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Adjacent Vertex-Distinguishing Edge-Coloring of Brick-Product

S. Anantharaman[1]

Keywords
adjacent vertex-distinguishing proper edge - coloring, brick - product, edge-coloring

Abstract — Let G be a finite, simple, undirected and connected graph. $\chi'_{as}(G)$ denotes the minimum number of colors required for a proper edge-coloring of G, in which no two adjacent vertices are incident to edges colored with the same set of colors. In this paper, I am compute sharp bound for adjacent vertex-distinguishing proper edge-coloring of brick-product.

Subject Classification (2020): 05C15, 05C32.

1. Introduction

I am refer the books [4, 11] for graph theoretical notation and terminology. Let G be a finite, simple, undirected and connected graph. Denote by $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. Let $\Delta(G)$ denotes the maximum degree of G. A *proper edge-coloring* σ is a mapping from $E(G)$ to the set of colors such that any two adjacent edges receive distinct colors. For any vertex v of G, let $S_\sigma(v)$ denote the set of the colors of all edges incident to v. A proper edge-coloring σ is said to an *adjacent vertex-distinguishing* (AVD) if $S_\sigma(u) \neq S_\sigma(v)$, for every adjacent vertices u and v. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G, denoted by $\chi'_{as}(G)$, is called the *adjacent vertex-distinguishing chromatic index* (*AVD chromatic index*) of G. Thus, $\chi'_{as}(G) \geq \chi'(G)$.

The concept of adjacent vertex-distinguishing edge-coloring has been introduce and studied in [19] Zhang et al. (2002) and pose the following conjecture.

Conjecture 1.1. (Zhang et al. [19]) For any connected graph G ($|V(G)| \geq 6$), there is $\chi'_{as}(G) \leq \Delta(G) + 2$.

If H is a subgraph of G, it is interesting that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is not always true. Let $K_{m,n}$ be the complete bipartite graph, then $\chi'_{as}(K_{2,3}) = 3$ and $K_{2,3} - e$ for any edge, then $\chi'_{as}(K_{2,3} - e) = 4$. Deletion of an edge of a graph may also decrease the coloring number of the graph. Let $n \geq 3$, then $\chi'_{as}(K_{1,n}) = n$ and $\chi'_{as}(K_{1,n} - e) = n - 1$.

The concept of adjacent vertex-distinguishing edge-coloring has been studied in many paper such as [1, 3,

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5–10, 12–20].

In [1] Anantharaman (2019) obtained exact values for adjacent vertex-distinguishing edge-coloring of strong product of some graphs. In [3] Axenovich et al. (2016) obtained upper bound for adjacent vertex-distinguishing edge-colorings of graphs. In [5] Balister et al. (2007) obtained upper bound for adjacent vertex-distinguishing edge-coloring some special graphs also consider 3-regular graphs. In [6] Baril et al. (2006) obtained exact values for adjacent vertex-distinguishing edge-coloring of meshes. In [7] Bu et al. (2011) finding adjacent vertex-distinguishing edge-colorings of planar graphs with girth at least six. In [8] Chen et al. (2015) obtained adjacent vertex-distinguishing proper edge-coloring of planar bipartite graphs with $\Delta = 9$, 10, or 11. In [9] Hatami (2005) prove that $\Delta + 300$ is a bound on the adjacent vertex-distinguishing edge chromatic number. In [10] Hocquard et al. (2011) compute adjacent vertex-distinguishing edge-coloring of graphs with maximum degree at least five[1]. In [12] Li et al. (2006) compute adjacent strong edge-coloring of $K(n,m)$. In [13] Lin et al. (2010) compute the adjacent vertex-distinguishing edge-coloring of graphs containing Hamiltonian path and graphs containing dominating path. In [14] Lin-zhong et al. (2003) compute on the adjacent strong edge-coloring of Halin Graphs. In [15] Omai et al. (2017) compute for some result for AVD-edge-coloring on power of path[1]. In [17] Wang et al. (2010) obtained adjacent vertex-distinguishing edge-colorings of graphs with smaller maximum average degree. In [18] Yu et al. (2016) compute adjacent vertex-distinguishing colorings by sum of sparse graphs. In [19] Zhang et al. (2002) obtained some standard result and pose the conjecture for adjacent Strong edge-coloring of graphs. In [20] Zhang et al. (2014) obtained improved upper bound on adjacent vertex-distinguishing chromatic index of a graph.

2. Brick-product

Let $\ell \geq 2$, $m \geq 1$ and $r \geq 0$ be integers such that $m + r$ is even. Let $C_{2\ell}$ be a cycle of length 2ℓ. The (m,r)-*brick-product of* $C_{2\ell}$, denoted by $Br(2\ell, m, r)$, is the graph with adjacency defined in two cases.

• For $m = 1$, $r \geq 3$ must be odd and $Br(2\ell, 1, r)$ is obtained from the cycle $C_{2\ell} = (v_0, v_1, v_2, \ldots, v_{2\ell-1}, v_0)$, by adding chords joining v_{2i} and v_{2i+r} for $i \in \{0, 1, \ldots, \ell-1\}$ where subscripts are taken modulo 2ℓ.

• For $m \geq 2$, $Br(2\ell, m, r)$ is obtained by first taking the vertex-disjoint union of m copies of $C_{2\ell}$ denoted by

$$C_{2\ell}(i) = (v_{i,0}, v_{i,1}, v_{i,2}, \ldots, v_{i,2\ell-1}, v_{i,0}), \quad i \in \{0, 1, \ldots, m-1\}.$$

Next, for each pair $(i,j) \in \{0, 1, \ldots, m-2\} \times \{0, 1, \ldots, 2\ell-1\}$ such that i and j have the same parity, an edge is added to join $v_{i,j}$ and $v_{i+1,j}$. Finally, for odd $j \in \{1, 3, 5, \ldots, 2\ell-1\}$, an edge is added to join $v_{0,j}$ and $v_{m-1,j+r}$, where the second subscript is modulo 2ℓ ([16]).

By definition, $Br(2\ell, m, r)$ is 3-regular. So $\chi'_{as}(Br(2\ell, m, r)) \geq \Delta + 1 = 4$. We show at most brick-product have $\chi'_{as}(Br(2\ell, m, r)) = 4$.

3. $\chi'_{as}(Br(2\ell, m, r))$ for $m \notin \{1, 2, 5\}$

Theorem 3.1. For $m \notin \{1,2,5\}$, $\chi'_{as}(Br(2\ell, m, r)) = 4$.

Proof.
Let $G = Br(2\ell, m, r)$. I am consider four cases.
Case 1. $m \equiv 0 \pmod 4$.
Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i \in \{0,4,8,\ldots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,5,9,\ldots,m-3\}$

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,6,10,\ldots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\ldots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{0,4,8,\ldots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{1,5,9,\ldots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{2,6,10,\ldots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{3,7,11,\ldots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i \in \{0,4,8,\ldots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,5,9,\ldots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,6,10,\ldots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\ldots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Case 2. $m \equiv 1 \pmod 4$.
Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:
for $i \in \{0,3,6\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,4,7\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{2,5,8\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\dots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{10,14,18,\dots,m-3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{11,15,19,\dots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{12,16,20,\dots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{0,6\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i = 3$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{1,7\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i = 4$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{2,8\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i = 5$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{9,13,17,\dots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{10,14,18,\dots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{11,15,19,\dots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{12,16,20,\dots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i \in \{0,6\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{1,7\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,8\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 3$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 5$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\ldots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{10,14,18,\ldots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{11,15,19,\ldots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{12,16,20,\ldots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 3. $m \equiv 2 \pmod 4$.

Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i \in \{0,3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,5\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\ldots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{7,11,15,\ldots,m-3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{8,12,16,\ldots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\ldots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i=0$, $\sigma(v_{i,j}v_{i+1,j})=4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i=1$, $\sigma(v_{i,j}v_{i+1,j})=2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i=2$, $\sigma(v_{i,j}v_{i+1,j})=3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i=3$, $\sigma(v_{i,j}v_{i+1,j})=4$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i=4$, $\sigma(v_{i,j}v_{i+1,j})=2$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i=5$, $\sigma(v_{i,j}v_{i+1,j})=3$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{6,10,14,\ldots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j})=4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{7,11,15,\ldots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j})=2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{8,12,16,\ldots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j})=1$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{9,13,17,\ldots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j})=3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i = 0$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 1$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 2$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 3$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 5$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\ldots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{7,11,15,\ldots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{8,12,16,\ldots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\ldots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 4. $m \equiv 3 \pmod 4$.

Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i = 0$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 1$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 2$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\dots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{4,8,12,\dots,m-3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{5,9,13,\dots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{6,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\dots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i = 0$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i = 1$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i = 2$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{3,7,11,\dots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{4,8,12,\dots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{5,9,13,\dots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{6,10,14,\dots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i = 0$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 1$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 2$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\dots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{4,8,12,\ldots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{5,9,13,\ldots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\ldots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Thus, $\chi'_{as}(Br(2\ell,m,r)) = 4$.

4. $\chi'_{as}(Br(2\ell,1,r))$

By definition, $r \in \{3,5,7,\ldots\}$. Also, $\ell \geq 3$.

Theorem 4.1. If $\ell \equiv 3 \pmod 6$ and $r \notin \{3,9,15,21,\ldots\}$, then
$\chi'_{as}(Br(2\ell,1,r)) = 4$.

Proof.
Define $\sigma : E(Br(2\ell,1,r)) \to \{1,2,3,4\}$ as follows:

$$\sigma(v_j v_{j+1}) = \begin{cases} 1 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 2 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Remaining edges are colored 4.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are:

$$S_\sigma(v_j) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Thus, $\chi'_{as}(Br(2\ell,1,r)) = 4$.

5. $\chi'_{as}(Br(2\ell,2,r))$

By the definition of $Br(2\ell,2,r)$, r is even.

Theorem 5.1. For $\ell \equiv 0 \pmod 3$, $\chi'_{as}(Br(2\ell,2,r)) = 4$.

Proof.

Let $G = Br(2\ell, 2, r)$. I am consider two cases.

Case 1. $r \notin \{4, 10, 16, \ldots\}$.

Define $\sigma : E(G) \to \{1, 2, 3, 4\}$ as follows:

$$\sigma(v_{0,j}v_{0,j+1}) = \begin{cases} 1 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 2 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 3 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}; \end{cases}$$

$$\sigma(v_{1,j}v_{1,j+1}) = \begin{cases} 3 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 1 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 2 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}. \end{cases}$$

Remaining edges are colored 4.

By the construction, σ is a proper edge-coloring.

The induced vertex-color sets are given below:

$$S_\sigma(v_{0,j}) = \begin{cases} \{1, 3, 4\} & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ \{1, 2, 4\} & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ \{2, 3, 4\} & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}; \end{cases}$$

$$S_\sigma(v_{1,j}) = \begin{cases} \{1, 2, 4\} & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ \{2, 3, 4\} & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ \{1, 3, 4\} & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 2. $r \in \{4, 10, 16, \ldots\}$.

Define $\sigma : E(G) \to \{1, 2, 3, 4\}$ as follows:

$$\sigma(v_{0,j}v_{0,j+1}) = \begin{cases} 1 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 2 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 3 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}; \end{cases}$$

$$\sigma(v_{1,j}v_{1,j+1}) = \begin{cases} 2 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 3 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 1 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}. \end{cases}$$

Remaining edges are colored 4.

By the construction, σ is a proper edge-coloring.

The induced vertex-color sets are given below:

$$S_\sigma(v_{0,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

$$S_\sigma(v_{1,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{2,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Thus, $\chi'_{as}(Br(2\ell,2,r)) = 4$.
This completes the proof.

6. $\chi'_{as}(Br(2\ell,5,r))$

By the definition of $Br(2\ell,5,r)$, r is odd.

Theorem 6.1. For $\ell \equiv 0 \pmod 3$, $\chi'_{as}(Br(2\ell,5,r)) = 4$.

Proof.
Let $G = Br(2\ell,5,r)$. I am consider two cases.
Case 1. $r \notin \{3,9,15,\ldots\}$.
Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:
for $i \in \{0,2,4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 2 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 1 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Edges $\{v_{0,j}v_{1,j}, v_{2,j}v_{3,j} : j \in \{1,3,5,\ldots,2\ell-1\}\} \cup$
$\{v_{1,j}v_{2,j}, v_{3,j}v_{4,j}, v_{0,j}v_{4,j+r} : j \in \{0,2,4,\ldots,2\ell-2\}\}$ are colored 4.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i \in \{0,2,4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 2. $r \in \{3,9,15,\ldots\}$.

Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i \in \{0,2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 2 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 1 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 3 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 1 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Remaining edges are colored 4.

By the construction, σ is a proper edge-coloring.

The induced vertex-color sets are given below:

for $i \in \{0,2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{2,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Thus, $\chi'_{as}(Br(2\ell,5,r)) = 4$. We finish this paper with the following problem.

i) For $\ell \equiv 3 \pmod 6$ and $r \in \{3,9,15,21,\dots\}$, compute $\chi'_{as}(Br(2\ell,1,r))$.
ii) For $\ell \not\equiv 3 \pmod 6$, compute $\chi'_{as}(Br(2\ell,1,r))$.
iii) For $\ell \not\equiv 0 \pmod 3$, compute $\chi'_{as}(Br(2\ell,2,r))$.
iv) For $\ell \not\equiv 0 \pmod 3$, compute $\chi'_{as}(Br(2\ell,5,r))$.

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Some Identities For "Hyperbolic" Trigonometric Functions

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Trigonometric functions, hyperbolic trigonometric functions, generalized Fibonacci and Lucas polynomials

Abstract — In this article, we give proofs of some properties provided by "hyperbolic" trigonometric functions defined in [1].

Subject Classification (2020): 11B39,11P99,33B10.

1. Introduction

An arithmetical function is a complex valued function defined on the set of positive integers and the set of these functions is denoted by A. The (Dirichlet) convolution $\left(g*h\right)$ of g and h is defined by $\left(g*h\right)(n) = \sum_{d|n} g(d)h\left(\frac{n}{d}\right)$ for all $g, h \in A$. Rearick [2] introduced the notions of Logarithm and Exponential operators of arithmetic functions. These operators were inverses of one another. The Logarithm operator takes Dirichlet products to sums in A, and the Exponential operator takes sums to Dirichlet products. Inspired by Rearick's work Li and MacHenry introduced LOG and EXP operators. The LOG operates on generalized Fibonacci polynomials($F_{k,n}(t)$) giving generalized Lucas polynomials($G_{k,n}(t)$). The EXP is the inverse of LOG[1]. Then Li and MacHenry defined the "Hyperbolic" SINE and "Hyperbolic" COSINE functions with the help of the EXP operator. First, let's give the definitions necessary to make sense of these definitions.

Definition 1.1. [1] An isobaric polynomial is a polynomial in the variables $t_1, t_2, \ldots, t_k$ for $k \in \{1, 2, \ldots\}$, with coefficients in $\mathbb{Z}$, of the form

$$P_{k,n}(t_1, t_2, \ldots, t_k) = \sum_{\alpha \vdash n} C_\alpha t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k}$$

where $\alpha = \{\alpha_1, \alpha_2, \ldots \alpha_k\}$ and $\alpha \vdash n$ means that $\sum_{j=1}^{k} j\alpha_j = n$.

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Definition 1.2. [1] A weighted isobaric polynomial given by the following explicit expression:

$$P_{w,k,n}(t_1,t_2,\dots,t_k)=\sum_{\alpha\vdash n}\binom{|a|}{\alpha_1,\alpha_2,\dots,\alpha_k}\frac{\sum_{j=1}^{k}w_j\alpha_j}{|\alpha|}t_1^{\alpha_1}t_2^{\alpha_2}\dots t_k^{\alpha_k}$$

where w is the weight vector $(w_1,w_2,\dots,w_k)$, $w_j\in\mathbb{Z}$ and $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_k$.

$F_{k,n}(t)$ and $G_{k,n}(t)$, are defined inductively by as follows:

$$F_{k,0}(t)=1,\ F_{k,n+1}(t)=t_1F_{k,n}(t)+\cdots+t_kF_{k,n-k+1}(t)(n>1),$$

and

$$\begin{aligned}G_{k,0}(t)&=k,\ G_{k,1}(t)=t_1,\ G_{k,n}(t)=G_{k-1,n}(t)(1\le n\le k),\\ G_{k,n}(t)&=t_1G_{k,n-1}(t)+\cdots+t_kG_{k,n-k}(t)(n>k),\end{aligned}$$

where the vector $t=(t_1,t_2,\dots,t_k)$ and t_i $(1\le i\le k)$ are constant coefficients of the core polynomial

$$P(x;t_1,t_2,\dots,t_k)=x^k-t_1x^{k-1}-\cdots-t_k.$$

Li and MacHenry [1] defined two operators $\mathscr{L}$(LOG) and $\mathscr{E}$(EXP).

Definition 1.3. [1] For a fixed k and $n\ge 1$,

$$\mathscr{L}(P_n)=-t_{n-1}P_1-2t_{n-2}P_2-\ldots-(n-1)t_1P_{n-1}+nP_n$$

where P_n is weighted isobaric polynomial and $t_i=0$ for $i>k$.

Definition 1.4. [1] For a fixed k, $\mathscr{E}(G_{k,0})=1$,

$$\mathscr{E}(G_{k,n})=\frac{1}{n}(F_{k,n-1}G_{k,1}+F_{k,n-2}G_{k,2}+\ldots+F_{k,1}G_{k,n-1}+G_{k,n}).$$

Lemma 1.5. [1]$\mathscr{L}$ and $\mathscr{E}$ are inverses of one another on F and G, i.e.,

$$\begin{aligned}\mathscr{L}(F_n)&=G_n\\ \mathscr{E}(G_n)&=F_n\end{aligned}$$

Definition 1.6. [1] "Hyperbolic" SINE and "Hyperbolic" COSINE functions are defined as;

$$\begin{aligned}C(G)&=\frac{1}{2}(\mathscr{E}(G)+\overline{\mathscr{E}(G)})\\ S(G)&=\frac{1}{2}(\mathscr{E}(G)-\overline{\mathscr{E}(G)}).\end{aligned}$$

2. "Hyperbolic" Trigonometric Operators

The purpose of this article is to give proofs of some properties provided by "Hyperbolic" trigonometric functions defined in [1].

Theorem 2.1. [1] Let δ be the function whose values are $(1,0,0,...,0,...)$,

$$C(G)^{*2} - S(G)^{*2} = \delta.$$

Theorem 2.2. [1] Let F and G be induced by the core $[t_{1,...,}t_k]$, $F^{'}$ and $G^{'}$ be induced by the core $\left[t^{'}_{1,...,}t^{'}_k\right]$ with $\mathscr{L}(F) = G$ and $\mathscr{L}(F^{'}) = G^{'}$, then

$$\begin{aligned} C(G+G^{'}) &= C(G)*C(G^{'}) + S(G)*S(G^{'}), \\ S(G+G^{'}) &= S(G)*C(G^{'}) + C(G)*S(G^{'}). \end{aligned}$$

Theorem 2.3. Let F and G be induced by the core $[t_{1,...,}t_k]$, with $\mathscr{L}(F) = G$ then,

$$S(2G) = 2(S(G)*C(G))$$

Proof.

$$\begin{aligned} 2(S(G)*C(G)) &= 2(\frac{1}{2}(\mathscr{E}(G) - \overline{\mathscr{E}(G)})*\frac{1}{2}(\mathscr{E}(G) + \overline{\mathscr{E}(G)})) \\ &= \frac{1}{2}(\mathscr{E}(G) - \overline{\mathscr{E}(G)})*(\mathscr{E}(G) + \overline{\mathscr{E}(G)}) \\ &= \frac{1}{2}(\mathscr{E}(G)*\mathscr{E}(G) + \mathscr{E}(G)*\overline{\mathscr{E}(G)} - \overline{\mathscr{E}(G)}*\mathscr{E}(G) - \overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)}) \\ &= \frac{1}{2}(\mathscr{E}(2G) - \overline{\mathscr{E}(2G)}) \\ &= S(2G). \end{aligned}$$

Theorem 2.4. Let F and G be induced by the core $[t_{1,...,}t_k]$, with $\mathscr{L}(F) = G$ then,

$$C(2G) = 2(C(G))^{*2} - \delta.$$

Proof.

$$\begin{aligned} 2(C(G))^{*2} - \delta &= 2\left(\frac{1}{2}(\mathscr{E}(G) + \overline{\mathscr{E}(G)})\right)^{*2} - \delta \\ &= \frac{1}{2}\left(\mathscr{E}(G) + \overline{\mathscr{E}(G)}\right)^{*2} - \delta \\ &= \frac{1}{2}(\mathscr{E}(G)*\mathscr{E}(G) + 2\mathscr{E}(G)*\overline{\mathscr{E}(G)} + \overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)} - 2\delta) \\ &= \frac{1}{2}(\mathscr{E}(2G) + 2\delta + \overline{\mathscr{E}(2G)} - 2\delta) \\ &= \frac{1}{2}(\mathscr{E}(2G) + \overline{\mathscr{E}(2G)}) \\ &= C(2G). \end{aligned}$$

Theorem 2.5. Let F and G be induced by the core $\left[t_{1,\dots,}t_k\right]$, with $\mathscr{L}(F)=G$ then,

$$C(2G)=(C(G))^{*2}+(S(G))^{*2}$$

Proof.

$$\begin{aligned}
(C(G))^{*2}+(S(G))^{*2} &= \left(\frac{1}{2}(\mathscr{E}(G)+\overline{\mathscr{E}(G)})\right)^{*2}+\left(\frac{1}{2}(\mathscr{E}(G)-\overline{\mathscr{E}(G)})\right)^{*2}\\
&= \frac{1}{4}\left(\left(\mathscr{E}(G)+\overline{\mathscr{E}(G)}\right)^{*2}+\left(\mathscr{E}(G)-\overline{\mathscr{E}(G)}\right)^{*2}\right)\\
&= \frac{1}{4}(\mathscr{E}(G)*\mathscr{E}(G)+2\mathscr{E}(G)*\overline{\mathscr{E}(G)}+\overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)}\\
&\quad +\mathscr{E}(G)*\mathscr{E}(G)-2\mathscr{E}(G)*\overline{\mathscr{E}(G)}+\overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)})\\
&= \frac{1}{4}(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}+\mathscr{E}(2G)+\overline{\mathscr{E}(2G)})\\
&= \frac{1}{4}(2\left(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)}\right))\\
&= \frac{1}{2}(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)})\\
&= C(2G).
\end{aligned}$$

Theorem 2.6. Let F and G be induced by the core $\left[t_{1,\dots,}t_k\right]$, with $\mathscr{L}(F)=G$ then,

$$C(2G)=2(S(G))^{*2}+\delta$$

Proof.

$$\begin{aligned}
2(S(G))^{*2}+\delta &= 2\left(\frac{1}{2}(\mathscr{E}(G)-\overline{\mathscr{E}(G)})\right)^{*2}+\delta\\
&= \frac{1}{2}(\left(\mathscr{E}(G)-\overline{\mathscr{E}(G)}\right)^{*2})+\delta\\
&= \frac{1}{2}(\mathscr{E}(G)*\mathscr{E}(G)-2\mathscr{E}(G)*\overline{\mathscr{E}(G)}+\overline{\mathscr{E}(G)}*\overline{\mathscr{E}(G)}+2\delta)\\
&= \frac{1}{2}(\mathscr{E}(2G)-2\delta+\overline{\mathscr{E}(2G)}+2\delta)\\
&= \frac{1}{2}(\mathscr{E}(2G)+\overline{\mathscr{E}(2G)})\\
&= C(2G).
\end{aligned}$$

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A New Moving Frame For Trajectories on Regular Surfaces

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Abstract — In this study, we introduce a new moving frame on regular surfaces for trajectories with non-vanishing angular momentum and give the angular velocity vector for this frame. Then, we consider the special trajectories generated by Smarandache curves according to this frame in three-dimensional Euclidean space and investigate the Serret-Frenet apparatus of them. Moreover, we provide an illustrative example explaining how this frame is constructed and how the aforementioned special trajectories are generated. This moving frame is a new contribution to the field and we expect that it will be useful in some specific applications of differential geometry and kinematics in the future.

Subject Classification (2020): 70B99, 70B05, 57R25, 53A05, 53A04.

1. Introduction

In differential geometry, the theory of surfaces in 3-dimensional Euclidean space has an important place. Although the theory of surfaces in 3-dimensional Euclidean space had already been developed widely when the Serret-Frenet frame was introduced by Serret and Frenet, Serret-Frenet frame helped developing this theory further by researchers. This theory is still an issue of interest despite its long history. The approaches followed by Serret and Frenet led to the success of adapting the method of moving frames to the surface curves. This was carried out by Jean Gaston Darboux. He introduced a moving frame which is constructed on a surface. It is called as Darboux frame. At all non-umbilic points of a surface, Darboux frame exists. Thus, it exists at all the points of a curve on a regular surface [9, 12]. Darboux frame is a useful tool for investigating the theory of surfaces. From the discovery of this frame until now, many researchers have carried out lots of interesting studies on this theory by using this frame. Some of these studies can be found in [2, 7, 8, 10, 14, 17].

In Euclidean 3-space, a point particle of constant mass moving on a regular surface curve has a position vector according to Darboux frame of this curve. So, an arbitrary point of the trajectory can be represented by the aforesaid particle. As a result of this case, there is a very close relationship between the differential geometry of the trajectory, the differential geometry of the surface and the kinematics of the moving particle. This relationship has motivated us to prepare this study. In this study, a new moving frame on regular

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surfaces for trajectories with non-vanishing angular momentum has been constructed by considering the Darboux frame of the trajectory. It is expected that this moving frame will enable more convenient observation environment for the researchers studying on modern robotics. Note that we carried out a similar study [11] for trajectories, not necessarily lying on a surface, by considering Serret-Frenet frame. The present study includes similar techniques and approaches given in [11].

Let E^3 be endowed with the standard inner product $\langle \mathbf{D}, \mathbf{E} \rangle = d_1 e_1 + d_2 e_2 + d_3 e_3$ where $\mathbf{D} = (d_1, d_2, d_3)$, $\mathbf{E} = (e_1, e_2, e_3)$ are arbitrary vectors in this space. The norm of the vector $\mathbf{D}$ is stated as $\|\mathbf{D}\| = \sqrt{\langle \mathbf{D}, \mathbf{D} \rangle}$. If a differentiable curve $\chi = \chi(s) : I \subset \mathbb{R} \to E^3$ satisfies the equality $\left\| \frac{d\chi}{ds} \right\| = 1$ for all $s \in I$, this curve is called a unit speed curve. In this case, s is said to be arc-length parameter of χ. A differentiable curve is called regular curve if its derivative is nonzero along the curve. Regular curves can be reparameterized by the arc-length [13]. In the rest of the paper, the differentiation with respect to the arc-length parameter s will be shown with a dash.

The Serret-Frenet frame of the curve $\chi = \chi(s)$ is denoted by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$. The unit vectors $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are called the unit tangent, unit principal normal and unit binormal vectors, respectively. On the other hand, the Serret-Frenet formulas are given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{1.1}$$

where $\kappa(s) = \left\| \mathbf{T}'(s) \right\|$ is the curvature function and $\tau(s) = -\left\langle \mathbf{B}'(s), \mathbf{N}(s) \right\rangle$ is the torsion function [13].

Suppose that $\chi : I \subset R \to M \subset E^3$ is a unit speed curve which lies on a regular surface M. In that case, there exists Darboux frame denoted by $\{\mathbf{T}, \mathbf{Y}, \mathbf{U}\}$ along the curve χ. $\mathbf{T}$ is the unit tangent vector of χ, $\mathbf{U}$ is the unit normal vector of M restricted to χ and $\mathbf{Y}$ is the unit vector given by $\mathbf{Y} = \mathbf{U} \times \mathbf{T}$. The derivative formulas of Darboux frame are as follows:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{Y}' \\ \mathbf{U}' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{Y} \\ \mathbf{U} \end{pmatrix}. \tag{1.2}$$

Here, the functions k_g, k_n and τ_g are called geodesic curvature, normal curvature and geodesic torsion of the curve χ, respectively [6, 9].

This study is organized as follows. In Section 2, we explain how our frame is constructed and give the relation matrix between this frame and Darboux frame. Afterwards, we obtain derivative formulas and complete the set of apparatus of this frame. Also, angular velocity vector is obtained for this frame. In Section 3, we study the special trajectories generated by Smarandache curves according to this frame in three-dimensional Euclidean space.

2. Positional Adapted Frame on Regular Surfaces

In E^3, let a point particle of constant mass m move on a curve which lies on a regular surface M. Denote by $\mathbf{x}$ the position vector of this particle relative to fixed origin O at time t. Let the curve $\chi = \chi(s)$ be the unit speed parametrization of the trajectory of the particle where the arc-length s of χ corresponds to time t. In that case, the unit tangent vector, velocity vector and linear momentum vector at the point $\chi(s)$ (at time t) are given by

$$\begin{aligned} \mathbf{T}(s) &= \frac{d\mathbf{x}}{ds} \\ \mathbf{v}(t) &= \frac{d\mathbf{x}}{dt} = \left(\frac{ds}{dt}\right)\mathbf{T}(s) \\ \mathbf{p}(t) &= m\mathbf{v}(t) = m\left(\frac{ds}{dt}\right)\mathbf{T}(s), \end{aligned} \tag{2.1}$$

respectively [4]. Also, we can write

$$\mathbf{x} = \left\langle \chi(s), \mathbf{T}(s)\right\rangle \mathbf{T}(s) + \left\langle \chi(s), \mathbf{Y}(s)\right\rangle \mathbf{Y}(s) + \left\langle \chi(s), \mathbf{U}(s)\right\rangle \mathbf{U}(s) \tag{2.2}$$

at the point $\chi(s)$ (at time t) with respect to Darboux frame. By vector product of $\mathbf{x}$ and $\mathbf{p}(t)$, the angular momentum vector (at time t) of the particle about O is found as:

$$\mathbf{H}^O = m\left\langle \chi(s), \mathbf{U}(s)\right\rangle \left(\frac{ds}{dt}\right)\mathbf{Y}(s) - m\left\langle \chi(s), \mathbf{Y}(s)\right\rangle \left(\frac{ds}{dt}\right)\mathbf{U}(s). \tag{2.3}$$

Throughout the paper, we suppose that angular momentum vector of the aforementioned particle never vanishes. In other words, we restrict ourselves to the trajectories having non-vanishing angular momentum. This assumption ensures that the coefficient functions $\left\langle \chi(s), \mathbf{Y}(s)\right\rangle$ and $\left\langle \chi(s), \mathbf{U}(s)\right\rangle$ of the position vector are not zero simultaneously. That is, we ensure that the tangent line never passes through the origin along the trajectory. Let us return to the position vector. The opposite of this vector is given as in the following:

$$-\mathbf{x} = \left\langle -\chi(s), \mathbf{T}(s)\right\rangle \mathbf{T}(s) + \left\langle -\chi(s), \mathbf{Y}(s)\right\rangle \mathbf{Y}(s) + \left\langle -\chi(s), \mathbf{U}(s)\right\rangle \mathbf{U}(s). \tag{2.4}$$

The projections of it on the instantaneous planes $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ yield two vectors playing important roles to construct a new moving frame on M along χ. These roles are stated in detail below.

The vector, whose starting point is $\chi(s)$ and endpoint is the foot of perpendicular (from O to $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$), can be given by

$$\mathbf{r}(s) = \left\langle -\chi(s), \mathbf{T}(s)\right\rangle \mathbf{T}(s) + \left\langle -\chi(s), \mathbf{Y}(s)\right\rangle \mathbf{Y}(s) \tag{2.5}$$

and corresponds to the aforementioned projection on $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. On the other hand the vector, whose starting point is $\chi(s)$ and endpoint is the foot of the perpendicular (from origin to $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$), can be given by

$$\mathbf{r}^*(s) = \left\langle -\chi(s), \mathbf{T}(s)\right\rangle \mathbf{T}(s) + \left\langle -\chi(s), \mathbf{U}(s)\right\rangle \mathbf{U}(s) \tag{2.6}$$

and corresponds to the aforementioned projection on $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. From the Equation 2.5 and Equation 2.6, we can get the vector

$$\mathbf{r}(s) - \mathbf{r}^*(s) = \left\langle -\chi(s), \mathbf{Y}(s)\right\rangle \mathbf{Y}(s) + \left\langle \chi(s), \mathbf{U}(s)\right\rangle \mathbf{U}(s) \tag{2.7}$$

whose starting point is $\chi(s)$ and which lies on the instantaneous plane $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$. We must emphasize that the vector $\mathbf{r}(s)-\mathbf{r}^*(s)$ is equivalent to the vector whose starting point is the aforesaid foot on $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ and endpoint is the other aforesaid foot on $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ (see Figure 1).

Let us talk about the determination of unit vector in direction $\mathbf{r}(s)-\mathbf{r}^*(s)$. If both planes $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ do not contain the origin, the foots are distinct from each other and from the origin. Therefore, two distinct foots generate the non-zero vector $\mathbf{r}(s)-\mathbf{r}^*(s)$. In this case, the desired unit vector can be obtained. When only one of the planes $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ passes through the origin, the foot of the perpendicular on the plane, containing origin, is taken as the origin. Certainly, the other foot is distinct from the origin. In that case, the desired unit vector is determined similarly. The case both the planes $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ include the origin simultaneously causes not to be determined of the desired unit vector because the both of the aforesaid foots correspond to the origin. This situation occurs only when the tangent line contains the origin. Fortunately, the assumption on the angular momentum vector averts this. Let the unit vector in direction $\mathbf{r}(s)-\mathbf{r}^*(s)$ be denoted by $\mathbf{H}(s)$. Namely,

$$\mathbf{H}(s)=\frac{\mathbf{r}(s)-\mathbf{r}^*(s)}{\|\mathbf{r}(s)-\mathbf{r}^*(s)\|}=\frac{\langle-\chi(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)+\frac{\langle\chi(s),\mathbf{U}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s). \tag{2.8}$$

By vector product $\mathbf{H}(s)$ and $\mathbf{T}(s)$, we can get the another basis vector. We show it with $\mathbf{G}(s)$. Then we obtain

$$\mathbf{G}(s)=\mathbf{H}(s)\wedge\mathbf{T}(s)=\frac{\langle\chi(s),\mathbf{U}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)+\frac{\langle\chi(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s). \tag{2.9}$$

This completes the positively oriented orthonormal moving frame $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s)\}$.

Since the vectors $\mathbf{Y}(s),\mathbf{U}(s),\mathbf{G}(s)$ and $\mathbf{H}(s)$ lie on the plane $\{\mathbf{T}(s)\}^{\perp}$, there is a relation between the Darboux frame and this frame as follows:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cos\Omega(s)&-\sin\Omega(s)\\ 0&\sin\Omega(s)&\cos\Omega(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{Y}(s)\\ \mathbf{U}(s)\end{pmatrix} \tag{2.10}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{U}(s)$ and $\mathbf{H}(s)$ which is positively oriented from $\mathbf{U}(s)$ to $\mathbf{H}(s)$ (see Figure 1). By using the Equation 1.2 and Equation 2.10, we can write

$$\begin{aligned}
\mathbf{G}'(s) &= (\cos\Omega(s)\mathbf{Y}(s)-\sin\Omega(s)\mathbf{U}(s))'\\
&= -\Omega'(s)\sin\Omega(s)\mathbf{Y}(s)+\cos\Omega(s)\left(-k_g(s)\mathbf{T}(s)+\tau_g(s)\mathbf{U}(s)\right)\\
&\quad -\Omega'(s)\cos\Omega(s)\mathbf{U}(s)+\sin\Omega(s)\left(k_n(s)\mathbf{T}(s)+\tau_g(s)\mathbf{Y}(s)\right)\\
&= \left(-k_g(s)\cos\Omega(s)+k_n(s)\sin\Omega(s)\right)\mathbf{T}(s)+\left(\tau_g(s)-\Omega'(s)\right)[\sin\Omega(s)\mathbf{Y}(s)+\cos\Omega(s)\mathbf{U}(s)]\\
&= \left(-k_g(s)\cos\Omega(s)+k_n(s)\sin\Omega(s)\right)\mathbf{T}(s)+\left(\tau_g(s)-\Omega'(s)\right)\mathbf{H}(s)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{H}'(s) &= (\sin\Omega(s)\mathbf{Y}(s)+\cos\Omega(s)\mathbf{U}(s))' \\
&= \Omega'(s)\cos\Omega(s)\mathbf{Y}(s)+\sin\Omega(s)\left(-k_g(s)\mathbf{T}(s)+\tau_g(s)\mathbf{U}(s)\right) \\
&\quad -\Omega'(s)\sin\Omega(s)\mathbf{U}(s)-\cos\Omega(s)\left(k_n(s)\mathbf{T}(s)+\tau_g(s)\mathbf{Y}(s)\right) \\
&= \left(-k_g(s)\sin\Omega(s)-k_n(s)\cos\Omega(s)\right)\mathbf{T}(s)+\left(\Omega'(s)-\tau_g(s)\right)[\cos\Omega(s)\mathbf{Y}(s)-\sin\Omega(s)\mathbf{U}(s)] \\
&= \left(-k_g(s)\sin\Omega(s)-k_n(s)\cos\Omega(s)\right)\mathbf{T}(s)+\left(\Omega'(s)-\tau_g(s)\right)\mathbf{G}(s).
\end{aligned}$$

In that case, differentiating the vector $\mathbf{G}(s)\wedge\mathbf{H}(s)$ gives us the following:

$$\begin{aligned}
\mathbf{T}'(s) &= (\mathbf{G}(s)\wedge\mathbf{H}(s))' \\
&= \mathbf{G}'(s)\wedge\mathbf{H}(s)+\mathbf{G}(s)\wedge\mathbf{H}'(s) \\
&= \left[\left(-k_g(s)\cos\Omega(s)+k_n(s)\sin\Omega(s)\right)\mathbf{T}(s)+\left(\tau_g(s)-\Omega'(s)\right)\mathbf{H}(s)\right]\wedge\mathbf{H}(s) \\
&\quad +\mathbf{G}(s)\wedge\left[\left(-k_g(s)\sin\Omega(s)-k_n(s)\cos\Omega(s)\right)\mathbf{T}(s)+\left(\Omega'(s)-\tau_g(s)\right)\mathbf{G}(s)\right] \\
&= \left(k_g(s)\cos\Omega(s)-k_n(s)\sin\Omega(s)\right)\mathbf{G}(s)+\left(k_g(s)\sin\Omega(s)+k_n(s)\cos\Omega(s)\right)\mathbf{H}(s).
\end{aligned}$$

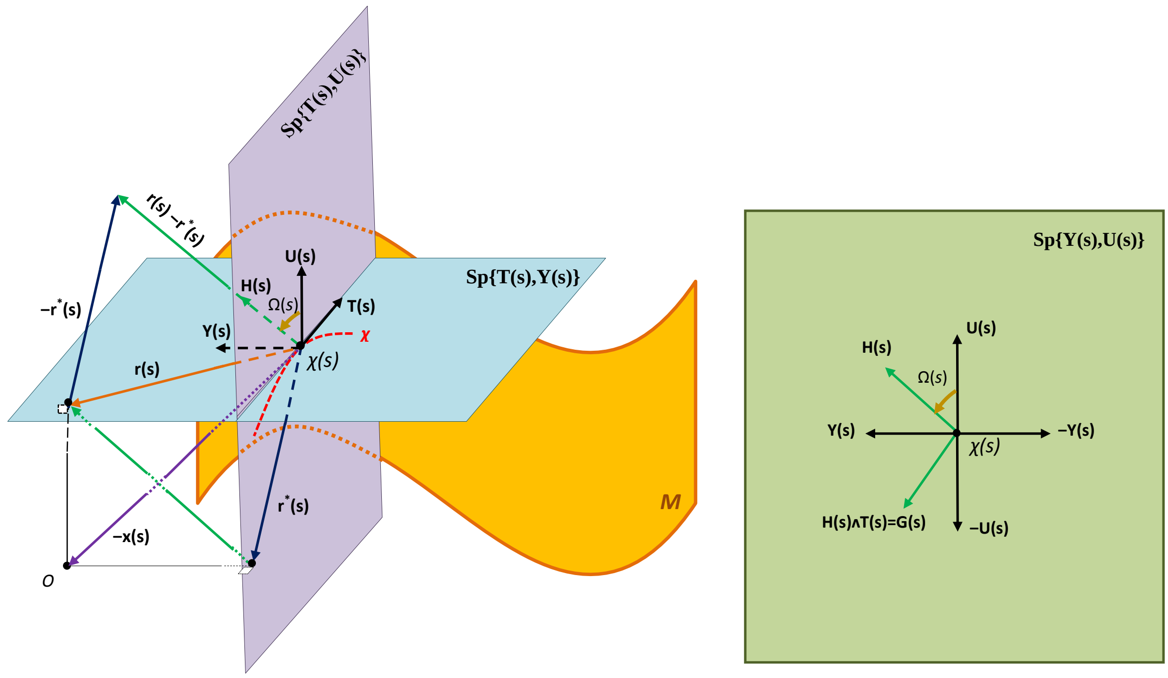
Therefore, the derivative formulas are given by

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{G}'(s)\\ \mathbf{H}'(s)\end{pmatrix}=\begin{pmatrix}0 & k_1(s) & k_2(s)\\ -k_1(s) & 0 & k_3(s)\\ -k_2(s) & -k_3(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix} \tag{2.11}$$

where

$$\begin{aligned}
k_1(s) &= k_g(s)\cos\Omega(s)-k_n(s)\sin\Omega(s) \\
k_2(s) &= k_g(s)\sin\Omega(s)+k_n(s)\cos\Omega(s) \\
k_3(s) &= \tau_g(s)-\Omega'(s).
\end{aligned} \tag{2.12}$$

Based on the relationship of the frame $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s)\}$ to the position vector, we call it as "Positional Adapted Frame on Regular Surface". We will use the abbreviation PAFORS for it in the rest of the study. Also, we call the set $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s),k_1(s),k_2(s),k_3(s)\}$ as PAFORS apparatus of the regular surface curve $\chi=\chi(s)$.



(a) **(b)**

Figure 1. An illustration for explaining the construction of PAFORS

From the Equation 2.8, Equation 2.9 and Equation 2.10, the followings can be written easily:

$$\sin\Omega(s) = \frac{-\langle \chi(s), \mathbf{Y}(s)\rangle}{\sqrt{\langle \chi(s), \mathbf{Y}(s)\rangle^2 + \langle \chi(s), \mathbf{U}(s)\rangle^2}} \tag{2.13}$$

$$\cos\Omega(s) = \frac{\langle \chi(s), \mathbf{U}(s)\rangle}{\sqrt{\langle \chi(s), \mathbf{Y}(s)\rangle^2 + \langle \chi(s), \mathbf{U}(s)\rangle^2}}. \tag{2.14}$$

Then we obtain

$$\tan\Omega(s) = -\frac{\langle \chi(s), \mathbf{Y}(s)\rangle}{\langle \chi(s), \mathbf{U}(s)\rangle}. \tag{2.15}$$

Taking into consideration Figure 1 and Equations 2.13, 2.14, 2.15, the rotation angle $\Omega(s)$ is determined as

$$\Omega(s) = \begin{cases} \arctan\left(-\frac{\langle \chi(s), \mathbf{Y}(s)\rangle}{\langle \chi(s), \mathbf{U}(s)\rangle}\right) \ if \ \langle \chi(s), \mathbf{U}(s)\rangle > 0 \\ \arctan\left(-\frac{\langle \chi(s), \mathbf{Y}(s)\rangle}{\langle \chi(s), \mathbf{U}(s)\rangle}\right) + \pi \ if \ \langle \chi(s), \mathbf{U}(s)\rangle < 0 \\ -\frac{\pi}{2} \ if \ \langle \chi(s), \mathbf{U}(s)\rangle = 0\,, \ \langle \chi(s), \mathbf{Y}(s)\rangle > 0 \\ \frac{\pi}{2} \ if \ \langle \chi(s), \mathbf{U}(s)\rangle = 0\,, \ \langle \chi(s), \mathbf{Y}(s)\rangle < 0. \end{cases} \tag{2.16}$$

When $\langle \chi(s), \mathbf{U}(s)\rangle = 0, \langle \chi(s), \mathbf{Y}(s)\rangle > 0$, PAFORS apparatus $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s), k_1(s), k_2(s), k_3(s)\}$ correspond to $\{\mathbf{T}(s), \mathbf{U}(s), -\mathbf{Y}(s), k_n(s), -k_g(s), \tau_g(s)\}$. Similar to above, in the case $\langle \chi(s), \mathbf{U}(s)\rangle = 0$, $\langle \chi(s), \mathbf{Y}(s)\rangle < 0$, $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s), k_1(s), k_2(s), k_3(s)\}$ correspond to the apparatus $\{\mathbf{T}(s), -\mathbf{U}(s), \mathbf{Y}(s), -k_n(s), k_g(s), \tau_g(s)\}$.

Now, we will get the angular velocity vector for PAFORS. A better insight into the structure of the derivative formulas, given in (2.11), is presented by the help of the angular velocity vector $\omega(s)$. The evolution of PAFORS $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ is specified by its angular velocity via

$$\begin{aligned} \mathbf{T}'(s) &= \omega(s) \wedge \mathbf{T}(s) \\ \mathbf{G}'(s) &= \omega(s) \wedge \mathbf{G}(s) \\ \mathbf{H}'(s) &= \omega(s) \wedge \mathbf{H}(s). \end{aligned} \tag{2.17}$$

Let us obtain the vector $\omega(s)$. Assume that it is written with respect to PAFORS as follows:

$$\omega(s) = \lambda_1(s)\mathbf{T}(s) + \lambda_2(s)\mathbf{G}(s) + \lambda_3(s)\mathbf{H}(s)$$

where $\lambda_1(s)$, $\lambda_2(s)$ and $\lambda_3(s)$ are real-valued functions of s. In this case, (2.17) becomes

$$\begin{aligned} \mathbf{T}'(s) &= -\lambda_2(s)\mathbf{H}(s) + \lambda_3(s)\mathbf{G}(s) \\ \mathbf{G}'(s) &= \lambda_1(s)\mathbf{H}(s) - \lambda_3(s)\mathbf{T}(s) \\ \mathbf{H}'(s) &= -\lambda_1(s)\mathbf{G}(s) + \lambda_2(s)\mathbf{T}(s). \end{aligned} \tag{2.18}$$

By comparing (2.11) with (2.18) we find

$$\begin{aligned} \lambda_1(s) &= k_3(s) \\ \lambda_2(s) &= -k_2(s) \\ \lambda_3(s) &= k_1(s). \end{aligned}$$

Consequentially, the angular velocity vector is given as

$$\omega(s) = \left[\tau_g(s) - \Omega'(s)\right]\mathbf{T}(s) - \left[k_g(s)\sin\Omega(s) + k_n(s)\cos\Omega(s)\right]\mathbf{G}(s) + \left[k_g(s)\cos\Omega(s) - k_n(s)\sin\Omega(s)\right]\mathbf{H}(s)$$

for PAFORS.

3. Some Special Trajectories Generated by Smarandache Curves According to PAFORS

In the study [1], author defined special Smarandache curves in the Euclidean space. Author considered a unit speed regular curve $\gamma = \gamma(s)$ with its Serret-Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and defined **TN**, **NB**, **TNB**$-$ Smarandache curves as follows:

$$\begin{aligned} \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{N}) \\ \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{B}) \\ \beta(s^*) &= \frac{1}{\sqrt{3}}(\mathbf{T} + \mathbf{N} + \mathbf{B}), \end{aligned}$$

respectively. There can be found some studies [1, 3, 5, 15, 16, 18] on Smarandache curves in the literature.

In this section, we investigate special trajectories generated by Smarandache curves according to PAFORS in 3-dimensional Euclidean space.

3.1. Special Trajectories Generated by TG Smarandache Curves

Definition 3.1. In E^3, assume that a point particle P of constant mass moves on the regular surface M along the trajectory $\chi = \chi(s)$ which is a unit speed curve. Let PAFORS be shown with $\{\mathbf{T}_\chi, \mathbf{G}_\chi, \mathbf{H}_\chi\}$ for $\chi = \chi(s)$. Then, special trajectories generated by $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache curves may be defined as follows:

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\chi + \mathbf{G}_\chi\right). \tag{3.1}$$

For convenience, we call these trajectories as $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache trajectories.

Note that PAFORS apparatus of $\chi = \chi(s)$ will be denoted by $\{\mathbf{T}_\chi(s), \mathbf{G}_\chi(s), \mathbf{H}_\chi(s), k_1(s), k_2(s), k_3(s)\}$ in the rest of the paper.

Now, we will discuss Serret-Frenet apparatus of $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache trajectories. Differentiating the Equation 3.1 with respect to the arc-length parameter s of $\chi = \chi(s)$, we obtain

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\mathbf{H}_\chi\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\mathbf{H}_\chi\right). \tag{3.2}$$

From the Equation 3.2, one can easily find

$$\frac{ds^*}{ds} = \sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}}. \tag{3.3}$$

Thus, we can rewrite the Equation 3.2 as

$$\mathbf{T}_\sigma\sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\mathbf{H}_\chi\right). \tag{3.4}$$

The Equation 3.4 gives us the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_1}^2 + (k_2+k_3)^2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\mathbf{H}_\chi\right). \tag{3.5}$$

Differentiating the last equation with respect to the arc-length parameter s of $\chi = \chi(s)$, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_1}^2 + (k_2+k_3)^2\right)^{-3/2}\left(\xi_1\mathbf{T}_\chi + \xi_2\mathbf{G}_\chi + \xi_3\mathbf{H}_\chi\right) \tag{3.6}$$

where

$$\begin{aligned}
\xi_1 &= (k_2+k_3)\left[k_1k'_2 + k_1k'_3 - {k_1}^2k_2 - {k_1}^2k_3 - k'_1(k_2+k_3) - k_2\left(2{k_1}^2 + (k_2+k_3)^2\right)\right] - 2{k_1}^4 \\
\xi_2 &= (k_2+k_3)\left[-k_1k'_2 - k_1k'_3 - {k_1}^2k_2 - {k_1}^2k_3 + k'_1(k_2+k_3) - k_3\left(2{k_1}^2 + (k_2+k_3)^2\right)\right] - 2{k_1}^4 \\
\xi_3 &= k_1(k_2+k_3)\left[-2k'_1 - {k_2}^2 + {k_3}^2\right] + 2{k_1}^2\left[k'_2 + k'_3 + k_1k_3 - k_1k_2\right].
\end{aligned}$$

Considering the Equation 3.3 in the Equation 3.6, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2k_1{}^2 + (k_2+k_3)^2\right)^{-2}\left(\xi_1\mathbf{T}_\chi + \xi_2\mathbf{G}_\chi + \xi_3\mathbf{H}_\chi\right).$$

In that case, the curvature and principal normal vector of σ are obtained as in the following:

$$\kappa_\sigma = \left\|\frac{d\mathbf{T}_\sigma}{ds^*}\right\| = \frac{\sqrt{2\left(\xi_1{}^2+\xi_2{}^2+\xi_3{}^2\right)}}{\left(2k_1{}^2+(k_2+k_3)^2\right)^2}$$

and

$$\mathbf{N}_\sigma = \frac{1}{\sqrt{\xi_1{}^2+\xi_2{}^2+\xi_3{}^2}}\left(\xi_1\mathbf{T}_\chi + \xi_2\mathbf{G}_\chi + \xi_3\mathbf{H}_\chi\right).$$

Where

$$\begin{aligned}\zeta_1 &= k_1\xi_3 - k_2\xi_2 - k_3\xi_2\\ \zeta_2 &= k_2\xi_1 + k_3\xi_1 + k_1\xi_3\\ \zeta_3 &= -k_1\xi_2 - k_1\xi_1,\end{aligned}$$

we can get the binormal vector of σ as

$$\begin{aligned}\mathbf{B}_\sigma &= \frac{1}{\sqrt{\left(2k_1{}^2+(k_2+k_3)^2\right)\left(\xi_1{}^2+\xi_2{}^2+\xi_3{}^2\right)}}\begin{vmatrix}\mathbf{T}_\chi & \mathbf{G}_\chi & \mathbf{H}_\chi\\ -k_1 & k_1 & k_2+k_3\\ \xi_1 & \xi_2 & \xi_3\end{vmatrix}\\ &= \frac{1}{\sqrt{\left(2k_1{}^2+(k_2+k_3)^2\right)\left(\xi_1{}^2+\xi_2{}^2+\xi_3{}^2\right)}}\left(\zeta_1\mathbf{T}_\chi + \zeta_2\mathbf{G}_\chi + \zeta_3\mathbf{H}_\chi\right)\end{aligned}$$

by vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$.

3.2. Special Trajectories Generated by TH Smarandache Curves

Definition 3.2. In E^3, suppose that a point particle P of constant mass moves on the regular surface M along the trajectory $\chi = \chi(s)$ which is a unit speed curve. Let PAFORS be denoted by $\{\mathbf{T}_\chi, \mathbf{G}_\chi, \mathbf{H}_\chi\}$ for $\chi = \chi(s)$. In this case, special trajectories generated by $\mathbf{T}_\chi\mathbf{H}_\chi$−Smarandache curves may be defined by

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\chi + \mathbf{H}_\chi\right). \tag{3.7}$$

For convenience, we call these trajectories as $\mathbf{T}_\chi\mathbf{H}_\chi$−Smarandache trajectories.

Now, we will investigate Serret-Frenet apparatus of $\mathbf{T}_\chi\mathbf{H}_\chi$−Smarandache trajectories. Differentiating the Equation 3.7 with respect to the arc-length parameter s of $\chi = \chi(s)$, we find

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\chi + (k_1-k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right)$$

and so

$$\mathbf{T}_\sigma \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\chi + (k_1-k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right). \tag{3.8}$$

From the Equation 3.8, one can easily obtain

$$\frac{ds^*}{ds} = \sqrt{{k_2}^2 + \frac{(k_1-k_3)^2}{2}}. \tag{3.9}$$

Therefore we can rewrite the Equation 3.8 as in the following:

$$\mathbf{T}_\sigma\sqrt{{k_2}^2 + \frac{(k_1-k_3)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\chi + (k_1-k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right). \tag{3.10}$$

The Equation 3.10 yields the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_2}^2 + (k_1-k_3)^2}}\left(-k_2\mathbf{T}_\chi + (k_1-k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right). \tag{3.11}$$

Differentiating the Equation 3.11 with respect to s, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_2}^2 + (k_1-k_3)^2\right)^{-3/2}\left(\mu_1\mathbf{T}_\chi + \mu_2\mathbf{G}_\chi + \mu_3\mathbf{H}_\chi\right) \tag{3.12}$$

where

$$\begin{aligned}
\mu_1 &= (k_3-k_1)\left[-k_2\left({k'}_1-{k'}_3\right) + 2k_1{k_2}^2 - {k'}_2\,(k_3-k_1) - {k_2}^2\,(k_3-k_1) + k_1(k_3-k_1)^2\right] - 2{k_2}^4 \\
\mu_2 &= k_2\,(k_1-k_3)\left[-2{k'}_2 - {k_1}^2 + {k_3}^2 k_2\left({k'}_1-{k'}_3\right)\right] + 2{k_2}^2\left[{k'}_1 - {k'}_3 - k_1k_2 - k_2k_3\right] \\
\mu_3 &= (k_3-k_1)\left[k_2\left({k'}_1-{k'}_3\right) - 2k_3{k_2}^2 + {k'}_2\,(k_3-k_1) - {k_2}^2\,(k_3-k_1) - k_3(k_3-k_1)^2\right] - 2{k_2}^4.
\end{aligned}$$

Taking into consideration the Equation 3.9 in the Equation 3.12, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2{k_2}^2 + (k_1-k_3)^2\right)^{-2}\left(\mu_1\mathbf{T}_\chi + \mu_2\mathbf{G}_\chi + \mu_3\mathbf{H}_\chi\right).$$

In this case, the curvature and principal normal vector of σ are obtained as follows:

$$\kappa_\sigma = \left\|\frac{d\mathbf{T}_\sigma}{ds^*}\right\| = \frac{\sqrt{2\left({\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2\right)}}{\left(2{k_2}^2 + (k_1-k_3)^2\right)^2}$$

and

$$\mathbf{N}_\sigma = \frac{1}{\sqrt{{\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2}}\left(\mu_1\mathbf{T}_\chi + \mu_2\mathbf{G}_\chi + \mu_3\mathbf{H}_\chi\right).$$

Where

$$\begin{aligned}\eta_1 &= k_1\mu_3 - k_3\mu_3 - k_2\mu_2 \\ \eta_2 &= k_2\mu_1 + k_2\mu_3 \\ \eta_3 &= -k_2\mu_2 + k_3\mu_1 - k_1\mu_1,\end{aligned}$$

we can immediately obtain the binormal vector of σ as

$$\begin{aligned}\mathbf{B}_\sigma &= \frac{1}{\sqrt{\left(2k_2{}^2+(k_1-k_3)^2\right)\left(\mu_1{}^2+\mu_2{}^2+\mu_3{}^2\right)}}\begin{vmatrix}\mathbf{T}_\chi & \mathbf{G}_\chi & \mathbf{H}_\chi \\ -k_2 & k_1-k_3 & k_2 \\ \mu_1 & \mu_2 & \mu_3\end{vmatrix} \\ &= \frac{1}{\sqrt{\left(2k_2{}^2+(k_1-k_3)^2\right)\left(\mu_1{}^2+\mu_2{}^2+\mu_3{}^2\right)}}\left(\eta_1\mathbf{T}_\chi+\eta_2\mathbf{G}_\chi+\eta_3\mathbf{H}_\chi\right)\end{aligned}$$

by vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$.

3.3. Special Trajectories Generated by GH Smarandache Curves

Definition 3.3. In E^3, assume that a point particle P of constant mass moves on the regular surface M along the trajectory $\chi = \chi(s)$ which is a unit speed curve. Let $\{\mathbf{T}_\chi, \mathbf{G}_\chi, \mathbf{H}_\chi\}$ be PAFORS for $\chi = \chi(s)$. Then, special trajectories generated by $\mathbf{G}_\chi\mathbf{H}_\chi$−Smarandache curves can be defined as follows:

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{G}_\chi + \mathbf{H}_\chi\right). \tag{3.13}$$

For convenience, we call these trajectories as $\mathbf{G}_\chi\mathbf{H}_\chi$−Smarandache trajectories.

Now, we will investigate Serret-Frenet apparatus of $\mathbf{G}_\chi\mathbf{H}_\chi$−Smarandache trajectories. Differentiating the Equation 3.13 with respect to the arc-length parameter s of $\chi = \chi(s)$, we get

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right). \tag{3.14}$$

From the Equation 3.14, one can easily obtain

$$\frac{ds^*}{ds} = \sqrt{k_3{}^2 + \frac{(k_1+k_2)^2}{2}}. \tag{3.15}$$

Therefore we can rewrite the Equation 3.14 as in the following:

$$\mathbf{T}_\sigma\sqrt{k_3{}^2 + \frac{(k_1+k_2)^2}{2}} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right). \tag{3.16}$$

The Equation 3.16 yields the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2k_3{}^2 + (k_1 + k_2)^2}} \left((-k_1 - k_2)\, \mathbf{T}_\chi - k_3 \mathbf{G}_\chi + k_3 \mathbf{H}_\chi \right). \tag{3.17}$$

Differentiating the Equation 3.17 with respect to s, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*} \frac{ds^*}{ds} = \left(2k_3{}^2 + (k_1 + k_2)^2 \right)^{-3/2} \left(\upsilon_1 \mathbf{T}_\chi + \upsilon_2 \mathbf{G}_\chi + \upsilon_3 \mathbf{H}_\chi \right) \tag{3.18}$$

where

$$\begin{aligned}
\upsilon_1 &= k_3 (k_1 + k_2) \left[2k'_3 + k_1{}^2 - k_2{}^2 \right] + 2k_3{}^2 \left[k_1 k_3 - k_2 k_3 - k'_1 - k'_2 \right] \\
\upsilon_2 &= (k_1 + k_2) \left[k_3 \left(k'_1 + k'_2 \right) - 2k_1 k_3{}^2 - k'_3 (k_1 + k_2) - k_3{}^2 (k_1 + k_2) - k_1 (k_1 + k_2)^2 \right] - 2k_3{}^4 \\
\upsilon_3 &= (k_1 + k_2) \left[-k_3 \left(k'_1 + k'_2 \right) - 2k_2 k_3{}^2 + k'_3 (k_1 + k_2) - k_3{}^2 (k_1 + k_2) - k_2 (k_1 + k_2)^2 \right] - 2k_3{}^4.
\end{aligned}$$

Taking into consideration the Equation 3.15 in the Equation 3.18, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2} \left(2k_3{}^2 + (k_1 + k_2)^2 \right)^{-2} \left(\upsilon_1 \mathbf{T}_\chi + \upsilon_2 \mathbf{G}_\chi + \upsilon_3 \mathbf{H}_\chi \right).$$

In this case, the curvature and principal normal vector of σ are obtained as follows:

$$\begin{aligned}
\kappa_\sigma &= \frac{\sqrt{2 \left(\upsilon_1{}^2 + \upsilon_2{}^2 + \upsilon_3{}^2 \right)}}{\left(2k_3{}^2 + (k_1 + k_2)^2 \right)^2} \\
\mathbf{N}_\sigma &= \frac{1}{\sqrt{\upsilon_1{}^2 + \upsilon_2{}^2 + \upsilon_3{}^2}} \left(\upsilon_1 \mathbf{T}_\chi + \upsilon_2 \mathbf{G}_\chi + \upsilon_3 \mathbf{H}_\chi \right).
\end{aligned}$$

By vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$, we can immediately obtain the binormal vector of σ as

$$\begin{aligned}
\mathbf{B}_\sigma &= \frac{1}{\sqrt{\left(2k_3{}^2 + (k_1 + k_2)^2 \right) \left(\upsilon_1{}^2 + \upsilon_2{}^2 + \upsilon_3{}^2 \right)}} \begin{vmatrix} \mathbf{T}_\chi & \mathbf{G}_\chi & \mathbf{H}_\chi \\ -k_1 - k_2 & -k_3 & k_3 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \end{vmatrix} \\
&= \frac{1}{\sqrt{\left(2k_3{}^2 + (k_1 + k_2)^2 \right) \left(\upsilon_1{}^2 + \upsilon_2{}^2 + \upsilon_3{}^2 \right)}} \left(\psi_1 \mathbf{T}_\chi + \psi_2 \mathbf{G}_\chi + \psi_3 \mathbf{H}_\chi \right)
\end{aligned}$$

where

$$\begin{aligned}
\psi_1 &= -k_3 \upsilon_3 - k_3 \upsilon_2 \\
\psi_2 &= k_3 \upsilon_1 + k_2 \upsilon_3 + k_1 \upsilon_3 \\
\psi_3 &= -k_1 \upsilon_2 - k_2 \upsilon_2 + k_3 \upsilon_1.
\end{aligned}$$

Note that the torsions of $\mathbf{T}_\chi \mathbf{G}_\chi$, $\mathbf{T}_\chi \mathbf{H}_\chi$, $\mathbf{G}_\chi \mathbf{H}_\chi$-Smarandache trajectories can be obtained by following the similar steps given in this section. We leave this to the readers.

Example 3.4. In E^3, assume that a point particle P of constant mass moves on the regular surface

$$M = \left\{ (x, y, z) : x^2 + y^2 = 64,\ z \geq 0 \right\}$$

along the trajectory

$$\delta:(0,255)\to M\subset E^3,\ \ \delta(t)=\left(8\cos\frac{t}{17},8\sin\frac{t}{17},\frac{t}{17}\right).$$

Reparameterization of $\delta=\delta(t)$ in terms of arc-length parameter is given as follows:

$$\chi(s)=\left(8\cos\frac{s}{\sqrt{65}},8\sin\frac{s}{\sqrt{65}},\frac{s}{\sqrt{65}}\right)$$

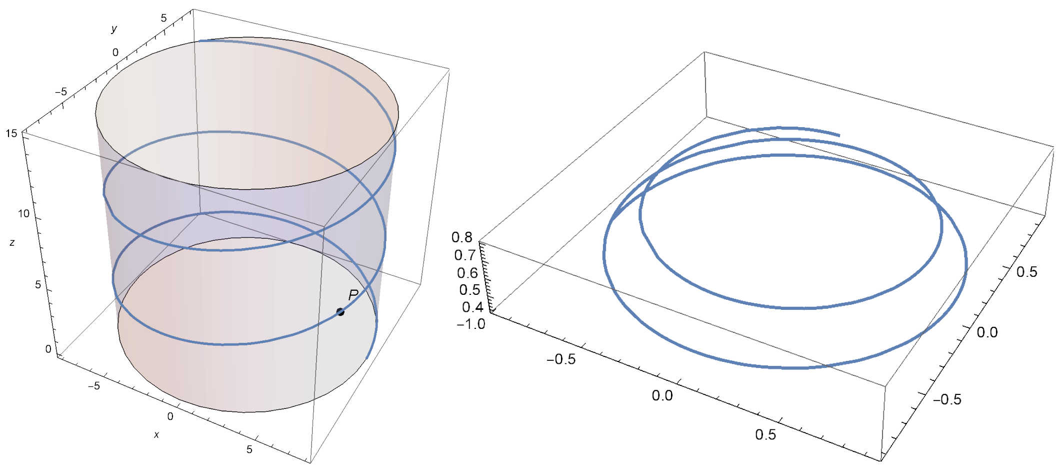
where $s=\frac{\sqrt{65}}{17}t$. One can easily calculate Darboux apparatus of this trajectory as in the following:

$$\begin{aligned}
\mathbf{T}(s) &= \left(\frac{-8}{\sqrt{65}}\sin\frac{s}{\sqrt{65}},\frac{8}{\sqrt{65}}\cos\frac{s}{\sqrt{65}},\frac{1}{\sqrt{65}}\right)\\
\mathbf{U}(s) &= \left(\cos\frac{s}{\sqrt{65}},\sin\frac{s}{\sqrt{65}},0\right)\\
\mathbf{Y}(s) &= \left(\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}},\frac{-1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}},\frac{8}{\sqrt{65}}\right)\\
k_g(s) &= 0\\
k_n(s) &= \frac{-8}{65}\\
\tau_g(s) &= \frac{1}{65}.
\end{aligned}$$

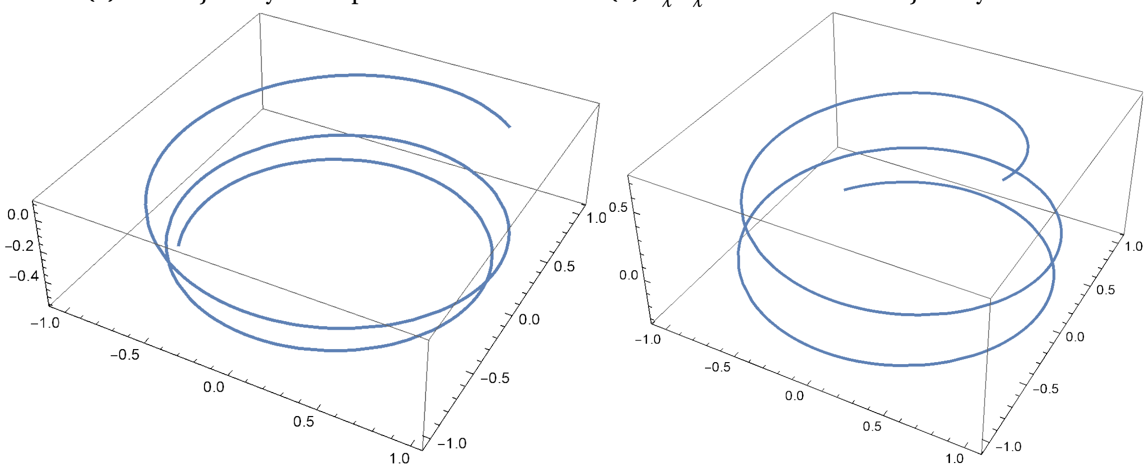
Then, $\left\langle \chi(s),\mathbf{Y}(s)\right\rangle=\frac{8}{65}s$ and $\left\langle \chi(s),\mathbf{U}(s)\right\rangle=8$ are obtained. Since $\left\langle \chi(s),\mathbf{U}(s)\right\rangle>0$ for all the values of parameter, we get $\Omega(s)=\arctan\left(-\frac{s}{65}\right)$. The above information yields the PAFORS apparatus as follows:

$$\begin{aligned}
\mathbf{T}(s) &= \left(\frac{-8}{\sqrt{65}}\sin\frac{s}{\sqrt{65}},\frac{8}{\sqrt{65}}\cos\frac{s}{\sqrt{65}},\frac{1}{\sqrt{65}}\right)\\
\mathbf{G}(s) &= \begin{pmatrix}\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)-\cos\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{-1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)-\sin\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{8}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)\end{pmatrix}\\
\mathbf{H}(s) &= \begin{pmatrix}\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)+\cos\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{-1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)+\sin\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{8}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)\end{pmatrix}\\
k_1(s) &= \frac{8}{65}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)\\
k_2(s) &= -\frac{8}{65}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)\\
k_3(s) &= \frac{1}{65}+\frac{65}{s^2+65^2}
\end{aligned}$$

in the light of the Equation 2.10 and Equation 2.12. We can give the following figure for this example:



(a) The trajectory of the particle P **(b)** $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache trajectory



(c) $\mathbf{T}_\chi\mathbf{H}_\chi-$Smarandache trajectory **(d)** $\mathbf{G}_\chi\mathbf{H}_\chi-$Smarandache trajectory

Figure 2. An illustration including special Smarandache trajectories

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Weighted Integral Transforms Involving Convolution With Some Subclasses of Analytic Functions

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Abstract − Let $\mathscr{A}$ represent the class of analytic functions f defined in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = f'(0) - 1 = 0$ and let $\mathscr{P}$ represent the well-known class of Carathéodory functions p such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0,\ z \in \mathbb{U}$. A functions p analytic in $\mathbb{U}$ such that $p(0) = 1$ belongs to the class $\mathscr{P}_k$ for $k \geq 2$, if and only if

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \mathrm{d}\alpha(\theta) \qquad (z \in \mathbb{U}),$$

where $\alpha(\theta) : 0 \leq \theta \leq 2\pi$ is a function of bounded variation with $\int_0^{2\pi} d\alpha(\theta) = 2\pi$ and $\int_0^{2\pi} |\mathrm{d}\alpha(\theta)| \leq k\pi$. For some $\eta \in \mathbb{R}, \varsigma < 1, k \geq 2$ and $\gamma \geq 0$, let $\mathscr{R}_k^{\eta}(\gamma,\varsigma)$ denote the class of functions $f \in \mathscr{A}$ satisfying the condition: $e^{i\eta}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \varsigma\right) \in \mathscr{P}_k \quad (z \in \mathbb{U})$. For $f \in \mathscr{R}_k^{\eta}(\gamma,\varsigma)$, we define the integral transform $\Im_m(f)(z) = \int_0^1 m(t)\frac{f(tz)}{t} dt$, where m is a non-negative real-valued weight function with $\int_0^1 m(t)\,dt = 1$. The main objective of this paper is to study conditions for invariance of the integral transforms $\Im_m$ and other relevant properties in connection with functions in the class $\mathscr{R}_k^{\eta}(\gamma,\varsigma)$. Also by varying parameters, we encompass a large number of previously known results.

Subject Classification (2020): 30C45, 30C80.

1. Introduction and Definitions

Let $\mathscr{H}(\mathbb{U})$ represent the class of all analytic functions f defined in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and for a positive integer n and $a \in \mathbb{C}$, let

$$\mathscr{H}[a,n] := \left\{ f \in \mathscr{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \qquad (z \in \mathbb{U}) \right\}.$$

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Also, the subclass $\mathcal{A}$ of the class $\mathcal{H}[a,n]$ is defined as:

$$\mathcal{A} := \left\{ f \in \mathcal{H}[0,1] : f'(0) = 1 \right\}. \tag{1.1}$$

The class of univalent functions is represented by $\mathcal{S}$ and it is a subclass of the class $\mathcal{A}$, whereas, $\mathcal{S}^*, \mathcal{C}, \mathcal{K}$ and $\mathcal{Q}$ are the well-known classes of starlike, convex, close-to-convex and quasi-convex functions respectively.

For $f,\ g \in \mathcal{A}$, we define the Hadamard product or convolution $f * g$ by

$$\left(f * g\right)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in \mathbb{U}),$$

where f is defined by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{U}).$$

Let $\mathcal{P}$ denote the well-known class of *Carathéodory functions* p such that $p \in \mathcal{H}(\mathrm{U})$, with

$$p(0) = 1 \text{ and } \operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}).$$

Also $\mathcal{P}(\varsigma)$ represents the class of *Carathéodory functions* p such that $p \in \mathcal{H}(\mathrm{U})$ with

$$p(0) = 1 \text{ and } \operatorname{Re} p(z) > \varsigma \qquad (0 \le \varsigma < 1, z \in \mathbb{U}).$$

For details of these classes, we refer [7]. The function $p \in \mathcal{P}_k$, if and only if it satisfies the conditions $p(0) = 1$ and

$$p(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \mathrm{d}\alpha(\theta) \qquad (z \in \mathbb{U}),$$

where $\alpha(\theta) : 0 \le \theta \le 2\pi$ is a function of bounded variation satisfies the conditions

$$\int_0^{2\pi} \mathrm{d}\alpha(\theta) = 2\pi \text{ and } \int_0^{2\pi} |\mathrm{d}\alpha(\theta)| \le k\pi.$$

or equivalently, $p \in \mathcal{P}_k$ if and only if there exist $p_1,\ p_2 \in \mathcal{P}$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (k \ge 2, z \in \mathbb{U}).$$

Let p be an analytic function defined in the open unit disk $\mathbb{U}$. Then $p \in \mathcal{P}_k(\varsigma)$, if and only if $p(0) = 1$ and

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (0 \le \varsigma < 1, k \ge 2, z \in \mathbb{U}),$$

where $p_1,\ p_2 \in \mathcal{P}(\varsigma)$. For detail of the classes $\mathcal{P}_k$ and $\mathcal{P}_k(\varsigma)$, see [17] and [18] respectively.

For some $\eta \in \mathbb{R}, \varsigma < 1, k \ge 2$ and $\gamma \ge 0$, let $\mathcal{R}_k^{\eta}(\gamma, \varsigma)$ denote the class of functions $f \in \mathcal{A}$ satisfying the condition:

$$e^{i\eta}\left(\left(1-\gamma\right)\frac{f(z)}{z} + \gamma f'(z) - \varsigma\right) \in \mathcal{P}_k \qquad (z \in \mathbb{U}). \tag{1.2}$$

where $\mathscr{A}$ is defined by (1.1). For various related classes, we refer [1, 5, 9, 11, 13, 14].

The well-known Gaussian hypergeometric function $\mathscr{F}$ is defined as:

$$\mathscr{F}\left(\alpha,\beta;\lambda;z\right) := \sum_{n=0}^{\infty} \frac{(\alpha)_n \left(\beta\right)_n}{(\lambda)_n \, n!} z^n \qquad (z \in \mathbb{U}), \tag{1.3}$$

where $\alpha, \beta, \lambda \in \mathbb{C}$, $\lambda \notin \{0, -1, -2, ...\}$. Here for $\alpha \neq 0$, we have

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1), & n = 1,2,3.... \\ 1, & n = 0 \end{cases}$$

If $\operatorname{Re}\lambda > \operatorname{Re}\beta > 0$, then

$$\mathscr{F}\left(\alpha,\beta;\lambda;z\right) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma\left(\lambda-\beta\right)} \int_0^1 t^{\beta-1}(1-t)^{\lambda-\beta-1}(1-tz)^{-\alpha}\,\mathrm{d}t \qquad (z \in \mathbb{U}).$$

Moreover, for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}(\lambda+1) > \operatorname{Re}\left(\alpha+\beta\right)$, we have

$$\mathscr{F}\left(\alpha,\beta;\lambda;z\right) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma\left(\beta\right)\Gamma\left(\lambda-\alpha-\beta+1\right)} \int_0^1 \lambda_1(t)\frac{1}{(1-tz)}\mathrm{d}t \qquad (z \in \mathbb{U}),$$

where

$$\lambda_1(t) = t^{\beta-1}(1-t)^{\lambda-\alpha-\beta}\mathscr{F}\left(\lambda-\alpha, 1-\alpha; \lambda-\alpha-\beta+1; 1-t\right),$$

for detail, see [3, 8]. For special choices of parameters, $\mathscr{F}\left(\alpha,\beta;\lambda;z\right)$ contains Noor integral operator [12, 15], Ruscheweyh derivative [23] and others. For a function $f \in \mathscr{R}_k^{\eta}(\gamma,\varsigma)$, we define the integral transform

$$\Im_m\left(f\right)(z) = \int_0^1 m(t)\frac{f(tz)}{t}\mathrm{d}t, \tag{1.4}$$

where m is a non-negative real-valued integrable weight function such that $\int_0^1 m(t)\,\mathrm{d}t = 1$ and $f \in \mathscr{R}_k^{\eta}(\gamma,\varsigma)$ satisfies (1.2). The operator $\Im_m\left(f\right)$ contain Libera, Bernardi, and Komatu operators as special cases. For $f \in \mathscr{R}_2^{\eta}(\gamma,\varsigma)$, $\Im_m\left(f\right)$ has been investigated by various authors, for reference, see [3, 10, 19−−22].

2. A Set of Preliminary Results

To establish our main results, we will use the following lemmas.

Lemma 2.1. [20] Let $\varsigma_1, \varsigma_2 < 1$ and let the functions p and q be analytic in $\mathbb{U}$ with $p(0) = q(0) = 1$. Then the conditions

$$\operatorname{Re} p(z) > \varsigma_1 \qquad (z \in \mathbb{U}) \ \text{ and } \ \operatorname{Re} e^{i\eta} q(z) - \varsigma_2 > 0 \qquad (z \in \mathbb{U})$$

imply

$$\operatorname{Re}\left(e^{i\eta}\left(p * q\right)(z) - \delta\right) > 0 \qquad (z \in \mathbb{U}),$$

where $1-\delta = 2(1-\varsigma_1)(1-\varsigma_2)$.

Lemma 2.2. Let $\varsigma_1 < 1$, $\gamma \geq 1$ and $\varsigma = \varsigma\left(\varsigma_1, \gamma\right)$ be such that

$$\varsigma = 1 - \frac{1-\varsigma_1}{2}\{1 - \frac{1}{\gamma}\int_0^1 \frac{m(t)}{1+t}\mathrm{d}t + (\frac{1}{\gamma}-1)\int_0^1 m(t)(\int_0^1 \frac{\mathrm{d}u}{1+tu^{\gamma}})\mathrm{d}t\}^{-1}. \tag{2.1}$$

If $\mathscr{F}\left(\alpha,\beta;\lambda;z\right)=\mathscr{F}\left(2,\frac{1}{\gamma},1+\frac{1}{\gamma};z\right)$, then

$$\operatorname{Re}\int_0^1 m(t)\,\mathscr{F}\left(2,\frac{1}{\gamma},1+\frac{1}{\gamma};tz\right)\mathrm{d}t>1-\frac{1-\varsigma_1}{2(1-\varsigma)},$$

where m is a real-valued non-negative weight function with $\int_0^1 m(t)\,\mathrm{d}t=1$ and $\mathscr{F}\left(\alpha,\beta;\lambda;z\right)$ is defined by (1.3). The value of ς is sharp.

Lemma 2.3. Let $0<\alpha\leq 1$ and $\beta<\lambda-\alpha\leq\frac{1}{\alpha}$. Then

$$\operatorname{Re}M(z)=\operatorname{Re}\left\{(1-\alpha)\mathscr{F}\left(\alpha,\beta;\gamma;z\right)+\alpha\mathscr{F}\left(\alpha+1,\beta;\lambda;z\right)\right\}\geq M(-1)=\varsigma_1\qquad(z\in\mathbb{U}).$$

This result is sharp.

Lemma 2.4. Let $-1<\alpha<0$ and $\beta>\alpha$. Then for

$$M(z)=\begin{cases}\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1-tz}\mathrm{d}t, \text{ for } \beta\neq\alpha\\ (1+\alpha)^2\int\limits_0^1\frac{t^{\alpha}(1+\alpha\log t)}{1-tz}\mathrm{d}t, \text{ for } \beta=\alpha\end{cases}\qquad(z\in\mathbb{U}),$$

we have

$$\operatorname{Re}M(z)>M(-1)=\varsigma_1=\begin{cases}\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1+t}\mathrm{d}t, \text{ for } \beta\neq\alpha\\ (1+\alpha)^2\int\limits_0^1\frac{t^{\alpha}(1+\alpha\log t)}{1+t}\mathrm{d}t, \text{ for } \beta=\alpha\end{cases}\qquad(z\in\mathbb{U}).$$

These inequalities are sharp.

Lemma 2.5. Let $-1<\alpha\leq 0$, $q>1$ and

$$M(z)=\frac{(1+\alpha)^q}{\Gamma(q)}\int\limits_0^1 t^{\alpha}\log\left(\frac{1}{t}\right)^{q-2}\frac{q-1-\alpha\log\left(\frac{1}{t}\right)}{1-tz}\mathrm{d}t\qquad(z\in\mathbb{U}).$$

Then

$$\operatorname{Re}M(z)\geq M(-1)=\varsigma_1=\frac{(1+\alpha)^q}{\Gamma(q)}\int\limits_0^1\log(\frac{1}{t})^{q-2}\left(q-1-\alpha\log(\frac{1}{t})\right)\frac{t^{\alpha}}{1+t}\mathrm{d}t.$$

For the proof of Lemma 2 to Lemma 5, we refer, [4].

3. Main Results

In the following theorem, we find the conditions such that $\Im_m(f)\in\mathscr{R}_k(1,\varsigma_1)$ whenever $f\in\mathscr{R}_k^{\eta}(\gamma,\varsigma)$.

Theorem 3.1. Let $\varsigma_1<1$, $\gamma\geq 1$, $k\geq 2$ and let $\varsigma=\varsigma\left(\varsigma_1,\gamma\right)$ be defined by (2.1). If $f\in\mathscr{R}_k^{\eta}(\gamma,\varsigma)$, then $\Im_m(f)$ defined by (1.4) also belongs to the class $\mathscr{R}_k(1,\varsigma_1)$. The value of ς is sharp.

Proof.

Let

$$\left(1-\gamma\right)\frac{f(z)}{z}+\gamma f'(z)=p(z)\qquad(z\in\mathbb{U}),\tag{3.1}$$

where $p(0)=1$. If $f\in\mathscr{R}_k^{\eta}(\gamma,\varsigma)$, then by (3.1), we have

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\in\mathscr{P}_k(\varsigma)\qquad(z\in\mathbb{U}),\tag{3.2}$$

$p_i \in \mathscr{P}(\varsigma), i = 1, 2$ and conversely. For $\gamma \neq 0$, from (3.2), we write

$$\left\{1 + \left(1+\gamma\right) z + \left(1+2\gamma\right) z^2 + ...\right\} * \frac{f(z)}{z} = \left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z).$$

On further simplification, we obtain

$$f'(z) = \left[\left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z)\right] * \sum_{n=0}^{\infty} \frac{n+1}{1+n\gamma} z^n \qquad (z \in \mathbb{U}), \tag{3.3}$$

which is equivalent to

$$f'(z) = \left[\left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z)\right] * \mathscr{F}(2, \frac{1}{\gamma}; 1+\frac{1}{\gamma}; z) \qquad (z \in \mathbb{U}) \tag{3.4}$$

where $\mathscr{F}(2, \frac{1}{\gamma}; 1+\frac{1}{\gamma}; z)$ is defined by (1.3). For $\gamma = 0$, we write

$$\begin{aligned} f'(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) \left(zp_1(z)\right)' - \left(\frac{k}{4}-\frac{1}{2}\right) \left(zp_2(z)\right)' \\ &= \left[\left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z)\right] * \mathscr{F}(2, 1; 1; z). \end{aligned}$$

This is the limiting case of (3.3) for $\gamma \longrightarrow 0$. Differentiating (1.4) and then simplifying, we have

$$\Im'_m(f)(z) = \frac{\mathrm{d}}{\mathrm{d}z} \int_0^1 m(t) \frac{f(tz)}{t} \mathrm{d}t = f'(z) * \int_0^1 \frac{m(t)}{1-tz} \mathrm{d}t \qquad (z \in \mathbb{U}), \tag{3.5}$$

where m a non-negative real-valued weight function such that $\int_0^1 m(t)dt = 1$. Both (3.4) and (3.5) yield

$$\Im'_m(f)(z) = k_1 p_1(z) * \int_0^1 m(t)\mathscr{F}(2, \frac{1}{\gamma}; 1+\frac{1}{\gamma}; tz)\mathrm{d}t - k_2 p_2(z) * \int_0^1 m(t)\mathscr{F}(2, \frac{1}{\gamma}; 1+\frac{1}{\gamma}; tz)\mathrm{d}t. \tag{3.6}$$

For $\gamma = 0$, we have

$$\Im'_m(f)(z) = \left(\frac{k}{4}+\frac{1}{2}\right) \left[p_1(z) * \int_0^1 \frac{m(t)}{(1-tz)^2} \mathrm{d}t\right] - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z) * \int_0^1 \frac{m(t)}{(1-tz)^2} \mathrm{d}t,$$

which is just the limiting case of (3.5) for $\gamma \longrightarrow 0$ and m a non-negative real-valued weight function such that $\int_0^1 m(t)dt = 1$. For $\gamma \geq 1$, using Lemma 2, we write

$$\mathrm{Re} \int_0^1 m(t)\mathscr{F}(2, \frac{1}{\gamma}; 1+\frac{1}{\gamma}; tz)\mathrm{d}t > \varsigma_1 = 1 - \frac{1-\rho}{2(1-\varsigma)}, \varsigma_1 < 1 \qquad (z \in \mathbb{U}),$$

where ς is given by (2,1) the condition mentioned above in the statement of the theorem and m a non-negative real-valued weight function such that $\int_0^1 m(t)dt = 1$. Again using Lemma 1, we obtain

$$p_i(z) * \int_0^1 m(t)\mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; tz)\mathrm{d}t \in \mathscr{P}(\varsigma_1) \text{ for } i = 1, 2 \qquad (z \in \mathbb{U}). \tag{3.7}$$

From (3.5), (3.6) and (3.7), we obtain $\Im_m(f) \in \mathscr{R}_k(1, \rho)$. To prove the sharpness, we consider the function $f \in \mathscr{R}_k(\gamma, \varsigma)$ determined by the relation

$$(1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) = (1 - \varsigma) \frac{1 + kz + z^2}{1 - z^2} + \varsigma \qquad (z \in \mathbb{U}).$$

On simplification, we obtain

$$f'(z) = 1 + (1 - \varsigma)k \left\{ \frac{2z}{1 + \gamma} + \frac{4z^3}{1 + 3\gamma} + ... \right\} + (1 - \varsigma) \left\{ \frac{6z^2}{1 + 2\gamma} + \frac{10z^4}{1 + 4\gamma} + ... \right\}.$$

This implies that

$$f(z) = z + (1 - \varsigma) \sum_{n=1}^{\infty} \left[\frac{k}{1 + (2n - 1)\gamma} z^{2n} + \frac{2}{1 + 2n\gamma} z^{2n+1} \right] \qquad (z \in \mathbb{U}). \tag{3.8}$$

Now, using (3.8) in (1.4), we have

$$\Im_m f(z) = z + k(1 - \varsigma) \sum_{n=1}^{\infty} \frac{\mu_n}{1 + (2n - 1)\gamma} z^{2n} + 2(1 - \varsigma) \sum_{n=1}^{\infty} \frac{\upsilon_n}{1 + 2n\gamma} z^{2n+1} \qquad (z \in \mathbb{U}), \tag{3.9}$$

where

$$\mu_n = \int_0^1 m(t)\, t^{2n-1}\mathrm{d}t \quad \text{and} \quad \upsilon_n = \int_0^1 m(t)\, t^{2n}\mathrm{d}t.$$

The function given in (3.9) is the required extremal function for the parameter ς.

Theorem 3.2. Let $0 < \alpha \leq 1$, $\beta < \lambda - \alpha \leq \frac{1}{\alpha}$ and let $\mathfrak{F}$ be the convolution operator defined as:

$$\mathfrak{F}(z) := f(z) * z\mathscr{F}(\alpha, \beta; \lambda; z) \qquad (z \in \mathbb{U}). \tag{3.10}$$

Suppose that $f \in \mathscr{R}_k(0, \varsigma)$. Then,

$$\mathfrak{F} \in \mathscr{R}_k(1, \gamma = 1 - 2(1 - \varsigma)\, 1 - \varsigma_1))$$

with

$$\varsigma_1 = M(-1) = (1 - \alpha)\mathscr{F}(\alpha, \beta; \lambda; -1) + \alpha\mathscr{F}(\alpha + 1, \beta; \lambda; -1).$$

In particular

(i) $e^{i\eta}\left(\frac{f(z)}{z} - \frac{1-2\varsigma_1}{2(1-\varsigma_1)}\right) \in \mathscr{P}_k$ implies that $e^{i\eta}\, \mathfrak{F}'(z) \in \mathscr{P}_k$

and

(ii) $e^{i\eta}\left(\frac{f(z)}{z} - \frac{1}{2}\right) \in \mathscr{P}_k$ yields$\left(e^{i\eta}\, \mathfrak{F}'(z) - \varsigma_1\right) \in \mathscr{P}_k$.

Proof.

Rewriting (3.10), we have $\mathfrak{F}(z) := f(z) * z\mathscr{F}(\alpha,\beta;\lambda;z)$, where $f \in \mathscr{R}_k(0,\varsigma)$. This implies that

$$\mathfrak{F}'(z) = \frac{f(z)}{z} * \left(z\mathscr{F}(\alpha,\beta;\lambda;z)\right)' = \frac{f(z)}{z} * M(z), \tag{3.11}$$

where $M(z) = \left(z\mathscr{F}(\alpha,\beta;\lambda;z)\right)'$. Now, taking derivative of hypergeometric function and using

$$\lambda\mathscr{F}(\alpha+1,\beta;\lambda;z) = \beta z\mathscr{F}(\alpha+1,\beta+1;\lambda+1;z) + \lambda\mathscr{F}(\alpha,\beta;\lambda;z),$$

we obtain

$$M(z) = (1-\alpha)\mathscr{F}(\alpha,\beta;\lambda;z) + \alpha\mathscr{F}(\alpha+1,\beta;\lambda;z) \qquad (z \in \mathbb{U}).$$

For $\lambda > \alpha+\beta$, we write

$$M(z) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda-\alpha-\beta)}\int_0^1 m_1(t)\frac{1}{1-tz}\mathrm{d}t \qquad (z \in \mathbb{U}),$$

where

$$m_1(t) = \frac{(1-\alpha)t^{\beta-1}(1-t)^{\lambda-\alpha-\beta}}{\lambda-\alpha-\beta}\mathscr{F}(\lambda-\alpha,1-\alpha;\lambda-\alpha-\beta+1;1-t) \\ + t^{\beta-1}(1-t)^{\lambda-\alpha-\beta-1}\mathscr{F}(\lambda-\alpha-1,-\alpha;\lambda-\alpha-\beta;1-t).$$

For $\beta < \lambda-\alpha \le 1$ and $\alpha \in (0,1]$, using Lemma 3, we see that

$$\operatorname{Re} M(z) > M(-1) = \varsigma_1, \tag{3.12}$$

where

$$\varsigma_1 = (1-\alpha)\mathscr{F}(\alpha,\beta;\lambda;-1) + \alpha\mathscr{F}(\alpha+1,\beta;\lambda;-1).$$

For $f \in \mathscr{R}_k(0,\varsigma)$, we have

$$\frac{f(z)}{z} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \qquad (z \in \mathbb{U}),$$

where $p_i \in \mathscr{P}(\varsigma)$ for $i = 1,2$. This implies that

$$\frac{f(z)}{z} * M(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) * M(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) * M(z) \qquad (z \in \mathbb{U}). \tag{3.13}$$

Using (3.12) and Lemma 1, we write

$$p_i * M \in \mathscr{P}(\gamma) \text{ for } i = 1,2 \qquad (z \in \mathbb{U}), \tag{3.14}$$

where $\gamma = 1-2(1-\varsigma)1-\varsigma_1)$. On combining (3.11), (3.13) and (3.14), we obtain

$$\mathfrak{F}'(z) = \frac{f(z)}{z} * M(z) \in \mathscr{P}_k(\gamma).$$

This implies that $\mathfrak{F} \in \mathscr{R}_k(1,\gamma)$. Let $f \in \mathscr{R}_k(0,\varsigma)$. For the extremal function which gives the sharpness, consider

$$\frac{f(z)}{z}=\left(\frac{k}{4}+\frac{1}{2}\right)\left[1+2(1-\varsigma)\frac{z}{1-z}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[1-2(1-\varsigma)\frac{z}{1+z}\right] \qquad (z\in\mathbb{U})$$

and

$$M(z)=1+2(1-\varsigma_1)\frac{z}{1-z}.$$

Now

$$\frac{f(z)}{z}*M(z)=M(z)*\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left(1+\frac{2(1-\varsigma)z}{1-z}\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(1-\frac{2(1-\varsigma)z}{1+z}\right)\right\}$$

which on simplification yields

$$\frac{f(z)}{z}*M(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\left[1+\frac{4(1-\varsigma)(1-\varsigma_1)z}{1-z}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[1-\frac{4(1-\varsigma)(1-\varsigma_1)z}{1+z}\right]. \tag{3.15}$$

Thus from (3.15), we obtain the required extremal function.

Theorem 3.3. Let $-1<\alpha<0,\ \beta>\alpha$ and $f\in\mathscr{R}_k(0,\varsigma)$. Then

$$\mathscr{G}\in\mathscr{R}_k(1,1-2(1-\varsigma)(1-\varsigma_1)),$$

where

$$\mathscr{G}(z)=\sum_{n=1}^{\infty}\frac{(1+\alpha)(1+\beta)}{(n+\alpha)(n+\beta)}z^n*f(z)=\mathscr{G}(f)(z) \qquad (z\in\mathbb{U}), \tag{3.16}$$

and

$$\varsigma_1=\begin{cases}\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int_0^1\frac{\beta t^\beta-\alpha t^\alpha}{1+t}\mathrm{d}t, & \text{for } \beta\neq\alpha,\\ (1+\alpha)^2\int_0^1\frac{t^\alpha(1+\alpha\log t)}{1+t}\mathrm{d}t, & \text{for } \beta=\alpha.\end{cases}$$

This result is sharp.

Proof.

Let $\alpha\in(-1,0),\beta>\alpha$ and $\mathscr{G}$ be defined by (3.16). Then

$$\mathscr{G}'(z)=\frac{1}{(1-z)^2}*\sum_{n=0}^{\infty}\frac{(1+\alpha)(1+\beta)}{(n+\alpha)(n+\beta)}z^n*\frac{f(z)}{z} \qquad (z\in\mathbb{U})$$

or

$$\mathscr{G}'(z)=\sum_{n=0}^{\infty}\frac{(1+\alpha)(1+\beta)(n+1)}{(n+\alpha)(n+\beta)}z^n*\frac{f(z)}{z}=\frac{f(z)}{z}*M(z) \qquad (z\in\mathbb{U}), \tag{3.17}$$

where

$$M(z)=\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\left[-\alpha\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha+1)}+\beta\sum_{n=0}^{\infty}\frac{z^n}{(n+\beta+1)}\right] \qquad (z\in\mathbb{U}).$$

The function M can also be written as

$$M(z)=\frac{1}{\beta-\alpha}(1+\alpha)(1+\beta)\int_0^1\frac{\beta t^\beta-\alpha t^\alpha}{1-tz}\mathrm{d}t \qquad (z\in\mathbb{U}). \tag{3.18}$$

Using Lemma 4, from (3.18), we have

$$\operatorname{Re}M(z)>M(-1)=\varsigma_1=\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int_0^1\frac{\beta t^\beta-\alpha t^\alpha}{1+t}\mathrm{d}t. \tag{3.19}$$

Now for $f \in \mathscr{R}_k(0,\varsigma)$, consider

$$\frac{f(z)}{z} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \qquad (z \in \mathbb{U}),$$

where $p_i \in \mathscr{P}(\varsigma)$ for $i = 1,2$. This implies that

$$\frac{f(z)}{z} * M(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) * M(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) * M(z) \qquad (z \in \mathbb{U}). \tag{3.20}$$

Using (3.19) and Lemma 1, we write

$$p_i(z) * M(z) \in \mathscr{P}(1-2(1-\varsigma)1-\varsigma_1)) \text{ for } i = 1,2 \qquad (z \in \mathbb{U}). \tag{3.21}$$

On combining (3.17), (3.20) and (3.21), we obtain

$$\mathscr{G}'(z) = \frac{f(z)}{z} * M(z) \in \mathscr{P}_k(1-2(1-\varsigma)1-\varsigma_1)) \qquad (z \in \mathbb{U}).$$

This implies that

$$\mathscr{G} \in \mathscr{R}_k(1, 1-2(1-\varsigma)1-\varsigma_1)).$$

For $\beta = \alpha$, the similar result for the conditions described in the theorem can be obtained by taking the limit $\beta \longrightarrow \alpha$ in the previous case $\alpha < \beta$. Sharpness can be obtained as in previous theorems.

Theorem 3.4. Let $-1 < \alpha \leq 0,\ q > 1$ and $f \in \mathscr{R}_k(0,\varsigma)$. Then the the operator $\mathfrak{F}_{\alpha,q}$ defined by

$$\mathfrak{F}_{\alpha,q}(z) = \mathfrak{F}_{\alpha,q}(f)(z) = \sum_{n=1}^{\infty} \frac{(1+\alpha)^q}{(n+\alpha)^q} z^n * f(z) \qquad (z \in \mathbb{U})$$

is in the class $\mathscr{R}_k(1, 1-2(1-\varsigma)(1-\varsigma_1))$ with

$$\varsigma_1 = M(-1) = \frac{(1+\alpha)^q}{\Gamma(q)} \int_0^1 \log\left(\frac{1}{t}\right)^{q-2}\left(q-1-\alpha\log\left(\frac{1}{t}\right)\right)\frac{t^\alpha}{1+t}\mathrm{d}t.$$

Proof.

For $q > 0$ and $\alpha > -1$, the operator $\mathfrak{F}_{\alpha,q}$ is defined as

$$\mathfrak{F}_{\alpha,q}(f)(z) = \frac{(1+\alpha)^q}{\Gamma(q)} \int_0^1 \log\left(\frac{1}{t}\right)^{q-1} t^{\alpha-1} f(tz)\,\mathrm{d}t \qquad (z \in \mathbb{U}).$$

Now for $-1 < \alpha \leq 0,\ q > 1$ and $f \in \mathscr{R}_k(0,\varsigma)$,

$$\mathfrak{F}_{\alpha,q}(f)(z) = \sum_{n=1}^{\infty} \frac{(1+\alpha)^q}{(n+\alpha)^q} z^n * f(z) \qquad (z \in \mathbb{U})$$

or

$$\mathfrak{F}'_{\alpha,q}(f)(z) = \sum_{n=1}^{\infty} \frac{n(1+\alpha)^q}{(n+\alpha)^q} z^{n-1} * \frac{f(z)}{z} = M(z) * \frac{f(z)}{z} \qquad (z \in \mathbb{U}). \tag{3.22}$$

By Lemma 5, we see that

$$\operatorname{Re} M(z) > M(-1) = \frac{(1+\alpha)^q}{\Gamma(q)} \int_0^1 \log\left(\frac{1}{t}\right)^{q-2}\left(q-1-\alpha\log\left(\frac{1}{t}\right)\right)\frac{t^\alpha}{1+t}\mathrm{d}t.$$

Now for $f \in \mathscr{R}_k(0,\varsigma)$, consider

$$\frac{f(z)}{z} = \left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z) \qquad (z \in \mathbb{U}),$$

where $p_i \in \mathscr{P}(\varsigma)$ for $i = 1,2$. This implies that

$$\frac{f(z)}{z} * M(z) = \left(\frac{k}{4}+\frac{1}{2}\right) p_1(z) * M(z) - \left(\frac{k}{4}-\frac{1}{2}\right) p_2(z) * M(z) \qquad (z \in \mathbb{U}). \tag{3.24}$$

Using (3.23) and Lemma 1, we write

$$p_i * M \in \mathscr{P}(1-2(1-\varsigma)1-\varsigma_1)) \text{ for } i = 1,2. \tag{3.25}$$

On combining (3.22), (3.24) and (3.25), we obtain

$$\mathfrak{F}'_{\alpha,q}(f)(z) = \frac{f(z)}{z} * M(z) \in \mathscr{P}_k(1-2(1-\varsigma)1-\varsigma_1)).$$

This implies that $\mathfrak{F}'_{\alpha,q} \in \mathscr{R}_k(1,\gamma)$. The sharpness of the above result is straight forward.

For special choices of parameter, we also refer [2, 6, 16, 24].

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