-\rho,\\
\mathscr{G}\left(x_m-\xi_2,\tfrac{\lambda}{2}\right)&>\tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_2,\tfrac{\lambda}{2}\right)>1-\rho
\end{aligned}$$

(e.g., consider $\max\left\{\mathscr{G}\left(x_k-\xi_1,\tfrac{\lambda}{2}\right),\mathscr{G}\left(x_k-\xi_2,\tfrac{\lambda}{2}\right):k\in I_r\right\}$ and select that k as m for which the maximum occurs).
Then, we get

$$\begin{aligned}
\mathscr{G}(\xi_1-\xi_2,\lambda)\quad&\ge\mathscr{G}\left(x_m-\xi_1,\tfrac{\lambda}{2}\right)*\mathscr{G}\left(x_m-\xi_2,\tfrac{\lambda}{2}\right)\\
&>(1-\rho)*(1-\rho)>1-\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get $\mathscr{G}(\xi_1-\xi_2,\lambda)=1$ for all $\lambda>0$, which yields that $\xi_1=\xi_2$. On the other hand, if we take $r\in\left(K^c_{\mathscr{B}1}(\rho,\lambda)\cup K^c_{\mathscr{B}2}(\rho,\lambda)\right)$, then we can write

$$\mathscr{B}(\xi_1-\xi_2,\lambda)\le\mathscr{B}\left(x_m-\xi_1,\frac{\lambda}{2}\right)\Diamond\mathscr{B}\left(x_m-\xi_2,\frac{\lambda}{2}\right)\le\rho\Diamond\rho<\varepsilon.$$

Therefore, we can see that $\mathscr{B}(\xi_1-\xi_2,\lambda)<\varepsilon$. For all $\lambda>0$, we obtain $\mathscr{B}(\xi_1-\xi_2,\lambda)=0$, which implies that $\xi_1=\xi_2$. Again, for the situation $r\in\left(K^c_{\mathscr{Y}1}(\rho,\lambda)\cap K^c_{\mathscr{Y}2}(\rho,\lambda)\right)$, then, utilizing a same method, it can be proved that $\mathscr{Y}(\xi_1-\xi_2,\lambda)<\varepsilon$ for all $\lambda>0$ and arbitrary $\varepsilon>0$, and thus $\xi_1=\xi_2$. Hence, in all cases, we conclude that the $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-limit is unique. □

Theorem 2.7. *If $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi$, then $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi$.*

Proof. Let $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi$. Then, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$$

for all $r\ge r_0$. Therefore, we obtain

$$\begin{aligned}
T&=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)\le 1-\varepsilon\\ \text{or } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)\ge\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)\ge\varepsilon\end{array}\right\}\\
&\subseteq\{1,2,...,k_0-1\}.
\end{aligned}$$

If we accept $\mathscr{I}$ as admissible ideal, we get $T\in\mathscr{I}$. Hence, $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi$. □

Theorem 2.8. *If $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x=\xi$, then there is a subsequence (x_{p_k}) of x such that $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x_{p_k}=\xi$.*

Proof. Take $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x=\xi$. Then, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$$

for all $r\geq r_0$. Obviously, for each $r\geq r_0$, we choose $p_k\in I_r$ such that

$$\begin{aligned}\mathscr{G}\left(x_{p_k}-\xi,\lambda\right) &> \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon,\\ \mathscr{B}\left(x_{p_k}-\xi,\lambda\right) &< \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\\ \mathscr{Y}\left(x_{p_k}-\xi,\lambda\right) &< \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon.\end{aligned}$$

It follows that $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x_{p_k}=\xi$. $\square$

Definition 2.9. *Take an NNS V. A sequence $x=(x_k)$ is named to be lacunary Cauchy with regards to the NN $\mathscr{N}$ (LCa-NN) if, for every $\varepsilon\in(0,1)$ and $\lambda>0$, there are $r_0,p\in\mathbb{N}$ satisfying*

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-x_p,\lambda\right)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-x_p,\lambda\right)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-x_p,\lambda\right)<\varepsilon$$

for all $r\geq r_0$.

Definition 2.10. *Let V be an NNS. A sequence $x=(x_k)$ is called to be lacunary $\mathscr{I}$-Cauchy ($\mathscr{I}_\theta$-Cauchy) with regards to the NN $\mathscr{N}$ ($\mathscr{I}_\theta$Ca-NN) if, for every $\varepsilon\in(0,1)$ and $\lambda>0$, there is $p\in\mathbb{N}$ satisfying*

$$\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}\left(x_k-x_p,\lambda\right)>1-\varepsilon\\ \text{and } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}\left(x_k-x_p,\lambda\right)<\varepsilon,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}\left(x_k-x_p,\lambda\right)<\varepsilon\end{array}\right\}\in\mathscr{F}(\mathscr{I}).$$

Definition 2.11. *Take an NNS V. A sequence $x=(x_k)$ is named to be $\mathscr{I}_\theta^*$-Cauchy with regards to the NN $\mathscr{N}$ if there is a set $M=\{p_1<p_2<\ldots<p_k<..\}$ of $\mathbb{N}$ such that the set $M'=\{r\in\mathbb{N}:p_k\in I_r\}\in\mathscr{F}(\mathscr{I})$ and the subsequence $\left(x_{p_k}\right)$ is a lacunary Cauchy sequence with regards to the NN $\mathscr{N}$.*

The following theorems are similar of previous theorems, so the proof follows easily.

Theorem 2.12. *If a sequence $x=(x_k)$ in NNS is lacunary Cauchy with regards to NN $\mathscr{N}$, then it is $\mathscr{I}_\theta$-Cauchy with regards to the same.*

Theorem 2.13. *If a sequence $x=(x_k)$ in NNS is lacunary Cauchy with regards to NN $\mathscr{N}$, then there is a subsequence of x which is ordinary Cauchy with regards to the same.*

Theorem 2.14. *If a sequence $x=(x_k)$ in NNS is $\mathscr{I}_\theta^*$-Cauchy with regards to NN $\mathscr{N}$, then it is $\mathscr{I}_\theta$-Cauchy as well.*

Theorem 2.15. *If a sequence $x=(x_k)$ in NNS is $\mathscr{I}_\theta$-convergent with regards to NN $\mathscr{N}$, then it is $\mathscr{I}_\theta$-Cauchy with regards to NN $\mathscr{N}$.*

Proof. Let $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi$. Select $\varepsilon>0$. Then, for a given $\rho\in(0,1)$, $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\Diamond\rho<\varepsilon$. Then, for $\lambda>0$, we get,

$$K(\rho,\lambda)=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)\leq 1-\rho\\ \text{or } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)\geq\rho,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)\geq\rho\end{array}\right\}\in\mathscr{I} \tag{2.1}$$

which gives that

$$\emptyset\neq K^c(\rho,\lambda)=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\rho\\ \text{and } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\rho,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\rho\end{array}\right\}\in\mathscr{F}(\mathscr{I}).$$

Let $m\in K^c(\rho,\lambda)$. But then, for every $\lambda>0$ we have, $\mathscr{G}(x_m-\xi,\lambda)>1-\rho$ and $\mathscr{B}(x_m-\xi,\lambda)<\rho$, $\mathscr{Y}(x_m-\xi,\lambda)<\rho$. If we take

$$B(\rho,\lambda)=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(x_k-x_m,\lambda)\leq 1-\varepsilon\\ \text{or} \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(x_k-x_m,\lambda)\geq\varepsilon,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(x_k-x_m,\lambda)\geq\varepsilon\end{array}\right\},$$

then to demonstrate the result it is sufficient to prove $B(\rho,\lambda)$ is included in $K(\rho,\lambda)$. Let $k\in B(\rho,\lambda)$, then we get $\mathscr{G}\left(x_k-x_m,\frac{\lambda}{2}\right)\leq 1-\varepsilon$ or $\mathscr{B}\left(x_k-x_m,\frac{\lambda}{2}\right)\geq\varepsilon$, $\mathscr{Y}\left(x_k-x_m,\frac{\lambda}{2}\right)\geq\varepsilon$, for $\lambda>0$. We have three possible cases.

Case (i) We first think that $\mathscr{G}(x_k - x_m, \lambda) \leq 1-\varepsilon$. Then, we have $\mathscr{G}\left(x_k - \xi, \frac{\lambda}{2}\right) \leq 1-\rho$ and therefore, $k \in K(\rho,\lambda)$. As otherwise i.e., if $\mathscr{G}\left(x_k - \xi, \frac{\lambda}{2}\right) > 1-\rho$, then we get

$$1-\varepsilon \geq \mathscr{G}(x_k - x_m, \lambda) \geq \mathscr{G}\left(x_k - \xi, \frac{\lambda}{2}\right) * \mathscr{G}\left(x_m - \xi, \frac{\lambda}{2}\right)$$
$$> (1-\rho)*(1-\rho) > 1-\varepsilon$$

which is not possible. So, $B(\rho,\lambda) \subset K(\rho,\lambda)$.
Case (ii) If $\mathscr{B}(x_k - x_m, \lambda) \geq \varepsilon$, then we get $\mathscr{B}\left(x_k - \xi, \frac{\lambda}{2}\right) > \rho$ and therefore $k \in K(\rho,\lambda)$. As otherwise i.e., if $\mathscr{B}\left(x_k - \xi, \frac{\lambda}{2}\right) < \rho$, then we obtain

$$\varepsilon \leq \mathscr{B}\left(x_k - x_m, \frac{\lambda}{2}\right) \geq \mathscr{B}\left(x_k - \xi, \frac{\lambda}{2}\right) \Diamond \mathscr{B}\left(x_m - \xi, \frac{\lambda}{2}\right)$$
$$< \rho \Diamond \rho < \varepsilon;$$

which is not possible. Hence, $B(\rho,\lambda) \subset K(\rho,\lambda)$. The last case, again we get $B(\rho,\lambda) \subset K(\rho,\lambda)$. Thus, in all cases we obtain $B(\rho,\lambda) \subset K(\rho,\lambda)$. By 2.1, $B(\rho,\lambda) \in \mathscr{I}$. This shows that (x_k) is $\mathscr{I}_\theta$-Cauchy sequence with regards to NN $\mathscr{N}$. □

Definition 2.16. *Let V be an NNS and take $x = (x_k)$ in NNS.*

(a) An element $\xi \in F$ is named to be lacunary $\mathscr{I}$-limit point of $x = (x_k)$ if there is set $M = \{p_1 < p_2 < ... < p_k < ..\} \subset \mathbb{N}$ such that the set

$$M' = \{r \in \mathbb{N} : p_k \in I_r\} \notin \mathscr{I}$$

and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim x_{p_k} = \xi$.

(b) An element $\xi \in F$ is called to be lacunary $\mathscr{I}$-cluster point of $x = (x_k)$ if, for every $\lambda > 0$ and $\varepsilon \in (0,1)$, we get

$$\left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1-\varepsilon \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \end{array} \right\} \notin \mathscr{I}.$$

Let $\Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$ demonstrate the set of all lacunary $\mathscr{I}$-limit points and $\Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$ indicate the set of all lacunary $\mathscr{I}$-cluster points in NNS, respectively.

Theorem 2.17. *For each sequence $x = (x_k)$ in NNS, we have $\Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x) \subseteq \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$.*

Proof. Let $\xi \in \Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$. So, there is a set $M \subset \mathbb{N}$ such that the set $M' \notin \mathscr{I}$, where M and M' are as in Definition 2.16, satisfies $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim x_{p_k} = \xi$. Hence, for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}\left(x_{p_k} - \xi, \lambda\right) > 1-\varepsilon \text{ and } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}\left(x_{p_k} - \xi, \lambda\right) < \varepsilon, \ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}\left(x_{p_k} - \xi, \lambda\right) < \varepsilon$$

for all $r \geq r_0$. Therefore,

$$B = \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1-\varepsilon \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \end{array} \right\}$$
$$\supseteq M' \setminus \left\{p_1, p_2, ..., p_{k_0}\right\}.$$

Now, with $\mathscr{I}$ being admissible, we must have $M' \setminus \left\{p_1, p_2, ..., p_{k_0}\right\} \notin \mathscr{I}$ and as such $B \notin \mathscr{I}$. Hence, $\xi \in \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$. □

Theorem 2.18. *For each sequence $x = (x_k)$ in NNS, the set $\Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$ is closed in NNS with regards to the usual topology induced by the NN $\mathscr{N}$.*

Proof. Let $y \in \overline{\Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)}$. Take $\lambda > 0$ and $\varepsilon \in (0,1)$. Then, there is $\xi_0 \in \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x) \cap B(y,\varepsilon,\lambda)$. Select $\delta > 0$ such that $B(\xi_0,\delta,\lambda) \subseteq B(y,\varepsilon,\lambda)$. We obtain

$$G = \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - y, \lambda) > 1-\varepsilon \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - y, \lambda) < \varepsilon, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - y, \lambda) < \varepsilon \end{array} \right\}$$
$$\supseteq \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi_0, \lambda) > 1-\delta \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi_0, \lambda) < \delta, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi_0, \lambda) < \delta \end{array} \right\} = H.$$

Thus, $H \notin \mathscr{I}$, and so $G \notin \mathscr{I}$. Hence, $y \in \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$. □

Theorem 2.19. *The following situations are equivalent.*

(a) $\xi \in \Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$.

(b) There are two sequences $y=(y_k)$ and $z=(z_k)$ in NNS such that $x=y+z$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim y=\xi$ and

$$\{r\in\mathbb{N}: k\in I_r,\, z_k\neq\theta\}\in\mathscr{I},$$

where θ indicates zero element of NNS.

Proof. Presume that (a) holds. Then there are M and M' are as above such that $M'\notin\mathscr{I}$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x_{p_k}=\xi$. Take the sequences y and z as follows:

$$y_k=\begin{cases} x_k, & \text{if } k\in I_r \text{ such that } r\in M' \\ \xi, & \text{otherwise}\end{cases}$$

and

$$z_k=\begin{cases} \theta, & \text{if } k\in I_r \text{ such that } r\in M' \\ x_k-\xi, & \text{otherwise.}\end{cases}$$

It sufficies to think the case $k\in I_r$ such that $r\in\mathbb{N}\setminus M'$. For each $\lambda>0$ and $\varepsilon\in(0,1)$, we get $\mathscr{G}(y_k-\xi,\lambda)=1>1-\varepsilon$ and $\mathscr{B}(y_k-\xi,\lambda)=0<\varepsilon$, $\mathscr{Y}(y_k-\xi,\lambda)=0<\varepsilon$. Thus, in this statement,

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(y_k-\xi,\lambda)=1>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(y_k-\xi,\lambda)=0<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(y_k-\xi,\lambda)=0<\varepsilon.$$

Hence, $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim y=\xi$. Now,

$$\{r\in\mathbb{N}: k\in I_r,\ z_k\neq\theta\}\subset\mathbb{N}\setminus M'.$$

But $\mathbb{N}\setminus M'\in\mathscr{I}$, and so

$$\{r\in\mathbb{N}: k\in I_r,\ z_k\neq\theta\}\in\mathscr{I}.$$

Now, assume that (b) holds. Let $M'=\{r\in\mathbb{N}: k\in I_r,\ z_k=\theta\}$. Then, obviously $M'\in\mathscr{F}(\mathscr{I})$ and so it is an infinite set. Construct the set $M=\{p_1<p_2<...<p_k<...\}\subset\mathbb{N}$ such that $p_k\in I_r$ and $z_{p_k}=\theta$. Since $x_{p_k}=y_{p_k}$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim y=\xi$ we get $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x_{p_k}=\xi$. □

Definition 2.20. *A mapping $T:V\to V$ is called to be continuous at $y_0\in F$ with regards to the NN $\mathscr{N}$ if for every $\varepsilon>0$ and $\alpha\in(0,1)$, there are $\delta>0$ and $\beta\in(0,1)$ such that, for all $y\in F$, $\mathscr{G}(y-y_0,\delta)>1-\beta$ and $\mathscr{B}(y-y_0,\delta)<\beta$, $\mathscr{Y}(y-y_0,\delta)<\beta$ give that $\mathscr{G}(T(y)-T(y_0),\varepsilon)>1-\alpha$ and $\mathscr{B}(T(y)-T(y_0),\varepsilon)<\alpha$, $\mathscr{Y}(T(y)-T(y_0),\varepsilon)<\alpha$. If T is continuous on all point of V, then T is called to be continuous on V.*

Definition 2.21. *A mapping $T:V\to V$ is called to be sequentially continuous at $y_0\in F$ with regards to the NN $\mathscr{N}$ if for any sequence $\{y_k\}$, with $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim y_k=y_0$ implies that $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim T(y_k)=T(y_0)$. If T is sequentially continuous at all point of V, then T is said to be sequentially continuous on V.*

Theorem 2.22. *A mapping $T:V\to V$ is continuous with regards to the NN $\mathscr{N}$ iff it is sequentially continuous with regards to the same.*

Definition 2.23. *A lineer operator $T:V\to V$ is called to preserve $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-convergence in NNS if $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x_k=\xi$ gives that $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim T(x_k)=T(\xi)$ for each sequence $x=(x_k)$ in NNS which is $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-convergent to $\xi\in F$.*

Theorem 2.24. *A linear operator $T:V\to V$ preserves $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-convergence in V iff T is continuous on V.*

Proof. Let $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x_k=\xi$. If T is continuous, then for every $\varepsilon>0$ and $\alpha\in(0,1)$, there are $\delta>0$ and $\beta\in(0,1)$ such that, for $y\in F$, if $y\in B(\xi,\beta,\delta)$, then $T(y)\in B(T(\xi),\alpha,\varepsilon)$. But then, we obtain

$$\begin{aligned} C(\delta,\beta)&=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(x_k-\xi,\delta)>1-\beta \\ \text{and } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(x_k-\xi,\delta)<\beta,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(x_k-\xi,\delta)<\beta \end{array}\right\} \\ &\subseteq\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(T(x_k)-T(\xi),\varepsilon)>1-\alpha \\ \text{and } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(T(x_k)-T(\xi),\varepsilon)<\alpha,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(T(x_k)-T(\xi),\varepsilon)<\alpha \end{array}\right\} \\ &=D(\varepsilon,\alpha). \end{aligned}$$

Since $C(\delta,\beta)\in\mathscr{F}(\mathscr{I})$, we get $D(\varepsilon,\alpha)\in\mathscr{F}(\mathscr{I})$. Hence $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim T(x_k)=T(\xi)$.

To demonstrate the converse, assume T be not continuous at same $\xi \in F$. Then, there is some $\varepsilon > 0$ and $\alpha \in (0,1)$ such that $\delta > 0$ and $\beta \in (0,1)$, if $y \in B(\xi, \beta, \delta)$, then $T(y) \notin B(T(\xi), \alpha, \varepsilon)$, where $y \in F$. Now we get a sequence $x = (x_k)$ such that $(\mathscr{G}, \mathscr{B}, \mathscr{Y})^{\theta} - \lim x_k = \xi$ but $(\mathscr{G}, \mathscr{B}, \mathscr{Y})^{\theta} - \lim T(x_k) \neq T(\xi)$. Then, we obtain

$$\begin{aligned} C'(\delta,\beta) &= \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi, \delta) > 1 - \beta \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi, \delta) < \beta,\ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi, \delta) < \beta \end{array} \right\} \\ &\subseteq \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(T(x_k) - T(\xi), \varepsilon) \leq 1 - \alpha \\ \text{or } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(T(x_k) - T(\xi), \varepsilon) \geq \alpha,\ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(T(x_k) - T(\xi), \varepsilon) \geq \alpha \end{array} \right\} \\ &= D'(\varepsilon, \alpha). \end{aligned}$$

Now, $C'(\delta,\beta) \in \mathscr{F}(\mathscr{I})$, and as a result $D'(\varepsilon,\alpha) \in \mathscr{F}(\mathscr{I})$. Therefore $\mathscr{I}_{\theta}^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim T(x_k) \neq T(\xi)$. $\square$

3. Conclusion

We have examined lacunary ideal convergence of sequences in NNS. The fundamental characteristic features of this type of convergence in NNS has been studied. The notions of lacunary $\mathscr{I}$-convergence, lacunary $\mathscr{I}$-Cauchy and lacunary $\mathscr{I}^*$-Cauchy for sequences in NNS are investigated and noteworthy results are established. The results of the paper are expected to be a source for researchers in the areas of convergence methods for sequences and applications in NNS. In future studies on this topic, it is also possible to work with the idea of "Lacunary ideal convergence in Probabilistic metric space" using neutrosophic probability.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Controllability and Accumulation of Errors Arising in a General Iteration Method

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Abstract

In this paper, we propose and analyze a three-step general iteration method which is a special case of an iteration method proposed in (S. Thianwan and S. Suantai, Convergence criteria of a new three-step iteration with errors for nonexpansive nonself-mappings, Comput. Math. Appl. 52 (2006), 1107-1118). Here we intend to study directly the accumulation, estimation and control of random errors in the newly proposed general iteration method. We give conditions under which the accumulated-error in our iteration method is bounded and controllable in a permissible range.

1. Introduction

The tools of fixed point theory are successfully applied to the solutions of a wide variety of problems arising in many disciplines of science. In particular, fixed point iteration methods have attracted the attention of researchers and in parallel with the extension of the application areas of fixed point theory, a great deal of effort has been devoted to the study of some important features of iteration methods (see, for instance, [1]-[9]).
Errors usually occur in the iterative calculations and so consideration of error estimates is of utmost importance in the study of iteration methods. A quick look at literature reveals that many paper have been devoted to the study of iteration methods with errors where the errors are calculated indirectly. There are only a few papers concerning direct estimation and control of errors of the iteration methods (see, e.g., [10]-[12]).
Throughout this exposition, we assume that $(B,\|\cdot\|)$ is an arbitrary real Banach space, S a nonempty closed and convex subset of B, $T:S\to S$ an operator, and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$,$\{\lambda_n\}_{n=0}^{\infty}$,$\{\mu_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\alpha_n+\beta_n+\lambda_n\}_{n=0}^{\infty}$, $\{b_n+c_n+\mu_n\}_{n=0}^{\infty}$, $\{a_n+\gamma_n\}_{n=0}^{\infty}\subseteq[0,1]$ are parameter sequences satisfying certain control condition(s) and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ are bounded sequences in S.
In 2006, Thianwan and Suantai [13] defined a three-step iteration method on S with error terms as:

$$\begin{cases} x_0\in S, \\ x_{n+1}=(1-\alpha_n-\beta_n-\lambda_n)x_n+\alpha_nTy_n+\beta_nTz_n+\lambda_nw_n \\ y_n=(1-b_n-c_n-\mu_n)x_n+b_nTz_n+c_nTx_n+\mu_nv_n \\ z_n=(1-a_n-\gamma_n)x_n+a_nTx_n+\gamma_nu_n, \text{ for all } n\in\mathbb{N}. \end{cases} \tag{1.1}$$

The iteration method (1.1) has been used for approximation of fixed points of various nonlinear mappings (see, for instance,

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[14, 15]). If we put $\lambda_n=\mu_n=\gamma_n=0$ for all $n\in\mathbb{N}$ in (1.1), then we obtain

$$\begin{cases} x_0\in S, \\ x_{n+1}=(1-\alpha_n-\beta_n)x_n+\alpha_nTy_n+\beta_nTz_n \\ y_n=(1-b_n-c_n)x_n+b_nTz_n+c_nTx_n \\ z_n=(1-a_n)x_n+a_nTx_n, \text{ for all } n\in\mathbb{N}. \end{cases} \tag{1.2}$$

Remark 1.1. *The iteration method (1.1) reduces to:*
(i) Noor iteration method [16] if $c_n=\beta_n=\gamma_n=\lambda_n=\mu_n=0$ for all $n\in\mathbb{N}$,
(ii) Ishikawa iteration method [17] if $a_n=c_n=\beta_n=\gamma_n=\lambda_n=\mu_n=0$ for all $n\in\mathbb{N}$,
(iii) Mann iteration method [18] if $a_n=b_n=c_n=\beta_n=\gamma_n=\lambda_n=\mu_n=0$ for all $n\in\mathbb{N}$.

2. Main results

Here we intend to study directly the accumulation, estimation and control of random errors in the iteration method (1.2). Define the errors of Tx_n, Ty_n and Tz_n by

$$u_n=Tx_n-\overline{Tx_n},\ v_n=Tz_n-\overline{Tz_n} \text{ and } w_n=Ty_n-\overline{Ty_n} \tag{2.1}$$

for all $n\in\mathbb{N}$, where $\overline{Tx_n}$, $\overline{Ty_n}$ and $\overline{Tz_n}$ are the exact values of Tx_n, Ty_n and Tz_n respectively, that is, Tx_n, Ty_n and Tz_n are approximate values of $\overline{Tx_n}$, $\overline{Ty_n}$ and $\overline{Tz_n}$, respectively. The theory of errors implies that $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are bounded. Set

$$B=\max\{B_u,B_v,B_w\} \tag{2.2}$$

where $B_u=\sup_{n\in\mathbb{N}}\|u_n\|$, $B_v=\sup_{n\in\mathbb{N}}\|v_n\|$ and $B_w=\sup_{n\in\mathbb{N}}\|w_n\|$ are the bounds on the absolute errors of $\{Tx_n\}_{n=0}^{\infty}$, $\{Tz_n\}_{n=0}^{\infty}$ and $\{Ty_n\}_{n=0}^{\infty}$, respectively.
The main part of accumulation of errors from (1.2) comes essentially from u_n, v_n and w_n; hence we can set

$$\begin{cases} \overline{x_0}\in S, \\ \overline{x_{n+1}}=(1-\alpha_n-\beta_n)\overline{x_n}+\alpha_n\overline{Ty_n}+\beta_n\overline{Tz_n} \\ \overline{y_n}=(1-b_n-c_n)\overline{x_n}+b_n\overline{Tz_n}+c_n\overline{Tx_n} \\ \overline{z_n}=(1-a_n)\overline{x_n}+a_n\overline{Tx_n}, \text{ for all } n\in\mathbb{N}. \end{cases} \tag{2.3}$$

where $\overline{x_n}$, $\overline{y_n}$ and $\overline{z_n}$ are exact values of x_n, y_n and z_n, respectively. Clearly, the errors of last iteration will affect the next $(n+1)$ steps. So, utilizing (1.2), (2.1) and (2.3), we have

$$\begin{aligned} x_0 &= \overline{x_0}; \\ z_0 &= (1-a_0)x_0+a_0Tx_0 \\ &= (1-a_0)\overline{x_0}+a_0\overline{Tx_0}+a_0u_0=\overline{z_0}+a_0u_0; \\ y_0 &= (1-b_0-c_0)x_0+b_0Tz_0+c_0Tx_0 \\ &= (1-b_0-c_0)\overline{x_0}+b_0\overline{Tz_0}+c_0\overline{Tx_0}+b_0v_0+c_0u_0 \\ &= \overline{y_0}+b_0v_0+c_0u_0; \\ x_1 &= (1-\alpha_0-\beta_0)x_0+\alpha_0Ty_0+\beta_0Tz_0 \\ &= (1-\alpha_0-\beta_0)\overline{x_0}+\alpha_0\overline{Ty_0}+\beta_0\overline{Tz_0}+\alpha_0w_0+\beta_0v_0 \\ &= \overline{x_1}+\alpha_0w_0+\beta_0v_0; \end{aligned}$$

$$\begin{aligned} z_1 &= \overline{z_1}+(1-a_1)(\alpha_0w_0+\beta_0v_0)+a_1u_1; \\ y_1 &= \overline{y_1}+(1-b_1-c_1)(\alpha_0w_0+\beta_0v_0)+b_1v_1+c_1u_1; \\ x_2 &= \overline{x_2}+(1-\alpha_1-\beta_1)(\alpha_0w_0+\beta_0v_0)+\alpha_1w_1+\beta_1v_1; \end{aligned}$$

$$\begin{aligned} z_2 &= \overline{z_2}+(1-a_2)(1-\alpha_1-\beta_1)(\alpha_0w_0+\beta_0v_0) \\ &\quad +(1-a_2)(\alpha_1w_1+\beta_1v_1)+a_2u_2; \\ y_2 &= \overline{y_2}+(1-b_2-c_2)[(1-\alpha_1-\beta_1)(\alpha_0w_0+\beta_0v_0) \\ &\quad +(\alpha_1w_1+\beta_1v_1)]+b_2v_2+c_2u_2; \\ x_3 &= \overline{x_3}+(1-\alpha_2-\beta_2)(1-\alpha_1-\beta_1)(\alpha_0w_0+\beta_0v_0) \\ &\quad +(1-\alpha_2-\beta_2)(\alpha_1w_1+\beta_1v_1)+\alpha_2w_2+\beta_2v_2; \end{aligned}$$

Repeating the above process, we obtain

$$x_{n+1}=\overline{x_{n+1}}+\sum_{k=0}^{n}\left(\alpha_k w_k+\beta_k v_k\right)\left[\prod_{i=k+1}^{n}\left(1-\alpha_i-\beta_i\right)\right],$$

$$\begin{aligned} y_n &= \overline{y_n}+b_n v_n+c_n u_n+\left(1-b_n-c_n\right)\sum_{k=0}^{n-1}\left(\alpha_k w_k+\beta_k v_k\right)\left[\prod_{i=k+1}^{n-1}\left(1-\alpha_i-\beta_i\right)\right] \\ &= \overline{y_n}+b_n v_n+c_n u_n+\left(1-b_n-c_n\right)\left(x_n-\overline{x_n}\right), \end{aligned}$$

and

$$\begin{aligned} z_n &= \overline{z_n}+a_n u_n+\left(1-a_n\right)\sum_{k=0}^{n-1}\left(\alpha_k w_k+\beta_k v_k\right)\left[\prod_{i=k+1}^{n-1}\left(1-\alpha_i-\beta_i\right)\right] \\ &= \overline{z_n}+a_n u_n+\left(1-a_n\right)\left(x_n-\overline{x_n}\right) \text{ for all } n\in\mathbb{N}. \end{aligned}$$

Define

$$Q_n^{(1)}:=x_{n+1}-\overline{x_{n+1}}=\sum_{k=0}^{n}\left(\alpha_k w_k+\beta_k v_k\right)\left[\prod_{i=k+1}^{n}\left(1-\alpha_i-\beta_i\right)\right], \tag{2.4}$$

$$Q_n^{(2)}:=y_n-\overline{y_n}=b_n v_n+c_n u_n+\left(1-b_n-c_n\right)Q_{n-1}^{(1)}, \tag{2.5}$$

and

$$Q_n^{(3)}:=z_n-\overline{z_n}=a_n u_n+\left(1-a_n\right)Q_{n-1}^{(1)} \text{ for all } n\in\mathbb{N}. \tag{2.6}$$

Obviously, the errors of iteration method, after $(n+1)$ times iterations, are added up to $Q_n^{(1)}$, $Q_n^{(2)}$ and $Q_n^{(3)}$.

Now, we are in a position to give the following result.

Theorem 2.1. *Let S, T, B, $Q_n^{(1)}$, $Q_n^{(2)}$ and $Q_n^{(3)}$ be as above.*

(i) If $\sum\limits_{i=0}^{\infty}\left(\alpha_i+\beta_i\right)=+\infty$, then the accumulation of errors in (1.2) is bounded and does not exceed the number B;

(ii) If $\sum\limits_{i=0}^{\infty}\left(\alpha_i+\beta_i\right)<+\infty$, $\lim_{n\to\infty}a_n=0$ and $\lim_{n\to\infty}\left(b_n+c_n\right)=0$, then random errors of (1.2) are controllable.

Proof. (i) It is well known that $\sum\limits_{i=0}^{\infty}\left(\alpha_i+\beta_i\right)=+\infty$ implies $\prod\limits_{i=0}^{\infty}\left(1-\alpha_i-\beta_i\right)=0$ (see, e.g., (Remark 2.1 of [19])). From (2.2), (2.4)-(2.6) we have

$$\begin{aligned}
\left\|Q_n^{(1)}\right\| &= \left\|(\alpha_0 w_0+\beta_0 v_0)\prod_{i=1}^{n}(1-\alpha_i-\beta_i)\right.\\
&\quad +(\alpha_1 w_1+\beta_1 v_1)\prod_{i=2}^{n}(1-\alpha_i-\beta_i)\\
&\quad \left.+\cdots+(\alpha_{n-1}w_{n-1}+\beta_{n-1}v_{n-1})\prod_{i=n}^{n}(1-\alpha_i-\beta_i)+\alpha_n w_n+\beta_n v_n\right\|\\
&\leq \left\|(\alpha_0 w_0+\beta_0 v_0)\prod_{i=1}^{n}(1-\alpha_i-\beta_i)\right\|\\
&\quad +\left\|(\alpha_1 w_1+\beta_1 v_1)\prod_{i=2}^{n}(1-\alpha_i-\beta_i)\right\|\\
&\quad +\cdots+\left\|(\alpha_{n-1}w_{n-1}+\beta_{n-1}v_{n-1})\prod_{i=n}^{n}(1-\alpha_i-\beta_i)\right\|\\
&\quad +\|\alpha_n w_n+\beta_n v_n\|\\
&\leq (\alpha_0\|w_0\|+\beta_0\|v_0\|)\prod_{i=1}^{n}(1-\alpha_i-\beta_i)\\
&\quad +(\alpha_1\|w_1\|+\beta_1\|v_1\|)\prod_{i=2}^{n}(1-\alpha_i-\beta_i)\\
&\quad +\cdots+(\alpha_{n-1}\|w_{n-1}\|+\beta_{n-1}\|v_{n-1}\|)\prod_{i=n}^{n}(1-\alpha_i-\beta_i)\\
&\quad +\alpha_n\|w_n\|+\beta_n\|v_n\|\\
&\leq B\left\{(\alpha_0+\beta_0)\prod_{i=1}^{n}(1-\alpha_i-\beta_i)+(\alpha_1+\beta_1)\prod_{i=2}^{n}(1-\alpha_i-\beta_i)\right.\\
&\quad \left.+\cdots+(\alpha_{n-1}+\beta_{n-1})\prod_{i=n}^{n}(1-\alpha_i-\beta_i)+\alpha_n+\beta_n\right\}\\
&= B\left\{\prod_{i=0}^{n}(1-\alpha_i-\beta_i)+(\alpha_0+\beta_0)\prod_{i=1}^{n}(1-\alpha_i-\beta_i)\right.\\
&\quad +(\alpha_1+\beta_1)\prod_{i=2}^{n}(1-\alpha_i-\beta_i)+\cdots+(\alpha_{n-1}+\beta_{n-1})\prod_{i=n}^{n}(1-\alpha_i-\beta_i)\\
&\quad \left.+\alpha_n+\beta_n-\prod_{i=0}^{n}(1-\alpha_i-\beta_i)\right\}\\
&= B\left[1-\prod_{i=0}^{n}(1-\alpha_i-\beta_i)\right]\leq B\left[1-\prod_{i=0}^{\infty}(1-\alpha_i-\beta_i)\right]=B, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
\left\|Q_n^{(2)}\right\| &= \left\|b_n v_n+c_n u_n+(1-b_n-c_n)Q_{n-1}^{(1)}\right\|\\
&\leq b_n\|v_n\|+c_n\|u_n\|+(1-b_n-c_n)\left\|Q_{n-1}^{(1)}\right\|\\
&\leq B(b_n+c_n)+(1-b_n-c_n)B=B, \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
\left\|Q_n^{(3)}\right\| &= \left\|a_n u_n+(1-a_n)Q_{n-1}^{(1)}\right\|\\
&\leq a_n\|u_n\|+(1-a_n)\left\|Q_{n-1}^{(1)}\right\|\\
&\leq a_n B+(1-a_n)B=B \text{ for all } n\in\mathbb{N}. \qquad (2.9)
\end{aligned}$$

Hence, we have $\max_{n\in\mathbb{N}}\left\{\left\|Q_n^{(1)}\right\|,\left\|Q_n^{(2)}\right\|,\left\|Q_n^{(3)}\right\|\right\}\leq B$.

(ii) Indeed, $\sum\limits_{i=0}^{\infty}(\alpha_i+\beta_i)<+\infty$ implies that $\prod\limits_{i=0}^{\infty}(1-\alpha_i-\beta_i)\in(0,1)$. Let $1-\prod\limits_{i=0}^{\infty}(1-\alpha_i-\beta_i)=\ell\in(0,1)$. Thus, from (2.7), we obtain

$$\left\|Q_n^{(1)}\right\|\leq B\left[1-\prod_{i=0}^{\infty}(1-\alpha_i-\beta_i)\right]\leq \ell B \text{ for all } n\in\mathbb{N}. \tag{2.10}$$

On the other hand, the condition $\lim_{n\to\infty}(b_n+c_n)=0$ implies the existence of an $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $b_n+c_n\leq \ell/(1-\ell)$. Using this fact together with (2.8) and (2.10), we get

$$\begin{aligned}\left\|Q_n^{(2)}\right\| &\leq (b_n+c_n)B+(1-b_n-c_n)\left\|Q_{n-1}^{(1)}\right\|\\ &\leq (b_n+c_n)B(1-\ell)+B\ell\\ &\leq \frac{\ell}{1-\ell}B(1-\ell)+B\ell=2B\ell \text{ for all } n\geq n_0.\end{aligned} \tag{2.11}$$

Similarly, the condition $\lim_{n\to\infty}a_n=0$ implies the existence of an $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ we have $a_n\leq \ell/(1-\ell)$. Hence, from (2.9) and (2.10), we have

$$\begin{aligned}\left\|Q_n^{(3)}\right\| &\leq a_n\|u_n\|+(1-a_n)\left\|Q_{n-1}^{(1)}\right\|\\ &\leq a_nB(1-\ell)\ +B\ell\\ &\leq \frac{\ell}{1-\ell}B(1-\ell)\ +B\ell=2B\ell \text{ for all } n\geq n_0.\end{aligned} \tag{2.12}$$

Thus, we conclude that $\left\|Q_n^{(1)}\right\|$, $\left\|Q_n^{(2)}\right\|$ and $\left\|Q_n^{(3)}\right\|$ can be controlled for suitable choice of the parameter sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ for all $n\geq n_0$. □

Example 2.2. *Let $\alpha_n+\beta_n=\frac{1}{(n^2+4n+3)^2}$ for all $n\in\mathbb{N}$. Then, we have by the Wolfram Mathematica 9 software package that $\sum\limits_{i=0}^{\infty}(\alpha_i+\beta_i)=\frac{1}{48}\left(4\pi^2-33\right)<+\infty$ and $\ell=1-\prod\limits_{i=0}^{\infty}(1-\alpha_i-\beta_i)=1+\frac{2\sqrt{2}\sin(\sqrt{2}\pi)}{\pi}\approx 0.132183\in(0,1)$ which implies together with (2.10)-(2.12) that $\left\|Q_n^{(1)}\right\|\leq\left(1+\frac{2\sqrt{2}\sin(\sqrt{2}\pi)}{\pi}\right)B$, $\left\|Q_n^{(2)}\right\|\leq 2\left(1+\frac{2\sqrt{2}\sin(\sqrt{2}\pi)}{\pi}\right)B$ and $\left\|Q_n^{(3)}\right\|\leq 2\left(1+\frac{2\sqrt{2}\sin(\sqrt{2}\pi)}{\pi}\right)B$ for all $n\in\mathbb{N}$.*
Especially, for any $\varepsilon\in(0,1)$, if $\alpha_n+\beta_n=\frac{5^{n+2}}{7^{n+3}}\varepsilon$ for all $n\in\mathbb{N}$, then

$$\prod_{i=0}^{\infty}(1-\alpha_i-\beta_i)\geq 1-\sum_{i=0}^{\infty}(\alpha_i+\beta_i)=1-\frac{25}{98}\varepsilon,$$

which yields $\ell<\frac{25}{98}\varepsilon$, so that

$$\left\|Q_n^{(1)}\right\|\leq\frac{25}{98}\varepsilon B \text{ for all } n\in\mathbb{N},$$

$$\left\|Q_n^{(2)}\right\|\leq\frac{25}{49}\varepsilon B \text{ for all } n\geq n_0,$$

and

$$\left\|Q_n^{(3)}\right\|\leq\frac{25}{49}\varepsilon B \text{ for all } n\geq n_0,$$

where n_0 belongs to $\mathbb{N}$ and the inequalities $a_n\leq\frac{\varepsilon}{3.92-\varepsilon}$ and $b_n+c_n\leq\frac{\varepsilon}{3.92-\varepsilon}$ hold. Hence, the random errors is controllable in a permissible range for suitable choice of the parameter sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ for all $n\geq n_0$.

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On Weak Projection Invariant Semisimple Modules

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Abstract

We introduce and investigate the notion of weak projection invariant semisimple modules. We deal with the structural properties of this new class of modules. In this trend we have indecomposable decompositions of the special class of the former class of modules via some module theoretical properties. As a consequence, we obtain when the finite exchange property implies full exchange property for the latter class of modules.

1. Introduction

All rings are associative with unity and modules are unital right modules. Let R be a ring and M a right R-module. Recall that M is called CS (or, *extending*) if every submodule of M is essentially contained in a direct summand of M. This kind of modules are important generalizations of injective, semisimple and uniform modules. There have been several generalizations of CS modules as well as some classes of modules which are related to the direct summands of the module in literature (see [1]-[4]).

A submodule N of M is called *projection invariant*, if $f(N) \subseteq N$ for all $f^2 = f \in End(M_R)$ (see [3, 5, 6]). Note that torsion subgroup of a group, socle of a module and the radical of a ring are all projection invariant submodules of the corresponding modules, respectively. Recall from [6], a module M is called π-extending if every projection invariant submodule of M is essential in a direct summand of M. It is well-known that a CS-module is π-extending [3].

In this paper, we introduce and investigate the notion of weak projection invariant semisimple modules which is a generalization of semisimple and projection invariant semisimple modules [7]. We call a module M is weak projection invariant semisimple, denoted by $w\pi$-semisimple, provided that each semisimple projection invariant submodule of M is a direct summand of M. It is clear that the class of the $w\pi$-semisimple modules is contained in the class of π-extending modules. We deal with structural module properties of $w\pi$-semisimple modules. Moreover, we define special class of $w\pi$-semisimple modules and obtain indecomposable decomposition for the aforementioned modules via Abelian endomorphism rings over rings with ascending chain condition on the right annihilators. As a consequence, we obtain that the finite exchange property implies full exchange property.

Let $X \subseteq M$, then $X \leq M$, $SocM$ and $End(M_R)$ denote X is a submodule of M, the socle of M and the endomorphism ring of M_R, respectively. Recall that a module M over a ring R is said to have (finite) exchange property if for any (finite) index set I, whenever $M \oplus Y = \bigoplus\limits_{i\in I} A_i$ for modules Y and A_i, then $M \oplus Y = M \oplus (\bigoplus\limits_{i\in I} B_i)$ for submodules B_i of A_i [8]. A family $\{N_i : i \in I\}$ of independent submodules of a module M is said to be a local summand if for any finite subset F of I, $\bigoplus\limits_{i\in F} N_i$ is a direct summand of M [3, 9]. Recall further that a ring R is called Abelian if every idempotent of R is central [3, 10].

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Since $w\pi$-semisimple modules are based on semisimple projection invariant submodules, we start with the following basic result.

Lemma 1.1. *(i) If A is projection invariant in B and B is projection invariant in M then A is projection invariant in M.*
(ii) If $M = \bigoplus_{i\in I} M_i$, and X is a semisimple projection invariant submodule of M, then $X = \bigoplus_{i\in I}(X \cap M_i)$ and $X \cap M_i$ is semisimple projection invariant submodule of M_i for all $i \in I$.

Proof. Immediate by definitions (see [11, p.50]). □

In [7, Lemma 1.2], the author attempts to obtain the following statement: Let M_R be a module and $N \leq K \leq M_R$. If N is projection invariant in M and K/N is projection invariant in M/N, then K is projection invariant in M. However, the proof therein is inconsistent. Since $f(N)$ would be nonzero, the function $\theta : M/N \to M$, $\theta(m+N) = f(m)$ for all $m \in M$ is not well-defined. Let us make it clear by the following example.

Example 1.2. *Let $M = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$ be the $\mathbb{Z}$-module, and $N = (2\mathbb{Z}/4\mathbb{Z}) \oplus 0$ be the submodule of $M_{\mathbb{Z}}$. Now, it is easy to see that $End(M_{\mathbb{Z}}) \cong \begin{bmatrix} \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \\ \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \end{bmatrix}$. Let $f^2 = f = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} \in End(M_{\mathbb{Z}})$. So, let us consider $\theta : M/N \to M$, $\theta(m+N) = f(m)$ for all $m \in M$. Since $(\bar{1}+\bar{3}) - (\bar{3}+\bar{3}) \in N$, $(\bar{1}+\bar{3})+N = (\bar{3}+\bar{3})+N$. But $\theta((\bar{1}+\bar{3})+N) = f(\bar{1}+\bar{3}) = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}\begin{bmatrix} \bar{1} \\ \bar{3} \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}$, and $\theta((\bar{3}+\bar{3})+N) = f(\bar{3}+\bar{3}) = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}\begin{bmatrix} \bar{3} \\ \bar{3} \end{bmatrix} = \begin{bmatrix} \bar{2} \\ \bar{0} \end{bmatrix}$. Hence $\theta((\bar{1}+\bar{3})+N) \neq \theta((\bar{3}+\bar{3})+N)$.*

Notice that Proposition 2.3 (ii), Corollaries 2.4, 2.5 and one part of the proof of Theorem 2.6 in [7] use [7, Lemma 1.2]. By the previous example, the aforementioned results are also invalid.

2. Main results

In this section, we introduce and investigate the class of weak projection invariant semisimple modules. We focus on some structural properties of weak projection invariant semisimple modules as well as indecomposable decompositions for the special class of the weak projection invariant semisimple modules via some module theoretical conditions.

Definition 2.1. *We call an R-module M weak projection invariant semisimple, denoted by $w\pi$-semisimple, if each semisimple projection invariant submodule of M is a direct summand of M.*

Observe that any semisimple module and π-semisimple module is $w\pi$-semisimple. Moreover, any module which has zero socle (for example, a polynomial ring $R[x]$ over any ring R) is clearly a $w\pi$-semisimple module. Next, we provide $w\pi$-semisimple modules which are not π-semisimple.

Example 2.2. *(i) Let M be the $\mathbb{Z}$-module $\mathbb{Z}$. Obviously, $M_{\mathbb{Z}}$ is $w\pi$-semisimple. However, $M_{\mathbb{Z}}$ is not π-semisimple. For example, $N = 2\mathbb{Z}$ is a projection invariant in $M_{\mathbb{Z}}$ which is not a direct summand of $M_{\mathbb{Z}}$.*
(ii) Let M be the $\mathbb{Z}[x]$-module $\mathbb{Z}[x]$. Then $SocM = 0$. Hence M is a $w\pi$-semisimple module. Since M is uniform, it is not π-semisimple.
(iii) [4, Example 2.4(ii)]. Let D be a simple domain which is not a division ring. Take $R = \begin{bmatrix} D & D\oplus D \\ 0 & D \end{bmatrix}$ then $I = \begin{bmatrix} 0 & 0\oplus D \\ 0 & 0 \end{bmatrix}$ is an ideal of R. Thus, I is a projection invariant submodule of R_R which is not a direct summand of R_R. It follows that R_R is not π-semisimple. However, $Soc(R_R) = 0$, and hence R_R is $w\pi$-semisimple.

Example 2.2 sheds light on the natural question, namely, when a $w\pi$-semisimple module is a π-semisimple. The second part of the following result provides an answer.

Proposition 2.3. *(i) Assume that M_R is an indecomposable module. Then M_R is semisimple if and only if M_R is $w\pi$-semisimple and $SocM$ is essential in M.*
(ii) If M_R is a $w\pi$-semisimple module with essential socle then M_R is π-semisimple.

Proof. (i) ($\Rightarrow$) This implication is clear.
($\Leftarrow$) Let $X \leq M$. Since M is indecomposable, X is projection invariant in M. It follows that $SocX$ is projection invariant in M, by Lemma 1.1 (i). By hypothesis, $SocX$ is a direct summand of M. Hence $SocX = 0$ or $SocX = M$. Therefore $X = 0$ or M. Thus, X is a direct summand of M. So, M is semisimple.
(ii) Let X be any projection invariant submodule of M_R. Then $SocX$ is projection invariant in M, by Lemma 1.1 (i). It follows that $SocX$ is a direct summand of M. On the other hand,

$$SocX = X \cap SocM \leq X \cap M = X$$

gives that $SocX$ is essential in X. Thus $SocX = X$ i.e., X is a direct summand of M. So, M_R is π-semisimple. □

Corollary 2.4. *If M_R is a $w\pi$-semisimple module with essential socle then M_R is π-extending.*

Proof. Let X be a projection invariant submodule of M. By Proposition 2.3, X is a direct summand of M. Since X is essential in itself, M_R is a π-extending module. □

Lemma 2.5. *Let M_R be $w\pi$-semisimple and N a projection invariant submodule of M. Then N is $w\pi$-semisimple.*

Proof. Let X be any semisimple projection invariant submodule of N. By Lemma 1.1 (i), X is projection invariant in M. Therefore $M = X \oplus X'$ for some X' submodule of M. Now, by Lemma 1.1 (ii), $N = (N \cap X) \oplus (N \cap X') = X \oplus (N \cap X')$. Thus X is a direct summand of N which yields that N is $w\pi$-semisimple. □

Lemma 2.6. *Let M_R be a $w\pi$-semisimple module. Then the following statements hold:*

(i) Every fully invariant submodule of M_R is $w\pi$-semisimple.
(ii) If $End(M_R)$ is Abelian then every direct summand of M_R is $w\pi$-semisimple.

Proof. (i) Since every fully invariant submodule is projection invariant, the proof follows from Lemma 2.5.
(ii) Let M_R be a $w\pi$-semisimple module with an Abelian endomorphism ring. Let $K = eM$ for some $e^2 = e \in End(M_R)$. Thus $g(eM) \subseteq eM$ for all $g^2 = g \in End(M_R)$. Hence K_R is a projection invariant submodule of M_R. By Lemma 2.5, K_R is a $w\pi$-semisimple module. □

Proposition 2.7. *Let $M = M_1 \oplus M_2$ such that M_2 is semisimple fully invariant submodule of M. If M_R is $w\pi$-semisimple, then both M_1 and M_2 are $w\pi$-semisimple.*

Proof. It is clear that M_2 is $w\pi$-semisimple. Let X be a semisimple projection invariant submodule of M_1. Then $X \oplus M_2$ is a semisimple projection invariant submodule of M (see [6, Lemma 4.13]). By hypothesis, $X \oplus M_2$ is a direct summand of M. Hence $M = X \oplus M_2 \oplus L$ for some submodule L of M. Now, the modular law gives that

$$M_1 = M_1 \cap (X \oplus M_2 \oplus L) = X \oplus (M_1 \cap (M_2 \oplus L)).$$

Hence X is a direct summand of M_1 which yields that M_1 is $w\pi$-semisimple. □

Theorem 2.8. *Let $M = \underset{i\in I}{\oplus} M_i$ where M_i's are fully invariant submodules of M for $i \in I$. If M_i is $w\pi$-semisimple for all $i \in I$, then M is $w\pi$-semisimple.*

Proof. Assume each M_i is $w\pi$-semisimple for all $i \in I$ and $M = \underset{i\in I}{\oplus} M_i$. Let N be a semisimple projection invariant submodule of M. Then $N = \underset{i\in I}{\oplus}(N \cap M_i)$ where $N \cap M_i$ is a semisimple projection invariant submodule of M_i for all $i \in I$, from Lemma 1.1(ii). By assumption, M_i is $w\pi$-semisimple which gives that $N \cap M_i$ is a direct summand of M_i for all $i \in I$. It follows that N is a direct summand of M. Thus, M is $w\pi$-semisimple. □

Observe that if M is a $w\pi$-semisimple module in the previous result then by Lemma 2.5, each M_i is also $w\pi$-semisimple for all $i \in I$. Our next aim is to obtain an indecomposable decomposition for special $w\pi$-semisimple modules. To do this, let us give the following definition.

Definition 2.9. *We call an R-module M $w\pi^*$-semisimple provided that whenever any semisimple projection invariant submodule is contained as projection invariant in a projection invariant submodule of M then the larger submodule is a direct summand of M.*

It can be seen easily that any $w\pi^*$-semisimple module is $w\pi$-semisimple (any projection invariant submodule has a semisimple projection invariant submodule, namely, its socle). However, there are several $w\pi$-semisimple modules which are not $w\pi^*$-semisimple. For example, let M be the $\mathbb{Z}$-module $\mathbb{Z}$ (see, Example 2.2 (i)).

Lemma 2.10. *Let R be a ring and M an R-module such that R satisfies ascending chain condition on right annihilators of the form $r(m)$ $(m \in M)$. If M is $w\pi^*$-semisimple with an Abelian endomorphism ring then M has an indecomposable decomposition.*

Proof. Let $\{X_\lambda : \lambda \in I\}$ be an independent family of submodules of M and $X = \underset{\lambda\in I}{\oplus} X_\lambda$ be a local summand of M. Now, let us define the canonical projection $\pi_k : X \to \underset{k\in I, k\neq\lambda}{\oplus} X_k$. Then $f(X) = f(\underset{\lambda\in I}{\oplus} X_\lambda) = \underset{\lambda\in I}{\oplus} f(X_\lambda) = \underset{\lambda\in I}{\oplus} f(ker\pi_\lambda)$ where $f^2 = f \in End(M_R)$. By the assumption that $End(M_R)$ is Abelian, $f(ker\pi_\lambda) \subseteq ker\pi_\lambda$. Thus $f(X) \subseteq X$. It follows that X is projection invariant in M_R. Since $SocX$ is projection invariant in X, by $w\pi^*$-semisimple, X is a direct summand of M. Hence [9, Theorem 2.17] yields that M has an indecomposable decomposition. □

Next, we have the following result.

Theorem 2.11. *Let R be a ring and M an R-module such that R satisfies ascending chain condition on right annihilators of the form $r(m)$ $(m \in M)$. If M is $w\pi^*$-semisimple with an Abelian endomorphism ring then M is a direct sum of uniform submodules.*

Proof. Observe that being $w\pi^*$-semisimple implies π-extending, and an indecomposable module is uniform by [6, Proposition 3.8]. Now, we conclude the proof by Lemma 2.10 and Lemma 2.6 (ii). $\square$

It would be worthy construct an example which shows that being Abelian endomorphism ring in the previous theorem is not superfluous. Incidentally, let us give the example.

Example 2.12. *Let R be the real field and n be any odd integer with $n \geq 1$. Let S be the polynomial ring $R[x_1,\dots,x_n]$ indeterminates $x_1,x_2,\dots,x_n$ over R. Let R be the ring S/Ss, where $s = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$. Then the free R-module $M = \bigoplus_{i=1}^{n} R$ contains a submodule K_R which is indecomposable and has uniform dimension $n-1$ (see [12, Corollary 16]). Note that K_R is not uniform. Since $SocM = 0$, then $Soc(K_R) = 0$. Now, let $Y = \begin{bmatrix} S & K \\ 0 & R \end{bmatrix}$ be the split null extension ring where $S = End(K_R)$. Observe that ${}_SK$ is faithful. Therefore $Soc(Y_Y) = 0$. Hence Y is a $w\pi^*$-semisimple module. Moreover, $Y = \begin{bmatrix} S & K \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}$ and $\begin{bmatrix} S & K \\ 0 & 0 \end{bmatrix}$ is indecomposable with uniform dimension $n-1$. It follows that Y has no decomposition into uniform submodules. It can be seen that Y_Y is Noetherian. However, $End(Y_Y) \cong Y$ is not Abelian. For, let $a = \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix}$, and $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two elements of $End(Y_Y)$ where $f \in S$ and $0 \neq x \in K_R$. Then $ab = \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ and $ba = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.*

Now, we have the following consequences of the Theorem 2.11. The first one is the result on exchange property of modules which was pointed out in the introduction and the last is based on locally Noetherian modules. Recall that a module is called locally Noetherian provided that every finitely generated submodule is Noetherian (see [3]).

Corollary 2.13. *Let R be a right Noetherian ring and M an R-module with an Abelian endomorphism ring. If M is $w\pi^*$-semisimple then the finite exchange property implies full exchange property.*

Proof. By Theorem 2.11 and [8, Corollary 6]. $\square$

Corollary 2.14. *Let M be a locally Noetherian module with an Abelian endomorphism ring. If M is $w\pi^*$-semisimple then the finite exchange property implies full exchange property.*

Proof. Let $m \in M$. Then $R/r(m) \cong mR$ is right Noetherian module. It follows that R satisfies ascending chain condition on right annihilators of the form $r(m)$ $(m \in M)$. Thus Theorem 2.11 gives the result. $\square$

Finally, we have the next result on endomorphism ring of a $w\pi^*$-semisimple module. First, recall that a ring R is π-Baer if the right annihilator of a projection invariant left ideal of R is of the form eR for some $e^2 = e \in R$ (see [5, 13]).

Theorem 2.15. *Assume that M is a $w\pi^*$-semisimple module. Then the endomorphism ring of M is a π-Baer ring.*

Proof. Let $S = End(M_R)$ and I be a projection invariant left ideal of S. We want to show that $r_S(I) = eS$ for some $e^2 = e \in S$. It can be seen that $r_M(I)$ is a projection invariant submodule of M_R. Hence $Soc(r_M(I))$ is a projection invariant submodule of $r_M(I)$. By hypothesis, $r_M(I) = eM$ for some $e^2 = e \in S$. Thus $IeM = 0$, so $Ie = 0$, as ${}_SM$ faithful. Therefore $eS \subseteq r_S(I)$. Now, let $a \in r_S(I)$. Hence $Ia = 0$ which gives that $l(aM) = 0$. It follows that $aM \subseteq r_M(I) = eM$. Thus $a \in eS$, so $r_S(I) \subset eS$. Therefore, S is a π-Baer ring. $\square$

Corollary 2.16. *If M is a π-semisimple module then the endomorphism ring of M is a π-Baer ring.*

Proof. Since π-semisimple implies $w\pi^*$-semisimple the result follows from Theorem 2.15. $\square$

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Hermite-Hadamard Type Inequalities for the Functions Whose Absolute Values of First Derivatives are p-Convex

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Abstract

In this paper, we extend some estimates of a Hermite-Hadamard type inequality for functions whose absolute values of the first derivatives are p-convex. By means of the obtained inequalities, some bound functions involving beta functions and hypergeometric functions are derived as applications. Also, we suggest an upper bound for error in numerical integration of p-convex functions via composite trapezoid rule.

1. Introduction

Some features of sets and functions make them more important than the others in mathematics, so this kind of sets and functions attract great interest, especially, if they are useful to handle optimization problems. One of them is convexity. Since the discovery of the convex sets and functions, it has been so extended and generalized in many ways that a lof of convexity types have been defined, from quasiconvexity to B-convexity, B^{-1}-convexity, p-convexity etc (See [1]- [12] and the references therein). On the other hand, in researching new types of convexity, many inequalities valid for convex functions and on convex sets such as Jensen, Ostrowski, and Hermite-Hadamard are adapted to new convexity types such as s-convex functions, p-convex function [13] -[16]. In this study, we focus on p-convex functions and Hermite-Hadamard type inequalities.
Some of the studies on p-convex sets and their properties can be seen in [17] -[21]. p-convex functions are shortly introduced in [17] and its main characteristics are given in [22].

Definition 1.1. *[17] Let $U \subseteq \mathbb{R}$ and $0 < p \leq 1$. If for each $x, y \in U$, $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, $\lambda x + \mu y \in U$, then U is called a p-convex set in $\mathbb{R}$.*

It is clear that any interval of real numbers including zero or accepting zero as a boundary point is a p-convex set. Using Theorem 3.2 in [22], we can give the following definition of p-convex function:

Definition 1.2. *Let $U \subseteq \mathbb{R}$ a p-convex set and let $f : U \to \mathbb{R}$ be a function. f is said to be a p-convex function if the following inequality*

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$$

is satisfied for all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$ and for each $x, y \in U$.

Although the definition of p-convexity coincides with the classic convexity for $p = 1$, there are some cases that distinguish it from the classical convexity for $0 < p < 1$; for example, the single point set is convex but this is not p-convex. On the other hand, any open or closed interval is convex, but in order to be p-convex, it must be in the form of any interval of real numbers including zero or accepting zero as a boundary point.

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Let $U \subseteq \mathbb{R}$ be a p-convex set and $k \in \mathbb{R}$. If we define $f, g, h : U \to \mathbb{R}$ such that $f(x) = |x|$, $g(x) = kx$ and $h(x) = kx^2$ then f, g and the derivative of h are p-convex functions.
Hermite-Hadamard inequality is well-known inequality that is given by Hermite, ten years later obtained by Hadamard as follows:
Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This theorem says that the average integral of a convex function interpolates between the image of the average of endpoints and the average of the images of the endpoints. It is obtained for p-convex functions in [23] as follows:

Theorem 1.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable p-convex function. For $a, b \in \mathbb{R}_+$ with $a < b$, the following inequality holds:*

$$2^{\frac{1}{p}-1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)(b-a) \leq \int_a^b f(x)dx \leq \frac{1}{2p}\left\{p\left[bf(b)+af(a)\right]+\left[bf(a)+af(b)\right]B\left(\frac{1}{p},\frac{1}{p}\right)\right\}.$$

In this paper, we obtain some bounds for the difference between the average integral and left expression and for the difference between the average integral and right expression in the inequality (1.1).
Also, let us state the necessary inequalities and formulas to be used throughout the paper. The Beta function is defined as follows:

$$B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1}dt \quad \text{for } \alpha_1, \alpha_2 > 0,$$

and $B(\alpha_1, \alpha_2)$ satisfies the properties below:

$$B(\alpha_1, \alpha_2) = B(\alpha_2, \alpha_1) \text{ and } B(\alpha_1 + 1, \alpha_2) = \frac{\alpha_1}{\alpha_1 + \alpha_2}B(\alpha_1, \alpha_2).$$

2. Main results

2.1. Hermite-Hadamard type inequalities

For the sake of clarity, throughout this section $D[a,b]$ denotes the class of real valued differentiable functions for $a, b \in \mathbb{R}$ with $a < b$.
An upper bound for the right Hermite-Hadamard inequality for p-convex functions will be found by means of the lemma below:

Lemma 2.1. *Let $p \in (0,1]$ and $f \in D[a,b]$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2p(a-b)}\int_0^1 \left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt.$$

Proof. If we apply the partial integration formula and change the variable as $x = t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a$, we get the desired result as follows:

$$\frac{1}{2p(a-b)}\int_0^1 \left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt$$

$$= \frac{1}{2(a-b)}\left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f\left(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a\right)\Big|_0^1 + \frac{1}{p(a-b)}\int_0^1 f\left(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a\right)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt$$

$$= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 2.2. *Let $f \in D[a,b]$ such that $|f'| \in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{3}{2(p+1)(b-a)}\left(|a|+|b|\right)^2\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{2.1}$$

Proof. From Lemma 2.1, triangle inequality and the p-convexity of $|f'|$,

$$\begin{aligned}
\left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{2p(b-a)}\int_0^1\left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|dt\\
&\leq \tfrac{1}{2p(b-a)}\int_0^1\left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|)dt\\
&\leq \tfrac{1}{2p(b-a)}3(|a|+|b|)(|a|+|b|)\left(\int_0^1 t^{\frac{1}{p}}|f'(b)|dt+\int_0^1(1-t)^{\frac{1}{p}}|f'(a)|dt\right)\\
&\leq \tfrac{3}{2(p+1)(b-a)}(|a|+|b|)^2(|f'(a)|+|f'(b)|).
\end{aligned}$$

□

Surely, the sharper versions for the inequality (2.1) and next inequalities to be presented throughout the paper can be obtained. To exemplify, we present only the following two theorems as sharper version for only the theorem above.

Theorem 2.3. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq{}& \frac{1}{12(b-a)}\left[|b|(7|b|+3|a|)\left|f'(b)\right|+|a|(7|a|+3|b|)\left|f'(a)\right|\right]\\
&+\frac{1}{4p(b-a)}(|a|+|b|)\left(\left|af'(b)\right|+\left|bf'(a)\right|\right)B(\frac{1}{p},\frac{1}{p})\\
&+\frac{1}{3p(b-a)}\left[|a|(|a|+3|b|)\left|f'(b)\right|+|b|(|b|+3|a|)\left|f'(a)\right|\right]B(\frac{1}{p},\frac{2}{p}). \qquad (2.2)
\end{aligned}$$

Proof. Let $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$ and $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$, then

$$\begin{aligned}
\left|\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{2p(b-a)}\int_0^1|h(t)g(t)|\left|t^{\frac{1}{p}}f'(b)+(1-t)^{\frac{1}{p}}f'(a)\right|dt\\
&\leq \tfrac{1}{2p(b-a)}\int_0^1|h(t)g(t)|\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt.
\end{aligned}$$

Using triangle inequality, we have

$$\begin{aligned}
|h(t)g(t)| &= \left|(ab+b^2)t^{\frac{1}{p}-1}-(a^2+ab)(1-t)^{\frac{1}{p}-1}-2b^2t^{\frac{2}{p}-1}+2ab(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}-t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}\right|\\
&\leq \left(|ab|+b^2\right)t^{\frac{1}{p}-1}+\left(a^2+|ab|\right)(1-t)^{\frac{1}{p}-1}+2b^2t^{\frac{2}{p}-1}+2|ab|(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}. \qquad (2.3)
\end{aligned}$$

If we multiply (2.3) with $\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)$ then expand and integrate on $[0,1]$ with respect to t, we get

$$\begin{aligned}
\int_0^1|h(t)g(t)|\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt \leq{}& \tfrac{1}{6}p\left[|b|(7|b|+3|a|)|f'(b)|+|a|(7|a|+3|b|)|f'(a)|\right]\\
&+\tfrac{1}{2}(|a|+|b|)(|a||f'(b)|+|b||f'(a)|)B(\tfrac{1}{p},\tfrac{1}{p})\\
&+\tfrac{2}{3}\left[|a|(|a|+3|b|)|f'(b)|+|b|(|b|+3|a|)|f'(a)|\right]B(\tfrac{1}{p},\tfrac{2}{p}).
\end{aligned}$$

When this inequality is used in the first inequality of the proof, (2.2) is obtained. □

Theorem 2.4. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2(p+1)(b-a)}\max\{|g(0)|,|g(1)|,|g(t_1)|\}\max\{|h(0)|,|h(1)|,|h(t_2)|\}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

where $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$ and $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$ as in the proof of the above theorem and for $a\neq 0$,

$$t_1=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-2p}}\right)^{-1} \quad and \quad t_2=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-p}}\right)^{-1}$$

for $a=0$, t_1,t_2 equal to 0 or 1.

Proof. From Lemma 2.1, as in the proof of Theorem 2.2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2p(b-a)}\int_0^1 |h(t)g(t)|\left|t^{\frac{1}{p}}f'(b)+(1-t)^{\frac{1}{p}}f'(a)\right|dt$$

Let $a \neq 0$. In search of extremum points of $g(t)$ and $h(t)$ it is seen that while $\frac{b}{a}<0$ and $\frac{b}{a}>0$ $g(t)$ and $h(t)$ have one extremum point in [0,1], i.e., $g(t)$ and $h(t)$ are unimodal functions on [0,1], respectively. In other cases $g(t)$ and $h(t)$ will be monotone functions. So $g(t)$ and $h(t)$ take extremum values either at the points $t_1=\left(1+\left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$ and $t_2=\left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$ for proper values of a,b, respectively, or at the points $t=0$ or $t=1$ in common.
If we take $\left|\frac{b}{a}\right|$ in the expression of t_1 and t_2, we can express the largest values that can be reached in the [0,1] interval, regardless of the sign of $\frac{b}{a}$ as follows. Thus, $|g(t)|\leq \max\{|g(0)|,|g(1)|,|g(t_1)|\}$ and $|h(t)|\leq \max\{|h(0)|,|h(1)|,|h(t_2)|\}$ is derived. For the case $a=0$, extremum values are obtained for $t=0$, $t=1$, which is included in the inequality above. In a similar way in the proof of Theorem 2.2, by using the p-convexity of $|f'|$, we get the desired result. □

By making use of the Hölder inequality, some kind of extensions of the above theorems can be obtained as in the following theorems.

Theorem 2.5. *Let $s>1$, $f\in D[a,b]$ such that $|f'|^s\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{3}{2p(b-a)}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2\left(\left|f'(a)\right|^s+\left|f'(b)\right|^s\right)^{\frac{1}{s}}.$$

Proof. From Lemma 2.1, triangle and Hölder inequality and the p-convexity of $|f'|^s$,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}\right. &\left.-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2p(b-a)}\int_0^1\left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|dt\\
&=\frac{1}{2p(b-a)}\left(\int_0^1\left(\left|a+b-2t^{\frac{1}{p}}b-2(1-t)^{\frac{1}{p}}a\right)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|\right)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}}\\
&\leq\frac{1}{2p(b-a)}\left(\int_0^1(|a|+|b|+2|b|+2|a|)^{\frac{s}{s-1}}(|b|+|a|)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left(t^{\frac{1}{p}}|f'(b)|^s+(1-t)^{\frac{1}{p}}|f'(a)|^s\right)dt\right)^{\frac{1}{s}}\\
&\leq\frac{3}{2p(b-a)}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2\left(|f'(a)|^s+|f'(b)|^s\right)^{\frac{1}{s}}.\end{aligned}$$

□

Theorem 2.6. *Let $s>1$, $f\in D[a,b]$ such that $|f'|^s\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2(p+1)(b-a)}\max\{|g(0)|,|g(1)|,|g(t_1)|\}\max\{|h(0)|,|h(1)|,|h(t_2)|\}\left(|f'(a)|^s+|f'(b)|^s\right)^{\frac{1}{s}}$$

where $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$, $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$ and for $a\neq 0$, $t_1=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-2p}}\right)^{-1}$, $t_2=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-p}}\right)^{-1}$ for $a=0$, t_1,t_2 equal to 0 or 1.

Proof. By applying the Hölder inequality as in the proof of Theorem 2.5, and then using the findings about the maximum of $h(t)$ and $g(t)$ from the proof of Theorem 2.4, the desired inequality is obtained. □

An upper bound for the left Hermite-Hadamard inequality for p-convex functions will be found using the following lemma.

Lemma 2.7. *Let $p\in(0,1]$ and $f\in D[a,b]$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\\
&=\frac{1}{p(b-a)}\int_0^1\left[t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right]f'(t^{\frac{1}{p}}\frac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right]dt\\
&+\frac{1}{p(b-a)}\int_0^1\left[b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\frac{a+b}{2})\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right]dt.\end{aligned}\tag{2.4}$$

Proof. If we apply partial integration to the integrals on the right side of equality (2.4) and make the necessary variable substitution, we get equality (2.4).

$$\frac{1}{p(b-a)}\int_0^1\left[t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right]f'(t^{\frac{1}{p}}\frac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right]dt$$

$$+\frac{1}{p(b-a)}\int_0^1\left[b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\frac{a+b}{2})\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right]dt$$

$$=\frac{1}{(b-a)}\left[\left(\frac{a+b}{2}-a\right)f(\tfrac{a+b}{2})-\int_a^{\frac{a+b}{2}}f(x)dx\right]+\frac{1}{(b-a)}\left[\left(b-\frac{a+b}{2}\right)f(\tfrac{a+b}{2})-\int_{\frac{a+b}{2}}^b f(x)dx\right]$$

$$=f(\tfrac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 2.8. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|f(\tfrac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{(p+1)(b-a)}\left[\left(\frac{3|a|+|b|}{2}\right)^2|f'(a)|+\frac{5a^2+6|ab|+5b^2}{2}\left|f'(\tfrac{a+b}{2})\right|+\left(\frac{|a|+3|b|}{2}\right)^2|f'(b)|\right].$$

Proof. From Lemma 2.7, triangle inequality and the p-convexity of $|f'|$,

$$\begin{aligned}\left|f(\tfrac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le\frac{1}{p(b-a)}\int_0^1\left|t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right|\left|t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right|\left|f'(t^{\frac{1}{p}}\frac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|dt\\ &+\frac{1}{p(b-a)}\int_0^1\left|b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\frac{a+b}{2})\right|dt\\ &\le\frac{1}{p(b-a)}\int_0^1\left(\frac{|a|+|b|}{2}+|a|\right)^2(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|)dt\\ &+\frac{1}{p(b-a)}\int_0^1\left(\frac{|a|+|b|}{2}+|b|\right)^2(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|)dt\\ &\le\frac{1}{(p+1)(b-a)}\left[\left(\frac{3|a|+|b|}{2}\right)^2|f'(a)|+\frac{5a^2+6|ab|+5b^2}{2}\left|f'(\tfrac{a+b}{2})\right|+\left(\frac{|a|+3|b|}{2}\right)^2|f'(b)|\right].\end{aligned}$$

□

Theorem 2.9. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Let*

$$g_1(t)=t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\ ,\ \ g_2(t)=b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2},$$

$$h_1(t)=t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\ \ \text{and}\ \ h_2(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}.$$

Then,

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{(p+1)(b-a)}\left(w_1|f'(a)|+(w_1+w_2)f'(\frac{a+b}{2})+w_2|f'(b)|\right)$$

where

$$w_1=\max\{|g_1(0)|,|g_1(1)|,|g_1(t_1)|\}\cdot\max\{|h_1(0)|,|h_1(1)|,|h_1(s_1)|\},$$
$$w_2=\max\{|g_2(0)|,|g_2(1)|,|g_2(t_2)|\}\cdot\max\{|h_2(0)|,|h_2(1)|,|h_2(s_2)|\}$$

and for a,b which makes t_1,t_2,s_1,s_2 defined,

$$t_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{p-1}}\right)^{-1},\quad t_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{p-1}}\right)^{-1},\quad s_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},\quad s_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{2p-1}}\right)^{-1}$$

for a,b which makes any of t_1,t_2,s_1,s_2 undefined, that one will be zero or one.

Proof. When their first derivatives of these functions are investigated, it is seen that $g_1(t)$, $g_2(t)$, $h_1(t)$, $h_2(t)$ with respect to values of a,b,p are either monotonic functions or unimodal functions on $[0,1]$, the maximum values of $|g_1(t)|$, $|g_2(t)|$, $|h_1(t)|$, $|h_2(t)|$ are attained at either boundary points of $[0,1]$ or extremum points. The extremum points for these functions with respect to values of a,b making the following values defined are

$$t_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{p-1}}\right)^{-1},\ t_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{p-1}}\right)^{-1},\ s_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},\ s_2=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},$$

respectively. For the values of a and b that makes $\frac{a+b}{2a}$ or $\frac{a+b}{2b}$ negative, these functions will be monotone function on $[0,1]$. Therefore for $i=1,2$

$$|g_i(t)| \leq \max\{|g_i(0)|, |g_i(1)|, |g_i(t_i)|\} \text{ and } |h_i(t)| \leq \max\{|h_i(0)|, |h_i(1)|, |h_i(s_i)|\}.$$

From Lemma 2.7, we have

$$\begin{aligned}
&\left|f(\tfrac{a+b}{2}) - \tfrac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \tfrac{1}{p(b-a)}\int_0^1 |g_1(t)||h_1(t)|\left|f'(t^{\frac{1}{p}}\tfrac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|dt + \tfrac{1}{p(b-a)}\int_0^1 |g_2(t)||h_2(t)|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2})\right|dt \\
&\leq \tfrac{1}{p(b-a)}\int_0^1 \max\{|g_1(0)|,|g_1(1)|,|g_1(t_1)|\}\max\{|h_1(0)|,|h_1(1)|,|h_1(s_1)|\}(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|)dt \\
&\quad +\tfrac{1}{p(b-a)}\int_0^1 \max\{|g_2(0)|,|g_2(1)|,|g_2(t_2)|\}\max\{|h_2(0)|,|h_2(1)|,|h_2(s_2)|\}(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|)dt \\
&\leq \tfrac{1}{(p+1)(b-a)}\left(w_1|f'(a)|+(w_1+w_2)f'(\tfrac{a+b}{2})+w_2|f'(b)|\right) \quad \square
\end{aligned}$$

Theorem 2.10. *Let $f \in D[a,b]$ such that $|f'| \in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\begin{aligned}
\left|f(\tfrac{a+b}{2}) - \tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{5}{12(b-a)}\left(2a^2|f'(a)|+(a+b)^2\left|f'(\tfrac{a+b}{2})\right|+2b^2|f'(b)|\right) \\
&+\tfrac{1}{4p(b-a)}\left(|a|(|a|+|b|)|f'(a)|+2(a^2+b^2)\left|f'(\tfrac{a+b}{2})\right|+|b|(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{1}{p}) \\
&+\tfrac{1}{12p(b-a)}\left((7|a|+|b|)(|a|+|b|)|f'(a)|+2(5a^2+6|a||b|+5b^2)\left|f'(\tfrac{a+b}{2})\right|+(|a|+7|b|)(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{2}{p}).
\end{aligned}$$

Proof. Let $g_1(t), g_2(t), h_1(t)$ and $h_2(t)$ functions as in Theorem 2.9. Using triangle inequality, we can write the followings

$$\begin{aligned}
|g_1(t)h_1(t)| &= \left|\left(t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right)\left(t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right)\right| \\
&= \left|(\frac{a+b}{2})^2 t^{\frac{2}{p}-1}+a(\frac{a+b}{2})(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}-t^{\frac{1}{p}-1}-t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})-a^2((1-t)^{\frac{2}{p}-1}+(1-t)^{\frac{1}{p}-1})\right| \\
&\leq (\frac{a+b}{2})^2 t^{\frac{2}{p}-1}+|a|\left|\frac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+t^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+a^2((1-t)^{\frac{2}{p}-1}+(1-t)^{\frac{1}{p}-1}) \quad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
|g_2(t)h_2(t)| &= \left|\left(b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right)\left(t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right)\right| \\
&= \left|b^2(t^{\frac{2}{p}-1}-t^{\frac{1}{p}-1})+b(\frac{a+b}{2})(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+(1-t)^{\frac{1}{p}-1}-t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})-(\frac{a+b}{2})^2(1-t)^{\frac{2}{p}-1}\right| \\
&\leq b^2(t^{\frac{2}{p}-1}+t^{\frac{1}{p}-1})+|b|\left|\frac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+(\frac{a+b}{2})^2(1-t)^{\frac{2}{p}-1}. \quad (2.6)
\end{aligned}$$

If we multiply (2.5) and (2.6) inequalities with $\left(t^{\frac{1}{p}}\left|f'(\frac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|\right)$ and $\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\frac{a+b}{2})\right|\right)$, respectively and integrate on [0,1], then, use Lemma 2.7 and the p-convexity of $|f'|$, we have the following

$$
\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{p(b-a)}\int_0^1 |g_1(t)h_1(t)|\left(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt \\
&+\tfrac{1}{p(b-a)}\int_0^1 |g_2(t)h_2(t)|\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|\right)dt \\
&\leq \tfrac{1}{p(b-a)}\int_0^1\left[\left|\tfrac{a+b}{2}\right|^2 t^{\frac{2}{p}-1}+|a|\left|\tfrac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+t^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+a^2((1-t)^{\frac{2}{p}-1}+(1-t)^{\frac{1}{p}-1})\right] \\
&\times\left(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt \\
&+\tfrac{1}{p(b-a)}\int_0^1\left[b^2(t^{\frac{2}{p}-1}+t^{\frac{1}{p}-1})+|b|\left|\tfrac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+2(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})\ +\left|\tfrac{a+b}{2}\right|^2(1-t)^{\frac{2}{p}-1}\right] \\
&\times\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|\right)dt \\
&\leq \tfrac{5}{12(b-a)}\left(2a^2|f'(a)|+(a+b)^2\left|f'(\tfrac{a+b}{2})\right|+2b^2|f'(b)|\right) \\
&+\tfrac{1}{4p(b-a)}\left(|a|(|a|+|b|)|f'(a)|+2(a^2+b^2)\left|f'(\tfrac{a+b}{2})\right|+|b|(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{1}{p}) \\
&+\tfrac{1}{12p(b-a)}\left((7|a|+|b|)(|a|+|b|)|f'(a)|+2(5a^2+6|a||b|+5b^2)\left|f'(\tfrac{a+b}{2})\right|+(|a|+7|b|)(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{2}{p}).
\end{aligned}
$$

□

Theorem 2.11. *Let $f\in D[a,b]$ such that $|f'|^s\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$
\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|a|+\left|\tfrac{a+b}{2}\right|\right)\left(|a|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(a)|^s\right)^{\frac{1}{s}} \\
&+\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|b|+\left|\tfrac{a+b}{2}\right|\right)\left(|b|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(b)|^s\right)^{\frac{1}{s}}.
\end{aligned}
$$

Proof. From Lemma 2.7, Hölder inequality, triangle inequality and the p-convexity of $|f'|^s$, we have

$$
\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{p(b-a)}\left(\int_0^1|g_1(t)h_1(t)|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}\tfrac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}} \\
&+\tfrac{1}{p(b-a)}\left(\int_0^1|g_2(t)h_2(t)|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2})\right|^s dt\right)^{\frac{1}{s}} \\
&\leq \tfrac{1}{p(b-a)}\left(\int_0^1\left|\left(t^{\frac{1}{p}}\tfrac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right)\left(t^{\frac{1}{p}-1}\tfrac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right)\right|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}\tfrac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}} \\
&+\tfrac{1}{p(b-a)}\left(\int_0^1\left|\left(b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2}\right)\left(t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\tfrac{a+b}{2}\right)\right|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2})\right|^s dt\right)^{\frac{1}{s}} \\
&\leq \tfrac{1}{p(b-a)}\left(\int_0^1\left(\left|\tfrac{a+b}{2}\right|+2|a|\right)^{\frac{s}{s-1}}\left(\left|\tfrac{a+b}{2}\right|+|a|\right)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|^s+(1-t)^{\frac{1}{p}}|f'(a)|^s\right)dt\right)^{\frac{1}{s}} \\
&+\tfrac{1}{p(b-a)}\left(\int_0^1\left(2|b|+\left|\tfrac{a+b}{2}\right|\right)^{\frac{s}{s-1}}\left(|b|+\left|\tfrac{a+b}{2}\right|\right)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left(t^{\frac{1}{p}}|f'(b)|^s+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|^s\right)dt\right)^{\frac{1}{s}} \\
&\leq \tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|a|+\left|\tfrac{a+b}{2}\right|\right)\left(|a|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(a)|^s\right)^{\frac{1}{s}} \\
&+\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|b|+\left|\tfrac{a+b}{2}\right|\right)\left(|b|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(b)|^s\right)^{\frac{1}{s}}.
\end{aligned}
$$

□

Additionally, we will use the following lemma to obtain a similar result to the right side of the Hermite-Hadamard inequality for p-convex functions given in Theorem 1.3.

Lemma 2.12. *Let $p \in (0,1]$ and $f \in D[a,b]$. If $f' \in L[a,b]$, then the following equality holds:*

$$bf(b) - af(a) - \int_a^b f(x)dx = \frac{1}{p}\int_0^1 \left[b^2 t^{\frac{2}{p}-1} + \frac{a^2}{t-1}(1-t)^{\frac{2}{p}} + abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} + ab\frac{t^{\frac{1}{p}}}{t-1}(1-t)^{\frac{1}{p}}\right] f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)dt.$$

Proof. If we apply the partial integration formula and change the variable as $x = t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a$, then we get the desired equality.

$$\frac{1}{p}\int_0^1 \left[b^2 t^{\frac{2}{p}-1} - a^2(1-t)^{\frac{2}{p}-1} + abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} - abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right] f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)dt$$

$$= \int_0^1 \left(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a\right) f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)\frac{1}{p}(t^{\frac{1}{p}-1}b - (1-t)^{\frac{1}{p}-1}a)dt$$

$$= \left[(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)\right]_0^1 - \int_0^1 f(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)\frac{1}{p}(t^{\frac{1}{p}-1}b - (1-t)^{\frac{1}{p}-1}a)dt$$

$$= bf(b) - af(a) - \int_a^b f(x)dx.$$

□

Theorem 2.13. *Let $f \in D[a,b]$. If $|f'|$ is p-convex on $\mathbb{R}$, then the following inequality holds:*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \le \frac{1}{p+1}(|a|+|b|)^2\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Proof. Using Lemma 2.12 above and convexity of $|f'|$, we have

$$\left|(bf(b) - af(a)) - \int_a^b f(x)dx\right| = \frac{1}{p}\left|\int_0^1 \left[b^2 t^{\frac{2}{p}-1} - a^2(1-t)^{\frac{2}{p}-1} + abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} - abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right] f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)dt\right|$$

$$\le \frac{1}{p}\int_0^1 \left|b^2 t^{\frac{2}{p}-1} - a^2(1-t)^{\frac{2}{p}-1} + abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} - abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right| \left|f'(t^{\frac{1}{p}}b + (1-t)^{\frac{1}{p}}a)\right|dt$$

$$\le \frac{1}{p}\int_0^1 \left(\left|b^2 t^{\frac{2}{p}-1}\right| + \left|a^2(1-t)^{\frac{2}{p}-1}\right| + \left|abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}\right| + \left|abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right|\right)\left(t^{\frac{1}{p}}|f'(b)| + (1-t)^{\frac{1}{p}}|f'(a)|\right)dt$$

$$\le \frac{1}{p}\int_0^1 (|a|+|b|)^2\left(t^{\frac{1}{p}}|f'(b)| + (1-t)^{\frac{1}{p}}|f'(a)|\right)dt$$

$$= \frac{1}{p+1}(|a|+|b|)^2(|f'(a)| + |f'(b)|).$$

□

Theorem 2.14. *Let $f \in D[a,b]$, $s \in (1,\infty)$ such that $\frac{1}{s} < p$ and $|f'|^s \in L[a,b]$. If $|f'|^s$ is p-convex on $\mathbb{R}$, then the following inequality holds:*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \le \frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2\left(|f'(b)|^s + |f'(a)|^s\right)^{\frac{1}{s}}.$$

Proof. From Lemma 2.12, Hölder inequality, triangle inequality and the p-convexity of $|f'|^s$ we can write the following:

$$\left| bf(b)-af(a)-\int_a^b f(x)dx \right| = \frac{1}{p}\left| \int_0^1 \left[b^2 t^{\frac{2}{p}-1} + \frac{a^2}{t-1}(1-t)^{\frac{2}{p}} + abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} + ab\frac{t^{\frac{1}{p}}}{t-1}(1-t)^{\frac{1}{p}} \right] f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)dt \right|$$

$$\leq \frac{1}{p}\left(\int_0^1 \left| b^2 t^{\frac{2}{p}-1} - \frac{a^2}{1-t}(1-t)^{\frac{2}{p}} + abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} - ab\frac{t^{\frac{1}{p}}}{1-t}(1-t)^{\frac{1}{p}} \right|^{\frac{s}{s-1}} dt \right)^{\frac{s-1}{s}} \left(\int_0^1 \left| f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a) \right|^s dt \right)^{\frac{1}{s}}$$

$$\leq \frac{1}{p}\left(\int_0^1 \left[b^2 t^{\frac{2}{p}-1} + a^2(1-t)^{\frac{2}{p}-1} + |ab|t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}} + |ab|t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1} \right]^{\frac{s}{s-1}} dt \right)^{\frac{s-1}{s}} \left(\int_0^1 (t^{\frac{1}{p}}|f'(b)|^s + (1-t)^{\frac{1}{p}}|f'(a)|^s)dt \right)^{\frac{1}{s}}$$

$$\leq \frac{1}{p}\left(\int_0^1 \left[b^2 + 2|ab| + a^2 \right]^{\frac{s}{s-1}} dt \right)^{\frac{s-1}{s}} \left(\int_0^1 (t^{\frac{1}{p}}|f'(b)|^s + (1-t)^{\frac{1}{p}}|f'(a)|^s)dt \right)^{\frac{1}{s}}$$

$$= \frac{1}{p}\left(\frac{p}{p+1} \right)^{\frac{1}{s}} (|a|+|b|)^2 (|f'(b)|^s + |f'(a)|^s)^{\frac{1}{s}}.$$

□

2.2. Applications

By using p-convexity of the function and the derived inequalities, some bounds and inequalities involving Beta functions can be obtained. To do this we use the function $f(x)=x^2$. Let us show that it is p-convex function on any p-convex set of real number:

$$\begin{aligned}
f(t^{\frac{1}{p}}x+(1-t)^{\frac{1}{p}}y) &= (t^{\frac{1}{p}}x+(1-t)^{\frac{1}{p}}y)^2 \\
&= (t^{\frac{1}{p}})^2x^2 + 2t^{\frac{1}{p}}(1-t)^{\frac{1}{p}}xy + ((1-t)^{\frac{1}{p}})^2y^2 \\
&\leq (t^{\frac{1}{p}})^2x^2 + t^{\frac{1}{p}}(1-t)^{\frac{1}{p}}(x^2+y^2) + ((1-t)^{\frac{1}{p}})^2y^2 \\
&= t^{\frac{1}{p}}x^2(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}) + (1-t)^{\frac{1}{p}}y^2(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}) \\
&\leq t^{\frac{1}{p}}x^2(t+1-t) + (1-t)^{\frac{1}{p}}y^2(t+1-t) \\
&= t^{\frac{1}{p}}x^2 + (1-t)^{\frac{1}{p}}y^2 \\
&= t^{\frac{1}{p}}f(x) + (1-t)^{\frac{1}{p}}f(y).
\end{aligned}$$

Moreover by making use of some theorems in Section 2.1, we suggest an upper bound for error in numerical integration of p-convex functions via composite trapezoid rule.
Using Theorem 2.3 and Theorem 2.10, we obtain two similar results involving Beta functions.

Proposition 2.15. *Let $a,b\in\mathbb{R}$ with $a<b$ and $\alpha>1$. Then*

$$4ab(3a^2+2ab+3b^2)\alpha\,\mathrm{B}(\alpha,2\alpha) + 3ab(a+b)^2\alpha\,\mathrm{B}(\alpha,\alpha) + (5ab^3+a^3b+8a^4+6b^4) \geq 0.$$

Proof. Applying Theorem 2.3 for $f(x)=\frac{x^3}{3}$, whose derivative is p-convex function, then making substitution $\alpha=\frac{1}{p}$ with $0<p<1$, we have the desired inequality. □

Proposition 2.16. *Let $a,b\in\mathbb{R}$ and $\alpha>1$. Then*

$$47a^4+16a^3b+30a^2b^2+24ab^3+43b^4 \geq \alpha(a+b)^2\left[6\left(2ab-3(a^2+b^2)\right)\mathrm{B}(\alpha,\alpha) + 2\left(6ab-19(a^2+b^2)\right)\mathrm{B}(\alpha,2\alpha)\right].$$

Proof. Applying the same ideas in proof of Proposition 2.15 to Theorem 2.10 yield to the desired inequality. □

Making some algebraic manipulations in both propostion above, we can get an inequality with respect to one variable. Considerig these propositions for positive numbers a,b with $a<b$, dividing both side by b^4, taking $t=\frac{a}{b}$ $(0<t<1)$, multiplying both side of inequality with $(1-t)^{\alpha}$ $(\alpha>1)$, then integrating both side with respect to t on $[0,1]$, we have the following corollaries corresponding to Proposition 2.15 and Proposition 2.16, respectively:

Corollary 2.17. *Let $\alpha>1$. Then*

$$\begin{aligned}
4\alpha\left(\tfrac{7}{\alpha+1}-\tfrac{14}{\alpha+2}+\tfrac{9}{\alpha+3}-\tfrac{2}{\alpha+4}\right)\mathrm{B}(\alpha,2\alpha) \quad &+3\alpha\left(-\tfrac{4}{\alpha+1}+\tfrac{8}{\alpha+2}-\tfrac{5}{\alpha+3}+\tfrac{1}{\alpha+4}\right)\mathrm{B}(\alpha,\alpha) \\
&\leq \left(\tfrac{20}{\alpha+1}-\tfrac{40}{\alpha+2}+\tfrac{51}{\alpha+3}-\tfrac{33}{\alpha+4}+\tfrac{8}{\alpha+5}\right).
\end{aligned}$$

Corollary 2.18. *For $\alpha>1$,*

$$6\left(-\tfrac{16}{\alpha+1}-\tfrac{32}{\alpha+2}+\tfrac{32}{\alpha+3}+\tfrac{16}{\alpha+4}-\tfrac{3}{\alpha+5}\right)\alpha\,\mathrm{B}(\alpha,\alpha)\quad -2\left(\tfrac{128}{\alpha+1}-\tfrac{256}{\alpha+2}+\tfrac{236}{\alpha+3}-\tfrac{108}{\alpha+4}+\tfrac{19}{\alpha+5}\right)\alpha\,\mathrm{B}(\alpha,2\alpha)$$
$$\leq\left(\tfrac{160}{\alpha+1}-\tfrac{320}{\alpha+2}+\tfrac{360}{\alpha+3}-\tfrac{204}{\alpha+4}+\tfrac{47}{\alpha+5}\right).$$

Using the inequalities obtained via Hölder inequality, we can have the following generalized inequalities with respect to s.

Proposition 2.19. *Let $a,b\in(0,\infty)$ with $a<b$ and $0<p$, $\alpha<1$. Then,*

$$\left|\frac{a^{2\alpha+1}+b^{2\alpha+1}}{2}-\frac{b^{2\alpha+2}-a^{2\alpha+2}}{2(\alpha+1)(b-a)}\right|\leq\frac{3}{2p}\left(\frac{p}{p+1}\right)^{\alpha}\frac{(a+b)^2\left(a^2+b^2\right)^{\alpha}}{(b-a)}.$$

Proof. In Theorem 2.5, let $f(x)=\frac{s}{s+2}x^{\frac{2}{s}+1}$ on $[0,\infty)$ and $a<b$. Then $|f'(x)|^s$ is p-convex. We have

$$\left|\frac{s}{2(s+2)}(a^{\frac{2}{s}+1}+b^{\frac{2}{s}+1})-\frac{1}{b-a}\frac{s}{s+2}\frac{s}{2(s+1)}(b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2})\right|\leq\frac{3}{2p(b-a)}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(a+b)^2\left(a^2+b^2\right)^{\frac{1}{s}}. \tag{2.7}$$

The substitution $\alpha=\frac{1}{s}$ and algebraical manipulations yield to desired inequality. □

Some algebraic manipulations in proposition above yield to the inequality involving a hypergeometric function.

Proposition 2.20. *For $s>1$ and $0<p<1$,*

$$\frac{1}{4}\frac{-9s^3+34s+16s^2+12}{(3s+2)(s+2)(s+1)}\leq\frac{3}{2p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}\frac{1}{s+1}\left(\frac{2}{(3s+2)}\left((s+1)^2\cdot{}_2F_1\left(\frac{1}{2},-\frac{1}{s};\frac{3}{2};-1\right)+2^{\frac{1}{s}}s(4s+3)\right)-s\right)$$

where ${}_2F_1$ is hypergeometric function, i.e.

$${}_2F_1(\alpha,\beta;\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt,\ \ (\gamma>\beta>0).$$

Proof. It is clear that the expression inside the absolute value in (2.7) is less then or equal to right side. Multiplying this inequality with $b-a$, dividing both side by $b^{\frac{2}{s}+2}$, taking $t=\frac{a}{b}$ and integrating both side with respect to t on $[0,1]$, we have desired result. □

Proposition 2.21. *Let $0<p,\alpha<1$ and $a,b\in(0,\infty)$ with $a<b$. Then*

$$\left|\left(\tfrac{a+b}{2}\right)^{2\alpha+1}-\tfrac{1}{1+\alpha}\left(\tfrac{b^{2\alpha+2}-a^{2\alpha+2}}{b-a}\right)\right|$$
$$\leq\tfrac{1}{2^{2\alpha+1}}\tfrac{2\alpha+1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\alpha}\left(\left(2ab+5a^2+b^2\right)^{\alpha}(3a+b)(5a+b)+(a+3b)(a+5b)\left(2ab+a^2+5b^2\right)^{\alpha}\right).$$

Proof. In Theorem 2.11, let $f(x)=\frac{s}{s+2}x^{\frac{2}{s}+1}$ on $[0,\infty)$ and $a<b$. Then $|f'(x)|^s$ is p-convex. We have

$$\left|\tfrac{s}{2(s+2)}\left(\tfrac{a+b}{2}\right)^{\frac{2}{s}+1}-\tfrac{s}{s+2}\tfrac{s}{2(s+1)}\left(\tfrac{b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2}}{b-a}\right)\right|\leq\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2a+\tfrac{a+b}{2}\right)\left(a+\tfrac{a+b}{2}\right)\left(\left(\tfrac{a+b}{2}\right)^2+a^2\right)^{\frac{1}{s}}$$
$$+\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2b+\tfrac{a+b}{2}\right)\left(b+\tfrac{a+b}{2}\right)\left(\left(\tfrac{a+b}{2}\right)^2+b^2\right)^{\frac{1}{s}}.$$

The substitution $\alpha=\frac{1}{s}$ and algebraical manipulations yield to desired inequality. □

Proposition 2.22. *Let $0<\alpha<p<1$ and $a,b\in(0,\infty)$ with $a<b$. Then*

$$\frac{b^{2\alpha+2}-a^{2\alpha+2}}{2(\alpha+1)}\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\alpha}(a+b)^2\left(b^2+a^2\right)^{\alpha}.$$

Proof. In Theorem 2.14, let $f(x)=\frac{s}{s+2}x^{\frac{2}{s}+1}$ on $[0,\infty)$ and $a<b$. Then $|f'(x)|^s$ is p-convex. We have

$$\left|\frac{s}{(s+2)}(b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2})-\frac{s}{s+2}\frac{s}{2(s+1)}(b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2})\right|\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(a+b)^2\left(b^2+a^2\right)^{\frac{1}{s}}.$$

The substitution $\alpha=\frac{1}{s}$ and algebraical manipulations yield to desired inequality. □

When the same idea in the proof of Proposition 2.20 is applied to the inequality in Propositon 2.22, we have the following result involving a hypergeometric function.

Corollary 2.23. *For* $s>1$ *and* $p\in(0,1]$ *with* $p>\frac{1}{s}$,

$$\frac{s+1}{s}\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}\left(2(\frac{s+1}{s})^2{}_2F_1(\frac{1}{2},-\frac{1}{s};\frac{3}{2};-1)+2^{\frac{1}{s}+3}+2(3\,2^{\frac{1}{s}}-1)\frac{1}{s}-3\right).$$

Moreover by making use of some theorems in main results, we can find an upper bound for error in numerical integration of p-convex functions via composite trapezoid rule.
Let f be an integrable function on $[a,b]$ and P be a partition of the interval $[a,b]$, i.e. $P: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ and $\Delta x_i=x_i-x_{i-1}$. Then

$$\int\limits_a^b f(x)dx=\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\Delta x_k+E(f,P) \tag{2.8}$$

where $E(f,P)$ is called the error of integral with respect to P. There are some ways to estimate an upper bound for $E(f,P)$. For p-convex functions, we suggest the following proposition:

Proposition 2.24. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be differentiable function and* $|f'|\in L[a,b]$ *and p-convex function on* $\mathbb{R}$. *Suppose that P is a partition of* $[a,b]$. *Then,*

$$|E(f,P)|\leq\frac{3}{2(p+1)}\sum_{k=0}^{n-1}(|x_k|+|x_{k+1}|)^2\left(\left|f'(x_k)\right|+\left|f'(x_{k+1})\right|\right).$$

Proof. Applying Theorem 2.2 on $[x_k,x_{k+1}]$, we have

$$\left|\frac{f(x_k)+f(x_{k+1})}{2}-\frac{1}{x_{k+1}-x_k}\int\limits_{x_k}^{x_{k+1}}f(x)dx\right|\leq\frac{3}{2(p+1)(x_{k+1}-x_k)}(|x_k|+|x_{k+1}|)^2\left(\left|f'(x_k)\right|+\left|f'(x_{k+1})\right|\right). \tag{2.9}$$

Then using (2.8) and (2.9), we get the desired result as follows:

$$\begin{aligned}|E(f,P)| &=\left|\sum_{k=0}^{n-1}\tfrac{f(x_k)+f(x_{k+1})}{2}\Delta x_k-\int\limits_a^b f(x)dx\right|\\ &=\left|\sum_{k=0}^{n-1}\left(\tfrac{f(x_k)+f(x_{k+1})}{2}\Delta x_k-\int\limits_{x_k}^{x_{k+1}}f(x)dx\right)\right|\\ &\leq\sum_{k=0}^{n-1}\left|\tfrac{f(x_k)+f(x_{k+1})}{2}\Delta x_k-\int\limits_{x_k}^{x_{k+1}}f(x)dx\right|\\ &=\sum_{k=0}^{n-1}\Delta x_k\left|\tfrac{f(x_k)+f(x_{k+1})}{2}-\tfrac{1}{x_{k+1}-x_k}\int\limits_{x_k}^{x_{k+1}}f(x)dx\right|.\end{aligned}$$

□

Proposition 2.25. *Let* $f\in D[a,b]$ *such that* $|f'|^s\in L[a,b]$ *and p-convex function on* $\mathbb{R}$. *Suppose that P is a partition of* $[a,b]$. *Then*

$$|E(f,P)|\leq\frac{3}{2p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}\sum_{k=0}^{n-1}(|x_k|+|x_{k+1}|)^2\left(\left|f'(x_k)\right|^s+\left|f'(x_{k+1})\right|^s\right)^{\frac{1}{s}}.$$

Proof. Applying Theorem 2.5 in a similar way to proof of the proposition. □

3. Conclusion

In this article, some upper boundaries related to Hermite-Hadamard type inequalities for the functions of real numbers whose derivatives are p-convex are obtained and by means of these results some interesting applications are given. Basically, setting three integral equalities containing the derivative of a function, we present new inequalities involving p-convex functions. Then, these are extended to the powers of the derivative of the function via the Hölder inequality. For the applications section, it has been shown that $f(x)=x^2$ is p-convex and through this, the inequalities related to Beta and Hypergeometric functions are obtained. In addition, an upper bound has been obtained for the errors in numerical integration via the composite trapezoid rule of the functions whose derivative and some powers of derivative are p-convex. This study is based on the fact that p-convex functions are defined on real numbers and some applications are obtained via only few examples of functions. In the future, more interesting inequalities regarding special functions can be obtained through different examples of p-convex functions. The introduction of p-convex functions and their properties for n dimensional case are given in [22]. By making use of that study, the existence of similar results can be investigated for multiple integrals.

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Perrin n-Dimensional Relations

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Abstract

This work aims, to perform a complexity in the Perrin sequence, to present the two-dimensional, three-dimensional, and n-dimensional recurrence relations of this sequence. Thus, from the one-dimensional relationship of this sequence, we will discuss the increase of its dimensionality and the insertion of imaginary units in the Perrin sequence, which is a recursive sequence of third order and presents large similarities with the Padovan sequence, differing only its initial values. Moreover, we will present a relationship between the Perrin numbers and the Padovan numbers, which will be used to perform the complexity of this sequence.

1. Introduction

Perrin sequence was discovered by French engineer François Olivier Raoul Perrin (1841-1910), [1] affirms that this sequence was implicitly mentioned by mathematician Édouard Lucas in 1876, but only in 1899, François Perrin defined the Perrin sequence. This is linear sequence of third order is very similar to the Padovan sequence, since their recurrence formula are equal, changing only its initial values, so we will denote the Perrin numbers by $Pe(n)$ and Padovan numbers by $Pd(n)$.
The recurrence of the Perrin sequence is given by $Pe(n) = Pe(n-2) + Pe(n-3)$, $n \geq 3$, being $Pe(0) = 3$, $Pe(1) = 0$ and $Pe(2) = 2$ its initial values, presenting the first ones as: 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17. On the other hand, the Padovan initial values are $Pd(0) = Pd(1) = Pd(2) = 1$, maintaining the same recurrence than the Perrin sequence, we have that the first terms of this sequence is given by: 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12. Thus, from the similarity between these two sequences, [2] presents a relation between the Perrin numbers and the Padovan numbers given by:

$$Pe(n) = 2Pd(n-4) + 3Pd(n-5), n \geq 5.$$

It is noteworthy that this equation is derived from their respective matrix formulas of the Padovan and Perrin numbers [1] and [3]-[5] and will be used in the following sections.
In this sense, in this work, we will present the process of complexity of the Perrin sequence, which is associated with the insertion of the imaginary unit, the dimensional increase and its corresponding algebraic representation. The complexity process of this sequence is based on the work of [2] and [6]-[9]. Thus, the following sections will present the two-dimensional, three-dimensional, and n-dimensional relations of the Perrin sequence.

2. Two-dimensional Perrin relations

Initially, Harman [6] explores the two-dimensional relations or Gaussian numbers, denoted by $(n, m) = n + mi$, where n and m are integers and $i^2 = -1$. Thus, from the one-dimensional Perrin recurrence and based on [2], in this section, we will increase

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the dimensionality of this sequence and insert the imaginary unit i, presenting then the two-dimensional relation of the Perrin sequence.

Definition 2.1. *The numbers described in the form $Pe(n,m)$ will be represented by the numbers of the two-dimensional Perrin sequence, thus satisfying their respective two-dimensional recurrence conditions, where $n,m \in \mathbb{N}$:*

$$\begin{cases} Pe(n+1,m) & = Pe(n-1,m)+Pe(n-2,m) \\ Pe(n,m+1) & = Pe(n,m-1)+Pe(n,m-2) \end{cases}$$

Presenting the initial values defined as: $Pe(0,0)=3, Pe(1,0)=0, Pe(0,1)=3+2i, Pe(0,2)=3+3i, Pe(2,0)=2, Pe(2,1)=2+2i, Pe(1,2)=3i, Pe(2,2)=2+3i, Pe(1,1)=2i$ where $i^2=-1$ and $Pe(0)=3, Pe(1)=0, Pe(2)=2$.

Lemma 2.2. *Given the following properties:*
(i) $Pe(n,0)=Pe(n)$,
(ii) $Pe(0,m)=3Pd(m)+Pe(m+1)i$,
(iii) $Pe(n,1)=Pe(n)+2Pd(n)i$,
(iv) $Pe(1,m)=Pe(m+1)i$.

Proof. According to $Pe(n+1,m)=Pe(n-1,m)+Pe(n-2,m)$, having defined the initial values and applying the second principle of finite induction on n, where it fixes $m=0$ and varies $n=0,1,2,\ldots,k$, we observe:

$$\begin{aligned} Pe(n+1,0) &= Pe(n-1,0)+Pe(n-2,0): \\ Pe(3,0) &= Pe(1,0)+Pe(0,0)=3=Pe(3); \\ Pe(4,0) &= Pe(2,0)+Pe(1,0)=2=Pe(4); \\ Pe(5,0) &= Pe(3,0)+Pe(2,0)=5=Pe(5); \\ &\vdots \\ Pe(k-3,0) &= Pe(k-5,0)+Pe(k-6,0)=Pe(k-3); \\ Pe(k-2,0) &= Pe(k-4,0)+Pe(k-5,0)=Pe(k-2); \\ Pe(k-1,0) &= Pe(k-3,0)+Pe(k-4,0)=Pe(k-1); \\ Pe(k,0) &= Pe(k-2,0)+Pe(k-3,0) \\ &= Pe(k-2)+Pe(k-3)=Pe(k). \end{aligned}$$

Thus, we can verify the property $Pe(n,0)=Pe(n)$. Analogously, one can prove the validity $Pe(0,m)=3Pd(m)+Pe(m+1)i$, considering the relation $Pe(n,m+1)=Pe(n,m-1)+Pe(n,m-2)$ also verifying the relation of the Perrin sequence with the Padovan sequence, and the initial Padovan numbers $Pd(0)=Pd(1)=Pd(2)=1$. By analysing the recursiveness for $n=0$ and varying $m=0,1,2,3,\ldots,k$, we see that:

$$\begin{aligned} Pe(0,m+1) &= Pe(0,m-1)+Pe(0,m-2): \\ Pe(0,3) &= Pe(0,1)+Pe(0,0)=6+2i=3Pd(3)+Pe(4)i; \\ Pe(0,4) &= Pe(0,2)+Pe(0,1)=6+5i=3Pd(4)+Pe(5)i; \\ Pe(0,5) &= Pe(0,3)+Pe(0,2)=9+5i=3Pd(5)+Pe(6)i; \\ &\vdots \\ Pe(0,k-3) &= Pe(0,k-5)+Pe(0,k-6)=3Pd(k-3)+Pe(k-2)i; \\ Pe(0,k-2) &= Pe(0,k-4)+Pe(0,k-5)=3Pd(k-2)+Pe(k-1)i; \\ Pe(0,k-1) &= Pe(0,k-3)+Pe(0,k-4)=3Pd(k-1)+Pe(k)i; \\ Pe(0,k) &= Pe(0,k-2)+Pe(0,k-3) \\ &= 3Pd(k-2)+Pe(k-1)i+3Pd(k-3)+Pe(k-2)i=3Pd(k)+Pe(k+1)i. \end{aligned}$$

Validating the property $Pe(0,m)=3Pd(m)+Pe(m+1)i$. To demonstrate the following property, the same principle of induction is used, with: $Pe(n,m+1)=Pe(n,m-1)+Pe(n,m-2)$ and with the initial values established at the beginning. We also verify a relationship between the Perrin sequence and the Padovan sequence, thus fixing $m=1$ and varying $n=0,1,2,3,\ldots,k$, it follows that:

$$\begin{aligned}
Pe(n+1,1) &= Pe(n-1,1)+Pe(n-2,1):\\
Pe(3,1) &= Pe(1,1)+Pe(0,1)=3+4i=Pe(3)+2Pd(3)i;\\
Pe(4,1) &= Pe(2,1)+Pe(1,1)=2+4i=Pe(4)+2Pd(4)i;\\
Pe(5,1) &= Pe(3,1)+Pe(2,1)=5+6i=Pe(5)+2Pd(5)i;\\
&\vdots\\
Pe(k-3,1) &= Pe(k-5,1)+Pe(k-6,1)=Pe(k-3)+2Pd(k-3)i;\\
Pe(k-2,1) &= Pe(k-4,1)+Pe(k-5,1)=Pe(k-2)+2Pd(k-2)i;\\
Pe(k-1,1) &= Pe(k-3,1)+Pe(k-4,1)=Pe(k-1)+2Pd(k-1)i;\\
Pe(k,1) &= Pe(k-2,1)+Pe(k-3,1)\\
&= Pe(k-2)+2Pd(k-2)i+Pe(k-3)+2Pd(k-3)i=Pe(k)+2Pd(k)i.
\end{aligned}$$

Proving that $Pe(n,1)=Pe(n)+2Pd(n)i$. Concluding the properties demonstrations, we have that, analogously, we can consider the relation $Pe(n,m+1)=Pe(n,m-1)+Pe(n,m-2)$, the values established initially, and fixing $n=1$ and varying $m=0,1,2,3,\dots,k$. We note that:

$$\begin{aligned}
Pe(1,m+1) &= Pe(1,m-1)+Pe(1,m-2):\\
Pe(1,3) &= Pe(1,1)+Pe(1,0)=2i=Pe(4)i;\\
Pe(1,4) &= Pe(1,2)+Pe(1,1)=5i=Pe(5)i;\\
Pe(1,5) &= Pe(1,3)+Pe(1,2)=5i=Pe(6)i;\\
&\vdots\\
Pe(1,k-3) &= Pe(1,k-5)+Pe(1,k-6)=Pe(k-2)i;\\
Pe(1,k-2) &= Pe(1,k-4)+Pe(1,k-5)=Pe(k-1)i;\\
Pe(1,k-1) &= Pe(1,k-3)+Pe(1,k-6)=Pe(k)i;\\
Pe(1,k) &= Pe(1,k-2)+Pe(1,k-3)\\
&= Pe(k-1)i+Pe(k-2)i=Pe(k+1)i.
\end{aligned}$$

□

Theorem 2.3. *For the two integers, $n,m\in\mathbb{N}$, the numbers in the form $Pe(n,m)$ are described by:*

$$Pe(n,m)=Pe(n)Pd(m)+Pe(m+1)Pd(n)i.$$

Proof. Fixing the value of natural number n, we can carry out the demonstration by induction on m. For $m=0$, there is the property $Pe(n,0)=Pe(n)$ and $Pe(n,1)=Pe(n)+2Pd(n)i$, previously validated by Lemma 2.2, where $Pe(0)=3,Pe(1)=0,Pe(2)=2$, whose initial values were defined previously. For this, some values of $Pe(n,m)$ will be calculated, varying m. For $Pe(n,2)$, we use the recurrence $Pe(n,m+1)=Pe(n,m-1)+Pe(n,m-2)$ with the initial values established, with $m=2$ fixed and $n=0,1,2,3,\dots,k$, we have that:

$$\begin{aligned}
Pe(n+1,2) &= Pe(n-1,2)+Pe(n-2,2):\\
Pe(3,2) &= Pe(1,2)+Pe(0,2)=3+6i=Pe(3)+3Pd(3)i;\\
Pe(4,2) &= Pe(2,2)+Pe(1,2)=2+6i=Pe(4)+3Pd(4)i;\\
Pe(5,2) &= Pe(3,2)+Pe(2,2)=5+9i=Pe(5)+3Pd(5)i;\\
&\vdots\\
Pe(k-3,2) &= Pe(k-5,2)+Pe(k-6,2)=Pe(k-3)+3Pd(k-3)i;\\
Pe(k-2,2) &= Pe(k-4,2)+Pe(k-5,2)=Pe(k-2)+3Pd(k-2)i;\\
Pe(k-1,2) &= Pe(k-3,2)+Pe(k-4,2)=Pe(k-1)+3Pd(k-1)i;\\
Pe(k,2) &= Pe(k-2,2)+Pe(k-3,2)\\
&= Pe(k-2)+3Pd(k-2)i+Pe(k-3)+3Pd(k-3)i=Pe(k)+3Pd(k)i.
\end{aligned}$$

Rewriting the properties seen in the previous Lemma (i) and (ii), we have that:

$$\begin{aligned}
Pe(n,0) &= Pe(n)+Pe(1)Pd(n)i;\\
Pe(n,1) &= Pe(n)+Pe(2)Pd(n)i;
\end{aligned}$$

However, assuming that for $m = 1, 2, \ldots, k$, the following identities are valid:

$$\begin{aligned}
Pe(n,0) &= Pe(n)Pd(0) + Pe(1)Pd(n)i;\\
Pe(n,1) &= Pe(n)Pd(1) + Pe(2)Pd(n)i;\\
Pe(n,2) &= Pe(n)Pd(2) + Pe(3)Pd(n)i;\\
&\vdots\\
Pe(n,k-3) &= Pe(n)Pd(k-3) + Pe(k-2)Pd(n)i;\\
Pe(n,k-2) &= Pe(n)Pd(k-2) + Pe(k-1)Pd(n)i;\\
Pe(n,k-1) &= Pe(n)Pd(k-1) + Pe(k)Pd(n)i;
\end{aligned}$$

$$\begin{aligned}
Pe(n,k) &= Pe(n,k-2) + Pe(n,k-3)\\
&= Pe(n)Pd(k-2) + Pe(k-1)Pd(n)i + Pe(n)Pd(k-3) + Pe(k-2)Pd(n)i\\
&= Pe(n)Pd(k) + Pe(k+1)Pd(n)i
\end{aligned}$$

Demonstrating for $m = k+1$, from the recurrence $Pe(n,k+1) = Pe(n,k-1) + Pe(n,k-2)$, we have that:

$$\begin{aligned}
Pe(n,k+1) &= Pe(n,k-1) + Pe(n,k-2)\\
&= Pe(n)Pd(k-1) + Pe(k)Pd(n)i + Pe(n)Pd(k-2) + Pe(k-1)Pd(n)i\\
&= Pe(n)Pd(k+1) + Pe(k+2)Pd(n)i
\end{aligned}$$

□

3. The three-dimensional Perrin relations

In this section, the three-dimensional Perrin relations will be presented, denoted by $Pe(n,m,p)$, from its one-dimensional and two-dimensional recurrence. For this, we will increase the dimensionality of this sequence and insert the imaginary unit i and j, where $i^2 = j^2 = -1$.

Definition 3.1. *The Perrin numbers, we can consider the initial values, defined as:* $Pe(0,0,0) = 3 = Pe(0), Pe(1,0,0) = 0 = Pe(1), Pe(2,0,0) = 2 = Pe(2), Pe(0,1,0) = 3+2i, Pe(0,2,0) = 3+3i, Pe(0,0,1) = 3+2j, Pe(0,0,2) = 3+3j, Pe(1,0,1) = 2j, Pe(1,0,2) = 3j, Pe(1,1,0) = 2i, Pe(1,2,0) = 3i, Pe(0,1,1) = 3+2i+2j, Pe(0,1,2) = 3+2i+3j, Pe(0,2,1) = 3+3i+2j, Pe(0,2,2) = 3+3i+3j, Pe(2,1,1) = 2+2i+2j, Pe(2,2,1) = 2+3i+2j, Pe(2,0,1) = 2+2j, Pe(2,0,2) = 2+3j, Pe(2,1,0) = 2+2i, Pe(2,2,0) = 2+3i, Pe(1,1,1) = 2i+2j, Pe(1,2,1) = 3i+2j, Pe(1,1,2) = 2i+3j$ *in which* $i^2 = j^2 = -1$*, forming the numbers as* $Pe(n,m,p)$ *satisfying the following three-dimensional recurrence conditions, where* $n,m,p \geqslant 0$*:*

$$\begin{cases}
Pe(n,m,p) = Pe(n-2,m,p) + Pe(n-3,m,p)\\
Pe(n,m,p) = Pe(n,m-2,p) + Pe(n,m-3,p)\\
Pe(n,m,p) = Pe(n,m,p-2) + Pe(n,m,p-3)
\end{cases}$$

Lemma 3.2. *The following properties are valid for Perrin numbers:*
(a) $Pe(n,0,0) = Pe(n)$*,*
(b) $Pe(n,1,0) = Pe(n) + 2Pd(n)i$*,*
(c) $Pe(n,0,1) = Pe(n) + 2Pd(n)j$*,*
(d) $Pe(n,1,1) = Pe(n) + 2Pd(n)i + 2Pd(n)j$*.*

Proof. To demonstrate property (a) $Pe(n,0,0) = Pe(n)$, we will consider the relation $Pe(n,m,p) = Pe(n-2,m,p) + Pe(n-3,m,p)$ and the initial values first defined. Thus, for $m = p = 0$ and varying $n = (0,1,2,3,\ldots,k)$. We can see that:

$$\begin{aligned}
Pe(n,0,0) &= Pe(n-2,0,0) + Pe(n-3,0,0):\\
Pe(3,0,0) &= Pe(1,0,0) + Pe(0,0,0) = 3 = Pe(3);\\
Pe(4,0,0) &= Pe(2,0,0) + Pe(1,0,0) = 2 = Pe(4);\\
Pe(5,0,0) &= Pe(3,0,0) + Pe(2,0,0) = 5 = Pe(5);\\
&\vdots\\
Pe(k-3,0,0) &= Pe(k-5,0,0) + Pe(k-6,0,0) = Pe(k-3);\\
Pe(k-2,0,0) &= Pe(k-4,0,0) + Pe(k-5,0,0) = Pe(k-2);\\
Pe(k-1,0,0) &= Pe(k-3,0,0) + Pe(k-4,0,0) = Pe(k-1);\\
Pe(k,0,0) &= Pe(k-2,0,0) + Pe(k-3,0,0)\\
&= Pe(k-2) + Pe(k-3) = Pe(k).
\end{aligned}$$

Now, to validate property (b) $Pe(n,1,0) = Pe(n) + 2Pd(n)i$, we use the relation $Pe(n,m,p) = Pe(n-2,m,p) + Pe(n-3,m,p)$. Therefore, the recursiveness for $m = p = 1$, where $n = (0,1,2,3,\dots,k)$, is such that:

$$\begin{aligned}
Pe(n,1,0) &= Pe(n-2,1,0) + Pe(n-3,1,0):\\
Pe(3,1,0) &= Pe(1,1,0) + Pe(0,1,0) = 3+4i = Pe(3) + 2Pd(3)i;\\
Pe(4,1,0) &= Pe(2,1,0) + Pe(1,1,0) = 2+4i = Pe(4) + 2Pd(4)i;\\
Pe(5,1,0) &= Pe(3,1,0) + Pe(2,1,0) = 5+6i = Pe(5) + 2Pd(5)i;\\
&\vdots\\
Pe(k-3,1,0) &= Pe(k-5,1,0) + Pe(k-6,1,0) = Pe(k-3) + 2Pd(k-3)i;\\
Pe(k-2,1,0) &= Pe(k-4,1,0) + Pe(k-5,1,0) = Pe(k-2) + 2Pd(k-2)i;\\
Pe(k-1,1,0) &= Pe(k-3,1,0) + Pe(k-4,1,0) = Pe(k-1) + 2Pd(k-1)i;\\
Pe(k,1,0) &= Pe(k-2,1,0) + Pe(k-3,1,0)\\
&= Pe(k-2) + 2Pd(k-2)i + Pe(k-3) + 2Pd(k-3)i = Pe(k) + 2Pd(k)i.
\end{aligned}$$

To demonstrate the third property (c) $Pe(n,0,1) = Pe(n) + 2Pd(n)j$ with $Pe(n,m,p) = Pe(n-2,m,p) + Pe(n-3,m,p)$ we evaluate the recurrence for $m = 0$ and $p = 1$, where $n = (0,1,2,3,\dots,k)$. Thus:

$$\begin{aligned}
Pe(n,0,1) &= Pe(n-2,0,1) + Pe(n-3,0,1):\\
Pe(3,0,1) &= Pe(1,0,1) + Pe(0,0,1) = 3+4j = Pe(3) + 2Pd(3)j;\\
Pe(4,0,1) &= Pe(2,0,1) + Pe(1,0,1) = 2+4j = Pe(4) + 2Pd(4)j;\\
Pe(5,0,1) &= Pe(3,0,1) + Pe(2,0,1) = 5+6j = Pe(5) + 2Pd(5)j;\\
&\vdots\\
Pe(k-3,0,1) &= Pe(k-5,0,1) + Pe(k-6,0,1) = Pe(k-3) + 2Pd(k-3)j;\\
Pe(k-2,0,1) &= Pe(k-4,0,1) + Pe(k-5,0,1) = Pe(k-2) + 2Pd(k-2)j;\\
Pe(k-1,0,1) &= Pe(k-3,0,1) + Pe(k-4,0,1) = Pe(k-1) + 2Pd(k-1)j;\\
Pe(k,0,1) &= Pe(k-2,0,1) + Pe(k-3,0,1)\\
&= Pe(k-2) + 2Pd(k-2)j + Pe(k-3) + 2Pd(k-3)j\\
&= Pe(k) + 2Pd(k)j.
\end{aligned}$$

Finally, by induction, we can demonstrate the property (d) $Pe(n,1,1) = Pe(n) + 2Pd(n)i + 2Pd(n)j$ through the recurrence $Pe(n,m,p) = Pe(n-2,m,p) + Pe(n-3,m,p)$ for $m = 1$ and $p = 1$, with variation $n = (0,1,2,3,\dots,k)$. With this, we can see that:

$$\begin{aligned}
Pe(n,1,1) &= Pe(n-2,1,1) + Pe(n-3,1,1):\\
Pe(3,1,1) &= Pe(1,1,1) + Pe(0,1,1) = 3+4i+4j = Pe(3) + 2Pd(3)i + 2Pd(3)j;\\
Pe(4,1,1) &= Pe(2,1,1) + Pe(1,1,1) = 2+4i+4j = Pe(4) + 2Pd(4)i + 2Pd(4)j;\\
Pe(5,1,1) &= Pe(3,1,1) + Pe(2,1,1) = 5+6i+6j = Pe(5) + 2Pd(5)i + 2Pd(5)j;\\
&\vdots\\
Pe(k-3,1,1) &= Pe(k-5,1,1) + Pe(k-6,1,1) = Pe(k-3) + 2Pd(k-3)i + 2Pd(k-3)j;\\
Pe(k-2,1,1) &= Pe(k-4,1,1) + Pe(k-5,1,1) = Pe(k-2) + 2Pd(k-2)i + 2Pd(k-2)j;\\
Pe(k-1,1,1) &= Pe(k-3,1,1) + Pe(k-4,1,1) = Pe(k-1) + 2Pd(k-1)i + 2Pd(k-1)j;\\
Pe(k,1,1) &= Pe(k-2,1,1) + Pe(k-3,1,1)\\
&= Pe(k-2) + 2Pd(k-2)i + 2Pd(k-2)j + Pe(k-3) + 2Pd(k-3)i + 2Pd(k-3)j\\
&= Pe(k) + 2Pd(k)i + 2Pd(k)j.
\end{aligned}$$

Thus, the properties described above are verified. □

Lemma 3.3. *Given the following properties:*
(a) $Pe(0,m,0) = 3Pd(m) + Pe(m+1)i$,
(b) $Pe(0,m,1) = 3Pd(m) + Pe(m+1)i + 2Pd(m)j$,
(c) $Pe(1,m,0) = Pe(m+1)i$,
(d) $Pe(1,m,1) = Pe(m+1)i + 2Pd(m)j$.

Proof. (a) Given the recurrence, the second principle of induction on m at $n = p = 0$ applies. Thus, varying $m = 1,2,3,\dots,k$, we see that:

$$\begin{aligned}
Pe(0,m,0) &= Pe(0,m-2,0)+Pe(0,m-3,0): \\
Pe(0,3,0) &= Pe(0,1,0)+Pe(0,0,0) = 6+2i = 3Pd(3)+Pe(4)i; \\
Pe(0,4,0) &= Pe(0,2,0)+Pe(0,1,0) = 6+5i = 3Pd(4)+Pe(5)i; \\
Pe(0,5,0) &= Pe(0,3,0)+Pe(0,2,0) = 9+5i = 3Pd(5)+Pe(6)i; \\
&\vdots \\
Pe(0,k-3,0) &= Pe(0,k-5,0)+Pe(0,k-6,0) = 3Pd(k-3)+Pe(k-2)i; \\
Pe(0,k-2,0) &= Pe(0,k-4,0)+Pe(0,k-5,0) = 3Pd(k-2)+Pe(k-1)i; \\
Pe(0,k-1,0) &= Pe(0,k-3,0)+Pe(0,k-4,0) = 3Pd(k-1)+Pe(k)i; \\
Pe(0,k,0) &= Pe(0,k-2,0)+Pe(0,k-3,0) \\
&= 3Pd(k-2)+Pe(k-1)i+3Pd(k-3)+Pe(k-2)i \\
&= 3Pd(k)+Pe(k+1)i.
\end{aligned}$$

Validating the property (a) $Pe(0,m,0) = 3Pd(m)+Pe(m+1)i$.

(b) For the demonstration of item (b), the same principle on m at $n = 0$ and $p = 1$ follows. Hence, $m = k$:

$$\begin{aligned}
Pe(0,m,1) &= Pe(0,m-2,1)+Pe(0,m-3,1): \\
Pe(0,3,1) &= Pe(0,1,1)+Pe(0,0,1) = 6+2i+4j = 3Pd(3)+Pe(4)i+2Pd(3)j; \\
Pe(0,4,1) &= Pe(0,2,1)+Pe(0,1,1) = 6+5i+4j = 3Pd(4)+Pe(5)i+2Pd(4)j; \\
Pe(0,5,1) &= Pe(0,3,1)+Pe(0,2,1) = 9+5i+6j = 3Pd(5)+Pe(6)i+2Pd(5)j; \\
&\vdots \\
Pe(0,k-3,1) &= Pe(0,k-5,1)+Pe(0,k-6,1) = 3Pd(k-3)+Pe(k-2)i+2Pd(k-3)j; \\
Pe(0,k-2,1) &= Pe(0,k-4,1)+Pe(0,k-5,1) = 3Pd(k-2)+Pe(k-1)i+2Pd(k-2)j; \\
Pe(0,k-1,1) &= Pe(0,k-3,1)+Pe(0,k-4,1) = 3PdF(k-1)+Pe(k)i+2Pd(k-1)j; \\
Pe(0,k,1) &= Pe(0,k-2,1)+Pe(0,k-3,1) \\
&= 3Pd(k-2)+Pe(k-1)i+2Pd(k-2)j+3Pd(k-3)+Pe(k-2)i+2Pd(k-3)j \\
&= 3Pd(k)+Pe(k+1)i+2Pd(k)j.
\end{aligned}$$

Validating the property (b) $Pe(0,m,1) = 3Pd(m)+Pe(m+1)i+2Pd(m)j$.

(c) Following the same principle on m at $n = 1$ and $p = 0$. Therefore, for $m = k$, we have that:

$$\begin{aligned}
Pe(1,m,0) &= Pe(1,m-2,0)+Pe(1,m-3,0): \\
Pe(1,3,0) &= Pe(1,1,0)+Pe(1,0,0) = 2i = Pe(4)i; \\
Pe(1,4,0) &= Pe(1,2,0)+Pe(1,1,0) = 5i = Pe(5)i; \\
Pe(1,5,0) &= Pe(1,3,0)+Pe(1,2,0) = 5i = Pe(6)i; \\
&\vdots \\
Pe(1,k-3,0) &= Pe(1,k-5,0)+Pe(1,k-6,0) = Pe(k-2)i; \\
Pe(1,k-2,0) &= Pe(1,k-4,0)+Pe(1,k-5,0) = Pe(k-1)i; \\
Pe(1,k-1,0) &= Pe(1,k-3,0)+Pe(1,k-4,0) = Pe(k)i; \\
Pe(1,k,0) &= Pe(1,k-2,0)+Pe(1,k-3,0) \\
&= Pe(k-1)i+Pe(k-2)i = Pe(k+1)i.
\end{aligned}$$

Demonstrating the property (c) $Pe(1,m,0) = Pe(m+1)i$.

(d) Analogously, the same principle on m at $n = p = 1$ follows. Therefore, for $m = k$, we have that:

$$\begin{aligned}
Pe(1,m,1) &= Pe(1,m-2,1)+Pe(1,m-3,1):\\
Pe(1,3,1) &= Pe(1,1,1)+Pe(1,0,1) = 2i+4j = Pe(4)i+2Pd(3)j;\\
Pe(1,4,1) &= Pe(1,2,1)+Pe(1,1,1) = 5i+4j = Pe(5)i+2Pd(4)j;\\
Pe(1,5,1) &= Pe(1,3,1)+Pe(1,2,1) = 5i+6j = Pe(6)i+2Pd(5)j;\\
&\vdots\\
Pe(1,k-3,1) &= Pe(1,k-5,1)+Pe(1,k-6,1) = Pe(k-2)i+2Pd(k-3)j;\\
Pe(1,k-2,1) &= Pe(1,k-4,1)+Pe(1,k-5,1) = Pe(k-1)i+2Pd(k-2)j;\\
Pe(1,k-1,1) &= Pe(1,k-3,1)+Pe(1,k-4,1) = Pe(k)i+2Pd(k-1)j;\\
Pe(1,k,1) &= Pe(1,k-2,1)+Pe(1,k-3,1)\\
&= Pe(k-1)i+2Pd(k-2)j+Pe(k-2)i+2Pd(k-3)j\\
&= Pe(k+1)i+2Pd(k)j.
\end{aligned}$$

Therefore, property (d) $Pe(1,m,1) = Pe(m+1)i+2Pd(m)j$. $\square$

Lemma 3.4. *The following identities are valid:*
(a) $Pe(0,0,p) = 3Pd(p)+Pe(p+1)j$,
(b) $Pe(0,1,p) = 3Pd(p)+2Pd(p)i+Pe(p+1)j$,
(c) $Pe(1,0,p) = Pe(p+1)j$,
(d) $Pe(1,1,p) = 2Pd(p)i+Pe(p+1)j$.

Proof. (a) By applying the second principle of induction on p for $n = m = 0$ and varying $p = 1,2,3,\ldots,k$, we have that:

$$\begin{aligned}
Pe(0,0,p) &= Pe(0,0,p-2)+Pe(0,0,p-3):\\
Pe(0,0,3) &= Pe(0,0,1)+Pe(0,0,0) = 6+2j = 3Pd(3)+Pe(4)j;\\
Pe(0,0,4) &= Pe(0,0,2)+Pe(0,0,1) = 6+5j = 3Pd(4)+Pe(5)j;\\
Pe(0,0,5) &= Pe(0,0,3)+Pe(0,0,2) = 9+5j = 3Pd(5)+Pe(6)j;\\
&\vdots\\
Pe(0,0,k-3) &= Pe(0,0,k-5)+Pe(0,0,k-6) = 3Pd(k-3)+Pe(k-2)j;\\
Pe(0,0,k-2) &= Pe(0,0,k-4)+Pe(0,0,k-5) = 3Pd(k-2)+Pe(k-1)j;\\
Pe(0,0,k-1) &= Pe(0,0,k-3)+Pe(0,0,k-4) = 3Pd(k-1)+Pe(k)j;\\
Pe(0,0,k) &= Pe(0,0,k-2)+Pe(0,0,k-3)\\
&= 3Pd(k-2)+Pe(k-1)j+3Pd(k-3)+Pe(k-2)j\\
&= 3Pd(k)+Pe(k+1)j.
\end{aligned}$$

The property (a) $Pe(0,0,p) = 3Pd(p)+Pe(p+1)j$ is, thus, validated.
(b) By using the second principle of induction on p for $n = 0$ and $m = 1$ and varying $p = 1,2,3,\ldots,k$, we have that:

$$\begin{aligned}
Pe(0,1,p) &= Pe(0,1,p-2)+Pe(0,1,p-3):\\
Pe(0,1,3) &= Pe(0,1,1)+Pe(0,1,0) = 6+4i+2j = 3Pd(3)+2Pd(3)i+Pe(4)j;\\
Pe(0,1,4) &= Pe(0,1,2)+Pe(0,1,1) = 6+4i+5j = 3Pd(4)+2Pd(4)i+Pe(5)j;\\
Pe(0,1,5) &= Pe(0,1,3)+Pe(0,1,2) = 9+6i+5j = 3Pd(5)+2Pd(5)i+Pe(6)j;\\
&\vdots\\
Pe(0,1,k-3) &= Pe(0,1,k-5)+Pe(0,1,k-6) = 3Pd(k-3)+2Pd(k-3)i+Pe(k-2)j;\\
Pe(0,1,k-2) &= Pe(0,1,k-4)+Pe(0,1,k-5) = 3Pd(k-2)+2Pd(k-2)i+Pe(k-1)j;\\
Pe(0,1,k-1) &= Pe(0,1,k-3)+Pe(0,1,k-4) = 3Pd(k-1)+2Pd(k-1)i+Pe(k)j;\\
Pe(0,1,k) &= Pe(0,1,k-2)+Pe(0,1,k-3)\\
&= 3Pd(k-2)+2Pd(k-2)i+Pe(k-1)j+3Pd(k-3)+2Pd(k-3)i+Pe(k-2)j\\
&= 3Pd(k)+2Pd(k)i+Pe(k+1)j.
\end{aligned}$$

Thus, the property (b) $Pe(0,1,p) = 3Pd(p)+2Pd(p)i+Pe(p+1)j$ is validated.

(c)Through the second principle of induction on p for $n=1$ and $m=0$ and varying $p=1,2,3,\ldots,k$, we have that:

$$\begin{aligned}
Pe(1,0,p) &= Pe(1,0,p-2)+Pe(1,0,p-3):\\
Pe(1,0,3) &= Pe(1,0,1)+Pe(1,0,0)=2j=Pe(4)j;\\
Pe(1,0,4) &= Pe(1,0,2)+Pe(1,0,1)=5j=Pe(5)j;\\
Pe(1,0,5) &= Pe(1,0,3)+Pe(1,0,2)=5j=Pe(6)j;\\
&\vdots\\
Pe(1,0,k-3) &= Pe(1,0,k-5)+Pe(1,0,k-6)=Pe(k-2)j;\\
Pe(1,0,k-2) &= Pe(1,0,k-4)+Pe(1,0,k-5)=Pe(k-1)j;\\
Pe(1,0,k-1) &= Pe(1,0,k-3)+Pe(1,0,k-4)=Pe(k)j;\\
Pe(1,0,k) &= Pe(1,0,k-2)+Pe(1,0,k-3)\\
&= Pe(k-1)j+Pe(k-2)j=Pe(k+1)j.
\end{aligned}$$

Validating property (c) $Pe(1,0,p)=Pe(p+1)j$.

(d) According to the second principle of induction on p for $n=m=1$ and varying $p=1,2,3,\ldots,k$, we have that:

$$\begin{aligned}
Pe(1,1,p) &= Pe(1,1,p-2)+Pe(1,1,p-3):\\
Pe(1,1,3) &= Pe(1,1,1)+Pe(1,1,0)=4i+2j=2Pd(3)i+Pe(4)j;\\
Pe(1,1,4) &= Pe(1,1,2)+Pe(1,1,1)=4i+5j=2Pd(4)i+Pe(5)j;\\
Pe(1,1,5) &= Pe(1,1,3)+Pe(1,1,2)=6i+5j=2Pd(5)i+Pe(6)j;\\
&\vdots\\
Pe(1,1,k-3) &= Pe(1,1,k-5)+Pe(1,1,k-6)=2Pd(k-3)i+Pe(k-2)j;\\
Pe(1,1,k-2) &= Pe(1,1,k-4)+Pe(1,1,k-5)=2Pd(k-2)i+Pe(k-1)j;\\
Pe(1,1,k-1) &= Pe(1,1,k-3)+Pe(1,1,k-4)=2Pd(k-1)i+Pe(k)j;\\
Pe(1,1,k) &= Pe(1,1,k-2)+Pe(1,1,k-3)\\
&= 2Pd(k-2)i+Pe(k-1)j+2Pd(k-3)i+Pe(k-2)j\\
&= 2Pd(k)i+Pe(k+1)j.
\end{aligned}$$

Demonstrating property (d) $Pe(1,1,p)=2Pd(p)i+Pe(p+1)j$. □

Theorem 3.5. *For the three integers, $n,m,p\in\mathbb{N}$, the numbers in the form $Pe(n,m,p)$ are described by:*

$$Pe(n,m,p) = Pe(n)Pd(m)Pd(p)+Pd(n)Pe(m+1)Pd(p)i+Pd(n)Pd(m)Pe(p+1)j.$$

Proof. Hence, for $p=0$ and $m=2$, we have that:

$$\begin{aligned}
Pe(3,2,0) &= Pe(1,2,0)+Pe(0,2,0)=3+6i=Pe(3)+3Pd(3)i;\\
Pe(4,2,0) &= Pe(2,2,0)+Pe(1,2,0)=2+6i=Pe(4)+3Pd(4)i;\\
Pe(5,2,0) &= Pe(3,2,0)+Pe(2,2,0)=5+9i=Pe(5)+3Pd(5)i;\\
&\vdots\\
Pe(n-3,2,0) &= Pe(n-5,2,0)+Pe(n-6,2,0)=Pe(n-3)+3Pd(n-3)i;\\
Pe(n-2,2,0) &= Pe(n-4,2,0)+Pe(n-5,2,0)=Pe(n-2)+3Pd(n-2)i;\\
Pe(n-1,2,0) &= Pe(n-3,2,0)+Pe(n-4,2,0)=Pe(n-1)+3Pd(n-1)i;\\
Pe(n,2,0) &= Pe(n-2,2,0)+Pe(n-3,2,0)\\
&= Pe(n-2)+3Pd(n-2)i+Pe(n-3)+3Pd(n-3)i\\
&= Pe(n)+3Pd(n)i.
\end{aligned}$$

In addition, are stimulated other properties inherent to this process, for $m = 1,2,3,\ldots,k$, and we obtain:

$$\begin{aligned}
Pe(n,0,0) &= Pe(n)+Pe(1)Pd(n)i;\\
Pe(n,1,0) &= Pe(n)+Pe(2)Pd(n)i;\\
Pe(n,2,0) &= Pe(n)+Pe(3)Pd(n)i;\\
Pe(n,3,0) &= Pe(n,1,0)+Pe(n,0,0) = 2Pe(n)+Pe(4)Pd(n)i\\
&= Pd(3)Pe(n)+Pe(4)Pd(n)i;\\
&\vdots\\
Pe(n,k-3,0) &= Pe(n,k-5,0)+Pe(n,k-6,0) = Pd(k-3)Pe(n)+Pe(k-2)Pd(n)i;\\
Pe(n,k-2,0) &= Pe(n,k-4,0)+Pe(n,k-5,0) = Pd(k-2)Pe(n)+Pe(k-1)Pd(n)i;\\
Pe(n,k-1,0) &= Pe(n,k-3,0)+Pe(n,k-4,0) = Pd(k-1)Pe(n)+Pe(k)Pd(n)i;\\
Pe(n,k,0) &= Pe(n,k-2,0)+Pe(n,k-3,0)\\
&= Pd(k-2)Pe(n)+Pe(k-1)Pd(n)i+Pd(k-3)Pe(n)+Pe(k-2)Pd(n)i\\
&= Pd(k)Pe(n)+Pe(k+1)Pd(n)i.
\end{aligned}$$

With this, the veracity of Theorem 3.5, is proven, through its application to $m = 1,2,3,\ldots,k$, in the situation presented below:

$$\begin{aligned}
Pe(n,m,0) &= Pd(m)Pe(n)+Pe(m+1)Pd(n)i;\\
Pe(n,m,1) &= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+2Pd(n)j;\\
Pe(n,m,2) &= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+3Pd(n)j;\\
&\vdots\\
Pe(n,m,k-3) &= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+Pe(k-2)Pd(n)j;\\
Pe(n,m,k-2) &= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+Pe(k-1)Pd(n)j;\\
Pe(n,m,k-1) &= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+Pe(k)Pd(n)j;\\
&;\\
Pe(n,m,k) &= Pe(n,m,k-2)+Pe(n,m,k-3)\\
&= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+Pe(k-1)Pd(n)j\\
&+ Pd(m)Pe(n)+Pe(m+1)Pd(n)i+Pe(k-2)Pd(n)j\\
&= Pd(m)Pe(n)+Pe(m+1)Pd(n)i+Pe(k+1)Pd(n)j.
\end{aligned}$$

□

4. The n-dimensional Perrin relations

From the two-dimensional and three-dimensional relations discussed above, it is possible to generalise the insertion of imaginary units up to the n order, thus obtaining the n-dimensional relations of the Perrin sequence. Thus, we have a generalised expression for the hypercomplex numbers described in the form $Pe(n_1,n_2,\ldots,n_n)$, where n represents the amount of imaginary variables inserted.

Theorem 4.1. *Given $Pe(n_1,n_2,\ldots,n_n)$ the numbers in Perrin n-dimensional form, with $n \in \mathbb{N}$ and the imaginary units represented by $\mu_1 = i, \mu_2 = j, \ldots, \mu_n$. Hence, they are given by:*

$$Pe(n_1,n_2,\ldots,n_n) = Pe(n_1)Pd(n_2)\cdots Pd(n_n)+Pd(n_1)Pe(n_2+1)\cdots Pd(n_n)\mu_1+\cdots+Pd(n_1)Pd(n_2)\cdots Pe(n_n+1)\mu_n$$

Proof. Thus, it has already been demonstrated that the two-dimensional and three-dimensional relations are valid, which can be verified by the inductive process:

$$\begin{aligned}
Pe(n_1,n_2) &= Pe(n_1)Pd(n_2)+Pe(n_2+1)Pd(n_1)\mu_1\\
Pe(n_1,n_2,n_3) &= Pe(n_1)Pd(n_2)Pd(n_3)+Pd(n_1)Pe(n_2+1)Pd(n_3)\mu_1+Pd(n_1)Pd(n_2)Pe(n_3+1)\mu_2\\
Pe(n,n_2,n_3,n_4) &= Pe(n_1)Pd(n_2)Pd(n_3)Pd(n_4)+Pd(n_1)Pe(n_2+1)Pd(n_3)Pd(n_4)\mu_1\\
&\quad +Pd(n_1)Pd(n_2)Pe(n_3+1)Pd(n_4)\mu_2+Pd(n_1)Pd(n_2)Pd(n_3)Pe(n_4+1)\mu_3\\
&\vdots\\
Pe(n_1,n_2,n_3,\ldots,n_n) &= Pe(n_1)Pd(n_2)Pd(n_3)\cdots Pd(n_n)+Pd(n_1)Pe(n_2+1)Pd(n_3)\cdots Pd(n_n)\mu_1+\cdots+\\
&\quad Pd(n_1)Pd(n_2)Pd(n_3)\cdots Pe(n_n+1)\mu_n
\end{aligned}$$

□

5. Conclusion

Starting from the one-dimensional Perrin recursive model and the relationship between the Perrin numbers and the Padovan numbers, the recurrence two-dimensional, three-dimensional relations, and the inductive n-dimensional relation are explored concerning aspects of the complexity of the Perrin model. The process of complexity of the Perrin sequence was discussed through investigations around the addition of the imaginary unit, the dimensional increase, and its corresponding algebraic representations.
For future work, it is possible to develop mathematical properties around numbers $Pe(n,m)$, $Pe(n,m,p)$ and $Pe(n_1,n_1,n_2,\ldots,n_n)$, extend it to the integer indexes, and identify possible applications in the domain of science.

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Author's contributions

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Stability Analysis of a Mathematical Model SI_uI_aQR for Covid-19 with the Effect of Contamination Control (Filiation) Strategy

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Abstract

In this study, using a system of delay nonlinear ordinary differential equations, we introduce a new compartmental epidemic model considered effect of filiation (contamination) control strategy to the spread of Covid-19. Firstly, the formulation of this new SI_uI_aQR epidemic model with delay process and the parameters arised from isolation and filiation is formed. Then the disease-free and endemic equilibrium points of the model are obtained. Also, the basic reproduction number $\mathscr{R}_0$ is found by using the next generation matrix method and the results on stabilities of the disease-free and endemic equilibrium points are investigated. Finally some examples are presented to show the effect of filiation control strategy.

1. Introduction

In December 2019, Hubei province in Wuhan, China, became the centre of an outbreak of Covid-19. Then the disease caused by the new type coronavirus has affected hundreds of countries by spreading rapidly all around the world. The World Health Organization (WHO) declared the epidemic as a pandemic (global epidemic) on March 11, 2020 due to the fact that it caused the loss of thousands of lives. Since it could not be found exactly being effective drugs or vaccine in today's stages for Covid-19, to control of spreading of the disease, the governments with the support of its health authorities have immediately tried to made effective control measures including procedures such as isolating of people suspected to have the disease, closely tracking of contacts, collecting of epidemiological and clinical data from patients, boosting of diagnostic and providing treatment services.

Mathematical modelling has an important role in understanding of problems and phenomenons in some areas such as medicine, biology and epidemiology, [1]-[3]. Especially, epidemic diseases such as Covid-19, the global agenda in 2020, are some of the main threats that are seriously affecting humanity. Various studies have been introduced to overcome the problems caused by such diseases for a long time, [4]-[6]. Modelling of epidemic diseases as mathematically is quite important in terms of controlling and reducing effects of the outbreaks, [7]-[10]. Looking at the literature, there are many compartmental models determined the basic principles for the spread of a disease in a population. Kermack and McKendrick with their study [7] have pioneered these compartmental mathematical models which are used extensively. They have tried to explain the spreading of an infectious disease in the course of time for a closed population. In the model, the population exposed to an infectious disease has been divided into three groups. First group (S : Susceptibles) consists from individuals who are not yet infected and have not immune to the pathogen. In the other group (I : Infectious) consisting of infectious individuals, the members can be transmitted the disease to the susceptible individuals via effective contacts. The last group (R : Recovered) is formed from individuals who recovered and have immune against the pathogen. This model is called as "SIR" model with the initials of the group names.

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After, many authors have studied intensively on this model and in detail to carry further forward this model, [11]-[15].

In recent years, various complemental models have been considered in order to explain the more complex phenomena in diseases. In this sense, by giving some additional circumstances which will have more reality to the basic epidemic models, many studies have been obtained. For instance, a lot of *SEIR* epidemic models constructed by taking into account a latent compartment (E : Exposed) have been considered. This compartment consists of individuals who are infected but are not yet infectious although an effective contact has occurred between the infectious. Many *SEIR* models have been studied with various meaningful details, [16]-[22].

On the other hand, "quarantine" is another factor affecting to control of the spreading of diseases. This transaction is one of commonly used method for preventing and controlling spread of diseases. In today's society, with increasing the effectiveness of the quarantine process, spreading of some diseases has significantly decreased. This fact entails to consider the quarantine process in the models by giving an additional compartment Q represented the quarantined individuals. Recently, the epidemic models with quarantine has been investigated by some authors, [23]-[26].

During the early stages of the Covid-19 outbreak, a lot of studies, such as [27]-[33],[35], on its transmission dynamics has been revealed. In this study, using a system of delay nonlinear ordinary differential equations, we aim to introduce a time delay compartmental epidemic model considered the effect of filiation control strategy via quarantine in spreading of Covid-19 and other diseases. In the literature, there exist some compartmental models including the quarantine class consisting of some of the exposed or infectious, [33, 34]. However, in the model presented in this study depends on the fact that individuals who contacted with the infectious but whose exposure status is not yet known are quarantined. The model differs from many studies in the literature with this feature. Considering that the latent period for Covid-19 can be completed before the incubation period and the rate of asymptomatic infectious is quite high, it can be seen that the model is competent in modeling diseases such as Covid-19.

In the next section, firstly, the formulation of this new SI_uI_aQR epidemic model with delay process and the parameters arised from isolation and filiation is formed. Then the feasible region which is being positive invariant set for the model and guaranteeing the boundedness of the functions is determined.

In the third section, disease-free and endemic equilibrium points of the model are obtained. After the basic reproduction number $\mathscr{R}_0$ is found by using the next generation matrix method, the local stabilities of the disease-free and endemic equilibrium points are proved using the corresponding characteristic equation. Then the global stability of disease-free equilibrium point is handled via LaSalle's Invariance Principle associated with the Lyapunov function. In the last section, some examples are presented to show the effect of filiation control strategy.

2. Description of the model

In this part instruction of the model, defining the parameters and the transitions between the compartments are introduced.

As it is known that some individuals may have no symptoms throughout their infectiousness and these individuals are called as asymptomatic infectious. We use the notations I_u (who are unaware of their infectious) and I_a (who are aware of their infectious) to denote the compartment of asymptomatic and symptomatic infectious, respectively. Indeed I_a consists of symptomatics, and some of asymptomatic individuals whose positivity is known via test (i.e. confirmed cases). Further, asymptomatic individuals are unaware of the fact that own being infectious, and what is worse they may not avoid contact with susceptibles.

Also we assume that the all members of I_a have been isolated during the treatment in a hospital or home, and any members of I_a have not contacted with susceptibles. So the class I_a can be seen as the *treatment compartment.* Thus it is assumed that the disease spreads only via I_u. As a result of this fact, it is very important detecting of asymptomatics (members of I_u) for the course of the disease.

On the other hand we should mention the latent and incubation period. Incubation period is the time elapsed between exposure to a pathogen and when symptoms and signs are first apparent. Depending on the disease, the person may or may not be infectious during the incubation period. The latent period is the time interval between when an individual or host is infected by a pathogen and when he or she becomes infectious, i.e. capable of transmitting pathogens to other susceptible individuals.

According to our model it is assumed that all new cases born from the contact between susceptibles and asymptomatics infectious (I_u). The individuals who are infected at time t are *asymptomatic at the rate r*. So, the number of new individuals who are symptomatic (i.e, known to be positive) is $(1-r)\beta S(t) I_u(t)$ at each time t. Where β is the effective contact rate between susceptibles and asymptomatics infectious (I_u). Also, when an individual who became infectious is detected, the persons who are contacted to him in the past several days should be taken to quarantine from S and controlled during one incubation period.

In serious cases, quarantine of individuals suspected of being exposed to an epidemic disease is one of the most important and effective public health measures used in struggle against the disease. We assume that the number of individuals who contact with an individual identified as positive yet is m and the part of them at the rate q are taken to quarantine. We are called q as "*filiation (chain of contamination) control rate*" such as $0 \leq q \leq 1$. So $(1-r)\, mq\beta S(t)\, I_u(t)$ is the number of new individuals transferred to Q from S. Also some of individuals, who are not quarantined although they contact with newly positive cases, may be asymptomatic (at the rate p). We represent this transmission with $(1-r)\, m\,(1-q)\, p\beta S(t)\, I_u(t)$.

On the other hand it is assumed that all individuals in the quarantine (Q) do not contact with each other and susceptibles. In addition, according to our model, the individuals who have completed the quarantine process (this is one incubation period) are tested at the rate y and it is assumed that there exist positive cases at the rate p of them.

We should immediately note that, of course there will the individuals who have not been tested but have the risk in the sense of becoming positive. (Even if they may think that no there is any positivity risk). So these individuals will take part in I_u. Thus it is assumed that the individuals whose quarantine process has been completed but who have not been tested have join to I_u at the rate $(1-y)p$ (the rate of positivity of not tested individuals) and to S at the rate $(1-y)(1-p)$ (the rate of negativity of not tested individuals). In addition, taking into account that rate of individuals who have been tested and negative is $y(1-p)$, the individuals whose quarantine process has been completed turn to S at the rate $(1-p)$.

Also, taking into account that the time taking in quarantine is τ and some individuals will death at the rate d with natural causes (not caused by disease), it is obtained that the total number of individuals who leave from Q at time t is $(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau}$. This number is obtained by the solution of following initial value problem

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\overline{Q}(t,\tau) = -d\overline{Q}(t,\tau), \quad \overline{Q}(t,0) = (1-r)\, mq\beta S(t) I_u(t).$$

Where $\overline{Q}(t,\tau)$ describes the number of individuals who are joined to quarantine at time $t-\tau$ and still surviving at the time t.

Under these assumptions, the system of ordinary differential equations which reflects the model is as follows. In order to better understand, in the first stage, the transitions between compartments, simplifications have not been made intentionally in the equations.

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda-(1-r)\,\beta S(t)\, I_u(t)-[r+(1-r)\, m\,(1-q)\, p]\,\beta S(t)\, I_u(t)-(1-r)\, mq\beta S(t)\, I_u(t) \\
&\quad -dS(t)+(1-p)\,(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau}, \\
\frac{dI_u(t)}{dt} &= [r+(1-r)\, m\,(1-q)\, p]\,\beta S(t)\, I_u(t)+(1-y)\, p\,(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau} \\
&\quad -(d+\theta)\, I_u(t), \\
\frac{dI_a(t)}{dt} &= (1-r)\,\beta S(t)\, I_u(t)+yp\,(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau} \\
&\quad -(d+\mu+\gamma)\, I_a(t), \\
\frac{dQ(t)}{dt} &= (1-r)\, mq\beta S(t)\, I_u(t)-(1-y)\, p\,(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau} \\
&\quad -(1-p)\,(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau} \\
&\quad -yp\,(1-r)\, mq\beta S(t-\tau)\, I_u(t-\tau)\, e^{-d\tau}-dQ(t), \\
\frac{dR(t)}{dt} &= \theta I_u(t)+\gamma I_a(t)-dR(t),
\end{aligned} \tag{2.1}$$

Where $S(t)$, $I_u(t)$, $I_a(t)$, $Q(t)$ and $R(t)$ denote the number of the susceptible, unaware infectious, aware infectious, quarantined and recovered individuals at time t, respectively. The total population size at time t is $N(t)$ and $N(t) = S(t)+I_a(t)+I_u(t)+Q(t)+R(t)$ for all $t>0$, such that all these functions are nonnegative. Also, all newborn individuals is be included the population by entering with the rate Λ to the compartment S. μ represents the death rate derived from the disease. θ denotes the transition rate to R from I_u. On the other hand, all parameters in the model are nonnegative constants. After necessary simplifications and abbreviations, the transition diagram between compartments of the model is as follows.

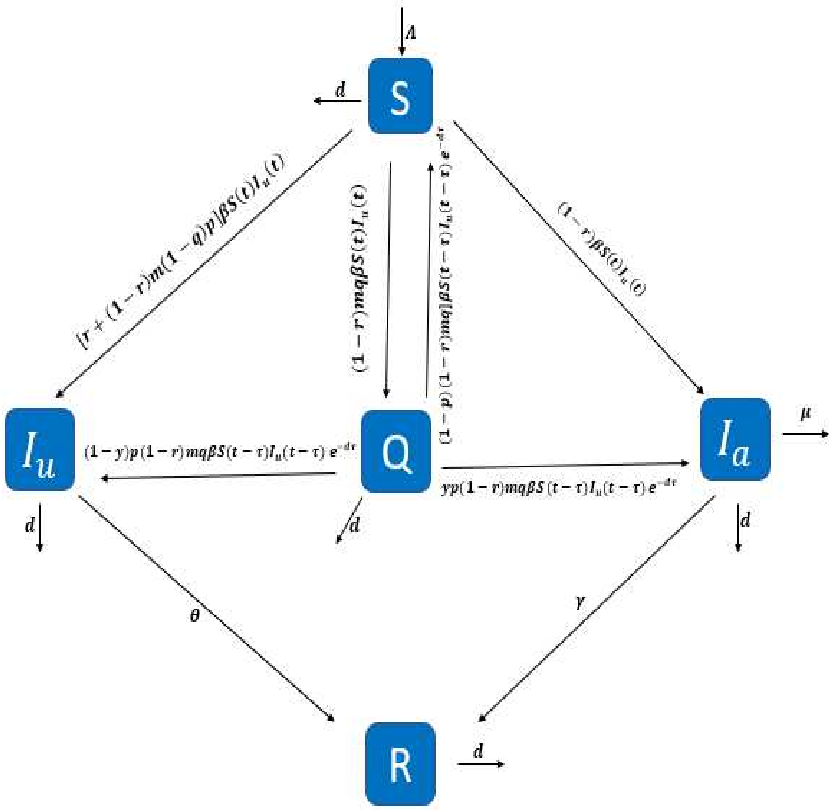

Figure 2.1: Transition diagram of the SI_uI_aQR model for spreading of the disease

Now, we determine the feasible region which is being positive invariant set for system (2.1).
Summing equations in (2.1), we obtain

$$\begin{aligned}\frac{dS}{dt}+\frac{dI_u}{dt}+\frac{dI_a}{dt}+\frac{dQ}{dt}+\frac{dR}{dt} &= \frac{dN}{dt}\\ &= \Lambda - dN(t) - \mu I_a(t)\\ &\le \Lambda - dN(t). \end{aligned} \tag{2.2}$$

If we use the fact that

$$x(t)=\frac{\Lambda}{d}\left(1-e^{-dt}\right)+x(0)\,e^{-dt}$$

is the solution of the equation

$$x'(t)=\Lambda - dx(t)$$

then we get the maximal solution of (2.2) as

$$N(t)=\frac{\Lambda}{d}\left(1-e^{-dt}\right)+N(0)\,e^{-dt}$$

for all $t\ge 0$. Then we can say $N(0)\le\frac{\Lambda}{d}$ implies $N(t)\le\frac{\Lambda}{d}$, for all $t\ge 0$. This means that all solutions of system (2.1) are eventually confined in this region bounded with Λ/d. So

$$\Gamma=\left\{(S,I_u,I_a,Q,R)\in C\left([-\tau,\infty),\left[0,\frac{\Lambda}{d}\right]^5\right): N(t)\le\frac{\Lambda}{d}\right\}$$

is positively invariant set for the model (2.1) and to concentrate on this restricted area will be enough for analysing of the model.
It can be seen that functions I_a, Q and R do not appear in the other equations of the system (2.1). Also there is no nonlinear relationship between these functions. So the dynamics of (2.1) are the same as the following reduced system (2.3), and it is sufficient to study on the system (2.3). Dynamics and behaviour of functions S and I_u determine the state of the others. Also

taking into account that the disease spreads only contacts between S and I_u, this reducing is meaningful epidemiologically as well.

$$\begin{cases} \frac{dS}{dt} = \Lambda - [1+(1-r)m(q+p(1-q))]\beta S(t) I_u(t) - dS(t) & \\ \quad +(1-p)(1-r)mq\beta S(t-\tau) I_u(t-\tau) e^{-d\tau} & \\ \frac{dI_u}{dt} = [r+(1-r)m(1-q)p]\beta S(t) I_u(t) - (d+\theta) I_u(t) & t \geq 0 \\ \quad +(1-y)p(1-r)mq\beta S(t-\tau) I_u(t-\tau) e^{-d\tau} & \\ S(t) = g_1(t) & \\ I_u(t) = g_2(t) & -\tau \leq t \leq 0 \end{cases} \tag{2.3}$$

Where $g_i \in C\left([-\tau,0],\left[0,\frac{\Lambda}{d}\right]\right)$, $i=1,2$ and $(S,I_u) \in C\left([-\tau,\infty),\left[0,\frac{\Lambda}{d}\right]^2\right)$. If we choose $x=(x_1,x_2)$, $x^t(\theta)=x(t+\theta)$, $g=(g_1,g_2)$ and $f:\Omega \to \left[0,\frac{\Lambda}{d}\right]^2$ such that $\Omega \subset C\left([-\tau,0],\left[0,\frac{\Lambda}{d}\right]^2\right)$ then finding the solution of the system (2.3) is equivalent to solving the following equation

$$\begin{aligned} x'(t) &= f\left(x^t\right),\ t \geq 0 \\ x_0 &= g. \end{aligned} \tag{2.4}$$

Where f is defined by

$$\begin{aligned} f_1(x) &= \Lambda - [1+(1-r)m(q+p(1-q))]\beta x_1(0) x_2(0) - dx_1(0) + (1-p)(1-r)mq\beta x_1(-\tau) x_2(-\tau) e^{-d\tau} \\ f_2(x) &= [r+(1-r)m(1-q)p]\beta x_1(0) x_2(0) - (d+\theta) x_2(0) + (1-y)p(1-r)mq\beta x_1(-\tau) x_2(-\tau) e^{-d\tau} \end{aligned}$$

for $x=(x_1,x_2)=(S,I_u) \in \Omega$. Also, as known $C\left([-\tau,0],\left[0,\frac{\Lambda}{d}\right]^n\right)$ is a Banach space of continuous functions, and $\|\cdot\|_C$ denotes the norm on $C\left([-\tau,0],\left[0,\frac{\Lambda}{d}\right]^n\right)$ and is defined by

$$\|x\|_C = \sup\left\{\sum_{i=1}^{n} |x_i(t)| : -\tau \leq t \leq 0\right\}.$$

3. Analysis of the model

In this section, we interest with qualitative analysis of the model (2.3). We firstly show the uniqueness of the solution of the model (2.3).

Theorem 3.1. *There exists a unique solution of the eqution (2.3) with initial function $x_1(t)=g_1(t)$, $x_2(t)=g_2(t)$ for $-\tau \leq t \leq 0$.*

Proof. It sufficient to show that f, given in (2.4), is Lipschitz continuous in every compact subset $M \subset \Omega$. Let $x=(x_1,x_2)$, $y=(y_1,y_2) \in M$, then we can write from the description of f

$$\begin{aligned}
& \|f(x)-f(y)\| \\
= {} & |f_1(x)-f_1(y)| + |f_2(x)-f_2(y)| \\
= {} & [1+(1-r)m(q+p(1-q))]\beta |x_1(0)x_2(0) - y_1(0)y_2(0)| \\
& + d|y_1(0)-x_1(0)| \\
& + (1-p)(1-r)mq\beta e^{-d\tau} |x_1(-\tau)x_2(-\tau) - y_1(-\tau)y_2(-\tau)| \\
& + [r+(1-r)m(1-q)p]\beta |x_1(0)x_2(0) - y_1(0)y_2(0)| \\
& + (d+\theta)|y_2(0)-x_2(0)| \\
& + (1-y)p(1-r)mq\beta e^{-d\tau} |x_1(-\tau)x_2(-\tau) - y_1(-\tau)y_2(-\tau)| \\
\leq {} & [1+(1-r)m(q+p(1-q))]\beta (|x_2(0)| + |y_1(0)|) \|x-y\|_C \\
& + (d+\theta)\|x-y\|_C \\
& + (1-p)(1-r)mq\beta e^{-d\tau} (|x_2(-\tau)| + |y_1(-\tau)|) \|x-y\|_C \\
& + [r+(1-r)m(1-q)p]\beta (|x_2(0)| + |y_1(0)|) \|x-y\|_C \\
& + (1-y)p(1-r)mq\beta e^{-d\tau} (|x_2(-\tau)| + |y_1(-\tau)|) \|x-y\|_C \\
= {} & [1+r+(1-r)m(q+2p(1-q))]\beta (|x_2(0)| + |y_1(0)|) \|x-y\|_C \\
& + (d+\theta)\|x-y\| \\
& + [(1-r)mq(1-p+p(1-y))]\beta e^{-d\tau} (|x_2(-\tau)| + |y_1(-\tau)|) \|x-y\|_C
\end{aligned} \tag{3.1}$$

Taking into account the fact $|x_i(t)| \leq \frac{\Lambda}{d}$ for $-\tau \leq t \leq 0$, $i = 1,2$ then we conlude

$$\|f(x) - f(y)\| \leq \left(\frac{2\Lambda}{d}(A+B) + d + \theta\right)\|x-y\|_C$$

from (3.1), where $A = [1 + r + (1-r)m(q + 2p(1-q))]\beta$ and $B = [(1-r)mq(1-p+p(1-y))]\beta e^{-d\tau}$.
So if we take

$$l \geq \frac{2\Lambda}{d}(A+B) + d + \theta,$$

the inequality

$$\|f(x) - f(y)\| \leq l\|x-y\|_C$$

hold in every compact subset $M \subset \Omega$. This completes the proof. □

3.1. Disease-free equilibrium point and basic reproduction number

The disease-free equilibrium point of the model (2.3) is easily found as

$$\varepsilon_0 = \left(S^0, I_u^0\right) = \left(\frac{\Lambda}{d}, 0\right).$$

The number of secondary infections produced by a single infected individual introduced into a population is a threshold value and very significant for providing information about the course of the disease in the population. This number represented by $\mathscr{R}_0$ is also known as the basic reproduction number. Now, let us get the basic reproduction number $\mathscr{R}_0$ of system (2.3) by means of the next generation matrix method.
Let $X = (I_u, S)^T$. Then the system (2.3) can be written in the form

$$\frac{dX}{dt} = \mathscr{F}(X) - \mathscr{V}(X).$$

such that

$$\mathscr{F}(X) = \begin{bmatrix} [r + (1-r)m(1-q)p]\beta S(t) I_u(t) + (1-y)p(1-r)mq\beta S(t-\tau) I_u(t-\tau) e^{-d\tau} \\ 0 \end{bmatrix}$$

and

$$\mathscr{V}(X) = \begin{bmatrix} (d+\theta) I_u(t) \\ [1 + (1-r)m(q + p(1-q))]\beta S(t) I_u(t) + dS(t) - \Lambda - (1-p)(1-r)mq\beta S(t-\tau) I_u(t-\tau) e^{-d\tau} \end{bmatrix}.$$

The basic reproduction number belonging to the model (2.3) is based on the linearization of the system about disease-free equilibrium. The jacobian matrices of $\mathscr{F}(X)$ and $\mathscr{V}(X)$ at the disease-free equilibrium $\varepsilon_0 = \left(\frac{\Lambda}{d}, 0\right)$ are respectively found as

$$d\mathscr{F}(\varepsilon_0) = \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} \\ \mathscr{F}_{21} & \mathscr{F}_{22} \end{bmatrix},$$

$$d\mathscr{V}(\varepsilon_0) = \begin{bmatrix} \mathscr{V}_{11} & \mathscr{V}_{12} \\ \mathscr{V}_{21} & \mathscr{V}_{22} \end{bmatrix}.$$

Where

$$\begin{aligned} \mathscr{F}_{11} &= [r + (1-r)m(1-q)p]\beta S^0 + (1-y)p(1-r)mq\beta S^0 e^{-d\tau} \\ \mathscr{F}_{12} &= [r + (1-r)m(1-q)p]\beta I_u^0 + (1-y)p(1-r)mq\beta I_u^0 e^{-d\tau} \\ \mathscr{F}_{21} &= 0 \\ \mathscr{F}_{22} &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathscr{V}_{11} &= d + \theta, \\ \mathscr{V}_{12} &= 0, \\ \mathscr{V}_{21} &= [1 + (1-r)m(q + p(1-q))]\beta S^0 - (1-p)(1-r)mq\beta S^0 e^{-d\tau}, \\ \mathscr{V}_{22} &= [1 + (1-r)m(q + p(1-q))]\beta I_u^0 + d - (1-p)(1-r)mq\beta I_u^0 e^{-d\tau}. \end{aligned}$$

Then

$$F=\mathscr{F}_{1\times 1}=\left[\left(r+(1-r)m(1-q)p\right)\beta S^0+(1-y)p(1-r)mq\beta S^0 e^{-d\tau}\right],$$

$$V=\mathscr{V}_{1\times 1}=[d+\theta]$$

and

$$FV^{-1}=\left[\frac{\left[r+(1-r)m(1-q)p+(1-y)p(1-r)mqe^{-d\tau}\right]\beta S^0}{d+\theta}\right].$$

Following Diekmann and Heesterbeek [36], the matrix FV^{-1} is referred to as the next generation matrix for the system at the disease-free equilibrium and the basic reproduction number is defined as the spectral radius of the matrix FV^{-1}. Now let us find maximum of the eigenvalues of this matrix. The characteristic polynomial of FV^{-1} is

$$\det\left(\lambda I_1-FV^{-1}\right)=\lambda-\frac{\left[r+(1-r)m(1-q)p+(1-y)p(1-r)mqe^{-d\tau}\right]\beta S^0}{d+\theta}.$$

Then, the spectral radius of the next generation matrix is

$$\rho\left(FV^{-1}\right)=\frac{\left[r+(1-r)m(1-q)p+(1-y)p(1-r)mqe^{-d\tau}\right]\beta S^0}{d+\theta}.$$

Taking into $S^0=\Lambda/d$ account that, the basic reproduction number of the model (2.3) is found as

$$\mathscr{R}_0=\frac{\beta\Lambda\left\{r+(1-r)m\left[(1-q)p+(1-y)pqe^{-d\tau}\right]\right\}}{d(d+\theta)}.$$

Now, let us open another matter and consider its results. It is clear that

$$\frac{\partial\mathscr{R}_0}{\partial y}=-p(1-r)mqe^{-d\tau}\leq 0$$

and

$$\frac{\partial\mathscr{R}_0}{\partial q}=m(1-r)p\left[(1-y)e^{-d\tau}-1\right]\leq 0,$$

taking into account that $(1-y)e^{-d\tau}<1$. So we can say that the test rate y and the filiation (chain of contamination) control rate q have opposite effects on $\mathscr{R}_0$.
On the other hand

$$\frac{\partial\mathscr{R}_0}{\partial r}=\frac{\beta\Lambda\left[1-mp\left(1-q+(1-y)qe^{-d\tau}\right)\right]}{d(d+\theta)}$$

and so, if $1>mp\left[1-q\left(1+(1-y)e^{-d\tau}\right)\right]$ then increasing of asymptomatic individuals increases the value $\mathscr{R}_0$. As a result of this fact, it should be aimed that the following inequality is hold

$$mp\left[1-q\left(1+(1-y)e^{-d\tau}\right)\right]\geq 1. \tag{3.2}$$

Hence, it can be concluded what is relation of the test rate y and the filiation control rate q according to other parameters from (3.2). In other words, this relation can give an answer to the question: "What should filiation control rate q is required with the test rate y so that the disease brought under control and does not turn into an epidemic?"

3.2. Existence of endemic equilibrium point

It can be seen that, from in subsection 3.1, the system (2.3) always has a disease-free equilibrium point. Now, we investigate the existence of endemic equilibrium point of the system (2.3).
If we take $S(t)=S^*$ and $I_u(t)=I_u^*\neq 0$, the endemic equilibrium point of the system (2.3) can be calculated via following system of algebraic equations

$$\begin{aligned}0 &= \Lambda-[1+(1-r)m(q+p(1-q))]\beta S^*I_u^*-dS^*+(1-p)(1-r)mq\beta S^*I_u^*e^{-d\tau},\\ 0 &= [r+(1-r)m(1-q)p]\beta S^*I_u^*+(1-y)p(1-r)mq\beta S^*I_u^*e^{-d\tau}-(d+\theta)I_u^*.\end{aligned} \tag{3.3}$$

From the second equation of the system (3.3), we write

$$I_u^*\left\{[r+(1-r)m(1-q)p]\beta S^*+(1-y)p(1-r)mq\beta S^*e^{-d\tau}-(d+\theta)\right\}=0.$$

Since $I_u^*\neq 0$ for endemic equilibrium point, we can say

$$[r+(1-r)m(1-q)p]\beta S^*+(1-y)p(1-r)mq\beta S^*e^{-d\tau}-(d+\theta)=0$$

and obtain

$$S^*=\frac{(d+\theta)}{\beta\{r+(1-r)m[(1-q)p+(1-y)pqe^{-d\tau}]\}}=\frac{\Lambda}{d\mathscr{R}_0}. \tag{3.4}$$

Substituting expression in (3.4) into first equation of (3.3) and arranging, we get

$$\begin{aligned} I_u^* &= \frac{\Lambda-dS^*}{\beta S^*[1+(1-r)m(q+p(1-q))-(1-p)(1-r)mqe^{-d\tau}]} \\ &= \frac{d(\mathscr{R}_0-1)}{\beta\{1+(1-r)m[q(1-e^{-d\tau})+p(1-q)+pqe^{-d\tau}]\}}. \end{aligned}$$

Then taking into account that $q,r<1$ and $e^{-d\tau}<1$, it can be obtain $I_u^*>0$ for $\mathscr{R}_0>1$.
Therefore, we say the system (2.3) has a unique endemic equilibrium point $\varepsilon_*=(S^*,I_u^*)$ when $\mathscr{R}_0>1$.
$\varepsilon_*=(S^*,I_u^*)$ can be written as

$$(S^*,I_u^*)=\left(\frac{\Lambda}{d\mathscr{R}_0},\frac{d(\mathscr{R}_0-1)}{\beta\{1+(1-r)m[q(1-e^{-d\tau})+p(1-q)+pqe^{-d\tau}]\}}\right)$$

according to $\mathscr{R}_0$.

3.3. Stabilities of the equilibrium points

In this part it is examined the stability behaviour of system (2.3). Firstly, for local stabilities of disease-free and endemic equilibrium points, the characteristic equations which correspond to Jacobian matrices at the equilibrium points are analysed. Next by using the Lyapunov functional technique, global stability of disease-free equilibrium point is proved.

Theorem 3.2. *The disease-free equilibrium ε_0 of the system (2.3) is locally asymptotically stable in the positively invariant region Γ for $\mathscr{R}_0<1$, and unstable for $\mathscr{R}_0>1$.*

Proof. The Jacobian matrix at the disease-free equilibrium point $\varepsilon_0=\left(S^0,I_u^0\right)$ of the system (2.3) is

$$J(\varepsilon_0)=\begin{bmatrix} J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix},$$

where

$$\begin{aligned} J_{11} &= -[1+(1-r)m(q+p(1-q))]\beta I_u^0-d+(1-p)(1-r)mq\beta I_u^0e^{-d\tau},\\ J_{12} &= -[1+(1-r)m(q+p(1-q))]\beta S^0+(1-p)(1-r)mq\beta S^0e^{-d\tau},\\ J_{21} &= [r+(1-r)m(1-q)p]\beta I_u^0+(1-y)p(1-r)mq\beta I_u^0e^{-d\tau},\\ J_{22} &= [r+(1-r)m(1-q)p]\beta S^0+(1-y)p(1-r)mq\beta S^0e^{-d\tau}-(d+\theta). \end{aligned}$$

Taking into account $\left(S^0,I_u^0\right)=\left(\frac{\Lambda}{d},0\right)$, the characteristic equation which is correspond to this matrix is

$$(-d-\lambda)\left(\beta\frac{\Lambda}{d}\left[(r+(1-r)m(1-q)p)+(1-y)p(1-r)mqe^{-d\tau}\right]-(d+\theta)-\lambda\right)=0. \tag{3.5}$$

This equation always have negative eigenvalue $-d$. The other eigenvalue of characteristic equation (3.5) is determined by

$$\begin{aligned} \lambda_2 &= (d+\theta)\left(\frac{\beta\Lambda\left\{r+(1-r)m\left[(1-q)p+(1-y)pqe^{-d\tau}\right]\right\}}{d(d+\theta)}-1\right)\\ &= (d+\theta)(\mathscr{R}_0-1). \end{aligned}$$

If $\mathscr{R}_0<1$, then two roots of Eq. (3.5) are negative. If $\mathscr{R}_0=1$, then we say that one of roots of Eq. (3.5) is zero. In the case $\mathscr{R}_0>1$, one of roots of Eq. (3.5) has positive real parts. Therefore, the disease-free equilibrium point ε_0 is locally asymptotically stable for $\mathscr{R}_0<1$, is stable for $\mathscr{R}_0=1$, and is unstable for $\mathscr{R}_0>1$. □

Theorem 3.3. *The endemic equilibrium ε_* is locally asymptotically stable in the positively invariant region Γ for $\mathscr{R}_0 > 1$.*

Proof. The Jacobian matrix at the endemic equilibrium point ε_* of the system (2.3) is

$$J(\varepsilon_*) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},$$

where

$$\begin{aligned}
J_{11} &= -[1+(1-r)m(q+p(1-q))]\beta I_u^* - d + (1-p)(1-r)mq\beta I_u^* e^{-d\tau}, \\
J_{12} &= -[1+(1-r)m(q+p(1-q))]\beta S^* + (1-p)(1-r)mq\beta S^* e^{-d\tau}, \\
J_{21} &= [r+(1-r)m(1-q)p]\beta I_u^* + (1-y)p(1-r)mq\beta I_u^* e^{-d\tau}, \\
J_{22} &= [r+(1-r)m(1-q)p]\beta S^* + (1-y)p(1-r)mq\beta S^* e^{-d\tau} - (d+\theta).
\end{aligned}$$

If we take into account that

$$(S^*, I_u^*) = \left(\frac{\Lambda}{d\mathscr{R}_0}, \frac{d(\mathscr{R}_0 - 1)}{\beta\{1+(1-r)m[q(1-e^{-d\tau}) + p(1-q) + pqe^{-d\tau}]\}}\right),$$

and make necessary arrangements, we obtain the followings:

$$\begin{aligned}
J_{11} &= -[1+(1-r)m(q+p(1-q))]\beta I_u^* - d + (1-p)(1-r)mq\beta I_u^* e^{-d\tau} \\
&= -\beta I_u^* \left\{1+(1-r)m\left[q\left(1-e^{-d\tau}\right) + p(1-q) + pqe^{-d\tau}\right]\right\} - d \\
&= d(1-\mathscr{R}_0) - d \\
&= -d\mathscr{R}_0,
\end{aligned}$$

$$\begin{aligned}
J_{12} &= -[1+(1-r)m(q+p(1-q))]\beta S^* + (1-p)(1-r)mq\beta S^* e^{-d\tau} \\
&= -\beta S^* \left\{1+(1-r)m\left[q\left(1-e^{-d\tau}\right) + p(1-q) + pqe^{-d\tau}\right]\right\} \\
&= -\frac{\Lambda(\mathscr{R}_0 - 1)}{I_u^* \mathscr{R}_0},
\end{aligned}$$

$$\begin{aligned}
J_{21} &= [r+(1-r)m(1-q)p]\beta I_u^* + (1-y)p(1-r)mq\beta I_u^* e^{-d\tau} \\
&= \beta I_u^* \left\{r+(1-r)m\left[(1-q)p + (1-y)pqe^{-d\tau}\right]\right\} \\
&= I_u^* \frac{d(d+\theta)\mathscr{R}_0}{\Lambda}
\end{aligned}$$

and

$$\begin{aligned}
J_{22} &= [r+(1-r)m(1-q)p]\beta S^* + (1-y)p(1-r)mq\beta S^* e^{-d\tau} - (d+\theta) \\
&= \beta S^* \left\{r+(1-r)m\left[(1-q)p + (1-y)pqe^{-d\tau}\right]\right\} - (d+\theta) \\
&= S^* \frac{d(d+\theta)\mathscr{R}_0}{\Lambda} - (d+\theta) \\
&= 0.
\end{aligned}$$

After from the simplification, the corresponding characteristic equation for $J(\varepsilon_*)$ is found as

$$\lambda^2 + C_1\lambda + C_2 = 0, \tag{3.6}$$

where

$$C_1 = d\mathscr{R}_0$$

and

$$C_2 = d(d+\theta)(\mathscr{R}_0 - 1).$$

Then we can say

$$C_1 = d\mathscr{R}_0 > 0, \text{ for } \mathscr{R}_0 > 1$$

and

$$C_2 = d(d+\theta)(\mathscr{R}_0 - 1) > 0, \text{ for } \mathscr{R}_0 > 1.$$

Therefore, we obtain $tr(J(\varepsilon_*)) = -C_1 < 0$ and $\det(J(\varepsilon_*)) = C_2 > 0$. So, each of the eigenvalues of $J(\varepsilon_*)$ (i.e. two roots of the equation (3.6)) have negative real parts. Consequently, if $\mathscr{R}_0 > 1$ then endemic equilibrium $\varepsilon_* = (S^*, I_u^*)$ is locally asymptotically stable. □

Theorem 3.4. *The disease-free equilibrium ε_0 of the system (2.3) is globally asymptotically stable in the positively invariant region Γ for $\mathscr{R}_0 < 1$.*

Proof. Let us define the following function as a candidate for Lyapunov function.

$$W(t) = I_u(t) + (1-y)p(1-r)mqe^{-d\tau}\int_{t-\tau}^{t}\beta S(x) I_u(x)\,dx.$$

Differentiating $W(t)$ according to time t, we get

$$\begin{aligned}
&\dot{W}(t)\\
=\;& [r+(1-r)m(1-q)p]\beta S(t) I_u(t) + (1-y)p(1-r)mq\beta S(t-\tau) I_u(t-\tau)e^{-d\tau} - (d+\theta)I_u(t)\\
&+(1-y)p(1-r)mqe^{-d\tau}\beta S(t)I_u(t) - (1-y)p(1-r)mqe^{-d\tau}\beta S(t-\tau)I_u(t-\tau)\\
=\;& I_u(t)\left\{[r+(1-r)m(1-q)p]\beta S(t) + (1-y)p(1-r)mqe^{-d\tau}\beta S(t) - (d+\theta)\right\}\\
=\;& I_u(t)\left\{\beta S(t)\left[r+(1-r)m\left((1-q)p+(1-y)pqe^{-d\tau}\right)\right] - (d+\theta)\right\}\\
\leq\;& I_u(t)\left[\frac{\beta\Lambda\left\{r+(1-r)m\left[(1-q)p+(1-y)pqe^{-d\tau}\right]\right\}}{d} - (d+\theta)\right]\\
=\;& I_u(t)\left[(d+\theta)\left(\frac{\beta\Lambda\left\{r+(1-r)m\left[(1-q)p+(1-y)pqe^{-d\tau}\right]\right\}}{d(d+\theta)} - 1\right)\right]\\
=\;& I_u(t)(d+\theta)(\mathscr{R}_0 - 1).
\end{aligned}$$

Hence, it can be concluded that $W(t) > 0$ and $\dot{W}(t) < 0$ when $\mathscr{R}_0 < 1$ and for all points which is different from equilibrium points. So W is a Lyapunov function for the system (2.3) on the set Γ. Now, let us define the set $\Phi = \left\{(S,I) : \dot{W}(t) = 0\right\}$ and let ϕ be largest invariant subset of Φ. It can be easily seen that $\phi = \{\varepsilon_0\}$ and ϕ is invariant. We say that ε_0 is globally asymptotically stable in Γ by aid of LaSalle's Invariance Principle [37] well-known from global stability thecniques. □

4. Conclusion

This study describes and analyses an SI_uI_aQR mathematical model that investigates the effect of quarantine on spreading of the Covid-19. To avoid second major or interim sub-waves of Covid-19 pandemic, one of the most effective methods that will minimize the harm and spread of the outbreak is to quarantine the exposed people and monitor the individuals they are in contact with. With this study, which aims to evaluate the effect of quarantine on the transmission of the Covid-19, it is thought that a different perspective and contribution will be provided to the literature.
Let us start to present the examples with the course of Q, I_u and I_a such that the estimated parameters are as follows

Parameters	Value (Estimated)
Λ	4000 (per day)
β	1.1×10^{-9}
r	0.5
m	5 (individual)
q	0.3
p	0.2
d	0.000015
y	0.7
μ	0.02
τ	12 (day)
θ	0.04
γ	0.2

with the initial conditions $S(0) = 7 \times 10^7$, $I_u(0) = 5000$, $I_a(0) = 500$, $Q(0) = 0$ and $R(0) = 0$.

According to the above parameters, the dynamical behaviours of the model has been described in Figure 4.1.

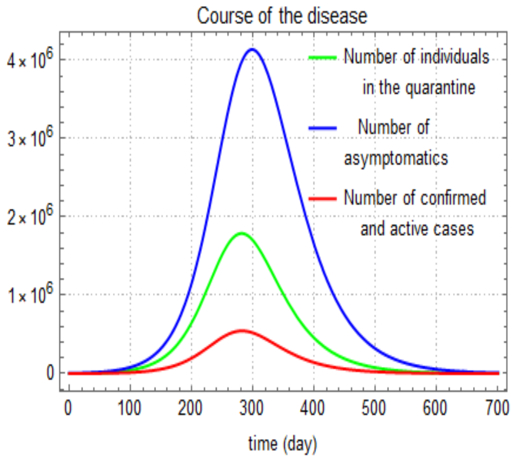

Figure 4.1: The dynamical behaviours of the compartments I_u, I_a and Q.

The other figures respectively reflect that the effect of filiation control rate q, of the rate of tests y for finding the infected individuals, and of the average number contacted persons m with the new cases, to spread of Covid-19. The figures, prepared to visualize the effect of these parameters in (2.3) has been generated using the Wolfram Mathematica 12.1 with NDSolve code. The control of filiation is a method of tracking from whom the virus is transmitted to the positive case and who the case has infected. In other words, in every positive case, it is the method of tracking the infection of the virus and determining the chain of spread. In the monitoring method of chain of contamination, the followings of contacted cases are provided. The individuals those who come in contact with positive cases are reached in a short time and asked to isolate themselves, and their evaluations and tests are made by visiting their locations. So, it is provided to prevent their infection potentials.
With the help of filiation studies, the cases are detected early and the risk of transmitting the disease to others is minimized. Again, the early detection of the patients by means of the filiation provides an early start of the treatment process and increases the recovery rate, [38]. Therefore the case detection and control works, and filiation activities belong to these play a very important role in struggle with Covid-19. Hence, all countries that are trying to control the disease should particular importance to the filiation activities for the strict monitoring and isolation of people having contact with cases within the community in addition to put communal limitations to keep the disease limited. With the filiation works, it is clear that great advances will be recorded. In our model, q corresponds to filiation control rate and the reader can see the effect of q (in a short period) on the spread of the Covid-19, in Figure 4.2.

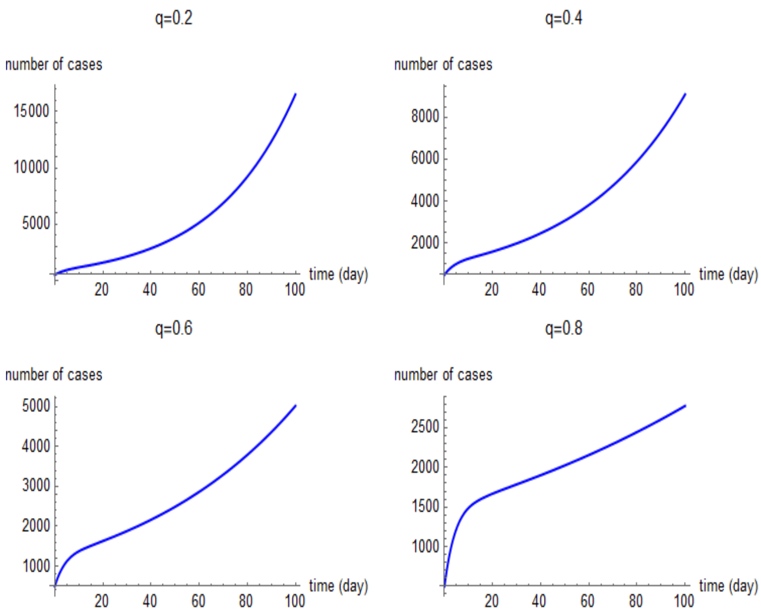

Figure 4.2: The effect of filiation control rate q in a short period for $y = 1$.

It can be seen that the increasing tendency of the disease decreases as q increases. Also the following figure shows the course of number of infectious (confirmed cases) according to q.

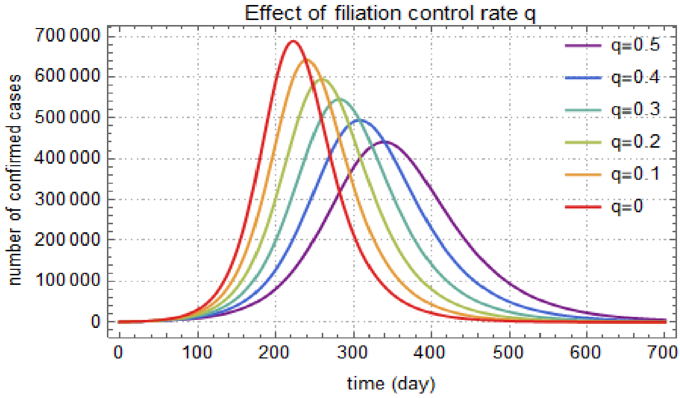

Figure 4.3: The effect of filiation control rate q in a long period.

Another big problem in the spread of disease is the number of contacts within the population. Unfortunately, no one can be sure that the people with whom they are in contact are not positive. So it should be avoided from dispensable contact. The following figure shows the result of this fact.

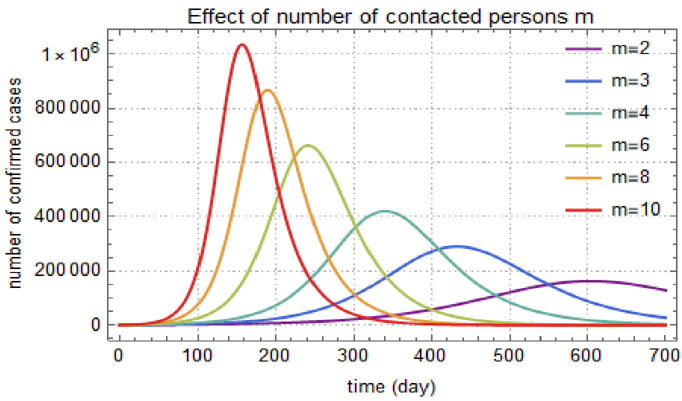

Figure 4.4: The effect of the average number of contacted individuals for $m=2, m=3, m=4, m=6, m=8, m=10$.

Let us come to the rate of test y. The rate of tests for determining the infected individuals is crucial in reducing the size of I_u. Let us note immediately that, as known, the potential danger group here is the asymptomatic individuals in the circulation in the community. Since it is assumed that the members of I_a will be isolated during the treatment in a hospital or home, they don't transmit the disease to susceptible. The rate of test at the end of the quarantine process and the rate of positive test within total tests determine the percentage of individuals that moved to the classes I_a, I_u and S from Q. In this regard, the rate of tests to be performed is reasonably significant. The following figure shows how the compartments can be influenced from evolution in y, for $q = 0.5$ and different values of the parameter y.

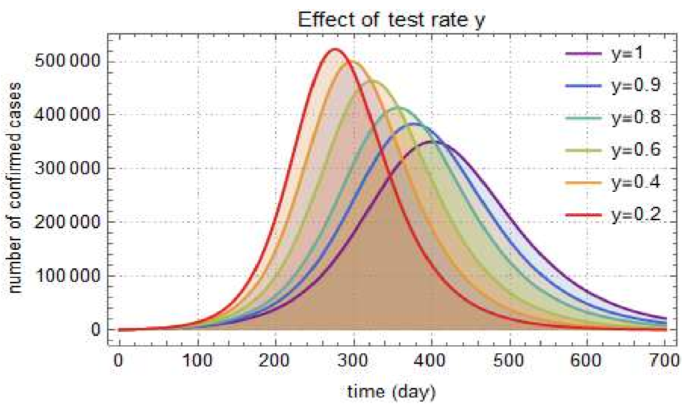

Figure 4.5: The effect of test rates for $y = 0.2, y = 0.4, y = 0.4, y = 0.6, y = 0.8, y = 0.9, y = 1$.

Finally we want to present, according to q, the bound of ratio of asymptomatics to confirmed cases.

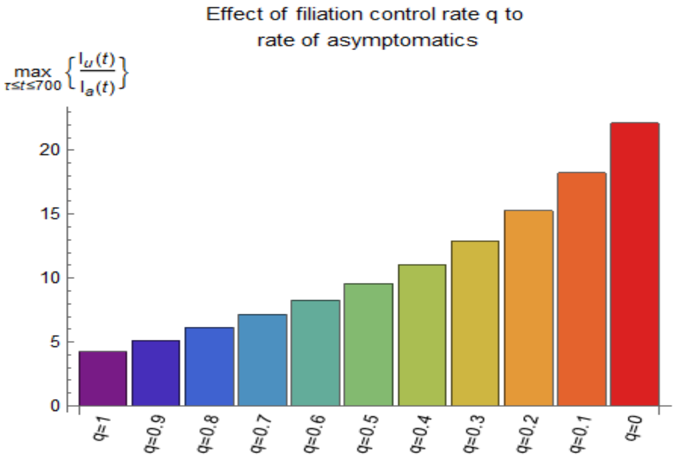

Figure 4.6: Different scenario of maximum rate between I_u and I_a after one incubation period from the initial of the disease.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Compact and Matrix Operators on the Space $\left|\overline{N}_p^{\phi}\right|_k$

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Abstract

In this paper, determining the operator norm, we give certain characterizations of matrix transformations from the space $\left|\overline{N}_p^{\phi}\right|_k$, the space of all series summable by the absolute weighted mean summability method, to one of the classical sequence spaces c_0, c, l_∞. Also, we obtain the necessary and sufficient conditions for each matrix in these classes to be compact and establish a number of estimates or identities for the Hausdorff measures of noncompactness of the matrix operators in these classes.

1. Introduction

The summability theory has an important role in analysis, applied mathematics and engineering sciences, and has been studied by many authors for a long time. One of the main subjects in the summability theory is the theory of sequence spaces that concerns with the generalization of the notions of convergence for sequences and series. The main purpose is to assign a limit value for non-convergent series or sequences by using a transformation which is given by the most general linear mappings of infinite matrices. In this concept, the literature has still enlarged, concerned with characterizing completely all matrices which transform one given sequence space into another and also, many sequence spaces defined as domain of special matrices such as Euler, Nörlund, Hausdorff, Cesàro and weighted mean matrices and related matrix operators have been investigated by several authors (see, [1, 2]). On the other hand, from a different point of view, using the concept of absolute summability, several new spaces of series summable by the absolute summability methods have taken place in the literature (see, for instance, [3]-[7]). In a recent paper, the sequence space $\left|\overline{N}_p^{\phi}\right|_k$ has introduced and studied by Sarıgöl [8, 9], Mohapatra and Sarıgöl [10].

The present paper aims to characterize the infinite matrix classes $\left(\left|\overline{N}_p^{\phi}\right|_k, c\right)$, $\left(\left|\overline{N}_p^{\phi}\right|_k, c_0\right)$, $\left(\left|\overline{N}_p^{\phi}\right|_k, l_\infty\right)$ and to determine the operator norms for $1 \le k < \infty$. Further, the necessary and sufficient conditions for each matrix in these classes to be compact are obtained and certain identities or estimates for the Hausdorff measure of noncompactness are established.

A vector subspace of ω, the space of all sequences of real or complex numbers, is called a *sequence space*. The sequence spaces Φ, l_∞, c, c_0, b_s, c_s and l_k $(k \ge 1)$ stand for the sets of all finite, bounded, convergent and null sequences and the sets of all bounded, convergent and k-absolutely convergent series, respectively.

Let Λ and Γ be two arbitrary sequence spaces and $R = (r_{nv})$ be an infinite matrix of complex components. The transform sequence $R(\lambda)$ of the sequence $\lambda = (\lambda_v)$ is deduced by the usual matrix product and the components of $R(\lambda)$ are written as

$$R_n(\lambda) = \sum_{v=0}^{\infty} r_{nv}\lambda_v,$$

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provided that the series converges for all $n \in \mathbb{N}$. If the sequence $R(\lambda)$ exists and $R(\lambda) \in \Gamma$ for $\lambda \in \Lambda$, then, it is said that R is a matrix mapping from Λ into Γ. The collection of all such infinite matrices is denoted by (Λ, Γ).
The set

$$\Lambda_R = \{\lambda \in \omega : R(\lambda) \in \Lambda\}$$

is called domain of an infinite matrix R in the space Λ. Note that it is also a sequence space.
The β-dual of $\Lambda \subset \omega$ is the set

$$\Lambda^\beta = \left\{ a : \forall \lambda \in \Lambda, \sum_{v=0}^{\infty} a_v \lambda_v \text{ converges} \right\}.$$

Let Λ and Γ be Banach spaces. By $\mathscr{B}(\Lambda, \Gamma)$, we mean the set of all bounded (continuous) linear operators L from Λ to Γ. $\mathscr{B}(\Lambda, \Gamma)$ is also a Banach space with the operator norm given by

$$\|L\| = \sup_{\lambda \in S_\Lambda} \|L(\lambda)\|_\Gamma$$

for all $L \in \mathscr{B}(X, Y)$. Here, S_Λ represents the unit sphere in Λ, i.e.,

$$S_\Lambda = \{\lambda \in \Lambda : \|\lambda\| = 1\}.$$

If $a \in \omega$ and $\Lambda \supset \Phi$ is a BK-space, a Banach space on which all coordinate functional $p_n(\lambda) = \lambda_n$ are continuous for all n, then

$$\|a\|^*_\Lambda = \sup_{\lambda \in S_\Lambda} \left| \sum_{k=0}^{\infty} a_k \lambda_k \right|$$

provided the expression on the right side is defined and finite which is the case whenever $a \in \Lambda^\beta$.
If, for each $\lambda \in \Lambda$,

$$\left\| \lambda - \sum_{j=0}^{m} \lambda_j e^{(j)} \right\| \to 0 \text{ as } m \to \infty$$

then it is said that the BK-space Λ has AK property, and in this case we write $\lambda = \sum_{j=0}^{\infty} \lambda_j e^{(j)}$ where $e^{(j)}$ is a sequence whose only non-zero term is one in jth place for $j \in \mathbb{N}$.
Throughout the whole paper, assume that $\phi = (\phi_n)$ is a sequence of positive constants and $R = (r_{nv})$ is an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$. Also, k^* is the conjugate of k, that is, $1/k + 1/k^* = 1$ for $k > 1$ and $1/k^* = 0$ for $k = 1$.
Let $\sum \lambda_n$ be an infinite series with its partial sum s_n. The series $\sum \lambda_v$ is said to be summable $|R, \phi_n|_k$, if (see[11]).

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |\Delta R_n(s)|^k < \infty,$$

where $1 \le k < \infty$ and $\Delta R_n(s) = R_n(s) - R_{n-1}(s)$. In the special case, when R is a weighted mean matrix, the summability method $|R, \phi_n|_k$ is reduced to $|\overline{N}, p_n, \phi|_k$ [12]. In recent paper, $\left|\overline{N}_p^\phi\right|_k$ has been generated from the space l_k as a set of all series summable by the absolute weighted mean method by Mohapatra and Sarıgöl [10] and Sarıgöl [8, 9]. The space $\left|\overline{N}_p^\phi\right|_k$ can be expressed as

$$\left|\overline{N}_p^\phi\right|_k = \left\{ \lambda = (\lambda_v) : \sum_{n=1}^{\infty} \phi_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v \right|^k < \infty \right\},$$

or equivalently, according to notation of domain, $\left|\overline{N}_p^\phi\right|_k = (l_k)_{T^{(p)}}$ where the matrix $T^{(p)}$ is given by

$$t_{nv}^{(p)} = \begin{cases} 1, \; n = 0, \; v = 0 \\ \phi_n^{1/k^*} \frac{p_n P_{v-1}}{P_n P_{n-1}}, \; 1 \le v \le n \\ 0, \qquad v > n, \end{cases}$$

whose inverse $S^{(p)}$ is

$$s_{nv}^{(p)} = \begin{cases} 1, \quad n = 0, v = 0 \\ -\phi_{n-1}^{-1/k^*} \frac{P_{n-2}}{p_{n-1}}, \; v = n-1 \\ \phi_n^{-1/k^*} \frac{P_n}{p_n}, \quad v = n \\ 0, \quad v \ne n-1, n. \end{cases} \tag{1.1}$$

Besides, it is obvious that the space $\left|\overline{N}_p^\phi\right|_k$ is a BK-space with the norm $\|\lambda\|_{\left|\overline{N}_p^\phi\right|_k} = \left\|T^{(p)}(\lambda)\right\|_{l_k}$ and it is also linearly isomorphic to the space l_k for $1 \le k < \infty$ [9].
We recall the following lemmas which are useful in proving our results:

Lemma 1.1. *[13] Let U be a triangle. Then,*

(i) For $\Lambda,\Gamma \subset \omega$, $R \in (\Lambda,\Gamma_U)$ iff $B = UR \in (\Lambda,\Gamma)$.

(ii) If Λ,Γ are BK-spaces and $R \in (\Lambda,\Gamma_U)$, then $\|L_R\| = \|L_B\|$.

Lemma 1.2. *[14] The following statements hold:*

1. $R \in (l,c) \Leftrightarrow$ *(i)* $\lim_n r_{nv}$ *exists for all* $v \geq 0$, *(ii)* $\sup_{n,v} |r_{nv}| < \infty$ *and* $R \in (l,l_\infty) \Leftrightarrow$ *(ii) holds.*
2. *If* $1 < k < \infty$, *then,* $R \in (l_k,c) \Leftrightarrow (i)$ *holds,* $(iii) \sup_n \sum_{v=0}^{\infty} |r_{nv}|^{k^*} < \infty$ *and* $R \in (l_k,l_\infty) \Leftrightarrow (iii)$ *holds.*
3. $R \in (l,c_0) \Leftrightarrow$ *(ii) holds, (iv)* $\lim_n r_{nv} = 0$ *for all* $v \geq 0$.
4. *If* $1 < k < \infty$, *then,* $R \in (l_k,c_0) \Leftrightarrow$ *(iii) and (iv) hold.*

Lemma 1.3. *[15] Let $1 \leq k < \infty$. Then, $R \in (l,l_k)$ iff*

$$\|R\|_{(l,l_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |r_{nv}|^k \right\}^{1/k}.$$

Lemma 1.4. *[16] Let $1 < k < \infty$. Then, $R \in (l_k,l)$ iff*

$$\|R\|'_{(l_k,l)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |r_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty.$$

Since

$$\|R\|_{(l_k,l)} \leq \|R\|'_{(l_k,l)} \leq 4\,\|R\|_{(l_k,l)},$$

there exists $1 \leq \xi \leq 4$ such that $\|R\|'_{(l_k,l)} = \xi\,\|R\|_{(l_k,l)}$ where

$$\|R\|_{(l_k,l)} = \sup_{N \in \mathfrak{F}} \left\{ \sum_{v=0}^{\infty} \left| \sum_{n \in N}^{\infty} r_{nv} \right|^{k^*} \right\}^{1/k^*}$$

and $\mathfrak{F}$ represents the collection of all finite subsets of $\mathbb{N}$.

Lemma 1.5. *[13] Let $1 < k < \infty$ and k^* denote the conjugate of k. Then, we have $l_k^\beta = l_{k^*}$ and $l_\infty^\beta = c^\beta = c_0^\beta = l$, $l^\beta = l_\infty$. Also, if $\Lambda \in \{l_\infty,c,c_0,l,l_k\}$ then, we have*

$$\|a\|_\Lambda^* = \|a\|_{\Lambda^\beta}$$

for all $a \in \Lambda^\beta$, where $\|.\|_{\Lambda^\beta}$ is the natural norm on Λ^β.

Lemma 1.6. *[17] Let $\Lambda \supset \Phi$ be a BK-space and $\Gamma \in \{c,c_0,\ell_\infty\}$. If $R \in (\Lambda,\Gamma)$, then*

$$\|L_R\| = \|R\|_{(\Lambda,l_\infty)} = \sup_n \|R_n\|_\Lambda^* < \infty.$$

The Hausdorff measure of noncompactness χ was introduced by Goldenstein, Gohberg and Markus [18]. Using the Hausdorff measure of noncompactness, some compact operators on various sequence spaces are characterized by many authors. For example, Mursaleen and Noman in [19, 20], Malkowsky and Rakocevic in [21] have used the Hausdorff measure of noncompactness method to characterize the class of compact operators on some known spaces, (see also [2, 4, 6, 17], [21]-[25]).
Let (Λ,d) be a metric space and $H,M \subset \Lambda$. If there exists an $h \in H$ such that $d(h,m) < \varepsilon$ for every $m \in M$, then it is said that H is an ε-net of M; if H is finite, then the ε-net H of M is called a finite ε-net of M. Let Q be a bounded subset of the metric space Λ. Then, the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon-\text{net in } \Lambda\},$$

and χ is called the Hausdorff measure of noncompactness.
Let Λ,Γ be Banach spaces. A linear operator L from Λ into Γ is called compact if its domain is all of Λ and, for every bounded sequence (λ_n) in Λ, $(L(\lambda_n))$ has a convergent subsequence in Γ. The class of all compact operators in $\mathscr{B}(\Lambda,\Gamma)$ is denoted by $\mathscr{C}(\Lambda,\Gamma)$.
The following lemmas give a calculation method for the Hausdorff measure of noncompactness of a bounded subset and the necessary and sufficient conditions a linear operator to be compact.

Lemma 1.7. *[26] Let Λ be one of the spaces c_0 or l_k for $1 \leq k < \infty$ and Q be a bounded subset of Λ. If $P_r : \Lambda \to \Lambda$ is the operator described by $P_r(\lambda) = (\lambda_0, \lambda_1, ...\lambda_r, 0, 0, ...)$ for all $\lambda \in \Lambda$, then*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{\lambda\in Q}\|(I-P_r)(\lambda)\|\right).$$

Assume that χ_1, χ_2 are two Hausdorff measures on the spaces Λ, Γ and Q is a bounded subset of Λ. The linear operator $L : \Lambda \to \Gamma$ is said to be (χ_1, χ_2)- bounded if $L(Q)$ is a bounded subset of Γ and there exists a positive constant M such that $\chi_2(L(Q)) \leq M\chi_1(Q)$ for every Q. If an operator L is (χ_1, χ_2)- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded sets } Q \subset \Lambda\}$$

is called the (χ_1, χ_2)-measure noncompactness of L. In particular, for $\chi_1 = \chi_2 = \chi$, it is written by $\|L\|_{(\chi,\chi)} = \|L\|_{\chi}$.

Lemma 1.8. *[27] $L \in \mathscr{B}(\Lambda, \Gamma)$ and S_Λ be the unit sphere in X. Then,*

$$\|L\|_{\chi} = \chi(L(S_\Lambda))$$

and

$$L \text{ is compact } \Leftrightarrow \|L\|_{\chi} = 0.$$

Lemma 1.9. *[28] Let Λ be a normed sequence space, $U = (u_{nv})$ be an infinite triangle matrix, χ_U and χ denote the Hausdorff measures of noncompactness on M_{Λ_U} and M_Λ, the collections of all bounded sets in Λ_U and Λ, respectively. Then, $\chi_U(Q) = \chi(U(Q))$ for all $Q \in M_{\Lambda_U}$.*

Lemma 1.10. *[20] Let $\Lambda \supset \Phi$ be a BK-space with AK property or $\Lambda = l_\infty$. If $R \in (\Lambda, c)$, then we have*

$$\lim_{n\to\infty} r_{nk} = \alpha_k \text{ exists for all } k,$$

$$\alpha = (\alpha_k) \in \Lambda^{\beta},$$

$$\sup_n \|R_n - \alpha\|^*_\Lambda < \infty,$$

$$\lim_{n\to\infty} R_n(\lambda) = \sum_{k=0}^{\infty} \alpha_k\lambda_k \text{ for every } \lambda = (\lambda_k) \in \Lambda.$$

Lemma 1.11. *[20] Let $X \supset \Phi$ be a BK-space. Then,*
(a) If $R \in (\Lambda, c_0)$, then

$$\|L_R\|_{\chi} = \lim_{r\to\infty}\left(\sup_{n>r}\|R_n\|^*_\Lambda\right).$$

(b) If Λ has AK property or $\Lambda = l_\infty$ and $R \in (\Lambda, c)$, then

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n>r}\|R_n - \alpha\|^*_\Lambda\right) \leq \|L_R\|_{\chi} \leq \lim_{r\to\infty}\left(\sup_{n>r}\|R_n - \alpha\|^*_\Lambda\right)$$

where $\alpha = (\alpha_k)$ defined by $\alpha_k = \lim_{n\to\infty} r_{nk}$, for all $n \in \mathbb{N}$.
(c) If $R \in (\Lambda, l_\infty)$, then

$$0 \leq \|L_R\|_{\chi} \leq \lim_{r\to\infty}\left(\sup_{n>r}\|R_n\|^*_\Lambda\right).$$

2. Matrix and compact operators on space $\left|\overline{N}^{\phi}_{p}\right|_k$

In this section, by computing the operator norms we characterize infinite matrix classes $\left(\left|\overline{N}^{\phi}_{p}\right|_k, c\right)$, $\left(\left|\overline{N}^{\phi}_{p}\right|_k, c_0\right)$, $\left(\left|\overline{N}^{\phi}_{p}\right|_k, l_\infty\right)$ and also compact matrix classes $\mathscr{C}\left(\left|\overline{N}^{\phi}_{p}\right|_k, c\right)$, $\mathscr{C}\left(\left|\overline{N}^{\phi}_{p}\right|_k, c_0\right)$, $\mathscr{C}\left(\left|\overline{N}^{\phi}_{p}\right|_k, l_\infty\right)$. Moreover, we establish some identities or estimates for the Hausdorff measure of noncompactness.
For simplicity of notation, in what follows, we use

$$\sigma_{nv} = \Delta r_{nv}\frac{P_v}{p_v} + r_{n,v+1},$$

where $\Delta r_{nv} = r_{nv} - r_{n,v+1}$.

Lemma 2.1. *Let* $1<k<\infty$. *Then,*
(i) *If* $a=(a_v)\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$, *then,* $\tilde{a}^{(k)}=(\tilde{a}_v^{(k)})\in l_{k^*}$ *for all* $\lambda\in\left|\overline{N}_p^{\phi}\right|_k$
(ii) *If* $a=(a_v)\in\left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$, *then,* $\tilde{a}^{(1)}=(\tilde{a}_v^{(1)})\in l_{\infty}$ *for all* $\lambda\in\left|\overline{N}_p^{\phi}\right|$
and the equality

$$\sum_{v=0}^{\infty}a_v\lambda_v=\sum_{v=0}^{\infty}\tilde{a}_v^{(k)}y_v \tag{2.1}$$

holds, where $y=T^{(p)}(\lambda)$ *and*

$$\tilde{a}_v^{(k)}=\phi_v^{-1/k^*}\left(\Delta a_v\frac{P_v}{p_v}+a_{v+1}\right)\ \textit{for}\ v>0,a_0=\tilde{a}_0^{(k)}.$$

Proof. (i) Let $a=(a_v)\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$. By (1.1), the equation (2.1) is immediately obtained. Also, it follows from Lemma 1.5 that $\tilde{a}^{(k)}\in l_{k^*}$ whenever $a\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$, which completes the proof.
The proof of (ii) is left to reader. □

Lemma 2.2. *Let* $1<k<\infty$. *Then, we have* $\|a\|^*_{\left|\overline{N}_p^{\phi}\right|_k}=\left\|\tilde{a}^{(k)}\right\|_{l_{k^*}}$ *for all* $a\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$ *and* $\|a\|^*_{\left|\overline{N}_p^{\phi}\right|}=\left\|\tilde{a}^{(1)}\right\|_{\infty}$ *for all* $a\in\left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$.

Proof. Take $a\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$. It is obvious from Lemma 2.1 that $\tilde{a}^{(k)}\in l_{k^*}$. Also, it follows from Lemma 1.5 and Lemma 2.1 that

$$\|a\|^*_{\left|\overline{N}_p^{\phi}\right|_k}=\sup_{\lambda\in S_{\left|\overline{N}_p^{\phi}\right|_k}}\left|\sum_{v=0}^{\infty}a_v\lambda_v\right|=\sup_{y\in S_{l_k}}\left|\sum_{v=0}^{\infty}\tilde{a}_v^{(k)}y_v\right|=\left\|\tilde{a}^{(k)}\right\|^*_{l_k}=\left\|\tilde{a}^{(k)}\right\|_{l_{k^*}}.$$

For $a\in\left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$, the proof is similar, so it is left to reader. □

Theorem 2.3. *Let* $1\leq k<\infty$, Λ *be arbitrary sequence space. Further, let* $B=(b_{nj})$ *be a matrix satisfying*

$$b_{nj}=\phi_n^{1/k^*}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}r_{vj}. \tag{2.2}$$

Then, $R\in\left(\Lambda,\left|\overline{N}_p^{\phi}\right|_k\right)$ *iff* $B\in(\Lambda,l_k)$.

Proof. Let $\lambda\in\Lambda$. Then, it follows from (2.2) that

$$\sum_{j=0}^{\infty}b_{nj}\lambda_j=\phi_n^{1/k^*}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\sum_{j=0}^{\infty}\lambda_jr_{vj},$$

which implies that $B_n(\lambda)=T_n^{(p)}(R(\lambda))$. This gives that $R_n(\lambda)\in\left|\overline{N}_p^{\phi}\right|_k$ for all $\lambda\in\Lambda$ iff $B(\lambda)\in l_k$ for all $\lambda\in\Lambda$. So, the proof of the theorem is completed. □

Let us define the matrix $\tilde{R}^{(k)}=\left(\tilde{r}_{nv}^{(k)}\right)$ with $\tilde{r}_{nv}^{(k)}=\frac{1}{\phi_v^{1/k^*}}\sigma_{nv}$ for $v>0,\tilde{r}_{n0}^{(k)}=r_{n0}$. It is clear that the matrices R and $\tilde{R}^{(k)}$ are connected by (2.1).

Theorem 2.4. (i) *Let* $1<k<\infty$ *and* $\Lambda\in\{c_0,c,l_\infty\}$. *Then,*

$$R\in\left(\left|\overline{N}_p^{\phi}\right|_k,\Lambda\right)\Rightarrow\|L_R\|=\sup_n\left\|\tilde{R}_n^{(k)}\right\|_{l_{k^*}}=\sup_n\left(\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}\right)^{1/k^*},$$

$$R\in\left(\left|\overline{N}_p^{\phi}\right|,\Lambda\right)\Rightarrow\|L_R\|=\sup_n\left\|\tilde{R}_n^{(1)}\right\|_{l_\infty}=\sup_{n,v}\left|\tilde{r}_{nv}^{(1)}\right|.$$

(ii) *Let* $1<k<\infty$. *Then, there exists* $1\leq\xi\leq4$ *such that*

$$R\in\left(\left|\overline{N}_p^{\phi}\right|_k,l\right)\Rightarrow\|L_R\|=\frac{1}{\xi}\left\|\tilde{R}^{(k)}\right\|'_{(l_k,l)}=\frac{1}{\xi}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|\right)^{k^*}\right\}^{1/k^*},$$

$$R\in\left(\left|\overline{N}_p^{\phi}\right|,l_k\right)\Rightarrow\|L_R\|=\left\|\tilde{R}^{(1)}\right\|_{(l,l_k)}=\sup_v\left\{\sum_{n=0}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^k\right\}^{\frac{1}{k}},$$

$$R\in\left(\left|\overline{N}_p^{\phi}\right|,l\right)\Rightarrow\|L_R\|=\left\|\tilde{R}_n^{(1)}\right\|_{(l,l)}=\sup_v\sum_{n=0}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|.$$

Proof. The proof of the theorem is obtained from Lemma 1.3, Lemma 1.4, Lemma 1.6, and Lemma 2.2. □

Theorem 2.5. *Let $1<k<\infty$. Then,*
a) $R\in(\left|\overline{N}_p^{\phi}\right|_k,c_0)$ iff

$$\lim_{n\to\infty}\tilde{r}_{nv}^{(k)}=0\ \textit{for all } v \tag{2.3}$$

$$\sup_n\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}<\infty \tag{2.4}$$

$$\sup_m\left\{\sum_{v=1}^{m-1}\frac{1}{\phi_v}\left|\sigma_{mv}\right|^{k^*}+\frac{1}{\phi_m}\left|r_{nm}\frac{P_m}{p_m}\right|^{k^*}\right\}<\infty \tag{2.5}$$

hold.
b) $R\in(\left|\overline{N}_p^{\phi}\right|_k,c)$ iff (2.4), (2.5) and

$$\lim_{n\to\infty}\tilde{r}_{nv}^{(k)}\ \textit{exists for all } v$$

hold.
c) $R\in(\left|\overline{N}_p^{\phi}\right|_k,l_\infty)$ iff (2.4), (2.5) hold.

Proof. Prove only the part (a) since the proofs of the other parts can be made the same way. $R\in(\left|\overline{N}_p^{\phi}\right|_k,c_0)$ if and only if $(r_{nv})_{v=0}^{\infty}\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$ and $R(\lambda)\in c_0$ for every $\lambda\in\left|\overline{N}_p^{\phi}\right|_k$. It is seen immediately from Theorem 2.1 in [10] $(r_{nv})_{v=0}^{\infty}\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$ if and only if (2.5) holds. Also, if any matrix $R\in(l_k,c_0)$, then the series $\sum_v r_{nv}\lambda_v$ converges uniformly in n and so

$$\lim_n\sum_v r_{nv}\lambda_v=\sum_v\lim_n r_{nv}\lambda_v. \tag{2.6}$$

On the other hand,

$$\lim_m\sum_{v=0}^{m}r_{nv}\lambda_v=\lim_m\sum_{v=0}^{m}b_{mv}^{(n)}y_v$$

where $B^{(n)}=(b_{mv}^{(n)})$ is defined by

$$b_{mv}^{(n)}=\begin{cases} r_{n0}, & v=0,\\ \frac{P_v}{\phi_v^{1/\mu_v^*}p_v}\left(r_{nv}-\frac{P_{v-1}}{P_v}r_{n,v+1}\right), & 1\le v<m-1\\ \frac{P_m r_{nm}}{\phi_m^{1/\mu_m^*}p_m}, & v=m, m\ge 1\\ 0, & v>m.\end{cases}$$

So, it follows from (2.6)

$$R_n(\lambda)=\lim_m\sum_{v=0}^{m}r_{nv}\lambda_v=\lim_m\sum_{v=0}^{m}b_{mv}^{(n)}y_v=\sum_{v=0}^{\infty}\tilde{r}_{nv}^{(k)}y_v=\tilde{R}_n^{(k)}(y).$$

It is clear that $R(\lambda)\in c_0$ for every $\lambda\in\left|\overline{N}_p^{\phi}\right|_k$ equals to $\tilde{R}^{(k)}(y)\in c_0$ for every $y\in l_k$ since $\left|\overline{N}_p^{\phi}\right|_k\cong l_k$. This means that $\tilde{R}^{(k)}\in(l_k,c_0)$. Applying Lemma 1.2 to the matrix $\tilde{R}^{(k)}$ the conditions (2.3) and (2.4) are obtained which completes the proof of the part (a).

□

Theorem 2.6. *The following statements hold:*
a) $R \in \left(\left|\overline{N}_p^{\phi}\right|, c_0\right)$ *iff*

$$\lim_{n\to\infty} \tilde{r}_{nv}^{(1)} = 0 \text{ for all } v \tag{2.7}$$

$$\sup_{n,v} \left|\tilde{r}_{nv}^{(1)}\right| < \infty \tag{2.8}$$

$$\sup_{v} \left\{ |\sigma_{nv}| + \left| r_{nv} \frac{P_v}{p_v} \right| \right\} < \infty, \text{ for all } n \tag{2.9}$$

hold.
b) $R \in \left(\left|\overline{N}_p^{\phi}\right|, c\right)$ *iff* (2.8), (2.9) *and*

$$\lim_{n\to\infty} \tilde{r}_{nv}^{(1)} \text{ exists for all } v$$

hold.
c) $R \in \left(\left|\overline{N}_p^{\phi}\right|, l_\infty\right)$ *iff* (2.8), (2.9) *hold.*

Proof. (b) Let $R \in \left(\left|\overline{N}_p^{\phi}\right|, c\right)$. $R \in \left(\left|\overline{N}_p^{\phi}\right|, c\right)$ if and only if $(r_{nv})_{v=0}^{\infty} \in \left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$ and $R(\lambda) \in c$ for every $\lambda \in \left|\overline{N}_p^{\phi}\right|$. It follows from Theorem 2.1 in [10], $(r_{nv})_{v=0}^{\infty} \in \left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$ if and only if (2.9) holds. Further, for any matrix $R \in (l, c)$, the series $\sum_v r_{nv}\lambda_v$ converges uniformly in n and so

$$\lim_n \sum_v r_{nv}\lambda_v = \sum_v \lim_n r_{nv}\lambda_v. \tag{2.10}$$

Also,

$$\lim_m \sum_{v=0}^{m} r_{nv}\lambda_v = \lim_m \sum_{v=0}^{m} d_{mv}^{(n)} y_v$$

where the matrix $D^{(n)} = (d_{mv}^{(n)})$ is given by

$$d_{mv}^{(n)} = \begin{cases} \frac{P_v}{p_v}\left(r_{nv} - \frac{P_{v-1}}{P_v} r_{n,v+1}\right), 0 \le v < m-1 \\ \frac{P_m}{p_m} r_{nm}, \qquad v = m, m \ge 1 \\ 0, \qquad v > m. \end{cases}$$

So, it is deduced from (2.10)

$$R_n(\lambda) = \lim_m \sum_{v=0}^{m} r_{nv}\lambda_v = \lim_m \sum_{v=0}^{m} d_{mv}^{(n)} y_v = \sum_{v=0}^{\infty} \tilde{r}_{nv}^{(1)} y_v = \tilde{R}_n^{(1)}(y).$$

It is obvious that $R(\lambda) \in c$ for every $\lambda \in \left|\overline{N}_p^{\phi}\right|$ if and only if $\tilde{R}^{(1)}(\lambda) \in c$ for every $y \in l$, i.e., $\tilde{R}^{(1)} \in (l, c)$. Applying Lemma 1.2 to the matrix $\tilde{R}^{(1)}$ the conditions (2.7), (2.8) are obtained. This completes the proof of the part (b). The other parts can be proved by the similar way with Lemma 1.2. □

Take the matrix $L = (l_{nj})$ defined by

$$l_{nj} = \begin{cases} 1, & 0 \le j \le n \\ 0, & j > n. \end{cases}$$

Then, since $b_s = \{l_\infty\}_L$ and $c_s = \{c\}_L$, the matrix classes $\left(\left|\bar{N}_p^{\phi}\right|_k, c_s\right)$ and $\left(\left|\bar{N}_p^{\phi}\right|_k, b_s\right)$ can be characterized as follows with Lemma 1.1:

Corollary 2.7. *Let* $1 < k < \infty$. $R \in \left(\left|\overline{N}_p^{\phi}\right|_k, c_s\right)$ *iff*

$$\lim_{n\to\infty} \tilde{r}(n, v) \text{exists for all } v$$

$$\sup_{n}\sum_{v=0}^{\infty}|\tilde{r}(n,v)|^{k^*}<\infty \tag{2.11}$$

$$\sup_{m}\left\{\sum_{v=1}^{m-1}|\tilde{r}(n,v)|^{k^*}+\frac{1}{\phi_m}\left|r(n,m)\frac{P_m}{p_m}\right|^{k^*}\right\}<\infty, \tag{2.12}$$

$R\in(\left|\overline{N}_p^{\phi}\right|_k,b_s)$ if and only if (2.11) and (2.12) hold where $r(n,v)=\sum_{j=0}^{n}r_{jv}$, $R(n,v)$ and $\tilde{R}(n,v)$ are connected by (2.1).

Theorem 2.8. *Suppose that $1<k<\infty$. Then,*
a) $R\in(\left|\overline{N}_p^{\phi}\right|_k,c_0)$ iff

$$\|L_R\|_{\chi}=\limsup_{n\to\infty}\left(\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}\right)^{1/k^*}$$

and $R\in\mathscr{C}(\left|\bar{N}_p^{\phi}\right|_k,c_0)$ iff $\limsup_{n\to\infty}\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}=0$.
b) $R\in(\left|\overline{N}_p^{\phi}\right|_k,c)$ iff

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}-\alpha_v\right|^{k^*}\right)^{1/k^*}\leq\|L_R\|_{\chi}\leq\limsup_{n\to\infty}\left(\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}-\alpha_v\right|^{k^*}\right)^{1/k^*}$$

and $R\in\mathscr{C}(\left|\overline{N}_p^{\phi}\right|_k,c_0)$ iff $\limsup_{n\to\infty}\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}-\alpha_v\right|^{k^*}=0$ where $\alpha_v=\lim_{n\to\infty}\tilde{r}_{nv}^{(k)}$.
c) $R\in(\left|\overline{N}_p^{\phi}\right|_k,l_\infty)$ iff

$$0\leq\|L_R\|_{\chi}\leq\limsup_{n\to\infty}\left(\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}\right)^{1/k^*}$$

also, if $\limsup_{n\to\infty}\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}=0$, then $R\in\mathscr{C}(\left|\overline{N}_p^{\phi}\right|_k,l_\infty)$.

Proof. To avoid repetition, only the proof of b is made and the proofs of (a) and (c) are left to the reader.
(b) Let $R\in\left(\left|\overline{N}_p^{\phi}\right|_k,c\right)$. To compute the Hausdorff measure of noncompactness of L_R, take the unit sphere $S_{\left|\overline{N}_p^{\phi}\right|_k}$ in the space $\left|\overline{N}_p^{\phi}\right|_k$. It is written from Lemma 1.8 that

$$\|L_R\|_{\chi}=\chi(RS_{\left|\overline{N}_p^{\phi}\right|_k}).$$

On the other hand, since $\left|\overline{N}_p^{\phi}\right|_k\cong l_k$, $R\in\left(\left|\overline{N}_p^{\phi}\right|_k,c\right)$ if and only if $\tilde{R}^{(k)}\in(l_k,c)$, and so

$$\|L_R\|_{\chi}=\chi(RS_{\left|\overline{N}_p^{\phi}\right|_k})=\chi(\tilde{R}^{(k)}T^{(p)}S_{\left|\overline{N}_p^{\phi}\right|_k})=\left\|L_{\tilde{R}^{(k)}}\right\|_{\chi}$$

which implies, by Lemma 1.11,

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{R}_n^{(k)}-\alpha\right\|_{l_k}^{*}\right)\leq\|L_R\|_{\chi}\leq\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{R}_n^{(k)}-\alpha\right\|_{l_k}^{*}\right), \tag{2.13}$$

where $\alpha_v=\lim_{n\to\infty}\tilde{r}_{nv}^{(k)}$, for all $v\geq 0$.
By Lemma 1.5, $\left\|\tilde{R}_n^{(k)}-\alpha\right\|_{l_k}^{*}=\left\|\tilde{R}_n^{(k)}-\alpha\right\|_{l_{k^*}}$. The last equality completes the first part of the proof of (b) with (2.13). Moreover, the compactness of L_R is immediately deduced from Lemma 1.8. So, the proof of (b) is completed. □

We have the following theorems by following the above lines:

Theorem 2.9. *(a) If $R\in\left(\left|\overline{N}_p^{\phi}\right|,c_0\right)$. Then*

$$\|L_R\|_{\chi}=\limsup_{n\to\infty}\left\|\tilde{R}_n^{(k)}\right\|_{l_\infty}=\limsup_{n\to\infty}\sup_{v}\left|\tilde{r}_{nv}^{(1)}\right|$$

and $R \in \mathscr{C}\left(\left|\overline{N}_p^{\phi}\right|, c_0\right)$ *iff* $\limsup_{n\to\infty} \sup_v \left|\tilde{r}_{nv}^{(1)}\right| = 0.$

(b) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|, c\right)$*, then*

$$\frac{1}{2}\limsup_{n\to\infty}\sup_v \left|\tilde{r}_{nv}^{(1)} - \alpha_v\right| \leq \|L_R\|_\chi \leq \limsup_{n\to\infty}\sup_v \left|\tilde{r}_{nv}^{(1)} - \alpha_v\right|$$

and $R \in \mathscr{C}\left(\left|\overline{N}_p^{\phi}\right|, c\right)$ *iff* $\limsup_{n\to\infty}\sup_v \left|\tilde{r}_{nv}^{(1)} - \alpha_v\right| = 0$ *where* $\alpha_v = \lim_{n\to\infty} \tilde{r}_{nv}^{(1)}$*, for all* $v \in \mathbb{N}$.

(c) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|, l_\infty\right)$*, then*

$$0 \leq \|L_R\|_\chi \leq \limsup_{n\to\infty}\sup_v \left|\tilde{r}_{nv}^{(1)}\right|$$

and $R \in \mathscr{C}\left(\left|\overline{N}_p^{\phi}\right|, c_0\right)$ *if* $\limsup_{n\to\infty}\sup_v \left|\tilde{r}_{nv}^{(1)}\right| = 0.$

Theorem 2.10. (a) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|, l_k\right)$, $1 \leq k < \infty$*, then*

$$\|L_R\|_\chi = \lim_{j\to\infty}\left\{\sup_v \left(\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^k\right)^{1/k}\right\},$$

and R *is a compact operator iff* $\lim_{j\to\infty}\sup_v \sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^k = 0.$

(b) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|_k, l\right)$, $1 < k < \infty$*, then there exists* $1 \leq \xi \leq 4$ *such that*

$$\|L_R\|_\chi = \frac{1}{\xi}\lim_{j\to\infty}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|\right)^{k^*}\right\}^{1/k^*},$$

and R *is compact a compact operator iff* $\lim_{j\to\infty}\sum_{v=0}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|\right)^{k^*} = 0.$

Proof. (a) Let $S_{\left|\overline{N}_p^{\phi}\right|}$ be a unit sphere in the space $\left|\overline{N}_p^{\phi}\right|$ and $R = \tilde{R}^{(1)} \circ T^{(p)}$. Since $\lambda \in S_{\left|\overline{N}_p^{\phi}\right|}$, $y = T^{(p)}(\lambda) \in S_l$. So, by Lemma 1.7, Lemma 1.9 and Lemma 1.3, it is written that

$$\begin{aligned}
\|R\|_\chi &= \chi\left(RS_{\left|\overline{N}_p^{\phi}\right|}\right) = \chi\left(\tilde{R}^{(1)} \circ T^{(p)}S_{\left|\overline{N}_p^{\phi}\right|_k}\right) \\
&= \lim_{j\to\infty}\left(\sup_{y \in T^{(p)}S_{\left|\overline{N}_p^{\phi}\right|}}\left\|(I-P_j)(\tilde{R}^{(1)}(y))\right\|\right) \\
&= \lim_{j\to\infty}\sup_v\left\{\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^k\right\}^{1/k}
\end{aligned}$$

which completes the proof of the first part with Lemma 1.8. The proof of (b) is similar, so it is omitted. □

3. Conclusion

The approach of constructing a lot of new sequence spaces by means of the matrix domain of some particular limitation methods have recently been employed by several authors in many research papers. Also, with a different point of view, using the concept of absolute summability method new sequence spaces have taken into the literature. For instance, in recent paper, $\left|\overline{N}_p^{\phi}\right|_k$ has been generated from the space l_k as a set of all series summable by the absolute weighted mean method by Mohapatra and Sarıgöl [10] and Sarıgöl [8, 9]. In the present study, as a continuation of these papers, certain compact and matrix operatos from this space to one of the classical sequence spaces c, l_∞, c_0 are characterized and their norms and Hausdorff measures of noncompactness are determined. So, it has been brought a different perspective and studying field.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Oscillatory Criteria of Nonlinear Higher Order Ψ-Hilfer Fractional Differential Equations

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Abstract

In this paper, we study the forced oscillatory theory for higher order fractional differential equations with damping term via Ψ-Hilfer fractional derivative. We get sufficient conditions which ensure the oscillation of all solutions and give an illustrative example for our results. The Ψ-Hilfer fractional derivative according to the choice of the Ψ function is a generalization of the different fractional derivatives defined earlier. The results obtained in this paper are a generalization of the known results in the literature, and present new results for some fractional derivatives.

1. Introduction

Arbitrary order differential and integration notions are notions that combine and generalize integer order derivatives and n-fold integrals. Fractional differential theory is a very good tool that can be used to describe the inherited properties of various items and operations. This is an important advantage for fractional derivatives compared to integer order derivatives. This advantage of fractional derivatives is used in mathematical modeling of the mechanical and electrical properties of objects, in many other fields such as fluid theory, electrical circuits, electro-analytical chemistry [1]-[6]. Many definitions of fractional derivatives and integrals have been made, for more details, we recommend the monographs [7]-[10]. In recent years, the behavior of solutions of fractional differential equations has been an attractive area for researchers. Especially, the oscillation behavior of solutions has been studied by many researchers [11]-[19]. We also refer the reader to the papers [20], [21] for the oscillation of dynamic equations on time scales and to the papers [22], [23] for the oscillation of functional differential equations.
In [14] the authors considered the oscillatory criteria of nonlinear fractional differential equations by taking fractional initial value problem

$$\begin{aligned} D_a^{\mu}x(t)+f_1(t,x) &= v(t)+f_2(t,x), \quad t>a,\ 0<\mu\leq 1 \\ \lim_{t\to a^+} J_a^{1-\mu}x(t) &= b, \end{aligned}$$

where D_a^{μ} shows μ order Riemann-Liouville fractional derivative, $J_a^{1-\mu}$ is $1-\mu$ order Riemann-Liouville fractional integral. Recently, in [24] Vivek et al. studied the oscillatory theory for Ψ-Hilfer fractional type fractional differential equations

$$\begin{aligned} {}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}x(t)+f_1(t,x) &= \omega(t)+f_2(t,x), \qquad 0<\mu<1,\ 0\leq\nu\leq 1 \\ I_{a^+}^{1-\eta;\Psi}x(t) &= b_1 \end{aligned}$$

where ${}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}$ denotes Ψ-Hilfer fractional derivative and $I_{a^+}^{1-\eta;\Psi}$ is the Ψ-Riemann-Liouville fractional integral with $\eta=\mu+\nu(1-\mu)$.

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In [18] the authors examined oscillation of the solutions of forced fractional differential equations with damping term via the Riemann-Liouville fractional derivative

$$\begin{aligned} \left(D_{0^+}^{1+\mu}y\right)(t)+p(t)\left(D_{0^+}^{\mu}y\right)(t)+q(t)f(y(t)) &= g(t), \quad t>0 \\ \left(I_{0^+}^{1-\mu}y\right)(0^+) &= b, \end{aligned}$$

where $\mu \in (0,1)$.
In this paper, inspired by the above articles, we studied the oscillation properties of forced fractional differential equations with damping term

$$\begin{aligned} D\left({}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)\right)+p(x){}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)+q(x)f(y(x)) &= g(x), \quad x>0 \\ \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^{m-i} I_{a^+}^{1-\eta;\Psi}y(x)|_a &= y_i,\ i=1,2,\dots,m \end{aligned} \tag{1.1}$$

where $m-1<\mu<m$, $0\le\nu\le1$ is a constant and $\eta=\mu+\nu(m-\mu)$, ${}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)$ is the Ψ-Hilfer fractional differential operator of order μ type ν of $y(x)$. Throughout this paper, we assume that
(A) $p(x)\in C(\mathbb{R}^+,\mathbb{R})$, $q(x)\in C(\mathbb{R}^+,\mathbb{R}^+)$, $f(x)\in C(\mathbb{R},\mathbb{R})$ and $f(y)/y>0$ for all $y\neq0$, $g(x)\in C(\mathbb{R}^+,\mathbb{R})$.

Definition 1.1 ([25])**.** *A solution $y(x)$ of problem (1.1) is said to be oscillatory if it has arbitrarily large zeros for $x\ge x_0$ there exists a sequence of zeros $\{x_n\}$ of y such that $\lim_{n\to\infty}x_n=\infty$. Otherwise, y is said to be non-oscillatory.*

2. Preliminaries

In this section, we mention some basic definitions and theorems which will be used in the study.

Definition 2.1 ([8])**.** *Let f be a function defined on $[a,b]$, $(-\infty<a<b<\infty)$. μ-th left-sided and right-sided Riemann-Liouville fractional integrals of f are given by*

$$I_{a^+}^{\mu}f(x):=\frac{1}{\Gamma(\mu)}\int_a^x(x-t)^{\mu-1}f(t)dt, \quad x>a,\ \mu>0$$

and

$$I_{b^-}^{\mu}f(x):=\frac{1}{\Gamma(\mu)}\int_x^b(t-x)^{\mu-1}f(t)dt, \quad x<b,\ \mu>0$$

respectively.

Definition 2.2 ([8])**.** *Assume $\lceil\mu\rceil=m$, $m\in\mathbb{N}_0$ and $f(x)\in C^m(a,b)$. Left-sided and right-sided Riemann-Liouville fractional derivatives of f of order μ, are defined respectively by*

$$\begin{aligned} D_{a^+}^{\mu}f(x) &= \left(\frac{d}{dx}\right)^m I_{a^+}^{m-\mu}f(x) \\ &= \frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\int_a^x(x-t)^{m-\mu-1}f(t)dt \end{aligned}$$

and

$$\begin{aligned} D_{b^-}^{\mu}f(x) &= (-1)^m\left(\frac{d}{dx}\right)^m I_{b^-}^{m-\mu}f(x) \\ &= \frac{(-1)^m}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\int_x^b(t-x)^{m-\mu-1}f(t)dt. \end{aligned}$$

In [26], Hilfer generalized the Riemann-Liouville fractional derivative operator by introducing a right-sided fractional derivative operator.

Definition 2.3. *Let $\lceil\mu\rceil=m$, $m\in\mathbb{N}_0$, $\nu\in[0,1]$ and $f(x)\in C^n(a,b)$. The left-sided and right sided Hilfer fractional derivatives of f of order μ and type ν are given by*

$$D_{a^+}^{\mu,\nu}f(x)=I_{a^+}^{\eta-\mu}\left(\frac{d}{dx}\right)^m I_{a^+}^{(1-\nu)(m-\mu)}f(x)$$

and

$$D_{b^-}^{\mu,\nu} f(x) = I_{b^-}^{\eta-\mu} \left(-\frac{d}{dx}\right)^m I_{b^-}^{(1-\nu)(m-\mu)} f(x)$$

where $\eta = \mu + \nu(m-\mu)$.

Due to a large number of definitions, the next definition is a significant approach because of the kernel has an arbitrary function Ψ.

Definition 2.4 ([8]). *Let f be a function defined on (a,b), $(-\infty \le a < b \le \infty)$ and $\mu > 0$ and also assume $\Psi(x)$ is a positive monotone and increasing function on $(a,b]$, $\Psi'(x)$ is continuous on (a,b). The left and right-sided fractional integrals of f with respect to Ψ of order μ are given by*

$$I_{a^+}^{\mu;\Psi} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (\Psi(x)-\Psi(t))^{\mu-1} f(t)\Psi'(t)dt$$

and

$$I_{b^-}^{\mu;\Psi} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (\Psi(t)-\Psi(x))^{\mu-1} f(t)\Psi'(t)dt.$$

Lemma 2.5 ([8]). *Assume $\mu > 0$ and $\nu > 0$. Then,*

$$I_{a^+}^{\mu;\Psi} I_{a^+}^{\nu;\Psi} f(x) = I_{a^+}^{\mu+\nu;\Psi} f(x)$$

and

$$I_{b^-}^{\mu;\Psi} I_{b^-}^{\nu;\Psi} f(x) = I_{b^-}^{\mu+\nu;\Psi} f(x)$$

semigroup property hold.

Definition 2.6 ([8]). *Assume f is a function defined on $[a,b]$, $\Psi(x) \neq 0$ and $\lceil \mu \rceil = m$, $m \in \mathbb{N}$. The right and left-sided Riemann-Liouville derivatives of f with respect to another function Ψ of order μ are given respectivly by*

$$\begin{aligned} D_{a^+}^{\mu;\Psi} f(x) &= \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{a^+}^{m-\mu;\Psi} f(x) \\ &= \frac{1}{\Gamma(m-\mu)} \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m \int_a^x (\Psi(x)-\Psi(t))^{m-\mu-1} \Psi'(t) f(t)dt \end{aligned}$$

and

$$\begin{aligned} D_{b^-}^{\mu;\Psi} f(x) &= \left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{b^-}^{m-\mu;\Psi} f(x) \\ &= \frac{1}{\Gamma(m-\mu)} \left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m \int_x^b (\Psi(t)-\Psi(x))^{m-\mu-1} \Psi'(t) f(t)dt. \end{aligned}$$

In [27], Sousa and Oliveria presented a new fractional derivative which unifies Hilfer fractional derivative and Riemann-Lioville derivative with respect to another function.

Definition 2.7. *Assume $\lceil \mu \rceil = m$, $m \in \mathbb{N}$ and $\nu \in [0,1]$. Also let $f \in C^n([a,b],\mathbb{R})$, $-\infty \le a < b \le \infty$, Ψ be an increasing function on $[a,b]$ and $\Psi'(x) \neq 0$, for all $x \in [a,b]$. The right and left-sided $\Psi-$Hilfer fractional derivatives of f of order μ and type ν, are given by*

$${}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} f(x) = I_{a^+}^{\nu(m-\mu);\Psi} \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{a^+}^{(1-\nu)(m-\mu);\Psi} f(x)$$

and

$${}^H\mathbb{D}_{b^-}^{\mu,\nu;\Psi} f(x) = I_{b^-}^{\nu(m-\mu);\Psi} \left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{b^-}^{(1-\nu)(m-\mu);\Psi} f(x).$$

Remark 2.8 ([27]). *The Ψ-Hilfer fractional derivative can be given as following for $\eta = \mu + \nu(m-\mu)$*

$${}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} f(x) = I_{a^+}^{\eta-\mu;\Psi} D_{a^+}^{\eta;\Psi} f(x)$$

and

$${}^H\mathbb{D}_{b^-}^{\mu,\nu;\Psi} f(x) = I_{b^-}^{\eta-\mu;\Psi} (-1)^m D_{b^-}^{\eta;\Psi} f(x).$$

Theorem 2.9 ([27]). *Let $f \in C^m[a,b]$, $\lceil \mu \rceil = m$ and $\nu \in [0,1]$. Then*

$$I_{a^+}^{\mu;\Psi}\,{}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} f(x) = f(x) - \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^{m-i} I_{a^+}^{(1-\nu)(m-\mu);\Psi} f(a) \tag{2.1}$$

and

$$I_{b^-}^{\mu;\Psi}\,{}^{H}\mathbb{D}_{b^-}^{\mu,\nu;\Psi} f(x) = f(x) - \sum_{i=1}^{m} \frac{(-1)^i\,(\Psi(b)-\Psi(x))^{\eta-i}}{\Gamma(\eta-i+1)} \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^{m-i} I_{b^-}^{(1-\nu)(m-\mu);\Psi} f(b).$$

3. Main results

Theorem 3.1. *Assume (A) and the following conditions meet*

$$\liminf_{x\to+\infty} (\Psi(x))^{1-\eta} I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) = -\infty, \tag{3.1}$$

$$\limsup_{x\to+\infty} (\Psi(x))^{1-\eta} I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) = \infty, \tag{3.2}$$

where $M \in \mathbb{R}$ is a constant and $V(t) = \exp\int_{x_0}^{t} p(\xi)d\xi$. Then each solution of (1.1) oscillates for every sufficiently large T.

Proof. To obtain contradiction, assume that $y(x)$ is a non-oscillatory solution of (1.1). We can suppose that there exist $T > 0$, $x_0 > x$ without losing any generality, such that $y(x) > 0$ for all $x \geq x_0$. According to (1.1) and *(A)*,

$$\begin{aligned}
\left[{}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\;V(x)\right]' &= D\left[{}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\right]V(x) + {}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\;p(x)V(x)\\
&= -q(x)f(y(x))V(x) + g(x)V(x)\\
&< g(x)V(x).
\end{aligned}$$

Integrating the inequality from x_0 to x, we get

$${}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\;V(x) < {}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x_0)\;V(x_0) + \int_{x_0}^{x} g(t)V(t)dt = M + \int_{x_0}^{x} g(t)V(t)dt,$$

where $M = {}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x_0)\;V(x_0)$. From (2.1) we can obtain

$$y(x) < \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i + I_{a^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right). \tag{3.3}$$

Multiplying the both sides of inequality (3.3) with $(\Psi(x))^{1-\eta}$ we obtain

$$\begin{aligned}
(\Psi(x))^{1-\eta} y(x) &\leq (\Psi(x))^{1-\eta} \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i\\
&\quad + (\Psi(x))^{1-\eta} I_{a^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right)\\
&\leq (\Psi(x))^{1-\eta} \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i\\
&\quad + (\Psi(x))^{1-\eta} \frac{1}{\Gamma(\mu)}\int_a^T (\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)} + \frac{1}{V(\tau)}\int_{x_0}^{\tau} g(t)V(t)dt\right)d\tau\\
&\quad + (\Psi(x))^{1-\eta} I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right),\; x \geq T.
\end{aligned}$$

Define

$$\Phi(x) = (\Psi(x))^{1-\eta} \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i$$

and

$$\Psi(x,T)=(\Psi(x))^{1-\eta}\frac{1}{\Gamma(\mu)}\int_a^T(\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)d\tau.$$

Then we get,

$$0<(\Psi(x))^{1-\eta}y(x)\leq\Phi(x)+\Psi(x,T)+(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right),\ x\geq T. \tag{3.4}$$

We take two cases as follows.

Case(1): Assume $0<\mu\leq 1$ and so on $0<\eta\leq 1$. Then $m=1$ and $|\Phi(x)|=\left|y_1(\Psi(x))^{1-\eta}\frac{(\Psi(x)-\Psi(a))^{\eta-1}}{\Gamma(\eta)}\right|$. For $x>T_1>T$, we get

$$|\Phi(x)|=\left|y_1\frac{1}{\Gamma(\eta)}\left(\frac{\Psi(x)-\Psi(a)}{\Psi(x)}\right)^{\eta-1}\right|\leq\frac{|y_1|}{\Gamma(\eta)}\left(\frac{\Psi(T_1)-\Psi(a)}{\Psi(T_1)}\right)^{\eta-1}:=c_1(T_1).$$

Furthermore we have

$$\begin{aligned}|\Psi(x,T)| &= \left|(\Psi(x))^{1-\eta}\frac{1}{\Gamma(\mu)}\int_a^T(\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)d\tau\right|\\ &\leq \frac{1}{\Gamma(\mu)}\int_a^T\left|\frac{(\Psi(x)-\Psi(\tau))^{\mu-1}}{(\Psi(x))^{\eta-1}}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)\right|d\tau\\ &\leq \frac{1}{\Gamma(\mu)}\int_a^T\left(\frac{\Psi(x)-\Psi(\tau)}{(\Psi(x))^{1-\nu}}\right)^{\mu-1}\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau\\ &\leq \frac{1}{\Gamma(\mu)}\int_a^T\left(\frac{\Psi(T_1)-\Psi(\tau)}{(\Psi(T_1))^{1-\nu}}\right)^{\mu-1}\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau\\ &:= c_2(T,T_1).\end{aligned}$$

Using inequality (3.4), we obtain

$$0<(\Psi(x))^{1-\eta}y(x)\leq c_1(T_1)+c_2(T,T_1)+(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right),$$

and then

$$(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-[c_1(T_1)+c_2(T,T_1)]. \tag{3.5}$$

Taking limit of (3.5) as $x\to\infty$ we get

$$\liminf_{x\to\infty}(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-[c_1(T_1)+c_2(T,T_1)]>-\infty$$

which contradicts the condition (3.1).

Case(2): Assume $\mu>1$. Then $m\geq 2$ and $m-1\leq\eta\leq m$. For $x\geq T_2$ we get

$$\begin{aligned}|\Phi(x)| &= \left|(\Psi(x))^{1-\eta}\sum_{i=1}^{m}\frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)}y_i\right|\\ &\leq \left(\frac{\Psi(x)-\Psi(a)}{\Psi(x)}\right)^{\eta-1}\sum_{i=1}^{m}\frac{(\Psi(x)-\Psi(a))^{1-i}}{\Gamma(\eta-i+1)}|y_i|\\ &\leq \sum_{i=1}^{m}\frac{(\Psi(x)-\Psi(a))^{1-i}}{\Gamma(\eta-i+1)}|y_i|\\ &\leq \sum_{i=1}^{m}\frac{(\Psi(T_2)-\Psi(a))^{1-i}}{\Gamma(\eta-i+1)}|y_i|:=c_3(T_2).\end{aligned}$$

Since $\eta=\mu+\nu(m-\mu)\geq\mu$, $\frac{(\Psi(x)-\Psi(a))^{\mu-1}}{(\Psi(x))^{\eta-1}}\leq 1$ for $m-1<\mu<m$. Then we have

$$\begin{aligned}
|\Psi(x,T)| &= \left|(\Psi(x))^{1-\eta}\frac{1}{\Gamma(\mu)}\int_a^T(\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)d\tau\right| \\
&\leq \frac{1}{\Gamma(\mu)}\int_a^T\frac{(\Psi(x)-\Psi(\tau))^{\mu-1}}{(\Psi(x))^{\eta-1}}\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau \\
&\leq \frac{1}{\Gamma(\mu)}\int_a^T\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau \\
&:= c_4(T).
\end{aligned}$$

Using inequality (3.4), we conclude that

$$(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-\left[c_3(T_2)+c_4(T)\right], \tag{3.6}$$

hence taking limit of (3.6) as $x\to\infty$ we obtain

$$\liminf_{x\to\infty}(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-\left[c_3(T_2)+c_4(T)\right]>-\infty,$$

which contradicts (3.1).

Consequently, we conclude that $y(x)$ is an oscillatory solution of (1.1). If $y(x)$ is eventually negative, a contradiction can be obtained with (3.2) similarly. □

Choosing a special Ψ function and specific μ and ν real numbers, the Ψ-Hilfer fractional derivative turn into 22 different fractional derivative which is defined before. Sousa and Oliveira remarked on all of these 22 different situations in [27].

Remark 3.2. *If we take limit $\nu\to 0$ and $\Psi(x)=x^\rho$, then we have the following fractional derivative*

$${}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)={}^H\mathbb{D}_{a^+}^{\mu,0;x^\rho}y(x)=\left(\frac{1}{x^{\rho-1}}\frac{d}{dx}\right)^m I_{a^+}^{m-\mu;x^\rho}y(x)$$

which is defined by Katugampola in [28].

Remark 3.3. *If we take limit $\nu\to 1$ and $\Psi(x)=x^\rho$, then we have Caputo type Katugampola fractional derivative which is defined in [29] as follows*

$${}^H\mathbb{D}_{a^+}^{\mu,1;x^\rho}y(x)=I_{a^+}^{(m-\mu);x^\rho}\left(\frac{1}{x^{\rho-1}}\frac{d}{dx}\right)^m y(x).$$

Remark 3.4. *If we take limit $\nu\to 0$ and $\Psi(x)=\ln x$, then we have Hadamard fractional derivative*

$${}^H\mathbb{D}_{a^+}^{\mu,0;\ln x}y(x)=\left(x\frac{d}{dx}\right)^m I_{a^+}^{m-\mu;\ln x}y(x).$$

Example 3.5. *Consider the initial value problem*

$$\begin{aligned}
D\left({}^H\mathbb{D}_{a^+}^{\frac{3}{2},0;\ln x}y(x)\right)-\frac{1}{x}\,{}^H\mathbb{D}_{a^+}^{\frac{3}{2},0;\ln x}y(x)+e^{x^2}y^3e^y &= x\sin(\ln x) \\
I_{a^+}^{\frac{3}{2};\ln x}y(t) &= b.
\end{aligned} \tag{3.7}$$

Here $\mu=3/2$, $\nu=0$, $\Psi=\ln x$, $p(x)=-1/x$, $q(x)=e^{x^2}$, $f(y)=y^3e^y$, $g(x)=x\sin(\ln x)$ and $V(x)=\exp\int_{x_0}^{x}p(t)dt=x_0/x$. Then

$$\begin{aligned}
\int_{x_0}^{x}g(t)V(t)dt &= \int_{x_0}^{x}t\sin(\ln t)\frac{x_0}{t}dt \\
&= x_0\int_{\ln x_0}^{\ln x}e^{\xi}\sin\xi\,d\xi \\
&= \frac{x_0}{2}\left[x\left(\sin(\ln x)-\cos(\ln x)\right)+x_0\left(\cos(\ln x_0)-\sin(\ln x_0)\right)\right] \\
&= \frac{x_0}{2}\left[\frac{2x}{\sqrt{2}}\sin\left(\ln x-\frac{\pi}{4}\right)+x_0\left(\cos(\ln x_0)-\sin(\ln x_0)\right)\right].
\end{aligned}$$

Set $x_0 = 1$. *Then, we can obtain*

$$\begin{aligned} I_{a^+}^{\frac{3}{2};\ln x}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) &= \frac{1}{\Gamma(3/2)}\int_a^x (\ln x-\ln t)^{1/2}\left(\frac{M}{V(t)}+\frac{1}{V(t)}\left[\frac{t}{\sqrt{2}}\sin\left(\ln t-\frac{\pi}{4}\right)+\frac{1}{2}\right]\right)\frac{dt}{t} \\ &= \frac{2}{\sqrt{\pi}}\int_a^x (\ln x-\ln t)^{1/2}\left(\left(M+\frac{1}{2}\right)t+\frac{t^2}{\sqrt{2}}\sin\left(\ln t-\frac{\pi}{4}\right)\right)\frac{dt}{t}. \end{aligned}$$

Set $\ln x-\ln t=\xi^2$. *Then the above integral can be written as the form:*

$$\begin{aligned} &\frac{2}{\sqrt{\pi}}\int_a^x (\ln x-\ln t)^{1/2}\left(\left(M+\frac{1}{2}\right)t+\frac{t^2}{\sqrt{2}}\sin\left(\ln t-\frac{\pi}{4}\right)\right)\frac{dt}{t} \\ &= \frac{2}{\sqrt{\pi}}\int_{\sqrt{\ln\frac{x}{a}}}^{0}\xi\left(\left(M+\frac{1}{2}\right)xe^{-\xi^2}+\frac{x^2e^{-2\xi^2}}{\sqrt{2}}\sin\left(\ln x-\xi^2-\frac{\pi}{4}\right)\right)(-2\xi)d\xi \\ &= \frac{2(2M+1)x}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{2\sqrt{2}x^2}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin\left(\ln x-\xi^2-\frac{\pi}{4}\right)d\xi \\ &= \frac{2(2M+1)x}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{2\sqrt{2}x^2}{\sqrt{\pi}}\sin\left(\ln x-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi \\ &\quad -\frac{2\sqrt{2}x^2}{\sqrt{\pi}}\cos\left(\ln x-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi. \end{aligned}$$

Letting $x\to+\infty$, *because of* $|\xi^2e^{-2\xi^2}\cos(\xi^2)|\le\xi^2e^{-2\xi^2}$, $|\xi^2e^{-2\xi^2}\sin(\xi^2)|\le\xi^2e^{-2\xi^2}$ *and*

$$\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}d\xi=\lim_{x\to\infty}\left[-\frac{\xi e^{-2\xi^2}}{4}\Big|_0^{\sqrt{\ln\frac{x}{a}}}+\frac{1}{4}\int_0^{\sqrt{\ln\frac{x}{a}}}e^{-2\xi^2}d\xi\right]=0+\frac{1}{4}\frac{\sqrt{2\pi}}{4}=\frac{\sqrt{2\pi}}{16},$$

we know that $\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi$ *and* $\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi$ *are convergent. Thus, we can set*

$$\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi=K \text{ and } \lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi=L.$$

Selecting sequence $\{x_k\}=\{e^{\frac{5\pi}{2}+\frac{\pi}{4}+2k\pi-\arctan\frac{-L}{K}}\}$, $\lim_{k\to\infty}x_k=\infty$, *then we calculate*

$$\begin{aligned} \lim_{k\to\infty}\Bigg\{(\ln x_k)^{-1/2}x_k\Bigg[&\frac{2M+1}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{\sqrt{2}x_k}{\sqrt{\pi}}\Bigg(\sin\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi \\ &-\cos\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi\Bigg)\Bigg]\Bigg\}. \end{aligned}$$

Firstly, let compute the following limit.

$$\begin{aligned} &\lim_{k\to\infty}\left(\sin\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi-\cos\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi\right) \\ &= K\cdot\lim_{k\to\infty}\sin\left(\frac{5\pi}{2}+2k\pi-\arctan\frac{-L}{K}\right)-L\cdot\lim_{k\to\infty}\cos\left(\frac{5\pi}{2}+2k\pi-\arctan\frac{-L}{K}\right) \\ &= K\sin\left(\frac{5\pi}{2}-\arctan\frac{-L}{K}\right)-L\cos\left(\frac{5\pi}{2}-\arctan\frac{-L}{K}\right) \\ &= \sqrt{K^2+L^2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\lim_{k\to\infty}\Bigg\{(\ln x_k)^{-1/2}x_k\Bigg[\frac{2M+1}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{\sqrt{2}x_k}{\sqrt{\pi}}\Bigg(\sin\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi \\ &\qquad -\cos\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi\Bigg)\Bigg]\Bigg\} \\ &= (+\infty)\left[\frac{2M+1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{4}+(+\infty)\sqrt{K^2+L^2}\right] \\ &= \infty. \end{aligned}$$

Then we obtain

$$\limsup_{x\to+\infty}(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)=\infty.$$

Similarly, selecting the sequence $x_l=\{e^{\frac{3\pi}{2}+\frac{\pi}{4}+2l\pi-\arctan\frac{-L}{K}}\}$*, we can obtain*

$$\liminf_{x\to+\infty}(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)=-\infty.$$

Therefore, all solutions of (3.7) *are oscillatory by Theorem 3.1.*

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