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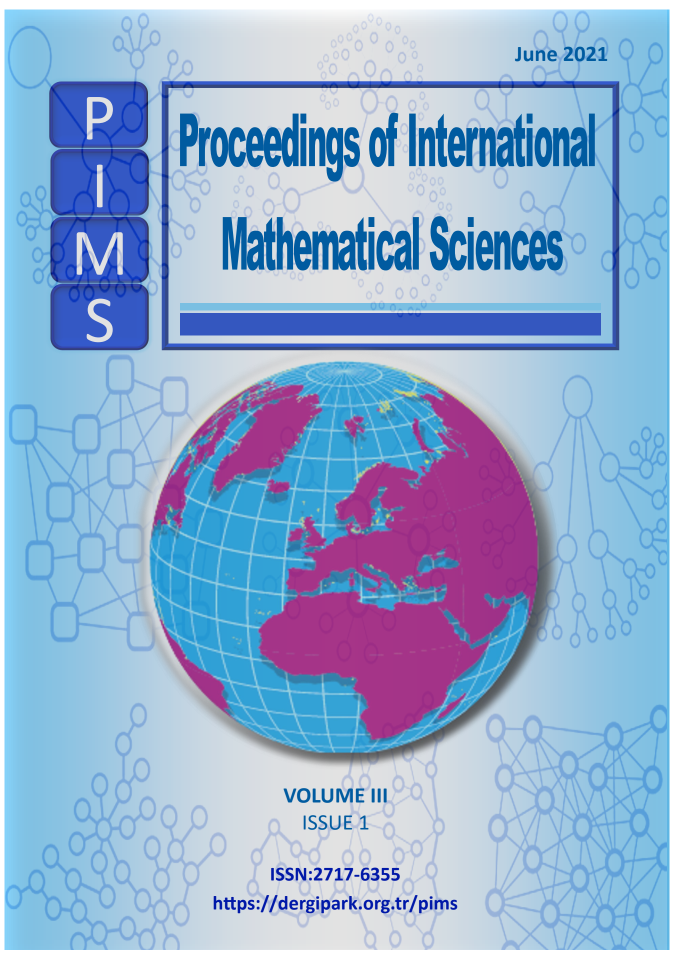

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A NEW VARIATION ON ABSOLUTE SUMMABILITY

ŞEBNEM YILDIZ
DEPARTMENT OF MATHEMATICS, KIRŞEHIR AHI EVRAN UNIVERSITY, TURKEY

Abstract. In [4], Bor has proved a main theorem dealing with absolute weighted arithmetic mean summability factors of infinite series by using a positive non-decreasing sequence. In this paper, we have extended this result to absolute matrix summability method by using an almost increasing sequence in place of a positive non-decreasing sequence. Also, some new and known results are also obtained.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums be denoted by (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [9])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0.$$

A series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$

If we take $\alpha = 1$, then we have the $|C,1|_k$ summability. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

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defines the sequence (w_n) of the weighted arithmetic mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [11]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all n, then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.
Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix with nonzero diagonal entries. Then A defines a sequence-to-sequence transformation, mapping of the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [15])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability is reduced to the $|A, p_n|_k$ summability (see [17]). If we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then $\varphi - |A, p_n|_k$ summability is reduced to the $|\bar{N}, p_n|_k$ summability. If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then

$\varphi - |A, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

1. Known Result

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (z_n) and two positive constants A and B such that $Az_n \leq b_n \leq Bz_n$ (see [1]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. The following theorem concerning on absolute summability factors of infinite series has been obtained.
Theorem 2.1 [4] Let (X_n) be a positive non-decreasing sequence and let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n), \tag{1.1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{1.2}$$

If the sequences (X_n) , (β_n), and (λ_n) satisfy the conditions

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.3}$$

$$\beta_n \to 0 \quad \text{as} \quad n \to \infty, \tag{1.4}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n < \infty, \tag{1.5}$$

$$|\lambda_n| X_n = O(1), \tag{1.6}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{1.7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.
Remark 2.1 It should be noted that, under the conditions on the sequence (λ_n) we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

2. Main Result

The aim of this paper is to generalize Theorem 2.1 by using an almost increaing sequence for $\varphi - |A, p_n|_k$ summability method, which is more general matrix summability method than $|\bar{N}, p_n|_k$ summability method. Some papers have been done dealing with absolute summability methods (see [5]-[8], [18]-[24]).
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, .. \tag{2.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{2.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{2.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{2.4}$$

Let ω be the class of all matrices $A = (a_{nv})$ satisfying

$$A \quad \text{is a positive normal matrix,} \tag{2.5}$$

$$\bar{a}_{n0} = 1, \; n = 0, 1, ... \tag{2.6}$$

$$a_{n-1,v} \geq a_{nv}, \quad n \geq v + 1. \tag{2.7}$$

With this notation we have the following theorem.
Theorem 3.1 Let $A \in \omega$ satisfying

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{2.8}$$

$$1 = O(na_{nn}), \tag{2.9}$$

$$\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}| = O(a_{nn}). \tag{2.10}$$

Let (X_n) be an almost increasing sequence and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (1.2)-(1.6) of Theorem 2.1, and the condition

$$\sum_{n=1}^{m} (\frac{\varphi_n p_n}{P_n})^{k-1} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty \tag{2.11}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\varphi - |A, p_n|_k$, $k \geq 1$.

Remark 3.1 It is noted that by using the conditions (2.8) and (2.9), we have

$$P_n = O(np_n) \tag{2.12}$$

We need the following lemmas for the proof of Theorem 3.1.
Lemma 3.1 [12] Under the conditions of Theorem 2.1, we have

$$nX_n\beta_n = O(1),$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

Lemma 3.2 [14] If the condition (1.2) of Theorem 2.1 and (2.12) are satisfied, then $\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right)$.
Lemma 3.3 [16] Let $A \in \omega$ and by using (2.1) and (2.2), we have that

$$\sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \leq 1,$$

and by using $\Delta_v(\hat{a}_{nv}) = a_{nv} - a_{n-1,v}$, we get

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \leq a_{nn},$$

and

$$\sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \leq a_{vv}.$$

3. Proof of Theorem 3.1

Let (V_n) denotes the A-transform of the series $\sum a_n \frac{P_n \lambda_n}{np_n}$. Then

$$\bar{\Delta} V_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \frac{P_v \lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v$$

$$\bar{\Delta}V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v p_v} \right) s_v + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} s_n,$$

by the formula for the difference of products of sequences (see [11]) we have

$$\begin{aligned}
\bar{\Delta}V_n &= \frac{a_{nn} P_n \lambda_n}{n p_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{v p_v} \Delta_v(\hat{a}_{nv}) s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) s_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_{v+1}}{(v+1) p_{v+1}} \Delta \lambda_v s_v \\
\bar{\Delta}V_n &= V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \mid V_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{3.1}$$

Firstly, by using condition (2.8), (2.11) and (2.12) and applying Abel's transformation, we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid V_{n,1} \mid^k &\leq \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^k \left(\frac{P_n}{p_n} \right)^k |\lambda_n|^k \frac{|s_n|^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\frac{P_n}{p_n} \right)^k |\lambda_n|^k \frac{|s_n|^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\frac{P_n}{p_n} \right)^{k-1} |\lambda_n|^k \frac{|s_n|^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} n^{k-1} |\lambda_n|^k \frac{|s_n|^k}{n^k} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\lambda_n|^k |s_n|^k \frac{1}{n} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} |\lambda_n|^{k-1} |\lambda_n| \frac{|s_n|^k}{n} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \frac{1}{X_n^{k-1}} |\lambda_n| \frac{|s_n|^k}{n}
\end{aligned}$$

$$= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{|s_v|^k}{vX_v^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\frac{|s_n|^k}{nX_n^{k-1}}$$
$$= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m$$
$$= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m$$
$$= O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. By applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, and as in $V_{n,1}$, we have that

$$\sum_{n=2}^{m+1}\varphi_n^{k-1}\mid V_{n,2}\mid^k = \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v}{vp_v}\Delta_v(\hat{a}_{nv})s_v\right|^k$$
$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}a_{vv}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right)\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v^k}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}v^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v^k}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{1}{vX_v^{k-1}}|\lambda_v||s_v|^k$$
$$= O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1, Lemma 3.1 and Lemma 3.3. Also, by using the the fact that $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ and Lemma 3.3, again as in $V_{n,1}$, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid V_{n,3} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta\left(\frac{P_v}{vp_v}\right) \lambda_v s_v \right|^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} a_{vv}^{1-k} |\hat{a}_{n,v+1}||\lambda_v|^k |s_v|^k \frac{1}{v^k} \right\} \times \left\{ \sum_{v=1}^{n-1} a_{vv} |\hat{a}_{n,v+1}| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{k-1} \sum_{v=1}^{n-1} a_{vv}^{1-k} |\hat{a}_{n,v+1}||\lambda_v|^k |s_v|^k \frac{1}{v^k} \\
&= O(1) \sum_{v=1}^{m} a_{vv}^{1-k} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n}\right)^{k-1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} a_{vv}^{1-k} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} a_{vv}^{1-k} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} v^{k-1} |\lambda_v|^k |s_v|^k \frac{1}{v^k} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} |\lambda_v|^{k-1} |\lambda_v||s_v|^k \frac{1}{v} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} \frac{1}{vX_v^{k-1}} |\lambda_v||s_v|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1. Finally, by virtue of the hypotheses of Theorem 3.1, Lemma 3.1, Lemma 3.3 and considering the fact that $v\beta_v = O(\frac{1}{X_v})$, we have that

$$\sum_{n=2}^{m+1}\varphi_n^{k-1} \mid V_{n,4} \mid^k = \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\frac{P_{v+1}}{(v+1)p_{v+1}}\Delta\lambda_v s_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}a_{vv}^{1-k}|\hat{a}_{n,v+1}|(\beta_v)^k|s_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}|\hat{a}_{n,v+1}|\right\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}a_{vv}^{1-k}|\hat{a}_{n,v+1}|(\beta_v)^k|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}a_{vv}^{1-k}(\beta_v)^k|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}a_{vv}^{1-k}(\beta_v)^k|s_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}a_{vv}^{1-k}(\beta_v)^k|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}(v\beta_v)^{k-1}\beta_v|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}v\beta_v\frac{|s_v|^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{\varphi_r p_r}{P_r}\right)^{k-1}\frac{|s_r|^k}{rX_r^{k-1}} + O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{|s_v|^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m$$

$$= O(1)\sum_{v=1}^{m-1}|(v+1)\Delta\beta_v - \beta_v|X_v + O(1)m\beta_m X_m$$

$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m$$

$$= O(1) \quad \text{as} \quad m\to\infty,$$

This completes the proof of Theorem 3.1.

4. Conclusions

1. If we take (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we have Theorem 2.1.
2. If we take (X_n) as a positive non-decreasing sequence, $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then we obtain a new result concerning the $|C,1|_k$ summability (see [13]).

3. If we take $\varphi_n = \frac{P_n}{p_n}$, then we have a new result concerning the $|A, p_n|_k$ summability.

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Şebnem YILDIZ,
Department of Mathematics, Kirşehir Ahi Evran University,Turkey, https://orcid.org/0000-0003-3763-0308
Email address: sebnemyildiz@ahievran.edu.tr

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COLD WEATHER TEAMS IN THE NATIONAL FOOTBALL LEAGUE AND HOME-FIELD ADVANTAGE

BRANDON JOLY* AND MEHMET DIK**
*BELOIT, WISCONSIN, USA.
**BELOIT, WISCONSIN, USA.

Abstract. The National Football League (NFL) has long had this idea of home-field advantage when teams play at home where they win more of their games. However, home games entail many things such as home fan attendance and weather. In considering the mean winning percentage of home games played during the winter months of December, January, and February by Cold Weather Teams, and the mean winning percentage of all home games played and all December, January, and February home games played, the null hypothesis would be that there is no difference in the means. By rejecting the null hypothesis, it would lead to the conclusion that cold weather plays a factor in determining NFL games. This hypothesis testing is important because if teams in "colder weather climates" gain a significant advantage during the colder months of December, January, and February, the impacts of playing in an open grass stadium during these months will have a greater impact on the playoffs and draft, which is decided based on final NFL standings. The finding is that Cold Weather Teams have a significant advantage when compared to the other data found, which suggests that cold weather tends to impact NFL games more than expected.

1. Introduction

Before the start of the 2021 league year and since the 2002 season, the NFL has been composed of 32 teams, each team playing 16 games, 8 of which are considered home games [1]. From 2002-2019, the NFL would play a total of 266 games that included a home team, which included playoff games, but excluded the Super Bowl which is played on a neutral field; however, in 2020, the NFL expanded the playoff field by one team in the American Football Conference (AFC) and National Football Conference (NFC) to create 268 games that included a home team [2]. However, the 2020 NFL season is full of oddities, including changes to the schedule of 11 teams due to the Covid-19 pandemic [3], resulting in the Denver Broncos without a starting quarterback (QB) play a game [4], and it is also the first time the host

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city of Super Bowl game was the home stadium of a team that participated, which brings the number of home games of the first team to 269 [5]. In completion, this research project is trying to reject the null hypothesis that there is not a significant advantage for the Cold Weather Teams, during the months of December, January, and February.

2. Method and Materials

The research project defines home games as games that are played at a team's home-field or was voluntarily played at a neutral site in agreement with the NFL that the game would count as a home game for that team in order to include games played in London, Canada, and other places. The definition used to determine "cold weather games" was if the game was played in December, January, or February of any given league season. In reference to "cold weather teams" this is for teams that play in an outdoor environment (not a dome) above the 39°N latitude line. These teams include the Chicago Bears (Memorial Stadium used in 2002 at 40.099365234°N [6], Soldier Field used from 2003-Present at 41.8625332°N) [7], Green Bay Packers (Lambeau Field used from 1957-Present at 44.501389°N)[8], Kansas City Chiefs (Arrowhead Stadium used from 1972-Present at 39.048889°N) [9], Buffalo Bills (Bills Stadium used from 1973-Present at 42.774°N [10] and "Toronto Series" from 2008-2011 at the Rogers Centre, Toronto Canada [11] at 43.641438°N)[12], Cleveland Browns (FirstEnergy Stadium used from 1999-Present at 41.506111°N)[13], Cincinnati Bengals (Paul Brown Stadium used from 2000-Present at 39.095°N)[14], Denver Broncos (Empower Field at Mile High used from 2001-Present at 39.743952°N)[15], New England Patriots (Gillette Stadium used from 2002-Present at 42.091°N)[16], Pittsburgh Steelers (Heinz Field used from 2001-Present at 40.446667°N)[17], Philadelphia Eagles (Veterans Stadium used from 1971-2002 at 39.906667°N[18], Lincoln Financial Field used from 2003-Present at 39.900833°N)[19], Seattle Seahawks (Lumen Field used from 2002-Present at 47.5952°N) [20], Baltimore Ravens (M&T Bank Stadium used from 1998-Present at 39.278056°N)[21], New York Jets and Giants (Giants Stadium used from 1984-2009 at 40.812222°N [22], MetLife Stadium used from 2010-Present at 40.813528°N)[23], and Minnesota Vikings (TCF Bank Stadium [64] used from 2014-2015 at 44.976°N)[24]. The research only takes into account home games from the 2002 season and onwards because that is when the Houston Texans became the 32nd team in the NFL [25].

3. Results

Between the years 2002-2020, there were 5057 games in this time period. The overall record of all teams was 5046-5046-22 which gives you an overall winning percentage of 50% or 0.50. This will allow this study to assume the odds of winning or losing a football game in the NFL is 50/50 probability and is random which is defined by "events that cannot be predicted with certainty, but the relative frequency with which they occur in a long series of trials is often remarkably stable," [26]. The table and graph below shows this phenomenon is associated with NFL teams and their winning percentage, home-field winning percentage, and cold weather winning percentage from 2002-2019 and 2002-2020. *Winning percentage was calculated by taking (wins + (ties*0.5))/(total number of games).*

Table 1: NFL Teams Winning Percentages Table.

NFL Winning Percentages [27]						
Team	2002-2019	2002-2020	Home-field 2002-2019	Home-field 2002-2020	Cold Weather 2002-2019	Cold Weather 2002-2020
Atlanta Falcons (Atl) [28]	0.528	0.514	0.587	0.570	0.587	0.562
Arizona Cardinals (Ari) [29]	0.448	0.450	0.568	0.564	0.640	0.623
Baltimore Ravens (Bal) [30]	0.580	0.585	0.720	0.715	0.673	0.691
Buffalo Bills (Buf) [31]	0.434	0.456	0.521	0.545	0.489	0.531
Carolina Panthers (Car) [32]	0.522	0.511	0.567	0.551	0.660	0.633
Cincinnati Bengals (Cin) [33]	0.463	0.453	0.537	0.529	0.553	0.540
Chicago Bears (Chi) [34]	0.481	0.481	0.547	0.538	0.500	0.491
Cleveland Browns (Cle) [35]	0.310	0.331	0.378	0.398	0.351	0.364
Dallas Cowboys (Dal) [36]	0.540	0.531	0.584	0.580	0.462	0.488
Denver Broncos (Den) [37]	0.563	0.550	0.647	0.627	0.66	0.635
Detroit Lions (Det) [38]	0.356	0.353	0.444	0.428	0.333	0.310
Houston Texans (Hou) [39]	0.453	0.443	0.540	0.525	0.542	0.510
Green Bay Packers (GB) [40]	0.606	0.615	0.697	0.703	0.745	0.750
Indianapolis Colts (Ind) [41]	0.620	0.621	0.697	0.699	0.759	0.768

Continuation of Table 6

Team	2002-2019	2002-2020	Home-field 2002-2019	Home-field 2002-2020	Cold Weather 2002-2019	Cold Weather 2002-2020
Jacksonville Jaguars (Jax) [42]	0.398	0.381	0.483	0.464	0.500	0.479
Kansas City Chiefs (KC) [43]	0.525	0.544	0.592	0.605	0.630	0.644
Las Vegas Raiders (LVR) [44]	0.360	0.367	0.432	0.422	0.422	0.396
Los Angeles Chargers (LAC) [45]	0.537	0.532	0.591	0.586	0.588	0.593
Los Angeles Rams (LAR) [46]	0.419	0.430	0.476	0.490	0.422	0.447
Miami Dolphins (Mia) [47]	0.438	0.448	0.503	0.510	0.551	0.558
Minnesota Vikings (Min) [48]	0.515	0.511	0.649	0.635	0.592	0.588
New England Patriots (NE) [49]	0.761	0.746	0.838	0.829	0.838	0.829
New York Giants (NYG) [50]	0.488	0.483	0.490	0.484	0.404	0.400
New York Jets (NYJ) [51]	0.452	0.435	0.531	0.510	0.581	0.578
Philadelphia Eagles (Phi) [52]	0.580	0.565	0.609	0.601	0.600	0.596
Pittsburgh Steelers (Pit) [53]	0.637	0.640	0.706	0.710	0.724	0.710
San Francisco 49ers (SF) [54]	0.460	0.456	0.557	0.535	0.630	0.592
Seattle Seahawks (Sea) [55]	0.578	0.585	0.703	0.707	0.702	0.689

Continuation of Table 6						
Team	2002-2019	2002-2020	Home-field 2002-2019	Home-field 2002-2020	Cold Weather 2002-2019	Cold Weather 2002-2020
Tampa Bay Buccaneers (TB) [56]	0.416	0.438	0.449	0.462	0.333	0.375
Tennessee Titans (Ten) [57]	0.485	0.494	0.534	0.535	0.571	0.558
Washington Football Team (WFT) [58]	0.401	0.402	0.445	0.439	0.311	0.292
New Orleans Saints (NO) [59]	0.574	0.583	0.612	0.617	0.509	0.517
End of Table-*all decimals were rounded to the nearest thousandth*						

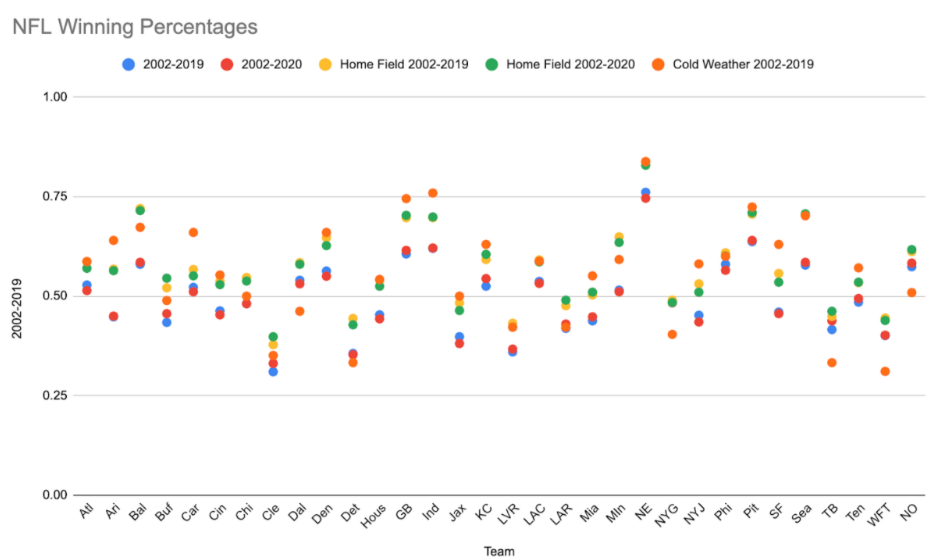

FIGURE 1. Graph: Scatterplot showing teams winning percentages.

This data was used and then compiled on a season by season basis to construct the following table and graph.

Table 2: Entire NFL Winning Percentage Season By Season Table.

Entire NFL Winning Percentage Season By Season		
Year	Home Games	Cold Weather Team Games at Home During December, January, and February
2002	0.5883458647	0.6829268293
2003	0.6127819549	0.7105263158
2004	0.5676691729	0.625
2005	0.5827067669	0.6363636364
2006	0.5413533835	0.575
2007	0.5714285714	0.6315789474
2008	0.5695488722	0.6129032258
2009	0.5751879699	0.5526315789
2010	0.5526315789	0.5909090909
2011	0.5751879699	0.619047619
2012	0.5733082707	0.5853658537
2013	0.5996240602	0.6097560976
2014	0.5770676692	0.6216216216
2015	0.5413533835	0.5652173913
2016	0.5864661654	0.6888888889
2017	0.5714285714	0.5609756098
2018	0.5977443609	0.6341463415
2019	0.5206766917	0.6341463415
2020	0.5	0.5909090909
End of Table		

The overall winning percentages for home games including the 2020 NFL season is 0.5686177576 and without the 2020 season is 0.5724728488. Similarly, the overall winning percentages for Cold Weather Teams including the 2020 season is 0.617076326 and without 0.6186556927. These numbers will vary from the mean due to a different number of games being played every season. For this set of data we found the mean ($\bar{x}$), population standard deviation (σ), and sample standard deviation (s) for home games, Cold Weather Team games at home during December, January, and February, and all team games at home during December, January, and February for with data that included and was without the 2020 NFL season.

Table of Standard Deviations and Arithmetic Means with the 2020 Season			
Mathematical Method	Home games	Cold Weather Team Games at Home During December, January	All Team Games at Home During December, January, and February
$\bar{x}$ arithmetic mean	0.5686584883	0.6172586568	0.566979456
s sample standard deviation	0.02750025926	0.04325904819	0.05386899073
σ population standard deviation	0.02676678683	0.04210526565	0.05243222539

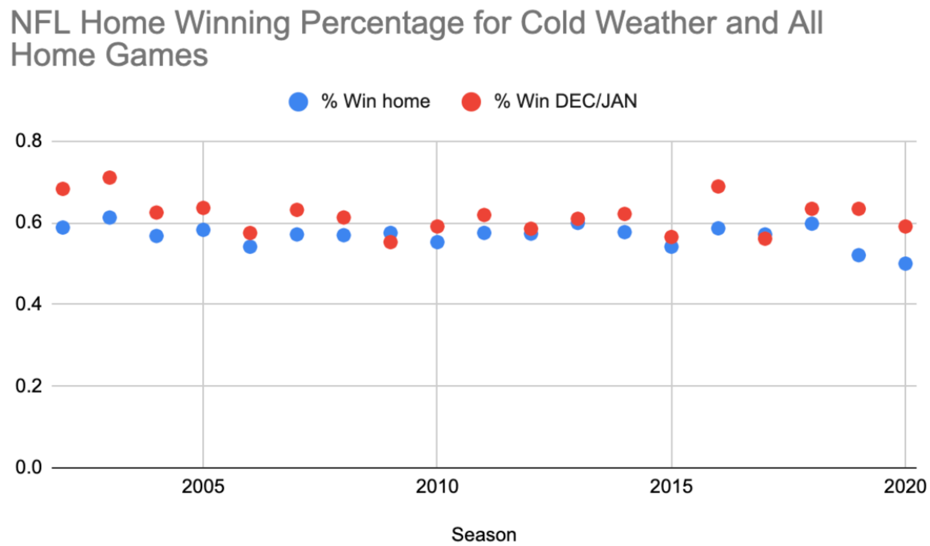

FIGURE 2. Graph 2: This is the NFL winning percentages for all home teams and cold weather teams including the 2020 season.

Table of Standard Deviations and Arithmetic Means without the 2020 Season			
Mathematical Method	Home games	Cold Weather Team Games at Home During December, January	All Team Games at Home During December, January, and February
$\bar{x}$ arithmetic mean	0.5724728488	0.6187225216	0.5715965942
s sample standard deviation	0.0225400042	0.04402629073	0.0514169134
σ population standard deviation	0.0219049467	0.04278586389	0.04996825811

The formula used for the arithmetic mean ($\bar{x}$) was $\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i$, where n= number of values and $x_i=$ data set values.

The formula used for sample standard deviation (s) was s=$\sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^2}{N-1}}$ where N=the number of observations, $x_i=$ data set values, and ($\bar{x}$) is the arithmetic mean.

The formula used for population standard deviation (σ) was $\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i-\mu)^2}{N-1}}$ N=the number of observations, $x_i=$ data set values, and (μ) is the population mean.

In this case, $\mu = \bar{x}$ because the study has a set number of data points. For the purpose of this study, the study will run 4 hypotheses tests because the 2020 NFL season was considered an outlier. An outlier is "data values that are very different from other measurements in the data set." [60] The study assumes based on Las Vegas and the gambling markets assumption in making their lines and odds that home-field advantage exists within the NFL and is shown through home teams having a winning percentage of 0.5686177576 and 0.5724728488 (when you don't include the 2020 NFL season) [61]. The tests run will be a right-tailed dependent samples paired differences test because the samples can be deemed to be dependent on the other since a home game of cold weather teams during December, January, and February will also be included in the sample of all team games at home during December, January, and February and all home games. Thus, this clearly passes the definition of dependent samples being "each data value in one sample can be paired with a corresponding data value in the other sample," [60]. Using the test outlines and tables provided by Brasse and Brasse (2009), the tests were done as follows.

3.1. **Test 1: Cold Weather Teams Games at Home During December, January, and February, and All Team Games at Home During December, January, and February (including the 2020 NFL Season).** .

$H_0 : \mu_d = 0$, $H_1 : \mu_d > 0$, $\alpha = 0.01$

The study begins by pairing the seasons together as a pair. For instance, 2002 Cold Weather Team Winning Percentage with All Team Home Games During December, January, and February Winning Percentage which leaves the following table.

Table 3: Paired Differences Table for Test 1.

Paired Differences Table			
Season	Cold Weather Team Games at Home During December, January, and February (B)	All Team Games at Home during December, January, and February (A)	D=B-A
2002	0.6829268293	0.6136363636	0.06929046563
2003	0.7105263158	0.6623376623	0.04818865345
2004	0.625	0.5888888889	0.03611111111
2005	0.6363636364	0.5444444444	0.09191919192
2006	0.575	0.4494382022	0.1255617978
2007	0.6315789474	0.606741573	0.02483737433
2008	0.6129032258	0.64	-0.02709677419
2009	0.5526315789	0.5444444444	0.008187134503
2010	0.5909090909	0.5222222222	0.06868686869
2011	0.619047619	0.5888888889	0.03015873016
2012	0.5853658537	0.5730337079	0.01233214579
2013	0.6097560976	0.5862068966	0.02354920101
2014	0.6216216216	0.5466666667	0.07495495495
2015	0.5652173913	0.5161290323	0.04908835905
2016	0.6888888889	0.6	0.08888888889
2017	0.5609756098	0.6153846154	-0.05440900563
2018	0.6341463415	0.5730337079	0.0611126336
2019	0.6341463415	0.5172413793	0.1169049622
2020	0.5909090909	0.4838709677	0.1070381232

From the table, the study finds that $\bar{d} = 0.05027920086$ and $s_d = 0.04720562542$. n=19.

$t = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} = \frac{0.05027920086}{\frac{0.04720562542}{\sqrt{19}}} = 4.64241$

d.f.=n-1=19-1=18

The P-value falls to the right of 3.922 from the "Critical Values for Student's t distribution." 3.922 < P-Value for the sample t.

Since the interval containing the P-Value of sample t lies to the right of $\alpha = 0.01$, we reject H_0.

At the 1% level of significance and even the 0.1% level of significance, we conclude that the Cold Weather Team Games at Home During December, January have an advantage over All Team Games at Home During December, January, and February which includes the 2020 season.

3.2. Test 2: Cold Weather Teams Games at Home During December, January, and February, and All Team Games at Home During December, January, and February (without the 2020 NFL Season). .

$H_0 : \mu_d = 0$, $H_1 : \mu_d > 0$, $\alpha = 0.01$

The study begins by pairing the seasons together as a pair. For instance, 2002 Cold Weather Team Winning Percentage with All Team Home Games During December, January, and February Winning Percentage which leaves the following table.

Table 4: Paired Differences Table for Test 2.

Paired Differences Table			
Season	Cold Weather Team Games at Home During December, January, and February (B)	All Team Games at Home during December, January, and February (A)	D=B-A
2002	0.6829268293	0.6136363636	0.06929046563
2003	0.7105263158	0.6623376623	0.04818865345
2004	0.625	0.5888888889	0.03611111111
2005	0.6363636364	0.5444444444	0.09191919192
2006	0.575	0.4494382022	0.1255617978
2007	0.6315789474	0.606741573	0.02483737433
2008	0.6129032258	0.64	-0.02709677419
2009	0.5526315789	0.5444444444	0.008187134503
2010	0.5909090909	0.5222222222	0.06868686869
2011	0.619047619	0.5888888889	0.03015873016
2012	0.5853658537	0.5730337079	0.01233214579
2013	0.6097560976	0.5862068966	0.02354920101
2014	0.6216216216	0.5466666667	0.07495495495
2015	0.5652173913	0.5161290323	0.04908835905
2016	0.6888888889	0.6	0.08888888889
2017	0.5609756098	0.6153846154	-0.05440900563
2018	0.6341463415	0.5730337079	0.0611126336
2019	0.6341463415	0.5172413793	0.1169049622

From the table, the study finds that $\bar{d} = 0.0471259274$ and $s_d = 0.04646955209$. n=18.

$t = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} = \frac{0.0471259274}{\frac{0.04646955209}{\sqrt{18}}} = 4.30345$

d.f.=n-1=18-1=17

The P-value falls to the right of 3.965 from the "Critical Values for Student's t distribution." 3.965< P-Value for the sample t.

Since the interval containing the P-Value of sample t lies to the right of $\alpha = 0.01$, we reject H_0.

At the 1% level of significance and even the 0.1% level of significance, we conclude that the Cold Weather Team Games at Home During December, January have an advantage over All Team Games at Home During December, January, and February which does not include the 2020 season.

3.3. **Test 3:Cold Weather Teams Games at Home During December, January, and February, and All Home Games (including the 2020 NFL Season).** .

$H_0 : \mu_d = 0$, $H_1 : \mu_d > 0$, $\alpha = 0.01$

The study begins by pairing the seasons together as a pair. For instance, 2002 Cold Weather Team Winning Percentage with All Team Home Games Winning Percentage which leaves the following table.

Table 5: Paired Differences Table for Test 3.

Paired Differences Table			
Season	Cold Weather Team Games at Home During December, January, and February (B)	All Team Games at Home (A)	D=B-A
2002	0.6829268293	0.5883458647	0.09458096461
2003	0.7105263158	0.6127819549	0.0977443609
2004	0.625	0.5676691729	0.05733082707
2005	0.6363636364	0.5827067669	0.05365686945
2006	0.575	0.5413533835	0.03364661654
2007	0.6315789474	0.5714285714	0.06015037594
2008	0.6129032258	0.5695488722	0.04335435363
2009	0.5526315789	0.5751879699	-0.02255639098
2010	0.5909090909	0.5526315789	0.03827751196
2011	0.619047619	0.5751879699	0.04385964912
2012	0.5853658537	0.5733082707	0.01205758298
2013	0.6097560976	0.5996240602	0.01013203741
2014	0.6216216216	0.5770676692	0.04455395245
2015	0.5652173913	0.5413533835	0.02386400785
2016	0.6888888889	0.5864661654	0.1024227235
2017	0.5609756098	0.5714285714	-0.01045296167
2018	0.6341463415	0.5977443609	0.03640198056
2019	0.6341463415	0.5206766917	0.1134696497
2020	0.5909090909	0.5	0.09090909091

From the table, the study finds that $\bar{d} = 0.04860016852$ and $s_d = 0.03813895447$. n=19.

$t = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} = \frac{0.04860016852}{\frac{0.03813895447}{\sqrt{19}}} = 5.55579$

d.f.=n-1=19-1=18

The P-value falls to the right of 3.922 from the "Critical Values for Student's t distribution." 3.922 $<$ P-Value for the sample t.

Since the interval containing the P-Value of sample t lies to the right of $\alpha = 0.01$, we reject H_0.

At the 1% level of significance and even the 0.1% level of significance, we conclude that the Cold Weather Team Games at Home During December, January have an advantage over All Home Games which includes the 2020 season.

3.4. **Test 4:Cold Weather Teams Games at Home During December, January, and February, and All Home Games (without the 2020 NFL Season).** .

$H_0 : \mu_d = 0$, $H_1 : \mu_d > 0$, $\alpha = 0.01$

The study begins by pairing the seasons together as a pair. For instance, 2002 Cold Weather Team Winning Percentage with All Team Home Games Winning Percentage which leaves the following table.

Table 6: Paired Differences Table for Test 4.

Paired Differences Table			
Season	Cold Weather Team Games at Home During December, January, and February (B)	All Team Games at Home (A)	D=B-A
2002	0.6829268293	0.5883458647	0.09458096461
2003	0.7105263158	0.6127819549	0.0977443609
2004	0.625	0.5676691729	0.05733082707
2005	0.6363636364	0.5827067669	0.05365686945
2006	0.575	0.5413533835	0.03364661654
2007	0.6315789474	0.5714285714	0.06015037594
2008	0.6129032258	0.5695488722	0.04335435363
2009	0.5526315789	0.5751879699	-0.02255639098
2010	0.5909090909	0.5526315789	0.03827751196
2011	0.619047619	0.5751879699	0.04385964912
2012	0.5853658537	0.5733082707	0.01205758298
2013	0.6097560976	0.5996240602	0.01013203741
2014	0.6216216216	0.5770676692	0.04455395245
2015	0.5652173913	0.5413533835	0.02386400785
2016	0.6888888889	0.5864661654	0.1024227235
2017	0.5609756098	0.5714285714	-0.01045296167
2018	0.6341463415	0.5977443609	0.03640198056
2019	0.6341463415	0.5206766917	0.1134696497

From the table, the study finds that $\bar{d} = 0.04624967283$ and $s_d = 0.03780207387$. n=18.

$t = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} = \frac{0.04624967283}{\frac{0.03780207387}{\sqrt{18}}} = 5.18993$

d.f.=n-1=18-1=17

The P-value falls to the right of 3.965 from the "Critical Values for Student's t distribution." 3.965 $<$ P-Value for the sample t.

Since the interval containing the P-Value of sample t lies to the right of $\alpha = 0.01$, we reject H_0.

At the 1% level of significance and even the 0.1% level of significance, we conclude that the Cold Weather Team Games at Home During December, January have an advantage over All Home Games which does not include the 2020 season.

4. Discussion

These results mean that although prior studies conclusions have shown that the impact of home-field advantage decreases year over year, it has been shown that NFL teams playing in an outdoor environment (not a dome) above the 39°N latitude line have a statistically significant advantage over teams whose home stadiums is below the the 39°N latitude line or play in a controlled playing environment (a dome) [62,63]. The results suggest that Cold Weather Teams have been more successful on average at home than non cold weather teams. This would imply these teams win more division titles, Wild Card Round games, Divisional Round games, Conference Championships, and participate in more Super Bowls. As for this data, of the 19 years studied, 15 of 19 of the years included Cold Weather Teams winning the Super Bowl or World Championship Game of the NFL (New England Patriots with 5 Super Bowls in 2003, 2004, 2014, 2016, and 2018 seasons, New York Giants with 2 Super Bowls in 2007 and 2011 seasons, Pittsburgh Steelers with 2 Super Bowls in 2005 and 2008, and 6 teams with 1 Super Bowl title), 10 of 19 Super Bowl Runner Ups, 28 of 38 American Football Conference (AFC) Championship Game teams, and 17 of 38 National Football Conference (NFC) Championship Game Teams. These results matter because this could change how betting services such as DraftKings and FanDuel set the lines for games in December, January, and February. These results could also be used so that the average fan could understand how much weather and more specifically colder weather and snow plays a factor in the outcome of a game. The data gathered is very reliable as it is winning percentages that are based off of NFL data. The potential mistake comes as there are a different amount of cold weather games every year which could cause varying differences in the means. Other reasons for limitations on results are that Colder Weather Teams have been very successful which could have skewed the results as playoff games are held at the stadium of the higher seeded team and these teams were successful during December, January, and February to reach these points. The limitation to only 19 years also affects the results and this data could be expanded to the entire Super Bowl era of the NFL beginning in 1967 for a larger sample size of results. More research could go into what types of weather have a larger impact on games or try to determine the full causes of weather on playing capacity of NFL players.

5. Conclusion

By the results of this article, the research project rejected the null hypothesis and left no doubt that Cold Weather Teams have a mathematically significant advantage over all other NFL teams during December, January, and February and in comparison to all other times of the year. However, the NFL is constantly changing to bigger and better stadiums with teams shifting to more indoor environments for playing games. For that reason, these results may be ever changing and with players such as Tom Brady who has played through the duration of this study with mainly the New England Patriots and contributed to 5 Super Bowls during this period, the ebb and flow of the NFL success was not able to be adequately seen. Yet, the NFL's Cold Weather Teams thrived in late season games and provided for lots of success which resulted in the conclusion of a statistical advantage for them. 2020 was the perfect year for weather to play the biggest factor as many teams had the second biggest d value because fans and their impact was in a sense controlled. Since 2020 had a big d value and all of the paired differences tests rejected the null hypothesis, this research project ultimately concluded that Cold Weather Teams during December, January, and February have a statistically significant advantage over other teams.

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Brandon Joly,
700 College Street Beloit College Box 693, Beloit, Wisconsin, USA 53511, Phone: (920)639-3849
Email address: jolybs@beloit.edu

Mehmet Dik,
700 College Street, Beloit, Wisconsin, USA 53511, Phone: (815)226-4135
Email address: mdik@rockford.edu

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NUMERICAL SIMULATION OF THE REDUCED FIELD INFLUENCE ON THE EVOLUTION OF NITROGEN OXIDES PRESENT IN THE MIXTURE $N_2/O_2/H_2O/CO_2$ BATHED IN AN OUT-OF- EQUILIBRIUM PLASMA

A. K. FEROUANI *, AND M. LEMERINI **
* ECOLE SUPÉRIEURE DES SCIENCES APPLIQUÉES, B.P. 165, 13000 TLEMCEN, ALGERIA.
** L.P.T, FACULTÉ DES SCIENCES, UNIVERSITÉ A BELKAID, 13000 TLEMCEN, ALGERIA.

Abstract. The industrial and technological development of the last century has led to increasing energy consumption, which has led to an increasing increase in the discharge of gaseous pollutants into the atmosphere. In these discharges, nitrogen oxides account for a large part of the environmental pollutants and are there fore directly or indirectly responsible for certain diseases when their concentration in the air is high. In this work, we propose to numerically simulate the evolution of the density of NO_x nitrogen oxides present in the gas mixture $N_2/O_2/H_2O/CO_2$ which is subjected to different values of the reduced electric field: 100 to 200 Td (1Td = 10^{-17} V.cm^2). We are particularly interested in the NO, NO_2 and NO_3 species that are the main components of nitrogen oxides. The simulation runs from 10^{-9} to 10^{-3} s. The model takes into account twenty species reacting with each other following two hundred chemical reactions. The results obtained clearly show the effectiveness of the reduced electric field in the destruction of nitrogen oxides.

1. Introduction

Generally, the reduced electric field (E/n ratio) is considered a critical parameter controlling the electron energy in plasma discharges. The chemistry of plasma can be different based on the value of E/n ratio (where E is the electric field and n is the gas density). For several years, the common catalytic and thermal methods used to eliminate or at least to reduce the level of NO_x that existed in industrial flue gas and/or generated by the vehicles will not permit us to respect the limits of gases emission [1]. There are several important parameters for the decomposition of NO: initial NO concentration, composition and temperature of the carrier gas and the value of the reduced electric field (E/n). In order to choose the optimal

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operating conditions for a non-thermal plasma process, it is of great importance to understand the chemical mechanisms responsible for the decomposition of pollutant molecules in a plasma. In the past few years, simulation of plasma processing using detailed chemistry has become an important tool for investigating this problem. The influence of different parameters (concentration, gas, temperature...) has been examined theoretically and experimentally [2].

Discharges and post-discharges in pure N_2 and O_2 or in their mixtures with inert gases are receiving increasing attention, due to their importance for understanding atmospheric and ionospheric physics and to their use as active laser media and as sources of active species (N and O atoms, metastable atoms and molecules, ions) for applications in plasma chemistry. Such applications include surface treatments, coating processes, metal nitriding, TiN deposition, oxidation and etching of polymers and semiconductors [3].

During the past decade, the removal of NO, (including NO and NO_2) and SO_2 has become a central scientific concern because of its key role in many global environmental problems such as acid rain or photochemical smog formation. There are also adverse effects on human health: In higher concentrations these chemical species may cause bronchitis or pneumonia [4]. In addition, SO_2 and NO contribute to the degradation of visibility since they form accumulation-mode aerosol particles containing sulfates and nitrates [5]. While the emission of SO_2 can be limited by using low-sulphur fuels, NOx, remains a serious hazard to the human health, is an ozone precursor, and is one of the most difficult air pollutants to suppress [6]. A major source of NO_x, emissions are exhaust gases of motor vehicle, especially from diesel engines. Diesel engines have much better fuel economy than cars with gasoline engines. They also have lower emissions of hydrocarbons and carbon-monoxide, but NO_x, and partliculate, emissions are much higher [7].

In this work, we simulate the time behavior of different species and their reaction rates model based on chemical kinetics equations. The analyze concerns twenty chemical species (molecules i.e. N_2, O_2, H_2O, CO_2, OH, HO_2, HNO_3 and O_3, atoms i.e. N, O and H, nitric oxides i.e. NO, NO_2, NO_3 and N_2O_5, and negative ions i.e. O_2^-, O_3^-, O_4^-, NO_2^- and NO_3^-), in the mixture (i.e. N_2: 70%, O_2: 20%, H_2O: 5% and CO_2: 5%). These different species react following 100 selected chemical reactions and the analyze concerns six values of the reduced electric field (100, 120, 140, 160, 180, 200 Td). In this numerical simulation we suppose various effects induced by the passage of a corona discharge [8] in a mixed gas. For the sake of simplification, we assume that the gas has no convective movement gradients and the pressure remains constant.

2. Basic formulae

The mathematical model used in the present work consists of a system of equations that takes into account the variation of the density and the chemical kinetics of the environment.

We developed a zero order numerical code to resolve the transport equations for neutral and charged particles. The algorithm is based on the time integration of the system of equations under consideration [9].

$$\frac{d\,n_i}{d\,t} = \sum_{j=1}^{j_{\max}} F_{ij},\ j \in [1, ..., j_{\max}] \tag{2.1}$$

where

$$F_{ij} = \varphi_{ij} - \chi_{ij} \tag{2.2}$$

n_i means the species densities vector, and F_{ij} mean the source term vector depending on the rate coefficient and corresponding to the contributions from different processes. φ_{ij} and χ_{ij} represent respectively the gain and loss of species i due to the chemical reactions. The solution of such a system requires the knowledge of the initial concentrations.

The total density n component is expressed in terms of pressure p equation can be written as (ideal gas law):

$$p = n\,k_\beta\,T \tag{2.3}$$

where T the absolute temperature (is in Kelvins) and k_β is Boltzmann constant (is in J/K). However, the reactivity of the gas were taken into account to the source term F_{ij} (density conservation) Eq (2.1).

$$\varphi_{ij} = \sum_{\gamma} K_\gamma(T)(n_i\,n_j)_\gamma \tag{2.4}$$

and

$$\chi_{ij} = \sum_{\eta} K_\eta(T)(n_i\,n_j)_\eta \tag{2.5}$$

$K_\gamma(T)$ and $K_\eta(T)$ are the coefficients of the chemical reaction number γ or η, $(n_i\,n_j)$ means the product of densities of species i and j.

These coefficients satisfy Arrhenius formula:

$$K_\gamma(T) = \kappa_1 \exp\left(\frac{\theta_\gamma}{T}\right) \tag{2.6}$$

and

$$K_\eta(T) = \kappa_2 \exp\left(\frac{\theta_\eta}{T}\right) \tag{2.7}$$

where κ_1 and κ_2 are the constant factors and θ_γ and θ_η are the activation energy of the chemical reaction.

In table 1, we present the chemical reactions considered in this paper.

TABLE 1. The main plasma reactions to generate the main radical to remove NO_x and their rate constants. $x[y]$ denotes $x \times 10^y$.

	Reaction	Rate constants	References
R_1	$NO + NO_3 \rightarrow NO_2 + NO_2$	K_1=2.00[-11]	[10]
R_2	$NO + O_3 \rightarrow O_2 + NO_2$	K_2=1.80[-12]	[10]
R_3	$NO + O_3^- \rightarrow NO_2^- + O_2$	K_3=2.00[-12]	[10]
R_4	$NO + O_3^- \rightarrow NO_3^- + O$	K_4=1.00[-10]	[10]
R_5	$NO + O_4^- \rightarrow NO_3^- + O_2$	K_5=2.50[-10]	[10]
R_6	$NO + HO_2 \rightarrow NO_2 + OH$	K_6=13.5[-11]	[10]
R_7	$NO_2 + O_2^- \rightarrow NO_2^- + O_2$	K_7=7.00[-10]	[9]
R_8	$NO_2 + OH \rightarrow HNO_3$	K_8=13.5[-11]	[9]
R_9	$NO_2 + O_3^- \rightarrow NO_2^- + O_3$	K_9=7.00[-10]	[9]
R_{10}	$NO_2 + N \rightarrow NO + NO$	K_{10}=2.30[-12]	[9]
R_{11}	$NO_3 + OH \rightarrow HO_2 + NO_2$	K_{11}=2.35[-11]	[9]
R_{12}	$NO_3 + HO_2 \rightarrow HNO_3 + O_2$	K_{12}=4.05[-12]	[9]
R_{13}	$NO_3 + NO_3 \rightarrow NO_2 + NO_2 + O_2$	K_{13}=1.20[-15]	[11]
R_{14}	$NO_3 + O \rightarrow NO_2 + O_2$	K_{14}=1.70[-11]	[11]
R_{15}	$N + O_2 \rightarrow O + NO$	K_{15}=8.90[-17]	[10]
R_{16}	$N + NO_2 \rightarrow N_2 + O_2$	K_{16}=7.00[-13]	[10]
R_{17}	$N + NO_3^- \rightarrow NO + NO_2 + e^-$	K_{17}=5.00[-10]	[10]
R_{18}	$NO_2 + NO_3 + O_2 \rightarrow N_2O_5 + O_2$	K_{18}=3.70[-30]	[11]
R_{19}	$O_3 + H \rightarrow OH + O_2$	K_{19}=2.80[-11]	[12]
R_{20}	$OH + H_2 \rightarrow H_2O + H$	K_{20}=6.70[-15]	[12]
R_{21}	$OH + O_3 \rightarrow HO_2 + O_2$	K_{21}=6.50[-14]	[12]
R_{22}	$OH + HO_2 \rightarrow H_2O + O_2$	K_{22}=1.10[-10]	[12]
R_{23}	$OH + HNO_3 \rightarrow NO_3 + H_2O$	K_{23}=1.30[-13]	[12]
R_{24}	$H_2O + e^- \rightarrow OH + H + e^-$	K_{24}=2.60[-12]	[12]
R_{25}	$CO_2 + e^- \rightarrow CO + O + e^-$	K_{25}=8.70[-10]	[12]

3. Numerical results and discussion

In figures 1, we have showing the time evolution of rate coefficient of these reactions at various values of reduced electric fields (100, 120, 140, 160, 180 and 200 Td). We notice that NO can react with the oxidizing radicals such as O_3, O_3^-, O_4^- and HO_2 to form especially NO_2^-, NO_3^- and OH. We note firstly that the effectiveness of these reactions is higher at the beginning than at the end. Secondly, plus the value of the reduced electric field is more important reaction is effective. For example, at 100 Td the rate coefficient does not vary significantly, but at 200 Td we have a significant reduction. We notice to all of these curves that the reactions become less effective after $t = 4 \times 10^{-4}$ s.

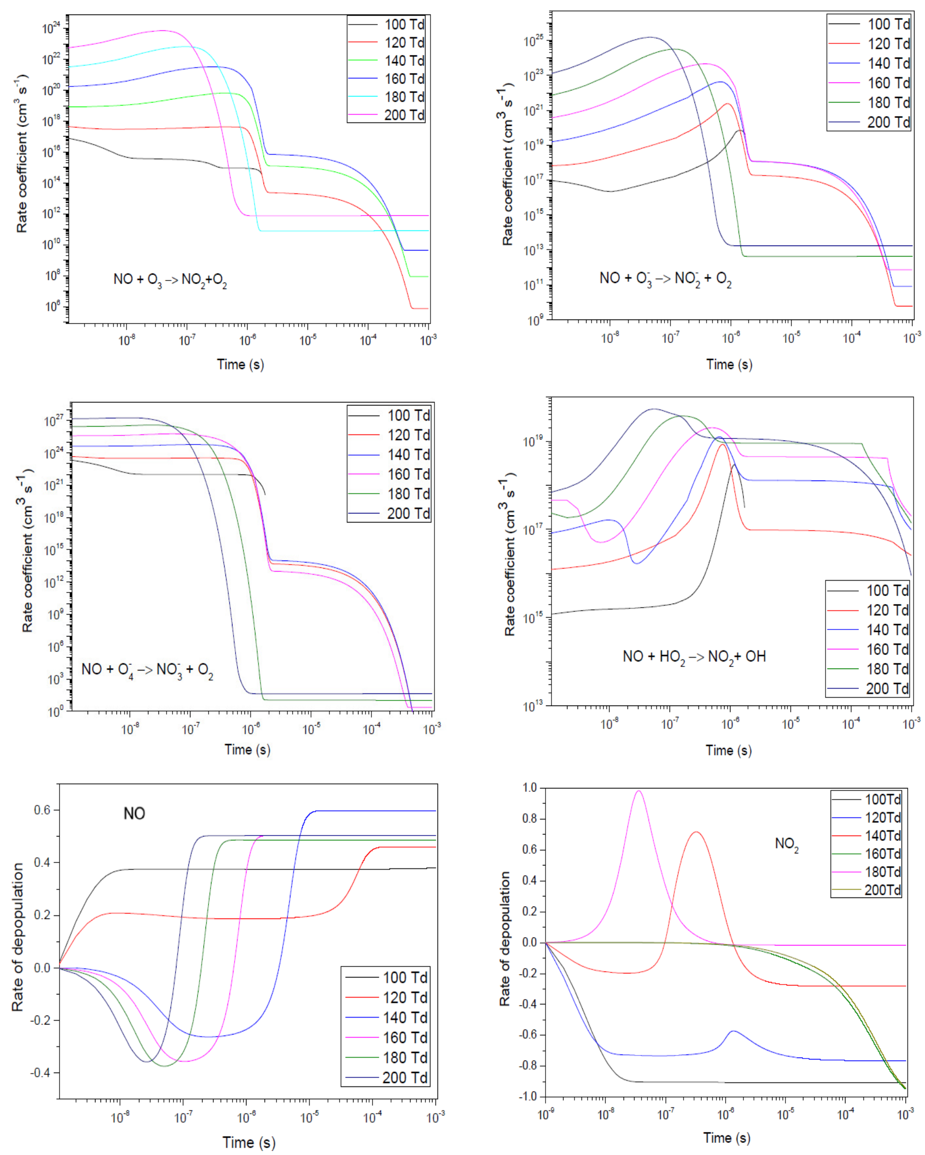

Fig 1: Time evolution of rate coefficient for four reaction selected as a function of the time. Also we presented at the end the time evolution of depopulation rate specie in mixture specie in mixture $N_2/O_2/H_2O/CO_2$ as a function of the time, shown for different reduced E in the range of 100−200 Td. At the end of this figure we presented: *left* is NO and to the *right* NO_2.

In end figure 1 and left, we plot the time evolution of depopulation rate $(n_0 - n)/n_0$, where n_0 means the initial density of NO specie in mixture $N_2/O_2/H_2O/CO_2$

for different values reduced E in the range of $100 - 200$ Td. We presented in this figure the results of the competition between all inherent reactions to oxide nitrogen reduction. We clearly observe the influence of the reduced E on NO reduction, we presented also in this figure the results of the competition between all inherent reactions to oxide nitrogen reduction. What we could notice for low values of the reduced E in the range of $100 - 130$ Td an average reduction of $\sim$ 15% which explains by the overall reaction, while for high value 180 Td, we also observed that the rate coefficient of reduction reached $\sim$ 70%. Finally, NO reduction largely depends on the radical concentration of O_3, O_3^-, O_4^- and HO_2. In the beginning, the NO consumption is not significant because the O_3^-, O_4^- and HO_2 radicals generated react mostly with NO_x and their concentration remains low. In the right shows the time evolution of depopulation rate of NO_2 for various values of reduced E (100−200 Td). Unlike the previous result for oxide nitrogen, we observe for NO_2 a different behavior. Firstly, we notice in the beginning from 10^{-9} s to 10^{-8} s, a significant reduction ($\sim$ 85 % an average) especially for 100 and 140 Td which stabilizes at this value until the end. Secondly at 160 and 200 Td there is a different behavior, for example when the reduced $E \simeq 160$ Td the depopulation rate decreases and reaches approximately $\sim$ 30 % at $t \simeq 5 \times 10^{-8}$ s. Then there is an increase that reaches $\sim$ 65 % at the moment $t \simeq 5 \times 10^{-8}$ s followed by a reduction $\sim$ 30 % till the end.

4. Conclusion

The objective of this work is to contribute to the understanding of the reaction mechanisms that compete in the creation and consumption of nitrogen oxides present in the $N_2/O_2/H_2O/CO_2$ gas mixture at atmospheric pressure. In particular, we were interested in studying:

The main pollutants that are present in most industrial and automotive gaseous effluents (nitrogen monoxide NO, nitrogen dioxide NO_2...).

The influence of the most important reactions that are effective in the temporal evolution of these species.

the results obtained enabled us to highlight the influence of chemical reactivity on the spatio-temporal evolution of these species. In particular, we have shown, on the one hand, the predominant role of certain chemical reactions in the conversion of nitrogen oxides, and on the other hand, the difference in the contribution of these reactions between the different species. Indeed, if we compare the participation of the different chemical reactions on the conversion of the three species we can summarize the results obtained as follows:

For nitrogen monoxide is the reaction:

$NO + O_4^- \rightarrow NO_3^- + O_2$.

containing the radical O_4^- which dominates all other reactions.

For nitrogen dioxide, these are the two reactions: $NO_2 + O_2^- \rightarrow NO_2^- + O_2$ and $NO_2 + O_3^- \rightarrow NO_2^- + O_3$.

containing O_2^- and O_3^- radicals that play a significant role in conversion. For nitrogen trioxide, this is mainly the reaction: $NO_3 + OH \rightarrow HO_2 + NO_2$.

containing the OH radical which gives a better conversion.

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A. K. Ferouani,
Ecole Supérieure en Sciences Appliquées, Tlemcen, 13000, Algeria.
Email address: `e mail: ferouani_karim@yahoo.fr`

M. Lemerini
L.P.T Laboratory, University A. Belkaid of Tlemcen, 13000, Algeria.
Email address: `e mail: mlemerini@yahoo.fr`

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DEVELOPMENT OF AN INFORMATION SYSTEM FOR STORING DIGITIZED WORKS OF THE ALMATY ACADEMGORODOK RESEARCH INSTITUTES

NURLAN TEMIRBEKOV*, DOSSAN BAIGEREYEV**, ALMAS TEMIRBEKOV***, AND BAKYTZHAN OMIRZHANOVA****
*KAZAKHSTAN ENGINEERING TECHNOLOGICAL UNIVERSITY, ALMATY, KAZAKHSTAN,
**S. AMANZHOLOV EAST KAZAKHSTAN STATE UNIVERSITY, UST-KAMENOGORSK, KAZAKHSTAN, +77053312023
***AL-FARABI NATIONAL UNIVERSITY, ALMATY, KAZAKHSTAN,
****KAZAKH RESEARCH INSTITUTE OF PROCESSING AND FOOD INDUSTRY, ALMATY, KAZAKHSTAN

Abstract. The present article describes the architecture of the integrated distributed information system created for storing digitized works of employees of Almaty Akademgorodok research institutes (Kazakhstan) and providing access to them using Web technology. Comparative analysis of two data storage systems for storing digitized works, Ceph and GlusterFS, is provided. The description of the software part of the information system is provided which consists of four subsystems: repository of digital objects, subsystem for managing current research information, subsystem of integration of distributed information resources, subsystem of access to distributed information resources based on Web technologies. The relation between the subsystems and their integration is described. The work defines the requirements to the repository of digital objects. The requirements for the repository of digital objects are defined; a comparative analysis of open source software used for these purposes is made.

1. Introduction

For decades, scientists of Almaty Academgorodok research institutes have been conducting enormous research in leading areas of the agro-industrial, processing, microbiological, seismological and other areas producing hundreds of thousands of articles, technical reports, and other documents. The latter also include digital research materials in the form of statistical, cartographic, multimedia data which are derived using radars, telescopes, and satellites. However, it should be recognized that the results of these studies remain inaccessible to the vast majority of

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researchers and employees of the agro-industrial complex in the age of the information explosion. One of the reasons for this problem is the lack of a publicly available single repository of information and knowledge base in the field of agriculture. In addition, important works created more than half a century ago and stored in the archives of libraries in paper form take an unpresentable form over the years.

For this reason, a team of scientists of Kazakhstan Engineering Technological University and Academset LLP has created an integrated distributed information system of Academgorodok, acagor.kz, the main objectives of which are: (1) providing reliable storage of the results of intellectual creative activity of employees of research institutes including digitized works, geographic materials (maps, satellite images, field observations), audio and video recordings; (2) management of current research information; (3) external and internal integration of information resources; (4) providing a single user interface for all functions and modules of a distributed information system, providing a "transparent" search and user access to documents; (5) optical character recognition of handwritten manuscripts created in past centuries based on neural networks.

Therefore, the objective of the work is not only to preserve the rich heritage of the research institutes, but also to provide access to them and the ability to quickly search for the necessary information. The creation of such a specialized information resource could not only act as a reservoir of valuable research results, but also unite employees of research institutes, agricultural sector workers and other users working in this field.

This article presents a description of the information system for the implementation of the above tasks. The structure of the article is as follows. Section II provides the description of the data storage system. Section III provides the description of the software part of the information system. Section IV outlines plans for further work on the modernization of the information system.

2. Data Storage

At the initial stage, a NAS type architecture was chosen when conducting tests on a relatively small amount of data. However, the exponential growth of stored information led to a revision of the data warehouse architecture.

To organize the storage of data, it was decided to use distributed file systems. The choice of a distributed file system is made on the basis of compliance with the following criteria: (1) high reliability of storage; (2) high availability of data; (3) fault tolerance; (4) decentralization; (5) scalability; (6) low unit cost of storage; (7) ease of deployment and operation.

Ceph [1] and GlusterFS [2] were considered as data storage systems. Both of the systems provide high performance and scalability. However, these systems are architecturally opposite.

GlusterFS works in user space using FUSE technology, therefore it does not require support from the operating system kernel and works on top of existing file systems. The strengths of GlusterFS include simpler deployment. Unlike Ceph, GlusterFS does not require a separate server to store metadata, it is stored along with the data in the extended file attributes. Due to the lack of binding to the centralized meta-data server, the file system provides almost unlimited scalability. However, GlusterFS has fewer options and is less flexible compared to Ceph.

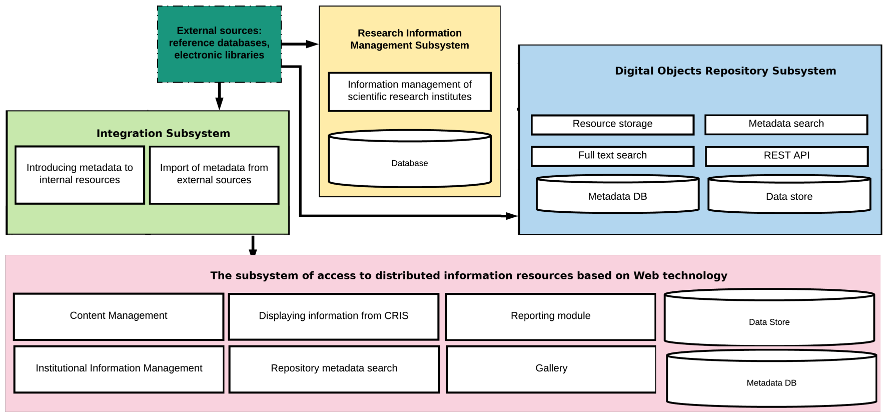

Figure 1. Relation between subsystems of the information system.

Unfortunately, our tests conducted on the basis of three servers showed a relatively slow work of the FUSE driver when working with small files, as well as poor responsiveness when rebuilding. Testing of the file system was carried out on the basis of servers with the following characteristics: Intel Xeon E5 2V Core, 8 GB DDR3 ECC, 500 GB HDD RAID 5.

The advantage of Ceph over GlusterFS is the lack of single points of failure and almost zero maintenance costs for recovery operations. In Ceph, the data array is automatically rebalanced when adding or removing new nodes. It occurs almost imperceptibly to clients and ensures a high survivability of the system.

According to the developers, the current version of CephFS is not stable for production use, but the test, deployed on the basis of three physical nodes and one virtual machine, showed no problems. Four servers were used in the test data warehouse: an administrative node (hub.acagor.kz) and three servers with data (node1.acagor.kz, node2.acagor.kz, node3.acagor.kz). Testing was performed within one month with the emergency shutdown of one of the servers. In this case, the rebalancing of the cluster occurs without a second downtime and is transparent to clients.

As a result of numerous studies, preference is given to Ceph object storage network. Ceph showed more acceptable results in terms of performance compared to GlusterFS when working with a digital repository of DSpace (see below) due to its assets storage features.

3. Architecture of the Information System

The information system comprises the following subsystems which are presented in Fig. 1: (1) Subsystem of repository of digital objects, (2) Subsystem for managing current research information; (3) Subsystem of integration of distributed information resources; (4) Subsystem of access to distributed information resources based on web technologies. Detailed information about the subsystems is given below.

3.1. **Subsystem of Repository of Digital Objects.** The first subsystem is intended for long-term storage of information described in the Introduction. The following system requirements were defined to the software underlying the subsystem: (1) the ability to store various types of resources including images, maps, audio and video recordings; (2) flexible storage organization: possibility of arbitrary grouping of resources by various criteria; (3) the ability to authenticate users and manage user roles through LDAP; (4) the ability to access external scientometric databases (i.e. Web of Science, Scopus etc.); (5) the ability to integrate with internal resources through APIs; (6) text recognition and full-text search; (7) open source software.

The capabilities of many software and technical solutions were analyzed including Ambra, Digital Commons, DSpace, ePrints, Evergreen ILS, Greenstone, Fedora Commons, Invenio, RODA, and VuFind. The experience of using each of these systems by scientists from various countries was also studied [3, 4, 5, 6, 7, 8]. Analyzing the advantages and disadvantages of the listed systems, Greenstone, ePrints, and DSpace were selected as the most satisfactory.

The strengths of Greenstone include the hierarchical structuring of each document, the automatic extraction of metadata from the document when it is uploaded. However, this system supports only a limited number of formats. Storage of geographical maps, as well as other results of scientific activities that have a more complex structure is not provided. ePrints supports more metadata formats, but does not support the extended Dublin kernel. The system supports various user roles. The strengths of DSpace include a more sophisticated system of user rights compared to the systems considered: various research institutes can have their own areas within the system. In each institute, certain employees that are responsible for pre-moderation may be appointed. DSpace, like the other systems considered, provides interfaces for integration with other subsystems based on open international standards. DSpace supports more than 70 formats of information resources.

As a result of the analysis, DSpace was chosen as a underlying subsystem. However it did not fully satisfy some of the requirements. Changes were made to its configuration during its compilation in order to adapt to the conditions established in the Republic of Kazakhstan. The standard DSpace metadata scheme based on the DCMI scheme is expanded by the following fields: "Journal in the list of the Committee for the Control of Education and Science of the Republic of Kazakhstan", "Full bibliographic reference in accordance with state standard" and others. In addition, reporting subsystem did not cover the requirements.

To store the repository data, the PostgreSQL database management system is used. In developing the information system architecture, the possibility of using its cluster version, Posgtres-XL, was considered. However, in later versions of the DSpace digital repository, metadata and content are stored in archival information packets, AIPs, and the database is used as a data cache. After analyzing the amount of information that DSpace stores in the database, it was concluded that the use of cluster DBMS is impractical under current conditions.

3.2. **Current Research Information Management (CRIS) Subsystem.** A literature review showed that there are few CRIS systems that can be integrated to DSpace. There is a powerful system created for this purpose at the Institute of Computational Technologies of the Siberian Branch of Russian Academy of Science (SB RAS) [9].

As a result of the study, an extension of the DSpace system, DSpace-CRIS was chosen as a research management system. This system allows to store information on research organizations, information about the employee of research organizations, various spellings of researcher's name, including different languages, links to profiles in various databases (Scopus, Researcher ID, ORCID), information on scientific activities (participation in funded projects, conferences, internships, etc.). The system is integrated with a DSpace instance, which allows to view the publication of scientists. The CRIS-system allows to export information about the publications of the scientist in popular formats.

3.3. **The Subsystem of Access to Distributed Information Resources Based on Web Technologies.** This subsystem is designed to provide a single user interface for all functions and modules included in the distributed information system. To date, it successfully solves the following tasks: (1) providing detailed information about the activities of research institutes and their employees; (2) flexible search both in the repository of digital objects and the external databases; (3) the opportunity to discuss topical issues on the forum, providing users the opportunity to share their work with colleagues; (4) organization of conferences including submissions of participants' papers (and storing in the repository), sending the papers to reviewers, booking a hotel, etc. The subsystem was developed using the Django web framework and runs on the Gunicorn WSGI HTTP server with the nginx HTTP server installed as a reverse proxy server.

3.4. **Integration Subsystem.** As integrating software, the ZooSPACE distributed information system was chosen [9]. The literature review did not reveal more suitable software for this purpose. The ZooSPACE distributed information system integrates data from various information sources, providing access to heterogeneous distributed information in accordance with standard protocols (SRW/SRU, Z39.50). The system operates on the basis of original ZooPARK-ZS servers, LDAP servers and Apache WEB servers, providing end-to-end information retrieval in heterogeneous databases, extracting information in standard schemes and formats and displaying it. To implement search in the digital object repository, a web portal was integrated with DSpace using the DSpace REST API, which provides a programming interface to communities, collections, item metadata and files. As a result of the integration metadata and links to materials uploaded to the repository subsystem are displayed on the scientist's profile page and on the information page of research institutes. Search by metadata is available. Additionally, filtering by keywords, institutes, date and language of publication was implemented.

4. Conclusion

The developed information system fully provides the necessary computational resources for research and educational processes, simplifying the prospect of its further development, and allows to build an advanced IT infrastructure for managing intellectual capital, an electronic library, which will store all the books and scientific works of Kazakhstan Engineering Technological University and research institutes of the Almaty Academgorodok.

Currently, the authors of the papers are working on full-text search in the repository of digital objects. Analysis of existing optical character recognition software (including proprietary) for digitization of texts revealed significant difficulties. This is partly due to the quality of the source materials, as well as the difficulty of recognizing characters of some Asian alphabets. In addition, work is underway on the use of neural networks for the recognition of manuscripts and old printed texts. The results of the research in this area will be presented in future papers.

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Nurlan Temirbekov,
93A, al-Farabi ave., 050060 Almaty, Kazakhstan, Phone: +77772794876
Email address: temirbekov@rambler.ru

Dossan Baigereyev,
34, 30th Guardian Division str., 070000 Ust-Kamenogorsk, Kazakhstan, Phone: +77053312023
Email address: dbaigereyev@gmail.com

Almas Temirbekov,
71, al-Farabi ave., 050040 Almaty, Kazakhstan, Phone: +77052500653
Email address: almas_tem@mail.ru

Bakytzhan Omirzhanova,
238G, Gagarin str., 050060 Almaty, Kazakhstan, Phone: +77770645051
Email address: omirzhanova61@mail.ru

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DERIVATION OF BLACK-SCHOLES EQUATION USING ITÔ'S LEMMA

BRANDON WASHBURN* AND MEHMET DIK**
*BELOIT COLLEGE, BELOIT, WI, UNITED STATES. ORCID NUBMER: 0000-0002-5815-3873
**ROCKFORD UNIVERSITY, ROCKFORD, IL UNITED STATES.

Abstract. The Black-Scholes Equation is arguably the most influential financial equation, as it is an effective example of how to eliminate risk from a financial portfolio by using a hedged position. Hedged positions are used by many firms, mutual funds and finance companies to increase the value of financial assets over time. The derivation of the Black-Scholes equation is often considered difficult to understand and overly complicated, when in reality most confusion arises from misunderstandings in notation or lack of intuition around the mathematical processes involved. This paper aims to take a simple look at the derivation of the Black-Scholes equation as well as the reasoning behind it.

1. Introduction

1.1. **A Brief History of Black-Scholes Equation.** The Black-Scholes Equation is a partial differential equation that gives the theoretical price of a European-style option for a certain security. First derived in 1968 by Fischer Black and Myron Scholes, it was revolutionary in presenting ideas of how to eliminate unpredictable terms from mathematical models.

Their results were published in the Journal of Political Economy in 1973 as an article title "The Pricing of Options and Corporate Liabilities". Fischer Black unfortunately passed away in 1995. Myron Scholes is still alive and would go on to win the Nobel Prize in economics in 1997 along with contributor Robert C. Merton [6] for their work, however Black would be recognized for his contributions by the Nobel Committee. [2]

1.2. **A Brief History of Itô/Stochastic Calculus.** Keyosi Itô was a Japanese mathematician who pioneered many of the mathematical concepts behind the

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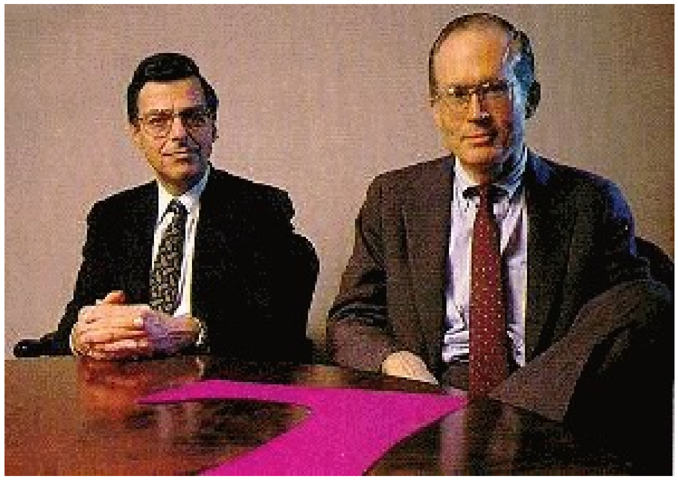

FIGURE 1. Myron Scholes and Fischer Black[8]

FIGURE 2. Kiyosi Itô at Kyoto University in 1995[1]

Black-Scholes Equation, the most important of which is Itô's Lemma. His insightful techniques allowed for the creation of his own field of calculus, Itô Calculus. In this field of mathematics, he explored stochastic processes, processes that are based in randomness and could not be quantified by normal arithmetic means. As stated by the National Academy of Science, "If one disqualifies the Pythagorean Theorem from contention, it is hard to think of a mathematical result which is better known and more widely applied in the world today than "Ito's Lemma." This result holds the same position in stochastic analysis that Newton's fundamental theorem holds in classical analysis. That is, it is the sine qua non of the subject." His work has implications across other fields such as stochastic control theory in engineering, conformal form theory in physics, modeling of population genetics in biology and more. [1] Kiyosi Itô unfortunately passed away in 2008 at the age of 93.

2. Intuition of Stochastic Processes

In this section, we aim to cover the basic intuition behind a stochastic process, enough to understand an Itô Process and Itô's Lemma. We are only going to cover Standard Brownian Motion to understand the theoretical processes behind stochastic calculus. Standard Brownian motion is sometimes referred to as a Wiener Process.

2.1. **Basic Definitions of Standard Brownian Motion.** There are a few basic definitions that we should cover to begin:

(1) We define Brownian Motion at time t as B_t
(2) Standard Brownian Motion begins at the origin. Written in our notation, $B_0 = 0$
(3) The difference of Brownian Motion from time t_1 to time t_2 can be written as $B_2 - B_1$ where B_1 is the Brownian Motion at time t_1 and B_2 is the Brownian Motion at time t_2
(4) $B_2 - B_1$ is normally distributed with a $\mu = 0$ and a $\sigma = t_2 - t_1$
(5) The derivative of B_t can be written as dB_t and the integral (without limits of integration) of dB_t can be written as B_t

2.2. **Intuition of Brownian Motion.** In Figure 3 we can see an example of $f'(x)$, a real valued continuous function. There are many things we can do with this function. We are able to integrate $f'(x)$ from point a to point b, we can take the derivative of $f'(x)$ at any point and we can show $f'(x)$ is continuous at any point.

Now let's take a look at Brownian Motion. Using computer software, it is possible to simulate 2-dimensional Brownian Motion from one time period to another. In Figure 4, we examine a single simulated Brownian Motion walk with standard parameters from time 0 to time 1. This motion does not follow typical or predictable paths compared to the path of a function. Here, our sample points fluctuate in an unpredictable and random way, based on the normal distribution. Furthermore, this motion will be different every time we simulate it. In theory, it should be impossible to replicate the same motion more than once[7]. This leads us into several problems. We are no longer able to quantify the motion using a function, we cannot prove that the motion is continuous from point a to point b, we cannot integrate via Newtonian means with respect to time from point a to point b and we cannot take the derivative with respect to time at any point.

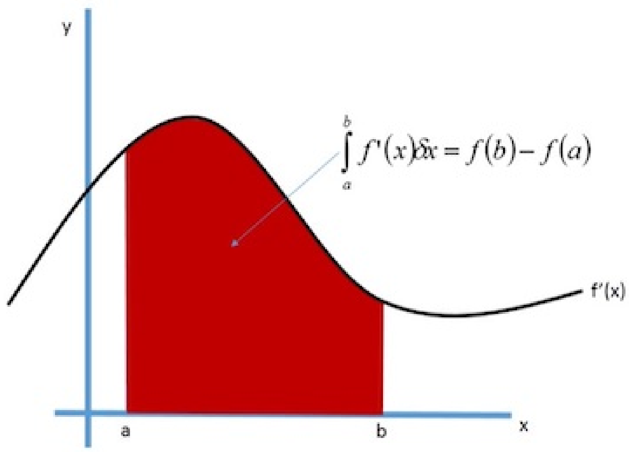

FIGURE 3. A real valued function $f'(x)$ with a smooth curve[3]

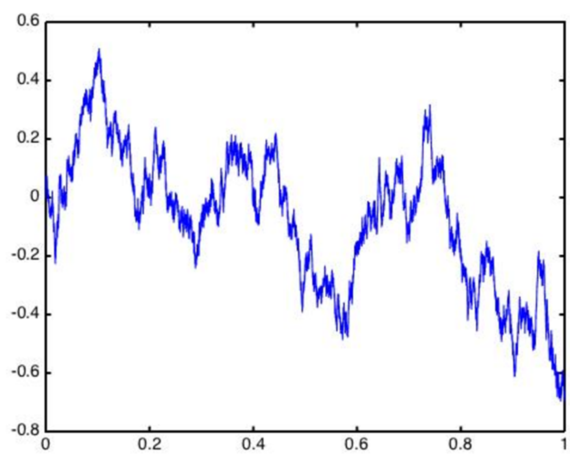

FIGURE 4. Graph of Standard Brownian Motion from $t = 0$ to $t = 1$[5]

An interesting thing to note is that while single simulations of Brownian Motion do not yield any sort of pattern, multiple simulations do. As the number of simulations increases we start to see the emergence of the normal distribution curve, all be it horizontally.

2.3. **Derivatives with Respect to Brownian Motion.** Intuitively B_t is not differentiable, at least not in a normal sense. But if we examine B_t in a similar sense to Newtonian calculus, we can arrive at some interesting results.

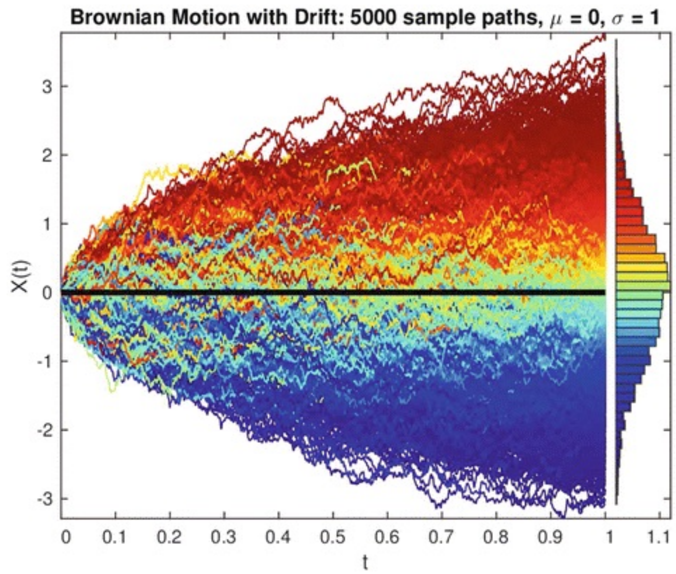

FIGURE 5. 5000 sample paths of Standard Brownian Motion[7]

In Newtonian calculus, we can take the derivative of a function

$$g(t) = g(0)e^{rt}$$

very easily. The derivative is

$$\frac{dg}{dt} = g(0)e^{rt} * r$$

Now if we add a Brownian Motion with a scalar beta term to $g(t)$ to create a new equation $s(t)$

$$s(t) = s(0)e^{rt+\beta B_t}$$

Upon attempting to differentiate $s(t)$ we find

$$\frac{ds}{dt} = [r + \beta\frac{dB_t}{dt}]s(0)e^{rt+\beta B_t} = [r + \beta\frac{dB_t}{dt}]s(t) = rs(t) + \beta s(t)\frac{dB_t}{dt}$$

However, at this time, we have no way of quantifying

$$\frac{dB_t}{dt}$$

We can further attempt at a separation of variables

$$\frac{ds}{dt} = rs(t) + \beta s(t)\frac{dB_t}{dt}$$

Becomes

$$ds = rs(t)dt + \beta s(t)\frac{dB_t}{dt}dt$$

The $dt's$ cancel in the second term to give us

$$ds = rs(t)dt + \beta s(t)dB_t$$

Once again we have a term we are unfamiliar with

$$dB_t$$

So as of now, it does not appear we can take the derivative of a process involving a Brownian Motion term. We will return to this discussion later.

2.4. **Integration with Respect to Brownian Motion.** Since we cannot take the derivative of terms with Brownian Motion, or a stochastic process, let's examine how integration with respect to Brownian Motion theoretically should behave.

2.4.1. *Integration of a Constant Function.* Let $x(t) = k$ a constant. If we wish to take the integral from 0 to some time t_1 with respect to Standard Brownian Motion we can write this as

$$\int_0^{t_1} x(t) dB_t$$

Evaluating this integral via Newtonian means,

$$\int_0^{t_1} x(t) dB_t = \int_0^{t_1} k dB_t = k \int_0^{t_1} dB_t = k[B_t \Big|_0^{t_1}] = k[B_{t_1} - B_0] = kB_{t_1}$$

Here we see that integrating a constant with respect to Brownian Motion gives us the constant multiplied by the Brownian Motion at the upper limit of the integral.

2.4.2. *Integration of a Step Function.* Let $\Phi(t)$ be a step function with n steps. Let k_i be the value of the step function at $i = 1, 2, 3, ..., n$ Taking the integral of a step function is effectively taking the the sum of the integral of each constant with respect to Brownian Motion. So we can write

$$\int_a^b \phi(t) dB_t$$

as

$$\sum_{i=1}^{n} \int_a^b k_i dB_t$$

Evaluating this as a sum of all of the integrals of each constant k_i we can see

$$\sum_{i=1}^{n} \int_a^b k_i dB_t = \sum_{i=1}^{n} k_i [B_b - B_a]$$

and from our definitions we know that

$$[B_b - B_a] \sim \mathcal{N}(0, (b-a))$$

Leading us to the result that the integral of a function with a term involving Brownian Motion will have a distribution.

2.4.3. *Integration of a Well-Behaved Function.* Normally in Newtonian calculus, integration begins with a Riemann sum. If we have a continuous real valued function $f(t)$ and we wish to find the area underneath the curve from a to b, the Riemann sum can be written as

$$Area = \sum_{a}^{b} f(t) * \Delta t$$

If we send the length of the partitions of this sum, Δt, to zero

$$\lim_{\Delta t \to 0} \sum_{a}^{b} f(t) * \Delta t = \int_{a}^{b} f(t)dt$$

The result is our standard Newtonian integration. We can apply this concept to Brownian Motion in a similar way. With a step function, our integral could be defined as

$$\int_{a}^{b} \phi(t) dB_t = \sum_{i=1}^{n} k_i [B_b - B_a]$$

Where

$$[B_b - B_a] \sim \mathcal{N}(0, K^2(b-a))$$

Effectively if we wanted to evaluate an integral of $f(t)$ with respect to Brownian Motion, we can again partition our function $f(t)$ into sections with widths of Δt. A well-behaved function evaluated via Riemann sum can be compared to a step function being evaluated by our process in 2.4.2. By the same process as before, if we wish to evaluate

$$\int_{a}^{b} f(t) dB_t$$

We can approximate it to a step function, and partition $f(t)$ into n partitions, each with length Δt. As we take smaller and smaller partitions, we will get a closer and closer approximation of the function. In theory then,

$$\lim_{\Delta t \to 0} \int_{a}^{b} f(t) dB_t \approx \int_{a}^{b} f(t)dt \sim \mathcal{N}(0, \int_{a}^{b} [f(t)]^2 dt)$$

However, because it is impossible to fully quantify a function to Brownian Motion, this will only give us an approximation of the integral.

2.4.4. *Integration of a Random Variable.* Let Y_t be a random variable defined by:

$$Y_t = \begin{cases} y_0 & 0 < t < t_1 \\ y_1 & t_1 < t < 1 \end{cases}$$

Integrating this random variable with respect to Brownian Motion, if $t < t_1$ then

$$\int_{0}^{t} Y_t dB_t = y_0 B_t$$

If $t > t_1$ then our integral is

$$\int_0^t Y_t dB_t = y_0 B_{t_1} + y_1(B_t + B_{t_1})$$

Combining these results, out entire integral can be written over our interval (0,1) as

$$\int_0^1 Y_t dB_t = \begin{cases} y_0 B_t & 0 < t < t_1 \\ y_0 B_t + y_1(B_1 + B_t) & t_1 < t < 1 \end{cases}$$

And we are able to quantify our integral with respect to Brownian Motion of a random variable.

3. Itô Calculus[1]

The ideas and processes of all of these examples highlight a reoccurring theme. That theme is that each process much be broken into a section we can evaluate via Newtonian means and one that must be evaluated by stochastic means. This comes from applying a discrete scenario where we can effectively separate the randomness of our Brownian Motion terms with the certainty of our t terms. This idea of separation is the methodology and thought process behind the Itô Process.

3.1. **Itô Process.** An Itô Process is an advanced technique that separates a stochastic process x_t into a a sum of two integrals, one with respect to time and one with respect to Brownian Motion.

$$x_t = x_0 + \int_0^t \sigma_t dB_t + \int_0^t \mu_t dt$$

Where x_0 is a constant term, σ is a process contingent on time that can be integrated with respect to Brownian Motion, and μ is a process contingent on time that can be integrated with respect to time.

3.2. **Itô Differential Equation.** Taking the "derivative" of the Itô Process gives us the following differential equation:

$$dx_t = \mu_t dt + \sigma_t dB_t$$

However, this is only an equation, it cannot be evaluated since we cannot take the derivative with respect to Brownian Motion by normal means. This will serve as our derivative of a stochastic process for the sake of making the calculations we intend to do.

3.3. **Itô's Lemma Pre-Remarks.** Itô's Lemma is an equation that can be compared to the "chain rule" of Newtonian calculus. The formal proof of Itô's Lemma is beyond the the scope of this paper, however, I have included an informal proof using a Taylor power series approximation that provides the contextual intuition behind the formal proof.

3.3.1. *Itô's Lemma.* If $f(t, x_t)$ is a twice differentiable scalar function, where x_t is defined as an Itô Process. Then

$$df = (\mu_t \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2)dt + \frac{\partial f}{\partial x}\sigma_t dB_t$$

Itô's Lemma is essential in the derivation of the Black-Scholes Equation.

3.3.2. *Taylor Series Approximation of Itô's Lemma.* Let x_t be an Itô Process that satisfies our requirements for an Itô Differential Equation.

$$dx_t = \mu_t dt + \sigma_t dB_t$$

Normally, if $f(t,x)$ is a two-variable and twice differentiable scalar function, we can write its Taylor expansion as

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2$$

If we equate x_t for x and $\mu_t dt + \sigma_t dB_t$ for dx, we can rewrite df as

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t^2 dt + 2\mu_t\sigma_t dt dB_t + \sigma_t^2 dB_t^2)$$

which means

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\mu_t dt + \frac{\partial f}{\partial x}\sigma_t dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mu_t^2 dt + \frac{\partial^2 f}{\partial x^2}\mu_t\sigma_t dt dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dB_t^2$$

Taking the limit of df as $dt \to 0$, dt^2 and $dtdB_t$ will tend to zero faster than dt and dB_t^2, leading us to the result

$$\lim_{dt\to 0} df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\mu_t dt + \frac{\partial f}{\partial x}\sigma_t dB_t + 0 + 0 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dB_t^2$$

If we replace dB_t^2 with dt (from the quadratic variance of a Wiener process)[9]

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\mu_t dt + \frac{\partial f}{\partial x}\sigma_t dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2 dt$$

Finally factoring out our dt and dB_t terms, and separating them

$$df = (\mu_t\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma_t^2)dt + \frac{\partial f}{\partial x}\sigma_t dB_t$$

We arrive at Itô's Lemma.

4. Black-Scholes Model

The Black-Scholes Model is sometimes referred to the Black-Scholes-Merton Model because of the contributions made by American mathematician Robert C. Merton. He had built upon the ideas presented originally by Myron Scholes and Fischer Black in 1968, adding further mathematical implications and applications to the model. This specific derivation of the equation comes from Hull[4].

4.1. **Assumptions of the Model.** The following assumptions are made in the Black-Scholes Model:

(1) The short term interest rate, r, is known and constant through time. All securities share this short term interest rate
(2) The stock price follows Brownian Motion and the variance rate of return on the stock is constant.
(3) The parameters μ and σ are contingent on S, the stock price
(4) The stock does not pay dividends or other distributions
(5) Short selling is allowed
(6) There are no arbitrage opportunities and all security trading is continuous

(7) The option has a maturity at time period t (European Option)

4.2. **Black-Scholes Equation.** If S is a stock price that follows an Itô Process, then the value of an option, f, of S is quantified by the following equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2 = rf$$

4.3. **Derivation.** Let S be a stock price that follows an Itô Process. Thus S has a Itô Differential Equation of

$$dS = \mu S dt + \sigma S dz \tag{4.1}$$

Suppose f is a twice differentiable function of the price of a call option or other derivative contingent on S. Using Itô's Lemma we can write df as

$$df = (\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dz \tag{4.2}$$

Notice how μ and σ are no longer contingent on t, but are contingent on S. (1) and (2) can also be written discretely over a time period Δt as

$$\Delta S = \mu S\Delta + \sigma S\Delta z \tag{4.3}$$

and

$$\Delta f = (\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z \tag{4.4}$$

respectively.

(1) Now S and f both follow the same Itô Process. Thus, if we select a portfolio, we can eliminate the Itô Process and effectively price an option. The portfolio we select will consist of short 1 derivative and long $\frac{\delta f}{\delta S}$ shares of stock. It will become clear shortly why we select this portfolio.
(2) We define Π as the value of our portfolio. By definition,

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{4.5}$$

We can also write our the discrete version of this equation over the time interval Δt as

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{4.6}$$

Now we will substitute our (3) and (4) into (6).

$$\Delta\Pi = -[(\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z] + \frac{\partial f}{\partial S}[\mu S\Delta + \sigma S\Delta z]$$

Distributing everything out,

$$\Delta\Pi = -\frac{\partial f}{\partial x}\mu S\Delta t - \frac{\partial f}{\partial t}\Delta t - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Delta t - \frac{\partial f}{\partial S}\sigma S\Delta z + \frac{\partial f}{\partial S}\mu S\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z$$

Canceling out like terms,

$$\Delta\Pi = -\frac{\partial f}{\partial t}\Delta t - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Delta t$$

Factoring out a Δt,

$$\Delta\Pi = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t \tag{4.7}$$

Without a Δz term (stochastic variable), this portfolio is effectively riskless during the time period Δt. Since there are no arbitrage opportunities, security trading is continuous and all securities share the same short term constant interest rate, our portfolio we have created will earn instantaneous rates of return over the time period Δt. Thus we can write $\Delta\Pi$ as

$$\Delta\Pi = r\Pi\Delta t \tag{4.8}$$

Now we can substitute our (5) and (7) into (8) and the result is the following equation

$$(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t = r(-f + \frac{\partial f}{\partial S}S)\Delta t$$

Canceling out a Δt on both sides of the equation and multiplying through by -1,

$$\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = r(f - \frac{\partial f}{\partial S}S)$$

which when rearranged becomes

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2 = rf \tag{4.9}$$

and our derivation is complete.

4.4. **Verification of the Model.** Now that we have our equation, does it actually work? Let's see through an example. This specific example comes from Hull, John C. (2008) of page 332.

4.4.1. *Example.* A forward contract (option) on a non dividend paying stock is a derivative dependent on the stock. We can define the value of the forward contract, f, at a general time t in terms of the stock price S as

$$f = S - Ke^{-r(T-t)} \tag{4.10}$$

Where K is our delivery price of the option, and the interest on the stock compounds continuously. Now recall our equation (9)

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2 = rf$$

We need to find our $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial S}$, and $\frac{\partial^2 f}{\partial S^2}$ in order to verify this equation. Taking the partial derivative of (10) with respect to t we find

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}$$

When we take the partial derivative of (10) with respect to S we find

$$\frac{\partial f}{\partial S} = 1$$

and taking the second partial of this we find

$$\frac{\partial^2 f}{\partial S^2} = 0$$

When we substitute these results into the left side of (9) the equation becomes

$$-rKe^{-r(T-t)} + rS$$

Factoring out an r our result is

$$r(S - Ke^{-r(T-t)}) \tag{4.11}$$

We can clearly see that equation (11) is simply rf. Thus our equation (9), the Black-Scholes Equation, holds.

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Brandon Washburn,
5028 S 36th St, Greenfield, WI 53221, Phone: 414-333-7564 Orcid nubmer: 0000-0002-5815-3873
Email address: 17bwashburn@gmail.com

Mehmet Dik,
5050 E State St, Rockford, IL Phone: 815-226-4135 Orchid number: 0000-0003-0643-2771
Email address: mdik@rockford.edu