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COTTON TENSOR ON SASAKIAN 3-MANIFOLDS ADMITTING ETA RICCI SOLITONS

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Abstract. The object of the present paper is to characterize Cotton tensor on a 3-dimensional Sasakian manifold admitting η-Ricci solitons. After introduction, we study 3-dimensional Sasakian manifolds and introduce a new notion, namely, Cotton pseudo-symmetric manifolds. Next we deal with the study of Cotton tensor on a Sasakian 3-manifold admitting η-Ricci solitons. Among others we prove that such a manifold is a manifold of constant scalar curvature and Einstein manifold with some appropriate conditions. Also, we classify the nature of the soliton metric. Finally, we give an important remark.

1. Introduction

In differential geometry, the Weyl conformal curvature tensor vanishes on a 3-dimensional pseudo-Riemannian manifold and hence one can consider an another type of conformal invariant, which is the Cotton tensor. Cotton tensor C is a tensor of type (1,2), defined by

$$C(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{(Xr)Y - (Yr)X\},$$

for any smooth vector fields X, Y. Therefore, in a 3-dimensional pseudo-Riemannian manifold Cotton tensor vanishes if the metric be conformally flat and the idea is given by Eisenhart. At the present time, the 3-dimensional spaces becoming onto the dignity of interest, as the Cotton tensor restricts the relation between the Ricci tensor and the energy-momentum tensor of matter in the Einstein equations and plays an important role in the Hamiltonian formalism of general relativity.
The notion of Ricci flow was introduced [17] by R. S. Hamilton in 1982 to find a

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canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g = -2S, \tag{1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$, where ψ_t are homeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [12]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (2) are interesting and useful in physics and are often referred as quasi-Einstein ([13], [14]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo homeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([15], [16], [18], [19], [21], [29]) and many others.

The notion of η-Ricci soliton, which is a generalization of Ricci soliton, was introduced by CHM and Kiaora [11]. This notion has also been studied in [12] for Hope hyperuricemia in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3}$$

where S is the Ricci tensor associated to g. In this connection we may mention the works of Ayar et al. [2], Blaga ([3], [4], [5]), Prakasha et al. [24], Kar et al. ([20], [23]) and Turan et al. [27]. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton(g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named proper η-Ricci soliton.

In this paper, after introduction, in section 2, we study 3-dimensional Sasakian manifold. Section 3 deals with Cotton tensor on a Sasakian 3-manifold admitting η-Ricci solitons. In section 4, we prove that a Cotton flat Sasakian 3-manifold admitting η-Ricci solitons is a manifold of constant scalar curvature 6 and an Einstein manifold. We classify Sasakian 3-manifolds admitting η-Ricci solitons satisfying $Q \cdot C = 0$ and show that such manifolds are the manifolds of constant scalar curvature in section 5. After these, in section 6 we characterize concircularly-Cotton semisymmetric Sasakian 3-manifolds admitting η-Ricci solitons and establish a result. Then in section 7, we introduce a new notion call Cotton pseudo-symmetric manifold and accordingly we study Sasakian 3-manifolds admitting η-Ricci solitons . We complete our paper with a valuable remark.

2. Three dimensional Sasakian manifolds

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([7], [8])

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0. \tag{4}$$

The first and one of the remaining three relations in (4) imply the other two relations in (4). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{5}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η), structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{6}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{7}$$

and

$$g(X, \xi) = \eta(X), \tag{8}$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).
An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{9}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.
If the characteristic vector field ξ is a Killing vector field , the contact metric manifold (M, η, ξ, ϕ, g) is called K-contact manifold. This is the case if and only if $h = 0$. The contact structure on M is said to be normal if the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, \frac{fd}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where f is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian metrics are K-contact and K-contact 3-metrics are Sasakian. For a Sasakian manifold, the following hold ([7], [8]):

$$\nabla_X \xi = -\phi X, \tag{10}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{11}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{12}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{13}$$

$$Q\xi = 2n\xi, \tag{14}$$

where ∇, R and Q denotes respectively, the Riemannian connection, curvature tensor and the $(1,1)$-tensor metrically equivalent to the Ricci tensor of g. The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{15}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.
It is known that the Ricci tensor of a three dimensional Sasakian manifold is given by [9]

$$S(X,Y) = \frac{1}{2}\{(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)\}, \tag{16}$$

where r is the scalar curvature which need not be constant, in general. So, g is Einstein (hence has constant curvature 1) if and only if $r = 6$.
As a consequence of (16), we have

$$S(X,\xi) = 2\eta(X). \tag{17}$$

Contact metric manifolds have also been studied by several authors such as ([9]-[14], [20]- [29]) and many others.

Definition 1. *In a n-dimensional Riemannian manifold the concircular curvature tensor of type (1,3) is defined by*

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{18}$$

Then in a 3-dimensional Riemannian manifold the concircular curvature tensor is given by

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{19}$$

Definition 2. *A Riemannian manifold is said to be concircularly flat if the concircular curvature tensor $\mathcal{Z}$ vanishes.*

Let us consider a Riemannian manifold (M,g) and let the Levi-Civita connection ∇ of (M,g). A Riemannian manifold is called locally symmetric [10] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). A Riemannian or a semi-Riemannian manifold $(M,g), n \geq 3$, is called semisymmetric if

$$R.R = 0 \tag{20}$$

holds, where R denotes the curvature tensor of the manifold. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semisymmetric Riemannian manifolds were first studied by Cartan, Lichnerowich, Couty and Sinjukov. A fundamental study on Riemannian semisymmetric manifolds was made by Szabó [26], Boeckx et al [6], Kowalski [22] and Prakasha et al. [25]. A semi-Riemannian manifold $(M, g), n \geq 3$, is said to be Ricci-semisymmetric if on M we have

$$R.S = 0, \tag{21}$$

where S is the Ricci tensor. Alegre et al. [1] have studied semi-Riemannian generalized Sasakian space-forms.

The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci semisymmetric manifolds were investigated by several authors.
For a $(0, k+2)$-tensor field $Q(g, T)$ associated with any $(0, k)$-tensor field T on a Riemannian manifold (M, g) is defined as follows [28]:

$$\begin{aligned} (Q(g,T))(X_1,...,X_k;X,Y) &= ((X \wedge_g Y).T)(X_1,...,X_k) \\ &= -T((X \wedge_g Y))X_1,X_2,...,X_k) \\ &\quad -... - T(X_1,...X_{k-1},(X \wedge_g Y)X_k), \end{aligned} \tag{22}$$

where $X \wedge Y$ is the endomorphism given by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{23}$$

We define the subsets U_R, U_S of a Riemannian Manifold M by $U_R = \{x \in M : R - \frac{r}{n(n-1)}G \neq 0 \quad at \quad x\}$ and $U_S = \{x \in M : S - \frac{r}{n}g \neq 0 \quad at \quad x\}$ respectively, where $G(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. Evidently we have $U_S \subset U_R$. A Riemannian manifold is said to be pseudo-symmetric [28] if at every point of M the tensor $R.R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$R.R = f_R Q(g,R)$$

on U_R, where f_R is some function on U_R. Clearly, every semi-symmetric manifold is pseudo-symmetric but the converse is not true [28].
A Riemannian manifold M is said to Ricci pseudo-symmetric if $R.S$ and $Q(g, S)$ on M are linearly dependent. This is equivalent to

$$R.S = f_S Q(g,S)$$

holds on U_S, where f_S is a function defined on U_S.

In the present work we introduce a new notion, namely Cotton pseudo-symmetric manifold for the first time as follows:

Definition 3. *A Riemannian manifold M is said to Cotton pseudo-symmetric if $R.C$ and $Q(g, C)$ on M are linearly dependent. This is equivalent to*

$$R.C = f_S Q(g,C)$$

holds on U_S, where f_S is a function defined on U_S.

Lemma 4. *(Proposition 2.1 of [23]) The Ricci tensor of a three dimensional Sasakian manifold admitting η-Ricci soliton is of the form:*

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{24}$$

As a consequence of the above Lemma we have

$$QX = -\lambda X - \mu\eta(X)\xi. \tag{25}$$

Lemma 5. *(Proposition 2.2 of [23]) For an η-Ricci soliton on a three dimensional Sasakian manifold we have*

$$\lambda + \mu = -2. \tag{26}$$

In view of (25) and (26) we have

$$Q\xi = 2\xi. \tag{27}$$

On contraction, (24) gives

$$r = -3\lambda - \mu. \tag{28}$$

We use the above Lemmas in the next sections to develop our results.

3. Cotton tensor on Sasakian 3-manifolds admitting η-Ricci solitons

In this section, we consider a skewsymmetric tensor of type (1,2) on Sasakian 3-manifold, called Cotton tensor C, defined by

$$C(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{(Xr)Y - (Yr)X\}, \tag{29}$$

for all smooth vector fields X, Y.
Making use of (7), (10), (12) and (25) in (29) we get

$$C(X,Y) = \mu[\eta(Y)\phi X - \eta(X)\phi Y + 2g(\phi X, Y)\xi] - \frac{1}{4}[(Xr)Y - (Yr)X]. \tag{30}$$

The Cotton tensor can also be exhibited as a tensor of type (0,3) as follows:

$$C(X,Y,Z) = g(C(X,Y),Z). \tag{31}$$

By the virtue of (30) and (31), it follows that

$$\begin{aligned} C(X,Y,Z) = \mu[&2g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)] \\ &- \frac{1}{4}[(Xr)g(Y,Z) - (Yr)g(X,Z)]. \end{aligned} \tag{32}$$

As a consequence of (30) and (32), we derived the following results:

$$C(X,\xi) = \mu\phi X - \frac{1}{4}(Xr)\xi, \tag{33}$$

$$\eta(C(X,Y)) = 2\mu g(\phi X,Y) - \frac{1}{4}[(Xr)\eta(Y) - (Yr)\eta(X)], \tag{34}$$

$$\eta(C(X,\xi)) = -\frac{1}{4}(Xr), \tag{35}$$

$$C(\phi X, Y) = \mu[3\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \eta(Y)X] - \frac{1}{4}[((\phi X)r)Y - (Yr)\phi X], \tag{36}$$

$$\eta(C(\phi X, Y)) = -2\mu g(X,Y) + 2\mu\eta(X)\eta(Y) - \frac{1}{4}((\phi X)r)\eta(Y), \tag{37}$$

$$\eta(C(\phi X, \phi Y)) = -2\mu g(X, \phi Y), \tag{38}$$

$$\eta(C(\phi X, \xi)) = -\frac{1}{4}(\phi X)r, \tag{39}$$

$$C(\phi X, \phi Y, \phi Z) = -\frac{1}{4}((\phi X)r)[g(Y,Z) - \eta(Y)\eta(Z)] + \frac{1}{4}((\phi Y)r)[g(X,Z) - \eta(X)\eta(Z)]. \tag{40}$$

4. Cotton flat Sasakian 3-manifolds admitting η-Ricci solitons

In this section we characterize Cotton flat Sasakian 3-manifolds admitting η-Ricci solitons. Then we have

$$C(X,Y,Z) = 0. \tag{41}$$

By the virtue of (32) and (41) we get

$$\mu[2g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)] -\frac{1}{4}[(Xr)g(Y,Z) - (Yr)g(X,Z)] = 0. \tag{42}$$

Replacing Z by ξ in the above equation we find

$$2\mu g(\phi X, Y) = \frac{1}{4}[(Xr)\eta(Y) - (Yr)\eta(X)]. \tag{43}$$

Putting $Y = \xi$ in (43) gives

$$Xr = 0 \tag{44}$$

and hence r becomes constant.

Since r is constant, from (43) it follows that

$$2\mu g(\phi X, Y) = 0. \tag{45}$$

Substituting Y by ϕY in (45) and then in the light of (6), after contraction, we obtain

$$\mu = 0. \tag{46}$$

Thus η-Ricci soliton is not proper and so we have the following:

Theorem 6. *A Cotton flat Sasakian 3-manifold does not admit proper η-Ricci soliton.*

Making use of (46) in (26) entails that

$$\lambda = -2. \tag{47}$$

Thus the η-Ricci soliton is shrinking and hence we can state the following:

Theorem 7. *An η-Ricci soliton (g, ξ, λ, μ) on a Cotton flat Sasakian 3-manifold is shrinking.*

Making use of (46) and (47) in (28) yields

$$r = 6. \tag{48}$$

Therefore we are in a position to state the following:

Theorem 8. *A Cotton flat Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ) is of constant scalar curvature* 6.

In view of the Theorem 8, from (16) we get

$$S(X, Y) = 2g(X, Y), \tag{49}$$

that is, the manifold becomes Einstein manifold. Thus we can conclude the following:

Corollary 9. *A Cotton flat Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ) is an Einstein manifold.*

5. Sasakian 3-manifolds admitting η-Ricci solitons satisfying $Q \cdot C = 0$

In the present section, we classify Sasakian 3-manifolds admitting η-Ricci solitons satisfying $Q \cdot C = 0$. Then we have

$$(Q \cdot C)(X, Y) = 0, \tag{50}$$

for any smooth vector fields X, Y.
From (50) we get

$$QC(X, Y) - C(QX, Y) - C(X, QY) = 0. \tag{51}$$

With the help of (25), (26), (30), (33) and (34) in the preceding equation yields

$$-2\mu\eta(Y)\phi X + 2\mu\eta(X)\phi Y - 4(\mu + 1)\mu g(\phi X, Y)\xi - \frac{\lambda}{4}[(Xr)Y - (Yr)X] = 0. \tag{52}$$

Taking inner product of the above with an arbitrary smooth vector field Z and then contracting X and Z and using $\phi\xi = \operatorname{Tr}\phi = 0$, we obtain

$$\lambda(Yr) = 0 \tag{53}$$

from which it follows that either $\lambda = 0$ or r is constant. Hence we have the following:

Theorem 10. *Let M^3 be a Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ) satisfying $Q \cdot C = 0$. Then either g is steady or M^3 is a manifold of constant scalar curvature.*

6. Concircularly Cotton semisymmetric Sasakian 3-manifolds admitting η-Ricci solitons

This section deals with the study of Concircularly Cotton semisymmetric Sasakian 3-manifolds admitting η-Ricci solitons. Then we have the following:

$$(\mathcal{Z}(X,Y)\cdot C)(U,V)=0, \tag{54}$$

which implies that

$$\mathcal{Z}(X,Y)C(U,V)-C(\mathcal{Z}(X,Y)U,V)+C(\mathcal{Z}(X,Y)V,U)=0. \tag{55}$$

Using (15), (24) and (25) in (19) we get

$$\begin{aligned}\mathcal{Z}(X,Y)Z &= 2(\lambda+\frac{r}{3})[g(X,Z)Y-g(Y,Z)X]\\ &\quad +\mu[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\ &\quad -g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi].\end{aligned} \tag{56}$$

As a consequence of (56) we derived the following:

$$\mathcal{Z}(X,\xi)Z=(2\lambda+\frac{2r}{3}+\mu)[g(X,Z)\xi-\eta(Z)X] \tag{57}$$

and

$$\mathcal{Z}(X,\xi)\xi=(2\lambda+\frac{2r}{3}+\mu)[\eta(X)\xi-X]. \tag{58}$$

With the help of (56), from (55) it follows that

$$\begin{aligned}&2(\lambda+\frac{r}{3})[g(X,C(U,V))Y-g(Y,C(U,V))X]\\ &+\mu[\eta(X)\eta(C(U,V))Y-\eta(Y)\eta(C(U,V))X\\ &-g(Y,C(U,V))\eta(X)\xi+g(X,C(U,V))\eta(Y)\xi]\\ &-C(\mathcal{Z}(X,Y)U,V)+C(\mathcal{Z}(X,Y)V,U)=0.\end{aligned} \tag{59}$$

Putting $Y=V=\xi$ in the above equation we have

$$\begin{aligned}&(2\lambda+\frac{2r}{3}+\mu)[g(X,C(U,\xi))-\eta(C(U,\xi))X]\\ &+C(\mathcal{Z}(X,\xi)\xi,U)-C(\mathcal{Z}(X,\xi)U,\xi)=0.\end{aligned} \tag{60}$$

On the application of (57) and (58), the above equation reduces to the following equation

$$\begin{aligned}&(2\lambda+\frac{2r}{3}+\mu)[g(C(U,\xi),X)\xi-\eta(C(U,\xi))X-\eta(X)C(U,\xi)\\ &-C(X,U)+\eta(U)C(X,\xi)]=0.\end{aligned} \tag{61}$$

Using (30), (33) and (35) in the last equation gives

$$(2\lambda+\frac{2r}{3}+\mu)[3\mu g(X,\phi U)\xi+\frac{1}{4}(Xr)U-\frac{1}{4}(Xr)\eta(U)\xi]=0. \tag{62}$$

Substituting $U = \phi U$ in (62) and the using (4) yields

$$(2\lambda + \frac{2r}{3} + \mu)[-3\mu g(X,U)\xi + 3\mu\eta(X)\eta(U)\xi + \frac{1}{4}(Xr)\phi U] = 0. \tag{63}$$

Taking inner product of (63) with ξ and then contracting X, U we obtain

$$(2\lambda + \frac{2r}{3} + \mu)\mu = 0. \tag{64}$$

By the virtue of (26) and (64) we get

$$(\lambda + \frac{2r}{3} - 2)(\lambda + 2) = 0, \tag{65}$$

which implies that $r = \frac{3}{2}(2-\lambda)$ or $\lambda = -2$. Hence we can state our next theorem as follows:

Theorem 11. *Let M^3 be a Concircularly Cotton semisymmetric Sasakian 3-manifold admitting η-Ricci solitons (g, ξ, λ, μ). Then either M^3 is a manifold of constant scalar curvature or the metric g is shrinking.*

7. Cotton pseudo-symmetric Sasakian 3-manifolds admitting η-Ricci solitons

This section is devoted to study of a Sasakian 3-manifold admitting η-Ricci solitons satisfying the curvature property

$$(R(U,V)\cdot C)(X,Y,Z) = f_C \mathcal{Q}(g,C)(X,Y,Z;U,V), \tag{66}$$

where we assume that $f_C \neq 1$.
From (66) we get

$$\begin{aligned} & -C(R(U,V)X,Y,Z) - C(X,R(U,V)Y,Z) - C(X,Y,C(U,V)Z) \\ = \; & f_C((U\wedge_g V)\cdot C)(X,Y,Z), \end{aligned} \tag{67}$$

from which it follows that

$$\begin{aligned} & C(R(U,V)X,Y,Z) + C(X,R(U,V)Y,Z) + C(X,Y,R(U,V)Z) \\ = \; & f_C[C((U\wedge_g V)X,Y,Z) + C(X,(U\wedge_g V)Y,Z) \\ & + C(X,Y,(U\wedge_g V)Z)]. \end{aligned} \tag{68}$$

In view of (23) and (68) we get

$$\begin{aligned} & C(R(U,V)X,Y,Z) + C(X,R(U,V)Y,Z) + C(X,Y,R(U,V)Z) \\ = \; & f_C[g(V,X)C(U,Y,Z) - g(U,X)C(V,Y,Z) \\ & + g(V,Y)C(X,U,Z) - g(U,Y)C(X,V,Z) \\ & + g(V,Z)C(X,Y,U) - g(U,Z)C(X,Y,V)]. \end{aligned} \tag{69}$$

Replacing X, Z and U by ξ in the preceding equation we find

$$\begin{aligned} & \eta(C(R(\xi,V)\xi,Y)) + \eta(C(\xi,R(\xi,V)Y)) + C(\xi,Y,R(\xi,V)\xi) \\ = \; & f_C[\eta(V)\eta(C(\xi,Y)) - \eta(C(V,Y)) - \eta(Y)\eta(C(\xi,V)) \end{aligned}$$

$$+\eta(V)\eta(C(\xi,Y))-C(\xi,Y,V)]. \tag{70}$$

Substituting $Y=\phi Y$ and $V=\phi V$ in (70) we obtain

$$\begin{aligned} & -\eta(C(R(\phi V,\xi)\xi,\phi Y))-\eta(C(R(\xi,\phi V)\phi Y,\xi))+C(\xi,\phi Y,R(\xi,\phi V)\xi) \\ = \;& f_C[\eta(C(\phi Y,\phi V))+C(\phi Y,\xi,\phi V)]. \end{aligned} \tag{71}$$

Using (13) and (35) in (71) we have

$$(1-f_C)\eta(C(\phi Y,\phi V))+\frac{1}{4}(R(\xi,\phi V)\phi Y)r+(f_C-1)C(\xi,\phi Y,\phi V)=0. \tag{72}$$

From (15), (24) and (25) it follows that

$$\begin{aligned} R(X,Y)Z \;=\; & -(2\lambda+\frac{r}{2})[g(Y,Z)X-g(X,Z)Y] \\ & +\mu[\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X \\ & -g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi]. \end{aligned} \tag{73}$$

The equations (6) and (73) we obtain the followings:

$$R(\xi,\phi V,\phi Y)=-(2\lambda+\frac{r}{2}+\mu)[g(V,Y)-\eta(V)\eta(Y)]\xi \tag{74}$$

and

$$(R(\xi,\phi V)\phi Y)r=0. \tag{75}$$

Using (32), (38), (73), (74) and (75) in (72), we observe that

$$\mu(f_C-1)g(Y,\phi V)=0. \tag{76}$$

Replacing Y by ϕY in (76) and the using (6), we get

$$\mu(f_C-1)[g(Y,V)-\eta(Y)\eta(V)]=0. \tag{77}$$

On contraction over Y and V in (77) yields

$$\mu(f_C-1)=0, \tag{78}$$

which implies that

$$\mu=0. \tag{79}$$

In view of (26) and (79), we have

$$\lambda=-2. \tag{80}$$

Thus we can state our next theorem as follows:

Theorem 12. *An η-Ricci soliton (g,ξ,λ,μ) on a Cotton pseudo-symmetric Sasakian 3-manifold is shrinking.*

In view of (79) and (80), from (28) we infer

$$r=6. \tag{81}$$

Thus we can state the following:

Theorem 13. *A Cotton pseudo-symmetric Sasakian 3-manifold admitting an η-Ricci soliton (g,ξ,λ,μ) is a manifold of constant scalar curvature 6.*

In light of the Theorem 13, from (16) we observe that

$$S(X,Y) = 2g(X,Y) \tag{82}$$

that is, the manifold becomes Einstein. Therefore, we have the following:

Theorem 14. *A Cotton pseudo-symmetric Sasakian 3-manifold admitting an η-Ricci soliton (g, ξ, λ, μ) is an Einstein manifold.*

8. Conclusion

We know that ϕ-sectional curvature (sectional curvature with respect to a plane section orthogonal to ξ) of a 3-dimensional Sasakian manifold M^3 is equal to $\frac{r-4}{2}$. In view of the Theorem **8** and Theorem **12**, we can conclude that r is constant. Hence the ϕ-sectional curvature is constant and so M^3 is a 3-dimensional Sasakian space-form (see Blair [8]). Therefore we can make the following:

Remark 15. *A Sasakian* 3*-manifold admitting an η-Ricci soliton which is Cotton flat or Cotton pseudo-symmetric becomes a Sasakian-space-form.*

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(α, β)-CUTS AND INVERSE (α, β)-CUTS IN BIPOLAR FUZZY SOFT SETS

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Abstract. Bipolar fuzzy soft set theory, which is a very useful hybrid set in decision making problems, is a mathematical model that has been emphasized especially recently. In this paper, the concepts of (α, β)-cuts, first type semi-strong (α, β)-cuts, second type semi-strong (α, β)-cuts, strong (α, β)-cuts, inverse (α, β)-cuts, first type semi-weak inverse (α, β)-cuts, second type semi-weak inverse (α, β)-cuts and weak inverse (α, β)-cuts of bipolar fuzzy soft sets were introduced together with some of their properties. In addition, some distinctive properties between (α, β)-cuts and inverse (α, β)-cuts were established. Moreover, some related theorems were formulated and proved. It is further demonstrated that both (α, β)-cuts and inverse (α, β)-cuts of bipolar fuzzy soft sets were useful tools in decision making.

1. INTRODUCTION

Many mathematical models have been introduced to the literature in order to express the uncertainty problems encountered in the most accurate way. For example; the fuzzy sets put forward by Zadeh [1] is a theory that allows the abandonment of strict rules in classical mathematics in expressing uncertainty. After this theory was introduced, the theories of fuzzy sets and fuzzy systems developed rapidly. As is well known, the cut set (or level set) of fuzzy set [1] is an important concept in theory of fuzzy sets and systems, which plays a significant role in fuzzy algebra [7,8], fuzzy reasoning [9, 10], fuzzy measure [11, 12, 13] and so on. The cut set allows us to express fuzzy sets as classical sets. Based on the cut sets, the decomposition theorems and representation theorems can be established [14]. The cut sets on fuzzy sets are described in [15] by using the neighborhood relations between fuzzy point and fuzzy set. It is pointed out that there are four kinds of definitions of cut

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sets on fuzzy sets, each of which has similar properties.

Fuzzy set is a type of important mathematical structure to represent a collection of objects whose boundary is vague. There are several types of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets [16], interval-valued fuzzy sets [17], vague sets [18], etc. Bipolar-valued fuzzy set is another an extension of fuzzy set whose membership degree range is different from the above extensions. In 2000, Lee [19] initiated an extension of fuzzy set named bipolar-valued fuzzy set. Bipolar-valued fuzzy sets membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree 0 indicate that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ assigne that elements some what satisfy the property, and the membership degrees on $[-1, 0)$ assigne that elements somewhat satisfy the implicit counterproperty [19]. However, it was not practical to express an uncertainty problem using fuzzy sets and its extensions.

Realizing the inadequacy of fuzzy set theory and extensions in expressing uncertainty problems, Molodsov [2] thought that this deficiency was due to the lack of a parameterization tool. Therefore, he [2] proposed the soft set theory in 1999 and gave some relevant features. Such theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. Especially with the introduction of soft sets to the literature, the construction of hybrid set types has accelerated. This is due to the easy and practical applicability of the parameter tool. It is also because the hybrid set is more successful in expressing uncertainty, as it retains the properties of the set types that compose it. One of these hybrid sets is the bipolar fuzzy soft set, a combination of bipolar fuzzy set and soft set provided by Abdullah et al. [20]. As another example, the bipolar soft set with applications in decision making popularized by Shabir et al. [4] and discussed exhaustively by Karaaslan et al. [21] are another hybrid set model. This mathematical approach has managed to attract the attention of researchers since it was built with the contribution of a parameterization tool to this theory by addressing bipolar fuzzy sets, which is an effective generalization of fuzzy sets. In addition, we can easily say that the studies with hybrid cluster models introduced for the solution of uncertainty problems are increasing day by day [22, 23, 24, 27, 28].

In this paper, the concepts of (α, β)-cuts, first type semi-strong (α, β)-cuts, second type semi-strong (α, β)-cuts, strong (α, β)-cuts for bipolar fuzzy soft sets were introduced and some of their properties were examined. Moreover, the concepts of inverse (α, β)-cuts, first type semi-weak inverse (α, β)-cuts, second type semi-weak inverse (α, β)-cuts and weak inverse (α, β)-cuts for bipolar fuzzy soft sets were identified and some of their distinctive features were investigated. Thanks to these cuts, bipolar fuzzy soft sets can be expressed as bipolar soft sets, which in turn can assist us in the decision making process. In addition, related examples are given in the paper in order to better understand this situation.

Throughout this study, let $U = \{u_1, u_2, ..., u_m\}$ be a non-empty universe set and $E = \{x_1, x_2, ..., x_n\}$ be a set of parameters. Also, let $P(U)$ denote the power set of U and $A \subseteq E$.

2. PRELIMINARIES

Here, we remind some basic information from the literature for subsequent use.

2.1. **Fuzzy Sets.** It is possible to express definite expressions in classical mathematics with values of 0 ("false") and 1 ("true"). However, in real life this situation may not always be possible. For example; the FS theory (Zadeh 1965) put forward to present human thoughts expresses this situations in the interval $[0, 1]$ with the help of membership functions for better outcome. Zadeh expressed this set theory as follows,

Definition 1. *[1] A FS X over U is a set defined by a function μ_X representing a mapping*

$$\mu_X : U \to [0, 1]$$

μ_X is called the membership function of X, and the value $\mu_X(u)$ is called the grade of membership of $u \in U$. The value represents the degree of u belonging to the FS X. Thus, a FS X over U can be represented as follows:

$$X = \{(u, \mu_X(u)) : \mu_X(u) \in [0, 1], u \in \mathcal{U}\}$$

State that the set of all the FSs over U will be denoted by $F(U)$.

With Zadeh's [1] min-max system, FS union, intersection, and complement operations are defined below.

The union of two FSs M and N is a FS in U, denoted by $M \cup N$, whose membership grade is $\mu_{M \cup N}(u) = \mu_M(u) \vee \mu_N(u) = max\{\mu_M(u), \mu_N(u)\}$ for each $u \in U$. So

$$M \cup N = \Big\{ \Big(u, \mu_{M \cup N}(u)\Big) : \mu_{M \cup N}(u) = max\{\mu_M(u), \mu_N(u)\}, \forall u \in U \Big\}.$$

The intersection of two FSs M and N is a FS in U, denoted by $M \cap N$, whose membership grade is $\mu_{M \cap N}(u) = \mu_M(u) \wedge \mu_N(u) = min\{\mu_M(u), \mu_N(u)\}$ for each $u \in U$. So

$$M \cap N = \Big\{ \Big(u, \mu_{M \cap N}(u)\Big) : \mu_{M \cap N}(u) = min\{\mu_M(u), \mu_N(u)\}, \forall u \in U \Big\}.$$

Let D be a FS defined over U. Then its complement, denoted by D^c, is defined in terms of membership grade as $\mu_{D^c}(u) = 1 - \mu_D(u)$ for each $u \in U$.

$$D^c = \Big\{ \Big(u, \mu_{D^c}(u)\Big) : u \in U \Big\}.$$

Definition 2. *[1] Let $X \in F(U)$ and $\alpha \in [0,1]$. Then the non-fuzzy set (or crisp set) $X_\alpha = \{u \in U : \mu_X(u) \geq \alpha\}$ is called the α-cut or α-level set of X.*

If the weak inequality $\geq$ is replaced by the strict inequality $>$, the it is called the strong α-cut of X, denoted by X_{α^+}. That is, $X_{\alpha^+} = \{u \in U : \mu_X(u) > \alpha\}$.

Definition 3. *[3] Let $X \in F(U)$ and $\alpha \in [0,1]$. Then the non-fuzzy set $X_\alpha^{-1} = \{u \in U : \mu_X(u) < \alpha\}$ is called an inverse α-cut or inverse α-level set of X.*

If the strict inequality $<$ is replaced by the weak inequality $\leq$, the it is called the weak inverse α-cut of X, denoted by $X_{\alpha^-}^{-1}$. That is, $X_{\alpha^-}^{-1} = \{u \in U : \mu_X(u) \leq \alpha\}$.

2.2. **Bipolar Fuzzy Sets.**

Definition 4. *[25, 26] Let U be any nonempty set. Then a bipolar fuzzy set, is an object of the form*

$$\chi = \{(u, < \mu_\chi^+(u), \mu_\chi^-(u) >) : u \in U\}$$

and $\mu_\chi^+ : U \to [0,1]$ and $\mu_\chi^- : U \to [-1,0]$, $\mu_\chi^+(u)$ is a positive material and $\mu_\chi^-(u)$ is a negative material of $u \in U$. For simplicity, we donate the bipolar fuzzy set as $\chi =< \mu_\chi^+, \mu_\chi^- >$ in its place of $\chi = \{(u, < \mu_\chi^+(u), \mu_\chi^-(u) >) : u \in U\}$.

Definition 5. *[25, 26] Let $\chi_1 =< \mu_{\chi_1}^+, \mu_{\chi_1}^- >$ and $\chi_2 =< \mu_{\chi_2}^+, \mu_{\chi_2}^- >$ be two bipolar fuzzy sets, on U. Then we define the following operations.*
(i) $\chi_1^c = \{< 1 - \mu_{\chi_1}^+(u), -1 - \mu_{\chi_1}^-(u) >\}$,
(ii) $\chi_1 \cup \chi_2 =< max(\mu_{\chi_1}^+(u), \mu_{\chi_2}^+(u)), min(\mu_{\chi_1}^-(u), \mu_{\chi_2}^-(u)) >$,
(iii) $\chi_1 \cap \chi_2 =< min(\mu_{\chi_1}^+(u), \mu_{\chi_2}^+(u)), max(\mu_{\chi_1}^-(u), \mu_{\chi_2}^-(u)) >$.

2.3. **Soft Sets and Bipolar Soft Sets.**

Definition 6. *[2] Let U be an initial universe, E be the set of parameters, $A \subset E$ and $P(U)$ is the power set of U. Then (F, A) is called a soft set, where $F : A \to P(U)$.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A), or as the set of ϵ-approximate elements of the soft set.

Definition 7. *[5] Let $E = \{x_1, x_2, ..., x_n\}$ be a set of parameters. The NOT set of E denoted by $\neg E$ is defined by $\neg E = \{\neg x_1, \neg x_2, ..., \neg x_n\}$ where, $\neg x_i =$ not x_i for all i.*

Definition 8. *[4] A triplet (F, G, A) is called a bipolar soft set over U, where F and G are mappings, given by $F : A \to P(U)$ and $G : \neg A \to P(U)$ such that $F(x) \cap G(\neg x) = \emptyset$ (Empty Set) for all $x \in A$.*

Definition 9. *[6] Let (F, G, A) be a BSS over U. The presentation of*

$$(F, G, A) = \{(x, F(x), G(\neg x)) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E \text{ and } F(x), G(\neg x) \in P(U)\}$$

is said to be a short expansion of BSS (F, G, A).

Example 10. *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of five cars under consideration and $A = \{x_1 = Expensive, x_2 = Modern\ Technology, x_3 = Comfortable, x_4 = Fast\} \subseteq E$ be the set of parameters. Then*

$\neg A = \{x_1 = Cheap, x_2 = Classic\ Technology,\ x_3 = Not\ Comfortable, x_4 = Slow\} \subseteq \neg E$.

Suppose that a BSS (F, G, A) is given as follows.

$$F(x_1) = \{u_2, u_4\},\quad F(x_2) = \{u_1, u_4, u_5\},\quad F(x_3) = \{u_1, u_3, u_4\},\quad F(x_4) = \{u_3, u_5\},$$

$$G(\neg x_1) = \{u_1, u_5\},\quad G(\neg x_2) = \{u_2, u_3\},\quad G(\neg x_3) = \{u_5\},\quad G(\neg x_4) = \{u_2, u_4\}.$$

Then the short expansion of BSS (F, G, A) is denoted by

$$(F, G, A) = \left\{ \begin{array}{c} (x_1, \{u_2, u_4\}, \{u_1, u_5\}), (x_2, \{u_1, u_4, u_5\}, \{u_2, u_3\}), \\ (x_3, \{u_1, u_3, u_4\}, \{u_5\}), (x_4, \{u_3, u_5\}, \{u_2, u_4\}) \end{array} \right\}$$

Definition 11. *[4] For two bipolar soft sets (F, G, A) and (F_1, G_1, B) over a universe U, we say that (F, G, A) is a bipolar soft subset of (F_1, G_1, B), if,*
(1) $A \subseteq B$ *and*
(2) $F(e) \subseteq F_1(e)$ *and* $G_1(\neg x) \subseteq G(\neg x)$ *for all* $x \in A$.
This relationship is denoted by $(F, G, A)\tilde{\subseteq}(F_1, G_1, B)$. Similarly (F, G, A) is said to be a bipolar soft superset of (F_1, G_1, B), if (F_1, G_1, B) is a bipolar soft subset of (F, G, A). We denote it by $(F, G, A)\tilde{\supseteq}(F_1, G_1, B)$.

Definition 12. *[4] Two bipolar soft sets (F, G, A) and (F_1, G_1, B) over a universe U are said to be equal if (F, G, A) is a bipolar soft subset of (F_1, G_1, B) and (F_1, G_1, B) is a bipolar soft subset of (F, G, A).*

Definition 13. *[4] The complement of a bipolar soft set (F, G, A) is denoted by $(F, G, A)^c$ and is defined by $(F, G, A)^c = (F^c, G^c, A)$ where F^c and G^c are mappings given by $F^c(x) = G(\neg x)$ and $G^c(\neg x) = F(x)$ for all $x \in A$.*

Definition 14. *[4] Extended Union of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C) over U, where $C = A \cup B$ and for all $x \in C$,*

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ F_1(x) & \text{if } x \in B - A \\ F(x) \cup F_1(x) & \text{if } x \in A \cap B \end{cases}$$

$$I(\neg x) = \begin{cases} G(\neg x) & \text{if } \neg x \in (\neg A) - (\neg B) \\ G_1(\neg x) & \text{if } \neg x \in (\neg B) - (\neg A) \\ G(\neg x) \cap G_1(\neg x) & \text{if } \neg x \in (\neg A) \cap (\neg B) \end{cases}$$

We denote it by $(F, G, A)\tilde{\cup}(F_1, G_1, B) = (H, I, C)$.

Definition 15. *[4] Extended Intersection of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C) over U,* *where $C = A \cup B$ and for all $x \in C$,*

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ F_1(x) & \text{if } x \in B - A \\ F(x) \cap F_1(x) & \text{if } x \in A \cap B \end{cases}$$

$$I(\neg x) = \begin{cases} G(x) & \text{if } x \in (\neg A) - (\neg B) \\ G_1(x) & \text{if } x \in (\neg B) - (\neg A) \\ G(x) \cup G_1(x) & \text{if } x \in (\neg A) \cap (\neg B) \end{cases}$$

We denote it by $(F, G, A) \tilde{\cap} (F_1, G_1, B) = (H, I, C)$.

Definition 16. *[4] Restricted Union of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C), where $C = A \cap B$ is non-empty and for all $x \in C$*

$$H(x) = F(x) \cup G(x) \quad \text{and} \quad I(\neg x) = F_1(\neg x) \cap G_1(\neg x)$$

We denote it by $(F, G, A) \cup_{\mathfrak{R}} (F_1, G_1, B) = (H, I, C)$.

Definition 17. *[4] Restricted Intersection of two bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the bipolar soft set (H, I, C), where $C = A \cap B$ is non-empty and for all $x \in C$*

$$H(x) = F(x) \cap G(x) \quad \text{and} \quad I(\neg x) = F_1(\neg x) \cup G_1(\neg x)$$

We denote it by $(F, G, A) \cap_{\mathfrak{R}} (F_1, G_1, B) = (H, I, C)$.

2.4. Bipolar Fuzzy Soft Sets.

Definition 18. *[20] Define $f : A \to BF^U$, where BF^U is the collection of all bipolar fuzzy subsets of U. Then (f, A), denoted by f_A, is said to be a bipolar fuzzy soft set over a universe U. It is defined by*

$$f_A = \left\{ \left(u, \mu^{+}_{(f_A)_x}(u), \mu^{-}_{(f_A)_x}(u) \right) : \forall u \in U, x \in A \right\}$$

Example 19. *Let $U = \{u_1, u_2, u_3, u_4\}$ be the set of four computers under consideration and $A = \{x_1 = Modern\ Technology, x_2 = Cost, x_3 = Fast\} \subseteq E$ be the set of parameters. Then,*

$$f_A = \left\{ \begin{array}{c} f(x_1) = \left\{ \begin{array}{c} (u_1, 0.45, -0.2), (u_2, 0.6, -0.43), \\ (u_3, 0.7, -0.35), (u_4, 0.55, -0.25) \end{array} \right\}, \\ f(x_2) = \left\{ \begin{array}{c} (u_1, 0.34, -0.65), (u_2, 0.32, -0.22), \\ (u_3, 0.48, -0.24), (u_4, 0.64, -0.8) \end{array} \right\}, \\ f(x_3) = \left\{ \begin{array}{c} (u_1, 0.9, -0.15), (u_2, 0.72, -0.34), \\ (u_3, 0.34, -0.56), (u_4, 0.24, -0.87) \end{array} \right\} \end{array} \right\}$$

Definition 20. *[20] Let U be a universe and E a set of attributes. Then, (U, E) is the collection of all bipolar fuzzy soft sets on U with attributes from E and is said to be bipolar fuzzy soft class.*

Definition 21. *[20] Let f_A and g_B be two bipolar fuzzy soft sets over a common universe U. We say that f_A is a bipolar fuzzy soft subset of g_B, if*
(i) $A \subseteq B$ and
(ii) For all $x \in A$, $f(x)$ is a bipolar fuzzy subset of $g(x)$. We write $f_A \widehat{\subseteq} g_B$.

Moreover, we say that f_A and g_B are bipolar fuzzy soft equal sets if f_A is a bipolar fuzzy soft subset of g_B and g_B is a bipolar fuzzy soft subset of f_A.

Definition 22. *[20] The complement of a bipolar fuzzy soft set f_A is denoted ${f_A}^c$ and is defined by ${f_A}^c = \Big\{ \Big(u, 1 - \mu^{+}_{(f_A)_x}(u), -1 - \mu^{-}_{(f_A)_x}(u)\Big) : \forall u \in U, x \in A \Big\}$.*

It should be noted that $1 - f(x)$ denotes the fuzzy complement of $f(x)$ for $x \in A$.

Definition 23. *[20] Let f_A and g_B be two bipolar fuzzy soft sets over a common universe U. Then*
(i) The union of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cup} g_B$ over U, where $C = A \cup B$, $h : C \to BF^U$ and

$$h(e) = \begin{cases} f(x) & \text{if } x \in A \setminus B \\ g(x) & \text{if } x \in B \setminus A \\ f(x) \cup g(x) & \text{if } x \in A \cap B \end{cases}$$

for all $x \in C$.
(ii) The restricted union of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cup}_{\mathcal{R}} g_B$ over U, where $C = A \cap B \neq \emptyset$, $h : C \to BF^U$ and $h(x) = f(x) \cup g(x)$ for all $x \in C$.
(iii) The extended intersection of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cap} g_B$ over U, where $C = A \cup B$, $h : C \to BF^U$ and

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \setminus B \\ g(x) & \text{if } x \in B \setminus A \\ f(x) \cap g(x) & \text{if } x \in A \cap B \end{cases}$$

for all $x \in C$.
(iv) The restricted intersection of bipolar fuzzy soft sets f_A and g_B is defined as the bipolar fuzzy soft set $h_C = f_A \widehat{\cap}_{\mathcal{R}} g_B$ over U, where $C = A \cap B \neq \emptyset$, $h : C \to BF^U$ and $h(x) = f(x) \cap g(x)$ for all $x \in C$.

3. (α, β)-cuts and its Properties in Bipolar Fuzzy Soft Sets

In this section, the concepts of (α, β)-cuts and strong (α, β)-cuts of BFSSs were introduced together with some of their properties.

Definition 24. *Let f_A be a BFSS over U and $\alpha \in [0, 1]$, $\beta \in [-1, 0]$. Then the (α, β)-cut or (α, β)-level BSS of f_A denoted by $[f_A]_{(\alpha,\beta)}$ is defined as*

$$[f_A]_{(\alpha,\beta)} = \Big\{ \Big(x, \widehat{F}^{(\alpha,\beta)}_{[f_A]}(x), \widehat{G}^{(\alpha,\beta)}_{[f_A]}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E \Big\}$$

where

$$\widehat{F}^{(\alpha,\beta)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha,\beta)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

The first type semi-strong (α,β)-cut, denoted by $[f_A]_{(\alpha^+,\beta)}$ is defined as

$$[f_A]_{(\alpha^+,\beta)}=\left\{\left(x,\widehat{F}^{(\alpha^+,\beta)}_{[f_A]}(x),\widehat{G}^{(\alpha^+,\beta)}_{[f_A]}(\neg x)\right):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\right\}$$

where

$$\widehat{F}^{(\alpha^+,\beta)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha^+,\beta)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)\leq\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

The second type semi-strong (α,β)-cut, denoted by $[f_A]_{(\alpha,\beta^+)}$ is defined as

$$[f_A]_{(\alpha,\beta^+)}=\left\{\left(x,\widehat{F}^{(\alpha,\beta^+)}_{[f_A]}(x),\widehat{G}^{(\alpha,\beta^+)}_{[f_A]}(\neg x)\right):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\right\}$$

where

$$\widehat{F}^{(\alpha,\beta^+)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha,\beta^+)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)\geq\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

The strong (α,β)-cut, denoted by $[f_A]_{(\alpha^+,\beta^+)}$ is defined as

$$[f_A]_{(\alpha^+,\beta^+)}=\left\{\left(x,\widehat{F}^{(\alpha^+,\beta^+)}_{[f_A]}(x),\widehat{G}^{(\alpha^+,\beta^+)}_{[f_A]}(\neg x)\right):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\right\}$$

where

$$\widehat{F}^{(\alpha^+,\beta^+)}_{[f_A]}(x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)\geq\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\},$$

$$\widehat{G}^{(\alpha^+,\beta^+)}_{[f_A]}(\neg x)=\left\{u:\left[\mu^{+}_{[f_A]_x}(u)>\alpha\right]\wedge\left[\mu^{-}_{[f_A]_x}(u)<\beta\right]\wedge\left[\mu^{+}_{[f_A]_x}(u)<\left|\mu^{-}_{[f_A]_x}(u)\right|\right]\right\}.$$

Example 25. *Let $U=\{u_1,u_2,u_3\}$, $A=\{x_1,x_2,x_3\}\subseteq E$ and BFSS f_A over U be*

$$f_A=\left\{\begin{array}{c} f(x_1)=\left\{\ (u_1,0.56,-0.42),(u_2,0.75,-0.5),(u_3,0.5,-0.3)\ \right\},\\ f(x_2)=\left\{\ (u_1,0.8,-0.15),(u_2,0.4,-0.56),(u_3,0.64,-0.15)\ \right\},\\ f(x_3)=\left\{\ (u_1,0.35,-0.6),(u_2,0.1,-0.5),(u_3,0.56,-0.2)\ \right\} \end{array}\right\}$$

For example; let U be the supplier firms that apply to become a supplier of a pharmaceutical company and $A\subseteq E$ is the set of parameters that the company wants from the supplier. This BFSS is represented in tabular form as follows:

TABLE 1. Representation of BFSS f_A

$U\setminus E$	$Experienced=x_1$	$Cheap=x_2$	$Fast=x_3$
u_1	$<0.56,-0.42>$	$<0.8,-0.15>$	$<0.35,-0.6>$
u_2	$<0.75,-0.5>$	$<0.4,-0.56>$	$<0.64,-0.15>$
u_3	$<0.35,-0.6>$	$<0.1,-0.5>$	$<0.56,-0.2>$

Example 26. *Then if* $\alpha = 0.56$ *and* $\beta = -0.5$*, we have*

$$[f_A]_{(0.56,-0.5)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2, u_3\}), (x_3, \{u_2, u_3\}, \{u_1\})\},$$

$$[f_A]_{(0.56^+,-0.5)} = \{(x_1, \{u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2, u_3\}), (x_3, \{u_2\}, \{u_1\})\},$$

$$[f_A]_{(0.56,-0.5^+)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2\}), (x_3, \{u_2, u_3\}, \{u_1\})\}$$

and

$$[f_A]_{(0.56^+,-0.5^+)} = \{(x_1, \{u_2\}, \{u_3\}), (x_2, \{u_1\}, \{u_2\}), (x_3, \{u_2\}, \{u_1\})\}.$$

Then if $\alpha = 0.35$ *and* $\beta = -0.6$*, we have*

$$[f_A]_{(0.35,-0.6)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{u_1\})\},$$

$$[f_A]_{(0.35^+,-0.6)} = \{(x_1, \{u_1, u_2\}, \{u_3\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{u_1\})\},$$

$$[f_A]_{(0.35,-0.6^+)} = \{(x_1, \{u_1, u_2\}, \{\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{\})\}$$

and

$$[f_A]_{(0.35^+,-0.6^+)} = \{(x_1, \{u_1, u_2\}, \{\}), (x_2, \{u_1, u_2\}, \{\}), (x_3, \{u_2, u_3\}, \{\})\}.$$

Remark 27. (α, β)*-cut can be use to make a decision. For example, let's assume that the pharmaceutical company will consider the most suitable supplier firm as the firm that provides the most number of parameters under* (α, β)*. For this, the mapping* $\Theta_{[f_A]_{(\alpha,\beta)}}$ *is defined by* $\Theta_{[f_A]_{(\alpha,\beta)}} : U \to [-n, n]$ *for all* $u_i \in U$ *as follows: (*$1 \leq i \leq s(E) = n$ *and* $1 \leq j \leq s(U) = m$*)*

$$\Theta_{[f_A]_{(\alpha,\beta)}}(u_i) = \sum_{j=1}^{n} \Upsilon^{ij}_{[f_A]_{(\alpha,\beta)}} \tag{1}$$

$$\Upsilon^{ij}_{[f_A]_{(\alpha,\beta)}} = \begin{cases} 1, & \text{if } u_i \in \widehat{F}^{(\alpha,\beta)}_{[f_A]}(x_j) \\ -1, & \text{if } u_i \in \widehat{G}^{(\alpha,\beta)}_{[f_A]}(x_j) \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Here, the value $\Theta_{[f_A]_{(\alpha,\beta)}}(u_i)$ *is called the "total score" for the objects and the greater the total score of an object, the more recommended it is to select that object. Under these conditions, the calculation of the total scores for* $\alpha = 0.56$ *and* $\beta = -0.5$ *given in Example 25 is as follows;*

$$\Theta_{[f_A]_{(0.56,-0.5)}}(u_1) = \Upsilon^{11}_{[f_A]_{(0.56,-0.5)}} + \Upsilon^{12}_{[f_A]_{(0.56,-0.5)}} + \Upsilon^{13}_{[f_A]_{(0.56,-0.5)}} = 1+1+(-1) = 1,$$

$$\Theta_{[f_A]_{(0.56,-0.5)}}(u_2) = 1, \quad \Theta_{[f_A]_{(0.56,-0.5)}}(u_3) = -1.$$

Similarly, for $\alpha = 0.35$ *and* $\beta = -0.6$

$$\Theta_{[f_A]_{(0.35,-0.6)}}(u_1) = 1, \quad \Theta_{[f_A]_{(0.35,-0.6)}}(u_2) = 3, \quad \Theta_{[f_A]_{(0.35,-0.6)}}(u_3) = 2.$$

As can be seen, it is not possible to choose the best element for $\alpha = 0.56$ *and* $\beta = -0.5$*, because there are two supplier firms that have the highest total score. However, the total scores calculated for* $\alpha = 0.35$ *and* $\beta = -0.6$ *indicate that the most suitable supplier firm for the pharmaceutical company is* u_2*.*

Proposition 28. *Let $\alpha \in [0,1]$, $\beta \in [-1,0]$ and f_A, g_B be BFSSs over U, the following properties hold:*
(i) $[f_A]_{(\alpha^+,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha^+,\beta)} \widetilde{\subseteq} [f_A]_{(\alpha,\beta)}$ *and* $[f_A]_{(\alpha^+,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha,\beta)}$.
(ii) $[f_A]_{(\alpha^+,\beta)} \widetilde{\cap} [f_A]_{(\alpha,\beta^+)} = [f_A]_{(\alpha^+,\beta^+)}$.
(iii) If $\alpha_1 \leq \alpha_2$ *and* $\beta_1 \geq \beta_2$*, then* $[f_A]_{(\alpha_2,\beta_2)} \widetilde{\subseteq} [f_A]_{(\alpha_1,\beta_1)}$.
(iv) $[f_A \widehat{\cup} g_B]_{(\alpha,\beta)} = [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$.
(v) $[f_A \widehat{\cap} g_B]_{(\alpha,\beta)} = [f_A]_{(\alpha,\beta)} \widetilde{\cap} [g_B]_{(\alpha,\beta)}$.

Proof. (i) Let $(x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha^+,\beta^+)}$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) < \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) < \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha^+,\beta)}$
Therefore $[f_A]_{(\alpha^+,\beta^+)} \widetilde{\subseteq} [f_A]_{(\alpha^+,\beta)}$. Similarity, for $(x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha^+,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) > \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)}$
Therefore $[f_A]_{(\alpha^+,\beta)} \widetilde{\subseteq} [f_A]_{(\alpha^+,\beta^+)}$. It is proved similarly in the other part.
(ii) Straighforward.
(iii) It is clear from Definition 11 and Definition 24.
(iv) Let $(x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]_{(\alpha,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \left|\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_j) < \left|\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_j)\right|\right]$, $\forall x \in A \cup B$
$\Rightarrow \Bigg[\left[\mu^+_{[f_A]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_i) \geq \left|\mu^-_{[f_A]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[f_A]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[f_A]_x}(u_j) < \left|\mu^-_{[f_A]_x}(u_j)\right|\right]$, $\forall x \in A\Bigg]$ or $\Bigg[\left[\mu^+_{[g_B]_x}(u_i) \geq \alpha\right] \wedge \left[\mu^-_{[g_B]_x}(u_i) \leq \beta\right] \wedge \left[\mu^+_{[g_B]_x}(u_i) \geq \left|\mu^-_{[g_B]_x}(u_i)\right|\right]$ and $\left[\mu^+_{[g_B]_x}(u_j) \geq \alpha\right] \wedge \left[\mu^-_{[g_B]_x}(u_j) \leq \beta\right] \wedge \left[\mu^+_{[g_B]_x}(u_j) < \left|\mu^-_{[g_B]_x}(u_j)\right|\right]$, $\forall x \in B\Bigg]$

$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)}$ or $(x,\{u_i\},\{u_j\}) \in [g_B]_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$
Therefore, $[f_A \widehat{\cup} g_B]_{(\alpha,\beta)} \widetilde{\subset} [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$.

Conversely, suppose $(x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]_{(\alpha,\beta)}$ or $(x,\{u_i\},\{u_j\}) \in [g_B]_{(\alpha,\beta)}$
$\Rightarrow \Big[\Big[\mu^+_{[f_A]_x}(u_i) \geq \alpha\Big] \wedge \Big[\mu^-_{[f_A]_x}(u_i) \leq \beta\Big] \wedge \Big[\mu^+_{[f_A]_x}(u_i) \geq \Big|\mu^-_{[f_A]_x}(u_i)\Big|\Big]$ and $\Big[\mu^+_{[f_A]_x}(u_j) \geq \alpha\Big] \wedge \Big[\mu^-_{[f_A]_x}(u_j) \leq \beta\Big] \wedge \Big[\mu^+_{[f_A]_x}(u_j) < \Big|\mu^-_{[f_A]_x}(u_j)\Big|\Big], \forall x \in A\Big]$ or $\Big[\Big[\mu^+_{[g_B]_x}(u_i) \geq \alpha\Big] \wedge \Big[\mu^-_{[g_B]_x}(u_i) \leq \beta\Big] \wedge \Big[\mu^+_{[g_B]_x}(u_i) \geq \Big|\mu^-_{[g_B]_x}(u_i)\Big|\Big]$ and $\Big[\mu^+_{[g_B]_x}(u_j) \geq \alpha\Big] \wedge \Big[\mu^-_{[g_B]_x}(u_j) \leq \beta\Big] \wedge \Big[\mu^+_{[g_B]_x}(u_j) < \Big|\mu^-_{[g_B]_x}(u_j)\Big|\Big], \forall x \in B\Big]$
$\Rightarrow \Big[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \alpha\Big] \wedge \Big[\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_i) \leq \beta\Big] \wedge \Big[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_i) \geq \Big|\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_i)\Big|\Big]$ and $\Big[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_j) \geq \alpha\Big] \wedge \Big[\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_j) \leq \beta\Big] \wedge \Big[\mu^+_{[f_A \widehat{\cup} g_B]_x}(u_j) < \Big|\mu^-_{[f_A \widehat{\cup} g_B]_x}(u_j)\Big|\Big]$, $\forall x \in A \cup B$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]_{(\alpha,\beta)}$
Therefore, $[f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)} \widetilde{\subset} [f_A \widehat{\cup} g_B]_{(\alpha,\beta)}$. Thus $[f_A \widehat{\cup} g_B]_{(\alpha,\beta)} = [f_A]_{(\alpha,\beta)} \widetilde{\cup} [g_B]_{(\alpha,\beta)}$.

(v) It is proved similar to step (iv). □

4. Inverse (α,β)-cuts and its Properties in Bipolar Fuzzy Soft Sets

In this section, the concepts of inverse (α,β)-cuts and weak inverse (α,β)-cuts of BFSSs were introduced together with some of their properties.

Definition 29. *Let f_A be a BFSS over U and $\alpha \in [0,1]$, $\beta \in [-1,0]$. Then the inverse (α,β)-cut or inverse (α,β)-level BSS of f_A denoted by $[f_A]^{-1}_{(\alpha,\beta)}$ is defined as*

$$[f_A]^{-1}_{(\alpha,\beta)} = \Big\{\Big(x, \widehat{F}^{(\alpha,\beta)}_{[f_A]^{-1}}(x), \widehat{G}^{(\alpha,\beta)}_{[f_A]^{-1}}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E\Big\}$$

where

$$\widehat{F}^{(\alpha,\beta)}_{[f_A]^{-1}}(x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) < \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) > \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) \geq \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\},$$

$$\widehat{G}^{(\alpha,\beta)}_{[f_A]^{-1}_x}(\neg x) = \Big\{u : \Big[\mu^+_{[f_A]^{-1}_x}(u) < \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u) > \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u) < \Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}.$$

The first type semi-weak inverse (α,β)-cut, denoted by $[f_A]^{-1}_{(\alpha^-,\beta)}$ is defined as

$$[f_A]^{-1}_{(\alpha^-,\beta)} = \Big\{\Big(x, \widehat{F}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(x), \widehat{G}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(\neg x)\Big) : x \in A \subseteq E, \neg x \in \neg A \subseteq \neg E\Big\}$$

where

$$\widehat{F}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(x)=\Big\{u:\Big[\mu^+_{[f_A]^{-1}_x}(u)\le\alpha\Big]\wedge\Big[\mu^-_{[f_A]^{-1}_x}(u)>\beta\Big]\wedge\Big[\mu^+_{[f_A]^{-1}_x}(u)\ge\Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\},$$

$$\widehat{G}^{(\alpha^-,\beta)}_{[f_A]^{-1}}(\neg x)=\Big\{u:\Big[\mu^+_{[f_A]^{-1}_x}(u)\le\alpha\Big]\wedge\Big[\mu^-_{[f_A]^{-1}_x}(u)>\beta\Big]\wedge\Big[\mu^+_{[f_A]^{-1}_x}(u)<\Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}.$$

The second type semi-weak inverse (α,β)-cut, denoted by $[f_A]^{-1}_{(\alpha,\beta^-)}$ is defined as

$$[f_A]^{-1}_{(\alpha,\beta^-)}=\Big\{\Big(x,\widehat{F}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(x),\widehat{G}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(\neg x)\Big):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\Big\}$$

where

$$\widehat{F}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(x)=\Big\{u:\Big[\mu^+_{[f_A]^{-1}_x}(u)<\alpha\Big]\wedge\Big[\mu^-_{[f_A]^{-1}_x}(u)\ge\beta\Big]\wedge\Big[\mu^+_{[f_A]^{-1}_x}(u)\ge\Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\},$$

$$\widehat{G}^{(\alpha,\beta^-)}_{[f_A]^{-1}}(\neg x)=\Big\{u:\Big[\mu^+_{[f_A]^{-1}_x}(u)<\alpha\Big]\wedge\Big[\mu^-_{[f_A]^{-1}_x}(u)\ge\beta\Big]\wedge\Big[\mu^+_{[f_A]^{-1}_x}(u)<\Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}.$$

The weak inverse (α,β)-cut, denoted by $[f_A]^{-1}_{(\alpha^-,\beta^-)}$ is defined as

$$[f_A]^{-1}_{(\alpha^-,\beta^-)}=\Big\{\Big(x,\widehat{F}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(x),\widehat{G}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(\neg x)\Big):x\in A\subseteq E,\neg x\in\neg A\subseteq\neg E\Big\}$$

where

$$\widehat{F}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(x)=\Big\{u:\Big[\mu^+_{[f_A]^{-1}_x}(u)\le\alpha\Big]\wedge\Big[\mu^-_{[f_A]^{-1}_x}(u)\ge\beta\Big]\wedge\Big[\mu^+_{[f_A]^{-1}_x}(u)\ge\Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\},$$

$$\widehat{G}^{(\alpha^-,\beta^-)}_{[f_A]^{-1}}(\neg x)=\Big\{u:\Big[\mu^+_{[f_A]^{-1}_x}(u)\le\alpha\Big]\wedge\Big[\mu^-_{[f_A]^{-1}_x}(u)\ge\beta\Big]\wedge\Big[\mu^+_{[f_A]^{-1}_x}(u)<\Big|\mu^-_{[f_A]^{-1}_x}(u)\Big|\Big]\Big\}.$$

Example 30. *Consider the BFSS f_A as given in Example 25.*
Then if $\alpha=0.56$ and $\beta=-0.5$, we have

$$[f_A]^{-1}_{(0.56,-0.5)}=\{(x_1,\{u_3\},\{u_1\}),(x_2,\{u_2,u_3\},\{u_1\}),(x_3,\{u_1\},\{u_2,u_3\})\},$$

$$[f_A]^{-1}_{(0.56^-,-0.5)}=\{(x_1,\{u_1,u_3\},\{\}),(x_2,\{u_2,u_3\},\{u_1\}),(x_3,\{u_1,u_3\},\{u_2\})\},$$

$$[f_A]^{-1}_{(0.56,-0.5^-)}=\{(x_1,\{u_3\},\{u_1,u_2\}),(x_2,\{u_2\},\{u_1,u_3\}),(x_3,\{u_1\},\{u_2,u_3\})\}$$

and

$$[f_A]^{-1}_{(0.56^-,-0.5^-)}=\{(x_1,\{u_1,u_3\},\{u_2\}),(x_2,\{u_2\},\{u_1,u_3\}),(x_3,\{u_1,u_3\},\{u_2\})\}.$$

Then if $\alpha=0.35$ and $\beta=-0.6$, we have

$$[f_A]^{-1}_{(0.35,-0.6)}=\{(x_1,\{\},\{u_1,u_2\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{\},\{u_2,u_3\})\},$$

$$[f_A]^{-1}_{(0.35^-,-0.6)}=\{(x_1,\{u_3\},\{u_1,u_2\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{u_1\},\{u_2,u_3\})\},$$

$$[f_A]^{-1}_{(0.35,-0.6^-)}=\{(x_1,\{\},\{u_1,u_2,u_3\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{\},\{u_1,u_2,u_3\})\}$$

and

$$[f_A]^{-1}_{(0.35^-,-0.6^-)}=\{(x_1,\{\},\{u_1,u_2,u_3\}),(x_2,\{\},\{u_1,u_2,u_3\}),(x_3,\{\},\{u_1,u_2,u_3\})\}.$$

Remark 31. *Inverse (α, β)-cut can be use to know the most unfavorable selection. For example, let's assume that the pharmaceutical company will consider the unsuitable supplier firm as the firm that provides the least number of parameters under inverse (α, β). For this, let's create a similar mapping given in Remark 27 and the mapping $\Theta^{-1}_{[f_A]_{(\alpha,\beta)}}$ is defined by $\Theta^{-1}_{[f_A]_{(\alpha,\beta)}} : U \to [-n, n]$ for all $u_i \in U$ as follows: ($1 \le i \le s(E) = n$ and $1 \le j \le s(U) = m$)*

$$\Theta^{-1}_{[f_A]_{(\alpha,\beta)}}(u_i) = \sum_{j=1}^{n} \Upsilon^{ij}_{[f_A]^{-1}_{(\alpha,\beta)}} \tag{3}$$

$$\Upsilon^{ij}_{[f_A]^{-1}_{(\alpha,\beta)}} = \begin{cases} 1, & \text{if } u_i \in \widehat{F}^{(\alpha,\beta)}_{[f_A]^{-1}}(x_j) \\ -1, & \text{if } u_i \in \widehat{G}^{(\alpha,\beta)}_{[f_A]^{-1}}(x_j) \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Here, the value $\Theta_{[f_A]^{-1}_{(\alpha,\beta)}}(u_i)$ is called the "inverse total score" for the objects and the smaller the inverse total score of an object, the more recommended it is not to select that object. Under these conditions, the calculation of the total scores for $\alpha = 0.56$ and $\beta = -0.5$ given in Example 30 is as follows;

$$\begin{aligned} \Theta^{-1}_{[f_A]_{(0.56,-0.5)}}(u_1) &= \Upsilon^{11}_{[f_A]^{-1}_{(0.56,-0.5)}} + \Upsilon^{12}_{[f_A]^{-1}_{(0.56,-0.5)}} + \Upsilon^{13}_{[f_A]^{-1}_{(0.56,-0.5)}} \\ &= (-1) + (-1) + 1 = -1, \end{aligned}$$

$$\Theta^{-1}_{[f_A]_{(0.56,-0.5)}}(u_2) = 0, \quad \Theta^{-1}_{[f_A]_{(0.56,-0.5)}}(u_3) = 1.$$

Similarly, for $\alpha = 0.35$ and $\beta = -0.6$

$$\Theta^{-1}_{[f_A]_{(0.35,-0.6)}}(u_1) = -2, \quad \Theta^{-1}_{[f_A]_{(0.35,-0.6)}}(u_2) = -3, \quad \Theta^{-1}_{[f_A]_{(0.35,-0.6)}}(u_3) = -2.$$

As can be seen, the inverse total scores calculated for $\alpha = 0.35$ and $\beta = -0.6$ indicate that the unsuitable supplier firm for the pharmaceutical company is u_2. Moreover, the inverse total scores calculated for $\alpha = 0.56$ and $\beta = -0.5$ indicate that the unsuitable supplier firm for the pharmaceutical company is u_1. It means that the unsuitable object can change for the selected inverse (α, β)-cuts. In this case, we should pay attention to the selection of inverse (α, β)-cuts in order for the decision making process to function properly.

Remark 32. *Items (iv) and (v) given in Proposition 28 are not generally correct for inverse (α, β)-cuts. For this, let's examine Example 33 and 34:*

Example 33. $\big[$***Counter Example for** (iv):*$\big]$
Let $U = \{u_1, u_2, u_3\}$, $A = \{x_1, x_2, x_3\} \subseteq E$, $B = \{x_2, x_3, x_4\} \subseteq E$ and BFSS f_A, g_B over U be

$$f_A = \left\{ \begin{array}{c} f(x_1) = \left\{ \ (u_1, 0.56, -0.42), (u_2, 0.75, -0.5), (u_3, 0.5, -0.3) \ \right\}, \\ f(x_2) = \left\{ \ (u_1, 0.8, -0.15), (u_2, 0.4, -0.56), (u_3, 0.64, -0.15) \ \right\}, \\ f(x_3) = \left\{ \ (u_1, 0.35, -0.6), (u_2, 0.1, -0.5), (u_3, 0.56, -0.2) \ \right\} \end{array} \right\}$$

$$g_B = \left\{ \begin{array}{c} g(x_2) = \left\{ \ (u_1, 0.64, -0.2), (u_2, 0.57, -0.55), (u_3, 0.6, -0.65) \ \right\}, \\ g(x_3) = \left\{ \ (u_1, 0.51, -0.24), (u_2, 0.7, -0.2), (u_3, 0.7, -0.52) \ \right\}, \\ g(x_4) = \left\{ \ (u_1, 0.18, -0.62), (u_2, 0.33, -0.6), (u_3, 0.5, -0.3) \ \right\} \end{array} \right\}$$

Then $h_C = f_A \widehat{\cup} g_B$, *where* $C = A \cup B = \{e_1, e_2, e_3, e_4\}$

$$h_C = \left\{ \begin{array}{c} h(x_1) = \left\{ \ (u_1, 0.56, -0.42), (u_2, 0.75, -0.5), (u_3, 0.5, -0.3) \ \right\}, \\ h(x_2) = \left\{ \ (u_1, 0.8, -0.2), (u_2, 0.57, -0.56), (u_3, 0.64, -0.65) \ \right\}, \\ h(x_3) = \left\{ \ (u_1, 0.51, -0.6), (u_2, 0.7, -0.5), (u_3, 0.7, -0.52) \ \right\}, \\ h(x_4) = \left\{ \ (u_1, 0.18, -0.62), (u_2, 0.33, -0.6), (u_3, 0.5, -0.3) \ \right\} \end{array} \right\}.$$

Then

$$[f_A]^{-1}_{(0.56,-0.5)} = \{(x_1, \{u_3\}, \{u_1\}), (x_2, \{u_2, u_3\}, \{u_1\}), (x_3, \{u_1\}, \{u_2, u_3\})\},$$

$$[g_B]^{-1}_{(0.56,-0.5)} = \{(x_2, \{\}, \{u_1\}), (x_3, \{u_1\}, \{u_2\}), (x_4, \{u_1, u_2, u_3\}, \{\})\},$$

and

$$[h_C]^{-1}_{(0.56,-0.5)} = \{(x_1, \{u_3\}, \{u_1\}), (x_2, \{\}, \{u_1\}), (x_3, \{u_1\}, \{\}), (x_4, \{u_1, u_2, u_3\}, \{\})\}.$$

Also,

$$[f_A]^{-1}_{(0.56,-0.5)} \widetilde{\cup} [g_B]^{-1}_{(0.56,-0.5)} = \{(x_1, \{u_3\}, \{u_1\}), (x_2, \{u_2, u_3\}, \{u_1\}), (x_3, \{u_1\}, \{u_2, u_3\}), (x_4, \{u_1, u_2, u_3\}, \{\})\}.$$

Thus $[f_A]^{-1}_{(0.56,-0.5)} \widetilde{\cup} [g_B]^{-1}_{(0.56,-0.5)} \neq [f_A \widehat{\cup} g_B]^{-1}_{(0.56,-0.5)}$.

Example 34. $\big[$***Counter Example for*** (v)*:*$\big]$
Consider the BFSS f_A *and* g_B *as given in Example 33. In this case,* $h_C = f_A \widehat{\cap} g_B$, *where* $C = A \cap B = \{e_2, e_3\}$

$$h_C = \left\{ \begin{array}{c} h(x_2) = \left\{ \ (u_1, 0.64, -0.15), (u_2, 0.4, -0.55), (u_3, 0.6, -0.15) \ \right\}, \\ h(x_3) = \left\{ \ (u_1, 0.51, -0.15), (u_2, 0.4, -0.2), (u_3, 0.64, -0.15) \ \right\} \end{array} \right\}.$$

Then

$$[h_C]^{-1}_{(0.56,-0.5)} = \{(x_2, \{u_2\}, \{u_1, u_3\}), (x_3, \{u_1, u_2\}, \{u_3\})\}$$

and

$$[f_A]^{-1}_{(0.56,-0.5)} \widetilde{\cap} [g_B]^{-1}_{(0.56,-0.5)} = \{(x_1, \{u_3\}, \{u_1\}), (x_2, \{\}, \{u_1\}), (x_3, \{u_1\}, \{u_2\}), (x_4, \{u_1, u_2, u_3\}, \{\})\}.$$

Thus $[f_A]^{-1}_{(0.56,-0.5)} \widetilde{\cap} [g_B]^{-1}_{(0.56,-0.5)} \neq [f_A \widehat{\cap} g_B]^{-1}_{(0.56,-0.5)}$.

Proposition 35. *Let* $\alpha \in [0,1]$, $\beta \in [-1,0]$ *and* f_A, g_B *be BFSSs over* U, *the following properties hold:*
(i) $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta^-)}$ *and* $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha,\beta^-)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta^-)}$.
(ii) $[f_A]^{-1}_{(\alpha^-,\beta)} \widetilde{\cap} [f_A]^{-1}_{(\alpha,\beta^-)} = [f_A]^{-1}_{(\alpha^-,\beta^-)}$.
(iii) If $\alpha_1 \leq \alpha_2$ *and* $\beta_1 \geq \beta_2$, *then* $[f_A]^{-1}_{(\alpha_1,\beta_1)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha_2,\beta_2)}$.

(iv) $[f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subset} [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cup} [g_B]^{-1}_{(\alpha,\beta)}$.
(v) $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cap} [g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subset} [f_A \widehat{\cap} g_B]^{-1}_{(\alpha,\beta)}$.

Proof. (i) Let $(x, \{u_i\}, \{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x, \{u_i\}, \{u_j\}) \in [f_A]^{-1}_{(\alpha^-,\beta)}$
Therefore $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta)}$. Similarity, for $(x, \{u_i\}, \{u_j\}) \in [f_A]^{-1}_{(\alpha^-,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow \left[\mu^+_{[f_A]^{-1}_x}(u_i) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) \geq \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and $\left[\mu^+_{[f_A]^{-1}_x}(u_j) \leq \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) \geq \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A$
$\Rightarrow (x, \{u_i\}, \{u_j\}) \in [f_A]^{-1}_{(\alpha^-,\beta^-)}$
Therefore $[f_A]^{-1}_{(\alpha^-,\beta)} \widetilde{\subseteq} [f_A]^{-1}_{(\alpha^-,\beta^-)}$. It is proved similarly in the other part.
(ii) Straighforward.
(iii) It is clear from Definition 11 and Definition 29.
(iv) Let $(x, \{u_i\}, \{u_j\}) \in [f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)}$
$\Rightarrow \left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i)\right|\right]$
and $\left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \left|\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A \cup B$
$\Rightarrow \Big[\left[\mu^+_{[f_A]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \left|\mu^-_{[f_A]^{-1}_x}(u_i)\right|\right]$ and

$\left[\mu^+_{[f_A]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[f_A]^{-1}_x}(u_j) < \left|\mu^-_{[f_A]^{-1}_x}(u_j)\right|\right]$, $\forall x \in A\Big]$

or $\Big[\left[\mu^+_{[g_B]^{-1}_x}(u_i) < \alpha\right] \wedge \left[\mu^-_{[g_B]^{-1}_x}(u_i) > \beta\right] \wedge \left[\mu^+_{[g_B]^{-1}_x}(u_i) \geq \left|\mu^-_{[g_B]^{-1}_x}(u_i)\right|\right]$ and

$\left[\mu^+_{[g_B]^{-1}_x}(u_j) < \alpha\right] \wedge \left[\mu^-_{[g_B]^{-1}_x}(u_j) > \beta\right] \wedge \left[\mu^+_{[g_B]^{-1}_x}(u_j) < \left|\mu^-_{[g_B]^{-1}_x}(u_j)\right|\right]$, $\forall x \in B\Big]$
$\Rightarrow (x, \{u_i\}, \{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)}$ or $(x, \{u_i\}, \{u_j\}) \in [g_B]^{-1}_{(\alpha,\beta)}$
$\Rightarrow (x, \{u_i\}, \{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cup} [g_B]^{-1}_{(\alpha,\beta)}$
Therefore, $[f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subset} [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cup} [g_B]^{-1}_{(\alpha,\beta)}$.

(v) Let $(x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cap} [g_B]^{-1}_{(\alpha,\beta)}$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A]^{-1}_{(\alpha,\beta)}$ and $(x,\{u_i\},\{u_j\}) \in [g_B]^{-1}_{(\alpha,\beta)}$

$$\Rightarrow \Bigg[\Big[\mu^+_{[f_A]^{-1}_x}(u_i) < \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u_i) > \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u_i) \geq \Big|\mu^-_{[f_A]^{-1}_x}(u_i)\Big|\Big] \text{ and}$$

$$\Big[\mu^+_{[f_A]^{-1}_x}(u_j) < \alpha\Big] \wedge \Big[\mu^-_{[f_A]^{-1}_x}(u_j) > \beta\Big] \wedge \Big[\mu^+_{[f_A]^{-1}_x}(u_j) < \Big|\mu^-_{[f_A]^{-1}_x}(u_j)\Big|\Big], \forall x \in A\Bigg]$$

$$\text{and } \Bigg[\Big[\mu^+_{[g_B]^{-1}_x}(u_i) < \alpha\Big] \wedge \Big[\mu^-_{[g_B]^{-1}_x}(u_i) > \beta\Big] \wedge \Big[\mu^+_{[g_B]^{-1}_x}(u_i) \geq \Big|\mu^-_{[g_B]^{-1}_x}(u_i)\Big|\Big] \text{ and}$$

$$\Big[\mu^+_{[g_B]^{-1}_x}(u_j) < \alpha\Big] \wedge \Big[\mu^-_{[g_B]^{-1}_x}(u_j) > \beta\Big] \wedge \Big[\mu^+_{[g_B]^{-1}_x}(u_j) < \Big|\mu^-_{[g_B]^{-1}_x}(u_j)\Big|\Big], \forall x \in B\Bigg]$$

$$\Rightarrow \Big[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) < \alpha\Big] \wedge \Big[\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) > \beta\Big] \wedge \Big[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i) \geq \Big|\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_i)\Big|\Big]$$
$$\text{and } \Big[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \alpha\Big] \wedge \Big[\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) > \beta\Big] \wedge \Big[\mu^+_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j) < \Big|\mu^-_{[f_A \widehat{\cup} g_B]^{-1}_x}(u_j)\Big|\Big],$$
$\forall x \in A \cup B$
$\Rightarrow (x,\{u_i\},\{u_j\}) \in [f_A \widehat{\cup} g_B]^{-1}_{(\alpha,\beta)}$
Therefore, $[f_A]^{-1}_{(\alpha,\beta)} \widetilde{\cap} [g_B]^{-1}_{(\alpha,\beta)} \widetilde{\subseteq} [f_A \widehat{\cap} g_B]^{-1}_{(\alpha,\beta)}$. $\square$

5. Results and Conclusion

The concepts of (α,β)-cut, first type semi-strong (α,β)-cut, second type semi-strong (α,β)-cut, strong (α,β)-cut, inverse (α,β)-cut, first type semi-weak inverse (α,β)-cut, second type semi-weak inverse (α,β)-cut and weak inverse (α,β)-cut for bipolar fuzzy soft sets were introduced and their applications were highlighted. It is shown that (α,β)-cut of bipolar fuzzy soft sets can be used to determine the best choice while inverse (α,β)-cuts of bipolar fuzzy soft sets can be used to determine unfavorable alternative. Moreover, some related results were presented. I think that these concepts proposed for better management of decision-making processes for uncertainty problems may be useful in the future.

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S-δ-CONNECTEDNESS IN S-PROXIMITY SPACES

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Abstract. New types of connectedness in S-proximity spaces, named as an S-δ-connectedness, local S-δ-connectedness are introduced. Also, their interrelationships are studied. In an S-proximity space (X, δ_X), the S-δ-connectedness of a subset U of X with respect to δ_X may not be same as the S-δ-connectedness of U with respect to its subspace proximity δ_U. Further, S-δ-component and S-δ-treelike spaces are also defined and a number of results are given.

1. Introduction

In 1908, Reisz [13] discussed the idea of proximity (now it is called an E-proximity) and although this idea was revived by Wallace [17, 18]. But the real beginning of E-proximity was due to Efremovič [5, 6] who gave axioms of it as a natural generalization of metric space and topological group. Smirnov [14, 15] demonstrated that a completely regular space always has a compatible E-proximity relation and vice versa. Also, he found the relationship between E-proximity space and uniform space. Several generalizations of E-proximity were defined and studied. The notion of Čech proximity spaces was given by E. Čech [2], later elaborated in [10], [11] and [12]. An S-proximity was introduced independently by Krishna Murti [7], Szymanski [16], Wallace [17, 18].

Mrówka *et al.* [9] defined the notion of δ-connectedness in E-proximity spaces and after that in 1987, the concepts of local δ-connectedness, δ-component and δ-quasi components were introduced by Dimitrijević *et al.* [3]. Dimitrijević *et al.* [4] also studied δ-treelike proximity spaces. Recently, Modak *et al.* [8] introduced the weaker form of connectedness (Cl-Cl-connectedness) in topological spaces.

In this paper, we introduce a new type of δ-connectedness (named as S-δ-connectedness) in S-proximity spaces and show that S-δ-connectedness is different

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from δ-connectedness [9] in the category of S-proximity spaces. And it become identical in the categories of L-proximity spaces and E-proximity spaces. We give a characterization for an S-proximity space X to be S-δ-connected and several other properties analogous to δ-connectedness are justified. Relation among different types of connectedness are shown. In the last section, S-δ-component, local S-δ-connectedness and S-δ-treelike spaces are defined and their properties are studied.

Throughout this paper, $(A, B) \in \delta$ ($(A, B) \notin \delta$) denotes A, B are near (δ-separated). We write an S-proximity space as X instead of (X, δ) whenever there is no confusion of the S-proximity δ. $Cl_X(.)$ and $int_X(.)$ are used to denote closure and interior, respectively, with respect to topology $\mathcal{T}_\delta$ generated by δ in X.

2. Preliminaries

In this section, we recall some important definitions and results that will be used in subsequent sections.

Definition 1. *[10] For a nonempty set X, a Čech proximity (or basic proximity) on X is a binary relation δ on the power set of X, $\mathcal{P}(X)$, that satisfies the following axioms for all $A, B, C \in \mathcal{P}(X)$:*

(i) *If $(A, B) \in \delta$, then $(B, A) \in \delta$.*
(ii) *If $(A, B) \in \delta$, then $A \neq \phi$ and $B \neq \phi$.*
(iii) *If $A \cap B \neq \phi$, then $(A, B) \in \delta$.*
(iv) *$(A, B \cup C) \in \delta$ if and only if $(A, B) \in \delta$ or $(A, C) \in \delta$.*

The set X together with a Čech proximity δ is called a Čech proximity space (X, δ).

Definition 2. *[10] A Čech proximity space X is called separated if we have $(\{x\}, \{y\}) \in \delta$, then $x = y$ for all $x, y \in X$.*

Definition 3. *[10, 12] For $A, B, C \in \mathcal{P}(X)$, a Čech proximity δ on a set X is:*

(i) *E-proximity if $(A, B) \notin \delta$, then there is some $E \subset X$ with $(A, E) \notin \delta$ and $(X \backslash E, B) \notin \delta$.*
(ii) *L-proximity if $(A, B) \in \delta$ and $(\{b\}, C) \in \delta$ for each $b \in B$, then $(A, C) \in \delta$.*
(iii) *S-proximity if $(\{x\}, B) \in \delta$ and $(\{b\}, C) \in \delta$ for each $b \in B$, then $(x, C) \in \delta$.*

A Čech proximity space (X, δ) is called an E-proximity space (or a L-proximity space, an S-proximity space respectively) if the Čech proximity δ satisfies the E-proximity axiom (or L-proximity axiom, S-proximity axiom respectively.).

Definition 4. *[10, 12] Let (X, δ) be an S-proximity space and $\mathcal{T}$ be a topology on X. Then δ is compatible with $\mathcal{T}$ if and only if the generated topology $\mathcal{T}_\delta$ and $\mathcal{T}$ are equal.*

Definition 5. *[10] Let (X, δ) be a Čech proximity space. Then a subset V of X is said to be a δ-neighbourhood of $U \subset X$ if $(U, X \backslash V) \notin \delta$.*

Definition 6. *[10,12] Let (X,δ) and $(Y,\delta^{'})$ be two E-proximity spaces, a function $f:(X,\delta)\longrightarrow(Y,\delta^{'})$ is δ-continuous (or p-continuous) if for all $A,B\subset X$ such that $(A,B)\in\delta$, implies $(f(A),f(B))\in\delta^{'}$.*

Definition 7. *[9] Let (X,δ) be an E-proximity space. Then X is said to be δ-connected if every δ-continuous map from X to discrete proximity space is constant.*

Theorem 8. *[9] Let (X,δ) be an E-proximity space. Then the following statements are equivalent:*

(i) *X is δ-connected.*
(ii) *$(A,X\backslash A)\in\delta$ for each nonempty subset A with $A\neq X$.*
(iii) *For every δ-continuous real-valued function f, the image $f(X)$ is dense in some interval of $\mathbb{R}$.*
(iv) *If $X=A\cup B$ and $(A,B)\notin\delta$, then either $A=\phi$ or $B=\phi$.*

However, if X is not δ-connected, then by Theorem 8 (iv) we have $X=A\cup B$ with $(A,B)\notin\delta$ where $A,B\subset X$ are nonempty. Here, the pair (A,B) forms a δ-separation for X.

Definition 9. *[3] Let (X,δ) be an E-proximity space and $Y\subset X$. Then Y is δ-connected, if it is δ-connected with respect to the subspace proximity of Y.*

Definition 10. *[3] An E-proximity space X is locally δ-connected if for every point x of X and for every δ-neighborhood U of x, there exists some δ-connected δ-neighborhood V of x such that $x\in V\subset U$.*

Definition 11. *[7,10] Let (X,δ_X) and (Y,δ_Y) be S-proximity spaces. Then a map $f:X\longrightarrow Y$ is said to be S-δ-continuous if $(A,B)\notin\delta_X$ implies $(f(A),f(B))\notin\delta_Y$, for all $A,B\subset X$.*

Definition 12. *[8] Let $(X,\mathcal{T})$ be a topological space. A pair of non-empty subsets A,B of X is called $Cl-Cl$-separated (weak separated) if $Cl(A)\cap Cl(B)=\phi$. A subset U of a space X is said to be $Cl-Cl$-connected (weak connected) if U is not the union of two $Cl-Cl$-separated (weak separated) sets in X.*

Definition 13. *[4] If an E-proximity space X can be written as $X=P\cup Q$ with $(P,Q)\notin\delta$, then the pair (P,Q) is said to be a separation for X and write it as $X=P+Q$. If P contains some set A and Q contains B, then it can be written as $X=P(A)+Q(B)$.*

Definition 14. *[4] Let X be an E-proximity space. Then it is called δ-treelike if it is δ-connected, and for each pair (x,y) of distinct points in X there is a δ-connected set V such that $X\backslash V=P(x)+Q(y)$.*

3. S-δ-connectedness

In this section, we define S-δ-connectedness in S-proximity spaces and give characterizations of it.

Recall that every discrete proximity is an S-proximity and induces the discrete topology.

Definition 15. *An S-proximity space X is said to be S-δ-connected if every S-δ-continuous map from X to discrete space is constant.*

Next, we give a characterization for an S-proximity space to be S-δ-connected.

Theorem 16. *For an S-proximity space X, the following statements are equivalent:*

(i) *X is S-δ-connected.*

(ii) *$(Cl_X(A), X\backslash A) \in \delta$ for all nonempty proper subset A of X.*

(iii) *If $X = P \cup Q$ with $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, then either $P = \phi$ or $Q = \phi$.*

Proof. $(i) \Rightarrow (ii)$. If $(Cl_X(A), X\backslash A) \notin \delta$ for some nonempty proper subset A of X, then the map $f : X \longrightarrow \{0, 1\}$ defined as $f(A) = \{0\}$ and $f(X\backslash A) = \{1\}$ is non-constant, S-δ-continuous map. Therefore, X is not S-δ-connected.

$(ii) \Rightarrow (iii)$. If $X = P \cup Q$, where P, Q are nonempty subsets such that $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, then $Q = X\backslash P$. Thus, $(Cl_X(P), X\backslash P) \notin \delta$, a contradiction.

$(iii) \Rightarrow (i)$. If X is not S-δ-connected, then the map $f : (X, \delta) \longrightarrow \{0, 1\}$ defined as $f(P) = \{0\}$ and $f(Q) = \{1\}$ is non-constant, surjective, S-δ-continuous map. Therefore, $X = P \cup Q$, where P, Q are nonempty subsets such that $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, a contradiction. $\square$

Definition 17. *Let X be an S-proximity space. A pair (P, Q) of two nonempty subsets of X is said to be S-δ-separated in X if $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$.*

Every S-δ-separated sets are always δ-separated. However, converse need not be true.

Example 18. *Let $X = \mathbb{R}$ be the real line. For $P, Q \subset X$, define a binary relation δ on $\mathcal{P}(X)$ as:*

$$(P, Q) \in \delta \text{ if and only if } (\bar{P} \cap Q) \cup (P \cap \bar{Q}) \neq \phi$$

Here $\bar{P}$ and $\bar{Q}$ denote the closure of P and Q in X, respectively. Then δ is a compatible S-proximity on X which is not an L-proximity. The pair $P = (1, 2)$ and $Q = (2, 3)$ is δ-separated, but not S-δ-separated in X.

Definition 19. *Let (X, δ_X) be an S-proximity space and $U \subset X$. Then U is said to be S-δ-connected in X (that is, with respect to δ_X) if it cannot be written as the union of a pair of two S-δ-separated sets in X. If U is not S-δ-connected, then it is called S-δ-disconnected and the pair of two S-δ-separated sets is called S-δ-separation for U in X.*

By an S-δ-connected subset U of an S-proximity space (X, δ_X), we mean it is an S-δ-connected with respect to δ_X (that is, with respect to the proximity of X not subspace proximity of U).

Since every S-δ-separation for a set always forms δ-separation, therefore every δ-connected set is S-δ-connected. But converse need not be true.

Example 20. *Let $X = \mathbb{R}$ be the real line and δ be a S-proximity on X defined as in Example 18. Let $U = (1, 2) \cup (2, 3)$. Then U is S-δ-connected, but not δ-connected subset of X.*

Thus, S-δ-connectedness is different from δ-connectedness in general. Next, we know that δ-connectedness [9] of a subset U in E-proximity space (X, δ_X) is same as the δ-connectedness of U with respect to subspace proximity δ_U. But, it is not true for the case of an S-δ-connectedness. In Example 20, note that U is S-δ-connected with respect to δ_X, and with respect to δ_U, it is not S-δ-connected as $Cl_U((0, 1)) = (0, 1)$ and $Cl_U((1, 2)) = (1, 2)$ with respect to δ_U. But, if U is S-δ-connected with respect to δ_U, then it is also S-δ-connected with respect to δ_X.

Remark 21. *The notions of δ-connectedness and S-δ-connectedness are equivalent in the category of L-proximity spaces, as for every L-proximity space X, we have $(P, Q) \in \delta$ if and only if $(Cl_X(P), Cl_X(Q) \in \delta$ for all non-empty P, Q in X.*

Since every E-proximity is an L-proximity, so above remark holds for E-proximity spaces.

Recall that if for all $A, B \subset X$, $(A, B) \in \delta_1$ implies $(A, B) \in \delta_2$, then $\delta_1 > \delta_2$.

Corollary 22. *Let δ_1, δ_2 be two S-proximities on X such that $\delta_1 > \delta_2$. If X is S-δ_1-connected, then so is S-δ_2-connected.*

Theorem 23. *Let X be an S-proximity space. Suppose M is an S-δ-connected subset of X and (P, Q) be a pair of S-δ-separated sets in X such that $M \subset P \cup Q$. Then either $M \subset P$ or $M \subset Q$.*

Proof. If possible, $M \nsubseteq P$ and $M \nsubseteq Q$. M is S-δ-connected set such that $M \subset P \cup Q$. Therefore, $M = (M \cap P) \cup (M \cap Q)$. Also by hypothesis $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. If $(Cl_X(P), Q) \notin \delta$, then $(Cl_X(M \cap P), M \cap Q) \notin \delta$. On the other hand, if $(P, Cl_X(Q)) \notin \delta$, then $(Cl_X(M \cap Q), M \cap P) \notin \delta$. Thus, the pair $M \cap P$ and $M \cap Q$ forms an S-δ-separation for X. □

Theorem 24. *Let M, N are two S-δ-connected subsets of an S-proximity space X. If (M, N) is not S-δ-separated, then $M \cup N$ is S-δ-connected in X.*

Proof. Suppose (P, Q) be an S-δ-separation for $M \cup N$. Therefore, $M \cup N = P \cup Q$ where $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Since M and N are S-δ-connected. Thus, by Theorem 23, two case arises:

Case (i). If $M \subset P$ and $N \subset Q$, then $(Cl_X(M), N) \notin \delta$ or $(M, Cl_X(N)) \notin \delta$, because $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Hence, (M, N) is S-δ-separated which is a contradiction.

Case (ii). If $M \subset Q$ and $N \subset P$, then $(Cl_X(N), M) \notin \delta$ or $(N, Cl_X(M)) \notin \delta$, because $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Hence, (M, N) is S-δ-separated which is a contradiction. □

Theorem 25. *Let $\{W_j : j \in J\}$ be a nonempty family of S-δ-connected subsets of an S-proximity space X. If there exists some $j_0 \in J$ such that $(W_{j_0}, W_j) \in \delta$ for all $j \in J$, then $\bigcup_{j\in J} W_j$ is also S-δ-connected in X.*

Proof. If possible, there exists an S-δ-separation (P, Q) such that $\bigcup_{j\in J} W_j = P \cup Q$ with $(Cl(P), Q) \notin \delta$ or $(P, Cl(Q)) \notin \delta$. Therefore, $W_{j_0} \subset P \cup Q$ which implies either $W_{j_0} \subset P$ or $W_{j_0} \subset Q$. If $W_{j_0} \subset P$, then $W_j \subset P$ for all $j \in J$ because $(W_{j_0}, W_j) \in \delta$ for all $j \in J$. Thus $\bigcup_{j\in J} W_j \subset P$ so $Q = \phi$. Similarly, if $W_{j_0} \subset Q$, then $P = \phi$. Thus, $\bigcup_{j\in J} W_j$ is S-δ-connected. □

Corollary 26. *If $\{W_j : j \in J\}$ is a nonempty family of S-δ-connected subsets of an S-proximity space X and $\bigcap_{j\in J} W_j \neq \phi$, then $\bigcup_{j\in J} W_j$ is also S-δ-connected in X.*

Proof. Since $\bigcap_{j\in J} W_j \neq \phi$, therefore $(W_i, W_j) \in \delta$ for all $i, j \in J$. So for some fix $j_0 \in J$, $(W_{j_0}, W_j) \in \delta$ for all $j \in J$. Thus, by Theorem 25, $\bigcup_{j\in J} W_j$ is S-δ-connected in X. □

Corollary 27. *If Y is an S-δ-connected subset of an S-proximity space X, then every subset Z such that $Y \subset Z \subset Cl_X(Y)$ is also S-δ-connected in X.*

Proof. Note that $\{Y \cup \{z\} : z \in Z\}$ is a family of S-δ-connected sets such that Y is near to each of the set. Therefore, by Theorem 25, Z is S-δ-connected. □

Corollary 28. *If an S-proximity space X contains some S-δ-connected dense subset, then X is S-δ-connected.*

Proof. Let Y be an S-δ-connected dense subset of X. Then, $Cl_X(Y) = X$. Therefore, by Corollary 27, X is S-δ-connected. □

Lemma 29. *Let X be an S-proximity space and $\{M_i : i \in I\}$ be a nonempty family of S-δ-connected subsets of X. If M is S-δ-connected in X such that $M \cap M_i \neq \phi$ for all $i \in I$, then $M \cup (\bigcup_{i\in I} M_i)$ is also S-δ-connected in X.*

Proof. By Theorem 25, $(M, M_i) \in \delta$ for all $i \in I$. Hence the proof follows. □

Corollary 30. *In an S-proximity space X, if any two points can be joined by an S-δ-connected subset of X, then X is S-δ-connected.*

Proof. Fix a point x_0 in X and let M_x be an S-δ-connected subset of X which joins x_0 and x. By Lemma 29, $M = \{x_0\}$ and $M \cap M_x \neq \phi$ for all $x \in X$. Thus, $M \cup (\bigcup_{x\in X} M_x) = X$ is S-δ-connected. □

Theorem 31. *The S-δ-continuous image of S-δ-connected space is S-δ-connected.*

Proof. Let $f : (X, \delta) \longrightarrow (Y, \delta')$ be S-δ-continuous, surjective map and X is S-δ-connected space. It is to show that Y is also an S-δ-connected space. On contrary, suppose Y is not S-δ-connected space. So, there exists a pair (P, Q) in Y such that $Y = P \cup Q$ with $(Cl_Y(P), Q) \notin \delta'$ or $(P, Cl_Y(Q)) \notin \delta'$. If $(Cl_Y(P), Q) \notin \delta'$, then $(f^{-1}(Cl_Y(P)), f^{-1}(Q)) \notin \delta$. Since S-δ-continuity of f implies continuity with respect to $\mathcal{T}_\delta$, so $Cl_X(f^{-1}(P)) \subset f^{-1}(Cl_Y(P))$. Thus, $(Cl_X(f^{-1}(P)), f^{-1}(Q)) \notin \delta$. Hence, $(f^{-1}(P), f^{-1}(Q))$ forms an S-δ-separation for X, a contradiction. A similar case for $(P, Cl_Y(Q)) \notin \delta'$. □

As every S-δ-continuous map is continuous, so every weak connected [8] space is S-δ-connected.

Example 32. *The set of rationals $\mathbb{Q}$ is an S-δ-connected in $\mathbb{R}$, but is not weak connected.*

However, compact Hausdorff S-δ-connected space is weak connected as every continuous map with compact Hausdorff domain is S-δ-continuous. Thus, we have the following diagram of implications.

$$\begin{array}{ccc} \delta-\text{connected} & \Longleftarrow & \text{connected} \\ \Downarrow & & \Downarrow \\ S-\delta-\text{connected} & \Longleftarrow & \text{weak connected} \end{array}$$

Following example concludes that a locally δ-connected space may not be an S-δ-connected.

Example 33. *Let $\mathbb{R}$ be the real line and δ be a compatible S-proximity defined as in Example 18. Let $X = (-1, 0) \cup (2, 3)$. Then the pair (P, Q) where $P = (-1, 0)$ and $Q = (2, 3)$, is S-δ-separation for X. Therefore X is not S-δ-connected in $\mathbb{R}$, but it is locally δ-connected.*

An S-δ-connected space may not be locally δ-connected.

Example 34. *The closed Topologist's Sine curve $T = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$ with subspace E-proximity induced by $\mathbb{R}^2$ is S-δ-connected in $\mathbb{R}^2$, but not locally δ-connected.*

Theorem 35. *Suppose $\{(X_i, \delta_i) : i \in I\}$ be a nonempty family of S-proximity spaces. Then the product $(X, \delta) = \prod\{(X_i, \delta_i) : i \in I\}$ is S-δ-connected if and only if X_i is S-δ-connected for each $i \in I$.*

Proof. Let $\prod_{i \in I} X_i$ be S-δ-connected. Since S-δ-continuous image of S-δ-connected set is S-δ-connected, therefore X_i is S-δ-connected for each $i \in I$ as projections are S-δ-continuous, surjective maps.

Conversely, assume that each X_i is S-δ-connected. Firstly, take $I = \{1, 2\}$. Then in $X_1 \times X_2$, any two distinct points (x_1, x_2) and (y_1, y_2) can be connected by the S-δ-connected set $(X_1 \times \{x_2\}) \cup (\{y_1\} \times X_2)$. Therefore, $X_1 \times X_2$ is S-δ-connected. Using induction, it can be shown that any finite product of S-δ-connected set is S-δ-connected. Now, for an arbitrary product, choose $x_i \in X_i$ for all $i \in I$. Consider a family $\mathcal{F}$ consisting of all finite subsets of the set I and put $K_F = \prod_{i \in I} L_i$ for all $F \in \mathcal{F}$ with $L_i = X_i$ if $i \in F$ and $L_i = \{x_i\}$ if $i \notin F$. Then, $\{K_F : F \in \mathcal{F}\}$ is a family of S-δ-connected sets by induction hypothesis. Therefore, $K = \bigcup_{F\in\mathcal{F}} K_F$ is S-δ-connected as $\bigcap_{F\in\mathcal{F}} K_F \neq \phi$. Since K is dense in $\prod_{i\in I} X_i$, therefore by Corollary 28, $\prod_{i\in I} X_i$ is S-δ-connected. $\square$

Definition 36. *For given S-proximity spaces (X, δ) and $(Y, \delta^{'})$, S-δ-continuous map $f : X \longrightarrow Y$ is said to be S-δ-monotone if for every $y \in Y$, the pullback $f^{-1}(y)$ is S-δ-connected in X.*

Definition 37. *A map $f : (X, \delta) \longrightarrow (Y, \delta^{'})$ is said to be δ_b-map if for every pair of subsets A, B of Y, the following two axioms hold:*

(i) *If $(Cl_X f^{-1}(A), f^{-1}(B)) \notin \delta$, then $(Cl_Y(A), B) \notin \delta^{'}$.*
(ii) *If $(f^{-1}(A), Cl_X f^{-1}(B)) \notin \delta$, then $(A, Cl_Y(B)) \notin \delta^{'}$.*

Following theorem shows that if a map is S-δ-monotone, surjective, δ_b-map, then inverse image of S-δ-connected set is S-δ-connected.

Theorem 38. *Let $f : (X, \delta) \longrightarrow (Y, \delta^{'})$ be a δ_b-map, S-δ-monotone, surjective map. Then for each S-δ-connected subset U of Y, $f^{-1}(U)$ is S-δ-connected in X.*

Proof. Let $f^{-1}(U)$ be not S-δ-connected. Then, $f^{-1}(U) = P \cup Q$ with $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. As f is S-δ-monotone, so for each $y \in U$, $f^{-1}(y)$ is S-δ-connected. Thus, $f^{-1}(y) \subset P$ or $f^{-1}(y) \subset Q$ for all $y \in U$. Now, let us define $A = \{y \in U : f^{-1}(y) \subset P\}$ and $B = \{y \in U : f^{-1}(y) \subset Q\}$. Note that $P = f^{-1}(A)$, $Q = f^{-1}(B)$ and $U = A \cup B$. Since f is δ_b- map with $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, therefore (A, B) forms an S-δ-separation for U. $\square$

Definition 39. *In an S-proximity space X, a finite sequence $U_1, U_2, \cdots, U_n$ of subsets of X is called an S-δ-chain if $(Cl_X(U_i), U_{i+1}) \in \delta$ and $(U_i, Cl_X(U_{i+1})) \in \delta$ for all $i = 1, 2, \cdots, n-1$.*

A family $\mathcal{F}$ of subsets of X is said to be S-δ-chained in X if for every pair (U, V) of elements of $\mathcal{F}$, there is an S-δ-chain in $\mathcal{F}$ joining U and V.

Theorem 40. *Suppose $\{U_i\}_{i=1}^n$ be a finite family of S-δ-connected subsets of an S-proximity space X and forms an S-δ-chain, then $\bigcup_{i=1}^n U_i$ is S-δ-connected in X.*

Proof. The Proof follows by induction on n as it holds for $n = 2$ by Theorem 24. $\square$

Theorem 41. *For an S-δ-chained family $\mathcal{F} = \{U_i : i \in I\}$ in X, if each member U_i is S-δ-connected in X, then $U = \bigcup_{i\in I} U_i$ is also S-δ-connected in X.*

Proof. Let $x, y \in U$ be arbitrary. So, there is some $i, j \in I$ such that $x \in U_i$ and $y \in U_j$. Thus by hypothesis, there is an S-δ-chain joining U_i and U_j. Therefore, by Theorem 40, union of all the members of this S-δ-chain is S-δ-connected. Thus, x and y can be joined by an S-δ-connected set. Hence, by Corollary 30, U is S-δ-connected in X. $\square$

Definition 42. *In an S-proximity space X, a cover $\mathcal{F}$ is said to be an S-δ-cover if $(Cl_X(M), N) \in \delta$ and $(M, Cl_X(N)) \in \delta$ for $M, N \subset X$, then there is some $U \in \mathcal{F}$ such that $M \cap U \neq \phi$ and $N \cap U \neq \phi$.*

Theorem 43. *In an S-δ-connected space X, every S-δ-cover is an S-δ-chained family.*

Proof. Assume that $\mathcal{F} = \{U_i : i \in I\}$ be any S-δ-cover of X. Suppose there exist $i, j \in I$ such that U_i and U_j can not be joined by any S-δ-chain in $\mathcal{F}$. Now, consider P as the union of all the members of $\mathcal{F}$ which are joinable with U_i by some S-δ-chain $\mathcal{F}' \subset \mathcal{F}$, and Q as the union of rest of the elements of $\mathcal{F}$. Then note that $X = P \cup Q$. Now it is to show that X is not S-δ-connected, that is, $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Again on the contrary, let $(Cl_X(P), Q) \in \delta$ and $(P, Cl_X(Q)) \in \delta$. Then there exists $U \in \mathcal{F}$ such that $U \cap P \neq \phi$ and $U \cap Q \neq \phi$. Thus, there is some $U_m \subset P$ and $U_n \subset Q$ such that $U \cap U_m \neq \phi$ and $U \cap U_n \neq \phi$. So, U_n can be joined with U_i using an S-δ-chain $\mathcal{F}'' \subset \mathcal{F}$, which is absurd. $\square$

Theorem 44. *Let X be an S-δ-connected, separated S-proximity space. If for some $x \in X$, $X \setminus \{x\} = P \cup Q$ where (P, Q) is S-δ-separated in X, then $(\{x\}, Cl_X(P)) \in \delta$ and $(\{y\}, Cl_X(Q)) \in \delta$.*

Proof. If $(\{x\}, Cl_X(P)) \notin \delta$, then $(\{x\}, P) \notin \delta$. Since pair (P, Q) is S-δ-separated in X and X is separated, therefore it is easy to conclude that X is not S-δ-connected, a contradiction. Similarly, conclude that $(\{y\}, Cl_X(Q)) \in \delta$. $\square$

4. Local S-δ-connectedness

In this section, local S-δ-connectedness is defined and it's several properties are studied.

Definition 45. *The S-δ-component of a subset U in an S-proximity space X is defined as the union of all S-δ-connected subsets of X containing U and it is denoted by $C^*_\delta(U)$.*

Every δ-component is contained in some S-δ-component. Any S-δ-component being union of S-δ-connected sets with nonempty intersection is S-δ-connected. An S-δ-component being a maximal S-δ-connected set is $\mathcal{T}_\delta$-closed.

Analogously, the S-δ-component of a point x can be defined as the union of all S-δ-connected subsets of X containing x. Note that S-δ-components of any two distinct points of X are either same or δ-far sets in X.

In the next theorem, we show that the S-δ-component of product S-proximity is exactly the product of S-δ-components of each S-proximity.

Theorem 46. *Suppose $\{(X_i, \delta_i) : i \in I\}$ be a nonempty family of S-proximity spaces. Then the S-δ-component of the product $(X, \delta) = \prod\{(X_i, \delta_i) : i \in I\}$ coincides with the product $\prod\{C^*_{\delta_i}(x_i) : i \in I\}$ of each S-δ-component of the point $x_i \in X_i$.*

Proof. Let $C^*_\delta(x)$ be the S-δ-component of x in X and for each $i \in I$, $C^*_{\delta_i}(x_i)$ be the S-δ-component of x_i in X_i. Then, $\prod\{C^*_{\delta_i}(x_i) : i \in I\}$ being the product of the S-δ-connected sets is S-δ-connected. Therefore it is contained in $C^*_\delta(x)$. Conversely, for each $i \in I$, $p_i C^*_\delta(x)$ being S-δ-continuous image of S-δ-connected set is S-δ-connected. Therefore, $p_i C^*_\delta(x) \subset C^*_{\delta_i}(x_i)$ for each $i \in I$. Hence, $C^*_\delta(x) \subset \prod\{p_i C^*_\delta(x) : i \in I\} \subset \prod\{C^*_{\delta_i}(x_i) : i \in I\}$. □

Next, we show that S-δ-component is preserved under an S-δ-monotone, surjective, δ_b-map

Theorem 47. *Suppose $f : (X, \delta) \longrightarrow (Y, \delta')$ be S-δ-monotone, surjective and δ_b- map. Then C^* is an S-δ-component of $W \subset Y$ if and only if $f^{-1}(C^*)$ is an S-δ-component of $f^{-1}(W)$.*

Proof. Assume that C^* is S-δ-component of subspace $W \subset Y$. Obviously, $f^{-1}(C^*)$ is S-δ-connected by Theorem 38. Now, suppose there is some S-δ-connected set M in $f^{-1}(W)$ such that $f^{-1}(C^*) \subset M \subset f^{-1}(W)$. Since the map f is surjective, therefore $C^* \subset f(M) \subset W$. As f is S-δ-continuous being S-δ-monotone, so $f(M)$ is S-δ-connected. Thus, $f(M) = C^*$ which implies $f^{-1}(C^*) = M$.

Conversely, let $f^{-1}(C^*)$ be an S-δ-component of $f^{-1}(W)$. Therfore, $f^{-1}(C^*)$ is S-δ-connected subset of $f^{-1}(W)$ and f is S-δ-continuous being S-δ-monotone. Thus, C^* is S-δ-connected subset of W. Now, suppose that N be an S-δ-connected set such that $C^* \subset N \subset W$. Then, $f^{-1}(C^*) \subset f^{-1}(N) \subset f^{-1}(W)$ and $f^{-1}(N)$ is S-δ-connected by Theorem 38. Hence, by hypothesis, $f^{-1}(C^*) = f^{-1}(N)$ which implies $C^* = N$. □

Definition 48. *Let X be an S-proximity space. Then X is locally S-δ-connected at $x \in X$, if every δ-neighbourhood of x contains some S-δ-connected δ-neighbourhood of x. We call X is locally S-δ-connected if it is locally S-δ-connected for all $x \in X$. Further, a subset $Y \subset X$ is locally S-δ-connected if Y is locally S-δ-connected in the subspace S-proximity of X.*

Now, we show that locally S-δ-connectedness and S-δ-connectedness are two independent concepts.

Example 49. (a). *Let X be any discrete proximity space with $|X| \geq 2$. Then X is locally S-δ-connected, but it is not S-δ-connected.*

(b). *Suppose X be an S-proximity space defined as in Example* 33*. Then X is locally S-δ-connected, but not S-δ-connected.*

Example 50. *The closed Topologist's sine curve $T = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$ with subspace E-proximity induced by $\mathbb{R}^2$ is S-δ-connected, but not locally S-δ-connected.*

Example 51. *The subspace $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$ with S-proximity defined as in Example* 18*. Then X is not locally S-δ-connected.*

Theorem 52. *Suppose $x \in P \cap Q$, where P and Q are locally S-δ-connected sets at x. Then $P \cup Q$ is also locally S-δ-connected at x.*

Proof. Let W be a δ-neighbourhood of the point x. Then, $W_P = W \cap P$ and $W_Q = W \cap Q$ are δ-neighbourhoods of the point x in P and Q respectively. Using hypothesis, there exist some S-δ-connected δ-neighbourhoods M_P and M_Q of x such that $M_P \subset W_P$ and $M_Q \subset W_Q$. Then, $x \in M_P \cup M_Q \subset W_P \cup W_Q$ such that $M_P \cup M_Q$ is S-δ-connected set. Also, $(\{x\}, (P\backslash M_P) \cup (Q\backslash M_Q)) \notin \delta$ which implies $(\{x\}, (P \cup Q)\backslash(M_P \cup M_Q) \notin \delta$. Therefore, $M_P \cup M_Q$ is a δ-neighbourhood of x. □

Theorem 53. *If an S-proximity space X is locally S-δ-connected, then S-δ-component of every $\mathcal{T}_\delta$-open subspace of X is $\mathcal{T}_\delta$-open.*

Proof. Assme that X is locally S-δ-connected and W be $\mathcal{T}_\delta$-open subspace in X. Let C^* be an S-δ-component of W. If $y \in C^*$, then $(\{y\}, X\backslash W) \notin \delta$. Therefore W is a δ-neighbourhood of y. Since X is locally S-δ-connected, then there exists an S-δ-connected δ-neighbourhood M of y such that $y \in M \subset W$. But C^* is maximal S-δ-connected set containing y, so $y \in M \subset C^*$. Therefore, C^* is $\mathcal{T}_\delta$-open. □

Corollary 54. *If X is locally S-δ-connected space, then S-δ-components of X are clopen sets in the induced topology $\mathcal{T}_\delta$.*

Corollary 55. *If an S-proximity space X is locally S-δ-connected and compact, then it has at most finite number of S-δ-components.*

Definition 56. *Let U be a subset of an S-proximity space X. Then it is called an S-δ-treelike in X if it is S-δ-connected and for each pair of points $x, y \in U$ there exists an S-δ-connected set $V \subset U$ in X such that $U\backslash V = P \cup Q$ where $x \in P$, $y \in Q$ and the pair (P, Q) is S-δ-separated in X.*

Example 20 shows that there exists an S-δ-treelike S-proximity space which is not δ-treelike [4], and from Example 32 we conclude that there exists an S-δ-treelike S-proximity space which is not treelike [1] (Topologically).

Theorem 57. *If an S-proximity space X is S-δ-treelike, then it is separated.*

Proof. Suppose X is not separated. So, there exist two distinct points x, y in X such that $(\{x\}, \{y\}) \in \delta$. Thus, $\{x, y\}$ is S-δ-connected in X. Since X is an S-δ-treelike space, therefore there exists an S-δ-connected set U in X such that $X\backslash U = P \cup Q$ where $x \in P$, $y \in Q$ and the pair (P, Q) is S-δ-separated in X. Then the pair $P \cap \{x, y\}$ and $Q \cap \{x, y\}$ forms an S-δ-separation for $\{x, y\}$, a contradiction. □

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THE EXISTENCE OF THE BOUNDED SOLUTIONS OF A SECOND ORDER NONHOMOGENEOUS NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper, we consider a second order nonlinear differential equation and establish two new theorems about the existence of the bounded solutions of a second order nonlinear differential equation. In these theorems, we use different Lyapunov functions with different conditions but we get the same result. In addition, two examples are given to support our results with some figures.

1. Introduction

For more than sixty years, a great deal of work has been done by various authors to investigate the autonomous and non-autonomous second order nonlinear ordinary differential equations (ODEs) ([1]- [5], [7]- [14], [16], [17], [19]) and references cited therein.

In investigating the qualitative properties of solutions for second order ODEs, the fixed point method, perturbation theory, variations of parameter formulas, etc. have been used to get information without solving the equations. Moreover, in some of these works, the authors have been studied the Lyapunov direct or second method by constructing different Lyapunov functions or using existing Lyapunov functions.

As far as we know, it should be noted in the relevant literature that so far, the second method of Lyapunov is the most effective tool for studying qualitative

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features of nonlinear higher order equations without getting solutions of the equations. This method needs the creation of an appropriate function or functionality that gives concrete results for the problem being studied.

In 1995, Meng [6] dealt with the ordinary linear differential equation of second order

$$x''(t) + p(t)x'(t) + [q_1(t) + q_2(t)]x(t) = f(t),$$

and in 2002, Yuangong and Fanwei [18] considered the second order time lag nonlinear differential equation

$$(r(t)x'(t))' + p(t)x'(t) + [q_1(t) + q_2(t)]x(t) = f(t, x(t)).$$

The authors got some interesting results on the boundedness and square integrability of solutions of the ODEs.

In 2019, Tunç and Mohammed [15] considered two different models for nonlinear of second order

$$x''(t) + p(t)g(x') + q_1(t)h(x) + q_2(t)x = f(t, x, x')$$

and

$$x''(t) + \Phi(t, x, x') + q_1(t)x + q_2(t)\theta(x) = q(t, x, x').$$

They investigate asymptotic boundedness of solutions of the ODEs as $t \to \infty$.

In this paper, motivated by the work of Tunc and Mohammed [15], we deal with the following second order nonlinear differential equation:

$$x'' + f(t, x, x') + q_1(t)\varphi(x) + q_2(t)\psi(x) = g(t, x, x'), \tag{1}$$

where $x \in \mathbb{R} = (-\infty, \infty)$, $t \in \mathbb{R}^+ = [0, \infty)$. $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $q_1, q_2 \in C^1(\mathbb{R}^+, \mathbb{R})$, $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$ and $f(t, x, 0) = 0$, $\varphi(0) = 0$, $\psi(0) = 0$. Under the assumptions, the existence of the solutions of Eq. (1) is guaranteed. In addition, we assume that the functions f, φ, ψ and g fulfill the Lipschitz condition with respect to x and its derivative x'. So, the solutions of Eq. (1) are uniqueness.

Eq. (1) can be written as

$$\begin{aligned} x' &= y \\ y' &= -f(t, x, x') - q_1(t)\varphi(x) - q_2(t)\psi(x) + g(t, x, x'). \end{aligned} \tag{2}$$

Let

$$\varphi^*(x) = \begin{cases} x^{-1}\varphi(x), & x \neq 0 \\ \varphi'(0), & x = 0, \end{cases}$$

$$\psi^*(x) = \begin{cases} x^{-1}\psi(x), & x \neq 0 \\ \psi'(0), & x = 0 \end{cases}$$

and

$$f^*(t, x, y) = \begin{cases} y^{-1}f(t, x, y), & y \neq 0 \\ f'_y(t, x, 0), & y = 0. \end{cases}$$

2. Main Results

The following assumptions are needed to formulate our main results.

(A1) $f(t,x,0)=0,\ y^{-1}f(t,x,y)\geq f_0\geq 1$ for all $t\in\mathbb{R}^+,\ x\in\mathbb{R},\ y\in\mathbb{R}-\{0\}$.
(A2) $\varphi(0)=0,\ x^{-1}\varphi(x)\geq\varphi_0\geq 1$ for all $x\in\mathbb{R}-\{0\}$.
(A3) $\psi(0)=0,\ x^{-2}\psi^2(x)\leq 1$ for all $x\in\mathbb{R}-\{0\}$.
(A4) $\psi(0)=0,\ x^{-1}\psi(x)\geq\psi_0\geq 1$ for all $x\in\mathbb{R}-\{0\}$.
(A5) $q_1(t)>0,\ q_2(t)>0,\ q_1'(t)>0, \forall t\in\mathbb{R}^+$.
(A6) The functions $g_1(t),\Theta(t),h(t)$ are continuous such that

$$|g(t,x,y)|\leq|g_1(t)|, \forall t\in\mathbb{R}^+,\ \forall x,\ y\in\mathbb{R}, \tag{3}$$

$$\Theta(t)=\frac{1}{2}(q_1'(t)+2q_1(t)),\ \forall t\in\mathbb{R}^+,$$

$$\int_a^\infty\frac{q_2^2(s)}{h^2(s)\Theta(s)}ds<\infty,\ \int_a^\infty\frac{g_1^2(s)}{\Theta(s)}ds<\infty,$$

$$h^2(t)\geq 1,\ \forall t\in\mathbb{R}^+.$$

Theorem 1. *If the conditions $(A1),(A2),(A3),(A5)$ and $(A6)$ hold, any solution of Eq. (1) satisfies*

$$|x(t)|\leq O(1),\ \left|\frac{dx}{dt}\right|\leq O(\sqrt{q_1(t)}),\ t\to\infty.$$

Proof. We establish the following Lyapunov function because we use the Lyapunov second method

$$V(x,y)=2\int_0^x\varphi(\zeta)d\zeta+\frac{1}{q_1(t)}y^2. \tag{4}$$

From $(A1)$, $(A2)$, $(A5)$ and $(A6)$, we get $V(x,y)=0$ if and only if $x=0$ and $y=0$. From $(A2)$ and $q_1(t)>0$, we have

$$V(x,y)\geq x^2+\frac{1}{q_1(t)}y^2\geq 0.$$

Differentiating the Lyapunov function V in (4) along the solutions of the system (2) and using $(A1)$, we obtain

$$\begin{aligned}\frac{d}{dt}V&=-\frac{q_1'(t)}{q_1^2(t)}y^2-\frac{2}{q_1(t)}yf(t,x,y)-2\frac{q_2(t)}{q_1(t)}y\psi(x)+\frac{2}{q_1(t)}yg(t,x,y)\\&\leq-\frac{q_1'(t)}{q_1^2(t)}y^2-\frac{2}{q_1(t)}y^2-2\frac{q_2(t)}{q_1(t)}y\psi(x)+\frac{2}{q_1(t)}yg(t,x,y)\\&=-\frac{2}{q_1^2(t)}\left[\frac{1}{2}q_1'(t)+q_1(t)\right]y^2-2\frac{q_2(t)}{q_1(t)}y\psi(x)+\frac{2}{q_1(t)}yg(t,x,y).\end{aligned}$$

Since

$$\Theta(t)=\frac{1}{2}(q_1'(t)+2q_1(t)), \tag{5}$$

we have

$$\frac{d}{dt}V \le -\frac{2\Theta(t)}{q_1^2(t)}y^2 - 2\frac{q_2(t)}{q_1(t)}y\psi(x) + \frac{2}{q_1(t)}yg(t,x,y).$$

We assume that $a > 0,\ b, x \in \mathbb{R}$. If we use the inequality

$$-ax^2 + bx \le -\frac{a}{2}x^2 + \frac{b^2}{2a}, \tag{6}$$

to the terms

$$-\frac{2\Theta(t)}{q_1^2(t)}y^2 + \frac{2}{q_1(t)}yg(t,x,y),$$

and from $(A5)$, $(A6)$, we get

$$\frac{d}{dt}V \le -\frac{\Theta(t)}{q_1^2(t)}y^2 - 2\frac{q_2(t)}{q_1(t)}y\psi(x) + \frac{g_1^2(t)}{\Theta(t)}. \tag{7}$$

Let

$$W(x,y) = -\frac{\Theta(t)}{q_1^2(t)}y^2 - 2\frac{q_2(t)}{q_1(t)}y\psi(x).$$

Rearranging $W(x,y)$, we have

$$W(x,y) = -\frac{\Theta(t)}{q_1^2(t)}\left[h(t)y + \frac{q_1(t)q_2(t)}{h(t)\Theta(t)}\psi(x)\right]^2 + \frac{q_2^2(t)}{h^2(t)\Theta(t)}\psi^2(x) + \frac{\Theta(t)}{q_1^2(t)}\left(h^2(t)-1\right)y^2.$$

Since the first term of $W(x,y)$ is negative, it is clear that

$$W(x,y) \le \frac{q_2^2(t)}{h^2(t)\Theta(t)}\psi^2(x) + \frac{\Theta(t)}{q_1^2(t)}(h^2(t)-1)y^2. \tag{8}$$

From (7) and (8)

$$\frac{d}{dt}V \le \frac{q_2^2(t)}{h^2(t)\Theta(t)}\psi^2(x) + \frac{\Theta(t)}{q_1^2(t)}\left(h^2(t)-1\right)y^2 + \frac{g_1^2(t)}{\Theta(t)}. \tag{9}$$

We assume that

$$\frac{q_2^2(t)}{h^2(t)\Theta(t)} = \frac{\Theta(t)}{q_1(t)}\left(h^2(t)-1\right).$$

Hence

$$h^2(t) = \frac{\Theta^2(t) + \sqrt{\Theta^4(t) + 4q_1(t)q_2^2(t)\Theta^2(t)}}{2\Theta^2(t)}.$$

So, it can be seen that $h^2(t) \ge 1$ for $t \in \mathbb{R}^+$. Thus, we obtain

$$W(x,y) \le \frac{q_2^2(t)}{h^2(t)\Theta(t)}\left[\psi^2(x) + \frac{1}{q_1(t)}y^2\right]. \tag{10}$$

From (9) and (10)

$$\frac{d}{dt}V \le \frac{q_2^2(t)}{h^2(t)\Theta(t)}\left[\psi^2(x) + \frac{1}{q_1(t)}y^2\right] + \frac{g_1^2(t)}{\Theta(t)}. \tag{11}$$

Also, from $(A3)$, we know that

$$\psi^2(x) + \frac{1}{q_1(t)} y^2 \leq x^2 + \frac{1}{q_1(t)} y^2 \leq V(t).$$

And applying the inequality to (11), we can derive

$$\frac{d}{dt} V - \frac{q_2^2(t)}{h^2(t)\Theta(t)} V \leq \frac{g_1^2(t)}{\Theta(t)}.$$

Multiplying the inequality by

$$\exp\left(-\int_{t_0}^{t} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right)$$

and integrating this inequality from t_0 to t, we get

$$V(t) \leq V(t_0) \exp\left(\int_{t_0}^{t} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right) + \int_{t_0}^{t} \left[\frac{g_1^2(s)}{\Theta(s)} \exp\left(\int_{s}^{t} \frac{q_2^2(\eta)}{h^2(\eta)\Theta(\eta)} d\eta\right)\right] ds.$$

Hence we can take

$$V(t) \leq V(t_0) \exp\left(\int_{t_0}^{\infty} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right) + \int_{t_0}^{\infty} \left[\frac{g_1^2(s)}{\Theta(s)} \exp\left(\int_{s}^{\infty} \frac{q_2^2(\eta)}{h^2(\eta)\Theta(\eta)} d\eta\right)\right] ds.$$

Because of $(A6)$, we can assume that

$$V(t_0) \exp\left(\int_{t_0}^{\infty} \frac{q_2^2(s)}{h^2(s)\Theta(s)} ds\right) + \int_{t_0}^{\infty} \left[\frac{g_1^2(s)}{\Theta(s)} \exp\left(\int_{s}^{\infty} \frac{q_3^2(\eta)}{h^2(\eta)\Theta(\eta)} d\eta\right)\right] ds = A,$$

where $A > 0,\ A \in \mathbb{R}$. So, we have

$$V(t) \leq A$$

and

$$x^2 + \frac{1}{q_1(t)} y^2 \leq V(t) \leq A.$$

Therefore, we find

$$|x(t)| \leq \sqrt{A}, \quad |y(t)| \leq \sqrt{A q_1(t)}.$$

Hence

$$|x(t)| \leq O(1), \quad |y(t)| \leq O(\sqrt{q_1(t)}), \quad t \to \infty.$$

□

The result of the following theorem is the same as the result of Theorem 1 but we use different Lyapunov function and some different conditions in Theorem 2.

Theorem 2. *If the conditions* $(A1)$, $(A2)$, $(A4)$, $(A5)$ *and* $(A6)$ *hold, any solution of Eq. (1) satisfies*

$$|x(t)| \leq O(1), \quad \left|\frac{dx}{dt}\right| \leq O(\sqrt{q_1(t)}), \quad t \to \infty.$$

Proof. We determine the Lyapunov function as follows

$$V(x,y) = 2\int_0^x \left[\varphi(\zeta) + \frac{q_2(t)}{q_1(t)}\psi(\zeta)\right] d\zeta + \frac{1}{q_1(t)}y^2. \tag{12}$$

From $(A1)$, $(A2)$, $(A4)$, $(A5)$ and $(A6)$, we get $V(x,y) = 0$ if and only if $x = 0$ and $y = 0$. From $(A2)$, $(A4)$, $q_1(t) > 0$ and $q_2(t) > 0$, we have

$$V(x,y) \geq \left(1 + \frac{q_2(t)}{q_1(t)}\right) x^2 + \frac{1}{q_1(t)}y^2 \geq 0.$$

Differentiating the Lyapunov function V in (12) along the solutions of the system (2) and using $(A1)$, we find

$$\begin{aligned}
\frac{d}{dt}V &= -\frac{2}{q_1(t)}yf(t,x,y) + \frac{2}{q_1(t)}yg(t,x,y) - \frac{q_1'(t)}{q_1^2(t)}y^2 \\
&\leq -\frac{2y^2}{q_1(t)} + \frac{2}{q_1(t)}yg(t,x,y) - \frac{q_1'(t)}{q_1^2(t)}y^2 \\
&= -\frac{2}{q_1^2(t)}\left[\frac{1}{2}q_1'(t) + q_1(t)\right] y^2 + \frac{2}{q_1(t)}yg(t,x,y).
\end{aligned}$$

Defining $\Theta(t)$ as in (5), we have

$$\frac{d}{dt}V \leq -\frac{2\Theta(t)}{q_1^2(t)}y^2 + \frac{2}{q_1(t)}yg(t,x,y).$$

Let $a > 0,\ b, x \in \mathbb{R}$. From the inequality (6) and $(A6)$, we get

$$\frac{d}{dt}V \leq -\frac{\Theta(t)}{q_1^2(t)}y^2 + \frac{g_1^2(t)}{\Theta(t)}.$$

Since the first term of the inequality is negative, we can write

$$\frac{d}{dt}V \leq \frac{g_1^2(t)}{\Theta(t)}.$$

Integrating this inequality from t_0 to t, we get

$$V(t) \leq V(t_0) + \int_{t_0}^{t} \frac{g_1^2(s)}{\Theta(s)} ds.$$

Hence we can take

$$V(t) \leq V(t_0) + \int_{t_0}^{\infty} \frac{g_1^2(s)}{\Theta(s)} ds.$$

Because of $(A6)$, we can assume that

$$V(t_0) + \int_{t_0}^{\infty} \frac{g_1^2(s)}{\Theta(s)} ds = B,\ B > 0,\ B \in \mathbb{R}.$$

So, we have

$$V(t) \leq B.$$

From $(A2)$, $(A4)$ and $(A5)$, we know that

$$x^2 + \frac{1}{q_1(t)} y^2 \le V(t) \le B.$$

Therefore, we find

$$|x(t)| \le \sqrt{B}, \quad |y(t)| \le \sqrt{Bq_1(t)}.$$

Hence

$$|x(t)| \le O(1), \ |y(t)| \le O(\sqrt{q_1(t)}), \ t \to \infty.$$

$\square$

Remark 3. *If it is taken $f(t,x,x') = p(t)g(x')$ and $\psi(x) = x$ in Eq. (1) or $\varphi(x) = x$ in Eq. (1), Theorem 1 or Theorem 2 in [15] is obtained, respectively.*

3. Examples

Example 4. *As a special case of Eq. (1), we consider the following second order nonlinear ODE*

$$x'' + 6x' + x'e^{-t-x^2} + 5e^{3t}(5 + sinx)x + 2e^{2t}(1 - e^{-x^2})x = \frac{cost}{e^{3t}(1 + 2e^{x^4})} \tag{13}$$

or

$$x' = y$$

$$y' = -6x' - x'e^{-t-x^2} - 5e^{3t}(5 + sinx)x - 2e^{2t}(1 - e^{-x^2})x + \frac{cost}{e^{3t}(1 + 2e^{x^4})}.$$

It is clear that the conditions $(A1)$, $(A2)$, $(A3)$, $(A5)$ and $(A6)$ are satisfied. So, from Theorem 1, all solutions of Eq. (13) satisfy

$$|x(t)| \le O(1), \ \left|\frac{dx}{dt}\right| \le O(\sqrt{5}e^{3t}), \ t \to \infty$$

as shown in Fig. 1 obtained by using the adaptive MATLAB solver ode45.

Example 5. *Taking $f(t,x,x') = 5x'e^t \sin^2 x$, $q_1(t) = 2e^{3t}$, $\varphi(x) = xe^{x^2}$, $q_2(t) = 5e^{4t}$, $\psi(x) = (3 + sinx)x$ and $g(t,x,x') = \dfrac{sinx'}{e^{6t}(2 + e^{x^2})}$ in Eq. (1), we get the following second order nonlinear ODE*

$$x'' + 5x'e^t \sin^2 x + 2e^{3t}xe^{x^2} + 5e^{4t}(3 + sinx)x = \frac{sinx'}{e^{6t}(2 + e^{x^2})} \tag{14}$$

or

$$x' = y$$

$$y' = -5x'e^t \sin^2 x - 2e^{3t}xe^{x^2} - 5e^{4t}(3 + sinx)x + \frac{sinx'}{e^{6t}(2 + e^{x^2})}.$$

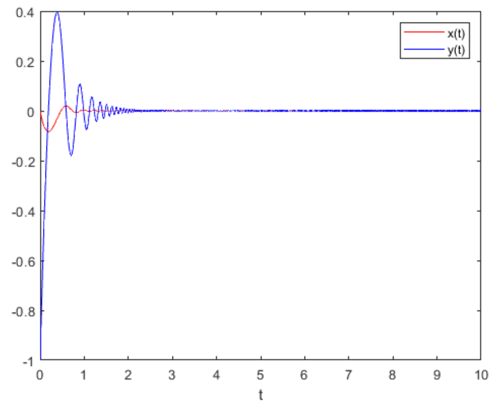

FIGURE 1. The solution of Eq. (13) with the initial conditions $x(0) = 0,\ y(0) = -1$ in $t \in [0, 10]$.

It is clear that the conditions $(A1)$, $(A2)$, $(A4)$, $(A5)$ *and* $(A6)$ *are satisfied. So, from Theorem 2, all solutions of Eq. (14) satisfy*

$$|x(t)| \leq O(1), \quad \left|\frac{dx}{dt}\right| \leq O(\sqrt{2}e^{-3t}), \quad t \to \infty$$

*as shown in Fig. 2 obtained by using the adaptive MATLAB solver ode*45*.*

4. Conclusion

We have presented a new second order nonlinear differential equation (1) to study the existence of the bounded solutions of the equation by using the Lyapunov direct or second method. Additionally, we give two examples to support our main results. Also, MATLAB has been used to draw two figures. Fig. 1 in first example shows the solution $(x(t), y(t))$ of Eq. (13) with the initial conditions $x(0) = 0,\ y(0) = -1$ in $t \in [0, 10]$. The solution is bounded since the conditions of Theorem 1 are satisfied. Fig. 2 exemplifies the solution $(x(t), y(t))$ of Eq. (14) with the initial conditions $x(0) = 1,\ y(0) = 0$ in $t \in [0, 7]$. The solution is bounded since the conditions of Theorem 2 are satisfied. Moreover, taking $f(t, x, x') = p(t)g(x')$ and $\psi(x) = x$ or $\varphi(x) = x$ in Eq. (1), Theorem 1 or Theorem 2 in [15] is gotten, respectively. So, Eq. (1) is a generalization of Eq. (6) and Eq. (7) in [15].

Declaration of Competing Interest The author has no competing interest to declare.

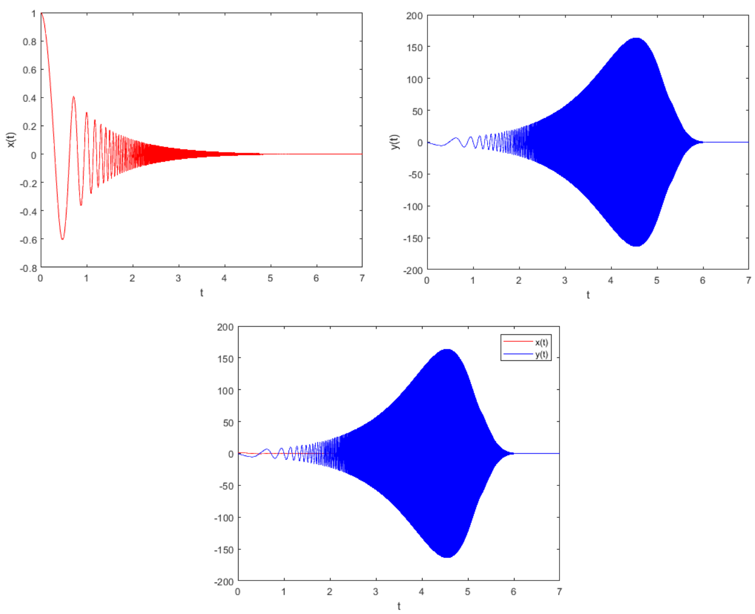

FIGURE 2. The solution of Eq. (14) with the initial conditions $x(0) = 1,\ y(0) = 0$ in $t \in [0, 7]$.

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THE GENERALIZED LUCAS HYBRINOMIALS WITH TWO VARIABLES

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Abstract. Özdemir defined the hybrid numbers as a generalization of complex, hyperbolic and dual numbers. In this research, we define the generalized Lucas hybrinomials with two variables. Also, we get the Binet formula, generating function and some properties for the generalized Lucas hybrinomials. Moreover, Catalan's, Cassini's and d'Ocagne's identities for these hybrinomials are obtained. Lastly, by the help of the matrix theory we derive the matrix representation of the generalized Lucas hybrinomials.

1. Introduction

Many researchers have studied on applications of the Fibonacci and the Lucas numbers for a long time in engineering, arts, physics and nature. These sequences have taken a huge interest of many authors.

The Fibonacci numbers are defined recursively by

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with initial values $F_0 = 0$ and $F_1 = 1$ [4].

The Lucas numbers are defined with the same recurrence relation of the Fibonacci numbers with initial values $L_0 = 2$ and $L_1 = 1$ [4].

For the variable x, Catalan defined the Lucas polynomials with the recurrence relation

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x),\ n \geq 2$$

with $L_0(x) = 2$ and $L_1(x) = x$ [4].

Bergum and Hoggatt studied on the generalized Lucas polynomials and defined these polynomials recursively by

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$$V_n(x,y) = xV_{n-1}(x,y) + yV_{n-2}(x,y),\ n \geq 2$$

with the initial conditions $V_0(x,y) = 2$ and $V_1(x,y) = x$ [1].

After that, Swamy obtained some identities and properties for the generalized Lucas sequence [7].

The first few terms of this sequence are

n	$V_n(x,y)$
0	2
1	x
2	$x^2 + 2y$
3	$x^3 + 3xy$
4	$x^4 + 4x^2y + 2y^2$
5	$x^5 + 5x^3y + 5xy^2$
6	$x^6 + 6x^4y + 9x^2y^2 + 2y^3$

For simplicity, we will use V_n instead of $V_n(x,y)$.

The characteristic equation of this sequence is

$$v^2 - xv - y = 0$$

with the roots

$$\alpha = \frac{x + \sqrt{x^2+4y}}{2},\ \beta = \frac{x - \sqrt{x^2+4y}}{2}. \tag{1}$$

Lemma 1. *[7] The roots α and β defined in (1) satisfy the following properties*

- $\alpha + \beta = x$
- $\alpha - \beta = \sqrt{x^2+4y}$
- $\alpha\beta = -y$

Lemma 2. *[7]For $n \geq 0$ the Binet formula for the generalized Lucas polynomials is*

$$V_n = \alpha^n + \beta^n.$$

The hybrid numbers were defined by Özdemir as a generalization of complex, hyperbolic and dual numbers [5]. The set of hybrid numbers is

$$K = \{a + bi + c\varepsilon + dh :\ a,b,c,d \in \mathbb{R}\}.$$

Let $Z_1 = a_1 + b_1 i + c_1\varepsilon + d_1 h$ and $Z_2 = a_2 + b_2 i + c_2\varepsilon + d_2 h$ be any two hybrid numbers. Then the main operations on hybrid numbers are defined as follows:

$$Z_1 = Z_2 \text{ if and only if } a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$$

$$Z_1 + Z_2 = (a_1 + a_2) + (b_1 + b_2)\, i + (c_1 + c_2)\,\varepsilon + (d_1 + d_2)\, h$$

$$Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)\, i + (c_1 - c_2)\,\varepsilon + (d_1 - d_2)\, h$$

$sZ_1 = sa_1 + sb_1 i + sc_1 \varepsilon + sd_1 h$, where $s \in \mathbb{R}$.

By using the following multiplication table, one can find the product of any two hybrid numbers:

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$h+1$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

By using the elements of integer sequences as coefficients of hybrid numbers, many authors defined new type of hybrid numbers ([8, 9, 3, 2, 10, 11, 12, 14]). The most exciting studies among the cited works are the Fibonacci hybrid numbers and the Lucas hybrid numbers which were defined as

$$FH_n = F_n + iF_{n+1} + \varepsilon F_{n+2} + hF_{n+3}$$

and

$$LH_n = L_n + iL_{n+1} + \varepsilon L_{n+2} + hL_{n+3}$$

respectively.

After that, for $n \geq 0$ Szynal-Liana and Włoch [13] defined the Fibonacci and the Lucas hybrinomials as

$$FH_n(x) = F_n(x) + iF_{n+1}(x) + \varepsilon F_{n+2}(x) + hF_{n+3}(x)$$

and

$$LH_n(x) = L_n(x) + iL_{n+1}(x) + \varepsilon L_{n+2}(x) + hL_{n+3}(x)$$

respectively.

2. Main Results

Definition 3. *For $n \geq 0$ the Lucas hybrinomials with two variables x and y, called the generalized Lucas hybrinomials defined by*

$$VH_n(x,y) = V_n + iV_{n+1} + \varepsilon V_{n+2} + hV_{n+3}$$

where V_n is the nth generalized Lucas polynomial.

For simplicity, we will use VH_n instead of $VH_n(x,y)$.

Theorem 4. *For the variables x and y, the generalized Lucas hybrinomials provides the recurrence relation*

$$VH_n = xVH_{n-1} + yVH_{n-2}, \; n \geq 2$$

with the initial conditions

$VH_0 = 2 + ix + \varepsilon(x^2 + 2y) + h(x^3 + 3xy)$ *and*

$VH_1 = x + i(x^2 + 2y) + \varepsilon(x^3 + 3xy) + h(x^4 + 4x^2 y + 2y^2)$.

Proof. For $n = 2$,we get

$$\begin{aligned}
VH_2 &= xVH_1 + yVH_0 \\
&= x\left(x + i(x^2+2y) + \varepsilon(x^3+3xy) + h(x^4+4x^2y+2y^2)\right) \\
&\quad + y\left(2 + ix + \varepsilon(x^2+2y) + h(x^3+3xy)\right) \\
&= x^2 + 2y + i(x^3+3xy) + \varepsilon(x^4+4x^2y+2y^2) + h(x^5+5x^3y+5xy^2) \\
&= V_2 + iV_3 + \varepsilon V_4 + hV_5.
\end{aligned}$$

For $n > 2$, using the definition of the generalized Lucas polynomials, we obtain

$$\begin{aligned}
VH_n &= V_n + iV_{n+1} + \varepsilon V_{n+2} + hV_{n+3} \\
&= (xV_{n-1} + yV_{n-2}) + i(xV_n + yV_{n-1}) \\
&\quad + \varepsilon\left(xV_{n+1} + yV_n\right) + h\left(xV_{n+2} + yV_{n+1}\right) \\
&= x\left(V_{n-1} + iV_n + \varepsilon V_{n+1} + hV_{n+2}\right) \\
&\quad + y\left(V_{n-2} + iV_{n-1} + \varepsilon V_n + hV_{n+1}\right) \\
&= xVH_{n-1} + yVH_{n-2}.
\end{aligned}$$

So, the proof is completed. □

For $y = 1$, we obtain the Lucas hybrinomials.
For $x = y = 1$, we obtain the Lucas hybrid numbers.

Theorem 5. *For any integer $n \geq 0$, the Binet formula for the generalized Lucas hybrinomials is defined as*

$$VH_n = \alpha^n(1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3) + \beta^n(1 + i\beta + \varepsilon\beta^2 + h\beta^3)$$

where $\alpha = \dfrac{x + \sqrt{x^2+4y}}{2}$ *and* $\beta = \dfrac{x - \sqrt{x^2+4y}}{2}$.

Proof. Using the definition of the generalized Lucas hybrinomials and the Binet formula for the generalized Lucas polynomials, we get

$$\begin{aligned}
VH_n &= V_n + iV_{n+1} + \varepsilon V_{n+2} + hV_{n+3} \\
&= \alpha^n + \beta^n + i\left(\alpha^{n+1} + \beta^{n+1}\right) + \varepsilon\left(\alpha^{n+2} + \beta^{n+2}\right) + h\left(\alpha^{n+3} + \beta^{n+3}\right) \quad \square \\
&= \alpha^n(1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3) + \beta^n(1 + i\beta + \varepsilon\beta^2 + h\beta^3)
\end{aligned}$$

For expressing the notations simply, let

$$\begin{aligned}
\widehat{\alpha} &= 1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3 \\
\widehat{\beta} &= 1 + i\beta + \varepsilon\beta^2 + h\beta^3.
\end{aligned}$$

Then, we can write the Binet formula for the generalized Lucas hybrinomials as

$$VH_n = \alpha^n\widehat{\alpha} + \beta^n\widehat{\beta}.$$

Theorem 6. *The generating function for the generalized Lucas hybrinomials is*

$$\sum_{n=0}^{\infty} VH_n t^n = \frac{2+ix+\varepsilon(x^2+2y)+h(x^3+3xy)+(-x+i2y+\varepsilon xy+h\left(x^2y+2y^2\right))t}{1-xt-yt^2}.$$

Proof. Suppose that the formal power series representation of the generating function for the generalized Lucas hybrinomials is

$$G(t) = \sum_{n=0}^{\infty} VH_n t^n = VH_0 + VH_1 t + VH_2 t^2 + \cdots \tag{2}$$

Then, multiplying the equation (2) by $-xt$ and $-yt^2$ respectively, we have
$-G(t)xt = -VH_0xt - VH_1xt^2 - VH_2xt^3 - \cdots$
and
$-G(t)yt^2 = -VH_0yt^2 - VH_1yt^3 - VH_2yt^4 - \cdots$
By using the above equations and the fact that for $n \geq 2$ the coefficients of t^n are zero by the recurrence relation of the generalized Lucas hybrinomials, we obtain

$$G(t)\left(1-xt-yt^2\right) = VH_0 + \left(VH_1 - VH_0x\right)t. \tag{3}$$

Finally, by substituting VH_0 and VH_1 in the equation (3), we get

$$G(t) = \frac{2+ix+\varepsilon(x^2+2y)+h(x^3+3xy)+(-x+i2y+\varepsilon xy+h\left(x^2y+2y^2\right))t}{1-xt-yt^2}.$$

□

Lemma 7. *[6]For any integer $n \geq 2$, the generalized Lucas polynomials provides the summation formula*

$$\sum_{m=1}^{n-1} V_m = \frac{V_n + yV_{n-1} - x - 2y}{x+y-1}.$$

Theorem 8. *For any integer $n \geq 2$, the generalized Lucas hybrinomials provides the summation formula*

$$\sum_{m=1}^{n-1} VH_m = \frac{VH_n + yVH_{n-1} - VH_1 - yVH_0}{x+y-1}.$$

Proof. By using the definition of the generalized Lucas hybrinomials, we have

$$\begin{aligned}\sum_{m=1}^{n-1} VH_m &= VH_1 + VH_2 + \cdots + VH_{n-1} \\ &= V_1 + iV_2 + \varepsilon V_3 + hV_4 \\ &\quad + V_2 + iV_3 + \varepsilon V_4 + hV_5 \\ &\quad \vdots\end{aligned}$$

$$
\begin{aligned}
&+V_{n-1}+iV_n+\varepsilon V_{n+1}+hV_{n+2}\\
=\,&V_1+V_2+\cdots+V_{n-1}\\
&+i\left[V_2+V_3+\cdots+V_n+(V_1-V_1)\right]\\
&+\varepsilon\left[V_3+V_4+\cdots+V_{n+1}+(V_1+V_2-V_1-V_2)\right]\\
&+h[V_4+V_5+\cdots+V_{n+2}+(V_1+V_2+V_3-V_1-V_2-V_3)]
\end{aligned}
$$

By using the previous lemma, we have

$$
\begin{aligned}
\sum_{m=1}^{n-1} VH_m = \frac{V_n+yV_{n-1}-x-2y}{x+y-1}
&+i\left(\frac{V_{n+1}+yV_n-x-2y}{x+y-1}-V_1\right)\\
&+\varepsilon\left(\frac{V_{n+2}+yV_{n+1}-x-2y}{x+y-1}-V_1-V_2\right)\\
&+h\left(\frac{V_{n+3}+yV_{n+2}-x-2y}{x+y-1}-V_1-V_2-V_3\right).
\end{aligned}
$$

Substituting V_1, V_2, V_3 and making the fractions common denominator, we obtain

$$
\begin{aligned}
\sum_{m=1}^{n-1} VH_m = \frac{V_n+yV_{n-1}-x-2y}{x+y-1}
&+i\left(\frac{V_{n+1}+yV_n-x-2y-x\,(x+y-1)}{x+y-1}\right)\\
&+\varepsilon\frac{1}{x+y-1}\left(V_{n+2}+yV_{n+1}-x-2y-x\,(x+y-1)\right.\\
&\qquad\left.-\left(x^2+2y\right)(x+y-1)\right)\\
&+h\frac{1}{x+y-1}\left(V_{n+3}+yV_{n+2}-x-2y-x\,(x+y-1)\right.\\
&\qquad\qquad\left.-\left(x^2+2y\right)(x+y-1)-\left(x^3+3xy\right)(x+y-1)\right).
\end{aligned}
$$

Finally, we get the result as

$$
\begin{aligned}
\sum_{m=1}^{n-1} VH_m = \frac{V_n+iV_{n+1}+\varepsilon V_{n+2}+hV_{n+3}}{x+y-1}
&+y\frac{V_{n-1}+iV_n+\varepsilon V_{n+1}+hV_{n+2}}{x+y-1}\\
&-\frac{1}{x+y-1}\left(x+2y+i\left(x^2+2y+yx\right)+\varepsilon(x^3+3xy+y(x^2+2y))\right.\\
&\qquad\left.+h(x^4+4x^2y+2y^2+y\left(x^3+3xy\right))\right)
\end{aligned}
$$

$$= \frac{VH_n + yVH_{n-1} - VH_1 - yVH_0}{x+y-1}.$$

□

Theorem 9 (Catalan Identity)**.** *For the nonnegative integers n and r with $n \geq r$, we have*

$$VH_{n-r}VH_{n+r} - (VH_n)^2 = (-y)^n \widehat{\alpha}\widehat{\beta}\left(\frac{\beta^r}{\alpha^r} - 1\right) + (-y)^n \widehat{\beta}\widehat{\alpha}\left(\frac{\alpha^r}{\beta^r} - 1\right).$$

Proof. By using the Binet formula for the generalized Lucas hybrinomials, we have

$$\begin{aligned}
&VH_{n-r}VH_{n+r} - (VH_n)^2 \\
&\quad = \left(\alpha^{n-r}\widehat{\alpha} + \beta^{n-r}\widehat{\beta}\right)\left(\alpha^{n+r}\widehat{\alpha} + \beta^{n+r}\widehat{\beta}\right) - \left(\alpha^n\widehat{\alpha} + \beta^n\widehat{\beta}\right)\left(\alpha^n\widehat{\alpha} + \beta^n\widehat{\beta}\right) \\
&\quad = \alpha^{n-r}\beta^{n+r}\widehat{\alpha}\widehat{\beta} + \beta^{n-r}\alpha^{n+r}\widehat{\beta}\widehat{\alpha} - \alpha^n\beta^n\widehat{\alpha}\widehat{\beta} - \beta^n\alpha^n\widehat{\beta}\widehat{\alpha} \\
&\quad = \alpha^n\beta^n\widehat{\alpha}\widehat{\beta}\left(\frac{\beta^r}{\alpha^r} - 1\right) + \beta^n\alpha^n\widehat{\beta}\widehat{\alpha}\left(\frac{\alpha^r}{\beta^r} - 1\right) \\
&\quad = (-y)^n \widehat{\alpha}\widehat{\beta}\left(\frac{\beta^r}{\alpha^r} - 1\right) + (-y)^n \widehat{\beta}\widehat{\alpha}\left(\frac{\alpha^r}{\beta^r} - 1\right).
\end{aligned}$$

□

Theorem 10 (Cassini Identity)**.** *For any nonnegative integer n, we have*

$$VH_{n-1}VH_{n+1} - (VH_n)^2 = (-y)^n \widehat{\alpha}\widehat{\beta}\left(\frac{\beta}{\alpha} - 1\right) + (-y)^n \widehat{\beta}\widehat{\alpha}\left(\frac{\alpha}{\beta} - 1\right)$$

Proof. Since the Cassini identity is a special case of the Catalan identity, by taking $r = 1$ in the Catalan identity theorem can be proved easily. □

Theorem 11 (d'Ocagne Identity)**.** *For the nonnegative integers m and n with $m \geq n$, we have*

$$VH_mVH_{n+1} - VH_{m+1}VH_n = (-y)^n (\alpha - \beta)\left(\beta^{m-n}\widehat{\beta}\widehat{\alpha} - \alpha^{m-n}\widehat{\alpha}\widehat{\beta}\right).$$

Proof. By using the Binet formula for the generalized Lucas hybrinomials, we have

$$\begin{aligned}
&VH_mVH_{n+1} - VH_{m+1}VH_n \\
&\quad = \left(\alpha^m\widehat{\alpha} + \beta^m\widehat{\beta}\right)\left(\alpha^{n+1}\widehat{\alpha} + \beta^{n+1}\widehat{\beta}\right) - \left(\alpha^{m+1}\widehat{\alpha} + \beta^{m+1}\widehat{\beta}\right)\left(\alpha^n\widehat{\alpha} + \beta^n\widehat{\beta}\right) \\
&\quad = \alpha^{m+n+1}\widehat{\alpha}^2 + \alpha^m\beta^{n+1}\widehat{\alpha}\widehat{\beta} + \beta^m\alpha^{n+1}\widehat{\beta}\widehat{\alpha} + \beta^{m+n+1}\widehat{\beta}^2 \\
&\qquad -\alpha^{m+n+1}\widehat{\alpha}^2 - \alpha^{m+1}\beta^n\widehat{\alpha}\widehat{\beta} - \beta^{m+1}\alpha^n\widehat{\beta}\widehat{\alpha} - \beta^{m+n+1}\widehat{\beta}^2 \\
&\quad = \left(\alpha^m\beta^{n+1} - \alpha^{m+1}\beta^n\right)\widehat{\alpha}\widehat{\beta} + \left(\beta^m\alpha^{n+1} - \beta^{m+1}\alpha^n\right)\widehat{\beta}\widehat{\alpha} \\
&\quad = \alpha^m\beta^n(\beta - \alpha)\widehat{\alpha}\widehat{\beta} + \beta^m\alpha^n(\alpha - \beta)\widehat{\beta}\widehat{\alpha} \\
&\quad = (-y)^n (\alpha - \beta)\left(\beta^{m-n}\widehat{\beta}\widehat{\alpha} - \alpha^{m-n}\widehat{\alpha}\widehat{\beta}\right).
\end{aligned}$$

□

Theorem 12. *For any nonnegative integer n, we have*

$$\begin{bmatrix} VH_{n+2} & VH_{n+1} \\ VH_{n+1} & VH_n \end{bmatrix} = \begin{bmatrix} VH_2 & VH_1 \\ VH_1 & VH_0 \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}^n$$

Proof. We prove the theorem using induction method on n.

For $n = 0$, the result is obvious.

Assume that for any $n \geq 0$ the theorem holds

$$\begin{bmatrix} VH_{n+2} & VH_{n+1} \\ VH_{n+1} & VH_n \end{bmatrix} = \begin{bmatrix} VH_2 & VH_1 \\ VH_1 & VH_0 \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}^n .$$

We must show that for $n+1$ the theorem holds

$$\begin{bmatrix} VH_{n+3} & VH_{n+2} \\ VH_{n+2} & VH_{n+1} \end{bmatrix} = \begin{bmatrix} VH_2 & VH_1 \\ VH_1 & VH_0 \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}^{n+1} .$$

By using the induction hypothesis, we have

$$\begin{aligned} \begin{bmatrix} VH_2 & VH_1 \\ VH_1 & VH_0 \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} VH_2 & VH_1 \\ VH_1 & VH_0 \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}^{n} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \\ &= \begin{bmatrix} VH_{n+2} & VH_{n+1} \\ VH_{n+1} & VH_n \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \\ &= \begin{bmatrix} xVH_{n+2} + yVH_{n+1} & VH_{n+2} \\ xVH_{n+1} + yVH_n & VH_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} VH_{n+3} & VH_{n+2} \\ VH_{n+2} & VH_{n+1} \end{bmatrix} \end{aligned}$$

which completes the proof. □

3. Conclusion

In the present work, we define the generalized Lucas hybrinomials with two variables x and y. Then, Binet formula, generating function and some properties of the generalized Lucas hybrinomials are obtained. Moreover, we obtain Catalan, Cassini and d'Ocagne identities for these hybrinomials. Finally, we derive the generalized Lucas hybrinomials by the help of matrix theory.

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FIXED POINT THEOREMS TO GENERALIZED $\mathtt{F}_{\Re}$- CONTRACTION MAPPINGS WITH APPLICATIONS TO NONLINEAR MATRIX EQUATIONS

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Abstract. In the present paper, we introduce the notion of generalized $\mathtt{F}_{\Re}$-contraction and establish some fixed point results for such mappings, which extend and generalize the result of Alam and Imdad [1], Sawangsup et al. [23] and many others. Our results reveal that the assumption of M-closedness of underlying binary relation is not a necessary condition for the existence of fixed points in relational metric spaces. We also derive some N-order fixed point theorems from our main results. As an application of our main result, we find a solution to a certain class of nonlinear matrix equations.

1. Introduction

It is widely known that the Banach contraction principle (BCP) [7] is the first metric fixed point theorem and one of the most powerful and versatile result in the field of nonlinear analysis. It asserts that every contraction mapping on a complete metric space possesses a unique fixed point. Several extensions of this principle were considered by many authors to various generalized contractions and different type of spaces (see [1], [3], [4], [5], [6], [8], [10], [12], [18], [20], [21], [26]). Wardowski [26] generalized the Banach contraction principle by introducing the notion of $\mathtt{F}$-contraction on metric spaces. The result of Wardowski was further extended and generalized by several authors (see [10], [11], [12], [17], [19], [27] and references therein) by improving the condition of $\mathtt{F}$-contraction .

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Another important generalization of the BCP was obtained by Alam and Imdad [1] in 2015. They generalized the BCP to complete metric spaces endowed with an arbitrary binary relation. Subsequently, Sawangsup et al. [23] introduced the notion of $\mathtt{F}_{\Re}$-contraction in relational metric space by modifying the condition of $\mathtt{F}$-contraction. They also introduced the notion of $\mathtt{F}_{\Re^N}$- contraction and established some multidimensional fixed point results of N-order.

In the present paper, we improve the idea of Sawangsup et al. [23] by introducing the notion of generalized $\mathtt{F}_{\Re}$-contraction mappings and prove some fixed point results for such mappings. Our results generalize the result of Alam and Imdad [1], Wardowski [26], Sawangsup et al. [23] and many others in the existing literature. We also introduce the notions of multidimensional generalized $\mathtt{F}_{\Re^N}$-contraction and $\mathtt{F}_{\Re^N}$-graph contraction and prove some multidimensional results for the existence of fixed points of N-order. Our results do not force the underlying binary relation to be M-closed for the existence of fixed points in relational metric spaces. Moreover, we furnish some examples to demonstrate the usefulness of our main results. As an application, we apply our result to find a solution of a class of non-linear matrix equations.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of positive integers, the set of non-negative integers, the set of real numbers and the set of positive real numbers, respectively.

Definition 1. *[26] Let $\mathcal{F}$ denotes the family of all functions $\mathtt{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:*

($\mathtt{F}_1$) $\mathtt{F}$ is strictly increasing, i.e., for all $\varrho, \mu \in \mathbb{R}^+$ such that $\varrho < \mu, \mathtt{F}(\varrho) < \mathtt{F}(\mu)$;
($\mathtt{F}_2$) for each sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of positive numbers we have $\lim_{n\to\infty} \varrho_n = 0$ iff $\lim_{n\to\infty} \mathtt{F}(\varrho_n) = -\infty$;
($\mathtt{F}_3$) there exists $k \in (0, 1)$ such that $\lim_{\varrho\to 0^+} \varrho^k \mathtt{F}(\varrho) = 0$.

Example 2. *[26] Let $\mathtt{F}_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3, 4$ by:*

(i) $\mathtt{F}_1(\varrho) = \log(\varrho)$ for all $\varrho > 0$;
(ii) $\mathtt{F}_2(\varrho) = \varrho + \log(\varrho)$ for all $\varrho > 0$;
(iii) $\mathtt{F}_3(\varrho) = -\frac{1}{\sqrt{\varrho}}$ for all $\varrho > 0$;
(iv) $\mathtt{F}_4(\varrho) = \log(\varrho^2 + \varrho)$ for all $\varrho > 0$.

Definition 3. *[26] Let (X, d) be a metric space and $M : X \to X$ be a mapping. The mapping M is said to be a $\mathtt{F}$-contraction if there exists $\tau > 0$ and $\mathtt{F} \in \mathcal{F}$ such that*

$$d(M\nu, M\rho) > 0 \Longrightarrow \tau + \mathtt{F}(d(M\nu, M\rho)) \le \mathtt{F}(d(\nu, \rho)), \quad \nu, \rho \in X.$$

We accept the following relation-theoretic notations and definitions in our subsequent discussions.

Definition 4. *[1] Let X be a non-empty set. A binary relation $\Re$ on X is a subset of $X \times X$. We say that ν relates to ρ under $\Re$ if and only if $(\nu, \rho) \in \Re$.*

Definition 5. *[1] Let $\Re$ be a binary relation on X. If either $(\nu, \rho) \in \Re$ or $(\rho, \nu) \in \Re$ then we say ν and ρ are $\Re$-comparable and we denote it by $[\nu, \rho] \in \Re$.*

Definition 6. *[1] A binary relation $\Re$ defined on a non-empty set X is called*

(a) reflexive if $(\nu, \nu) \in \Re$ for all $\nu \in X$,
(b) irreflexive if $(\nu, \nu) \notin \Re$ for all $\nu \in X$,
(c) symmetric if $(\nu, \rho) \in \Re$ implies $(\rho, \nu) \in \Re$,
(d) antisymmetric if $(\nu, \rho) \in \Re$ and $(\rho, \nu) \in \Re$ implies $\nu = \rho$,
(e) transitive if $(\nu, \rho) \in \Re$ and $(\rho, z) \in \Re$ implies $(\nu, z) \in \Re$,
(f) complete, connected or dichotomous if $[\nu, \rho] \in \Re$ for all $\nu, \rho \in X$,
(g) weakly complete, weakly connected or trichotomous if $[\nu, \rho] \in \Re$ or $\nu = \rho$ for all $\nu, \rho \in X$.

Definition 7. *[1] Let X be a non-empty set and $\Re$ be a binary relation on X. A sequence $\{\nu_n\} \in X$ is called $\Re$-preserving if*

$$(\nu_n, \nu_{n+1}) \in \Re, \quad \text{for all } n \in \mathbb{N}_0.$$

Definition 8. *[1] Let (X, d) be a metric space and $\Re$ be a binary relation on X. If for any $\Re$-preserving sequence $\{\nu_n\}$ on X such that*

$$\{\nu_n\} \xrightarrow{d} \nu,$$

*there exists a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \nu] \in \Re$, for all $k \in \mathbb{N}_0$, then the binary relation $\Re$ is called d-*self-closed* on X.*

Definition 9. *[1, 22] Let X be a non-empty set and M be a self-mapping on X. A binary relation $\Re$ is called M-closed, if for $\nu, \rho \in X$ with*

$$(\nu, \rho) \in \Re \implies (M\nu, M\rho) \in \Re$$

and the mapping M is also called comparative mapping on X, under binary relation $\Re$.

Definition 10. *[14] Let $\Re$ be a binary relation on X and $M : X \to X$ be a mapping. We denote the relational graph of mapping M under the binary relation $\Re$ on X, by $G(M; \Re)$ and defined as:*

$$G(M; \Re) = \{(\nu, M\nu) \in \Re : \nu \in X\}.$$

Definition 11. *[14] Let $\Re$ be a binary relation on X and $M : X \to X$ be a self-mapping. By $X(M; \Re)$, we denotes the set of all those $\nu \in X$ for which $(\nu, M\nu) \in G(M; \Re)$, that is,*

$$X(M; \Re) = \{\nu \in X : (\nu, M\nu) \in G(M; \Re)\}.$$

The above Definition 11 is equivalent to the Definition 2.12 of Shukla and Rodríguez-López [25] which states that $X(M;\Re)$ is a set of all those points ν in X for which $(\nu, M\nu) \in \Re$, that is,

$$X(M;\Re) = \{\nu \in X : (\nu, M\nu) \in \Re\}.$$

Definition 12. *[14] Let (X,d) be a metric space, $\Re$ be a binary relation on X and $M : X \to X$ be a mapping. A binary relation $\Re$ is called M_G-d-closed if the following condition holds:*

$$(\nu,\rho) \in G(M;\Re), \quad d(M\nu, M\rho) \le d(\nu,\rho) \implies (M\nu, M\rho) \in G(M;\Re).$$

Remark 13. *We notice that the condition of M_G-d-closedness is weaker than the condition of M-closedness. The following example illustrates this fact.*

Example 14. *Let $X = [0,1]$ equipped with usual metric $d(\nu,\rho) = |\nu - \rho|$. Let a binary relation $\Re$ and a self-map M on X be defined as $\Re = \{(0,0),(1,0),(1,1),(1/3,1)\}$ and*

$$M(\nu) = \begin{cases} \nu/4, & \text{if } \nu \in [0,1/3], \\ 1, & \text{if } \nu \in (1/3,1]. \end{cases}$$

Then $G(M;\Re) = \{(0,0),(1,1)\}$ and for each $(\nu,\rho) \in G(M;\Re)$, we have $d(M\nu, M\rho) = d(\nu,\rho)$ and $(M\nu, M\rho) \in G(M;\Re)$. Hence the binary relation $\Re$ is M_G-d-closed. But $\Re$ is not M-closed in X because $(1/3,1) \in \Re$ and $(M1/3, M1) = (1/12,1) \notin \Re$.

Definition 15. *[2] Let (X,d) be a metric space and $\Re$ be a binary relation on X. A self-mapping M on X is called* $\Re$-continuous mapping *at point $\nu \in X$ if for any $\Re$-preserving sequence $\{\nu_n\}$ such that $\{\nu_n\} \xrightarrow{d} \nu$, we have $\{M(\nu_n)\} \xrightarrow{d} M(\nu)$. Moreover, M is called* $\Re$-continuous *if it is $\Re$-continuous at each point of X.*

By above definition, it is clear that every continuous mapping is $\Re$-continuous and under universal relation the definition of $\Re$-continuity coincides with the definition of continuity.

Definition 16. *[16] A self-mapping M of a metric space (X,d) is called k- continuous, $k = 1,2,3\ldots,$ at a point $\nu \in X$ if $\{M^k\nu_n\} \to M\nu$, whenever $\{\nu_n\}$ is a sequence in X such that $\{M^{k-1}\nu_n\} \to \nu$ in X. Moreover, M is called k-continuous if it is k-continuous at each point of X.*

It is obvious by the definition of k-continuity that every continuous mapping M of a metric space (X,d) is k-continuous and the notion of continuity coincides with the notion of 1-continuity. However, k-continuity of a function (for $k \ge 2$) does not imply the continuity of the function (see Example 1.2 in [16]).

Definition 17. *[13] Let (X,d) be a metric space endowed with a binary relation $\Re$. A mapping $M : X \to X$ is called $(\Re,k)$-continuous at a point $\nu \in X$ if whenever $\{\nu_n\}$ is $\Re$-preserving sequence in X such that $\{M^{k-1}\nu_n\} \xrightarrow{d} \nu$, we have*

$\{M^k(\nu_n)\} \xrightarrow{d} M\nu$. Moreover, if M is a $(\Re, k)$-continuous at each point of X then M is called $(\Re, k)$-continuous.

By the definition of $(\Re, k)$-continuity, it is clear that every $\Re$-continuous mapping is a $(\Re, k)$-continuous mapping and both the definitions coincide for $k = 1$. Also every k-continuous mapping is $(\Re, k)$-continuous and for universal relation the definition of $(\Re, k)$-continuity is equivalent to the definition of k-continuity introduced by Pant and Pant in [16].

Remark 18. *Every continuous, k-continuous and $\Re$-continuous mapping is a $(\Re, k)$-continuous mapping but converse may not be true. The following example illustrates that $(\Re, k)$-continuity does not imply $\Re$-continuity and k-continuity as well.*

Example 19. *Let $X = [-1, 2]$ be a metric space equipped with a usual metric $d(\nu, \rho) = |\nu - \rho|$. Let $\Re = \{(\frac{1}{2^n}, \frac{1}{2^{n+1}}) : n \in \mathbb{N}\}$ be a binary relation on X and M be a self-mapping on X, defined as*

$$M(\nu) = \begin{cases} 1/3, & \text{if } \nu \in [-1, 0], \\ 1/2, & \text{if } \nu \in (0, 1], \\ \nu, & \text{if } \nu \in (1, 2]. \end{cases}$$

Clearly, M is not a continuous mapping in X and the sequence $\{\nu_n\} = \{\frac{1}{2^n}\}, n \in \mathbb{N}$ is $\Re$-preserving in X as $(\nu_n, \nu_{n+1}) \in \Re$, for all $n \in \mathbb{N}$. Since $\{\nu_n\} \to 0$ as $n \to \infty$ then $\{M\nu_n\} \to 1/2 \neq M0$. Hence, M is not a $\Re$-continuous mapping in X. Now, for each $k = 2, 3, 4, \ldots$,

$$M^k(\nu) = \begin{cases} 1/2, & \text{if } \nu \in [-1, 1], \\ \nu, & \text{if } \nu \in (1, 2]. \end{cases}$$

Since $M^k(\nu)$ is continuous everywhere in X, except at $\nu = 1$. Also, there does not exist any $\Re$-preserving sequence $\{\nu_n\}$ in X such that $\{M^{k-1}\nu_n\} \to 1$ as $n \to \infty$. So M is obviously a $(\Re, k)$-continuous mapping in X. However, for $\{\nu_n\} = \{1+\frac{1}{n}\}, n \in \mathbb{N}$, $\{M^{k-1}\nu_n\} \to 1$ and $\{M^k\nu_n\} \to 1 \neq M1$ yields M is not a k-continuous mapping in X.

Hence, the mapping M is a $(\Re, k)$-continuous mapping in X, but M is neither a continuous nor a k-continuous and also not a $\Re$-continuous mapping in X.

Definition 20. *[2] Let (X, d) be a metric space and $\Re$ be a binary relation on X. If every $\Re$-preserving Cauchy sequence converges in X, then we say that (X, d) is $\Re$-complete .*

Every complete metric space is $\Re$-complete under an arbitrary binary relation $\Re$ and both the definitions coincide under the universal relation.

Definition 21. *[15] Let $\Re$ be a binary relation on a non-empty set X and $\nu, \rho \in X$. A path of length $k \in \mathbb{N}$ in $\Re$ from ν to ρ is a finite sequence $\{z_0, z_1, \ldots, z_k\} \subseteq X$ satisfying the following conditions:*

(1) $z_0 = \nu$ and $z_k = \rho$;
(2) $(z_i, z_{i+1}) \in \Re$ for all $i \in \{0, 1, 2, \ldots, k-1\}$.

We denote by $\gamma(\nu; \rho; \Re)$, the family of all paths in $\Re$ from ν to ρ.

3. Main Results

Firstly, we introduce the notion of generalized $\mathtt{F}_{\Re}$-contraction mapping and $\mathtt{F}_{\Re}$-graph contraction mapping. Then, we will state our main results.

Definition 22. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose M be a self-mapping on X and A is any non-empty subset of $X(M; \Re)$. Then, the mapping M is called a generalized $\mathtt{F}_{\Re}$-contraction with respect to A, if for each $\nu, \rho \in A$ with $(\nu, \rho) \in \Re$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(M\nu, M\rho) > 0 \implies \tau + \mathtt{F}(d(M\nu, M\rho)) \leq \mathtt{F}(d(\nu, \rho)). \tag{1}$$

If we take $A = X(M; \Re)$ in the above definition then we get the following definition, which is a special case of the Definition 22.

Definition 23. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. A self-mapping M on X is called a generalized $\mathtt{F}_{\Re}$-contraction with respect to $X(M; \Re)$ or $\mathtt{F}_{\Re}$-graph contraction, if for each $\nu, \rho \in X(M; \Re)$ with $(\nu, \rho) \in \Re$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that*

$$d(M\nu, M\rho) > 0 \implies \tau + \mathtt{F}(d(M\nu, M\rho)) \leq \mathtt{F}(d(\nu, \rho)). \tag{2}$$

Clearly condition (1) and condition (2) is weaker than the condition of $\mathtt{F}_{\Re}$-contraction due to Sawangsup et al. [23].

Now, we state our first result for a generalized $\mathtt{F}_{\Re}$-contraction mapping in a relational metric space.

Theorem 24. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose $M : X \to X$ be a mapping and there exists a non-empty subset A of $X(M; \Re)$ such that the following conditions hold:*

(a) $M(A) \subseteq A$,
(b) M is $(\Re, k)$-continuous mapping or $\Re$ is d-self closed,
(c) M is a generalized $\mathtt{F}_{\Re}$-contraction with respect to A,
(d) there exists $\mathrm{Y} \subseteq A$ such that $M(A) \subseteq \mathrm{Y} \subseteq A$ and (Y, d) is $\Re$-complete.

Then, for each $\nu_0 \in A$, there exists a Picard sequence $\{\nu_n\}$ of M, starting from $\nu_1 = \nu_0$ which converges to the fixed point of M.

Proof. Let A be a non-empty subset of $X(M; \Re)$ and $\nu_0 \in A$. Then by virtue of subset A, we have $(\nu_0, M\nu_0) \in \Re$. If $\nu_0 = M\nu_0$ then the proof is complete. So in view of condition (a), there exists a point say ν_1 in A such that $\nu_1 = M\nu_0$. Again, since $\nu_1 \in A$ so $(\nu_1, M\nu_1) \in \Re$. If $\nu_1 = M\nu_1$ then ν_1 is a fixed point of M and the proof is complete. Therefore $\nu_1 \neq M\nu_1$ and by assumption (a), there exists a point say $\nu_2 \in A$ such that $\nu_2 = M\nu_1$. Continuing this process again and again, we get a $\Re$-preserving Cauchy sequence of points $\{\nu_n\}$ in A such that

$$\nu_{n+1} = M\nu_n \quad \text{and} \quad (\nu_n, \nu_{n+1}) \in \Re, \qquad \text{for all } n \in \mathbb{N}_0.$$

We denote $\zeta_n = d(\nu_{n+1}, \nu_n)$, $n \in \mathbb{N}_0$ and assume that $\nu_{n+1} \neq \nu_n$ for $n \in \mathbb{N}$. Then $\zeta_n > 0$, for $n \in \mathbb{N}$ and

$$\mathtt{F}(\zeta_n) \leq \mathtt{F}(\zeta_{n-1}) - \tau \leq \mathtt{F}(\zeta_{n-2}) - 2\tau \leq \cdots \leq \mathtt{F}(\zeta_0) - n\tau. \tag{3}$$

From (3), we get $\lim\limits_{n\to\infty} \mathtt{F}(\zeta_n) = -\infty$ and together with ($\mathtt{F}_2$), we have

$$\lim_{n\to\infty} \zeta_n = 0. \tag{4}$$

From ($\mathtt{F}_3$), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \zeta_n^k \mathtt{F}(\zeta_n) = 0. \tag{5}$$

By (3), the following inequality holds

$$\zeta_n^k \mathtt{F}(\zeta_n) - \zeta_n^k \mathtt{F}(\zeta_0) \leq \zeta_n^k (\mathtt{F}(\zeta_0) - n\tau) - \zeta_n^k \mathtt{F}(\zeta_0) = -\zeta_n^k n\tau \leq 0, \tag{6}$$

for all $n \in \mathbb{N}$. Making $n \to \infty$ in (6) and using (5), we obtain

$$\lim_{n\to\infty} n\zeta_n^k = 0. \tag{7}$$

From (7), we observe that there exists $n_1 \in \mathbb{N}$ such that $n\zeta_n^k \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\zeta_n \leq \frac{1}{n^{1/k}}, \tag{8}$$

for $n \geq n_1$. In order to prove that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy, consider $m, n \in \mathbb{N}$ with $m > n > n_1$. From (8) and triangle inequality, we get

$$d(\nu_m, \nu_n) \leq \zeta_{m-1} + \zeta_{m-2} + \cdots + \zeta_n < \sum_{i=n}^{\infty} \zeta_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

Now it follows, from the above inequality and by the convergence of $\sum\limits_{i=n}^{\infty} \dfrac{1}{i^{1/k}}$, that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is a Cauchy in A. Since $\{\nu_n\}_{n\in\mathbb{N}} \subseteq M(A) \subseteq \mathrm{Y}$ therefore $\{\nu_n\}_{n\in\mathbb{N}}$ is a $\Re$-preserving Cauchy sequence in Y. Since (Y, d) is a $\Re$-complete metric space so there exists a point say $\nu^* \in \mathrm{Y} \subseteq A$ such that $\lim\limits_{n\to\infty} \nu_n = \nu^*$.

We now assume that M is a $(\Re, k)$-continuous mapping. Since the sequence $\{\nu_n\} = \{M^{k-1}(\nu_{n-k+1})\}$ converges to ν^* then $(\Re, k)$-continuity of M implies that $\{M^k(\nu_{n-k+1})\}$ converges to $M(\nu^*)$. Hence, from the above we conclude that $M(\nu^*) = \nu^*$, that is, ν^* is a fixed point of the function M.

Alternately, we assume that $\Re$ is d-self-closed. Since $\{\nu_n\}$ is a $\Re$-preserving sequence in A such that

$$\{\nu_n\} \xrightarrow{d} \nu^*$$

and $\nu^* \in A$, therefore by assumption of d-self-closedness, there exists a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ with $[\nu_{n_k}, \nu^*] \in \Re$ for all $k \in \mathbb{N}_0$. From contraction condition (22), we obtain

$$\mathtt{F}\big(d(\nu_{n_k+1}, M\nu^*)\big) = \mathtt{F}\big(d(M\nu_{n_k}, M\nu^*)\big) \leq \mathtt{F}\big(d(\nu_{n_k}, \nu^*)\big) - \tau$$
$$\implies d(\nu_{n_k+1}, M\nu^*) < d(\nu_{n_k}, \nu^*) \to 0 \text{ as } k \to \infty,$$

which yields $\nu_{n_k+1} \xrightarrow{d} M(\nu^*)$, that is, M has a fixed point at ν^* in X. $\square$

The following example illustrates our Theorem 24.

Example 25. *Let $X = (-1, 2]$ be a metric space equipped with a usual metric $d(\nu, \rho) = |\nu - \rho|$. Let $\mathcal{L} = \big\{(\frac{1}{4^n}, \frac{1}{4^{n+1}}) : n \in \mathbb{N}\big\}$ and $\Re = \{(0,0), (0,1), (1,1), (0, \frac{3}{2}), (0, \frac{1}{4}), (1, \frac{1}{6}), (\frac{1}{4}, \frac{1}{6}), (\frac{1}{6}, \frac{1}{6})\} \cup \mathcal{L}$ be a binary relation on X. We define a self-mapping M on X as*

$$M(\nu) = \begin{cases} \frac{1}{4}, & \text{if } \nu \in (-1, 0], \\ \frac{1}{6}, & \text{if } \nu \in (0, 1], \\ \nu, & \text{if } \nu \in (1, 2], \end{cases}$$

then it is easy to see that $X(M; \Re) = \{0, \frac{1}{4}, \frac{1}{6}, 1\}$. Suppose that $A = \{0, \frac{1}{4}, \frac{1}{6}\} \subset X(M; \Re)$ and $Y = \{1/4, 1/6\}$. Then clearly $Y = M(A) \subseteq A$ and Y is $\Re$-complete. Since $\{\nu_n\} = \{\frac{1}{4^n} : n \in \mathbb{N}\}$ is a $\Re$-preserving sequence in X and $\{\nu_n\} \to 0$ but $\{M\nu_n\} \to \frac{1}{6} \neq M0$. Therefore, M is neither a continuous nor a $\Re$-continuous mapping in X. Now, for each $k = 2, 3, 4, \ldots$,

$$M^k(\nu) = \begin{cases} \frac{1}{6}, & \text{if } \nu \in (-1, 1], \\ \nu, & \text{if } \nu \in (1, 2]. \end{cases}$$

As $M^k(\nu)$ is continuous everywhere in X, except $\nu = 1$ and there does not exist any $\Re$-preserving sequence $\{\nu_n\}$ in X such that $\{M^{k-1}\nu_n\} \to 1$ as $n \to \infty$. Then, it is obvious by Definition 17 that M is a $(\Re, k)$-continuous mapping in X. However, for $\{\nu_n\} = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$, we have $\{M^{k-1}\nu_n\} \to 1$ and $\{M^k\nu_n\} \to 1 \neq M1$ which implies M is not a k-continuous mapping in X. Now, we will prove that M is a generalized $\mathtt{F}_\Re$-contraction mapping with respect to A. For this, we take $\tau = 1, \mathtt{F} \in \mathcal{F}$ given by $\mathtt{F}(\varrho) = \varrho + ln(\varrho), \varrho > 0$ and $\nu, \rho \in A$ with $(\nu, \rho) \in \Re$ such that $d(M\nu, M\rho) > 0$, we have only one choice for such (ν, ρ) in $\Re$, that is, $(\nu, \rho) = (0, 1/4)$. Then from (1), we obtain

$$\frac{d(M\nu, M\rho)}{d(\nu, \rho)} e^{[d(M\nu, M\rho) - d(\nu, \rho)]} = \frac{d(M0, M\frac{1}{4})}{d(0, \frac{1}{4})} e^{[d(M0, M\frac{1}{4}) - d(0, \frac{1}{4})]} = \frac{1}{3} e^{-\frac{1}{6}} < e^{-1}.$$

Hence, all the assumptions of Theorem 24 are hold and M has infinite fixed points in X.

Remark 26. *It is noticeable that the binary relation used in the Example 25 is not M-closed even though M has infinite fixed points in X, which reveals that the assumption of M-closedness of the underlying binary relation is not a necessary* *condition for the existence of fixed points in relational metric spaces. Thus in Example 25, the fixed point results of Sawangsup et al. [23], Alam and Imdad [1], Samet and Turinici [22] and many others does not work but our result is still valid therein.*

Remark 27. *We also notice that, the binary relation $\Re$ used in Example 25 is not one of the earlier known standard binary relation such as reflexive , symmetric, transitive, anti-symmetric, complete or weakly complete. Therefore, theorems contained in [1, 2, 7, 10, 11] can not be apply in the above example. Thus, Theorem 24 extends all the classical results to an arbitrary binary relation.*

We get the following corollary as a direct consequence of Theorem 24 by taking $\tau = \log \frac{1}{\varrho}$ and $\mathtt{F} = \log \nu$ in Theorem 24.

Corollary 28. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose $M : X \to X$ be a mapping and there exists a non-empty subset A of $X(M; \Re)$ such that the following conditions hold:*

(a) $M(A) \subseteq A$,
(b) M is $(\Re, k)$-continuous mapping or $\Re$ is d-self closed,
(c) there exists $\varrho \in [0, 1)$ such that

$$d(M\nu, M\rho) \leq \varrho\ d(\nu, \rho), \text{ for all } \nu, \rho \in A \text{ such that } (\nu, \rho) \in \Re.$$

(d) there exists $\mathrm{Y} \subseteq A$ such that $M(A) \subseteq \mathrm{Y} \subseteq A$ and (Y, d) is $\Re$-complete.

Then M has a fixed point in X.

Now we prove fixed point theorem for $\mathtt{F}_{\Re}$-graph contraction mappings in relational metric spaces.

Theorem 29. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose M be a self-mapping on X and $X(M; \Re)$ be a non-empty set such that the following conditions are satisfied:*

(a) $\Re$ is M_G-d-closed;
(b) M is $(\Re, k)$-continuous or $\Re$ is d-self closed;
(c) M is $\mathtt{F}_{\Re}$-graph contraction on X,
(d) there exists $\mathrm{Y} \subseteq X(M; \Re)$ such that $M(X(M; \Re)) \subseteq \mathrm{Y} \subseteq X(M; \Re)$ and (Y, d) is $\Re$-complete.

Then, for each $\nu_0 \in X(M; \Re)$, there exists a Picard sequence $\{\nu_n\}$ of M, starting from $\nu_1 = \nu_0$ which converges to the fixed point of M.

Proof. Suppose $X(M; \Re)$ be a non-empty and ν_0 be any point in $X(M; \Re)$. Then by virtue of $X(M; \Re)$, we have $(\nu_0, M\nu_0) \in \Re$. If $\nu_0 = M\nu_0$ then ν_0 is a fixed point of M and the proof is completed. Therefore, we assume that $\nu_0 \neq M\nu_0$ and $M\nu_0 = \nu_1$ (say). Now as $(\nu_0, \nu_1) = (\nu_0, M\nu_0) \in G(M; \Re)$ and M is a $\mathtt{F}_{\Re}$-graph contraction, we have

$$d(M\nu_0, M\nu_1) \leq d(\nu_0, \nu_1). \tag{9}$$

In view of assumption (a) and from condition (9), we get $(M\nu_0, M\nu_1) = (\nu_1, M\nu_1) \in \Re$. Again, if $\nu_1 = M\nu_1$ then the proof is complete, otherwise there exists a point say ν_2 in X, such that $\nu_2 = M\nu_1$ and $\nu_1 \neq \nu_2$. Continuing this process again and again, we get a $\Re$-preserving Cauchy sequence of points $\{\nu_n\}$ in X such that

$$\nu_{n+1} = M\nu_n \quad \text{and} \quad (\nu_n, \nu_{n+1}) \in \mathbb{R}, \quad \text{for all } n \in \mathbb{N}_0.$$

If we take $\nu_n = \nu_{n+1}$ for some $n \in \mathbb{N}$, then ν_n is called fixed point of M. Therefore, we assume that $\nu_n \neq \nu_{n+1}$ for $n \in \mathbb{N}$, that is, $d(\nu_n, \nu_{n+1}) \neq 0$ for $n \in \mathbb{N}$. Now proceeding the proof of Theorem 24, we get the conclusion. $\square$

The following example illustrates the utility of Theorem 29.

Example 30. *Let $X = (-1, 3]$ be a metric space equipped with a usual metric $d(\nu, \rho) = |\nu - \rho|$ and $\mathcal{P} = \big\{(\frac{1}{n}, \frac{1}{n+1}) : n \in \mathbb{N}\big\}$. Let a binary relation $\Re$ and a self-map M on X is defined as $\Re = \big\{(0,0), (0, \frac{1}{6}), (\frac{1}{6}, \frac{1}{8}), (\frac{1}{8}, \frac{1}{8}), (1, \frac{1}{8}), (1, 2)\big\} \cup \mathcal{P}$ and*

$$M(\nu) = \begin{cases} \frac{1}{6}, & \text{if } \nu \in (-1, 0], \\ \frac{1}{8}, & \text{if } \nu \in (0, 1], \\ 2, & \text{if } \nu \in (1, 3]. \end{cases}$$

Then, clearly $X(M; \Re) = \{0, \frac{1}{6}, \frac{1}{8}, 1\}$ and $G(M; \Re) = \{(0, \frac{1}{6}), (\frac{1}{6}, \frac{1}{8}), (\frac{1}{8}, \frac{1}{8}), (1, \frac{1}{8})\}$. For each $(\nu, \rho) \in G(M; \Re)$, we have $d(M\nu, M\rho) \leq d(\nu, \rho)$ and $(M\nu, M\rho) \in G(M; \Re)$ which yields the binary relation $\Re$ on X is M_G-d-closed. However, $\Re$ is not M-closed in X as $(0, 0) \in \Re$ but $(M0, M0) = (\frac{1}{6}, \frac{1}{6}) \notin \Re$. Since $\{\nu_n\} = \{\frac{1}{n}\}$, $n \in \mathbb{N}$ is a $\Re$-preserving sequence in X as $(\nu_n, \nu_{n+1}) \in \Re$ and $\{\nu_n\} \to 0$ then $\{M\nu_n\} \to \frac{1}{8} \neq M0$. Thus, M is neither a continuous nor a $\Re$-continuous mapping in X. Now, for each $k = 2, 3, 4, \ldots$,

$$M^k(\nu) = \begin{cases} \frac{1}{8}, & \text{if } \nu \in (-1, 1], \\ 2, & \text{if } \nu \in (1, 3]. \end{cases}$$

As $M^k(\nu)$ is continuous everywhere in X, except $\nu = 1$ and there does not exist any $\Re$-preserving sequence $\{\nu_n\}$ in X such that $\{M^{k-1}\nu_n\} \to 1$ as $n \to \infty$. So M is obviously a $(\Re, k)$-continuous mapping in X. However, for $\{\nu_n\} = \{1 + \frac{1}{n}\}$, $n \in \mathbb{N}$, $\{M^{k-1}\nu_n\} \to 1$ and $\{M^k\nu_n\} \to 1 \neq M1$, yields M is not a k-continuous mapping in X. Hence, the mapping M is a $(\Re, k)$-continuous mapping in X, but M is neither a continuous nor a k-continuous and also not a $\Re$-continuous mapping in X. Now, we will show that M is a generalized $\mathtt{F}_\Re$-graph contraction mapping with $\tau = 1$ and $\mathtt{F} \in \mathcal{F}$ defined by

$$\mathtt{F}(\varrho) = \varrho + ln(\varrho), \text{ for all } \varrho > 0.$$

For any $\nu, \rho \in X(M;\Re)$ with $(\nu,\rho) \in \Re$ and $d(M\nu, M\rho) > 0$, we have only one choice for $(\nu,\rho) = (0, \frac{1}{6})$ in $\Re$. Then from (23),

$$\frac{d(M\nu, M\rho)}{d(\nu,\rho)}\ e^{\{d(M\nu,M\rho)-d(\nu,\rho)\}} = \frac{d(M0, M\frac{1}{6})}{d(0,\frac{1}{6})}\ e^{\{d(M0,M\frac{1}{6})-d(0,\frac{1}{6})\}} = \frac{1}{4}e^{-\frac{1}{8}} < e^{-1}.$$

This yields M is a $\mathtt{F}_{\Re}$-graph contraction with $\tau = 1$. Hence, all the conditions of Theorem 29 are hold and M has two fixed points at points $\nu = \frac{1}{8}$ and $\nu = 2$.

A generalized version of relation-theoretic contraction principle due to Alam and Imdad [1] is derived from Theorem 29 by taking $\tau = \log\frac{1}{k}$ and $\mathtt{F} = \log \nu$ in Theorem 29.

Corollary 31. *Let (X,d) be a metric space and $\Re$ be a binary relation on X. Suppose M be a self-mapping on X and $X(M;\Re)$ be a non-empty set such that the following conditions are satisfied:*

(a) $\Re$ is M_G-d-closed,
(b) M is $(\Re,k)$-continuous or $\Re$ is d-self-closed,
(c) there exists $k \in [0,1)$ such that

$$d(M\nu, M\rho) \leq k\ d(\nu,\rho), \text{ for all } \nu,\rho \in X(M;\Re) \text{ with } (\nu,\rho) \in \Re.$$

(d) there exists $\mathrm{Y} \subseteq X(M;\Re)$ such that $M(X(M;\Re)) \subseteq \mathrm{Y} \subseteq X(M;\Re)$ and (Y, d) is $\Re$-complete.

Then M has a fixed point.

Remark 32. *We notice that Theorem 24 and Theorem 29 remain valid if we replace the assumption of $(\Re;k)$-continuity of M either by continuity of M, k-continuity of M or $\Re$-continuity of M (without altering the rest of the hypothesis).*

The following theorem guarantees the uniqueness of fixed points of Theorem 29 in a relational metric space.

Theorem 33. *In addition to the hypothesis of Theorem 29, suppose that $\Re$ is a transitive relation on X and $\gamma(\nu,\rho,\Re)$ is non-empty, for all $\nu,\rho \in X(M;\Re)$. Then, M has a unique fixed point in $X(M;\Re)$.*

Proof. Let ν^* and ρ^* be two distinct fixed points of M in $X(M;\Re)$ then $\nu^* = M\nu^*, \rho^* = M\rho^*$. Since $\gamma(\nu^*,\rho^*,\Re)$ is non-empty, there is a path (say $\{z_0, z_1, \ldots, z_k\}$) of some finite length k in $\Re$ from ν to ρ, so that

$$z_0 = \nu^*,\ z_k = \rho^*,\ (z_i, z_{i+1}) \in \Re, \quad \text{for each } i = 0,1,2,\ldots,k-1.$$

By transitivity of $\Re$, we get

$$(\nu^*, z_1) \in \Re, \quad (z_1,z_2) \in \Re, \ldots, (z_{k-1},\rho^*) \in \Re \implies (\nu^*,\rho^*) \in \Re.$$

The condition (23) implies that

$$\tau + \mathtt{F}(d(\nu^*,\rho^*)) = \tau + \mathtt{F}(d(M\nu^*, M\rho^*)) \leq \mathtt{F}(d(\nu^*,\rho^*))$$

which is not possible. Thus, M has a unique fixed point in $X(M;\Re)$. $\square$

4. Multidimensional results for the existence of fixed points of N-order

In this section, we drive some multidimensional results or N-order fixed point theorems from our main results by using very simple tools. Let $\Re$ be a binary relation on X and we denote by $\Re^N$ the binary relation on the product space X^N defined by:

$$\big((\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)\big)\in\Re^N \iff \begin{array}{l}(\nu_1,\rho_1)\in\Re,(\nu_2,\rho_2)\in\Re,\\ (\nu_3,\rho_3)\in\Re,\dots,(\nu_N,\rho_N)\in\Re.\end{array}$$

Suppose $M: X^N\to X$ is a mapping and by $X^N(M;\Re^N)$, we denote the set of all points $(\nu_1,\nu_2,\dots,\nu_N)\in X^N$ such that

$$\left(\begin{array}{r}(\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\nu_2,\nu_3,\dots,\nu_N,\nu_1)\\ ,\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big)\end{array}\right)\in\Re^N,$$

that is,

$$\big(\nu_i,M(\nu_i,\nu_{i+1},\dots,\nu_N,\nu_1,\nu_2,\dots,\nu_{i-1})\big)\in\Re,\ \text{for each } i\in\{1,2,\dots,N\}.$$

In addition, we denote by $\mathcal{S}_M^N: X^N\to X^N$ the mapping

$$\mathcal{S}_M^N(\nu_1,\nu_2,\dots,\nu_N)=\left(\begin{array}{r}M(\nu_1,\nu_2,\dots,\nu_N),M(\nu_2,\nu_3,\dots,\nu_N,\nu_1)\\ ,\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\end{array}\right),$$

for all $(\nu_1,\nu_2,\dots,\nu_N)\in X^N$.

Definition 34. *[24] Let $\Re$ be a binary relation defined on a non-empty set X and $(\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\in X^N$. Then $(\nu_1,\nu_2,...,\nu_N)$ and $(\rho_1,\rho_2,...,\rho_N)$ are $\Re^N$-comparative if either $\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\big)\in\Re^N$ or $\big((\rho_1,\rho_2,...,\rho_N),(\nu_1,\nu_2,...,\nu_N)\big)\in\Re^N$. We denote it by $\big[(\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)\big]\in\Re^N$.*

Definition 35. *[24] Let X be a non-empty set and $\Re$ be a binary relation on X. A sequence $\{(\nu_n^1,\nu_n^2,\dots,\nu_n^N)\}\subset X^N$ is called $\Re^N$-preserving if*

$$\big((\nu_n^1,\nu_n^2,\dots,\nu_n^N),(\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N)\big)\in\Re^N \text{ for all } n\in\mathbb{N}.$$

Definition 36. *[23] Let $M: X^N\to X$ be a mapping. A binary relation $\Re$ on X is called M_N-closed, if for any $(\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)\in X^N$,*

$$\left\{\begin{array}{c}(\nu_1,\rho_1)\in\Re\\(\nu_2,\rho_2)\in\Re\\ \cdot\\ \cdot\\ \cdot\\(\nu_N,\rho_N)\in\Re\end{array}\right\}\Rightarrow\left\{\begin{array}{c}\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\rho_1,\rho_2,\dots,\rho_N)\big)\in\Re\\ \big(M(\nu_2,\nu_3,\dots,\nu_1),M(\rho_2,\rho_3,\dots,\rho_1)\big)\in\Re\\ \cdot\\ \cdot\\ \cdot\\ \big(M(\nu_N,\nu_1,\dots,\nu_{N-1}),M(\rho_N,\rho_1,\dots,\rho_{N-1})\big)\in\Re\end{array}\right\}.$$

Definition 37. *If $M : X^N \to X$ is a mapping. Then, we denote the relational graph of the mapping M under the binary relation $\Re^N$ on X^N, by $G^N(M;\Re^N)$ and defined as:*

$$G^N(M;\Re^N) = \big\{\big((\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\nu_2,\nu_3,\dots,\nu_1), \dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big)\big) \in \Re^N : (\nu_1,\nu_2,\dots,\nu_N) \in X^N\big\}.$$

Definition 38. *Let (X,d) be a metric space, $\Re$ be a binary relation on X and $M : X^N \to X$ be a mapping. By $X^N(M;\Re^N)$, we denote the set of all those $(\nu_1,\nu_2,\dots,\nu_N) \in X^N$, for which*

$$\left(\begin{array}{r} (\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\nu_2,\nu_3,...,\nu_1) \\ ,\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big) \end{array} \right) \in G^N(M;\Re^N),$$

that is,

$$X^N(M;\Re^N) = \{(\nu_1,\nu_2,\dots,\nu_N) \in X^N : \big((\nu_1,\nu_2,\dots,\nu_N),\big(M(\nu_1,\nu_2,\dots,\nu_N), M(\nu_2,\nu_3,\dots,\nu_1),\dots,M(\nu_N,\nu_1,\dots,\nu_{N-1})\big)\big) \in G^N(M;\Re^N)\}.$$

Definition 39. *Let (X,d) be a metric space, $\Re$ be a binary relation on X and $M : X^N \to X$ be a mapping. A binary relation $\Re$ is called M_G^N-d-closed if for every $\big((\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)\big) \in G^N(M;\Re^N)$ with*

$$\left\{ \begin{array}{rl} d\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\rho_1,\rho_2,\dots,\rho_N)\big) & \le d((\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)) \\ d\big(M(\nu_2,\nu_3\dots,\nu_1),M(\rho_2,\rho_3,\dots,\rho_1)\big) & \le d((\nu_2,\nu_3,\dots,\nu_1),(\rho_2,\rho_3,\dots,\rho_1)) \\ & \vdots \\ d\left(\begin{array}{c} M(\nu_N,\nu_1,...,\nu_{N-1}), \\ M(\rho_N,\rho_1,...,\rho_{N-1}) \end{array} \right) & \le d\left(\begin{array}{c} (\nu_N,\nu_1,\dots,\nu_{N-1}), \\ (\rho_N,\rho_1,\dots,\rho_{N-1}) \end{array} \right) \end{array} \right\}$$

$$\Longrightarrow \left\{ \begin{array}{c} \big(M(\nu_1,\nu_2,\dots,\nu_N),M(\rho_1,\rho_2,\dots,\rho_N)\big) \in G^N(M;\Re^N) \\ \big(M(\nu_2,\nu_3,\dots,\nu_1),M(\rho_2,\rho_3,\dots,\rho_1)\big) \in G^N(M;\Re^N) \\ . \\ . \\ . \\ \big(M(\nu_N,\nu_1,\dots,\nu_{N-1}),M(\rho_N,\rho_1,\dots,\rho_{N-1})\big) \in G^N(M;\Re^N) \end{array} \right\}.$$

Remark 40. *It is obvious from the above definition that the condition of M_G^N-d-closedness is weaker than the condition of M_N-closedness of underlying relation in relational metric spaces.*

Definition 41. *Let X be a non-empty set and $\Re$ be a binary relation on X. A mapping $M : X^N \to X$ is said to be a $(\Re^N,k)$-continuous at $(\nu^1,\nu^2,\dots,\nu^N) \in X^N$ if for any $\Re^N$-preserving sequence $\big\{(\nu_n^1,\nu_n^2,\dots,\nu_n^N)\big\}$ in X^N such that*

$$\big\{M^{k-1}(\nu_n^1,\nu_n^2,...,\nu_n^N),M^{k-1}(\nu_n^2,\nu_n^3,...,\nu_n^1),...,M^{k-1}(\nu_n^N,\nu_n^1,...,\nu_n^{N-1})\big\} \xrightarrow{d} (\nu^1,\nu^2,\dots,\nu^N),$$

we have

$$\{M^k(\nu_n^1,\nu_n^2,\ldots,\nu_n^N),M^k(\nu_n^2,\nu_n^3,\ldots,\nu_n^1),\ldots,M^k(\nu_n^N,\nu_n^1,\ldots,\nu_n^{N-1})\}\xrightarrow{d}$$
$$\{M(\nu^1,\nu^2,\ldots,\nu^N),M(\nu^2,\nu^3,\ldots,\nu^1),\ldots,M(\nu^N,\nu^1,\ldots,\nu^{N-1})\}.$$

Then mapping M is called $(\Re^N,k)$-continuous if it is $(\Re^N,k)$-continuous at each point of X^N.

Lemma 42. *[23] Given $N\geq 2$ and $M:X^N\to X$ be a given mapping. A point $(\nu^1,\nu^2,\ldots,\nu^N)\in X^N$ is an N-order fixed point of M if and only if it is a fixed point of $\mathcal{S}_M^N$.*

Lemma 43. *[23] Given $N\geq 2$ and $M:X^N\to X$, a point $(\nu_1,\nu_2,\ldots,\nu_N)\in X^N(M;\Re^N)$ if and only if $(\nu_1,\nu_2,\ldots,\nu_N)\in X^N(\mathcal{S}_M^N;\Re^N)$.*

Lemma 44. *[23] Let (X,d) be a metric space and $D^N:X^N\times X^N\to\mathbb{R}$ be defined by*

$$D^N(U,V)=\sum_{i=1}^{N}d(u_i,v_i)$$

for all $U=(u_1,u_2,\ldots,u_N)$, $V=(v_1,v_2,\ldots,v_N)\in X^N$. Then the following properties hold:

(1) (X^N,D^N) is also a metric space.
(2) Let $\{U_n=(u_n^1,u_n^2,\ldots,u_n^N)\}$ be a sequence in X^N and $U=(u_1,u_2,\ldots,u_N)\in X^N$. Then $U_N\xrightarrow{D_N}U$ if and only if $\{u_n^i\}\xrightarrow{d}u_i$ for all $i\in\{1,2,3,\ldots,N\}$.
(3) If $\{U_n=(u_n^1,u_n^2,\ldots,u_n^N)\}$ is a sequence on X^N, then $\{U_n\}$ is a D^N-Cauchy sequence if and only if $\{u_n^i\}$ is a Cauchy sequence for all $i\in\{1,2,3,\ldots,N\}$.
(4) (X,d) is complete if and only if (X^N,D^N) is complete.

Definition 45. *Let (X^N,D^N) be a metric space and $\Re$ be a binary relation on X. If every $\Re^N$-preserving Cauchy sequence converges in X^N then we say that (X^N,D^N) is $\Re^N$-complete.*

Every complete metric space is $\Re^N$-complete under any binary relation $\Re^N$ on X^N and both the definitions coincide under the universal relation.

Definition 46. *[23] Let X be a non-empty set and $\Re$ be a binary relation on X. A path of length $k\in\mathbb{N}$ in $\Re^N$ from $(\nu_1,\nu_2,\ldots,\nu_N)\in X^N$ to $(\rho_1,\rho_2,\ldots,\rho_N)\in X^N$ is a finite sequence $\{(z_0^1,z_0^2,\ldots,z_0^N),(z_1^1,z_1^2,\ldots,z_1^N),\ \ldots,(z_k^1,z_k^2,\ldots,z_k^N)\}\subset X^N$ satisfying the following conditions:*

(i) $(z_0^1,z_0^2,\ldots,z_0^N)=(\nu_1,\nu_2,\ldots,\nu_N)$ and $(z_k^1,z_k^2,\ldots,z_k^N)=(\rho_1,\rho_2,\ldots,\rho_N)$;
(ii) $\big((z_i^1,z_i^2,\ldots,z_i^N),(z_{i+1}^1,z_{i+1}^2,\ldots,z_{i+1}^N)\big)\in\Re^N$ for all $i=0,1,2,...,k-1$.

Clearly, a path of length k involves $k+1$ elements of X^N, although they are not necessarily distinct. Moreover, let $\gamma\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N),\Re^N\big)$ be the class of all paths in $\Re^N$ from $(\nu_1,\nu_2,\ldots,\nu_N)$ to $(\rho_1,\rho_2,\ldots,\rho_N)$.

Now, we introduce the notion of generalized $\mathtt{F}_{\Re^N}$-contraction mapping and $\mathtt{F}_{\Re^N}$-graph contraction mapping for $N \geq 2$.

Definition 47. *Let (X,d) be a metric space endowed with a binary relation $\Re$ and A^N is a non-empty subset of $X^N(M;\Re^N)$. A mapping $M : X^N \to X$ is called a generalized $\mathtt{F}_{\Re^N}$-contraction with respect to A^N, if for each $(\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N) \in A^N$ with $\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\big) \in \Re^N$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that*

$$d\big(M(\nu_1,\nu_2,...,\nu_N),M(\rho_1,\rho_2,...,\rho_N)\big) > 0 \quad \Longrightarrow$$

$$\tau + \mathtt{F}\begin{pmatrix} d\big(M(\nu_1,\nu_2,...,\nu_N),M(\rho_1,\rho_2,...,\rho_N)\big)+ \\ d\big(M(\nu_2,\nu_3,...,\nu_1),M(\rho_2,\rho_3,...,\rho_1)\big)+ \\ . \\ . \\ . \\ d\big(M(\nu_N,\nu_1,...,\nu_{N-1}),M(\rho_N,\rho_1,...,\rho_{N-1})\big) \end{pmatrix} \leq \mathtt{F}\Big(\sum_{i=1}^{N} d(\nu_i,\rho_i)\Big).$$

Definition 48. *Let (X,d) be a metric space endowed with a binary relation $\Re$ and $X^N(M;\Re^N)$ be a non-empty subset of X. A mapping $M : X^N \to X$ is called a $\mathtt{F}_{\Re^N}$-graph contraction, if for each $(\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N) \in X^N(M;\Re^N)$ with $\big((\nu_1,\nu_2,\dots,\nu_N),(\rho_1,\rho_2,\dots,\rho_N)\big) \in \Re^N$, there exist $\mathtt{F} \in \mathcal{F}$ and $\tau > 0$ such that $d\big(M(\nu_1,\nu_2,\dots,\nu_N),M(\rho_1,\rho_2,\dots,\rho_N)\big) > 0 \Longrightarrow$*

$$\tau + \mathtt{F}\begin{pmatrix} d\big(M(\nu_1,\nu_2,...,\nu_N),M(\rho_1,\rho_2,...,\rho_N)\big)+ \\ d\big(M(\nu_2,\nu_3,...,\nu_1),M(\rho_2,\rho_3,...,\rho_1)\big)+ \\ . \\ . \\ . \\ d\big(M(\nu_N,\nu_1,...,\nu_{N-1}),M(\rho_N,\rho_1,...,\rho_{N-1})\big) \end{pmatrix} \leq \mathtt{F}\Big(\sum_{i=1}^{N} d(\nu_i,\rho_i)\Big). \quad (10)$$

Now using Theorem 24, we will prove a multidimensional result which conforms the existence of fixed points of N-order.

Theorem 49. *Let (X,d) be a metric space and $\Re$ be a binary relation on X. Suppose that $M : X^N \to X$ be a mapping and there exists a non-empty subset A^N of $X^N(M;\Re^N)$ such that the following conditions hold:*

(a) $M(A^N) \subseteq A^N$;
(b) M is $(\Re^N,k)$-continuous mapping;
(c) M is a generalized $\mathtt{F}_{\Re^N}$-contraction with respect to A^N;
(d) there exists $\mathrm{Y}^N \subseteq A^N$ such that $M(A^N) \subseteq \mathrm{Y}^N \subseteq A^N$ and (Y^N, D^N) is $\Re^N$-complete.

Then M has a fixed point of N-order.

Proof. Let A^N be a non-empty subset of $X^N(M, \Re^N)$ and $(\nu_0^1, \nu_0^2, \ldots, \nu_0^N) \in A^N$. Then by the virtue of subset A^N, we have

$$\left(\begin{array}{r} (\nu_0^1, \nu_0^2, \ldots, \nu_0^N), (M(\nu_0^1, \nu_0^2, \ldots, \nu_0^N), M(\nu_0^2, \nu_0^3, \ldots, \nu_0^1), \\ \ldots, M(\nu_0^N, \nu_0^1, \ldots, \nu_0^{N-1})) \end{array} \right) \in \Re^N.$$

If $(\nu_0^1, \nu_0^2, \ldots, \nu_0^N) = \left(\begin{array}{r} M(\nu_0^1, \nu_0^2, \ldots, \nu_0^N), M(\nu_0^2, \nu_0^3, \ldots, \nu_0^N, \nu_0^1), \\ \ldots, M(\nu_0^N, \nu_0^1, \ldots, \nu_0^{N-1}) \end{array} \right)$, then proof is complete. So in view of assumption (a), there exists $(\nu_1^1, \nu_1^2, \ldots, \nu_1^N)$ in A^N such that

$$(\nu_1^1, \nu_1^2, \ldots, \nu_1^N) = \left(\begin{array}{r} M(\nu_0^1, \nu_0^2, \ldots, \nu_0^N), M(\nu_0^2, \nu_0^3, \ldots, \nu_0^N, \nu_0^1), \\ \ldots, M(\nu_0^N, \nu_0^1, \ldots, \nu_0^{N-1}) \end{array} \right).$$

Again, since $(\nu_1^1, \nu_1^2, \ldots, \nu_1^N) \in A^N$ so

$$\left(\begin{array}{r} (\nu_1^1, \nu_1^2, \ldots, \nu_1^N), (M(\nu_1^1, \nu_1^2, \ldots, \nu_1^N), M(\nu_1^2, \nu_1^3, \ldots, \nu_1^N, \nu_1^1), \\ \ldots, M(\nu_1^N, \nu_1^1, \ldots, \nu_1^{N-1})) \end{array} \right) \in \Re^N.$$

If $(\nu_1^1, \nu_1^2, \ldots, \nu_1^N) = \left(\begin{array}{r} M(\nu_1^1, \nu_1^2, \ldots, \nu_1^N), M(\nu_1^2, \nu_1^3, \ldots, \nu_1^N, \nu_1^1), \\ \ldots, M(\nu_1^N, \nu_1^1, \ldots, \nu_1^{N-1}) \end{array} \right)$, then the proof is complete. Otherwise we will continue this process again and again and obtain a $\Re^N$-preserving sequence of points $\{(\nu_n^1, \nu_n^2, \ldots, \nu_n^N)\}$ in A^N such that

$$(\nu_{n+1}^1, \nu_{n+1}^2, \ldots, \nu_{n+1}^N) = \left(\begin{array}{r} M(\nu_n^1, \nu_n^2, \ldots, \nu_n^N), M(\nu_n^2, \nu_n^3, \ldots, \nu_n^N, \nu_n^1), \\ \ldots, M(\nu_n^N, \nu_n^1, \ldots, \nu_n^{N-1}) \end{array} \right)$$

and

$$\big((\nu_n^1, \nu_n^2, \ldots, \nu_n^N), (\nu_{n+1}^1, \nu_{n+1}^2, \ldots, \nu_{n+1}^N)\big) \in \Re^N, \text{ for all } n \in \mathbb{N}.$$

Since M is $(\Re^N, k)$-continuous, we get $\mathcal{S}_M^N$ is also $(\Re^N, k)$-continuous. From the generalized $\mathtt{F}_{\Re^N}$-contractive condition of M, we deduce that $\mathcal{S}_M^N$ is also a generalized $\mathtt{F}_{\Re^N}$-contraction. Applying Theorem 24, there exists $\hat{\mathrm{Z}} = (\nu_1^*, \nu_2^*, \ldots, \nu_N^*) \in X^N$ such that $\mathcal{S}_M^N(\hat{\mathrm{Z}}) = \hat{\mathrm{Z}}$, i.e., $(\nu_1^*, \nu_2^*, \ldots, \nu_N^*)$ is a fixed point of $\mathcal{S}_M^N$. Using Lemma 42, we have $(\nu_1^*, \nu_2^*, \ldots, \nu_N^*)$ is a fixed point of N-order of M. This completes the proof. □

If we take $\tau = \log \frac{1}{\varrho}$ and $\mathtt{F} = \log \nu$ in Theorem 49 then we get the following corollary as a direct consequence of Theorem 49.

Corollary 50. *Let (X, d) be a metric space and $\Re$ be a binary relation on X. Suppose that $M : X^N \to X$ be a mapping and there exists a non-empty subset A^N of $X^N(M; \Re^N)$ such that the following conditions hold:*

(a) $M(A^N) \subseteq A^N$,
(b) M is $(\Re^N, k)$-continuous mapping,

(c) there exists $\varrho \in [0,1)$ such that

$$\sum_{i=1}^{N} d\begin{pmatrix} M(\nu_i,\nu_{i+1},...,\nu_N,\nu_1,...,\nu_{i-1}), \\ M(\rho_i,\rho_{i+1},...,\rho_N,\rho_1,...,\rho_{i-1}) \end{pmatrix} \leq \varrho \sum_{i=1}^{N} d(\nu_i,\rho_i),$$

for each $(\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N) \in A^N$ such that $\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\big) \in \Re^N$, then M has a fixed point of N-order.

(d) there exists $\mathrm{Y}^N \subseteq A^N$ and $M(A^N) \subseteq \mathrm{Y}^N \subseteq A^N$, so that (Y^N, D^N) is $\Re^N$-complete.

Using similar technique as in the proof of Theorem 49, we obtain the following multidimensional result for the existence of fixed points of N-order.

Theorem 51. *Let (X,d) be a metric space and $\Re$ be a binary relation on X. Suppose $X^N(M;\Re^N)$ be a non-empty and $M : X^N \to X$ be a mapping such that the following conditions hold:*

(a) $\Re$ is M_G^N-d-closed;
(b) M is $(\Re^N,k)$-continuous;
(c) M is $\mathtt{F}_{\Re^N}$-graph contraction on X^N;
(d) there exists $\mathrm{Y}^N \subseteq X^N(M;\Re^N)$ such that $M(X^N(M;\Re^N)) \subseteq \mathrm{Y}^N \subseteq X^N(M;\Re^N)$ and (Y^N,D^N) is $\Re^N$-complete,

then M has a fixed point of N-order.

Proof. Suppose $X^N(M;\Re^N)$ be a non-empty set and $(\nu_0^1,\nu_0^2,\dots,\nu_0^N) \in X^N(M;\Re^N)$. Then, we have

$$\big((\nu_0^1,\nu_0^2,\dots,\nu_0^N),(M(\nu_0^1,\nu_0^2,\dots,\nu_0^N),\dots,M(\nu_0^N,\nu_0^1,\dots,\nu_0^{N-1}))\big) \in \Re^N.$$

Now in view of assumption (a) and from $\mathtt{F}_{\Re^N}$-graph contraction condition (10), we have

$$\big((\nu_1^1,\nu_1^2,\dots,\nu_1^N),(\nu_2^1,\nu_2^2,\dots,\nu_2^N)\big) =$$
$$\begin{pmatrix} \big(M(\nu_0^1,\nu_0^2,\dots,\nu_0^N), M(\nu_0^2,\nu_0^3,\dots,\nu_0^N,\nu_0^1),\dots,M(\nu_0^N,\nu_0^1,\dots,\nu_0^{N-1}) \big), \\ \big(M(\nu_1^1,\nu_1^2,\dots,\nu_1^N), M(\nu_1^2,\nu_1^3,\dots,\nu_1^N,\nu_1^1),\dots,M(\nu_1^N,\nu_1^1,\dots,\nu_1^{N-1} \big) \end{pmatrix}.$$

Continuing this process again and again, we get a $\Re^N$-preserving Cauchy sequence of points $(\nu_n^1,\nu_n^2,\dots,\nu_n^N)$ in X^N such that

$$(\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N) = \begin{pmatrix} M(\nu_n^1,\nu_n^2,\dots,\nu_n^N), M(\nu_n^2,\nu_n^3,\dots,\nu_n^N,\nu_n^1), \\ \dots, M(\nu_n^N,\nu_n^1,\dots,\nu_n^{N-1}) \end{pmatrix}$$

and

$$\big((\nu_n^1,\nu_n^2,\dots,\nu_n^N),(\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N)\big) \in \Re^N, \text{ for all } n \in \mathbb{N}.$$

If we take $(\nu_n^1,\nu_n^2,\dots,\nu_n^N) = (\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N)$ for some $n \in \mathbb{N}$, then $\{(\nu_n^1,\nu_n^2,\dots,\nu_n^N)\}$ is called a fixed point of M. Therefore we assume $(\nu_n^1,\nu_n^2,\dots,\nu_n^N) \neq (\nu_{n+1}^1,\nu_{n+1}^2,\dots,\nu_{n+1}^N)$ for all $n \in \mathbb{N}$. Now proceeding the proof of Theorem 49 we get the conclusion. $\square$

Corollary 52. *Let (X,d) be a metric space and $\Re$ be a binary relation on X. Suppose $X^N(M;\Re^N)$ be a non-empty set and $M : X^N \to X$ be a mapping such that the following conditions hold:*

(a) $\Re$ is M_G^N-d-closed,
(b) M is $(\Re^N, k)$-continuous,
(c) there exists $\varrho \in [0,1)$ such that

$$\sum_{i=1}^{N} d\left(\begin{array}{c} M(\nu_i,\nu_{i+1},...,\nu_N,\nu_1,...,\nu_{i-1}), \\ M(\rho_i,\rho_{i+1},...,\rho_N,\rho_1,...,\rho_{i-1}) \end{array}\right) \le \varrho \sum_{i=1}^{N} d(\nu_i,\rho_i),$$

for all $\big((\nu_1,\nu_2,...,\nu_N),(\rho_1,\rho_2,...,\rho_N)\big) \in G^N(M;\Re^N)$,
(d) there exists $\mathrm{Y}^N \subseteq X^N(M;\Re^N)$ such that $M(X^N(M;\Re^N)) \subseteq \mathrm{Y}^N \subseteq X^N(M;\Re^N)$ and (Y^N, D^N) is $\Re^N$-complete.

Then M has a fixed point of N-th order.

5. Application to nonlinear matrix equations

In this section, we follow the following notations:

- $\mathcal{X}_n$ denotes the set of all $n \times n$ Complex matrices;
- $\mathcal{H}_n \subset \mathcal{X}_n$ is the set of all $n \times n$ Hermitian matrices;
- $\mathcal{P}_n \subset \mathcal{H}_n$ is the set of all $n \times n$ positive definite matrices;
- $\mathcal{H}_n^+ \subset \mathcal{H}_n$ is the set of all $n \times n$ positive semidefinite matrices.

and for $U, V \in \mathcal{X}_n$, we denote the following notations:

- $U \succ 0 \Longleftrightarrow U \in \mathcal{P}_n$;
- $U \succeq 0 \Longleftrightarrow U \in \mathcal{H}_n^+$;
- $U - V \succ 0 \Longleftrightarrow U \succ V$;
- $U - V \succeq 0 \Longleftrightarrow U \succeq V$.

Let B^* is the conjugate transpose of B and $\lambda^+(B^*B)$ is the largest eigenvalue of B^*B. We use the symbol $\|.\|$ for the spectral norm of B and defined by $\|B\| = \sqrt{\lambda^+(B^*B)}$.

The symbol $\|.\|_{tr}$ is used for the metric induced by trace norm and it is defined by $\|B\|_{tr} = \sum_{j=1}^{n} s_j(B)$, where $s_j(B), j = 1,2,...,n$, are the singular values of $B \in \mathcal{X}_n$. Hence, $(\mathcal{H}_n, \|.\|_{tr})$ forms a complete metric space. See ([8], [9], [18]) for more details. Moreover, the binary relation $\preceq$ on $\mathcal{H}_n$ defined by:

$$U \preceq V \Longleftrightarrow V \succeq U$$

for all $U, V \in \mathcal{H}_n$.

In this section, we apply Theorem 24 to establish a solution of the nonlinear matrix equation.

$$U = Q + \sum_{i=1}^{n} A_i^* \mathcal{G}(U) A_i \tag{11}$$

where A_i is an any $n \times n$ matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}$ is continuous order preserving mapping (i.e., if $U, V \in \mathcal{H}_n$ with $U \preceq V$ implies that $\mathcal{G}(U) \preceq \mathcal{G}(V)$) with $\mathcal{G}(0) = 0$.

Now we state the following lemmas which are very useful in this sequel:

Lemma 53. *If $U, V \in \mathcal{H}_n^+$ such that $U \succeq 0$ and $V \succeq 0$, Then*

$$0 \leq tr(UV) \leq \|U\| tr(V).$$

Lemma 54. *If $U \in \mathcal{H}_n$ and $U \prec I$, then $\|U\| < 1$.*

Theorem 55. *Consider the matrix equation* (11) *and suppose that there is a positive numbers k and τ such that*

(i) For every $U, V \in \mathcal{H}_n^+$ with $U \preceq V$ and $\sum_{i=1}^n A_i^* \mathcal{G}(U) A_i \neq \sum_{i=1}^n A_i^* \mathcal{G}(V) A_i$, we have

$$|tr(\mathcal{G}(V) - \mathcal{G}(U))| \leq \frac{|tr(V-U)|}{k(1+\tau\sqrt{tr(V-U)})^2}; \tag{12}$$

(ii) $\sum_{i=1}^m A_i A_i^* \prec k I_n$ and $\sum_{i=1}^m A_i^* \mathcal{G}(U) A_i \succ 0$.

Then the matrix equation (11) has a solution. Moreover, the iteration

$$U_n = Q + \sum_{i=1}^n A_i^* \mathcal{G}(U_{n-1}) A_i \tag{13}$$

where $U_0 \in \mathcal{H}_n$ such that $U_0 \preceq Q + \sum_{i=1}^n A_i^* \mathcal{G}(U_0) A_i$, converges in the sense of trace norm $\|.\|_{tr}$, to the solution of the nonlinear matrix equation (11).

Proof. We define a mapping $M : \mathcal{H}_n \to \mathcal{H}_n$ by

$$M(U) = Q + \sum_{i=1}^n A_i^* \mathcal{G}(U) A_i$$

for all $U \in \mathcal{H}_n$ and a set by

$$\mathcal{H}_n^+(M, \preceq) = \{A \in \mathcal{H}^+ : A \preceq M(A) \text{ or } M(A) - A \succeq 0\}.$$

Then M is well defined mapping, $\mathcal{H}_n^+(M, \preceq)$ is a non-empty set as $Q \in \mathcal{H}^+$ and $M(Q) - Q = \sum_{i=1}^n A_i^* \mathcal{G}(Q) A_i \succeq 0$. It is easy to verify that for every positive semidefinite matrix B, $M(B)$ is also positive semidefinite matrix and $\mathcal{H}_n^+(M, \preceq)$ is $\preceq$-complete. Now, we will prove that the set $\mathcal{H}_n^+(M, \preceq)$ is invariant under the mapping M, that is $M(\mathcal{H}_n^+(M, \preceq)) \subseteq \mathcal{H}_n^+(M, \preceq)$. For this, it is sufficient to prove that $M(B) \in \mathcal{H}_n^+(M, \preceq)$ for every $B \in \mathcal{H}_n^+(M, \preceq)$. Let $B \in \mathcal{H}_n^+(M, \preceq)$ then $M(B) - B \succeq 0$ and

$$M(M(B)) - M(B) = \sum_{i=1}^n A_i^* \big(\mathcal{G}(M(B)) - \mathcal{G}(B)\big) A_i \succeq 0, \tag{14}$$

that is $M(B) \preceq M(M(B))$, which implies $M(B) \in \mathcal{H}_n^+(M, \preceq)$.

Next, we will show that M is a generalized $\mathtt{F}_{\preceq}$-contraction mapping with respect to $\mathcal{H}_n^+(M,\preceq)$. For this, let $\tau > 0$ be any real number and $\mathtt{F} : \mathbb{R}^+ \to \mathbb{R}$ be mapping defined as

$$\mathtt{F}(\varrho) = -\frac{1}{\sqrt{\varrho}} \text{ for all } \varrho \in \mathbb{R}^+.$$

Then from (12), for each $U, V \in \mathcal{H}_n^+(M,\preceq)$ with $U \preceq V$ and $\mathcal{G}(U) \preceq \mathcal{G}(V)$, we have

$$\begin{aligned}
\|M(V) - M(U)\|_{tr} &= tr(M(V) - M(U)) \\
&= tr\Big(\sum_{i=1}^{m} A_i^*(\mathcal{G}(V) - \mathcal{G}(U))A_i\Big) \\
&= \sum_{i=1}^{m} tr(A_i^*(\mathcal{G}(V) - \mathcal{G}(U)A_i) \\
&= \sum_{i=1}^{m} tr(A_iA_i^*(\mathcal{G}(V) - \mathcal{G}(U))) \\
&= tr\Big(\big(\sum_{i=1}^{m} A_iA_i^*\big)(\mathcal{G}(V) - \mathcal{G}(U))\Big) \\
&\le \Big(\|\sum_{i=1}^{m} A_iA_i^*\|\Big)\|\mathcal{G}(V) - \mathcal{G}(U)\|_{tr} \\
&\le \frac{\|\sum_{i=1}^{m} A_iA_i^*\|}{k}\left(\frac{\|V-U\|_{tr}}{\big(1+\tau\sqrt{\|V-U\|_{tr}}\big)^2}\right) \\
&< \left(\frac{\|V-U\|_{tr}}{\big(1+\tau\sqrt{\|V-U\|_{tr}}\big)^2}\right)
\end{aligned}$$

and so

$$\frac{\big(1+\tau\sqrt{\|V-U\|_{tr}}\big)^2}{\|V-U\|_{tr}} \le \frac{1}{\|M(V) - M(U)\|_{tr}}.$$

This implies that

$$\left(\tau + \frac{1}{\sqrt{\|V-U\|_{tr}}}\right)^2 \le \frac{1}{\|M(V) - M(U)\|_{tr}}$$

or

$$\tau + \frac{1}{\sqrt{\|V-U\|_{tr}}} \le \frac{1}{\sqrt{\|M(V) - M(U)\|_{tr}}}.$$

This yields that

$$\tau - \frac{1}{\sqrt{\|M(V) - M(U)\|_{tr}}} \le -\frac{1}{\sqrt{\|V-U\|_{tr}}}.$$

Hence

$$\tau + \mathtt{F}(\|M(V) - M(U)\|_{tr}) \leq \mathtt{F}(\|V - U\|_{tr}),$$

which shows that M is a generalized $\mathtt{F}_{\preceq}$-contraction with respect to $\mathcal{H}_n^+(M, \preceq)$. Since all the assumptions of Theorem 24 are satisfied therefore there exists $Z \in \mathcal{H}_n$ such that $M(Z) = Z$, i.e., the matrix equation (11) has a solution. □

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THE TRIPLE ZERO GRAPH OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with non-zero identity. We define the set of triple zero elements of R by $TZ(R) = \{a \in Z(R)^* :$ there exist $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$. In this paper, we introduce and study some properties of the triple zero graph of R which is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$, and two vertices a and b are adjacent if and only if $ab \neq 0$ and there exists a non-zero element c of R such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$. We investigate some properties of the triple zero graph of a general ZPI-ring R, we prove that $diam(TZ\Gamma(R)) \in \{0, 1, 2\}$ and $gr(TZ\Gamma(R)) \in \{3, \infty\}$.

1. Introduction

Throughout this paper, all rings are commutative with identity and $Z(R)$ denotes the set of zero-divisors of a ring R. The concept of the zero-divisor graph of a commutative ring was introduced by I. Beck [9]. He let all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. In [3], all elements of a commutative ring R are vertices, and distinct vertices a and b are adjacent if and only if $ab = 0$. This graph is denoted by $\Gamma_0(R)$. Then D.F. Anderson and P.S. Livingston [4] introduced a (induced) zero-divisor subgraph $\Gamma(R)$ of $\Gamma_0(R)$. The zero-divisor graph $\Gamma(R)$ introduced in [13] and [4] is as follows: Two distinct vertices $x, y \in Z(R)^* = Z(R)\backslash\{0\}$ are adjacent if and only if $xy = 0$. In [4], D.F. Anderson and P.S. Livingston have shown that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $gr((\Gamma(R)) \in \{3, 4, \infty\}$. The zero-divisor graph of a commutative ring in the sense of Anderson–Livingston has been studied extensively by several authors, [1], [2], [5], [6], [14], [15]. Since then, the concept of the zero-divisor graph of ring has been playing a vital role in its expansion.

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We define the set of the triple zero elements of R by $TZ(R) = \{a \in Z(R)^* :$ there exist $b, c \in R \backslash \{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$. It is clear that every triple zero element of R is a zero-divisor of R, but the converse is not true in general. For example, the element 2 is a zero-divisor of $\mathbb{Z}_6$, but clearly it is not a triple zero element. In this paper, motivated from zero-divisor graphs, we introduce the triple zero graph of a commutative ring. Our starting point is the following definition: The triple zero graph of R is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$. If two distinct elements a and b are adjacent, then (a, b) is an edge and we will denote it by $a \sim b$. Two distinct vertices a and b are adjacent if and only if $ab \neq 0$ and there exists an element $c \in R \backslash \{0\}$ such that $ac \neq 0$, $bc \neq 0$ and $abc = 0$. The relation "$\sim$" is always symmetric, but neither reflexive nor transitive in general. For instance, let $R = \mathbb{Z}_{36}$. Then clearly 2, 3, $6 \in TZ(R)$ with $6 \nsim 6$, and also $2 \sim 3$, $2 \sim 9$, but $3 \nsim 9$.

Recall from [8] that I is said to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. As defined in [7], I is said to to be a weakly 2-absorbing ideal *of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$, $ac \in I$, or $bc \in I$. From these definitions, note that $\{0\}$ is always a weakly 2-absorbing ideal of R. If 0 is not a 2-absorbing ideal, then there are some triple zero elements of R. The concept of (weakly) 2-absorbing ideals and the zero-divisor graphs motivated us to define the triple zero divisor graph and also investigate the relations between triple zero graph of a ring R and 2-absorbing ideals of R.

Among many results in this paper, in Section 2, we justify some properties of the triple zero graph of commutative rings. In Theorem 1, we show that a proper ideal I of a ring R is 2-absorbing if and only if $TZ\Gamma(R/I) = \emptyset$. In Theorem 11, we characterize triangle free triple zero graphs of general ZPI-rings. In [11], the authors define 3-zero-divisor hypergraph regarding to an ideal with vertices $\{x \in R \backslash I : xyz \in I$ for some $y, z \in R \backslash I$ such that $xy \notin I$, $yz \notin I$, $xz \notin I\}$ where distinct vertices are adjacent if and only if $xyz \in I$, $xy \notin I$, $yz \notin I$ and $xz \notin I$. They conclude that diameter of this graph is at most 4. In Section 3, we study the triple zero graph of general ZPI-rings. The graph properties of the triple zero graph of general ZPI-rings such as diameter and girth are investigated. We obtain that the triple zero graph of a zero dimensional general ZPI-ring is always connected with diameter at most 2 and girth 3 if it is determined. (Corollary 12). Furthermore, we give some characterizations for the triple zero graph of $\mathbb{Z}_n$ where $n > 1$ and justify the diameter and girth of $TZ\Gamma(\mathbb{Z}_n)$. (Theorem 13, Theorem 14 and Corollary 15)

For the sake of completeness, we state some definitions and notation used throughout. Let G be a (undirected) graph. The order of G, denoted by $|G|$, is equal to the cardinality of the vertex set. The graph G is connected if there is a path between any two distinct vertices. For vertices a and b of G, we say that the distance between a and b, $d(a, b)$ is the length of a shortest path from a to b. If there is no path between a and b, then $d(a, b) = \infty$, and $d(a, a) = 0$. A graph G is said to be totally disconnected if it has no edges. The diameter of G is defined by $diam(G) = \sup\{d(a, b) : a$

and b are vertices of $G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. If G contains no cycles, then $gr(G)=\infty$. A cycle of length three is commonly called a triangle. A triangle-free graph is an undirected graph in which no three vertices form a triangle of edges. A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n. A complete bipartite graph is a graph G which may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We denote the complete bipartite graph by $K_{m,n}$ where A and B are partitions with $\mid A \mid= m$ and $\mid B \mid= n$. If one of the vertex sets is a singleton, then we call G a star graph. A star graph is clearly $K_{1,n}$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively. For general background and terminology, the reader may consult [10].

2. Properties of The Triple Zero Graph

Theorem 1. *Let R be a commutative ring and I be a proper ideal of R. Then the following statements hold:*

1. $TZ\Gamma(R/I)=\emptyset$ if and only if I is a 2-absorbing ideal of R.
2. $TZ\Gamma(R)=\emptyset$ if and only if $\{0\}$ is a 2-absorbing ideal of R.
3. If (R,M) is a quasi-local ring with $M^2=0$, then $TZ\Gamma(R)=\emptyset$.

Proof. Suppose that I is not a 2-absorbing ideal of R. Then there exist some (not necessarily distinct) elements a,b,c of R with $abc\in I$ but neither $ab\in I$ nor $ac\in I$ nor $bc\in I$. Hence $(a+I)(b+I)(c+I)=I$ but neither $(a+I)(b+I)=I$ nor $(a+I)(c+I)=I$ nor $(b+I)(c+I)=I$. Thus $a,b,c\in TZ(R/I)$; and so $TZ\Gamma(R/I)\neq\emptyset$. Conversely, if $TZ\Gamma(R/I)\neq\emptyset$, then there are some (not necessarily distinct) elements $a+I,b+I,c+I$ of R/I satisfying $(a+I)(b+I)(c+I)=I$ but neither $(a+I)(b+I)=I$ nor $(a+I)(c+I)=I$ nor $(b+I)(c+I)=I$. It implies that $ab,ac,bc\notin I$ and $abc\in I$. Hence I is not a 2-absorbing ideal of R.

(2) It is clearly a particular case putting $I=0$ in (1).

(3) Suppose that (R,M) is a quasi-local ring with $M^2=0$. Hence 0 is a 2-absorbing ideal of R by [7, Corollary 3.3]. Thus $TZ\Gamma(R)=\emptyset$ by (2). □

The following example shows that the converse of Theorem 1 (3) does not hold.

Example 2. *Consider $R=\mathbb{Z}_2\times\mathbb{Z}_2$. Then clearly $TZ\Gamma(R)=\emptyset$ but since R has two maximal ideals $0\times\mathbb{Z}_2$ and $\mathbb{Z}_2\times 0$, it is not a quasi-local ring.*

Let $R=\mathbb{Z}_p[X]/\langle X^n\rangle$, where p is prime and $n\geq 3$. We denote $a(X)$ as the congruence class of polynomials congruent to $a(X) \bmod \langle X^n\rangle$. It is well-known that an element of $\mathbb{Z}_p[X]/\langle X^n\rangle$ is of the form $a(X)=a_0+a_1X+a_2X^2+\cdots+a_kX^k$ of degree $k\leq n$ where $a_i\in\mathbb{Z}_p$ for $i\in\{1,2,...,k\}$. Now we determine the vertex set of the graph $TZ(\mathbb{Z}_p[X]/\langle X^n\rangle)$.

Theorem 3. *Let $a(X) = a_0 + a_1X + a_2X^2 + \cdots + a_kX^k \in \mathbb{Z}_p[X]/\langle X^n\rangle$ where $n \geq 3$. Then $a(X)$ is a vertex of the graph $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^n\rangle)$ if and only if $a_0 = 0(\operatorname{mod} p)$ and of the form in one of the following types:*

(1) $a_1 = a_2 = ... = a_{k-1} = 0$ and $k \leq n - 2$.
(2) $a_i \neq 0$ for some $r = 1, 2, ..., k - 1$ and $k \leq n - 1$.

Proof. Let $a(X) \in TZ(\mathbb{Z}_p[X]/\langle X^n\rangle)$. Then there exists non-zero $b(X)$, $c(X) \in \mathbb{Z}_p[X]/\langle X^n\rangle$ such that $a(X)b(X)c(X) = 0 \operatorname{mod}\langle X^n\rangle$, $a(X)b(X) \neq 0 \operatorname{mod}\langle X^n\rangle$, $a(X)c(X) \neq 0 \operatorname{mod}\langle X^n\rangle$ and $b(X)c(X) \neq 0 \operatorname{mod}\langle X^n\rangle$. Let $b(X) = b_0 + b_1X + b_2X^2 + \cdots + b_tX^t$, $c(X) = c_0 + c_1X + c_2X^2 + \cdots + c_sX^s$ where b_j and c_r are the first non-zero (i.e., $b_j, c_r \neq 0(\operatorname{mod} p)$) coefficients in the polynomials $b(X)$ and $c(X)$, respectively. Then the coefficient of X_{j+r} in the product $a(X)b(X)c(X)$ is $a_0b_jc_r$. Since $a(X)b(X)c(X) = 0 \operatorname{mod}\langle X^n\rangle$ and $j, r < n$, we must have $a_0b_ic_j = 0(\operatorname{mod} p)$. Observe that since b_j, c_r are non-zero elements of $\mathbb{Z}_p$, we have $b_jc_r \neq 0$. Thus $a_0 = 0(\operatorname{mod} p)$.

Case I. Suppose that $a_1 = a_2 = ... = a_{k-1} = 0$. Then $a_kX^kb_jX^jc_rX^r = 0 \operatorname{mod}\langle X^n\rangle$ which implies that $k + j + r = n$. Since $j, r \geq 1$, we conclude that $k \leq n - 2$.

Case II. Suppose that $a_i \neq 0$ for some $i = 1, 2, ..., k - 1$. Then we show that k can be $n - 1$. Assume that $\deg(a(X)) = k = n - 1$. Then, clearly $a(X)$ X $X = 0 \operatorname{mod}\langle X^n\rangle$ and X $X \neq 0 \operatorname{mod}\langle X^n\rangle$. Since $a_iX^iX \neq 0 \operatorname{mod}\langle X^n\rangle$ where $i = 1, 2, ..., k - 1$, we conclude that $a(X)$ $X \neq 0 \operatorname{mod}\langle X^n\rangle$.

Conversely, assume that $a_0 = 0$ $(\operatorname{mod} p)$. If (1) holds, then $a(X) = a_kX^k$ and $k \leq n - 2$. Then $a(X)$ X^j $X^r = 0 \operatorname{mod}\langle X^n\rangle$ for all $j, r \geq 1$ such that $j + r = n - k$ but neither $a(X)$ $X^j = 0 \operatorname{mod}\langle X^n\rangle$ nor $a(X)$ $X^r = 0 \operatorname{mod}\langle X^n\rangle$ nor X^j $X^r = 0 \operatorname{mod}\langle X^n\rangle$. Hence $a(X)$ is a triple zero element of $\mathbb{Z}_p[X]/\langle X^n\rangle$. Suppose that (2) holds. We may assume that $a_1 \neq 0$ (mod p). Then $a(X)$ X^j $X^r = 0 \operatorname{mod}\langle X^n\rangle$ for all $j, r \geq 1$ such that $j + r = n - 1$. Since a_1X $X^j \neq 0 \operatorname{mod}\langle X^n\rangle$ and a_1X $X^r \neq 0 \operatorname{mod}\langle X^n\rangle$, we conclude that $a(X)$ $X^j \neq 0 \operatorname{mod}\langle X^n\rangle$ and $a(X)$ $X^r \neq 0 \operatorname{mod}\langle X^n\rangle$. Thus $a(X)$ is a triple zero element of $\mathbb{Z}_p[X]/\langle X^n\rangle$. □

Theorem 4. *Let $R = \mathbb{Z}_p[X]/\langle X^3\rangle$. Then $TZ\Gamma(R)$ is a complete graph with $p^2 - p$ vertices, i.e., $TZ\Gamma(R) \cong K_{p^2-p}$. In particular, if $p = 2$, then $TZ\Gamma(R) \cong K_2$.*

Proof. From Theorem 3, the vertices of $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^3\rangle)$ of the type $nX + mX^2$, where n, m are integers with $1 \leq n \leq p$ and $0 \leq m \leq p$. Hence, the number of the vertices of $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^3\rangle)$ is $p^2 - p$. Observe that all vertices of this graph are adjacent, thus it is the complete graph K_{p^2-p}. For $p = 3$, this graph is illustrated by Figure 2. In the particular case, since X X $(X + X^2) = 0$ but X $X \neq 0$ and $X(X + X^2) \neq 0$, X and $(X + X^2)$ are the only distinct adjacent vertices of $TZ\Gamma(\mathbb{Z}_2[X]/\langle X^3\rangle)$. □

We are unable to answer the following question which may be inspiring for the possible other work:

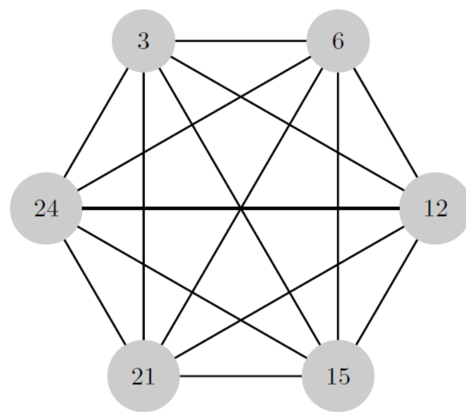

FIGURE 1. $TZ\Gamma(\mathbb{Z}_{27})$

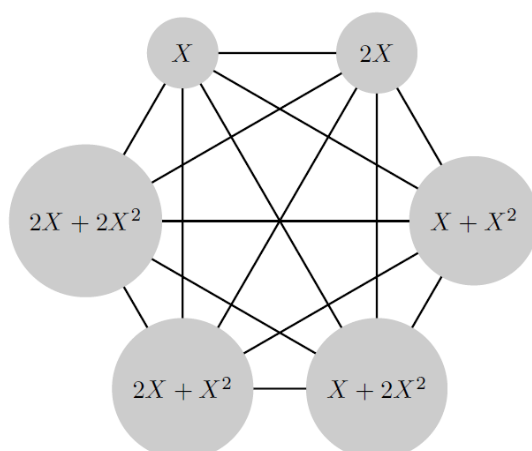

FIGURE 2. $TZ\Gamma(\mathbb{Z}_3[X]/\langle X^3\rangle)$

Question. Let $R = \mathbb{Z}_p[X]/\langle X^n\rangle$ where p is a prime number and $n \geq 3$. Can we have a general characterization for the triple zero graph of R?

We recall that an n-gon is a regular polygon with n sides. In the next example, we show that there are triple zero graphs with cycles of arbitrary specified length.

Example 5. *Let T be an integral domain and $n \geq 3$ is an integer. Consider $R = T[X_1, X_2, \cdots, X_n]/(X_1X_2X_3, X_3X_4X_5, \cdots, X_{n-1}X_nX_1)$. Then $TZ\Gamma(R)$ is a connected graph which has an n-gon, an $n/2$-gon and has triangles more than n.*

Proof. Observe that $X_1 \sim X_2 \sim X_3$, $X_3 \sim X_4 \sim X_5, \cdots, X_{n-1} \sim X_n \sim X_1$ are some of the triangles, and it is easy to see that $(X_k + X_kX_{k+1}) \sim (X_{k+1} + X_kX_{k+1}) \sim X_{k+2}$ is another triangle for each k, where k is odd, and $k < n-2$. Also $X_1 \sim X_3 \sim \cdots \sim X_{n-1} \sim X_1$ is an $n/2-gon$ and $X_1 \sim X_2 \sim \cdots \sim X_{n-1} \sim X_n \sim X_1$ is an n-gon. $\square$

3. Triple Zero Graph of General ZPI-rings

A ring is called a general ZPI-ring (resp. ZPI-ring) if each ideal (resp. each nonzero ideal) I of R is uniquely expressible as product of prime ideals of R. Dedekind domains are indecomposable general ZPI-rings. For a general background, the reader may refer to [12]. In this section, we study the graph theoretical properties of the triple zero graph for general ZPI-rings. First we need to prove the following lemma which is a generalization of [8, Theorem 3.15].

Lemma 6. *Let R be a zero dimensional Noetherian ring which is not a field. Then the following statements are equivalent:*

(1) R is a general ZPI-ring.
(2) If I is a 2-absorbing ideal of R, then I is a maximal ideal of R or $I = M^2$ for some maximal ideal M of R or $I = MM'$ for some maximal ideals M, M' of R.
(3) If I is a 2-absorbing ideal of R, then I is a prime ideal of R or $I = P^2$ for some prime ideal P of R or $I = PQ$ for some prime ideals P, Q of R.

Proof. (1)$\Rightarrow$(2) Let I be a 2-absorbing ideal of R. Since maximal ideals coincide with prime ideals, $\sqrt{I} = M$ for some maximal ideal M of R with $M^2 \subseteq I$ or $\sqrt{I} = M \cap M' = MM'$ for some maximal ideals M, M' of R with $MM' \subseteq I$ by [8, Theorem 2.4]. Thus, we have either $I = M$ is maximal or $I = M^2$ for some maximal ideal M of R or $I = MM'$ for some maximal ideals M, M' of R.

(2)$\Rightarrow$(3) is straightforward.

(3)$\Rightarrow$(1) Suppose that (3) holds. Assume that there is an ideal J of R which satisfies $M^2 \subseteq I \subseteq M$. Then I is an M-primary ideal of R; so I is a 2-absorbing ideal by [8, Theorem 3.1]. Hence $I = M$ or $I = M^2$ from our assumption (3). Thus there are no ideals properly between M and M^2. From [12, (39.2) Theorem], R is a general ZPI-ring. $\square$

Theorem 7. *Let R be a zero dimensional general ZPI-ring. Then $TZ\Gamma(R) = \emptyset$ if and ony if either R is an integral domain or $0 = P^2$ where P is a prime ideal of R or $0 = PQ$ where P and Q are prime ideals of R.*

Proof. If R is an integral domain or $0 = P^2$ where P is a prime ideal of R or $0 = PQ$ where P and Q are prime ideals of R, then it is easy to verify that there is no triple zero elements of R; so $TZ\Gamma(R) = \emptyset$. Conversely, suppose that $TZ\Gamma(R) = \emptyset$. Then 0 is a 2-absorbing ideal of R by Theorem 1. From Lemma 6, either 0 is prime, $0 = P^2$ for some prime ideal P or $0 = PQ$ for some prime ideals P,Q of R, so we are done. □

We recall that a special primary is an indecomposable general ZPI-ring which is a local ring with maximal ideal M such that each proper ideal of R is a power of M.

Lemma 8. *[12] An indecomposable general ZPI-ring with identity is either a Dedekind domain or a special primary ring.*

Theorem 9. *Let R be a general ZPI-ring and $0 = P^3$ where P is a prime ideal of R such that $P^2 \neq 0$. Then $TZ\Gamma(R)$ is a complete graph on $|P| - \left|P^2\right|$ vertices; i.e. $TZ\Gamma(R) \cong K_{|P|-|P^2|}$*

Proof. Suppose that $0 = P^3$ where P is a prime ideal of R. It is well-known that a ring R is indecomposable if and only if 1 is the only non-zero idempotent element of R. Let $0 \neq a \in R$ and $a^2 = a$. Hence $a - a^2 = a(1-a) = 0 \in P$ implies $a \in P$ or $(1-a) \in P$. If $a \in P$, then we get $0 = a^3 = a^2 = a$, a contradiction. Thus $(1-a) \in P$. It follows $0 = (1-a)^3 = 1 - 2a^2 + 2a - a^3 = 1 - a$, and so $a = 1$. Therefore, R is a indecomposable ring which is clearly not a domain as $0 = P^3$ and P is nonzero. Hence, we conclude from Lemma 8 that R is a special primary ring. Let M be the unique maximal ideal of R. Since every ideal, in particular, the zero ideal is a power of M, we have $M \subseteq \sqrt{0}$. Since $0 = P^3$, clearly we have $P = \sqrt{0} = M$.

Now, we show that a is a vertex of $TZ\Gamma(R)$ if and only if $a \in P\backslash P^2$. Let a be a vertex of $TZ\Gamma(R)$. Then, there exist $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$. If $a \notin P$, then a is unit and $bc = 0$ which is a contradiction. Thus $TZ(R) \subseteq P$. If $a \in P^2$, then since $b \in TZ(R) \subseteq P$, we conclude $ab \in P^3 = 0$, a contradiction. Therefore, $a \in P\backslash P^2$. Conversely, if $a \in P\backslash P^2$, then the claim follows from $a^3 = 0$ and $a^2 \in P^2 \neq 0$. Suppose a and b are any two distinct vertices. Since $a^2b = ab^2 = 0$ and ab, a^2, b^2 are nonzero, a and b are adjacent. Thus, $TZ\Gamma(R)$ is a complete graph on $|P| - \left|P^2\right|$ vertices. □

Theorem 10. *Let $0 = P^2Q$ where P and Q are prime ideals of a general ZPI-ring R. Then $TZ\Gamma(R)$ is a connected graph with diameter 2 and girth 3.*

Proof. Suppose that $0 = P^2Q$. Let a be a vertex of $TZ\Gamma(R)$. We show that $a \in Q\backslash P$ or $a \in P\backslash(P^2 \cup Q)$. Since $a \in TZ(R)$, there exist $b, c \in R\backslash P^2Q$ such that $abc \in P^2Q$ and $ab, bc, ac \notin P^2Q$. Hence, we have either $a \in P$ or $b \in P$ or $c \in P$, and $a \in Q$ or $b \in Q$ or $c \in Q$.

Case I. Let $a \in P \cap Q$. If $a \in P^2$, then $a \in P^2 \cap Q = P^2Q = 0$ as P^2 and Q are coprime, a contradiction. So, assume that $a \in (P\backslash P^2) \cap Q$. If $b \in P$ or $c \in P$, then

$ab = 0$ or $ac = 0$, a contradiction. If $b \in Q\backslash P$ and $c \in Q\backslash P$, then we get $abc \notin P^2Q$ which is again a contradiction. Thus, $TZ(R) \subseteq (P\backslash Q) \cup (Q\backslash P)$.

Case II. Let $a \in P\backslash Q$. Suppose that $a \in P^2$. If $b \in Q\backslash P$ or $c \in Q\backslash P$, then we have either $ab = 0$ or $ac = 0$, a contradiction. If $b, c \in P\backslash Q$, then $abc \notin Q$, and so $abc \notin P^2Q$, a contradiction.

Therefore, we conclude that $a \in P\backslash(P^2 \cup Q)$ or $a \in Q\backslash P$.

Observe that all pairs are adjacent except for the elements of $Q\backslash P$. In fact, if an element $x \in TZ(R)$ satisfies $a_1b_1x = 0$, where $a_1, a_2 \in Q\backslash P$, we conclude that $x \in P^2$, a contradiction. Thus $TZ\Gamma(R)$ is a connected graph with $diam(TZ\Gamma(R)) = 2$ and $gr(TZ\Gamma(R)) = 3$. $\square$

In the next theorem, we give a necessary and sufficient conditions for $TZ\Gamma(R)$ to be triangle free.

Theorem 11. *Let R be a zero dimensional general ZPI-ring. $TZ\Gamma(R)$ is triangle free if and only if one of the following statements is hold:*

(1) *R is an integral domain.*
(2) *$0 = PQ$ for some distinct prime ideals P and Q of R.*
(3) *$0 = P^2$ for some prime ideal P of R.*
(4) *$0 = P^3$ for some prime ideal P of R such that $|P| = 4$ and $\left|P^2\right| = 2$.*

Proof. ($\Rightarrow$). We investigate the following cases separately.

Case I. Suppose that 0 is divisible by at least three prime ideals of R, say P, Q and T. Then $p \sim q \sim t$ where $p \in P$, $q \in Q$, $t \in T$ forms a triangle.

Case II. If 0 is divisible by P^2 and Q, where P and Q are distinct prime ideals of R, then we obtain the triangle $p \sim q \sim kp$, where $p \in P$, $q \in Q$ and $1 \neq k \in R\backslash Q$.

Case III. Suppose that $0 = P^n$, where P is prime and $n \geq 3$. If $n = 3$, then this graph is complete by Theorem 9. If $0 = P^n$ $(n \geq 4)$, then $p \sim p^2 \sim kp$, where $p \in P$ and $1 \neq k \in R\backslash P$ forms a triangle.

($\Leftarrow$). If (1), (2) or (3) holds, then $TZ\Gamma(R) = \emptyset$ by Theorem 7. If (4) holds, then there are the only two vertices connected by an edge by Theorem 9; so $TZ\Gamma(R) \cong K_2$. $\square$

So we conclude the following result.

Corollary 12. *The diameter of the triple zero graph of a zero dimensional general ZPI-ring R is an element of $\{0, 1, 2\}$ and the girth of the triple zero graph of R is 3 or undefined.*

In the following result, we characterize the triple zero graph of $\mathbb{Z}_n$ and calculate $|TZ\Gamma(\mathbb{Z}_n)|$ cardinality of the vertex set for some particular cases.

Theorem 13. *Let $R = \mathbb{Z}_n$ where n is a positive integer. Then the following statements hold:*

(1) *If $n = p$ or $n = p^2$ or $n = pq$, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$.*

(2) *If* $n = p^3$ *where* p *is prime, then* $TZ\Gamma(\mathbb{Z}_n)$ *is a complete graph on* $p^2 - p$ *vertices.*

(3) *If* $n = p^2q$ *where* p *and* q *are distinct prime integers, then* $TZ\Gamma(\mathbb{Z}_n)$ *is a connected graph with diameter* 2 *and girth* 3*.*

Proof. (1) is clear by Theorem 7.

(2) The vertices of $TZ\Gamma(\mathbb{Z}_n)$ are kp, where $k \in \mathbb{Z}^*_{p^2} = \{k \in \mathbb{Z} : (k, p^2) = 1, k < p^2\}$. So the number of vertices can be calculated by Euler's function $\phi(p^2) = p(p-1)$. Since $(kp)(mp)(tp) = 0$ for all $k, m, t \in \mathbb{Z}^*_{p^2}$ and neither $(kp)(mp) = 0$ nor $(kp)(tp) = 0$ nor $(mp)(tp) = 0$, there is an edge between all vertices. Thus the graph is complete; so it is K_{p^2-p}.

(3) Suppose that $n = p^2q$. Then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph with diameter 2 and girth 3 by Theorem 10. Observe that the vertices of this graph are of the form kq where $k \in \mathbb{Z}^*_{p^2} = \{k \in \mathbb{Z} : (k, p^2) = 1,\ k < p^2\}$ and of the form sp where $s \in \Omega = \{s \in \mathbb{Z} : (s, p) = (s, q) = 1 \text{ and } s < pq\}$. So the number of vertices is $|\Omega| + \phi(p^2) = |\Omega| + p^2 - p$. Moreover, the number of edges can be calculated as $\binom{|\Omega|}{2} + (p^2 - p)\,|\Omega|$. □

Theorem 14. *Let* $n > 0$ *and* $R = \mathbb{Z}_n$*. Then the following statements are equivalent:*

(1) $TZ\Gamma(\mathbb{Z}_n)$ *is triangle free.*

(2) *Either* $n = p$, $n = p^2$, $n = pq$, *or* $n = 8$, *where* p *and* q *are distinct prime integers.*

Proof. We investigate the following cases separately.

Case I. Suppose that n is divisible by at least three primes, say p, q, and r. Then $p \sim q \sim (n/pq)$ forms a triangle.

Case II. If n is divisible by p^2 and q, where p and q are distinct prime integers, then we obtain the triangle $p \sim q \sim kp$, where $(k, q) = 1$ and $k < pq$.

Case III. Suppose that $n = p^n$, where p is prime and $n \geq 3$. If $n = 3$, then this graph is complete by Theorem 9. If $n = p^n$, where $n \geq 3$, except from $p = 2$, then $p \sim p^2 \sim kp$, where $(k, p) = 1$, $k < p^{n-3}$ forms a triangle. Thus, $n = p$, $n = p^2$, $n = pq$, or $n = 8$.

Conversely, if $n = p$, $n = p^2$ or $n = pq$, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$ by Theorem 7. If $n = 8$, then 2 and 6 are the only vertices connected by an edge; and so the claim is clear. □

So we conclude the following result which shows that $TZ\Gamma(\mathbb{Z}_n)$ is connected with diameter at most 2.

Corollary 15. *The diameter of the triple zero graph of* $\mathbb{Z}_n$ *is an element of* $\{0, 1, 2\}$ *and the girth of the triple zero graph of* $\mathbb{Z}_n$ *is* 3 *or undefined.*

Now we can summarize these results by the table below. Let p and q be distinct prime integers and $\Omega = \{s \in \mathbb{Z} : (s, p) = (s, q) = 1 \text{ and } s < pq\}$.

TABLE 1. $TZ\Gamma(\mathbb{Z}_n)$ Summary Table

n	Number of vertices	Number of edges	Diam	Girth	Remarks
p or p^2or pq	0	0	0	∞	$TZ\Gamma(\mathbb{Z}_n) = \emptyset$
8	2	1	1	∞	2∼6
$p^3(p \geq 3)$	$p^2{-}p$	$\binom{p^2-p}{2}$	2	3	K_{p^2-p}
p^2q	$\|\Omega\| + p^2{-}p$	$\binom{\|\Omega\|}{2} + (p^2{-}p)\,\|\Omega\|$	2	3	Connected
All others			2	3	Connected

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FINITE BLASCHKE PRODUCTS AND THE GOLDEN RATIO

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Abstract. Geometric properties of finite Blaschke products have been intensively studied by many different aspects. In this paper, our aim is to study geometric properties of finite Blaschke products related to the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$. Mainly, we focus on the relationships between the zeros of canonical finite Blaschke products of lower degree and the golden ratio. We show that the geometric notions such as "golden triangle, "golden ellipse" and "golden rectangle" are closely related to the geometry of finite Blaschke products.

1. INTRODUCTION

The golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ is the positive root of the quadratic equation $x^2 - x - 1 = 0$. So we have

$$\alpha^2 = \alpha + 1. \tag{1}$$

It is well-known that the golden ratio is almost everywhere in nature and science [12]. This ratio appears in modern research in many fields. For example, in [19], the golden ratio is used in graphs; in [10], it is proved that in any dimension all solutions between unity and the golden ratio to the optimal spherical code problem for N spheres are also solutions to the corresponding DLP (the densed local packing problem) problem. Some geometric applications of the golden ratio and its generalizations have been used to introduce new types of manifolds (see, for example, [2], [3], [11], [13], [14] and the references therein).

The rational function

$$B(z) = \beta \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i} z}$$

is called a finite Blaschke product of degree n for the unit disc where $|\beta| = 1$ and $|a_i| < 1$, $1 \leq i \leq n$. Finite Blaschke products and geometric properties of them have

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been extensively studied by many different aspects (see, for example, [4], [5], [6], [7], [8], [9], [15], [16], [17], [18]). Mainly, in this paper, we study on the connection between geometric properties of Blaschke products and the golden ratio.

It is known that Blaschke products of the following form are called as canonical:

$$B(z) = z\prod_{j=1}^{n-1}\frac{z-a_j}{1-\overline{a}_j z}, |a_j| < 1 \text{ for } 1 \leq j \leq n-1. \tag{2}$$

Note that the canonical Blaschke products correspond to finite Blaschke products vanishing at the origin. It is well-known that every Blaschke product B of degree n with $B(0) = 0$, is associated with a unique Poncelet curve (for more details, see [4], [5] and [8]). From [4] we know that the Poncelet curve associated with a Blaschke product of degree 3 is an ellipse.

In this paper, we investigate the relationships between the zeros of canonical finite Blaschke products of lower degree and the golden ratio. We see that some geometric notions such as "golden triangle, "golden ellipse" and "golden rectangle" are closely related to the geometry of finite Blaschke products.

2. BLASCHKE PRODUCTS OF DEGREE TWO

Let AB be a line segment and C be a point on the line segment AB such that AC is the greater part of AB. Recall that we say the point C divides the line segment AB in the golden ratio if $\frac{AC}{BC} = \alpha$ [12].

In this section we consider a finite Blaschke product B of degree two of the form

$$B_a(z) = z\frac{z-a}{1-\overline{a}z}, \tag{3}$$

with $a \neq 0$, $|a| < 1$. From [4], we know that there exist two distinct points z_1 and z_2 on $\partial\mathbb{D}$ that $B_a(z)$ maps to λ, for any point λ on the unit circle $\partial\mathbb{D}$, and that the line joining z_1 and z_2 passes through a, the nonzero zero of B_a. Conversely, let L be any line through the point a, then for the points z_1 and z_2 at which L intersects $\partial\mathbb{D}$ we have $B_a(z_1) = B_a(z_2)$.

Now we ask the following questions:

1) Does the point a divide the line segment $[z_1, z_2]$ joining z_1 and z_2 in the golden ratio?

2) If it does, what is the number of these line segments?

The answers of these questions are given in the following theorem.

Theorem 1. *Let $B_a(z) = z\frac{z-a}{1-\overline{a}z}$ be a Blaschke product with $a \neq 0$, $|a| < 1$. There are infinitely many values of a such that there is a line segment with endpoints on the unit circle divided by a in the golden ratio. Furthermore the number of such line segments is at most two for a fixed a.*

Proof. Let a be a fixed point such that $a \neq 0$, $|a| < 1$ and consider the finite Blaschke product $B_a(z) = z\frac{z-a}{1-\overline{a}z}$. The ratio of the length of the longer part to length of the smaller part of the segment $[z_1, z_2]$ divided by the point a gives rise

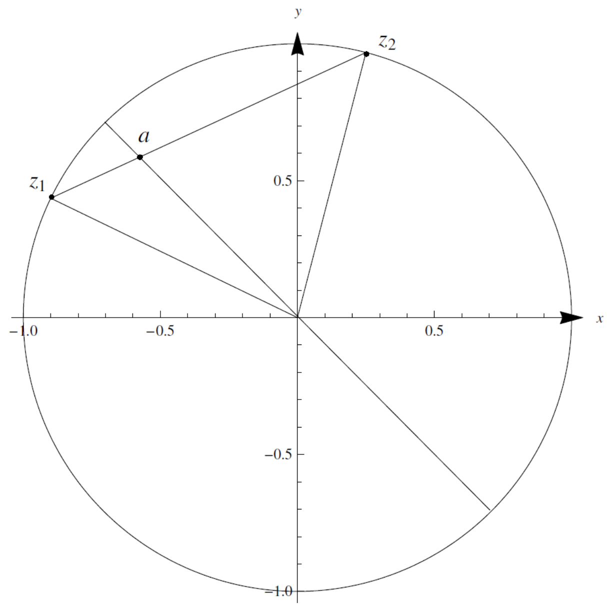

FIGURE 1. A Blaschke product of degree 2

to a continuous function of the angle θ between the segments $[0, a]$ and $[z_1, z_2]$. For $\theta = 0$ the ratio is $\frac{1+|a|}{1-|a|}$ and for $\theta = \frac{\pi}{2}$ the ratio is 1. Applying the well known secant property of a circle to Figure 1, it should be

$$(1 - |a|)(1 + |a|) = l\alpha.l$$

where l is the length of the segment $[z_1, a]$ and $l\alpha$ is the length of the segment $[a, z_2]$ [1]. Then we get

$$l = \sqrt{\frac{1 - |a|^2}{\alpha}} \tag{4}$$

Since nonlinear three distinct points determine a triangle, if the points $0, z_1, z_2$ form a triangle it should be

$$0 < l + l\alpha < 2. \tag{5}$$

If we substitute the equation (4) in (5), we get

$$\sqrt{\frac{1 - |a|^2}{\alpha}}(\alpha + 1) < 2.$$

Then we get

$$\frac{1 + |a|}{1 - |a|} > \alpha.$$

If $\alpha = \frac{1+|a|}{1-|a|}$, then the line passing through the points z_1, z_2 and a is the diameter of the unit circle.

For this reason, as long as $\frac{1+|a|}{1-|a|} \geq \alpha$, there is a segment divided by a in the golden ratio. Now we find the number of the segments divided by a in the golden ratio for a such a.

Let a be chosen such that $\frac{1+|a|}{1-|a|} \geq \alpha$ and z_1 be chosen such that the point a divides the line segment $[z_1, z_2]$ in the golden ratio. Then by definition we have

$$\frac{|z_2 - a|}{|z_1 - a|} = \alpha. \tag{6}$$

Using the fact that $|z| = 1$ for $z \in \partial\mathbb{D}$, we can write

$$B(z) = \frac{z - a}{\overline{z} - \overline{a}},\ z \in \partial\mathbb{D}.$$

Also, we know that $B(z_1) = B(z_2)$ and so we obtain

$$\frac{(z_1 - a)}{(\overline{z}_1 - \overline{a})} = \frac{z_2 - a}{\overline{z}_2 - \overline{a}} \tag{7}$$

From the equation (6), we have

$$\frac{(z_2 - a)(\overline{z}_2 - \overline{a})}{(z_1 - a)(\overline{z}_1 - \overline{a})} = \alpha^2 \tag{8}$$

and from the equation (7), we find

$$\overline{z}_2 - \overline{a} = \frac{(z_2 - a)(\overline{z}_1 - \overline{a})}{(z_1 - a)} \tag{9}$$

After substitute (9) into (8), we get the equation

$$- z_2^2 + 2az_2 + \alpha a^2 - 2a\alpha^2 z_1 + \alpha^2 z_1^2 = 0. \tag{10}$$

Clearly the last equation (10) has at most two roots with respect to z_2. Hence there are at most two line segments $[z_1, z_2]$ passing through the point a and divided by a in the golden ratio. This fact can be also seen by some geometric arguments. □

Example 2. *Let us consider the Blaschke product* $B(z) = z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$. *Let* z_1 *and* z_2 *be two distinct points satisfying* $B(z_1) = B(z_2)$. *If the point* $a = \frac{1}{2}$ *divides the line segment* $[z_1, z_2]$ *in the golden ratio, from the common solutions of the equations (8) and (7), we obtain Figure 2. There the dashed line segments show the line segments which are divided by the point* $a = \frac{1}{2}$ *in the golden ratio.*

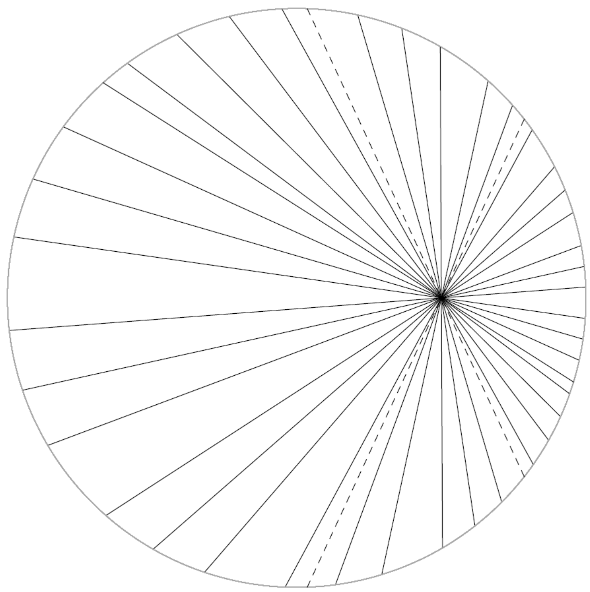

FIGURE 2. Blaschke product of degree 2 with $a = \frac{1}{2}$.

3. BLASCHKE PRODUCTS OF DEGREE THREE

In this section, we consider a finite Blaschke product B of degree three of the form

$$B(z) = z\frac{(z-a_1)(z-a_2)}{(1-\overline{a}_1 z)(1-\overline{a}_2 z)},$$

with distinct zeros at the points 0, a_1 and a_2. It is well-known that for any specified point λ of the unit circle $\partial\mathbb{D}$, there exist 3 distinct points z_1, z_2 and z_3 of $\partial\mathbb{D}$ such that $B(z_1) = B(z_2) = B(z_3) = \lambda$.

We know the following theorem for a Blaschke product of degree three.

Theorem 3. (*See [4] Theorem 1*) *Let B be a Blaschke product of degree three with distinct zeros at the points 0, a_1 and a_2. For λ on the unit circle, let z_1, z_2 and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation*

$$|z-a_1| + |z-a_2| = |1-\overline{a_1}a_2|. \tag{11}$$

Conversely, every point on E is the point of tangency of a line segment joining two distinct points z_1 and z_2 on the unit circle for which $B(z_1) = B(z_2)$.

The ellipse E in (11) is called a Blaschke 3-ellipse associated with the Blaschke product $B(z)$ of degree 3. There are many studies on the ellipse E given in (11) (see [5], [6], [7], [9], [16] and [17] for more details). For any $\lambda \in \partial\mathbb{D}$, we know that E circumscribed in the triangle $\Delta(z_1, z_2, z_3)$, where z_1, z_2 and z_3 are the points mapped to λ under B.

A golden triangle is an isosceles triangle such that the ratio of one its lateral sides to the base is the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$. A golden ellipse is an ellipse such that the ratio of the major axis to the minor axis is the golden ratio $\frac{1+\sqrt{5}}{2}$ (see [12] for more details).

We have the following questions:

1) Are there any Blaschke 3-ellipses which are circumscribed (at least) one golden triangle?

2) Can a Blaschke 3-ellipse be a golden ellipse? If so, what is the number of these ellipses?

We begin with answering of the first question.

Theorem 4. *There are infinitely many golden triangles whose three vertices lie on the unit circle.*

Proof. Without loss of generality, let x and y be chosen so that $x, y > 0$ and such that the triangle with vertices at the points $1, -x + iy, -x - iy$ is inscribed in the unit circle. We try to determine the values of x and y such that $x^2 + y^2 = 1$. By the definition of a golden triangle it is sufficient to show that there are values of x and y on the unit circle such that

$$2\alpha y = \sqrt{y^2 + (x+1)^2}. \tag{12}$$

Squaring both sides of (12) and using the fact that $x^2 + y^2 = 1$, we obtain $2y^2\alpha^2 = x + 1$. Then we have

$$2(1 - x^2)\alpha^2 - x - 1 = 0$$

and so

$$2x^2\alpha^2 + x + (1 - 2\alpha^2) = 0.$$

Solving this quadric equation for x and y, we obtain $x = 0,809017$ and $y = 0.587785$ where $y = \sqrt{1 - x^2}$. So we have one golden triangle such that its vertices are on the unit circle. Then there are infinitely many golden triangles with vertices on the unit circle by rotation. $\square$

Now we can construct some examples using some results from [5] and [9]. Recall that two sets $\{z_1, z_2, ..., z_n\}$ and $\{w_1, w_2, ..., w_n\}$ of points from $\partial\mathbb{D}$ are interspersed if $0 \le \arg(z_1) < \arg(w_1) < ... < \arg(z_n) < \arg(w_n) < 2\pi$ (see [5] for more details).

From [6], we know that the ellipses inscribed in triangles with vertices on the unit circle are precisely Blaschke 3-ellipses.

Example 5. *Let* $\Delta(z_1, z_2, z_3)$ *be a golden triangle on the unit circle. From Theorem* 2.1 *in [9], we know that the Steiner ellipse* E *inscribed in this golden triangle has*

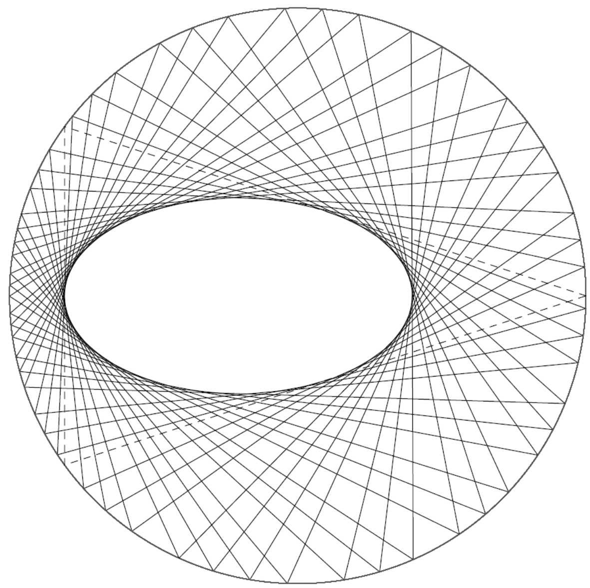

FIGURE 3. Blaschke product B of degree 3 whose Poncelet curve inscribed in (at least) one golden triangle. The dashed triangle is the golden triangle.

foci a_1 and a_2 with the following equation :

$$a_1 = \frac{1}{3}(z_1 + z_2 + z_3) + \sqrt{(\frac{1}{3}(z_1 + z_2 + z_3))^2 - \frac{1}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3)}$$

and

$$a_2 = \frac{1}{3}(z_1 + z_2 + z_3) - \sqrt{(\frac{1}{3}(z_1 + z_2 + z_3))^2 - \frac{1}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3)}.$$

Then this Steiner ellipse E is the Poncelet curve of the Blaschke product $B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-a_2}{1-\overline{a_2}z}$.

Example 6. *Let z_1, z_2, z_3 and w_1, w_2, w_3 be triples of points which form the golden triangles $\Delta(z_1, z_2, z_3)$ and $\Delta(w_1, w_2, w_3)$ on the unit circle so that $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ are interspersed sets of the points. From Corollary 10 on page 97 in [5], we know that there exists a Blaschke product B of degree 3 which maps 0 to 0 such that $B(z_j) = B(z_k)$ and $B(w_j) = B(w_k)$ for all j and k $(1 \leq j, k \leq 3)$. Since we can choose the triples z_1, z_2, z_3 and w_1, w_2, w_3 by infinitely many different ways then clearly there are infinitely many Blaschke ellipses each of which has at least two golden triangle circumscribing them and having the vertices on the unit circle.*

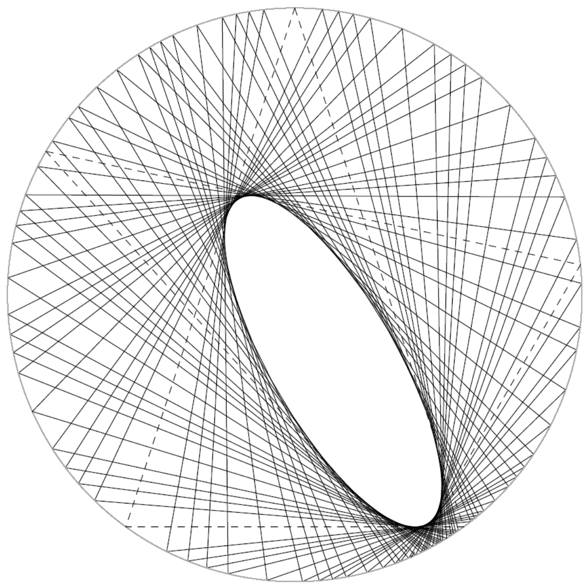

FIGURE 4. Blaschke product B of degree 3 whose Poncelet curve inscribed in (at least) two golden triangles. The dashed triangles are the golden triangles.

We have seen examples of Blaschke products of degree three of which Poncelet curves inscribed in at least one or two golden triangles.

Now we consider the answer of our second question.

Theorem 7. *There are infinitely many golden ellipses which are Blaschke ellipses in the unit disc.*

Proof. Let us take a golden ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the unit disc. Then by definition $\frac{a}{b}=\alpha$ and so $a=b\alpha$. Recall that we have the equation $a^2=b^2+c^2$ where the point c is the positive focus of the ellipse. So this ellipse has foci $-c$ and c. By the last equation and $\alpha^2=\alpha+1$, we find $b^2\alpha=c^2$. Combining $a=b\alpha$ and $b^2\alpha=c^2$ we find $a=\pm c\sqrt{\alpha}$. Now we consider the Blaschke product associated with this ellipse. If this ellipse is a Blaschke ellipse, it must be $2a=1+c^2$ by the definition of a Blaschke ellipse. Hence we find $c^2\pm 2\sqrt{\alpha}c+1=0$. As these equations have only one positive root $c=\frac{1}{2}(-\sqrt{2(-1+\sqrt{5})}+\sqrt{2(1+\sqrt{5})})$, there is one golden ellipse which is a Blaschke ellipse. Since every rotation of this golden ellipse is again golden, clearly we have infinitely many golden Blaschke ellipses in the unit disc. $\square$

We give the following definition.

Definition 8. *Let B be a finite Blaschke product of degree n of the canonical form. If the Poncelet curve associated with B is an ellipse and this ellipse is a golden ellipse, then B is called as a golden Blaschke product.*

Example 9. *Let us consider the Blaschke product*

$$B_1(z) = z\frac{(z-a_1)(z-a_2)}{(1-\overline{a}_1 z)(1-\overline{a}_2 z)},$$

where

$$a_1 = \frac{1}{2}(-\sqrt{2(-1+\sqrt{5})} + \sqrt{2(1+\sqrt{5})})$$

and $a_2 = -a_1$. By the proof of Theorem 7, we know that the Blaschke 3-ellipse E associated with B_1 is a golden ellipse. So $B_1(z)$ is a golden Blaschke product. The image of this golden Blaschke ellipse under the rotation transformation $f(z) = (\frac{1}{2} + i\frac{\sqrt{3}}{2})z$ is another golden Blaschke ellipse. Clearly we find the equation of $f(E)$ as

$$\left| z - (\frac{1}{2} + i\frac{\sqrt{3}}{2})a_1 \right| + \left| z - (\frac{1}{2} + i\frac{\sqrt{3}}{2})a_2 \right| = \left| 1 - \overline{a_1} a_2 \right|.$$

More precisely, this image ellipse $f(E)$ is the Poncelet curve of the following Blaschke product :

$$B_2(z) = z\frac{(z - (\frac{1}{2} + i\frac{\sqrt{3}}{2})a_1)(z - (\frac{1}{2} + i\frac{\sqrt{3}}{2})a_2)}{(1 - (\frac{1}{2} - i\frac{\sqrt{3}}{2})\overline{a}_1 z)(1 - (\frac{1}{2} - i\frac{\sqrt{3}}{2})\overline{a}_2 z)}.$$

4. BLASCHKE PRODUCTS OF DEGREE FOUR

A golden rectangle is a rectangle such that the ratio of the length x of the longer side to the length y of the shorter side is the golden ratio $\frac{1+\sqrt{5}}{2}$ (see [12] for more details).

We give the following theorem.

Theorem 10. *There are infinitely many golden rectangles whose four vertices lie on the unit circle.*

Proof. Without loss of generality, let x and y be chosen so that $x, y > 0$ and such that the rectangle with vertices at the points $x+iy, x-iy, -x-iy, -x+iy$ is inscribed in the unit circle. We try to determine the values of x and y such that $x^2+y^2 = 1$. So, it is sufficient to show that there are values of x and y on the unit circle such that

$$2x = 2\alpha y.$$

We get $x = \alpha y$ and using the facts that $x^2 + y^2 = 1$ and $\alpha^2 = \alpha + 1$ we obtain $y^2(\alpha^2+1) = 1$ and hence

$$y = \frac{1}{\sqrt{\alpha+2}} = 0.525731 \text{ and } x = \frac{\alpha}{\sqrt{\alpha+2}} = 0.850651.$$

So, we have one golden rectangle such that its vertices are on the unit circle. Then, there are infinitely many golden triangles with vertices on the unit circle by rotation. □

Example 11. *Let z_1, z_2, z_3, z_4 and w_1, w_2, w_3, w_4 be eight points which form the golden rectangles (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) on the unit circle so that $\{z_1, z_2, z_3, z_4\}$ and $\{w_1, w_2, w_3, w_4\}$ are interspersed sets of the points. From Corollary* 10 *on page* 97 *in [5], we know that there exists a Blaschke product B of degree* 4 *which maps* 0 *to* 0 *such that $B(z_j) = B(z_k)$ and $B(w_j) = B(w_k)$ for all j and k $(1 \leq j, k \leq 4)$. Then clearly there are infinitely many Poncelet curves associated with a finite Blaschke product of degree* 4 *each of which has at least two golden rectangle circumscribing them and having the vertices on the unit circle.*

Using the following lemmas, we construct examples of finite Blaschke products of degree 4 whose Poncelet curves are ellipses and each of them have at least one golden rectangle.

Lemma 12. (*See [7] Lemma* 5) *For any quadrilateral that is inscribed in the unit circle, an ellipse is inscribed in it if and only if the ellipse is associated with the composition of two Blaschke products of degree* 2.

Lemma 13. (*See [7] Lemma* 6) *For four mutually distinct points $z_1, \ldots, z_4$ on the unit circle* $(0 \leq \arg z_1 < \arg z_2 < \arg z_3 < \arg z_4 < 2\pi)$, *there exists an ellipse that is inscribed in the quadrilateral with vertices $z_1, \ldots, z_4$. Moreover, for each quadrilateral, inscribed ellipses form a real-valued one-parameter family.*

Now we give the following theorem.

Theorem 14. *Let Q be any golden rectangle inscribed in the unit circle. Then there is at least one ellipse E inscribed in Q such that E is a Poncelet curve of a finite Blaschke product B of degree* 4.

Proof. Let Q be any golden rectangle with the vertices z_1, z_2, z_3, z_4 on the unit circle. By Lemma 13 there exists an ellipse E inscribed in Q. We know that the two foci a and b of an ellipse inscribed in any rectangle whose vertices are z_1, z_2, z_3, z_4 satisfy the equations

$$\begin{gathered}
[(((-z_2 + z_1)\, z_3 - z_1 z_2)\, z_4 + z_1 z_2 z_3)\, \overline{a} + z_2 z_4 - z_1 z_3]\, a^2 \\
-[z_1 z_2 z_3 z_4\, (z_4 - z_3 + z_2 - z_1)\, \overline{a}^2 - (z_3 + z_1)\,(z_4 + z_2)\,(z_2 z_4 - z_1 z_3)\, \overline{a} \\
+ z_2 z_4\,(z_4 + z_2) - z_1 z_3\,(z_1 + z_3)]a + z_1 z_2 z_3 z_4\,(z_2 z_4 - z_1 z_3)\, \overline{a}^2 \\
-[\left(z_2^2 z_3 + z_1 z_2^2\right) z_4^2 - z_1^2 z_3^2 z_4 - z_1^2 z_2 z_3^2]\overline{a} \\
+ (z_2 z_4 - z_1 z_3)\,(z_2 z_4 + z_1 z_3) = 0
\end{gathered}$$

and

$$\begin{gathered}
(z_4 - z_3 + z_2 - z_1)\, ab - (z_2 z_4 - z_1 z_3)\,(a + b) \\
+[(z_2 - z_1)\, z_3 + z_1 z_2] z_4 - z_1 z_2 z_3 = 0
\end{gathered}$$

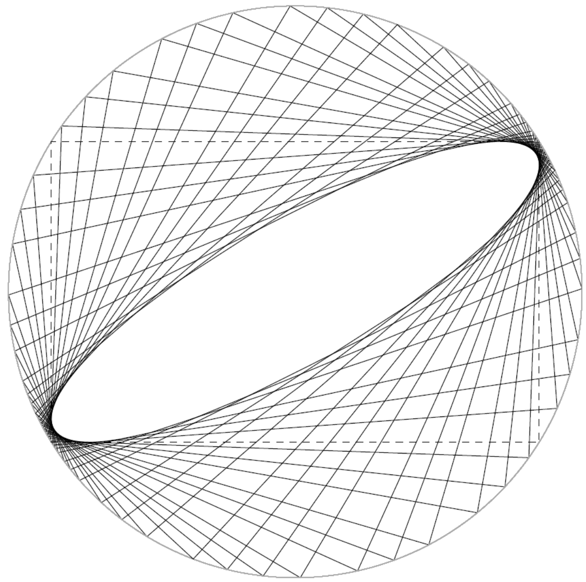

FIGURE 5. Blaschke product B of degree 4 whose Poncelet curve is an ellipse inscribed in (at least) one golden rectangle. The dashed rectangle is the golden rectangle.

given in [7]. Then by the proof of Lemma 12, E has the following equation

$$E : |z-a| + |z-b| = |1-\overline{a}b| \sqrt{\frac{|a|^2 + |b|^2 - 2}{|a|^2 |b|^2 - 1}}$$

and E is the Poncelet curve of the finite Blaschke product B of the following form:

$$B(z) = z \frac{z-\beta}{1-\overline{\beta}z} \frac{z^2 + (\overline{\beta}\alpha - \beta)z - \alpha}{1 - (-\overline{\alpha}\beta + \overline{\beta})z - \overline{\alpha}z^2},$$

where $\alpha = -ab$ and $\beta = \frac{a+b-ab(\overline{a}+\bar{b})}{1-|ab|^2}$. □

5. BLASCHKE PRODUCTS OF HIGHER DEGREE

We know that regular pentagon and regular decagon have the same properties of the golden ratio among polygons (see [12] for more details). It is not known the equation of the Poncelet curves of Blaschke products of degree 5 or 10, so we cannot obtain similar theorems to the ones given in the previous sections. In these two cases, by the similar arguments used in the Example 6 and Example 11,

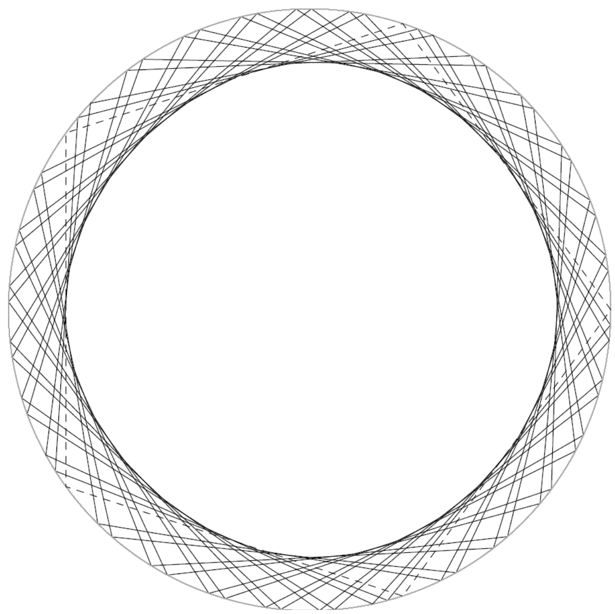

FIGURE 6. Blaschke product B of degree 5 whose Poncelet curve inscribed in (at least) one golden pentagon. The dashed pentagon is the golden pentagon.

we can obtain finite Blaschke products of degree 5 and 10 whose Poncelet curves circumscribed by at least two regular pentagon and regular decagon, respectively.

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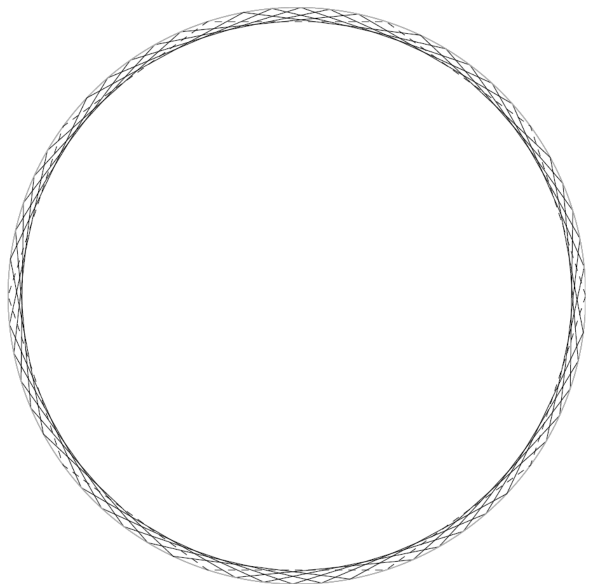

FIGURE 7. Blaschke product B of degree 10 whose Poncelet curve inscribed in (at least) one golden decagon. The dashed decagon is the golden decagon.

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SOLUTION OF FRACTIONAL KINETIC EQUATIONS INVOLVING GENERALIZED HURWITZ-LERCH ZETA FUNCTION USING SUMUDU TRANSFORM

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Abstract. Fractional kinetic equations (FKEs) comprising a large array of special functions have been extensively and successfully applied in specification and solving many significant problems of astrophysics and physics. In this present work, our aim is to demonstrate solutions of (FKEs) of the generalized Hurwitz-Lerch Zeta function by applying the Sumudu transform. In addition to these, solutions of (FKEs) in special conditions of generalised Hurwitz-Lerch Zeta function have been derived.

1. Introduction

The Hurwitz-Lerch Zeta function is defined by [34, 35]:

$$\Phi(\zeta, m, \alpha) = \sum_{n=0}^{\infty} \frac{\zeta^n}{(n+\alpha)^m} \tag{1}$$

$$\big(\alpha \in \mathbb{C} \setminus \mathbb{Z}_0;\ m \in \mathbb{C} \text{ when } |\zeta| < 1;\ \Re(m) > 1 \text{ when } |\zeta| = 1\big).$$

Many researchers studied many different generalisations and extensions of the Hurwitz-Lerch Zeta function by inserting certain additional parameters to the series representation of the Hurwitz-Lerch Zeta function. The interested readers can refer to these earlier publications for further researches and applications [13, 14, 15, 18, 20, 21, 22, 25, 26, 33, 36, 38, 42].

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In 2011, Srivastava et. al [41, p.491, Eq.(1.20)] introduced and studied the following extension of the generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\,\mu;\,\omega}^{(\sigma,\,\rho,\,\kappa)}(\zeta, m, a) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}}{(\omega)_{\kappa n}\,n!} \frac{\zeta^n}{(n+a)^m}, \tag{2}$$

$$(\lambda,\, \mu \,\in\, \mathbb{C};\, a,\, \omega \,\in\, \mathbb{C}\setminus\mathbb{Z}_0^-;\, \sigma,\, \rho,\, \kappa \,\in\, \mathbb{R}^+;\, \kappa-\sigma-\rho > -1 \,\text{when}\, m,\, \zeta \,\in\, \mathbb{C};$$
$$\kappa-\sigma-\rho = -1 \,\text{and}\, m \,\in\, \mathbb{C} \,\text{when}\, |\zeta| <\, \delta^\star \,=\, \sigma^{-\sigma}\rho^{-\rho}\kappa^{\kappa};$$
$$\kappa-\sigma-\rho = -1 \,\text{and}\, \Re(m+\omega-\lambda-\mu) > 1 \,\text{when}\, |\zeta| =\, \delta^\star).$$

1.1. **Fractional Kinetic Equations.** In [23] one determinated the fractional differential equation for the rate of change of reaction. The destruction rate and the production rate follow:

$$\frac{d\mathfrak{g}}{d\mathfrak{x}} = -\mathfrak{d}(\mathfrak{g}_{\mathfrak{x}}) + \mathfrak{p}(\mathfrak{g}_{\mathfrak{x}}), \tag{3}$$

where $\mathfrak{g} = \mathfrak{g}(\mathfrak{x})$ the rate of the reaction, $\mathfrak{d} = \mathfrak{d}(\mathfrak{g})$ the rate of destruction, $\mathfrak{p} = \mathfrak{p}(\mathfrak{g})$ the rate of production and $\mathfrak{g}_{\mathfrak{x}}$ denotes the function defined by $\mathfrak{g}_{\mathfrak{x}}(\mathfrak{x}^\star) = \mathfrak{g}(\mathfrak{x}-\mathfrak{x}^\star), \mathfrak{x}^\star > 0$.

The special condition of equation (3) for spatial fluctuations and inhomogeneities in $\mathfrak{g}(\mathfrak{x})$ the quantities are ignored, that is the equation

$$\frac{d\mathfrak{g}}{d\mathfrak{x}} = -\mathfrak{c}_i\, \mathfrak{g}_i(\mathfrak{x}) \tag{4}$$

with the initial condition that $\mathfrak{g}_i(\mathfrak{x} = 0) = \mathfrak{g}_0$ is the number of density of the species i at time $\mathfrak{x} = 0$ and $\mathfrak{c}_i > 0$. If we shift the index i and integrate the standard kinetic equation (4), we have

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 = -\mathfrak{c}\;{}_0\mathcal{D}_t^{-1}\,\mathfrak{g}(\mathfrak{x}) \tag{5}$$

where ${}_0\mathcal{D}_{\mathfrak{x}}^{-1}$ is the special condition of the Riemann-Liouville integral operator ${}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}$ given as [40],

$${}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} f(\mathfrak{x}) = \frac{1}{\Gamma(\xi)} \int_0^{\mathfrak{x}} (\mathfrak{x}-s)^{\xi-1} f(s)\,ds, \tag{6}$$
$$(\mathfrak{x} > 0,\, \Re(\xi) > 0).$$

The fractional generalisation of the standard kinetic equation (5) is studied by Haubold and Mathai as follows [23]:

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 = -\mathfrak{c}^{\nu}\;{}_0\mathcal{D}_{\mathfrak{x}}^{-1}\,\mathfrak{g}(\mathfrak{x}) \tag{7}$$

and acquired the solution of (4) as follows:

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\xi k+1)} (\mathfrak{c}\mathfrak{x})^{\xi k}. \tag{8}$$

In addition to that, Saxena and Kalla [30] take into consideration the following fractional kinetic equation:

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, f(\mathfrak{x}) = -\mathfrak{c}^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-1}\, \mathfrak{g}(\mathfrak{x}) \;\; (\Re(\xi) > 0), \tag{9}$$

where $\mathfrak{g}(\mathfrak{x})$ denotes the number density of a given species at time $\mathfrak{x}$, $\mathfrak{g}_0 = \mathfrak{g}(0)$ is the number of density of that species at time $\mathfrak{x} = 0$, $\mathfrak{c}$ is a constant and $f \in L(0, \infty)$.

By taking advantage of the Laplace transform [19, 37, 39] to the equation (9),

$$\mathfrak{L}\{\mathfrak{g}(\mathfrak{x}); p\} = \mathfrak{g}_0\, \frac{F(p)}{1 + \mathfrak{c}^{\nu}\, p^{-\nu}} = \mathfrak{g}_0 \left(\sum_{n=0}^{\infty} (-\mathfrak{c}^{\nu})^n\, p^{-\nu n} \right) F(p), \tag{10}$$

$$\left(n \in \mathfrak{g}_0, \left| \frac{\mathfrak{c}}{p} \right| < 1 \right).$$

The extension and generalisation of (FKEs) comprising many fractional operators were found in [1, 2, 3, 5, 16, 17, 23, 24, 28, 29, 30, 31, 32, 43].

1.2. **Sumudu Transform.** The Sumudu transform is extensively used to solve several type of problems in science and engineering and it was introduced by Watagula [44, 45]. For details, the reader is referred to [4, 7, 8, 9, 10, 11, 12].

Suppose that $\mathcal{U}$ be the class of exponentially bounded function $f : \Re \to \Re$, that is,

$$f(\zeta) < \begin{cases} \mathcal{M} \exp\left(-\dfrac{\zeta}{\eta_1} \right) & (\zeta \leqq 0); \\ \mathcal{M} \exp\left(\dfrac{\zeta}{\eta_2} \right) & (\zeta \geqq 0), \end{cases}$$

where $\mathcal{M}$, η_1 and η_2 are positive real constants. The Sumudu transform defined on the set $\mathcal{U}$ is given as follows [44, 45]:

$$\mathcal{G}(u) = \mathcal{S}\{f(\zeta); u\} = \int_0^{\infty} e^{-\zeta}\, f(u\zeta)\, d\zeta \qquad (-\eta_1 < u < \eta_2). \tag{11}$$

The main goal of this work is to demonstrate the generalized (FKEs) involving generalised Hurwitz-Lerch Zeta function (2). Here, we conceive the Sumudu transform methodology to arrive at the solutions.

2. Main Results

Here, we will explain the solution of the generalised (FKEs) which by considering generalized Hurwitz-Lerch Zeta function (2).

Theorem 1. *If $\mathfrak{b} > 0$, $\xi > 0$; λ, μ, $\delta \in \mathbb{C}$, and $\mathfrak{b} \neq \delta$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; σ, ρ, $\kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\lambda, \mu; \omega}^{(\sigma, \rho, \kappa)}(\mathfrak{b}^{\xi} \mathfrak{x}^{\nu}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{12}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\,t^{\xi n-1}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,\mathcal{E}_{\xi,\xi n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}), \tag{13}$$

where $\mathcal{E}_{\xi,\xi n}(.)$ is the Mittag-Leffler function [27].

Proof. The Sumudu transform of the Riemann-Liouville fractional integral operator is defined by [24, p. 460, Eq. (2.10)]:

$$\mathcal{S}\left[{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} f(\mathfrak{x}); u\right] = \mathcal{S}\left[\frac{\mathfrak{x}^{\xi-1}}{\Gamma(\xi)}; u\right] \cdot \mathcal{S}\left[f(\mathfrak{x}); u\right] = u^{\xi}\,G(u). \tag{14}$$

Now, taking advantage of the Sumudu transform to the both sides of (12), we have

$$\begin{aligned}
\mathcal{S}\{\mathfrak{g}(\mathfrak{x}); u\} &= \mathfrak{g}_0\,\mathcal{S}\{\,\Phi_{\lambda,\,\mu;\,\omega}^{(\sigma,\,\rho,\,\kappa)}(\mathfrak{b}^{\nu}\mathfrak{x}^{\nu}, m, a); u\} - \delta^{\xi}\,\mathcal{S}\{{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}); u\}\\
\mathfrak{g}(u) &= \mathfrak{g}_0\left\{\int_0^{\infty} e^{-\mathfrak{x}} \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,(\mathfrak{b}^{\nu}(u\mathfrak{x})^{\xi})^n}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\right\} d\mathfrak{x} - \delta^{\xi} u^{\xi}\,\mathfrak{g}(u)\\
\mathfrak{g}(u) &+ \delta^{\xi} u^{\xi}\,\mathfrak{g}(u)\\
&= \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,u^{\nu n} \int_0^{\infty} e^{-\mathfrak{x}}\,\mathfrak{x}^{\xi n}\,d\mathfrak{x}\\
&= \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,u^{\xi n}\,\Gamma(\xi n+1)\\
N(u) &= \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,u^{\xi n} \sum_{r=0}^{\infty}\left[-(\delta u)^{\xi}\right]^r.
\end{aligned} \tag{15}$$

Taking the inverse Sumudu transform of (15), and by applying

$$\mathcal{S}^{-1}\{u^{\xi}; \mathfrak{x}\} = \frac{\mathfrak{x}^{\xi-1}}{\Gamma(\xi)},\ \ (\Re(\xi) > 0), \tag{16}$$

we have

$$\begin{aligned}
\mathcal{S}^{-1}\{\mathfrak{g}(u)\} = {}& \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\\
&\times \mathcal{S}^{-1}\left[\sum_{r=0}^{\infty} \delta^{\xi r}\,u^{\xi(n+r)}\right]\\
\mathfrak{g}(\mathfrak{x}) = {}& \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\,\mathfrak{x}^{\xi n-1}}{(\omega)_{\kappa n}\,n!\,(n+a)^m} \sum_{r=0}^{\infty} (-1)^r \delta^{\xi r}\,\frac{\mathfrak{x}^{\xi r}}{\Gamma(\xi n+\xi r)}.
\end{aligned} \tag{17}$$

So, we can be yield the required result (13). $\square$

Theorem 2. *If $\mathfrak{b} > 0$, $\xi > 0$; λ, $\mu \in \mathbb{C}$ be such that a, $\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; σ, ρ, $\kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\lambda,\,\mu;\,\omega}^{(\sigma,\,\rho,\,\kappa)}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\mathfrak{b}^\xi\, {}_0\mathcal{D}_\mathfrak{x}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{18}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\, (\mu)_{\rho n}\, \Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n}\, \mathfrak{x}^{\xi n - 1}}{(\omega)_{\kappa n}\, n!\, (n+a)^m}\, \mathcal{E}_{\xi, \xi n}(-\mathfrak{b}^\xi\, \mathfrak{x}^\xi), \tag{19}$$

where $\mathcal{E}_{\xi,\xi n}(.)$ is the Mittag-Leffler function [27].

Proof. The proof of Theorem 2 is parallel to the proof of Theorem 1, thus the details are omitted. □

Theorem 3. *If $\xi > 0$; λ, μ, $\delta \in \mathbb{C}$ be such that a, $\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; σ, ρ, $\kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\lambda,\,\mu;\,\omega}^{(\sigma,\,\rho,\,\kappa)}(\mathfrak{x}, m, a) = -\delta^\xi\, {}_0\mathcal{D}_\mathfrak{x}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{20}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\, (\mu)_{\rho n}\, \Gamma(n + 1)\, \mathfrak{x}^{n - 1}}{(\omega)_{\kappa n}\, n!\, (n+a)^m}\, \mathcal{E}_{\xi, n}(-\delta^\xi\, \mathfrak{x}^\xi), \tag{21}$$

where $\mathcal{E}_{\xi,n}(.)$ is the Mittag-Leffler function [27].

Proof. Theorem 3 can be easily acquired from Theorem 1, so the details are omitted. □

2.1. **Special Conditions.** Choosing $\lambda = \sigma = 1$ in the equation (2), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\mu;\,\omega}^{\rho;\,\kappa}(\zeta, m, a)$ introduced and studied by Lin and Srivastava [25].

Applying $\lambda = \sigma = 1$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 4. *If $\mathfrak{b} > 0$, $\xi > 0$; μ, $\delta \in \mathbb{C}$, and $\mathfrak{b} \neq \delta$ be such that a, $\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; ρ, $\kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\mu;\,\omega}^{(\rho,\,\kappa)}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\delta^\xi\, {}_0\mathcal{D}_\mathfrak{x}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{22}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}\, \Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_{\kappa n}\, (n+a)^m}\, \mathcal{E}_{\xi, \xi n}(-\delta^\xi\, \mathfrak{x}^\xi). \tag{23}$$

Corollary 5. *If $\mathfrak{b} > 0$, $\xi > 0$; $\mu \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\rho, \kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\mathfrak{b}^\xi \, {}_0\mathcal{D}_\mathfrak{x}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{24}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} \, \Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_{\kappa n} \, (n+a)^m} \, \mathcal{E}_{\xi,\xi n}(-\mathfrak{b}^\nu \, \mathfrak{x}^\xi). \tag{25}$$

Corollary 6. *If $\mu, \delta \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\rho, \kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{x}, m, a) = -\delta^\xi \, {}_0\mathcal{D}_\mathfrak{x}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{26}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} \, \Gamma(n + 1) \mathfrak{x}^{n - 1}}{(\omega)_{\kappa n} \, (n+a)^m} \, \mathcal{E}_{\xi,n}(-\delta^\xi \, \mathfrak{x}^\xi). \tag{27}$$

Setting $\sigma = \rho = \kappa = 1$ in the equation (2), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\omega}(\zeta, m, a)$ introduced and studied by Garg et. all [20].

Applying $\sigma = \rho = \kappa = 1$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 7. *If $\mathfrak{b} > 0$, $\xi > 0$; $\lambda, \mu, \delta \in \mathbb{C}$, and $\mathfrak{b} \neq \delta$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the following given equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\delta^\xi \, {}_0\mathcal{D}_\mathfrak{x}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{28}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (\mu)_n \, \Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_n \, n! \, (n+a)^m} \, \mathcal{E}_{\xi,\xi n}(-\delta^\xi \, \mathfrak{x}^\xi). \tag{29}$$

Corollary 8. *If $\mathfrak{b} > 0$, $\xi > 0$; $\lambda, \mu \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\mathfrak{b}^\xi \, {}_0\mathcal{D}_\mathfrak{x}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{30}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (\mu)_n \, \Gamma(\xi n + 1) \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_n \, n! \, (n+a)^m} \, \mathcal{E}_{\xi,\xi n}(-\mathfrak{b}^\xi \, \mathfrak{x}^\xi). \tag{31}$$

Corollary 9. *If $\lambda, \mu, \delta \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}(\mathfrak{x}, m, a) = -\delta^\xi \, {}_0\mathcal{D}_\mathfrak{x}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{32}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n\,(\mu)_n\,\Gamma(n+1)\mathfrak{x}^{n-1}}{(\omega)_n\, n!\,(n+a)^m}\,\mathcal{E}_{\xi,n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}). \tag{33}$$

Upon taking $\sigma = \rho = \kappa = 1$ and $\lambda = \omega$ in the equation (2), which is the generalized Hurwitz-Lerch Zeta function $\Phi^{\star}_{\mu}(\zeta, m, a)$ introduced and studied by Goyal and Laddha [21, p.100, Eq.(1.5)].

Applying $\sigma = \rho = \kappa = 1$ and $\lambda = \omega$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 10. *If* $\mathfrak{b} > 0$, $\xi > 0$; μ, $\delta \in \mathbb{C}$, *and* $\mathfrak{b} \neq \delta$ *be such that* $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi^{\star}_{\mu}(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\delta^{\xi}\; {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}) \tag{34}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n-1}}{n!\,(n+a)^m}\,\mathcal{E}_{\xi,\xi n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}). \tag{35}$$

Corollary 11. *If* $\mathfrak{b} > 0$, $\xi > 0$; $\mu \in \mathbb{C}$ *be such that* $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi^{\star}_{\mu}(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\mathfrak{b}^{\xi}\; {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}) \tag{36}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n-1}}{n!\,(n+a)^m}\,\mathcal{E}_{\xi,\xi n}(-\mathfrak{b}^{\xi}\,\mathfrak{x}^{\xi}). \tag{37}$$

Corollary 12. *If* λ, μ, $\delta \in \mathbb{C}$ *be such that* $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi^{\star}_{\mu}(\mathfrak{x}, m, a) = -\delta^{\xi}\; {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}) \tag{38}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n\,\Gamma(n+1)\mathfrak{x}^{n-1}}{n!\,(n+a)^m}\,\mathcal{E}_{\xi,n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}). \tag{39}$$

Upon taking $\sigma = \rho = \mu = 1$ and $\zeta = \frac{\zeta}{\lambda}$. Then, the limit case of (2) when $\lambda \to \infty$, would yield the Mittag-Leffler type function $\mathcal{E}^{(a)}_{\kappa,\omega}(m;\mathfrak{x})$ studied by Barnes [6], that is,

$$\mathcal{E}^{(a)}_{\kappa,\omega}(m;\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{(n+a)^m\,\Gamma(\omega+\kappa n)}, \tag{40}$$
$$(a,\,\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(\kappa) > 0;\ m,\,\zeta \in \mathbb{C}).$$

Applying $\sigma = \rho = \mu = 1$ and $\zeta = \frac{\zeta}{\lambda}$. Then, the limit case of (2) when $\lambda \to \infty$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 13. *If $\mathfrak{b} > 0$, $\xi > 0$; κ, $\delta \in \mathbb{C}$, and $\mathfrak{b} \neq \delta$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \mathcal{E}^{(a)}_{\kappa,\omega}(m; \mathfrak{b}^{\xi}\mathfrak{x}^{\xi}) = -\delta^{\xi} \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{41}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(n+a)^m \, \Gamma(\omega + \kappa n)} \, \mathcal{E}_{\xi, \xi n}(-\delta^{\xi} \, \mathfrak{x}^{\xi}). \tag{42}$$

Corollary 14. *If $\mathfrak{b} > 0$, $\xi > 0$; $\kappa \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \mathcal{E}^{(a)}_{\kappa,\omega}(m; \mathfrak{b}^{\xi}\mathfrak{x}^{\xi}) = -\mathfrak{b}^{\xi} \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{43}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(n+a)^m \, \Gamma(\omega + \kappa n)} \, \mathcal{E}_{\xi, \xi n}(-\mathfrak{b}^{\xi} \, \mathfrak{x}^{\xi}). \tag{44}$$

Corollary 15. *If κ, $\delta \in \mathbb{C}$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \mathcal{E}^{(a)}_{\kappa,\omega}(m; \mathfrak{x}) = -\delta^{\xi} \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{45}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(n + 1) \mathfrak{x}^{n - 1}}{(n+a)^m \, \Gamma(\omega + \kappa n)} \, \mathcal{E}_{\xi, n}(-\delta^{\xi} \, \mathfrak{x}^{\xi}). \tag{46}$$

Finally, upon setting λ, μ, ω, σ, ρ, $\kappa = 1$ in the equation (2), which gives the equation (1) [34,35].

Choosing λ, μ, ω, σ, ρ, $\kappa = 1$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 16. *If $\mathfrak{b} > 0$; δ, $\xi \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $\mathfrak{b} \neq \delta$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\delta^{\xi} \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{47}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(n+a)^m} \, \mathcal{E}_{\xi, \xi n}(-\delta^{\xi} \, \mathfrak{x}^{\xi}). \tag{48}$$

Corollary 17. *If $\mathfrak{b} > 0$; $\xi \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\mathfrak{b}^{\xi} \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{49}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n+1)\, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n-1}}{(n+a)^m} \mathcal{E}_{\xi,\xi n}(-\mathfrak{b}^{\xi}\, \mathfrak{x}^{\xi}). \tag{50}$$

Corollary 18. *If $\delta \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi(\mathfrak{x}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{51}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(n+1) \mathfrak{x}^{n-1}}{(n+a)^m} \mathcal{E}_{\xi,n}(-\delta^{\xi}\, \mathfrak{x}^{\xi}). \tag{52}$$

3. Numerical Result and Graphic

In this section, we present the 2D plots of Equation (13) for special values such as: λ, μ, ω, ρ, κ, σ, a, $m = 1$, $\delta = 4$, $\mathfrak{g}_0 = 3$ and $\xi = 0.4, 0.5, 0.6$.

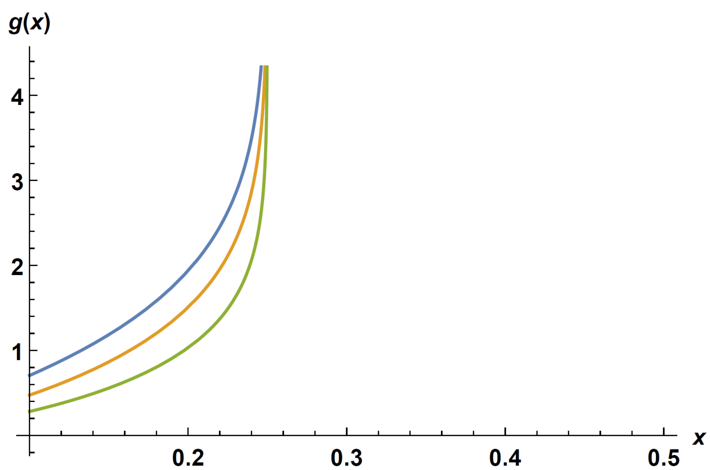

Figure 1. Solution of the FKE for GHLZ

4. Conclusions

The fractional kinetic equation involving the generalized Hurwitz-Lerch Zeta function is studied using the Sumudu transform. The results obtained in this study have remarkable significance as the solution of the equations are general and can be reproduced many new and known solutions of (FKEs) involving various type of special functions.

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SOME FIXED POINT THEOREMS ON COMPLEX VALUED MODULAR METRIC SPACES WITH AN APPLICATION

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Abstract. In this article, we introduce the notion of complex valued modular metric spaces. We also a prove generalization of Banach Fixed Point Theorem, which is one of the most simple and significant tests for existence and uniqueness of solution of problems arising in mathematics and engineering for complex valued modular metric spaces. In addition, we express some results related to these spaces. Finally, we give an application of our results to digital programming.

1. INTRODUCTION AND PRELIMINARIES

In 2011, Azam et al. [6] introduced the notion of complex valued metric spaces and they gave generalization of Banach contraction mapping principle [10]. Then, this space has been studied by many authors. After that, they obtained various fixed point theorems on this spaces [2, 7, 15, 16, 22, 24, 31, 32, 33, 34]. On the other hand, a lot of researchers have contributed introducing different concepts on these structures. And they extended them to b-metric, rectangular metric, generalized metric spaces, etc. [1, 4, 5, 20, 25, 26, 35].

In 1950, Nakano introduced modular spaces [30]. In 2008, Chistyakov introduced the notion of modular metric spaces, which has a physical interpretation [11] and he gave the fundamental properties of modular metric spaces [12]. In 2011, Mongkolkeha and et. al. proved contraction-type fixed point theorems on modular metric spaces [23]. Since the 2010, many researchers as Kumam, Cho, Alaca,

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Khamsi, Mutlu have contributed to develop these structures introducing various fixed point theorems on modular metric spaces [9, 3, 8, 13, 14, 17, 18, 19, 27, 28].

The aim of this paper is to introduced the concept of complex valued modular metric spaces, which is more general than well-know modular metric spaces, and give some fixed point theorems under the contraction condition in these spaces. Further, we discuss some results and an application related to these new spaces in digital programming.

Complex valued modular metric spaces form a special class of cone modular metric space. This idea is useful in defining rational expressions which are not meaningful in cone modular metric spaces. Thus, many results of analysis cannot be generalized to cone modular metric spaces. So the complex valued modular metric spaces are important spaces.

Let $z_1, z_2 \in \mathbb{C}$, $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ where $a_1, b_2, a_1, b_2 \in \mathbb{R}$ and $\precsim$ be a partial order on $\mathbb{C}$. Then $z_1 \precsim z_2 \Leftrightarrow a_1 \leq a_2$ and $b_1 \leq b_2$. Therefore, it is obvious that $z_1 \precsim z_2$, if

(i) $a_1 = a_2$ and $b_1 = b_2$ or;

(ii) $a_1 < a_2$ and $b_1 = b_2$ or;

(iii) $a_1 = a_2$ and $b_1 < b_2$ or;

(iv) $a_1 < a_2$ and $b_1 < b_2$.

Specially, $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of conditions (ii), (iii), (iv) is satisfied. Also, $z_1 \prec z_2$ if only the condition (iv) is satisfied.

Definition 1. *[29] Let X be a linear space on $\mathbb{R}$ (or $\mathbb{C}$). If a functional $\rho : X \to [0, \infty]$ holds the following conditions, we call that ϕ is a modular on X:* (1) $\rho(0) = 0$*;*
(2) *If $x \in X$ and $\rho(\alpha x) = 0$ some numbers $\alpha > 0$, then $x = 0$;*
(3) *$\rho(-x) = \rho(x)$, for all $x \in X$;*
(4) *$\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for some $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.*

2. MAIN RESULTS

Let $X \neq \emptyset$, $\lambda \in (0, \infty)$ and $\omega : (0, \infty) \times X \times X \to \mathbb{C}$ is a function. Throughout this article, the value $\omega(\lambda, x, y)$ is denoted as $\omega_\lambda(x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2. *Let $X \neq \emptyset$. The function $\omega : (0, \infty) \times X \times X \to \mathbb{C}$ is called a complex valued metric modular on X, if*

(CM1) $\omega_\lambda(x, y) = 0 \Leftrightarrow x = y$*;*

(CM2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *;*

(CM3) $\omega_{\lambda+\mu}(x, y) \precsim \omega_\lambda(x, z) + \omega_\mu(z, y)$

for all $x, y, z \in X$ and $\lambda, \mu > 0$.

If instead of $(CM1)$, we only have the condition

(CM1*) $\omega_\lambda(x, x) = 0$ *for all $\lambda > 0$, $x \in X$,*

then λ is said to be a complex valued metric pseudo-modular on X.

Definition 3. *Let $\omega : (0,\infty) \times X \times X \to \mathbb{C}$ be a complex valued metric (pseudo-) modular on X. For any $x_0 \in X$, the sets*

$$X_\omega = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, x_0) = 0\}$$

and

$$X^*_\omega = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < +\infty\}$$

are said to be complex valued modular spaces (around x_0).

If ω is complex valued metric modular on X, the complex valued modular spaces X_ω can be equipped with a metric, generated by ω and given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \precsim \lambda\} \text{ for any } x, y \in X_\omega.$$

Example 4. *Let (X, d) be a complex valued metric space. Then the functional $\omega : (0,\infty) \times X \times X \to \mathbb{C}$ defined by*

$$\omega_\lambda(x, y) = \frac{d(x, y)}{\lambda}$$

is a complex valued modular metric on X. Indeed, complex valued metric spaces are also complex valued modular metric spaces.

Definition 5. *Let X_ω be a complex valued modular metric space and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence on X_ω. Then,*

(1) *$\{a_n\}_{n\in\mathbb{N}}$ is called a complex valued modular convergent sequence to $a \in X_\omega$, if for every $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(a_n, a) \prec \epsilon$ for all $\lambda > 0$ and $n \geq n_0$. And this is denoted with $a_n \to a$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = a$.*

(2) *$\{a_n\}_{n\in\mathbb{N}}$ is called a complex valued modular Cauchy sequence, if for every $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(a_n, a_{n+m}) \prec \epsilon$ for all $\lambda > 0$ and $n \geq n_0$ as $m \in \mathbb{N}$. This is denoted with $\lim_{n\to\infty} \omega_\lambda(a_n, a_{n+m}) = 0$ for all $\lambda > 0$ and $m \succ 0$.*

(3) *X_ω is called a complete complex valued modular metric space, if every modular Cauchy sequence $\{a_n\}$ on X_ω converges to $a \in X_\omega$.*

(4) *The set $K \subseteq X_\omega$ is called closed, if the limit of a complex valued modular convergent sequence on K still in K.*

(5) *The set $K \subseteq X_\omega$ is called bounded, if*

$$\delta_\omega(K) = \sup\{\omega_\lambda(x, y) |\ x, y \in K\} < \infty$$

for all $\lambda > 0$.

Lemma 6. *Let X_ω be a complex valued modular metric space and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence on X_ω. Then $\{a_n\}$ converges to $a \in X_\omega$ if and only if $\omega_\lambda(a_n, a) \to 0$ as $n \to \infty$.*

Lemma 7. *Let $\omega : (0,\infty) \times X \times X \to \mathbb{C}$ be a complex valued modular metric space and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence on X_ω. Then, $\{a_n\}$ is a complex valued Cauch sequence on X_ω if and only if $\omega_\lambda(a_n, a_{n+m}) \to 0$ as $m, n \to \infty$ for all $m \in \mathbb{N}$.*

Lemma 8. *Let w and z be complex numbers. If $w \succsim 0$, $|z| < 1$ and $w \precsim zw$, then $w = 0 \in \mathbb{C}$.*

Proof. Let $w = a + ib,\ z = c + id$ where $a, b, c, d \in \mathbb{R}$. By properties of complex numbers, we have

$$w \succsim 0 \Rightarrow a \geq 0,\ b \geq 0 \tag{1}$$

and

$$|z| < 1 \Rightarrow \sqrt{c^2 + d^2} < 1 \Rightarrow |c^2 + d^2| < 1.$$

Also, since $zw = (ac - bd) + i(ad + bc)$, $w \precsim zw$ implies

$$a \leq ac - bd \text{ and } b \leq ad + bc. \tag{2}$$

We assume that $a \neq 0$. Since $a > 0$ and $|c| \leq |c^2 + d^2| < 1$, we get $ac < a$. From (2), we have $bd < 0$. This implies $b > 0$ and $d < 0$. Then we obtain that $ad < 0$ which contradicts with $b(1 - c) \leq ad$ for $|c| < 1$. Thus, $a = 0$. As $a = 0,\ 0 < 1 - c$, from (2) $b(1 - c) \leq 0$ and $b = 0$. So, $w = a + ib = 0 \in \mathbb{C}$. □

Theorem 9. *Let X_ω be a complete complex valued modular metric space. Suppose that $T : X_\omega \to X_\omega$ is a mapping satisfying*

$$\omega_\lambda(Tx, Ty) \precsim z\ \omega_\lambda(x, y),\ z \in \mathbb{C} \text{ as } |z| < 1 \tag{3}$$

for all $\lambda > 0$ and $x, y \in X_\omega$. Then T has a unique fixed point on X_ω.

Proof. Let $x_0 \in X_\omega$ be arbitrary. We define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n = T^n x_0$ for all $n \geq 0$. Using (3), we have

$$\omega_\lambda(x_n, x_{n+1}) = \omega_\lambda(Tx_{n-1}, Tx_n) \precsim z\ \omega_\lambda(x_{n-1}, x_n) \precsim \cdots \precsim z^n\ \omega_\lambda(x_0, x_1) \tag{4}$$

for $\lambda > 0$ and $n \geq 0$.

Using (4) and axiom (iii) in the definition of complex valued metric spaces, we obtain that

$$\begin{aligned} \omega_\lambda(x_n, x_{n+s}) &\precsim \sum_{j=n}^{n+s-1} \omega_{\frac{\lambda}{s}}(x_j, x_{j+1}) \\ &\precsim \sum_{j=n}^{n+s-1} z^j\ \omega_{\frac{\lambda}{s}}(x_0, x_1) \\ &\precsim \tfrac{z^n}{1-z} \omega_{\frac{\lambda}{s}}(x_0, x_1) \end{aligned}$$

some $\lambda > 0$, $s > 0$ and $n \in \mathbb{N}$.

Now, we take limit as $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} \omega_\lambda(x_n, x_{n+s}) &\precsim \lim_{n\to\infty} \tfrac{z^n}{1-z} \omega_{\frac{\lambda}{s}}(x_0, x_1) \\ &= \tfrac{\omega_{\frac{\lambda}{s}}(x_0, x_1)}{1-z} \lim_{n\to\infty} z^n \end{aligned}$$

We know that $|z^n| = |z|^n \to 0$. Then $z^n \to 0 \in \mathbb{C}$. So, we obtain that

$$0 \precsim \lim_{n\to\infty} \omega_\lambda(x_n, x_{n+s}) = 0. \tag{5}$$

for all $\lambda > 0$ and $s > 0$. From (5), we can say that $\{x_n\}$ is a Cauchy sequence. As X_ω is a complete complex valued modular metric space, there is at least one point $p \in X_\omega$ such that $lim_{n\to\infty}\omega_\lambda(x_n, p) = 0$.

We show that p is a fixed point of T. By using (3) and the axiom (iii) in the definition of complex valued modular metrics, we get

$$\begin{aligned} \omega_\lambda(p, Tp) &\precsim \omega_{\frac{\lambda}{2}}(p, Tx_n) + \omega_{\frac{\lambda}{2}}(Tx_n, Tp) \\ &\precsim \omega_{\frac{\lambda}{2}}(p, x_{n+1}) + z\ \omega_{\frac{\lambda}{2}}(x_n, p) \end{aligned} \tag{6}$$

for all $\lambda > 0$, $n \geq 0$ and $z \in \mathbb{C}$ with $|z| < 1$. If we take limit as $n \to \infty$ in (6) for $\lambda > 0$ and $z \in \mathbb{C}$, since $x_n \to p$, we obtain that

$$0 \precsim \lim_{n\to\infty} \omega_\lambda(p, Tp) \precsim 0. \tag{7}$$

Equation (7) implies $\omega_\lambda(p, Tp) = 0$. So, $Tp = p$.

In this sequel of the proof, we show the uniqueness of the fixed point p of the mapping T. We assume the existence of a point r which is another fixed point of T as $p \neq r$. From (3), we get

$$\omega_\lambda(p, r) = \omega_\lambda(Tp, Tr) \precsim z\ \omega_\lambda(p, r)$$

Since $\omega_\lambda(p, r)$, $z \in \mathbb{C}$ and $|z| < 1$, by Lemma 8, we obtain that $\omega_\lambda(p, r) = 0$ for all $\lambda > 0$. So, $p = r$. □

Now, as a corollary of this theorem, we express a generalization of the Banach fixed point principle in complex valued modular metric spaces.

Corollary 10. *Let X_ω be a complete complex valued modular metric space, z be a complex number such that $Imz = 0$ and $|z| < 1$. If $T : X_\omega \to X_\omega$ is a mapping satisfying*

$$\omega_\lambda(Tx, Ty) \precsim z\ \omega_\lambda(x, y)$$

for all $\lambda > 0$ and $x, y \in X_\omega$, then T has a unique fixed point.

Theorem 11. *Let X_ω be a complete complex valued modular metric space. If*

$$\omega_\lambda(T^n x, T^n y) \precsim z\ \omega_\lambda(x, y)$$

for all $\lambda > 0$, $n > 0$, $z \in \mathbb{C}$ and $x, y \in X_\omega$ as $|z| < 1$, then T has a unique fixed point.

Proof. Since

$$\omega_\lambda(T^n x, T^n y) \precsim z\ \omega_\lambda(x, y),$$

from Theorem 9, there exists a unique fixed point p of T^n on X_ω. Then $T^n p = p$ as $p \in X_\omega$. Then, we have

$$T^n(Tp) = T(T^n p) = Tp.$$

Hence, Tp is further fixed point of T^n. Since p is a unique fixed point of T^n, $Tp = p$. So, p is a fixed point of the mapping T. We assume that there exists another fixed point r of T. So, $Tr = r$. Therefore, $T^n(Tr) = Tr$, which contradicts with the uniqueness of fixed point p for T^n. Then, p is a unique fixed point of T. □

Example 12. *Let $X = \mathbb{C}$. The mapping $\omega : (0,\infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is defined by*

$$\omega_\lambda(z_1, z_2) = \frac{|a_1 - a_2| + i|b_1 - b_2|}{\lambda}$$

for all $\lambda > 0$ where $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then, it can be shown that

$$isacompletecomplexvaluedmodularmetricspace.We define amapping$$

T:$C_\omega \to \mathbb{C}_\omega$ such that $Tk = \frac{k}{3}$ and we take $z = \frac{1}{3} \in \mathbb{C}$. Then, for all $z_1, z_2 \in \mathbb{C}$ and $\lambda > 0$, we have

$$\omega_\lambda(Tz_1, Tz_2) = \omega_\lambda(\frac{z_1}{3}, \frac{z_2}{3}) = \frac{|a_1 - a_2| + i|b_1 - b_2|}{3\lambda}$$

and

$$\omega_\lambda(z_1, z_2) = \frac{|a_1 - a_2| + i|b_1 - b_2|}{\lambda}.$$

Hence, $\omega_\lambda(Tz_1, Tz_2) \precsim z\omega_\lambda(z_1, z_2)$. From Theorem 9, T has a fixed point, which is immediately seen to be $0 \in \mathbb{C}$.

Let X_ω be a complex valued modular metric space, $K \subseteq X_\omega$, $\psi : K \to \mathbb{C}$ be a function and $\{x_n\}$ be a sequence in K. ψ is called lower semi-continuous (l.s.c.) on K if

$$\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0 \text{ and } \lim_{n\to\infty} \inf(\psi(x_n)) = h \text{ imply } \psi(x) \le h$$

for all $\{x_n\} \subseteq K$ and $\lambda > 0$.

Theorem 13. *Let X_ω be a complete complex valued modular metric space and $\psi : X_\omega \to \mathbb{C}$ be a lower semi-continuous function on X_ω. If any mapping $T : X_\omega \to X_\omega$ satisfying*

$$\omega_\lambda(x, Tx) \le \psi(x) - \psi(Tx) \tag{8}$$

for all $\lambda > 0$ and $x, y \in X_\omega$, then T has a fixed point in X_ω.

Proof. For each $x \in X_\omega$ denote,

$$\begin{aligned} M(x) &= \{y \in X_\omega : \omega_\lambda(x, y) \precsim \psi(x) - \psi(y) \text{ for all } \lambda > 0\}, \\ \alpha(x) &= \inf\{\psi(y) : y \in M(x)\}. \end{aligned}$$

Let $x \in M(x)$. Then, $M(x)$ is not empty and $0 \le \alpha(x) \le \psi(x)$. We take an arbitrary point $x \in X_\omega$. Now, we form a sequence $\{x_n\}$ on X_ω as follows:
Let $x_1 = x$ and when $x_1, x_2, \ldots, x_n$ have been chosen, choose $x_{n+1} \in M(x_n)$ such that

$$\psi(x_{n+1}) \le \alpha(x_n) + \frac{1}{n}$$

for all $n \ge 1$. By doing so, we get a sequence $\{x_n\}$ satisfying the condition

$$\begin{aligned} \omega_\lambda(x_n, x_{n+1}) &\precsim \psi(x_n) - \psi(x_{n+1}), \\ \alpha(x_n) &\le \psi(x_{n+1}) \le \alpha(x_n) + \tfrac{1}{n} \end{aligned} \tag{9}$$

for all $n \geq 0$ and $\lambda > 0$. Then, $\{\psi(x_n)\}$ is a nonincreasing sequence and it is bounded from below by zero. So, the sequence $\{\psi(x_n)\}$ is convergent to a number $D \geq 0$. By virtue of (9), we get

$$D = \lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} \alpha(x_n). \tag{10}$$

Now, let $k \in \mathbb{N}$ be arbitrary. From (9) and (10), there exists a number N_k such that

$$\psi(x_n) < D + \frac{1}{k} \quad \text{for all} \quad n \geq N_k.$$

Since $\psi(x_n)$ monotone, we get

$$D \leq \psi(x_m) \leq \psi(x_n) < D + \frac{1}{k}$$

for $m \geq n \geq N_k$. Then, we obtain that

$$\psi(x_n) - \psi(x_m) < \frac{1}{k} \quad \text{for all} \quad m \geq n \geq N_k. \tag{11}$$

Preserving the generality, suppose that $m > n$ and $m, n \in \mathbb{N}$. From (11), we get

$$\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) \precsim \psi(x_n) - \psi(x_{n+1})$$

for all $\frac{\lambda}{m-n} > 0$. Now, we obtain that

$$\begin{array}{rcl} \omega_\lambda(x_n, x_m) & \precsim & \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ & \precsim & \sum_{j=n}^{m-1} [\psi(x_j) - \psi(x_{j+1})] \\ & = & \psi(x_n) - \psi(x_m) \end{array} \tag{12}$$

for all $m, n \geq N_k$. Then, by (11),

$$\omega_\lambda(x_n, x_m) \prec \frac{1}{k} \quad \text{for all} \quad m \geq n \geq N_k. \tag{13}$$

Letting k or m, n to tend to infinity in (13), we conclude that

$$\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0.$$

Then, $\{x_n\}_{n\in\mathbb{N}}$ is a complex valued modular Cauchy sequence. Hence, from the completeness of X_ω, there exist a point $p \in X_\omega$ such that $x_n \to p$ as $n \to \infty$. Since ψ is lower semi-continuous, using the equation (12), we have

$$\begin{array}{rcl} \psi(p) & \leq & lim_{m\to\infty} inf \psi(x_m) \\ & \precsim & lim_{m\to\infty} inf(\psi(x_n) - \omega_\lambda(x_n, x_m)) \\ & = & \psi(x_n) - \omega_\lambda(x_n, p) \end{array}$$

and hence

$$\omega_\lambda(x_n, p) \precsim \psi(x_n) - \psi(p).$$

Thus, $p \in M(x_n)$ for all $n \geq 0$ and $\alpha(x_n) \leq \psi(p)$. So, by (10), we have $D \leq \psi(p)$. Moreover, by lower semi-continuity of ψ and (10), we get

$$\psi(p) = \lim_{n\to\infty} \psi(x_n) = S.$$

So, $\psi(p) = S$. From 8, we know that $Tp \in M(p)$. Since $p \in M(p)$ for $n \in \mathbb{N}$, we get

$$\begin{aligned} \omega_\lambda(x_n, Tp) &\precsim \omega_{\frac{\lambda}{2}}(x_n, p) + \omega_{\frac{\lambda}{2}}(p, Tp) \\ &\precsim \psi(x_n) - \psi(p) + \psi(p) - \psi(Tp) \\ &= \psi(x_n) - \psi(Tp). \end{aligned}$$

Then $Tp \in M(x_n)$ and implies $\alpha(x_n) \leq \psi(Tp)$. Thus, we obtain $S \leq \psi(Tp)$. Since $\psi(Tp) \leq \psi(p)$ by (8) and $\psi(p) = S$, we get

$$\psi(p) = S \leq \psi(Tp) \leq \psi(p).$$

Therefore, $\psi(Tp) = \psi(p)$. Then from (8), we get

$$\omega_\lambda(p, Tp) \precsim \psi(p) - \psi(Tp) = 0.$$

Thus, $Tp = p$. □

Example 14. *Let $X = \mathbb{C}$. We define the mapping $\omega : (0, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by*

$$\omega_\lambda(z_1, z_2) = \frac{|a_1 - a_2| + i|b_1 - b_2|}{\lambda}$$

for all $\lambda > 0$ where $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. $\mathbb{C}_\omega$ is a complete modular metric space. Define $T : \mathbb{C}_\omega \to \mathbb{C}_\omega$ by $Tz = \frac{z}{4}$ and $\psi : \mathbb{C}_\omega \to \mathbb{C}$ by $\psi(z) = |a| + i|b|$ where $z = a + ib$. Then for all $z = a + ib \in \mathbb{C}$ and $\lambda > 0$, we have

$$\omega_\lambda(z, Tz) = \frac{|a - \frac{a}{4}| + i|b - \frac{b}{4}|}{\lambda} = \frac{\frac{3}{4}|a| + i\frac{3}{4}|b|}{\lambda} \leq \frac{3}{4}(|a| + i|b|)$$

and

$$\psi(z) - \psi(Tz) = (|a| + i|b|) - \left(\frac{|a|}{4} + i\frac{|b|}{4}\right) = \frac{3}{4}(|a| + i|b|).$$

Hence, $\omega_\lambda(z, Tz) \leq \psi(z) - \psi(T)$. From Theorem 13, the mapping T has a fixed point.

3. AN APPLICATION TO DYNAMIC PROGRAMMING

In the section, we express an application of Theorem 9 to dynamic programming which is a powerful tecnique for solving some complex problems in mathematics, economics, computer science and bioinformatics.

Let X_ω be a complete complex valued modular metric space induced by $\omega : (0, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $S \subseteq X_\omega$, Z be a Banach space and $P \subseteq Z$.

We consider the functional equation

$$q(x) = \sup_{y \in P}\{f(x, y) + H(x, y, q(\varphi(x, y)))\} \tag{14}$$

where $x \in S$, $\varphi : S \times P \to S$, $f : S \times P \to \mathbb{C}$ and $H : S \times P \times \mathbb{C} \to \mathbb{C}$. We show that existence of unique solution of the functional equation (14). We suppose that $B(S)$ is the set of all bounded complex valued function on S. We define

$$\|k\| = \sup_{x \in S} |k(x)|$$

for an arbitrary $k \in B(S)$. We take complex valued metric modular ω on $B(S)$ as

$$\omega_\lambda(k, g) = \sup_{x \in Z} \left\{ \left| \frac{k(x) - g(x)}{\lambda} \right| + i \left| \frac{k(x) - g(x)}{\lambda} \right| \right\} \tag{15}$$

for all $k, g \in B(S)$ and $\lambda > 0$. On the other hand, we take a Cauchy sequence $\{k_n\}_{n \in \mathbb{N}}$ in $B(S)$. Then $\{k_n\}_{n \in \mathbb{N}}$ is convergent to a function $t \in B(S)$.

Theorem 15. *Let $f : S \times P \to \mathbb{C}$ and $H : S \times P \times \mathbb{C} \to \mathbb{C}$ be bounded. We suppose that $T : B(S) \to B(S)$ defined by*

$$T(k)(x) = \sup_{y \in P} \{ f(x, y) + H(x, y, k(\varphi(x, y))) \}$$

for all $k \in B(S)$ and $x \in S$. If

$$\left| \frac{H(x, y, k(x)) - H(x, y, g(x))}{\lambda} \right| + i \left| \frac{H(x, y, k(x)) - H(x, y, g(x))}{\lambda} \right| \precsim z \omega_\lambda(k, g) \tag{16}$$

for all $\lambda > 0$, $x \in S$, $y \in P$, $k, g \in B(S)$ and a arbitrary complex number z where $|z| < 1$, the functional equation (14) has a unique solution.

Proof. Let $x \in S$ and $k, g \in B(S)$. Then there exist $y_1, y_2 \in P$ and a complex number $\delta > 0$ such that

$$T(k)(x) \precsim f(x, y_1) + H(x, y_1, k(\varphi(x, y_1))) + \delta \tag{17}$$

$$T(g)(x) \precsim f(x, y_2) + H(x, y_2, g(\varphi(x, y_2))) + \delta \tag{18}$$

$$T(k)(x) \succsim f(x, y_1) + H(x, y_1, k(\varphi(x, y_1))) \tag{19}$$

$$T(g)(x) \succsim f(x, y_2) + H(x, y_2, g(\varphi(x, y_2))). \tag{20}$$

From (17) and (20), we obtain that

$$\begin{aligned} T(k)(x) - T(g)(x) &\precsim H(x, y_1, k(\varphi(x, y_1))) - H(x, y_2, g(\varphi(x, y_2))) + \delta \\ &\precsim \left| H(x, y_1, k(\varphi(x, y_1))) - H(x, y_2, g(\varphi(x, y_2))) \right| + \delta. \end{aligned}$$

So, for $\lambda > 0$

$$\frac{T(k)(x) - T(g)(x)}{\lambda} \precsim \left| \frac{H(x, y_1, k(\varphi(x, y_1))) - H(x, y_2, g(\varphi(x, y_2)))}{\lambda} \right| + \frac{\delta}{\lambda}. \tag{21}$$

Similarly, combining (18) and (19) we have

$$\frac{T(g)(x) - T(k)(x)}{\lambda} \precsim \left| \frac{H(x, y_1, k(\varphi(x, y_1))) - H(x, y_2, g(\varphi(x, y_2)))}{\lambda} \right| + \frac{\delta}{\lambda}. \tag{22}$$

Therefore, from (21) and (22),

$$\left|\frac{T(k)(x)-T(g)(x)}{\lambda}\right| \precsim \left|\frac{H(x,y_1,k(\varphi(x,y_1)))-H(x,y_2,g(\varphi(x,y_2)))}{\lambda}\right| + \frac{\delta}{\lambda} \quad (23)$$

for all $\lambda > 0$. Since $\frac{\delta}{\lambda} > 0$ in inequality (23), we can ignore the contrary the $\frac{\delta}{\lambda}$. Therefore,

$$\left|\frac{T(k)(x)-T(g)(x)}{\lambda}\right| \precsim \left|\frac{H(x,y_1,k(\varphi(x,y_1)))-H(x,y_2,g(\varphi(x,y_2)))}{\lambda}\right|. \quad (24)$$

From inequality (24), we easily obtain that

$$\begin{aligned}\left|\frac{T(k)(x)-T(g)(x)}{\lambda}\right| + i\left|\frac{T(k)(x)-T(g)(x)}{\lambda}\right| \precsim & \left|\frac{H(x,y_1,k(\varphi(x,y_1)))-H(x,y_2,g(\varphi(x,y_2)))}{\lambda}\right| \\ & + i\left|\frac{H(x,y_1,k(\varphi(x,y_1)))-H(x,y_2,g(\varphi(x,y_2)))}{\lambda}\right|\end{aligned}$$

From (15) and (16), we get

$$\omega_\lambda(T(k),T(g)) \precsim z\omega_\lambda(k,g).$$

Then, from Theorem 9, T has a unique fixed point $t \in B(S)$. That is, the functional equation (14) has a unique solution. $\square$

Open problem How can we obtain coupled fixed point theorems and common fixed point theorems in these metric spaces?

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ON THE RESOLVENT OF SINGULAR q-STURM-LIOUVILLE OPERATORS

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Abstract. In this paper, we investigate the resolvent operator of the singular q-Sturm-Liouville problem defined as

$$-\frac{1}{q}D_{q^{-1}}\left[D_q y\left(x\right)\right]+\left[r\left(x\right)-\lambda\right]y\left(x\right)=0,$$

with the boundary condition

$$y\left(0,\lambda\right)\cos\beta+D_{q^{-1}}y\left(0,\lambda\right)\sin\beta=0,$$

where $\lambda\in\mathbb{C}$, r is a real-valued function defined on $[0,\infty)$, continuous at zero and $r\in L^1_{q,loc}[0,\infty)$. We give a representation for the resolvent operator and investigate some properties of this operator. Furthermore, we obtain a formula for the Titchmarsh-Weyl function of the singular q-Sturm-Liouville problem.

1. Introduction

Quantum (or q) calculus is a very interesting field in mathematics. It has numerous in statistic physics, quantum theory, the calculus of variations and number theory; see, e.g., [12, 1, 11, 14, 15, 18, 21, 24]). The first results in q-calculus belong to the Euler. In 2005, Annaby and Mansour investigated q-Sturm-Liouville problems [10]. Later in [9], the authors studied the Titchmarsh-Weyl theory for q-Sturm-Liouville equations. In [3,4], the authors proved the existence of a spectral function for q-Sturm-Liouville operator.

In this article, we investigate the following q-Sturm-Liouville problem defined as

$$-\frac{1}{q}D_{q^{-1}}D_q y\left(x\right)+u\left(x\right)y\left(x\right)=\lambda y\left(x\right),\tag{1}$$

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where $0 < x < \infty$. The resolvent operator for this problem is constructed. Using the spectral function, an integral representation is obtained. Furthermore, some properties of this operator are investigated. A formula for the Titchmarsh-Weyl function of Eq. (1) is given. Historically, in 1910, H. Weyl was first obtained a representation theorem for the resolvent of Sturm-Liouville problem defined by

$$-\left(py'\right)' + qy = \lambda y,\ x \in (0, \infty),$$

where p, q are real-valued and $p^{-1}, q \in L^1_{loc}[0, \infty)$. Similar representation theorems were proved in [25, 20, 2, 5, 6, 7].

2. Preliminaries

In this section, we give a brief introduction to quantum calculus and refer the interested reader to [17, 8, 12].

Let $0 < q < 1$ and let $A \subset \mathbb{R}$ is a q-geometric set, i.e., $qx \in A$ for all $x \in A$. The *Jackson q-derivative is* defined by

$$D_q y\left(x\right) = \mu^{-1}\left(x\right)\left[y\left(qx\right) - y\left(x\right)\right],$$

where $\mu\left(x\right) = qx - x$ and $x \in A$. We note that there is a connection the Jackson q-derivative between and q-deformed Heisenberg uncertainty relation (see [23]). The q-derivative at zero is defined as

$$D_q y\left(0\right) = \lim_{n \to \infty} \left[q^n x\right]^{-1}\left[y\left(q^n x\right) - y\left(0\right)\right] \quad (x \in A), \tag{2}$$

if the limit in (2) exists and does not depend on x. The *Jackson q-integration* is given by

$$\int_0^x f\left(t\right) d_q t = x\left(1 - q\right) \sum_{n=0}^{\infty} q^n f\left(q^n x\right) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f\left(t\right) d_q t = \int_0^b f\left(t\right) d_q t - \int_0^a f\left(t\right) d_q t,$$

where $a, b \in A$. The q-integration for a function over $[0, \infty)$ defined by the formula ([13])

$$\int_0^{\infty} f\left(t\right) d_q t = \sum_{n=-\infty}^{\infty} q^n f\left(q^n\right).$$

Let f be a function on A and let $0 \in A$. For every $x \in A$, if

$$\lim_{n \to \infty} f\left(xq^n\right) = f\left(0\right),$$

then f is called q-regular at zero. Throughout the paper, we deal only with functions q-regular at zero.

The following relation holds

$$\int_0^a g(t) D_q f(t) d_q t + \int_0^a f(qt) D_q g(t) d_q t = f(a) g(a) - f(0) g(0),$$

where f and g are q-regular at zero.

Let $L_q^2[0,\infty)$ be the Hilbert space consisting of all functions f satisfying ([9])

$$\|f\| := \sqrt{\int_0^\infty |f(x)|^2 d_q x} < +\infty$$

with the inner product

$$(f,g) := \int_0^\infty f(x) \overline{g(x)} d_q x.$$

The q-Wronskian of the functions $y(.)$ and $z(.)$ is defined by the formula

$$W_q(y,z)(x) := y(x) D_q z(x) - z(x) D_q y(x),$$

where $x \in [0,\infty)$.

3. Main Results

Consider the q-Sturm-Liouville equation

$$L(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + r(x) y(x) = \lambda y(x), \tag{3}$$

satisfying the conditions

$$y(0,\lambda)\cos\beta + D_{q^{-1}} y(0,\lambda)\sin\beta = 0, \tag{4}$$

$$y\left(q^{-n},\lambda\right)\cos\alpha + D_{q^{-1}} y\left(q^{-n},\lambda\right)\sin\alpha = 0,\ \alpha,\beta \in \mathbb{R},\ n \in \mathbb{N} := \{1,2,...\}, \tag{5}$$

where $\lambda \in \mathbb{C}$, r is a real-valued function defined on $[0,\infty)$, continuous at zero and $r \in L_{q,loc}^1[0,\infty)$.

Let $\varphi(x,\lambda)$ and $\theta(x,\lambda)$ be the solutions of the Eq. (3) satisfying the following conditions

$$\begin{array}{l} \varphi(0,\lambda) = \sin\beta,\ D_{q^{-1}}\varphi(0,\lambda) = -\cos\beta, \\ \theta(0,\lambda) = \cos\beta,\ D_{q^{-1}}\theta(0,\lambda) = \sin\beta. \end{array} \tag{6}$$

Lemma 1 ([9]). *Let $\lambda \notin \mathbb{R}$ and let*

$$\chi_{q^{-n}}(x,\lambda) = \theta(x,\lambda) + l\left(\lambda, q^{-n}\right)\varphi(x,\lambda) \in L_q^2(0,\infty),$$

where $n \in \mathbb{N}$. Then we have

$$\chi_{q^{-n}}(x,\lambda) \quad \to \quad \chi(x,\lambda),$$

$$\int_0^{q^{-n}} \left|\chi_{q^{-n}}(qt,\lambda)\right|^2 d_q x \quad \to \quad \int_0^\infty |\chi(x,\lambda)|^2 d_q x,\ n \to \infty.$$

Putting

$$G_{q^{-n}}(x,t,\lambda)=\begin{cases}\chi_{q^{-n}}(x,\lambda)\varphi(t,\lambda), & t\leq x\\ \varphi(x,\lambda)\chi_{q^{-n}}(t,\lambda), & t>x,\end{cases}$$

$$y(x,\lambda):=\left(R_{q^{-n}}f\right)(x,\lambda)=$$

$$\int_0^{q^{-n}}G_{q^{-n}}(x,t,\lambda)f(t)\,d_qt,\ \ (\lambda\in\mathbb{C},\ \ \operatorname{Im}\lambda\neq 0), \tag{7}$$

where $f\in L_q^2[0,q^{-n}]$. Now, we shall show that the equality (7) satisfies the equation $L(y)-\lambda y(x)=f(x)$, $x\in(0,q^{-n})$ ($\lambda\in\mathbb{C}$, $\operatorname{Im}\lambda\neq 0$) and the boundary conditions (4)-(5). From (7), we get

$$\begin{aligned} y(x,\lambda) &= q\chi_{q^{-n}}(x,\lambda)\int_0^x\varphi(qt,\lambda)f(qt)\,d_qt\\ &\quad+q\varphi(x,\lambda)\int_x^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt. \end{aligned}\tag{8}$$

From (8), it follows that

$$\begin{aligned} D_qy(x,\lambda) &= qD_q\chi_{q^{-n}}(x,\lambda)\int_0^x\varphi(qt,\lambda)f(qt)\,d_qt\\ &\quad+qD_q\varphi(x,\lambda)\int_x^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt, \end{aligned}$$

and

$$\begin{aligned} D_{q^{-1}}D_qy(x,\lambda) &= qD_{q^{-1}}D_q\chi_{q^{-n}}(x,\lambda)\int_0^x\varphi(qt,\lambda)f(qt)\,d_qt\\ &\quad+qD_{q^{-1}}D_q\varphi(x,\lambda)\int_x^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt\\ &\quad-qW_q\left(\chi_{q^{-n}},\varphi\right)f(x). \end{aligned}$$

Hence, by $W_q\left(\varphi,\chi_{q^{-n}}\right)=1$ ($n\in\mathbb{N}$), we deduce that

$$\begin{aligned} &-\frac{1}{q}D_{q^{-1}}D_qy(x,\lambda)\\ &= (\lambda-r(x))\,q\chi_{q^{-n}}(x,\lambda)\int_0^x\varphi(qt,\lambda)f(qt)\,d_qt\\ &\quad+(\lambda-r(x))\,q\varphi(x,\lambda)\int_x^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt+f(x)\\ &= (\lambda-r(x))\,y(x,\lambda)+f(x), \end{aligned}$$

i.e., the function $y(x,\lambda)$ satisfies the equation $L(y)-\lambda y(x)=f(x)$, $x\in(0,q^{-n})$.

Moreover,

$$\begin{aligned} y(0,\lambda) &= q\varphi(0,\lambda)\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt \\ &= q\cos\beta\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt, \\ D_{q^{-1}}y(0,\lambda) &= qD_{q^{-1}}\varphi(0,\lambda)\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt \\ &= -q\sin\beta\int_0^{q^{-n}}\chi_{q^{-n}}(qt,\lambda)f(qt)\,d_qt, \end{aligned}$$

i.e., $y(x,\lambda)$ satisfies (4). Similarly, we may infer that $y(x,\lambda)$ satisfies (5).

Note that the problem (3)-(5) has a purely discrete spectrum [10].

Let $\lambda_{m,q^{-n}}$ be the eigenvalues of the problem (3)-(5). Let $\varphi_{m,q^{-n}}$ be the corresponding eigenfunctions and

$$\alpha_{m,q^{-n}}:=\left\|\varphi_{m,q^{-n}}\right\|=\left(\int_0^{q^{-n}}\varphi^2_{m,q^{-n}}(x)\,d_qx\right)^{\frac{1}{2}},$$

where $\varphi_{m,q^{-n}}(x):=\varphi_{m,q^{-n}}\left(x,\lambda_{m,q^{-n}}\right)$ and $m\in\mathbb{N}$.

Then we have the following *Parseval equality* (see [8])

$$\int_0^{q^{-n}}|f(x)|^2\,d_qx=\sum_{m=1}^{\infty}\frac{1}{\alpha^2_{m,q^{-n}}}\left\{\int_0^{q^{-n}}f(x)\,\varphi_{m,q^{-n}}(x)\,d_qx\right\}^2, \tag{9}$$

where $f(.)\in L_q^2[0,q^{-n}]$.

Now, let us define the nondecreasing step function $\varrho_{q^{-n}}$ on $[0,\infty)$ by

$$\varrho_{q^{-n}}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_{m,q^{-n}}<0}\frac{1}{\alpha^2_{m,q^{-n}}}, & \text{for } \lambda\le 0 \\ \sum_{0\le\lambda_{m,q^{-n}}<\lambda}\frac{1}{\alpha^2_{m,q^{-n}}} & \text{for } \lambda>0. \end{cases}$$

It follows from (9) that

$$\int_0^{q^{-n}}|f(x)|^2\,d_qx=\int_{-\infty}^{\infty}F^2(\lambda)\,d\varrho_{q^{-n}}(\lambda), \tag{10}$$

where

$$F(\lambda)=\int_0^{q^{-n}}f(x)\,\varphi(x,\lambda)\,d_qx.$$

Lemma 2. *Let $\kappa > 0$. Then the following relation holds*

$$\overset{\kappa}{\underset{-\kappa}{V}} \left\{ \varrho_{q^{-n}}(\lambda) \right\} = \sum_{-\kappa \le \lambda_{m,q^{-n}} < \kappa} \frac{1}{\alpha^2_{m,q^{-n}}} = \varrho_{q^{-n}}(\kappa) - \varrho_{q^{-n}}(-\kappa) < \Upsilon, \tag{11}$$

where $\Upsilon = \Upsilon(\kappa)$ is a positive constant not depending on q^{-n}.

Proof. Let $\sin\beta \neq 0$. Since $\varphi(x,\lambda)$ is continuous at zero, by condition $\varphi(0,\lambda) = \sin\beta$, there exists a positive number h and nearby 0 such that

$$|\varphi(x,\lambda)| > \frac{1}{\sqrt{2}} |\sin\beta|, \ 0 \le x \le h$$

and

$$\left(\frac{1}{h} \int_0^h \varphi(x,\lambda)\, d_q x \right)^2 > \left(\frac{1}{\sqrt{2}h} \sin\beta \int_0^h d_q x \right)^2 = \frac{1}{2}\sin^2\beta. \tag{12}$$

Let us define $f_h(x)$ by

$$f_h(x) = \begin{cases} 0, & x > h \\ \frac{1}{h}, & 0 \le x \le h. \end{cases}$$

It follows from (10) and (12) that

$$\begin{aligned} \int_0^h f_h^2(x)\, d_q x &= \frac{1}{h} = \int_{-\infty}^{\infty} \left(\frac{1}{h}\int_0^h \varphi(x,\lambda)\, d_q x \right)^2 d\varrho_{q^{-n}}(\lambda) \\ &\ge \int_{-\kappa}^{\kappa} \left(\frac{1}{h}\int_0^h \varphi(x,\lambda)\, d_q x \right)^2 d\varrho_{q^{-n}}(\lambda) \\ &> \frac{1}{2}\sin^2\beta \left\{ \varrho_{q^{-n}}(\kappa) - \varrho_{q^{-n}}(-\kappa) \right\}, \end{aligned}$$

which proves the inequality (11).

Let $\sin\beta = 0$ and

$$f_h(x) = \begin{cases} 0, & x > h \\ \frac{1}{h^2}, & 0 \le x \le h. \end{cases}$$

By (10), we can get the desired result. □

We now return to the formula (7), whose right-hand side has been called the resolvent. The resolvent is known to exist for all λ which are not eigenvalues of the problem (3)-(5). Now, we will get the expansion of the resolvent.

Since the function $y(x,\lambda)$ satisfies the equation $L(y) - \lambda y(x) = f(x)$, $x \in (0, q^{-n})$ ($\lambda \in \mathbb{C}$, $\lambda \neq \lambda_{m,q^{-n}}$, $m \in \mathbb{N}$) and the boundary conditions (4), (5), via the q-integration by parts, we find (the operator A generated by the expression L and the boundary conditions (4), (5) is a self-adjoint (see [10]))

$$(Ay, \varphi_{m,q^{-n}})$$

$$
\begin{aligned}
&= \int_0^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q y(x,\lambda) + r(x)\, y(x,\lambda) \right] \varphi_{m,q^{-n}}(x)\, d_q x \\
&= (y, A\varphi_{m,q^{-n}}) \\
&= \int_0^{q^{-n}} y(x,\lambda) \left[-\frac{1}{q} D_{q^{-1}} D_q \varphi_{m,q^{-n}}(x) + r(x)\, \varphi_{m,q^{-n}}(x) \right] d_q x \\
&= \lambda_{m,q^{-n}} \int_0^{q^{-n}} y(x,\lambda)\, \varphi_{m,q^{-n}}(x)\, d_q x.
\end{aligned}
$$

The set of all eigenfunctions $\frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}}$ ($m \in \mathbb{N}$) of the self-adjoint operator A form an orthonormal basis for $L_q^2(0,q^{-n})$ (see [10]). Then, the function $y(.,\lambda) \in L_q^2(0,q^{-n})$ ($\lambda \in \mathbb{C}$, $\lambda \neq \lambda_{m,q^{-n}}$, $m \in \mathbb{N}$) can be expanded into Fourier series of eigenfunctions $\frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}}$ ($m \in \mathbb{N}$) of the problem (3)-(5) (or of the operator A). Then we have

$$
y(x,\lambda) = \sum_{m=1}^{\infty} t_m(\lambda) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}},
$$

where $t_m(\lambda)$ is the Fourier coefficient, i.e.,

$$
t_m(\lambda) = \int_0^{q^{-n}} y(x,\lambda) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} d_q x, \ m \in \mathbb{N}.
$$

Since $y(x,\lambda)$ ($\lambda \in \mathbb{C}$, $\lambda \neq \lambda_{m,q^{-n}}$, $m \in \mathbb{N}$) satisfies the equation

$$
-\frac{1}{q} D_{q^{-1}} D_q y(x,\lambda) + (r(x) - \lambda)\, y(x,\lambda) = f(x), \ x \in (0,q^{-n}),
$$

we get

$$
\begin{aligned}
a_m &:= \int_0^{q^{-n}} f(x) \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} d_q x \\
&= \int_0^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q y(x,\lambda) + (r(x) - \lambda)\, y(x,\lambda) \right] \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} d_q x \\
&= \int_0^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} D_q \varphi_{m,q^{-n}}(x) + (r(x) - \lambda) \varphi_{m,q^{-n}}(x) \right] \frac{y(x,\lambda)}{\alpha_{m,q^{-n}}} d_q x \\
&= \int_0^{q^{-n}} \left[\lambda_{m,q^{-n}} \varphi_{m,q^{-n}}(x) - \lambda \varphi_{m,q^{-n}}(x) \right] \frac{y(x,\lambda)}{\alpha_{m,q^{-n}}} d_q x \\
&= \lambda_{m,q^{-n}} t_m(\lambda) - \lambda t_m(\lambda), \ m \in \mathbb{N}.
\end{aligned}
$$

Thus, we have

$$t_m(\lambda) = \frac{a_m}{\lambda_{m,q^{-n}} - \lambda},$$

and

$$\begin{aligned} y(x,\lambda) &= \int_0^{q^{-n}} G_{q^{-n}}(x,t,\lambda) f(t)\, d_q t \\ &= \sum_{m=1}^{\infty} \frac{a_m}{\lambda_{m,q^{-n}} - \lambda} \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}} \ (\lambda \in \mathbb{C},\ \lambda \neq \lambda_{m,q^{-n}},\ m \in \mathbb{N}). \end{aligned}$$

Then

$$\begin{aligned} y(x,z) &= \left(R_{q^{-n}} f\right)(x,z) \\ &= \sum_{m=1}^{\infty} \frac{\varphi_{m,q^{-n}}(x)}{\alpha^2_{m,q^{-n}}\left(\lambda_{m,q^{-n}} - z\right)} \int_0^{q^{-n}} f(t)\, \varphi_{m,q^{-n}}(t)\, d_q t \\ &= \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} \left\{ \int_0^{q^{-n}} f(t)\, \varphi_{m,q^{-n}}(t,\lambda)\, d_q t \right\} d\varrho_{q^{-n}}(\lambda). \end{aligned} \tag{13}$$

Lemma 3. *The following formula holds*

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda) < K, \tag{14}$$

where x is a fixed number and z is a non-real number.

Proof. Let $f(t) = \frac{\varphi_{m,q^{-n}}(t)}{\alpha_{m,q^{-n}}}$. By (13), we conclude that

$$\frac{1}{\alpha_{m,q^{-n}}} \int_0^{q^{-n}} G_{q^{-n}}(x,t,z)\, \varphi_{m,q^{-n}}(t)\, d_q t = \frac{\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}\left(\lambda_{m,q^{-n}} - z\right)}. \tag{15}$$

Under (15) and (9), we see that

$$\begin{aligned} \int_0^{q^{-n}} \left|G_{q^{-n}}(x,t,z)\right|^2 d_q t &= \sum_{m=1}^{\infty} \frac{\left|\varphi_{m,q^{-n}}(x)\right|^2}{\alpha^2_{m,q^{-n}}\left|\lambda_{m,q^{-n}} - z\right|^2} \\ &= \int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda). \end{aligned}$$

It follows from Lemma 1 that the last integral is convergent. The proof is complete □

Now, we present below for the convenience of the reader.

Theorem 4 ([19]). *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real non-decreasing function on a finite interval $[a,b]$. Then*
(i) there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a non-decreasing function w such that

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda),$$

where $a \leq \lambda \leq b$.

(ii) suppose

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),$$

where $a \leq \lambda \leq b$. Then, we have

$$\lim_{n\to\infty} \int_a^b f(\lambda)\, dw_n(\lambda) = \int_a^b f(\lambda)\, dw(\lambda),$$

where $f \in C[a,b]$.

By Lemma 2 and Theorem 4, one can find a sequence $\{q^{-n_k}\}$ such that

$$\lim_{k\to\infty} \varrho_{q^{-n_k}}(\lambda) \to \varrho(\lambda),$$

where $\varrho(\lambda)$ is a monotone function.

Lemma 5. *Let $z \notin \mathbb{R}$. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho(\lambda) \leq K, \tag{16}$$

where x is a fixed number.

Proof. Let $\eta > 0$. It follows from (14) that

$$\int_{-\eta}^{\eta} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda) < K.$$

Then

$$\int_{-\infty}^{\infty} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho(\lambda) = \lim_{\substack{\eta\to\infty \\ n\to\infty}} \int_{-\eta}^{\eta} \left| \frac{\varphi(x,\lambda)}{\lambda - z} \right|^2 d\varrho_{q^{-n}}(\lambda) < K.$$

□

Lemma 6. *Let $\eta > 0$. Then we have*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty. \tag{17}$$

Proof. Let $\sin\beta \neq 0$. From (16), we deduce that

$$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|\lambda - z|^2} < \infty.$$

Let $\sin\beta = 0$. Hence we see that

$$\frac{1}{\alpha_{m,q^{-n}}}\int_0^{q^{-n}} \varphi_{m,q^{-n}}(t)\, D_{q,x}\left[G_{q^{-n}}(x,t,z)\right] d_q t = \frac{D_{q,x}\varphi_{m,q^{-n}}(x)}{\alpha_{m,q^{-n}}\left(\lambda_{m,q^{-n}} - z\right)}.$$

It follows from (9) that

$$\int_0^{q^{-n}} \left|D_{q,x}\left[G_{q^{-n}}(x,t,z)\right]\right|^2 d_q t = \int_{-\infty}^{\infty} \left|\frac{D_{q,x}\varphi(x,\lambda)}{\lambda - z}\right|^2 d\varrho_{q^{-n}}(\lambda).$$

Proceeding similarly, we can get the desired result. □

Lemma 7. *Let*

$$G(x,t,z) = \begin{cases} \chi(x,z)\,\varphi(t,z), & x \geq t \\ \varphi(x,z)\,\chi(t,z), & x < t, \end{cases}$$

and let $f(.) \in L_q^2[0,\infty)$. *Then we have*

$$\int_0^{\infty} \left|(Rf)(x,z)\right|^2 d_q x \leq \frac{1}{v^2}\int_0^{\infty} \left|f(x)\right|^2 d_q x,$$

where

$$(Rf)(x,z) = \int_0^{\infty} G(x,t,z)\, f(t)\, d_q t,$$

and $z = u + iv$.

Proof. See [9]. □

Now we shall state the main result of this paper.

Theorem 8. *The following relation holds*

$$(Rf)(x,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F(\lambda)\, d\varrho(\lambda), \tag{18}$$

where $f(.) \in L_q^2[0,\infty)$,

$$F(\lambda) = \lim_{\xi\to\infty}\int_0^{q^{-\xi}} f(x)\,\varphi(x,\lambda)\, d_q x,$$

and $z \notin \mathbb{R}$.

Proof. Define the function $f_\xi(x)$ as

$$f_\xi(x) = \begin{cases} f_\xi(x), & x \in [0, q^{-\xi}], \\ 0, & x \notin [0, q^{-\xi}] \end{cases} \quad (q^{-\xi} < q^{-n})$$

such that $f_\xi(x)$ satisfies (4). By (13), we conclude that

$$\left(R_{q^{-n}} f_\xi\right)(x,z)$$

$$= \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_{q^{-n}}(\lambda) = \int_{-\infty}^{-a} \frac{\varphi(x,\lambda)}{\lambda - z} F_\xi(\lambda)\, d\varrho_{q^{-n}}(\lambda)$$

$$
\begin{aligned}
& +\int_{-a}^{a}\frac{\varphi(x,\lambda)}{\lambda-z}F_{\xi}(\lambda)\,d\varrho_{q^{-n}}(\lambda)+\int_{a}^{\infty}\frac{\varphi(x,\lambda)}{\lambda-z}F_{\xi}(\lambda)\,d\varrho_{q^{-n}}(\lambda) \\
= \quad & I_1+I_2+I_3, \qquad (19)
\end{aligned}
$$

where

$$
F_{\xi}(\lambda)=\int_{0}^{q^{-\xi}}f(x)\,\varphi(x,\lambda)\,d_qx,
$$

and $a>0$.

It follows from (13) that

$$
\begin{aligned}
|I_1| &= \left|\int_{-\infty}^{-a}\frac{\varphi(x,\lambda)}{\lambda-z}F_{\xi}(\lambda)\,d\varrho_{q^{-n}}(\lambda)\right| \\
&\le \sum_{\lambda_{k,q^{-n}}<-a}\frac{\left|\varphi_{k,q^{-n}}(x)\right|\left|\int_0^{q^{-\xi}}f_{\xi}(t)\,\varphi_{k,q^{-n}}(t)\,d_qt\right|}{\alpha_{k,q^{-n}}^2\left|\lambda_{k,q^{-n}}-z\right|} \\
&\le \left(\sum_{\lambda_{k,q^{-n}}<-a}\frac{\varphi_{k,q^{-n}}^2(x)}{\alpha_{k,q^{-n}}^2\left|\lambda_{k,q^{-n}}-z\right|^2}\right)^{1/2} \\
&\quad\times\left(\sum_{\lambda_{k,q^{-n}}<-a}\frac{1}{\alpha_{k,q^{-n}}^2}\left[\int_0^{q^{-\xi}}f_{\xi}(x)\,\varphi_{k,q^{-n}}(x)\,d_qx\right]^2\right)^{1/2}. \qquad (20)
\end{aligned}
$$

Using the q-integration-by-parts formula in the integral below, we have

$$
\begin{aligned}
& \int_0^{q^{-\xi}}f_{\xi}(x)\,\varphi_{k,q^{-n}}(x)\,d_qx \\
= \quad & \frac{1}{\lambda_{k,q^{-n}}}\int_0^{q^{-\xi}}f_{\xi}(x)\left\{-\frac{1}{q}D_{q^{-1}}D_q\varphi_{k,q^{-n}}(x)+r(x)\,\varphi_{k,q^{-n}}(x)\right\}d_qx \\
= \quad & \frac{1}{\lambda_{k,q^{-n}}}\int_0^{q^{-\xi}}\left\{-\frac{1}{q}D_{q^{-1}}D_qf_{\xi}(x)+r(x)\,f_{\xi}(x)\right\}\varphi_{k,q^{-n}}(x)\,d_qx. \qquad (21)
\end{aligned}
$$

From Lemma 3, we get

$$
|I_1|\le\frac{K^{1/2}}{a}\left(\begin{array}{c}\sum_{\lambda_{k,q^{-n}}<-a}\frac{1}{\alpha_{k,q^{-n}}^2} \\ \times\left[\int_0^{q^{-\xi}}\left\{-\frac{1}{q}D_{q^{-1}}D_qf_{\xi}(x)+r(x)\,f_{\xi}(x)\right\}\varphi_{k,q^{-n}}(x)\,d_qx\right]^2\end{array}\right)^{1/2}.
$$

Application of Bessel inequality yields

$$|I_1| \leq \frac{K^{1/2}}{a}\left[\int_0^{q^{-\xi}}\left\{-\frac{1}{q}D_{q^{-1}}D_q f_\xi(x)+r(x)f_\xi(x)\right\}^2 d_qx\right]^{1/2}=\frac{C}{a}.$$

Likewise, we show that $|I_3| \leq \frac{C}{a}$. Then $I_1, I_3 \to 0$, as $a \to \infty$, uniformly in q^{-n}. By virtue of (19) and Theorem 4, we see that

$$(Rf_\xi)(x,z)=\int_{-\infty}^{\infty}\frac{\varphi(x,\lambda)}{\lambda - z}F_\xi(\lambda)\,d\varrho(\lambda). \tag{22}$$

We can find a sequence $\{f_\xi(x)\}_{\xi=1}^{\infty}$ which satisfies the previous conditions and tend to $f(x)$ as $\xi \to \infty$, since $f(.) \in L_q^2[0,\infty)$. It follows from (9) that the sequence of Fourier transform converges to the transform of $f(x)$. Using Lemmas 5 and 7, one can pass to the limit $\xi \to \infty$ in (22). □

Remark 9. *The following formula holds.*

$$\int_0^{\infty}(Rf)(x,z)\,g(x)\,d_qx=\int_{-\infty}^{\infty}\frac{F(\lambda)\,G(\lambda)}{\lambda - z}d\varrho(\lambda), \tag{23}$$

where

$$G(\lambda)=\lim_{\xi\to\infty}\int_0^{q^{-\xi}} g(x)\,\varphi(x,\lambda)\,d_qx,$$

and

$$F(\lambda)=\lim_{\xi\to\infty}\int_0^{q^{-\xi}} f(x)\,\varphi(x,\lambda)\,d_qx.$$

Now, we will study some properties of the resolvent operator. We give the following definition and theorems.

Definition 10. *Let $M(x,t)$ be a complex-valued function, where $x,t \in (a,b)$. If*

$$\int_a^b\int_a^b |M(x,t)|^2\,d_qx\,d_qt < +\infty,$$

then $M(x,t)$ is called the q-Hilbert-Schmidt kernel.

Theorem 11 ([22])**.** *Let us define the operator A as*

$$A\{x_i\}=\{y_i\},$$

where

$$y_i=\sum_{k=1}^{\infty}a_{ik}x_k,\ i\in\mathbb{N}. \tag{24}$$

If

$$\sum_{i,k=1}^{\infty}|a_{ik}|^2<+\infty \tag{25}$$

then A is a compact operator in the sequence space l^2.

Theorem 12. *Let the limit circle case holds for Eq. (3) and*

$$G(x,t)=G(x,t,0)=\begin{cases}\varphi(x)\chi(t), & x<t\\ \chi(x)\varphi(t), & x\geq t.\end{cases} \tag{26}$$

Then the function $G(x,t)$ defined by (26) is a q-Hilbert-Schmidt kernel.

Proof. It follows from (26) that

$$\int_0^\infty d_qx\int_0^x |G(x,t)|^2\,d_qt<+\infty,$$

and

$$\int_0^\infty d_qx\int_x^\infty |G(x,t)|^2\,d_qt<+\infty,$$

since the integrals

$$\int_0^\infty |G(x,t)|^2\,d_qx$$

and

$$\int_0^\infty |G(x,t)|^2\,d_qt$$

exist and are a linear combination of the products $\varphi(x)\chi(t)$, and these products belong to $L_q^2[0,\infty)\times L_q^2[0,\infty)$. Then

$$\int_0^\infty\int_0^\infty |G(x,t)|^2\,d_qxd_qt<+\infty. \tag{27}$$

□

Theorem 13. *Let us define the operator R as*

$$(Rf)(x)=\int_0^\infty G(x,t)\,f(t)\,d_qt$$

Under the assumptions of Theorem 12, R is a compact operator.

Proof. Let $\phi_i=\phi_i(t)$ $(i\in\mathbb{N})$ be a complete, orthonormal basis of $L_q^2[0,\infty)$. By Theorem 12, we can define

$$\begin{aligned} x_i &= (f,\phi_i)=\int_0^\infty \overline{\phi_i(t)}f(t)\,d_qt,\\ y_i &= (g,\phi_i)=\int_0^\infty \overline{\phi_i(t)}g(t)\,d_qt,\\ a_{ik} &= \int_0^\infty\int_0^\infty \overline{\phi_k(t)\phi_i(x)}G(x,t)\,d_qxd_qt, \end{aligned}$$

where $i,k\in\mathbb{N}$. Then, $L_q^2[0,\infty)$ is mapped isometrically l^2. Therefore, the operator R transforms into A defined by (24) in l^2 by this mapping, and (27) is translated into (25). It follows from Theorem 11 that A is compact operator. Consequently, R is a compact operator. □

Now, we will give some auxiliary lemmas.

Lemma 14. *The following equalities hold.*

$$\lim_{x\to\infty} W_q\left(\chi(x,\lambda), \chi(x,\lambda')\right) = 0, \tag{28}$$

$$\int_0^\infty \chi(x,\lambda), \chi(x,\lambda')\, d_qx = \frac{m(\lambda) - m(\lambda')}{\lambda - \lambda'}, \tag{29}$$

where λ *and* λ' *are any fixed nonreal numbers.*

Proof. See [9]. □

Using (29) and setting $\lambda = u + iv$ and $\lambda' = \overline{\lambda}$, we obtain

$$\int_0^\infty |\chi(x,\lambda)|^2\, d_qx = -\frac{\operatorname{Im}\{m(\lambda)\}}{v}. \tag{30}$$

Lemma 15. *For fixed* u_1 *and* u_2*, we have*

$$\int_{u_1}^{u_2} -\operatorname{Im}\{m(u+i\delta)\}\, du = O(1), \quad \text{as } \delta \to 0. \tag{31}$$

Proof. Let $\sin\beta \neq 0$. It follows from (9) and (18) that

$$\int_0^\infty |\chi(t,z)|^2\, d_qt = \int_{-\infty}^\infty \frac{d\varrho(\lambda)}{(u-\lambda)^2 + v^2}, \tag{32}$$

where $z = u + iv$.

Let $\sin\beta = 0$. If the equality (15) is q-differentiated throughout with respect to x, and the limit is taken as $n \to \infty$, then we can get the desired result.

By virtue of (30) and (32), we conclude that

$$-\operatorname{Im}\{m(u+i\delta)\} = \delta \int_{-\infty}^\infty \frac{d\varrho(\lambda)}{(u-\lambda)^2 + \delta^2}.$$

Then we have

$$-\int_{u_1}^{u_2} \operatorname{Im}\{m(u+i\delta)\}\, du = \delta \int_{u_1}^{u_2} du \int_{-\infty}^\infty \frac{d\varrho(\lambda)}{(u-\lambda)^2 + \delta^2}.$$

Let (a,b) be a finite interval where $a < u_1$ and $b > u_2$. From (17), we see that

$$\delta \int_{u_1}^{u_2} du \int_{-\infty}^a \frac{d\varrho(\lambda)}{(u-\lambda)^2 + \delta^2} = O(1),$$

$$\delta \int_{u_1}^{u_2} du \int_b^\infty \frac{d\varrho(\lambda)}{(u-\lambda)^2 + \delta^2} = O(1).$$

Hence, we get

$$\delta \int_{u_1}^{u_2} du \int_a^b \frac{d\varrho(\lambda)}{(u-\lambda)^2 + \delta^2} = \int_a^b d\varrho(\lambda) \int_{\frac{u_1-\lambda}{\delta}}^{\frac{u_2-\lambda}{\delta}} \frac{dv}{1+v^2} = O(1).$$

□

Assume that $\sigma(\lambda) = \sigma_1(\lambda) + i\sigma_2(\lambda)$ is a complex bounded variation on the entire line. Set

$$\begin{aligned} \varphi(z) &= \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z},\ \psi(\sigma,\tau) = \frac{sgn\tau}{\pi}\frac{\varphi(z) - \varphi(\overline{z})}{2i} \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\tau|\, d\sigma(\lambda)}{(\lambda-\sigma)^2 + \tau^2},\ z = \sigma + i\tau. \end{aligned}$$

Theorem 16 ([20])**.** *Let the points a, b are points of continuity of $\sigma(\lambda)$. Then we obtain*

$$\sigma(b) - \sigma(a) = \lim_{\tau\to 0}\int_a^b -\psi(\sigma,\tau)\, d\sigma.$$

Theorem 17. *Let the endpoints of $\Delta = (\lambda, \lambda+\Delta)$ be the points of continuity of $\varrho(\lambda)$. Then, we deduce that*

$$\varrho(\lambda+\Delta) - \varrho(\lambda) = \frac{1}{\pi}\lim_{\delta\to 0}\int_{\Delta} -\operatorname{Im}\{m(u+i\delta)\}\, du. \tag{33}$$

Proof. Let $f(.), g(.) \in L_q^2[0,\infty)$ vanish outside a finite interval. By (23), we deduce that

$$\begin{aligned} y(\lambda) &= \int_0^{\infty} (Rf)(x,z)\, g(x)\, d_q x \\ &= \int_{-\infty}^{\infty} \frac{F(\lambda)G(\lambda)}{\lambda - z} d\varrho(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{\lambda - z}, \end{aligned}$$

where

$$\rho(\Delta) = \int_{\Delta} F(\lambda)G(\lambda)\, d\varrho(\lambda).$$

It follows from Theorem 16 that

$$\rho(\Delta) = -\frac{1}{\pi}\lim_{\delta\to 0}\int_{\Delta} \operatorname{Im}\{\psi(u+i\delta)\}\, du. \tag{34}$$

Furthermore, we have

$$\begin{aligned} &\operatorname{Im}\{\psi(u+i\delta)\} = \int_0^{\infty} g(x)\, d_q x \\ &\times\{\int_0^x [\theta(x,u+i\delta) + m(u+i\delta)\varphi(x,u+i\delta)]\,\varphi(t,u+i\delta)\, f(t)\, d_q t \\ &+ \int_x^{\infty} [\theta(t,u+i\delta) + m(u+i\delta)\varphi(t,u+i\delta)]\,\varphi(x,u+i\delta)\, f(t)\, d_q t\}, \end{aligned}$$

where $\theta(x,u)$, $\varphi(x,u)$, $g(x)$ and $f(x)$ are real-valued functions. It follows from (34) and Lemma 15 that

$$\rho(\Delta) = \frac{1}{\pi}\lim_{\delta\to 0}\int_{\Delta} -\operatorname{Im}\{m(u+i\delta)\}\, G(u)\, F(u)\, du. \tag{35}$$

If we choose $g(x)$ and $f(x)$ conveniently, we can make $G(u)$ and $F(u)$ differ as little from unity in the fixed interval Δ. From Lemma 15 and (33), we get the desired result. □

Theorem 18. *Let $z \notin \mathbb{R}$. Then we have*

$$m(z) = -\cot\beta + \int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{\lambda - z}. \tag{36}$$

Proof. It follows from (18) that

$$G(x,t,z) = \int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)\varphi(t,\lambda)\, d\varrho(\lambda)}{\lambda - z}, \tag{37}$$

since $f(x)$ is an arbitrary function. By definition, we get

$$G(x,t,z) = \begin{cases} [\theta(t,z) + m(z)\varphi(t,z)]\varphi(x,z), & t > x \\ [\theta(x,z) + m(z)\varphi(x,z)]\varphi(t,z), & t \le x. \end{cases}$$

By virtue of (6) and (37), we conclude that

$$\begin{aligned} G(0,0,z) &= \sin\beta\{\cos\beta + m(z)\sin\beta\} \\ &= \int_{-\infty}^{\infty} \frac{\sin^2\beta}{\lambda - z} d\varrho(\lambda), \end{aligned}$$

i.e.,

$$m(z) = -\cot\beta + \int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{\lambda - z}.$$

□

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FRACTIONAL VARIATIONAL PROBLEMS ON CONFORMABLE CALCULUS

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Abstract. In this paper, we deal with the variational problems defined by an integral that include fractional conformable derivative. We obtained the optimality results for variational problems with fixed end-point boundary conditions and variable end-point boundary conditions. Then, we studied on the variational problems with integral constraints and holonomic constraints, respectively.

1. Introduction

Origin of fractional calculus dates back to 1600's, firstly seen in a letter from Leibnitz to L'Hospital. So far, a number of famous mathematicians such as Abel, Fourier, Liouville, Leibnitz, Weyl and Riemann made contributions to this theory. Probably, Abel has given the first applications of fractional calculus in 1823. Especially in last decades, fractional calculus find ample applications in various fields of science (see [13, 22, 23, 27, 28]). Recently, fractional order Black-Scholes equation is studied in [11], fractional Harry-Dym equation is studied in [12]. There are several definitions of fractional derivatives and fractional integrals, such as Atangana-Baleanu, Riemann-Liouville, Grunwald-Letnikov, Caputo, Riesz, Riesz-Caputo, Hadamard-Hilfer, Caputo-Fabrizio, and Weyl, etc. We refer to monographs [15, 20, 24] for definitions and properties of most common fractional derivatives. Recently Khalil et. al. [19] gave a new well-behaved fractional derivative definition; named as conformable fractional derivative. This new definition has many similar properties with ordinary integer order derivative such as constant function rule, linearity, product and quotient rules and Leibnitz rule (see [1]). Conformable fractional differential equations are studied widely in the literature. We refer to [9]

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for Lie symmetry analysis; to [8] for boundary value problems; to [18] for numerical solutions conformable differential equations; to [14] for Fourier transform, etc.

Calculus of variations is a subject which is concerned with finding the maxima and minima of functionals and plays important role in many problems arising in mechanics, geometry, analysis etc. We refer to monograph [17] for the basic concepts of this theory. In 1996, Riewe [25] noted that the traditional Lagrangian and Hamiltonian mechanics can not be used with non-conservative forces. In order to deal with Lagrangians involving nonconservative forces, Riewe [26] generalized the usual variational methods by using Riemann-Liouville type operators and introduced the fractional order calculus of variations. For different definitions on fractional derivatives, different approaches have been developed to generalize calculus of variations to fractional case. Agarwal [2, 3, 4] studied variational methods for Riemann-Liouville, Caputo and Riesz fractional derivatives. Almeida [5, 6] considered variational problems involving Riesz-Caputo and Caputo-Katugampola fractional derivatives. Zhang et. al. [29] and Bastos [7] studied calculus of variations with Caputo-Fabrizio derivatives. Chatibi et. al. [10] investigated variational methods for Atangana-Baleanu fractional derivatives. Lazo and Torres [21] and Eroğlu and Yapışkan [16] studied variational methods for conformable fractional derivatives.

In this paper, we consider more general variational problems with conformable fractional derivative and extend the results given in [21]. More specially, we investigate variable end-point variational problems and variational problems with subsidiary conditions.

2. Preliminaries

In this section, we introduce definitions and basic properties concerning the conformable fractional derivative that will be needed in our proofs.

$0 < \alpha \leq 1$ order left-conformable fractional derivative of the function $h : [a, \infty) \to \mathbb{R}$ is defined by

$$\left(T_\alpha^a h\right)(t) := \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon(t-a)^{1-\alpha}) - h(t)}{\varepsilon}.$$

If $(T_\alpha h)$ exists on the interval (a, b), then $\left(T_\alpha^a h\right)(a) = \lim\limits_{t \to a^+} \left(T_\alpha^a h\right)(t)$.

Similarly, $0 < \alpha \leq 1$ order right-conformable fractional derivative of the function h is defined by

$$\left({}_\alpha^b T h\right)(t) := \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon(b-t)^{1-\alpha}) - h(t)}{\varepsilon}.$$

If $\left({}_\alpha^b T h\right)$ exists on the interval (a, b), then $\left({}_\alpha^b T h\right)(a) = \lim\limits_{t \to b^-} \left({}_\alpha^b T h\right)(t)$.

We remark that, additionally if h is differentiable, then $\left(T_\alpha^a h\right)(.) = (.-a)^{1-\alpha} h'(.)$ and $\left({}_\alpha^b T h\right)(.) = -(b-.)^{1-\alpha} f'(.)$ for all $t \in (a, b)$. As in the case Caputo derivative, conformable derivative of the constant function is zero (see [1, 19]).

$0 < \alpha \leq 1$ order left and right conformable fractional integrals of the function h are defined by

$$\left(I_\alpha^a h\right)(t) := \int_a^t h(s) d^\alpha(s,a)$$

and

$$\left({}_\alpha^b I h\right)(t) := \int_t^b h(s) d^\alpha(b,s)$$

respectively, where $d^\alpha(s,a) = (s-a)^{\alpha-1}ds$ and $d^\alpha(b,s) = (b-s)^{\alpha-1}ds$ (see [1]).

Let $0 < \alpha \leq 1$. If $h : [a,\infty) \to \mathbb{R}$ is continuous, then the identity $\left(T_\alpha^a I_\alpha^a h\right)(t) = h(t)$ holds for all $t > a$. And, if $h : (a,\infty) \to \mathbb{R}$ is continuous, then the identity $\left(I_\alpha^a T_\alpha^a h\right)(t) = h(t) - h(a)$ holds for all $t > a$ (see [1,19]).

For the differentiable functions $h, g : [a,b] \to \mathbb{R}$, the conformable integration by parts formula reads as follows (see [1])

$$\int_a^b h(t)\left(T_\alpha^a g\right)(t) d^\alpha(t,a) = (hg)(t)|_{t=a}^{t=b} - \int_a^b g(t)\left(T_\alpha^a h\right)(t) d^\alpha(t,a). \tag{1}$$

In the following, we give the fundamental lemma of fractional variational calculus and the definition of jointly-convex functions that will be used in the sequel .

Lemma 1 ([21]). *Let the functions $\varphi, \xi : [a,b] \to \mathbb{R}$ be continuous and the the equality*

$$\int_a^b \varphi(t)\xi(t) d^\alpha(t,a) = 0$$

holds for all $\xi \in \mathcal{C}[a,b]$ satisfying $\xi(a) = \xi(b) = 0$. Then

$$\varphi(t) = 0$$

for all $t \in [a,b]$.

Definition 2 ([7]). *Let $F(x_1,x_2,x_3)$ be continuous function for its second and third arguments. If the inequality*

$$F(x_1, x_2 + h_1, x_3 + h_2) - F(x_1,x_2,x_3) \geq (\leq) \partial_2 F(x_1,x_2,x_3) h_1 + \partial_3 F(x_1,x_2,x_3) h_2$$

is hold for all $(x_1,x_2,x_3) \in A$ and all $h_1, h_2 \in \mathbb{R}$, then we say that function F is jointly-convex (or jointly-concave) in $A \subseteq \mathbb{R}^3$.

3. Main results

In this study, we consider the functional

$$\mathbb{J}[x] := \int_a^b L(t, x(t), T_\alpha^a x(t)) d^\alpha(t,a). \tag{2}$$

Throughout the paper, we assume that $x \in \mathcal{C}^1[a,b]$, $T_\alpha^a(\partial_3 L(t, x(t), T_\alpha^a x(t)))$ is continuous, and $L \in \mathcal{C}^1_{2,3}\left([a,b] \times \mathbb{R}^2, \mathbb{R}\right)$, where $\mathcal{C}^1_{2,3}$ denotes the family of functions that continuously differentiable for its second and third arguments, and ∂_i denotes the partial derivative of the function for its $i-$th argument.

One can find necessary optimality conditions for the problem of finding local minimizers of the functional (2) in the following result.

Theorem 3 ([13]). *Let $0 < \alpha \leq 1$ and $x_a, x_b \in \mathbb{R}$ be fixed. If x is a minimizer of the* (2) *on the set*

$$S := \left\{x \in \mathcal{C}^1[a,b] : x(a) = x_a,\ x(b) = x_b\right\}, \tag{3}$$

then we say that $x(t)$ is a solution of the equation

$$\partial_2(L_{\alpha,x}) - T_\alpha^a(\partial_3(L_{\alpha,x})) = 0, \tag{4}$$

where $L_{\alpha,x} := L(., x(.), T_\alpha^a x(.))$.

Definition 4. *Equation* (4) *is named as the Euler-Lagrange equation for* (2)*, and its solutions are named as the extremals of* (2)*.*

Equation (4) provides only a necessary condition for the function $x(t)$ to be an extremal of (2). By using notion of jointly-convex functions given above and conformable integration by parts formula (1), we can give a sufficient condition as follows.

Theorem 5. *If the function L is jointly-convex in $[a,b] \times \mathbb{R}^2$, then every solution of the Euler-Lagrange equation* (4) *minimizes the functional $\mathbb{J}$ on the set S.*

Proof. Assume that function $x(t)$ is a solution of (4). Let $x + \epsilon\xi$ be a variation of x, with $1 \gg |\epsilon|$ and $\xi \in \mathcal{C}^1[a,b]$ with $\xi(a) = \xi(b) = 0$. Since $x(t)$ is a solution of (4) and L is jointly-convex, we have

$$\begin{aligned}
& \mathbb{J}[x+\epsilon\xi] - \mathbb{J}[x] \\
= & \int_a^b (L_{\alpha,x+\epsilon\xi}) d^\alpha(t,a) - \int_a^b (L_{\alpha,x}) d^\alpha(t,a) \\
\geq & \ \epsilon \int_a^b [\partial_2(L_{\alpha,x})\xi(t) + \partial_3(L_{\alpha,x}) T_\alpha^a \xi(t)] d^\alpha(t,a).
\end{aligned}$$

$$= \epsilon \int_a^b \partial_2 (L_{\alpha,x}) \xi(t) d^\alpha(t,a) + \epsilon \int_a^b \partial_3 (L_{\alpha,x}) T_\alpha^a \xi(t) d^\alpha(t,a)$$

Using (1) for the second term of the inequality, we can write

$$\begin{aligned}
& \mathbb{J}[x+\epsilon\xi] - \mathbb{J}[x] \\
\geq\ & \epsilon \int_a^b \partial_2 (L_{\alpha,x}) \xi(t) d^\alpha(t,a) + \ \epsilon\xi(t)\partial_3\partial_3 (L_{\alpha,x})\Big|_{t=a}^{t=b} \\
& -\epsilon \int_a^b T_\alpha^a (\partial_3 (L_{\alpha,x})) \xi(t) d^\alpha(t,a) \\
=\ & \epsilon \int_a^b [\partial_2 (L_{\alpha,x}) - T_\alpha^a (\partial_3 (L_{\alpha,x}))] \xi(t) d^\alpha(t,a) \\
=\ & 0.
\end{aligned}$$

Hence, we can say that $x(t)$ is a local minimizer of the functional $\mathbb{J}$. □

In Theorems 3 and 5, we introduced the variational problems with fixed end-point. In the following result, we will study the variational problems with variable end-point. Because of the absence of at least one of the end-point conditions (3) in such problems, we need additional conditions, which are named transversality conditions in the literature.

Theorem 6. *Assume that $x(t)$ is a minimizer of the functional $\mathbb{J}$. Then, $x(t)$ is a solution of the Euler-Lagrange equation (4).*
If $x(a)$ is absent, then

$$\partial_3 (L_{\alpha,x})\big|_{t=a} = 0.$$

If $x(b)$ is absent, then

$$\partial_3 (L_{\alpha,x})\big|_{t=b} = 0.$$

Proof. Let $x+\epsilon\xi$ be a variation of x, with $1 \gg |\epsilon|$ and $\xi \in \mathcal{C}^1[a,b]$. Let the functional j defined in a neighborhood of zero by

$$j[\epsilon] := \mathbb{J}[x+\epsilon\xi].$$

Since x is a minimizer of $\mathbb{J}$, then $\epsilon = 0$ will be a minimizer of j and so we can conclude that $j'[0] = 0$. Using (1), we can calculate $j'[\epsilon]$ as

$$\frac{\partial}{\partial\epsilon} j[\epsilon] = \frac{\partial}{\partial\epsilon}\left(\int_a^b L_{\alpha,x+\epsilon\xi} d^\alpha(t,a)\right)$$

$$= \int_a^b \left[\partial_2 \left(L_{\alpha,x+\epsilon\xi}\right)\xi(t) + \partial_3 L\left(L_{\alpha,x+\epsilon\xi}\right) T_\alpha^a \xi\left(t\right)\right] d^\alpha(t,a)$$

$$= \int_a^b \partial_2 \left(L_{\alpha,x+\epsilon\xi}\right)\xi(t) d^\alpha(t,a) + \ \partial_3 \left(L_{\alpha,x+\epsilon\xi}\right)\xi\left(t\right)\Big|_{t=a}^{t=b}$$

$$- \int_a^b T_a^\alpha \left(\partial_3 \left(L_{\alpha,x+\epsilon\xi}\right)\right)\xi(t) d^\alpha(t,a).$$

Using the fact that $j'[0] = 0$, we get

$$\int_a^b \left[\partial_2 \left(L_{\alpha,x}\right) - \partial_3 \left(L_{\alpha,x}\right)\right]\xi(t) d^\alpha(t,a) + \ \partial_3 \left(L_{\alpha,x}\right)\xi(t)\Big|_{t=a}^{t=a} = 0. \tag{5}$$

Also, since x is a minimizer, the relation

$$\partial_2 \left(L_{\alpha,x}\right) - \partial_3 \left(L_{\alpha,x}\right) = 0$$

holds for all $t \in [a,b]$. Therefore, from (5) we have

$$\partial_3 \left(L_{\alpha,x}\right)\xi(t)\big|_{t=a}^{t=a} = 0.$$

If $x(a)$ is not fixed, then $\xi(a)$ will be free. Hence taking the end-point conditions as $\xi(a) \neq 0$ and $\xi(b) = 0$, we obtain that

$$\partial_3 \left(L_{\alpha,x}\right)\xi(t)\big|_{t=a} = 0.$$

If $x(b)$ is not fixed, then $\xi(b)$ will be free. Hence taking the end-point conditions as $\xi(b) \neq 0$ and $\xi(a) = 0$, we obtain that

$$\partial_3 \left(L_{\alpha,x}\right)\xi(t)\big|_{t=b} = 0.$$

Thus, the proof is complete. $\square$

Now we consider variational problems with constraints, i.e. subsidiary conditions. Let $l \in \mathbb{R}$ fixed, $G \in \mathcal{C}_{2,3}^1\left([a,b] \times \mathbb{R}^2, \mathbb{R}\right)$, and $T_\alpha^a\left(\partial_3\left(G_{\alpha,x}(t)\right)\right)$ is continuous.

Theorem 7. *Assume that x is a minimizer of functional* (2)*, defined on the set* (3) *subject to the additional restriction*

$$\mathbb{I}\left[x\right] := \int_a^b G_{\alpha,x} d^\alpha\left(t,a\right) = l. \tag{6}$$

If x is not an extremal of $\mathbb{I}$, then there exists a $\mu \in \mathbb{R}$ such that x is a solution of the equation

$$\partial_2 \left(K_{\alpha,x}\right) - \partial_3 \left(K_{\alpha,x}\right) = 0 \tag{7}$$

where $K : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $K = L + \mu G$.

Proof. Consider a variation of x with two parameters $x+\epsilon_1\xi_1+\epsilon_2\xi_2$; with $1 \gg |\epsilon|$ and $\xi_i \in \mathcal{C}^1[a,b]$ satisfying $\xi_i(a)=\xi_i(b)=0$; for $i=1,2$. In the neighborhood of zero, let define the bivariate functions k and j^* as

$$k(\epsilon_1,\epsilon_2)=\mathbb{I}(x+\epsilon_1\xi_1+\epsilon_2\xi_2)$$

and

$$j^*(\epsilon_1,\epsilon_2)=\mathbb{J}(x+\epsilon_1\xi_1+\epsilon_2\xi_2).$$

Using integration by parts formula given by (1), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon_2}k(\epsilon_1,\epsilon_2)\\
=\;&\frac{\partial}{\partial\epsilon_2}\left(\int_a^b G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}d^{\alpha}(t,a)-\int_a^b G_{\alpha,x}d^{\alpha}(t,a)\right)\\
=\;&\int_a^b \partial_2\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\xi_2(t)\,d^{\alpha}(t,a)+\int_a^b \partial_3\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)T_{\alpha}^{a}\xi_2(t)\,d^{\alpha}(t,a)\\
=\;&\int_a^b \partial_2\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\xi_2(t)\,d^{\alpha}(t,a)+\;\partial_3\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\xi_2(t)\Big|_{t=a}^{t=b}\\
&-\int_a^b T_{\alpha}^{a}\left[\partial_3\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\right]\xi_2(t)\,d^{\alpha}(t,a)\\
=\;&\int_a^b \Big[\partial_2\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\;-T_{\alpha}^{a}\left[\partial_3\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\right]\Big]\,\xi_2(t)\,d^{\alpha}(t,a)\\
&+\;\partial_3\left(G_{\alpha,x+\epsilon_1\xi_1+\epsilon_2\xi_2}\right)\xi_2(t)\Big|_{t=a}^{t=b}.
\end{aligned}$$

Therefore, we have

$$\frac{\partial}{\partial\epsilon_2}k(\epsilon_1,\epsilon_2)\Big|_{(0,0)}=\int_a^b\Big[\partial_2(G_{\alpha,x})-T_{\alpha}^{a}\left[\partial_3(G_{\alpha,x})\right]\Big]\,\xi_2(t)\,d^{\alpha}(t,a)+\;\partial_3(G_{\alpha,x})\,\xi_2(t)\Big|_{t=a}^{t=b}.$$

From the hypothesis we know that x is not an extremal of $\mathbb{I}$, so we can conclude that there exists a function ξ_2 such that $\frac{\partial}{\partial\epsilon_2}k(\epsilon_1,\epsilon_2)\Big|_{(0,0)}\neq 0$. From the Implicit Function Theorem, we can say that there exists a unique function $\epsilon_2(.)$ defined in the neighborhood of zero such that $k(\epsilon_1,\epsilon_2(\epsilon_1))=0$ is satisfied.

Additionally, $(0,0)$ is a minimizer of j^*, with condition $k(.,.)=0$, and so we proved that $\nabla k(0,0)=0$. After that using the Lagrange multiplier rule, we conclude that there exists a $\mu\in\mathbb{R}$ such that $\nabla(j+\mu k)=0$ is satisfied. Differentiating the map $\epsilon \to j(\epsilon_1, \epsilon_2) + \mu k(\epsilon_1, \epsilon_2)$, and taking $(\epsilon_1, \epsilon_2) = (0, 0)$

$$\int_a^b \Big[\partial_2 (G_{\alpha,x}) - T_\alpha^a [\partial_3 (G_{\alpha,x})] \Big] \xi_2(t)\, d^\alpha(t,a) + \partial_3 (G_{\alpha,x}) \xi_2(t)\Big|_{t=a}^{t=b} = 0$$

is obtained. Finally, using the fundamental lemma, we obtain the desired result. □

Now we consider variational problems with holonomic constraints, i.e. the case when admissible functions lie on a certain surface. Let the function $L \in \mathcal{C}^1_{2,3,4,5}\left([a,b] \times \mathbb{R}^4, \mathbb{R}\right)$, and the functions $T_\alpha^a \left(\partial_i L\left(t, x_1(t), x_2(t), T_\alpha^a x_1(t), T_\alpha^a x_2(t)\right)\right)$ are continuous for $i = 4, 5$.

Consider the functional

$$\mathbb{J}[x_1, x_2] := \int_a^b L\left(t, x_1(t), x_2(t), T_\alpha^a x_1(t), T_\alpha^a x_2(t)\right) d^\alpha(t,a) \tag{8}$$

on the space

$$S^* := \left\{(x_1, x_2) : x_{1,2} \in \mathcal{C}^1[a,b],\ (x_1(a), x_2(a)) = x_a \text{ and } (x_1(a), x_2(b)) = x_b\right\}$$

where $x_a, x_b \in \mathbb{R}^2$ are fixed, and assume that the admissible functions of (8) lie on the surface

$$G(t, x_1(t), x_2(t)) = 0 \tag{9}$$

where $G \in \mathcal{C}^1_{2,3}\left([a,b] \times \mathbb{R}^2, \mathbb{R}\right)$.

For the sake of brevity, we denote $x(t) := (x_1(t), x_2(t))$ and $T_\alpha^a x(t) := (T_\alpha^a x_1(t), T_\alpha^a x_2(t))$ in the remaining part of this paper.

Theorem 8. *Let $x \in S^*$ be a minimizer of $\mathbb{J}$ given by (8) under the constraint (9). If*

$$\partial_3 G(t, x(t)) \neq 0,\ t \in [a,b],$$

then there is a continuous function $\gamma : [a,b] \to \mathbb{R}$ such that x satisfies

$$\partial_2 (L_{\alpha,x}) - T_\alpha^a (\partial_4 (L_{\alpha,x})) + \gamma(t)\, \partial_2 G(t, x(t)) = 0, \tag{10}$$

and

$$\partial_3 (L_{\alpha,x}) - T_\alpha^a (\partial_5 (L_{\alpha,x})) + \gamma(t)\, \partial_3 G(t, x(t)) = 0.$$

Proof. Consider a variation of x as $x + \epsilon\xi$ with $1 \gg |\epsilon|$ and $\xi \in \mathcal{C}^1[a,b]$ satisfying end-point conditions $\xi(a) = \xi(b) = 0$. Since

$$\partial_3 G(t, x(t)) \neq 0,$$

from the Implicit Function Theorem, we can say that there exists a subfamily of variations satisfying condition (9), i.e., there exists a unique function $\xi_2(\epsilon, \xi_1)$ such that $(x_1 + \epsilon\xi_1, x_2 + \epsilon\xi_2)$ satisfies (7). Therefore, we get

$$G(t, x_1(t) + \epsilon\xi_1(t), x_2(t) + \epsilon\xi_2(t)) = 0,\ t \in [a,b]. \tag{11}$$

Differentiating equation (11) with respect to ϵ and putting $\epsilon = 0$, we obtain

$$\partial_2 G\left(t, x_1\left(t\right), x_2\left(t\right)\right) \xi_1\left(t\right) + \partial_3 G\left(t, x_1\left(t\right), x_2\left(t\right)\right) \xi_2\left(t\right) = 0,$$

i.e.

$$\partial_2 G\left(t, x\left(t\right)\right) \xi_1\left(t\right) + \partial_3 G\left(t, x\left(t\right)\right) \xi_2\left(t\right) = 0. \tag{12}$$

Now, define the function

$$\gamma(t) = -\frac{\partial_3\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_5\left(L_{\alpha,x}\right)\right)}{\partial_3 G\left(t, x\left(t\right)\right)}. \tag{13}$$

Using equations (12) and (13), we obtain

$$\gamma\left(t\right) \partial_2 G\left(t, x\left(t\right)\right) \xi_1\left(t\right) = \left[\partial_3\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_5\left(L_{\alpha,x}\right)\right)\right] \xi_2\left(t\right). \tag{14}$$

On the other hand, using the fact that if x is a minimizer of $\mathbb{J}$, then first variation of $\mathbb{J}$ is equal to zero, we have

$$\begin{aligned}\int_a^b \Big[& \partial_2\left(L_{\alpha,x}\right) \xi_1\left(t\right) + \partial_3\left(L_{\alpha,x}\right) \xi_2\left(t\right) \\ & + \partial_4\left(L_{\alpha,x}\right) T_\alpha^a \xi_1\left(t\right) + \partial_5\left(L_{\alpha,x}\right) T_\alpha^a \xi_2\left(t\right) \Big] d^\alpha(t,a) = 0.\end{aligned}$$

Using conformable integration by parts, we obtain

$$\begin{aligned}\int_a^b \Big[& \Big[\partial_2\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_4\left(L_{\alpha,x}\right)\right)\Big] \xi_1\left(t\right) \\ & + \Big[\partial_3\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_5\left(L_{\alpha,x}\right)\right) \Big] \xi_2\left(t\right)\Big] d^\alpha(t,a) = 0.\end{aligned}$$

Inserting (14) into the this integral, we get

$$\int_a^b \Big[\partial_2\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_4\left(L_{\alpha,x}\right)\right) + \gamma(t)\partial_2 G\left(t, x\left(t\right)\right)\Big] \xi_1\left(t\right) d^\alpha\left(t, a\right) = 0.$$

Since ξ_1 is an arbitrary function, we can conclude that x is a solution of

$$\partial_2\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_4\left(L_{\alpha,x}\right)\right) + \gamma(t)\partial_2 G\left(t, x\left(t\right)\right) = 0.$$

Following the same process, the second condition

$$\partial_3\left(L_{\alpha,x}\right) - T_\alpha^a\left(\partial_5\left(L_{\alpha,x}\right)\right) + \gamma(t)\partial_3 G\left(t, x\left(t\right)\right) = 0$$

can be obtained, and the proof is complete. □

Theorem 9. *Suppose that the function $L\left(t, x_1\left(t\right), x_2\left(t\right), y_1\left(t\right), y_2\left(t\right)\right)$ given by* (8) *is convex in $[a,b] \times \mathbb{R}^4$, $G \in \mathcal{C}^1_{2,3}$, and let γ be given by equation* (13). *If $\partial_3 G\left(t, x\left(t\right)\right) \neq 0$ for all $t \in [a,b]$ and x is a solution of the fractional Euler-Lagrange equation* (10), *then x minimizes $\mathbb{J}$ in S^*, subject to* (9).

Proof. If $x+\epsilon\xi$ is a variation of x, then we have

$$\begin{aligned}\mathbb{J}\left[x+\epsilon\xi\right]-\mathbb{J}\left[x\right] &\geq \int\limits_a^b \left\{\left[\partial_2\left(L_{\alpha,x}\right)-T_\alpha^a\left(\partial_4\left(L_{\alpha,x}\right)\right)\right]\epsilon\xi_1\left(t\right)\right. \\ &\quad +\left[\partial_3\left(L_{\alpha,x}\right)-T_\alpha^a\left(\partial_5 L_{\alpha,x}\right)\right]\epsilon\xi_2\left(t\right)d^\alpha\left(t,a\right).\end{aligned}$$

since the variation functions must satisfy the constraint (9), from (12) we have the

$$\xi_2\left(t\right)=-\frac{\partial_2 G\left(t,x\left(t\right)\right)\xi_1\left(t\right)}{\partial_3 G\left(t,x\left(t\right)\right)}$$

and from equation (13), we obtain

$$\mathbb{J}\left[x+\epsilon\xi\right]-\mathbb{J}\left[x\right]\geq\int\limits_a^b\left[\partial_2\left(L_{\alpha,x}\right)-T_\alpha^a\left(\partial_4\left(L_{\alpha,x}\right)\right)+\gamma(t)\partial_2 G\left(t,x\left(t\right)\right)\right]\epsilon\xi_1\left(t\right)d^\alpha\left(t,a\right),$$

which is zero by hypothesis. □

4. Conclusions

We have discussed the optimality conditions of the variational problems including conformable fractional derivatives. We obtained the optimality conditions for fixed end-point variational problems in Theorem 5, and for variable end-point variational problems in Theorem 6. Then, we have investigated the isoperimetric problem in Theorem 7, and variational problem with holonomic constraints in Theorem 8. Finally, in Theorem 9, we have given a sufficient condition for optimality results of variational problems.

It is known that conformable fractional derivative generalizes the ordinary derivative, i.e. if we take $\alpha=1$ in conformable derivative $T_\alpha^a h(t)$, we have ordinary derivative $Dh(t)$. Using this fact, It is clear that the results obtained in our study expand the results in the literature given before.

The problems we have dealt with include only one independent variable and one dependent variable and its derivative. As a possible extension of our results, one can study the problems involving more than one dependent variable and their derivatives. And problems with one dependent variable and its derivatives of different orders can be studied.

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NUMERICAL SOLUTION TO AN INTEGRAL EQUATION FOR THE KTH MOMENT FUNCTION OF A GEOMETRIC PROCESS

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Abstract. In this paper, an integral equation for the kth moment function of a geometric process is derived as a generalization of the lower-order moments of the process. We propose a general solution to solve this integral equation by using the numerical method, namely trapezoidal integration rule. The general solution is reduced to the numerical solution of the integral equations which will be given for the third and fourth moment functions to compute the skewness and kurtosis of a geometric process. To illustrate the numerical method, we assume gamma, Weibull and lognormal distributions for the first interarrival time of the geometric process.

1. INTRODUCTION

The geometric process (GP), which is a natural generalization of a renewal process (RP), is an important stochastic monotone model used in many areas of statistics and applied probability, especially for statistical analysis of series of events. Since its introduction by [9], many researcher and authors made a significance effort on GP by publishing more than 200 papers. For instance, the GP has been used as a model in modelling of an epidemic disease [8], software reliability [16,18], maintenance [23], warranty analysis [5,12] and electricity prices [7]. This process is defined as follows.

Definition 1. *Let $\{N(t),\ t \geq 0\}$ be a counting process (CP) and X_i be the interarrival time between $(i-1)$th and ith event of this process for $i = 1, 2, \ \ldots$. The CP $\{N(t),\ t \geq 0\}$ is called a GP with the ratio parameter a if there exists a real number $a > 0$ such that $a^{i-1}X_i,\ i = 1, 2, \ \ldots$ are independent and identically distributed*

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(iid) random variables with a distribution function F, where F is the distribution function of the first interarrival time X_1.

The GP is also called quasi-renewal process with ratio parameter $\alpha = 1/a$ by [18].

Let $\{N(t),\ t \geq 0\}$ be a GP with the ratio parameter a and F_i be the distribution function of X_i for $i = 1, 2,\ \ldots$. Then, it is easy to verify that $F_i(x) = F\left(a^{i-1}x\right)$ for $i = 1, 2,\ \ldots$. Further, it can be shown that GP is stochastically increasing if $a < 1$ and stochastically decreasing if $a > 1$. When $a = 1$, GP reduces to a RP.

By considering the distribution functions of the interarrival times, one of the important differences between RP and GP can be given as follows. In RP, the distribution function of the interarrival times remains same over i's, that is $F_i(x) = F(x)$ for $i = 1, 2, \ldots$. However, in GP, the distribution of the interarrival time X_i does not remain same over i's, that is $F_i(x) = F(a^{i-1}x)$ for $i = 1, 2, \ldots$. This provides some advantages to the GP in applications, in particular for reliability mathematics since it can be used as a model for deteriorating systems which may have decreasing working times between failures. In order to understand the place of the GP model in the literature, see the following papers, [1, 19, 20, 21, 22].

Let $\{N(t), t \geq 0\}$ be a GP with ratio parameter a. Set $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n = 1, 2, \ldots$. Thus, S_n is called nth arrival time of the GP. The distribution function of S_n is given by $F_1 * F_2 * \cdots * F_n(t)$, where * denotes the Stieltjes convolution and $F_i(t)$ for $i = 1, 2, \ldots$ is the distribution function of X_i. Further, since the events $(S_n \leq t)$ and $(N(t) \geq n)$ are equivalent, the probability distribution of the random variable $N(t)$ is given by

$$P(N(t) = n) = F_1 * \cdots * F_n(t) - F_1 * \cdots * F_{n+1}(t)$$

for each fixed $t \geq 0$.

Now, let give the following theorem which states the existence of the moments of the GP. The proof of this theorem can be found in many manuscripts and monographs, for example, [6, 11].

Theorem 2. *Consider the GP $\{N(t), t \geq 0\}$ with ratio parameter a. If $a \leq 1$, then $M_k(t) = E\left(N^k(t)\right) < \infty$ for all $t \geq 0$ and $k \geq 0$. If $a > 1$ and $F(t) > 0$ for all $t > 0$, then the moments of $N(t)$ are infinite.*

Let $\{N(t), t \geq 0\}$ be a GP with ratio parameter a. The mean value function of a GP, which is also called the geometric function, is given by $M_1(t) = E(N(t))$. $M_1(t)$ is the number of the events that have occurred by time t. The geometric function $M_1(t)$ satisfies the following integral equation.

$$M_1(t) = F(t) + \int_0^t M_1(a(t-x))dF(x), \quad t \geq 0. \tag{1}$$

The second moment function of a GP is given by $M_2(t) = E\left(N^2(t)\right)$. [15] show that $M_2(t)$ satisfies the following integral equation.

$$M_2(t) = 2M_1(t) - F(t) + \int_0^t M_2(a(t-x))dF(x), \quad t \geq 0. \tag{2}$$

According to the Theorem 1, for $a \leq 1$, the geometric function $M_1(t)$ and the second moment function $M_2(t)$ are finite for all $t \geq 0$. Furthermore, if F is continuous, then the integral equations (1) and (2) can be solved uniquely although $M_1(t)$ and $M_2(t)$ cannot be obtained in analytical forms. In the case of $a > 1$, $M_1(t)$ and $M_2(t)$ are infinite for all $t > 0$.

When the GP model is used as a model for the data sets come from series of event, the distribution function of the first interarrival time is assumed to be one of four common lifetime distributions as exponential, gamma, Weibull and lognormal. See [13] for the details how to discriminate the lifetime distributions in GP model. Under a lifetime distribution assumption, it is of importance to calculate the moment functions of the GP. Many researchers and authors made some studies on the first and second moment functions of the GP by considering these lifetime distributions. [6] deal with the boundary problem for $M_1(t)$. [17] propose a numerical method, namely trapezoidal integration rule, for $M_1(t)$ with the help of the integral equation (1). In addition, [4] and [3] suggest power series expansions for $M_1(t)$ depending on the integral equation (1). [2] obtain the numerical calculation and Monte Carlo estimation of the variance function, which is $M_2(t) - M_1^2(t)$, by using the convolution of the distribution functions. Alternative methods for computing $M_2(t)$ are given in [15]. They adapt the Tang and Lam's method to $M_2(t)$ and propose a power series expansion for $M_2(t)$ with the help of the integral equation (2). Further, [14] show the asymptotic solution of the integral equation for the second moment function. To the best of our knowledge, in the literature, there is no study about the higher-order moment functions of the GP. However, in order to calculate, for instance, the skewness and kurtosis of the process, the third and fourth moment functions are required. Moreover, to compare the estimators proposed for some parameters of GP and to examine some theoretical properties of the process, higher-order moment functions should be known.

The rest of the paper organized as follows. In section 2, firstly, we obtain the integral equations for the third and fourth moment functions of the GP. Then, a generalization of the integral equation for kth moment function of the GP is given. We adapt the Tang and Lam's numerical method for the kth moment function of the GP with the help of the integral equation given for this function. Then, we reduce this general approximation to third and fourth moment functions of the GP in Section 3. Illustrative examples are provided in Section 4. Conclusion remarks are presented in Section 5.

2. INTEGRAL EQUATIONS FOR THE MOMENT FUNCTIONS OF THE GP

In this section, firstly, integral equations for the third and fourth moment functions of the GP are obtained. Then, in general, we present an integral equation for the kth moment function of the GP.

Let $\{N(t), t \geq 0\}$ be a GP with ratio parameter a and let us assume that the first interarrival time X_1 follows the distribution function F. The third moment function of the GP is given by $M_3(t) = E\left(N^3(t)\right),\ t \geq 0$. Conditioning on the first interarrival time X_1, we have

$$M_3(t) = E\left(N^3(t)\right) = \int_0^\infty E\left(N^3(t) \mid X_1 = x\right) dF(x).$$

Since $E(N^3(t) \mid X_1 = x) = E(1 + N(a(t-x)))^3,\ x < t$ and $E(N^3(t) \mid X_1 = x) = 0,\ x \geq t$, we rewrite the equality as follows.

$$\begin{aligned}
M_3(t) &= \int_0^t E(1 + N(a(t-x)))^3 dF(x) \\
&= \int_0^t dF(x) + 3\int_0^t E(N(a(t-x)))\, dF(x) \\
&+ 3\int_0^t E\left(N^2(a(t-x))\right) dF(x) + \int_0^t E\left(N^3(a(t-x))\right) dF(x) \\
&= F(t) + 3\int_0^t M_1(a(t-x))dF(x) + 3\int_0^t M_2(a(t-x))\, dF(x) \\
&+ \int_0^t M_3(a(t-x))\, dF(x).
\end{aligned}$$

Using the integral equations given in (1) and (2), we have

$$\begin{aligned}
M_3(t) = F(t) &+ 3(M_1(t) - F(t)) + 3(M_2(t) - 2M_1(t) + F(t)) \\
&+ \int_0^t M_3(a(t-x))dF(x).
\end{aligned}$$

Simplifying the expression, we obtain

$$M_3(t) = 3M_2(t) - 3M_1(t) + F(t) + \int_0^t M_3(a(t-x))dF(x),\ t \geq 0. \tag{3}$$

The fourth moment function of the GP is defined by $M_4(t) = E\left(N^4(t)\right),\ t \geq 0$. Using similar arguments in the derivation of the integral equation for $M_3(t)$, we have

$$\begin{aligned}
M_4(t) &= 4M_3(t) - 6M_2(t) + 4M_1(t) - F(t) \\
&+ \int_0^t M_4(a(t-x))dF(x),\ t \geq 0.
\end{aligned} \tag{4}$$

Following theorem states the kth moment function of a GP model.

Theorem 3. *For any $k \in \mathbb{N}$, the kth moment function $M_k(t)$ of the GP is given by*

$$M_k(t) = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k}{j} M_j(t) + \int_0^t M_k(a(t-x))dF(x), \ t \geq 0, \tag{5}$$

where $M_0(t) = F(t)$.

Proof. The proof of (5) is given by using the mathematical induction. It is obvious that (5) holds for $k = 1$ when we consider equation (1). Now, let us assume that (5) holds for any integer k and show that it also holds for integer $k+1$. The $(k+1)$th moment function of the GP is defined by $M_{k+1}(t) = E\left(N^{k+1}(t)\right), \ t \geq 0$. Conditioning on the first interarrival time X_1, we have

$$\begin{aligned} M_{k+1}(t) &= \int_0^t E(1 + N(a(t-x)))^{k+1} dF(x) \\ &= \sum_{j=0}^{k+1} \binom{k+1}{j} \int_0^t M_j(a(t-x))dF(x). \end{aligned}$$

The following equation can be written by separating the $(k+1)$th term.

$$M_{k+1}(t) = \sum_{j=0}^{k} \binom{k+1}{j} \int_0^t M_j(a(t-x))dF(x) + \int_0^t M_{k+1}(a(t-x))dF(x).$$

Since we assume that (5) holds for any integer k, we have

$$\begin{aligned} M_{k+1}(t) &= \sum_{j=0}^{k} \binom{k+1}{j} \left(M_j(t) - \sum_{i=0}^{j-1} (-1)^{j-1-i} \binom{j}{i} M_i(t) \right) \\ &\quad + \int_0^t M_{k+1}(a(t-x))dF(x) \\ &= \sum_{j=0}^{k} \binom{k+1}{j} \left(\sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} M_i(t) \right) \\ &\quad + \int_0^t M_{k+1}(a(t-x))dF(x). \end{aligned}$$

When we rearrange the terms, we obtain

$$M_{k+1}(t) = \sum_{i=0}^{k} \left(\sum_{j=i}^{k} (-1)^{j-i} \binom{k+1}{j} \binom{j}{i} \right) M_i(t) + \int_0^t M_{k+1}(a(t-x))dF(x).$$

Since the identities $\binom{k+1}{j}\binom{j}{i} = \binom{k+1}{i}\binom{k+1-i}{j-i}$ and $\sum_{j=i}^{k}(-1)^{j-i}\binom{k+1-i}{j-i} = (-1)^{k-i}$ hold,

$$M_{k+1}(t) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k+1}{i}M_i(t) + \int_0^t M_{k+1}(a(t-x))dF(x).$$

Hence, the proof is completed. □

According to the Theorem 1, for $a \leq 1$, the function $M_k(t)$ is finite for all $t \geq 0$. Furthermore, if F is continuous, then the integral equation (5) can be solved uniquely although this function does not have an analytical form. We discuss this problem in the next section. In the case of $a > 1$, $M_k(t)$ is infinite for all $t > 0$.

3. NUMERICAL SOLUTION

In this section, we give a method based on the trapezoidal integration rule for the numerical solution of the integral equation (5). This solution is obtained by recursive calculations with respect to k. Now, let us remind the trapezoidal integration rule as follows.

According to the trapezoidal integration rule, an integral $\int_a^b g(t)\,dt$ can be calculated numerically as

$$\int_a^b g(t)\,dt \approx \sum_{i=1}^{n}\frac{g(t_{i-1})+g(t_i)}{2}(t_i - t_{i-1}) = \frac{h}{2}g(t_0) + h\sum_{i=1}^{n-1}g(t_i) + \frac{h}{2}g(t_n),$$

where $\{t_0, t_1, \ldots, t_n\}$ is a partition of the interval $[a,b]$ such that $a = t_0 < t_1 < \cdots < t_n = b$, $t_i = a + ih$, $i = 0,1,\ldots,n$ and $h = \frac{b-a}{n}$. The approximation gives more precise results as the number of the partition increases.

Let $\{N(t),\ t \geq 0\}$ be a GP with ratio parameter $a < 1$. Assume that the first interarrival time X_1 has probability density function f. Then, the integral equation (5) can be written as

$$M_k(t) = \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k}{j}M_j(t) + \int_0^t M_k(a(t-x))f(x)\,dx,\ t \geq 0. \tag{6}$$

If we substitute $s = a(t-x)$, we have

$$M_k(t) = \sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k}{j}M_j(t) + \frac{1}{a}\int_0^{at} M_k(s)f\left(t - \frac{s}{a}\right)ds. \tag{7}$$

Assume that $T > 0$, $t \in [0,T]$ and $f(0) = 0$. Let $\{t_0, t_1, \ldots, t_n\}$ be a partition of the interval $[0,T]$ such that $0 = t_0 < t_1 < \cdots < t_n = T$. Take the step width $h = \frac{T}{n}$ and $t_i = ih$ for $i = 0,1,\ldots,n$. By (7), we have

$$
\begin{aligned}
M_k(t_i) = &\sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k}{j} M_j(t_i) \\
&+ \frac{1}{a}\int_0^{t_{\lfloor ai \rfloor}} M_k(s) f\left(t_i - \frac{s}{a}\right) ds \\
&+ \frac{1}{a}\int_{t_{\lfloor ai \rfloor}}^{at_i} M_k(s) f\left(t_i - \frac{s}{a}\right) ds.
\end{aligned} \tag{8}
$$

where $\lfloor . \rfloor$ is the greatest integer function. Since at_i does not have to belong to this partition, the interval $[0, at_i]$ is divided into two subsets as $[0, t_{\lfloor ai \rfloor}]$ and $[t_{\lfloor ai \rfloor}, at_i]$. Now, let us define I_1 and I_2 as

$$I_1 = \frac{1}{a}\int_0^{t_{\lfloor ai \rfloor}} M_k(s) f\left(t_i - \frac{s}{a}\right) ds$$

and

$$I_2 = \frac{1}{a}\int_{t_{\lfloor ai \rfloor}}^{at_i} M_k(s) f\left(t_i - \frac{s}{a}\right) ds.$$

Considering the trapezoidal integration rule with the partition $\{0, h, 2h, \ldots, [ai]\, h\}$ of the interval $[0, t_{\lfloor ai \rfloor}]$, we obtain

$$
\begin{aligned}
I_1 &= \frac{1}{a}\int_0^{t_{\lfloor ai \rfloor}} M_k(s) f\left(t_i - \frac{s}{a}\right) ds \\
&\approx \frac{h}{2a} M_k(t_0)\, f\left(t_i - \frac{t_0}{a}\right) + \frac{h}{a}\sum_{j=1}^{\lfloor ai \rfloor -1} M_k(t_j)\, f\left(t_i - \frac{t_j}{a}\right) \\
&+ \frac{h}{2a} M_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \\
&= \frac{h}{a}\sum_{j=1}^{\lfloor ai \rfloor -1} M_k(t_j)\, f\left(t_i - \frac{t_j}{a}\right) \\
&+ \frac{h}{2a} M_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right),
\end{aligned} \tag{9}
$$

where $M_k(0) = 0$. To calculate the integral I_2, it can be again used the trapezoidal integration rule with two points (the values of the lower and upper bounds) on the interval $[t_{\lfloor ai \rfloor}, at_i]$. Hence,

$$
\begin{aligned}
I_2 &= \frac{1}{a}\int_{t_{\lfloor ai \rfloor}}^{at_i} M_k(s) f\left(t_i - \frac{s}{a}\right) ds \\
&\approx \frac{at_i - t_{\lfloor ai \rfloor}}{2a}\left(M_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) + M_k(at_i)\, f(0)\right)
\end{aligned}
$$

$$= \frac{at_i - t_{\lfloor ai \rfloor}}{2a} \left(M_k \left(t_{\lfloor ai \rfloor} \right) f \left(t_i - \frac{t_{\lfloor ai \rfloor}}{a} \right) \right). \tag{10}$$

By using the equations (9) and (10) into the equation (8), we have

$$\begin{aligned} M_k(t_i) \approx & \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k}{j} M_j(t_i) \\ & + \frac{h}{a} \sum_{j=1}^{\lfloor ai \rfloor - 1} M_k(t_j) f\left(t_i - \frac{t_j}{a}\right) \\ & + \frac{h}{2a} M_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \\ & + \frac{at_i - t_{\lfloor ai \rfloor}}{2a} \left(M_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \right). \end{aligned}$$

Denote the approximate value of $M_k(t_i)$ as $\tilde{M}_k(t_i)$. Then, for any given $k \geq 1$ and $i = 0, 1, \ldots, n$, the values of $\tilde{M}_k(t_i)$ can be recursively calculated as

$$\begin{aligned} \tilde{M}_k(t_i) = & \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k}{j} \tilde{M}_j(t_i) \\ & + \frac{h}{a} \sum_{j=1}^{\lfloor ai \rfloor - 1} \tilde{M}_k(t_j) f\left(t_i - \frac{t_j}{a}\right) \\ & + \frac{h}{2a} \tilde{M}_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \\ & + \frac{at_i - t_{\lfloor ai \rfloor}}{2a} \left(\tilde{M}_k\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \right), \end{aligned} \tag{11}$$

where $\tilde{M}_k(0) = M_k(0) = 0$.

Let us reduce this general solution to the numerical solution of integral equations (3) and (4). By (11), the values of $\tilde{M}_3(t_i)$ and $\tilde{M}_4(t_i)$ can be calculated as

$$\begin{aligned} \tilde{M}_3(t_i) = & \, 3\tilde{M}_2(t_i) - 3\tilde{M}_1(t_i) + F(t_i) \\ & + \frac{h}{a} \sum_{j=1}^{\lfloor ai \rfloor - 1} \tilde{M}_3(t_j) f\left(t_i - \frac{t_j}{a}\right) \\ & + \frac{h}{2a} \tilde{M}_3\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \\ & + \frac{at_i - t_{\lfloor ai \rfloor}}{2a} \left(\tilde{M}_3\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right) \right) \end{aligned}$$

and

$$\begin{aligned}\tilde{M}_4(t_i) &= 4\tilde{M}_3(t_i) - 6\tilde{M}_2(t_i) + 4\tilde{M}_1(t_i) - F(t_i)\\ &\quad + \frac{h}{a}\sum_{j=1}^{\lfloor ai \rfloor - 1} \tilde{M}_4(t_j)\, f\left(t_i - \frac{t_j}{a}\right)\\ &\quad + \frac{h}{2a}\tilde{M}_4\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right)\\ &\quad + \frac{at_i - t_{\lfloor ai \rfloor}}{2a}\left(\tilde{M}_4\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right)\right)\end{aligned}$$

respectively, where $M_1(t_i)$ and $M_2(t_i)$ can be approximately calculated with the help of the formula given in (11) as

$$\begin{aligned}\tilde{M}_1(t_i) &= F(t_i) + \frac{h}{a}\sum_{j=1}^{\lfloor ai \rfloor - 1} \tilde{M}_1(t_j)\, f\left(t_i - \frac{t_j}{a}\right)\\ &\quad + \frac{h}{2a}\tilde{M}_1\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right)\\ &\quad + \frac{at_i - t_{\lfloor ai \rfloor}}{2a}\left(\tilde{M}_1\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right)\right) \qquad (12)\end{aligned}$$

and

$$\begin{aligned}\tilde{M}_2(t_i) &= 2\tilde{M}_1(t_i) - F(t_i)\\ &\quad + \frac{h}{a}\sum_{j=1}^{\lfloor ai \rfloor - 1} \tilde{M}_2(t_j)\, f\left(t_i - \frac{t_j}{a}\right)\\ &\quad + \frac{h}{2a}\tilde{M}_2\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lceil ai \rfloor}}{a}\right)\\ &\quad + \frac{at_i - t_{\lfloor ai \rfloor}}{2a}\left(\tilde{M}_2\left(t_{\lfloor ai \rfloor}\right) f\left(t_i - \frac{t_{\lfloor ai \rfloor}}{a}\right)\right). \qquad (13)\end{aligned}$$

Note that these formulas in (12) and (13) are given previously by [17] and [15], respectively.

4. ILLUSTRATIVE EXAMPLES

In this section, we consider gamma, Weibull and lognormal distributions for the first interarrival time of the GP to illustrate the numerical method developed in previous section. As indicated in [10], the ratio parameter a satisfies the condition $0.95 \leq a \leq 1.05$ for many real data sets fitted by the GP. Further, in the applications of the GP, we mostly encounter with values of a which is less than 1. For this reason, the ratio parameter of the GP is taken as $a = 0.95$ in each example. It is worth to noting that similar results are obtained for the different values of a. The value of T is chosen to be at least $10E(X_1)$.

In each example, we calculate the approximate values of the skewness and kurtosis of the GP model as

$$\tilde{S}(t) = \frac{\tilde{M}_3(t) - 3\tilde{M}_2(t)\tilde{M}_1(t) + 2\tilde{M}_1^3(t)}{\left(\tilde{M}_2(t) - \tilde{M}_1^2(t)\right)^{3/2}},\ t \geq 0$$

and

$$\tilde{K}(t) = \frac{\tilde{M}_4(t) - 4\tilde{M}_3(t)\tilde{M}_1(t) + 6\tilde{M}_2(t)\tilde{M}_1^2(t) - 3\tilde{M}_1^4(t)}{\left(\tilde{M}_2(t) - \tilde{M}_1^2(t)\right)^{2}},\ t \geq 0$$

respectively.

Example 1. (Gamma distribution)

Let $\{N(t),\ t \geq 0\}$ be a GP with ratio parameter $a = 0.95$ and assume that the first interarrival time X_1 follows gamma distribution $\Gamma(2,1)$. Since $T = 10E(X_1) = 20$, we divide the interval $[0,20]$ into $n = 2000$ subintervals with the equal width $h = 0.01$. The following Table 1 presents the approximate values of $M_3(t)$, $M_4(t)$, $S(t)$ and $K(t)$.

Table 1. Results for $X_1 \sim \Gamma(2,1)$

t	$\tilde{M}_3(t)$	$\tilde{M}_4(t)$	$\tilde{S}(t)$	$\tilde{K}(t)$
0.1	0.0047	0.0047	14.5437	213.1503
0.5	0.1016	0.1148	3.0185	10.9469
1	0.3954	0.5569	1.4545	4.2591
3	4.7898	11.9093	0.4792	3.1260
5	17.2850	58.7025	0.3523	3.0903
8	57.5463	267.5775	0.2646	3.0412
10	101.6607	552.6060	0.2279	3.0256
15	280.9871	2041.6477	0.1696	3.0086
20	565.2640	5051.7662	0.1350	3.0045

Example 2. (Weibull distribution)

Let $\{N(t),\ t \geq 0\}$ be a GP with ratio parameter $a = 0.95$ and assume that the first interarrival time X_1 has Weibull distribution W(2,1). Since $T = 10 > 10E(X_1) = 5\sqrt{\pi}$, the interval $[0, 10]$ is divided into $n = 1000$ subintervals with the equal width $h = 0.01$. Thus, the approximate values of $M_3(t)$, $M_4(t)$, $S(t)$ and $K(t)$ are given in the Table 2 below.

TABLE 2. Results for $X_1 \sim$ W(2,1).

t	$\tilde{M}_3(t)$	$\tilde{M}_4(t)$	$\tilde{S}(t)$	$\tilde{K}(t)$
0.1	0.0101	0.0102	9.9125	99.8624
0.5	0.2833	0.3573	1.5956	4.4056
1	1.4831	2.6215	0.4613	2.9561
2	9.6617	26.4035	0.4614	3.2020
3	29.7577	108.7189	0.3757	3.1660
4	65.4750	297.7482	0.3335	3.1183
5	119.3496	645.6989	0.3003	3.0952
8	403.4629	3151.2045	0.2350	3.0699
10	699.8581	6486.4930	0.2059	3.0786

Example 3. (Lognormal distribution)

Let $\{N(t),\ t \geq 0\}$ be a GP with ratio parameter $a = 0.95$ and assume that the first interarrival time X_1 has lognormal distribution LN(0,1). Taking $T = 18 > 10E(X_1) = 10e^{1/2}$, the interval $[0, 18]$ is divided into $n = 1800$ subintervals with the equal width $h = 0.01$. Thus, we obtain the approximate values of $M_3(t)$, $M_4(t)$, $S(t)$ and $K(t)$ in the Table 3 below.

TABLE 3. Results for $X_1 \sim$ LN(0,1).

t	$\tilde{M}_3(t)$	$\tilde{M}_4(t)$	$\tilde{S}(t)$	$\tilde{K}(t)$
0.1	0.0106	0.0107	9.5450	92.2791
0.5	0.3473	0.4711	1.5313	4.3893
1	1.4488	2.8111	0.8825	3.2820
3	16.3445	58.9722	0.4168	2.8515
5	52.0086	255.8734	0.2412	2.7747
8	151.0759	997.1935	0.0829	2.7883
10	249.7131	1898.0273	0.0096	2.8269
15	613.5389	6027.1855	-0.1164	2.9489
18	911.0975	10036.7429	-0.1682	3.0225

It can be concluded from Tables 1-3 that the shape of the distribution of the GP converges to the normal distribution when the value of t gets closer to the mean of the first interarrival time.

5. CONCLUSIONS

Integral equations satisfied by the third and fourth moment functions of a GP are derived. Further, we present an integral equation for the kth moment function as a generalization of the lower-order moment functions of the GP. In general manner, a numerical method established for solving the integral equation given for the kth moment function is presented. Then, we reduce this general structure to the solutions of the third and fourth integral equations to obtain their solutions. By using the solutions of the integral equation (3) and (4), skewness and kurtosis of the GP model are calculated for some lifetime distributions. According to the numerical calculations, as t gets closer to $E(X_1)$, the shape of the distribution of the GP converges to the normal distribution. Note that more precise results can be obtained by taking smaller step width in numerical calculation of the integral equations since the accuracy of the approximation depends on the step width.

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HYPERSURFACE FAMILIES WITH SMARANDACHE CURVES IN GALILEAN 4-SPACE

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Abstract. In this paper, we study the hypersurface families with Smarandache curves in 4-dimensional Galilean space G_4 and give the conditions for different Smarandache curves to be parameter and the curve which generates the Smarandache curves is geodesic on a hypersurface in G_4. Also, we investigate three types of marching-scale functions for one of these hypersurfaces and construct an example for it.

1. PRELIMINARIES

In physics, geodesics which are defined as a parallel transport of a tangent vector in a linear (affine) connection on the manifold M are very important for general relativity. Because, the geodesic equation which is given with a set of initial conditions is very useful in theoretical foundations of general relativity. Also, in general, relativity gravity can be regarded as not a force but a consequence of a curved spacetime geometry where the source of curvature is the stress–energy tensor. For example, the path of a planet orbiting a star is the projection of a geodesic of the curved four-dimensional spacetime geometry around the star onto three-dimensional space.

Furthermore, as an alternative definition of a geodesic line can be defined as the shortest curve connecting two points on a manifold. A curve $\gamma(\nu)$ on a hypersurface $\varphi(\nu,\mu,\sigma)$ is geodesic iff the normal $N(\nu)$ of the curve $\gamma(\nu)$ and the normal $\eta(\nu,\mu_0,\sigma_0)$ of the hypersurface $\varphi(\nu,\mu,\sigma)$ at any point on the curve $\gamma(\nu)$ are parallel to each other and a curve $\gamma(\nu)$ on the hypersurface $\varphi(\nu,\mu,\sigma)$ is asymptotic iff

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the normal $N(\nu)$ of the curve $\gamma(\nu)$ and the normal $\eta(\nu,\mu_0,\sigma_0)$ of the hypersurface $\varphi(\nu,\mu,\sigma)$ at any point on the curve $\gamma(\nu)$ are perpendicular.

The problem of constructing a family of surfaces from a given spatial geodesic curve firstly has been studied by Wang et al. in 2004 and in that study, the authors have derived a parametric representation for a surface pencil whose members share the same geodesic curve as an isoparametric curve [15]. After this study, in 2008 the generalization of the Wangs' assumption to more general marching-scale functions has been given by Kasap et al [7]. By using these studies, the problem of finding a surface pencil from a given spatial asymptotic curve has been investigated in [4] and the necessary and sufficient condition for the given curve to be the asymptotic curve for the parametric surface has been stated in [1]. Also, the problem of finding a hypersurface family from a given asymptotic curve in R^4 has been handled in [5].

Surfaces with common geodesic and family of surface with a common null geodesic in Minkowski 3-space have been studied in [8] and [13], respectively.

The Galilean space G_3 is a Cayley-Klein space equipped with the metric of signature $(0,0,+,+)$. The absolute figure of the Galilean space consists of an ordered triple $\{\omega,f,I\}$ in which ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of f.

In the Galilean n-space, there are just two types of vectors. A vector $u=(u_1,u_2,...,u_n)$ is said to be non-isotropic, if $u_1\neq 0$ and it is said to be isotropic otherwise.

If $u=(u_1,u_2,u_3,u_4)$, $v=(v_1,v_2,v_3,v_4)$ and $w=(w_1,w_2,w_3,w_4)$ are three vectors in Galilean space G_4, then the Galilean scalar product of x and y is given by

$$\langle u,v\rangle=\begin{cases} u_1v_1, \text{ if } u_1\neq 0 \text{ or } v_1\neq 0\\ u_2v_2+u_3v_3+u_4v_4, \text{ if } u_1=0 \text{ and } v_1=0\end{cases} \tag{1}$$

and for $e_2=(0,1,0,0)$, $e_3=(0,0,1,0)$ and $e_4=(0,0,0,1)$, the Galilean cross product of u, v and w is defined by

$$u\times v\times w=\begin{vmatrix} 0 & e_2 & e_3 & e_4\\ u_1 & u_2 & u_3 & u_4\\ v_1 & v_2 & v_3 & v_4\\ w_1 & w_2 & w_3 & w_4\end{vmatrix}. \tag{2}$$

Let γ be an admissible curve of the class C^∞ in G_4, parameterized by the invariant arc-length parameter ν, given by

$$\gamma(\nu)=(\nu,f(\nu),g(\nu),h(\nu)). \tag{3}$$

Then the Frenet frame is

$$\begin{aligned} T(\nu)&=\gamma'(\nu)=(1,f'(\nu),g'(\nu),h'(\nu)),\\ N(\nu)&=\tfrac{\gamma''(\nu)}{\kappa_1(\nu)}=\tfrac{1}{\kappa_1(\nu)}(0,f''(\nu),g''(\nu),h''(\nu)),\\ B_1(\nu)&=\tfrac{1}{\kappa_2(\nu)}\left(0,(\tfrac{f''(\nu)}{\kappa_1(\nu)})',(\tfrac{g''(\nu)}{\kappa_1(\nu)})',(\tfrac{h''(\nu)}{\kappa_1(\nu)})'\right),\\ B_2(\nu)&=\pm T(\nu)\times N(\nu)\times B_1(\nu)\end{aligned} \tag{4}$$

and the first, second and third curvatures of the curve $\gamma(\nu)$ are given by

$$\begin{array}{l} \kappa_1(\nu) = \sqrt{f''(\nu)^2 + g''(\nu)^2 + h''(\nu)^2}, \\ \kappa_2(\nu) = \sqrt{\langle N'(\nu), N'(\nu)\rangle}, \\ \kappa_3(\nu) = \langle B_1'(\nu), B_2(\nu)\rangle\,, \end{array} \tag{5}$$

respectively and where T, N, B_1 and B_2 are called tangent vector, principal normal vector, first binormal vector and second binormal vector of $\gamma(\nu)$. We must note that, throughout this study, we will assume that $\kappa_1(\nu) \neq 0$ and $\kappa_2(\nu) \neq 0$ at everywhere.

Also, Frenet formulas are given by

$$\begin{array}{l} T'(\nu) = \kappa_1(\nu)N(\nu), \\ N'(\nu) = \kappa_2(\nu)B_1(\nu), \\ B_1'(\nu) = -\kappa_2(\nu)N(\nu) + \kappa_3(\nu)B_2(\nu), \\ B_2'(\nu) = -\kappa_3(\nu)B_1(\nu). \end{array} \tag{6}$$

The equation of a hypersurface in G_4 can be given by the parametrization

$$\varphi(\nu,\mu,\sigma) = \left((\varphi(\nu,\mu,\sigma))_1\,, (\varphi(\nu,\mu,\sigma))_2\,, (\varphi(\nu,\mu,\sigma))_3\,, (\varphi(\nu,\mu,\sigma))_4\right), \tag{7}$$

where $(\varphi(\nu,\mu,\sigma))_i \in C^3$, $i = 1,2,3,4$. The normal of this hypersurface is calculated as follows

$$\eta(\nu,\mu,\sigma) = \varphi_\nu \times \varphi_\mu \times \varphi_\sigma, \tag{8}$$

where $\varphi_i = \frac{\partial\varphi(\nu,\mu,\sigma)}{\partial i}$, $i \in \{\nu,\mu,\sigma\}$.

For more information about 4-dimensional Galilean space, we refer to [6], [11], [12], [16], [17] and etc.

If $\gamma(\nu)$ is a isoparametric curve on the hypersurface $\varphi(\nu,\mu,\sigma)$, then there exists a pair of parameters $\mu_0 \in [T_1,T_2]$ and $\sigma_0 \in [M_1,M_2]$, such that $\gamma(\nu) = \varphi(\nu,\mu_0,\sigma_0)$.

If the curve is both an asymptotic and parameter (isoparametric) curve on φ, then it is called isoasymptotic on the hypersurface φ. Similarly, if the curve is both a geodesic and parameter (isoparametric) curve on the hypersurface φ, then it is called isogeodesic on the hypersurface φ.

On the constructions of surface families with common geodesic and asymptotic curves in Galilean Space G_3 and an approach for hypersurface family with common geodesic curve in the Galilean Space G_4 have been handled in [9], [10] and [11], respectively.

On the other hand, the geometry of Smarandache curves has been very popular topic for differential geometers, recently. Let $\gamma(\nu)$ be an admissible curve in G_4 and $\{T,N,B_1,B_2\}$ be its moving Frenet frame. Then TN, TB_1, TB_2, NB_1, NB_2, B_1B_2, TNB_1, TNB_2, TB_1B_2, NB_1B_2 and TNB_1B_2-Smarandache curves are defined by $r_{TN} = \frac{T+N}{\|T+N\|}$, $r_{TB_1} = \frac{T+B_1}{\|T+B_1\|}$, $r_{TB_2} = \frac{T+B_2}{\|T+B_2\|}$, $r_{NB_1} = \frac{N+B_1}{\|N+B_1\|}$, $r_{NB_2} = \frac{N+B_2}{\|N+B_2\|}$, $r_{B_1B_2} = \frac{B_1+B_2}{\|B_1+B_2\|}$, $r_{TNB_1} = \frac{T+N+B_1}{\|T+N+B_1\|}$, $r_{TNB_2} = \frac{T+N+B_2}{\|T+N+B_2\|}$, $r_{TB_1B_2} = \frac{T+B_1+B_2}{\|T+B_1+B_2\|}$, $r_{NB_1B_2} = \frac{N+B_1+B_2}{\|N+B_1+B_2\|}$, $r_{TNB_1B_2} = \frac{T+N+B_1+B_2}{\|T+N+B_1+B_2\|}$, respectively.

The problem of constructing a family of surfaces from a given some special Smarandache asymptotic curves in Euclidean 3-space has been analyzed in [14] and surfaces using Smarandache asymptotic curves in Galilean space have been studied in [2].

In the present study, we investigate the hypersurface families with Smarandache curves in 4-dimensional Galilean space G_4.

2. HYPERSURFACE FAMILIES WITH SMARANDACHE CURVES IN 4-DIMENSIONAL GALILEAN SPACE G_4

Let $\varphi(\nu,\mu,\sigma)$ be a parametric hypersurface which is defined by a given curve $\gamma(\nu)$ as follows

$$\varphi(\nu,\mu,\sigma)=\gamma(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right], \tag{9}$$

where $L_1\leq\nu\leq L_2$, $T_1\leq\mu\leq T_2$ and $M_1\leq\sigma\leq M_2$. Also, $x(\nu,\mu,\sigma)$, $y(\nu,\mu,\sigma)$, $z(\nu,\mu,\sigma)$ and $m(\nu,\mu,\sigma)$ which are the values of the marching-scale functions indicate and the values of these functions are C^1-functions and $\{T,N,B_1,B_2\}$ is the Frenet frame associated with the curve γ in G_4.

Throughout this study, for simplicity, we will denote $x(\nu,\mu,\sigma)=x$, $x(\nu,\mu_0,\sigma_0)$ $=x_0$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial\nu}=x_\nu$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial\mu}=x_\mu$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial\sigma}=x_\sigma$ and $\frac{\partial x(\nu,\mu,\sigma)}{\partial\nu}|_{(\nu,\mu_0,\sigma_0)}=(x_v)_0$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial\mu}|_{(\nu,\mu_0,\sigma_0)}=(x_\mu)_0$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial v}|_{(\nu,\mu_0,\sigma_0)}=(x_\sigma)_0$. Similar abbreviations for $y(\nu,\mu,\sigma)$, $z(\nu,\mu,\sigma)$ and $m(\nu,\mu,\sigma)$ will be used, too.

CASE 1.

In this case, by taking the TN-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TN}(\nu,\mu,\sigma)$ which is given with the aid of the TN-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TN}(\nu,\mu,\sigma)=r_{TN}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right]. \tag{10}$$

From (10), we have

$$\begin{aligned} (\varphi_{TN})_\nu &= x_\nu T+(\kappa_1+x\kappa_1+y_\nu-z\kappa_2)N \\ &\quad +(\kappa_2+y\kappa_2+z_\nu-m\kappa_3)B_1+(m_\nu+z\kappa_3)B_2, \\ (\varphi_{TN})_\mu &= x_\mu T+y_\mu N+z_\mu B_1+m_\mu B_2, \\ (\varphi_{TN})_\sigma &= x_\sigma T+y_\sigma N+z_\sigma B_1+m_\sigma B_2, \end{aligned} \tag{11}$$

where we denote $\frac{\partial\varphi_{TN}(\nu,\mu,\sigma)}{\partial\nu}=(\varphi_{TN})_\nu$, $\frac{\partial\varphi_{TN}(\nu,\mu,\sigma)}{\partial\mu}=(\varphi_{TN})_\mu$, $\frac{\partial\varphi_{TN}(\nu,\mu,\sigma)}{\partial\sigma}=(\varphi_{TN})_\sigma$.

Thus, if we use (11) in (8), by obtaining the normal of the hypersurface (10), we can state the following theorem:

Theorem 1. *$\gamma(\nu)$ is not a geodesic curve where TN-Smarandache curve r_{TN} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ in G_4.*

Proof. If TN-Smarandache curve is a isoparametric curve on $\varphi_{TN}(\nu,\mu,\sigma)$, then there exists a pair of parameters $\mu=\mu_0$ and $\sigma=\sigma_0$ such that

$$\varphi_{TN}(\nu,\mu_0,\sigma_0)=T(\nu)+N(\nu),$$

that is

$$x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0, \tag{12}$$

where $L_1\leq\nu\leq L_2$, $T_1\leq\mu_0\leq T_2$ and $M_1\leq\sigma_0\leq M_2$. Here we must note that, from (12), we have

$$x_v(\nu,\mu_0,\sigma_0)=y_v(\nu,\mu_0,\sigma_0)=z_v(\nu,\mu_0,\sigma_0)=m_v(\nu,\mu_0,\sigma_0)=0. \tag{13}$$

So, from (2), (11), (12) and (13), the normal of the hypersurface (10) for $\mu=\mu_0$ and $\sigma=\sigma_0$ is obtained as

$$\begin{aligned}\eta_{TN}(\nu,\mu_0,\sigma_0)&=(\eta_{TN})_1(\nu,\mu_0,\sigma_0)T(\nu)+(\eta_{TN})_2(\nu,\mu_0,\sigma_0)N(\nu)\\&\quad+(\eta_{TN})_3(\nu,\mu_0,\sigma_0)B_1(\nu)+(\eta_{TN})_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned} \tag{14}$$

where

$$\begin{aligned}&(\eta_{TN})_1(\nu,\mu_0,\sigma_0)=0,\\&(\eta_{TN})_2(\nu,\mu_0,\sigma_0)=\kappa_2\left((m_\sigma)_0(x_\mu)_0-(x_\sigma)_0(m_\mu)_0\right),\\&(\eta_{TN})_3(\nu,\mu_0,\sigma_0)=\kappa_1\left((m_\mu)_0(x_\sigma)_0-(x_\mu)_0(m_\sigma)_0\right),\\&(\eta_{TN})_4(\nu,\mu_0,\sigma_0)=\kappa_1\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)\\&\qquad\qquad+\ \kappa_2\left((y_\mu)_0(x_\sigma)_0-(x_\mu)_0(y_\sigma)_0\right).\end{aligned} \tag{15}$$

Also, from the definition of a given curve $\gamma(\nu)$ on the hypersurface $\varphi(\nu,\mu,\sigma)$ to be geodesic, it must be

$$(\eta_{TN})_2(\nu,\mu_0,\sigma_0)\neq 0$$

and

$$(\eta_{TN})_3(\nu,\mu_0,\sigma_0)=(\eta_{TN})_4(\nu,\mu_0,\sigma_0)=0.$$

So, from (15) the curve $\gamma(\nu)$ is a geodesic on the hypersurface $\varphi_{TN}(\nu,\mu,\sigma)$ in G_4 if

$$\left\{\begin{array}{l}x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\\kappa_1=0,\ \kappa_2\neq 0,\\(y_\mu)_0(x_\sigma)_0=(x_\mu)_0(y_\sigma)_0,\ (m_\mu)_0(x_\sigma)_0\neq(x_\mu)_0(m_\sigma)_0\end{array}\right. \tag{16}$$

satisfied, where $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$. Since $\kappa_1(v)\neq 0$, the proof completes. □

CASE 2.

Here, by taking the NB_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ which is given with the aid of the NB_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{NB_2}(\nu,\mu,\sigma)=r_{NB_2}(\nu)+\left[\begin{array}{c}x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\+z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right].$$

If NB_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{NB_2}(\nu,\mu_0,\sigma_0) = &\left(\eta_{NB_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{NB_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\ &+ \left(\eta_{NB_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{NB_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned} \tag{17}$$

where

$$\begin{aligned}&\left(\eta_{NB_2}\right)_1(\nu,\mu_0,\sigma_0) = 0,\\ &\left(\eta_{NB_2}\right)_2(\nu,\mu_0,\sigma_0) = \left(\tfrac{\kappa_2-\kappa_3}{\sqrt{2}}\right)\left((x_\mu)_0\,(m_\sigma)_0 - (m_\mu)_0\,(x_\sigma)_0\right),\\ &\left(\eta_{NB_2}\right)_3(\nu,\mu_0,\sigma_0) = 0,\\ &\left(\eta_{NB_2}\right)_4(\nu,\mu_0,\sigma_0) = \left(\tfrac{\kappa_3-\kappa_2}{\sqrt{2}}\right)\left((y_\sigma)_0\,(x_\mu)_0 - (x_\sigma)_0\,(y_\mu)_0\right).\end{aligned} \tag{18}$$

Theorem 2. *$\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0,\\ \kappa_2 \neq \kappa_3,\\ (y_\mu)_0\,(x_\sigma)_0 = (x_\mu)_0\,(y_\sigma)_0\,, \quad (m_\mu)_0\,(x_\sigma)_0 \neq (x_\mu)_0\,(m_\sigma)_0 \end{cases} \tag{19}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

Proof. If $x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0$ satisfies for a pair of parameters $\mu=\mu_0$ and $\sigma=\sigma_0$ on hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$, then NB_2-Smarandache curve is a isoparametric curve such that

$$\varphi_{NB_2}(\nu,\mu_0,\sigma_0) = \frac{N(\nu)+B_2(\nu)}{\sqrt{2}},$$

where $L_1\le\nu\le L_2$, $T_1\le\mu_0\le T_2$ and $M_1\le\sigma_0\le M_2$.

Also, from the definition of a given curve $\gamma(\nu)$ on the hypersurface $\varphi(\nu,\mu,\sigma)$ to be geodesic where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric, it must be

$$\left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0) \neq 0$$

and

$$\left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0) = \left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0) = 0.$$

So, using these conditions with (19) in (18), the proof completes. □

For the purposes of simplification and better analysis, Wang et al. have studied the case when the marching-scale functions can be decomposed into two factors in Euclidean 3-space. The factor-decomposition form possesses an evident advantage: the designer can select different sets of functions to adjust the shape of the surface until they are gratified with the design, and the resulting surface is guaranteed to belong to the isogeodesic surface pencil with the curve as the common geodesic [15]. Also in [3] and [11], the three types of the marching-scale function which have three parameters have been studied in 4-dimensional Galilean and Euclidean spaces, respectively. In this study, we have used the marching-scale functions which have been given in these studies. Now, for simplicity, let we investigate three types of marching-scale functions for this hypersurface.

Marching-scale functions of type 1

Let us choose

$$\begin{cases} x(\nu,\mu,\sigma)=p(\nu)X(\mu,\sigma), \\ y(\nu,\mu,\sigma)=q(\nu)Y(\mu,\sigma), \\ z(\nu,\mu,\sigma)=w(\nu)Z(\mu,\sigma), \\ m(\nu,\mu,\sigma)=l(\nu)M(\mu,\sigma), \end{cases} \tag{20}$$

where $L_1 \leq \nu \leq L_2$, $T_1 \leq \mu \leq T_2$, $M_1 \leq \sigma \leq M_2$; $p(\nu)$, $q(\nu)$, $w(\nu)$, $l(\nu)$, $X(\mu,\sigma)$, $Y(\mu,\sigma)$, $Z(\mu,\sigma)$, $M(\mu,\sigma) \in C^1$ and $p(\nu)$, $q(\nu)$, $w(\nu)$, $l(\nu)$, $\forall \nu \in [L_1, L_2]$ are not identically zero. By using (19), if the conditions

$$\begin{cases} X\left(\mu_0,\sigma_0\right)=Y\left(\mu_0,\sigma_0\right)=Z\left(\mu_0,\sigma_0\right)=M\left(\mu_0,\sigma_0\right)=0, \\ \kappa_2 \neq \kappa_3, \\ \frac{\partial Y(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial X(\mu_0,\sigma_0)}{\partial \sigma}=\frac{\partial X(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial Y(\mu_0,\sigma_0)}{\partial \sigma}, \\ \frac{\partial M(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial X(\mu_0,\sigma_0)}{\partial \sigma}\neq\frac{\partial X(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial M(\mu_0,\sigma_0)}{\partial \sigma}, \end{cases} \tag{21}$$

satisfy, then $\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4. Here, $\mu_0 \in [T_1,T_2]$, $\sigma_0 \in [M_1,M_2]$.

Marching-scale functions of type 2

If we take

$$\begin{cases} x(\nu,\mu,\sigma)=p(\nu,\mu)X(\sigma), \\ y(\nu,\mu,\sigma)=q(\nu,\mu)Y(\sigma), \\ z(\nu,\mu,\sigma)=w(\nu,\mu)Z(\sigma), \\ m(\nu,\mu,\sigma)=l(\nu,\mu)M(\sigma), \end{cases} \tag{22}$$

where $L_1 \leq \nu \leq L_2$, $T_1 \leq \mu \leq T_2$, $M_1 \leq \sigma \leq M_2$ and $p(\nu,\mu)$, $q(\nu,\mu)$, $w(\nu,\mu)$, $l(\nu,\mu)$, $X(\mu)$, $Y(\mu)$, $Z(\mu)$, $M(\mu) \in C^1$, by using (19), if the conditions

$$\begin{cases} p(\nu,\mu_0)X(\sigma_0)=q(\nu,\mu_0)Y(\sigma_0)=w(\nu,\mu_0)Z(\sigma_0)=l(\nu,\mu_0)M(\sigma_0)=0, \\ \kappa_2 \neq \kappa_3, \\ \frac{\partial q(\nu,\mu_0)}{\partial \mu}Y(\sigma_0)p(\nu,\mu_0)\frac{\partial X(\sigma_0)}{\partial \sigma}=\frac{\partial p(\nu,\mu_0)}{\partial \mu}X(\sigma_0)q(\nu,\mu_0)\frac{\partial Y(\sigma_0)}{\partial \sigma}, \\ \frac{\partial l(\nu,\mu_0)}{\partial \mu}M(\sigma_0)p(\nu,\mu_0)\frac{\partial X(\sigma_0)}{\partial \sigma}\neq\frac{\partial p(\nu,\mu_0)}{\partial \mu}X(\sigma_0)l(\nu,\mu_0)\frac{\partial M(\sigma_0)}{\partial \sigma}, \end{cases} \tag{23}$$

satisfy, then $\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4. Here, $\mu_0 \in [T_1,T_2]$, $\sigma_0 \in [M_1,M_2]$.

Marching-scale functions of type 3

For

$$\begin{cases} x(\nu,\mu,\sigma) = p(\nu,\sigma)X(\mu), \\ y(\nu,\mu,\sigma) = q(\nu,\sigma)Y(\mu), \\ z(\nu,\mu,\sigma) = w(\nu,\sigma)Z(\mu), \\ m(\nu,\mu,\sigma) = l(\nu,\sigma)M(\mu), \end{cases} \tag{24}$$

where $L_1 \le \nu \le L_2$, $T_1 \le \mu \le T_2$, $M_1 \le \sigma \le M_2$ and $p(\nu,\sigma)$, $q(\nu,\sigma)$, $w(\nu,\sigma)$, $l(\nu,\sigma)$, $X(\mu)$, $Y(\mu)$, $Z(\mu)$, $M(\mu) \in C^1$, from (19), if the conditions

$$\begin{cases} p(\nu,\sigma_0)X(\mu_0) = q(\nu,\sigma_0)Y(\mu_0) = w(\nu,\sigma_0)Z(\mu_0) = l(\nu,\sigma_0)M(\mu_0) = 0, \\ \kappa_2 \ne \kappa_3, \\ \frac{\partial Y(\mu_0)}{\partial \mu} q(\nu,\sigma_0) \frac{\partial p(\nu,\sigma_0)}{\partial \sigma} X(\mu_0) = \frac{\partial X(\mu_0)}{\partial \mu} p(\nu,\sigma_0) \frac{\partial q(\nu,\sigma_0)}{\partial \sigma} Y(\mu_0), \\ \frac{\partial M(\mu_0)}{\partial \mu} l(\nu,\sigma_0) \frac{\partial p(\nu,\sigma_0)}{\partial \sigma} X(\mu_0) \ne \frac{\partial X(\mu_0)}{\partial \mu} p(\nu,\sigma_0) \frac{\partial l(\nu,\sigma_0)}{\partial \sigma} M(\mu_0), \end{cases} \tag{25}$$

satisfy, then $\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4. Here, $\mu_0 \in [T_1, T_2], \sigma_0 \in [M_1, M_2]$.

Now, let we construct an example for this hypersurface family.

Example 3. *Let $\gamma(\nu)$ be a curve which is parametrized by*

$$\gamma(\nu) = (\nu, \sin v, \cos v, \nu^3). \tag{26}$$

From (4) and (5), it is easy to obtain that

$$\begin{cases} T(\nu) = (1, \cos v, -\sin v, 3\nu^2), \\ N(\nu) = \frac{1}{\sqrt{1+36\nu^2}}(0, -\sin v, -\cos v, 6v), \\ B_1(\nu) = \frac{1}{\sqrt{(1+36\nu^2)(37+36\nu^2)}}(0, 36v\sin v - (36v^2+1)\cos v, \\ \qquad\qquad 36v\cos v + (36v^2+1)\sin v, 6), \\ B_2(\nu) = \frac{1}{\sqrt{37+36\nu^2}}(0, 6(\cos v + v\sin v), 6(v\cos v - \sin v), 1) \end{cases} \tag{27}$$

and

$$\kappa_1 = \sqrt{1+36\nu^2}, \quad \kappa_2 = \frac{\sqrt{37+36\nu^2}}{1+36\nu^2}, \quad \kappa_3 = \frac{6v\sqrt{1+36\nu^2}}{37+36\nu^2}. \tag{28}$$

Now, if we choose $\sigma_0 = \frac{\pi}{2}$, $\mu_0 = 0$, $x(\nu,\mu,\sigma) = \nu^3(\sigma - \frac{\pi}{2})$, $y(\nu,\mu,\sigma) = \nu\mu^2\sin\sigma$, $z(\nu,\mu,\sigma) = \cos\sigma$ and $m(\nu,\mu,\sigma) = \nu\mu\sigma^2$, then (26) is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve (26) is isoparametric on the hypersurface

$$\varphi_{NB_2}(\nu,\mu,\sigma) = (\nu^3(\sigma - \frac{\pi}{2}),$$

$$\frac{-\sin v}{\sqrt{1+36\nu^2}} + \frac{6(\cos v + v\sin v)}{\sqrt{(37+36\nu^2)}} + \nu^3(\sigma - \frac{\pi}{2})\cos v - \frac{\nu\mu^2\sin\sigma\sin v}{\sqrt{1+36\nu^2}}$$

$$+ \frac{\cos\sigma(36v\sin v - (36v^2+1)\cos v)}{\sqrt{(1+36\nu^2)(37+36\nu^2)}} + \frac{6(\cos v + v\sin v)\nu\mu\sigma^2}{\sqrt{(37+36\nu^2)}},$$

$$\frac{-\cos v}{\sqrt{1+36\nu^2}}+\frac{6(v\cos v-\sin v)}{\sqrt{(37+36\nu^2)}}-\nu^3(\sigma-\frac{\pi}{2})\sin v-\frac{\nu\mu^2\sin\sigma\cos v}{\sqrt{1+36\nu^2}} \tag{29}$$
$$+\frac{\cos\sigma(36v\cos v+(36v^2+1)\sin v)}{\sqrt{(1+36\nu^2)(37+36\nu^2)}}+\frac{6(v\cos v-\sin v)\nu\mu\sigma^2}{\sqrt{(37+36\nu^2)}},$$
$$\frac{6v}{\sqrt{1+36\nu^2}}+\frac{1}{\sqrt{(37+36\nu^2)}}+3v^5(\sigma-\frac{\pi}{2})+\frac{6\nu^2\mu^2\sin\sigma}{\sqrt{1+36\nu^2}}$$
$$+\frac{6\cos\sigma}{\sqrt{(1+36\nu^2)(37+36\nu^2)}}+\frac{\nu\mu\sigma^2}{\sqrt{(37+36\nu^2)}}).$$

in G_4.

Different projections from four-space to three-spaces of the hypersurface (29) for $\sigma=\pi/2$ *can be seen in the Fig.1:*

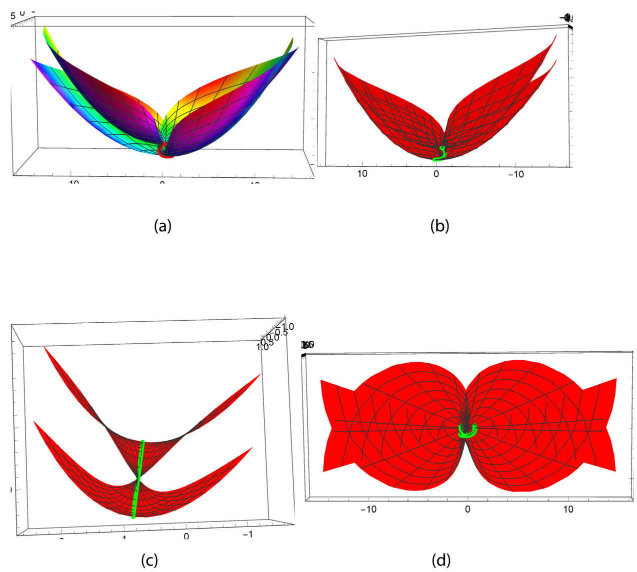

FIGURE 1. Projections of hypersurface family with parameter NB_2-Smarandache curve into $x_2x_3x_4$, $x_1x_3x_4$, $x_1x_2x_4$ and $x_1x_2x_3$-spaces in (a), (b), (c) and (d), respectively.

From now on, we'll give the parametric hypersurfaces given by different Smarandache curves of curve $\gamma(\nu)$ and their normal vector fields. Also, we'll state the theorems which give us the conditions for which $\gamma(\nu)$ is a geodesic curve where

Smarandache curves of the curve $\gamma(\nu)$ is isoparametric on these hypersurfaces. One can prove these theorems and investigate the conditions for different types of marching-scale functions with the similar methods given in the above case.

CASE 3.

Here, by taking the TB_1-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ which is given with the aid of the TB_1-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TB_1}(\nu,\mu,\sigma)=r_{TB_1}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right].$$

If TB_1-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TB_1}(\nu,\mu_0,\sigma_0)&=\left(\eta_{TB_1}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TB_1}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\&\quad+\left(\eta_{TB_1}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TB_1}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}\left(\eta_{TB_1}\right)_1(\nu,\mu_0,\sigma_0)&=0,\\ \left(\eta_{TB_1}\right)_2(\nu,\mu_0,\sigma_0)&=\kappa_3\left((x_\sigma)_0\left(z_\mu\right)_0-\left(z_\sigma\right)_0\left(x_\mu\right)_0\right),\\ \left(\eta_{TB_1}\right)_3(\nu,\mu_0,\sigma_0)&=(\kappa_1-\kappa_2)\left(\left(m_\mu\right)_0(x_\sigma)_0-\left(x_\mu\right)_0\left(m_\sigma\right)_0\right)\\&\quad+\kappa_3\left(\left(x_\mu\right)_0\left(y_\sigma\right)_0-\left(y_\mu\right)_0(x_\sigma)_0\right),\\ \left(\eta_{TB_1}\right)_4(\nu,\mu_0,\sigma_0)&=(\kappa_1-\kappa_2)\left(\left(x_\mu\right)_0\left(z_\sigma\right)_0-\left(z_\mu\right)_0(x_\sigma)_0\right).\end{aligned}$$

Thus,

Theorem 4. *$\gamma(\nu)$ is a geodesic curve where TB_1-Smarandache curve r_{TB_1} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \kappa_1=\kappa_2,\ \ \kappa_3\neq 0,\\ \left(y_\mu\right)_0(x_\sigma)_0=\left(x_\mu\right)_0\left(y_\sigma\right)_0,\ \ \left(z_\mu\right)_0(x_\sigma)_0\neq\left(x_\mu\right)_0\left(z_\sigma\right)_0 \end{array}\right. \tag{30}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 4.

Here, by taking the TB_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TB_2}(\nu,\mu,\sigma)$ which is given with the aid of the TB_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TB_2}(\nu,\mu,\sigma)=r_{TB_2}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right].$$

If TB_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TB_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TB_2}(\nu,\mu_0,\sigma_0) &= \left(\eta_{TB_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TB_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\ &\quad+\left(\eta_{TB_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TB_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}&\left(\eta_{TB_2}\right)_1(\nu,\mu_0,\sigma_0)=0,\\ &\left(\eta_{TB_2}\right)_2(\nu,\mu_0,\sigma_0)=\kappa_3\left((x_\sigma)_0(m_\mu)_0-(m_\sigma)_0(x_\mu)_0\right),\\ &\left(\eta_{TB_2}\right)_3(\nu,\mu_0,\sigma_0)=\kappa_1\left((m_\mu)_0(x_\sigma)_0-(x_\mu)_0(m_\sigma)_0\right),\\ &\left(\eta_{TB_2}\right)_4(\nu,\mu_0,\sigma_0)=\kappa_1\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)+\kappa_3\left((y_\sigma)_0(x_\mu)_0-(x_\sigma)_0(y_\mu)_0\right).\end{aligned}$$

For the curve $\gamma(\nu)$ to be a geodesic where TB_2-Smarandache curve r_{TB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ in G_4, the following conditions must hold:

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \kappa_1=0,\ \kappa_3\neq 0,\\ (y_\mu)_0(x_\sigma)_0=(x_\mu)_0(y_\sigma)_0,\ \ (m_\mu)_0(x_\sigma)_0\neq(x_\mu)_0(m_\sigma)_0\end{array}\right., \qquad (31)$$

where $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$. But, from our assumption that $\kappa_1(v)\neq 0$, we have a contradiction. So, we have

Theorem 5. *$\gamma(\nu)$ is not a geodesic curve where TB_2-Smarandache curve r_{TB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_2}(\nu,\mu,\sigma)$ in G_4.*

CASE 5.

Here, by taking the NB_1-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ which is given with the aid of the NB_1-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{NB_1}(\nu,\mu,\sigma)=r_{NB_1}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right]. \qquad (32)$$

If NB_1-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{NB_1}(\nu,\mu_0,\sigma_0) &= \left(\eta_{NB_1}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\ &\quad+\left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}
&\left(\eta_{NB_1}\right)_1(\nu,\mu_0,\sigma_0)=0,\\
&\left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0)=\tfrac{\kappa_2}{\sqrt{2}}\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)\\
&\qquad\qquad-\tfrac{\kappa_3}{\sqrt{2}}\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right),\\
&\left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0)=\tfrac{\kappa_2}{\sqrt{2}}\left((m_\sigma)_0(x_\mu)_0-(x_\sigma)_0(m_\mu)_0\right)\\
&\qquad\qquad+\tfrac{\kappa_3}{\sqrt{2}}\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right),\\
&\left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0)=\tfrac{\kappa_2}{\sqrt{2}}\left((x_\sigma)_0(z_\mu)_0-(z_\sigma)_0(x_\mu)_0\right)\\
&\qquad\qquad+\tfrac{\kappa_2}{\sqrt{2}}\left((x_\sigma)_0(y_\mu)_0-(y_\sigma)_0(x_\mu)_0\right).
\end{aligned}\tag{33}$$

Hence,

Theorem 6. *$\gamma(\nu)$ is a geodesic curve where NB_1-Smarandache curve r_{NB_1} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0)\neq 0,\ \ \left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0)=\left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0)=0\end{cases}\tag{34}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 6.

Here, by taking the B_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{B_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the B_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{B_1B_2}(\nu,\mu,\sigma)=r_{B_1B_2}(\nu)+\begin{bmatrix} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{bmatrix}.$$

If B_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{B_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{B_1B_2}(\nu,\mu_0,\sigma_0)&=\left(\eta_{B_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{B_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\&\quad+\left(\eta_{B_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{B_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}
&\left(\eta_{B_1B_2}\right)_1(\nu,\mu_0,\sigma_0)=0,\\
&\left(\eta_{B_1B_2}\right)_2(\nu,\mu_0,\sigma_0)=\frac{\kappa_3}{\sqrt{2}}\left((x_\sigma)_0\left((m_\mu)_0+(z_\mu)_0\right)-(x_\mu)_0\left((m_\sigma)_0+(z_\sigma)_0\right)\right),\\
&\left(\eta_{B_1B_2}\right)_3(\nu,\mu_0,\sigma_0)=\frac{\kappa_2}{\sqrt{2}}\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)+\frac{\kappa_3}{\sqrt{2}}\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right),\\
&\left(\eta_{B_1B_2}\right)_4(\nu,\mu_0,\sigma_0)=\frac{\kappa_2}{\sqrt{2}}\left((z_\mu)_0(x_\sigma)_0-(x_\mu)_0(z_\sigma)_0\right)-\frac{\kappa_3}{\sqrt{2}}\left((y_\mu)_0(x_\sigma)_0-(x_\mu)_0(y_\sigma)_0\right).
\end{aligned}$$

So, we can state the following Theorem:

Theorem 7. *$\gamma(\nu)$ is a geodesic curve where B_1B_2-Smarandache curve $r_{B_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{B_1B_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0, \\ \kappa_3\neq 0, \\ (x_\mu)_0\left((m_\sigma)_0+(z_\sigma)_0\right)\neq (x_\sigma)_0\left((m_\mu)_0+(z_\mu)_0\right), \\ \left(\eta_{B_1B_2}\right)_3(\nu,\mu_0,\sigma_0)=\left(\eta_{B_1B_2}\right)_4(\nu,\mu_0,\sigma_0)=0 \end{array}\right. \tag{35}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 7.

Here, by taking the TNB_1-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TNB_1}(\nu,\mu,\sigma)$ which is given with the aid of the TNB_1-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TNB_1}(\nu,\mu,\sigma)=r_{TNB_1}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right].$$

If TNB_1-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TNB_1}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TNB_1}(\nu,\mu_0,\sigma_0)=&\left(\eta_{TNB_1}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TNB_1}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\ &+\left(\eta_{TNB_1}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TNB_1}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}\left(\eta_{TNB_1}\right)_1(\nu,\mu_0,\sigma_0)=&\,0,\\ \left(\eta_{TNB_1}\right)_2(\nu,\mu_0,\sigma_0)=&\,\kappa_3\left((z_\mu)_0(x_\sigma)_0-(x_\mu)_0(z_\sigma)_0\right)-\kappa_2\left((m_\mu)_0(x_\sigma)_0-(x_\mu)_0(m_\sigma)_0\right),\\ \left(\eta_{TNB_1}\right)_3(\nu,\mu_0,\sigma_0)=&\,(\kappa_2-\kappa_1)\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)\\ &+\kappa_3\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right),\\ \left(\eta_{TNB_1}\right)_4(\nu,\mu_0,\sigma_0)=&\,(\kappa_1-\kappa_2)\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)\\ &+\kappa_2\left((y_\mu)_0(x_\sigma)_0-(x_\mu)_0(y_\sigma)_0\right).\end{aligned}$$

Hence,

Theorem 8. *$\gamma(\nu)$ is a geodesic curve where TNB_1-Smarandache curve r_{TNB_1} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_1}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0, \\ \left(\eta_{TNB_1}\right)_2(\nu,\mu_0,\sigma_0)\neq 0,\ \ \left(\eta_{TNB_1}\right)_3(\nu,\mu_0,\sigma_0)=\left(\eta_{TNB_1}\right)_4(\nu,\mu_0,\sigma_0)=0 \end{array}\right. \tag{36}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 8.

Here, by taking the TNB_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ which is given with the aid of the TNB_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TNB_2}(\nu,\mu,\sigma)=r_{TNB_2}(\nu)+\begin{bmatrix} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{bmatrix}.$$

If TNB_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TNB_2}(\nu,\mu_0,\sigma_0)&=\left(\eta_{TNB_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TNB_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\&\quad+\left(\eta_{TNB_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TNB_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}&\left(\eta_{TNB_2}\right)_1(\nu,\mu_0,\sigma_0)=0,\\&\left(\eta_{TNB_2}\right)_2(\nu,\mu_0,\sigma_0)=(\kappa_2-\kappa_3)\left((x_\mu)_0\,(m_\sigma)_0-(m_\mu)_0\,(x_\sigma)_0\right),\\&\left(\eta_{TNB_2}\right)_3(\nu,\mu_0,\sigma_0)=-\kappa_1\left((x_\mu)_0\,(m_\sigma)_0-(m_\mu)_0\,(x_\sigma)_0\right)\\&\left(\eta_{TNB_2}\right)_4(\nu,\mu_0,\sigma_0)=\kappa_1\left((x_\mu)_0\,(z_\sigma)_0-(z_\mu)_0\,(x_\sigma)_0\right)\\&\qquad\qquad+(\kappa_3-\kappa_2)\left((x_\mu)_0\,(y_\sigma)_0-(y_\mu)_0\,(x_\sigma)_0\right).\end{aligned}$$

For the curve $\gamma(\nu)$ to be a geodesic where TNB_2-Smarandache curve r_{TNB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ in G_4, the following conditions must hold:

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \kappa_1=0,\ \kappa_2\neq\kappa_3,\\ (x_\mu)_0\,(y_\sigma)_0=(y_\mu)_0\,(x_\sigma)_0,\ (x_\mu)_0\,(m_\sigma)_0\neq(m_\mu)_0\,(x_\sigma)_0\end{array}\right.,\tag{37}$$

where $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$. Since $\kappa_1(v)\neq 0$, we have

Theorem 9. *$\gamma(\nu)$ is not a geodesic curve where TNB_2-Smarandache curve r_{TNB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ in G_4.*

CASE 9.

Here, by taking the TB_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TB_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the TB_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TB_1B_2}(\nu,\mu,\sigma)=r_{TB_1B_2}(\nu)+\begin{bmatrix} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{bmatrix}.$$

If TB_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TB_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\eta_{TB_1B_2}(\nu,\mu_0,\sigma_0) = \left(\eta_{TB_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{TB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) + \left(\eta_{TB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{TB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),$$

where

$$\begin{aligned}
\left(\eta_{TB_1B_2}\right)_1(\nu,\mu_0,\sigma_0) &= 0,\\
\left(\eta_{TB_1B_2}\right)_2(\nu,\mu_0,\sigma_0) &= \kappa_3\left((x_\sigma)_0((m_\mu)_0+(z_\mu)_0)-(x_\mu)_0((m_\sigma)_0+(z_\sigma)_0)\right),\\
\left(\eta_{TB_1B_2}\right)_3(\nu,\mu_0,\sigma_0) &= (\kappa_2-\kappa_1)\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)\\
&\quad+\kappa_3\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right),\\
\left(\eta_{TB_1B_2}\right)_4(\nu,\mu_0,\sigma_0) &= (\kappa_1-\kappa_2)\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)\\
&\quad+\kappa_3\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right).
\end{aligned}$$

Thus,

Theorem 10. *$\gamma(\nu)$ is a geodesic curve where TB_1B_2-Smarandache curve $r_{TB_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_1B_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases}
x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\
\kappa_3\neq 0,\\
\left(\eta_{TB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)=\left(\eta_{TB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)=0,\\
(x_\mu)_0((m_\sigma)_0+(z_\sigma)_0)\neq(x_\sigma)_0((m_\mu)_0+(z_\mu)_0)
\end{cases} \tag{38}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 10.

Here, by taking the NB_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{NB_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the NB_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{NB_1B_2}(\nu,\mu,\sigma)=r_{NB_1B_2}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right].$$

If NB_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{NB_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\eta_{NB_1B_2}(\nu,\mu_0,\sigma_0) = \left(\eta_{NB_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{NB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) + \left(\eta_{NB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{NB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),$$

where

$$\begin{aligned}
\left(\eta_{NB_1B_2}\right)_1(\nu,\mu_0,\sigma_0) &= 0,\\
\left(\eta_{NB_1B_2}\right)_2(\nu,\mu_0,\sigma_0) &= \left(\frac{\kappa_2-\kappa_3}{\sqrt{3}}\right)\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_3}{\sqrt{3}}\left((z_\mu)_0(x_\sigma)_0-(x_\mu)_0(z_\sigma)_0\right),\\
\left(\eta_{NB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)=&\frac{\kappa_2}{\sqrt{3}}\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)\\
&+\frac{\kappa_3}{\sqrt{3}}\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right),\\
\left(\eta_{NB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)=&\frac{\kappa_2}{\sqrt{3}}\left((z_\mu)_0(x_\sigma)_0-(x_\mu)_0(z_\sigma)_0\right)\\
&+(\frac{\kappa_3-\kappa_2}{\sqrt{3}})\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right).
\end{aligned}
$$

So,

Theorem 11. *$\gamma(\nu)$ is a geodesic curve where NB_1B_2-Smarandache curve $r_{NB_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_1B_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$
\left\{\begin{array}{l}
x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\
\left(\eta_{NB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)\neq 0,\ \ \left(\eta_{NB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)=\left(\eta_{NB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)=0
\end{array}\right. \tag{39}
$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 11.

Here, by taking the TNB_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TNB_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the TNB_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$
\varphi_{TNB_1B_2}(\nu,\mu,\sigma)=r_{TNB_1B_2}(\nu)+\left[\begin{array}{c}x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\+z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right].
$$

If TNB_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TNB_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$
\begin{aligned}
\eta_{TNB_1B_2}(\nu,\mu_0,\sigma_0)=&\left(\eta_{TNB_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TNB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\
&+\left(\eta_{TNB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TNB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),
\end{aligned}
$$

where

$$
\begin{aligned}
\left(\eta_{TNB_1B_2}\right)_1(\nu,\mu_0,\sigma_0)=&\,0,\\
\left(\eta_{TNB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)=&\,\kappa_2\left((m_\sigma)_0(x_\mu)_0-(x_\sigma)_0(m_\mu)_0\right)\\
&+\kappa_3\left(\left((x_\sigma)_0(m_\mu)_0-(m_\sigma)_0(x_\mu)_0\right)-\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)\right),\\
\left(\eta_{TNB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)=&\,(\kappa_2-\kappa_1)\left((x_\mu)_0(m_\sigma)_0-(m_\mu)_0(x_\sigma)_0\right)\\
&+\kappa_3\left((x_\mu)_0(y_\sigma)_0-(y_\mu)_0(x_\sigma)_0\right),\\
\left(\eta_{TNB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)=&\,(\kappa_1-\kappa_2)\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)
\end{aligned}
$$

$$+ (\kappa_3 - \kappa_2) \left((x_\mu)_0 (y_\sigma)_0 - (y_\mu)_0 (x_\sigma)_0 \right).$$

Finally, we get

Theorem 12. *$\gamma(\nu)$ is a geodesic curve where TNB_1B_2-Smarandache curve $r_{TNB_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_1B_2}(\nu, \mu, \sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu, \mu_0, \sigma_0) = y(\nu, \mu_0, \sigma_0) = z(\nu, \mu_0, \sigma_0) = m(\nu, \mu_0, \sigma_0) = 0, \\ (\eta_{TNB_1B_2})_2 (\nu, \mu_0, \sigma_0) \neq 0, \ \ (\eta_{TNB_1B_2})_3 (\nu, \mu_0, \sigma_0) = (\eta_{TNB_1B_2})_4 (\nu, \mu_0, \sigma_0) = 0 \end{cases} \tag{40}$$

are satisfied. Here, $\nu \in [L_1, L_2]$, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

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A NOTE ON THE GENERATING SETS FOR THE MAPPING CLASS GROUPS

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Abstract. In this short note, we obtain generating sets with two elements for the mapping class group of closed, oriented surfaces of genus three and four, containing elements of the lowest order known so far.

1. Introduction

Let Σ_g denote a closed, oriented surface of genus g. Let $\mathrm{MCG}(\Sigma_g)$ denote the group of isotopy classes of orientation preserving self homeomorphisms, which is called the mapping class group of the surface Σ_g.

Dehn, [1] proved that the mapping class group of a compact, closed surface Σ_g, $\mathrm{MCG}(\Sigma_g)$, is generated by twists, which are now called Dehn twists. Lickorish [6], unaware of Dehn's work, proved that the group is generated by 3g-1 Dehn twists about the non–separating curves that are given in Figure 1.

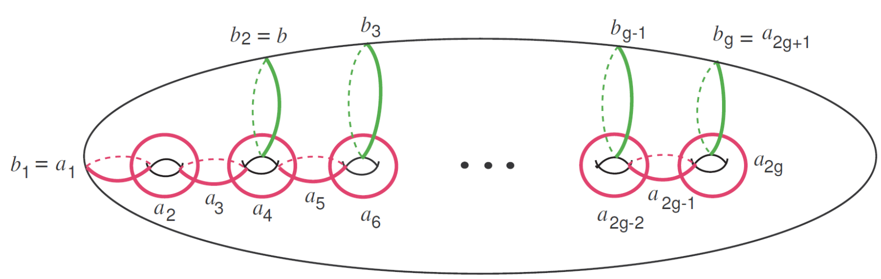

Figure 1. Lickorish generators

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Following these developments, Humphries [3] showed that, one can obtain the Dehn twists about the non–seperating curves $b_3, b_4, \ldots, b_g$, from the remaining collection of Lickorish generators, see Figure 1. Hence, Humphries proved that 2g+1 of Lickorish generators are sufficient to generate the MCG(Σ_g). These generators are the Dehn twists about the non–seperating curves of Figure 2. In the same paper he showed that this number is in fact minimal.

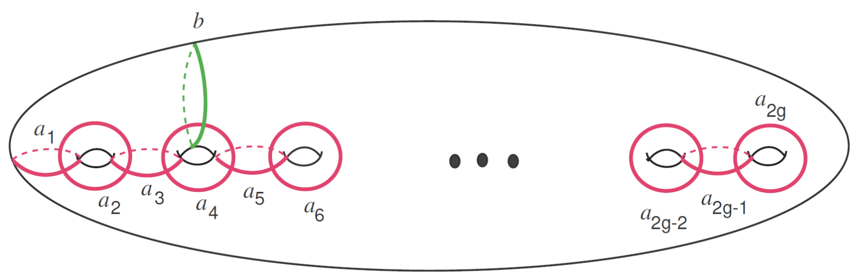

FIGURE 2. Humphries generators

These generators are of infinite order, hence the next question that comes to minds: Can we have generators of finite order? Let A_i and B_i denote Dehn twist about simple closed curves a_i and b_i given in Figure 3, respectively Wajnryb [7] proved that MCG (Σ_g) is generated by $\langle S, B_{g-1}B_g^{-1} \rangle$, where $S = A_{2g}A_{2g-1} \cdots A_2A_1$, and is of order $4g + 2$, see Figure 3.

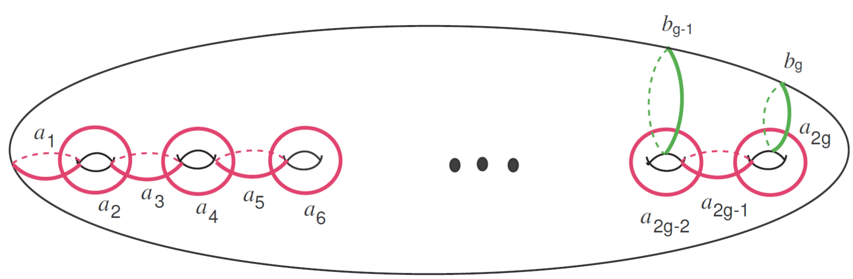

FIGURE 3. Wajnryb generators

Then Korkmaz improved this result, by the following theorem:

Theorem 1. *[4] Suppose that $g \geq 2$ and Σ_g is a closed oriented surface of genus g. The mapping class group* MCG (Σ_g) *of Σ_g is generated by S and B.*

In Theorem 1, Korkmaz proved that MCG(Σ_g) of a closed surface Σ_g is generated by two elements. One of the generators is B, Dehn twist about the curve b, hence it is of infinite order. The other generator is $S = A_{2g}A_{2g-1}\cdots A_2A_1$ and is of order $4g+2$, see Figure 2 for the curves $b, a_1, a_2, \ldots, a_{2g}$. In this note we lower this order using the homeomorphism $Q = A_{2g+1}A_{2g}\cdots A_2A_1$ for a genus 4 surface and $Q' = A_{2g+1}A_{2g}\cdots A_2A_1A_1$ for a genus 3 surface. In the following theorem we show that MCG(Σ_4) is generated by $\langle Q, B\rangle$ where $Q = A_9A_8\cdots A_2A_1$, and is of order 10 (that is $2g+2$ for $g=4$).

Theorem 2. *Mapping class group of a closed, oriented genus 4 surface is generated by Q, an element of order 10, and the Dehn twist B.*

Our second result gives even a lower order generator $Q' = A_7A_6\cdots A_2A_1A_1$, which is of order 7 (that is $2g+1$ for $g=3$), for the group of mapping classes of the closed genus 3 surface:

Theorem 3. *Mapping class group of a closed, oriented genus 3 surface is generated by Q', an element of order 7, and the Dehn twist B.*

Remark 4. *Even though we tried to generalize Theorem 2 and Theorem 3 for higher genus surfaces, it was not possible with the current technique. One needs other approaches to prove such generalizations.*

2. Preliminaries

In the next section, we will prove Theorem 2 and Theorem 3, using some basic properties of the group of mapping classes. In this section we will review the basic properties that we need, without giving their proofs. For the proofs see [2], [5].

Convention: In this note we consider simple closed curves and homeomorphisms up to isotopy and the usual composition of functions, meaning that, if there are several number of functions (Dehn twists) to be composed, we first apply the function on the right then continue from right to left. Throughout the paper, when we use an equality sign between the curves or homeomorphisms, we mean the equivalence up to isotopy.

Notation: We use lower case letters $(a_i, b, b_j, \ldots)$ for the simple closed curves, and capital letters $(A_i, B, B_j, \ldots)$ to denote the Dehn twists about these curves $(a_i, b, b_j, \ldots)$.

Relations:

(1) Let c, d be two simple closed curves on an oriented surface Σ_g and let H be an orientation preserving self homeomorphism of the surface such that $H(c) = d$. Then
$$HCH^{-1} = D.$$

(2) **Commutativity relation:** Let c, d be two disjoint simple closed curves on an oriented surface Σ_g, then the Dehn twists around the curves c and d commute:
$$CD = DC$$

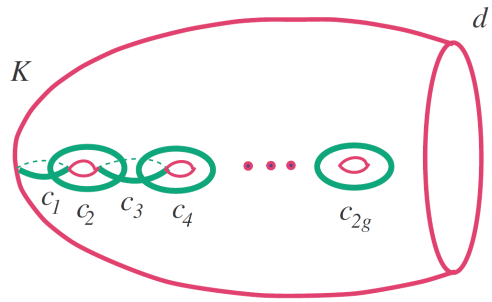

FIGURE 4. K, the neighborhood of union of even number of curves in the chain

(3) **Braid relation:** Let c, d be two simple closed curves, intersecting transversely at one point, on an oriented surface Σ_g, then the Dehn twists around the curves c and d satisfy the relation below:

$$CDC = DCD$$

(4) **Chain relation:**
First we define the chain, then we will give the chain relation.

Definition 5. Chain: Let $c_1, c_2, ..., c_n$ be a sequence of simple closed curves on an orientable surface. If only the consecutive ones intersect transversely at one point, and the others are disjoint, then this sequence of simple closed curves $c_1, c_2, ..., c_n$ is called a chain.

Let K be a tubular neighborhood of union of curves in the chain. There are two cases according to the parity of number of curves (n is even or odd) in the chain:

- If $n = 2g$, then K is an orientable genus g surface of one boundary component, call that boundary component d, see Figure 4.
- If $n = 2g+1$, then K is an orientable genus g surface of two boundary components, call them d_1 and d_2.

Then we have the following relations, which are called the chain relations in MCG(K):

- If $n = 2g$, then we have $(C_1C_2 \cdots C_{2g})^{4g+2} = D$
- If $n = 2g+1$, then we have $(C_1C_2 \cdots C_{2g+1})^{2g+2} = D_1D_2$.

3. PROOFS

We will start this section with the proof of Theorem 2. In Theorem 2, we show that MCG(Σ_4) $=< Q, B >$, where $Q = A_9A_8 \cdots A_2A_1$, see Figure 1 for the curves $a_1, a_2, \ldots, a_9$. Then we show that the order of the element Q is 10, in Lemma 8.

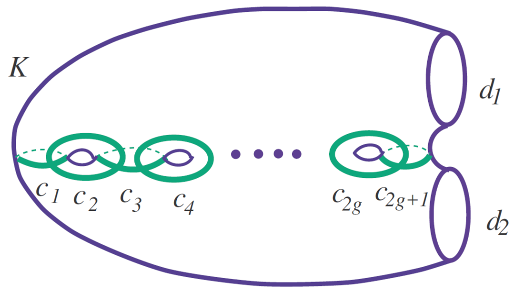

FIGURE 5. K, the neighborhood of union of odd number of curves in the chain

Hence, it is a lower order element than the one in Theorem 1, in which Korkmaz proved that $\mathrm{MCG}(\Sigma_4) =< S, B >$ and $S = A_8A_7 \cdots A_2A_1$. The element S is of order 14.

In order to prove these results we need some preparatory results. Let $G =< Q, B >$ be the subgroup of $\mathrm{MCG}(\Sigma_4)$ generated by Q and B. Let C_1, C_2, C_3, D_1, D_2, E_1, E_2, F_1 and B_3 be the Dehn twists around the curves c_1, c_2, c_3, d_1, d_2, e_1, e_2, f_1 and b_3, respectively, which are shown in Figure 6.

Lemma 6. *The Dehn twists C_1, C_2, C_3, D_1, D_2, E_1, E_2, F_1 and B_3 are contained in $G =< Q, B >$.*

Proof of Lemma 6. We will show that we can get the Dehn twists C_1, C_2, C_3, D_1, D_2, E_1, E_2, F_1, B_3 from the homeomorphisms Q and B by applying the Relation (1) several times. First, applying the homeomorphism Q^{-1} to the simple closed curve b, we get the simple closed curve c_1. Hence, we have

- $Q^{-1}(b) = c_1$ and by Relation (1), we can write $C_1 = Q^{-1}BQ$. Since Q and B are already in the group G, we conclude that $C_1 \in G$.

Similarly, we repeat this process:

- $Q^{-1}(c_1) = d_1$, using Relation (1) and since C_1, $Q \in G$ we have $D_1 \in G$.
- $Q^{-1}(d_1) = c_2$ which implies $C_2 \in G$.
- $Q^{-1}(c_2) = d_2$ which implies $D_2 \in G$.
- $Q^{-1}(d_2) = c_3$ which implies $C_3 \in G$.
- $Q^{-1}(c_3) = b_3$ which implies $B_3 \in G$.
- $Q^{-1}(b_3) = e_1$ which implies $E_1 \in G$.
- $Q^{-1}(e_1) = f_1$ which implies $F_1 \in G$.
- $Q^{-1}(f_1) = e_2$ which implies $E_2 \in G$.

□

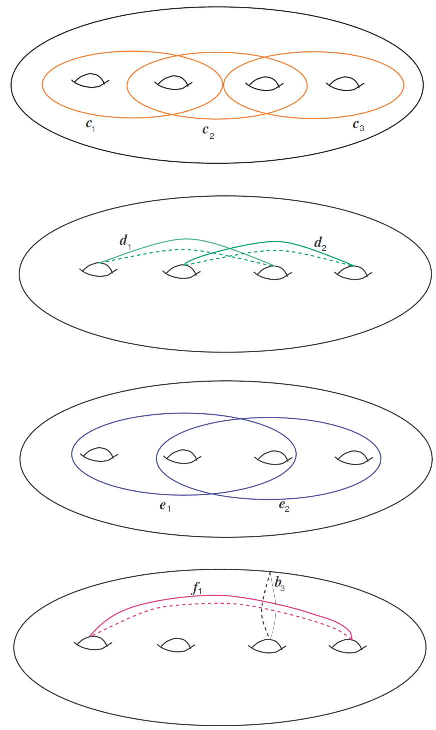

FIGURE 6. $c_1, c_2, c_3, d_1, d_2, e_1, e_2, f_1, b_3$, curves on a genus 4 surface

The main idea of the proof of Theorem 2 is to show that the G-orbit of the curve b contains the simple closed curves $a_1, a_2, \dots, a_8$. As a result of this, all the Humphries generators $A_1, A_2, \dots, A_8$ and B of $\mathrm{MCG}(\Sigma_4)$ are contained in G, hence we conclude that $\mathrm{MCG}(\Sigma_4) = G$.

Proof of Theorem 2. Let h be the self homeomorphism of the surface Σ_4 which is given by the product of the Dehn twists: $C_2 D_1 B B_3^{-1}$. We will apply the homeomorphism h to the curve c_2, and obtain the non-separating simple closed curve a_3, which is shown in detail in Figure 7. Using Relation (1) and Lemma 6, we can say that the Dehn twist A_3 is in G. Then applying the homeomorphism Q^{-1} to the curve a_3, we get:

- $Q^{-1}(a_3) = a_4$, using Relation (1) we say that $Q^{-1}A_3Q = A_4$ and since $Q,\ A_3 \in G$ we have $A_4 \in G$. Using repeatedly this process, we get the following:
- $Q^{-1}(a_4) = a_5$ which implies that $A_5 \in G$.
- $Q^{-1}(a_5) = a_6$ which implies that $A_6 \in G$.

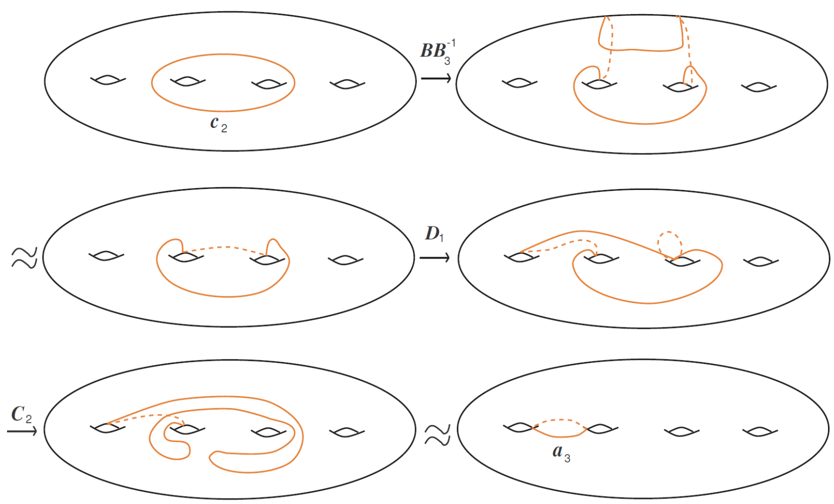

FIGURE 7. $h(c_2) = a_3$

- $Q^{-1}(a_6) = a_7$ which implies that $A_7 \in G$.
- $Q^{-1}(a_7) = a_8$ which implies that $A_8 \in G$.

Similarly applying the homeomorphism Q and Q^2 to the curve a_3, we get a_2 and a_1:

- $Q(a_3) = a_2$ which implies that $A_2 \in G$.
- $Q^2(a_3) = a_1$ which implies that $A_1 \in G$.

Therefore, we get the result that all Humphries generators of $\mathrm{MCG}(\Sigma_4)$ are contained in G, and hence, conclude that $G = \mathrm{MCG}(\Sigma_4)$. □

Now we are going to prove Theorem 3, which implies that $\mathrm{MCG}(\Sigma_3) = \langle Q', B\rangle$, where $Q' = A_7A_6A_5A_4A_3A_2A_1A_1$. The idea in the proof of Theorem 3 is similar to the one in the proof of Theorem 2. In order to prove Theorem 3, we need the following lemma. For the simple closed curves that we use in Lemma 7, see Figure 8.

Lemma 7. *The Dehn twists C_1, C_2, D_1 are in the group $G' = \langle Q', B\rangle$.*

Proof of Lemma 7. We start with the simple closed curve b, and apply the homeomorphism Q'. We get $Q'(b) = c_2$ and by Relation (1) we have, $C_2 = Q'BQ'^{-1}$. Since Q' and B are in the group G', we deduce that $C_2 \in G'$.

Similarly,

$Q'(c_2) = d_1$ which implies that $D_1 \in G'$.

$Q'(d_1) = c_1$ which implies that $C_1 \in G'$.

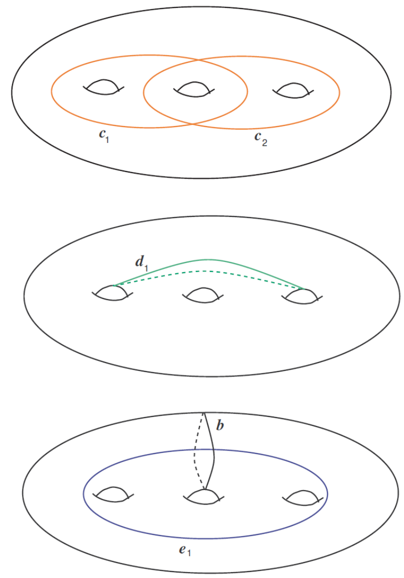

FIGURE 8. $c_1,\ c_2,\ d_1,\ e_1,\ b$ curves on a genus 3 surface

□

Proof of Theorem 3. Let h be the self-homeomorphism of the genus 3 surface to itself, given by $h = C_1^{-1}(Q')^{-1}$. We apply the homeomorphism h to the simple closed curve b, and see that $h(b) = a_1$, as in Figure 9. Using Relation (1), we get that $A_1 \in \langle Q', B\rangle = G'$. Note that for a genus 3 surface, the homeomorphism Q is given by $Q = A_7A_6A_5A_4A_3A_2A_1$ and $Q = Q'A_1^{-1}$, hence, $Q \in G'$.

- $Q^{-1}(a_1) = a_2$, using Relation (1) we say that $Q^{-1}A_1Q = A_2$ and since Q, $A_1 \in G'$ we have $A_2 \in G'$. Upon repeating this process we get the following results:
- $Q^{-1}(a_2) = a_3$ which implies that $A_3 \in G'$.
- $Q^{-1}(a_3) = a_4$ which implies that $A_4 \in G'$.
- $Q^{-1}(a_4) = a_5$ which implies that $A_5 \in G'$.
- $Q^{-1}(a_5) = a_6$ which implies that $A_6 \in G'$.

Therefore all the Humphries generators of $\mathrm{MCG}(\Sigma_3)$, are contained in G', hence, we conclude that $\mathrm{MCG}(\Sigma_3) = G'$.

□

Lemma 8. *The order of the element* $Q = A_9A_8\cdots A_2A_1$ *is* 10.

Proof of Lemma 8. The curves defining Q form a chain on a closed genus 4-surface Σ_4. Then using the chain relation we get, $Q^{10} = id$. Hence the order of Q is at

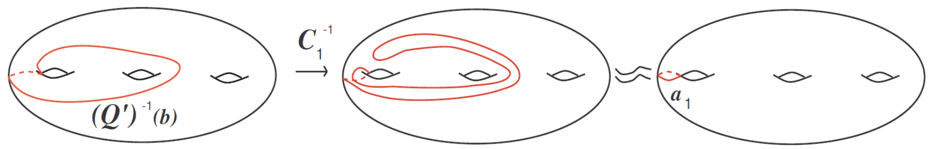

FIGURE 9. $h(b) = a_1$

most 10. Now, take the simple closed curve e_2 on Σ_4, see Figure 6. From the Proof of Lemma 6, we observe that $Q^i(e_2) \neq e_2$ for $1 \leq i \leq 9$. Therefore the order of Q is 10. $\square$

Lemma 9. *The order of the element $Q' = A_7A_6 \cdots A_2A_1A_1$ is 7.*

Proof of Lemma 9. Using commutativity and braid relation, we have $(Q')^7 = Q^8$, where $Q = A_7A_6A_5A_4A_3A_2A_1$ on a closed genus 3-surface Σ_3:

$$\begin{aligned}
(Q')^7 &= A_7A_6A_5A_4A_3A_2A_1A_1(Q')^6 \\
&= (Q)A_1(Q')^6 \\
&= (Q)A_1(A_7A_6A_5A_4A_3A_2A_1A_1)(Q')^5 \\
&= (Q)^2A_2A_1(Q')^5 \\
&= (Q)^3A_3A_2A_1(Q')^4 \\
&= (Q)^4A_4A_3A_2A_1(Q')^3 \\
&= (Q)^5A_5A_4A_3A_2A_1(Q')^2 \\
&= (Q)^6A_6A_5A_4A_3A_2A_1(Q') \\
&= (Q)^7A_7A_6A_5A_4A_3A_2A_1 \\
&= (Q)^8
\end{aligned}$$

Moreover since Q is a chain on a closed Σ_3 surface, from the chain relation we have, $Q^8 = id$, which implies that, $(Q')^7 = Q^8 = id$, hence, the order of Q' is at most 7.

On the other hand, one can easily check that $(Q')^i(a_7) \neq a_7$ for $1 \leq i \leq 6$ which is shown in the Figure 10. Therefore the order of Q' is 7. $\square$

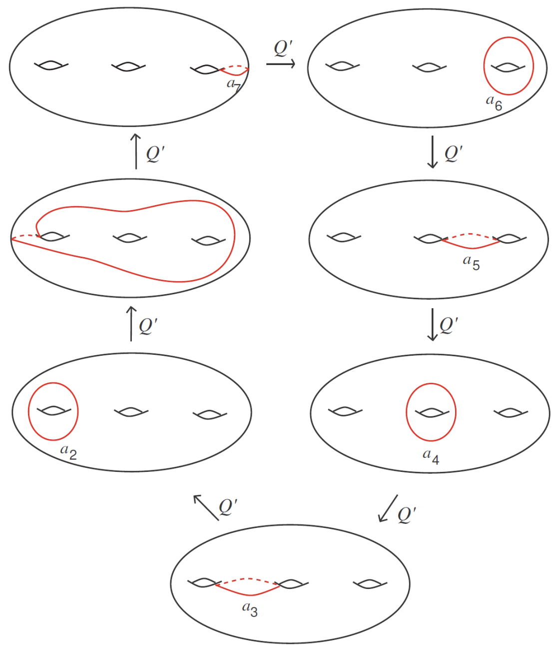

FIGURE 10. $(Q')^i\,(a_7) \neq a_7$ for $1 \leq i \leq 6$.

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SOME PROPERTIES OF CONVOLUTION IN SYMMETRIC SPACES AND APPROXIMATE IDENTITY

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Abstract. This paper deals with the symmetric space of functions and its subspace where continuous functions are dense is considered. Main properties of convolution which plays a vital role in harmonic analysis, as in other areas of mathematics are established in this space. Following the classical case, it is proved that the convolution can be approximated by linear combinations of shifts in a subspace of the considered space. An approximate identity for the convolution is also considered in that subspace.

1. Introduction

Convolution operation plays a vital role in harmonic analysis, as in other areas of mathematics. This is mainly due to the fact that many key operators like Hilbert transform, Poisson integrals, Dirichlet integrals, different types of potentials including Riesz potential, singular integrals, etc are expressed in terms of convolution. Involved in the above operators, convolution operation plays a key role also in approximation theory. Therefore, to have knowledge of basic properties of convolution in various Banach function spaces is very important and useful in the study of the problems of harmonic analysis, approximation theory, theory of partial differential equations, etc.

Recent years have seen an increased interest in different function spaces, such as Lebesgue spaces with variable summability index, Orlicz spaces, Morrey spaces, grand-Lebesgue spaces, etc. Some problems of harmonic analysis and approximation theory have been considered in [1-14]. Basicity of the classical exponential system, as well as its perturbations in the subspaces of Morrey space of functions defined on $[-\pi, \pi]$ was investigated in [6, 11, 13, 15] by the method of boundary

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value problems for analytic functions on a complex domain. In [12, 16, 23], an analogue of the classical Young inequality and some properties of the convolution of periodic functions belonging to Morrey type spaces have been obtained. In [12], it was proved that the convolution in the subspace of Morrey space can be approximated by finite linear combinations of shifts. In the same work, the validity of classical facts about approximate identities was also proved in Morrey space. Note that the spaces considered in the above works are all Banach function spaces (see, e.g., [17;18]). Moreover, all of them, except for Lebesgue spaces with variable summability index, are symmetric. Therefore, a question naturally arises: do the similar results hold for symmetric spaces? Some analogues of Young inequality for some symmetric spaces have been obtained in [20-22].

In this work, we consider a symmetric space of functions and its subspace where continuous functions are dense. We establish main properties of convolution in this space. We prove that the convolution can be approximated by the linear combinations of shifts in a subspace of this space. Approximate identity is also considered in that subspace.

2. Needful information

We will use the following standard notations and concepts. $R_+ = (0, +\infty)$; $\chi_M(\cdot)$ is the characteristic function of the set M; R is the set of real numbers; C is the complex plane; $\omega = \{z \in C : |z| < 1\}$ is a unit disk in C; $\gamma = \partial\omega$ is a unit circle; $\bar{M}$ is the closure of the set M with respect to appropriate norm; $(\bar{\cdot})$ is the complex conjugate. By $[X]$ we denote the algebra of linear bounded operators acting in a Banach space X.

We will need some concepts and facts from the theory of Banach function spaces (see e.g. [24;25]). Let $(R; \mu)$ be a measure space, and M^+ be the cone of μ-measurable functions on R whose values lie in $[0, +\infty]$. Denote the characteristic function of a μ-measurable subset E of R by χ_E .

Definition 1. *A mapping $\rho : M^+ \to [0, +\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N in\ M^+$, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset R$, the following properties hold:*

(P1)$\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f \mu$-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f \mu$-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;

(P5) $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, for some constant $C_E : 0 < C_E < +\infty$ depending on E and ρ, but independent of f .

Let M denote the collection of all extended scalar-valued (real or complex) μ-measurable functions and $M_0 \subset M$ denote the subclass of functions that are finite μ-a.e.

Definition 2. *Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in M for which $\rho(|f|) < +\infty$ is called a Banach function space. For each $f \in X$, define $\|f\|_X = \rho(|f|)$.*

The following theorem is true.

Theorem 3. *Let ρ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as above. Then under the natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$M_s \subset X \subset M_0$$

hold, where M_s is the set of μ-simple functions. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges pointwise μ-a.e. to f.

Let

$$\rho'(g) = \sup\left\{\int_\gamma f(\tau)\, g(\tau)\, |dt| : f \in M^+; \rho(f) \le 1\right\}, \forall g \in M^+.$$

A space

$$X' = \{g \in M : \rho'(|g|) < +\infty\},$$

is called an associate space (Kothe dual) of X.

The functions $f; g \in M_0$ are called equimeasurable if

$$|\{\tau \in \gamma : |f(\tau)| > \lambda\}| = |\{\tau \in \gamma : |g(\tau)| > \lambda\}|, \forall \lambda \ge 0.$$

Banach function norm $\rho : M^+ \to [0, \infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in M_0^+$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentz-Orlicz spaces are r.i.s.

Definition 4. *Let X be a Banach function space. The closure of the set of simple functions M_s in X is denoted by X_b.*

To obtain our main results, we will significantly use the following fact from the monograph [17, p.13].

Recall that a closed linear subspace B of the dual space X^* of a Banach space X is said to be norm-fundamental if

$$\|f\|_X = \sup\{|L(f)| : L \in B \wedge \|L\|_{X^*} \le 1\}$$

for every $f \in X$. Thus, B is norm-fundamental if it contains sufficiently many functionals to reproduce the norm of every element of X. The following theorem is true.

Theorem 5. *The associate space X' of a Banach function space X is canonically isometrically isomorphic to a closed norm-fundamental subspace of the Banach space X^* of X, i.e.*

$$\|f\|_X = \sup_{g\in X'} \left\{ \left| \int_{-\pi}^{\pi} fg dt \right| : \|g\|_{X'} \leq 1 \right\}, \forall f \in X.$$

In the sequel, we will assume that all the considered functions are defined on $[-\pi, \pi]$ and periodically continued to the whole real axis. By T_s we will denote the shift operator, i.e. $(T_s f)(x) = f(x+s)$, $\forall s; x \in (-\pi, \pi]$.

We will also use the following lemma of [17, p.157].

Lemma 6. *Let X be a r.i.s. on γ and X_b be the closure of simple functions in X. The following assertions are equivalent:*

(1) *X_b is the closure of continuous functions;*
(2) *translation is continuous in X_b, that is*

$$\lim_{s\to 0} \|T_s f - f\|_X = 0, \forall f \in X_b;$$

(3) *X_b is the closure in X of trigonometric polynomials.*

3. Main results

3.1. **Convolution.** Let X be a Banach function space with the norm $\|\cdot\|_X$ invariant with respect to shift on $[-\pi, \pi]$ (we will assume that the functions from X and X' are periodically continued to the whole axis R with period 2π). We will call such space a norm-invariant space for short. Let $f \in X$ and $g \in X'$. Consider the convolution

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy, x \in [-\pi, \pi].$$

As $X \subset L_1$ and $X' \subset L_1$, the existence of the convolution $(f * g)(x)$ a.e. $x \in [-\pi, \pi]$ is beyond any doubt. Applying Hölder's inequality, we obtain

$$|(f * g)(x)| \leq \|f(x-\cdot)\|_X \|g\|_{X'} = \|f\|_X \|g\|_{X'}, \quad \text{a.e. } x \in (-\pi, \pi).$$

Consequently,

$$\|f * g\|_\infty \leq \|f\|_X \|g\|_{X'}. \tag{1}$$

Let T_δ be a shift operator, i.e. $(T_\delta f)(x) = f(\delta + x)$, $x \in [-\pi, \pi]$. It is not difficult to see that

$$T_\delta (f * g)(x) = (f * g)(x + \delta) = \int_{-\pi}^{\pi} f(x + \delta - t)\, g(t) dt = (T_\delta f * g)(x).$$

In view of the periodicity of the functions f and g, we also have

$$T_\delta (f * g)(x) = \int_{-\pi}^{\pi} f(x + \delta - t)\, g(t)\, dt = |t - \delta = \tau| = \int_{-\pi-\delta}^{\pi-\delta} f(x - \tau)\, g(\delta + \tau)\, d\tau =$$

$$= \int_{-\pi}^{\pi} f(x-\tau)(T_\delta g)(\tau)\, d\tau = (f * T_\delta g)(x).$$

Denote by X_s (X'_s) the subspace of functions from X (from X') whose shifts are continuous in X (in X'). Applying inequality (1), we obtain

$$\|T_\delta (f*g) - f*g\|_\infty = \|T_\delta f * g - f*g\|_\infty = \|(T_\delta f - f)*g\| \le \|T_\delta f - f\|_X \|g\|_{X'}.$$

Similarly we have

$$\|T_\delta (f*g) - f*g\|_\infty \le \|f\|_X \|T_\delta g - g\|_{X'}.$$

These relations directly imply the validity of the following theorem.

Theorem 7. *Let X be a norm-invariant Banach function space. Then*

$$\|f*g\|_\infty \le \|f\|_X \|g\|_{X'}, \forall f \in X, \forall g \in X'.$$

Moreover, the convolution operation $f*g$ is continuous in L_∞ if either $f \in X_s$ or $g \in X'_s$.

Let X be a norm-invariant Banach function space and $f : [-\pi,\pi] \to R$ be some simple function, i.e. let $[-\pi,\pi] = \bigcup_{k=1}^{r} E_k$ be some division of segment $[-\pi,\pi]$ and $f(x) = c_k$, $\forall x \in E_k$, $k = \overline{1,r}$. Take an arbitrary function $g \in X_b$ and consider

$$(f*g)(x) = \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy = \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy =$$

$$= \sum_{k=1}^{r} c_k \int_{E_k} g(x-y)\, dy, \forall x \in [-\pi,\pi].$$

Consequently,

$$\|f*g\|_X \le \sum_{k=1}^{r} |c_k| \left\| \int_{E_k} g(x-y)\, dy \right\|_X \le \sum_{k=1}^{r} |c_k| \int_{E_k} \|g(\cdot - y)\|_X\, dy =$$

$$= \sum_{k=1}^{r} |c_k| \|g\|_X \int_{E_k} 1dy = \|f\|_{L_1} \|g\|_X, \forall f \in S[-\pi,\pi],$$

where $S[-\pi,\pi]$ is a set of all simple functions on $[-\pi,\pi]$. So the following inequality holds:

$$\|f*g\|_X \le \|f\|_{L_1} \|g\|_X, \forall f \in S[-\pi,\pi]. \tag{2}$$

Let $f \in L_1(-\pi,\pi)$ be an arbitrary function. Consider

$$\forall \{f_n\}_{n\in N} \subset S[-\pi,\pi] : \|f_n - f\|_{L_1} \to 0, n \to \infty.$$

Since $S[-\pi,\pi]$ is dense in $L_1(-\pi,\pi)$, the choice of such a sequence is always possible. Then it follows directly from the inequality (2) that the sequence $\{f_n * g\}_{n\in N}$ is fundamental in X. Assume

$$(f*g)_1 = \lim_{n\to\infty} f_n * g.$$

By virtue of inequality (2), the definition of $(f*g)_1$ does not depend on the choice of the sequence $\{f_n\}_{n\in N} \subset S[-\pi,\pi]$. On the other hand, it is clear that the sequence $\{f_n*g\}_{n\in N}$ converges to $(f*g)_1$ also in $L_1(-\pi,\pi)$. As $f;g\in L_1(-\pi,\pi)$, by classical facts (see, e.g., [19]), there exists a convolution $f*g$ and, moreover, $f_n*g\to f*g$, $n\to\infty$, in $L_1(-\pi,\pi)$. Then it is clear that $(f*g)_1(x)=(f*g)(x)$ a.e. $x\in[-\pi,\pi]$. So the following theorem is true.

Theorem 8. *Let X be a norm-invariant Banach function space and $f\in L_1(-\pi,\pi)\wedge g\in X$ be arbitrary functions. Then $f*g\in X$ and the following inequality holds:*

$$\|f*g\|_X \le \|f\|_{L_1}\|g\|_X\,,\forall f\in L_1(-\pi,\pi)\,,\forall g\in X. \tag{3}$$

Denote by M the space of measures on $[-\pi,\pi]$, i.e. M contains a distribution $F\in D$ (D is a space of distributions on $[-\pi,\pi]$) satisfying the inequality

$$|F(u)|\le c\|u\|_\infty\,,\forall u\in C_0^\infty,$$

where C_0^∞ are infinitely differentiable functions with compact support on $T=(-\pi,\pi)$. Such measures are called Radon measures. It is known (see Riesz-Markov-Kakutani theorem for compact space) that every functional (distribution) can be represented as an integral with respect to the unique regular Borel measure μ on T:

$$F(u)=\mu(u)=\int_T u(x)\,d\mu(x)\ .$$

M is a Banach space with respect to the norm

$$\|\mu\|_1=\sup\{|\mu(u)|:u\in C[-\pi,\pi]\,,\|u\|_\infty\le 1\}\,.$$

For more details on these facts we refer the reader to [19].

Let $\mu\in M$ and $f,g\in C[-\pi,\pi]$ be arbitrary functions. Then, as shown in the monograph [19] (see p. 93), we have the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu*f)\,gdx=\mu\left(\breve{f}*g\right),$$

where $\breve{f}(t)=f(-t)$. It directly follows

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu*f)\,gdx\right|\le\|\mu\|_1\|f*g\|_\infty\,.$$

Applying Theorem 7, we obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu*f)\,gdx\right|\le\|\mu\|_1\|f\|_X\|g\|_{X'}\,. \tag{4}$$

Passing to the limit, we see that the inequality (4) holds for $\forall f\in X_b$ and $\forall g\in(X')_b$. So the following lemma is true.

Lemma 9. *Let X be a norm-invariant Banach function space and $\mu \in M$ be a Radon measure. Then the following inequality holds*

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mu * f)\, g dx\right| \leq \|\mu\|_1 \|f\|_X \|g\|_{X'}, \forall f \in X_b, \forall g \in (X')_b. \tag{5}$$

Now let's assume that X has an absolutely continuous norm. Then, as is known (see, e.g., [17], Theorem 4.1, p. 20), $X = X_b$ and $X' = X^*$ (X^* is a conjugate space). Lemma 9 and the inequality (5) imply that $\mu * f \in X$ and

$$\|\mu * f\|_X \leq \|\mu\|_1 \|f\|_X. \tag{6}$$

So the following theorem is true.

Theorem 10. *Let X be a Banach function space with absolutely continuous and invariant norm. Then for $\forall \mu \in M$ and $\forall f \in X$ the relation $\mu * f \in X$ and the inequality* (6) *hold.*

In the sequel, we will need some direct corollaries of Theorem 8. Let all conditions of this theorem hold. Then from the inequality (3) we obtain

$$\|f * g\|_X \leq C \|f\|_X \|g\|_X, \forall f \in X_b, \forall g \in X, \tag{7}$$

and

$$\|f * g\|_X \leq C \|f\|_{(X')_b} \|g\|_X, \forall f \in (X')_b, \forall g \in X, \tag{8}$$

where C is a constant depending only on X. As $L_1(-\pi,\pi)$ is dense in X_b (in $(X')_b$), these inequalities follow from (3) by passage to the limit. So the following statement is true.

Proposition 11. *Let X be a norm-invariant Banach function space. Then for $\forall f \in X_b$ (or $\forall f \in (X')_b$) and $\forall g \in X : f * g \in X$ and the inequalities* (7), (8) *hold.*

A question naturally arises: does Proposition 11 hold for $\forall f \in X$? It is absolutely clear that $\forall f; g \in X$ the convolution $f * g$ is defined like an element of the space $L_1(-\pi,\pi)$. Let $S' = \{\vartheta \in X' : \|\vartheta\|_{X'} \leq 1\}$. Then, by Theorem 5, we have

$$\begin{aligned}\|f * g\|_X &= \sup_{\vartheta \in S'}\left|\int_{-\pi}^{\pi}(f * g)(x)\,\vartheta(x)\,dx\right| = \sup_{\vartheta \in S'}\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-t)\,g(t)\,\vartheta(x)\,dtdx\right| = \\ &= \sup_{\vartheta \in S'}\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-t)\,\vartheta(x)\,dxg(t)\,dt\right| \leq \int_{-\pi}^{\pi}\sup_{\vartheta \in S'}\left|\int_{-\pi}^{\pi} f(x-t)\,\vartheta(x)\,dx\right| |g(t)|\,dt = \\ &= \int_{-\pi}^{\pi}\|f(\cdot - t)\|_X |g(t)|\,dt = \|f\|_X \int_{-\pi}^{\pi}|g(t)|\,dt = \|f\|_X \|g\|_{L_1}.\end{aligned}$$

So the following lemma is true.

Lemma 12. *Let X be a norm-invariant Banach function space. Then for $\forall f; g \in X$: $f * g$ belongs to X and*

$$\|f * g\|_X \leq \|f\|_X \|g\|_{L_1}, \forall f; g \in X,$$

The following main theorem follows direcly from this lemma.

Theorem 13. *Let X be a norm-invariant Banach function space. Then for $\forall f; g \in X$: $f * g \in X$ and*

$$\|f * g\|_X \leq C \|f\|_X \|g\|_X, \forall f; g \in X,$$

where C is a constant independent of f and g.

3.2. **Approximation of convolution by shifts.** Let's prove the theorem below following the classical case.

Theorem 14. *Let X be a r.i.s. with Boyd indices $\alpha_X; \beta_X \in (0,1)$. Let $f \in L_1(-\pi,\pi)$ and $g \in E$, where E denotes any one of the spaces $C[-\pi,\pi]$ or X_b. Then the convolution $f * g$ in E can be approximated by finite linear combinations of shifts g, i.e. $\forall \varepsilon > 0$, $\exists \{a_k\}_1^n \subset [-\pi,\pi] \wedge \{\lambda_k\}_1^n \subset R$:*

$$\left\| f * g - \sum_{k=1}^{n} \lambda_k T_{a_k} g \right\|_E < \varepsilon.$$

Proof. The case of $E = C[-\pi,\pi]$ is known (see, e.g.,[19]). Consider the case of $E = X_b$. Following the classical scheme, as a subset S_0, such that the finite linear combinations of elements from S_0 are dense in L_1, we take a set of functions f, each of which coincides on $[-\pi,\pi]$ with the characteristic function of some interval $M = [a,b]$, $-\pi < a < b < \pi$, and continues further on periodically.

Let $\forall \varepsilon > 0$ be arbitrary. Let's divide M into a finite number of subintervals I_k of length $|I_k| < \delta$. Take $\forall a_k \in I_k$. Let $f(x) = \chi_M(x)$. We have

$$(f * g)(x) - \sum_k |I_k| g(x - a_k) = \int_{\bigcup_k I_k} g(x-y)\,dy -$$

$$-\sum_k \int_{I_k} g(x-a_k)\,dy = \sum_k \int_{I_k} [g(x-y) - g(x-a_k)]\,dy = \sum_k h_k(x),$$

where

$$h_k(x) = \int_{I_k} [g(x-y) - g(x-a_k)]\,dy.$$

Consequently

$$\left\| (f * g)(\cdot) - \sum_k |I_k| g(\cdot - a_k) \right\|_X \leq \sum_k \|h_k\|_X.$$

We have

$$\|h_k\|_X = \sup_{\vartheta \in S^1} \left| \int_{-\pi}^{\pi} h_k(t)\vartheta(t)\,dt \right| = \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{I_k} [g(t-x) - g(t-a_k)]\,dx\vartheta(t)\,dt \right| =$$

$$= \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [g(t-x) - g(t-a_k)]\chi_{I_k}(x)\vartheta(t)\,dxdt \right| =$$

$$= \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[g\left(t - x\right) - g\left(t - a_k\right) \right] \vartheta\left(t\right) dt \chi_{I_k}\left(x\right) dx \right| \leq$$

$$\leq \int_{-\pi}^{\pi} \sup_{\vartheta \in S'} \left| \int_{-\pi}^{\pi} \left[g\left(t - x\right) - g\left(t - a_k\right) \right] \vartheta\left(t\right) dt \chi_{I_k}\left(x\right) dx \right| =$$

$$= \int_{I_k} \left\| g\left(\cdot - x\right) - g\left(\cdot - a_k\right) \right\|_X dx.$$

So the following relation is valid

$$\left\| h_k \right\|_X \leq \int_{I_k} \left\| g\left(\cdot - x\right) - g\left(\cdot - a_k\right) \right\|_X dx. \tag{9}$$

In the sequel, we will assume that X is a r.i.s. with Boyd indies α_X; $\beta_X \in (0,1)$. Then it follows from Corollary 6.11 of [17] (see p. 165) that trigonometric polynomials are dense in X_b, and hence, by Lemma 6, the shifts are continuus in X_b. Therefore, for $\forall \varepsilon > 0$, $\exists \delta > 0$:

$$\left\| T_x g - T_{a_k} g \right\|_X < \varepsilon, \forall x \in I_k.$$

Considering this relation in (9), we obtain

$$\left\| h_k \right\|_X \leq \left| I_k \right| \varepsilon,$$

and hence

$$\left\| \left(f * g\right)\left(\cdot\right) - \sum_k \left| I_k \right| g\left(\cdot - a_k\right) \right\|_X \leq \left| I \right| \varepsilon \leq 2\pi\varepsilon.$$

Since $\sum_k \left| I_k \right| T_{a_k} g$ is a finite linear combination of shifts g, it is clear that $f * g \in \bar{V}_g$, where $\bar{V}_g$ is a closed linear subspace in E, generated by shifts $T_a g$ of the function g. Let $f \in L_1\left(-\pi, \pi\right)$ be an arbitrary element. Then for $\forall \varepsilon > 0$ there exist a partition of $[-\pi, \pi]$ into a finite number of intervals M_k and a number λ_k such that the inequality

$$\left\| f\left(\cdot\right) - \sum_k \lambda_k \chi_{M_k}\left(\cdot\right) \right\|_{L_1} < \varepsilon$$

holds. It follows directly from the previous result that $F * g \in \bar{V}_g$, where $F\left(\cdot\right) = \sum_k \lambda_k \chi_{M_k}\left(\cdot\right)$. Then there exists a finite linear combination of shifts $\sum_n \mu_n T_{a_n} g$ such that

$$\left\| F * g - \sum_n \mu_n T_{a_n} g \right\|_X < \varepsilon.$$

By Lemma 12, we obtain

$$\left\| f * g - \sum_n \mu_n T_{a_n} g \right\|_X \leq \left\| f * g - F * g \right\|_X + \left\| F * g - \sum_n \mu_n T_{a_n} g \right\|_X \leq$$

$$\leq \varepsilon + \left\| f - F \right\|_{L_1} \left\| g \right\|_X \leq \varepsilon \left(1 + \left\| g \right\|_X\right).$$

The arbitrariness of $\varepsilon > 0$ implies $f * g \in \bar{V}_g$. The theorem is proved. □

3.3. **Approximate identity.** Let's consider the approximate identities for convolutions in the space X_b. By the approximate identity (for convolution) we mean $\left\{K_n^{(\cdot)}\right\}_{n\in N} \subset L_1(-\pi,\pi)$ such that

$\alpha)\sup\limits_n \|K_n\|_{L_1} < +\infty$;

$\beta)\lim\limits_n \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$;

$\gamma)\lim\limits_{n\to\infty}\int_{\delta\le|x|\le\pi} |K_n(x)\,dx| = 0\ ,\ \ \forall\delta\in(0,\pi)\,.$

The following theorem is true.

Theorem 15. *Let X be a r.i.s. with Boyd indices $\alpha_X;\beta_X \in (0,1)$ and $\{K_n\}_{n\in N}$ be an approximate identity. Then*

$$\lim_{n\to\infty}\|K_n * f - f\|_X = 0, \forall f \in X_b.$$

Proof. Take $\forall f \in X_b$. Let $\varepsilon > 0$ be an arbitrary number. It is clear that $\exists g \in C[-\pi,\pi]$:

$$\|f-g\|_X < \varepsilon.$$

We have

$$\|K_n * f - K_n * g\|_X \le \|K_n\|_{L_1}\|f-g\|_X \le M\varepsilon,$$

where $M = \sup\limits_n \|K_n\|_{L_1}$. As is known (see, e.g., [19]),

$$\lim_{n\to\infty}\|K_n * g - g\|_\infty = 0.$$

Then $\exists n_0 \in N$:

$$\|K_n * g - g\|_\infty < \varepsilon, \forall n \ge n_0.$$

We have

$$\|K_n * g - g\|_X \le \varepsilon\|1\|_X = C\varepsilon, \forall n \ge n_0.$$

Hence

$$\|K_n * f - f\|_X \le \|K_n * f - K_n * g\|_X + \|K_n * g - g\|_X + \\ + \|g-f\|_X \le (M+C+1)\,\varepsilon, \forall n \ge n_0.$$

The theorem is proved. □

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AN EXCEEDANCE MODEL BASED ON BIVARIATE ORDER STATISTICS

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Abstract. In hydrologic risk analysis, the use of exceedance statistics are very important. In this sense, we construct a random threshold model based on bivariate order statistics. The exact distribution of exceedance statistics is calculated under some well-known copulas such as independent and Farlie-Gumbel-Morgenstern (FGM) copulas. Furthermore, numerical results are provided for expected value and variance of exceedance statistics under independent and Farlie-Gumbel-Morgenstern copulas. The application of the model in hydrology is also discussed.

1. Introduction

Exceedance statistics and random threshold models are very useful tools in real life applications. There have been many studies about the applications of exceedances in different areas such as hydrology, actuarial sciences and medicine, see [13, 15, 12] and [19], respectively.

Eryilmaz [11], construct a random threshold model by using univariate order statistics. The distribution of the longest run statistics are derived. Then the use of the model in hydrology is discussed. For univariate random threshold models we refer to [6, 2, 21, 17, 18] and [4].

Theoretical properties and application areas of bivariate random threshold models have been discussed in many publications. In [10], marginal distribution and joint distribution of the new sample rank of rth order statistics and its concomitant are obtained. The application of the model in hydorology is discussed based on numerical results. Bayramoglu and Giner [5], construct a random threshold

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model based on order statistics from independent but not necessarily identically distributed (INID) random variables. Asymptotic distributions of exceedance statistic is derived based on hypergeometric function and incomplete beta functions. Bayramoglu and Eryilmaz [4], compose a random threshold model based on two sets of exchangeable random vectors. The reliability applications of the model are discussed under the FGM distribution. In [7] and [9] bivariate random threshold models are composed based on concomitants of order statistics. Then the exact and asymptotic distribution of exceedance statistics are obtained. Applications in medicine, economics and air pollution are discussed. In [8], a statistical test is introduced for checking the equality of two copulas based on a bivariate random threshold model.

In hydrological analysis, if the flood peak and flood volume exceed critical values within a certain period, they create a risky situation. Therefore, the use of exceedance statistics and random thresholds in calculating these risk probabilities is quite important. In this study, a bivariate random threshold model based on bivariate order statistics is considered. Here we have a training sample which consists of bivariate random variables that represent flood peak and flood volume of n hydrological stations in a certain location, in the past year. We also have a bivariate control sample which consists of bivariate random variables that represent flood peak and flood volume of m hydrological stations in the same location, in the coming year. Then by using the minimum flood peak and minimum flood volume in training sample, the random threshold model is constructed. The use of the model in hydrological risk analysis is also discussed.

This paper is organized as follows: In section 2, the problem statement is provided. Then the exact distribution of exceedance statistics are obtained in terms of copula functions. Expected value and variance of exceedance statistics are provided as numerically for independent and Farlie-Gumbel -Morgenstern copulas. Lastly, Section 3 concludes the paper.

2. Model Description

Let $T_1 = \{(X_k, Y_k), k = 1, 2, ..., n\}$ be a sequence of independent random variables with joint cumulative distribution function (CDF) $F(x, y) = C_1(F_X(x), F_Y(y))$, where $C_1(u, v)$, $(u, v) \in [0, 1]^2$ is a connecting copula and $F_X(x), F_Y(y)$ are the marginal CDF' s of X and Y, respectively. Furthermore, let $T_2 = \{(X_k^{'}, Y_k^{'}), k = 1, 2, ..., m\}$ be another sequence of independent random variables with joint CDF $G(x, y) = C_2(F_X(x), F_Y(y))$, where $C_2(u, v)$, $(u, v) \in [0, 1]^2$ is a connecting copula and $F_X(x), F_Y(y)$ are the marginal CDF' s of X and Y, respectively. Let $f(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$, $g(x, y) = \frac{\partial^2 G(x,y)}{\partial x \partial y}$, $f_X(x) = \frac{dF_X(x)}{dx}$, $f_Y(y) = \frac{dF_Y(y)}{dy}$, $\overline{F}_X(x) = 1 - F_X(x)$, and $\overline{F}_Y(y) = 1 - F_Y(y)$. Here X_k and Y_k denote flood peak and flood volume of n stations in past for a certain location, respectively. Furthermore, $X_k^{'}$

and $Y_k^{'}$ denote flood peak and flood volume of the future m stations in the same location, respectively. Here we call T_1 as training sample and T_2 as control sample.

We define the rth bivariate order statistics of T_1 as $(X_{r:n}, Y_{r:n})$, where $1 \leq r \leq n$, $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ and $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ are the order statistics of $\{X_k, k = 1, 2, ..., n\}$ and $\{Y_k, k = 1, 2, ..., n\}$, respectively. For $r = 1$, $(X_{1:n}, Y_{1:n})$ denotes the smallest flood peak and flood volume in the past, respectively. Then the exceedance statistic $M_m\left(1\right)$ is defined as follows

$$M_m\left(1\right) = \sum_{k=1}^{m} \delta_k,$$

where

$$\delta_k = \begin{cases} 1, & \text{if } \left(X_k^{'}, Y_k^{'}\right) \in A_1 \\ 0, & \text{otherwise} \end{cases}$$

and $A_1 = \left(-\infty, X_{1:n}\right] \times \left(-\infty, Y_{1:n}\right]$. The set A_1 is constructed from training sample T_1.

Here $M_m\left(1\right)$ denotes the number of nonhazardous stations in the future observations. For example, if $M_m\left(1\right) = 4$ it means that there can be 4 nonhazardous stations in the future observations.

In Corollary 1, the probability mass function (PMF) of $M_m\left(1\right)$ is given by using the distribution of bivariate order statistics, see [3] and [14]. For $1 \leq r, s \leq n$, the joint probability density function (PDF) of $X_{r:n}$ and $Y_{s:n}$ is

$$\begin{aligned} f_{X_{r:n},Y_{s:n}}(t,s) = {} & \sum_{t_1=a_1}^{a_2} p_1 \left[F\left(t,s\right)\right]^{t_1} \left[\left(F_X\left(t\right) - F\left(t,s\right)\right)\right]^{r-1-t_1} \left[F_Y\left(s\right) - F\left(t,s\right)\right]^{s-1-t_1} \\ & \times \left[\overline{F}\left(t,s\right)\right]^{n-r-s+t_1+1} f\left(t,s\right) + \sum_{t_4=d_1}^{d_2} \sum_{t_2=c_1}^{c_2} \sum_{t_1=b_1}^{b_2} p_2 \left[F\left(t,s\right)\right]^{t_1} \\ & \times \left[\left(F_X\left(t\right) - F\left(t,s\right)\right)\right]^{r-1-t_1-t_2} \left[\left(F_Y\left(s\right) - F\left(t,s\right)\right)\right]^{s-1-t_1-t_4} \\ & \times \left[\overline{F}\left(t,s\right)\right]^{n-r-s+t_1+t_2+t_4} \left[F^{\cdot,1}\left(t,s\right)\right]^{t_2} \left[f_Y\left(s\right) - F^{\cdot,1}\left(t,s\right)\right]^{1-t_2} \\ & \times \left[F^{1,\cdot}\left(t,s\right)\right]^{t_4} \left[f_X\left(t\right) - F^{1,\cdot}\left(t,s\right)\right]^{1-t_4}, \end{aligned} \tag{1}$$

where $a_1 = max(0, r+s-n-1)$, $a_2 = min(r-1, s-1)$, $b_1 = max(0, r+s-n-t2-t4)$, $b_2 = min(r-t2-1, s-t4-1)$, $c_1 = max(0, r-n+1)$, $c_2 = min(1, r-1)$, $d_1 = max(0, s-n+1)$, $d_2 = min(1, s-1)$

$$\begin{aligned} \overline{F}\left(t,s\right) &= 1 - F_X(t) - F_Y(s) + F(t,s) \\ F^{1,\cdot}\left(t,s\right) &= \frac{\partial F\left(t,s\right)}{\partial t} \\ F^{\cdot,1}\left(t,s\right) &= \frac{\partial F\left(t,s\right)}{\partial s} \end{aligned}$$

and the constants p_1 and p_2 are

$$p_1 = \frac{n!}{t_1!\,(r-1-t_1)!\,(s-1-t_1)!\,(n-r-s+t_1-1)!},$$

$$p_2 = \frac{n!}{t_1!\,(r-1-t_1-t_2)!\,(s-1-t_1-t_4)!\,(n-r-s+t_1+t_2+t_4)!}.$$

Corollary 1. *The PMF of $M_k(1)$ is*

$$P\{M_m(1)=l\} = \binom{m}{l} G(x,y)^l (1-G(x,y))^{m-l} f_{X_{1:n},Y_{1:n}}(x,y)\,dxdy \qquad (2)$$

where $f_{X_{1:n},Y_{1:n}}(x,y)$ is the PDF of $X_{1:n}$ and $Y_{1:n}$ in training sample T_1.

Then by using the formula of bivariate order statistics, Equation (2) can be written as follows

$$\begin{aligned} P\{M_m(1)=l\} &= \binom{m}{l}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y)^l (1-G(x,y))^{m-l}\left\{n\left(\overline{F}(x,y)\right)^{n-1}\right. \\ &\quad \times f(x,y) + n(n-1)\left(\overline{F}(x,y)\right)^{n-2}(f_Y(y) \\ &\quad \left.-F^{\cdot,1}(x,y))\left(f_X(x)-F^{1,\cdot}(x,y)\right)\right\}dxdy \qquad (3) \end{aligned}$$

Proof. The proof of Corollary 1 is similar to proof of Theorem 1, in [7].

$P\{M_m(1)=l\} = P\{$l of the sample values in T_2 are in $(-\infty, X_{1:n}]\times(-\infty, Y_{1:n}]\}$

Define the events E_{i_j} and $E_{i_j}^c$ as follows

$E_{i_j} = \{X_{i_j} < X_{1:n}, Y_{i_j} < Y_{1:n}\}$ and $E_{i_j}^c = \{X_{i_j} < X_{1:n}, Y_{i_j} > Y_{1:n}\} \cup \{X_{i_j} > X_{1:n}, Y_{i_j} < Y_{1:n}\} \cup \{X_{i_j} > X_{1:n}, Y_{i_j} > Y_{1:n}\}$, $1 \le i,j \le m$. Then

$$P\{M_m(1)=l\} = \sum_{i_1,i_2,\ldots,i_m} P(E_{i_1}E_{i_2}\ldots E_{i_l}E_{i_{l+1}}^c\ldots E_{i_m}^c) \qquad (4)$$

By conditioning of the integral on $X=x$ and $Y=y$ in Equation (4) and using the distribution of bivariate order statistics, the proof is completed. □

When we apply the probability integral transformation $F(t)=u$, $F(s)=v$ and $F(t,s) = C_1\left(F^{-1}(t), F^{-1}(s)\right)$ and $G(t,s) = C_2\left(F^{-1}(t), F^{-1}(s)\right)$ in Equation (3), we have

$$\begin{aligned} P\{M_m(1)=l\} &= \binom{m}{l}\int_0^1\int_0^1 C_2(u,v)^l (1-C_2(u,v))^{m-l} \\ &\quad \times\{n(\widehat{C}_1(1-u,1-v))^{n-1}c_1(u,v) \\ &\quad + n(n-1)\left(\widehat{C}_1(1-u,1-v)\right)^{n-2} \\ &\quad \times(1-C_1^{\cdot}(u,v))(1-C_1^{\cdot\cdot}(u,v))\}dudv, \qquad (5) \end{aligned}$$

where

$$\widehat{C}_1(1-u,1-v) = 1-u-v+C_1(u,v)$$

$$C_1^{\cdot\cdot}(u,v) = \frac{\partial C_1(u,v)}{\partial u}$$
$$C_1^{\cdot}(u,v) = \frac{\partial C_1(u,v)}{\partial v}$$
$$c_1(u,v) = \frac{\partial^2 C_1(u,v)}{\partial u \partial v}.$$

Let $C_1(u,v) = C_2(u,v)$, then

$$\begin{aligned} P\{M_m(1) = l\} &= \binom{m}{l} \int_0^1 \int_0^1 C_1(u,v)^l (1 - C_1(u,v))^{m-l} \\ &\times \{n \left(\widehat{C}_1(1-u,1-v)\right)^{n-1} c_1(u,v) \\ &+ n(n-1)\left(\widehat{C}_1(1-u,1-v)\right)^{n-2} \\ &\times (1 - C_1^{\cdot}(u,v))(1 - C_1^{\cdot\cdot}(u,v))\} dudv. \end{aligned} \tag{6}$$

For the ease of the calculations we firstly consider the case $C_1(u,v) = C_2(u,v) = uv$. Then we have

$$\begin{aligned} P\{M_m(1) = l\} = \binom{m}{l} \int_0^1 \int_0^1 (uv)^l (1-uv)^{m-l} \Big\{ n(1-u-v+uv)^{n-1} \\ + n(n-1)(1-u-v+uv)^{n-2}(1-u)(1-v) \Big\} dudv, \end{aligned} \tag{7}$$

In Table 1, the numerical values of $P\{M_m(1) = l\}$ are provided for $C_1(u,v) = C_2(u,v) = uv$ by using Equations (6) and (7).

TABLE 1. Numerical values of $P\{M_m(1) = l\}$ for different values of n and $m = 5$.

(m,n)	l	0	1	2	3	4	5
$(5,5)$	$P\{M_m(1) = l\}$	0.88	0.1	0.015	0.0021	0.00024	0.000016
$(5,10)$	$P\{M_m(1) = l\}$	0.96	0.037	0.002	0.0001	4.4×10^{-6}	1.1×10^{-6}
$(5,20)$	$P\{M_m(1) = l\}$	0.989	0.0110	0.000178	3.01×10^{-6}	4.25×10^{-8}	3.54×10^{-10}

We can interpret Table 1, as follows. For example if there have been 10 stations in a certain region in the past, after a couple of years at the same location we can observe 5 stations. So under $C_1(u,v) = C_2(u,v) = uv$, the probability of observing 1 nonhazardous station in the coming years is 0.037. In other words, the probability that only 1 station will be less than the minimum flood peak and minimum flood volume observed in the past year is 0.037, in the coming years.

In Table 2, the numerical values of $P\{M_m(1) = l\}$ are provided under $C_1(u,v) = C_2(u,v) = uv$ for $m = 10$ and some values of n. Similar to Table 1, we can do the same interpretations with Table 2. When $m = 10$ and $n = 5$, probability of observing 0 nonhazardous stations in the coming years is 0.796.

In Tables 1 and 2, it can be easily seen that while n increases, $P\{M_m(1)=l\}$ also increases for fixed values of m and l. It is clear that as l increases $P\{M_m(1)=l\}$ decreases for fixed values of m and n. Furthermore, while m increases, $P\{M_m(1)=l\}$ decreases for fixed values of n and l.

Table 2. Numerical values of $P\{M_m(1)=l\}$ for different values of n and $m=10$.

(m,n)	l	0	1	2	3	4	5
$(10,5)$	$P\{M_m(1)=l\}$	0.796	0.151	0.038	0.011	0.00305913	0.00085
$(10,10)$	$P\{M_m(1)=l\}$	0.926	0.0657	0.00695	0.000853	0.000109	0.0000138
$(10,20)$	$P\{M_m(1)=l\}$	0.978	0.0211	0.000739	0.0000316	1.47×10^{-6}	6.84×10^{-8}
(m,n)	l	6	7	8	9	10	
$(10,5)$	$P\{M_m(1)=l\}$	0.00022	0.000051	9.70281×10^{-6}	1.38612×10^{-6}	1.10889×10^{-7}	
$(10,10)$	$P\{M_m(1)=l\}$	1.62×10^{-6}	1.68×10^{-7}	1.43×10^{-8}	8.79×10^{-10}	2.93×10^{-11}	
$(10,20)$	$P\{M_m(1)=l\}$	3.02×10^{-9}	1.18×10^{-10}	3.81×10^{-12}	8.86×10^{-14}	1.11×10^{-15}	

The FGM copula is highly preferred in applications due to its closed form structure that facilitates theoretical calculations. In addition, it has become one of the preferred distributions in applications in the field of hydrology, since it includes both negative and positive dependency structure, see [20], [1], and [16]. For this reason, some numerical results in this paper have been calculated under the FGM copula.

Let $C_1(u,v)=uv$ and $C_2(u,v)=uv\left(1+\theta\left(1-u\right)\left(1-v\right)\right),\ \theta\in[-1,1]$, then

$$\begin{aligned} P\{M_m(1)=l\} = \binom{m}{l}\int_0^1\int_0^1 & \left[uv\left(1+\theta\left(1-u\right)\left(1-v\right)\right)\right]^l \\ & \times\left[1-uv\left(1+\theta\left(1-u\right)\left(1-v\right)\right)\right]^{m-l} \\ & \times\{n\left(1-u-v+uv\right)^{n-1} \\ & +n\left(n-1\right)\left(1-u-v+uv\right)^{n-2}\left(1-u\right)\left(1-v\right)\}dudv, \end{aligned} \tag{8}$$

In Table 3, the numerical values of $P\{M_m(1)=l\}$ is provided for $C_1(u,v)=uv$ and $C_2(u,v)=uv\left(1+\theta\left(1-u\right)\left(1-v\right)\right),\ \theta\in[-1,1]$ by using equation (5).

In Table 3, similar to Tables 1 and 2, as n increases $P\{M_m(1)=l\}$ also increases for fixed values of θ,m and l. For $l=0$, fixed values of m and n when the dependence parameter θ increases $P\{M_m(1)=l\}$ decreases. But for $l=1,...,5$ and fixed values of m and n, $P\{M_m(1)=l\}$ increases. As in Tables 1 and 2, while l increases, $P\{M_m(1)=l\}$ decreases for fixed values of m,n and θ.

In Table 4, the numerical values of $P\{M_m(1)=l\}$ is provided for $C_1(u,v)=C_2(u,v)=uv\left(1+\theta\left(1-u\right)\left(1-v\right)\right),\ \theta\in[-1,1]$ by using Equation (6). In Table 4, similar to Table 3 when $l=0$ and θ increases $P\{M_m(1)=0\}$ decreases for fixed values of m and n. But for $l=1,...,5$ and fixed values of m and n, $P\{M_m(1)=l\}$ increases. In Tables 5-7, the expected values of $M_m(1)$ are calculated by using Equations (5) and (6) under

$$C_1(u,v)=C_2(u,v)=uv, \tag{9}$$

TABLE 3. Numerical values of $P\{M_m(1) = l\}$ for different values of n and $m = 5$.

(m,n)	θ/l	0	1	2	3	4	5
$(5,5)$	-1	0.940	0.0533	0.00604	0.000765	0.0000836	5.61×10^{-6}
	-0.5	0.909563	0.0790261	0.00996047	0.00129879	0.000141936	9.3751×10^{-6}
	0.5	0.853735	0.122215	0.0204795	0.00316417	0.000380553	0.000026234
	1	0.828	0.140	0.0267	0.00456	0.000591	0.0000429
$(5,10)$	-1	0.988	0.0118	0.000370	0.0000146	5.31×10^{-7}	1.21×10^{-8}
	-0.5	0.974123	0.0248275	0.00100337	0.0000443511	1.68777×10^{-6}	3.89844×10^{-8}
	0.5	0.947961	0.0486292	0.00319862	0.000200988	9.89698×10^{-6}	2.75831×10^{-7}
	1	0.935	0.0595	0.00469	0.000345	0.0000195	6.10×10^{-7}
$(5,20)$	-1	0.998	0.00194	0.0000122	1.10×10^{-7}	10^{-9}	6.08×10^{-12}
	-0.5	0.993419	0.00650981	0.0000701828	8.60826×10^{-7}	9.39504×10^{-9}	6.37741×10^{-11}
	0.5	0.984326	0.0153334	0.000333565	7.29272×10^{-6}	1.28804×10^{-7}	1.30955×10^{-9}
	1	0.980	0.0196	0.000534	0.0000144	3.07×10^{-7}	3.73×10^{-9}

TABLE 4. Numerical values of $P\{M_m(1) = l\}$ for different values of n and $m = 5$.

(m,n)	θ/l	0	1	2	3	4	5
$(5,5)$	-1	0.946472	0.0487	0.00435425	0.000407848	0.0000315723	1.44141×10^{-6}
	-0.5	0.913422	0.076709	0.00876889	0.00100174	0.0000937063	5.20157×10^{-6}
	0.5	0.849052	0.124131	0.0224053	0.00384628	0.000524333	0.0000415192
	1	0.818045	0.143125	0.0312117	0.00646731	0.00105374	0.0000981558
$(5,10)$	-1	0.988971	0.0107548	0.000266524	2.01062×10^{-7}	2.01062×10^{-7}	3.14295×10^{-9}
	-0.5	0.975021	0.0240525	0.000890853	0.0000348152	1.14229×10^{-6}	2.22888×10^{-8}
	0.5	0.946488	0.04975	0.00349145	0.000258435	0.0000154613	3.1657×10^{-7}
	1	0.931939	0.061969	0.00556312	0.000492862	0.0000348765	1.40955×10^{-6}

$$C_1(u,v) = uv, C_2(u,v) = uv\left(1 + \theta\left(1-u\right)\left(1-v\right)\right), \tag{10}$$

and

$$C_1(u,v) = C_2(u,v) = uv\left(1 + \theta\left(1-u\right)\left(1-v\right)\right), \tag{11}$$

respectively. In Table 5, we can interpret $E\left(M_m(1)\right)$ as follows. When $m = n = 5$ (The number of stations is not changed in a certain location), expected number of nonhazardous stations is 0.139.

TABLE 5. Expected values of $M_m(1)$ for $C_1(u,v) = C_2(u,v) = uv$

m	n	$E\left(M_m(1)\right)$	m	n	$E\left(M_m(1)\right)$
5	5	0.139	5	10	0.0413
10	5	0.278	10	10	0.0826
20	5	0.556	20	10	0.165
50	5	1.39	50	10	0.413

In Table 5, we can clearly see that when m increases $E\left(M_m(1)\right)$ also increases for fixed values of n. For fixed values of m, as n increases $E\left(M_m(1)\right)$ decreases.

From Table 6, we can easily observe that as θ increases, $E(M_m(1))$ also increases for fixed values of m and n. As n increases $E(M_m(1))$ also increases for fixed values of m and θ. From Table 7, we can see that for fixed values of n and θ, when m increases $E(M_m(1))$ also increases. For fixed values of m and θ, as n increases $E(M_m(1))$ decreases. Furthermore similar to Table 7, as θ increases $E(M_m(1))$ also increases for fixed values of m and n. In Tables 8-10, the variance of $M_m(1)$

TABLE 6. Expected value of $M_m(1)$ for $C_1(u,v) = uv$, $C_2(u,v) = uv(1+\theta(1-u)(1-v))$

θ	m	n	$E(M_m(1))$	θ	m	n	$E(M_m(1))$
	5	5	0.0680		5	5	0.210
	10	5	0.0126		10	5	0.0700
-1	5	10	0.136	1	5	10	0.420
	10	10	0.0253		10	10	0.140
	20	20	0.00787		20	20	0.0828
	5	5	0.103458		5	5	0.17432
-0.5	5	10	0.206916	0.5	5	10	0.348639
	10	5	0.0269743		10	5	0.0556703
	10	10	0.0539486		10	10	0.111357

TABLE 7. Expected value of $M_m(1)$ for $C_1(u,v) = C_2(u,v) = uv(1+\theta(1-u)(1-v))$

θ	m	n	$E(M_m(1))$	θ	m	n	$E(M_m(1))$
	5	5	0.0587987		5	5	0.229656
	10	5	0.117597		10	5	0.459311
-1	5	10	0.0113121	1	5	10	0.0747203
	5	5	0.0976528		5	5	0.182785
-0.5	10	5	0.195306	0.5	10	5	0.365559
	5	10	0.0259433		5	10	0.0575716

are calculated by using Equations (5) and (6) under

$$C_1(u,v) = C_2(u,v) = uv, \tag{12}$$

$$C_1(u,v) = uv, C_2(u,v) = uv(1+\theta(1-u)(1-v)), \tag{13}$$

and

$$C_1(u,v) = C_2(u,v) = uv(1+\theta(1-u)(1-v)), \tag{14}$$

respectively.

In Table 8, it is obvious that for fixed values of n, when m increases variance of $M_m(1)$ also increases. For fixed values of m, as n increases the variance of $M_m(1)$

decreases. Similary, in Tables 9 and 10, for fixed values of θ and n, as m increases variance of $M_m(1)$ increases. Furthermore for fixed values of m and n, when θ increases, the variance of $M_m(1)$ increases.

TABLE 8. Variances of $M_m(1)$ for $C_1(u,v) = C_2(u,v) = uv$

m	n	$V(M_m(1))$	m	n	$V(M_m(1))$
5	5	0.165	5	10	0.044
10	5	0.404	10	10	0.097
20	5	1.1	20	10	0.226

TABLE 9. Variances of $M_m(1)$ for $C_1(u,v) = uv$, $C_2(u,v) = uv\,(1+\theta\,(1-u)\,(1-v))$

θ	m	n	$V\,(M_m(1))$	θ	m	n	$V\,(M_m(1))$
	5	5	0.0812		5	5	0.254
	10	5	0.198		10	5	0.642
-1	5	10	0.133	1	5	10	0.0768
	10	10	0.0284		10	10	0.173
	5	5	0.121		5	5	0.201
-0.5	5	10	0.0286	0.5	5	10	0.0603
	10	5	0.297		10	5	0.52
	10	10	0.061		10	10	0.134

TABLE 10. Variances of $M_m(1)$ for $C_1(u,v) = C_2(u,v) = uv\,(1+\theta\,(1-u)\,(1-v))$

θ	m	n	$V\,(M_m(1))$	θ	m	n	$V\,(M_m(1))$
	5	5	0.0669047		5	5	0.292749
	5	10	0.0117666		5	10	0.0836673
-1	10	5	0.1558	1	10	5	0.769603
	5	5	0.112893		5	5	0.224385
-0.5	5	10	0.027275	0.5	5	10	0.0629825
	10	5	0.268656		10	5	0.5694260

3. CONCLUSION

In this study, a bivariate exceedance model is constructed based on bivariate order statistics. In this model, we compose a bivariate random threshold model by using the past flood peak and flood volume of the hydrological stations. Probability of exceedance statistics are calculated under some well-known copulas for small sample sizes. Then the numerical values of expected values of exceedance statistics are provided for independent and FGM copulas. Because of the complexity of the calculations, the numerical results are provided for small sample sizes. As a further study, we need to investigate the properties of exceedance statistics under different bivariate distributions by using some real data sets in hydorology. The results obtained using real data sets can be compared with the theoretical results in this article.

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ON THE RELIABILITY CHARACTERISTICS OF THE STANDARD TWO-SIDED POWER DISTRIBUTION

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Abstract. In this study, the standard two-sided power (STSP) distribution is considered with regard to statistical reliability analysis in detail. For this purpose, along with the reliability and hazard functions of the distribution, particular reliability indices that are useful in maintenance and replacement policies are obtained and they are evaluated with their plots. The STSP distribution is classified based on aging according to various cases of its parameters. Then, we studied the classical and Bayesian estimations of the reliability and hazard functions. In Bayesian estimation, symmetric and different asymmetric loss functions are considered. For obtaining the Bayes estimates, Monte Carlo Markov Chain simulation using the Gibbs algorithm is performed. Various simulation schemes are performed for comparing the performances of the estimators. Further, the Bayesian predictions of the future observations based on the observed samples are obtained. A real data example is used to illustrate the theoretical outcomes.

1. Introduction

Lifetime, survival time or failure time data is encountered in many study fields such as reliability assesment in engineering, clinical trial studies in medicine, biomedical engineering, social studies and etc. In this purpose; lifetimes of peoples, components, patients, industrial robots, animals, plants, cogs, softwares and etc. are considered with probability distributions. In statistical literature, there are many different probability distributions for modelling lifetime data. In reliability theory, a finite upper limit to the lifetime data does not frequently consider and thereby many lifetime distributions are defined over the range $(0, \infty)$ [14]. The commonly used lifetime distributions are the exponential, Weibull, lognormal, gamma and pareto

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etc. distributions. On the other hand, in many cases, the lifetime distributions are needed to consider on a finite range. For example, the pressure, strength, length, temperature, weight, or voltage of material can take any value on a finite range (e.g. $150 - 250$ MPa). Also, the existence of the censoring or truncation causes to reduce lifetimes on a finite range. In these cases, finite range distributions could be considered for modelling them. In the reliability studies, distributions on finite ranges are considered for failure data [1] in various studies. As a special case, finite ranges can be occur over the range $[0, 1]$ and used for modeling uncertainty about the probability of success of an experiment. In these cases, beta distributions could be considered as the most used lifetime distribution. The Beta distributions are quite useful to modeling many uncertainties since their versatile structure [10]. On the other hand, the standard two-sided power distribution, denoted by STSP, is introduced by van Dorp and Kotz [21] and it has the following probability density function (pdf) and the reliability function

$$f(x|\alpha,\beta) = \begin{cases} \alpha(\frac{x}{\beta})^{\alpha-1} & ,0 < x \leq \beta \\ \alpha(\frac{1-x}{1-\beta})^{\alpha-1} & ,\beta \leq x < 1 \end{cases} \tag{1}$$

$$R(x) = P(X > x) = \begin{cases} 1 - \beta(\frac{x}{\beta})^{\alpha} & ,0 < x \leq \beta \\ (1-\beta)(\frac{1-x}{1-\beta})^{\alpha} & ,\beta \leq x < 1 \end{cases} \tag{2}$$

while the hazard(failure rate, hazard rate or force of mortality) function is given by

$$\lambda(x) = \frac{f(x)}{R(x)} = \begin{cases} \alpha/\left\{\left(\frac{\beta}{x}\right)^{\alpha-1} - x\right\} & ,0 < x \leq \beta \\ \alpha/\{1-x\} & ,\beta \leq x < 1 \end{cases} \tag{3}$$

where $\alpha > 0$ is the shape and $0 < \beta < 1$ is the reflection parameters. The STSP distribution is proposed as a peaked alternative of beta distribution by Kotz and van Dorp [12]. Since the STSP distribution is defined on a finite range and has similar flexibility, the STSP distribution is a beta-like distribution. The parameters of the distribution determine the shapes of the distribution and similar to the beta case. For example, the STSP distribution is unimodal in the case of $0 < \beta < 1$ & $\alpha > 0$ and U shaped for $0 < \beta < 1$ & $0 < \alpha < 1$. It has relations with some other distributions according to its special cases. For instance; the uniform distribution on $(0, 1)$ for $\alpha = 1$ and the triangular model for $\alpha = 2$ are obtained. In the case of $\beta = 0.5$, the STSP distribution is symmetric and the left-skewed and right-skewed distributions occurs when $\beta > 0.5$ and $\beta < 0.5$, respectively, for $\alpha > 1$. The STSP distribution is intelligibly more flexible than the power function distribution which is a special case of the distribution in the case of $\beta = 1$ (see Fig. 1). In this way, the STSP distribution can be used in reliability and life testing experiments on $[0, 1]$ range of finite-range datasets. Particularly, when these types of lifetime data have any threshold point, they are convenient for modelling by a two-sided distribution. Mance, Barker and Chimka [13] studied some features of two-sided power distribution (TSP) which is an extension of the STSP distribution in reliability analysis, firstly. They introduced the reliability and hazard functions of

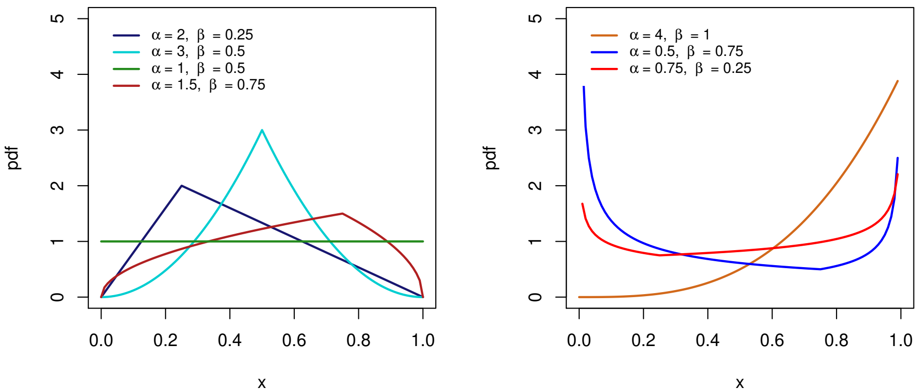

FIGURE 1. Plots of probability denstiy function of the STSP distribution for various choices of its parameters.

the TSP distribution and presented their plots with usefulness in engineering. Using analytical estimation procedure, they obtained the TSP parameters and compared the distribution with the Weibull distribution. Recently, Çetinkaya and Genç [8], [9] studied the STSP distribution under moments of order statistics and stress-strength reliability.

As a further study, we consider the STSP distribution under statistical reliability context. Fundamental reliability indices such as reliability and hazard functions are given and their plots are interpreted according to changing in parameters of the distribution. Following, some reliability indices which are useful in maintenance and replacement policies in engineering are given. Further, we considered the classifying of the STSP distribution based on notions of aging according to various cases of its parameters. Otherwise, as a diagnostics test if a data comes from the STSP distribution, we examined the hazard plot. After these main reliability indices, we obtanined the classical and Bayesian estimations of the reliability and hazard functions based on the symmetrical and asymmetrical loss functions. A real dataset is used to illustrate the outcomes and all estimates are compared. In the last section, Bayes prediction of a future sample based on current available sample is obtained.

2. Reliability characteristics

The STSP distribution is a two-sided distribution and quite useful on the finite range. The reliability graph of the STSP distribution is both convex and concave,or likely S-shaped, depending on different cases of its parameters (see Fig. 2-3). In

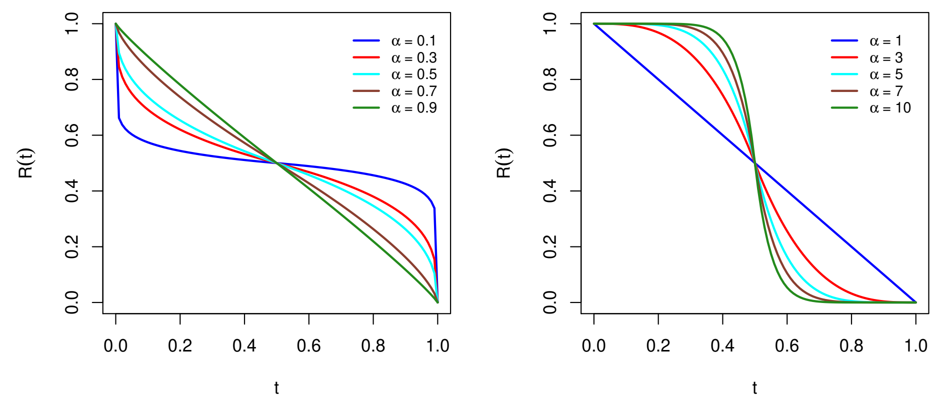

FIGURE 2. Reliability function plots of the symmetrical STSP distribution ($\beta = 0.5$) for different shape parameters.

symmetrical case, that is if $\beta = 0.5$, in the case of $\alpha < 1$, it is convex for the smaller values than β and concave for bigger values than β. On the conversely, in the case of $\alpha > 1$, it is concave for the smaller values than β and convex for bigger values than β. If the STSP distribution is not in symmetrical case, that is if $\beta \neq 0.5$, it is convex for small β values and it turns to concave with increasing β for $\alpha > 1$. On the other hand, it is convex for large β values and it turns to concave with decreasing β for $\alpha < 1$. While $\alpha = 1$, the STSP distribution has constant decreasing reliability.

Concave reliability curve imply low failure in early and useful life along with rapid increase in later life. On the contrary, convex reliability curve imply high failure in early and useful life along with rapid decrease in later life, the convexity or concavity of a reliability curve is depend on environmental conditions and genetic structure of the observations.

In parallel to its reliability function, the STSP distribution has both increasing and decreasing failure rate based on different cases of its parameters (see Fig. 4). On the other hand, the hazard function (Eq.3) shows that for any case of parameters the STSP distribution does not have constant hazard where imminent risk of failure does not change with time. It is clearly seen that, the failure rate of the STSP distribution is increasing for $\alpha > 1$ values and in the form of bathtube curve for $\alpha < 1$. Also, $\lambda(t)$ is not differantiable in the $t = \beta$ point so there is a cusp as seen in Fig. 4. Detailed comments about behaviour of the hazard function are given in the next section.

In statistical reliability studies, there are some indices to compare survival random

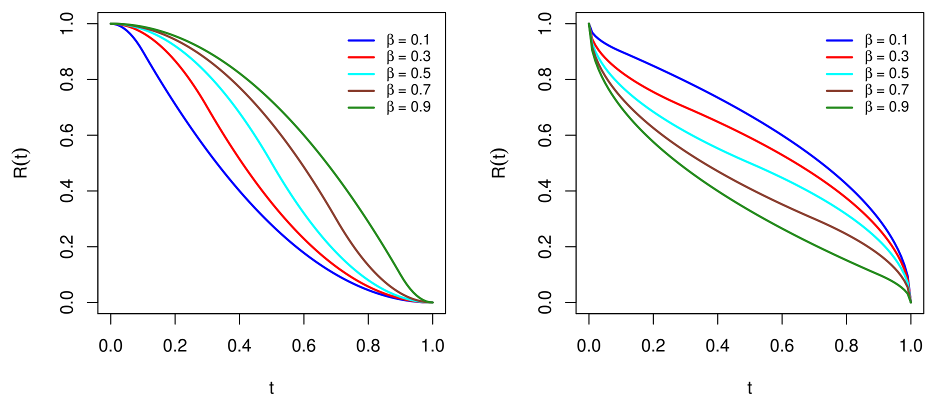

FIGURE 3. Reliability function plots of the STSP distribution for different reflection parameters in the case of $\alpha > 1(\alpha = 2)$ on left and $\alpha < 1(\alpha = 0.5)$ on right.

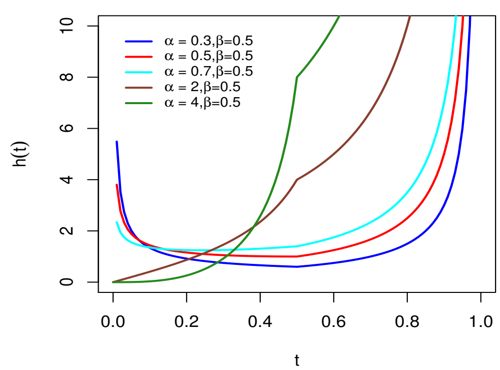

FIGURE 4. Hazard function plots of the symmetrical STSP distribution ($\beta = 0.5$) for different shape parameters.

variables. Also, these indices are quite useful for maintenence and replacement policies.

Firstly, mean time to failure (MTTF) is the length of lifetime a component is expected to failure. MTTF is one of various methods to assess the reliability of a component. The mean time to failure (MTTF) of the STSP distribution can be obtained by using the pdf (1) of the distribution as in the following.

$$MTTF = E(X) = \frac{\beta(\alpha-1)+1}{\alpha+1}$$

Mean residual life time (MRL) at age-t can be considered as another reliability index. Resiual lifetime at age t is about the question of a component how much life does it have left in on avarage while the experimental component still alive and under observation at time t [18]. Mean time to failure for the STSP distribution can be easily obtained as in the following. Firstly, the conditional density of the X given $X > t$ is obtained by

$$f(X|X>t) = \begin{cases} \frac{\alpha(\frac{x}{\beta})^{\alpha-1}}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,0<t<x<\beta \ (t<\beta, CaseI) \\ \frac{\alpha(\frac{1-x}{1-\beta})^{\alpha-1}}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,0<t<\beta<x<1 \ (t<\beta, CaseI) \\ \frac{\alpha}{1-x}\left(\frac{1-x}{1-t}\right)^{\alpha} & ,\beta<t<x<1 (t>\beta, CaseII) \end{cases}$$

Then, mean residual lifetime at age-t can be obtained by using

$r(t) = E(X-t|X>t) = \int (x-t)f(x|x>t)dx$ and equally

$E(X-t|X>t) = \frac{\int_t^1 R(x)dx}{R(t)} = \frac{\int_t^\beta R_1(x)dx + \int_\beta^1 R_2(x)dx}{R(t)}$ for $t \le \beta$

$E(X-t|X>t) = \frac{\int_t^1 R(x)dx}{R(t)} = \frac{\int_t^1 R_2(x)dx}{R(t)}$ for $t > \beta$.

where $R_1(x)$ and $R_2(x)$ are the two sides of the reliability function (2), respectively. Thus, under the STSP distribution mean residual lifetime at age-t is obtained as in the following

$$r(t) = E(X-t|X>t) = \begin{cases} \frac{\left[1+\beta(\alpha-1)+\beta t\left(\frac{t}{\beta}\right)^{\alpha}\right](\alpha+1)^{-1}-t}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,t \le \beta \\ \frac{1-t}{\alpha+1} & ,t>\beta \end{cases}$$

Together with the hazard plot, MRL plot is a useful and good indication to investigate the behaviour of lifetime data [15]. The MRL plot which are given in Fig. 5 shows that the MRL of a lifetime data under the STSP distribution brings with convex curve to concave curve with increasing shape parameter α. Similar to results which are obtained with hazard plot, for $\alpha < 1$, MRL is rapidly increasing in early life as parallel to rapidly decreasing failure. Then, MRL is rapidly decreasing in wear out stage after a stationary process in useful lifetime on peak. Examples can be increased for all possible conditions of the parameters α and β.

Further, when a component has already reached given age t, life expectancy at age t is named as mean life expectancy at age-t and denoted by $E(X|X>t) = t+r(t)$. If a component has a lifetime under the STSP distribution, the mean life expectancy at age t it is obtained as in the following

$$E(X|X>t)=t+r(t)=\begin{cases}\frac{\left[1+\beta(\alpha-1)-\alpha\beta t\left(\frac{t}{\beta}\right)^{\alpha}\right](\alpha+1)^{-1}}{1-\beta(\frac{t}{\beta})^{\alpha}} & ,t\le\beta\\ \frac{\alpha t+1}{\alpha+1} & ,t>\beta\end{cases}$$

Similar to MRL plot, the plots of the mean life expectancy at age-t are given in Fig. 5. The behaviour of the mean life expectancy shows consistents results with the hazard (Fig. 4) and MRL (Fig. 5) plots.

There is an other index for replacement policies is computation of the probability of that an A-year-old component reaches age-B. Under the STSP distribution, it can be obtained easily as in the following

$$e^{-\int_A^B \lambda(x)dx}=\begin{cases}\frac{\beta^{\alpha-1}-B^{\alpha}}{\beta^{\alpha-1}-A^{\alpha}} & ,A<B\le\beta\\ \frac{\beta^{\alpha-1}(1-B)^{\alpha}}{(\beta^{\alpha-1}-A^{\alpha})(1-\beta)^{\alpha-1}} & ,A\le\beta<B\\ \left(\frac{1-B}{1-A}\right)^{\alpha} & ,\beta\le A<B\end{cases}$$

Additionally, the expected service life (ESL) of a component under a replacement policy [3] whereby the component is replaced when it reaches age t is defined as the expected value of the mixture random variable, namely $Z=min\{X,t\}$ and $ESL(t)$ is given as in the following [18].

$$ESL(t)=\int_0^t xf(x)dx+\int_t^1 tf(x)dx$$

For the STSP distribution the expected service life of a component is considered for two cases as given below

If $t\le\beta$,
$ESL(t)=\int_0^t xf_1(x)dx+\int_t^{\beta} tf_2(x)dx+\int_{\beta}^1 tf_2(x)dx=\int_0^t xf_1(x)dx+tR_1(t)$

If $t>\beta$,
$ESL(t)=\int_0^{\beta} xf_1(y)dx+\int_{\beta}^t xf_2(x)dx+\int_t^1 tf_2(x)dx$
$ESL(t)=\int_0^{\beta} xf_1(x)dx+\int_{\beta}^t xf_2(x)dx+tR_2(t)$

where $f_1(x)$ and $f_2(x)$ are the two sides of the pdf (1) of the STSP distribution. Thus, ESL(t) under the STSP distribution is obtained as in the following

$$ESL(t)=\begin{cases}t-\frac{\beta t\left(\frac{t}{\beta}\right)^{\alpha}}{\alpha+1} & ,t\le\beta\\ \frac{\alpha\beta+(1-\beta)\left[1-(1-t)\left(\frac{1-t}{1-\beta}\right)^{\alpha}\right]}{\alpha+1} & ,t>\beta\end{cases}$$

The plots of EST(t) for different cases of the parameters are given in Fig. 6 and Fig. 7. In symmetrical case, that is if $\beta=0.5$, is changing to concave curve with increasing α. For fixed $\alpha>1$, ESL(t) has larger values and similar concavity with increasing β. On the contrary, for $\alpha<1$, ESL(t) has smaller values and similar

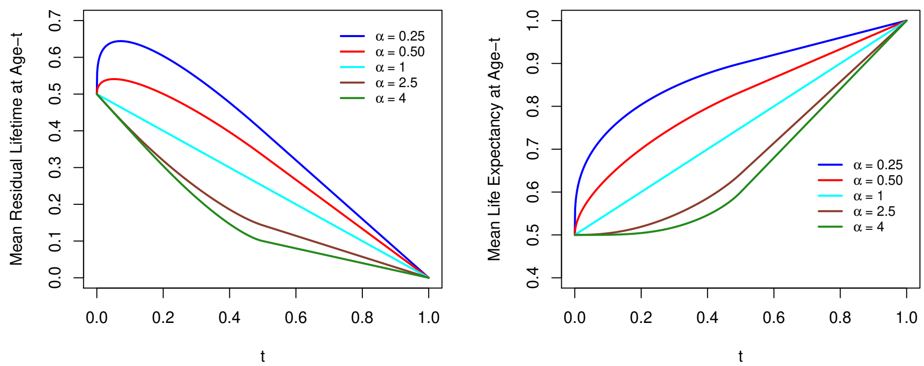

FIGURE 5. Plots of mean residual lifetime (left) and mean life expectancy (right) at age-t for the symmetrical STSP distribution

concavity with increasing β.

All these indices which are given and interpreted above is quite useful to evaluate the behaviour of a lifetime data. In engineering. maintenance and replacement policies of components and systems have been considered, seriously.

2.1. **Classifiying the distribution based on notions of aging.** Many lifetime distributions are considered under particular replacement policies. The maintanence policies are useful to reduce the deficit of the system failures and provide operational sustainability. In this purpose, the STSP distribution has been evaluated based on its aging. Firstly, the behaviour of the hazard function is considered and life characteristics for a lifetime data from the STSP distribution is determined as in the following and summarized in Table 1.

Theorem 1. *In the case of $x \leq \beta$, $\lambda(x)$ is increasing namely it has increasing failure rate (IFR) for $\alpha > 1$ and either decreasing on $x \leq min\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta\right)$ and increasing on $\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha} \leq x \leq \beta$ for $\alpha < 1$.*

Proof. If $x \leq \beta$, then
$\lambda^{'}(x) = \alpha\left[\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta} - 1\right]^{-2}\frac{1}{x^2}\left[\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta}(\alpha - 1) + 1\right]$
Note that; $\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta} > 1$. So,
the sign of $\lambda^{'}(x)$ depends on the sign of $\left(\frac{\beta}{x}\right)^{\alpha}\frac{1}{\beta}(\alpha - 1) + 1$.

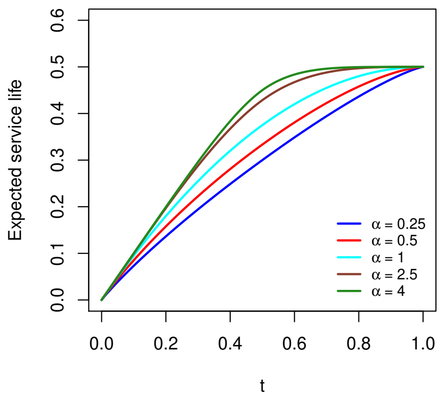

FIGURE 6. Plots of expected service life (ESL) for the symmetrical STSP distribution for various α values

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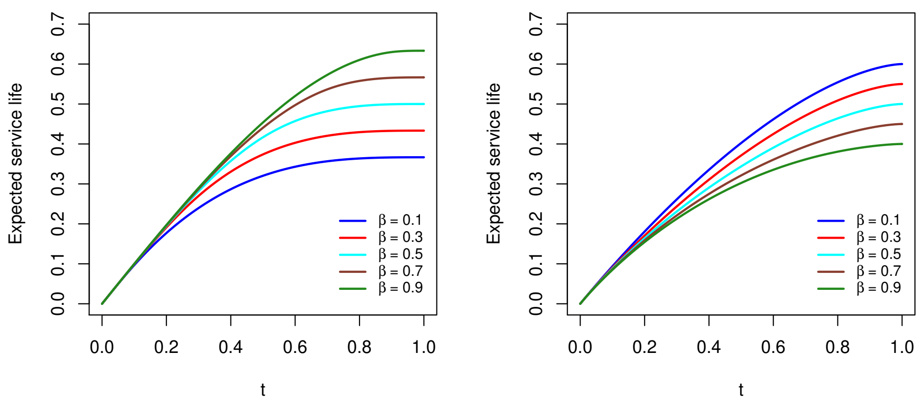

FIGURE 7. Plots of expected service life (ESL) for various β values in the case of $\alpha > 1(\alpha = 2)$ on the left and $\alpha < 1(\alpha = 0.5)$ on the right)

.

TABLE 1. Life characteristics for a lifetime data from the STSP distribution.

Parameters	**Domain**	**Failure Type**
$\alpha < 1$	$x \leq min\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta \right)$	Decreasing Hazard
$\alpha < 1$	$\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha} \leq x \leq \beta$	Increasing Hazard
$\alpha \geq 1$	$x \leq \beta$	Increasing Hazard
$\alpha \geq 0$	$x \geq \beta$	Increasing Hazard

For $\alpha > 1$, $\lambda(x)$ is increasing on $(0, \beta)$

$\left(\frac{\beta}{x}\right)^{\alpha} \frac{1}{\beta}(\alpha - 1) + 1 > 0 \Longleftrightarrow x > (1 - \alpha)\beta^{1-1/\alpha}$

For $\alpha < 1$, $\lambda(x)$ is either increasing or decreasing on $(0, \beta)$

$\left(\frac{\beta}{x}\right)^{\alpha} \frac{1}{\beta}(\alpha - 1) + 1 > 0 \Longleftrightarrow x > \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}$

Thus, $\lambda(x)$ is increasing on $\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta \right)$, if $1 < \alpha + \beta$.

So if $1 - \beta < \alpha < 1$ then $\lambda(x)$ is increasing on $\left(\left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}, \beta \right)$

$\left(\frac{\beta}{t}\right)^{\alpha} \frac{1}{\beta}(\alpha - 1) + 1 < 0 \Longleftrightarrow t < \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}$

So $\lambda(x)$ is decreasing on $\left(0, \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}\right)$, if $\alpha + \beta < 1$

So if $\alpha < 1 - \beta < 1$ then $\lambda(x)$ is decreasing on $\left(0, \left(\frac{1-\alpha}{\beta^{1-\alpha}}\right)^{1/\alpha}\right)$. $\square$

Theorem 2. *In the case of $x > \beta$, $\lambda(x)$ is an increasing function and namely it has IFR on $(\beta, 1)$ for both $\alpha > 1$ and $\alpha < 1$.*

Proof. $\lambda^{'}(x) = \frac{\alpha}{(1-x)^2}$. In this way, $\lambda^{'}(x) > 0$ for all $\alpha > 0$ values. $\square$

In the hazard function of the STSP distribution for $\alpha > 1$ values of shape parameter $\lambda_1^{'}(\beta) \neq \lambda_2^{'}(\beta)$ and it is not differentiable in the $x = \beta$ point so there is a cusp as seen in Fig. 4 (Here, $\lambda_1^{'}(.)$ and $\lambda_2^{'}(.)$ denotes to two side of the hazard function (3)).

If a lifetime distribution, has a hazard function with non-decreasing avarage, it is increasing failure rate avarage (IFRA) class of lifetime distribution. This class could be alternately defined by a condition intuitively related to wear out for each $x \geq 0$ [4]. An IFR limetime distribution is also IFRA. The both proporties of a lifetime distribution are notions of aging. The IFR, the IFRA or the NBU class of distributions have a number of benefits. For instance, the distribution or reliability functions of these distributions can be bounded from lower and upper in terms of their mean or quantiles. Many other useful properties of these class of distributions are elaborated by Barlow and Proschan [1] such as relating to the reliability of a simple system, a coherent system, a system subject to cumulative shocks and etc. [19].

An IFRA component Tends to more survive any shorter period and on the contrary, less surviving any longer period. The IFRA class contains the exponential survival probabilities. It contains all IFR survival probabilities. Birnbaum et al. [4] mentioned that the IFRA class is closed under the formation of coherent systems and that it is essentially the smallest class containing the exponentials which is so closed.

Remark 1. *A distribution has IFRA (Increasing failure rate avarage) if* $-(1/x)\ln R(x)$ *is increasing in* $x \geq 0$*. Similarly a distribution has DFRA (Decreasing failure rate avarage) if* $-(1/x)\ln R(x)$ *is decreasing in* $x \geq 0$ *[1].*

Theorem 3. *The STSP distribution is an IFRA class of distribution for* $\alpha > 1$ *in the both* $x \leq \beta$ *and* $x \geq \beta$ *cases.*

Proof.

$$\psi_1(x) = -\frac{\ln R_1(x)}{x} = -\frac{\ln\left[1-\beta(\frac{x}{\beta})^{\alpha}\right]}{x} = \frac{\ln\left[\frac{1}{1-\beta(\frac{x}{\beta})^{\alpha}}\right]}{x}$$

Using the expansion of $\ln\left[\frac{1}{1-\beta(\frac{x}{\beta})^{\alpha}}\right]$ as

$$\psi_1(x) = \frac{\beta(\frac{x}{\beta})^{\alpha} + \frac{\left[\beta(\frac{x}{\beta})^{\alpha}\right]^2}{2} + \frac{\left[\beta(\frac{x}{\beta})^{\alpha}\right]^3}{3} + \cdots}{x}$$

Then,

$$\psi_1^{'}(x) = (\alpha-1)\beta^{1-\alpha}x^{\alpha-2} + \frac{(2\alpha-1)\beta^{2-2\alpha}x^{2\alpha-1}}{2} + \frac{(3\alpha-1)\beta^{3-3\alpha}x^{3\alpha-2}}{3} + \cdots$$

It is clearly seen that for $\alpha > 1$, $\psi_1^{'}(x) > 0$. Thus, the STSP distribution is IFRA in the case of $x \leq \beta$ if and only if $\alpha > 1$.

On the other hand

$$\begin{aligned}\psi_2(x) &= -\frac{\ln R_2(x)}{x} = -\frac{\ln\left[\frac{(1-x)^{\alpha}}{(1-\beta)^{\alpha-1}}\right]}{x} = \frac{\ln\left[\frac{(1-\beta)^{\alpha-1}}{(1-x)^{\alpha}}\right]}{x} \\ &= \frac{\ln(1-\beta)^{\alpha-1} - \ln(1-x)^{\alpha}}{x} = \frac{\ln(1-\beta)^{\alpha-1} + \alpha\ln\left(\frac{1}{1-x}\right)}{x}\end{aligned}$$

Using the expansion of $\ln\left(\frac{1}{1-y}\right)$ as

$$\psi_2(x) = \frac{(\alpha-1)\ln(1-\beta) + \alpha\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)}{x}$$

Then

$$\psi_2^{'}(x) = \frac{(1-\alpha)\ln(1-\beta)}{x^2} + \frac{\alpha}{2} + \frac{2x\alpha}{3} + \frac{3x^2\alpha}{4}\cdots$$

In this equation $\ln(1-\beta) > 0$ for $\alpha > 1$ values. Thus, it makes $\frac{(1-\alpha)\ln(1-\beta)}{x^2} \geq 0$ and $\psi_2^{'}(x) > 0$ □

2.2. **Hazard plot.** A hazard plot is a simple plot of the points $\left(a_j, x_j\right)$, where $a_j = \sum_{i=1}^{j} \frac{1}{n-i+1}$ are called the hazard plot scores [18]. For using a hazard plot to determine if a data comes from the STSP distribution, note that, cumulative hazard function of the STSP distribution

$$H(x) = -\ln\{R(x)\} \begin{cases} \ln\left[1-\beta(\frac{x}{\beta})^{\alpha}\right]^{-1} & ,0 < x \leq \beta \\ \ln\left[\frac{(1-\beta)^{\alpha-1}}{(1-x)^{\alpha}}\right] & ,\beta \leq x < 1 \end{cases}$$

Therefore, if a data comes from the STSP distribution the relationship between $\ln(a_j)$ and $\ln(x_j)$ should be a 45^0 line similarly to hazard plot for the Weibull distribution. Many engineers regard hazard plot as a simpler diagnostic test than a probability plot [18].

3. Classical estimation

In this section, we have obtained the maximum likelihood estimation (MLE) of the reliability and hazard functions of the STSP distribution. Let us suppose that $x_1, x_1, \ldots, x_n$ is the independent and identical (IID) random samples from $STSP(\alpha, \beta)$. Then the likelihood function is given by

$$L(\alpha,\beta) = \alpha^n \left\{ \frac{\prod_{i=1}^{r} x_{(i)} \prod_{i=r+1}^{n} (1-x_{(i)})}{\beta^r (1-\beta)^{n-r}} \right\}^{\alpha-1}$$

where $x_{(r)} \leq \beta < x_{(r+1)}$ with $x_{(0)} \equiv 0$ and $x_{(n+1)} \equiv 1$.

The maximum likelihood estimators of the parameters are obtained by van Dorp and Kotz [21], and they are given by

$$\hat{\beta} = X_{(\hat{r})}$$

$$\hat{\alpha} = -\frac{n}{\log M(\hat{r})}$$

where $\hat{r} = arg\ max_{\{r\in 1,2,\cdots,n\}} M(r)$ and

$$M(r) = \prod_{i=1}^{r-1} \frac{X_{(i)}}{X_{(r)}} \prod_{i=r+1}^{n} \frac{1 - X_{(i)}}{1 - X_{(r)}}$$

Thus, by using the invariance property of the MLEs, the maximum likelihood estimators of the reliability function and hazard function can be obtained by replacing the parameters in Eq.(2) and Eq.(3) with their estimates and denoted by $\hat{R}_{ML}$ and $\hat{\lambda}_{ML}$.

4. Bayesian estimation

In this section, we provide Bayes estimates of reliability function $R(x)$ and hazard function $\lambda(x)$. Under considering different loss functions, these estimates are obtained and compared with respect to their expected risks (ER). In Bayesian estimation, squared error loss function (SELF) is the most commonly used loss function due to it is symmetrical and it provides equal distance to the losses through overestimation and underestimation. However, in some situations such as reliability and hazard estimates overestimation is more considerable than underestimation or vice-vera [16]. In this purpose, Linex loss function (LLF) defined by Varian [22] and general entropy loss function (GELF) defined by Calabria and Pulcini [5] are considered as asymmetric loss functions which are defined as, respectively,

SELF $\Longrightarrow L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$
LLF $\Longrightarrow L_2(\hat{\theta}, \theta) = e^{p(\hat{\theta}-\theta)} - p(\hat{\theta} - \theta) - 1, \quad p \neq 0$
GELF $\Longrightarrow L_3(\hat{\theta}, \theta) = \left(\frac{\hat{\theta}}{\theta}\right)^c - c\log\left(\frac{\hat{\theta}}{\theta}\right) - 1$

where p and c reflects the departure from the symmetry, $\hat{\theta}$ represents an estimate for parameter θ. Thus, Bayes estimates of the parameters under these loss functions can be obtained from their posterior distributions as in the following;

SELF $\Longrightarrow \hat{\theta}_{B1} = E(\theta|data)$
LLF $\Longrightarrow \hat{\theta}_{B2} = -\frac{1}{p}\log\{E(e^{-p\theta}|data)\}$
GELF $\Longrightarrow \hat{\theta}_{B3} = \{E(\theta^{-c}|data)\}^{-1/c}$

Under these loss functions, the Bayes estimators of reliability $R(x|\alpha, \beta)$ and hazard $\lambda(x|\alpha, \beta)$ functions which are given in Eq.(2) and Eq.(3), respectively, are expressed as in the following,

$$\hat{R}_{B1} = \int_0^\infty \int_0^1 R(data|\alpha, \beta)\ \pi(\alpha, \beta|data)d\beta d\alpha \tag{4}$$

$$\hat{R}_{B2} = -\frac{1}{p}\log\left\{\int_0^\infty \int_0^1 e^{-pR(data|\alpha,\beta)}\pi(\alpha, \beta|data)d\beta d\alpha\right\} \tag{5}$$

$$\hat{R}_{B3} = \left\{ \int_0^\infty \int_0^1 R(data|\alpha,\beta)^{-c} \pi(\alpha,\beta|data) d\beta d\alpha \right\}^{-1/c} \tag{6}$$

where $\pi(\alpha,\beta|data)$ is posterior distribution of the parameters. Estimators of the $\lambda(t)$, denoted by $\hat{\lambda}_{B1}$, $\hat{\lambda}_{B2}$ and $\hat{\lambda}_{B3}$, can be obtained by changing $R(data|\alpha,\beta)$ with Eq.(3), similarly.

However, the form of the $STSP$ distribution given in (1) is not proper for developing Bayesian models. Since the its support depends on the reflection parameter, posterior distributions of α and β, namely $\pi(\alpha,\beta|data)$ can not be obtained. Also, estimators given in (4),(5) and (6) can not be expressed in closed form and hence it can not be evaluated analytically. This fact was previously pointed out for the triangular distribution which is special form of the $STSP$ distribution ($\alpha = 2$ case) by Ho et al. [11]. To overcome this adversity and obtain a Bayesian inference for the STSP distribution, Çetinkaya and Genç [9] proposed a hierarchical model construction. This model provides conditional distributions of parameters to build a Markov Chain Monte Carlo (MCMC) algorithm using a Gibbs sampler as given in the following.

Çetinkaya and Genç [9] developed marginal densities by introducing an auxiliary or talent variable as in the following.

Let V be a random variable with parameter $\alpha > 1$. Suppose that V has the pdf

$$f_V(v;\alpha) = \alpha\big[1-(1-v)^{1/(\alpha-1)}\big], \quad 0 < v < 1.$$

Further, let the conditional distribution of X given $V = v$ be the uniform distribution represented by

$$U\big[\beta(1-v)^{1/(\alpha-1)}, 1-(1-\beta)(1-v)^{1/(\alpha-1)}\big].$$

Then the marginal distribution of X has the STSP distribution with pdf given in (1). Thus, this hierarchical model will simplify the computational procedures for Bayesian calculations. In order to implement a Gibbs sampler, Çetinkaya and Genç [9] are obtained the conditional distributions of α, β and v as in the following

$$\begin{aligned}
f(v|\alpha,\beta,x) &\propto f(v|\alpha)f(x|\alpha,\beta,v) \\
&\propto I\left(max\left\{ 1-\left(\frac{x}{\beta}\right)^{\alpha-1}, 1-\left(\frac{1-x}{1-\beta}\right)^{\alpha-1} \right\} < v < 1 \right) \\
f(\beta|\alpha,v,x) &\propto \pi(\beta)f(x|\beta,v,\alpha) \\
&\propto \pi(\beta) I\left(1-\frac{1-x}{(1-v)^{1/(\alpha-1)}} < \beta < \frac{x}{(1-v)^{1/(\alpha-1)}} \right) \\
f(\alpha|v,\beta,x) &\propto \pi(\alpha)f(v|\alpha)f(x|\beta,v,\alpha) \\
&\propto \pi(\alpha) I\left(1 < \alpha < min\left\{ \frac{\ln(1-v)}{\ln(\frac{x_<}{\beta})}+1, \frac{\ln(1-v)}{\ln(\frac{1-x_>}{1-\beta})}+1 \right\} \right)
\end{aligned}$$

where $I(.)$ denotes indicator function, $x^<$ denotes observations below β and $x^>$ observations above β, $\pi(\alpha)$ and $\pi(\beta)$ denotes prior distributions for the parameters. Thus, MCMC samples using Gibbs algorithm can be obtained by using the following steps;

Step 1: Assign initial $\alpha^{(0)}$ and $\beta^{(0)}$ values for α and β.
Step 2: Set t=1.
Step 3: Given $\alpha^{(t-1)}$ and $\beta^{(t-1)}$ and $\{x_1, x_2, \cdots, x_n\}$ generate $\{v_1, v_2, \cdots, v_n\}$ using Eq.(4).
Step 4: Considering uniform prior on $[0,1]$ for β, given $\alpha^{(t-1)}$, $\{x_1, x_2, \cdots, x_n\}$ and $\{v_1, v_2, \cdots, v_n\}$, generate $\beta^{(t)}$ using

$$I\left(max\left\{1 - \frac{1 - x_i}{(1 - v_i)^{1/(\alpha^{(t-1)}-1)}}, 0\right\} < \beta < min\left\{\frac{x_i}{(1 - v_i)^{1/(\alpha^{(t-1)}-1)}}, 1\right\}\right)$$

Step 5: Considering uniform prior on $[1, c]$ for α and choosing $c = 100$ generate α^t from the pdf $\left[(n+1)/(b^{n+1}-1)\right]\alpha^n$ using inverse transformation method, where

$$b = min\left\{1 + \frac{\ln(1 - v_i)}{\ln(\frac{x_i^<}{\beta^{(t)}})}, 1 + \frac{\ln(1 - v_i)}{\ln(\frac{1-x_i^>}{1-\beta^{(t)}})}, c\right\}$$

Step 6: Using Eq.(2) and Eq.(3), compute $R_B^{(t)}$ and $\lambda_B^{(t)}$ at $(\alpha^{(t)}, \beta^{(t)})$.
Step 7: Set $t = t + 1$.
Step 8: Repeat steps $2 - 7$, M times and obtain posterior samples $(R_B^{(t)} : t = 1, 2, \cdots, M)$ and $(\lambda_B^{(t)} : t = 1, 2, \cdots, M)$.

Finally, the posterior mean under mean sqaured error, linex loss and general entropy loss functions, say $\hat{R}_{B1}$, $\hat{R}_{B2}$, $\hat{R}_{B3}$ and $\hat{\lambda}_{B1}$, $\hat{\lambda}_{B2}$, $\hat{\lambda}_{B3}$, can be obtained as follows;

$$\hat{R}_{B1} = \frac{1}{M}\sum_{t=1}^{M} R_B^{(t)} \quad , \quad \hat{R}_{B2} = -\frac{1}{p}\ln\left\{\frac{1}{M}\sum_{t=1}^{M} e^{-pR_B^{(t)}}\right\}$$

$$\hat{R}_{B3} = \left\{\frac{1}{M}\sum_{t=1}^{M}(R_B^{(t)})^{-c}\right\}^{-1/c} \tag{7}$$

$\hat{\lambda}_{B1}$, $\hat{\lambda}_{B2}$, $\hat{\lambda}_{B3}$ are obtained similarly.

5. Simulation Studies

In this section, performances of the maximum likelihood and Bayes estimators under different loss functions are compared. According to various fixed point (t) and sample sizes, avarage estimates and corresponding expected risks (ER) of $R(t)$

are obtained and reported in Tables 2 and 3. Similar results are also obtained for $\lambda(t)$ and reported in Table 4 and 5.

The expected risks of estimates under all considered loss functions (SELF, LLF and GELF), when θ is estimated by $\hat{\theta}$, can be obtained by using the following equation,

$$\mathrm{ER}(\hat{\theta}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_i - \theta)^2,$$

where

$$\begin{aligned}
\hat{\theta} &= E(\theta|data), \qquad \text{for SELF} \\
\hat{\theta} &= -\frac{1}{p}\log\{E(e^{-p\theta}|data), \qquad \text{for LLF} \\
\hat{\theta} &= \{E(\theta^{-c}|data)\}^{-1/c}, \qquad \text{for GELF}
\end{aligned}$$

respectively. Choosen arbitrary values of the parameters (α, β) are taken as $(2.8, 0.8)$ and $(1.5, 0.5)$, respectively. The Bayes point estimates are obtained under SELF, LLF($p = -0.5, 0.5, 1$) and GELF($c = -0.5, 0.5, 1$) loss functions. We generate 2000 samples of size n (small sample size $n = 10$, moderate sample sizes $n = 20, 30$ and large sample sizes $n = 50, 100$). For Bayesian estimation, we run the Gibbs sampler to generate a Markov chain with 3500 observations using the given algorithm in Section 4. As burn-in period, we discard the first 500 values and take every third variate as a independent and identically distributed observation in thinning procedure. Thus, a sample of 1000 resulted which is used to calculate the posterior estimates. Then, the simulation is performed via MCMC for 2000 replicates. We report all the results of this simulation scheme in Table 2, 3 for reliability estimates. We observed that all the estimates are close to the actual values of $R(t)$. As expected, the ERs of all estimators decrease as sample size increases in all considered cases. In all cases $(t \leq \beta, t > \beta)$, maximum likelihood estimates tend to give overestimates. Being underestimating or overestimating is not only depend on loss parameters, it is also related to relation between t and β. Bayes estimates under squared error $\hat{R}_{B1}$ and Linex loss functions $\hat{R}_{B2}$ gives under estimates for $t \leq \beta$ and over estimates for $t > \beta$. Bayes estimates under general entropy loss function $\hat{R}_{B3}$ gives under estimates for $t \leq \beta$. On the other hand, for $t > \beta$ it gives under estimate for $c = 0.5$ and $c = 1$, overestimates for $c = -0.5$. Expected risks show that MLE and Bayes estimates under SELF have larger risks. Bayesian estimates under LLF and GELF gives better results in terms of expected risks. Especially, estimates give smallest risks for loss parameters $c = 0.5$ and $p = 0.5$. While loss parameter values converges to 1, risks are getting larger.

Furthermore, similar simulation scenario are applied for $\lambda(t)$ and reported in Table 4, 5. However, Linex loss function is not considered for hazard estimates, only SELF and GELF are used in Bayesian estimates in addition to MLE. Since the

TABLE 2. Avarage estimates and corresponding mean squared errors/risks of $R(t)$ for different choise of n and t when $\alpha = 2.8$ and $\beta = 0.8$ where actual $R(0.2) = 0.984$, $R(0.5) = 0.785$ and $R(0.9) = 0.029$.

t	n	$\hat{R}_{ML}$	$\hat{R}_{B1}$	$\hat{R}_{B2}(Linex)$			$\hat{R}_{B3}(GELF)$		
				$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	10	0.983448	0.970658	0.970911	0.970400	0.970138	0.970381	0.969809	0.969514
		0.000381	0.000818	0.000801	0.000836	0.000853	0.000837	0.000877	0.000898
	20	0.982662	0.975747	0.975863	0.975631	0.975512	0.975623	0.975369	0.975239
		0.000239	0.000437	0.000431	0.000443	0.000449	0.000444	0.000458	0.000465
	30	0.983106	0.978792	0.978854	0.978730	0.978668	0.978728	0.978596	0.978529
		0.000138	0.000209	0.000207	0.000212	0.000214	0.000212	0.000216	0.000219
	50	0.983434	0.980887	0.980916	0.980858	0.980828	0.980857	0.980796	0.980766
		0.000083	0.000111	0.000110	0.000111	0.000112	0.000111	0.000113	0.000113
	100	0.983348	0.982076	0.982088	0.982065	0.982053	0.982064	0.982040	0.982028
		0.000041	0.000050	0.000050	0.000050	0.000050	0.000050	0.000050	0.000050
0.5	10	0.805895	0.769507	0.772007	0.766961	0.764367	0.765790	0.757843	0.753595
		0.011199	0.011945	0.011693	0.012211	0.012491	0.012449	0.013632	0.014321
	20	0.795528	0.778710	0.779972	0.777433	0.776138	0.776950	0.773281	0.771367
		0.005247	0.005728	0.005655	0.005804	0.005886	0.005864	0.006173	0.006348
	30	0.793182	0.783086	0.783890	0.782275	0.781457	0.782004	0.779787	0.778650
		0.003232	0.003476	0.003450	0.003503	0.003532	0.003523	0.003631	0.003691
	50	0.790239	0.785236	0.785692	0.784777	0.784316	0.784638	0.783426	0.782812
		0.001903	0.001997	0.001990	0.002004	0.002012	0.002010	0.002038	0.002054
	100	0.787610	0.785392	0.785609	0.785173	0.784955	0.785111	0.784545	0.784261
		0.000964	0.001019	0.001017	0.001021	0.001023	0.001022	0.001029	0.001033
0.9	10	0.036569	0.029611	0.029755	0.029468	0.029327	0.025943	0.018368	0.014756
		0.002212	0.000556	0.000565	0.000547	0.000538	0.000495	0.000457	0.000478
	20	0.033779	0.034082	0.034217	0.033949	0.033817	0.031104	0.025323	0.022586
		0.001096	0.000477	0.000484	0.000470	0.000463	0.000408	0.000330	0.000318
	30	0.031665	0.034080	0.034189	0.033973	0.033866	0.031723	0.027279	0.025209
		0.000628	0.000401	0.000406	0.000396	0.000391	0.000347	0.000278	0.000262
	50	0.030256	0.033022	0.033088	0.032956	0.032891	0.031505	0.028678	0.027359
		0.000242	0.000236	0.000239	0.000234	0.000232	0.000209	0.000175	0.000166
	100	0.028860	0.030302	0.030325	0.030278	0.030255	0.029645	0.028380	0.027770
		0.000082	0.000090	0.000090	0.000090	0.000089	0.000085	0.000079	0.000077

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

Table 3. Avarage estimates and corresponding mean squared errors/risks of $R(t)$ for different choise of n and t when $\alpha = 1.5$ and $\beta = 0.5$ where actual $R(0.2) = 0.874$, $R(0.5) = 0.500$ and $R(0.9) = 0.045$.

t	n	$\hat{R}_{ML}$	$\hat{R}_{B1}$	$\hat{R}_{B2}(Linex)$			$\hat{R}_{B3}(GELF)$		
				$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	10	0.874192	0.880441	0.881192	0.879683	0.878918	0.879536	0.877680	0.876729
		0.008142	0.003009	0.002987	0.003033	0.003059	0.003050	0.003142	0.003195
	20	0.873982	0.872569	0.873088	0.872047	0.871521	0.871956	0.870711	0.870078
		0.004265	0.001863	0.001853	0.001873	0.001885	0.001878	0.001912	0.001930
	30	0.874393	0.869598	0.869997	0.869197	0.868793	0.869128	0.868176	0.867693
		0.002691	0.001442	0.001434	0.001450	0.001459	0.001453	0.001478	0.001491
	50	0.875334	0.868961	0.869237	0.868683	0.868404	0.868636	0.867980	0.867649
		0.001292	0.000944	0.000939	0.000950	0.000955	0.000951	0.000966	0.000975
	100	0.874437	0.869591	0.869733	0.869449	0.869306	0.869425	0.869091	0.868923
		0.000609	0.000606	0.000604	0.000609	0.000612	0.000610	0.000617	0.000621
0.5	10	0.501461	0.499542	0.502393	0.496691	0.493841	0.493212	0.479664	0.472407
		0.024020	0.011413	0.011422	0.011418	0.011438	0.011813	0.013025	0.013872
	20	0.498163	0.497941	0.499603	0.496279	0.494617	0.494423	0.487152	0.483395
		0.011601	0.005787	0.005783	0.005797	0.005812	0.005910	0.006251	0.006474
	30	0.502838	0.502116	0.503334	0.500899	0.499681	0.499604	0.494469	0.491846
		0.007896	0.00430	0.004307	0.004296	0.004295	0.004341	0.004465	0.004550
	50	0.497907	0.498000	0.498795	0.497205	0.496411	0.496369	0.493065	0.491393
		0.004915	0.003033	0.003030	0.003036	0.003041	0.003061	0.003133	0.003178
	100	0.499269	0.499646	0.500082	0.499209	0.498772	0.498762	0.496985	0.496091
		0.002365	0.001661	0.001661	0.001662	0.001662	0.001667	0.001684	0.001694
0.9	10	0.042168	0.039468	0.039622	0.039315	0.039163	0.035352	0.026263	0.021649
		0.001781	0.000528	0.000530	0.000527	0.000525	0.000587	0.000807	0.000959
	20	0.045160	0.046053	0.046180	0.045926	0.045800	0.043146	0.036971	0.033781
		0.001053	0.000394	0.000395	0.000392	0.000390	0.000400	0.000459	0.000513
	30	0.044577	0.047299	0.047403	0.047196	0.047092	0.045040	0.040359	0.037972
		0.000727	0.000357	0.000359	0.000356	0.000354	0.000354	0.000373	0.000396
	50	0.044447	0.048333	0.048408	0.048257	0.048182	0.046789	0.043676	0.042119
		0.000379	0.000279	0.000280	0.000278	0.000277	0.000270	0.000264	0.000268
	100	0.043858	0.046646	0.046685	0.046607	0.046568	0.045844	0.044252	0.043462
		0.000155	0.000165	0.000166	0.000165	0.000164	0.000161	0.000156	0.000156

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

TABLE 4. Avarage estimates and corresponding mean squared errors/risks of $\lambda(t)$ for different choise of n and t when $\alpha = 2.8$ and $\beta = 0.8$ where actual $\lambda(0.2) = 0.235$, $\lambda(0.5) = 1.530$ and $\lambda(0.9) = 28$.

t	n	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{B1}$	$\hat{\lambda}_{B2}$		
				$c = -0.5$	$c = 0.5$	$c = 1$
0.2	10	0.212659	0.313104	0.268433	0.179495	0.139538
		0.040227	0.059365	0.050593	0.042289	0.041897
	20	0.227699	0.279875	0.255613	0.206944	0.183191
		0.024096	0.032751	0.029739	0.026911	0.026982
	30	0.228798	0.260926	0.244609	0.211838	0.195616
		0.015070	0.017770	0.016607	0.015801	0.016135
	50	0.228658	0.247889	0.238138	0.218507	0.208703
		0.009430	0.010604	0.010247	0.010088	0.010284
	100	0.232933	0.242591	0.237783	0.228127	0.223305
		0.004708	0.005225	0.005131	0.005085	0.005132
0.5	10	1.527713	1.759105	1.692216	1.562116	1.497889
		0.645253	0.692297	0.626148	0.542050	0.522746
	20	1.523361	1.615436	1.587562	1.533140	1.506383
		0.227766	0.237356	0.224401	0.206981	0.202233
	30	1.508165	1.558570	1.542440	1.510611	1.494848
		0.103195	0.108809	0.105687	0.102085	0.101525
	50	1.515442	1.535820	1.527515	1.510964	1.502708
		0.056820	0.059281	0.058701	0.058129	0.058130
	100	1.526108	1.534501	1.530836	1.523501	1.519830
		0.024845	0.025726	0.025636	0.025553	0.025559
0.9	10	33.641810	34.949716	34.095302	32.380401	31.522558
		216.803883	232.935939	212.975812	177.496690	161.996142
	20	30.473946	30.805958	30.405633	29.597499	29.190319
		71.428341	71.478819	68.002931	62.007851	59.498323
	30	29.760384	29.848291	29.587420	29.059841	28.793272
		38.963410	40.020913	38.694668	36.473211	35.583057
	50	28.950512	28.947054	28.798466	28.498208	28.346406
		20.345468	21.397870	20.993493	20.333355	20.080180
	100	28.537238	28.599188	28.531815	28.396683	28.328937
		8.511153	9.343395	9.232614	9.041654	8.961717

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

TABLE 5. Avarage estimates and corresponding mean squared errors/risks of $\lambda(t)$ for different choise of n and t when $\alpha = 1.5$ and $\beta = 0.5$ where actual $\lambda(0.2) = 1.086$, $\lambda(0.5) = 3$ and $\lambda(0.9) = 15$.

t	n	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{B1}$	$\hat{\lambda}_{B2}$		
				$c = -0.5$	$c = 0.5$	$c = 1$
0.2	10	1.169160	1.138610	1.083326	0.969943	0.911189
		0.578881	0.282840	0.264464	0.256365	0.267456
	20	1.126438	1.130396	1.101100	1.042949	1.013914
		0.241296	0.119590	0.114397	0.110948	0.112758
	30	1.115528	1.136663	1.116351	1.076537	1.056966
		0.149128	0.083364	0.079938	0.076227	0.075905
	50	1.100535	1.127605	1.114830	1.089977	1.077876
		0.072287	0.047207	0.045266	0.042646	0.041937
	100	1.095927	1.119173	1.112864	1.100583	1.094604
		0.033968	0.029776	0.028847	0.027331	0.026735
0.5	10	3.163512	3.274372	3.189939	3.029371	2.953750
		2.061529	1.592072	1.445463	1.206419	1.113028
	20	2.955649	2.951840	2.909191	2.826440	2.786523
		0.805802	0.538925	0.521216	0.495661	0.487587
	30	2.891219	2.847796	2.817406	2.757840	2.728774
		0.535141	0.375209	0.373863	0.375581	0.378551
	50	2.873531	2.806055	2.786332	2.747164	2.727771
		0.324230	0.256955	0.261111	0.271222	0.277145
	100	2.866385	2.799225	2.787858	2.765075	2.753677
		0.175262	0.165486	0.169514	0.178249	0.182950
0.9	10	19.682280	20.553253	20.138921	19.332664	18.942773
		74.221823	77.820319	70.489912	57.178282	51.195827
	20	17.047429	17.258969	17.069952	16.697927	16.515474
		20.687151	19.057091	17.735260	15.325558	14.236774
	30	16.403576	16.389651	16.262977	16.011816	15.887572
		12.517064	11.323384	10.766275	9.747012	9.284522
	50	15.779786	15.640292	15.564618	15.413565	15.338271
		5.990385	5.627023	5.470833	5.190874	5.067049
	100	15.459294	15.358812	15.321466	15.246624	15.209147
		2.569578	2.791105	2.754111	2.688752	2.660413

*First rows in each coloumn represents the avarage estimates and the second rows represents the expected risks of the estimates.

second case ($t > \beta$) of the hazard function which is given in Eq. (3) is depend on only shape (α) parameter and dividing it to $(1-t)$ bring along large deviations even if small changes on α, the ERs under LLF do not provide consistent results. Also, many authors implied that LLF is not as appropriate for estimation of scale parameter as it is for location parameter and GELF is proposed as a suitable alternative to the modified LINEX loss function [2], [17]. Table 4, 5 show that the Bayes estimates under GELF has smaller expected risks and loss parameter $c = 0.5$ gives smallest risks for actual $\lambda > 1$ values of hazard function. On the contrary, MLE estimates has smaller risks while actual values converges to 0. In this case, ML gives better results than Bayes estimates in terms of ER. Similar to reliability estimates, the ERs of all hazard estimators decrease as sample size increases as expected.

6. Real Data Studies

In this section, a real data analysis is used to illustrate the proposed methods. In this purpose, breaking strengths of $1mm$ length single carbon fibers data, from Crowder [7], is used. We scaled the data by subtracting 2 and multiplying 5, respectively. Thus, the data lie in the interval $(0, 1)$. The sample size of the data is 58. The scaled data is given in Table 6.

Table 6. Re-scaled breaking strengths of $1mm$ length single carbon fibers data,($n = 58$).

0.0494	0.3570	0.5356	0.3362	0.5110	0.2656
0.2710	0.4222	0.6718	0.3824	0.5664	0.3456
0.3566	0.4914	0.1816	0.4432	0.7368	0.4126
0.4164	0.5272	0.3162	0.5084	0.2490	0.4652
0.4804	0.6268	0.3792	0.5476	0.3454	0.5264
0.5268	0.1684	0.4282	0.7142	0.4100	0.6086
0.6198	0.3144	0.5038	0.2252	0.4524	0.7996
0.8120	0,3572	0.5396	0.3452	0.5228	0.8120
0.1280	0.4236	0.6946	0.3928	0.5848	
0.2766	0.4932	0.2198	0.4502	0.7442	

We fit the STSP distribution to this dataset and we used maximum likelihood and Bayesian estimation methods. Estimations of the parameters α and β are reported in Table 7. Then, we applied to data Kolmogorov-Simirnov test to evaluate goodness of fit and test statistics are reported in Table 8, respectively. For sample size $n = 58$ and significance level $0,05$, the critical Kolmogorov-Simirnov test value is $D_{58,0.05} = 0,1783$. Thus, the null hypothesis that the data come from the STSP distribution cannot reject. Also, the QQ-plot and hazard plot, Fig. 8, support this observation.

Table 7. ML and Bayes estimates of the parameters for the real data set.

	MLE	SELF	LLF			GELF		
			$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
α	2.6704	2.6734	2.7060	2.6417	2.6111	2.6614	2.6373	2.6253
β	0.4164	0.4092	0.4097	0.4087	0.4083	0.4080	0.4056	0.4044

Estimates of reliability $R(t)$ and failure rate $\lambda(t)$ under maximum likelihood and Bayes method are obtained for different choise of t, say $t = 0.2, 0.4, 0.6, 0.8$, and reported in Table 9 and Table 10, respectively. We perform the algorithm which is given above for Bayes estimations with 100 000 iteration. We start the iteration with the maximum likelihood estimates of parameters and with these good starting values we prefer not to use burn-in operation. Also, we take every tenth variate as a independent and identically distributed observation in thinning procedure. Thus, a sample of 10 000 resulted which is used to calculate the posterior estimates. We used R program [20] to obtain the simulation results. Convergence of the simulated Markov chains is assessed by graphical methods.

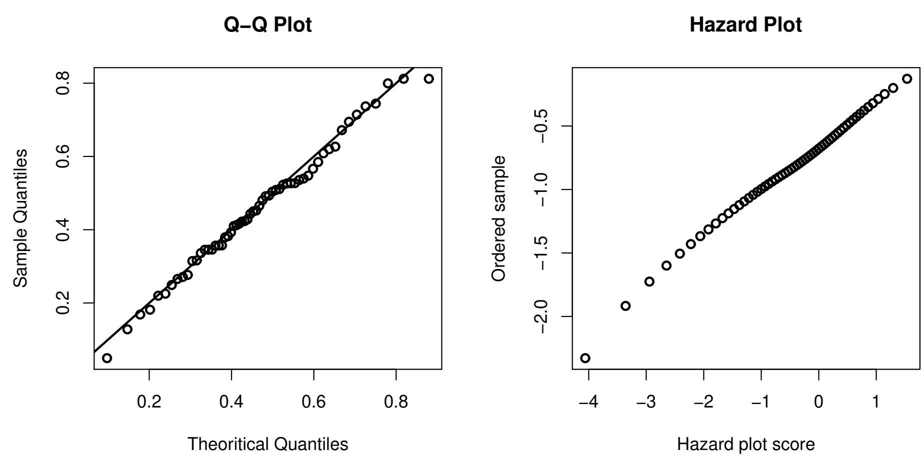

Figure 8. Q-Q and the hazard plots of the real dataset.

In this purpose, trace plots (Fig. 9, Fig. 10) which is a plot of the iteration number, t, against the value of the $R_B^{(t)}$ and $\lambda_B^{(t)}$ at each iteration. Also, density plots of the posterior distribution of the R and λ are drawn at the same time. It is observed that Markov chains fluctuates around their center with similar variation.

TABLE 8. Kolmogorov-Simirnov test statistics for the real data set. Kolmogorov-Simirnov critical test value $D_{58,0.05} = 0,1783$.

MLE	SELF	LLF			GELF		
		$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.1207	0.0862	0.1552	0.1034	0.0690	0.1379	0.1207	0.1379

TABLE 9. Reliability estimates of the real data set under various t values.

t	$\hat{R}_{ML}$	$\hat{R}_{B1}$	$\hat{R}_{B2}$			$\hat{R}_{B3}$		
			$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	0.9412	0.9363	0.9364	0.9362	0.9361	0.9362	0.9360	0.9359
0.4	0.6260	0.6137	0.6146	0.6129	0.6120	0.6123	0.6093	0.6078
0.6	0.2128	0.2142	0.2146	0.2139	0.2135	0.2125	0.2091	0.2074
0.8	0.0334	0.0354	0.0355	0.0354	0.0353	0.0341	0.0314	0.0300

The density plots seems in a symmetrical and unimodal shape. Morever, autocorrelation of the chains are evaluated and their plots are given in Fig. 11. The ACF plots show that thinning is succesful. Also, we computed the sample lag-t autocorrelation function by *autocorr* command in library *coda* [6] in R. For reliability estimates, the lag-10 autocorrelation is 0.02165095 and the lag-50 autocorrelation is -0.01679917. In addition to this, the lag-10 autocorrelation is 0.09367374 and the lag-50 autocorrelation is -0.02822016 for hazard estimates. Thus, we can say that convergence of the Markov chain is satisfactory.

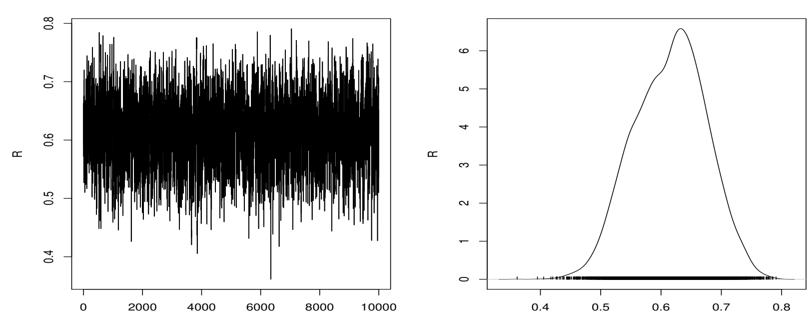

FIGURE 9. Trace plot of reliability estimates on the left and the density plot of the posterior distribution of reliability on the right.

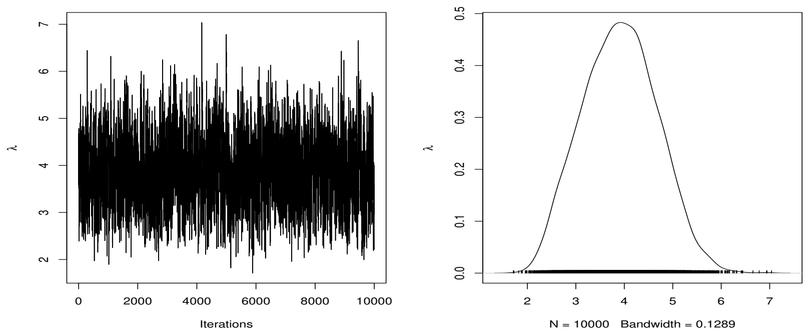

FIGURE 10. Trace plot of hazard estimates on the left and the density plot of the posterior distribution of hazard on the right.

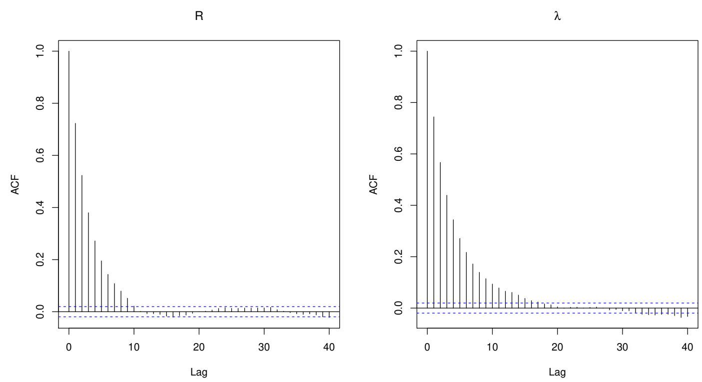

FIGURE 11. Autocorrelation plot for reliability estimates on the left and for hazard estimates on the right.

7. Bayesian Prediction

In this section, we studied Bayesian prediction of future ordered sample based on informative of current observed data. Let $y_{1:m}, y_{2:m}, \cdots, y_{m:m}$ be a future ordered observation independent of the given informative sample data $x_{1:n}, x_{2:n}, \cdots, x_{n:n}$. Then, Bayesian predictive density of the $s^{th}\{s = 1, 2, \cdots, m\}$ ordered future sample can be obtained by using

$$g_{s:m}(y|x) = \int_0^\infty \int_0^1 f_{s:m}(y|\alpha, \beta)\pi(\alpha, \beta|x)d\beta d\alpha$$

TABLE 10. Failure rate estimates of the real data set under various t values.

t	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{B1}$	$\hat{\lambda}_{B2}$			$\hat{\lambda}_{B3}$		
			$p=-0.5$	$p=0.5$	$p=1$	$c=-0.5$	$c=0.5$	$c=1$
0.2	0.8334	0.8852	0.8964	0.8745	0.8644	0.8732	0.8493	0.8374
0.4	3.9891	3.9164	4.0665	3.7712	3.6333	3.8779	3.7989	3.7584
0.6	6.6760	6.6874	6.8823	6.5091	6.3441	6.6597	6.6044	6.5767
0.8	13.3521	13.3790	14.2480	12.6505	12.0244	13.3201	13.2022	13.1433

where $\pi(\alpha,\beta|x)$ denotes the posterior density of the parameters and $f_{s:m}(y|\alpha,\beta)$ denotes the pdf of the s^{th} order statistic in the future sample as given in the following

$$f_{s:m}(y|\alpha,\beta)=\frac{m!}{(s-1)!(m-s)!}\big[F(y|\alpha,\beta)\big]^{s-1}\big[1-F(y|\alpha,\beta)\big]^{m-s}f(y|\alpha,\beta)$$

here $f(.|\alpha,\beta)$ denotes the pdf which is given in Eq. (1) and $F(.|\alpha,\beta)$ denotes the distribution function of the STSP distribution. Çetinkaya and Genç [8] studied the STSP distribution in detailed in terms of its order statistics. The density of the s^{th} order statistics is given as

$$f_{s:m}(y)=\alpha C_{m,s}\begin{cases}\beta^{(1-\alpha)s}\sum_{i=0}^{m-s}(-1)^i\binom{m-s}{i}\beta^{i(1-\alpha)}x^{\alpha(s+i)-1} & ,0<y\leq\beta\\ (1-\beta)^{\varphi_1}\sum_{i=0}^{s-1}(-1)^i\binom{r-1}{i}(1-\beta)^{i(1-\alpha)}(1-x)^{\varphi_2} & ,\beta\leq y<1\end{cases}$$

where $C_{m,s}=\frac{m!}{(s-1)!(m-s)!}$, $\varphi_1=(1-\alpha)(m-s+1)$ and $\varphi_2=\alpha(i+m-s+1)-1$. If we denote the predictive density of $y_{s:m}$ as $\hat{g}_{s:m}(y|x)$, it can be obtained by using

$$\hat{g}_{s:m}(y|x)=\int_0^{\infty}\int_0^1 f_{s:m}(y|\alpha,\beta)\pi(\alpha,\beta|x)d\beta d\alpha \tag{8}$$

However, it is be noted that Eq. (8) cannot be expressed in closed form and hence it cannot be evaluated analytically. Thus, we propose a simulation consistent estimator of $\hat{g}_{s:m}(y|x)$, which can be obtained by using Gibbs sampling MCMC method described in Section 4. Let suppose that MCMC sample $\{(\alpha_i,\beta_i);i=1,2,\cdots,M\}$ obtained from $\pi(\alpha,\beta|x)$ using the algorithm given in Section 4, then a simulation consistent estimator of $\hat{g}_{s:m}(y|x)$ can be obtained as

$$\hat{g}_{s:m}(y|x)=\frac{1}{M}\sum_{i=1}^{M}f_{s:m}(y|\alpha_i,\beta_i)$$

Further, a simulation consistent estimator of predictive distribution of s^{th} order statistics, say $\hat{G}_{s:m}(y|x)$, can be obtained as

$$\hat{G}_{s:m}(y|x)=\frac{1}{M}\sum_{i=1}^{M}F_{s:m}(y|\alpha_i,\beta_i)$$

where $F_{s:m}(y|\alpha,\beta)$ denotes the distribution function of the s^{th} order statistics, i.e. $F_{s:m}(y|\alpha,\beta) = C_{m,s}\int_0^y [F(z|\alpha,\beta)]^{s-1}[1-F(z|\alpha,\beta)]^{m-s} f(z|\alpha,\beta)dz$

$$= C_{m,s}\begin{cases} B(\beta(\frac{y}{\beta})^{\alpha}; s, m-s+1) & ,0<y\leq\beta \\ B(s,m-s+1) - B\biggl((1-\beta)(\frac{1-y}{1-\beta})^{\alpha}, m-s+1, s\biggr) & ,\beta\leq y<1 \end{cases}$$

here $B(a,b)$ denotes the beta function and $B(\cdot,a,b)$ denotes the incomplete beta function. It should be note that, $\hat{g}_{s:m}(y|x)$ is not a point prediction, it is a predictive density. The point prediction for the future observations under squared error loss function can be obtained as in the following

$$\hat{Y}_S = \int_0^1 y\hat{g}_{s:m}(y|x) = \frac{1}{M}\sum_{i=1}^{M}\int_0^1 y f_{s:m}(y|\alpha_i,\beta_i)dy = \frac{1}{M}\sum_{i=1}^{M}\mu_{s:m}$$

where $\mu_{s:m}$ is the first moment of s^{th} order statistics of the STSP distribution and it was given by Çetinkaya and Genç [8] as in the following

$$\mu_{s:m} = C_{m,s}\biggl[\beta^{1-1/\alpha}B(\beta;1/\alpha+s,m-s+1) + B(1-\beta;m-s+1,s) \\ -(1-\beta)^{1-1/\alpha}B(1-\beta;1/\alpha+m-s+1,s)\biggr]$$

Finally, point estimation under SELF, denoted by $\hat{Y}_S$, can be obtained as in the following;

$$\hat{Y}_S = \frac{1}{M}\sum_{i=1}^{M} C_{m,s}\biggl[\beta_i^{1-1/\alpha_i}B(\beta_i;1/\alpha_i+s,m-s+1) + B(1-\beta_i;m-s+1,s) \\ -(1-\beta_i)^{1-1/\alpha_i}B(1-\beta_i;1/\alpha_i+m-s+1,s)\biggr] \tag{9}$$

Further, point prediction under general entropy loss function, denoted by $\hat{Y}_G$ can be obtained as

$$\hat{Y}_G = \biggl[\int_0^1 y^{-c}\hat{g}_{s:m}(y|x)\biggr]^{-1/c} = \biggl[\frac{1}{M}\sum_{i=1}^{M}\int_0^1 y^{-c} f_{s:m}(y|\alpha_i,\beta_i)dy\biggr]^{-1/c}$$

Then, solution of this integral is obtained as

$$\int_0^1 y^{-c} f_{s:m}(y|\alpha,\beta)dy = \int_0^{\beta} y^{-c}\biggl[\beta\biggl(\frac{y}{\beta}\biggr)^{\alpha}\biggr]^{s-1}\biggl[1-\beta\biggl(\frac{y}{\beta}\biggr)^{\alpha}\biggr]^{m-s}\alpha\biggl(\frac{y}{\beta}\biggr)^{\alpha-1}dy \\ +\int_{\beta}^{1} y^{-c}\biggl[1-(1-\beta)\biggl(\frac{1-y}{1-\beta}\biggr)^{\alpha}\biggr]^{s-1}\biggl[(1-\beta)\biggl(\frac{1-y}{1-\beta}\biggr)^{\alpha}\biggr]^{m-s}\alpha\biggl(\frac{1-y}{1-\beta}\biggr)^{\alpha-1}dy$$

In the first integral, by change of variable $U = \beta\left(\frac{y}{\beta}\right)^{\alpha}$ and binomial expansion for $\left[1-(1-\beta)\left(\frac{1-y}{1-\beta}\right)^{\alpha}\right]^{s-1}$ and $1-y=v$ transformation in the second integral, the solution can be obtained as

$$\int_0^1 y^{-c} f_{s:m}(y|\alpha,\beta)dy = \beta^{c(1/\alpha-1)}B(\beta; s-c/\alpha, m-s+1) + \alpha\sum_{j=0}^{s-1}\binom{s-1}{j}$$
$$(-1)^j(1-\beta)^{(1-\alpha)(m-s+j+1)}B(1-\beta; \alpha(m-s+j+1, 1-c))$$

where $c<1$. Thus; $\hat{Y}_G$ can be obtained as

$$\hat{Y}_G = \left[\frac{1}{M}\sum_{i=1}^{M} C_{m,s}\left(\beta_i^{c(1/\alpha_i-1)}B(\beta_i; s-c/\alpha_i, m-s+1) + \alpha_i\sum_{j=0}^{s-1}\binom{s-1}{j}\right.\right.$$
$$\left.\left.(-1)^j(1-\beta_i)^{(1-\alpha_i)(m-s+j+1)}B(1-\beta_i; \alpha_i(m-s+j+1), 1-c)\right)\right]^{-1/c} \tag{10}$$

Moreover, we can construct a $100\gamma\%$ predictive interval for $y_{s:m}$. A symmetrical predictive interval for future sample can be obtained by solving the following non-linear equations for the lower bound L and upper bound U,

$$\begin{aligned}\frac{1+\gamma}{2} &= P(Y_{s:m} > L|data) = 1 - F^*_{s:m}(L|data) \Longrightarrow F^*_{s:m}(L|data) = \frac{1-\gamma}{2}\\ \frac{1-\gamma}{2} &= P(Y_{s:m} > U|data) = 1 - F^*_{s:m}(U|data) \Longrightarrow F^*_{s:m}(U|data) = \frac{1+\gamma}{2}\end{aligned} \tag{11}$$

It is not possible to obtain the solutions analytically and we need to apply suitable numerical techniques for solving these nonlinear equations.

Example. Under the future prediction framework, the prediction values of the first two and last two observations of future sample,$y_{1:m}$,$y_{2:m}$,$y_{m-1:m}$ and $y_{m:m}$, of size $m = 5, 10, 15, 20$ based on real data given in Sec. 6 are obtained with their constructed 95% symmetric predictive interval and reported in Table 11. We performed similar algorithm process which in given in Sec. 4 with the iteration number $M = 10000$ and we used Eq. (9) and Eq. (10) to obtain prediction and Eq. (11) for their predictive intervals. We take the first 500 values as burn-in period and take every third variable as a thinng procedure. Estimations are obtained under symmetric (SELF) and asymmetric (GELF) loss functions and they are represented with their expected risks. Under GELF, three different loss parameters ($c = -0.5, c = 0.5, c = 0.75$) are considered. For example, based on given real data set, prediction of the first observation of a future sample with size $m = 5$ is obtained as 0.263357 with ER 0.000419 under SELF and 0.252411 with ER 0.000964 under SELF ($c = -0.5$). Table 11 shows that predictions are closer to each other for the last observations. For all sample sizes and orders, GELF with $c = -0.5$ loss parameter has smallest expected risks. Prediction intervals are getting shorter by increasing sample size m for each order.

8. Conclusions

Mance et al. [13] first considered the TSP distribution under reliability properties. Recently, moments of order statistics and stress-strength reliability estimation under the STSP distribution were studied by Çetinkaya and Genç [8], [9]. In this study, the STSP distribution is considered as a further research in statistical reliability analysis. In this purpose, we introduced the importance of the distribution as defined on a finite range and two-sided distribution in reliability context. Particular reliability indices with their plots are presented. It has both convex and concave reliability curves according to various cases of its parameters. Also, it has bathtube failure rate for $\alpha < 1$ so it is useful for modelling early life, useful life and wear out proccesses of a component with only single model. By considering the behaviour of the hazard function, the STSP distribution is IFR class of distribution for $\alpha > 1$ and has better chance of surviving any shorter period and the worse chance of surviving any larger period. For the various cases of its parameters, it has both increasing and decreasing failure rate. We showed that the hazard plot is usable to determine if a data comes from the STSP distribution or not. Estimation of the reliability and hazard rate of the STSP distribution are obtained with maximum likelihood method and Bayesian estimation method under different loss functions. Loss functions are considered as symmetrical (SELF) and asymmetrical (LLF and GELF). Based on reliability and hazard estimation studies, our conclusions can be listed as follows;

- In all cases ($t \leq \beta, t > \beta$), maximum likelihood estimates tend to give overestimates.
- Being underestimating or overestimating is not only depend on loss parameters, it is also related to relation between t and β.
- Bayes estimates under squared error $\hat{R}_{B1}$ and Linex loss functions $\hat{R}_{B2}$ gives under estimates for $t \leq \beta$ and over estimates for $t > \beta$.
- Bayes estimates under general entropy loss function $\hat{R}_{B3}$ gives under estimates for $t \leq \beta$. On the other hand, for $t > \beta$ it gives under estimate for $c = 0.5$ and $c = 1$.
- Linex loss function is not proposed to obtain consistent estimations for hazard rate since the second case of the hazard function in Eq. 3 brings along large deviations even if small changes on α.
- For $\lambda > 1$ actual values of hazard function, Bayes estimates under GELF has smaller expected risks and loss parameter $c = 0.5$ gives smallest risks.
- MLEs of hazard rate have smaller risks while actual values converges to zero.
- While actual values of hazard rate λ converges to zero, ML gives better results than Bayes estimates in terms of expected risks.

TABLE 11. Bayesian point future predictions under SELF and GELF, their expected risks and corresponding predictive bounds for various sample size (m) and the first and last two ordered samples (r) based on given real dataset.

		Bayes Point Predictors				
			GELF			
m	r	SELF	$c=-0.5$	$c=0.5$	$c=0.75$	Prediction Interval
5	1	0.263357	0.252411	0.225168	0.216594	0.078675 0.445661
		(0.000419)	(0.000964)	(0.003864)	(0.012944)	(0.366985)
	2	0.368356	0.362191	0.348675	0.344976	0.191825 0.504161
		(0.000397)	(0.000400)	(0.000741)	(0.002049)	(0.312337)
	4	0.542520	0.537029	0.525876	0.523051	0.364299 0.763211
		(0.000536)	(0.000237)	(0.000346)	(0.000886)	(0.398912)
	5	0.663595	0.657488	0.644880	0.641650	0.432850 0.897675
		(0.000640)	(0.000190)	(0.000282)	(0.000726)	(0.464825)
10	1	0.211261	0.202036	0.179387	0.172369	0.062818 0.370456
		(0.000411)	(0.001357)	(0.004510)	(0.014746)	(0.307638)
	2	0.285261	0.279898	0.267940	0.264628	0.142756 0.419541
		(0.000430)	(0.000701)	(0.001144)	(0.003090)	(0.276785)
	9	0.638138	0.634510	0.627164	0.625311	0.468758 0.817628
		(0.000981)	(0.000300)	(0.000334)	(0.000786)	(0.348871)
	10	0.744783	0.740974	0.733117	0.731103	0.543045 0.928041
		(0.000817)	(0.000185)	(0.000213)	(0.000508)	(0.384996)
15	1	0.175294	0.166855	0.146081	0.139594	0.049536 0.317008
		(0.000378)	(0.001819)	(0.005730)	(0.018808)	(0.267472)
	2	0.243967	0.238845	0.227423	0.224256	0.120202 0.366792
		(0.000559)	(0.001252)	(0.001789)	(0.004687)	(0.246590)
	14	0.690972	0.688235	0.682689	0.681288	0.534454 0.848207
		(0.000893)	(0.000234)	(0.000250)	(0.000580)	(0.313753)
	15	0.774991	0.772149	0.766314	0.764824	0.595999 0.935209
		(0.000720)	(0.000151)	(0.000165)	(0.000385)	(0.339210)
20	1	0.166227	0.158357	0.139102	0.133144	0.048739 0.298049
		(0.000447)	(0.002244)	(0.005961)	(0.019004)	(0.249311)
	2	0.225341	0.220638	0.210224	0.207360	0.111514 0.340080
		(0.000470)	(0.001219)	(0.001751)	(0.004582)	(0.228566)
	19	0.721969	0.719759	0.715280	0.714149	0.578303 0.863639
		(0.000883)	(0.000211)	(0.000221)	(0.000507)	(0.285337)
	20	0.793014	0.790657	0.785832	0.784603	0.629258 0.939188
		(0.000756)	(0.000151)	(0.000160)	(0.000369)	(0.309930)

*First rows in each coloumn represents the point estimation values and prediction interval (last coloumn), the second rows in brackets represents the expected risks of the estimates and length of prediction interval (last coloumn).

All obtained results are illustrated with a real data example. The reliability and hazard rate estimates for various fixed point are obtained. Convergency of the obtained Markov chain is checked and consistent estimations are reported. Finally, we obtained the prediction of the future observations based on given datasets. For various sample size, the first two and last two observations are predicted with their prediction interval.

There are still some other problems concerning the STSP distribution. For example, censored or truncated sampling schemes may be considered in the frame of reliability estimation and prediction.

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PREDICTING CREDIT CARD CUSTOMER CHURN USING SUPPORT VECTOR MACHINE BASED ON BAYESIAN OPTIMIZATION

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Abstract. In this study, we have employed a hybrid machine learning algorithm to predict credit card customer churn. The proposed model is Support Vector Machine (SVM) with Bayesian Optimization (BO). BO is used to optimize the hyper-parameters of the SVM. Four different kernels are utilized. The hyper-parameters of the utilized kernels are calculated by the BO. The prediction power of the proposed models is compared by four different evaluation metrics. Used metrics are accuracy, precision, recall and F_1-score. According to each metrics linear kernel has the highest performance. It has accuracy of %91. The worst performance achieved by sigmoid kernel which has accuracy of %84.

1. Introduction

Customer churn is a business term expression which describes loss of customers. Firms invest in order not to lose their customers. Marketing departments continuously investigate the behavior of their existing customers and potential customers to understand the underlying causes of churn. These investigations are costly and time consuming. For that reason, in this study we propose a hybrid machine learning algorithm to predict customer churn of a bank by using the available data. We propose a model based on Support Vector Machine (SVM) which has many applications on regression and classifications. We utilized SVM as the classifier in this study because it ensure to use the technique called kernel transformations, projects the features space to a higher dimension, which makes it easier to find the bound between the classification objects. These kernels are non-linear so SVM can

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capture complex relations between the observations without making complex calculations. Some application areas of SVM are financial bubble detection [1], stock market movement forecasting [2], financial time series forecasting [3], oil price forecasting [4] and air pollution modelling [5].

The SVM has three hyper-parameters. The first one is C. It is the penalty parameter and it tells the magnitude of the margin of the hyperplane. Large values of C imply small margin while small values of C imply large margins. The second is the kernels. These can be radial basis, polynomial or sigmoid. The last one is the γ parameter. It decides the curvature of the hyperplane. A high value indicates more curvature while a low value represents less curvature. The parameters can not be predicted by the algorithm itself. They can be defined by the user or optimization algorithms can be employed to decide these parameters. In this study we use Bayesian Optimization to handle the hyper-parameter optimization problem.

[1] compares SVM with artificial neural networks (ANN), k-nearest neighbours (KNN) decision tress (DT), random forest (RF) and logistic regression (LR) to predict financial bubbles in the S&P 500 index. Their findings show that SVM is favourable among the others with almost %95 accuracy. [2] compares the performance of SVM with Linear Discriminant Analysis, Elman Backpropagation Neural Networks and Quadratic Discriminant Analysis to predict the markets movements of NIKKEI 225. Their results show that SVM outperforms the other classifiers. [3] compares SVM with multi-layer back-propagation (BP) neural network to forecast five futures contracts of Chicago Mercantile Market. The authors show that SVM outperforms BP based on weighted directional symmetry, mean absolute error, directional symmetry and normalized mean square error. [4] investigated the prediction power of SVM on oil price forecasting and compared it with auto regressive moving average (ARIMA) and BP. The findings show that the prediction power of SVM outperforms the others. Lastly, [5] use SVM to predict air pollution in the urban areas of Honk Kong and the proposed model compared with ANN. The findings reveal that SVM performs better than ANN. The literature above mentioned provides the necessary evidence of the performs of SVM in both classification and regression. For that reason, in this study we chose our classifier as SVM.

Summary of some related works which employ machine learning algorithms to predict customer churn are given in this paragraph. Customer churn prediction based on textual data is studied by [6]. The Convolution Neural Network (CNN) is proposed as the model. The data set contains structured information with textual information. The results show that using textual data as a feature of the model increases the performance of the proposed model. [7] use churn rate of the customer to predict the electricity sales of the power market. Credit card churn prediction is done by [8]. The used models are logistic regression and decision tree based methods. The comparison of the models show that logistic regression performs better than the tree algorithms. Extended SVM (E-SVM) and ANN are proposed by [9] to model customer churn in e-commerce sector. The results show that E-SVM has better performance based on accuracy, coverage rate, hit ratio and lift coefficient. Also, it is noted that the new algorithm handles data well when imbalanced is an issue. [10] propose SVM and RF to predict customer churn of telecom sector and the results reveal that the investigate learning models behave similarly. Ten different machine learning algorithms are compared by [11] to classify customer churn. The findings of the study indicate that best performance achived by RF and ADA boost with almost %96 accuracy and SVM with %94 accuracy. Some other recent machine learning approach on customer churn predictions are [12], [13], [14] and [15].

The remainder of this paper is organized as follows. Section 2 devoted to the methodology. Data and experimental results are given in Section 3 and finally Section 4 concludes the study.

2. Methodology

2.1. **Support Vector Machine.** Support vector machine is a supervised machine learning algorithm that can be used for regression or classification. It is introduced by [16]. The main idea under the algorithm is to find a hyperplane to separate a data set into multiple classes. For instance, if there are two linearly separable classes in a data set, multiple lines can divide the data into two parts. SVM proposes to find the line which maximize the margin between the closest data points. These data points are called support vectors. For more than two separable case the algorithm uses hyperplane for classification. If the data set contains classes which are not linearly separable than kernel tricks are used. It is the transformation of the features to the higher dimensions which makes it easier to separate.

Suppose it is given a data set which has n observations of d variables with features $(x_1, x_2, ..., x_d)$ where $x_i \in \mathbb{R}^n$ and labels $(y_1, y_2, ..., y_n)$ where $y_i \in \{-1, 1\}$. Define the linear classifier

$$y(x) = \text{sign}(w^T x + b), \tag{1}$$

where w is the weight vector and b is the bias term. If the data set is linearly separable than the hyperplane $w^T x + b$ separates the two class as:

$$\begin{aligned} w^T x + b &\geq 1 \quad \text{for} \quad y = 1, \\ w^T x + b &< 1 \quad \text{for} \quad y = -1. \end{aligned} \tag{2}$$

These two equations can be combined in one equations by multiplying both by y that is

$$y(w^T x + b) \geq 1. \tag{3}$$

The margin between the support vectors and the hyperplane is $\frac{2}{\|w\|}$. The optimal solution is found by maximizing the margin that is to minimize the length of w:

$$\min \frac{1}{2}\|w\|^2 \quad \text{s.t} \quad y(w^T x + b) \geq 1. \tag{4}$$

Solution for the above optimization problem can be obtained by using the Lagrange's method as

$$L(w, b, \alpha) = \frac{1}{2}\|w\|^2 - \sum_{i=1}^{n} \alpha_i (y_i(x_i w^T + b) - 1), \tag{5}$$

where

$$w = \sum_{i=1}^{n} \alpha_i y_i x_i, \tag{6}$$

and α is the non-negative Lagrange multiplier. The classifier for the linear case can be obtained as

$$f(x) = \operatorname{sign}\left(\sum_{i=1}^{n} \alpha_i y_i (x^T x_i) + b\right). \tag{7}$$

In the non linear case the classifier transformed to

$$f(x) = \operatorname{sign}\left(\sum_{i=1}^{n} \alpha_i y_i K(x, x_i) + b\right), \tag{8}$$

where $K(x_i, x_j)$ is the kernel function of the form

$$K(x_i, x_j) = \phi(x_i)^T \phi(x_j). \tag{9}$$

Mostly used kernels are:

$$\text{Radial basis kernel} : K(x_i, x_j) = \exp(-\gamma \|x_i - x_j\|^2),$$
$$\text{Polynomial kernel} : K(x_i, x_j) = (\gamma {x_i}^T x_j + r)^d,$$
$$\text{Sigmoid kernel} : K(x_i, x_j) = \tanh(\gamma {x_i}^T x_j + r).$$

2.2. **Bayesian Optimization.** Bayesian optimization is an iterative optimization which is very popular in hyper-parameter optimization of machine learning algorithms [17]. It searches and finds the candidate values based on previously obtained values. It contains two important elements called acquisition function and surrogate model [18]. The observed data points are fit into an objective function by the surrogate model. The acquisition function determines which points are used to balance the distribution of the surrogate model by evaluating the arrangements between exploration and exploitation [19]. Exploration is the process to search the upsampled area while the exploitation is the process of searching the most promising area in which the global minima or maxima may occurs.

In this paragraph we try to summarize Bayesian Optimization based on the work [17]. Firstly, the algorithm builds a surrogate model for the objective function. Secondly, using the surrogate model, it determines the optimal parameter values. Thirdly, the determined values are tested in the real objective function. Finally, the surrogate model is updated by the new results. These procedure repeats until the maximum number of iterations are achieved based on the initially surrogate model. Gaussian process can be given as a classic example of a surrogate model.

This algorithm is more efficient than grid search and random search, for that reason it is employed in this study.

2.3. **Evaluation Metrics.** We use 4 different evaluation metrics to test the performance of the proposed hybrid model. These are precision, recall, F_1-score and accuracy. Precision is the ratio of true positives to the sum of true positives and false positives.

$$\text{Precision} = \frac{\text{True Positive}}{\text{True Positive} + \text{False Positive}}.$$

It measures the classifier ability to not to label a sample as positive which is negative. Recall is the ratio of true positives to the sum of true positives and false negatives.

$$\text{Recall} = \frac{\text{True Positive}}{\text{True Positive} + \text{False Negatives}}.$$

It measures the classifier ability to identify the all positive sample points. F_1-score is the weighted average of precision and recall. It can take values between 0 and 1. The performance of the algorithm is at the best when takes value 1 or near to 1. In the same manner it is the worst when takes 0 or values very near to 0. It is calculated by the following formula:

$$F_1 = 2\frac{\text{Precision} * \text{Recall}}{\text{Precision} + \text{Recall}}.$$

Finally, accuracy is the fraction that the model predicts correctly. It is calculated as the ratio of sum of the total true positive and true negative to the total predictions. That is

$$\text{Accuracy} = \frac{\text{True Positive} + \text{True Negative}}{\text{True Positive} + \text{True Negative} + \text{False Positive} + \text{False Negative}}.$$

It can take values between 1 and 0. If the performance of the model is high, it will take values near to 1, otherwise near to 0.

3. Data and Analysis

The data set for this study is obtained from the Kaggle [20] which is a machine learning and data science community. The data set contains 20 variables and each contains 10127 observations with no missing values. The variable with their descriptions are given in Table 1.

The data set contains categorical and numerical variables. Categorical variables are: AF, EL, MS, IC, GN, ID and CC. AF is the target variable. Numerical variables are CA, DC, MB, TR, MI, CC12, CL, OB, TA, TT, TC, TC4, TR, RB and AU. Categorical variables are converted to binary and one hot encoding. The target variable AF takes 1 if it is existing customer and 0 otherwise. GN takes value 0 if it is male and 1 otherwise. The rest of the categorical variables are converted to the one hot encoding format.

TABLE 1. Variables with their descriptions

	Variables	Descriptions
1	ID	Unique identifier for the customer.
2	Attrition Flag (AF)	Customer activity. If the account is closed takes value 1 otherwise 0.
3	Customer Age (CA)	Demographic variable. Customer ages in years.
4	Gender (GN)	Demographic variable. M for male and F for female.
5	Dependent Count (DC)	Number of dependents.
6	Education Level (EL)	Demographic variable which takes the values of high school, graduate, college, post-graduate, doctorate, uneducated and unknown.
7	Marital Status (MS)	Demographic variable which can be married, single, divorced and unknown.
8	Income Category (IC)	Demographic variable which can be less than 40000$, between 40000$ and 60000$, between 60000$ and 80000$, between 800000$ and 120000$ and greater than 120000$.
9	Card Category (CC)	Product variable which represent the card category. It takes the values of blue, silver, gold and platinum.
10	Months on Book (MB)	Represents the time of period with the bank.
11	Months Inactive (MI)	Number of months in active in the last 12 months.
12	Contacts Count (CC12)	Number of contacts in the last 12 months.
13	Credit Limit (CL)	The amount of credit limit on the card.
14	Revolving Balance (RB)	Total revolving balance on the card.
15	Open to Buy Credit (OB)	Average of open to buy credit on line of 12 months.
16	Total Amount of Changes (TA)	Change in transaction from the first quarter to the fourth quarter.
17	Total Transaction Amount (TT)	Total transaction amount of the last 12 months.
18	Total Transaction Count (TC)	Total transaction count of the last 12 months.
19	Total Change in Transaction Count (TC4)	Change in transaction count from the first quarter to the fourth quarter.
20	Total Relationship Count (TR)	Total number of products held by the customer.
21	Average Utilization Ratio (AU)	Average credit card utilization ratio.

As an example consider the transformation of CC:

$$\text{Card Category} = \begin{cases} 1 \quad 0 \quad 0 \quad 0 & \text{if card category} = \text{blue}, \\ 0 \quad 1 \quad 0 \quad 0 & \text{if card category} = \text{silver}, \\ 0 \quad 0 \quad 1 \quad 0 & \text{if card category} = \text{gold}, \\ 0 \quad 0 \quad 0 \quad 1 & \text{if card category} = \text{platinum}. \end{cases}$$

For categorical variables which has more than 2 different observations, one hot encoding is used. It is used because there is no ordinary relations between the observations. Otherwise, algorithms would assume natural ordering between the categorical variables which leads poor performance.

According to the data 16% of the customer leaving the bank while 86% staying. The vast majority of the customers are married and female level is slightly higher than the male proportion by 3% . Mostly, blue credit cards are used and in general income levels are less than 40000$. More than 30% of the credit card users have graduate level. The age of the customers are between 26 and 73. Lastly, credit card limits are between 1.438 and 34.516. Correlation between the numerical variables are given in Figure 1. The colour codes of the figure is given in the right hand side of the table. Light red implies strong positive correlation while dark purple implies negative correlations. It is seen that there exists high positive correlation between MA - CA, OB - CL, TC4 - TA and AU - RB, high negative correlation between AU - CL and AU - OB.

The data set divided into test and train set. The test set contains the %20 of the data while the rest is the train set.

We have started our analysis with the linear kernel. The best parameter for C, the penalty parameter, is obtained as 37.5598. On the train set the algorithm with the given parameters has %91 accuracy. The other metrics are given in the Table 2.

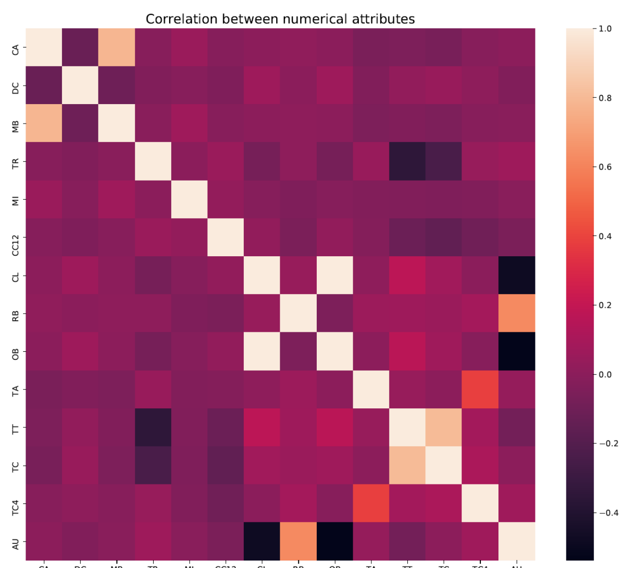

FIGURE 1. Correlation between numerical variables

Table 2. Evaluation Metrics for the Linear Kernel

Label	Precision	Recall	F_1-score	Support
0	0.78	0.60	0.68	319
1	0.93	0.97	0.95	1707
Macro Average	0.85	0.79	0.81	2026
Weighted Average	0.91	0.91	0.91	2026

As Table 2 shows linear kernel has weighted average, calculates the metrics for each label and takes the weighted average according to number of supports, of the precision, recall and F_1-score as 0.91 while it has accuracy of %91.

Secondly, polynomial kernel is utilized and by the help of the Bayesian optimization the best parameter for C is obtained as 0.28860 with $\gamma = 5.3504$. The accuracy of the train set with the given parameters are obtained as %87. The other metrics are given in Table 3.

Table 3. Evaluation Metrics for the Polynomial Kernel

Label	Precision	Recall	F_1-score	Support
0	0.61	0.63	0.62	319
1	0.93	0.92	0.93	1707
Macro Average	0.77	0.78	0.77	2026
Weighted Average	0.88	0.88	0.88	2026

As Table 3 shows polynomial kernel has weighted average of the precision as 0.77, recall as 0.78, F_1-score as 0.77 while it has accuracy of %88. It can be said that polynomial kernel is worse than the linear kernel according to the calculated metrics.

Thirdly, radial basis kernel is employed to predict credit card churns. The best parameter for C is obtained as 11.6085 with $\gamma = 3.2151$. The accuracy of the kernel in the train set is obtained as %86. The other metrics are given in Table 4.

Table 4. Evaluation Metrics for the Radial Kernel

Label	Precision	Recall	F_1-score	Support
0	0.62	0.31	0.42	319
1	0.88	0.96	0.92	1707
Macro Average	0.75	0.64	0.67	2026
Weighted Average	0.84	0.86	0.84	2026

As Table 4 shows radial kernel has weighted average of the precision as 0.75, recall as 0.64, F_1-score as 0.67 while it has accuracy of %86. It can be said that polynomial kernel is worse than the linear kernel and polynomial kernel according to the calculated metrics.

Lastly, sigmoid function is used as a kernel. The best parameters for the model are observed as $C = 45.4489$, $\gamma = 6.3796$. The model with these parameters have %83 accuracy. The metrics on the test set are given in Table 5.

Table 5. Evaluation Metrics for the Sigmoid Kernel

Label	Precision	Recall	F_1-score	Support
0	0.20	0.00	0.01	319
1	0.84	1.00	0.91	1707
Macro Average	0.52	0.50	0.46	2026
Weighted Average	0.74	0.84	0.77	2026

The worst result upon the investigated kernels are achieved by the sigmoid functions. The algorithm made 2026 forecasts and 2021 were identified as 1. As Table 5 shows it has accuracy of %84 while it has very low scores on precision, recall and F_1-score.

4. Conclusion

In this study, it is aimed to use a hybrid machine learning algorithm to classify the credit card churn of a bank. It is shown that the best kernel to predict churn behaviour of the customers is the SVM with linear kernel. Although, the data set is complex and contains many explanatory variables, a linear model fits the data better than the non-linear ones. The hyper-parameters of the algorithm is obtained by another algorithm called Bayesian optimization. Although, Bayesian optimization is not the only choice, it is utilized because of the flexibility and the speed of the algorithm. For the future studies the hyper-parameter optimizations tools can be compared and other machine learning and deep learning algorithms can be utilized to classify the churn behaviour of the customers.

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EMBEDDINGS BETWEEN WEIGHTED TANDORI AND CESÀRO FUNCTION SPACES

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Abstract. We characterize the weights for which the two-operator inequality

$$\left\| \left(\int_0^x f(t)^p v(t)^p dt \right)^{\frac{1}{p}} \right\|_{q,u,(0,\infty)} \le c \left\| \operatorname*{ess\,sup}_{t\in(x,\infty)} f(t) \right\|_{r,w,(0,\infty)}$$

holds for all non-negative measurable functions on $(0,\infty)$, where $0 < p < q \le \infty$ and $0 < r < \infty$, namely, we find the least constants in the embeddings between weighted Tandori and Cesàro function spaces. We use the combination of duality arguments for weighted Lebesgue spaces and weighted Tandori spaces with weighted estimates for the iterated integral operators.

1. INTRODUCTION

Given two function spaces X, Y and an operator T, a standard problem is characterizing the conditions for which T maps X into Y. If X and Y are (quasi) Banach spaces of measurable functions, a bounded operator $\mathrm{T} : X \to Y$ satisfies the inequality $\| \mathrm{T} f\|_Y \le c\|f\|_X$ for all $f \in X$ where $c \in (0,\infty)$. When T is the identity operator I, we say that X is embedded into Y and write $X \hookrightarrow Y$. The least constant c in the embedding $X \hookrightarrow Y$ is $\| \mathrm{I} \|_{X\to Y}$.

In this paper, we find the optimal constants in the embedding between weighted Tandori and Cesàro function spaces. We shall begin with the definitions of the function spaces considered in this paper.

Given a measurable function f on E, we set

$$\|f\|_{p,E} := \left(\int_E |f(x)|^p dx \right), \quad 1 \le p < \infty$$

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and

$$\|f\|_{\infty,E} := \operatorname*{ess\,sup}_{x\in E} |f(x)|, \quad p = \infty.$$

If w is a weight on E, that is, measurable, positive and finite a.e. on E, then we denote by $L_{p,w}(E)$[1], the weighted Lebesgue space, the set of measurable functions satisfying $\|f\|_{p,w,E} := \|fw\|_{p,E} < \infty$.

Let $0 < p, q \le \infty$, u be a non-negative measurable function and v be a weight, the weighted Cesàro space $\mathrm{Ces}_{p,q}(u,v)$ is the set of all measurable functions such that $\|f\|_{\mathrm{Ces}_{p,q}(u,v)} < \infty$, where

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} := \big\| \|f\|_{p,v,(0,x)} \big\|_{q,u,(0,\infty)};$$

and the weighted Copson space $\mathrm{Cop}_{p,q}(u,v)$ is the set of all measurable functions such that $\|f\|_{\mathrm{Cop}_{p,q}(u,v)} < \infty$, where

$$\|f\|_{\mathrm{Cop}_{p,q}(u,v)} := \big\| \|f\|_{p,v,(x,\infty)} \big\|_{q,u,(0,\infty)}.$$

The classical Cesàro spaces $\mathrm{Ces}_{1,p}(x^{-1},1)$, $1 \le p < \infty$ were defined by Shiue [20] in 1970. When $1 < p < \infty$ Hassard and Hussein [12] proved that $\mathrm{Ces}_{1,p}(x^{-1},1)$ are separable Banach spaces and Bennett [4] showed that the spaces $\mathrm{Ces}_{1,p}(x^{-1},1)$ and $\mathrm{Cop}_{1,p}(1,x^{-1})$ coincide. Dual spaces of the classical Cesàro function spaces were considered in [4, 21]. In [1], factorization theorems for classical Cesàro function spaces were given and based on these results the dual spaces of classical Cesàro function spaces were presented. One weighted Cesàro function spaces $\mathrm{Ces}_{1,p}(w^{\frac{1}{p}},1)$ and their duals were considered in [13]. Recently, in [3], factorization of the spaces $\mathrm{Ces}_{1,p}(x^{-1}w^{\frac{1}{p}},1)$ and $\mathrm{Cop}_{1,p}(w^{\frac{1}{p}},x^{-1})$ are given.

We do not aim to give a thorough set of references on the history of these spaces. Instead, we refer the interested reader to survey paper [2], where the comprehensive history on the structure of Cesàro and Copson function spaces are given.

In this paper our primary focus is the following inequality

$$\|f\|_{\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \le c\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)} \tag{1}$$

for all measurable functions where $0 < p_i, q_i \le \infty$, $i = 1, 2$.

There is more than one motivation to study inclusion between Cesàro and Copson spaces. First of all when $p_1 = q_1$ or $p_2 = q_2$, weighted Cesàro and Copson function spaces coincide with some weighted Lebesgue spaces (see [9, Lemmas 3.4-3.5]), thus inequality (1) is a generalization of the well-known weighted direct and reverse Hardy-type inequalities (e.g. [15, 7, 19]). Another justification is to give the characterization of pointwise multipliers between two spaces of Cesàro and Copson type, because it reduces to the characterization the embeddings between these spaces. In [11, Section 7] Grosse-Erdmann considered the multipliers between the spaces of p-summable sequences and Cesàro and Copson sequence spaces. He also introduced corresponding function spaces but the characterization of the multipliers

[1]When $E = (0,\infty)$, we simply write $L_{p,w}$ instead of $L_{p,w}(0,\infty)$.

between two spaces of Cesàro and Copson type remained open for both sequence and function spaces for a long time.

The characterization of the inequality (1) is given in one parameter case when $p_1 = p_2 = 1$, $q_1 = q_2 = p > 1$, $v_1(t) = t^{-\beta-1}$, $v_2(t) = t^{\alpha-1}$, $u_1(t) = t^{\beta-1/p}$ and $u_2(t) = t^{-\alpha-1/p}$, $t > 0$, $\alpha, \beta > 0$ in [5]. Moreover, it was shown that the inequality is reversed when $0 < p < 1$. In [6], inequality (1) is considered for two different parameters in the special case $p_1 = p_2 = 1$, $q_1 = p$, $q_2 = q$, $v_1(t) = t^{-1}$, $v_2(t) = 1$, $u_1(t)^p = v(t)$, $u_2(t)^q = w(t)t^{-q}$, $t > 0$, under the restriction $q \geq 1$ in order to characterize the embeddings between some Lorentz-type spaces. Recently, in [9] the two sided estimates for the best constant in (1) is given for four weights and four parameters $0 < p_1, p_2, q_1, q_2 < \infty$ under the restriction $p_2 \leq q_2$. Moreover, using these results, in [9, Theorems 3.11-3.12], the associate spaces of weighted Copson and Cesàro function spaces were characterized and in [10] pointwise multipliers between Cesàro and Copson function spaces are given for some ranges of parameters.

Furthermore, in 2015, Lesnik and Maligranda [16,17] began studying these spaces within an abstract framework, where they used a more general function space X instead of the weighted Lebesgue spaces. When X is a Banach space, they defined Cesàro space CX, Copson space C^*X and Tandori space $\widetilde{X}$ as the set of all measurable functions, respectively, with the following norms:

$$\|f\|_{CX} = \left\| \frac{1}{x} \int_0^x |f(t)| dt \right\|_X < \infty,$$

$$\|f\|_{C^*X} = \left\| \int_x^\infty \frac{|f(t)|}{t} dt \right\|_X < \infty,$$

$$\|f\|_{\widetilde{X}} = \left\| \operatorname*{ess\,sup}_{t \in (x,\infty)} |f(t)| \right\|_X < \infty.$$

In [18], they named $\widetilde{X}$ as the generalized Tandori spaces in honour of Tandori who provided dual spaces to the spaces $CL_\infty[0,1]$ in [22]. Their definition is related to our definition in the following way:

$$CL_{p,w} = \mathrm{Ces}_{1,p}(x^{-1}w(x), 1), \quad C^*L_{p,w} = \mathrm{Cop}_{1,p}(w, x^{-1}), \quad \widetilde{L}_{p,w} = \mathrm{Cop}_{\infty,p}(w, 1).$$

We should note that recently in [14] multipliers between CL_p and CL_q are given when $1 < q \leq p \leq \infty$.

We want to continue this research. In this paper, we will handle the inequality (1) when $p_1 = \infty$. In other words, we will consider the embeddings $\widetilde{L}_{r,w} \hookrightarrow \mathrm{Ces}_{p,q}(u, v)$, namely, we will give the characterization of the following inequality,

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} \leq C \|f\|_{\widetilde{L}_{r,w}} \tag{2}$$

for all measurable functions where $p, q, r \in (0, \infty)$ with $p < q$. The restriction on the parameters arises from the duality argument. The key ingredient of the proof is combining characterizations of the associate spaces of Tandori spaces, namely, the reverse Hardy-type inequality for supremal operators which was given in [19] with the characterizations of some iterated Hardy-type inequalities.

Throughout the paper, we put $0 \cdot \infty = \frac{0}{0} = 0$. We write $A \approx B$ if there exist positive constants α, β independent of relevant quantities appearing in expressions A and B such that

$$\alpha \leq \frac{A}{B} \leq \beta$$

holds.

The symbol $\mathfrak{M}$ will stand for the set of all measurable functions on $(0, \infty)$, and we denote the class of non-negative elements of $\mathfrak{M}$ by $\mathfrak{M}^+$.

We sometimes omit the differential element dx to make the formulas simpler when the expressions are too long.

The paper is structured as follows. In Section 2, we formulate the main results of this paper. In Section 3, we collect some properties and necessary background material. Finally, in the last section, we give the proofs of our main results.

2. MAIN RESULTS

It is convenient to start this section by recalling some properties of the weighted Cesàro and Copson spaces. Let $0 < p, q \leq \infty$. Assume that u is a non-negative measurable function and v is a weight. We will always assume that $\|u\|_{q,(t,\infty)} < \infty$ for all $t > 0$ and $\|u\|_{q,(0,t)} < \infty$ for all $t > 0$, when considering weighted Cesàro and Copson function spaces, respectively. Otherwise, these spaces consist only of functions equivalent to zero (see, [9, Lemmas 3.1-3.2]).

In this section, we will formulate the least constant in the embedding

$$\widetilde{L}_{r,w} \hookrightarrow \mathrm{Ces}_{p,q}(u, v). \tag{3}$$

Remark 1. *Observe that,*

$$\| \mathrm{I} \|_{\mathrm{Cop}_{\infty,r}(w,v_1) \to \mathrm{Ces}_{p,q}(u,v_2)} = \| \mathrm{I} \|_{\widetilde{L}_{r,w} \to \mathrm{Ces}_{p,q}(u,\frac{v_2}{v_1})}$$

holds. Therefore, it is enough to consider the three weighted case (3)*.*

Remark 2. *Note that, when $p = q$ or $r = \infty$, this problem is not interesting since it reduces to the characterizations of Hardy-type inequalities and can be found in [9], therefore we will consider the cases when $r < \infty$. On the other hand, we have the restriction $p < q$, which arises from the duality argument.*

Now we are in position to formulate the results of this paper. We begin with the cases where $q = \infty$.

Theorem 3. *Let $0 < p, r < \infty$. Assume that v is a weight, $w \in \mathfrak{M}^+$ such that $\|w\|_{r,(0,t)} < \infty$ for all $t \in (0, \infty)$ and $w \neq 0$ a.e. on $(0, \infty)$, and $u \in \mathfrak{M}^+$ such that $\|u\|_{\infty,(t,\infty)} < \infty$ for all $t \in (0, \infty)$.*

(i) If $r \leq p$, then

$$\| \mathrm{I} \|_{\widetilde{L}_{r,w} \to \mathrm{Ces}_{p,\infty}(u,v)} \approx I_1,$$

where

$$I_1 := \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x) \sup_{t\in(0,x)} \left(\int_0^t v^p\right)^{\frac{1}{p}} \left(\int_0^t w^r\right)^{-\frac{1}{r}} < \infty.$$

(ii) If $p < r$, *then*

$$\|\,\mathrm{I}\,\|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,\infty}(u,v)} \approx I_2 + I_3 + I_4$$

where

$$I_2 := \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\left(\int_0^x \left(\int_0^t v^p\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} < \infty,$$

$$I_3 := \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\left(\int_x^\infty \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} \left(\int_0^x v^p\right)^{\frac{1}{p}} < \infty,$$

and

$$I_4 := \left(\int_0^\infty w^r\right)^{-\frac{1}{r}} \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\left(\int_0^x v^p\right)^{\frac{1}{p}} < \infty.$$

When $q < \infty$, we consider the cases $r \le p$ and $p < r$ separately.

Theorem 4. *Let* $0 < r \le p < q < \infty$. *Assume that* $v \in \mathfrak{M}^+$, $w \in \mathfrak{M}^+$ *such that* $\|w\|_{r,(0,t)} < \infty$ *for all* $t \in (0,\infty)$ *and* $w \ne 0$ *a.e. on* $(0,\infty)$, *and* $u \in \mathfrak{M}^+$ *such that* $\|u\|_{q,(t,\infty)} < \infty$ *for all* $t \in (0,\infty)$. *Then*

$$\|\,\mathrm{I}\,\|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,q}(u,v)} \approx I_5 + I_6,$$

where

$$I_5 := \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \left(\int_0^t \left(\int_0^s v(y)^p dy\right)^{\frac{q}{p}} u(s)^q ds\right)^{\frac{1}{q}} < \infty,$$

and

$$I_6 := \sup_{t\in(0,\infty)} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}} \left(\int_0^t v(s)^p ds\right)^{\frac{1}{p}} \left(\int_t^\infty u(s)^q ds\right)^{\frac{1}{q}} < \infty.$$

Theorem 5. *Let* $0 < p < r < \infty$ *and* $0 < p < q < \infty$. *Assume that* $v \in \mathfrak{M}^+$, *such that* $v > 0$, $\|v\|_{p,(0,t)} < \infty$ *for all* $t \in (0,\infty)$ *and* $\|v\|_{p,(0,\infty)} = \infty$. *Suppose that* $w \in \mathfrak{M}^+$ *such that* $\|w\|_{r,(0,t)} < \infty$ *for all* $t \in (0,\infty)$ *and* $w \ne 0$ *a.e. on* $(0,\infty)$, *and* $u \in \mathfrak{M}^+$ *such that* $\|u\|_{q,(t,\infty)} < \infty$ *for all* $t \in (0,\infty)$. *Let*

- $\displaystyle\int_0^t \left(\int_0^s v^p\right)^{\frac{r}{r-p}} \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds < \infty$ *for all* $t \in (0,\infty)$,
- $\displaystyle\int_0^1 \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds = \infty$,

- $\int_t^\infty \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds < \infty$ *for all* $t \in (0,\infty)$,
- $\int_1^\infty \left(\int_0^s v^p\right)^{\frac{r}{r-p}} \left(\int_0^s w^r\right)^{-\frac{r}{r-p}} w(s)^r ds = \infty$

hold.

(i) If $r \le q$, *then*

$$\|\mathrm{I}\|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,q}(u,v)} \approx I_7 + I_8 + I_9,$$

where

$$I_7 := \left(\int_0^\infty w^r\right)^{-\frac{1}{r}} \left(\int_0^\infty \left(\int_0^y v(s)^p ds\right)^{\frac{q}{p}} u(y)^q dy\right)^{\frac{1}{q}} < \infty, \tag{4}$$

$$I_8 := \sup_{x\in(0,\infty)} \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} \left(\int_x^\infty u^q\right)^{\frac{1}{q}} < \infty,$$

and

$$I_9 := \sup_{x\in(0,\infty)} \left(\int_x^\infty \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r-p}{rp}} \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{q}{p}} u(t)^q dt\right)^{\frac{1}{q}} < \infty.$$

(ii) If $q < r$, *then*

$$\|\mathrm{I}\|_{\widetilde{L}_{r,w}\to \mathrm{Ces}_{p,q}(u,v)} \approx I_7 + I_{10} + I_{11},$$

where I_7 *is defined in* (4),

$$\begin{aligned} I_{10} := \Bigg(\int_0^\infty \left(\int_x^\infty u^q\right)^{\frac{r}{r-q}} \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{r}{r-p}} \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r(q-p)}{p(r-q)}} \\ \times \left(\int_0^x v^p\right)^{\frac{r}{r-p}} \left(\int_0^x w^r\right)^{-\frac{r}{r-p}} w(x)^r dx\Bigg)^{\frac{r-q}{rq}} < \infty, \end{aligned}$$

and

$$\begin{aligned} I_{11} := \Bigg(\int_0^\infty \left(\int_0^x \left(\int_0^t v^p\right)^{\frac{q}{p}} u(t)^q dt\right)^{\frac{r}{r-q}} \left(\int_x^\infty \left(\int_0^t w^r\right)^{-\frac{r}{r-p}} w(t)^r dt\right)^{\frac{r(q-p)}{p(r-q)}} \\ \times \left(\int_0^x w^r\right)^{-\frac{r}{r-p}} w(x)^r dx\Bigg)^{\frac{r-q}{rq}} < \infty. \end{aligned}$$

3. BACKGROUND MATERIAL

In this section we quote some known results. Let us start with the characterization of the reverse Hardy-type inequality for supremal operator, that is,

$$\left(\int_0^\infty f(t)^p \mathfrak{u}(t)^p dt\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \mathfrak{w}(t)^q \left(\operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right)^q dt\right)^{\frac{1}{q}} \tag{5}$$

for all non-negative measurable functions f on $(0,\infty)$ where $0<p,q<\infty$.

Theorem 6. *[19, Theorem 3.4] Let $0<p,q<\infty$. Assume that $\mathfrak{u}\in\mathfrak{M}^+$ and $\mathfrak{w}\in\mathfrak{M}^+$ such that $\int_0^t \mathfrak{w}^q<\infty$ for all $t\in(0,\infty)$ and $\mathfrak{w}\neq 0$ a.e. on $(0,\infty)$.*

(i) If $q\le p$, then inequality (5) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if $A_1<\infty$, where

$$A_1:=\sup_{t\in(0,\infty)}\bigg(\int_0^t \mathfrak{u}^p\bigg)^{\frac{1}{p}}\bigg(\int_0^t \mathfrak{w}^q\bigg)^{-\frac{1}{q}}. \tag{6}$$

Moreover, the least possible constant C in (5) satifies $C\approx A_1$.

(ii) If $p<q$, then inequality (5) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if $A_2<\infty$ and $A_3<\infty$, where

$$A_2:=\bigg(\int_0^\infty\bigg(\int_0^t \mathfrak{u}^p\bigg)^{\frac{q}{q-p}}\bigg(\int_0^t \mathfrak{w}^q\bigg)^{-\frac{q}{q-p}}\mathfrak{w}(t)^q dt\bigg)^{\frac{q-p}{pq}}, \tag{7}$$

and

$$A_3:=\bigg(\int_0^\infty \mathfrak{u}^p\bigg)^{\frac{1}{p}}\bigg(\int_0^\infty \mathfrak{w}^q\bigg)^{-\frac{1}{q}}. \tag{8}$$

Moreover, the least possible constant C in (5) satifies $C\approx A_2+A_3$.

We next recall the characterization of the weighted iterated inequality involving Hardy and Copson operators, that is,

$$\bigg(\int_0^\infty\bigg(\int_0^t\bigg(\int_s^\infty g\bigg)\mathfrak{v}(s)ds\bigg)^q\mathfrak{w}(t)^q dt\bigg)^{\frac{1}{q}}\le C\bigg(\int_0^\infty g(t)^p\mathfrak{u}(t)^p dt\bigg)^{\frac{1}{p}}. \tag{9}$$

Note that the characterization of inequality (9) is given in [8]. In the next theorem, we provide a modified version of [8, Theorem 3.1], using the gluing lemmas presented in the recent paper [10]. Denote by

$$\mathcal{V}(t):=\int_0^t \mathfrak{v}(s)ds,\ t>0.$$

Theorem 7. *Let $1<p<\infty$ and $0<q<\infty$. Assume that $\mathfrak{u}\in\mathfrak{M}^+$ and $\mathfrak{v},\mathfrak{w}\in\mathfrak{M}^+$ such that $\mathfrak{v}(t)>0$, $\mathcal{V}(t)<\infty$ for all $t\in(0,\infty)$ and $\mathcal{V}(\infty)=\infty$,*

- $\int_0^t \mathcal{V}(s)^q\mathfrak{w}(s)^q ds<\infty$ *for all* $t\in(0,\infty)$ *and* $\int_1^\infty \mathcal{V}(s)^q\mathfrak{w}(s)^q ds=\infty$,
- $\int_t^\infty \mathfrak{w}(s)^q ds<\infty$ *for all* $t\in(0,\infty)$ *and* $\int_0^1 \mathfrak{w}(s)^q ds=\infty$.

(i) If $p\le q$, then (9) holds for all non-negative measurable functions f on $(0,\infty)$ if and only if $B_1<\infty$ and $B_2<\infty$, where

$$B_1:=\sup_{x\in(0,\infty)}\bigg(\int_0^x \mathcal{V}(t)^q\mathfrak{w}(t)^q dt\bigg)^{\frac{1}{q}}\bigg(\int_x^\infty \mathfrak{u}(t)^{-\frac{p}{p-1}}dt\bigg)^{\frac{p-1}{p}},$$

and

$$B_2 := \sup_{x\in(0,\infty)} \left(\int_x^\infty \mathfrak{w}(t)^q dt\right)^{\frac{1}{q}} \left(\int_0^x \mathcal{V}(t)^{\frac{p}{p-1}} \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}.$$

Moreover, the least possible constant C in (9) *satifies $C \approx B_1 + B_2$.*

(ii) If $q < p$, then (9) *holds for all non-negative measurable functions f on* $(0,\infty)$ *if and only if $B_3 < \infty$ and $B_4 < \infty$, where*

$$B_3 := \left(\int_0^\infty \left(\int_x^\infty \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{q(p-1)}{p-q}} \left(\int_0^x \mathcal{V}(t)^q \mathfrak{w}(t)^q dt\right)^{\frac{q}{p-q}} \mathcal{V}(x)^q \mathfrak{w}(x)^q dx\right)^{\frac{p-q}{pq}},$$

and

$$B_4 := \left(\int_0^\infty \left(\int_x^\infty \mathfrak{w}(t)^q dt\right)^{\frac{q}{p-q}} \left(\int_0^x \mathcal{V}(t)^{\frac{p}{p-1}} \mathfrak{u}(t)^{-\frac{p}{p-1}} dt\right)^{\frac{q(p-1)}{p-q}} \mathfrak{w}(x)^q dx\right)^{\frac{p-q}{pq}}.$$

Moreover, the least possible constant C in (9) *satifies $C \approx B_3 + B_4$.*

Proof. The proof is the combination of [8, Theorem 3.1, (iii)] and [10, Lemma 2.7] for the first case and [8, Theorem 3.1, (iv)] and [10, Lemma 2.8] for the second case. □

4. PROOFS

Denote by

$$\mathcal{R}(p,r;v,w) := \sup_{f\in\mathfrak{M}^+} \frac{\|f\|_{p,v,(0,\infty)}}{\left\| \operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right\|_{r,w,(0,\infty)}}.$$

Proof of Theorem 3 Let $0 < p, r < \infty$. We have

$$C = \sup_{f\in\mathfrak{M}} \frac{\|f\|_{\mathrm{Ces}_{p,\infty}(u,v)}}{\|f\|_{\widetilde{L}_{r,w}}} = \sup_{f\in\mathfrak{M}^+} \frac{\operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\|f\|_{p,v,(0,x)}}{\left\| \operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right\|_{r,w,(0,\infty)}}.$$

Fix $x \in (0,\infty)$, then

$$C = \sup_{f\in\mathfrak{M}^+} \frac{\operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\|f\chi_{(0,x)}\|_{p,v,(0,\infty)}}{\left\| \operatorname*{ess\,sup}_{s\in(t,\infty)} f(s)\right\|_{r,w,(0,\infty)}}.$$

Observe that, interchanging supremum gives

$$C = \operatorname*{ess\,sup}_{x\in(0,\infty)} u(x)\mathcal{R}(p,r;\tilde{v},w),$$

where $\tilde{v}(t) = \chi_{(0,x)}(t)v(t)$, $t \in (0,\infty)$. Thus, the problem is reduced to the characterization of reverse Hardy-type inequalities for supremal operator. It remains to apply [Theorem 6, (i)] when $r \le p$ and [Theorem 6, (ii)] when $p < r$.

□

Proof of Theorem 4 Let $0 < r \le p < q < \infty$. We have

$$C = \sup_{f \in \mathfrak{M}} \frac{\|f\|_{\mathrm{Ces}_{p,q}(u,v)}}{\|f\|_{\widetilde{L}_{r,w}}}$$

Since $q/p \in (1,\infty)$, by the duality in weighted Lebesgue spaces, we have

$$\|f\|^p_{\mathrm{Ces}_{p,q}(u,v)} = \sup_{g \in \mathfrak{M}^+} \frac{\displaystyle\int_0^\infty \left(\int_0^t f(s)^p v(s)^p ds\right) g(t) dt}{\left(\displaystyle\int_0^\infty g(t)^{\frac{q}{q-p}} u(t)^{-\frac{qp}{q-p}} dt\right)^{\frac{q-p}{q}}}.$$

Interchanging supremum and Fubini's Theorem gives that

$$\begin{aligned} C &= \sup_{g \in \mathfrak{M}^+} \frac{1}{\left(\displaystyle\int_0^\infty g(t)^{\frac{q}{q-p}} u(t)^{-\frac{qp}{q-p}} dt\right)^{\frac{q-p}{qp}}} \sup_{f \in \mathfrak{M}^+} \frac{\left(\displaystyle\int_0^\infty f(s)^p v(s)^p \int_s^\infty g(t) dt\, ds\right)^{\frac{1}{p}}}{\left(\displaystyle\int_0^\infty \left(\operatorname*{ess\,sup}_{s \in (t,\infty)} f(s)\right)^r w(t)^r dt\right)^{\frac{1}{r}}} \\ &=: \sup_{g \in \mathfrak{M}^+} \frac{\mathcal{R}(p,r;\tilde{v},w)}{\|g\|^{\frac{1}{p}}} \end{aligned} \tag{10}$$

where, $\tilde{v}(s) = v(s)\big(\int_s^\infty g(t)dt\big)^{\frac{1}{p}}$, $s \in (0,\infty)$, and

$$\|g\| := \left(\int_0^\infty g(t)^{\frac{q}{q-p}} u(t)^{-\frac{qp}{q-p}} dt\right)^{\frac{q-p}{q}}.$$

Note that $\mathcal{R}(p,r;\tilde{v},w)$ is the best constant in the inequality

$$\left(\int_0^\infty h(s)^p v(s)^p \int_s^\infty g(t) dt\, ds\right)^{\frac{1}{p}} \le c \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{s \in (t,\infty)} h(s)\right)^r w(t)^r dt\right)^{\frac{1}{r}}, \ h \in \mathfrak{M}^+$$

for every fixed $g \in \mathfrak{M}^+$. Now, we can apply Theorem 6 by taking the parameters p, r, and weights

$$\mathfrak{w}(s) = w(s) \quad \mathfrak{u}(s) = v(s)\left(\int_s^\infty g\right)^{\frac{1}{p}}, \quad s > 0.$$

Since $r \le p$, according to the first case in Theorem 6,

$$\mathcal{R}(p,r;\tilde{v},w) \approx \sup_{t \in (0,\infty)} \left(\int_0^t v(s)^p \left(\int_s^\infty g\right) ds\right)^{\frac{1}{p}} \left(\int_0^t w(s)^r ds\right)^{-\frac{1}{r}}$$

holds. Thus,

$$C \approx \sup_{g \in \mathfrak{M}^+} \frac{\sup\limits_{t \in (0,\infty)} \left(\int_0^t v(s)^p \left(\int_s^\infty g \right) ds \right)^{\frac{1}{p}} \left(\int_0^t w(s)^r ds \right)^{-\frac{1}{r}}}{\|g\|^{\frac{1}{p}}}.$$

Interchanging suprema yields that

$$C \approx \sup_{t \in (0,\infty)} \left(\int_0^t w(s)^r ds \right)^{-\frac{1}{r}} \sup_{g \in \mathfrak{M}^+} \frac{\left(\int_0^\infty v(s)^p \left(\int_s^\infty g \right) \chi_{(0,t)}(s) ds \right)^{\frac{1}{p}}}{\|g\|^{\frac{1}{p}}}.$$

From Fubini's Theorem and duality in weighted Lebesgue spaces with $q/p \in (1,\infty)$ again, it follows that

$$\begin{aligned} C &= \sup_{t \in (0,\infty)} \left(\int_0^t w(s)^r ds \right)^{-\frac{1}{r}} \sup_{g \in \mathfrak{M}^+} \frac{\left(\int_0^\infty g(y) \left(\int_0^y v(s)^p \chi_{(0,t)}(s) ds \right) dy \right)^{\frac{1}{p}}}{\left(\int_0^\infty g(y)^{\frac{q}{q-p}} u(y)^{-\frac{qp}{q-p}} dy \right)^{\frac{q-p}{qp}}} \\ &\approx \sup_{t \in (0,\infty)} \left(\int_0^t w(s)^r ds \right)^{-\frac{1}{r}} \left(\int_0^\infty \left(\int_0^y v(s)^p \chi_{(0,t)}(s) ds \right)^{\frac{q}{p}} u(y)^q dy \right)^{\frac{1}{q}}. \end{aligned}$$

Observe that,

$$\begin{aligned} &\int_0^\infty \left(\int_0^y v(s)^p \chi_{(0,t)}(s) ds \right)^{\frac{q}{p}} u(y)^q dy \\ &\quad = \int_0^t \left(\int_0^y v(s)^p ds \right)^{\frac{q}{p}} u(y)^q dy + \left(\int_0^t v(s)^p ds \right)^{\frac{q}{p}} \left(\int_t^\infty u(y)^q dy \right). \end{aligned}$$

Thus we arrive at $C \approx I_5 + I_6$.

$\square$

Proof of Theorem 5 Let $0 < p < r < \infty$ and $0 < p < q < \infty$. Using the steps identical to the preceding proof, which relies on $q/p \in (1,\infty)$, duality in weighted Lebesgue spaces, and Fubini's Theorem one can see that (10) holds. Since $p < r$, applying the second case of Theorem 6, we obtain that

$$\begin{aligned} \mathcal{R}(p,r;\tilde{v},w) \approx &\left(\int_0^\infty \left(\int_0^t v(s)^p \left(\int_s^\infty g \right) ds \right)^{\frac{r}{r-p}} \left(\int_0^t w^r \right)^{-\frac{r}{r-p}} w(t)^r dt \right)^{\frac{r-p}{rp}} \\ &+ \left(\int_0^\infty v(s)^p \left(\int_s^\infty g \right) ds \right)^{\frac{1}{p}} \left(\int_0^\infty w^r \right)^{-\frac{1}{r}}. \end{aligned}$$

Then, $C \approx C_1 + C_2$, where

$$C_1 := \sup_{g \in \mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_0^t v(s)^p \left(\int_s^\infty g \right) ds \right)^{\frac{r}{r-p}} \left(\int_0^t w^r \right)^{-\frac{r}{r-p}} w(t)^r dt \right)^{\frac{r-p}{rp}}}{\|g\|^{\frac{1}{p}}}$$

and

$$C_2 := \left(\int_0^\infty w^r \right)^{-\frac{1}{r}} \sup_{g \in \mathfrak{M}^+} \frac{\left(\int_0^\infty v(s)^p \int_s^\infty g(y) dy \, ds \right)^{\frac{1}{p}}}{\|g\|^{\frac{1}{p}}}.$$

First observe that, using Fubini's Theorem and duality principle one more time, we have

$$C_2 = \left(\int_0^\infty w^r \right)^{-\frac{1}{r}} \left(\int_0^\infty \left(\int_0^y v(s)^p ds \right)^{\frac{q}{p}} u(y)^q dy \right)^{\frac{1}{q}},$$

and C_1^p is the best constant in the inequality (9) with parameters $p = \frac{q}{q-p}$ and $q = \frac{r}{r-p}$, and weights

$$\mathfrak{u}(s) = u(s)^{-p}, \quad \mathfrak{v}(s) = v(s)^p, \quad \mathfrak{w}(s) = \left(\int_0^s w^r \right)^{-1} w(s)^{r-p}, \quad s > 0.$$

It remains to apply Theorem 7. To this end we should again split this case into two parts.

(i) If $r \le q$, then applying the first case in Theorem 7, we obtain that $C_1 \approx I_8 + I_9$ and the result follows.

(ii) If $q < r$, then applying the second case in Theorem 7, we obtain that $C_1 \approx I_{10} + I_{11}$ and the result follows.

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ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract. In this study, by establishing a connection between normalized univalent functions in the unit disc and Pascal distribution series, we have obtained the necessary and sufficient conditions for these functions to belong to some subclasses of univalent functions of complex-order. We also determined some conditions by considering the integral operator for these functions.

1. Introduction

Let $\mathcal{A}$ stand for the standard class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

Moreover, let $\mathcal{S}$ be the class of functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$ (see [5]).

The necessary and sufficient condition for a function $f \in \mathcal{A}$ to be called starlike of complex order $\gamma\,(\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\})$ is $\frac{f(z)}{z} \neq 0$, $z \in \mathbb{U}$, and

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \qquad (z \in \mathbb{U}). \tag{2}$$

We denote the class of these functions with $\mathcal{S}^*(\gamma)$. The class $\mathcal{S}^*(\gamma)$ introduced by Nasr and Aouf [10].

The necessary and sufficient condition for a function $f \in \mathcal{A}$ to be called convex function of order $\gamma\,(\gamma \in \mathbb{C}^*)$, that is $f \in \mathcal{C}(\gamma)$ is $f'(z) \neq 0$ in $\mathbb{U}$ and

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad (z \in \mathbb{U}). \tag{3}$$

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The class $\mathcal{C}(\gamma)$ was introduced by Wiatrowski [15]. It follows from (2) and (3) that for a function $f \in \mathcal{A}$ we have the equivalence

$$f \in \mathcal{C}(\gamma) \Leftrightarrow zf' \in \mathcal{S}^*(\gamma).$$

For a function $f \in \mathcal{A}$, we say that it is close-to-convex function of order $\gamma\,(\gamma \in \mathbb{C}^*)$, that is $f \in \mathcal{R}(\gamma)$, if and only if

$$Re\left\{1 + \frac{1}{\gamma}\left(f'(z) - 1\right)\right\} > 0, \qquad (z \in \mathbb{U}).$$

The class $\mathcal{R}(\gamma)$ was studied by Halim [6] and Owa [11].

Let $\mathcal{T} \subset \mathcal{A}$ represent the functions of the form

$$f\,(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \geq 0). \tag{4}$$

Many important results for the class $\mathcal{T}$ have been given by Silverman [14]. A lot of consequences have obtained by researchers about the functions in the class $\mathcal{T}$. Using the functions of the form $f\,(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$, Altıntaş et al. [2] defined following subclasses of $\mathcal{A}(n)$, which generalizes the results of Nasr et al. and Wiatrowski [10, 15], and obtained several results for this class. It is clear that for $n = 1$, we obtain the class $\mathcal{T}$.

Definition 1. *[2] Let $\mathcal{S}_n(\gamma, \lambda, \beta)$ denote the subclass of $\mathcal{T}$ consisting of functions f which satisfy the inequality*

$$\left|\frac{1}{\gamma}\left(\frac{zf'^2 f''(z)}{\lambda zf'(z) + (1-\lambda)f(z)} - 1\right)\right| < \beta,$$

$$(z \in \mathbb{U}, \gamma \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1).$$

Also let $\mathcal{R}_n(\gamma, \lambda, \beta)$ denote the subclass of $\mathcal{T}$ consisting of functions f which satisfy the inequality

$$\left|\frac{1}{\gamma}\left(f'(z) + \lambda zf''(z) - 1\right)\right| < \beta,$$

$$(z \in \mathbb{U}, \gamma \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1).$$

We note that

$$\mathcal{S}_n(\gamma, 0, 1) \subset \mathcal{S}_n^*(\gamma) \quad \text{and} \quad \mathcal{R}_n(\gamma, 0, 1) \subset \mathcal{R}_n(\gamma).$$

Recently, it has been established a power series that its coefficients were probabilities of the elementary distributions such as Poisson, Pascal, Binomial, etc. Many researchers have obtained several results about some subclasses of univalent functions using these series. (see, for example [1, 3, 7, 8, 9, 12, 13])

A variable x is said to have the Pascal distribution if it takes on the values $0, 1, 2, 3, ...$ with the probabilities $(1-q)^r$, $\frac{qr(1-q)^r}{1!}$, $\frac{q^2 r(r+1)(1-q)^r}{2!}$, $\frac{q^3 r(r+1)(r+2)(1-q)^r}{3!}$,..., respectively, where q and r are parameters. Hence

$$P(X=k) = \binom{k+r-1}{r-1} q^k (1-q)^r, \qquad k \in \{0, 1, 2, ...\}.$$

Recently, El-Deeb et al. [4] introduced the following power series whose coefficients are probabilities of the Pascal distribution and stated some sufficient conditions for the Pascal distribution series and other related series to be in some subclasses of analytic functions.

$$\mathbf{K}_q^r(z) := z + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (1-q)^r z^k \tag{5}$$

$$(z \in \mathbb{U};\ r \geq 1;\ 0 \leq q \leq 1).$$

Now let us introduce the following new power series whose coefficients are probabilities of the Pascal distribution.

$$\mathbf{\Phi}_q^r(z) := 2z - \mathbf{K}_q^r(z) = z - \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} (1-q)^r z^k \tag{6}$$

$$(z \in \mathbb{U};\ r \geq 1;\ 0 \leq q \leq 1).$$

It is clear that $\mathbf{\Phi}_q^r(z)$ is in the class $\mathcal{T}$. Note that, by using ratio test we deduce that the radius of convergence of the power series $\mathbf{K}_q^r(z)$ and $\mathbf{\Phi}_q^r(z)$ are infinity.

We will need the following Lemmas from Altıntaş et al. [2] to prove our main results.

Lemma 2. *[2] Let the function $f \in \mathcal{A}(n)$, then f is in the class $\mathcal{S}_n(\gamma, \lambda, \beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} [\lambda(k-1)+1] (k+\beta|\gamma|-1) a_k \leq \beta|\gamma|. \tag{7}$$

Lemma 3. *[2] Let the function $f \in \mathcal{A}(n)$, then f is in the class $\mathcal{R}_n(\gamma, \lambda, \beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k [\lambda(k-1)+1] a_k \leq \beta|\gamma|. \tag{8}$$

Throughout this paper, we suppose that $n = 1$ for the functions in the classes $\mathcal{S}_n(\gamma, \lambda, \beta)$ and $\mathcal{R}_n(\gamma, \lambda, \beta)$ and we will write $\mathcal{S}_1(\gamma, \lambda, \beta) = \mathcal{S}(\gamma, \lambda, \beta)$ and $\mathcal{R}_1(\gamma, \lambda, \beta) = \mathcal{R}(\gamma, \lambda, \beta)$ for briefly.

In the present paper, we established necessary and sufficient conditions for the functions that coefficients consist of Pascal distribution series to be in $\mathcal{S}(\gamma, \lambda, \beta)$ and $\mathcal{R}(\gamma, \lambda, \beta)$. Also, we studied similar properties for integral transforms related to these series.

2. Main Results

Theorem 4. *$\mathbf{\Phi}_q^r(z)$ given by (6) is in the class $\mathcal{S}(\gamma,\lambda,\beta)$ if and only if*

$$\frac{q^2r(r+1)\lambda}{(1-q)^2}+\frac{qr(\lambda\beta|\gamma|+\lambda+1)}{1-q}\le \beta|\gamma|(1-q)^r. \tag{9}$$

Proof. To prove that $\mathbf{\Phi}_q^r\in\mathcal{S}(\gamma,\lambda,\beta)$, according to Lemma 2, it is sufficient to show that

$$\sum_{k=2}^{\infty}\left[\lambda(k-1)+1\right](k+\beta|\gamma|-1)\binom{k+r-2}{r-1}q^{k-1}(1-q)^r\le\beta|\gamma|. \tag{10}$$

We will use the following very known relation

$$\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}q^k=\frac{1}{(1-q)^r},\ 0\le q\le 1.$$

and the corresponding ones obtained by replacing the value of r with $r-1$,$r+1$ and $r+2$ in our proofs.

By making calculations on the left hand side of the inequality (10) we obtain,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[\lambda(k-1)+1\right](k+\beta|\gamma|-1)\binom{k+r-2}{r-1}q^{k-1}(1-q)^r\\
&=(1-q)^r\left[\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}\lambda(k-1)(k-2)+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}\beta|\gamma|\right.\\
&\qquad\left.+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}(k-1)(\lambda\beta|\gamma|+\lambda+1)\right]\\
&=(1-q)^r\left[q^2\sum_{k=3}^{\infty}\binom{k+r-2}{r+1}q^{k-3}\lambda r(r+1)+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}\beta|\gamma|\right.\\
&\qquad\left.+q\sum_{k=2}^{\infty}\binom{k+r-2}{r}q^{k-2}r(\lambda\beta|\gamma|+\lambda+1)\right]\\
&=(1-q)^r\left[q^2\sum_{k=0}^{\infty}\binom{k+r+1}{r+1}q^{k}\lambda r(r+1)+\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}q^{k}\beta|\gamma|-\beta|\gamma|\right.\\
&\qquad\left.+q\sum_{k=0}^{\infty}\binom{k+r}{r}q^{k}r(\lambda\beta|\gamma|+\lambda+1)\right]\\
&=\frac{q^2r(r+1)\lambda}{(1-q)^2}+\frac{qr(\lambda\beta|\gamma|+\lambda+1)}{1-q}+\beta|\gamma|\left[1-(1-q)^r\right].
\end{aligned}$$

Therefore the inequality (10) holds if and only if

$$\frac{q^2 r(r+1)\lambda}{(1-q)^2} + \frac{qr(\lambda\beta|\gamma| + \lambda + 1)}{1-q} + \beta|\gamma|\left[1 - (1-q)^r\right] \leq \beta|\gamma|,$$

which is equivalent to (9). This completes the proof. □

Upon letting $\lambda = 0$ and $\beta = 1$, Theorem 4 yields the following result.

Corollary 5. *$\mathbf{\Phi}_q^r(z)$ given by (6) is in the class $\mathcal{S}(\gamma, 0, 1) \subset \mathcal{S}^*(\gamma)$ if and only if*

$$\frac{qr}{(1-q)^{r+1}} \leq |\gamma|.$$

Taking $\lambda = 0$ and $\gamma = \beta = 1$, we obtain the following corollary.

Corollary 6. *$\mathbf{\Phi}_q^r(z)$ given by (6) is in the class $\mathcal{S}(1, 0, 1) \subset \mathcal{S}^*$ if and only if*

$$\frac{qr}{(1-q)^{r+1}} \leq 1.$$

Theorem 7. *$\mathbf{\Phi}_q^r(z)$ given by (6) is in the class $\mathcal{R}(\gamma, \lambda, \beta)$ if and only if*

$$\frac{q^2 r(r+1)\lambda}{(1-q)^2} + \frac{qr(1+2\lambda)}{1-q} + 1 - (1-q)^r \leq \beta|\gamma|. \tag{11}$$

Proof. To prove that $\mathbf{\Phi}_q^r \in \mathcal{R}(\gamma, \lambda, \beta)$, according to Lemma 3, it is sufficient to show that

$$\sum_{k=2}^{\infty} k\left[\lambda(k-1) + 1\right] \binom{k+r-2}{r-1} q^{k-1}(1-q)^r \leq \beta|\gamma|. \tag{12}$$

Now,using the same method as in the proof of Theorem 4, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} k\left[\lambda(k-1) + 1\right] \binom{k+r-2}{r-1} q^{k-1}(1-q)^r \\
&= (1-q)^r \Bigg[\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1} \lambda(k-1)(k-2) \\
&\qquad + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}(k-1)(1+2\lambda) + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}\Bigg] \\
&= (1-q)^r \Bigg[q^2 \sum_{k=3}^{\infty} \binom{k+r-2}{r+1} q^{k-3} \lambda r(r+1) + q \sum_{k=2}^{\infty} \binom{k+r-2}{r} q^{k-2} r(1+2\lambda) \\
&\qquad + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}\Bigg]
\end{aligned}$$

$$= (1-q)^r \left[q^2 \sum_{k=0}^{\infty} \binom{k+r+1}{r+1} q^k \lambda r(r+1) + q \sum_{k=0}^{\infty} \binom{k+r}{r} q^k r(1+2\lambda) \right.$$
$$\left. + \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} q^k - 1 \right]$$

$$= \frac{q^2 r(r+1)\lambda}{(1-q)^2} + \frac{qr(1+2\lambda)}{1-q} + 1 - (1-q)^r.$$

Therefore the inequality (12) holds if and only if

$$\frac{q^2 r(r+1)\lambda}{(1-q)^2} + \frac{qr(1+2\lambda)}{1-q} + 1 - (1-q)^r \leq \beta|\gamma|.$$

This completes the proof. □

As a special case of Theorem 7, if we put $\lambda = 0$ and $\beta = 1$, we arrive at the following result.

Corollary 8. *$\mathbf{\Phi}_q^r(z)$ given by (6) is in the class $\mathcal{R}(\gamma, 0, 1) \subset \mathcal{R}(\gamma)$ if and only if*

$$\frac{qr}{1-q} + 1 - (1-q)^r \leq |\gamma|.$$

Taking $\lambda = 0$ and $\gamma = \beta = 1$, we obtain the following corollary.

Corollary 9. *$\mathbf{\Phi}_q^r(z)$ given by (6) is in the class $\mathcal{R}(1, 0, 1) \subset \mathcal{R}(1)$ if and only if*

$$\frac{qr}{1-q} + 1 - (1-q)^r \leq 1.$$

3. Integral Operators

In this section, we will give analog results for the integral operators defined as follows:

$$H_q^r(z) = \int_0^z \frac{\mathbf{\Phi}_q^r(t)}{t} dt \tag{13}$$

where $\mathbf{\Phi}_q^r(t)$ is given by (6).

Theorem 10. *$H_q^r(z)$ given by (13) is in the class $\mathcal{S}(\gamma, \lambda, \beta)$ if and only if*

$$\frac{\lambda qr}{(1-q)} + \frac{(1-\lambda)(\beta|\gamma|-1)(1-q)}{q(r-1)} \left[1 - (1-q)^{r-1}\right] - \beta|\gamma|(1-q)^r + \lambda\beta|\gamma| + 1 - \lambda \leq \beta|\gamma|. \tag{14}$$

Proof. From (13), we can write

$$H_q^r(z)=\int_0^z \frac{\mathbf{\Phi}_q^r(t)}{t}dt = z-\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}(1-q)^r\frac{z^k}{k}. \tag{15}$$

According to Lemma 2, it is enough to show that

$$\sum_{k=2}^{\infty}\frac{[\lambda(k-1)+1]\,(k+\beta|\gamma|-1)}{k}\binom{k+r-2}{r-1}q^{k-1}(1-q)^r\le\beta|\gamma|. \tag{16}$$

Using the assumption (14), a simple computation shows that

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{[\lambda(k-1)+1]\,(k+\beta|\gamma|-1)}{k}\binom{k+r-2}{r-1}q^{k-1}(1-q)^r\\
&=(1-q)^r\left[\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}\lambda(k-1)+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}(\lambda\beta|\gamma|-\lambda+1)\right.\\
&\qquad\left.+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}\frac{(1-\lambda)(\beta|\gamma|-1)}{k}\right]\\
&=(1-q)^r\left[q\sum_{k=2}^{\infty}\binom{k+r-2}{r}q^{k-2}\lambda r+\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}q^{k-1}(\lambda\beta|\gamma|-\lambda+1)\right.\\
&\qquad\left.+\frac{(1-\lambda)(\beta|\gamma|-1)}{q(r-1)}\sum_{k=2}^{\infty}\binom{k+r-2}{r-2}q^{k}\right]\\
&=(1-q)^r\left\{\lambda qr\sum_{k=0}^{\infty}\binom{k+r}{r}q^k+(\lambda\beta|\gamma|-\lambda+1)\left[\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}q^k-1\right]\right.\\
&\qquad\left.+\frac{(1-\lambda)(\beta|\gamma|-1)}{q(r-1)}\left[\sum_{k=0}^{\infty}\binom{k+r-2}{r-2}q^k-1-q(r-1)\right]\right\}\\
&=\frac{\lambda qr}{(1-q)}+(\lambda\beta|\gamma|-\lambda+1)\left[1-(1-q)^r\right]\\
&\qquad+\frac{(1-\lambda)(\beta|\gamma|-1)}{q(r-1)}\left[(1-q)-(1-q)^r-q(r-1)(1-q)^r\right]\\
&=\frac{\lambda qr}{(1-q)}+\frac{(1-\lambda)(\beta|\gamma|-1)(1-q)}{q(r-1)}\left[1-(1-q)^{r-1}\right]-\beta|\gamma|(1-q)^r+\lambda\beta|\gamma|+1-\lambda.
\end{aligned}$$

From (14), we conclude that $H_q^r(z)\in\mathcal{S}(\gamma,\lambda,\beta)$. This completes the proof. □

Theorem 11. *$H_q^r(z)$ given by (13) is in the class $\mathcal{R}(\gamma,\lambda,\beta)$ if and only if*

$$\frac{qr\lambda}{(1-q)}+1-(1-q)^r\le\beta|\gamma|. \tag{17}$$

Proof. Since

$$H_q^r(z) = z - \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}(1-q)^r \frac{z^k}{k}, \tag{18}$$

according to Lemma 3, it is enough to show that

$$\sum_{k=2}^{\infty} \frac{k\left[\lambda(k-1)+1\right]}{k} \binom{k+r-2}{r-1} q^{k-1}(1-q)^r \leq \beta|\gamma|. \tag{19}$$

Using the assumption (17), some simple computations shows that

$$\sum_{k=2}^{\infty} \frac{k\left[\lambda(k-1)+1\right]}{k} \binom{k+r-2}{r-1} q^{k-1}(1-q)^r$$

$$= (1-q)^r \left[\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}\lambda(k-1) + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}\right]$$

$$= (1-q)^r \left[q\sum_{k=2}^{\infty} \binom{k+r-2}{r} q^{k-2}\lambda r + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} q^{k-1}\right]$$

$$= (1-q)^r \left[q\lambda r\sum_{k=0}^{\infty} \binom{k+r}{r} q^{k} + \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} q^{k} - 1\right]$$

$$= \frac{qr\lambda}{(1-q)} + 1 - (1-q)^r$$

From (17), we conclude that $H_q^r(z) \in \mathcal{R}(\gamma, \lambda, \beta)$. This completes the proof. □

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A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS DEFINED BY RAPID OPERATOR

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Abstract. We present and investigate a new subclass of meromorphic univalent functions described by the Rapid operator in this study. Coefficient inequalities is discussed, as well as distortion properties, closure theorems, Hadamard product. After this, integral transforms for the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ are obtained.

1. Introduction

Let Σ stands for the function class of the form

$$\aleph(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_\ell \hbar^\ell, \ \ell \in N = \{1, 2, 3, \cdots\} \tag{1}$$

analytic in the punctured unit disc $\Upsilon^* = \{\hbar \in C : 0 < |\hbar| < 1\} = \Upsilon \setminus \{0\}$.

A function $\aleph \in \Sigma$ given by (1) is said to be meromorphically starlike of order ϱ if it satisfies the following:

$$\Re\left\{-\left(\frac{\hbar\aleph'(\hbar)}{\aleph(\hbar)}\right)\right\} > \varrho, \ (\hbar \in \Upsilon)$$

for some $\varrho(0 \le \varrho < 1)$. We say that $\aleph$ is in the class $\Sigma^*(\varrho)$ of such functions.

Similarly a function $\aleph \in \Sigma$ given by (1) is said to be meromorphically convex of order ϱ if it satisfies the following:

$$\Re\left\{-\left(1 + \frac{\hbar\aleph''(\hbar)}{\aleph'(\hbar)}\right)\right\} > \varrho, \ (\hbar \in \Upsilon)$$

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for some $\varrho(0 \leq \varrho < 1)$. We say that $\aleph$ is in the class $\Sigma_\ell(\varrho)$ of such functions.

Akgul [1,2], Miller [8], Pommerenke [9], Royster [10], Aydogan and Sakar [4,5,11] and Venkateswarlu et al. [14, 15, 16] have all studied the class $\Sigma^*(\varrho)$ and numerous other subclasses of Σ extensively.

For functions $\aleph \in \Sigma$ given by (1) and $g \in \Sigma$ given by

$$g(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} b_\ell \hbar^\ell,$$

we define the Hadamard product of $\aleph$ and g by

$$(\aleph * g)(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_\ell b_\ell \hbar^\ell.$$

Jung et al. defined the integral operator on normalised analytic functions in [6] and Lashin [7] updated their operator for meromorphic functions in the following manner:

Lemma 1. *For $\aleph \in \Sigma$ given by (1), if the operator $S_\mu^\theta : \Sigma \to \Sigma$ is defined by*

$$S_\mu^\theta \aleph(\hbar) = \frac{1}{(1-\mu)^\theta \Gamma(\theta+1)} \int_0^\infty t^{\theta+1} e^{\frac{-t}{1-\mu}} \aleph(t\hbar) dt, \tag{2}$$

$(0 \leq \mu < 1, 0 \leq \theta \leq 1$ and $\hbar \in \Upsilon)$ then

$$S_\mu^\theta \aleph(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \phi_\ell(\theta, \mu) a_\ell \hbar^\ell \tag{3}$$

where $\phi_\ell(\theta,\mu) = (1-\mu)^{\ell+1} \frac{\Gamma(\ell+\theta+2)}{\Gamma(\theta+1)}$ and Γ is the familiar Gamma function.

Using the equation (3), it is easily seen that

$$\hbar(S_\mu^\theta \aleph(\hbar))' = \mu S_\mu^{\theta-1} \aleph(\hbar) - (\mu+1) S_\mu^\theta \aleph(\hbar), (0 \leq \mu \leq 1, 0 \leq \theta \leq 1). \tag{4}$$

We define a new subclass $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ of Σ based on Sivaprasad Kumar et al. [13] and Venkateswarlu et al. [14] $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ of Σ.

Definition 2. *For $0 \leq \vartheta < 1,\ \varrho \geq 0,\ 0 \leq \wp < \frac{1}{2}$, we let $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ be the subclass of Σ consisting of functions of the form (1) and satisfying the analytic condition*

$$-\Re\left(\frac{\hbar(S_\mu^\theta \aleph(\hbar))'^2 (S_\mu^\theta \aleph(\hbar))''}{(1-\wp) S_\mu^\theta \aleph(\hbar) + \wp\hbar(S_\mu^\theta \aleph(\hbar))'} + \vartheta\right) > \varrho \left|\frac{\hbar(S_\mu^\theta \aleph(\hbar))'^2 (S_\mu^\theta \aleph(\hbar))''}{(1-\wp) S_\mu^\theta \aleph(\hbar) + \wp\hbar(S_\mu^\theta \aleph(\hbar))'} + 1\right|. \tag{5}$$

The following lemmas are needed to prove our findings [3].

Lemma 3. *If η is a real number and ω is a complex number then*

$$\Re(\omega) \geq \eta \Leftrightarrow |\omega + (1-\eta)| - |\omega - (1+\eta)| \geq 0.$$

Lemma 4. *If ω is a complex number and η, ℓ are real numbers then*

$$-\Re(\omega) \geq \ell|\omega+1| + \eta \Leftrightarrow -\Re\left(\omega(1+\ell e^{i\theta}) + \ell e^{i\theta}\right) \geq \eta, \ (-\pi \leq \ \theta \leq \pi).$$

The key purpose of this paper is to look at some traditional geometric function theory properties for the class of geometric functions, such as coefficient bounds, distortion properties, closure theorems, Hadamard product, and integral transforms.

2. Coefficient estimates

We obtain required and adequate conditions for a function $\aleph$ to be in the class in this section.

Theorem 5. *Let $\aleph \in \Sigma$ be given by* (1). *Then $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ iff*

$$\sum_{\ell=1}^{\infty}[(1+(\ell-1)\wp)][\ell(\varrho+1)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)a_\ell \leq (1-\vartheta)(1-2\wp). \tag{6}$$

Proof. Let $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then by Definition 2 and using Lemma 4, It suffices to demonstrate that

$$-\Re\left\{\frac{\hbar(S_\mu^\theta\aleph(\hbar))'^2(S_\mu^\theta\aleph(\hbar))''}{(1-\wp)S_\mu^\theta\aleph(\hbar)+\wp\hbar(S_\mu^\theta\aleph(\hbar))'}(1+\varrho e^{i\theta})+\varrho e^{i\theta}\right\} \geq \vartheta, \ (-\pi \leq \ \theta \leq \pi). \tag{7}$$

For convenience

$$\begin{aligned} C(\hbar) =& -\left[\hbar(S_\mu^\theta\aleph(\hbar))'^2(S_\mu^\theta\aleph(\hbar))''\right](1+\varrho e^{i\theta}) \\ &- \varrho e^{i\theta}\left[(1-\wp)S_\mu^\theta\aleph(\hbar)+\wp\hbar(S_\mu^\theta\aleph(\hbar))'\right] \\ D(\hbar) =&(1-\wp)S_\mu^\theta\aleph(\hbar)+\wp\hbar(S_\mu^\theta\aleph(\hbar))'. \end{aligned}$$

That is, the equation (7) is equivalent to

$$-\Re\left(\frac{C(\hbar)}{D(\hbar)}\right) \geq \vartheta.$$

We only need to prove that in light of Lemma 3

$$|C(\hbar)+(1-\vartheta)D(\hbar)| - |C(\hbar)-(1+\vartheta)D(\hbar)| \geq 0.$$

Therefore

$$\begin{aligned} &|C(\hbar)+(1-\vartheta)D(\hbar)| \\ \geq&(2-\vartheta)(1-2\wp)\frac{1}{|\hbar|} - \sum_{\ell=1}^{\infty}[\ell-(1-\vartheta)][1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell \\ &- \varrho\sum_{\ell=1}^{\infty}(\ell+1)[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell \end{aligned}$$

and $|C(\hbar)-(1+\vartheta)D(\hbar)|$

$$\leq \vartheta(1-2\wp)\frac{1}{|\hbar|}+\sum_{\ell=1}^{\infty}[\ell+(1+\vartheta)][1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell$$
$$+\varrho\sum_{\ell=1}^{\infty}(\ell+1)[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell.$$

It is to show that

$$\begin{aligned}&|C(\hbar)+(1-\vartheta)D(\hbar)|-|C(\hbar)-(1+\vartheta)D(\hbar)|\\ \geq& 2(1-\vartheta)(1-2\wp)\frac{1}{|\hbar|}-2\sum_{\ell=1}^{\infty}[(\ell+\vartheta)(1+(\ell-1)\wp)]\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell\\ &-2\varrho\sum_{\ell=1}^{\infty}(\ell+1)(1+(\ell-1)\wp)\phi_\ell(\theta,\mu)a_\ell|\hbar|^\ell\\ \geq& 0,\ \text{by the given condition (6).}\end{aligned}$$

Conversely suppose $\aleph\in\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. Then by Lemma 3, we have (7).

The inequality (7) is reduced to when the values of $\hbar$ are chosen on the positive real axis

$$\Re\left\{\frac{[(1-2\wp)(1-\vartheta)(1+\varrho e^{i\theta})]\frac{1}{\hbar^2}+\sum\limits_{\ell=1}^{\infty}\{\ell+\varrho e^{i\theta}(\ell+1)+\vartheta\}[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)\hbar^{\ell-1}}{(1-2\wp)\frac{1}{\hbar^2}+\sum\limits_{\ell=1}^{\infty}[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell\hbar^{\ell-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta})\geq-|e^{i\theta}|=-1$, the above inequality is reduced to

$$\Re\left\{\frac{[(1-2\wp)(1-\vartheta)(1+\varrho e^{i\theta})]\frac{1}{r^2}+\sum\limits_{\ell=1}^{\infty}\{\ell+\varrho(\ell+1)+\vartheta\}[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)a_\ell r^{\ell-1}}{(1-2\wp)\frac{1}{r^2}+\sum\limits_{\ell=1}^{\infty}[1+\wp(\ell-1)]\phi_\ell(\theta,\mu)r^{\ell-1}}\right\}\geq 0.$$

We obtained the inequality (6) by letting $r\to 1^-$ and using the mean value theorem. □

Corollary 6. *If* $\aleph\in\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$ *then*

$$a_\ell\leq\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\vartheta+\varrho)]\phi_\ell(\theta,\mu)}. \tag{8}$$

The estimate is sharp for the function

$$\aleph(\hbar)=\frac{1}{\hbar}+\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\vartheta+\varrho)]\phi_\ell(\theta,\mu)}\hbar^\ell. \tag{9}$$

We get the following corollary by taking $\wp=0$ in Theorem 5.

Corollary 7. *If* $\aleph \in \Sigma^*(\vartheta, \varrho, \theta, \mu)$ *then*

$$a_\ell \leq \frac{1-\vartheta}{[\ell(1+\varrho)+(\vartheta+\varrho)]\phi_\ell(\theta,\mu)}. \tag{10}$$

3. Distortion theorem

Theorem 8. *If* $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ *then for* $0 < |\hbar| = r < 1$,

$$\frac{1}{r} - \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\ r \leq |\aleph(\hbar)| \leq \frac{1}{r} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)} r. \tag{11}$$

This estimate is sharp for the function

$$\aleph(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\ \hbar. \tag{12}$$

Proof. Since $\aleph(\hbar) = \frac{1}{\hbar} + \sum\limits_{\ell=1}^{\infty} a_\ell \hbar^\ell$, we have

$$|\aleph(\hbar)| = \frac{1}{r} + \sum_{\ell=1}^{\infty} a_\ell r^\ell \leq \frac{1}{r} + r\sum_{\ell=1}^{\infty} a_\ell. \tag{13}$$

Since $\ell \geq 1,\ (2\varrho+\vartheta+1)\phi_1(\theta,\mu) \leq [1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)$, using Theorem 5, we have

$$\begin{aligned}(2\varrho+\vartheta+1)\phi_1(\theta,\mu)\sum_{\ell=1}^{\infty} a_\ell &\leq \sum_{\ell=1}^{\infty}[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)\\ &\leq (1-\vartheta)(1-2\wp)\\ \Rightarrow \sum_{\ell=1}^{\infty} a_\ell &\leq \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}.\end{aligned}$$

Using the above inequality in (13), we have

$$\begin{aligned}|\aleph(\hbar)| &\leq \frac{1}{r} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)} r\\ \text{and } |\aleph(\hbar)| &\geq \frac{1}{r} - \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)} r.\end{aligned}$$

The estimate is sharp for the function $\aleph(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\hbar$. □

We omit the proof of the following corollary since it is similar to that of Theorem 8.

Corollary 9. *If* $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ *then*

$$\frac{1}{r^2} - \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\ \leq |\aleph'(\hbar)| \leq \frac{1}{r^2} + \frac{((1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}.$$

The estimate is sharp for the function given by (12).

4. Closure theorems

Let the function $\aleph_j$ be defined, for $j = 1, 2, \cdots, m$, by

$$\aleph_j(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_{\ell,j}\hbar^{\ell}, a_{\ell,j} \geq 0. \tag{14}$$

Theorem 10. *Let the functions $\aleph_j, j = 1, 2, \cdots, m$ defined by (14) be in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then the function h defined by*

$$h(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{\ell,j} \right) \hbar^{\ell} \tag{15}$$

also belongs to the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$.

Proof. Since $\aleph_j, j = 1, 2, \cdots, m$ are in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, it follows from Theorem 5, that

$$\sum_{\ell=1}^{\infty}[1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_\ell(\theta, \mu)a_{\ell,j} \leq (1 - \vartheta)(1 - 2\wp),$$

for every $j = 1, 2, \cdots, m$. Hence

$$\begin{aligned} &\sum_{\ell=1}^{\infty}[1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_\ell(\theta, \mu) \left(\frac{1}{m} \sum_{j=1}^{m} a_{\ell,j} \right) \\ &= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{\ell=1}^{\infty}[1 + \wp(\ell - 1)][\ell(1 + \varrho) + (\varrho + \vartheta)]\phi_\ell(\theta, \mu)a_{\ell,j} \right) \\ &\leq (1 - \vartheta)(1 - 2\wp). \end{aligned}$$

From Theorem (6), it follows that $h \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$.

Hence the proof. □

Theorem 11. *The class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ is closed under convex linear combinations.*

Proof. Let the functions $\aleph_j, j = 1, 2, \cdots, m$ defined by (14) be in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then one need only show that function

$$h(\hbar) = \varsigma\aleph_1(\hbar) + (1 - \varsigma)\aleph_2(\hbar),\ 0 \leq \varsigma \leq 1 \tag{16}$$

is in the class $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Since for $0 \leq \varsigma \leq 1$,

$$h(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty}[\varsigma a_{\ell,1} + (1 - \varsigma)a_{\ell,1}]\hbar^{\ell}, \tag{17}$$

with the assistance of the Theorem5, we have

$$\begin{aligned}&\sum_{\ell=1}^{\infty}[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)[\varsigma a_{\ell,1}+(1-\varsigma)a_{\ell,1}]\\ \leq&\varsigma(1-\vartheta)(1-2\wp)+(1-\varsigma)(1-\vartheta)(1-2\wp)\\ =&(1-\vartheta)(1-2\wp),\end{aligned}$$

which implies that $h \in \Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. □

Theorem 12. *Let $\xi \geq 0$. Then $\Sigma^{*\xi}(\vartheta,\varrho,\wp,\theta,\mu) \subseteq N(\varrho,\xi)$, where*

$$\xi = 1-\frac{2(1-\vartheta)(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)+(1-\vartheta)(1-2\wp)}. \tag{18}$$

Proof. If $\aleph \in \Sigma^{*\xi}(\vartheta,\varrho,\wp,\theta,\mu)$ then

$$\sum_{\ell=1}^{\infty}\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}a_\ell \leq 1.$$

We need to find the value of ξ such that

$$\sum_{\ell=1}^{\infty}\frac{[\ell(1+\varrho)+(\varrho+\xi)]\phi_\ell(\theta,\mu)}{1-\xi}a_\ell \leq 1.$$

Thus it is sufficient to show that

$$\frac{[\ell(1+\varrho)+(\varrho+\xi)]\phi_\ell(\theta,\mu)}{1-\xi} \leq \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}.$$

Then

$$\xi \leq 1-\frac{(\ell+1)(1-\vartheta)(1-2\wp)(1+\varrho)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]+(1-\vartheta)(1-2\wp)}.$$

Since

$$G(\ell) = 1-\frac{(\ell+1)(1-\vartheta)(1-2\wp)(1+\varrho)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]+(1-\vartheta)(1-2\wp)}$$

is an increasing function of $\ell,\ \ell \geq 1$, we obtain

$$\xi \leq G(1) = 1-\frac{2(1-\vartheta)(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)+(1-\vartheta)(1-2\wp)}.$$

□

Theorem 13. *Let $\aleph_0(\hbar) = \frac{1}{\hbar}$ and*

$$\aleph_\ell(\hbar) = \frac{1}{\hbar}+\sum_{\ell=1}^{\infty}\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}\hbar^\ell,\ \ \ell \geq 1. \tag{19}$$

Then $\aleph$ is in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$ iff can be expressed in the form

$$\aleph(\hbar) = \sum_{\ell=0}^{\infty}\omega_\ell\aleph_\ell(\hbar), \tag{20}$$

where $\omega_\ell \geq 0$ *and* $\sum\limits_{\ell=0}^{\infty} \omega_\ell = 1$.

Proof. Assume that

$$\begin{aligned}\aleph(\hbar) &= \sum_{\ell=0}^{\infty} \omega_\ell \aleph_\ell(\hbar)\\ &= \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)} \hbar^\ell.\end{aligned}$$

Then it follows that

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)} \hbar^\ell$$

$$= \sum_{\ell=1}^{\infty} \omega_\ell = 1 - \omega_0 \leq 1$$

which implies that $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$.

On the other side, assume that the function $\aleph$ defined by (1) be in the class $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then

$$a_\ell \leq \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}.$$

Setting

$$\omega_\ell = \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_\ell,$$

where

$$\omega_0 = 1 - \sum_{\ell=0}^{\infty} \omega_\ell,$$

$\aleph$ can be expressed in the form (20), as can be shown. □

Corollary 14. *The extreme points of the class* $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$ *are the functions* $\aleph_0(\hbar) = \frac{1}{\hbar}$ *and*

$$\aleph_\ell(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)} \hbar^\ell. \tag{21}$$

5. Modified Hadamard products

Let the functions $\aleph_j (j = 1, 2)$ defined by (14). The modified Hadamard product of $\aleph_1$ and $\aleph_2$ is defined by

$$(\aleph_1 * \aleph_2)(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} a_{\ell,1} a_{\ell,2} \hbar^\ell = (\aleph_2 * \aleph_1)(\hbar). \tag{22}$$

Theorem 15. *Let the function* $\aleph_j (j = 1, 2)$ *defined by* (14) *be in the class* $\Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. *Then* $\aleph_1 * \aleph_2 \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, *where*

$$\varphi = 1 - \frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu) + (1-\vartheta)^2(1-2\wp)}. \tag{23}$$

The estimate is sharp for the functions $\aleph_j (j = 1, 2)$ *given by*

$$\aleph_j(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\hbar, \ (j = 1, 2). \tag{24}$$

Proof. Using the same method that Schild and Silverman [12] used earlier, we need to find the largest real parameter φ such that

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\varphi)]\phi_\ell(\theta,\mu)}{(1-\varphi)(1-2\wp)} a_{\ell,1}a_{\ell,2} \le 1. \tag{25}$$

Since $\aleph_j \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, $j = 1, 2$, we readily see that

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_{\ell,1} \le 1$$

and

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_{\ell,2} \le 1.$$

By Cauchy- Schwarz inequality, we have

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} \sqrt{a_{\ell,1}a_{\ell,2}} \le 1. \tag{26}$$

Then merely demonstrating that is necessary

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\varphi)]\phi_\ell(\theta,\mu)}{(1-\varphi)(1-2\wp)} a_{\ell,1}a_{\ell,2}$$
$$\le \sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} \sqrt{a_{\ell,1}a_{\ell,2}}$$

or equivalently that

$$\sqrt{a_{\ell,1}a_{\ell,2}} \le \frac{[\ell(1+\varrho)+(\varrho+\vartheta](1-\varphi)}{[\ell(1+\varrho)+(\varrho+\varphi](1-\vartheta)}.$$

Hence, it light of the inequality (26), then merely demonstrating that is necessary

$$\frac{(1-\vartheta)(1-2\wp)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]\phi_\ell(\theta,\mu)} \le \frac{[\ell(1+\varrho)+(\varrho+\vartheta](1-\varphi)}{[\ell(1+\varrho)+(\varrho+\varphi](1-\vartheta)}. \tag{27}$$

It follows from (27) that

$$\varphi \le 1 - \frac{(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]^2\phi_\ell(\theta,\mu) + (1-\vartheta)^2(1-2\wp)}.$$

Now defining the function $E(\ell)$,

$$E(\ell) = 1 - \frac{(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta]^2\phi_\ell(\theta,\mu)+(1-\vartheta)^2(1-2\wp)}.$$

We see that $E(\ell)$ is an increasing of ℓ, $\ell \geq 1$. Therefore, we conclude that

$$\varphi \leq E(\ell) = 1 - \frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu)+(1-\vartheta)^2(1-2\wp)},$$

Hence the proof. □

The following theorem is obtained using arguments close to those used in the proof of 15,

Theorem 16. *Let the function* $\aleph_1$ *defined by* (14) *be in the class* $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. *Suppose also that the function* $\aleph_2$ *defined by* (14) *be in the class* $\Sigma^*(\rho,\vartheta,\varrho,\wp,\theta,\mu)$. *Then* $\aleph_1 * \aleph_2 \in \Sigma^*(\zeta,\vartheta,\varrho,\wp,\theta,\mu)$, *where*

$$\zeta = 1 - \frac{2(1-\vartheta)(1-\rho)(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)(2\varrho+\rho+1)\phi_1(\theta,\mu)+(1-\vartheta)(1-\rho)(1-2\wp)}. \tag{28}$$

The estimate is sharp for the functions $\aleph_j (j=1,2)$ *given by*

$$\aleph_1(\hbar) = \frac{1}{\hbar} + \frac{(1-\vartheta)(1-2\wp)}{(2\varrho+\vartheta+1)\phi_1(\theta,\mu)}\hbar$$

and

$$\aleph_2(\hbar) = \frac{1}{\hbar} + \frac{(1-\rho)(1-2\wp)}{(2\varrho+\rho+1)\phi_1(\theta,\mu)}\hbar.$$

Theorem 17. *Let the function* $\aleph_j (j=1,2)$ *defined by* (14) *be in the class* $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. *Then the function*

$$h(\hbar) = \frac{1}{\hbar} + \sum_{\ell=1}^{\infty}(a_{\ell,1}^2 + a_{\ell,2}^2)\hbar^\ell \tag{29}$$

belongs to the class $\Sigma^*(\varepsilon,\vartheta,\varrho,\wp,\theta,\mu)$, *where*

$$\varepsilon = 1 - \frac{4(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}. \tag{30}$$

The estimate is sharp for the functions $\aleph_j (j=1,2)$ *given by* (24).

Proof. By using Theorem 5, we obtain

$$\begin{aligned}&\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2 a_{\ell,1}^2\\ &\leq \sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}a_{\ell,1}\right\}^2 \leq 1\end{aligned} \tag{31}$$

and

$$\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2 a_{\ell,2}^2$$
$$\leq\sum_{\ell=1}^{\infty}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}a_{\ell,2}\right\}^2\leq 1. \tag{32}$$

It follows from (31) and (32) that

$$\sum_{\ell=1}^{\infty}\frac{1}{2}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2 (a_{\ell,1}^2+a_{\ell,2}^2)\leq 1.$$

Therefore, we need to find the largest ε such that

$$\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\varepsilon)]\phi_\ell(\theta,\mu)}{(1-\varepsilon)(1-2\wp)}$$
$$\leq\frac{1}{2}\left\{\frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)}\right\}^2,$$

that is

$$\varepsilon\ \leq 1-\frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]^2\phi_\ell(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}.$$

Since

$$G(\ell)=\ 1-\frac{2(1-\vartheta)^2(1-2\wp)(1+\varrho)(\ell+1)}{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]^2\phi_\ell(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}$$

is an increasing function of ℓ, $\ell\geq 1$, we obtain

$$\varepsilon\leq G(1)=\frac{4(1-\vartheta)^2(1-2\wp)(1+\varrho)}{(2\varrho+\vartheta+1)^2\phi_1(\theta,\mu)+2(1-\vartheta)^2(1-2\wp)}$$

and hence the proof. $\square$

6. Integral operators

Theorem 18. *Let the functions $\aleph$ given by* (1) *be in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$. Then the integral operator*

$$F(\hbar)=c\int_0^1 u^c\aleph(u\hbar)du,\ 0<u\leq 1,\ c>0 \tag{33}$$

is in the class $\Sigma^*(\vartheta,\varrho,\wp,\theta,\mu)$, where

$$\xi=1-\frac{2c(1-\vartheta)(1+\varrho)}{(c+2)(2\varrho+\vartheta+1)+c(1-\vartheta)}. \tag{34}$$

The estimate is sharp for the function $\aleph$ given by (12).

Proof. Let $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$. Then

$$\begin{aligned} F(\hbar) =& c \int_0^1 u^c \aleph(u\hbar) du \\ =& \frac{1}{\hbar} + \sum_{\ell=1}^{\infty} \frac{c}{\ell + c + 1} a_\ell \hbar^\ell. \end{aligned}$$

Thus it is enough to show that

$$\sum_{\ell=1}^{\infty} \frac{c[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\xi)]\phi_\ell(\theta,\mu)}{(\ell+c+1)(1-\xi)(1-2\wp)} a_\ell \leq 1. \tag{35}$$

Since $\aleph \in \Sigma^*(\vartheta, \varrho, \wp, \theta, \mu)$, then

$$\sum_{\ell=1}^{\infty} \frac{[1+\wp(\ell-1)][\ell(1+\varrho)+(\varrho+\vartheta)]\phi_\ell(\theta,\mu)}{(1-\vartheta)(1-2\wp)} a_\ell \leq 1. \tag{36}$$

From (35) and (36), we have

$$\frac{[\ell(1+\varrho)+(\varrho+\xi)]}{(\ell+c+1)(1-\xi)} \leq \frac{[\ell(1+\varrho)+(\varrho+\vartheta)]}{(1-\vartheta)}.$$

Then

$$\xi \leq 1 - \frac{c(1-\vartheta)(\ell+1)(1+\varrho)}{(\ell+c+1)[\ell(1+\varrho)+(\varrho+\vartheta)]+c(1-\vartheta)}.$$

Since

$$Y(\ell) = 1 - \frac{c(1-\vartheta)(\ell+1)(1+\varrho)}{(\ell+c+1)[\ell(1+\varrho)+(\varrho+\vartheta)]+c(1-\vartheta)}$$

is an increasing function of ℓ, $\ell \geq 1$, we obtain

$$\xi \leq Y(1) = 1 - \frac{2c(1-\vartheta)(1+\varrho)}{(c+2)(2\varrho+\vartheta+1)+c(1-\vartheta)}$$

and hence the proof. □

7. Conclusion

This research has introduced a new subclass of meromorphic functions defined by Rapid operator and studied some basic properties of geometric function theory. Accordingly, some results to coefficient estimates, distortion properties, closure theorems, hadamard product and integral transforms have been considered, inviting further research for this field of study.

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INVERSE STEREOGRAPHIC HYPERBOLIC SECANT DISTRIBUTION: A NEW SYMMETRIC CIRCULAR MODEL BY ROTATED BILINEAR TRANSFORMATIONS

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Abstract. The inverse stereographic projection (ISP), or equivalently, bilinear transformation, is a method to produce a circular distribution based on an existing linear model. By the genesis of the ISP method, many important circular models have been provided by many researchers. In this study, we propose a new symmetric unimodal/bimodal circular distribution by the rotated ISP method considering the hyperbolic secant distribution as a baseline distribution. Rotation means that fixing the origin and rotating all other points the same amount counterclockwise. Considering the effect of rotation on the circular distribution to be obtained with the bilinear transformation, it is seen that it actually induces a location parameter in the obtained circular probability distribution. We analyze some of the stochastic properties of the proposed distribution. The methods for the parameter estimation of the new circular model and the simulation-based compare results of these estimators are extensively provided by the paper. Furthermore, we compare the fitting performance of the new model according to its well-known symmetric alternatives, such as Von-Misses, and wrapped Cauchy distributions, on a real data set. From the information obtained by the analysis on the real data, we say that the fitting performance of the new distribution is better than its alternatives according to the criteria frequently used in the literature.

1. INTRODUCTION

Circular or directional data are observed in various fields of science. Data on angular observations can often be associated with compass measurements. Additionally, daily, weekly, or hourly observations obtained in the specific time period

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may be circular. Although it may seem attractive in some ways, processing and evaluating such data linearly can lead to false results. In directional data, the start and endpoints are neighbors despite having the furthest distance according to linear metric. As a simple example, the arithmetic mean of two angles 1 and 359 degrees is 180 degrees, although the circular average to be 0 degrees. Therefore, it requires a special class of distributions known as circular probability distributions to analyze such data.

Circular probability distributions are usually obtained by circularizing a known linear probability distribution. The two most common methods for circularization are wrapping and inverse stereographic projection (ISP). ISP method is based on bilinear transformations. Minh and Farnum [8] used bilinear transformations to map points on the unit circle in the complex plane into points on the real line. Thus, they used the stereographic projection as a transformation, to produces probability distributions on the real line by circular models. It was clear that by the inverse of this transformation (ISP), circular probability distributions could be obtained from probability distributions on the real line. Many studies on circular distributions obtained using the ISP method have been added to the literature. Yedlapalli, et al [11] used to transformation on double Weibull distribution to obtain a symmetric circular distribution. Kato and Jones [6] proposed a family of four-parameter distributions on the circle that contains the Von Mises and wrapped Cauchy distributions as special cases. Girija, et al [5] introduced stereographic double exponential distribution obtained by using double exponential (Laplace) distribution. The same authors introduced the stereographic logistic model [2] in a later study. Yedlapalli, et al [12] obtained semicircular (axial) model induced by using modified inverse stereographic projection on Quasi Lindley distribution. The projection method used in all these studies is based on the result obtained by Minh and Farnum [8] in a study in which they introduced the induction of linear models with Möbius transformations. Möbius transformation (bilinear, fractional linear or linear fractional transformation) provides very convenient methods of finding a one-to-one mapping of one domain into another. In a general form, Möbius transformation can be written as

$$\mathrm{T}\left(z\right)=\frac{az+b}{cz+d}, \tag{1}$$

where a,b,c and d are complex or real valued coefficients and $bc-ad\neq 0$. This transformation was proposed by Minh and Farnum [8] as a new method of generating probability distributions, which maps every point on a real line onto the point on a unit circle. Their construction proceeds as follows. In order for $T\left(z\right)$ to map the unit circle on the real line, the constraints $\mathrm{Im}\left(c\right)\neq 0$, $\overline{a}d=\overline{c}b$, and $a\neq 0$ must be provided. Dividing all coefficients in Eq.(1) by a and imposing the requirement $T\left(-1\right)=\infty$ yields the transformation of the form

$$\mathrm{T}\left(z\right)=\frac{cz+\overline{c}}{z+1}. \tag{2}$$

Finally, by taking $c = u - iv$ and $z = \cos(\theta) + i\sin(\theta)$, the transformation

$$\begin{aligned} x = T(\theta) &= \mathrm{T}(\cos(\theta) + i\sin(\theta)) \\ &= u + v\frac{\sin(\theta)}{1+\cos(\theta)} = u + v\tan\left(\frac{\theta}{2}\right) \end{aligned} \tag{3}$$

is obtained which is known as stereographic transformation. Inverse stereographic projection yields a circular model when it applied to a linear model. If a random variable is defined on the whole real line with probability density function (pdf) $f(.)$ and cumulative distribution function (cdf) $F(.)$ then $\Theta = T^{-1}(X)$ is a random point on the unit circle, with the pdf $g(.)$ and the cdf $G(.)$, respectively, defined as

$$g(\theta) = f(T(\theta))\left|\frac{d}{d\theta}T(\theta)\right| = f\left(u + v\tan\left(\frac{\theta}{2}\right)\right)\frac{v}{\cos(\theta)+1}, \tag{4}$$

$$G(\theta) = F(T(\theta)) = F\left(u + v\tan\left(\frac{\theta}{2}\right)\right), \tag{5}$$

where $\theta \in [-\pi, \pi)$, $u \in \mathbb{R}$ and $v > 0$. Multiplying the coefficients in Eq.(1) by k yields one to one and the same mapping, where k is an arbitrary (non-zero) complex number. Since three complex numbers are sufficient to pin down the mapping, i.e., there exist a unique Mobius transformation sending any three points (z_1, z_2, z_3) to any other three points (w_1, w_2, w_3) [9]. Consider the cross-ratio of three points

$$(z, z_1, z_2, z_3) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

where $z_i \neq z_j, i, j = 1, 2, 3$ and $i \neq j$. Then there is a unique Mobius transformation such that

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3).$$

Moreover, it is known that rotation is to fix the origin and spin all other points counter-clockwise by the same amount (see Fig.1). By this motivation, if we solve the equation

$$\left(z, e^{-i\alpha}, e^{-i(\alpha-\pi/2)}, e^{-i(\alpha+\pi/2)}\right) = (w, u, u+v, u-v)$$

with respect to w, we have

$$w = \mathrm{T}_\alpha(z) = u - iv\left(1 - \frac{2}{1 + e^{i\alpha}z}\right). \tag{6}$$

Note that, multiplication z by $e^{i\alpha}$ has a geometric effect of anti-clockwise rotation about the origin by an angle of $\alpha \in [-\pi, \pi)$. So, it is easy to see that $\mathrm{T}_\alpha(z) = \mathrm{T}\left(e^{i\alpha}z\right)$. Finally, by taking $z = \cos(\theta) + i\sin(\theta)$ in Eq.(6), we have

$$\begin{aligned} x = T_\alpha(\theta) &= \mathrm{T}_\alpha(\cos(\theta) + i\sin(\theta)) \\ &= u + v\tan\left(\frac{\theta+\alpha}{2}\right). \end{aligned} \tag{7}$$

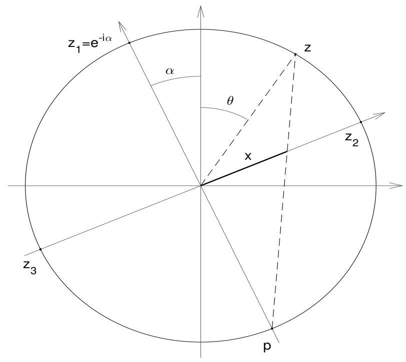

FIGURE 1. Rotation by α, cross-ratio points z_1, z_2, z_3 and pole (p).

Lemma 1. *Pole of transformation in Eq.(6) is* $z = -e^{-i\alpha}$.

Lemma 2. *Inverse transformation of* T_α *is* $T_\alpha^{-1}(x) = 2\tan^{-1}\left(\frac{x-u}{v}\right) - \alpha$.

Lemma 3. *Let X be a random variable defined on $(-\infty,\infty)$ with pdf $f(.)$ and cdf $F(.)$. Then $\Theta = T_\alpha^{-1}(X)$ is a circular random variable with pdf*

$$\begin{aligned} g(\theta;\alpha) &= f(T_\alpha(\theta))\left|\frac{d}{d\theta}T_\alpha(\theta)\right| \\ &= f\left(u + v\tan\left(\frac{\theta+\alpha}{2}\right)\right)\frac{v}{\cos(\theta+\alpha)+1}, \end{aligned} \tag{8}$$

and the corresponding cdf

$$G(\theta;\alpha) = F(T_\alpha(\theta)) = F\left(u + v\tan\left(\frac{\theta+\alpha}{2}\right)\right), \tag{9}$$

where $\alpha \in [-\pi,\pi)$, $v > 0$ and $u \in \mathbb{R}$.

The probability density function given by the Eq.(8) provides three properties: i) $g(\theta;\alpha) \geq 0$ for $\forall\theta \in \mathbb{R}$, ii) $g(.)$ is periodic with period 2π, iii) $\int_\Gamma g(\theta;\alpha)\,d\theta = 1$ where Γ is any interval of length 2π.

Proposition 4. *A rotation of Mobius transformation given by the Eq.(2) will induce a location parameter in the probability distribution given by the Eq.(4).*

Proof. Proof is clear from lemma 3. □

Corollary 5. *The quantile function of* $\Theta = T_\alpha^{-1}(X)$ *is*

$$Q(t) = 2\tan^{-1}\left(\frac{F^-(t) - t}{v}\right) - \alpha, \tag{10}$$

where $t \in (0,1)$, $F^{-}(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\}$ *and* $F(.)$ *is the cdf of random variable* X.

Proposition 6. *Let* X *be a symmetric random variable around* $E(X) = u$. *The random variable* Θ *defined as* $\Theta = T_{\alpha}^{-1}(X)$ *has a symmetrical distribution around* $-\alpha$.

Proof. If X is symmetric around $E(X) = u$ then, $F^{-}(1/2) = u$ and $F^{-}\left(\frac{1}{2} - r\right) + F^{-}\left(\frac{1}{2} + r\right) = 2u$, where $0 < r < 1/2$ and $F^{-}$is the quantile function of X. Thus $Q\left(\frac{1}{2}\right) = -\alpha$ and

$$\begin{aligned} Q\left(\tfrac{1}{2} - r\right) + Q\left(\tfrac{1}{2} + r\right) &= 2\tan^{-1}\left(\frac{F^{-}\left(\frac{1}{2}-r\right)-u}{v}\right) + 2\tan^{-1}\left(\frac{F^{-}\left(\frac{1}{2}+r\right)-u}{v}\right) - 2\alpha \\ &= 2\tan^{-1}\left(\frac{u-F^{-}\left(\frac{1}{2}+r\right)}{v}\right) + 2\tan^{-1}\left(\frac{F^{-}\left(\frac{1}{2}+r\right)-u}{v}\right) - 2\alpha \\ &= -2\alpha. \end{aligned}$$

Hence Θ has a symmetrical distribution around $-\alpha$. □

Corollary 7. *Since the distribution of* Θ *is symmetrical about* $-\alpha$

$$\begin{aligned} \mu &= \operatorname{atan}\left(E(\sin\Theta), E(\cos\Theta)\right) \\ &= -\alpha, \end{aligned}$$

$(E(\sin\Theta), E(\cos\Theta) < \infty)$ *where* $\operatorname{atan}(.,.)$ *is quadrant inverse tangent function defined as*

$$\operatorname{atan}(s,c) = \begin{cases} \tan^{-1}(s/c) & , c > 0, s \geq 0 \\ \pi/2 & , c = 0, s > 0 \\ \tan^{-1}(s/c) + \pi & , c < 0 \\ \tan^{-1}(s/c) + 2\pi & , c \geq 0, s < 0 \\ \textit{undefined} & , c = 0, s = 0 \end{cases}.$$

In the following section, we show an application of the T_{α}^{-1} transformation to hyperbolic secant distribution. We introduce the methods for estimating the location parameter induced by T_{α}^{-1} in the relevant subsections. Also, that section includes the basic properties of the obtained circular distribution and an application to a real-life data set.

2. Induce Inverse Stereographic Hyperbolic Secant Model with Rotated Bilinear Transformations

Suppose X follows hyperbolic secant distribution, then cdf and pdf of X are

$$F(x) = \frac{2}{\pi}\tan^{-1}\left(e^{\frac{\pi}{2}x}\right), \tag{11}$$

$$f(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi}{2}x\right), \quad x \in \mathbb{R} \tag{12}$$

respectively. This distribution is also called the inverse-cosh distribution because of the hyperbolic secant function is equivalent to the reciprocal hyperbolic cosine function. Note that the pdf given by Eq.(12) is symmetrical around $E(X) = 0$. By considering the Eq.(8) and Eq.(9) with Eq.(12) and Eq.(11), we obtain the cdf and pdf of the inverse stereographic hyperbolic secant distributed random variable $\Theta = T_{\alpha}^{-1}(X)$ as

$$G(\theta;\alpha,v) = \frac{2}{\pi}\tan^{-1}\left(e^{\frac{1}{2}\pi v \tan\left(\frac{\alpha+\theta}{2}\right)}\right), \tag{13}$$

$$g(\theta;\alpha,v) = \frac{v}{2(1+\cos(\alpha+\theta))}\operatorname{sech}\left[\frac{\pi v}{2}\tan\left(\frac{\alpha+\theta}{2}\right)\right], \tag{14}$$

respectively, where $v > 0$ is the scale parameter and $\alpha \in [-\pi, \pi)$ is the location parameter. In the rest of this paper, a random variable Θ having cdf as in Eq.(13) and pdf as in Eq.(14) will be denoted as $\Theta \sim ISHS(\alpha, v)$. Figure 2 illustrates the some of possible shapes of the pdf ofrandom variable $\Theta \sim ISHS(\alpha, v)$ for different values of the parameters α and v.

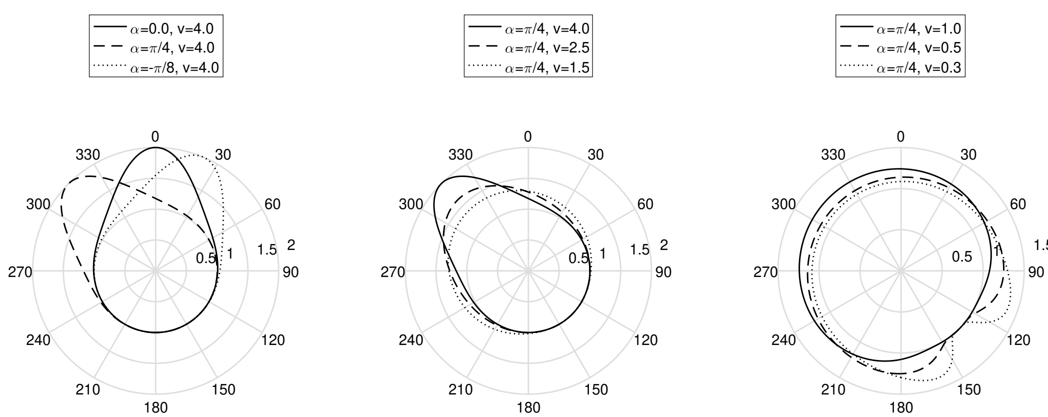

FIGURE 2. Pdf of $ISHS(\alpha, v)$ for different values of α and v.

Figure 2 shows that increasing α values cause counterclockwise rotation, and increasing v value causes an increase in angular concentration. The modality behavior of the ISHS distribution depends only the v parameter. For $v < 0.900316$, the distribution is bimodal. The modality behavior is studied more detailed in Subsection 2.3.

The inverse cdf of hyperbolic secant distribution is $F^{-1}(t) = -\frac{1}{\pi}\log\left(\cot^2\left(\frac{1}{2}\pi t\right)\right)$. Thus, the quantile function of $ISHS(\alpha, v)$ can be easily obtained from Eq.(10) as

$$Q(t) = -\alpha - 2\tan^{-1}\left[\frac{1}{\pi v}\log\left(\cot^2\left(\frac{\pi t}{2}\right)\right)\right], \tag{15}$$

where $t \in (0,1)$.

2.1. **Location, Dispersion and Median.** For a circular random variable, the pth cosine moment is defined as $c_p = E(\cos p\Theta)$, and the pth sine moment is defined as $s_p = E(\sin p\Theta)$ [7]. Thus, the mean direction is calculated as $\mu = \operatorname{atan}(s_1, c_1)$, where $\operatorname{atan}(.,.)$ is quadrant inverse tangent function. The explicit analytical forms of c_p and s_p values can not be obtained for random variable $\Theta \sim ISHS(\alpha, v)$. However, according to proposition 6 and corollary 7, it is clear that

$$\mu = \operatorname{atan}(s_1, c_1) = -\alpha.$$

The first trigonometric moments of Inverse Stereographic Hyperbolic Secant distribution are calculated numerically and presented in Figure 3. The following propositions give useful results for the location parameter of the $ISHS$ distribution.

Proposition 8. $\Theta \sim ISHS(\alpha, v) \Leftrightarrow -\Theta \sim ISHS(-\alpha, v)$.

Proposition 9. $\Theta \sim ISHS(\alpha, v) \Leftrightarrow \Theta + k \sim ISHS(\alpha - k, v)$

The length of mean direction vector is a measure of angular concentration around the mean and it is calculated as $\rho = \sqrt{c_1^2 + s_1^2}$. By using the value of ρ, the circular variance is calculated as $V = 1 - \rho$ and the circular standard deviation calculated as $\sigma = \sqrt{-2\ln\rho}$. These three characteristics are illustrated in Figure 4 for different values of v.

As a measure of asymmetry, the skewness coefficient for the circular distribution is calculated as $\gamma_1 = \bar{s}_2 V^{-3/2}$, where $\bar{s}_p$ denotes the pth central sine moment which is defined as $\bar{s}_p = E[\sin p(\Theta - \mu)]$. According to the following proposition, the skewness coefficient of the $ISHS(\alpha, v)$ distribution are zero for every $v > 0$.

Proposition 10. *All central sine moments of $ISHS(\alpha, v)$ distribution is zero.*

Proof. Since $g(\theta; \alpha, v)$ is periodic with period 2π and $\mu = -\alpha$, we have

$$\begin{aligned}\bar{s}_p = E[\sin p(\Theta - \mu)] &= \int_{\theta_0}^{\theta_0 + 2\pi} \sin[p(\theta - \mu)]\, g(\theta; \alpha, v)\, d\theta \\ &= \int_{-\pi-\alpha}^{\pi-\alpha} \sin[p(\theta + \alpha)]\, g(\theta; \alpha, v)\, d\theta.\end{aligned}$$

According to proposition 9 $g(\theta; \alpha, v) = g(\theta + \alpha; 0, v)$, and according to proposition 6 $g(\theta; 0, v)$ is an even function. Thus, we can write $\bar{s}_p$ as

$$\bar{s}_p = \int_{-\pi}^{\pi} \sin(p\theta)\, g(\theta; 0, v)\, d\theta = 0$$

since $\sin(p\theta)\, g(\theta; 0, v)$ is an odd function. $\square$

The kurtosis coefficient of a circular distribution is calculated as $\gamma_2 = \left(\bar{c}_2 - \rho^4\right)(1 - \rho)^{-2}$, where $\bar{c}_p$ denotes the pth central cosine moment and defined as $\bar{c}_p = E[\cos p(\Theta - \mu)]$. The change of the γ_2 value according to the parameter v of $ISHS(\alpha, v)$ distribution is shown in Figure 4.

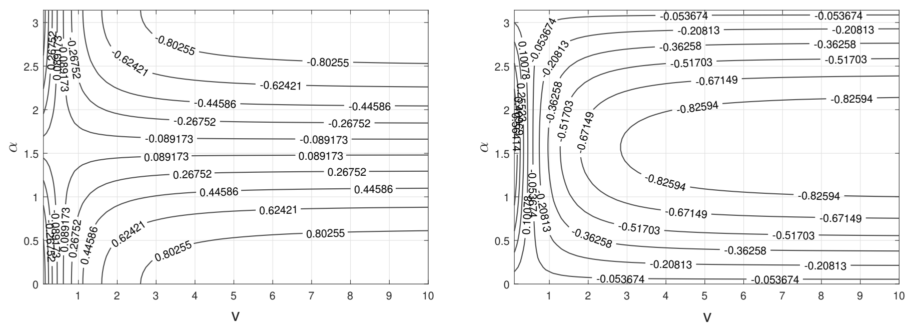

FIGURE 3. Contour plots for first cosine moment (left panel) and first sine moment (right panel) according to α and v.

The median direction (M) and the interquartile range (Iqr) of $ISHS(\alpha, v)$ distribution are easily obtained from Eq.(15), as follows, respectively:

$$M = Q\left(\frac{1}{2}\right) = -\alpha, \tag{16}$$

$$\begin{aligned} Iqr_\Theta &= Q(.75) - Q(.25) \\ &= 2\left(\tan^{-1}\left[\frac{2\log\left(\cot\left(\frac{\pi}{8}\right)\right)}{\pi v}\right] - \tan^{-1}\left[\frac{2\log\left(\tan\left(\frac{\pi}{8}\right)\right)}{\pi v}\right]\right) \\ &\simeq 4.\tan^{-1}\left(\frac{0.5611}{v}\right). \end{aligned}$$

2.2. **Entropy.** The entropy is a measure of variation or uncertainty of a random variable. Following the formal definition of the entropy, the entropy of the random variable $\Theta \sim ISHS(\alpha, v)$ is

$$H_\Theta = -\int_\Gamma g(\theta; \alpha, v) \ln g(\theta; \alpha, v)\, d\theta,$$

where Γ is any interval of length 2π. Since Θ is 2π periodic

$$\begin{aligned} H_\Theta &= -\int_\Gamma g(\theta; 0, v) \ln g(\theta; 0, v)\, d\theta \\ &= -v\int_\Gamma \frac{\operatorname{sech}\left(\frac{1}{2}\pi v \tan\left(\frac{\theta}{2}\right)\right)}{2\cos(\theta) + 2} \log\left(v\frac{\operatorname{sech}\left(\frac{1}{2}\pi v \tan\left(\frac{\theta}{2}\right)\right)}{2\cos(\theta) + 2}\right) d\theta. \end{aligned} \tag{17}$$

We could not get an explicit analytical form of the integral in Eq.(17). Therefore, we numerically calculated the H_Θ with respect to v and illustrated in Figure 4. Note that the entropy of the circular uniform distribution is $\ln(2\pi)$ and this is the maximum entropy any circular distribution may have. Figure 4 shows that the maximum value of the H_Θ is below this value. The entropy of the $ISHS$ distribution attains its maximum value when the circular variance is maximized or equivalently angular concentration minimized. Thus one can write

$$\begin{aligned} v^* &= \operatorname{argmax}_{v>0} H_\Theta = \operatorname{argmin}_{v>0} c_1 \\ &= \operatorname{argmin}_{v>0} \int_\Gamma \cos(\theta)\, g(\theta; 0, v)\, d\theta \\ &= \operatorname{argmin}_{v>0} \int_0^\pi \cos(\theta)\, g(\theta; 0, v)\, d\theta. \end{aligned}$$

Since the minimum value of the first cosine moment is zero, the value of v^* is obtained by solving the equation

$$\int_0^\pi \cos(\theta)\, g(\theta; 0, v)\, d\theta = 0$$

with respect to v. Using the bisection method, we observed that $v^* \simeq 0.521567$.

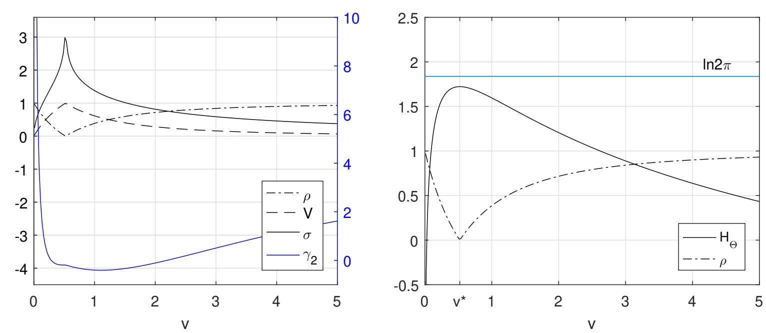

FIGURE 4. Values of ρ,V,σ (left axis) and γ_2 (right axis) according to v (left panel). Entropy and ρ values according to v (right panel).

2.3. **Modality.** The ISHS distribution is unimodal or bimodal depending on the value of the v parameter. Therefore, it will be sufficient to examine the modality behavior of the $g(\theta; 0, v)$ function, which is symmetric about 0 when $\Gamma = [-\pi, \pi)$. The first and second derivates of $g(\theta; 0, v)$ with respect to θ are

$$g'(\theta; 0, v) = -\frac{v \operatorname{sech}\left(\frac{1}{2}\pi v \tan\left(\frac{\theta}{2}\right)\right)\left(\pi v \tanh\left(\frac{1}{2}\pi v \tan\left(\frac{\theta}{2}\right)\right) - 2\sin(\theta)\right)}{4(\cos(\theta)+1)^2}$$

and

$$g''(\theta;0,v)=\frac{v\operatorname{sech}\left(\frac{1}{2}\pi v\tan\left(\frac{\theta}{2}\right)\right)}{8(\cos(\theta)+1)^3}$$
$$\times\left[4\cos(\theta)-2\cos(2\theta)+6+\pi v\left(\begin{array}{c}-2\pi v\operatorname{sech}^2\left(\frac{1}{2}\pi v\tan\left(\frac{\theta}{2}\right)\right)\\-6\sin(\theta)\tanh\left(\frac{1}{2}\pi v\tan\left(\frac{\theta}{2}\right)\right)\\+\pi v\end{array}\right)\right]$$

respectively. Since $g(\theta;0,v)$ is symmetric around 0, $\theta=0$ is a saddle point, ie $g'(0;0,v)=0$. If this point is a local minimum, then $g''(0;0,v)>0$. Thus, ISHS distribution is bimodal when $g''(0;0,v)=-64^{-1}v\left(\pi^2v^2-8\right)>0\Longleftrightarrow v<2\sqrt{2}/\pi\simeq 0.900316$.

2.4. **Order Statistics.** Let $\Theta_1,\Theta_2,...,\Theta_n$ be a random sample from $ISHS(\alpha,v)$ distribution and let $\Theta_{(1)}\leq\Theta_{(2)}\leq\ldots\leq\Theta_{(n)}$ denote the order statistic for this sample. Then, the pdf of the random variable $\Theta_{(i)}$, $i=1,2,...,n$ is obtained as

$$\begin{aligned}h_{\Theta_{(i)}}(\theta;\alpha,v)&=\frac{n!}{(i-1)!(n-i)!}G(\theta;\alpha,v)^{i-1}g(\theta;\alpha,v)(1-G(\theta;\alpha,v))^{n-i}\\&=\frac{2^{i-2}\pi^{1-i}vn!\tan^{-1}\left(e^{\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)}\right)^{i-1}}{(i-1)!(n-i)!(\cos(\alpha+\theta)+1)}\\&\quad\times\operatorname{sech}\left(\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)\right)\left(1-\frac{2\tan^{-1}\left(e^{\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)}\right)}{\pi}\right)^{n-i}.\end{aligned}\tag{18}$$

The pdf of first order (minimum) and nth order (maximum) statistics can be immediately calculated from Eq.(18) as

$$h_{\Theta_{(1)}}(\theta;\alpha,v)=\frac{\operatorname{sech}\left(\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)\right)}{2\cos(\alpha+\theta)+2}nv\left(1-\frac{2\tan^{-1}\left(e^{\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)}\right)}{\pi}\right)^{n-1}$$

and

$$h_{\Theta_{(n)}}(\theta;\alpha,v)=\frac{\operatorname{sech}\left(\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)\right)}{(\cos(\alpha+\theta)+1)}2^{n-2}\pi^{1-n}nv\tan^{-1}\left(e^{\frac{1}{2}\pi v\tan\left(\frac{\alpha+\theta}{2}\right)}\right)^{n-1},$$

respectively.

2.5. **Inference.** In this section, we consider estimating the unknown parameters of $ISHS(\alpha,v)$ distribution. We will use tree methods commonly used in the literature, such as, maximum likelihood (ml), weighted least-squares (ls) and moments estimation (me) methods. Finally, a Monte-Carlo simulation study will be given to show and compare the performance of ml, me and ls estimators.

2.5.1. *Maximum Likelihood Estimation.* Let $\Theta_1, \Theta_2, ..., \Theta_n$ be a random sample from $ISHS\left(\alpha, v\right)$ distribution. By considering the random variables $\Theta_i, i = 1, 2, ..., n$, The logarithmic likelihood function of α and v can be written as

$$L\left(\alpha, v; \theta_1, \theta_2, ..., \theta_n\right) = \sum_{i=1}^{n} \log\left[\frac{1}{4} v \sec^2\left(\frac{\alpha+\theta_i}{2}\right) \operatorname{sech}\left(\frac{1}{2}\pi v \tan\left(\frac{\alpha+\theta_i}{2}\right)\right)\right].$$

If the first derivatives of this log-likelihood function with respect to parameters α and v are taken and equalized them to zero, then we have the following normal equations

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \tan\left(\frac{1}{2}\left(\alpha+\theta_i\right)\right) - \frac{1}{4}\pi v \sum_{i=1}^{n} \sec^2\left(\frac{\alpha+\theta_i}{2}\right) \tanh\left(\frac{1}{2}\pi v \tan\left(\frac{\alpha+\theta_i}{2}\right)\right) = 0 \tag{19}$$

and

$$\frac{\partial L}{\partial v} = \frac{n}{v} - \frac{\pi}{2} \sum_{i=1}^{n} \tan\left(\frac{\alpha+\theta_i}{2}\right) \tanh\left(\frac{1}{2}\pi v \tan\left(\frac{\alpha+\theta_i}{2}\right)\right) = 0. \tag{20}$$

Let us denote the ml estimates of the parameters α and v as $\widehat{\alpha}_{ML}$ and $\widehat{v}_{ML}$, respectively. Hence, $\widehat{\alpha}_{ML}$ and $\widehat{v}_{ML}$ can obtained from the collective solution of Eq.(19) and Eq.(20). However, these equations do not have an analytical solution. So $\widehat{\alpha}_{ML}$ and $\widehat{v}_{ML}$ must be obtained numerically.

2.5.2. *Weighted Least Square Estimation.* A well-known modification of least square estimation method is the weighted least square, which has a lower bias than the ordinary least square estimation. Let us consider the ordered random sample $\theta_{(1)} < \cdots < \theta_{(n)}$ from $ISHS\left(\alpha, v\right)$ distribution. The weighted least square estimates of the parameters, say $\widehat{\alpha}_{LS}$ and $\widehat{v}_{LS}$ are obtained by minimizing

$$\sum_{j=1}^{n} \frac{\left(n+1\right)^2\left(n+2\right)}{j\left(n-j+1\right)} \left[\frac{2}{\pi} \tan^{-1}\left(e^{\frac{1}{2}\pi v \tan\left(\frac{\alpha+\theta_{(j)}}{2}\right)}\right) - \frac{j}{n+1}\right]^2, \tag{21}$$

with respect to α and v. Where $\frac{j}{n+1}$ is the expectation of the empirical distribution function of the ordered data, see Swain et al. [10]. Numerical methods can be used to minimize Eq.(21).

2.5.3. *Method of Moment Estimation.* Let us start by expressing the sample trigonometric moments for circular data [7]. The pth order sample cosine moment is defined as

$$\overline{C}_p = \frac{1}{n} \sum_{i=1}^{n} \cos\left(p\theta_i\right),$$

and sample sine moment is defined as

$$\overline{S}_p = \frac{1}{n} \sum_{i=1}^{n} \sin\left(p\theta_i\right).$$

Now consider the random sample $\theta_1, \theta_2, \ldots, \theta_n$ from the $ISHS\,(\alpha, v)$ distribution. Moment estimates of α and v ($\widehat{\alpha}_{ME}$ and $\widehat{v}_{ME}$) are obtained from the collective solution of equations

$$\overline{C}_1 - c_1 = 0, \tag{22}$$

and

$$\overline{S}_1 - s_1 = 0 \tag{23}$$

by using numerical methods, where $c_1 = E(\cos\Theta)$ and $s_1 = E(\sin\Theta)$.

2.5.4. *Monte-Carlo Simulation Study.* We perform some Monte-Carlo experiments to compare the performance of ml, ls, and me estimators in different sample sizes. We consider $n = 50, 100, 500,$ and 1000 sample sizes and the repitation of the simulation is set as 100 times in each sample size. The algorithm below has been run for different parameter sets and the results are shown in Table 1.

Step 1. Select n and set values of the parameters α and v.
Step 2. Generate n random numbers from $U\,(0,1) \to u_{n\times 1}$.
Step 3. Calculate $Q\,(u_{n\times 1}) \to \theta_{n\times 1}$, where $Q\,(.)$ as Eq.(15).
Step 4. Get $\widehat{\alpha}_{ML}$ and $\widehat{v}_{ML}$ from the collective solution of Eq.(19) and Eq.(20).
Get $\widehat{\alpha}_{LS}$ and $\widehat{v}_{LS}$ from minimazing Eq.(21).
Get $\widehat{\alpha}_{ME}$ and $\widehat{v}_{ME}$ from the collective solution of Eq.(22).and Eq.(23).
Step 5. Repeat Step 2 to Step 4 for $N = 100$ times.
Step 6. Calculate the $|Bias\,(.)|$ and $Mse\,(.)$ values of the $\widehat{\alpha}$ and $\widehat{v}$ estimators for each ml, ls, and me estimates.

As we discussed in relevant sections, the referred equations in Step 4 have no analytical solutions. We carried out the programming in Matlab and used the 'fsolve' subroutine to solve Eq.(19), Eq.(20), Eq.(22), and Eq.(23). For the minimization problem in Eq.(21), we used the 'fmincon' subroutine. In all routines, the initial values of parameters were taken as $-m_1 = -\operatorname{atan}\left(\overline{S}_1, \overline{C}_1\right)$ for α and 1 for v.

According to the results in Table 1, it is seen that the Bias and MSE values decrease to zero as the sample size increases for the estimation of parameters α and v by all three methods. This shows that the estimates are precise and accurate, hence, we say that it is consistent and unbiased. It is known that ml estimators are asymptotically unbiased estimators. So, the results in Table 1 agree with expectations for ml estimators. In addition, simulation results show that the other estimators have the same characteristics.

Table 1. Simulated Bias and MSE values of parameter estimates for different sample sizes and parameter values.

			$\alpha = \pi/2$				$\alpha = -\pi/4$			
			$\widehat{\alpha}$		$\widehat{v}$		$\widehat{\alpha}$		$\widehat{v}$	
	Method	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
v=3	ML	50	.0067	.0058	.0551	.1663	.0021	.0068	.0950	.2064
		100	.0051	.0031	.1007	.0840	.0098	.0033	.0533	.0954
		500	.0017	.0007	.0057	.0160	.0019	.0006	.0204	.0163
		1000	.0017	.0003	.0040	.0068	.0018	.0003	.0104	.0067
	ME	50	.0066	.0058	.0619	.1703	.0026	.0070	.1012	.2169
		100	.0057	.0031	.1007	.0831	.0075	.0032	.0605	.1005
		500	.0023	.0007	.0031	.0171	.0018	.0006	.0198	.0168
		1000	.0016	.0003	.0038	.0070	.0016	.0003	.0070	.0070
	LS	50	.0070	.0057	.0226	.1883	.0018	.0071	.0088	.2178
		100	.0052	.0030	.0745	.0910	.0081	.0032	.0177	.1029
		500	.0020	.0007	.0082	.0178	.0021	.0006	.0171	.0171
		1000	.0015	.0003	.0062	.0072	.0019	.0003	.0099	.0071
v=6	ML	50	.0049	.0015	.1384	.6167	.0064	.0014	.0302	.4931
		100	.0024	.0010	.0401	.2787	.0005	.0010	.0963	.3519
		500	.0021	.0001	.0189	.0554	.0024	.0002	.0165	.0556
		1000	.0003	.0001	.0072	.0264	.0002	.0001	.0085	.0284
	ME	50	.0049	.0017	.2071	.5840	.0047	.0017	.0538	.5626
		100	.0031	.0010	.0710	.2968	.0001	.0011	.1093	.4181
		500	.0020	.0001	.0502	.0677	.0024	.0002	.0132	.0626
		1000	.0005	.0001	.0270	.0305	.0007	.0001	.0029	.0321
	LS	50	.0049	.0016	.0183	.8181	.0062	.0015	.0939	.5589
		100	.0026	.0010	.0021	.3388	.0005	.0010	.0534	.3494
		500	.0022	.0001	.0037	.0595	.0024	.0002	.0130	.0605
		1000	.0003	.0001	.0049	.0292	.0003	.0001	.0159	.0292
v=0.75	ML	50	.0062	.0114	.0342	.0115	.0117	.0125	.0298	.0122
		100	.0105	.0068	.0126	.0043	.0008	.0056	.0086	.0042
		500	.0046	.0011	.0023	.0008	.0015	.0011	.0018	.0010
		1000	.0008	.0005	.0005	.0004	.0028	.0005	.0014	.0005
	ME	50	.0776	.4152	.0358	.0235	.0130	.4464	.0203	.0266
		100	.0825	.2560	.0102	.0092	.0658	.2579	.0059	.0097
		500	.0111	.0254	.0044	.0010	.0170	.0256	.0003	.0012
		1000	.0235	.0108	.0016	.0005	.0111	.0105	.0023	.0005
	LS	50	.0232	.0412	.0324	.0159	.0107	.0309	.0202	.0140
		100	.0219	.0147	.0074	.0049	.0058	.0139	.0045	.0050
		500	.0003	.0026	.0019	.0008	.0023	.0028	.0021	.0011
		1000	.0003	.0014	.0001	.0004	.0001	.0013	.0015	.0005

2.6. **Real Data Example.** In this section, we study the modeling behavior of the ISHS distribution on a real-life dataset. We consider the termite mounds data in Appendix B.13 (set 7) of Fisher [3]. The data consist of $n = 66$ termite mounds orientations of Amitermes laurensis in the Cape York Peninsula, North Queensland. We obtained the parameter estimates by using Matlab's 'fmincon' and 'fsolve' subroutines. In these subroutines, the parameter ranges were chosen as wide as possible to avoid local maxima. The initial values were set to $-m_1 = -\operatorname{atan}\left(\overline{S}_1, \overline{C}_1\right)$ for α and 1 for v. In order to make comparisons, we chosed the Von-Mises (VM) and Wrapped Cauchy (WC) distributions as well-known alternatives from the location family for modeling symmetrical circular data. Table 2 shows the parameter estimates for each models, and Figure 5 illustrates the fitted pdfs and cdfs. The ISHS parameters were estimated with three methods; ml, me and ls. Table 2 also includes the mean direction and resultant length estimates for each models, and the values of these characteristics obtained from the sample.

Table 2. Parameter estimates, estimated mean direction and resultant length for termit mounds data.

Model	Method	Parameters		Mean Direction	Res. Length	Iqr
		$\widehat{\alpha}$	$\widehat{v}$			
ISHS	ML	-3.0527	6.7146	3.0527 (174.91°)	0.9596	0.3335
	ME	-3.0381	6.4753	3.0381 (174.07°)	0.9569	0.3457
	LS	-3.0551	6.9872	3.0551 (175.04°)	0.9625	0.3205
		$\widehat{\mu}$	$\widehat{\kappa}$			
VM	ML	3.0381	11.8567	3.0381 (174.07°)	0.9569	0.3968
		$\widehat{\mu}$	$\widehat{\gamma}$			
WC	ML	3.0485	0.14748	3.0485 (174.66°)	0.8566	0.2974
Sample	-	-	-	3.0381 (174.07°)	0.9569	0.3491

Table 3 contains Log-likelihood (LL), the Akaike and Bayesian information criteria (AIC and BIC), Watson's $U^2(W^2)$ statistics values, Kolmogorov-Smirnov (KS) and Chi square tests statistics with p-values. Here, it is seen that the data fit all the distributions selected ($p > 0.05$). However, it can be said that the proposed ISHS model is the model that best fits the data since it has the smallest values in all model selection criteria.

TABLE 3. Summary of fits for termit mounds data.

Model		-LL	AIC	BIC	W^2	K-S (p)	Chi sq.(p)
ISHS	ML	10.559	25.118	29.497	.040	.06 (.9478)	3.24 (.3557)
	ME	10.710	25.419	29.799	.038	.06 (.9487)	3.70 (.2953)
	LS	10.628	25.255	29.634	.054	.09 (.6460)	3.05 (.3836)
VM	ML	13.537	31.073	35.453	.104	.11 (.3393)	6.21 (.1017)
WC	ML	16.768	37.537	41.916	.085	.10 (.4779)	5.52 (.1372)

Plots of the fitted densities are shown in Figure 5. Left panel of this figure represents the circular data plot, fitted pdfs of the ISHS distribution with ml, me and ls estimates, fitted vm and wc models. The arrow points out the sample mean resultant vector

$$m_1 = \operatorname{atan}\left(\overline{S}_1, \overline{C}_1\right)$$
$$= 3.0381\,(174.07^\circ)\,,$$

and resultant length

$$r_1 = \sqrt{\overline{C}_1^2 + \overline{S}_1^2}$$
$$= 0.4971,$$

where $\overline{C}_1$ and $\overline{S}_1$ the first order sample cosine and sine moments, respectively.

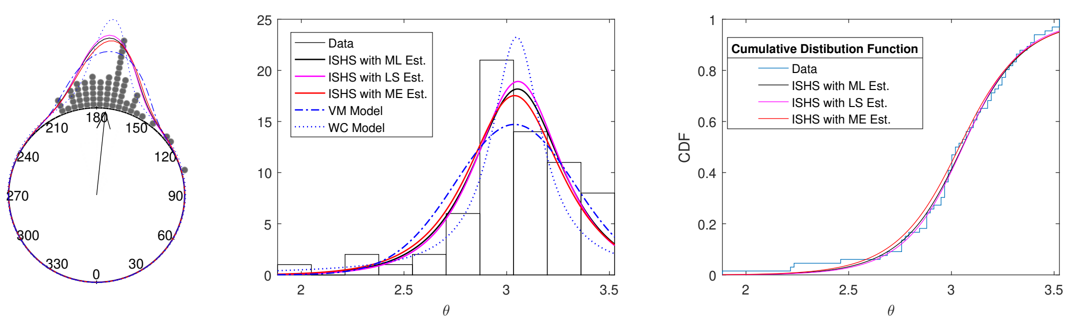

FIGURE 5. Plots for termite mounds data. Circular data plot, fitted circular pdfs (left) linear histogram and fitted pdfs (center), empirical cdf and fitted cdfs (rigth).

All models estimated the average orientation of the mounds to be almost south. The ISHS model with ME estimates gave the mean orientation and resultant length the same as in the sample. This is an expected result for moment estimators. Same thing valid for the VM model. However, when we compare the modeling performances with the values in Table 3, we see that the ISHS model is better than both the VM and the WC model.

3. Conclusion

After Minh and Farnum [8] introduced the ISP method, a number of researchers have introduced many circular distributions by employing the ISP method. In some of these (for example; [1], [4] and [6]), the authors added a location parameter to the circular distributions in their studies. In fact, the location parameter to be added to the circular probability distributions obtained by the ISP method corresponds to the rotation property of bilinear transforms. Here, the rotation means fixing the origin and rotating all other points by the same amount and counterclockwise. In this study, we considered rotation in bilinear transformations and used the rotated inverse stereographic projection (T_α^{-1}) to obtain a new circular model. Thus, we showed that the circular model to be obtained by the $T_\alpha^{-1}(X)$ transformation will naturally belong to the location family of the distributions. Before the section including the application of the method, we gave some propositions and theorems that are useful when the transformation is applied to especially symmetric distributions. In the study, we applied T_α^{-1} to the hyperbolic secant distribution. Thus, we obtained a symmetrical circular distribution with two parameters. One of these parameters is the location parameter and induced by rotated inverse stereographic projection T_α^{-1}. To estimate the unknown parameters of the introduced distribution, the maximum likelihood, the weighted least squares, and the moment estimators are obtained. By a conducted Monte Carlo simulation study, we show that, as the sample size increases, both Bias and MSE values decrease for all estimation methods. Finally, we used the introduced distribution on a real dataset. To compare the fitting performance, we considered the Von-Mises distribution (also known as the circular normal distribution) and Wrapped Cauchy distribution as well-known symmetric alternatives. We observed that the fitting performance of the obtained distribution according to the measures frequently used in the literature is better than both Von-Mises and Wrapped Cauchy distribution.

Declaration of Competing Interests The author declares that he has no known competing financial interests or personal relationships that could affect the work reported in this article.

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ON SOME WEAKER HESITANT FUZZY OPEN SETS

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Abstract. The purpose of this paper is to define and study some new types of hesitant fuzzy open sets namely, hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open in hesitant fuzzy topological space. Some properties and the relationships between these hesitant fuzzy sets are investigated. Furthermore, some relationships between them in hesitant fuzzy subspace are introduced.

1. Introduction

Hesitant fuzzy sets are very useful to deal with group decision making problems when experts have a hesitation among several possible memberships for an element to a set. During the evaluating process in practice, however, these possible memberships may be not only crisp values in $[0, 1]$, but also interval values. Then hesitant fuzzy set theory has many applications in various fields like decision making problems, decision support systems, clustering algorithms, algebras, etc. After that time, hesitant fuzzy set theory has been developed rapidly by some scholars in theory and practice. In 1965, Zadeh [16] introduced the concept of a fuzzy set as a generalization of a crisp set. Chang [3] defined initially the notion of fuzzy topological spaces. In 2010, Torra [14] introduced the notion of a hesitant fuzzy set as an extension of a fuzzy set. In 2011, Xia and Xu [15] applied a hesitant fuzzy set to decision making by defining "hesitant fuzzy information aggregation". Jun et al. [5] studied hesitant fuzzy bi-ideals in semigroups. Divakaran and John [4] introduced a basic version of hesitant fuzzy rough sets through hesitant fuzzy relations. On the other hand, Jun and Ahn [6] applied hesitant fuzzy sets to BCK/BCI-algebras. Kim et al. [7] gave characterizations of a hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal, and a hesitant fuzzy commutative ideal, respectively

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in BCK-algebras. Recently, Lee and Hur [10] defined a hesitant fuzzy topology and introduced the concepts of a hesitant fuzzy neighborhood, closure, interior, hesitant fuzzy subspace and obtained some of their properties. Also, they defined a hesitant fuzzy continuous mapping and investigated some of its properties. In 1965, Njastad [13] defined the class of α-open sets in topological spaces. In 1982, Mashhour et al [12] introduced the concept of preopen sets. The study of semiopen sets and their properties was initiated by Levine [11]. In 1996, Andrijevic' [2] introduced and studied a class of generalized open sets in a topological space called b-open sets, this class of sets contained in the class of β-open sets [1] and contains all semiopen sets and all preopen sets.

2. HESITANT FUZZY OPEN SETS

Definition 1. *[14] Let X be a reference set, and $P[0,1]$ denote the power set of $[0,1]$. Then, a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set in X.*

The hesitant fuzzy empty (resp. whole) set, denoted by h^0 (resp. h^1), is a hesitant fuzzy set in X defined as $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]$), for each $x \in X$. Especially, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$ [8].

Definition 2. *Assume that X is a nonempty set and $h, h_i \in HS(X)$ for i belong to the set of natural numbers $\boldsymbol{N}$. Then,*

(1) *h_1 is a subset of h_2, denoted by $h_1 \subseteq h_2$, if $h_1(x) \subseteq h_2(x)$, for each $x \in X$ [4].*
(2) *h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1(x) \subseteq h_2(x)$ and $h_2(x) \subseteq h_1(x)$ [4].*
(3) *the intersection of h_1 and h_2, denoted by $h_1 \widetilde{\cap} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
$(h_1 \widetilde{\cap} h_2)(x) = h_1(x) \cap h_2(x)$ *[8].*
(4) *the union of h_1 and h_2, denoted by $h_1 \widetilde{\cup} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
$(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x)$ *[8].*
(5) *the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: for each $x \in X$,*
$h^c(x) = h(x)^c = [0,1] \setminus h(x)$ *[8].*
(6) *the intersection of $\{h_i\}_{i \in \boldsymbol{N}}$, denoted by $\widetilde{\bigcap}_{i \in \boldsymbol{N}} h_i$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
$(\widetilde{\bigcap}_{i \in \boldsymbol{N}} h_i)(x) = \bigcap_{i \in \boldsymbol{N}} h_i(x)$ *[8].*
(7) *the union of $\{h_i\}_{i \in \boldsymbol{N}}$, denoted by $\widetilde{\bigcup}_{i \in \boldsymbol{N}} h_i$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
$(\widetilde{\bigcup}_{i \in \boldsymbol{N}} h_i)(x) = \bigcup_{i \in \boldsymbol{N}} h_i(x)$ *[8].*

Definition 3. *[9] Let $h \in HS(X)$. Then, h is called a hesitant fuzzy point with the support $x \in X$ and the value δ, denoted by x_δ, if $x_\delta : X \to P[0,1]$ is the mapping* *given by: for each* $y \in X$,

$$x_\delta(y) = \begin{cases} \delta \subseteq [0,1] & \text{if } y = x, \\ \phi & \text{otherwise.} \end{cases}$$

In particular, $H_P(X)$ *is called the set of all hesitant fuzzy points in* X*. If* $\delta \subseteq h(x)$*, then* x_δ *is said to belong to* h*, denoted by* $x_\delta \in h$*. It is obvious that* $h = \widetilde{\bigcup}_{x_\delta \in h} x_\delta$.

Definition 4. *[10] Let* X *be a nonempty set, and* $\tau \subseteq HS(X)$*. Then,* τ *is called a hesitant topology* (HFT) *on* X*, if it satisfies the following axioms:*

(1) $h^0, h^1 \in \tau$.
(2) *For any* $h_1, h_2 \in \tau$*, we have* $h_1 \widetilde{\cap} h_2 \in \tau$.
(3) *For each* $h_i \in \tau$*, we have* $\widetilde{\cup}_{i \in \boldsymbol{N}} h_i \in \tau$.

The pair (X, τ) *is called a hesitant fuzzy topological space. Each member of* τ *is called a hesitant fuzzy open set (HFOS) in* X*. A hesitant fuzzy set* h *in* X *is called a hesitant fuzzy closed set (HFCS) in* (X, τ)*, if* $h^c \in \tau$*. The set of all hesitant fuzzy closed sets is denoted by* $HFC(X)$.

Definition 5. *[10] Let* (X, τ) *be a hesitant fuzzy topological space, and* $h_A \in HS(X)$*. Then:*

(1) $int_H(h_A) = \widetilde{\bigcup}\{h_U \in \tau : h_U \subseteq h_A\}$.
(2) $cl_H(h_A) = \widetilde{\bigcap}\{h_F \in HFC(X) : h_A \subseteq h_F\}$.

3. Weaker hesitant fuzzy open sets

Definition 6. *Let* (X, τ) *be a hesitant fuzzy topological space. A subset* h *of* $HS(X)$ *is called:*

(1) *hesitant fuzzy* α*-open if* $h \subseteq int_H(cl_H(int_H(h)))$.
(2) *hesitant fuzzy preopen if* $h \subseteq int_H(cl_H(h))$.
(3) *hesitant fuzzy semiopen if* $h \subseteq cl_H(int_H(h))$.
(4) *hesitant fuzzy b-open if* $h \subseteq int_H(cl_H(h)) \widetilde{\cup} cl_H(int_H(h))$.
(5) *hesitant fuzzy* β*-open if* $h \subseteq cl_H(int_H(cl_H(h)))$.

Theorem 1. Let (X, τ) be a hesitant fuzzy topological space, then the following statements are hold:

(1) Every hesitant fuzzy open set is hesitant fuzzy α-open.
(2) Every hesitant fuzzy α-open set is hesitant fuzzy preopen.
(3) Every hesitant fuzzy α-open set is hesitant fuzzy semiopen.
(4) Every hesitant fuzzy preopen set is hesitant fuzzy b-open.
(5) Every hesitant fuzzy semiopen set is hesitant fuzzy b-open.
(6) Every hesitant fuzzy b-open set is hesitant fuzzy β-open.

Proof. (1) If h_A is hesitant fuzzy open, then $h_A = int_H(h_A) \subseteq int_H(cl_H(h_A)) = int_H(cl_H(int_H(h_A)))$. Thus, h_A is hesitant fuzzy α-open.

(2) If h_A is hesitant fuzzy α-open, then $h_A \subseteq int_H(cl_H(int_H(h_A))) \subseteq int_H(cl_H(h_A))$. Thus, h_A is hesitant fuzzy preopen.
(3) If h_A is hesitant fuzzy α-open, then $h_A \subseteq int_H(cl_H(int_H(h_A))) \subseteq cl_H(in_H(h_A))$. Thus, h_A is hesitant fuzzy semiopen.
(4) If h_A is hesitant fuzzy preopen, then $h_A \subseteq int_H(cl_H(h_A)) \subseteq int_H(cl_H(h_A)) \widetilde{\cup} cl_H(int_H(h_A))$. Thus, h_A is hesitant fuzzy b-open.
(5) If h_A is hesitant fuzzy semiopen, then $h_A \subseteq cl_H(int_H(h_A)) \subseteq int_H(cl_H(h_A)) \widetilde{\cup} cl_H(int_H(h_A))$. Thus, h_A is hesitant fuzzy b-open.
(6) If h_A is hesitant fuzzy b-open, then $h_A \subseteq int_H(cl_H(h_A)) \widetilde{\cup} cl_H(int_H(h_A)) \subseteq cl_H(int_H(cl_H(h_A))) \widetilde{\cup} cl_H(int_H(h_A)) = cl_H[int_H(cl_H(h_A)) \widetilde{\cup} int_H(h_A)] \subseteq cl_H[int_H(cl_H(h_A)) \widetilde{\cup} int_H(cl_H(h_A))] = cl_H(int_H(cl_H(h_A)))$. Thus, h_A is hesitant fuzzy β-open.

□

Remark 1. *The concepts of hesitant fuzzy preopen and hesitant fuzzy semiopen are independent.*

Remark 2. *The converse of the Theorem 1, need not be true as shown by the following examples.*

Example 1. *Consider the hesitant fuzzy sets in $X = \{a, b, c\}$ given by:*
$h_1(a) = [0.7, 1], h_1(b) = \{0.2, 0.5, 0.8\}, h_1(c) = [0.7, 1),$
$h_2(a) = [0.5, 1), h_2(b) = \{0.2, 0.5, 0.7\}, h_2(c) = (0.7, 1],$
$h_3(a) = [0.7, 1), h_3(b) = \{0.2, 0.5\}, h_3(c) = (0.7, 1),$ *and*
$h_4(a) = [0.5, 1], h_4(b) = \{0.2, 0.5, 0.7, 0.8\}, h_4(c) = [0.7, 1].$
Then, $\tau = \{h^0, h^1, h_1, h_2, h_3, h_4\}$ a hesitant topology on X. If h_A is the hesitant fuzzy set in X given by:
(1) $h_A(a) = [0.6, 1], h_A(b) = \{0.2, 0.5, 0.6, 0.8, 0.9\}, h_A(c) = [0.3, 1),$
then h_A is hesitant fuzzy α-open but h_A is not hesitant fuzzy open.
(2) $h_A(a) = [0, 1], h_A(b) = \phi, h_A(c) = \phi,$
then h_A is both hesitant fuzzy preopen and hesitant fuzzy b-open but h_A is neither hesitant fuzzy α-open nor hesitant fuzzy semiopen.

Example 2. *Consider the hesitant fuzzy sets in $X = \{a, b, c\}$ given by:*
$h_1(a) = \{0.4\}, h_1(b) = \{0.1\}, h_1(c) = \{0.8\},$
$h_2(a) = \{0.3\}, h_2(b) = \{0.2\}, h_2(c) = \{0.7\},$ *and*
$h_3(a) = \{0.3, 0.4\}, h_3(b) = \{0.1, 0.2\}, h_3(c) = \{0.7, 0.8\}.$
Then, $\tau = \{h^0, h^1, h_1, h_2, h_3\}$ a hesitant topology on X. If h_A is the hesitant fuzzy set in X given by:
$h_A(a) = \{0.4, 0.6\}, h_A(b) = \{0.1, 0.6\}, h_A(c) = \{0.6, 0.8\},$
then h_A is both hesitant fuzzy semiopen and hesitant fuzzy b-open but A is neither hesitant fuzzy α-open nor hesitant fuzzy preopen.

Example 3. *Consider the hesitant fuzzy sets in $X = \{a\}$ given by:*
$h_1(a) = \{0.1\},$

$h_2(a) = \{0.2\}$,
$h_3(a) = \{0.1, 0.2\}$,
$h_4(a) = \{0.3, 0.4\}$,
$h_5(a) = \{0.1, 0.3, 0.4\}$,
$h_6(a) = \{0.2, 0.3, 0.4\}$,
$h_7(a) = \{0.1, 0.2, 0.3, 0.4\}$,
$h_8(a) = \{0.1, 0.3, 0.4, 0.5\}$, *and*
$h_9(a) = \{0.1, 0.2, 0.3, 0.4, 0.5\}$.
Then, $\tau = \{h^0, h^1, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$ *is a hesitant topology on* X*. If* h_A *is the hesitant fuzzy set in* X *given by:*
$h_A(a) = \{0.2, 0.3, 0.5\}$,
then h_A *is hesitant fuzzy* β*-open but* A *is not hesitant fuzzy b-open.*

Remark 3. From Theorem 1, we obtain the following diagram of implications:

$$\begin{tikzcd}
\text{hesitant fuzzy open} \arrow[d] & \text{hesitant fuzzy preopen} \arrow[d] & \\
\text{hesitant fuzzy } \alpha\text{-open} \arrow[ur] \arrow[dr] & \text{hesitant fuzzy b-open} \arrow[r] & \text{hesitant fuzzy } \beta\text{-open} \\
& \text{hesitant fuzzy semiopen} \arrow[u] &
\end{tikzcd}$$

Theorem 2. Let (X, τ) be a hesitant fuzzy topological space and $h_A \in HS(X)$. Then:

(1) $cl_H(h_A)\widetilde{\cap}h_G \subseteq cl_H(h_A\widetilde{\cap}h_G)$, for every hesitant fuzzy open set h_G.
(2) $int_H(h_A\widetilde{\cup}h_F) \subseteq int_H(h_A)\widetilde{\cup}h_F$, for every hesitant fuzzy closed set h_F.

Proof. (1) Let $x_\delta \in cl_H(h_A)\widetilde{\cap}h_G$, then $x_\delta \in cl_H(h_A)$ and $x_\delta \in h_G$. If h_V is a hesitant fuzzy open set containing x_δ, then, $h_V\widetilde{\cap}h_G$ is also hesitant fuzzy open set containing x_δ. Since $x_\delta \in cl_H(h_A)$ implies $(h_V\widetilde{\cap}h_G)\widetilde{\cap}h_A \neq h^0$ and hence $h_V\widetilde{\cap}(h_G\widetilde{\cap}h_A) \neq h^0$. This is true for every h_V containing x_δ, so $x_\delta \in cl_H(h_G\widetilde{\cap}h_A)$. Therefore hesitant fuzzy $cl(h_A)\widetilde{\cap}h_G \subseteq cl_H(h_A\widetilde{\cap}h_G)$.
(2) Follows from (1) and so it is obvious.

□

Theorem 3. If $\{h_i : i \in \mathbf{N}\}$ is a collection of hesitant fuzzy b-open (resp. hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy β-open) sets of a hesitant fuzzy topological space (X, τ), then $\widetilde{\cup}_{i\in\mathbf{N}}h_i$ is a hesitant fuzzy b-open (resp. hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy β-open) set.

Proof. We prove only the first case since the other cases are similarly shown. Since $h_i \subseteq int_H(cl_H(h_i))\widetilde{\cup}cl_H(int_H(h_i))$ for every $i \in \mathbf{N}$, we have

$$\widetilde{\cup}_{i\in\mathbf{N}}h_i \subseteq \widetilde{\cup}_{i\in\mathbf{N}}[int_H(cl_H(h_i))\widetilde{\cup}cl_H(int_H(h_i))]$$

$$\subseteq [\widetilde{\cup}_{i\in\mathbf{N}} int_H(cl_H(h_i))]\widetilde{\cup}[\widetilde{\cup}_{i\in\mathbf{N}} cl_H(int_H(h_i))]$$

$$\subseteq [int_H(\widetilde{\cup}_{i\in\mathbf{N}} cl_H(h_i))]\widetilde{\cup}[cl_H(\widetilde{\cup}_{i\in\mathbf{N}} int_H(h_i))]$$

$$\subseteq [int_H(cl_H(\widetilde{\cup}_{i\in\mathbf{N}} h_i))]\widetilde{\cup}[cl_H(int_H(\widetilde{\cup}_{i\in\mathbf{N}} h_i))].$$

Therefore, $\widetilde{\cup}_{i\in\mathbf{N}} h_i$ is hesitant fuzzy b-open. □

Theorem 4. Let (X,τ) be a hesitant fuzzy topological space, $h_U \in \tau$ and $h_A \in HS(X)$.

(1) If h_A is hesitant fuzzy preopen, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy preopen.
(2) If h_A is hesitant fuzzy semiopen, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy semiopen.

Proof. (1) Since h_A is hesitant fuzzy preopen and h_U is hesitant fuzzy open, then, $h_A \subseteq int_H(cl_H(h_A))$ and $int_H(h_U) = h_U$ and so by Theorem 2 (1), $h_U\widetilde{\cap}h_A \subseteq int_H(h_U)\widetilde{\cap}int_H(cl_H(h_A)) = int_H(h_U\widetilde{\cap}cl_H(h_A)) \subseteq int_H(cl_H(h_U\widetilde{\cap}h_A))$. Therefore, $h_U\widetilde{\cap}h_A$ is hesitant fuzzy preopen.

(2) Since h_A is hesitant fuzzy semiopen, then by Theorem 2 (1), $h_U\widetilde{\cap}h_A \subseteq h_U\widetilde{\cap}cl_H(in_H(h_A)) \subseteq cl_H(h_U\widetilde{\cap}in_H(h_A)) = cl_H(in_H(h_U)\widetilde{\cap}in_H(h_A)) = cl_H(in_H(h_U\widetilde{\cap}h_A))$. Therefore, $h_U\widetilde{\cap}h_A$ is hesitant fuzzy semiopen.

□

Theorem 5. Let (X,τ) be a hesitant fuzzy topological space, $h_U \in \tau$ and $h_A \in HS(X)$. If h_A is hesitant fuzzy β-open, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy β-open.

Proof. Since h_A is hesitant fuzzy β-open, then

$$h_U\widetilde{\cap}h_A \subseteq h_U\widetilde{\cap}cl_H(int_H(cl_H(h_A)))$$

$$\subseteq cl_H(h_U\widetilde{\cap}int_H(cl_H(h_A)))$$

$$= cl_H(int_H(h_U)\widetilde{\cap}int_H(cl_H(h_A)))$$

$$= cl_H(int_H(h_U\widetilde{\cap}cl_H(h_A)))$$

$$\subseteq cl_H(int_H(cl_H(h_U\widetilde{\cap}h_A))).$$

This shows that $h_U\widetilde{\cap}h_A$ is hesitant fuzzy β-open. □

Theorem 6. Let (X,τ) be a hesitant fuzzy topological space, $h_U \in \tau$ and $h_A \in HS(X)$. If h_A is hesitant fuzzy b-open, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy b-open.

Proof. Since h_A is hesitant fuzzy b-open, then

$$h_U\widetilde{\cap}h_A \subseteq h_U\widetilde{\cap}[int_H(cl_H(h_A))\widetilde{\cup}cl_H(int_H(h_A))]$$

$$= [h_U\widetilde{\cap}int_H(cl_H(h_A))]\widetilde{\cup}[h_U\widetilde{\cap}cl_H(int_H(h_A))]$$

$$= [int_H(h_U)\widetilde{\cap}int_H(cl_H(h_A))]\widetilde{\cup}[h_U\widetilde{\cap}cl_H(int_H(h_A))]$$

$$\subseteq [int_H(h_U\widetilde{\cap}cl_H(h_A))]\widetilde{\cup}[cl_H(h_U\widetilde{\cap}int_H(h_A))]$$

$$\subseteq [int_H(cl_H(h_U \widetilde{\cap} h_A))] \widetilde{\cup} [cl_H(int_H(h_U \widetilde{\cap} h_A))].$$

This shows that $h_U \widetilde{\cap} h_A$ is hesitant fuzzy b-open. □

Remark 4. *We note that the intersection of two hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open) sets need not be hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open) as can be seen from the following examples:*

Example 4. *Consider the hesitant fuzzy sets in $X = \{a\}$ given by $h(a) = \{0.3, 0.6\}$. Then, $\tau = \{h^0, h^1, h\}$ is a hesitant topology on X. If $h_A(a) = \{0.1, 0.3\}$ and $h_B(a) = \{0.1, 0.6\}$, then h_A and h_B are hesitant fuzzy preopen (resp. hesitant fuzzy b-open and hesitant fuzzy β-open), but $h_A \widetilde{\cap} h_B = \{0.1\} = h_C$ which is not hesitant fuzzy preopen (resp. hesitant fuzzy b-open and hesitant fuzzy β-open).*

Example 5. *From Example 2, if h_A is the hesitant fuzzy set in X given by:*
$h_A(a) = \{0.4, 0.6\}, h_A(b) = \{0.1, 0.6\}, h_A(c) = \{0.6, 0.8\}$,
and h_B is the hesitant fuzzy set in X given by:
$h_B(a) = \{0.3, 0.6\}, h_B(b) = \{0.2, 0.6\}, h_B(c) = \{0.7, 0.9\}$,
then h_A and h_B are hesitant fuzzy semiopen, but $h_A \widetilde{\cap} h_B = h_C$ which is not hesitant fuzzy semiopen, where h_C is the hesitant fuzzy set in X given by:
$h_C(a) = \{0.6\}, h_C(b) = \{0.6\}, h_C(c) = \phi$.

Remark 5. *From Remark 4, we notice that the family of all hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open) sets need not be a topology in general.*

Theorem 7. Let (X, τ) be a hesitant fuzzy topological space. If h_A and h_B are hesitant fuzzy α-open, then $h_B \widetilde{\cap} h_A$ is also hesitant fuzzy α-open.

Proof. Since h_A and h_B are hesitant fuzzy α-open, then

$$h_B \widetilde{\cap} h_A \subseteq int_H(cl_H(int_H(h_B))) \widetilde{\cap} int_H(cl_H(int_H(h_A)))$$

$$\subseteq int_H[cl_H(int_H(h_B)) \widetilde{\cap} int_H(cl_H(int_H(h_A)))]$$

$$\subseteq int_H cl_H[int_H(h_B) \widetilde{\cap} int_H(cl_H(int_H(h_A)))]$$

$$\subseteq int_H cl_H[int_H(h_B) \widetilde{\cap} cl_H(int_H(h_A))]$$

$$\subseteq int_H cl_H cl_H[int_H(h_B) \widetilde{\cap} int_H(h_A)]$$

$$\subseteq int_H cl_H int_H(h_B \widetilde{\cap} h_A).$$

Thus, $h_B \widetilde{\cap} h_A$ is hesitant fuzzy α-open. □

Remark 6. *From the Theorems 3 and 7, we notice that the family of all hesitant fuzzy α-open is a topology.*

Theorem 8. Let (X,τ) be a hesitant fuzzy topological space and $h_A \in HS(X)$. If h_A is both hesitant fuzzy semiopen and hesitant fuzzy preopen, then h_A is hesitant fuzzy α-open.

Proof. By assumption, $h_A \subseteq cl_H(int_H(h_A))$ and $h_A \subseteq int_H(cl_H(h_A))$. Then, $h_A \subseteq int_H(cl_H(h_A)) \subseteq int_H(cl_H(cl_H(int_H(h_A)))) = int_H(cl_H(int_H(h_A)))$. Therefore, h_A is hesitant fuzzy α-open. □

Theorem 9. Let (X,τ) be a hesitant fuzzy topological space and h_A be hesitant fuzzy α-open.

(1) If h_B is hesitant fuzzy semiopen, then $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen.
(2) If h_B is hesitant fuzzy preopen, then $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen.

Proof. (1) By assumption, $h_A \subseteq int_H(cl_H(int_H(h_A)))$ and $h_B \subseteq cl_H(int_H(h_B))$, then by Theorem 2 (1), we have that

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(int_H(h_A))) \widetilde{\cap} cl_H(int_H(h_B)) \\ &\subseteq cl_H[int_H(cl_H(int_H(h_A))) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H(int_H(h_A)) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H[int_H(h_A) \widetilde{\cap} int_H(h_B)]] \\ &= cl_H(int_H(h_A \widetilde{\cap} h_B)). \end{aligned}$$

Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen.

(2) By assumption, $h_A \subseteq int_H(cl_H(int_H(h_A)))$ and $h_B \subseteq int_H(cl_H(h_B))$, then

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(int_H(h_A))) \widetilde{\cap} int_H(cl_H(h_B)) \\ &= int_H[int_H(cl_H(int_H(h_A))) \widetilde{\cap} int_H(cl_H(h_B))] \\ &\subseteq int_H[cl_H(int_H(h_A)) \widetilde{\cap} int_H(cl_H(h_B))] \\ &\subseteq int_H[cl_H[int_H(h_A) \widetilde{\cap} int_H(cl_H(h_B))]] \\ &\subseteq int_H[cl_H[int_H(h_A) \widetilde{\cap} cl_H(h_B)]] \\ &\subseteq int_H[cl_H[cl_H[int_H(h_A) \widetilde{\cap} h_B]]] \\ &\subseteq int_H(cl_H(cl_H(h_A \widetilde{\cap} h_B))) \\ &= int_H(cl_H(h_A \widetilde{\cap} h_B)). \end{aligned}$$

Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen.

□

Theorem 10. Let (X,τ) be a hesitant fuzzy topological space. If h_A is hesitant fuzzy preopen and h_B is hesitant fuzzy semiopen, then $h_A \widetilde{\cap} h_B$ is hesitant fuzzy β-open.

Proof. By assumption, $h_A \subseteq int_H(cl_H(h_A))$ and $h_B \subseteq cl_H(int_H(h_B))$, then by Theorem 2 (1), we have that

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(h_A)) \widetilde{\cap} cl_H(int_H(h_B)) \\ &\subseteq cl_H[int_H(cl_H(h_A)) \widetilde{\cap} int_H(h_B)] \end{aligned}$$

$$\begin{aligned}
&= cl_H[int_H[int_H(cl_H(h_A))\widetilde{\cap} int_H(h_B)]]\\
&\subseteq cl_H[int_H[cl_H(h_A)\widetilde{\cap} int_H(h_B)]]\\
&\subseteq cl_H[int_H[cl_H(h_A\widetilde{\cap} int_H(h_B))]]\\
&\subseteq cl_H(int_H(cl_H(h_A\widetilde{\cap} h_B))).
\end{aligned}$$

Therefore, $h_A\widetilde{\cap} h_B$ is hesitant fuzzy β-open. □

Theorem 11. Let (X, τ) be a hesitant fuzzy topological space and $h_A, h_B \in HS(X)$. Then,

(1) h_A is hesitant fuzzy semiopen if and only if there exists a hesitant fuzzy open set h_U such that $h_U \subseteq h_A \subseteq cl_H(h_U)$.
(2) h_B is hesitant fuzzy semiopen if h_A is hesitant fuzzy semiopen and $h_A \subseteq h_B \subseteq cl_H(h_A)$.
(3) h_A is hesitant fuzzy semiopen if and only if $cl_H(h_A) = cl_H(int_H(h_A))$.

Proof. (1) Let h_A be hesitant fuzzy semiopen, then $h_A \subseteq cl_H(int_H(h_A))$. Take $h_U = int_H(h_A)$, then h_U is hesitant fuzzy open such that $h_U = int_H(h_A) \subseteq h_A \subseteq cl_H(int_H(h_A)) = cl_H(h_U)$.

Conversely, since $h_U \subseteq h_A$ implies that $h_U = int_H(h_U) \subseteq int_H(h_A)$ and so $h_A \subseteq cl_H(h_U) = cl_H(int_H(h_U)) \subseteq cl_H(int_H(h_A))$. Thus, h_A is hesitant fuzzy semiopen.

(2) Since h_A is hesitant fuzzy semiopen, then by (1) there exists a hesitant fuzzy open set h_U such that $h_U \subseteq h_A \subseteq cl_H(h_U)$. Since $h_A \subseteq h_B$, so $h_U \subseteq h_B$. But $cl_H(h_A) \subseteq cl_H(h_U)$, then $h_B \subseteq cl_H(h_U)$. Hence, $h_U \subseteq h_B \subseteq cl_H(h_U)$. Thus, h_B is hesitant fuzzy semiopen.

(3) Let h_A be hesitant fuzzy semiopen, then $h_A \subseteq cl_H(int_H(h_A))$ which implies that $cl_H(h_A) \subseteq cl_H(int_H(h_A)) \subseteq cl_H(h_A)$ and hence $cl_H(h_A) = cl_H(int_H(h_A))$.

Conversely, since by Theorem 1, $int_H(h_A)$ is hesitant fuzzy semiopen such that $int_H(h_A) \subseteq h_A \subseteq cl_H(h_A) = cl_H(int_H(h_A))$ and therefore h_A is hesitant fuzzy semiopen.

□

Definition 7. *[10] Let (X, τ) be a hesitant fuzzy topological space and $h \in HS(X)$. Then, the collection $\tau_h = \{U\widetilde{\cap} h : U \in \tau\}$ is called a hesitant fuzzy subspace topology or hesitant fuzzy relative topology on h. The pair (h, τ_h) is called a hesitant fuzzy subspace, and each member of τ_h is called a hesitant fuzzy open set in h.*

Proposition 1. *[10] Let (X, τ) be a hesitant fuzzy topological space, $h, h_A \in HS(X)$ and $h_A \subseteq h$. Then, $cl_{\tau_h}(h_A) = h\widetilde{\cap} cl_H(h_A)$, where $cl_{\tau_h}(h_A)$ denotes the closure of h_A in (h, τ_h).*

Definition 8. *Let (X, τ) be a hesitant fuzzy topological space, $h, h_A \in HS(X)$ and $h_A \subseteq h$. Then, $int_{\tau_h}(h_A) = \widetilde{\bigcup}\{h_U \in \tau_h : h_U \subseteq h_A\}$.*

Theorem 12. Let (X, τ) be a hesitant fuzzy topological space and $h_A, h_B \in HS(X)$. If h_A is hesitant fuzzy preopen in X and h_B is hesitant fuzzy semiopen in X, then

(1) $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen in h_A.
(2) $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen in h_B.

Proof. By assumption, $h_A \subseteq int_H(cl_H(h_A))$ and $h_B \subseteq cl_H(int_H(h_B))$.

(1) Then,
$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(h_A)) \widetilde{\cap} cl_H(int_H(h_B)) \\ &\subseteq cl_H[int_H(cl_H(h_A)) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H(h_A) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H[h_A \widetilde{\cap} int_H(h_B)]] \\ &= cl_H[h_A \widetilde{\cap} int_H(h_B)]. \end{aligned}$$
Hence, $h_A \widetilde{\cap} h_B \subseteq cl_H(h_A \widetilde{\cap} int_H(h_B))$ and so $h_A \widetilde{\cap} h_B \subseteq cl_H(h_A \widetilde{\cap} int_H(h_B)) \widetilde{\cap} h_A = cl_{\tau_{h_A}}(h_A \widetilde{\cap} int_H(h_B))$. Since $h_A \widetilde{\cap} int_H(h_B)$ is a hesitant fuzzy open set in h_A, so $h_A \widetilde{\cap} h_B \subseteq cl_{\tau_{h_A}}(h_A \widetilde{\cap} int_H(h_B)) = cl_{\tau_{h_A}}(int_{\tau_{h_A}}(h_A \widetilde{\cap} int_H(h_B))) \subseteq cl_{\tau_{h_A}}(int_{\tau_{h_A}}(h_A \widetilde{\cap} h_B))$. Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen in h_A.

(2) Now,
$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(h_A)) \widetilde{\cap} h_B \\ &= int_{\tau_{h_B}}[int_H(cl_H(h_A)) \widetilde{\cap} h_B] \\ &\subseteq int_{\tau_{h_B}}[int_H(cl_H(h_A)) \widetilde{\cap} cl_H(int_H(h_B))] \\ &\subseteq int_{\tau_{h_B}}[cl_H[int_H(cl_H(h_A)) \widetilde{\cap} int_H(h_B)]] \\ &\subseteq int_{\tau_{h_B}}[cl_H[cl_H(h_A) \widetilde{\cap} int_H(h_B)]] \\ &\subseteq int_{\tau_{h_B}}[cl_H[cl_H[h_A \widetilde{\cap} int_H(h_B)]]] \\ &\subseteq int_{\tau_{h_B}}[cl_H[cl_H[h_A \widetilde{\cap} h_B]]] \\ &= int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B)). \end{aligned}$$
Since $int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B))$ is hesitant fuzzy open in h_B, then $int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B)) \widetilde{\cap} h_B = int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B) \widetilde{\cap} h_B)$, and hence $h_A \widetilde{\cap} h_B \subseteq int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B) \widetilde{\cap} h_B) = int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A \widetilde{\cap} h_B))$.

Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen in h_B. □

Theorem 13. Let (X, τ) be a hesitant fuzzy topological space, $h_A, h_B \in HS(X)$, $h_A \subseteq h_B$ and h_B be hesitant fuzzy semiopen in X. Then, h_A is hesitant fuzzy semiopen in X if and only if h_A is hesitant fuzzy semiopen in h_B.

Proof. Let h_A be hesitant fuzzy semiopen in X, then there is a hesitant fuzzy open set h_U such that $h_U \subseteq h_A \subseteq cl_H(h_U)$ implies that $h_U \subseteq h_A \subseteq h_B$. Hence,

$h_U \subseteq h_A \subseteq cl_H(h_U)\widetilde{\cap}h_B = cl_{\tau_{h_B}}(h_U)$. Since $h_U\widetilde{\cap}h_B = h_U$ is also hesitant fuzzy open in h_B, then h_A is hesitant fuzzy semiopen in h_B.

Conversely, let h_A be hesitant fuzzy semiopen in h_B. Then there is a hesitant fuzzy open set h_U in h_B such that $h_U \subseteq h_A \subseteq cl_{\tau_{h_B}}(h_U)$. Since h_U is hesitant fuzzy open in h_B, there exists a hesitant fuzzy open set h_V such that $h_U = h_V\widetilde{\cap}h_B$. Then, $h_V\widetilde{\cap}h_B = h_U \subseteq h_A \subseteq cl_{\tau_{h_B}}(h_U) = cl_{\tau_{h_B}}(h_V\widetilde{\cap}h_B) \subseteq cl_H(h_V\widetilde{\cap}h_B)$. By Theorem 4 (2), $h_V\widetilde{\cap}h_B$ is hesitant fuzzy semiopen, then by Theorem 11 (2), h_A is hesitant fuzzy semiopen in X. □

Theorem 14. Let (X, τ) be a hesitant fuzzy topological space, $h_A, h_B \in HS(X)$, $h_A \subseteq h_B$ and h_B be hesitant fuzzy preopen in X. Then, h_A is hesitant fuzzy preopen in X if and only if h_A is hesitant fuzzy preopen in h_B.

Proof. Suppose that h_A is hesitant fuzzy preopen in X, then $h_A = h_A\widetilde{\cap}h_B \subseteq int_H(cl_H(h_A))\widetilde{\cap}h_B$. Since $int_H(cl_H(h_A))\widetilde{\cap}h_B$ is hesitant fuzzy open in h_B, then $h_A \subseteq int_H(cl_H(h_A))\widetilde{\cap}h_B \subseteq int_{\tau_{h_B}}[int_H(cl_H(h_A))\widetilde{\cap}h_B] \subseteq int_{\tau_{h_B}}[cl_H(h_A)\widetilde{\cap}h_B] = int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$. Hence, h_A is hesitant fuzzy preopen in h_B.

Conversely, assume that h_A is hesitant fuzzy preopen in h_B. Then, $h_A \subseteq int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$. Since $int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$ is hesitant fuzzy open in h_B, so there a hesitant fuzzy open set h_U in X such that $int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A)) = h_U\widetilde{\cap}h_B$. By Theorem 9 (2), $int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$ is hesitant fuzzy preopen in X. Therefore,

$$\begin{aligned} h_A &\subseteq int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A)) \\ &\subseteq int_H(cl_H(int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A)))) \\ &= int_H(cl_H(int_{\tau_{h_B}}[cl_H(h_A)\widetilde{\cap}h_B])) \\ &\subseteq int_H(cl_H[cl_H(h_A)\widetilde{\cap}h_B]) \\ &\subseteq int_H(cl_H(cl_H(h_A))) \\ &= int_H(cl_H(h_A)). \end{aligned}$$

This shows that h_A is hesitant fuzzy preopen in X. □

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CHEBYSHEV INEQUALITY ON CONFORMABLE DERIVATIVE

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Abstract. Integral inequalities are very important in applied sciences. Chebyshev's integral inequality is widely used in applied mathematics. First of all, some necessary definitions and results regarding conformable derivative are given in this article. Then we give Chebyshev inequality for simultaneously positive (or negative) functions using the conformable fractional derivative. We used the Gronwall inequality to prove our results, unlike other studies in the literature.

1. Introduction

Various definitions are given in the literature for fractional derivatives [8, 14, 17, 20]. Some of which are Riemann-Liouville, Caputo, Grünwald-Letnikov, Riesz, Weyl fractional derivatives. Having more than one definition of derivative in fractional analysis ensures that the most suitable one is used according to the type of the problem and thus the best solution is obtained.

In [12], a new fractional derivative that is known as conformable derivative has been defined by Khalil. This new fractional derivative based on classical limit definition. Authors gave linearity condition, the product rule, the division rule, Rolle theorem and mean value theorem for this new definition of fractional derivative. They also defined the fractional integral of order $0 < \alpha \leq 1$ only.

In [1], definition of left and right conformable fractional integrals of any order $\alpha > 0$ has been given by Abdeljawad. He also gave chain rule, linear differential systems, Laplace transforms and exponential functions on a fractional version.

Conformable fractional derivative has been formulated in [1, 12] as

$$D^{\alpha}F(t) = \lim_{\epsilon \longrightarrow 0} \frac{F(t + \epsilon t^{1-\alpha}) - F(t)}{\epsilon},$$

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or in [11] as

$$D^{\alpha}F(t) = \lim_{\epsilon \to 0} \frac{F(te^{\epsilon t^{-\alpha}}) - F(t)}{\epsilon},\ D^{\ \alpha}F(0) = \lim_{t \to 0^+} D^{\alpha}F(t),$$

provided the limit exist; in both we have

$$D^{\alpha}F(t) = t^{1-\alpha}D^{'}(t),$$

where $F^{'}(t) = \lim_{\epsilon \to 0}[F(t+\epsilon) - F(t)]/\epsilon$.

In [2], Anderson and Ulness present an exact definition of a conformable fractional derivative of order α for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, where D^0 is the identity operator and D^1 is the classical differential operator.

Monotonicity is an important part of applications of derivatives. The monocity of a function gives an idea about behaviour of the function. Monotonic function is defined as a function that is either completely non-increasing or completely non-decreasing.

For monotonicity and convexity results for fractional integrals and some of their application we recommend the readers to refer the literature [18, 13, 7, 5, 19].

2. Preliminaries

Main definitions and results of conformable derivatives from [2] will be presented as follows:

Definition 1. *Let $\alpha \in [0,1]$. A differential operator D^{α} is conformable if and only if*

$$D^0F(t) = F(t) \text{ and } D^1F(t) = \frac{d}{dt}F(t) = F'(t), \tag{1}$$

where D^0 is the identity operator and D^1 is the classical differential operator.

Definition 2. *Let $\alpha \in [0,1]$ and let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that*

$$\lim_{\alpha \to 0^+} \kappa_1(\alpha,t) = 1, \quad \lim_{\alpha \to 0^+} \kappa_0(\alpha,t) = 0,\ \forall t \in \mathbb{R},$$

$$\lim_{\alpha \to 1^-} \kappa_1(\alpha,t) = 0, \quad \lim_{\alpha \to 1^-} \kappa_0(\alpha,t) = 1, \forall t \in \mathbb{R},$$

$$\kappa_1(\alpha,t) \neq 0\ ,\ \alpha \in [0,1), \quad \kappa_0(\alpha,t) \neq 0,\ \alpha \in (0,1]\ ,\ \forall t \in \mathbb{R}.$$

Then the following differential operator D^{α}, defined via

$$D^{\alpha}F(t) = \kappa_1(\alpha,t)F(t) + \kappa_0(\alpha,t)F'(t), \tag{3}$$

is conformable provided the function F is differentiable at t and $F^{'} = \frac{d}{dt}F$.

For more information on conformable derivative and integral, we refer [1, 2, 12, 11, 4].

Definition 3. ***(Partial Conformable Derivatives).*** *Let $\alpha \in [0,1]$, and let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2). Given a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial}{\partial t}F(t,s)$ exists for each fixed $s \in \mathbb{R}$, define the partial differential operator D_t^α via*

$$D_t^\alpha F(t,s) = \kappa_1(\alpha,t)F(t,s) + \kappa_0(\alpha,t)\frac{\partial}{\partial t}F(t,s). \tag{4}$$

Definition 4. ***(Conformable Exponential Function).*** *Let $\alpha \in (0,1]$, the points $s,t \in \mathbb{R}$ with $s \le t$, and the function $\rho : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), with ρ/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. After that the exponential function with respect to D^α in (3) is defined as follows*

$$e_\rho(t,s) = e^{\int\limits_s^t \frac{\rho(\tau)-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau}, \quad e_0(t,s) = e^{-\int\limits_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau}. \tag{5}$$

Lemma 5. ***(Basic Derivatives).*** *Let the conformable differential operator D^α be given as in (3), where $\alpha \in [0,1]$, $\rho : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), with ρ/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. Assume the functions F and H are differentiable as needed. Then*

(i) $D^\alpha[aF(t) + bH(t)] = aD^\alpha[F(t)] + bD^\alpha[H(t)]$ for all $a, b \in \mathbb{R}$**;**
(ii) $D^\alpha[c] = c\kappa_1(\alpha,t)$ for all constants $c \in \mathbb{R}$;
(iii) $D^\alpha[F(t)H(t)] = F(t)D^\alpha[H(t)] + H(t)D^\alpha[F(t)] - F(t)H(t)\kappa_1(\alpha,t)$;
(iv) $D^\alpha[F(t)/H(t)] = \frac{H(t)D^\alpha[F(t)] - F(t)D^\alpha[H(t)]}{H^2(t)} + \frac{F(t)}{H(t)}\kappa_1(\alpha,t)$;
(v) for $\alpha \in (0,1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies

$$D_t^\alpha[e_\rho(t,s)] = \rho(t)e_\rho(t,s), \tag{6}$$

for $e_\rho(t,s)$ given in (5);
(vi) for $\alpha \in (0,1]$ and for the exponential function e_0 given in (5), we have

$$D^\alpha[\int\limits_a^t \frac{F(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds] = F(t). \tag{7}$$

Definition 6. *Let $\alpha \in (0,1]$ and $t_0 \in \mathbb{R}$. In light of (5) and Lemma 1 (v) and (vi), define the antiderivative via*

$$\int D^\alpha F(t)d_\alpha t = F(t) + ce_0(t,t_0),\ c \in \mathbb{R}$$

Similarly, define the integral of F over [a,b] as

$$\int\limits_a^t F(s)e_0(t,s)d_\alpha s = \int\limits_a^t \frac{F(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds, \quad d_\alpha s = \frac{1}{\kappa_0(\alpha,s)}, \tag{8}$$

recall that

$$e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau} = e^{-\int_s^t \kappa_1(\alpha,\tau)d_\alpha\tau}$$

from (5).

Lemma 7. *Let the conformable differential operator D^α be given as in (3), the integral be given as in (8) with $\alpha \in (0,1]$. Let the functions κ_0, κ_1 be continuous and satisfy (2), and let F and H be differentiable as needed. Then*

(i) the derivative of the definite integral of F is given by

$$D^\alpha[\int_a^t F(s)e_0(t,s)d_\alpha s] = F(t);$$

(ii) the definite integral of the derivative of F is given by

$$\int_a^t D^\alpha[F(s)]e_0(t,s)d_\alpha s = F(s)e_0(t,s)\,|_{s=a}^t = F(t) - F(a)e_0(t,a);$$

(iii) an integration by parts formula is given by

$$\begin{aligned}\int_a^b F(t)D^\alpha[H(t)]e_0(b,t)d_\alpha t &= F(t)H(t)e_0(b,t)\,|_{t=a}^b \\ &\quad - \int_a^b H(t)(D^\alpha[F(t)] - \kappa_1(\alpha,,t)F(t))e_0(b,t)d_\alpha t;\end{aligned}$$

(iv) a version of the Leibniz rule for differentiation of an integral is given by

$$D^\alpha[\int_a^t F(t,s)e_0(t,s)d_\alpha s] = \int_a^t (D_t^\alpha[F(t,s)] - \kappa_1(\alpha,t)F(t,s))e_0(t,s)d_\alpha s + F(t,t),$$

using (4); or, if e_0 is absent,

$$D^\alpha(\int_a^t F(t,s)d_\alpha s) = F(t,t) + \int_a^t D_t^\alpha[F(t,s)]d_\alpha s.$$

Lemma 8. ***(Variation of Constants).*** *Assume κ_0, κ_1 satisfy (2). Let $f, \rho : [t_0, \infty] \to \mathbb{R}$ be continuous, let e_ρ be as in (5), and let $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$D^\alpha x(t) - \rho(t)x(t) = f(t),\ x(t_0) = x_0,$$

is given by

$$x(t) = x_0 e_\rho(t,t_0) + \int_{t_0}^t e_\rho(t,s)f(s)d_\alpha s,\ t \in [t_o, \infty). \tag{9}$$

Theorem 9. ***(Gronwall's Inequality).*** *Let ρ, x, f be continuous functions on $[t_0, \infty)$, with $\rho \geq 0$. Then*

$$x(t) \leq f(t) + \int_{t_0}^{t} \rho(s)x(s)e_0(t,s)d_\alpha s \quad \textit{for all } t \in [t_0, \infty),$$

implies

$$x(t) \leq f(t) + \int_{t_0}^{t} \rho(s)f(s)e_0(t,s)d_\alpha s \quad \textit{for all } t \in [t_0, \infty).$$

Corollary 10. *Let ρ, x be continuous functions on $[t_0, \infty)$, with $\rho \geqslant 0$. Then*

$$x(t) \leqslant \int_{t_0}^{t} \rho(s)x(s)e_0(t,s)d_\alpha s \textit{ for all } t \in [t_0, \infty),$$

implies $x(t) \leqslant 0$ for all $t \in [t_0, \infty)$.

3. Main Result

It is new to refer to inequalities as a mathematics discipline. A very small portion of these inequalities originated from the ancient traditions. In the 18th and early 19th century names such as Newton, Cauchy and Maclaurin started to work in this field. In this period, only Bernoulli and Cauchy-Schwarz-Bunyakovsky inequalities, which are mentioned with their own name, can be given as an example [9].

Towards the end of the 19th century, original products started to be given in the field of inequalities. Hölder and Minkovski could be shown among their pioneers. But the milestone in this area is the Chebyshev's paper [6]. Chebyshev submitted his paper to the Han'kovshov University's Editorial Committee in order to be published in the journal for the volumes in 1883. But the mentioned committee extremely impressed from this paper that they published it in the last volume of 1882 [9].

Theorem 11. ***(Chebyshev Inequality)****. Let f and g be two integrable functions on the $[0,1]$. If both functions are simultaneously increasing or decreasing for the same $x \in [0,1]$, then*

$$\int_0^1 f(x)g(x)dx \geqslant \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

If one of the functions is increasing, the other is decreasing for the same $x \in [0,1]$ values, then

$$\int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

(Chebyshev 1882).

Belarbi and Dahmani gave results on Chebyshev's inequality using the Riemann-Liouville integral in 2009 [4]. E.Set gave results on Chebyshev's inequality using conformable fractional integrals in 2019 [21]. For the background and summary on inequalities, we refer the readers to the references [3, 9, 10, 15].

Before giving Chebyshev inequality using conformable derivative, mentioning about following results [16] that play a key role in our proof will provide a better understanding:

Monotonicity

Let $a > 0$ and $\digamma : [a, b] \to \mathbb{R}$ be $\alpha-$differentiable on an interval $[a, b]$.

i. If $\digamma^{\alpha}(x) \geq 0$ for all $x \in [a, b]$, then $\digamma$ is nondecreasing on $[a, b]$.
ii. If $\digamma^{\alpha}(x) > 0$ for all $x \in [a, b]$, then $\digamma$ is increasing on $[a, b]$.
iii. If $\digamma^{\alpha}(x) \leq 0$ for all $x \in [a, b]$, then $\digamma$ is nonincreasing on $[a, b]$.
iv. If $\digamma^{\alpha}(x) < 0$ for all $x \in [a, b]$, then $\digamma$ is decreasing on $[a, b]$.
v. If $\digamma^{\alpha}(x) = 0$ for all $x \in [a, b]$, then $\digamma$ is constant on $[a, b]$.

Theorem 12. *Let f and g be two integrable functions on $[a, b]$. If both functions are simultaneously positive or negative for the same $x \in [a, b]$ values then*

$$\int_a^b f(x)g(x)e_0(t,x)d_\alpha x \geqslant \int_a^b f(x)e_0(t,x)d_\alpha x \int_a^b g(x)e_0(t,x)d_\alpha x.$$

If one of the functions for the same $x \in [a, b]$ values is positive and the other is negative then

$$\int_a^b f(x)g(x)e_0(t,x)d_\alpha x \leqslant \int_a^b f(x)e_0(t,x)d_\alpha x \int_a^b g(x)e_0(t,x)d_\alpha x.$$

Proof. Let f and g be two integrable functions on [a,b]. Let define

$$\digamma(x) = \int_a^x f(t)g(t)e_0(x,t)d_\alpha t - \int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t.$$

If we take the derivative of both sides, we have

$$\begin{aligned} D^\alpha \digamma(x) &= f(x)g(x) - f(x)\int_a^x g(t)e_0(x,t)d_\alpha t - g(x)\int_a^x f(t)e_0(x,t)d_\alpha t \\ &\quad + \kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t. \end{aligned}$$

$$D^{\alpha}F(x) = \frac{f(x)g(x)}{2} - f(x)\int_a^x g(t)e_0(x,t)d_\alpha t + \frac{f(x)g(x)}{2} - g(x)\int_a^x f(t)e_0(x,t)d_\alpha t$$
$$+\kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t.$$

$$D^{\alpha}F(x) = \frac{f(x)}{2}[g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t] + \frac{g(x)}{2}[f(x) - 2\int_a^x f(t)e_0(x,t)d_\alpha t]$$
$$+\kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t \int_a^x g(t)e_0(x,t)d_\alpha t.$$

a) (i) Let $g(x) > 0$, assume that

$$g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t \leqslant 0,$$

then

$$g(x) \leqslant 2\int_a^x g(t)e_0(x,t)d_\alpha t.$$

From Corollory 10 $g(x) \leqslant 0$. This is the contradiction. Then;

$$g(x) > 0,\ g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t > 0.$$

Using similar arguments, we can write

$$f(x) > 0,\ f(x) - 2\int_a^x f(t)e_0(x,t)d_\alpha t > 0.$$

(ii) Let $g(x) < 0, -g(x) = G(x), G(x) > 0$, assume that

$$g(x) - 2\int_a^x g(t)e_0(x,t)d_\alpha t \geq 0,$$

then

$$-G(x) + 2\int_a^x G(t)e_0(x,t)d_\alpha t \geq 0,$$

this implies

$$G(x) \leq 2 \int_a^x G(t) e_0(x,t) d_\alpha t.$$

From Corollory 10 $G(x) \leq 0$. This is the contradiction. Consequently,

$$g(x) < 0,\ g(x) - 2 \int_a^x g(t) e_0(x,t) d_\alpha t < 0.$$

Using similar arguments, we can write

$$f(x) < 0,\ f(x) - 2 \int_a^x f(t) e_0(x,t) d_\alpha t < 0.$$

Also we can say $\kappa_1(\alpha,t) \int_a^x f(t) e_0(x,t) d_\alpha t \int_a^x g(t) e_0(x,t) d_\alpha t \geq 0$. As a result of this part we have

$$\begin{aligned} D^\alpha F(x) &= \frac{f(x)}{2}[g(x) - 2\int_a^x g(t) e_0(x,t) d_\alpha t] + \frac{g(x)}{2}[f(x) - 2\int_a^x f(t) e_0(x,t) d_\alpha t] \\ &\quad + \kappa_1(\alpha,t) \int_a^x f(t) e_0(x,t) d_\alpha t \int_a^x g(t) e_0(x,t) d_\alpha t, \end{aligned}$$

is positive. So, the function $F(x)$ is increasing on $[a,b]$. Then,

$$F(b) \geq F(a) = 0.$$

This implies the first inequality in theorem is proved.

b)Let $f(x) > 0$, assume that

$$f(x) - 2 \int_a^x f(t) e_0(x,t) d_\alpha t \leq 0,$$

then

$$f(x) \leq 2 \int_a^x f(t) e_0(x,t) d_\alpha t.$$

From Corollory 10 $f(x) \leq 0$. This is the contradiction. Hence,

$$f(x) > 0,\ f(x) > 2 \int_a^x f(t) e_0(x,t) d_\alpha t.$$

Now, from part a, if

$$g(x)<0,\ g(x)-2\int_a^x g(t)e_0(x,t)d_\alpha t<0.$$

As a result of this part we have

$$\begin{aligned} D^\alpha F(x) &= \frac{f(x)}{2}[g(x)-2\int_a^x g(t)e_0(x,t)d_\alpha t]+\frac{g(x)}{2}[f(x)-2\int_a^x f(t)e_0(x,t)d_\alpha t] \\ &\quad +\kappa_1(\alpha,t)\int_a^x f(t)e_0(x,t)d_\alpha t\int_a^x g(t)e_0(x,t)d_\alpha t, \end{aligned}$$

is negative. So the function $F(x)$ is decreasing on [a,b]. Then,

$$F(b)\le F(a)=0.$$

This implies the second inequality in theorem is proved. □

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LOGARITHMIC COEFFICIENTS OF STARLIKE FUNCTIONS CONNECTED WITH k-FIBONACCI NUMBERS

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Abstract. Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disc $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. For a function $f \in \mathcal{S}$, the logarithmic coefficients δ_n $(n = 1, 2, 3, \ldots)$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U})$$

and it is known that $|\delta_1| \leq 1$ and $|\delta_2| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) = 0,635 \cdots$. The problem of the best upper bounds for $|\delta_n|$ of univalent functions for $n \geq 3$ is still open. Let $\mathcal{SL}^k$ denote the class of functions $f \in \mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \quad \tau_k = \frac{k - \sqrt{k^2 + 4}}{2} \qquad (z \in \mathbb{U}).$$

In the present paper, we determine the sharp upper bound for $|\delta_1|, |\delta_2|$ and $|\delta_3|$ for functions f belong to the class $\mathcal{SL}^k$ which is a subclass of $\mathcal{S}$. Furthermore, a general formula is given for $|\delta_n|$ $(n \in \mathbb{N})$ as a conjecture.

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

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if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U} \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U} \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha\,(0 \le \alpha < 1)$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike of order α by $\mathcal{S}^*(\alpha)$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

By means of the principle of subordination, Yılmaz Özgür and Sokól [13] defined the following class $\mathcal{SL}^k$ of functions $f \in \mathcal{S}$, connected with a shell-like region described by a function $\tilde{p}_k$ with coefficients depicted in terms of the k-Fibonacci numbers where k is a positive real number. The name attributed to the class $\mathcal{SL}^k$ is motivated by the shape of the curve

$$\Gamma = \left\{\tilde{p}_k\left(e^{i\varphi}\right) : \varphi \in [0, 2\pi) \setminus \{\pi\}\right\}.$$

The curve Γ has a shell-like shape and it is symmetric with respect to the real axis. For more details about the class $\mathcal{SL}^k$, please refer to [11, 13].

Definition 1. *[13] Let k be any positive real number. The function $f \in \mathcal{S}$ belongs to the class $\mathcal{SL}^k$ if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}_k(z) \qquad (z \in \mathbb{U}), \tag{2}$$

where

$$\tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = \frac{1 + \tau_k^2 z^2}{1 - (\tau_k^2 - 1) z - \tau_k^2 z^2} \tag{3}$$

with

$$\tau_k = \frac{k - \sqrt{k^2+4}}{2}. \tag{4}$$

For $k = 1$, the class $\mathcal{SL}^k$ reduces to the class $\mathcal{SL}$ which consists of functions $f \in \mathcal{A}$ defined by (1) satisfying

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

where

$$\tilde{p}(z) := \tilde{p}_1(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{5}$$

with

$$\tau := \tau_1 = \frac{1-\sqrt{5}}{2}. \tag{6}$$

This class was introduced by Sokól [10].

Definition 2. *[3] For any positive real number k, the k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \qquad (n \in \mathbb{N})$$

with initial conditions

$$F_{k,0} = 0, \qquad F_{k,1} = 1.$$

Furthermore n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}, \tag{7}$$

where τ_k is given by (4).

For $k = 1$, we obtain the classic Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}_0}$:

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{N}).$$

For more details about the k-Fibonacci sequences please refer to [7, 9, 12, 14].

Yılmaz Özgür and Sokól [13] showed that the coefficients of the function $\tilde{p}_k(z)$ defined by (3) are connected with k-Fibonacci numbers. This connection is pointed out in the following theorem.

Theorem 1. *[13] Let $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ be the sequence of k-Fibonacci numbers defined in Definition 2. If*

$$\tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2} := 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n, \tag{8}$$

then we have

$$\tilde{p}_{k,1} = k\tau_k, \quad \tilde{p}_{k,2} = \left(k^2+2\right)\tau_k^2, \quad \tilde{p}_{k,n} = (F_{k,n-1} + F_{k,n+1})\,\tau_k^n \qquad (n \in \mathbb{N}). \tag{9}$$

It can be found the more results related to Fibonacci numbers in [7, 12, 14].

Remark 1. *[13] For each $k>0$,*

$$\mathcal{SL}^k \subset \mathcal{S}^*(\alpha_k), \qquad \alpha_k = \frac{k}{2\sqrt{k^2+4}},$$

that is, $f \in \mathcal{SL}^k$ is a starlike function of order α_k, and so is univalent.

For a function $f \in \mathcal{S}$, the logarithmic coefficients δ_n $(n \in \mathbb{N})$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U}), \tag{10}$$

and play a central role in the theory of univalent functions. The idea of studying the logarithmic coefficients helped Kayumov [8] to solve Brennan's conjecture for conformal mappings. If $f \in \mathcal{S}$, then it is known that

$$|\delta_1| \leq 1$$

and

$$|\delta_2| \leq \frac{1}{2}\left(1+2e^{-2}\right) = 0,635\cdots$$

(see [2]). The problem of the best upper bounds for $|\delta_n|$ of univalent functions for $n \geq 3$ is still open.

The main purpose of this paper is to determine the upper bound for $|\delta_1|, |\delta_2|$ and $|\delta_3|$ for functions f belong to the univalent function class $\mathcal{SL}^k$. To prove our main results we need the following lemmas.

Lemma 1. *[11] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(z \in \mathbb{U})$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k-\sqrt{k^2+4}}{2},$$

then we have

$$|p_1| \leq k\,|\tau_k| \qquad \text{and} \qquad |p_2| \leq \left(k^2+2\right)\tau_k^2.$$

The above estimates are sharp.

Lemma 2. *[5] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(z \in \mathbb{U})$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k-\sqrt{k^2+4}}{2},$$

then we have

$$|p_3| \leq \left(k^3+3k\right)|\tau_k|^3.$$

The result is sharp.

Lemma 3. *[1] If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ $(z \in \mathbb{U})$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k-\sqrt{k^2+4}}{2},$$

then we have

$$\left|p_2 - \gamma p_1^2\right| \leq k\left|\tau_k\right| \max\left\{1,\ \left|k^2 + 2 - \gamma k^2\right| \frac{|\tau_k|}{k}\right\}$$

for all $\gamma \in \mathbb{C}$. *The above estimates are sharp.*

Lemma 4. *[2] Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. *Then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N}).$$

Lemma 5. *[4] Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. *Then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

for some x, $|x| \leq 1$, *and*

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z$$

for some z, $|z| \leq 1$.

Lemma 6. *[1] If the function* f *given by* (1) *is in the class* $\mathcal{SL}^k$, *then we have*

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} \tau_k^2\left(k^2 + 1 - \lambda k^2\right) & , \quad \lambda \leq \frac{2\left(k^2+1\right)\tau_k + k}{2k^2\tau_k} \\ \\ \frac{k|\tau_k|}{2} & , \quad \frac{2\left(k^2+1\right)\tau_k + k}{2k^2\tau_k} \leq \lambda \leq \frac{2\left(k^2+1\right)\tau_k - k}{2k^2\tau_k} \\ \\ \tau_k^2\left(\lambda k^2 - k^2 - 1\right) & , \quad \lambda \geq \frac{2\left(k^2+1\right)\tau_k - k}{2k^2\tau_k} \end{cases}.$$

If $\frac{2\left(k^2+1\right)\tau_k + k}{2k^2\tau_k} \leq \lambda \leq \frac{k^2+1}{k^2}$, *then*

$$\left|a_3 - \lambda a_2^2\right| + \left(\lambda - \frac{2\left(k^2+1\right)\tau_k + k}{2k^2\tau_k}\right)|a_2|^2 \leq \frac{k\left|\tau_k\right|}{2}.$$

Furthermore, if $\frac{k^2+1}{k^2} \leq \lambda \leq \frac{2\left(k^2+1\right)\tau_k - k}{2k^2\tau_k}$, *then*

$$\left|a_3 - \lambda a_2^2\right| + \left(\frac{2\left(k^2+1\right)\tau_k - k}{2k^2\tau_k} - \lambda\right)|a_2|^2 \leq \frac{k\left|\tau_k\right|}{2}.$$

Each of these results is sharp.

Lemma 7. *[6] If the function* f *given by* (1) *is in the class* $\mathcal{SL}^k$, *then*

$$\left|a_2 a_4 - a_3^2\right| \leq \tau_k^4.$$

The bound is sharp.

Lemma 8. *[6] If the function* f *given by* (1) *is in the class* $\mathcal{SL}^k$, *then*

$$\left|a_2 a_3 - a_4\right| \leq k\left|\tau_k\right|^3.$$

The bound is sharp.

2. The coefficients of $\log(f(z)/z)$

Theorem 2. *Let* $f \in \mathcal{SL}^k$ *be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then*

$$|\delta_1| \le \frac{k}{2}|\tau_k|, \qquad |\delta_2| \le \frac{k^2+2}{4}\tau_k^2, \qquad |\delta_3| \le \frac{k^3+3k}{6}|\tau_k|^3, \tag{11}$$

where τ_k *is defined by* (4). *Each of these results is sharp. The equalities are attained by the function* $\tilde{p}_k$ *given by* (3).

Proof. Firstly, by differentiating (10) and equating coefficients, we have

$$\delta_1 = \frac{1}{2}a_2,$$
$$\delta_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right),$$
$$\delta_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).$$

If $f \in \mathcal{SL}^k$, then by the principle of subordination, there exists a Schwarz function $\omega \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} = \tilde{p}_k(\omega(z)) \quad (z \in \mathbb{U}), \tag{12}$$

where the function $\tilde{p}_k$ is given by (8). Therefore, the function

$$g(z) := \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1z + c_2z^2 + \cdots \quad (z \in \mathbb{U}) \tag{13}$$

is in the class $\mathcal{P}$. Now, defining the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1z + p_2z^2 + \cdots, \tag{14}$$

it follows from (12) and (13) that

$$p(z) = \tilde{p}_k\left(\frac{g(z)-1}{g(z)+1}\right). \tag{15}$$

Note that

$$\omega(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots$$

and so

$$\begin{aligned}\tilde{p}_k(\omega(z)) = 1 &+ \frac{\tilde{p}_{k,1}c_1}{2}z + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,1} + \frac{1}{4}c_1^2\tilde{p}_{k,2}\right]z^2 \\ &+ \left[\frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\tilde{p}_{k,1} + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,2} + \frac{c_1^3}{8}\tilde{p}_{k,3}\right]z^3 + \cdots.\end{aligned} \tag{16}$$

Thus, by using (13) in (15), and considering the values $\tilde{p}_{k,j}$ $(j = 1, 2, 3)$ given in (9), we obtain

$$p_1 = \frac{k\tau_k}{2} c_1, \tag{17}$$

$$p_2 = \frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\left(k^2+2\right)\tau_k^2}{4} c_1^2, \tag{18}$$

$$p_3 = \frac{k\tau_k}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{\left(k^2+2\right)\tau_k^2}{2} c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\left(k^3+3k\right)\tau_k^3}{8} c_1^3. \tag{19}$$

On the other hand, a simple calculation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right) z^3 + \cdots,$$

which, in view of (14), yields

$$a_2 = p_1, \qquad a_3 = \frac{p_1^2 + p_2}{2}, \qquad a_4 = \frac{p_1^3 + 3p_1p_2 + 2p_3}{6}. \tag{20}$$

Substituting for a_2, a_3 and a_4 from (20), we obtain

$$\delta_1 = \frac{1}{2} p_1, \qquad \delta_2 = \frac{1}{4} p_2, \qquad \delta_3 = \frac{1}{6} p_3. \tag{21}$$

Using Lemma 1 and Lemma 2, we get the desired results. This completes the proof of theorem. □

Conjecture. Let $f \in \mathcal{SL}^k$ be given by (1) and the coefficients of $\log(f(z)/z)$ be given by (10). Then

$$|\delta_n| \leq \frac{F_{k,n-1} + F_{k,n+1}}{2n} |\tau_k|^n \qquad (n \in \mathbb{N}),$$

where $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ is the Fibonacci sequence given by (7).

This conjecture has been verified for the values $n = 1, 2, 3$ by the Theorem 2.

Letting $k = 1$ in Theorem 2, we obtain the following consequence.

Corollary 1. *Let $f \in \mathcal{SL}$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then*

$$|\delta_1| \leq \frac{1}{2}|\tau|, \qquad |\delta_2| \leq \frac{3}{4}\tau^2, \qquad |\delta_3| \leq \frac{2}{3}|\tau|^3,$$

where τ is defined by (6). *Each of these results is sharp. The equalities are attained by the function $\tilde{p}$ given by* (5).

Theorem 3. *Let $f \in \mathcal{SL}^k$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|\delta_2 - \gamma\delta_1^2\right| \leq \frac{k\,|\tau_k|}{4} \max\left\{1,\ \left|k^2 + 2 - \gamma k^2\right| \frac{|\tau_k|}{k}\right\}.$$

Proof. By using (21), the desired result is obtained from the equality

$$\delta_2 - \gamma\delta_1^2 = \frac{1}{4}\left(p_2 - \gamma p_1^2\right) \qquad (\gamma \in \mathbb{C})$$

and Lemma 3. □

Letting $k = 1$ in Theorem 3, we obtain the following consequence.

Corollary 2. *Let $f \in \mathcal{SL}$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then for any $\gamma \in \mathbb{C}$, we have*

$$\left|\delta_2 - \gamma\delta_1^2\right| \leq \frac{|\tau|}{4} \max\left\{1,\ |(3-\gamma)\,\tau|\right\}.$$

If we take $\gamma = 1$ in Theorem 3, then we obtain the following result.

Corollary 3. *Let $f \in \mathcal{SL}^k$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then*

$$\left|\delta_2 - \delta_1^2\right| \leq \begin{cases} \frac{\tau_k^2}{2} & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \\ \frac{k|\tau_k|}{4} & \quad k \geq \frac{2}{\sqrt{3}} \end{cases}.$$

Letting $k = 1$ in Corollary 3, we obtain the following consequence.

Corollary 4. *Let $f \in \mathcal{SL}$ be given by* (1) *and the coefficients of* $\log(f(z)/z)$ *be given by* (10). *Then*

$$\left|\delta_2 - \delta_1^2\right| \leq \frac{\tau^2}{2}.$$

3. The coefficients of the inverse function

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [2] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f)\,;\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, for a function $f \in \mathcal{A}$ given by (1) the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots =: w + \sum_{n=2}^{\infty} A_n w^n. \tag{22}$$

Since $\mathcal{SL}^k \subset \mathcal{S}$, the functions f belonging to the class $\mathcal{SL}^k$ are invertible.

Theorem 4. *Let $f \in \mathcal{SL}^k$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then we have*

$$|A_2| \leq k\,|\tau_k|$$

and

$$|A_3| \leq \frac{k\,|\tau_k|}{2} \max\left\{1,\ 2\left|1-k^2\right| \frac{|\tau_k|}{k}\right\}.$$

Each of these results is sharp.

Proof. Let the function $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{SL}^k$, and f^{-1} be the inverse function of f defined by (22). Then using (20), we obtain

$$A_2 = -a_2 = -p_1 \tag{23}$$

and

$$A_3 = 2a_2^2 - a_3 = -\frac{1}{2}\left(p_2 - 3p_1^2\right).$$

The upper bound for $|A_2|$ is clear from Lemma 1. Furthermore by considering Lemma 3 we obtain the upper bound of $|A_3|$ as

$$|A_3| \leq \frac{k\,|\tau_k|}{2} \max\left\{1,\ 2\left|1-k^2\right| \frac{|\tau_k|}{k}\right\}.$$

Finally, for the sharpness, we have by (8) that

$$\tilde{p}_k(z) = 1 + k\tau_k z + \left(k^2+2\right)\tau_k^2 z^2 + \cdots$$

and

$$\tilde{p}_k\left(z^2\right) = 1 + k\tau_k z^2 + \left(k^2+2\right)\tau_k^2 z^4 + \cdots.$$

From this equalities, we obtain

$$p_1 = k\tau_k \qquad \textit{and} \qquad p_2 = \left(k^2+2\right)\tau_k^2$$

and

$$p_1 = 0 \qquad \textit{and} \qquad p_2 = k\tau_k,$$

respectively. Thus, it is clear that the equality for $|A_2|$ is attained for the function $\tilde{p}_k(z)$; and the equality for the first value of $|A_3|$ is attained for the function $\tilde{p}_k(z^2)$, for the second value of $|A_3|$ is attained for the function $\tilde{p}_k(z)$. This evidently completes the proof of theorem. □

Remark 2. *It is worthy to note that the coefficient bound obtained for $|A_3|$ in Theorem 4 is the improvement of [11, Corollary 2.4].*

Theorem 5. *Let $f \in \mathcal{SL}$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then we have*

$$|A_2| \leq |\tau|, \qquad |A_3| \leq \frac{|\tau|}{2} \qquad \text{and} \qquad |A_4| \leq 2|\tau|^3.$$

Each of these results is sharp.

Proof. Let $f \in \mathcal{SL}$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then the upper bounds for $|A_2|$ and $|A_3|$ are obtained as a consequence of Theorem 4 when $k = 1$. From (22), we have

$$-A_4 = 5a_2^3 - 5a_2a_3 + a_4.$$

By using (20) in the above equality, we obtain

$$-A_4 = \frac{8}{3}p_1^3 - 2p_1p_2 + \frac{1}{3}p_3.$$

By (17)-(19), this equality gives

$$A_4 = -\frac{\tau}{6}\left(c_3 - c_1c_2 + \frac{1-6\tau^2}{4}c_1^3\right).$$

By means of Lemma 5, we get

$$\begin{aligned} A_4 &= \frac{\tau}{6}\left[\frac{1}{4}c_1\left(4-c_1^2\right)x^2 - \frac{1}{2}\left(4-c_1^2\right)\left(1-|x|^2\right)z + \frac{3\tau^2}{2}c_1^3\right] \\ &= \frac{\tau}{24}\left[6\tau^2c_1^3 + \left(4-c_1^2\right)\left\{c_1x^2 - 2\left(1-|x|^2\right)z\right\}\right]. \end{aligned}$$

As per Lemma 4, it is clear that $|c_1| \leq 2$. Therefore letting $c_1 = c$, we may assume without loss of generality that $c \in [0,2]$. Hence, by using the triangle inequality, it is obtained that

$$|A_4| \leq \frac{|\tau|}{24}\left[6\tau^2c^3 + \left(4-c^2\right)\left\{c|x|^2 + 2\left(1-|x|^2\right)\right\}\right].$$

Thus, for $\mu = |x| \leq 1$, we have

$$|A_4| \leq \frac{|\tau|}{24}\left[6\tau^2c^3 + \left(4-c^2\right)\left\{c\mu^2 + 2\left(1-\mu^2\right)\right\}\right] := F(c,\mu).$$

Now, we need to find the maximum value of $F(c,\mu)$ over the rectangle Π,

$$\Pi = \{(c,\mu) : 0 \leq c \leq 2,\ 0 \leq \mu \leq 1\}.$$

For this, first differentiating the function F with respect to c and μ, we get

$$\frac{\partial F(c,\mu)}{\partial c} = \frac{|\tau|}{24}\left[18\tau^2c^2 + \left(4-c^2\right)\left\{c\mu^2 + 2\left(1-\mu^2\right)\right\}\right]$$

and

$$\frac{\partial F(c,\mu)}{\partial \mu} = \frac{|\tau|}{12}\left(4-c^2\right)(c-2)\mu,$$

respectively. The condition $\frac{\partial F(c,\mu)}{\partial \mu}=0$ gives $c=2$ or $\mu=0$, and such points (c,μ) are not interior point of Π. So the maximum cannot attain in the interior of Π. Now to see on the boundary, by elementary calculus one can verify the following:

$$\max_{0\le\mu\le1} F(0,\mu)=F(0,0)=\frac{|\tau|}{3}, \qquad \max_{0\le\mu\le1} F(2,\mu)=F(2,0)=2\,|\tau|^3$$

$$\max_{0\le c\le2} F(c,0)=F(2,0)=2\,|\tau|^3, \qquad \max_{0\le c\le2} F(c,1)=F(2,1)=2\,|\tau|^3.$$

Comparing these results, we get

$$\max_{\Pi} F(c,\mu)=2\,|\tau|^3$$

(see Figure 1). Also note that

$$\tilde{p}\,(z)=1+\tau z+3\tau^2\,z^2+4\tau^3\,z^3+\cdots$$

by (8) with $k=1$. From this equality, we obtain

$$p_1=\tau, \qquad p_2=3\tau^2 \qquad \textit{and} \qquad p_3=4\tau^3.$$

On the other hand, the sharpness of the upper bounds of $|A_2|$ and $|A_3|$ is known from Theorem 4 and it is seen that the equality for $|A_4|$ is attained for the function $\tilde{p}(z)$. This evidently completes the proof of theorem. □

Theorem 6. *Let $f\in\mathcal{SL}^k$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then for any $\gamma\in\mathbb{C}$, we have*

$$\left|A_3-\gamma A_2^2\right|\le\frac{k\,|\tau_k|}{2}\max\left\{1,\,2\left|1-(1-\gamma)\,k^2\right|\frac{|\tau_k|}{k}\right\}.$$

Proof. By using (20), the desired result is obtained from the equality

$$A_3-\gamma A_2^2=-\frac{1}{2}\left[p_2-(3-2\gamma)\,p_1^2\right] \qquad (\gamma\in\mathbb{C})$$

and Lemma 3. □

Letting $k=1$ in Theorem 6, we obtain following consequence.

Corollary 5. *Let $f\in\mathcal{SL}$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then for any $\gamma\in\mathbb{C}$, we have*

$$\left|A_3-\gamma A_2^2\right|\le\frac{|\tau|}{2}\max\left\{1,\,2\,|\gamma\tau|\right\}.$$

If we take $\gamma=1$ in Theorem 6, then we obtain the following result.

Corollary 6. *Let $f\in\mathcal{SL}^k$ be given by (1), and f^{-1} be the inverse function of f defined by (22). Then*

$$\left|A_3-A_2^2\right|\le\begin{cases}\tau_k^2 & , \quad 0<k\le\frac{2}{\sqrt{3}}\\[2ex] \frac{k|\tau_k|}{2} & , \quad k\ge\frac{2}{\sqrt{3}}\end{cases}.$$

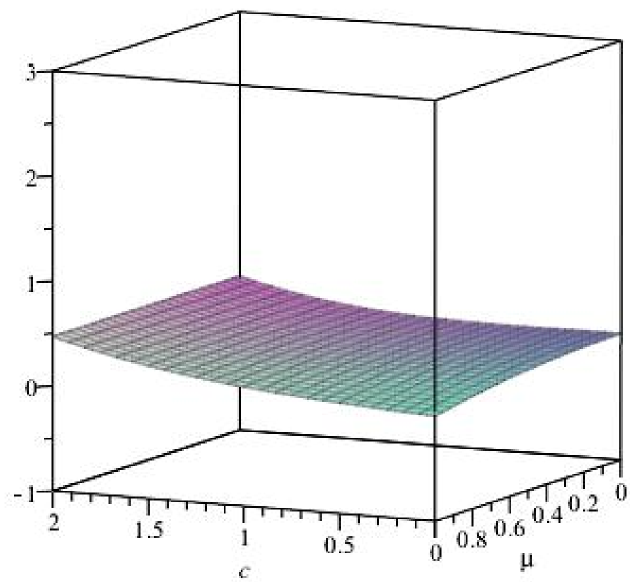

FIGURE 1. Mapping of $F(c, \mu)$ over Π

Letting $k = 1$ in Corollary 6, we obtain the following consequence.

Corollary 7. *Let $f \in \mathcal{SL}$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then*

$$\left|A_3 - A_2^2\right| \leq \tau^2.$$

Theorem 7. *Let $f \in \mathcal{SL}^k$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then*

$$\left|A_2A_4 - A_3^2\right| \leq \begin{cases} \left(1+k^2\right)\tau_k^4 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \\ \tau_k^4 + \frac{k^3|\tau_k|^3}{2} & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}$$

and

$$\left|A_2A_3 - A_4\right| \leq \begin{cases} 4k\left|\tau_k\right|^3 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \\ k\left|\tau_k\right|^3 + \frac{3k^2\tau_k^2}{2} & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}.$$

Proof. Let $f \in \mathcal{SL}^k$ be of the form (1) and its inverse f^{-1} be given by (22). Then we obtain

$$\left|A_2A_4 - A_3^2\right| = \left|a_2^2\left(a_2^2 - a_3\right) + \left(a_2a_4 - a_3^2\right)\right|$$

and

$$\left|A_2A_3 - A_4\right| = \left|3a_2\left(a_2^2 - a_3\right) - \left(a_2a_3 - a_4\right)\right|.$$

Hence, applying triangle inequality, we have

$$\left|A_2A_4 - A_3^2\right| \leq \left|a_2\right|^2\left|a_3 - a_2^2\right| + \left|a_2a_4 - a_3^2\right|$$

and

$$\left|A_2A_3 - A_4\right| \leq 3\left|a_2\right|\left|a_3 - a_2^2\right| + \left|a_2a_3 - a_4\right|,$$

respectively. On the other hand, from Lemma 6, we obtain

$$\left|a_3 - a_2^2\right| \leq \begin{cases} \tau_k^2 & , \quad 0 < k \leq \frac{2}{\sqrt{3}} \\ \\ \frac{k\left|\tau_k\right|}{2} & , \quad k \geq \frac{2}{\sqrt{3}} \end{cases}. \tag{24}$$

Furhermore, we get

$$\left|a_2\right| \leq k\left|\tau_k\right| \tag{25}$$

by using (23) together with Lemma 1. Now, by considering Lemma 7 and Lemma 8, we get the desired estimates. □

Letting $k = 1$ in Theorem 7, we obtain the following consequence.

Corollary 8. *Let $f \in \mathcal{SL}$ be given by* (1), *and f^{-1} be the inverse function of f defined by* (22). *Then*

$$\left|A_2A_4 - A_3^2\right| \leq 2\tau^4$$

and

$$\left|A_2A_3 - A_4\right| \leq 4\left|\tau\right|^3.$$

Declaration of Competing Interests The author declare that there is no competing interest.

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APPROXIMATION BY TRUNCATED LUPAŞ OPERATORS OF MAX-PRODUCT KIND

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Abstract. The goals of the present paper are to introduce truncated Lupaş type operators of max-product kind and give an estimation for the degree of approximation with respect to first modulus of continuity function. We prove that this estimate can not be improved; on the other hand, for some subclasses of functions, better degree of approximation is obtained. We also showed the piecewise convexity of the constructed operators on the interval $[0, 1]$.

1. Introduction

As it takes very important place in the approximation theory, the sequences of positive linear operators of discrete type have been studied by various authors in the last century. One of those operators that we deal with in this paper was constructed by A. Lupaş [23] in 1995. His starting point in this construction was the identity

$$\frac{1}{(1-a)^{\gamma}}=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!}a^k, \quad |a|<1.$$

With the help of this identity, he defined the following sequence of operators which is linear and positive:

$$L_n(f)(x)=(1-a)^{nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{k!}a^k f\left(\frac{k}{n}\right), \quad x\geq 0,$$

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with $f:[0,\infty)\to\mathbb{R}$. The notation here is the Pochhammer symbol and given by

$$(\gamma)_0=1,\ \ (\gamma)_k=\gamma\left(\gamma+1\right)...\left(\gamma+k-1\right),k\geq 1.$$

Imposing $L_n\left(e_1\right)=e_1$, one finds $a=1/2$ and therefore the operator turns into,

$$L_n\left(f\right)\left(x\right)=2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right),\ x\geq 0. \tag{1}$$

Agratini [2] studied the approximation properties of these operators by means of Korovkin's theorem and gave estimates for the rate of convergence of the operators. The well-known Korovkin's theorem, which gives a simple proof of Weierstrass theorem, is based on the approximation of functions by linear and positive operators. The underlying algebraic structure of these mentioned operators is linear over $\mathbb{R}$ and they are also linear operators. The idea of nonlinear positive operators was given by Bede et al. in [3]. They asked whether they could change the underlying algebraic structure to more general structures. In this sense they presented nonlinear Shepard-type operators by replacing the operations sum and product by max and product.

Following this paper Bede et. al. [4] defined and studied pseudo linear approximation operators. Based upon these studies, there appeared an open problem in the book of S.Gal [17] in which the max-product type Bernstein operators were introduced. Related to this open problem, a nonlinear modification of the classical Bernstein operators were first studied by Bede and Gal [5] in detail. The idea behind these studies were also applied to other well-known approximating operators. Same authors introduced the nonlinear versions of the previously defined operators and they studied the approximation order and shape-preserving properties of the stated operators.

The nonlinear Favard-Szász-Mirakjan operators of max-product type $F_n^{(M)}$ is given in [5] as

$$F_n^{(M)}\left(f\right)\left(x\right)=\frac{\bigvee\limits_{k=0}^{\infty}s_{n,k}\left(x\right)f\left(\frac{k}{n}\right)}{\bigvee\limits_{k=0}^{n}s_{n,k}\left(x\right)},x\in\left[0,\infty\right),\ n\in\mathbb{N},$$

where $s_{n,k}\left(x\right)=\frac{(nx)^k}{k!}$. Bede, Coroianu and Gal [6] introduced the truncated Favard-Szász-Mirakjan operators of max-product type as follows:

$$T_n^{(M)}\left(f\right)\left(x\right)=\frac{\bigvee\limits_{k=0}^{n}s_{n,k}\left(x\right)f\left(\frac{k}{n}\right)}{\bigvee\limits_{k=0}^{n}s_{n,k}\left(x\right)},\ x\in\left[0,1\right],\ n\in\mathbb{N}.$$

Recently, Güngör and İspir studied quantitative estimations for the generalized Szász operators of max product type in [18]. Also, they constructed nonlinear

Bernstein-Chlodowsky operators of max-product type in [19]. Holhos [20] studied weighted approximation of functions by Meyer-König and Zeller operators of max-product type. Coroianu and Gal [8, 9] introduced truncated max-product Kantorovich operators based on Fejer Kernel and generalized (φ, ψ)-kernels. By Costarelli and Vinti, the max-product neural networks operators were studied in [11]- [15]. Recently, in [1], the max-product of Bernstein operators for symmetric ranges are introduced by Acar et.al. and upper estimates of approximation error for some subclasses of functions are obtained. Also, they investigated the shape-preserving properties of the operators.

In this paper, the nonlinear truncated Lupaş operators of max-product type are introduced. We estimate the degree of approximation of the defined sequence of operators. More importantly, we show that the estimate with respect to the modulus of continuity function cannot be improved. On the other hand, for some subclasses of functions, better order of approximation is obtained. Finally, we proved that our sequence of operators is piecewise convex on the interval $[0, 1]$ for any arbitrary function f.

Before proceeding further, we will recall some general notations about the max-product type nonlinear operators. Considering the set of positive real numbers $\mathbb{R}_+$, we deal with the maximum "$\bigvee$" and the product "$\cdot$" operations. Then $(\mathbb{R}_+, \bigvee, \cdot)$ is called as Max-Product algebra.

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I) = \{f : I \to \mathbb{R}_+;\ f \text{ continuous and bounded on } I\}.$$

A discrete max-product type approximation operator $L_n : CB_+(I) \to CB_+(I)$, has a general form

$$L_n\left(f\right)\left(x\right) = \bigvee_{i=0}^{n} D_n(x, x_i)\dot{f}\left(x_i\right),$$

or

$$L_n\left(f\right)\left(x\right) = \bigvee_{i=0}^{\infty} D_n(x, x_i)\dot{f}\left(x_i\right)$$

where $n \in \mathbb{N}$, $f \in CB_+(I), D_n(\cdot, xi) \in CB_+(I)$ and $x_i \in I$, for all i. The above form of the operators are positive and nonlinear. These operators also satisfy the pseudolinearity condition which is of the form

$$L_n(a \cdot\ f \vee b \cdot\ g)(x) = a \cdot\ L_n(f)(x) \vee b \cdot\ L_n(g)(x), \forall a, b \in \mathbb{R}_+, f, g : I \to \mathbb{R}_+.$$

In order to give some properties of the operators L_n, we present the following auxiliary Lemma.

Lemma 1. *([5]) Let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties :*

(i) (Monotonicity)

If $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$;

(ii) (Subadditivity)

$L_n(f+g) \leq L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.
Then for all $f, g \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|L_n(f)(x) - L_n(g)(x)| \leq L_n(|f-g|)(x).$$

2. Construction of the Operators

Now, we define our truncated max-product type operators as follows:

$$V_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} v_{n,k}(x)\, f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} v_{n,k}(x)}, \ x \in [0,1], \ n \in \mathbb{N}, \tag{2}$$

where

$$v_{n,k}(x) = \frac{(nx)_k}{2^k k!}, \ (nx)_0 = 1, \ (nx)_k = nx(nx+1)\ldots(nx+k-1), k \geq 1.$$

We can write the following properties of the operators $V_n^{(M)}(f)$.

i) One can see that $\bigvee_{k=0}^{n} v_{n,k}(x) > 0$ for $x \in [0,1]$.

For any $f \in C_+[0,1]$, the space of all positive real-valued and continuous functions on $[0,1]$, $V_n^{(M)}(f) \in C_+[0,1]$. So, $V_n^{(M)} : C_+[0,1] \to C_+[0,1]$ is a sequence of positive operators and since $\bigvee_{k=0}^{n} v_{n,k}(x) = 1$ for $x = 0$, $V_n^{(M)}(f)(0) = f(0)$.

ii) For any $f \in C_+[0,1]$ and $\lambda \geq 0$,

$$V_n^{(M)}(\lambda f) = \lambda V_n^{(M)}(f). \tag{3}$$

Hence, the max-product operators $V_n^{(M)}$ given by (2) are positive homogenous.

iii) For $V_n^{(M)}$ the identity

$$V_n^{(M)}(e_0) = e_0, e_0(x) = 1 \tag{4}$$

holds.

iv) $V_n^{(M)}(f)$ satisfy the pseudo-linearity condition, i.e., for any $f, g \in C_+[0,1]$ and $\alpha, \beta \in \mathbb{R}_+$

$$V_n^{(M)}(\alpha f \vee \beta g)(x) = \alpha V_n^{(M)}(f)(x) \vee \beta V_n^{(M)}(g)(x). \tag{5}$$

From the above equality, we have

$$f \leq g \Longrightarrow V_n^{(M)}(f)(x) \leq V_n^{(M)}(g)(x). \tag{6}$$

So, $V_n^{(M)}(f)$ is a monotone operator.

v) For any $f, g \in C_+[0,1]$, we get

$$V_n^{(M)}(f+g)(x) \leq V_n^{(M)}(f)(x) + V_n^{(M)}(g)(x). \tag{7}$$

That is the sublinearity condition is satisfied by the operators $V_n^{(M)}(f)$.
vi) From the above properties and Lemma 1 we have

$$\left|V_n^{(M)}(f)(x)-V_n^{(M)}(g)(x)\right|\leq V_n^{(M)}(|f-g|)(x). \tag{8}$$

Now, we can write the following corollary.

Corollary 2. *For all $f\in C_+[0,1]$,*

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\leq\left[1+\frac{1}{\delta}V_n^{(M)}(\varphi_x)(x)\right]\omega_1(f,\delta), \tag{9}$$

where $\varphi_x(t)=|t-x|, t,x\in[0,1]$ and the modulus of continuity function of f is defined as

$$\omega_1(f,\delta)=\max_{\substack{t,x\in[0,1]\\|t-x|\leq\delta}}\{|f(t)-f(x)|\}.$$

Proof. For the proof, see for example [6]. □

3. Auxiliary Lemmas

In the current section we will give some auxiliary lemmas which we need for the proof of the main theorem.

Lemma 3. *Let $x\in\left[\frac{j+1}{n},\frac{j+2}{n}\right]$ and $j\in\{0,1,...,n-2\}$. Then we have,*

$$\bigvee_{k=0}^{n}v_{n,k}(x)=v_{n,j}(x).$$

Also, $\bigvee_{k=0}^{n}v_{n,k}(x)=1$, for $x\in\left[0,\frac{2}{n}\right]$.

Proof. In fact for fixed $n\in\mathbb{N}$ and $k\geq0$, the inequality

$$\begin{aligned}0&\leq v_{n,k+1}(x)\leq v_{n,k}(x)\\0&\leq nx+k\leq2(k+1)\end{aligned}$$

is equivalent to

$$0\leq x\leq\frac{k+2}{n}.$$

So, taking $k=0,1,...,n-2$, we get

$$\begin{aligned}0&\leq v_{n,1}(x)\leq v_{n,0}(x)\iff x\in\left[0,\frac{2}{n}\right],\\0&\leq v_{n,2}(x)\leq v_{n,1}(x)\iff x\in\left[0,\frac{3}{n}\right],\\&...\\0&\leq v_{n,k+1}(x)\leq v_{n,k}(x)\iff x\in\left[0,\frac{k+2}{n}\right],\\&...\end{aligned}$$

$$0 \quad \leq \quad v_{n,n-1}(x) \leq v_{n,n-2}(x) \iff x \in [0,1].$$

For $k = n-1$ we also have

$$0 \leq v_{n,n}(x) \leq v_{n,n-1}(x) \iff x \in \left[0, 1 + \frac{1}{n}\right].$$

From all these inequalities, we can write

$$x \in \left[0, \frac{2}{n}\right] \implies v_{n,k}(x) \leq v_{n,0}(x) = 1 \text{ for all } k = 0,1,...,n,$$

also

$$x \quad \in \quad \left[\frac{2}{n}, \frac{3}{n}\right] \implies v_{n,k}(x) \leq v_{n,1}(x) \text{ for all } k = 0,1,...,n,$$

$$x \quad \in \quad \left[\frac{3}{n}, \frac{4}{n}\right] \implies v_{n,k}(x) \leq v_{n,2}(x) \text{ for all } k = 0,1,...,n,$$

in general, for fixed $j = 0,1,...,n-2$,

$$x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right] \implies v_{n,k}(x) \leq v_{n,j}(x) \text{ for all } k = 0,1,...,n.$$

So, the proof is completed.

□

In order to proceed we need the following notations:

For each $k \in \{0,1,...,n\}$, $j \in \{0,1,...,n-2\}$, state

$$M_{k,n,j}(x) = \frac{v_{n,k}(x)\left|\frac{k}{n} - x\right|}{v_{n,j}(x)}, \; m_{k,n,j}(x) = \frac{v_{n,k}(x)}{v_{n,j}(x)}. \tag{10}$$

It is clear that if $k \geq j+2$ then

$$M_{k,n,j}(x) = \frac{v_{n,k}(x)\left(\frac{k}{n} - x\right)}{v_{n,j}(x)},$$

and if $k \leq j$ then

$$M_{k,n,j}(x) = \frac{v_{n,k}(x)\left(x - \frac{k}{n}\right)}{v_{n,j}(x)}.$$

Lemma 4. *For all $k \in \{0,1,...,n\}$, $j \in \{0,1,...,n-2\}$ and $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, we have*

$$m_{k,n,j} \leq 1.$$

Proof. If $k \geq j$ then, since $h(x) = \frac{1}{nx+k}$ is nonincreasing on $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, we have

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{2(k+1)}{nx+k} \geq \frac{2(k+1)}{k+j+2} \geq 1. \tag{11}$$

So, $m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq ... \geq m_{n,n,j}(x)$ is true.
If $k \leq j$ then,

$$\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{nx+k-1}{2k} \geq \frac{j+k}{2k} \geq 1, \tag{12}$$

which implies, $m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq ... \geq m_{0,n,j}(x)$ is true. Hence for all $k \in \{0,1,2,...,n\}$, $j \in \{0,1,2,...,n-2\}$, $n \in \mathbb{N}$ and $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, we can write

$$m_{k,n,j} \leq m_{j,n,j}(x) = 1.$$

□

Lemma 5. *Let* $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$,

i) If $k \in \{j+2,...,n-2\}$ *is such that* $,k - \sqrt{3k+2} \geq j$ *then* $M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$.

ii) If $k \in \{1,2,...,j\}$ *is such that* $k + \sqrt{3k} \leq j$, *then* $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)$.

Proof. i) Since $g(x) = \frac{1}{nx+k}\frac{k-nx}{k+1-nx}$ is nonincreasing, we can write $g(x) \geq g\left(\frac{j+2}{n}\right)$ and hence get,

$$\begin{aligned}\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &= \frac{2(k+1)}{nx+k}\frac{\frac{k}{n}-x}{\frac{k+1}{n}-x} \\ &\geq \frac{2(k+1)}{k+j+2}\frac{k-j-2}{k-j-1}\end{aligned}$$

Then, the condition $k - \sqrt{3k+2} \geq j$ implies $(k-j)^2 \geq 3k - j + 2$. This implies $2(k+1)(k-j-2) \geq (k+j+2)(k-j-1)$. So, we have

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq 1.$$

ii) Since $h(x) = (nx+k-1)\frac{x-\frac{k}{n}}{x-\frac{k-1}{n}}$ is nondecreasing, we can write $g(x) \geq g\left(\frac{j+1}{n}\right)$ and hence get,

$$\begin{aligned}\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} &= \frac{(nx+k-1)}{2k}\frac{x-\frac{k}{n}}{x-\frac{k-1}{n}} \\ &\geq \frac{j+k}{2k}\frac{j-k+1}{j-k+2}.\end{aligned}$$

Then, the condition $k + \sqrt{3k} \leq j$ implies $(j-k)^2 \geq 3k - j$. Since this implies $(j+k)(j-k+1) \geq 2k(j-k+2)$, we have

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq 1.$$

So, the proof is completed. □

4. Degree of Approximation by $V_n^{(M)}(f)$

Our aim is to estimate the degree of convergence of the sequence of max-product operators $V_n^{(M)}(f)$ given by (2) with respect to modulus of continuity function and then show that this estimate can not be improved.

Theorem 6. *Let $V_n^{(M)}(f)$, $n \in \mathbb{N}$ be defined by (2). For all $f \in C_+[0,1]$, the following inequality*

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 8\omega_1\left(f, \frac{1}{\sqrt{n}}\right), \; x \in [0,1]$$

holds.

Proof. One can see from Corollary 2 that, in order to reach the desired inequality, we have to estimate the term

$$V_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} v_{n,k}(x)\left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{n} v_{n,k}(x)}, x \in [0,1].$$

From Lemma 3 we can write, for $x \in \left[0, \frac{2}{n}\right]$,

$$V_n^{(M)}(\varphi_x)(x) = \bigvee_{k=0}^{n} v_{n,k}(x)\left|\frac{k}{n} - x\right| = \max_{k=0,1,\ldots,n}\{M_{k,n,0}(x)\}$$

where $M_{k,n,0}(x)$ is defined by (10).
If $k = 0$ then,

$$M_{0,n,0}(x) = x \leq \frac{2}{n}, \; x \in \left[0, \frac{2}{n}\right].$$

If $k \geq 1$ then,
for $x \in \left[0, \frac{1}{n}\right]$, since $(1)_k = k!$ and $k \leq 2^k$,

$$M_{k,n,0}(x) \leq \frac{(nx)_k}{2^k k!}\frac{k}{n} \leq \frac{k}{2^k n} \leq \frac{1}{n},$$

for $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$ and $k = 1$,

$$M_{1,n,0}(x) = \frac{(nx)_1}{2^1 1!}\left(x - \frac{1}{n}\right) \leq \frac{1}{n},$$

for $x \in \left[\frac{1}{n}, \frac{2}{n}\right]$ and $k = 2, 3, ..., n$

$$M_{k,n,0}(x) \leq \frac{(nx)_k}{2^k k!}\left(\frac{k}{n} - x\right) \leq \frac{(k+1)(k-1)}{2^k n} \leq \frac{k^2}{2^k n}.$$

If we take $g(x) = \frac{x^2}{2^x}$, since $g(x) \leq g\left(\frac{2}{\ln 2}\right) < 2$, $x \in [0, \infty)$, we can write

$$M_{k,n,0}(x) \leq \frac{k^2}{2^k n} \leq \frac{2}{n}.$$

Consequently, for $x \in \left[0, \frac{2}{n}\right]$

$$V_n^{(M)}(\varphi_x)(x) = \max_{k=0,1,\ldots,n} \{M_{k,n,0}(x)\} \leq \frac{2}{n} \ .$$

Considering Lemma 3 once more, we can write

$$V_n^{(M)}(\varphi_x)(x) = \max_{k=0,1,\ldots,n} \{M_{k,n,j}(x)\}\ ,\ j \in \{1,...,n-2\}\,.$$

Now, we will try to obtain an upper estimate for $M_{k,n,j}(x)$ with $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, $j \in \{1,...,n-2\}$ .We consider the following 3 cases:
i) If $k = j+1$; for $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, $M_{j+1,n,j}(x) = \frac{(nx+j)}{2j+2}\left(x - \frac{j+1}{n}\right) \leq \frac{1}{n}$.
ii) If $k \geq j+2$;
a) Firstly, we suppose that $k - \sqrt{3k+2} < j$. From Lemma 4, since $m_{k,n,j}(x) \leq 1$, we write

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)\left(\frac{k}{n} - x\right) \leq \frac{k}{n} - x \leq \frac{k}{n} - \frac{j+1}{n} \\ &\leq \frac{(\sqrt{3k+2}-1)}{n} \leq \frac{\sqrt{3n+2}}{n} \leq \frac{3}{\sqrt{n}}. \end{aligned}$$

b) Now, we suppose that $k - \sqrt{3k+2} \geq j$. Since $g(x) = x - \sqrt{3x+2}$ is nondecreasing on $\left[\frac{1}{12}, \infty\right)$ and since in this case $k \geq j+2$, we take $3 \leq k \leq n$. Since g is nondecreasing, there exists $\overline{k} \in \{3,...,n\}$ of maximum value such that $\overline{k} - \sqrt{3\overline{k}+2} < j$ and $\overline{k} + 1 - \sqrt{3\overline{k}+5} \geq j$.

$$\begin{aligned} M_{\overline{k}+1,n,j}(x) &= m_{\overline{k}+1,n,j}(x)\left(\frac{\overline{k}+1}{n} - x\right) \leq \frac{\overline{k}+1}{n} - x \\ &\leq \frac{\overline{k}+1}{n} - \frac{j+1}{n} < \frac{\sqrt{3\overline{k}+2}}{n} \leq \frac{3}{\sqrt{n}}. \end{aligned} \tag{13}$$

So, from Lemma 5, for $k \in \left\{\overline{k}+1, \overline{k}+2, ..., n\right\}$, we get $M_{\overline{k}+1,n,j}(x) \geq M_{\overline{k}+2,n,j}(x) \geq ... \geq M_{n,n,j}(x)$. Finally, by (13), we can write $M_{k,n,j}(x) \leq \frac{3}{\sqrt{n}}$, for $k \in \left\{\overline{k}+1, \overline{k}+2, ..., n\right\}$.
iii) $k \leq j$;
a) Firstly, we suppose that $k + \sqrt{3k} > j$. From Lemma 4 we get

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)\left(x - \frac{k}{n}\right) \leq \frac{j+2}{n} - \frac{k}{n} \\ &\leq \frac{\sqrt{3k}+2}{n} \leq \frac{\sqrt{3n}+2}{n} \leq \frac{4}{\sqrt{n}}. \end{aligned}$$

b) Now, we suppose that $k + \sqrt{3k} \leq j$. Since $h(x) = x + \sqrt{3x}$ is increasing on $[0, \infty)$, there exists $\tilde{k} \in \{1,...,n\}$ of minimum value such that $\tilde{k} + \sqrt{3\tilde{k}} > j$ and

$\tilde{k}-1+\sqrt{3\tilde{k}-3} \leq j$ and

$$\begin{aligned}
M_{\tilde{k}-1,n,j}(x) &= m_{\tilde{k}-1,n,j}(x)\left(x-\frac{\tilde{k}-1}{n}\right) \leq x-\frac{\tilde{k}-1}{n} \\
&\leq \frac{j+2}{n}-\frac{\tilde{k}-1}{n} < \frac{\tilde{k}+\sqrt{3\tilde{k}}+2}{n}-\frac{\tilde{k}-1}{n} \\
&= \frac{\sqrt{3\tilde{k}}+3}{n} \leq \frac{\sqrt{3n}+2}{n} \leq \frac{4}{\sqrt{n}}. \qquad (14)
\end{aligned}$$

We have $j \geq 1$ in this case. Also, since $j+\sqrt{3j} > j$ and $\tilde{k} \in \{0,1,...,n\}$ of minimum value such that $\tilde{k}+\sqrt{3\tilde{k}} > j$, $\tilde{k}-1 \leq j$. So, from Lemma 5, for $k \in \left\{1,2,...,\tilde{k}-1\right\}$, we get $M_{\tilde{k}-1,n,j}(x) \geq M_{\tilde{k}-2,n,j}(x) \geq ... \geq M_{0,n,j}(x)$. Finally, by (14), we can write $M_{k,n,j}(x) \leq \frac{4}{\sqrt{n}}$, for $k \in \left\{1,2,...,\tilde{k}-1\right\}$.
Considering all the estimates above, we can write

$$M_{k,n,j}(x) \leq \frac{4}{\sqrt{n}} \text{ for all } x \in \left[\frac{j+1}{n},\frac{j+2}{n}\right]$$

which implies

$$V_n^{(M)}(\varphi_x)(x) \leq \frac{4}{\sqrt{n}} \text{ for all } x \in [0,1]$$

Since the first modulus of continuity function satisfies the property $\omega_1(f,m\delta) \leq m\omega_1(f,\delta)$, for $m \in \mathbb{N}$, with all cases and subcases and taking $\delta = \frac{4}{\sqrt{n}}$ in (9), the proof is completed. □

Remark 7. *The estimate regarding the first order modulus of continuity function given in Theorem* 6 *cannot be improved for* $n \geq 4$. *Suppose that*

$$j_n = \left[\frac{n}{2}\right],\ k_n = j_n + \left[\sqrt{n}\right],\ x_n = \frac{j_n}{n},$$

With calculations, we can write

$$\begin{aligned}
M_{k_n,n,j_n}(x_n) &= \frac{(nx_n+j_n)...(nx_n+k_n-1)}{2^{k_n-j_n}(j_n+1)...k_n}\left|\frac{k_n}{n}-x_n\right| \\
&\geq \frac{(nx_n+j_n)^{k_n-j_n}}{2^{k_n-j_n}k_n^{k_n-j_n}}\left|\frac{k_n}{n}-x_n\right| \\
&= \frac{(2j_n)^{k_n-j_n}}{2^{k_n-j_n}k_n^{k_n-j_n}}\left|\frac{k_n}{n}-x_n\right| \\
&= \frac{(j_n)^{k_n-j_n}}{k_n^{k_n-j_n}}\frac{(k_n-j_n)}{n}
\end{aligned}$$

$$= \left(\frac{\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]+\left[\sqrt{n}\right]}\right)^{\left[\sqrt{n}\right]}\frac{\left[\sqrt{n}\right]}{n}.$$

From definition of the greatest integer function, we can write

$$\left(\frac{\frac{n}{2}-1}{\frac{n}{2}+\sqrt{n}}\right)^{\sqrt{n}} \leq \left(\frac{\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]+\left[\sqrt{n}\right]}\right)^{\left[\sqrt{n}\right]} \leq \left(\frac{\frac{n}{2}}{\frac{n}{2}+\sqrt{n}-2}\right)^{\sqrt{n}-1}.$$

One can obtain that $\lim\limits_{n\to\infty}\left(\frac{\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]+\left[\sqrt{n}\right]}\right)^{\left[\sqrt{n}\right]} = e^{-2}$ and there exists $n_0 \in \mathbb{N}$ such that

$$M_{k_n,n,j_n}(x_n) \geq \frac{\left[\sqrt{n}\right]}{e^3 n},\ n \geq n_0.$$

Also $\frac{\left[\sqrt{n}\right]}{\sqrt{n}} \geq \frac{\sqrt{n}-1}{\sqrt{n}} \geq \frac{1}{2}$ for $n \geq 4$. Therefore we can say

$$M_{k_n,n,j_n}(x_n) \geq \frac{1}{2e^3\sqrt{n}},\ n \geq n_0.$$

Now, we will show that better order of approximation can be obtained for some subclasses of functions f.

For $x = 0$, since $v_{n,k}(0) = 0$ for all $k \in \{1,...,n\}$ and $v_{n,0}(0) = 1$, $V_n^{(M)}(f)(0) - f(0) = 0$. So, we assume $x > 0$.

For any $k \in 0,1,...n$ and $j \in \{0,1,...,n-2\}$ consider the functions

$$f_{k,n,j}(x) = \frac{v_{n,k}(x)}{v_{n,j}(x)} f\left(\frac{k}{n}\right) = m_{k,n,j}(x) f\left(\frac{k}{n}\right).$$

For any $j \in \{0,1,...,n-2\}$ and $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$ we can write

$$V_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} \frac{v_{n,k}(x)}{v_{n,j}(x)} f\left(\frac{k}{n}\right) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Lemma 8. *For $f : [0,1] \longrightarrow [0,\infty)$ and $j \in \{0,1,...,n-2\}$, if*

$$V_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x), f_{j+2,n,j}(x)\},\ \textit{for all } x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right] \tag{15}$$

then

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f, \frac{1}{n}\right),\ \textit{for all } x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right].$$

Proof. We proceed in the same manner as Bede et.al. [6]. This time we have three cases to examine:

Case (i) Let $V_n^{(M)}(f)(x) = f_{j,n,j}(x) = f\left(\frac{j}{n}\right)$ for fixed $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$. Since $\frac{1}{n} \le x - \frac{j}{n} \le \frac{2}{n}$

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f, \frac{2}{n}\right).$$

Case (ii) Let $V_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$ for fixed $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$.
Subcase (a) If $V_n^{(M)}(f)(x) \le f(x)$ then

$$\begin{aligned}\left|V_n^{(M)}(f)(x) - f(x)\right| &= f(x) - f_{j+1,n,j}(x)\\ &\le f(x) - f_{j,n,j}(x) \le \omega_1\left(f, \frac{2}{n}\right).\end{aligned}$$

Subcase (b) If $V_n^{(M)}(f)(x) > f(x)$ then

$$\begin{aligned}\left|V_n^{(M)}(f)(x) - f(x)\right| &= f_{j+1,n,j}(x) - f(x)\\ &= m_{j+1,n,j}(x)\, f\left(\frac{j+1}{n}\right) - f(x)\\ &\le f\left(\frac{j+1}{n}\right) - f(x).\end{aligned}$$

Since $0 \le x - \frac{j+1}{n} \le \frac{1}{n}$

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f, \frac{1}{n}\right).$$

Case (iii) Let $V_n^{(M)}(f)(x) = f_{j+2,n,j}(x)$ for fixed $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$.
Subcase (a) If $V_n^{(M)}(f)(x) \le f(x)$ then

$$\begin{aligned}\left|V_n^{(M)}(f)(x) - f(x)\right| &= f(x) - f_{j+2,n,j}(x)\\ &\le f(x) - f_{j,n,j}(x) \le \omega_1\left(f, \frac{2}{n}\right).\end{aligned}$$

Subcase (b) If $V_n^{(M)}(f)(x) > f(x)$ then

$$\begin{aligned}\left|V_n^{(M)}(f)(x) - f(x)\right| &= f_{j+2,n,j}(x) - f(x)\\ &= m_{j+2,n,j}(x)\, f\left(\frac{j+2}{n}\right) - f(x)\\ &\le f\left(\frac{j+2}{n}\right) - f(x).\end{aligned}$$

Since $0 \le \frac{j+2}{n} - x \le \frac{1}{n}$

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f, \frac{1}{n}\right).$$

So the proof is completed. □

Theorem 9. *For $f:[0,1]\longrightarrow[0,\infty)$ is a nondecreasing function and the function $g:(0,1]\to[0,\infty)$ $g(x)=\frac{f(x)}{x}$ is nonincreasing, we have*

$$\left|V_n^{(M)}(f)(x)-f(x)\right|\leq 2\omega_1\left(f,\frac{1}{n}\right)\text{ for all } x\in[0,1].$$

Proof. From the monotonocity of f and $k\leq j$,

$$\begin{aligned} f_{k-1,n,j}(x) &= \frac{2^{j-k+1}j!}{(k-1)!(nx+k-1)\dots(nx+j-1)}f\left(\frac{k-1}{n}\right)\\ &\leq \frac{2^{j-k}j!}{k!(nx+k)\dots(nx+j-1)}\frac{2k}{(nx+k-1)}f\left(\frac{k}{n}\right)\\ &\leq \frac{2^{j-k}j!}{k!(nx+k)\dots(nx+j-1)}\frac{2k}{j+k}f\left(\frac{k}{n}\right)\\ &\leq f_{k,n,j}(x)\end{aligned}$$

So, we can write

$$\begin{aligned}\bigvee_{k=1}^{j}f_{k,n,j}(x) &= f_{j,n,j}(x),\\ V_n^{(M)}(f)(x) &= \bigvee_{k=j}^{n}f_{k,n,j}(x).\end{aligned}\tag{16}$$

Let $k\in\{0,1,...,n\}$ with $k\geq j$. Since g is nonincreasing and $x\in\left[\frac{j+1}{n},\frac{j+2}{n}\right]$, we have

$$\begin{aligned} f_{k+1,n,j}(x) &= \frac{j!(nx+j)\dots(nx+k)}{2^{k+1-j}(k+1)!}f\left(\frac{k+1}{n}\right)\\ &\leq \frac{j!(nx+j)\dots(nx+k)}{2^{k+1-j}(k+1)!}\frac{k+1}{k}f\left(\frac{k}{n}\right)\\ &= \frac{j!(nx+j)\dots(nx+k-1)}{2^{k-j}k!}\frac{(nx+k)}{2k}f\left(\frac{k}{n}\right)\\ &\leq f_{k,n,j}(x)\frac{k+j+2}{2k}.\end{aligned}$$

So, we get

$$f_{k+1,n,j}(x)\leq f_{k,n,j}(x),\text{ for } k\geq j+2,$$

from which, we have,

$$\bigvee_{k=j+2}^{n}f_{k,n,j}(x)=f_{j+2,n,j}(x),\ j\in\{0,1,...,n-2\}.\tag{17}$$

From (16) and (17), we obtain (15) for all $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, i.e.,

$$V_n^{(M)}(f)(x) = \max\left\{f_{j,n,j}(x), f_{j+1,n,j}(x), f_{j+2,n,j}(x)\right\}, \ j \in \{0, 1, ..., n-2\}.$$

By Lemma 8, the proof is completed. □

Lemma 10. *[7] If $f : [0,1] \longrightarrow [0,\infty)$ is a concave function then $g : (0,1] \to [0,\infty)$ $g(x) = \frac{f(x)}{x}$ is nonincreasing.*

Proof. For the proof see [7] □

Corollary 11. *For $f : [0,1] \longrightarrow [0,\infty)$ is a nondecreasing concave function, we have*

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f, \frac{1}{n}\right), \text{ for all } x \in [0,1].$$

Proof. The proof is completed by Theorem 9 and Lemma 10. □

The last theorem is about the piecewise convexity of the truncated Lupaş operators of max-product type on the interval $[0,1]$.

Theorem 12. *For any $f : [0,1] \longrightarrow \mathbb{R}_+$, $V_n^{(M)}(f)$ is convex on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, $j \in \{0, 1, ..., n-2\}$.*

Proof. For any $j \in \{0, 1, ..., n-2\}$ and $x \in \left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, since we can write $V_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x)$, we will show for any fixed j, $f_{k,n,j}$ is convex on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$. This imply that $V_n^{(M)}(f)$ is convex, as being the maximum of convex functions on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$.
Since $f \geq 0$ and $f_{k,n,j}(x) = \frac{(nx)_k j!}{2^{k-j}(nx)_j k!} f\left(\frac{k}{n}\right)$, we will prove that $g_{k,n,j}(x) = \frac{(nx)_k}{(nx)_j}$ are convex on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$.
For $k = j$, $g_{j,n,j}$ is constant so it is convex.
For $k = j+1$, $g_{j+1,n,j}(x) = nx + j$ is convex.
For $k = j-1$, $g_{j-1,n,j}(x) = \frac{1}{nx+j-1}$ and since $g''_{j-1,n,j}(x) = \frac{2n^2}{(nx+j-1)^3} > 0$ on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$, it is convex.
For $k \geq j+2$, $g_{k,n,j}(x) = (nx+j)...(nx+k-1)$ and $\ln(g_{k,n,j}(x)) = \ln(nx+j) + ... + \ln(nx+k-1)$. Since $g'_{k,n,j}(x) = n g_{k,n,j}(x)\left[\frac{1}{nx+j} + ... + \frac{1}{nx+k-1}\right]$ and $g''_{k,n,j}(x) = n g'_{k,n,j}(x)\left[\frac{1}{nx+j} + ... + \frac{1}{nx+k-1}\right] - n^2 g_{k,n,j}(x)\left[\frac{1}{(nx+j)^2} + ... + \frac{1}{(nx+k-1)^2}\right]$, we obtain
$g''_{k,n,j}(x) = n^2 g_{k,n,j}(x)\left\{\left(\frac{1}{nx+j} + ... + \frac{1}{nx+k-1}\right)^2 - \left(\frac{1}{(nx+j)^2} + ... + \frac{1}{(nx+k-1)^2}\right)\right\} > 0$.
For $k \leq j-2$, $g_{k,n,j}(x) = \frac{1}{(nx+k)...(nx+j-1)}$ and since
$g'_{k,n,j}(x) = -n g_{k,n,j}(x)\left[\frac{1}{nx+k} + ... + \frac{1}{nx+j-1}\right]$, we get

$g''_{k,n,j}(x) = n^2 g_{k,n,j}(x) \left\{ \left(\frac{1}{nx+k} + ... + \frac{1}{nx+j-1} \right)^2 + \left(\frac{1}{(nx+k)^2} + ... + \frac{1}{(nx+j-1)^2} \right) \right\} > 0.$

Hence, we see that all the functions $g_{k,n,j}$ are convex on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$. As being maximum of all these functions, $V_n^{(M)}(f)$ is convex on $\left[\frac{j+1}{n}, \frac{j+2}{n}\right]$. □

5. Conclusion

The nonlinear max product type operators have been studied by various authors for the last two decades. For example, in [16] Duman obtained convergence results for a sequence of max-product operators in the statistical sense. Karakuş and Demirci [22] examined the σ-statistical convergence of the max product type operators. For the future studies, one can examine whether the truncated Lupaş operators of max product kind can be generalized in the light of these studies or not. Also the statistical convergence of the constructed operators may be investigated.

Another study related to this topic is due to Holhoş. He examined the approximation properties of Meyer-König and Zeller and Favard-Szász-Mirakyan operators of max-product type in weighted space of functions in the papers [20] and [21], respectively. Taking these studies into account, Lupaş operators of max-product type may be constructed on an unbounded interval $[0, \infty)$ and weighted approximation results of the operators can be examined.

Very recently, Coroianu and Gal [10] have studied the Kantorovich type max-product operators. In view of this paper one can consider the Lupaş-Kantorovich operators of max-product type and analyze the approximating properties.

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A NEW APPROACH TO THE BI-UNIVALENT ANALYTIC FUNCTIONS RELATED WITH q-ANALOGUE OF NOOR INTEGRAL OPERATOR

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Abstract. Recently, q-analogue of Noor integral operator and other special operators became importance in the field of Geometric Function Theory. In this study, by connecting this operators and the principle of subordination we introduced an interesting class of bi-univalent functions and obtained coefficient estimates for this new class.

1. Introduction

Let $\mathcal{A}$ indicates the family of analytic functions having form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disk $\mathfrak{D} = \{z : |z| < 1, z \in \mathbb{C}\}$ and let $S = \{f \in \mathcal{A} : f \text{ is univalent in } \mathfrak{D}\}$.

According the Koebe one-quarter theorem [6], the image of $\mathfrak{D}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. That is, every such univalent function has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{D})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

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If f and f^{-1} are univalent, then we say that f is bi-univalent function in $\mathfrak{D}$. The class of bi-univalent functions defined in $\mathfrak{D}$ is symbolized by Σ.

One can see important examples in the class in [20]. Although the functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}.\right)$ are in Σ, well known Koebe function is not in Σ. For example, $z-\frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are in $\mathcal{S}$ but not in Σ [20].

Given f , $g \in \mathcal{A}, f$ is said to be subordinate to g, symbolized

$$f(z) \prec g(z), \tag{3}$$

such that there is an analytic function w defined on $\mathfrak{D}$ with

$$w(0)=0 \quad \text{and} \quad |w(z)|<1$$

fulfilling the following condition:

$$f(z)=g\left(w(z)\right).$$

The aforecited subclasses of Σ were constructed and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in several recent studies (see [7], [8], [10], [20] , [21], [22]). In very nearly, they have been followed by many works (see also [5], [9], [12], [14], [19]).

Now, we give some basic definitions.

Definition 1. *[13] For $q\in(0,1)$, the q-derivative of function $f\in\mathcal{A}$ is defined by*

$$\partial_q f(z)=\frac{f(qz)-f(z)}{(q-1)z}, z\neq 0 \tag{4}$$

and

$$\partial_q f(0)=f'(0).$$

Thus we have

$$\partial_q f(z)=1+\sum_{k=2}^{\infty}[k,q]\,a_k z^{k-1} \tag{5}$$

where $[k,q]$ is given by

$$[k,q]=\frac{1-q^k}{1-q}, \qquad [0,q]=0 \tag{6}$$

and the q-fractional is defined by

$$[k,q]!=\begin{cases}\prod_{n=1}^{k}[n,q]\,, & k\in\mathbb{N}\\ 1, & k=0\end{cases}. \tag{7}$$

Also, the $q-$generalized Pochhammer symbol for $\mathfrak{p}\geq 0$ is given by

$$[\mathfrak{p},q]_k=\begin{cases}\prod_{n=1}^{k}[\mathfrak{p}+n-1,q]\,, & k\in\mathbb{N}\\ 1, & k=0\end{cases}. \tag{8}$$

As $q \to 1$, then we get $[k,q] \to k$. Thus, by choosing the function $g(z) = z^k$, while $q \to 1$, then we obtain

$$\partial_q g(z) = \partial_q z^k = [k,q]\, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Recently, function $F^{-1}_{q,\mu+1}(z)$ is defined by Arif et al. [4] by

$$F^{-1}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = z\partial_q f(z), \quad (\mu > -1) \tag{9}$$

where

$$F_{q,\mu+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1,q]_{k-1}}{[k-1,q]!} z^k, \quad z \in \mathcal{D}. \tag{10}$$

To the series defined in (10) is convergent absolutely in $\mathfrak{D}$, by using the definition of q-derivative through convolution, let us explain the integral operator $\zeta_q^\mu : \mathfrak{D} \to \mathfrak{D}$ by

$$\zeta_q^\mu f(z) = F^{-1}_{q,\mu+1}(z) * f(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \quad (z \in \mathfrak{D}) \tag{11}$$

where

$$\phi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{12}$$

From (11), one can readily have the identity

$$[\mu+1,q]\zeta_q^\mu f(z) = [\mu,q]\zeta_q^{\mu+1} f(z) + q^\mu z \partial_q(\zeta_q^{\mu+1} f(z)). \tag{13}$$

We can state that

$$\zeta_q^0 f(z) = z\partial_q f(z),\ \zeta_q' f(z) = f(z) \tag{14}$$

also

$$\lim_{q\to 1^-} \zeta_q^\mu f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k. \tag{15}$$

This means that, by taking $q \to 1^-$, the operator defined in equation (11) reduces to the famous Noor integral operator given in ([15]). Moreover, for more detailed knowledge on the coefficient estimates of analytic bi-univalent functions given by q-analogue of differential and integral operators, see the work of [1], [2], [3], [4], [16], [17].

In this study, utilizing by the aforementioned works we introduce a general new subclass $\Im_\Sigma^{\mu,q}(\xi,\tau,\theta;k)$ of the function class Σ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in our new class $\Im_\Sigma^{\mu,q}(\xi,\tau,\theta;k)$. Also through this paper, f, g are given by (1) and (2) and ζ_q^μ is $q-$analogue of Noor integral operator.

2. The class $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$

Definition 2. *Let $k:\mathfrak{D}\to\mathbb{C}$ be a convex univalent function such that*

$$k(0)=1, k(\overline{z})=\overline{k(z)}, \quad (z\in\mathfrak{D};\mathcal{R}(k(z))>0). \tag{16}$$

For $f\in\Sigma$, the function f is said to be in the class of $\in \Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$ if the following conditions are satisfied:

$$e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}f(z)}{z}+\xi\partial_q(\zeta_q^{\mu}f(z))-1\right]\right)\prec k(z)\cos\theta+i\sin\theta,(z\in\mathfrak{D}),$$
$$e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}g(w)}{w}+\xi\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)\prec k(w)\cos\theta+i\sin\theta,(w\in\mathfrak{D}) \tag{17}$$

where $\xi\geq 1$, $\tau\neq 0$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Remark 3. *Choosing*

$$k(z)=\frac{1+\mathcal{A}z}{1+\mathcal{B}z},(-1\leq\mathcal{B}<\mathcal{A}\leq 1) \tag{18}$$

in the class $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$, we have $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;\mathcal{A},\mathcal{B})$ and defined as

$$e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}f(z)}{z}+\xi\partial_q(\zeta_q^{\mu}f(z))-1\right]\right)\prec \frac{1+\mathcal{A}z}{1+\mathcal{B}z}\cos\theta+i\sin\theta,(z\in\mathfrak{D}),$$
$$e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}g(w)}{w}+\xi\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)\prec \frac{1+\mathcal{A}w}{1+\mathcal{B}w}\cos\theta+i\sin\theta,(w\in\mathfrak{D}) \tag{19}$$

where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\xi\geq 1$.

Remark 4. *Choosing*

$$k(z)=\frac{1+(1-2\gamma)z}{1-z},(0\leq\gamma<1) \tag{20}$$

in the class$\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$, we have $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta,\gamma)$ and defined as

$$R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}f(z)}{z}+\xi\partial_q(\zeta_q^{\mu}f(z))-1\right]\right)\right\}>\gamma\cos\theta,(z\in\mathfrak{D}),$$
$$R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}g(w)}{w}+\xi\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)\right\}>\gamma\cos\theta,(w\in\mathfrak{D}) \tag{21}$$

where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\xi\geq 1$.

In the case of $k(z)=\frac{1+(1-2\gamma)z}{1-z}$, $(0\leq\gamma<1)$, by choosing different values instead of parameters, we obtain different subclasses:

1. Upon setting $q \to 1^-$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{\mu,1}(\xi,\tau,\theta,\gamma) = \Im_{\Sigma}^{\mu}(\xi,\tau,\theta,\gamma)$$
if the following inequalities hold:
$$R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta^{\mu}f(z)}{z}+\xi(\zeta^{\mu}f(z))'-1\right]\right)\right\} > \gamma\cos\theta, (z\in\mathfrak{D}),$$
$$R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta^{\mu}g(w)}{w}+\xi(\zeta^{\mu}g(w))'-1\right]\right)\right\} > \gamma\cos\theta, (w\in\mathfrak{D}). \quad (22)$$
2. Upon setting $q \to 1^-$and for $\tau = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{\mu,1}(\xi,1,\theta,\gamma)$$
if the following inequalities hold:
$$R\left\{e^{i\theta}\left[(1-\xi)\frac{\zeta^{\mu}f(z)}{z}+\xi(\zeta^{\mu}f(z))'\right]\right\} > \gamma\cos\theta, (z\in\mathfrak{D}),$$
$$R\left\{e^{i\theta}\left[(1-\xi)\frac{\zeta^{\mu}g(w)}{w}+\xi(\zeta^{\mu}g(w))'\right]\right\} > \gamma\cos\theta, (w\in\mathfrak{D}). \quad (23)$$
3. Upon setting $q \to 1^-$, for $\tau = 1$ and $\xi = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{\mu,1}(1,1,\theta,\gamma) = \Im_{\Sigma}^{\mu}(\theta,\gamma)$$
if the following inequalities hold:
$$R\left\{e^{i\theta}(\zeta^{\mu}f(z))'\right\} > \gamma\cos\theta, (z\in\mathfrak{D}),$$
$$R\left\{e^{i\theta}(\zeta^{\mu}g(w))'\right\} > \gamma\cos\theta, (w\in\mathfrak{D}). \quad (24)$$
4. Upon setting $q \to 1^-$, for $\tau = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{1,1}(\xi,1,\theta,\gamma) = \Im_{\Sigma}(\xi,\theta,\gamma)$$
if the following inequalities hold:
$$R\left\{e^{i\theta}\left[(1-\xi)\frac{f(z)}{z}+\xi(f(z))'\right]\right\} > \gamma\cos\theta, (z\in\mathfrak{D}),$$
$$R\left\{e^{i\theta}\left[(1-\xi)\frac{g(w)}{w}+\xi(g(w))'\right]\right\} > \gamma\cos\theta, (w\in\mathfrak{D}). \quad (25)$$
5. Upon setting $q \to 1^-$,for $\tau = 1, \xi = 1$ and $\mu = 0$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{0}(1,1;\gamma) = \Im_{\Sigma}(\gamma)$$
if the following inequalities hold:
$$R\left\{e^{i\theta}(z\partial f(z))'\right\} > \gamma\cos\theta, (z\in\mathfrak{D}),$$
$$R\left\{e^{i\theta}(w\partial g(w))'\right\} > \gamma\cos\theta, (w\in\mathfrak{D}). \quad (26)$$

6. Upon setting $q \to 1^{-}$, for $\tau = 1, \xi = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{1,1}(1,1,\theta,\gamma) = \Im_{\Sigma}(\theta,\gamma)$$
if the following inequalities hold:
$$\begin{aligned} R\left\{e^{i\theta}(f(z))'\right\} &> \gamma\cos\theta, (z \in \mathfrak{D}), \\ R\left\{e^{i\theta}(g(w))'\right\} &> \gamma\cos\theta, (w \in \mathfrak{D}). \end{aligned} \tag{27}$$
We state that
$$\begin{aligned} \Im_{\Sigma}(\xi 0,\gamma) &= \mathcal{B}_{\Sigma}(\alpha,\lambda) \qquad \text{(see [10])} \\ \Im_{\Sigma}(0,\gamma) &= \mathcal{H}_{\Sigma}(\alpha,\lambda) \qquad \text{(see [20])}. \end{aligned}$$

We need the following lemma to derive our main result.

Lemma 5. *[18] Let the function $k(z)$ defined with*
$$k(z) = \sum_{n=1}^{\infty} \mathcal{B}_n z^n$$
be convex in $\mathfrak{D}$. Assume also that the function $\Psi(z)$ given by
$$\Psi(z) = \sum_{n=1}^{\infty} c_n z^n$$
is holomorphic in $\mathfrak{D}$. If $\Psi(z) \prec k(z), (z \in \mathfrak{D})$, then
$$|c_n| \leq |\mathcal{B}_1|, (n \in \mathbb{N}). \tag{28}$$
Now, we give our general results involving the new class $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$.

Theorem 6. *Let $f \in \Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k), (\xi \geq 1, \theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and $\tau \neq 0$, with*
$$k(z) = 1 + B_1 z + B_2 z^2 + \cdots. \tag{29}$$
Then
$$|a_2| \leq \min\left\{\frac{|\tau B_1|\cos\theta}{(1+\xi q)\,\phi_1}, \sqrt[2]{\frac{|\tau B_1|\cos\theta}{(1+\xi q+\xi q^2)\phi_2}}\right\} \tag{30}$$
and
$$|a_3| \leq \frac{|\tau B_1|\cos\theta}{(1+\xi q+\xi q^2)\phi_2}. \tag{31}$$

Proof. According the inequality (17), we can write that
$$\begin{aligned} e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu} f(z)}{z} + \xi\partial_q(\zeta_q^{\mu} f(z)) - 1\right]\right) &= r(z)\cos\theta + i\sin\theta, (z \in \mathfrak{D}) \\ e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu} g(w)}{w} + \xi\partial_q(\zeta_q^{\mu} g(w)) - 1\right]\right) &= s(w)\cos\theta + i\sin\theta\ , (w \in \mathfrak{D}) \end{aligned} \tag{32}$$

where $r(z) \prec k(z)$ and $s(w) \prec k(w)$ have the following series expansions

$$r(z) = 1 + r_1 z + r_2 z^2 + ... \tag{33}$$

$$s(w) = 1 + s_1 w + s_2 w^2 + ... \tag{34}$$

respectively: By equating the coefficients of the two equations in (32), we have

$$e^{i\theta} \frac{1}{\tau} (1 + \xi q) \phi_1 a_2 = r_1 \cos\theta \quad ... \tag{35}$$

$$e^{i\theta} \frac{1}{\tau} (1 + \xi q + \xi q^2) \phi_2 a_3 = r_2 \cos\theta \ ... \tag{36}$$

and

$$- e^{i\theta} \frac{1}{\tau} (1 + \xi q) \phi_1 a_2 = s_1 \cos\theta \ ... \tag{37}$$

$$e^{i\theta} \frac{1}{\tau} \left[(1 + \xi q + \xi q^2) \phi_2 \left(2a_2^2 - a_3 \right) \right] = s_2 \cos\theta \ .... \tag{38}$$

From (35) and (37), we have

$$r_1 = -s_1 \quad ... \tag{39}$$

and

$$a_2^2 = \frac{(r_1^2 + s_1^2) e^{-2i\theta} \cos^2\theta}{2 \left[\frac{1}{\tau} (1 + \xi q) \phi_1 \right]^2} \quad ... \tag{40}$$

Also from (36) and (38), we get

$$a_2^2 = \frac{(r_2 + s_2) e^{-i\theta} \cos\theta}{2(1 + \xi q + \xi q^2) \phi_2} \ \tau \ .... \tag{41}$$

Due to the fact $r, s \in h(\mathfrak{D})$, by virtue of Lemma 5, we obtain

$$\begin{aligned} |r_k| &= \left| \frac{r^{(k)}(0)}{k!} \right| \leq |B_1|, \\ |s_k| &= \left| \frac{s^{(k)}(0)}{k!} \right| \leq |B_1|, (k \in \mathbb{N}). \end{aligned} \tag{42}$$

If we apply (42) and Lemma 5 for coefficients r_1, r_2, s_1 and s_2, from (40) and (41), we have

$$|a_2|^2 \leq \frac{|\tau B_1|^2 \cos^2\theta}{|(1 + \xi q) \phi_1|^2} \tag{43}$$

and

$$|a_2|^2 \leq \frac{|\tau B_1| \cos\theta}{|(1 + \xi q + \xi q^2) \phi_2|}. \tag{44}$$

Thus, we obtain desired result for $|a_2|$.

Next, in order to find the bound on the coefficient $|a_3|$, if we subtract (38), from (36), then we get

$$a_3 - a_2^2 = \frac{e^{-i\theta}(r_2 - s_2)\tau\cos\theta}{2(1+\xi q+\xi q^2)\phi_2}. \tag{45}$$

By substituting $\mathcal{A}_2^2$ from (41) into (45), it is obtained that

$$a_3 = \frac{e^{-2i\theta}(r_1^2 + s_1^2)\tau^2\cos^2\theta}{2(1+\xi q)^2\phi_1^2} + \frac{e^{-i\theta}(r_2 - s_2)\tau\cos\theta}{2(1+\xi q+\xi q^2)\phi_2}. \tag{46}$$

Taking absolute value of the equation (46), we get

$$|a_3| \leq \frac{\cos^2\theta \ \left|\tau B_1\right|^2}{(1+\xi q)^2\phi_1^2} + \frac{\cos\theta\left|\tau B_1\right|}{(1+\xi q+\xi q^2)\phi_2} \tag{47}$$

Thus,

$$|a_3| \leq \frac{\cos\theta\left|\tau B_1\right|}{(1+\xi q+\xi q^2)\phi_2}.$$

□

3. Corollaries and Consequences

According the Remark 1 and Remark 2, choosing

$$k(z) = \frac{1+\mathcal{A}z}{1+\mathcal{B}z}, (-1 \leq \mathcal{B} < \mathcal{A} \leq 1)$$

and

$$k(z) = \frac{1+(1-2\gamma)z}{1-z} \ (0 \leq \gamma < 1)$$

in Theorem 6, Corollaries 7, 8, and 9 can be readily deduced, respectively.

Corollary 7. *If* $f \in \Im_{\Sigma}^{\mu,q}(\xi,\theta,\tau;\mathcal{A},\mathcal{B})$, $(\theta \in (-\frac{\pi}{2},\frac{\pi}{2}), \xi \geq 1, \tau \neq 0, -1 \leq \mathcal{B} < \mathcal{A} \leq 1)$, *then we have*

$$|a_2| \leq \min\left\{\frac{|\tau|\left(\mathcal{A}-\mathcal{B}\right)\cos\theta}{(1+\xi q)\,\phi_1}, \sqrt{\frac{|\tau\mathcal{B}|\left(\mathcal{A}-\mathcal{B}\right)\cos\theta}{(1+\xi q+\xi q^2)\phi_2}}\right\}$$

and

$$|a_3| \leq \frac{(\mathcal{A}-\mathcal{B})\,|\tau|\cos\theta}{(1+\xi q+\xi q^2)\phi_2}.$$

Corollary 8. *If* $f \in \Im_{\Sigma}^{\mu}(\xi,\theta,\gamma)$, $(\theta \in (-\frac{\pi}{2},\frac{\pi}{2}), \xi \geq 1, 0 \leq \gamma < 1)$, *then we have*

$$|a_2| \leq \min\left\{\frac{2\tau(1-\gamma)\cos\theta}{(1+\xi)\,\phi_1}, \sqrt{\frac{2\,|\tau|\,(1-\gamma)\cos\theta}{(1+2\xi)\phi_2}}\right\}$$

and

$$|a_3| \leq \frac{2\tau(1-\gamma)\cos\theta}{((1+2\xi)\phi_2}.$$

When $\theta = 0$ in Corollary 8, we obtain a new result:

Corollary 9. *If* $f \in \Im_{\Sigma}^{\mu}(\xi,\gamma), \quad (\xi \geq 1, 0 \leq \gamma < 1)$*, then we have*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{(1+2\xi)\phi_2}}$$

and

$$|a_3| \leq \frac{2\tau(1-\gamma)}{((1+2\xi)\phi_2}.$$

Corollary 10. *If* $f \in \Im_{\Sigma}(\gamma)$*, then we have*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3\phi_2}}$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{3\phi_2}.$$

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DISCRETIZATION AND CHAOS CONTROL IN A FRACTIONAL ORDER PREDATOR-PREY HARVESTING MODEL

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Abstract. The study of interaction between predator and prey species is one of the important subjects in mathematical biology. Optimal strategy control plays a vital role in preserving animals from extinction. Harvesting of species is a vital issue for the conservation biologists. In this work, we investigate the bifurcation and chaos control of the two species interaction model of fractional order in discrete time with harvesting of both prey and predator species. Existence results and the stability conditions of the system are analyzed using the fixed points and jacobian matrix. The chaotic behavior of the system is discussed with bifurcation diagrams. Linear control and hybrid control methods are used in controlling the chaos of the system. Numerical experiments with different phase portraits are simulated for the better understanding of the qualitative behavior of the considered model.

1. Introduction

Modelling of real life phenomena by fractional order equations is more realistic and follows the laws of nature very well. Fractional calculus is used in modelling of physical and chemical phenomena like diffusion waves, nonlinear oscillations involved in earthquakes, hydrologic models, blood vessel models and various other interdisciplinary fields. In construction of biological models, fractional calculus relates the memory effects of biological populations very well rather than ordinary

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integer order calculus. Recently, models of species interaction and biological populations are developed using fractional calculus with discrete time [2, 21, 22, 24, 25].

Mathematical models of the species by Lotka in 1925 and Volterra in 1926 were the first models on the interactions involving multi species [11]. Later several models on interacting species were developed by Robert May in 1972, Holling and Tanner in 1975 and many other researchers proving the necessity in studying mathematical ecology [15, 26]. Biological populations with non overlapping generations are modeled with difference equations and some of the discrete time models are studied by [8, 9, 12, 13, 14].

The population dynamics gives an accurate and deep understanding of the factors threatening the existence of the species in ecosystem. Apart from the natural forces like drought and natural calamities there are some artificial factors that has human involvement such as hunting, human habitat and harvesting also paves way for the extinction of the species. Continuous harvesting results in unstability of ecosystem with loss in biodiversity. Thus, it is necessary to bring forth some conservation policy for optimal harvesting of species. Myerscough et al [20] reported the effects of predator harvesting on ecosystem, harvesting and its effects in aquasystem was studied in [16], [19] studied the prey-predator system with constant harvesting policy and the fractional order model of quadratic harvesting of scavenger was studied in [23].

The paper is structured with discretization of fractional order system in section 2 followed with analysis of stability condition in Section 3. The bifurcation analysis and chaos control in section 4 and 5 respectively. Section 6 provides some numerical examples with simulations.

2. Fractional Order System with Discretization

In the recent decades, Fractional order has emerged as one of the significant interdisciplinary subjects in physical & biological sciences and Engineering [4], [3], [1]. In this work, the biological system with harvesting of predator and prey species are considered. The non-dimensional form of system is

$$\begin{aligned}\frac{dx}{dt} &= sx(1-x) - \beta xy - cx\\ \frac{dy}{dt} &= -wy + \eta xy - fy\end{aligned} \tag{1}$$

In system (1), the prey and predator populations are represented by $x(t)$ and $y(t)$. All the system parameters s, β, c, w, η, f take positive real values that stand for growth rate of prey, interaction rate, harvesting effort of prey, mortality rate of predator, conversion rate of prey and harvesting effort of predator respectively.

Generalization of (1) to arbitrary order yield

$$\begin{aligned}D_t^v(x) &= sx(1-x) - \beta xy - cx\\ D_t^v(y) &= -wy + \eta xy - fy\end{aligned} \tag{2}$$

with $x(0) = x_0$ and $y(0) = y_0$, where $\upsilon \in (0,1)$ is non integer order and fractional order caputo derivative is ${}_aD_t^{\upsilon} f(t) = \frac{1}{\Gamma(1-\upsilon)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\upsilon-n+1}} d\tau$, for $n-1 < \upsilon < n$ [5].

2.1. **Discretization Process.** The discretization of the system (2) with initial point $x(0) = x_0$ and $y(0) = y_0$ is carried out using piecewise constant arguments method [2, 7]. The fractional order predator prey system with harvesting at discrete time is

$$\begin{aligned} x(t+1) &= x(t) + \frac{\rho^{\nu}}{\Gamma(1+\nu)} \left[sx(t)(1-x(t)) - \beta x(t)y(t) - cx(t) \right] \\ y(t+1) &= y(t) + \frac{\rho^{\nu}}{\Gamma(1+\nu)} \left[-wy(t) + \eta x(t)y(t) - fy(t) \right] \end{aligned} \tag{3}$$

where $\upsilon \in (0,1]$ and $\rho > 0$ is step size.

2.2. **Existence and Uniqueness of the Solution.** Let the region be defined by $\Theta \times (0, \mathfrak{T}]$ where

$$\Theta = \left\{ (x,y) \in \Re^2 : \max(|x|, |y| \le L) \right\}$$

The existence results are established with method as in [17]. Consider a mapping $\mathcal{H}(C) = (\mathcal{H}_1(C), \mathcal{H}_2(C))$ such that

$$\begin{aligned} \mathcal{H}_1(C) &= sx(1-x) - \beta xy - cx \\ \mathcal{H}_2(C) &= -wy + \eta xy - fy. \end{aligned} \tag{4}$$

Let $C, \bar{C} \in \Theta$. We have from (4) that

$$\begin{aligned} \left\| \mathcal{H}(C) - \mathcal{H}(\overline{C}) \right\| &= \left| \mathcal{H}_1(C) - \mathcal{H}_1(\overline{C}) \right| + \left| \mathcal{H}_2(C) - \mathcal{H}_2(\overline{C}) \right| \\ &= |sx(1-x) - \beta xy - cx - s\bar{x}(1-\bar{x}) + \beta \bar{x}\,\bar{y} + c\bar{x}| \\ &+ |-wy + \eta xy - fy + w\bar{y} - \eta \bar{x}\,\bar{y} + f\,\bar{y}| \\ &= |s(x-\bar{x}) - s(x-\bar{x})(x+\bar{x}) - \beta(xy - \bar{x}\,\bar{y}) - c(x-\bar{x})| \\ &+ |\eta(xy - \bar{x}\,\bar{y}) - w(y-\bar{y}) - f\,(y-\bar{y})| \\ &\le s|(x-\bar{x})| + 2sL|x-\bar{x}| + \beta L|(y-\bar{y})| + \beta L|(x-\bar{x}\,)| \\ &+ c|(x-\bar{x})| + \eta\,|(x-\bar{x}\,)| + \eta(y-\bar{y})| + w|(y-\bar{y})| + f|\,(y-\bar{y})| \\ &\le [(1+2L)s + (\beta+\eta)L + c]\,|x-\bar{x}| + [(\beta+\eta)L + w + f]\,|y-\bar{y}| \\ \left\| \mathcal{H}(C) - \mathcal{H}(\overline{C}) \right\| &\le \Omega \left\| C - \overline{C} \right\| \end{aligned}$$

and

$$\Omega = \max\left\{ (1+2L)s + (\beta+\eta)L + c, (\beta+\eta)L + w + f \right\}.$$

For $\Omega < 1$, we obtain $C = \mathcal{H}(C)$ and hence $\mathcal{H}(C)$ is a contraction mapping.

Theorem 1. *The sufficient condition for existence of unique solution of the fractional system* (2) *in* $\Theta \times (0, \mathfrak{T}]$ *is*

$$\Omega = \max\left\{ (1+2L)s + (\beta+\eta)L + c, (\beta+\eta)L + w + f \right\} < 1.$$

3. Equilibrium points and Stability of system (3)

This section investigates the stability results of the (3) using Jury conditions.

3.1. Equilibrium Points and its Existence. The positive equilibrium points of system (3) are obtained by solving

$$\begin{aligned} sx(1-x) - \beta xy - cx &= 0 \\ -wy + \eta xy - fy &= 0. \end{aligned} \tag{5}$$

(1) $ES_0 = (0, 0)$
(2) $ES_1 = \left(\frac{s-c}{s}, 0\right)$
(3) $\mathrm{ES}_2 = \left(x^*, \frac{s-c}{\beta} - \frac{s}{\beta}x^*\right)$.

where $x^* = \frac{w+f}{\eta}$.

Theorem 2. *The equilibrium points satisfy*

(1) *The trivial equilibrium point ES_0 always exists.*
(2) *The axial equilibrium steady state ES_1 exists if $s > c$*
(3) *The interior equilibrium point ES_2 exists if $\eta > \frac{s(w+f)}{s-c}$.*

3.2. Stability Results of System (3). Jacobian matrices are formulated at the steady states and jury conditions are employed to investigate the stability of the system (3). At (x, y), the Jacobian matrix is

$$J(x,y) = \begin{bmatrix} 1 + Q\left[s(1-2x) - \beta y - c\right] & -Q\beta x \\ Q\eta y & 1 + Q\left[\eta x - w - f\right] \end{bmatrix} \tag{6}$$

where $Q = \frac{\rho^{\upsilon}}{\Gamma(1+\upsilon)}$. Now the characteristic equation of (6) is

$$\Phi(m) = m^2 - Tm + D = 0 \tag{7}$$

where $T = 2 + Q[s - c - w - f + (\eta - 2s)x - \beta y]$ is the trace of (6) and determinant of (6) is $D = 1 + Q\left[s - c - w - f - \beta y + x(\eta - 2s)\right]$
$+ Q^2\left[(s - 2sx - c)(\eta x - w - f) + \beta y(w + f)\right]$.

Lemma 3. [18] *Let m_1, m_2 satisfy $\Phi(m) = 0$ and suppose that $\Phi(1) > 0$. Then (x^*, y^*) is*

(1) *stable if $|m_1| < 1$, $|m_2| < 1 \Leftrightarrow \Phi(-1) > 0$, $\Phi(0) < 1$.*
(2) *saddle point if $|m_1| < 1$, $|m_2| > 1$ (or $|m_1| > 1, |m_2| < 1$) $\Leftrightarrow \Phi(-1) < 0$.*
(3) *unstable if $|m_1| > 1$,$|m_2| > 1 \Leftrightarrow \Phi(-1) > 0$, $\Phi(0) > 1$.*
(4) $|m_1| = -1, |m_2| \neq 1 \Leftrightarrow \Phi(-1) = 0$.
(5) *m_1, m_2 are complex and $|m_1| = |m_2| \Leftrightarrow T^2 - 4D < 0$ and $\Phi(0) = 1$.*

Theorem 4. *The equilibrium point ES_0 is*

(a) *unstable for $|m_2| > 1$ i.e. $\rho > \left[\frac{2\Gamma(1+\upsilon)}{w+f}\right]^{\frac{1}{\upsilon}}$.*
(b) *saddle point for $|m_2| < 1$, i.e. $0 < \rho < \left[\frac{2\Gamma(1+\upsilon)}{w+f}\right]^{\frac{1}{\upsilon}}$.*

(c) *non-hyperbolic for* $\rho = \left[\frac{2\Gamma(1+\upsilon)}{c}\right]^{\frac{1}{\upsilon}}$.

Proof. At ES_0, (6) becomes

$$J_{ES_0} = J(0,0) = \begin{bmatrix} 1+Q(s-c) & 0 \\ 0 & 1-Q(w+f) \end{bmatrix}$$

whose eigenvalues are $m_1 = 1+Q(s-c)$ and $m_2 = 1-Q(w+f)$. Since $\frac{\rho^{\upsilon}}{\Gamma(1+\upsilon)} > 0$ for $0 < \upsilon \leq 1$.

(a) It is obvious that $|m_2| > 1$. Then ES_0 is source if $|1-Q(w+f)| > 1$ which yields $\rho > \left[\frac{2\Gamma(1+\upsilon)}{w+f}\right]^{\frac{1}{\upsilon}}$.

(b) $ES_0$is saddle point if $|1-Q(w+f)| < 1$, i.e. $0 < \rho < \left[\frac{2\Gamma(1+\upsilon)}{w+f}\right]^{\frac{1}{\upsilon}}$.

(c) This result is a consequence of (I) and (II).

∎

Theorem 5. *The axial equilibrium point* ES_1 *is*

(a) *stable for* $c < s < \frac{\eta c}{\eta-w-f}$ *and* $\rho <$ min $\left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}, \left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}}\right\}$.

(b) *unstable for* $\frac{\eta c}{\eta-w-f} < s < c$ *and* $\rho >$ max $\left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}, \left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}}\right\}$.

(c) *saddle for* $s >$ max $\left\{\frac{\eta c}{\eta-w-f}, c\right\}$ *and* $\left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}} < \rho < \left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}\right\}$.

(d) *non-hyperbolic for* $s = c$ *(or)* $s = \left\{\frac{\eta c}{\eta-w-f}\right\}$ *and* $\rho = \left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}}$ *(or)* $\rho = \left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}\right\}$.

Proof. For ES_1, Jacobian matrix is

$$J_{ES_1} = J\left(\frac{s-c}{s}, 0\right) = \begin{bmatrix} 1+Q(s-c) & -Q\frac{\beta(s-c)}{s} \\ 0 & 1-Q\left[w+f-\frac{\eta(s-c)}{s}\right] \end{bmatrix}$$

whose eigen values are $m_1 = 1-Q(s-c)$ and $m_2 = 1-Q\left[w+f-\frac{\eta(s-c)}{s}\right]$. Since $\frac{\rho^{\upsilon}}{\Gamma(1+\upsilon)} > 0$ for $0 < \alpha \leq 1$.

(a) ES_1 is stable if $|1-Q(s-c)| < 1$ and $\left|1-Q\left[w+f-\frac{\eta(s-c)}{s}\right]\right| < 1$ which yields

$$c < s < \frac{\eta c}{\eta-w-f} \text{ and } \rho < \min\left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}, \left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}}\right\}.$$

(b) ES_1 is unstable if $|1-Q(s-c)|>1$ and $\left|1-Q\left[w+f-\frac{\eta(s-c)}{s}\right]\right|>1$, i.e.

$$\frac{\eta c}{\eta-w-f} < s < c \text{ and } \rho > \max\left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}, \left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}}\right\}$$

(c) ES_1 is Saddle if $|1-Q(s-c)|<1$ and $\left|1-Q\left[w+f-\frac{\eta(s-c)}{s}\right]\right|>1$, i.e.

$$s > \max\left\{\frac{\eta c}{\eta-w-f}, c\right\} \text{ and } \left[\frac{2\Gamma(1+\upsilon)}{s(w+f)-\eta(s-c)}\right]^{\frac{1}{\upsilon}} < \rho < \left\{\left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}\right\}$$

(d) The proof follows from result (a) and (b).

∎

At ES_2, (6) becomes

$$J_{ES_2} = \begin{bmatrix} 1+Qa_{11} & -Qa_{12} \\ Qa_{21} & 1 \end{bmatrix} \tag{8}$$

The characteristic equation of J_{ES_2} is $\Phi(m)=m^2-Tm+D=0$, with $T=2+Qa_{11}$ and $D=1+Qa_{11}+Q^2a_{12}a_{21}$, where $Q=\frac{\rho^\upsilon}{\Gamma(1+\upsilon)}$, $a_{11}=-\frac{s(w+f)}{\eta}$, $a_{12}=\frac{\beta(w+f)}{\eta}$ and $a_{21}=\frac{\eta(s-c)-s(w+f)}{\beta}$. Eigen values are

$$m_{1,2}=1+\frac{QM}{2}\pm\frac{Q}{2}\sqrt{M^2-4N},$$

here $M=a_{11}$ and $N=a_{12}a_{21}$.

Theorem 6. *The interior equilibrium point ES_2 is a*

(a) *sink if one of the following conditions are satisfied:*
 (i) $\mathfrak{S}^*<0$ *and* $\rho<\rho_3$,
 (ii) $\mathfrak{S}^*\geq 0$ *and* $\rho<\rho_2$,

(b) *source if one of the following conditions are satisfied:*
 (i) $\mathfrak{S}^*<0$ *and* $\rho>\rho_3$,
 (ii) $\mathfrak{S}^*\geq 0$ *and* $\rho>\rho_1$,

(c) *saddle, if*
 (i) $\mathfrak{S}^*\geq 0$ *and* $\rho_2<\rho<\rho_1$,

(d) *non-hyperbolic, if one of the following conditions are satisfied:*
 (i) $\mathfrak{S}^*<0$ *and* $\rho=\rho_3$.
 (ii) $\mathfrak{S}^*>0$ *and* $\rho=\rho_1$ *or* $\rho=\rho_2$,

$\mathfrak{S}^*=(M^2-4N)$ *and* $\rho_1=\left\{\Gamma(1+\upsilon)\left[\frac{-M+\sqrt{M^2-4N}}{N}\right]\right\}^{\frac{1}{\upsilon}}$,
$\rho_2=\left\{\Gamma(1+\upsilon)\left[\frac{-M-\sqrt{M^2-4N}}{N}\right]\right\}^{\frac{1}{\upsilon}}$, $\rho_3=\left\{\left[\frac{-\Gamma(1+\upsilon)M}{N}\right]\right\}^{\frac{1}{\upsilon}}$

4. Bifurcation Analysis of (3)

The existence of bifurcation is investigated in this section.

4.1. **Periodic Doubling Bifurcation.** The parameter for analyzing existence of bifurcation is chosen as ρ. The equilibrium point ES_2 is said to undergo periodic doubling bifurcation if one of the eigenvalue is -1 and other shall not be 1 (or) -1 [10].

The quadratic equation obtained from (8) is

$$\Phi(m) = m^2 - (2 + QM)m + (1 + QM + Q^2N).$$

By Theorem (6), if $S^* \geq 0$ and $\rho = \left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}$, the eigen values are

$$m_1 = -1, m_2 = 1 - \frac{2(w+f)}{s-c} + \frac{2\eta}{s}.$$

Theorem 7. *The periodic doubling bifurcation occurs causing instability to* ES_1 *when* $S^* = \left(Q(s-c) - Q\left(w + f - \frac{\eta(s-c)}{s}\right)\right)^2 \geq 0$ *and* $\rho = \left[\frac{2\Gamma(1+\upsilon)}{s-c}\right]^{\frac{1}{\upsilon}}$, *and*

$$m_1 = -1, m_2 = 1 - \frac{2(w+f)}{s-c} + \frac{2\eta}{s} \neq \pm 1.$$

4.2. **Neimark Sacker Bifurcation.** Let ρ be the bifurcation parameter considered to analyzes Neimark-Sacker bifurcation. The occurance of this bifurcation is ensured when the eigenvalues at steady state ES_2 are complex conjugate with modulus equal to 1 [10]. The quadratic equation obtained from (8) is

$$\Phi(m) = m^2 - (2 + QM)m + (1 + QM + Q^2N).$$

From Theorem (6), if $S^* < 0$ and $\rho = \rho_3$, then

$$m_{1,2} = 1 - \frac{M^2}{2N} \pm i\frac{M}{2N}\sqrt{4N - M^2}.$$

are the corresponding eigen values.

Theorem 8. *The Neimark-Sacker bifurcation of system* (3)*occurs when* $S^* < 0$ *and* $\rho = \rho_3$, *and*

$$|m_{1,2}| = \left|1 - \frac{M^2}{2N} \pm i\frac{M}{2N}\sqrt{4N - M^2}\right| = 1.$$

5. Control Strategies

The system with linear feedback controller [6] is

$$\begin{aligned} x(t+1) &= x(t) + \frac{\rho^\upsilon}{\Gamma(1+\upsilon)}\left[sx(t)(1 - x(t)) - \beta x(t)y(t) - cx(t)\right] + R(t) \\ y(t+1) &= y(t) + \frac{\rho^\upsilon}{\Gamma(1+\upsilon)}\left[-wy(t) + \eta x(t)y(t) - fy(t)\right] \end{aligned} \tag{9}$$

where feedback control is $R(t) = -r_1\left(x(t) - \frac{s-c}{s}\right) - r_2 y(t)$ with r_1, r_2 being feedback gains. The Jacobian of system (9) at $\left(\frac{s-c}{s}, 0\right)$ is

$$J_1\left(\frac{s-c}{s},0\right)=\begin{bmatrix}1-QA-r_1 & -\frac{Q\beta A}{s}-r_2\\ 0 & 1-QB\end{bmatrix} \tag{10}$$

Here $A=(s-c)$ and $B=\left[w+f-\frac{\eta(s-c)}{s}\right]$. The corresponding characteristic equation of $J_1\left(\frac{s-c}{s},0\right)$ is

$$m^2-(2-Q(A+B)-r_1)m+(Q^2AB-Q(A+B+Br_1)+1-r_1)=0. \tag{11}$$

If m_1, m_2 are the eigenvalues of (11), then

$m_{1,2}=\frac{(2-Q(A+B)-r_1)\pm\sqrt{(2-Q(A+B)-r_1)^2-4(Q^2AB-Q(A+B-Br_1)+1-r_1)}}{2}$
and

$$m_1m_2=1-r_1-Q(A+B-Br_1)+Q^2AB \tag{12}$$

The equations $m_1=\pm1$ and $m_1m_2=1$ ensure that absolute values of the eigen values are less than 1. Suppose $m_1m_2=1$, then (12) becomes

$$l_1\,:\,Q^2AB-Q(A+B)=r_1-QBr_1$$

Suppose that $m_1=1$ or $m_1=-1$, then equation (11) yields

$$\begin{aligned}&l_2\,:\,-QA=r_1\\&l_3\,:\,Q^2AB-2Q(A+B)+4=2r_1-QBr_1\end{aligned}$$

The triangular region with lines l_1, l_2 and l_3 contains the eigenvalues.

Next, the system with hybrid controlled strategy is given by

$$\begin{aligned}x(t+1)&=\alpha x(t)+\frac{\alpha\rho^\nu}{\Gamma(1+\nu)}\left[sx(t)(1-x(t))-\beta x(t)y(t)-cx(t)\right]+(1-\alpha)x(t)\\ y(t+1)&=\alpha y(t)+\frac{\alpha\rho^\nu}{\Gamma(1+\nu)}\left[-wy(t)+\eta x(t)y(t)-fy(t)\right]+(1-\alpha)x(t)\end{aligned} \tag{13}$$

where $0<\alpha<1$. Parameter perturbation and feedback control are combined in (13) as control strategy and appropriate choice of α results in partial or completely elimination of Neimark sacker bifurcation. Jacobian of (13) at ES_2 is

$$J_2\left(ES_2\right)=\begin{bmatrix}1+\alpha Aa_{11} & -\alpha Aa_{12}\\ \alpha Aa_{21} & 1\end{bmatrix}. \tag{14}$$

where $A, a_{11}, a_{12}, a_{21}$ are given in (8). The presence of the roots of the (14) in the unit disk ensure the asymptotic stability of ES_2.

6. Numerical Experiments

This section illustrates the results obtained above with suitable examples.

Example 9. *Let* $\nu = 0.85, s = 2.3, \beta = 0.8, c = 0.05,\ w = 0.3, \eta = 0.81, f = 0.5$ *and* $0.75 \leq \rho \leq 1.31$ *of system* (3) *and* $x(0) = 0.55,\ y(0) = 0.45$*. We obtain* $ES_1 = (x^*, y^*) = (0.9782, 0)$*. Eigen values are* $m_1 = -1$ *and* $m_2 = 0.9932 \neq 1$*. The critical point of periodic doubling bifurcation given in Theorem* (7) *is* $\rho = 0.8151$*. Figure 1(a), 1(b) show flip bifurcation diagrams in* (ρ, x) *and lyapunov exponent. The periodic windows of the corresponding bifurcation diagrams are represented in 1(c), 1(d), 1(e) and 1(f) respectively.*

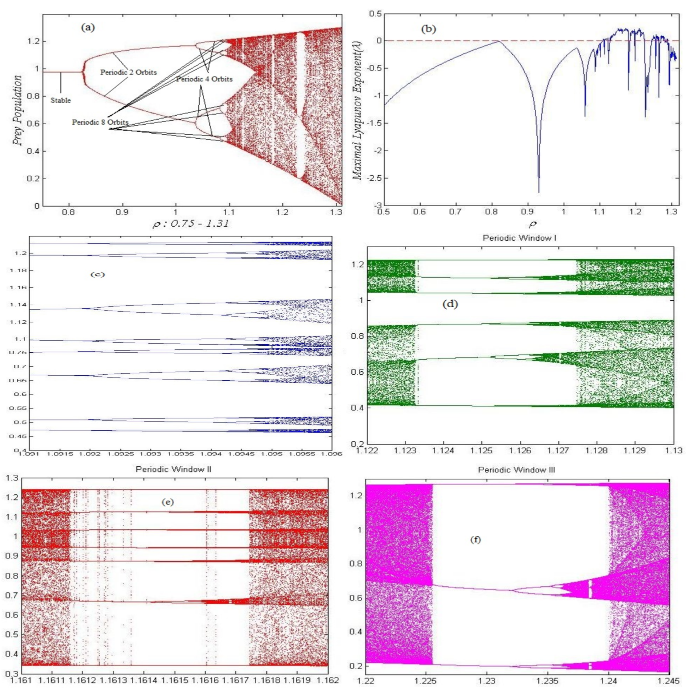

FIGURE 1. Flip bifurcation diagram in (ρ, x) plane and Maximum Lyapunov exponents of the system (3) with different periodic windows

Example 10. *Let* $\nu = 0.85, \rho = 0.82, s = 2.3, \beta = 0.8, c = 0.05, w = 0.3, \eta = 0.81, f = 0.5$ *and* $x(0) = 0.55, y(0) = 0.45$ *for the system* (3)*, periodic doubling bifurcation occurs as* ρ *varies in* $\rho \in [0.75, 1.31]$*. Moreover, Figure 2(a) displays the time plots for both prey and predator populations at* $\rho = 0.82$*.*

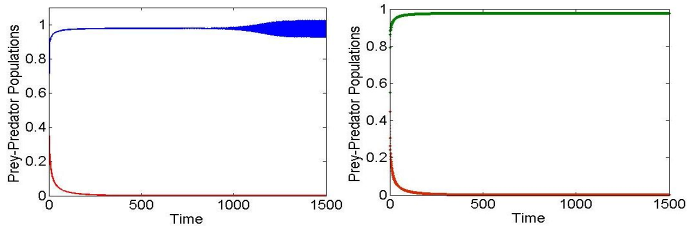

FIGURE 2. Time series for the system 3 and (15)

The controlled system (9) *for above values takes the form*

$$\begin{aligned} x(t+1) &= x(t) + \frac{\rho^{\upsilon}}{\Gamma(1+\upsilon)}\left[sx(t)(1-x(t)) - \beta x(t)y(t) - cx(t)\right] + R(t) \\ y(t+1) &= y(t) + \frac{\rho^{\upsilon}}{\Gamma(1+\upsilon)}\left[-wy(t) + \eta x(t)y(t) - fy(t)\right] \end{aligned} \tag{15}$$

where $R(t) = -r_1\left(x(t) - \frac{s-c}{s}\right) - r_2 y(t)$ *and* $r_1 = -0.07$ *and* $r_2 = 0.075$. *The plots for the system* (15) *with control terms are provided in Figure 2(b). It is clear that the equilibrium* ES_1 *is stable. The time plots at different* ρ *values are displayed in Figure 3 and Figure 4.*

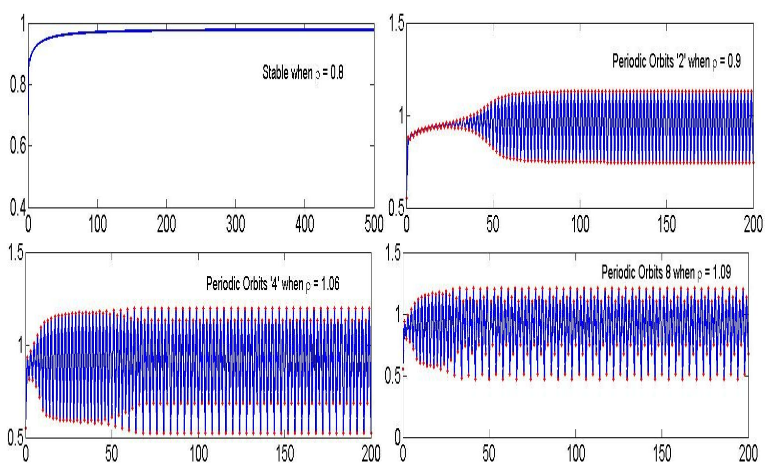

FIGURE 3. Different periodic orbits of the axial Bifurcation of the system (3)

Example 11. *Taking* $\nu = 0.85, s = 0.35, \beta = 0.4, c = 0.01, w = 0.01, \eta = 0.56, f = 0.12$ *and* $2 \leq \rho \leq 4.5$ *in system* (3) *and* $x(0) = 0.55, y(0) = 0.45$, *we get* $ES_2 = (x^*, y^*) = (0.2321, 0.6469)$. *For* $M = -0.0804; N = 0.0336, S^* = -0.1279, \rho_3 =$

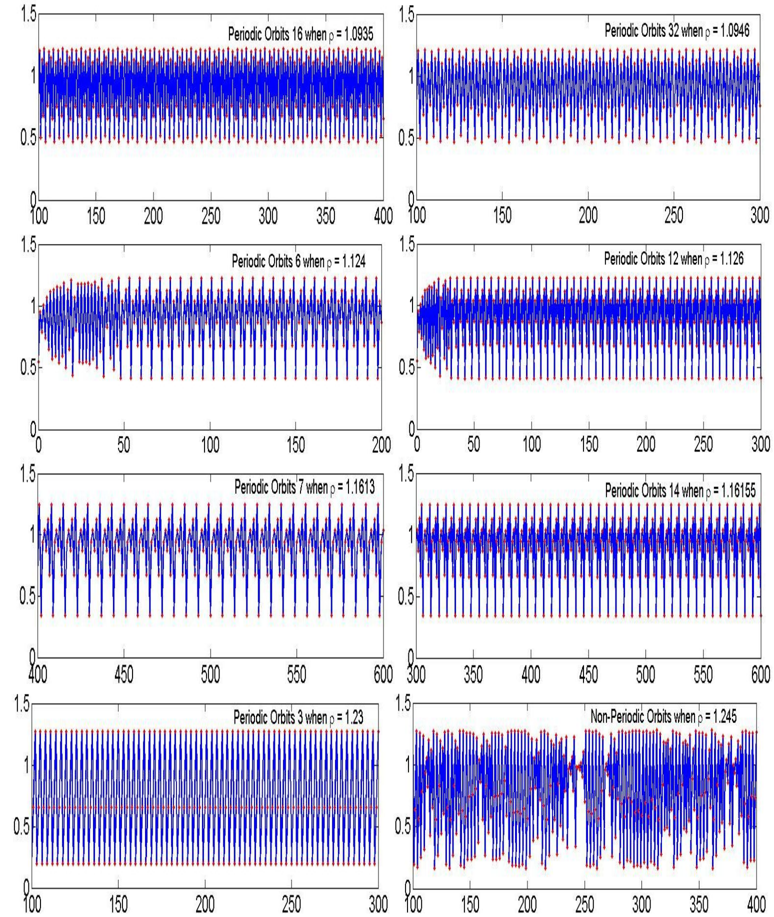

FIGURE 4. Different periodic orbits of the axial Bifurcation of the system (3)

2.6124, *the eigen values are* $m_{1,2} = 0.9038 \pm i0.4280$ *with* $|\eta_{1,2}| = 1$. *The critical value given in Theorem* (8) *is* $\rho_3 = 2.6124$.

The bifurcation diagrams in x and y plane are presented in Figure 5(a), 5(b). Phase trajectories obtained for various values of ρ are given in Figure 6 and Figure 7. System (3) at ES_2 is locally asymptotic stable for $\rho < \rho_3 = 2.6124$ and becomes unstable at $\rho = \rho_3$ followed by formation of invariant cycles for $\rho > \rho_3$ which are presented in Figure 5, Figure 6 and Figure 7.

Example 12. *Consider the values* $\nu = 0.85, \rho = 2.67, s = 0.35, \beta = 0.4, c = 0.01, w = 0.01, \eta = 0.56, f = 0.12$ *and* $x(0) = 0.55, y(0) = 0.45$. *The occurrence*

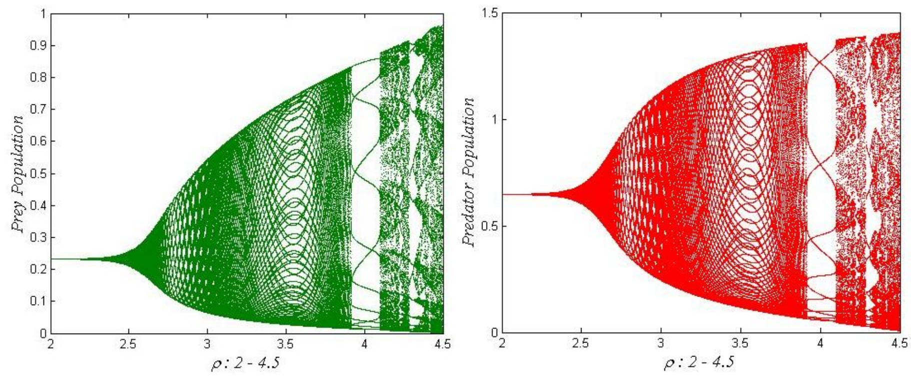

FIGURE 5. Neimark- Sacker Bifurcations for the system 3

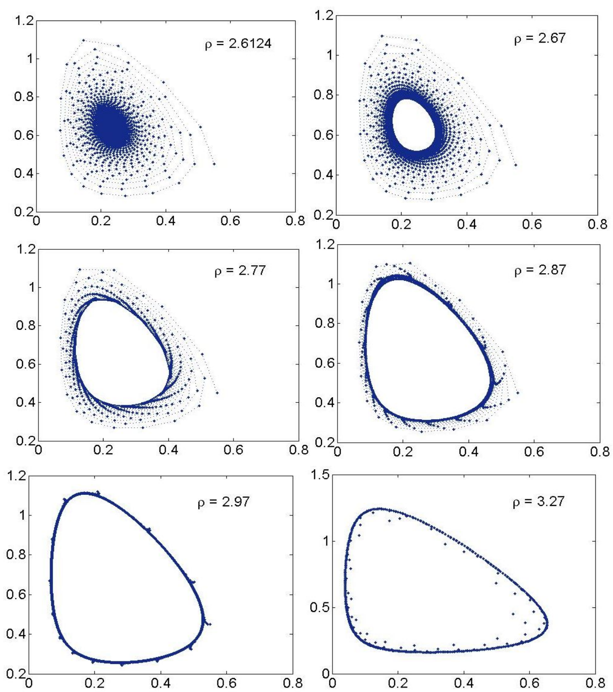

FIGURE 6. Phase portraits of system (3) for different values of ρ

of the Neimark-Sacker bifurcation for $\rho \in [2, 4.5]$ *of the system* (3) *is illustrated*

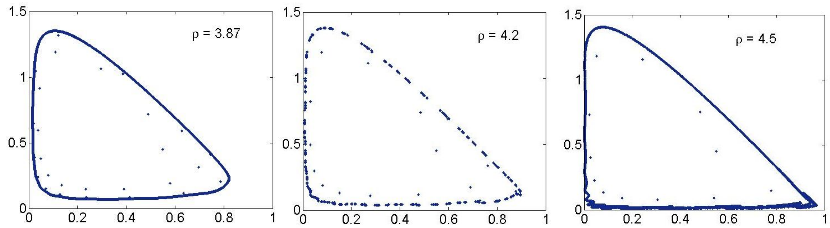

FIGURE 7. Phase portraits of system (3) for different values of ρ

in Example (11). *The unstable closed path formed at* $\rho = 2.67$ *enclosing unstable steady state* $ES_2 = (x^*, y^*) = (0.2321, 0.6469)$ *is presented in Figure-8.*

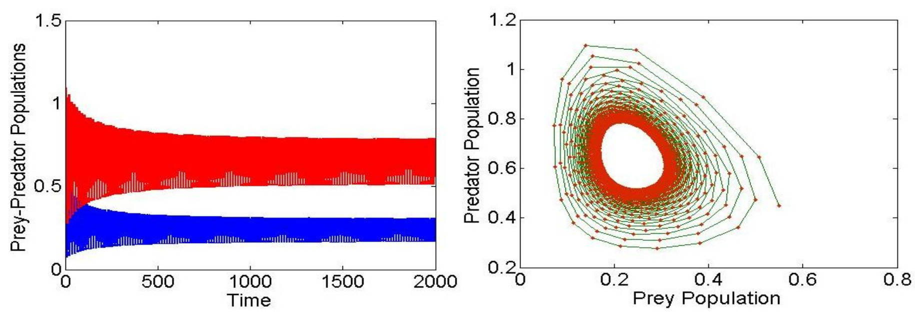

FIGURE 8. Phase portrait for the system 3 with time series plot

The controlled system (13) *for above values can be written as*

$$\begin{aligned} x(t+1) &= x(t) + \frac{\alpha\rho^{\nu}}{\Gamma(1+\nu)}\left[sx(t)(1-x(t)) - \beta x(t)y(t) - cx(t)\right] \\ y(t+1) &= y(t) + \frac{\alpha\rho^{\nu}}{\Gamma(1+\nu)}\left[-wy(t) + \eta x(t)y(t) - fy(t)\right] \end{aligned} \tag{16}$$

with $\nu = 0.85, \rho = 2.67, s = 0.35, \beta = 0.4, c = 0.01, w = 0.01, \eta = 0.56, f = 0.12$ *and* $0 < \alpha < 1$. *The stability of* ES_2 *is confirmed by the phase trajectory and time plots in Figure 9 for* (16) *at* $\alpha = 0.95$.

7. CONCLUSION

The qualitative study of the fractional order discrete equations of the prey- predator model with harvesting is carried out. The stability conditions and bifurcation analysis of the system is studied. The chaos control is performed with feedback control and numerical simulations for bifurcations with different phase trajectories are performed as well in accordance with the theoretical work. The periodic windows and different time plots are provided to understand the dynamics exhibited

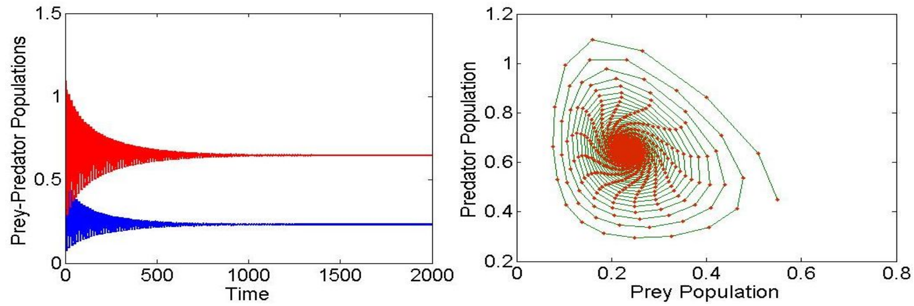

FIGURE 9. Phase portrait for the system 13 with time series plot

by the prey predator model.

Author Contribution Statements All authors jointly worked on the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

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THE LOMAX-LINDLEY DISTRIBUTION: PROPERTIES AND APPLICATIONS TO LIFETIME DATA

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Abstract. This paper introduces a new three-parameter distribution which is obtained by combining the Lomax and Lindley distributions in a serial system and is called the Lomax-Lindley distribution. The new distribution is quite flexible to model lifetime data. This distribution provides a simple form for hazard rate function which can be increasing, decreasing, bathtub-shaped and unimodal for different choices of the parameter values. Some statistical properties of the Lomax-Lindley distribution such as quantiles, moments, order statistics, Renyi entropy and mean deviations are discussed. The maximum likelihood estimators of its unknown parameters are obtained and the approximate confidence intervals of the parameters are provided. A Monte Carlo simulation study is conducted to investigate the performance of the maximum likelihood estimators and their corresponding confidence intervals. Finally, two real data sets having bathtub-shaped and unimodal hazard rate functions are analyzed and it is shown that the proposed distribution can provide a better fit than other distributions for both lifetime data.

1. Introduction

The Lomax (also known as the Pareto Type II) distribution has been introduced by Lomax [16] as a model for lifetime data analysis. The Lomax distribution is a heavy-tailed distribution and it has wide applications in many fields such as business, economics, actuarial modeling, queuing problems, biological sciences, reliability and life testing problems. For more details we refer to Arnold [5].

The cumulative distribution function (cdf) of the Lomax distribution with two parameters α and β is given by

$$F(x) = 1 - (1 + \beta x)^{-\alpha}, \quad x > 0, \quad \alpha, \beta > 0. \tag{1}$$

Keywords. Lomax distribution, Lindley distribution, maximum likelihood estimation, lifetime data analysis.

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Another distribution that has been extensively used over the past decades for modeling data in reliability, biology, insurance, finance and lifetime analysis is Lindley distribution. This distribution was introduced by Lindley [15] in the context of fiducial and Bayesian inference. Ghitany et al. [11] showed that the Lindley distribution is more flexible than the exponential distribution for modeling lifetime data. In recent years, Lindley distribution has been considered by several authors. See, for example, Gupta and Singh [12] and Maiti and Mukherjee [17].

The cdf of the Lindley distribution with parameter θ is given by

$$F(x) = 1 - \left(1 + \frac{\theta x}{\theta + 1}\right) e^{-\theta x}, \quad x > 0, \quad \theta > 0. \tag{2}$$

Although the Lomax and the Lindley distributions are very useful for modeling lifetime data, they have monotone hazard rate functions. The hazard rate function of Lomax distribution is decreasing and that of Lindley distribution is increasing. Therefore, these distributions may not provide a reasonable parametric fit for modeling phenomena with non-monotone hazard rates such as the bathtub-shaped and unimodal hazard rates which are often encountered in practice. In this regard, some researchers have considered modified forms and generalizations of these distributions to provide more flexibility for describing different types of data. For example, McDonald Lomax distribution by Lemonte and Cordeiro [14], Weibull-Lomax distribution by Tahir et al. [23], Burr X exponentiated Lomax distribution by Aboraya [1], new extended generalized lindley distribution by Maya and Irshad [19], odd log-logistic Lindley distribution by Ozel et al. [21], odd log-logistic Marshal-Olkin Lindley distribution by Alizadeh et al. [3] and exponentiated power Lindley power series class of distributions by Alizadeh et al. [2]. Most of these distributions have four or more parameters which cause estimation problems as a consequence of the number of parameters. On the other hand, the hazard rate functions in most of these models have the complex forms and therefore cannot have many applications for lifetime data analysis in practice. Therefore, major motivation of this study is to introduce a new flexible three-parameter distribution based on the Lomax and the Lindley distributions which its hazard rate function is simple and can cover monotone as well as non-monotone hazard rates.

To obtain more flexible models with simple hazard rate functions, a useful technique is combining the hazard rates of two distributions. For example, additive Weibull distribution by Xie and Lai [25], new modified Weibull distribution by Almalki and Yuan [4] and power-exponential hazard rate distribution by Tarvirdizade and Nematollahi [24] are some models introduced by using this technique. The goal of this article is to propose a new three-parameter lifetime distribution called the Lomax-Lindley (L-L) distribution using the combination of the Lomax and the Lindley distributions in a serial system. The new distribution can be used effectively for analyzing lifetime data since it provides a simple hazard function which can cover increasing, decreasing, bathtub-shaped and unimodal hazard rates. Some properties of the L-L distribution including the density and hazard rate functions, quantiles, moments, order statistics, Renyi entropy and mean deviations are presented. The method of maximum likelihood estimation (MLE) is used to estimate the unknown parameters of this new class of distributions. The flexibility of the new model is demonstrated by fitting the L-L distribution to two real data sets having bathtub-shaped and unimodal hazard rate functions.

The contents of this paper are organized as follows. In Section 2, we define the L-L distribution and and study the shape of the hazard rate function of this model. In Section 3, we consider some distributional properties of the new model. In Section 4, the MLEs of unknown parameters are discussed and their asymptotic confidence intervals are provided. In Section 5, a Monte Carlo simulation study is conducted. The applications of the L-L distribution are illustrated by analyzing two real data sets in Section 6. Finally, the conclusions of this paper are given in Section 7.

2. The L-L distribution

We define the cdf of the L-L distribution with three parameters α, β and θ as

$$F(x) = 1 - (1+\beta x)^{-\alpha}\left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}, \quad x > 0,\ \alpha > 0,\ \beta, \theta \geq 0, \tag{3}$$

where $\beta + \theta > 0$. The probability density function (pdf) of this distribution is given by

$$f(x) = \left(\frac{\theta^2(1+x)}{1+\theta} + \frac{\alpha\beta}{1+\beta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right)(1+\beta x)^{-\alpha}e^{-\theta x},$$
$$x > 0,\ \alpha > 0, \quad \beta, \theta \geq 0. \tag{4}$$

Henceforth, we denote a random variable X having pdf (4) by $X \sim$ L-L(α, β, θ). The L-L model in (4) can be interpreted as the lifetime of a serial system with two components, one following Lomax distribution with parameters α and β and another following Lindley distribution with parameter θ, and the system's lifetime is the minimum of the two components. Clearly, the L-L distribution contains Lomax and Lindley distributions as special cases.

The hazard rate function of the L-L distribution takes a simple form as

$$h(x) = \frac{\alpha\beta}{1+\beta x} + \frac{\theta^2(1+x)}{1+\theta(1+x)}. \tag{5}$$

To derive the shape of $h(x)$, we obtain the first derivative of (5) as

$$h'(x) = -\frac{\alpha\beta^2}{(1+\beta x)^2} + \frac{\theta^2}{[1+\theta(1+x)]^2}.$$

Setting $h'(x) = 0$, we obtain

$$x_0 = \frac{\sqrt{\alpha}\beta(1+\theta) - \theta}{\theta\beta(1-\sqrt{\alpha})}.$$

Then it can be seen that the hazard rate function has bathtub-shaped property if

$$\alpha < 1,\ \beta < \frac{1}{\sqrt{\alpha}},\ \theta < \frac{\sqrt{\alpha}\beta}{1-\sqrt{\alpha}\beta},$$

and the hazard rate function is unimodal if

$$\alpha > 1,\ \beta < \frac{1}{\sqrt{\alpha}},\ \theta > \frac{\sqrt{\alpha}\beta}{1-\sqrt{\alpha}\beta}.$$

For other values of the parameters α, β and θ, the hazard rate function can also be increasing or decreasing. Plots of increasing, bathtub-shaped, decreasing and unimodal hazard rate functions of the L-L distribution and the corresponding pdfs for different values of the parameters α, β and θ are displayed in Figures 1 and 2, respectively.

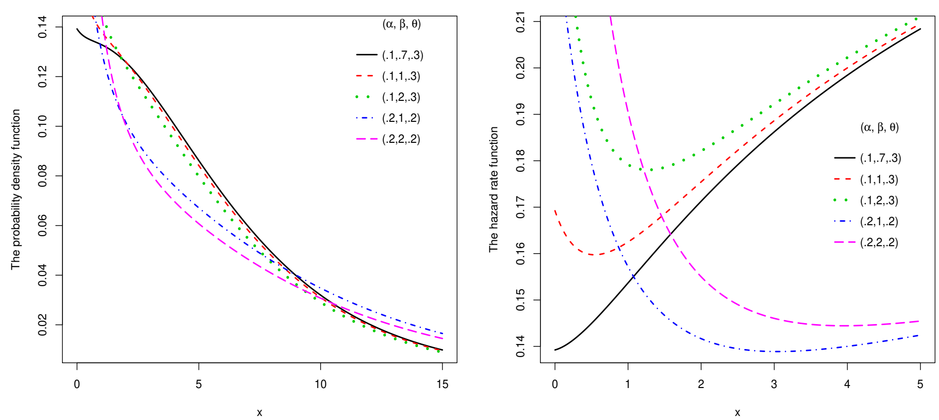

Figure 1. The increasing and bathtub-shaped hazard rate functions of the L-L distribution and the corresponding pdfs for different parameter values.

3. Properties of the L-L distribution

We discuss some of the basic statistical properties of the L-L distribution in this section, which consist of quantiles, moments, order statistics, Renyi entropy and mean deviations.

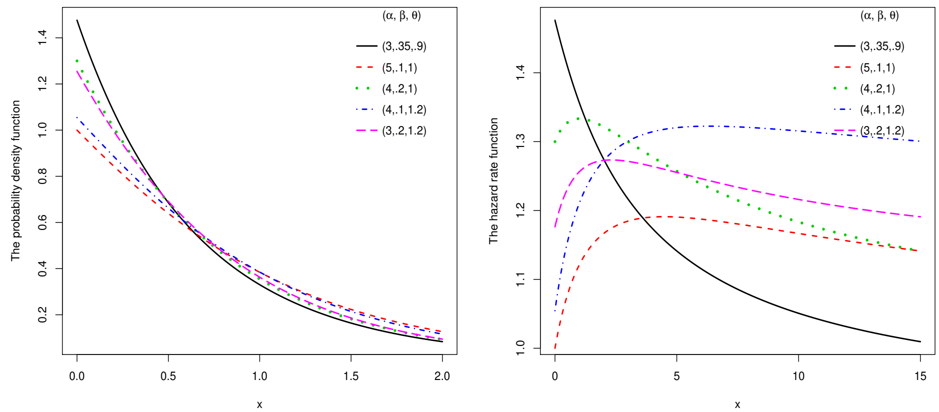

Figure 2. The decreasing and unimodal hazard rate functions of the L-L distribution and the corresponding pdfs for different parameter values.

3.1. **Quantiles.** The quantile function is one of the important functions in probability theory and statistical applications which can be used in data generation from a distribution. The qth quantile (x_q) of the L-L(α, β, θ) is obtained by solving $F(x_q) = q$ where $F(x)$ is given in (3). It can be easily shown that x_q is the real solution of the following equation

$$\alpha \log(1 + \beta x_q) - \log\left(1 + \frac{\theta x_q}{\theta + 1}\right) + \theta x_q + \log(1 - q) = 0.$$

The above equation has no closed form solution in x_q and therefore, a numerical technique such as the Newton-Raphson method can be employed to get the quantile. In particular, the median of the L-L(α, β, θ) is obtained for $q = 0.5$.

3.2. **Moments.** The rth moment (μ_r) of the L-L(α, β, θ) could be obtained from (4) and integration by parts as follow:

$$\begin{aligned}\mu_r = E(X^r) &= \int_0^\infty x^r \left(\frac{\theta^2(1+x)}{1+\theta} + \frac{\alpha\beta}{1+\beta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right)(1+\beta x)^{-\alpha} e^{-\theta x} dx \\ &= \int_0^\infty r x^{r-1}(1+\beta x)^{-\alpha}\left(1 + \frac{\theta x}{\theta+1}\right) e^{-\theta x} dx, \quad r = 1, 2, ... \qquad (6)\end{aligned}$$

Now, using the Taylor expansion

$$(1 + \beta x)^{-\alpha} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} (\beta x)^i, \qquad (7)$$

it follows from (6) that

$$\mu_r = \sum_{i=0}^{\infty} (-1)^i \beta^i \left(\begin{array}{c} \alpha+i-1 \\ i \end{array} \right) \int_0^{\infty} r x^{r+i-1} \left(1 + \frac{\theta x}{\theta+1}\right) e^{-\theta x} dx. \tag{8}$$

Finally, using the transformation $y = \theta x$, we obtain the rth moment of the L-L(α, β, θ) as

$$\mu_r = \sum_{i=0}^{\infty} (-1)^i \beta^i \left(\begin{array}{c} \alpha+i-1 \\ i \end{array} \right) \frac{r}{\theta^{r+i}} \left[\Gamma\left(r+i\right) + \frac{1}{\theta+1} \Gamma\left(r+i+1\right)\right], \tag{9}$$

where $\Gamma(.)$ is the gamma function.

Some of the most important characteristics of a distribution can be obtained through moments. For example, the measures of variance, skewness and kurtosis of the L-L distribution can be obtained according to the following relations, respectively,

$$\sigma^2 = \mu_2 - \mu_1^2, \quad SK = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\left(\mu_2 - \mu_1^2\right)^{3/2}}, \quad KU = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{\left(\mu_2 - \mu_1^2\right)^2}.$$

3.3. **Order statistics.** Order statistics make their appearance in many areas of statistical theory and practice. In this subsection, we provide an expression for the pdf of the ith order statistic ($X_{i:n}$) of a random sample $X_1, X_2, ..., X_n$ drawn from the L-L(α, β, θ). From Balakrishnan and Nagaraja [6], the pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) F(x)^{i-1} (1 - F(x))^{n-i}, \tag{10}$$

where $B(.,.)$ is the beta function. Using (3), (4) and the binomial expansion, we have

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)} \sum_{j=0}^{i-1} (-1)^j \left(\begin{array}{c} i-1 \\ j \end{array} \right) \left(\frac{\theta^2(1+x)}{1+\theta} + \frac{\alpha\beta}{1+\beta x} \left(1 + \frac{\theta x}{\theta+1}\right)\right) \\ &\times \left(1 + \frac{\theta x}{\theta+1}\right)^{n-i+j} (1+\beta x)^{-(n-i+j+1)\alpha} e^{-(n-i+j+1)\theta x} \\ &= \frac{1}{B(i,n-i+1)} \sum_{j=0}^{i-1} \sum_{k=0}^{n-i+j} (-1)^j \left(\begin{array}{c} i-1 \\ j \end{array} \right) \left(\begin{array}{c} n-i+j \\ k \end{array} \right) \left(\frac{\theta x}{\theta+1}\right)^k \\ &\times \left(\frac{\theta^2(1+x)}{1+\theta} + \frac{\alpha\beta}{1+\beta x} \left(1 + \frac{\theta x}{\theta+1}\right)\right) (1+\beta x)^{-(n-i+j+1)\alpha} e^{-(n-i+j+1)\theta x}. \end{aligned} \tag{11}$$

Some statistical properties of the L-L distribution such as moments, generating function and mean deviations of the order statistics can be derived using (11).

3.4. **Renyi entropy.** The entropy is an index for measuring variation or uncertainty of a random variable and has been used in many fields such as physics, engineering and economics among others. A popular entropy measure is Renyi entropy. Renyi entropy of a random variable with pdf $f(.)$ is defined as follows:

$$I_R(r) = \frac{1}{1-r} \log \int_0^\infty f^r(x)dx,$$

where $r > 0$ and $r \neq 1$. Let $X \sim$ L-L(α, β, θ). Then, using (4) and the binomial expansion, we have

$$\begin{aligned}
f^r(x) &= \sum_{i=0}^{r} \binom{r}{i} (\alpha\beta)^i \left(1 + \frac{\theta x}{\theta + 1}\right)^i \left(\frac{\theta^2(1+x)}{1+\theta}\right)^{r-i} (1+\beta x)^{-(r\alpha+i)} e^{-r\theta x} \\
&= \sum_{i=0}^{r} \sum_{j=0}^{r-i} \binom{r}{i} \binom{r-i}{j} (-1)^j (\alpha\beta)^i \theta^{r-i} \left(\frac{1}{\theta+1}\right)^j \left(1 + \frac{\theta x}{\theta+1}\right)^{r-j} \\
&\quad \times (1+\beta x)^{-(r\alpha+i)} e^{-r\theta x} \\
&= \sum_{i=0}^{r} \sum_{j=0}^{r-i} \sum_{k=0}^{r-j} \binom{r}{i} \binom{r-i}{j} \binom{r-j}{k} (-1)^j (\alpha\beta)^i \theta^{r-i} \left(\frac{1}{\theta+1}\right)^j \left(\frac{\theta x}{\theta+1}\right)^k \\
&\quad \times (1+\beta x)^{-(r\alpha+i)} e^{-r\theta x}.
\end{aligned}$$

Now, using the Taylor expansions of $(1+\beta x)^{-(r\alpha+i)}$, it follows that

$$\begin{aligned}
f^r(x) = \sum_{i=0}^{r} \sum_{j=0}^{r-i} \sum_{k=0}^{r-j} \sum_{\ell=0}^{\infty} \binom{r}{i} \binom{r-i}{j} \binom{r-j}{k} \binom{r\alpha+i+\ell-1}{\ell} \\
\times (-1)^{j+\ell} (\alpha\beta)^i \theta^{r-i} \left(\frac{1}{\theta+1}\right)^j \left(\frac{\theta x}{\theta+1}\right)^k (\beta x)^\ell e^{-r\theta x}.
\end{aligned}$$

Finally, by making the transformation $y = r\theta x$ and by identifying a gamma density inside the integral sign, Renyi entropy of X is given by

$$\begin{aligned}
I_R(r) = \frac{1}{1-r} \quad \log \quad & \sum_{i=0}^{r} \sum_{j=0}^{r-i} \sum_{k=0}^{r-j} \sum_{\ell=0}^{\infty} \binom{r}{i} \binom{r-i}{j} \binom{r-j}{k} \binom{r\alpha+i+\ell-1}{\ell} \\
& \times \frac{(-1)^{j+\ell} \alpha^i \beta^{i+\ell} \theta^{r-i+k}}{(\theta+1)^{j+k}} \frac{\Gamma(k+\ell+1)}{(r\theta)^{k+\ell+1}}. \qquad (12)
\end{aligned}$$

The values of some important measures of the L-L distribution such as the median, mean, variance, skewness, kurtosis and Renyi entropy ($r = 2$) for various choices of the parameters (α, β, θ) are presented in Table 1.

3.5. **Mean deviations.** The totality of deviations from the mean and median is an index for measuring the amount of scatter in a population. Let X be a random

TABLE 1. The values of some measures of the L-L distribution for different parameter values

α	β	θ	Median	Mean	Variance	Skewness	Kurtosis	$I_R(2)$
0.5	0.5	0.5	1.6632	2.3438	5.1731	1.8618	8.0781	1.5610
0.5	0.5	3	0.2716	0.3835	0.1392	1.8874	8.2468	-0.2493
0.5	1	0.5	1.2725	1.9780	4.5025	2.0830	9.2932	1.3022
0.5	1	3	0.2512	0.3602	0.1276	1.9563	8.6825	-0.3232
1	0.5	0.5	1.1298	1.7308	3.4137	2.1972	10.267	1.1925
1	0.5	3	0.2490	0.3547	0.1219	1.9365	8.5762	-0.3338
1	1	0.5	0.7273	1.2819	2.4594	2.6654	13.755	0.7655
1	1	3	0.2159	0.3155	0.1027	2.0620	9.4225	-0.4701
2	0.5	0.5	0.6514	1.0642	1.5340	2.6978	14.694	0.6536
2	0.5	3	0.2129	0.3074	0.0947	2.0165	9.1454	-0.4868
2	1	0.5	0.3668	0.6666	0.7861	3.5006	23.599	0.0906
2	1	3	0.1673	0.2500	0.0685	2.2203	10.670	-0.7202

variable with cdf (3), pdf (4), mean μ and median M. Then, the mean deviation from the mean and the mean deviation from the median are defined by

$$\delta(\mu) = \int_0^\infty |x-\mu|\, f(x)dx = 2\mu F(\mu) - 2I(\mu),$$

and

$$\delta(M) = \int_0^\infty |x-M|\, f(x)dx = 2MF(M) - M + \mu - 2I(M),$$

respectively, where

$$\begin{aligned} I(a) &= \int_0^a xf(x)dx = aF(a) - \int_0^a F(x)dx \\ &= -a(1+\beta a)^{-\alpha}\left(1+\frac{\theta a}{\theta+1}\right)e^{-\theta a} + \int_0^a (1+\beta x)^{-\alpha}\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}dx, \end{aligned}$$

and using (7) and the Taylor expansion of $e^{-\theta x}$,

$$\begin{aligned} &\int_0^a (1+\beta x)^{-\alpha}\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{\alpha+i-1}{i}\beta^i\theta^j\int_0^a x^{i+j}\left(1+\frac{\theta x}{\theta+1}\right)dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\binom{\alpha+i-1}{i}\beta^i\theta^j\left(\frac{a^{i+j+1}}{i+j+1}+\frac{\theta}{\theta+1}\frac{a^{i+j+2}}{i+j+2}\right). \end{aligned}$$

4. Maximum likelihood estimation

In this section, we apply the maximum likelihood method for estimating the unknown parameters α, β and θ. Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be n observations of a random sample from the L-L(α, β, θ). The likelihood function of this sample using (4) can be written as

$$L(\alpha,\beta,\theta|x) = \prod_{i=1}^{n}\left(\frac{\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i)}{1+\theta}\right)(1+\beta x_i)^{-(\alpha+1)}e^{-\theta x_i}, \quad (13)$$

and hence the log-likelihood function is given by

$$\begin{aligned}\ell(\alpha,\beta,\theta|x) \quad &= \quad \sum_{i=1}^{n}\log\left(\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i)\right)\\ &\quad -n\log(\theta+1)-(\alpha+1)\sum_{i=1}^{n}\log(1+\beta x_i)-\theta\sum_{i=1}^{n}x_i. \qquad (14)\end{aligned}$$

Setting the first partial derivatives of ℓ with respect to α, β and θ equal to zero, the likelihood equations are obtained in the following form

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n}\left(\frac{\beta(1+\theta+\theta x_i)}{\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i)}\right)-\sum_{i=1}^{n}\log(1+\beta x_i)=0, \quad (15)$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n}\left(\frac{\theta^2(1+x_i)x_i+\alpha(1+\theta+\theta x_i)}{\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i)}\right)-(\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i}=0, \quad (16)$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n}\left(\frac{2\theta(1+x_i)(1+\beta x_i)+\alpha\beta(1+x_i)}{\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i)}\right)-\frac{n}{\theta+1}-\sum_{i=1}^{n}x_i=0. \quad (17)$$

To find the MLEs of α, β and θ, say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$, we should solve the above system of non-linear equations (15)–(17) with respect to α, β and θ. These equations cannot be solved analytically and therefore, we have to solve the equations numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the MLEs of the parameters α, β and θ. The subroutines to solve non-linear optimization problem are available in R software. We maximize the log-likelihood function using *nlm*() package.

To obtain the confidence intervals for the parameters α, β and θ, the distributions of the MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$ are needed. Since the MLEs were not obtained in closed forms, then it is not possible to derive their exact distributions. Thus, for interval estimation of the parameters α, β and θ, we derive the approximate confidence intervals of the parameters based on the asymptotic distributions of their MLE which is need to calculate the Fisher information matrix. We obtain the observed information matrix since the expected information matrix is very complicated and will require numerical integration. The elements of the 3×3 observed information matrix I are given in the Appendix. Then the asymptotic distribution of the MLE of the parameters α, β and θ is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\theta} \end{pmatrix} \sim N\left(\begin{pmatrix} \alpha \\ \beta \\ \theta \end{pmatrix}, \mathbf{V} \right), \tag{18}$$

where $\mathbf{V}$ is the variance-covariance matrix and can be approximated by the reciprocal of the observed information matrix, i.e., $\mathbf{V} = I^{-1}$. Since $\mathbf{V}$ involves the parameters α, β and θ, we replace the parameters by the corresponding MLEs in order to obtain an estimate of $\mathbf{V}$, which is denoted by

$$\hat{V} = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} = \begin{pmatrix} \hat{I}_{11} & \hat{I}_{12} & \hat{I}_{13} \\ \hat{I}_{21} & \hat{I}_{22} & I_{23} \\ \hat{I}_{31} & \hat{I}_{32} & \hat{I}_{33} \end{pmatrix}^{-1}, \tag{19}$$

where $\hat{I}_{ij}$ is the (i, j)th element of the observed information matrix I when α, β and θ are replaced by $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$, respectively. Now, using (18), the $100(1-\gamma)\%$ approximate confidence intervals for the parameters α, β and θ are given, respectively, as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\hat{V}_{11}}, \quad \hat{\beta} \pm z_{\gamma/2}\sqrt{\hat{V}_{22}}, \quad \hat{\theta} \pm z_{\gamma/2}\sqrt{\hat{V}_{33}}, \tag{20}$$

where $z_{\gamma/2}$ is the $(1-\gamma/2)$ quantile of the standard normal distribution.

5. A simulation study

In this section, we perform a Monte Carlo simulation study to assess the performance of the point and interval estimates of the parameters presented in Section 4. The performance of the MLEs is compared in terms of their average estimates and mean squared errors (MSEs). We also compare the performance of the confidence intervals in terms of their expected length and coverage probability. To this end, the samples of size $n = 10, 30, 80, 150$ are generated from the L-L distribution with three different values for the parameters (α, β, θ), namely, (0.5, 0.5, 2), (0.5, 1, 2), and (1.5, 0.5, 2) which correspond to the increasing, bathtub-shaped and unimodal hazard rates, respectively. We report the average estimates and MSEs of the parameters in Table 2. The expected length and coverage probability of the confidence intervals for confidence level $(1-\gamma) = 0.95$ are also reported in Table 3. This simulation study is performed using the statistical software R and the number of Monte Carlo replications was 5000.

The results of Table 2 indicate that the MSEs for all the selected parameter values decrease with increasing the sample size, which confirm the consistency properties of the MLEs. Based on the results in Table 3, it is observed that increasing the sample size result in a decrease in the expected lengths of the intervals. Also, the assessment of the coverage probabilities show that the approximate confidence

TABLE 2. Average estimates and MSEs (in parentheses) of the parameters (α, β, θ)

n	$\alpha = 0.5$	$\beta = 0.5$	$\theta = 2$
10	0.1029 (0.4945)	0.0113 (0.2918)	2.3134 (0.6471)
30	0.0719 (0.3537)	0.0134 (0.2635)	2.2286 (0.2787)
80	0.0058 (0.2878)	0.0097 (0.2583)	2.2473 (0.1249)
150	0.0085 (0.2510)	0.0112 (0.2488)	2.2568 (0.0993)
n	$\alpha = 0.5$	$\beta = 1$	$\theta = 2$
10	0.0523 (0.4019)	0.0584 (1.3474)	2.7009 (1.3509)
30	0.0382 (0.3736)	0.0380 (0.9569)	2.4825 (0.3880)
80	0.0257 (0.3080)	0.0249 (0.8908)	2.4428 (0.2663)
150	0.0135 (0.2483)	0.0139 (0.7824)	2.4000 (0.1915)
n	$\alpha = 1.5$	$\beta = 0.5$	$\theta = 2$
10	0.7741 (6.5418)	0.1515 (0.8265)	2.9206 (1.9750)
30	0.2392 (2.3196)	0.0669 (0.2934)	2.7443 (0.7976)
80	0.1907 (1.9680)	0.0665 (0.2554)	2.6547 (0.5210)
150	0.2264 (1.8560)	0.0845 (0.2158)	2.6232 (0.4418)

TABLE 3. Expected lengths and coverage probabilities (in parentheses) of the parameters (α, β, θ)

n	$\alpha = 0.5$	$\beta = 0.5$	$\theta = 2$
10	2.3847 (0.874)	6.7524 (0.951)	3.4349 (0.848)
30	1.6174 (0.849)	6.3374 (0.964)	2.8423 (0.890)
80	1.3127 (0.870)	5.5297 (0.956)	2.3102 (0.862)
150	0.9047 (0.866)	4.5623 (0.978)	1.7505 (0.872)
n	$\alpha = 0.5$	$\beta = 1$	$\theta = 2$
10	2.5894 (0.824)	7.7381 (0.894)	5.6279 (0.944)
30	1.9630 (0.806)	6.3784 (0.892)	4.1238 (0.948)
80	1.1327 (0.778)	3.0566 (0.898)	2.4911 (0.934)
150	0.7995 (0.752)	1.9818 (0.887)	1.5862 (0.942)
n	$\alpha = 1.5$	$\beta = 0.5$	$\theta = 2$
10	3.6819 (0.786)	6.8136 (0.978)	5.7652 (0.928)
30	2.2182 (0.768)	5.4194 (0.971)	3.2770 (0.904)
80	1.2703 (0.757)	3.5746 (0.968)	1.8928 (0.913)
150	0.8794 (0.734)	2.6734 (0.973)	1.2594 (0.921)

intervals for most of the parameters provides the coverage probabilities smaller than their nominal level.

6. Applications

In this section, two applications to real data sets with bathtub-shaped and unimodal hazard rates are considered in order to illustrate the potentiality of the L-L distribution. We use goodness-of-fit tests including the Anderson-Darling (A-D) and Kolmogorov-Smirnov (K-S) tests to compare the L-L distribution with its sub-models, namely the Lindley and Lomax distributions, and the following three-parameter distributions:

(1) Exponential Lomax (E-Lo) distribution with the pdf

$$f(x)=\frac{\alpha\lambda}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}},\ x>0,\ \alpha,\beta,\lambda>0,$$

which was introduced by El-Bassiouny et al. [10].

(2) Generalized Lomax (G-Lo) distribution with the pdf

$$f(x)=\alpha\beta\gamma x^{\gamma-1}(1+\beta x^{\gamma})^{-(\alpha+1)},\ x>0,\ \alpha,\beta,\gamma>0,$$

which was introduced by Maurya et al. [18].

(3) Lindley-Lomax (Li-Lo) distribution with the pdf

$$f(x)=\frac{\alpha\theta^2}{\sigma(\theta+1)}\left[1+\alpha\log\left(1+\frac{x}{\sigma}\right)\right]\left(1+\frac{x}{\sigma}\right)^{-(\alpha\theta+1)},\ x>0,\ \alpha,\theta,\sigma>0,$$

which was introduced by Cakmakyapan and Ozel [7].

(4) Lindley Weibull (Li-W) distribution with the pdf

$$f(x)=\frac{\beta\theta^2}{(\theta+1)}\left(\alpha^{\beta}x^{\beta-1}+\alpha^{2\beta}x^{2\beta-1}\right)e^{-\theta(\alpha x)^{\beta}},\ x>0,\ \alpha,\beta,\theta>0,$$

which was introduced by Cordeiro et al. [8].

(5) Extended Generalized Lindley (EG-Li) distribution with the pdf

$$f(x)=\frac{\lambda^2(1+x)\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{\alpha-1}\left\{\alpha+(\gamma-\alpha)\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{\gamma}\right\}}{(1+\lambda)e^{\lambda x}\left\{\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{\alpha}+1-\left[1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]^{\gamma}\right\}^2},$$
$$x>0,\ \alpha,\gamma,\lambda>0,$$

which was introduced by Ranjbar et al. [22].

6.1. **Bathtub-shaped hazard rate lifetime data.** The first application consists the times between failures (in hours) of load-haul-dump (LHD) machine used to pick up rock or waste. The data has been obtained from Kumar et al. [13] and are presented in Table 4. The TTT-plot presented by Kumar et al. [13] for this data set exhibits a bathtub-shaped hazard rate function.

Table 4. Times between failures of LHD machine (LHD data)

110	13	72	4	45	56	19	27	36	90	19	7	2	118	44	8	277
4	8	10	79	103	6	18	147	96	22	3	24	3	9	99	82	121
54	79	99	18	5	21	1	3	5	1	59	22	17	35	35	29	

Table 5 present the MLEs of the model parameters as well as values of A-D statistics, K-S statistics and their corresponding p-values for all models. These results show that the L-L distribution has the lowest A-D and K-S values and, has the biggest p-value of K-S test statistic among all the fitted models. Hence, L-L distribution yields a better fit than Lindley, Lomax, E-Lo, G-Lo, Li-Lo, Li-W and EG-Li distributions under these criteria. Furthermore, Figure 3 show the empirical cdf versus fitted cdfs and the histogram of the data versus fitted pdfs for the LHD data. This figure confirms the goodness-of-fit of L-L distribution with respect to its sub-models and the other competitor distributions.

Substituting the MLE of the unknown parameters in (19), we obtain estimation of the variance-covariance matrix $\hat{V}$ as

$$\hat{V} = \begin{pmatrix} 0.1227 & -0.0316 & -0.0017 \\ -0.0316 & 0.0091 & 0.0004 \\ -0.0017 & 0.0004 & 4.4 \times 10^{-5} \end{pmatrix}.$$

Therefore, the approximate 95% confidence intervals of the parameters α, β and θ using (20) are given as (-0.2207, 1.1525), (-0.0966, 0.2789) and (0.0077, 0.0337), respectively.

6.2. **Unimodal hazard rate lifetime data.** As second application, we consider a clinical trial data set involving survival times (in days) of 45 head and neck cancer patients in arm B which was considered earlier by Efron [9]. The data are presented in Table 6. Mudholkar et al. [20] discussed that this data set indicates a unimodal hazard rate function.

Table 5. MLEs of parameters, A-D statistic, K-S statistic and corresponding p-value

Distribution	Estimates			A-D statistic	K-S statistic	p-value
Lindley(θ)	0.0432	-	-	8.1689	0.2383	0.0068
Lomax(α, β)	0.0788	7020	-	17.006	0.5027	0.0000
L-L(α, β, θ)	0.4658	0.0911	0.0207	**0.2864**	**0.0699**	**0.9672**
E-Lo(α, β, λ)	0.8621	0.0254	0.0016	0.4053	0.0823	0.8865
G-Lo(α, β, γ)	77.547	0.0005	0.8717	0.4166	0.0830	0.8806
Li-Lo(α, θ, σ)	0.3819	15.138	8203.02	0.6425	0.0928	0.7816
Li-W(α, β, θ)	0.2223	0.6868	0.3984	0.4360	0.0845	0.8670
EG-Li(α, γ, λ)	0.4311	0.2170	0.0198	0.4810	0.0874	0.8386

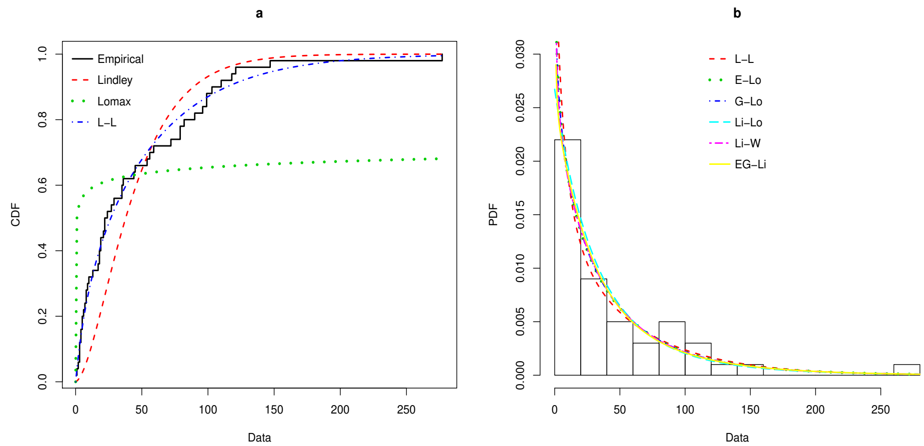

FIGURE 3. (a) The fitted cdfs and empirical cdf. (b) The fitted pdfs and histogram of the data for LHD data.

TABLE 6. Survival times of 45 head and neck cancer patients (Cancer data)

37	84	92	94	110	112	119	127	130	133	140	146
155	159	169	173	179	194	195	209	249	281	319	339
432	469	519	528	547	613	633	725	759	817	1092	1245
1331	1557	1642	1771	1776	1897	2023	2146	2297			

To compare the goodness of fit of L-L distribution with other distributions, the MLEs of the model parameters as well as values of A-D statistics, K-S statistics and their corresponding p-values for all models are calculated and the results are reported in Table 7. It is observed that the L-L distribution provides the lowest A-D and K-S values and, has the biggest p-value of K-S test statistic in comparison with its sub-models and the other competitor distributions and therefore, it could be chosen as the best model under these criteria. Figure 4 show the empirical cdf versus fitted cdfs and the histogram of the data versus fitted pdfs for the cancer data. This figure confirms the goodness-of-fit of L-L distribution with respect to all the fitted distributions.

Substituting the MLE of the unknown parameters in (19), we obtain estimation of the variance-covariance matrix $\hat{V}$ as

$$\hat{V} = \begin{pmatrix} 0.0937 & 8.6 \times 10^{-5} & -3.1 \times 10^{-5} \\ 8.6 \times 10^{-5} & 2.1 \times 10^{-7} & -4.9 \times 10^{-8} \\ -3.1 \times 10^{-5} & -4.9 \times 10^{-8} & 1.8 \times 10^{-7} \end{pmatrix}.$$

TABLE 7. MLEs of parameters, A-D statistic, K-S statistic and corresponding p-value

Distribution	Estimates			A-D statistic	K-S statistic	p-value
Lindley(θ)	0.0031	-	-	7.5923	0.3037	0.0003
Lomax(α, β)	0.0626	7266.3	-	16.594	0.5436	0.0000
L-L(α, β, θ)	1.5395	0.0013	0.0010	**1.0182**	**0.1258**	**0.4383**
E-Lo(α, β, λ)	0.9813	0.8707	0.0015	1.3949	0.1579	0.1904
G-Lo(α, β, γ)	1.1442	0.0002	1.4413	1.0320	0.1341	0.3608
Li-Lo(α, θ, σ)	146.12	0.0151	321.13	1.0544	0.1334	0.3663
Li-W(α, β, θ)	0.0001	0.9759	10.344	1.3961	0.1582	0.1887
EG-Li(α, γ, λ)	0.5304	0.1198	0.0012	1.2175	0.1410	0.3027

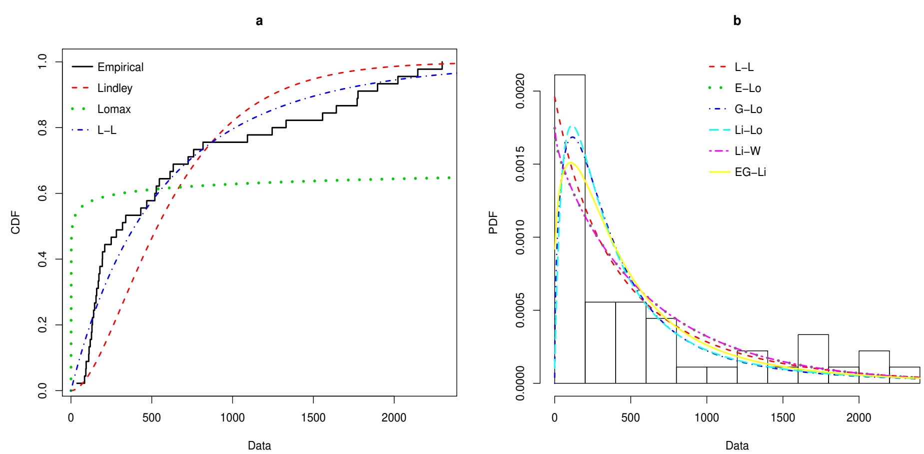

FIGURE 4. (a) The fitted cdfs and empirical cdf. (b) The fitted pdfs and histogram of the data for cancer data.

Therefore, the approximate 95% confidence intervals of the parameters α, β and θ using (20) are given as (0.9395, 2.1396), (0.0004, 0.0022) and (0.0002, 0.0019), respectively.

7. CONCLUSIONS

In this paper, we have proposed a new three-parameter lifetime distribution which is referred to as the the Lomax-Lindley distribution. This distribution is obtained by combining the Lomax and Lindley distributions in a serial system. The new distribution is quite flexible to model lifetime data since it provides a simple form for hazard rate function which can cover increasing, decreasing, bathtub-shaped and unimodal hazard rates. We have studied some important mathematical properties of new distribution, which consist of quantiles, moments, order statistics, Renyi entropy and mean deviations. The maximum likelihood estimation and asymptotic confidence intervals for the model parameters are also discussed and a simulation study is conducted to evaluate the performances of the point and interval estimates of the parameters. Two real data sets having bathtub-shaped and unimodal hazard rate functions are analyzed to show the superiority of the new distribution. It observed that the present distribution can provide a better fit than other competitor distributions for both lifetime data.

Appendix A

In Section 4, we used the observed information matrix I to construct the asymptotic confidence intervals for the parameters of the L-L distribution. The elements of this matrix are given by

$$\begin{aligned}
I_{11} &= \sum_{i=1}^{n}\left(\frac{\beta(1+\theta+\theta x_i)}{g_i}\right)^2, \\
I_{12} &= -\sum_{i=1}^{n}\frac{\theta^2(1+x_i)(1+\theta+\theta x_i)}{g_i^2}+\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i}, \\
I_{22} &= \sum_{i=1}^{n}\left(\frac{\theta^2(1+x_i)x_i+\alpha(1+\theta+\theta x_i)}{g_i}\right)^2-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)^2, \\
I_{13} &= \sum_{i=1}^{n}\frac{\beta\theta(1+x_i)(1+\beta x_i)(2+\theta+\theta x_i)}{g_i^2}, \\
I_{23} &= -\sum_{i=1}^{n}\frac{\alpha\theta^2(1+x_i)^2-2\alpha\theta(1+x_i)(1+\theta+\theta x_i)}{g_i^2}, \\
I_{33} &= -\sum_{i=1}^{n}\frac{2(1+x_i)(1+\beta x_i)g_i-[2\theta(1+x_i)(1+\beta x_i)+\alpha\beta(1+x_i)]^2}{g_i^2}-\frac{n}{(\theta+1)^2},
\end{aligned}$$

where

$$g_i(\alpha,\beta,\theta)=\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i).$$

Appendix B

Some programs developed in R for the L-L distribution fitting and estimation of its parameters are given as follows:

```
library(MASS)
# LHD data
x=c(110, 13, 72, 4, 45, 56, 19, 27, 36, 90, 19, 7, 2, 118, 44, 8, 277, 4, 8, 10, 79,
103, 6, 18, 147, 96, 22, 3, 24, 3, 9, 99, 82, 121, 54, 79, 99, 18, 5, 21, 1, 3, 5, 1, 59,
22, 17, 35, 35, 29)
n=50
```

```
logL=function(par){
al=par[1]
be=par[2]
te=par[3]
-sum(log((1+x)*(1+be*x)*te^ 2+al*be*(1+te+te*x))-log(te+1)-(al+1)
*log(1+be*x)-te*x)}
est=nlm(logL,c(1,.01,.01))
alpha=est $estimate[1]
beta=est$estimate[2]
theta=est$estimate[3]
print(c(alpha,beta,theta))

# Cumulative distribution function of the L-L distribution
F=function(z,al,be,te){
P=1-(1+te*z/(te+1))*exp(-te*z)/(1+be*z)^ al
return(P)
}
print(ks.test(x,"F",alpha,beta,theta))

# Elements of the observed information matrix
g=(1+x)*(1+beta*x)*theta^ 2+alpha*beta*(1+theta+theta*x)
I=matrix(c(rep(0,9)),ncol=3)
I[1,1]=sum((beta*(1+theta+theta*x)/g)^ 2)
I[1,2]=I[2,1]=- sum((1+x)*(1+theta+theta*x)*theta^ 2/g^ 2 -x/(1+beta*x))
I[1,3]=I[3,1]=sum(beta*theta*(1+x)*(1+beta*x)*(2+theta+theta*x)/g^ 2)
I[2,2]=sum((((1+x)*x*theta^ 2+alpha*(1+theta+theta*x))/g)^ 2-(alpha+1)
*(x/(1+beta*x))^ 2)
I[2,3]=I[3,2]=- sum((alpha*theta^ 2*(1+x)^ 2-2*alpha*theta*(1+x)
*(1+theta+theta*x))/g^ 2)
I[3,3]=- sum((2*(1+x)*(1+beta*x)*g-(2*theta*(1+x)*(1+beta*x)
+alpha*beta*(1+x))^ 2)/g^ 2)-n/(theta+1)^ 2
V=ginv(I)
print(V)
al=alpha-qnorm(0.975)*sqrt(V[1,1])
au=alpha+qnorm(0.975)*sqrt(V[1,1])
print(c(al,au))
bl=beta-qnorm(0.975)*sqrt(V[2,2])
bu=beta+qnorm(0.975)*sqrt(V[2,2])
print(c(bl,bu))
tl=theta-qnorm(0.975)*sqrt(V[3,3])
tu=theta+qnorm(0.975)*sqrt(V[3,3])
print(c(tl,tu))
```

Declaration of Competing Interests The author declares that he has no conflict of interest.

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A GRAPH ASSOCIATED TO A COMMUTATIVE SEMIRING

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Abstract. Let R be a commutative semiring with nonzero identity and H be an arbitrary multiplicatively closed subset R. The generalized identity-summand graph of R is the (simple) graph $G_H(R)$ with all elements of R as the vertices, and two distinct vertices x and y are adjacent if and only if $x+y \in H$. In this paper, we study some basic properties of $G_H(R)$. Moreover, we characterize the planarity, chromatic number, clique number and independence number of $G_H(R)$.

1. Introduction

Semirings provide useful instruments to solve problems in many areas of information sciences and applied mathematics such as optimization theory, graph theory, automata theory, coding theory and analysis of computer programs, because the structure of semiring provides a useful algebraic technique for investigating and modelling the key factors in these problems.

Over the last few years, the study of algebraic structures by graphs has been done and several interesting results have been obtained (see [1,2,4,5,10,11,13–17]). For instance, the total graph of a commutative ring R is a simple graph whose vertex set is R, and two distinct vertices a and b are adjacent if $a+b$ is a zero divisor of R (the set of all zero-divisor elements of R is denoted by $Z(R)$)(see [3,18]). Recently, in [9], the authors considered the *identity summand graph* of a commutative semiring R denoted by $\Gamma(R)$, as the simple graph with the set of vertices $\{x \in R \setminus \{1\} : x + y = 1 \text{ for some } y \in R \setminus \{1\}\}$, where two distinct vertices x and y are adjacent if and only if $x + y = 1$. Moreover, the identity-summand graph with respect to co-ideal I denoted by $\Gamma_I(R)$ is a graph with vertices as elements

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$S_I(R) = \{x \in R \setminus I : \ x + y \in I \text{ for some } y \in R \setminus I\}$, where two distinct vertices x and y are adjacent if and only if $x + y \in I$ [12].

Let H be a nonempty subset of a semiring R with nonzero identity. H is said to be *multiplicatively closed* if $xy \in H$, for all x and y of H. Also, a subset H of R is called saturated if $xy \in H$ if and only if $x, y \in H$. For a multiplicatively closed subset H of R, we define the *generalized identity-summand graph* of R, denoted by $G_H(R)$, as a simple graph, with vertex set R and two distinct vertices x and y being adjacent if and only if $x + y \in H$. Since the subsets $Z(R)$ of R is multiplicatively closed, $G_H(R)$ is a natural generalization of the total graph of R. Hence the total graph is a well-known graph of this type. Moreover, if H is a co-ideal of R, then $\Gamma_H(R)$ is a subgraph of $G_H(R)$.

We summarize the contents of this article as follows. In Section 2, we investigate the basic properties of generalized identity-summand graph, for instance , the degree of the vertices and connectivity. Also, We consider the possible integers for the diameter and the girth of the graph $G_H(R)$. We investigate the case that H is a saturated multiplicatively closed subset of R. We prove a subset H of R is saturated if and only if $R \setminus H$ is a union of some prime ideals. Therefore $R \setminus H = \bigcup_{j \in J} M_j$ for some prime ideals M_j with $j \in J$. Set $I := \bigcap_{j \in J} M_j$. If I is a Q-ideal of R, then set $\widetilde{H} := \{q + I \ : \ h \in q + I \text{ for some } h \in H\}$. We show that the newly constructed subset $\widetilde{H}$ is a saturated multiplicatively closed subset of R/I and study the relationship between the combinatorial properties of the graphs $G_H(R)$ and $G_{\widetilde{H}}(R/I)$. Further, we consider the graph $G_H(R)$, when it is complete, complete r-partite, complete 2-partite and regular graph. It is proved that $G_H(R)$ is complete 2-partite if and only if it is star graph. In Section 3, we consider and study the planar property, clique number, chromatic number and independence number of $G_H(R)$. We will show that $\omega(G_H(R)) = \chi(G_H(R))$ and completely determine the chromatic number, clique number and independence number of $G_H(R)$.

Now, we are going to recall some notations and definitions of graph theory from [6], which are needed in this paper. Let G be a graph. By $E(G)$ and $V(G)$ we will denote the set of all *edges* and *vertices*, respectively. A graph G is called *connected* provided that there exists a path between any two distinct vertices. Otherwise, G is said to be *disconnected*. The *distance* between two distinct vertices a and b is the length of the shortest path connecting them, denoted by $d(a, b)$, (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The *diameter* of a graph G, denoted by $diam(G)$, is equal to $sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of a graph G denoted $gr(G)$, is the length of a shortest cycle in G, provided that G contains a cycle; otherwise $gr(G) = \infty$. For a given vertex $x \in V(G)$, the *neighborhood set* of x is the set $N(x) = \{a \in V(G) : a \text{ is adjacent to } x\}$. A graph G is called *complete*, if every pair of distinct vertices is connected by a unique edge. The notation K_n will denote the complete graph on n vertices. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. A complete r-partite graph with part sizes $m_1, ..., m_r$ is denoted by $K_{m_1,m_2,...,m_r}$. We will sometimes call $K_{1,n}$ a *star graph*. Let G be a graph. A *coloring* of a graph G is an assignment one color to each vertex of G such that distinct colors are assigned to adjacent vertices. If one used n colors for the coloring of G, then it is referred to as an *n-coloring*. If G has n-coloring, then G is called *n-colorable*. The minimum positive integer n for which a graph G is n-colorable is called the *chromatic number* of G, and is denoted by $\chi(G)$. A graph G is said to be *totally disconnected*, if no two vertices of G are adjacent. Every complete subgraph of a graph G is called a *clique* of G, and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. In a graph $G = (V, E)$, a subset S of V is said to be an *independent set* provided that the subgraph induced by S is totally disconnected. The *independence number* is the maximum size of an independent set in G and denoted by $\alpha(G)$. A graph G is called a *null graph* if whose vertex-set is empty and a graph whose edge-set is empty is said to be an *empty graph*. Let G be a graph with edge set E. Also, suppose that there exists a family of edge-disjoint subgraphs $\{G_i\}_{i\in I}$ of G. Then we put $G = \oplus_{i\in I} G_i$. Furthermore, in the case that $G_i \cong H$ for every $i \in I$, we set $G = \oplus_{|I|} H$.

An algebraic system $(R, +, .)$ is called a commutative *semiring* provided that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exist $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. Throughout this paper, all semirings considered will be assumed to be commutative semirings with a non-zero identity. Let R be a semiring. A non-empty subset I of R is called *co-ideal* (resp. *ideal*), if it is closed under multiplication (rep. under addition) and satisfies the condition $r + a \in I$ (resp. $ra \in I$) for all $a \in I$ and $r \in R$ (so $0 \in I$ (resp. $1 \in I$) if and only if $I = R$). A co-ideal I of a semiring R is said to be a *strong co-ideal*, if $1 \in I$. A co-ideal (resp. ideal) I of R is called *k-ideal* or *subtractive*, if $ab \in I$ and $b \in I$ imply that $a \in I$ (resp. $a + b \in I$ and $a \in I$ imply that $b \in I$), for each $a, b \in R$. A proper ideal P of R is called *prime* if $xy \in P$, then $x \in P$ or $y \in P$. A proper co-ideal M of R is said to be *prime*, if $x + y \in M$, then $x \in M$ or $y \in M$ [8]. A semiring R is called *I-semiring*, if $r + 1 = 1$ for all $r \in R$. A semiring R is called *idempotent* if $x^2 = x$ for all $x \in R$. Let I be a proper ideal of R. Then I is said to be *maximal* if R is the only ideal having I. The notation $Jac(R)$ will denote the *jacobson radical* of R which is the intersection of all maximal ideals of R. Let I be an ideal of a semiring R. Then I is said to be a *partitioning ideal* (= Q-ideal) provided that there exists a subset Q of R such that

(1) $R = \cup\{q + I \ : \ q \in Q\}$,

(2) *If* $q_1, q_2 \in Q$, *then* $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ *if and only if* $q_1 = q_2$.

If I is a Q-ideal of a semiring R, the we set

$$R/I := \{q + I \ : \ q \in Q\}.$$

Thus R/I is a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$.

$(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$. Semiring R/I is said to be the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of R/I. Clearly, if R is an idempotent I-semiring, then so is R/I ([7]). Dual notion of Q-ideal (Q-co-ideal) was defined in [8].

2. Basic structure $G_H(R)$

Throughout this paper, R is a I-semiring and H is a multiplicatively closed subset of R.

Lemma 1. *The following statements hold:*

(i) If $0 \in H$, then $N(0) = H \setminus \{0\}$ and if $0 \notin H$, then $N(0) = H$.

(ii) If $1 \in H$, then $N(1) = R \setminus \{1\}$ and if $1 \notin H$, then $N(1) = \emptyset$.

Proof. (i) Since $0 + x = x \in H$ for all $x \in H$ and $0 \in H$, $N(0) = H \setminus \{0\}$. Otherwise, $N(0) = H$. This proves (i). Since $1 + x = 1$ for all $x \in R$, the statement (ii) holds. □

Theorem 1. *$G_H(R)$ is connected if and only if $1 \in H$. Moreover, if $G_H(R)$ is connected, then $diam(G_H(R)) \leq 2$.*

Proof. If $1 \in H$, then $deg(1) = |R| - 1$ by Lemma 1 (ii); so $G_H(R)$ is connected. Conversely, if $G_H(R)$ is connected, then $deg(1) \neq 0$ which implies that $1 \in H$ by Lemma 1 (ii). Finally, let x and y be distinct elements of R. If $x + y \in H$, then $x - y$ is a path in $G_H(R)$. So we may assume that $x + y \notin H$. Now the assertion follows the fact that $x - 1 - y$ is a path in $G_H(R)$. □

Proposition 1. *The following statements hold:*

(1) $G_H(R)$ is complete if and only if $R = H$ or $H = R \setminus \{0\}$.

(2) $G_H(R)$ is regular if and only if it is either complete or totally disconnected.

Proof. (1) Let $G_H(R)$ be complete. Thus 0 is connected to every element of $R \setminus \{0\}$, and so $0 + x \in H$ for every $x \in R \setminus \{0\}$. So $R \setminus \{0\} \subseteq H$. Therefore $R = H$ or $H = R \setminus \{0\}$. The converse is clear. Note that if $x + y = 0$, then $x = x + x + y = x(1 + 1) + y = x + y = 0$, because R is an I-semiring.

(2) Assume that $G_H(R)$ is regular and that is not totally disconnected. By Theorem 1, $1 \in H$; so $deg(1) = |R| - 1$. Then $G_H(R)$ is regular gives $deg(y) = |R| - 1$ for all $y \in R$; hence $G_H(R)$ is complete. The other implication is clear. □

In the following, the notation $max(R)$ denotes the set of all maximal ideals of R.

Theorem 2. *If $1 \in H$, then $gr(G_H(R)) \in \{3, \infty\}$.*

Proof. Assume that $|max(R)| \geq 2$ and let $M_1, M_2 \in max(R)$. Since $x + y = 1$, for some $x \in M_1$ and $y \in M_2$, we have $1 - x - y - 1$ is a cycle in $G_H(R)$; hence $gr(G_H(R)) = 3$. So we may assume that $|max(R)| = 1$. If $H = \{1\}$, then the graph $G_H(R)$ is a star graph which implies that $gr(G_H(R)) = \infty$. Now suppose that $|H| = 2$. If $H = \{0, 1\}$, then the graph $G_H(R)$ is a star graph which implies that $gr(G_H(R)) = \infty$, because $x + y = 0$ implies $x = 0$ and $y = 0$ for each $x, y \in R$. Otherwise, $H = \{1, r\}$, where $r \neq 0$. Then the cycle $1 - r - 0 - 1$ is the shortest cycle in the graph $G_H(R)$. So $gr(G_H(R)) = 3$. If $|H| \geq 3$, then there is an element $r \in H$ such that $r \neq 0, 1$. Now the cycle $1 - r - 0 - 1$ is the shortest cycle in the graph $G_H(R)$ which implies that $gr(G_H(R)) = 3$.

□

The remaining of this section, we assume that R is an idempotent I-semiring, H is a saturated subset of R and $H \neq R$. Note that if $0 \in H$, then $H = R$, and so, by Proposition 1, the graph $G_H(R)$ is complete.

Proposition 2. *Assume that $|R| \geq 3$. If $|H| \geq 2$, then every vertex of the graph $G_H(R)$ lies in a cycle of length 3, and so $gr(G_H(R)) = 3$.*

Proof. By assumption, there is an element $x \in H$ with $x \neq 1$. If $y \neq 1, x$ is an arbitrary element in R, then $x(x + y) = x + xy = x \in H$. Therefore $x + y \in H$ and we have the cycle $1 - y - x - 1$, as required. □

Theorem 3. *Let $|H| = 1$. Then the following hold:*

(i) $deg(a) = 1$ for all $a \in Jac(R)$.

(ii) If $|max(R)| \geq 2$, then every vertex in graph $G_H(R) \setminus Jac(R)$ lies in a cycle of length 3.

Proof. (i) Since $|H| = 1$, we have $H = \{1\}$. Let $x \in Jac(R)$. Since $1 + y = 1 \in H$, 1 is adjacent to every vertex y in $G_H(R)$ which implies that $deg(x) \geq 1$. Suppose the result is false. Let $deg(x) \geq 2$. So there is $1 \neq y \in R$ such that x and y are adjacent (note that $1 + x = 1 \in H = \{1\}$), so $x + y = 1$. One can find a maximal ideal M of R such that $y \in M$. Hence $1 = x + y \in M$, which is impossible. So $deg(a) = 1$ for all $a \in Jac(R)$.

(ii) Assume that x is an arbitrary vertex in $G_H(R) \setminus Jac(R)$. Thus $x \notin M$, for some maximal ideal M of R. Thus $xR + M = R$, and so there exist $r \in R, m \in M$ such that $xr + m = 1$. Hence $x + m = x + xr + m = 1 + x = 1$. If $m \in Jac(R)$, then $x + m \in M'$, for some maximal ideal M' of R (we can find the maximal ideal M' such that $x \in M'$), which is a contradiction. Hence, we can consider the cycle $x - m - 1 - x$ in $G_H(R) \setminus Jac(R)$. □

Lemma 2. *The following statements hold:*
(1) If I is an ideal of R and $a+b \in I$, for some $a, b \in R$, then $a, b \in I$.
(2) Every ideal of R is k-ideal.

Proof. (1) Let I be an ideal of R and $a+b \in I$, for some $a, b \in R$. Then

$$a = a(1+b) = a + ab = a(a+b) \in I.$$

Similarly, $b \in I$.
(2) It is clear from (1). □

Proposition 3. *(1) The following statements are equivalent on a subset H of R:*
(i) H is saturated.
(ii) $R \setminus H = \bigcup_{i \in \Lambda} M_i$, for some prime ideals M_i of R.
(2) H is a saturated multiplicatively closed subset of R if and only if H is a co-ideal of R. Moreover, $H = \bigcap_{j \in J} P_j$, where $\{P_j\}_{j \in J}$ is the set of all prime co-ideals of R containing H.
(3) P is a prime co-ideal of R if and only if $R \setminus P$ is a prime ideal of R.
(4) Let H be a subset of R. Then P is a minimal prime co-ideal of R containing H if and only if $R \setminus P$ is an ideal of R which is maximal with disjoint from H.

Proof. (1) $(i) \Rightarrow (ii)$ Let $x \in R \setminus H$. Set $\sum = \{I :\ I$ is an ideal of R, $I \cap H = \emptyset$ and$x \in I\}$. Since $Rx \in \sum$, $\sum \neq \emptyset$. By Zorn's Lemma, $\sum$ has a maximal element P. It can be easily seen that P is a prime ideal. Therefore every $x \notin H$ has been inserted in a prime ideal disjoint from H. This proves (2).
$(ii) \Rightarrow (i)$ It is clear.
(2) Let H be saturated. Then $R \setminus H = \bigcup_{i \in \Lambda} M_i$, for some prime ideals M_i of R, by (1). Let $a \in H$ and $r \in R$. If $r + a \notin H$, then $r + a \in M_i$, for some $i \in \Lambda$. Therefore by Lemma 2(1), $a \in M_i$, a contradiction. Therefore H is a co-ideal of R. The converse is clear from [12, Proposition 2.1(1)]. Therefore $H = \bigcap_{j \in J} P_j$, where $\{P_j\}_{j \in J}$ is the set of all prime strong co-ideals of R containing S, by [12, Theorem 4.6].
(3) Assume that P is a prime co-ideal of R. Let $x \in R - P$ and $r \in R$. If $rx \in P$, then $r, x \in P$, by [12, Proposition 2.1(1)], a contradiction. Thus $rx \in R - P$. Let $x, y \in R - P$. If $x + y \in P$, then either $x \in P$ or $y \in P$, which is impossible. Therefore $x + y \in R - P$. This implies that $R - P$ is an ideal of R. It is clear that $R - P$ is a prime ideal. Conversely, let T be a prime ideal of R. Let $x \in R - T$ and $r \in R$. If $r + x \in T$, then $r, x \in T$, by Lemma 2. Thus $r + x \in R - T$. Let $x, y \in R - T$. If $xy \in T$, then either $x \in T$ or $y \in T$. Therefore $xy \in R - T$. This implies that $R - T$ is a co-ideal of R. Also, It is clear that $R - T$ is a prime co-ideal. Therefore, if $R - P$ is a prime ideal of R, then P is a prime co-ideal of R.
(4) It is straightforward. □

Throughout the paper, by $min(H)$ and $max(H)$, we show the set of minimal prime co-ideals of R containing H and the set of ideals of R which are maximal with disjoint from H, respectively.

Proposition 4. *If $G_H(R)$ is complete r-partite, then $r = |H| + 1$.*

Proof. Assume that $G_H(R)$ is complete r-partite with parts V_i $(1 \leq i \leq r)$. Since H is a clique in $G_H(R)$, every element of H is in a part V_i, where $|V_i| = 1$. Let V_1 and V_2 be two parts of $G_H(R)$ and $a, b \in R \setminus H$ such that $a \in V_1$ and $b \in V_2$. As 0 is not adjacent to a, $0 \in V_1$. Therefore 0 and b are adjacent, which is a contradiction. Therefore every element of $R \setminus H$ is in one part and $R \setminus H$ is an ideal. Thus $r = |H| + 1$. □

Theorem 4. *The following statements are equivalent:*

(1) $gr(G_H(R)) = \infty$.

(2) $G_H(R)$ is a star graph.

(3) $H = \{1\}$ and $max(H) = \{R - \{1\}\}$.

(4) $G_H(R)$ is a complete bipartite.

Proof. (1) $\Rightarrow$ (3) Assume that $|H| \geq 2$ and $a, b \in H$. Then $a - b - 0 - a$ is a cycle in $G_H(R)$, a contradiction. Hence $H = \{1\}$. Let $|max(H)| \geq 2$ and $M_1, M_2 \in max(H)$. As $H = \{1\}$, every ideal which is maximal with respect to disjoint from H, is a maximal ideal of R. Therefore $M_1 + M_2 = R$ and $a + b = 1$ for some $a \in M_1, b \in M_2$. Therefore $a - b - 1 - a$ is a cycle in $G_H(R)$, which is a contradiction. Therefore $H = \{1\}$ and $max(H) = \{R - \{1\}\}$.

The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) By Proposition 4, $r = 2$. It is clear that $H = \{1\}$ and $R - \{1\}$ is a maximal ideal of R. Therefore $gr(G_H(R)) = \infty$. □

In the rest of this section, we will assume that $R \setminus H = \bigcup_{i \in \Lambda} M_i$ for some prime ideals M_i of R and $I := \bigcap_{i \in \Lambda} M_i$. Let I be a Q-ideal and $\widetilde{H} := \{q + I \ : \ h \in q + I \text{ for some } h \in H\}$.

Lemma 3. *Let I be a Q-ideal of R. Then $\widetilde{H}$ is a saturated multiplicatively closed subset of R/I.*

Proof. Let $q_1 + I$ and $q_2 + I$ be two elements of $\widetilde{H}$, where $h_1 \in q_1 + I$ and $h_2 \in q_2 + I$, for some $h_1, h_2 \in H$. If $(q_1 + I) \odot (q_2 + I) = q_3 + I$, where $q_1q_2 + I \subseteq q_3 + I$ and $q_3 \in Q$, then we have $h_1h_2 \in q_1q_2 + I \subseteq q_3 + I$. Thus $q_3 + I \in \widetilde{H}$. We show $\widetilde{H}$ is saturated. Let $(q_1 + I) \odot (q_2 + I) = q_3 + I \in \widetilde{H}$, where $q_1q_2 + I \subseteq q_3 + I$ and $q_3 \in Q$. Since $q_3 + I \in \widetilde{H}$, there exists $h \in H$ such that $h \in q_3 + I$. Thus $h = q_3 + i$ for some $i \in I$. As $h \in H$ and $i \in I$, $q_3 \in H$. Let $q_1q_2 = q_3 + j$ for some $j \in I$. Then $q_1q_2 \in H$, because H is a co-ideal, by Lemma 2. Therefore $q_1, q_2 \in H$ and so $q_1 + I, q_2 + I \in \widetilde{H}$. □

Lemma 4. *Let I be a Q-ideal of R. Then the following statements hold:*

(1) Let p_1 and p_2 be two elements of R with $p_1 \in q_1 + I$ and $p_2 \in q_2 + I$, where $q_1 + I \neq q_2 + I$. Then the following statements are equivalent:

(i) p_1 is adjacent to p_2 in $G_H(R)$.

(ii) $q_1 + I$ *is adjacent to* $q_2 + I$ *in* $G_{\widetilde{H}}(R/I)$.

(iii) each element of $q_1 + I$ *is adjacent to* $q_2 + I$.

(iv) there exists an element of $q_1 + I$ *which is adjacent to an element of* $q_2 + I$.

(2) If $q + I \in \widetilde{H}$, *then* $q \in Q \cap H$ *and* $q + I$ *is a clique in* $G_H(R)$.

(3) If $q + I \notin \widetilde{H}$, *then* $q + I$ *is an independent set in* $G_H(R)$.

Proof. (1) $(i) \Rightarrow (ii)$ By (i), $p_1 + p_2 \in H$. Let $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$. Therefore $p_1 + p_2 \in q_1 + q_2 + I \subseteq q_3 + I$ gives $q_3 + I \in \widetilde{H}$.

$(ii) \Rightarrow (iii)$ Let $q_1 + i_1 \in q_1 + I$ and $q_2 + i_2 \in q_2 + I$, where $i_1, i_2 \in I$. Assume that $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$. By (ii), $q_3 + I \in \widetilde{H}$. Thus there exists $h \in H$ such that $h \in q_3 + I$. Hence $h = q_3 + j$ for some $j \in I$. Therefore $q_3 \in H$. Let $q_1 + q_2 = q_3 + i$ for some $i \in I$. Then $q_1 + i_1 + q_2 + i_2 \in H$, because H is a co-ideal.

$(iii) \Rightarrow (iv)$ This implication is clear.

$(iv) \Rightarrow (i)$ Assume that $q_1 + i \in q_1 + I$ and $q_2 + i' \in q_2 + I$ are adjacent in $G_H(R)$, where $i, i' \in I$. Let $q_1 + i_1 \in q_1 + I$ and $q_2 + i_2 \in q_2 + I$, where $i_1, i_2 \in I$. As $q_1 + i + q_2 + i' \in H$ and $i, i' \in I$, we have $q_1 + q_2 \in H$. Therefore $p + q \in H$.

(2) Let $q + I \in \widetilde{H}$. Then $h = q + i$ for some $h \in H$ and $i \in I$. Therefore $q \in H$. Also, it is clear that $q + I$ is a clique in $G_H(R)$.

(3) If $q + I \notin \widetilde{H}$, then $q \notin H$. Let $q + i$ and $q + i'$ be arbitrary elements of $q + I$. Then $q + i + q + i' \notin H$, because $q, i, i' \in M$ for some $M \in max(H)$. Therefore $q + I$ is an independent set in $G_H(R)$. □

In the following, we investigate the relationship between the diameter and the girth of the graphs $G_H(R)$ and $G_{\widetilde{H}}(R/I)$.

Theorem 5. *The following statements hold:*

(1) $gr(G_H(R)) \leq gr(G_{\widetilde{H}}(R/I))$.

(2) $diam(G_{\widetilde{H}}(R/I)) \leq diam(G_H(R))$.

Proof. (1) If $G_{\widetilde{H}}(R/I)$ has no cycle, then there is nothing to prove. Hence assume that $q_1 + I - q_2 + I - ... - q_n + I - q_1 + I$ is a cycle in $G_{\widetilde{H}}(R/I)$. Then we have the cycle $q_1 - q_2 - ... - q_n - q_1$ in $G_H(R)$, by Lemma 4, which implies that $gr(G_H(R)) \leq gr(G_{\widetilde{H}}(R/I))$.

(2) If $n := diam(G_{\widetilde{H}}(R/I))$, then there are two vertices $q_1 + I$ and $q_2 + I$ of $G_{\widetilde{H}}(R/I)$ with $d(q_1 + I, q_2 + I) = n$. Assume that $q_1 + I - p_1 - ... - p_{n-2} + I - q_2 + I$ is a corresponding path of length n between $q_1 + I$ and $q_2 + I$ in $G_{\widetilde{H}}(R/I)$. In view of Lemma 4, $q_1 - p_1 - ... - p_{n-2} - q_2$ is a path of length n in $G_H(R)$. Therefore $diam(G_{\widetilde{H}}(R/I)) \leq diam(G_H(R))$. □

The following example shows that we may have strict inequality in parts (1), (2) of Theorem 5.

Example 1. *Let* $X = \{a, b, c\}$ *and* $R = (P(X), \cup, \cap)$ *a semiring, where* $P(X)$ *is the set of all subsets of* X. *If* $H = \{\{a\}, \{a, b\}, \{a, c\}, X\}$, *then* H *is a saturated* *multiplicatively closed subset of R and a minimal prime co-ideal of R. Therefore $I = R \setminus H$ is a maximal ideal of R. It can be verified that I is a Q-ideal of R and $Q = \{\emptyset, \{a\}\}$. By drawing $G_H(R)$ and $G_{\widetilde{H}}(R/I)$, one can see that $1 = diam(G_{\widetilde{H}}(R/I)) < diam(G_H(R)) = 2$ and $3 = gr(G_H(R)) < gr(G_{\widetilde{H}}(R/I)) = \infty$.*

In the following theorem, we provide a characterization of $G_H(R)$ in terms of $G_{\widetilde{H}}(R/I)$.

Theorem 6. *Let I be a Q-ideal of R. Then*

$$G_H(R) = (\oplus_{|I|^2} G_{\widetilde{H}}(R/I)) \oplus (\oplus_{|I|} K_{|Q \cap H|}).$$

Proof. If there exist $p, q \in Q$ such that $p + I$ and $q + I$ are adjacent in $G_{\widetilde{H}}(R/I)$, then in view of Lemma 4, each element of $p + I$ is adjacent to each element of $q + I$ in $G_H(R)$. Thus, each edge of $G_{\widetilde{H}}(R/I)$ corresponds to exactly $|I|^2$ edges in $G_H(R)$. Also, for each $p \in Q \cap H$, the coset $p + I$ forms a clique in $G_H(R)$. Hence $G_H(R) = (\oplus_{|I|^2} G_{\widetilde{H}}(R/I)) \oplus (\oplus_{|I|} K_{|Q \cap H|})$. □

3. Planarity, clique number, chromatic number and independence number of $G_H(R)$

In this section, we use the notations already established, so R is an idempotent I-semiring and H is a saturated proper subset of R. We will investigate clique number, independence number and planar property of the graph $G_H(R)$. A graph G is called planar, if it can be drawn in the plane (i.e. its edges intersect only at their ends). A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. An interesting characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [6].

Proposition 5. *The following hold:*
(i) If $0 \in H$, then $G_H(R)$ is planar if and only if $|R| \leq 4$.
(ii) If $|max(H)| \geq 4$ or $|H| \geq 4$, then $G_H(R)$ is not planar.
(iii) If $|H| = 3$, then $G_H(R)$ is planar if and only if $|R| \leq 5$.
(iv) Let $H = \{1\}$. Then $G_H(R)$ is planar if and only if $G_H(R) \setminus \mathrm{Jac}(R)$ is planar.

Proof. (i) Since $0 \in H$, $H = R$. It follows that $G_H(R)$ is complete. Now the assertion follows from Kuratowski's theorem.

(ii) If $|max(H)| \geq 4$, then $|min(H)| \geq 4$. Hence $\Gamma_H(R)$ is not planar, by [12, Theorem 4.10]. Therefore $G_H(R)$ is not planar. The other implication is clear.

(iii) Assume that $G_H(R)$ is planar and let $V_1 = H = \{x_1, x_2, x_3\}$. Suppose to the contrary that $|R| \geq 6$. Set $V_2 = \{y_1, y_2, y_3\} \subseteq R \setminus H$. It can be easily seen that one can find a copy of $K_{3,3}$ in $G_H(R)$, which is a contradiction. Conversely, assume that $|R| \leq 5$. If $|R| \leq 4$, we are done. If $|R| = 5$, then by Proposition 1, $G_H(R)$ is not K_5; hence $G_H(R)$ is planar.

(iv) Since by Theorem 3 (i), $\deg(a) = 1$ for all $a \in Jac(R)$, the result is clear. □

If $max(H) = \{M_1, M_2, ..., M_n\}$, then we denote $M_i \setminus (\cup_{j=1, j\neq i}^{n} M_j)$ by M_i' and $(M_i \cap M_j) \setminus (\cup_{s=1, s\neq i,j}^{n} M_s)$ by $M_{i,j}$ for each $1 \leq i \neq j \leq n$.

Theorem 7. *Let $H = \{1\}$. Then the graph $G_H(R)$ is planar if and only if one of the following statements holds:*

(1) $max(R) = \{M_1, M_2, M_3\}$, $|M_i'| = 1$ for each $1 \leq i \leq 3$ and $V(G_H(R)) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$, where V_i's are satisfying the following:

(i) $V_1 = M_1' \cup M_2' \cup M_3' \cup \{1\}$ is a clique in $G_H(R)$.

(ii) $V_2 = M_{1,2}$ and every element of V_2 is adjacent to 1 and $a \in M_3'$.

(iii) $V_3 = M_{1,3}$ and every element of V_3 is adjacent to 1 and $b \in M_2'$.

(iv) $V_4 = M_{2,3}$ and every element of V_4 is adjacent to 1 and $c \in M_1'$.

(v) $V_5 = M_1 \cap M_2 \cap M_3$ and every element of V_5 is adjacent to 1.

(2) $max(R) = \{M_1, M_2\}$, $V(G_H(R)) = V_1 \cup V_2 \cup V_3 \cup V_4$ where V_i's are satisfying the following:

(i) $V_1 = \{1\}$, and 1 is adjacent to every vertex of $G_H(R)$.

(ii) $V_2 = M_1'$, $V_3 = M_2'$ and either $|V_i| \geq 3$ and $|V_j| = 1$ $(i \neq j)$ or $|V_i| \leq 2$ for each $i = 1, 2$. Moreover, the subgraph generated by V_2, V_3 is complete 2-partite with parts V_2 and V_3 and every element of $V_2 \cup V_3$ is adjacent to 1.

(iii) $V_4 = M_1 \cap M_2$ and every element of V_4 is adjacent to 1.

(3) $R - \{1\}$ is a maximal ideal of R and $G_H(R)$ is a star graph.

Proof. Assume that the graph $G_H(R)$ is planar. Then $|max(R)| \leq 3$, by Proposition 5. Let $max(R) = \{M_1, M_2, M_3\}$. If $|M_i'| \geq 2$, for some $i \in \{1, 2, 3\}$, then there exist $x, y \in M_i'$. Let $z \in M_j'$ and $t \in M_k'$, where $1 \leq k, j \leq 3$ and $k \neq j$ are distinct from i. Set $S_1 := \{x, y, 1\}$ and $S_2 := \{z, t, zt\}$. As $x + z = x + t = 1$, we have $x + tz = 1$ (note that $zt \neq z$ and $zt \neq t$). Similarly, $y + z = y + t = y + tz = 1$. Hence, one can find a copy of $K_{3,3}$ in $G_H(R)$, which is impossible. Hence $|M_i'| = 1$ for each $i \in \{1, 2, 3\}$. It can be easily verified that (1) holds. If $max(R) = \{M_1, M_2\}$, then we will prove that (2) holds. If $|M_1'| \geq 3$, then there exist $x, y, z \in M_1'$. If $t, s \in M_2'$, then by setting $S_1 := \{x, y, z\}$ and $S_2 := \{t, s, 1\}$, the graph $G_H(R)$ has a subgraph isomorphic to $K_{3,3}$, a contradiction. Hence $|M_2'| = 1$. Similarly, if $|M_2'| \geq 3$, then $|M_1'| = 2$. Hence $|M_i'| \geq 3$ and $|M_j'| = 1$ $(i \neq j)$ or $|M_i'| \leq 2$ for each $i = 1, 2$. It is easy to see that (2) holds. If $|max(R)| = 1$, then by Theorem 4 , $G_H(R)$ is a star graph.

Conversely, if one of the conditions (1) or (2) or (3) holds, then it is easy to show $G_H(R)$ is a planar graph. □

Theorem 8. *Let $H = \{1, a\}$. Then the graph $G_H(R)$ is planar if and only if one of the following statements holds:*

(1) $max(R) = \{M_1, M_2\}$, $V(G_H(R)) = V_1 \cup V_2 \cup V_3 \cup V_4$ where V_i's are satisfying the following:

(i) $V_1 = \{1, a\}$, and every element of V_1 is adjacent to every vertex of $G_H(R)$.

(ii) $V_2 = M_1'$, $V_3 = M_2'$ and either $|V_i| = 1$ for each $i = 1, 2$ or $|V_i| = 2$ and $|V_j| = 1$ for each $i \neq j \in \{1, 2\}$. Moreover, the subgraph generated by V_2, V_3 is

complete 2-partite with parts V_2 and V_3 and every element of $V_2 \cup V_3$ is adjacent to 1 and a.

(iii) $V_4 = M_1 \cap M_2$ and every element of V_4 is adjacent to 1 and $\{a\}$.

(2) $max(H) = \{R - \{1, a\}\}$ and $G_H(R) \cong K_{1,1,|R-\{1\}|}$.

Proof. If $G_H(R)$ is planar, then $|max(R)| \leq 3$, by Proposition 5. If $max(H) = \{M_1, M_2, M_3\}$, then there exist $d \in M_1',\ b \in M_2',\ c \in M_3'$ such that $\{1, a, b, c, d\}$ is a clique in $G_H(R)$, which is impossible. Hence $|max(H)| \leq 2$. If $|M_i'| \geq 2$ for each $i = 1, 2$, then there exist $x, y \in M_1'$ and $t, z \in M_2'$. By setting $S_1 := \{x, y, a\}$ and $S_2 := \{1, t, z\}$, $G_H(R)$ has a subgraph isomorphic to $K_{3,3}$, which is a contradiction. Hence either $|M_i'| = 1$ for each $i = 1, 2$ or $|M_i'| = 2$ and $|M_j'| = 1$ for each $i \neq j \in \{1, 2\}$. Therefore (1) holds. If $|max(H)| = 1$, then it is easy to verified that $G_H(R)$ is complete 3-partite and $G_H(R) \cong K_{1,1,|R-\{1\}|}$. □

Theorem 9. *In the graph $G_H(R)$ we have the following equality:*

$$\omega(G_H(R)) = \chi(G_H(R)) = |H| + |max(H)|.$$

Proof. It is clear that $\omega(G) \leq \chi(G)$, for each graph G. We consider two cases:

Case 1: $\omega(G_H(R)) = \infty$. Then $\chi(G_H(R)) = \infty$. Assume that H and $max(H)$ are finite and $max(H) = \{M_1, ..., M_n\}$. Let $\mathcal{C}$ be a maximal clique in $G_H(R)$. Set for each $1 \leq i \leq n$, $I_i = \{a \in \mathcal{C} \setminus H :\ a \in M_i\}$. If $|I_i| \geq 2$, for some $1 \leq i \leq n$, then there exist $a, b \in \mathcal{C} \setminus H$. Therefore $a, b \in M_i$ and so $a + b \notin H$ contradicts $a, b \in \mathcal{C}$. Therefore $|I_i| \leq 1$ for each $1 \leq i \leq n$. As $\mathcal{C} \setminus H = \bigcup_{i=1}^n I_i$ and I_i is a finite set for each $1 \leq i \leq n$, $\mathcal{C} \setminus H$ is a finite set. Therefore $\mathcal{C}$ is a finite set, a contradiction. Therefore either H is infinite or $max(H)$ is infinite. This gives $\omega(G_H(R)) = \chi(G_H(R)) = |H| + |max(H)| = \infty$.

Case 2: $\omega(G_H(R)) < \infty$. As $\omega(\Gamma_H(R)) < \infty$ and H is a clique in $G_H(R)$, H is a finite set. Moreover, $\omega(\Gamma_H(R)) < \infty$, because $\Gamma_H(R)$ is a subgraph of $G_H(R)$. Therefore $min(H)$ is finite, and so $max(H)$ is finite. Assume that $max(H) = \{M_1, ..., M_n\}$. Let $a_i \in M_i \setminus (\bigcup_{i \neq j, j=1}^n M_j)$. If $a_i + a_j \notin H$, for some $1 \leq i, j \leq n$, then $a_i + a_j \in M_k$, for some $M_k \in max(H)$, and so by Lemma 2 we have $a_i, a_j \in M_k$, a contradiction. Therefore $a_i + a_j \in H$. Hence $|H| + |max(H)| \leq \omega(G_H(R))$. Let $|H| = m$ and $H = \{a_1, ..., a_m$. Define $f :\ V(G_H(R)) \to \{1, ..., n, n+1, ..., m\}$ by

$$f(a) = \begin{cases} n + i, & \text{if } a = a_i \in \{a_1, ..., a_m\} \\ i, & \text{if } a = a_i \in M_i - (\bigcup_{i \neq j, j=1}^n M_j) \\ j, & \text{if } a \in M_j \cap M_{j+s_1} \cap ... \cap M_{j+s_t}, \text{where } s_1, ..., s_t \in \mathbb{N}. \end{cases}$$

Let $a, b \in R$ be adjacent in $G_H(R)$. Then it is clear that $f(a) \neq f(b)$ provided that $(a, b \in H)$ or $(a \notin H, b \in H)$ or $(a \in H, b \notin H)$. Let $a \notin H$ and $b \notin H$. Then $a \in M_i$ and $b \in M_j$ for some $M_i, M_j \in max(H)$. If $i = j$, then $a + b \in M_i$ and $a + b \notin H$, a contradiction. Let $I = \{i :\ a \in M_i,\ 1 \leq i \leq n\}$ and $J = \{j :\ b \in M_j,\ 1 \leq j \leq n\}$. As $a + b \in H$, we have $I \cap J = \emptyset$. Therefore $f(a)$ and $f(b)$ are the least element of I and J, respectively. Thus $f(a) \neq f(b)$. This implies that $\chi(G_S(R)) \leq |H| + |max(H)|$ and so we have $\omega(G_H(R)) = \chi(G_H(R)) = |H| + |max(H)|$. □

Let $T \subseteq P(\{1, 2, ..., n\})$, where $P(\{1, 2, ..., n\}$ denotes the power set of $\{1, 2, ..., n\}$. We say that T satisfies the property (P), provided that:

(1) For each $I \in T$, $|I| \geq 2$.

(2) For each $I, J \in T$, $I \cap J \neq \emptyset$.

Set $\sum = \{T \subseteq P(\{1, 2, ..., n\} : \ T$ satisfies the property $(P)\}$.

Theorem 10. *Let $max(H) = \{M_1, M_2, ..., M_n\}$. Then*

$$\alpha(G_H(R)) = max\{\{|M_i|\}_{i=1}^n \cup \{| \cup_{I \in T} (\cap_{i \in I} M_i)|\}_{T \in \sum}\}.$$

Proof. It can be easily seen that M_i and $\cup_{I \in T}(\cap_{j \in I} M_j)$ are independent sets in $G_H(R)$, for each $1 \leq i \leq n$ and $T \in \sum$. Therefore, $\alpha(G_H(R)) \geq max\{\{|M_i|\}_{i=1}^n \cup \{| \cup_{I \in T} (\cap_{i \in I} M_i)|\}_{T \in \sum}\}$. Assume that Y is a maximal independent set of $G_H(R)$. For each $a \in Y$, set

$$I_a = \{i : \ a \in M_i, \ 1 \leq i \leq n\}.$$

Let $a \in Y$ and $I_a = \{i\}$, for some $1 \leq i \leq n$. If $b \in Y$, then $b + a \notin H$. Hence $b + a \in M_k$ for some $1 \leq k \leq n$. Hence $a, b \in M_k$, by Lemma 2. This implies that $b \in M_i$. Therefore, $Y \subseteq M_i$. As Y is a maximal independent set, we have $Y = M_i$ (M_i is independent set). Now, let $|I_a| \geq 2$, for each $a \in Y$. If there exist $a, b \in Y$ such that $I_a \cap I_b = \emptyset$, then $a + b \in H$, a contradiction. Thus, $I_a \cap I_b \neq \emptyset$. Set $T = \{I_a\}_{a \in Y}$. Then $T \in \sum$ and $Y \subseteq \cup_{I \in T}(\cap_{i \in I} M_i)$. Since Y is maximal, $Y = \cup_{I \in T}(\cap_{i \in I} M_i)$. This proves that $\alpha(G_H(R)) = max\{\{|M_i|\}_{i=1}^n \cup \{| \cup_{I \in T} (\cap_{i \in I} M_i)|\}_{T \in \sum}\}$. □

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AN EFFICIENT VARIANT OF DUAL TO PRODUCT AND RATIO ESTIMATORS IN SAMPLE SURVEYS

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Abstract. This manuscript considers a dual to product and ratio estimator for estimating the finite population mean of study variable on applying a simple transformation to the auxiliary variable by using its average values in the population that is generally available in practice. The mean square error (MSE) of the proposed estimator has been obtained to the first degree of approximation. The optimum values and range of suitably chosen scalar, under which the proposed estimator perform better, have been determined. A method to lower the MSE of the proposed estimator relative to that of the MSE of the linear regression estimator is developed for small sample sizes. Theoretical and empirical studies have been done to demonstrate the superiority of the proposed estimator over the other estimators.

1. Introduction

There are numerous number of ratio and product type estimators available in survey literature from the time ratio estimator was developed by Cochran [4], and the product estimator was defined by Robson [12] that was revisited by Murthy [11]. Ratio and product type estimators have been largely used due to computational simplicity, greater applicability to the general design and researchers' impulsive draw towards it. Most of the ratio and product type estimators recently developed are simply a modification of other existing estimators available in the literature.

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This has led to the accumulation of a large number of the ratio as well as product type estimators with cumbersome structure over the time. Often these estimators require the knowledge of other population parameters in advance or has to guess it with the experience gathered over the period of time in sample survey or estimate it through pilot survey or the sample itself and in optimum case the MSE of the proposed estimator is found generally equivalent to the MSE of the regression estimator. Moving in this direction, we have proposed the dual to product and ratio estimators and shown that how in optimal case their minimum MSE becomes nothing but MSE of regression estimator. We have carried out then the key study of developing the new estimator using the previously proposed dual to product and ratio estimators which will be called parent estimator for the newly developed estimator. The new estimator' MSE is improved to an extent that it becomes better or more efficient than the regression estimator. One more aspect of our method is the important role played by the bias of the estimator in improving MSE which was neglected before in the survey literature works in the area of ratio and product estimators.

Let $U = \{U_1, U_2, ..., U_N\}$ be a finite population of size N. Also, let Y and X be the study and auxiliary variables, respectively, taking the values y_i and x_i on the i^{th} unit U_i $(i = 1, 2, ..., N)$ of the population U. Assuming that the population mean $\bar{X}$ of the auxiliary variable X is known,the population mean $\bar{Y}$ of the study variable Y is estimated by selecting a sample of size n (with $n < N$) from the population U using *simple random sampling without replacement* (SRSWOR) scheme.

The ratio estimator of $\bar{Y}$ as developed by Cochran [4], and the product estimator of $\bar{Y}$ as developed by Murthy [11] are given, respectively, by

$$\bar{y}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{1}$$

$$\bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right) \tag{2}$$

with their respective *Biases* and *MSEs* to the first order of approximations as

$$Bias(\bar{y}_R) = \lambda\bar{Y}\left[{C_x}^2 - \rho_{yx}C_yC_x\right] \tag{3}$$

$$Bias(\bar{y}_P) = \lambda\bar{Y}\rho_{yx}C_yC_x \tag{4}$$

$$MSE(\bar{y}_R) = \lambda\bar{Y}^2C_y^2\left[1 + \left(\frac{C_x}{C_y}\right)^2 - 2\rho_{yx}\frac{C_x}{C_y}\right] \tag{5}$$

$$MSE(\bar{y}_P) = \lambda\bar{Y}^2C_y^2\left[1 + \left(\frac{C_x}{C_y}\right)^2 + 2\rho_{yx}\frac{C_x}{C_y}\right] \tag{6}$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ are the sample means of Y and X, respectively. Also, C_y and C_x represent the coefficients of variations of the variables Y and X, respectively. Moreover, ρ_{yx} denotes the correlation coefficient between the study variable Y and the auxiliary variable X. The notations used above are as follows:

$$\lambda = \frac{1-f}{n},\ f = \frac{n}{N},\ C_y^2 = \frac{S_y^2}{\bar{Y}^2},\ C_x^2 = \frac{S_x^2}{\bar{X}^2},\ \rho_{yx} = \frac{S_{yx}}{S_y S_x}$$

$$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})^2,\ S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \bar{X})^2$$

$$S_{yx} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})$$

The classical linear regression estimator for population mean $\bar{Y}$ is defined by

$$\hat{\bar{Y}}_{reg} = \bar{y} + b_{yx}(\bar{X} - \bar{x}) \tag{7}$$

where b_{yx} is the sample regression coefficient of Y on X.

Also, the *Bias* and *MSE* of $\hat{\bar{Y}}_{reg}$ to the first order of approximations are given, respectively, by

$$Bias(\hat{\bar{Y}}_{reg}) = -cov(\bar{x}\ ,\ b_{yx}) \tag{8}$$

$$MSE(\hat{\bar{Y}}_{reg}) = \lambda \bar{Y}^2 C_y^2 \left(1 - \rho_{yx}^2\right) \tag{9}$$

Srivenkataramana [15] and Bandyopadhya [2] suggested a dual to ratio and a dual to product estimators, respectively, for $\bar{Y}$ as

$$\bar{y}_R^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right) \tag{10}$$

$$\bar{y}_P^* = \bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right) \tag{11}$$

with their respective *Biases* and *MSEs* to the first order of approximations as

$$Bias(\bar{y}_R^*) = -g\lambda\bar{Y}\rho_{yx}C_yC_x \tag{12}$$

$$Bias(\bar{y}_P^*) = \lambda\bar{Y}\left[g^2{C_x}^2 + g\rho_{yx}C_yC_x\right] \tag{13}$$

$$MSE(\bar{y}_R^*) = \lambda\bar{Y}^2C_y^2\left[1 + g^2\left(\frac{C_x}{C_y}\right)^2 - 2g\rho_{yx}\frac{C_x}{C_y}\right] \tag{14}$$

$$MSE(\bar{y}_P^*) = \lambda \bar{Y}^2 C_y^2 \left[1 + g^2 \left(\frac{C_x}{C_y} \right)^2 + 2g\rho_{yx} \frac{C_x}{C_y} \right] \tag{15}$$

where $\bar{x}^* = (1+g)\bar{X} - g\bar{x}$ is an unbiased estimator of $\bar{X}$, and $g = n/(N-n)$.

Some recent developments towards the formulation of different classes of dual to product-cum-dual to ratio estimators have been made by Singh et al. [13], and Choudhury and Singh [3]. Moreover, Adebola et al. [?] developed a class of regression estimator with cum-dual ratio estimator as intercept. The recently developed estimators as described here are listed in Table 1.

TABLE 1. Recent developed estimators of $\bar{Y}$

Authors	**Estimators**
Singh *et al.* [13]	$\bar{y}_{PR}^* = \eta \bar{y} \left(\frac{a\bar{X}+b}{a\bar{x}^*+b} \right) + (1-\eta)\, \bar{y} \left(\frac{a\bar{x}^*+b}{a\bar{X}+b} \right)$
Choudhury and Singh [3]	$\bar{y}_{CS}^* = \bar{y} \left[\alpha \frac{\bar{X}}{\bar{x}^*} + (1-\alpha) \frac{\bar{x}^*}{\bar{X}} \right]$
Adebola *et al.* [?]	$\bar{y}_{Rd}^* = \bar{y} \frac{\bar{x}^*}{\bar{X}} + \alpha \left(\bar{X} - \bar{x}^* \right)$

In Table 1, η, a, b and α denote the scalars, which are suitably determined so as to minimize the *MSEs* of the concerned estimators. Also, the expressions for the *Biases* and *MSEs* of various estimators to the terms of order $o\left(n^{-1}\right)$ are given by

$$Bias\left(\bar{y}_{PR}^*\right) = \lambda \bar{Y} \left[\eta \left(\frac{a\bar{X}}{a\bar{X}+b} \right)^2 g^2 {C_x}^2 + (2\eta - 1) \left(\frac{a\bar{X}}{a\bar{X}+b} \right) g\rho_{yx} C_y C_x \right] \tag{16}$$

$$Bias\left(\bar{y}_{CS}^*\right) = \lambda \bar{Y} \left[(2\alpha - 1)\, g\rho_{yx} C_y C_x + \alpha g^2 {C_x}^2 \right] \tag{17}$$

$$Bias\left(\bar{y}_{Rd}^*\right) = -\lambda g \bar{Y} \rho_{yx} C_y C_x \tag{18}$$

$$MSE\left(\bar{y}_{PR}^*\right) = \lambda \bar{Y}^2 \left[C_y^2 + \left\{ \frac{a\bar{X}}{a\bar{X}+b} \right\}^2 g^2 (2\eta - 1)^2 C_x^2 + \frac{a\bar{X}}{a\bar{X}+b} g\,(2\eta - 1)\, \rho_{yx} C_y C_x \right] \tag{19}$$

$$MSE\left(\bar{y}_{CS}^*\right) = \bar{Y}^2 \lambda \left[C_y^2 + g\,(2\alpha - 1)\, C_x^2 \left\{ g\,(2\eta - 1) + \rho_{yx} \frac{C_y}{C_x} \right\} \right] \tag{20}$$

$$MSE\left(\bar{y}_{Rd}^*\right) = \lambda \left[\bar{Y}^2 C_y^2 - 2g\bar{Y} \rho_{yx} C_x C_y \left(\bar{Y} - \alpha \bar{X} \right) + g C_x^2 \left(\bar{Y} - \alpha \bar{X} \right)^2 \right] \tag{21}$$

Furthermore, the minimum attainable *MSEs* of the estimators $\bar{y}^*_{PR}$, $\bar{y}^*_{CS}$ and $\bar{y}^*_{Rd}$ are

$$MSE\left(\bar{y}^*_{PR}\right)_{min} = \lambda \bar{Y}^2 C_y^2 \left(1 - \rho_{yx}^2\right) \tag{22}$$

$$MSE\left(\bar{y}^*_{CS}\right)_{min} = \lambda \bar{Y}^2 C_y^2 \left(1 - \rho_{yx}^2\right) \tag{23}$$

$$MSE\left(\bar{y}^*_{Rd}\right)_{min} = \lambda \bar{Y}^2 C_y^2 \left(1 - \rho_{yx}^2\right) \tag{24}$$

Hence, we have

$$MSE\left(\bar{y}^*_{PR}\right)_{min} = MSE\left(\bar{y}^*_{CS}\right)_{min} = MSE\left(\bar{y}^*_{Rd}\right)_{min} = \lambda \bar{Y}^2 C_y^2 \left(1 - \rho_{yx}^2\right) \tag{25}$$

Table 1 and Eqs. (19) to (21) and Eq. (25) substantiate that the modified ratio and product type estimators are too complex in structure, demands advance knowledge of the scalars and the minimum *MSEs* of these estimators are equivalent to the *MSE* of linear regression estimator $\hat{\bar{Y}}_{reg}$ as given in Eq. (9). Thus, making their theoretical and practical relevance in the argument.

2. Proposed Estimator

We define an efficient variant of dual to product and ratio estimators for $\bar{Y}$ as

$$\hat{\bar{Y}}_{Md} = \bar{y}\left(\frac{\bar{X} + \theta \bar{x}^*}{\bar{x}^* + \theta \bar{X}}\right) \tag{26}$$

where θ is a scalar which is determined so as to minimize the *MSE* of the proposed estimator $\hat{\bar{Y}}_{Md}$. Also, it is worth noting that, for $\theta = 1$, $\hat{\bar{Y}}_{Md} = \bar{y}$ and that, for $\theta = 0$, $\hat{\bar{Y}}_{Md} = \bar{y}^*_P$. Moreover, if θ is very large, $\hat{\bar{Y}}_{Md}$ is almost the same as $\bar{y}^*_R$.

The *Bias* and *mean square error (MSE)* of the proposed estimator $\hat{\bar{Y}}_{Md}$ are obtained by considering

$$\bar{y} = \bar{Y}(1 + e_0),\ \bar{x} = \bar{X}(1 + e_1)$$

such that $E(e_0) = E(e_1) = 0$.

Also, on simplification, we get

$$E(e_0^2) = \lambda C_y^2\ ,\ E(e_1^2) = \lambda C_x^2\ ,\ E(e_0 e_1) = \lambda \rho_{yx} C_y C_x \tag{27}$$

Now, expressing Eq. (26) in terms of $e_0,\ e_1$ we get

$$\hat{\bar{Y}}_{Md} = \bar{Y}(1 + e_0)\left\{1 - \frac{\theta g e_1}{(1+\theta)}\right\}\left\{1 - \frac{g e_1}{(1+\theta)}\right\}^{-1} \tag{28}$$

Expanding the right hand side of Eq. (28), multiplying out, and retaining the terms up to second powers of e's, we get

$$\hat{\hat{Y}}_{Md} = \bar{Y}\left\{1 + e_0 + \frac{(1-\theta)}{(1+\theta)}ge_1 + \frac{(1-\theta)}{(1+\theta)}ge_0e_1 + \frac{(1-\theta)}{(1+\theta)^2}g^2e_1^2\right\} \tag{29}$$

or

$$\hat{\hat{Y}}_{Md} - \bar{Y} = \bar{Y}\left\{e_0 + \frac{(1-\theta)}{(1+\theta)}ge_1 + \frac{(1-\theta)}{(1+\theta)}ge_0e_1 + \frac{(1-\theta)}{(1+\theta)^2}g^2e_1^2\right\} \tag{30}$$

Taking the expectation in Eq. (30) and using results in Eq. (27), we get the *bias* of $\hat{\hat{Y}}_{Md}$ to the first degree of approximation as

$$Bias(\hat{\hat{Y}}_{Md}) = \lambda\bar{Y}\left\{\frac{(1-\theta)}{(1+\theta)}g\rho_{yx}C_yC_x + \frac{(1-\theta)}{(1+\theta)^2}g^2C_x^2\right\} \tag{31}$$

Again from Eq. (30), by neglecting the terms of e's having degree greater than one, we have

$$\hat{\hat{Y}}_{Md} - \bar{Y} = \bar{Y}\left[e_0 + \left(\frac{1-\theta}{1+\theta}\right)ge_1\right] \tag{32}$$

Squaring both sides of Eq. (32), taking the expectation and using results in Eq. (27), we obtain the *MSE* of $\hat{\hat{Y}}_{Md}$ to the first degree of approximation as

$$MSE(\hat{\hat{Y}}_{Md}) = \lambda\bar{Y}^2\left[C_y^2 + \left(\frac{1-\theta}{1+\theta}\right)^2 g^2C_x^2 + 2g\left(\frac{1-\theta}{1+\theta}\right)\rho_{yx}C_yC_x\right] \tag{33}$$

Minimization of $MSE(\hat{\hat{Y}}_{Md})$ in Eq. (33) with respect to θ yields the optimum value of θ as

$$\theta_{opt} = \frac{g + \rho_{yx}\frac{C_y}{C_x}}{g - \rho_{yx}\frac{C_y}{C_x}} \tag{34}$$

On substituting Eq. (34) in Eq. (33), the minimum attainable *MSE* of $\hat{\hat{Y}}_{Md}$ is obtained as

$$MSE(\hat{\hat{Y}}_{Md})_{min} = \lambda\bar{Y}^2C_y^2(1-\rho_{yx}^2) \tag{35}$$

Remark 1. *The minimum MSE of* $\hat{\hat{Y}}_{Md}$ *is same as that of the MSE of the linear regression estimator* $\hat{\hat{Y}}_{reg}$ *as given in Eq.* (9).

Even our proposed estimator's minimum *MSE* corroborate the results of the other modified ratio and product type estimators' minimum *MSEs* as given in Eq. (25). But now we will work out a simple condition on our proposed estimator in order to derive a new proposed estimator for which previously proposed estimator will be called parent estimator. Hence, using the parent estimator our new proposed estimator is

$$\hat{\bar{Y}}_w = w\hat{\bar{Y}}_{Md} \tag{36}$$

where w denotes the scalar which is to be suitably determined so as to minimize the *MSE* of the above concerned estimator.

Now, expanding Eq. (36) using Eq. (29), we get

$$\hat{\bar{Y}}_w = w\bar{Y}\left\{1 + e_0 + \frac{(1-\theta)}{(1+\theta)}ge_1 + \frac{(1-\theta)}{(1+\theta)}ge_0e_1 + \frac{(1-\theta)}{(1+\theta)^2}g^2e_1^2\right\} \tag{37}$$

or

$$\hat{\bar{Y}}_w - \bar{Y} = (w-1)\bar{Y} + w\bar{Y}\left\{e_0 + \frac{(1-\theta)}{(1+\theta)}ge_1 + \frac{(1-\theta)}{(1+\theta)}ge_0e_1 + \frac{(1-\theta)}{(1+\theta)^2}g^2e_1^2\right\} \tag{38}$$

Squaring both sides of Eq. (38), taking the expectation and using results in Eq. (27), we obtain the *MSE* of $\hat{\bar{Y}}_w$ to the first degree of approximation as

$$\begin{aligned} MSE(\hat{\bar{Y}}_w) = (w-1)^2\bar{Y}^2 + w^2\lambda\bar{Y}^2 &\left\{C_y^2 + \left(\frac{1-\theta}{1+\theta}\right)^2 g^2C_x^2 + 2g\left(\frac{1-\theta}{1+\theta}\right)\rho_{yx}C_yC_x\right\} \\ &+ 2w(w-1)\lambda\bar{Y}\left\{\frac{1-\theta}{1+\theta}g\rho_{yx}C_yC_x + \frac{1-\theta}{(1+\theta)^2}g^2C_x^2\right\} \end{aligned} \tag{39}$$

which can be rewritten as

$$MSE\left(\hat{\bar{Y}}_w\right) = (w-1)^2\bar{Y}^2 + w^2MSE(\hat{\bar{Y}}_{Md}) + 2w(w-1)\bar{Y}Bias(\hat{\bar{Y}}_{Md}) \tag{40}$$

From Eq. (40) it can be brought to notice that the *MSE* of the new proposed estimator contains the *MSE* and *Bias* of its parent estimator. Now differentiating Eq. (40) w.r.t w and equating it to zero, we get

$$w_{opt} = \frac{\bar{Y}^2 + \bar{Y}Bias(\hat{\bar{Y}}_{Md})}{\bar{Y}^2 + MSE(\hat{\bar{Y}}_{Md}) + 2\bar{Y}Bias(\hat{\bar{Y}}_{Md})} \tag{41}$$

and using it to find the minimum *MSE* of the new proposed estimator, we have

$$MSE(\hat{\bar{Y}}_w)_{min} = \frac{\bar{Y}^2\left(MSE(\hat{\bar{Y}}_{Md}) - Bias(\hat{\bar{Y}}_{Md})^2\right)}{\bar{Y}^2 + MSE(\hat{\bar{Y}}_{Md}) + 2\bar{Y}Bias(\hat{\bar{Y}}_{Md})} \tag{42}$$

From Eq. (42), we see that the numerator is nothing but variance of the parent estimator. The trade-off between bias and variance in order to increase the efficiency of the new proposed estimator is very effective here.

If we substitute in equation Eq. (42), the minimum attainable MSE of parent estimator $\hat{\hat{Y}}_{Md}$, we get

$$MSE(\hat{\hat{Y}}_w)_{min} = \frac{\bar{Y}^2\left(MSE(\hat{\hat{Y}}_{reg}) - B^2\right)}{\bar{Y}^2 + MSE(\hat{\hat{Y}}_{reg}) + 2\bar{Y}B} \tag{43}$$

where B represents the $Bias(\hat{\hat{Y}}_{Md})$ at the value of $\theta = \theta_{opt}$ as given in Eq. (34). That is

$$B = -\frac{\lambda\bar{Y}}{2}\left\{g\rho_{yx}C_yC_x + \rho_{yx}^2C_x^2\right\} \tag{44}$$

Theorem 1. *For small sample size, the proposed estimator* $\hat{\hat{Y}}_w$ *is more efficient than the regression estimator* $\hat{\hat{Y}}_{reg}$*. But as the sample size increases, i.e., as* $n \to N$ *the relative efficiency of the proposed estimator* $\hat{\hat{Y}}_w$ *is same as that of the regression estimator* $\hat{\hat{Y}}_{reg}$*.*

Proof. From the definition of relative efficiency RE, we get:

$$RE = \frac{MSE\left(\hat{\hat{Y}}_{reg}\right)}{MSE(\hat{\hat{Y}}_w)_{min}} = \frac{1}{\bar{Y}^2\left(1 - \frac{B^2}{MSE(\hat{\hat{Y}}_{reg})}\right)}\left(\bar{Y}^2 + MSE(\hat{\hat{Y}}_{reg}) + 2\bar{Y}B\right)$$

Now as $n \to N$ we have $\lambda \to 0$. As a result $\frac{B^2}{MSE(\hat{\hat{Y}}_{reg})} \to 0$, $MSE(\hat{\hat{Y}}_{reg}) \to 0$ and $B \to 0$. Therefore

$$RE \to 1$$

i.e., $MSE(\hat{\hat{Y}}_w)_{min} \to MSE\left(\hat{\hat{Y}}_{reg}\right)$. Hence the theorem. □

3. Bias and Efficiency Comparisons

It is well known that *Bias* and *MSE* of the usual unbiased estimator $\bar{y}$ for population mean in SRSWOR are

$$Bias\,(\bar{y}) = 0 \tag{45}$$

$$V\,(\bar{y}) = \lambda\bar{Y}^2{C_y}^2 \tag{46}$$

For making *Bias* comparisons of the proposed estimator $\hat{\hat{Y}}_{Md}$ with the existing estimators, we have from Eq. (31), and Eq. (3), Eq. (4), Eq. (12), Eq. (13), Eq. (16), Eq. (17), Eq. (18), and Eq. (45).

(i) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid B(\bar{y}) \mid$ or $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq 0$ if

$$\theta = 0 \tag{47}$$

(ii) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid Bias(\bar{y}_R) \mid$ if

$$\left[\frac{g^2\,(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}-\left(C_x-\rho_{yx}C_y\right)^2\right]\leq 0 \tag{48}$$

(iii) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid Bias(\bar{y}_P) \mid$ if

$$\left[\frac{g^2\,(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}-\left(\rho_{yx}C_y\right)^2\right]\leq 0 \tag{49}$$

(iv) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid bias(\bar{y}_R^*) \mid$ if

$$\left[\frac{(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}-\left(\rho_{yx}C_y\right)^2\right]\leq 0 \tag{50}$$

(v) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid Bias\,(\bar{y}_P^*) \mid$ if

$$\left[\frac{g^2\,(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}-\left(C_x+g\rho_{yx}C_y\right)^2\right]\leq 0 \tag{51}$$

(vi) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid Bias(\bar{y}_{PR}^*) \mid$ if

$$\left[\frac{(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}\right.$$
$$\left.-\left(\frac{a\bar{X}}{a\bar{X}+b}\right)^2\left(g\eta\left(\frac{a\bar{X}}{a\bar{X}+b}\right)C_x-(1-2\eta)\,\rho_{yx}C_y\right)^2\right]\leq 0 \tag{52}$$

(vii) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid Bias(\bar{y}_{CS}^*) \mid$ if

$$\left[\frac{(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}-\left(g\alpha C_x-(1-2\alpha)\,\rho_{yx}C_y\right)^2\right]\leq 0 \tag{53}$$

(viii) $\mid Bias(\hat{\hat{Y}}_{Md}) \mid \leq \mid Bias(\bar{y}_{Rd}^*) \mid$ if

$$\left[\frac{(1-\theta)^2\left(gC_x+\rho_{yx}C_y\,(1+\theta)\right)^2}{(1+\theta)^4}-\left(\rho_{yx}C_y\right)^2\right]\leq 0 \tag{54}$$

For making efficiency comparisons of the proposed estimator $\hat{\hat{Y}}_{Md}$ with the existing estimators, we have from Eq. (33), and Eq. (5), Eq. (6), Eq. (14), Eq. (15), and (46)

(i) $MSE(\hat{\hat{Y}}_{Md}) < V(\bar{y})$ if

$$min\left(0,-2\rho_{yx}\frac{C_y}{gC_x}\right)<\psi<max\left(0,-2\rho_{yx}\frac{C_y}{gC_x}\right) \tag{55}$$

where $\psi = \frac{1-\theta}{1+\theta}$.

(ii) $MSE(\hat{\hat{Y}}_{Md}) < MSE(\bar{y}_R)$ if

$$min\left\{-\frac{1}{g},\left(-2\rho_{yx}\frac{C_y}{gC_x}+\frac{1}{g}\right)\right\} < \psi < max\left\{-\frac{1}{g},\left(-2\rho_{yx}\frac{C_y}{gC_x}+\frac{1}{g}\right)\right\} \tag{56}$$

(iii) $MSE(\hat{\hat{Y}}_{Md}) < MSE(\bar{y}_P)$ if

$$min\left\{\frac{1}{g},\left(-2\rho_{yx}\frac{C_y}{gC_x}-\frac{1}{g}\right)\right\} < \psi < max\left\{\frac{1}{g},\left(-2\rho_{yx}\frac{C_y}{gC_x}-\frac{1}{g}\right)\right\} \tag{57}$$

(iv) $MSE(\hat{\hat{Y}}_{Md}) < MSE(\bar{y}_R^*)$ if

$$-1 < \psi < \left(1-2\rho_{yx}\frac{C_y}{gC_x}\right) \tag{58}$$

(v) $MSE(\hat{\hat{Y}}_{Md}) < MSE\left(\bar{y}_P^*\right)$ if

$$min\left\{1,\left(-2\rho_{yx}\frac{C_y}{gC_x}-1\right)\right\} < \psi < max\left\{1,\left(-2\rho_{yx}\frac{C_y}{gC_x}-1\right)\right\} \tag{59}$$

Now let us denote the estimators $\hat{\hat{Y}}_{Md}, \bar{y}_{PR}^*$, $\bar{y}_{CS}^*$ and $\bar{y}_{Rd}^*$ which attains minimum *MSEs* equivalent to *MSE* of linear regression estimator $\hat{\hat{Y}}_{reg}$ as T, and comparing it to the new proposed estimator $\hat{\hat{Y}}_w$, we have

(vi) $MSE(\hat{\hat{Y}}_w)_{min} < MSE(T)$ if

$$\left(MSE(T)+\bar{Y}B\right)^2 > 0 \tag{60}$$

where $MSE(T) = MSE(\hat{\hat{Y}}_{reg}) = \lambda\bar{Y}^2C_y^2\left(1-\rho_{yx}^2\right)$

4. Empirical Study

To examine the merits of the new proposed estimator $\hat{\hat{Y}}_w$ over other existing estimators, seven natural population data sets have been considered. The description of the populations and the values of various parameters are listed in Tables 2 and 3, respectively.

In Table 4, the effective ranges of ψ along with its optimum values are shown for which the proposed estimator $\hat{\hat{Y}}_{Md}$ is better than the other existing estimators. However, in practice, it may be difficult to determine the interval extremes depending on the unknown parameter values of the population.

The *percentage relative efficiencies (PREs)* are obtained for various suggested estimators of $\bar{Y}$ with respect to the usual unbiased estimator $\bar{y}$ using the formula

$$PRE(\phi,\ \bar{y}) = \frac{V(\bar{y})}{MSE(\phi)} \times 100$$

where ϕ is used in places of any estimator among $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$, $\bar{y}_P^*$, $\hat{\hat{Y}}_{Md}$ and $\hat{\hat{Y}}_w$, and the findings are presented in Table 5.

TABLE 2. Description of Populations

Populations	**Variables**
Population I	Y = Apple production amount in 1999
Kadilar and Cingi [9]	X = Apple production amount in 1998
	$N = 204,\ n = 50$
Population II	Y = Apple trees of bearing age in 1964
Sukhatme and Chand [16]	X = Bushels of apples harvested in 1964
	$N = 200,\ n = 20$
Population III	Y = Peach production in bushels in an orchard in 1946
Cochran [5]	X = Number of peach trees in the orchard in 1946
	$N = 256,\ n = 100$
Population IV	Y = Number of females employed
Singh [14]	X = Number of females in service
	$N = 61,\ n = 20$
Population V	Y = Number of agricultural laborers for 1971
Das [6]	X = Number of agricultural laborers for 1961
	$N = 278,\ n = 30$
Population VI	Y = Consumption per capita
Maddala [10]	X = Deflated prices of veal
	$N = 16,\ n = 4$
Population VII	Y = Percentage of hives affected by disease
Johnston [8]	X = Date of flowering of a particular summer species (number of days from January 1)
	$N = 10,\ n = 4$

TABLE 3. Parameters of populations

Populations	$\bar{Y}$	$\bar{X}$	C_y	C_x	ρ_{yx}	g
I	966	1014	2.4739	2.4866	0.94	0.3247
II	1031.82	2934.58	1.5977	2.0062	0.93	0.1111
III	56.47	44.45	1.42	1.40	0.887	0.6410
IV	7.46	5.31	0.7103	0.7574	0.7737	0.4878
V	39.0680	25.1110	1.4451	1.6198	0.7213	0.1209
VI	7.6375	75.4313	0.2278	0.0986	-0.6823	0.3333
VII	52	200	0.1562	0.0458	-0.94	0.6667

TABLE 4. Effective ranges of ψ under which $\hat{\hat{Y}}_{Md}$ is better than the other existing estimators

Population	**Range of ψ in which $\hat{\hat{Y}}_{Md}$ is better than**					**Optimum value of ψ**
	$\bar{y}$	$\bar{y}_R$	$\bar{y}_P$	$\bar{y}^*_R$	$\bar{y}^*_P$	(i.e., ψ_0)
I	(-5.7608, 0)	(-3.08, -2.6808)	(-8.8408, 3.08)	(-4.7608, -1)	(-6.7608, 1)	-2.8804
II	(-13.3315, 0)	(-9, -4.3315)	(-22.3315, 9)	(-12.3315, -1)	(-14.3315, 1)	-6.6657
III	(-2.8069, 0)	(-1.56, -1.2469)	(-4.3669, 1.56)	(-1.8069, -1)	(-3.8069, 1)	-1.4035
IV	(-2.975, 0)	(-2.05, -0.9250)	(-5.025, 2.05)	(-1.975, -1)	(-3.975, 1)	-1.4875
V	(-10.6452, 0)	(-8.2713, -2.3739)	(-18.9165, 8.2713)	(-9.6452, -1)	(-11.6452, 1)	-5.3226
VI	(0, 9.4598)	(-3.0003, 12.4601)	(3.0003, 6.4595)	(-1, 10.4598)	(1, 8.4598)	4.7299
VII	(0, 9.6125)	(-1.5, 11.1125)	(1.5, 8.1125)	(-1, 10.6125)	(1, 8.6125)	4.8062

5. Discussion and Conclusion

Section 3 examines how, within a very wide range of ψ, the proposed estimator $\hat{\hat{Y}}_{Md}$ behaves more efficiently than the other estimators namely $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}^*_R$ and $\bar{y}^*_P$. Table 4 provides the effective ranges of ψ along with its optimum values for which the proposed estimator $\hat{\hat{Y}}_{Md}$ is more efficient than the other existing estimators as far as the *MSE* criterion is considered. In section 2 we see that *MSE* of the estimator $\hat{\hat{Y}}_{Md}$ is equivalent to the *MSE* of $\hat{\bar{Y}}_{reg}$. But using the procedure to lower the *MSE* and forming the new proposed estimator by simply conditioning the parent estimator, we obtain a more efficient estimator than the linear regression estimator. The two estimators (linear regression estimator and new proposed estimator) needs an equal number of prior knowledge of population parameters (S_y and S_x) but the reason why the latter is more efficient is it utilizes the knowledge of *Bias* of

TABLE 5. *Percentage Relative Efficiencies (PREs)* of various estimators with respect to $\bar{y}$

Estimators	**Populations**						
	I	II	III	IV	V	VI	VII
$\bar{y}$	100	100	100	100	100	100	100
$\bar{y}_R$	828.89	414.66	448.4	205.35	156.39	*	*
$\bar{y}_P$	*	*	*	*	*	167.58	187.08
$\bar{y}_R^*$	202.85	131.59	359.38	214.74	121.53	*	*
$\bar{y}_P^*$	*	*	*	*	*	121.37	149.13
$\hat{\bar{Y}}_{Md}$	859.11	740.19	468.98	249.14	208.45	187.10	859.11
$\hat{\bar{Y}}_w$	1076.6	844.75	474.85	249.24	209.24	187.52	868.68

* Data is not applicable.

parent population. This is an additional work to see how different estimators with different *Bias* will affect the *MSEs*, which is a research question and is left by the authors for further work. In addition, our theoretical results is supported numerically based on the results obtained in Table 5 using the data sets as shown in Table 2 along with the required values of various parameters in Table 3. Table 5 exhibits that there is a considerable gain in efficiency by using proposed estimator $\hat{\bar{Y}}_w$ over the estimators $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$, $\bar{y}_P^*$, and $\hat{\bar{Y}}_{Md}$. Thus, the new proposed estimator is more appropriate, in comparison to all the other existing estimators, for estimating the unknown mean $\bar{Y}$ of the study variable Y. Hence, the proposed estimator $\hat{\bar{Y}}_w$ should be preferred in practice. The present study deals with the estimation of unknown mean $\bar{Y}$ under *SRSWOR* scheme. It can also be extended to double (or two-phase) sampling, two-stage sampling and other sampling designs.

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NEW INTEGRAL TYPE INEQUALITIES VIA RAINA–CONVEX FUNCTIONS AND ITS APPLICATIONS

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Abstract. In this work, we discuss and introduce the novel literature about Raina–convex function and its algebraic properties. In addition, we elaborate and investigate Hermite-Hadamard and Fejér–type inequalities for newly discussed definition. Finally, using the newly introduced definition, we find and prove amazing new integral type inequalities and applications for mean to positive real numbers. The amazing techniques and wonderful ideas of this paper may inspire and motivate for further activities and research in this direction furthermore.

1. Introduction

During the whole of the 20^{th} century, an enormous and extreme research activity was done and fruitful ideas and magnificent results were obtained in mathematical analysis, functional analysis , convex analysis, mathematical economics and non–linear optimization. But interesting and tremendous book namely "Inequalities", which is written by Hardy, Littlewood and Polya. This book has played an elegant role in popularization and importance of the subject of convex functions. The modern and amazing viewpoint on convexity entails a powerful, enlighten and distinguish interaction between analysis and geometry, which makes and enables the readers to shear a sense of excitement. The theory of convexity encompasses a large variety of classes of convex functions like functions, s–convex, p–convex, log–convex, h–convex, quasi convex and exponential type convex functions while it

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is good to understand and what they have in common, it is of equal importance to look inside their own field. The theory of convexity also played a magnificent act in the advances of theory of inequalities. Inequalities have a lot of applications in statistical problems, probability and numerical quadrature formulas. Due to rich and paramount history, convex analysis and inequalities have become an attractive, interesting and absorbing field for the researchers and for the attention of the reader, see [1, 3, 4, 8, 9, 16, 18–22].
In recent years, many researchers working in the direction of convexity and generalized convexity of Raina type using meaningful ideas and magnificent techniques to bring a new dimension to mathematical analysis and applied mathematics with different features in the literature. Interested readers see the references [2, 7, 14, 15]. So that is the main aim and motivation of our work. Before we start, we need the following necessary known definitions and literature and throughout the paper, "(H–H)" means Hermite–Hadamard inequality and "diff mapp" means differential mapping.

2. Preliminaries

In this section we recall some basic definitions.

Definition 1. *[10] A $\zeta : \mathscr{H} \to \mathscr{R}$ is called convex, if*

$$\zeta\left(\kappa\tau_1 + (1-\kappa)\tau_2\right) \leq \kappa\zeta\left(\tau_1\right) + (1-\kappa)\zeta\left(\tau_2\right), \tag{1}$$

holds $\forall\ \tau_1, \tau_2 \in \mathscr{H}$ and $\kappa \in [0,1]$.

The well known and remarkable inequality concerning convex function is Hermite–Hadamard inequality given as:

Theorem 1. *[6] If $\zeta : \mathscr{H} = [\tau_1, \tau_2] \to \mathscr{R}$ is a convex function, then*

$$\zeta\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\nu)d\nu \leq \frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}. \tag{2}$$

The double inequality (2) is in reverse order if ζ is a concave function.

Theorem 2. *If $\zeta : \mathscr{H} = [\tau_1, \tau_2] \to \mathscr{R}$ is a convex function, then*

$$\zeta\left(\frac{\tau_1+\tau_2}{2}\right)\int_{\tau_1}^{\tau_2}\xi(\nu)d\nu \leq \int_{\tau_1}^{\tau_2}\zeta(\nu)\xi(\nu)d\nu \leq \frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}\int_{\tau_1}^{\tau_2}\xi(\nu)d\nu. \tag{3}$$

In 1906, L. Fejér [5] proved the above integral inequality (3) which is known in the literature as Fejér inequality. Since the researchers have shown interest in the above inequality and as a result, various generalizations and improvements have have been appeared in the literature. This inequality has remained an area of great and vital field for research activities due to its widespread views and robustness

applications in the field of mathematical and convex analysis.
In 2005, Raina [12] introduced a class of functions defined formally by

$$\mathcal{F}^{\aleph}_{\varkappa,\lambda}(z) = \mathcal{F}^{\aleph(0),\,\aleph(1),\ldots}_{\varkappa,\lambda}(z) = \sum_{k=0}^{+\infty} \frac{\aleph(k)}{\Gamma(\varkappa k+\lambda)} z^k, \tag{4}$$

where $\aleph = (\aleph(0), \ldots, \aleph(k), \ldots)$ and $\varkappa, \lambda > 0, |z| < \mathscr{R}$.
If $\varkappa = 1, \lambda = 0$ and $\aleph(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \ldots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol α_k denote the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k = 0, 1, 2, \ldots,$$

and restrict its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k.$$

Also, if $\aleph = (1, 1, \ldots)$ with $\varkappa = \alpha$, $(Re(\alpha) > 0), \lambda = 1$, then

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1+\alpha k)}.$$

The above function is called a classical Mittag–Leffler function.

Theorem 3. *[11] Suppose $\zeta : \mathscr{H} \subseteq [0, \infty) \to \mathscr{R}$ be a diff mapp on $\mathscr{H}^o$ of $\mathscr{H}$ such that $\zeta'' \in L^1[\tau_1, \tau_2]$, where $\tau_1, \tau_2 \in \mathscr{H}$ with $\tau_1 < \tau_2$. If $|\zeta|$ is convex on $[\tau_1, \tau_2]$, then*

$$\left|\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \zeta(\nu)d\nu\right| \leq \frac{(\tau_2-\tau_1)^2}{192}\left\{|\zeta''(\tau_1)| + 6\left|\zeta''(\frac{\tau_1+\tau_2}{2})\right| + |\zeta''(\tau_2)|\right\}. \tag{5}$$

Theorem 4. *[11] Suppose $\zeta : \mathscr{H} \subseteq [0, \infty) \to \mathscr{R}$ be a diff mapp on $\mathscr{H}^o$ such that $\zeta'' \in L^1[\tau_1, \tau_2]$, where $\tau_1, \tau_2 \in \mathscr{H}$ with $\tau_1 < \tau_2$. If $|\zeta''|^\ell$ for $\ell \geq 1$ is convex on $[\tau_1, \tau_2]$, then*

$$\begin{aligned} &\left|\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \zeta(\nu)d\nu\right| \leq \frac{(\tau_2-\tau_1)^2}{48}\Big(\frac{3}{4}\Big)^{\frac{1}{\ell}} \qquad (6) \\ &\times \left\{ \left(\frac{|\zeta''(\tau_1)|^\ell}{3} + |\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell \right)^{\frac{1}{\ell}} + \left(\left|\zeta''(\frac{\tau_1+\tau_2}{2})\right|^\ell + \frac{|\zeta''(\tau_2)|^\ell}{3} \right)^{\frac{1}{\ell}} \right\}. \end{aligned}$$

Lemma 1. *[17] Suppose $\zeta : \mathscr{H} \subseteq \mathscr{R} \to \mathscr{R}$ be a diff mapp on $\mathscr{H}^o$ such that $\zeta' \in L^1[\tau_1, \tau_2]$, where $\tau_1, \tau_2 \in \mathscr{H}$ with $\tau_1 < \tau_2$. If $\alpha, \beta \in \mathscr{R}$, then*

$$\frac{\alpha\zeta(\tau_1) + \beta\zeta(\tau_2)}{2} + \frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\nu)d\nu = \frac{\tau_2-\tau_1}{4} \tag{7}$$

$$\times \int_0^1 \Bigg[(1-\alpha-\kappa)\zeta'\Big(\kappa\tau_1 + (1-\kappa)\frac{\tau_1+\tau_2}{2}\Big) + (\beta-\kappa)\zeta'\Big(\kappa\frac{\tau_1+\tau_2}{2} + (1-\kappa)\tau_2\Big)\Bigg]d\kappa.$$

Lemma 2. *[17] For $\mathscr{L} > 0$ and $0 \leq \varepsilon \leq 1$, we have*

$$\int_0^1 |\varepsilon-\kappa|^{\mathscr{L}} d\kappa = \frac{\varepsilon^{\mathscr{L}+1} + (1-\varepsilon)^{\mathscr{L}+1}}{\mathscr{L}+1} \tag{8}$$

$$\int_0^1 \kappa|\varepsilon-\kappa|^{\mathscr{L}} d\kappa = \frac{\varepsilon^{\mathscr{L}+2} + (\mathscr{L}+1+\varepsilon)(1-\varepsilon)^{\mathscr{L}+1}}{(\mathscr{L}+1)(\mathscr{L}+2)}$$

Owing to the aforementioned trend and inspired by the ongoing activities in this absorbing field, we organize the paper in the following pattern. Firstly, we introduce Raina–convex function and its properties. Secondly, we debate and investigate (H-H) and Fejér-type integral inequalities for Riana–convex functions. Furthermore, we find integral inequalities and applications about fractional calculus regarding Riana–convex functions.

3. Raina–Convex functions and its properties

In this section we are going to add a new definition namely Raina–convex function and will study some of their algebraic properties..

Definition 2. *A function $\zeta : \mathscr{H} \to \mathscr{R}$ is said to be Raina–convex function on $\mathscr{H}$, if the following inequality*

$$\zeta(\kappa\tau_1 + (1-\kappa)\tau_2) \leq \zeta(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2)) \tag{9}$$

holds $\forall$ $[\tau_1, \tau_2] \in \mathscr{H}$ and $\kappa \in [0,1]$, where $\varkappa, \lambda > 0$ and $\aleph = (\aleph(1), \aleph(2), ..., \aleph(\kappa))$ is a bounded sequence of positive real no.

Note that when we choose $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2)) = \zeta(\tau_1) - \zeta(\tau_2)$, then Raina–convex function collapse to the classical convex function.

Theorem 5. *Let $\zeta, \xi : \mathscr{H} = [\tau_1, \tau_2] \to \mathscr{R}$. If ζ and ξ are Raina–convex functions then*
(i) $\zeta + \xi$ is Raina–convex function.
(ii) For $c \in \mathscr{R}$ and $(c \geq 0)$ then $c\zeta$ is Raina–convex function.

Proof. (i) Let ζ and ξ be a Raina–convex functions, then

$$\begin{aligned}
&(\zeta+\xi)(\kappa\tau_1 + (1-\kappa)\tau_2) \\
&= \zeta(\kappa\tau_1 + (1-\kappa)\tau_2) + \xi(\kappa\tau_1 + (1-\kappa)\tau_2) \\
&\leq \zeta(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2)) + \xi(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\xi(\tau_1) - \xi(\tau_2))
\end{aligned}$$

$$\leq (\zeta+\xi)(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}((\zeta+\xi)(\tau_1) - (\zeta+\xi)(\tau_2)).$$

(ii) Let ζ be a Raina–convex function and $c \in \mathscr{R}$, then

$$\begin{aligned}
&(c\zeta)(\kappa\tau_1 + (1-\kappa)\tau_2)\\
&= c\big[\zeta(\kappa\tau_1 + (1-\kappa)\tau_2)\big]\\
&\leq c\big[\zeta(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2))\big]\\
&\leq (c\zeta)(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}((c\zeta)(\tau_1) - (c\zeta)(\tau_2)),
\end{aligned}$$

which completes the proof. □

Theorem 6. *Let $\zeta : \mathscr{H} \to \mathscr{J}$ be a Raina–convex function and $\xi : \mathscr{J} \to \mathscr{R}$ is non-decreasing function. Then $\xi \circ \zeta : \mathscr{H} \to \mathscr{R}$ is Raina–convex function.*

Proof. $\forall\ \tau_1, \tau_2 \in \mathscr{H}$ and $\kappa \in [0, 1]$, we have

$$\begin{aligned}
&(\xi \circ \zeta)\,(\kappa\tau_1 + (1-\kappa)\tau_2)\\
&= \xi(\zeta(\kappa\tau_1 + (1-\kappa)\tau_2))\\
&\leq \xi\left[\zeta(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2))\right]\\
&\leq \xi(\zeta(\tau_2)) + \kappa\xi\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2))\\
&=\ (\xi \circ \zeta)(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}((\xi \circ \zeta)(\tau_1) - (\xi \circ \zeta)(\tau_2)),
\end{aligned}$$

which completes the proof. □

Theorem 7. *Let $\zeta_i : \mathscr{H} = [\tau_1, \tau_2] \to \mathscr{R}$ be an arbitrary family of Raina–convex functions and let $\zeta(\tau) = \sup_i \zeta_i(\tau)$. If $\mathscr{H} = \{\tau \in [\tau_1, \tau_2] : \zeta(\tau) < +\infty\} \neq \emptyset$, then $\mathscr{H}$ is an interval and ζ is Raina–convex function.*

Proof. $\forall\ \tau_1, \tau_2 \in \mathscr{H}$ and $\kappa \in [0, 1]$, we have

$$\begin{aligned}
&\zeta(\kappa\tau_1 + (1-\kappa)\tau_2)\\
&= \sup_j \zeta_j(\kappa\tau_1 + (1-\kappa)\tau_2)\\
&\leq \sup_j \zeta_j\,(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\sup_j \zeta_j(\tau_1) - \sup_j \zeta_j(\tau_2))\\
&=\ \zeta\,(\tau_2) + \kappa\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2)) < +\infty,
\end{aligned}$$

which completes the proof. □

4. New version of H–H and Fejér–type inequalities

Theorem 8. *Let $\zeta : \mathscr{H} \subseteq \mathscr{R} \to \mathscr{R}$ be a Raina–convex function with $\zeta \in L^1[\tau_1.\tau_2]$, where $\tau_1, \tau_2 \in \mathscr{H}$ with $\tau_1 < \tau_2$, then*

$$\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2} \mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1+\tau_2-\mu) - \zeta(\mu))d\mu \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \zeta(\mu)d\mu \tag{10}$$

$$\leq \zeta(\tau_2) + \frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1) - \zeta(\tau_2)).$$

Proof. Using (9), with $\mu = \kappa\tau_1 + (1-\kappa)\tau_2$, $\nu = (1-\kappa)\tau_1 + \kappa\tau_2$ and $\kappa = \frac{1}{2}$, we find that

$$\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) \leq \zeta\Big((1-\kappa)\tau_1+\kappa\tau_2\Big) + \frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\kappa\tau_1+(1-\kappa)\tau_2) - \zeta((1-\kappa)\tau_1+\kappa\tau_2)\Big)$$

Thus by integrating, we obtain

$$\begin{aligned}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) &\leq \int_0^1 \zeta\Big((1-\kappa)\tau_1+\kappa\tau_2\Big)d\kappa + \frac{1}{2}\int_0^1 \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\kappa\tau_1+(1-\kappa)\tau_2) \\ &\qquad -\zeta((1-\kappa)\tau_1+\kappa\tau_2)\Big)d\kappa \\ &\leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu + \frac{1}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2} \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1+\tau_2-\mu)-\zeta(\mu)\Big)d\mu\end{aligned}$$

So that

$$\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2} \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1+\tau_2-\mu)-\zeta(\mu)\Big)d\mu \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu. \tag{11}$$

This completes the proof of left side of above inequality. For the right side using $\mu = \tau_1$ and $\nu = \tau_2$ in (9), and integrating over $[0,1]$, we have

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu \leq \zeta(\tau_2) + \frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1)-\zeta(\tau_2)). \tag{12}$$

By simplification, the inequalities (11) and (12), we get the inequality (10). □

Remark 1. *Taking $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu) = \mu-\nu$, we reduce* (10) *to inequality (2).*

Remark 2. *Under the assumption of Theorem 8, if we take $\aleph = (1,1,...)$ with $\varkappa = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function*

$$\begin{aligned}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{2(\tau_2-\tau_1)}\int_{\tau_1}^{\tau_2} E_\alpha(\zeta(\tau_1+\tau_2-\mu)-\zeta(\mu))d\mu &\leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu \\ &\leq \zeta(\tau_2) + \frac{1}{2}E_\alpha(\zeta(\tau_1)-\zeta(\tau_2)).\end{aligned} \tag{13}$$

Theorem 9. *Let ζ and ξ be non-negative generalized convex functions of Raina type with $\zeta\xi \in L^1[\tau_1,\tau_2]$, where $\tau_1,\tau_2 \in \mathscr{H}$, $\tau_1 < \tau_2$. Then*

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu \leq M'(\tau_1,\tau_2) \tag{14}$$

where

$$M'(\tau_1,\tau_2) = \zeta(\tau_2)\xi(\tau_2) + \frac{1}{2}\zeta(\tau_2)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big) + \frac{1}{2}\xi(\tau_2)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)$$

$$+\frac{1}{3}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big)$$

Proof. Since ζ and ξ be a Raina–convex functions, we have

$$\zeta\Big(\kappa\tau_1+(1-\kappa)\tau_2\Big)\le\zeta(\tau_2)+\kappa\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)$$
$$\xi\Big(\kappa\tau_1+(1-\kappa)\tau_2\Big)\le\xi(\tau_2)+\kappa\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big)$$

For all $\kappa\in[0,1]$. Since ζ and ξ are non–negative, we have

$$\zeta\Big(\kappa\tau_1+(1-\kappa)\tau_2\Big)\xi\Big(\kappa\tau_1+(1-\kappa)\tau_2\Big)\le\zeta(\tau_2)\xi(\tau_2)+\kappa\zeta(\tau_2)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big)$$
$$+\kappa\xi(\tau_2)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)+\kappa^2\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big)$$

integrating over $[0,1]$ both sides, we have

$$\int_0^1\zeta\Big(\kappa\tau_1+(1-\kappa)\tau_2\Big)\xi\Big(\kappa\tau_1+(1-\kappa)\tau_2\Big)d\kappa\le\zeta(\tau_2)\xi(\tau_2)$$
$$+\frac{1}{2}\zeta(\tau_2)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big)+\frac{1}{2}\xi(\tau_2)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)$$
$$+\frac{1}{3}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta\tau_2)\Big)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\xi(\tau_1)-\xi(\tau_2)\Big)$$

then

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu\le M'(\tau_1,\tau_2).$$

□

Remark 3. *Taking* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=\mu-\nu$ *in above inequality* (14)*, we get inequality* (1.4) *in [13].*

Remark 4. *Under the assumption of Theorem 9, if we take* $\aleph=(1,1,...)$ *with* $\varkappa=\alpha,\ \lambda=1$*, we get the following inequality involving classical Mittag–Leffler function*

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu\le M'(\tau_1,\tau_2)\tag{15}$$

where

$$M'(\tau_1,\tau_2)=\zeta(\tau_2)\xi(\tau_2)+\frac{1}{2}\zeta(\tau_2)E_\alpha\Big(\xi(\tau_1)-\xi(\tau_2)\Big)+\frac{1}{2}\xi(\tau_2)E_\alpha\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)$$
$$+\frac{1}{3}E_\alpha\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)E_\alpha\Big(\xi(\tau_1)-\xi(\tau_2)\Big)$$

Theorem 10. *Let* ζ *be a Raina–convex function with* $\zeta\in L^1[\tau_1,\tau_2]$*, where* $\tau_1,\tau_2\in\mathscr{H}$*,* $\tau_1<\tau_2$*, and* $\xi:\mathscr{H}=[\tau_1,\tau_2]\to\mathscr{R}$ *be non–negative, integrable symmetric about* $\frac{\tau_1+\tau_2}{2}$*, then*

$$\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu\le\Big[\zeta(\tau_2)+\frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)\Big]\int_{\tau_1}^{\tau_2}\xi(\mu)d\mu.\tag{16}$$

Proof. Since ζ be a Raina–convex function and ξ is non-negative integrable and symmetric about $\frac{\tau_1+\tau_2}{2}$, we find that

$$\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu=\frac{1}{2}\Big[\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu+\int_{\tau_1}^{\tau_2}\zeta(\tau_1+\tau_2-\mu)g(\tau_1+\tau_2-\mu)d\mu\Big]$$
$$=\frac{1}{2}\int_{\tau_1}^{\tau_2}\left[\Big(\zeta(\mu)+\zeta(\tau_1+\tau_2-\mu)\Big)\xi(\mu)d\mu\right]$$
$$=\frac{1}{2}\int_{\tau_1}^{\tau_2}\left[\zeta\Big(\frac{\tau_2-\mu}{\tau_2-\tau_1}\tau_1+\frac{\mu-\tau_1}{\tau_2-\tau_1}\tau_2\Big)+\zeta\Big(\frac{\mu-\tau_1}{\tau_2-\tau_1}\tau_1+\frac{\tau_2-\mu}{\tau_2-\tau_1}\tau_2\Big)\right]\xi(\mu)d\mu$$
$$\leq\frac{1}{2}\int_{\tau_1}^{\tau_2}\left[\Big(\zeta(\tau_2)+\frac{\tau_2-\mu}{\tau_2-\tau_1}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1)-\zeta(\tau_2))\Big)\right.$$
$$\left.+\Big(\zeta(\tau_2)+\frac{\mu-\tau_1}{\tau_2-\tau_1}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1)-\zeta(\tau_2))\Big)\right]\xi(\mu)d\mu$$
$$\leq\Big[\zeta(\tau_2)+\frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\zeta(\tau_1)-\zeta(\tau_2))\Big]\int_{\tau_1}^{\tau_2}\xi(\mu)d\mu,$$

which completes the proof. □

Remark 5. (i) *Taking* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=\mu-\nu$ *and* $\xi(x)=1$, *then inequality* (16) *reduce to the inequality (2).*
(ii) *Taking* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=\mu-\nu$, *then inequality* (16) *reduce to the inequality (3).*
(iii) *Under the assumption of Theorem 10, if we take* $\aleph=(1,1,...)$ *with* $\varkappa=\alpha,\ \lambda=1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\int_{\tau_1}^{\tau_2}\zeta(\mu)\xi(\mu)d\mu\leq\Big[\zeta(\tau_2)+\frac{1}{2}E_\alpha\Big(\zeta(\tau_1)-\zeta(\tau_2)\Big)\Big]\int_{\tau_1}^{\tau_2}\xi(\mu)d\mu. \tag{17}$$

5. New integral type inequalities via Raina-convex function

Theorem 11. *Suppose* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta'\in L^1[\tau_1,\tau_2]$, *where* $\tau_1,\tau_2\in\mathscr{H}$, $\tau_1<\tau_2$. *If* $|\zeta'(\mu)|^\ell$ *for* $\ell\geq 1$ *is Raina–convex function on* $[\tau_1,\tau_2]$ *and* $0\leq\alpha,\ \beta\leq 1$ *then*

$$\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2}+\frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$
$$\leq\frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}}\Bigg\{(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|^\ell+(4-9\alpha+12\alpha^2-2\alpha^3)$$
$$\times\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Big]^{\frac{1}{\ell}}+(1-2\beta+2\beta^2)^{1-\frac{1}{\ell}}\Big[(6-12\beta+12\beta^2)|\zeta'(\tau_2)|^\ell$$

$$+ (2-3\beta+2\beta^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big]^{\frac{1}{\ell}}\Bigg\}. \tag{18}$$

Proof. In case $\ell > 1$, using lemma (1), Raina–convexity of $|\zeta'(x)|^\ell$ on $[\tau_1,\tau_2]$ and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2}+\frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|\\
&\le \frac{\tau_2-\tau_1}{4}\Big[\int_0^1 |1-\alpha-\kappa|\Big|\zeta'(\kappa\tau_1+(1-\kappa)\frac{\tau_1+\tau_2}{2})\Big|d\kappa\\
&+\int_0^1|\beta-\kappa|\Big|\zeta'(\kappa\frac{\tau_1+\tau_2}{2}+(1-\kappa)\tau_2)\Big|d\kappa\Big]\\
&\le \frac{\tau_2-\tau_1}{4}\Big[\Big(\int_0^1|1-\alpha-\kappa|d\kappa\Big)^{1-\frac{1}{\ell}}\Big[\int_0^1|1-\alpha-\kappa|\Big(|\zeta'(\tau_2)|^\ell+(\frac{1+\kappa}{2})\\
&\times\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Big)d\kappa\Big]^{\frac{1}{\ell}}+(\int_0^1|\beta-\kappa|d\kappa)^{1-\frac{1}{\ell}}\\
&\times\Big[\int_0^1|\beta-\kappa|\Big(|\zeta'(\tau_2)|^\ell+\frac{\kappa}{2}\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell)\Big)d\kappa\Big]^{\frac{1}{\ell}}\Big]
\end{aligned} \tag{19}$$

using lemma (2), by simplifications we obtain

$$\begin{aligned}
&\int_0^1|1-\alpha-\kappa|\Bigg(|\zeta'(\tau_2)|^\ell+(\frac{1+\kappa}{2})\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Bigg)d\kappa\\
&=\Bigg(|\zeta'(\tau_2)|^\ell+\frac{1}{2}\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Bigg)\int_0^1|1-\alpha-\kappa|d\kappa\\
&+\frac{1}{2}\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\int_0^1\kappa|1-\alpha-\kappa|d\kappa\\
&=\Bigg(|\zeta'(\tau_2)|^\ell+\frac{1}{2}\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Bigg)\Big(\frac{1}{2}-\alpha+\alpha^2\Big)\\
&+\frac{1}{12}\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Big[(1-\alpha)^3+\alpha^2(3-\alpha)\Big]\\
&=\frac{1}{2}(1-2\alpha+2\alpha^2)|\zeta'(\tau_2)|^\ell+\frac{1}{12}(4-9\alpha+12\alpha^2-2\alpha^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1|\beta-\kappa|\Bigg(|\zeta'(\tau_2)|^\ell+(\frac{\kappa}{2})\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Bigg)d\kappa\\
&=|\zeta'(\tau_2)|^\ell\int_0^1|\beta-t|d\kappa+\frac{1}{2}\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\int_0^1\kappa|\beta-\kappa|d\kappa
\end{aligned}$$

$$= |\zeta'(\tau_2)|^\ell(\frac{1}{2}-\beta-\beta^2)+\frac{1}{12}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Big(\beta^3+(2+\beta)(1-\beta)^2\Big)$$
$$= \frac{1}{2}(1-2\beta+2\beta^2)\zeta'(\tau_2)|^\ell+\frac{1}{12}(2-3\beta+2\beta^3)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big),$$

The following above two inequalities substitute into inequality (19) and according lemma (2) result in inequality (18) for $\ell > 1$.
For $\ell = 1$, from lemma (1) and (2) it follows that

$$\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2}+\frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$
$$\leq \frac{\tau_2-\tau_1}{4}\left[\int_0^1|1-\alpha-\kappa|\left(|\zeta'(\tau_2)|+(\frac{1+\kappa}{2})\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|-|\zeta'(\tau_2)|)\right)d\kappa\right.$$
$$\left.+\int_0^1|\beta-\kappa|\Big(|\zeta'(\tau_2)|+\frac{\kappa}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|-|\zeta'(\tau_2)|\Big)\Big)d\kappa\right]$$
$$=\frac{\tau_2-\tau_1}{48}\Big\{(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|+(4-9\alpha+12\alpha^2-2\alpha^3)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|-|\zeta'(\tau_2)|)$$
$$+(6-12\beta+12\beta^2)|\zeta'(\tau_2)|+(2-3\beta+2\beta^3)\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|-|\zeta'(\tau_2)|)\Big\}. \qquad (20)$$

□

Remark 6. (i) *Choosing* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=(\mu-\nu)$, *then the inequality* (18) *collapse to the inequality* (3.1) *in [17].*
(ii) *Under the assumption of Theorem 11, if we take* $\aleph=(1,1,...)$ *with* $\varkappa=\alpha,\ \lambda=1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2}+\frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|\leq\frac{\tau_2-\tau_1}{8}$$
$$\times(\frac{1}{6})^{\frac{1}{\ell}}\Big\{(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|^\ell+(4-9\alpha+12\alpha^2-2\alpha^3)$$
$$\times E_\alpha\Big(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell\Big)\Big]^{\frac{1}{\ell}}+(1-2\beta+2\beta^2)^{1-\frac{1}{\ell}}\Big[(6-12\beta+12\beta^2)|\zeta'(\tau_2)|^\ell$$
$$+(2-3\beta+2\beta^3)E_\alpha(|\zeta'(\tau_1)|^\ell-|\zeta'(\tau_2)|^\ell)\Big]^{\frac{1}{\ell}}\Big\}. \qquad (21)$$

(iii) *Choosing* $\alpha=\beta$ *in above Theorem (11), we derive the following corollary,*

Corollary 1. *Let* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta'\in L_1[\tau_1,\tau_2]$, *where* $\tau_1,\tau_2\in\mathscr{H}$, $\tau_1<\tau_2$. *If* $|\zeta'(\mu)|^\ell$ *for* $\ell\geq1$ *is Raina–convex function on*

$[\tau_1, \tau_2]$ *and* $0 \leq \alpha \leq 1$*then*

$$\left|\frac{\alpha}{2}[\zeta(\tau_1)+\zeta(\tau_2)]+(1-\alpha)\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_1-\tau_2}\zeta(\mu)d\mu\right| \leq \frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}}$$
$$\times\Bigg\{(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|^{\ell} \tag{22}$$
$$+(4-9\alpha+12\alpha^2-2\alpha^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell})-|\zeta'(\tau_2)|^{\ell}\Big]^{\frac{1}{\ell}}$$
$$+\Big[(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|^{\ell}+(2-3\alpha+2\alpha^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\Bigg\}.$$

Remark 7. (i) *Choosing* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=\mu-\nu$ *in Corollary (1), then inequality (22) collapse to inequality* (3.5) *in [17].*
(ii) *Under the assumption of Corollary 1, if we take* $\aleph=(1,1,...)$ *with* $\varkappa=\alpha,\ \lambda=1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\left|\frac{\alpha}{2}[\zeta(\tau_1)+\zeta(\tau_2)]+(1-\alpha)\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_1-\tau_2}\zeta(\mu)d\mu\right| \leq \frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}} \tag{23}$$
$$\times\Bigg[(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|^{\ell}$$
$$+(4-9\alpha+12\alpha^2-2\alpha^3)E_{\alpha}(|\zeta'(\tau_1)|^{\ell})-|\zeta'(\tau_2)|^{\ell}\Big]^{\frac{1}{\ell}}$$
$$+\Big[(6-12\alpha+12\alpha^2)|\zeta'(\tau_2)|^{\ell}+(2-3\alpha+2\alpha^3)E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\Bigg].$$

(iii) *choosing* $\alpha=\beta=\frac{1}{2},\frac{1}{3}$, *respectively, in above Theorem (11), we can obtain the following inequality,*

Corollary 2. *Suppose* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta'\in L^1[\tau_1,\tau_2]$ *where* $\tau_1,\tau_2\in\mathscr{H}$, $\tau_1<\tau_2$. *If* $|\zeta'(\mu)|^{\ell}$ *for* $\ell\geq 1$ *is Raina–convex function on* $[\tau_1,\tau_2]$ *and* $0\leq\alpha,\ \beta\leq 1$, *then*

$$\left|\frac{1}{2}\Big[\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}+\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(x)dx\right| \leq \frac{\tau_2-\tau_1}{16}\Big(\frac{1}{12}\Big)^{\frac{1}{\ell}} \tag{24}$$
$$\times\Bigg\{\Big[12|\zeta'(\tau_2)|^{\ell}+9\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}$$
$$+\Big[12|\zeta'(\tau_2)|^{\ell}+3\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\Bigg\},$$

$$\left|\frac{1}{6}\Big[\zeta(\tau_1)+\zeta(\tau_2)+4\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(x)dx\right|$$
$$\leq \frac{5(\tau_2-\tau_1)}{72}\left(\frac{1}{90}\right)^{\frac{1}{\ell}}\left\{\Big[90|\zeta'(\tau_2)|^{\ell}+61\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\right.$$
$$\left.+\Big[90|\zeta'(\tau_2)|^{\ell}+29\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\right\}.$$

Remark 8. *Under the assumption of Corollary 2, if we take* $\aleph=(1,1,...)$ *with* $\varkappa=\alpha$, $\lambda=1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\left|\frac{1}{2}\Big[\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}+\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|\leq\frac{\tau_2-\tau_1}{16}\left(\frac{1}{12}\right)^{\frac{1}{\ell}} \tag{25}$$
$$\times\left\{\Big[12|\zeta'(\tau_2)|^{\ell}+9E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\right.$$
$$\left.+\Big[12|\zeta'(\tau_2)|^{\ell}+3E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\right\},$$
$$\left|\frac{1}{6}\Big[\zeta(\tau_1)+\zeta(\tau_2)+4\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$
$$\leq \frac{5(\tau_2-\tau_1)}{72}\left(\frac{1}{90}\right)^{\frac{1}{\ell}}\left\{\Big[90|\zeta'(\tau_2)|^{\ell}+61E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\right.$$
$$\left.+\Big[90|\zeta'(\tau_2)|^{\ell}+29E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\right\}.$$

If we choose $\ell=1$ in Corollary (2), then we take the following inequality,

Corollary 3. *Suppose* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta'\in L^1[\tau_1,\tau_2]$ *where* $\tau_1,\tau_2\in\mathscr{H}$, $\tau_1<\tau_2$. *If* $|\zeta'(\mu)|$ *is Raina–convex function on* $[\tau_1,\tau_2]$

$$\left|\frac{1}{2}\Big[\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}+\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \tag{26}$$
$$\leq\frac{\tau_2-\tau_1}{16}\left[2|\zeta'(\tau_2)|+\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|-|\zeta'(\tau_2)|\Big)\right]$$

$$\left|\frac{1}{6}\Big[\zeta(\tau_1)+\zeta(\tau_2)+4\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$

$$\leq \frac{5(\tau_2-\tau_1)}{72}\Big[2|\zeta'(\tau_2)| + \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)| - |\zeta'(\tau_2)|\Big)\Big].$$

Remark 9. *(i) Choosing $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu) = (\mu-\nu)$, then inequalities (24) and (26) reduce to inequalities (3.6) and (3.7) in [17].*
(ii) Under the assumption of Corollary 3, if we take $\aleph = (1,1,...)$ with $\varkappa = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\left|\frac{1}{2}\Big[\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2} + \zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big] - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \tag{27}$$
$$\leq \frac{\tau_2-\tau_1}{16}\left[2|\zeta'(\tau_2)| + E_\alpha\Big(|\zeta'(\tau_1)| - |\zeta'(\tau_2)|\Big)\right]$$

$$\left|\frac{1}{6}\Big[\zeta(\tau_1)+\zeta(\tau_2) + 4\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)\Big] - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$
$$\leq \frac{5(\tau_2-\tau_1)}{72}\Big[2|\zeta'(\tau_2)| + E_\alpha\Big(|\zeta'(\tau_1)| - |\zeta'(\tau_2)|\Big)\Big].$$

Theorem 12. *Suppose $\zeta : \mathscr{H} \subseteq \mathscr{R} \to \mathscr{R}$ be a diff mapp on $\mathscr{H}^o$ with $\zeta' \in L_1[\tau_1,\tau_2]$, where $\tau_1,\tau_2 \in \mathscr{H}$, $\tau_1 < \tau_2$. If $|\zeta'(\mu)|^\ell$ is Raina–convex function on $[\tau_1,\tau_2]$ and $0 \leq \alpha,\ \beta \leq 1$, then*

$$\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2} + \frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \tag{28}$$
$$\leq \frac{\tau_2-\tau_1}{4}\Big[\frac{1}{2(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}}$$
$$\times\Bigg\{\Big[\Big(2(\ell+2)(1-\alpha)^{\ell+1} + 2(\ell+2)\alpha^{\ell+1}\Big)|\zeta'(\tau_2)|^\ell$$
$$+\Big((\ell+3-\alpha)(1-\alpha)^{\ell+1} + (2\ell+4-\alpha)\alpha^{\ell+1}\Big)$$
$$\times\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell\Big)\Big]^{\frac{1}{\ell}} + \Big[\Big(2(\ell+2)(1-\beta)^{\ell+1} + 2(\ell+2)\beta^{\ell+1}\Big)|\zeta'(\tau_2)|^\ell$$
$$+\Big(\beta^{\ell+2} + (\ell+1+\beta)(1-\beta)^{\ell+1}\Big)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell\Big)\Big]^{\frac{1}{\ell}}\Bigg\}.$$

Proof. In case $\ell > 1$, using the property of Raina–convexity of $|\zeta'(\mu)|^\ell$ on $[\tau_1,\tau_2]$ and Hölder's inequality

$$\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2} + \frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \tag{29}$$

$$
\begin{aligned}
&\leq \frac{\tau_2-\tau_1}{4}\Big[\int_0^1 (1-\alpha-\kappa)|\zeta'(\kappa\tau_1+(1-\kappa)\frac{\tau_1+\tau_2}{2})|d\kappa \\
&+\int_0^1 |\beta-\kappa|\Big|\zeta'(\kappa\frac{\tau_1+\tau_2}{2}+(1-\kappa)\tau_2)\Big|d\kappa\Big] \leq \frac{\tau_2-\tau_1}{4}\Bigg[(\int_0^1 d\kappa)^{1-\frac{1}{\ell}}\Big[\int_0^1 |1-\alpha-\kappa|^{\ell}| \\
&\times\Big(|\zeta'(\tau_2)|^{\ell}+(\frac{1+\kappa}{2})\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell}\Big)\Big)d\kappa\Big]^{\frac{1}{\ell}} \\
&+(\int_0^1 d\kappa)^{1-\frac{1}{\ell}}\Big[\int_0^1 |\beta-\kappa|^{\ell}|\Big(|\zeta'(\tau_2)|^{\ell}+(\frac{\kappa}{2})\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell}\Big)\Big)d\kappa\Big]^{\frac{1}{\ell}}\Bigg] \\
&\leq \frac{\tau_2-\tau_1}{4}\Bigg[\Big[\int_0^1 |1-\alpha-\kappa|^{\ell}|\Big(|\zeta'(\tau_2)|^{\ell}+(\frac{1+\kappa}{2})\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell}\Big)\Big)d\kappa\Big]^{\frac{1}{\ell}} \\
&+\Big[\int_0^1 |\beta-\kappa|^{\ell}|\Big(|\zeta'(\tau_2)|^{\ell}+(\frac{\kappa}{2})\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell}\Big)\Big)d\kappa\Big]^{\frac{1}{\ell}}\Bigg].
\end{aligned}
$$

By lemma (2) we have

$$
\begin{aligned}
&\int_0^1 |1-\alpha-\kappa|^{\ell}\Big(|\zeta'(\tau_2)|^{\ell}+(\frac{1+\kappa}{2})\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big)d\kappa \\
&=\Big(|\zeta'(\tau_2)|^{\ell}+\frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big)\int_0^1 |1-\alpha-\kappa|^{\ell}d\kappa \\
&+\frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\int_0^1 \kappa|1-\alpha-\kappa|^{\ell}d\kappa \\
&=\Big(|\zeta'(\tau_2)|^{\ell}+\frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big)\Big(\frac{(1-\alpha)^{\ell+1}+\alpha^{\ell+1}}{\ell+1}\Big) \\
&+\frac{1}{2}\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell}\Big)\Big(\frac{(1-\alpha)^{\ell+2}+(\ell+2-\alpha)\alpha^{\ell+1}}{(\ell+1)(\ell+2)}\Big) \\
&=\frac{1}{2(\ell+1)(\ell+2)}\Big[2(\ell+2)(1-\alpha)^{\ell+1}+2(\ell+2)\alpha^{\ell+1}\Big]|\zeta'(\tau_2)|^{\ell} \\
&+\Big[2(\ell+2)(1-\alpha)^{\ell+1}+(\ell+2)\alpha^{\ell+1}+(1-\alpha)^{\ell+2}+(\ell+2-\alpha)\alpha^{\ell+1}\Big] \\
&\times\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})=\frac{1}{2(\ell+1)(\ell+2)}\Bigg[\Big[2(\ell+2)(1-\alpha)^{\ell+1} \\
&+2(\ell+2)\alpha^{\ell+1}\Big]|\zeta'(\tau_2)|^{\ell}+\Big[(\ell+3-\alpha)(1-\alpha)^{\ell+1} \\
&+(2\ell+4-\alpha)\alpha^{\ell+1}\Big]\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Bigg],
\end{aligned}
$$

and

$$\int_0^1 |\beta-\kappa|^\ell \Big(|\zeta'(\tau_2)|^\ell + (\frac{\kappa}{2})\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big)d\kappa$$
$$= |\zeta'(\tau_2)|^\ell \int_0^1 |\beta-\kappa|^\ell d\kappa + \frac{1}{2}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell) \int_0^1 \kappa|\beta-\kappa|^\ell d\kappa$$
$$= |\zeta'(\tau_2)|^\ell \Big(\frac{\beta^{\ell+1}+(1-\beta)^{\ell+1}}{\ell+1}\Big) + \frac{1}{2}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)$$
$$\times \Big(\frac{\beta^{\ell+2}+(\ell+1+\beta)(1-\beta)^{\ell+1}}{(\ell+1)(\ell+2)}\Big)$$
$$= \frac{1}{2(\ell+1)(\ell+2)}\Bigg[[2(\ell+2)(1-\beta)^{\ell+1}+2(\ell+2)\beta^{\ell+1}]|\zeta'(\tau_2)|^\ell$$
$$+ [\beta^{\ell+2}+(\ell+1+\beta)(1-\beta)^{\ell+1}]\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Bigg].$$

If we put the last two inequalities into inequality (29), as a result we obtain the inequality (28) for $\ell > 1$. If we put $\ell = 1$, then the proof is the identitical as that of (20), and the theorem is investigated. □

Remark 10. *(i) Choosing* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu) = \mu-\nu$, *then the inequality* (28) *reduces to the inequality* (3.8) *in [17].*
(ii) Under the assumption of Corollary 12, if we take $\aleph = (1,1,...)$ *with* $\varkappa = \alpha,\ \lambda = 1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\left|\frac{\alpha\zeta(\tau_1)+\beta\zeta(\tau_2)}{2} + \frac{2-\alpha-\beta}{2}\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \tag{30}$$
$$\leq \frac{\tau_2-\tau_1}{4}\Big[\frac{1}{2(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}}$$
$$\times \Bigg\{\Big[\Big(2(\ell+2)(1-\alpha)^{\ell+1}+2(\ell+2)\alpha^{\ell+1}\Big)|\zeta'(\tau_2)|^\ell$$
$$+ \Big((\ell+3-\alpha)(1-\alpha)^{\ell+1}+(2\ell+4-\alpha)\alpha^{\ell+1}\Big)$$
$$\times E_\alpha\Big(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell\Big)\Big]^{\frac{1}{\ell}} + \Big[\Big(2(\ell+2)(1-\beta)^{\ell+1}+2(\ell+2)\beta^{\ell+1}\Big)|\zeta'(\tau_2)|^\ell$$
$$+ \Big(\beta^{\ell+2}+(\ell+1+\beta)(1-\beta)^{\ell+1}\Big)E_\alpha\Big(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell\Big)\Big]^{\frac{1}{\ell}}\Bigg\}.$$

Similarly to Corollaries of Theorem (11), *we can obtain the following Corollaries of Theorem* (12) .

Corollary 4. *Suppose $\zeta : \mathscr{H} \subseteq \mathscr{R} \to \mathscr{R}$ be a diff mapp on $\mathscr{H}^o$ with $\zeta' \in L^1[\tau_1, \tau_2]$ where $\tau_1, \tau_2 \in \mathscr{H}$, $\tau_1 < \tau_2$. If $|\zeta'(\mu)|^\ell$ is Raina–convex function on $[\tau_1, \tau_2]$ for $\ell \geq 1$ and $0 \leq \alpha \leq 1$, then*

$$\left|\frac{\alpha}{2}[\zeta(\tau_1) + \zeta(\tau_2)] + (1-\alpha)\zeta(\frac{\tau_1+\tau_2}{2}) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \tag{31}$$

$$\leq \frac{\tau_2-\tau_1}{4}\Big[\frac{1}{2(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}}\Bigg\{\Big[\Big(2(\ell+2)(1-\alpha)^{\ell+1} + 2(\ell+2)\alpha^{\ell+1}\Big)|\zeta'(\tau_2)|^\ell$$

$$+ \Big((\ell+3-\alpha)(1-\alpha)^{\ell+1} + (2\ell+4-\alpha)\alpha^{\ell+1}\Big)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big]^{\frac{1}{\ell}}$$

$$+ \Big[\Big(2(\ell+2)(1-\alpha)^{\ell+1} + 2(\ell+2)\alpha^{\ell+1}\Big)|\zeta'(\tau_2)|^\ell$$

$$+ \Big(\alpha^{\ell+2} + (\ell+1+\alpha)(1-\alpha)^{\ell+1}\Big)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big]^{\frac{1}{\ell}}\Bigg\}.$$

Remark 11. *Choosing $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu) = \mu - \nu$, then inequality* (31) *collapse to inequality* (3.11) *in [17].*

Corollary 5. *Suppose $\zeta : \mathscr{H} \subseteq \mathscr{R} \to \mathscr{R}$ be a diff mapp on $\mathscr{H}^o$ with $\zeta' \in L^1[\tau_1, \tau_2]$ where $\tau_1, \tau_2 \in \mathscr{H}$, $\tau_1 < \tau_2$. If $|\zeta'(\mu)|^\ell$ is Raina–convex function on $[\tau_1, \tau_2]$ for $\ell \geq 1$ and $0 \leq \alpha$, $\beta \leq 1$, then*

$$\left|\frac{1}{2}\Big[\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2} + \zeta(\frac{\tau_1+\tau_2}{2})\Big] - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right| \leq \frac{\tau_2-\tau_1}{8}\Big[\frac{1}{4(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}} \tag{32}$$

$$\times\Bigg\{\Big[\Big((4\ell+8)|\zeta'(\tau_2)|^\ell + (3\ell+6)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big)\Big]^{\frac{1}{\ell}}$$

$$+ \Big[\Big((4\ell+8)|\zeta'(\tau_2)|^\ell + (\ell+2)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big)\Big]^{\frac{1}{\ell}}\Bigg\}$$

$$\left|\frac{1}{6}\Big[f(\tau_1) + \zeta(\tau_2) + 4\zeta(\frac{\tau_1+\tau_2}{2})\Big] - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$

$$\leq \frac{\tau_2-\tau_1}{12}\Big[\frac{1}{18(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}}\Bigg\{\Big[\Big((3\ell+6)2^{\ell+2} + 6(\ell+2)\Big)|\zeta'(\tau_2)|^\ell$$

$$+ \Big((3\ell+8)2^{\ell+1} + (6\ell+11)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big)\Big]^{\frac{1}{\ell}} + \Big[\Big((3\ell+6)2^{\ell+2} + 6(\ell+2)\Big)$$

$$|\zeta'(\tau_2)|^\ell + \Big(1 + (3\ell+4)2^{\ell+1}\Big)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta'(\tau_1)|^\ell - |\zeta'(\tau_2)|^\ell)\Big]^{\frac{1}{\ell}}\Bigg\}.$$

Remark 12. *(i) Choosing* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=\mu-\nu$*, then the inequality* (32) *reduces to the inequality* (3.12) *in [17].*
(ii) Choosing $\ell=1$ *in Corollary* (4)*, then we get Corollary* (3)*.*
(iii) Under the assumption of Corollary 5, if we take $\aleph=(1,1,...)$ *with* $\varkappa=\alpha,\ \lambda=1$*, we get the following inequality involving classical Mittag–Leffler function*

$$\left|\frac{1}{2}\Big[\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}+\zeta(\frac{\tau_1+\tau_2}{2})\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}f(\mu)d\mu\right|\leq\frac{\tau_2-\tau_1}{8}\Big[\frac{1}{4(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}}$$
$$\times\Bigg\{\Big[\Big((4\ell+8)|\zeta'(\tau_2)|^{\ell}+(3\ell+6)E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big)\Big]^{\frac{1}{\ell}}$$
$$+\Big[\Big((4\ell+8)|\zeta'(\tau_2)|^{\ell}+(\ell+2)E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big)\Big]^{\frac{1}{\ell}}\Bigg\}$$
$$\left|\frac{1}{6}\Big[\zeta(\tau_1)+\zeta(\tau_2)+4\zeta(\frac{\tau_1+\tau_2}{2})\Big]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|$$
$$\leq\frac{\tau_2-\tau_1}{12}\Big[\frac{1}{18(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}}\Bigg\{\Big[\Big((3\ell+6)2^{\ell+2}+6(\ell+2)\Big)|\zeta'(\tau_2)|^{\ell}$$
$$+\Big((3\ell+8)2^{\ell+1}+(6\ell+11)E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big)\Big]^{\frac{1}{\ell}}+\Big[\Big((3\ell+6)2^{\ell+2}+6(\ell+2)\Big)$$
$$|\zeta'(\tau_2)|^{\ell}+\Big(1+(3\ell+4)2^{\ell+1}\Big)E_{\alpha}(|\zeta'(\tau_1)|^{\ell}-|\zeta'(\tau_2)|^{\ell})\Big]^{\frac{1}{\ell}}\Bigg\}. \tag{33}$$

For further results, we highlight the below Lemma which is proved in [11].

Lemma 3. *[11] Suppose* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta'\in L^1[\tau_1,\tau_2]$ *where* $\tau_1,\tau_2\in\mathscr{H}$ *and* $\tau_1<\tau_2$*, then*

$$\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu-\zeta(\frac{\tau_1+\tau_2}{2})=\frac{(\tau_2-\tau_1)^2}{16} \tag{34}$$
$$\times\left[\int_0^1\kappa^2\zeta''(\kappa\frac{\tau_1+\tau_2}{2}+(1-\kappa)\tau_1)d\kappa+\int_0^1(\kappa-1)^2\zeta''(\kappa\tau_2+(1-\kappa)\frac{\tau_1+\tau_2}{2})d\kappa\right].$$

Theorem 13. *Suppose* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta''\in L^1[\tau_1,\tau_2]$*, where* $\tau_1,\tau_2\in\mathscr{H}$ *and* $\tau_1<\tau_2$*. If* $|\zeta''(\mu)|$ *is Raina–convex function on* $[\tau_1,\tau_2]$*, then*

$$\left|\zeta(\frac{\tau_1+\tau_2}{2})-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\right|\leq\frac{(\tau_2-\tau_1)^2}{16}\left[\frac{1}{3}\Big(|\zeta''(\tau_1)|+|\zeta''(\frac{\tau_1+\tau_2}{2})|\Big)\right. \tag{35}$$

$$+\frac{1}{4}\Big(\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})|-|\zeta''(\tau_1)|)\Big)+\frac{1}{3}\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\tau_2)|-|\zeta''(\frac{\tau_1+\tau_2}{2})|)\Bigg].$$

Proof. From lemma (3), we have

$$\Big|\zeta(\frac{\tau_1+\tau_2}{2})-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\Big|\leq\frac{(\tau_2-\tau_1)^2}{16}\Big[\int_0^1\kappa^2|\zeta''(\kappa\frac{\tau_1+\tau_2}{2}+(1-\kappa)\tau_1)|d\kappa$$
$$+\int_0^1(\kappa-1)^2|\zeta''(\kappa\tau_2+(1-\kappa)\frac{\tau_1+\tau_2}{2})|d\kappa\Big]$$
$$\leq\frac{(\tau_2-\tau_1)^2}{16}\Big[\int_0^1\kappa^2\Big(|\zeta''(\tau_1)|+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})|-|\zeta''(\tau_1)|)\Big)d\kappa\Big]$$
$$+\frac{(\tau_2-\tau_1)^2}{16}\Bigg[\int_0^1(\kappa-1)^2\Big(|\zeta''(\frac{\tau_1+\tau_2}{2})|+\kappa\,\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\tau_2)|-|\zeta''(\frac{\tau_1+\tau_2}{2})|)\Big)d\kappa\Bigg]$$
$$=\frac{(\tau_2-\tau_1)^2}{16}\Bigg[\frac{1}{3}|\zeta''(\tau_1)|+\frac{1}{3}|\zeta''(\frac{\tau_1+\tau_2}{2})|+\frac{1}{4}\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})|-|\zeta''(\tau_1)|)$$
$$+\frac{1}{12}\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\tau_2)|-|\zeta''(\frac{\tau_1+\tau_2}{2})|)\Bigg]=\frac{(\tau_2-\tau_1)^2}{16}\Bigg[\frac{1}{3}\Big(|\zeta''(\tau_1)|+|\zeta''(\frac{\tau_1+\tau_2}{2})|\Big)$$
$$+\frac{1}{4}\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})|-|\zeta''(\tau_1)|)+\frac{1}{3}\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\tau_2)|-|\zeta''(\frac{\tau_1+\tau_2}{2})|)\Bigg].$$

□

Remark 13. *(i) Choosing* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu)=\mu-\nu$, *then inequality* (35) *reduce to inequality* (5).
(ii) Under the assumption of Theorem 13, if we take $\aleph=(1,1,...)$ *with* $\varkappa=\alpha$, $\lambda=1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\Big|\zeta(\frac{\tau_1+\tau_2}{2})-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\Big|\leq\frac{(\tau_2-\tau_1)^2}{16}\Bigg[\frac{1}{3}\Big(|\zeta''(\tau_1)|+|\zeta''(\frac{\tau_1+\tau_2}{2})|\Big)$$
$$+\frac{1}{4}\Big(E_\alpha(|\zeta''(\frac{\tau_1+\tau_2}{2})|-|\zeta''(\tau_1)|)\Big)+\frac{1}{3}E_\alpha(|\zeta''(\tau_2)|-|\zeta''(\frac{\tau_1+\tau_2}{2})|)\Bigg]. \tag{36}$$

Theorem 14. *Suppose* $\zeta:\mathscr{H}\subseteq\mathscr{R}\to\mathscr{R}$ *be a diff mapp on* $\mathscr{H}^o$ *with* $\zeta''\in L^1[\tau_1,\tau_2]$, *where* $\tau_1,\tau_2\in\mathscr{H}$ *and* $\tau_1<\tau_2$. *If* $|\zeta''(\mu)|^\ell$ *for* $\ell\geq1$ *with* $\frac{1}{p}+\frac{1}{\ell}=1$ *is Raina–convex function on* $[\tau_1,\tau_2]$, *then*

$$\Bigg|\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big)-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\Big)\Bigg|\leq\frac{(\tau_2-\tau_1)^2}{16}(\frac{1}{3})^{\frac{1}{p}} \tag{37}$$

$$\times \left[\Big(\frac{1}{3} |\zeta''(\tau_1)|^\ell + \frac{1}{4}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell - |\zeta''(\tau_1)|^\ell) \Big)^{\frac{1}{\ell}} + \Big(\frac{1}{3} |\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell + \frac{1}{12}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\tau_2)|^\ell - |\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell) \Big)^{\frac{1}{\ell}} \right].$$

Proof. If $p \geq 1$, using lemma (3) and power Mean Inequality, then

$$\left|\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \zeta(\mu)d\mu\Big)\right| \leq \frac{(\tau_2-\tau_1)^2}{16}\left[\int_0^1 \kappa^2 \Big|\zeta''\Big(\kappa\frac{\tau_1+\tau_2}{2} + (1-\kappa)\tau_1\Big)\Big| d\kappa + \int_0^1 (\kappa-1)^2 \Big|\zeta''\Big(\kappa\tau_2 + (1-\kappa)\frac{\tau_1+\tau_2}{2}\Big)\Big| d\kappa\right]$$

$$\leq \frac{(\tau_2-\tau_1)^2}{16}(\int_0^1 \kappa^2 d\kappa)^{\frac{1}{p}} \Big[\int_0^1 \kappa^2 \Big|\zeta''\Big(\kappa\frac{\tau_1+\tau_2}{2} + (1-\kappa)\tau_1\Big)\Big|^\ell d\kappa\Big]^{\frac{1}{\ell}}$$

$$+ \frac{(\tau_2-\tau_1)^2}{16}(\int_0^1 (\kappa-1)^2 d\kappa)^{\frac{1}{p}} \left(\int_0^1 (\kappa-1)^2 \Big|\zeta''\Big(\kappa\tau_2 + (1-\kappa)\frac{\tau_1+\tau_2}{2}\Big)\Big|^\ell d\kappa\right)^{\frac{1}{\ell}}$$

Because $|\zeta''|^\ell$ is Raina–convex function, we have

$$\int_0^1 \kappa^2 \Big|\zeta''\Big(\kappa\frac{\tau_1+\tau_2}{2} + (1-\kappa)\tau_1\Big)\Big|^\ell d\kappa \leq \frac{1}{3}|\zeta''(\tau_1)|^\ell + \frac{1}{4}\Big(\mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell - |\zeta''(\tau_2)|^\ell)\Big)$$

and

$$\int_0^1 (\kappa-1)^2 \Big|\zeta''(\kappa\tau_2 + (1-\kappa)\frac{\tau_1+\tau_2}{2})\Big|^\ell d\kappa \leq \frac{1}{3}|\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell + \frac{1}{12}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|\zeta''(\tau_2)|^\ell - |\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell\Big)$$

Therefore we have

$$\left|\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \zeta(\mu)d\mu\right| \leq \frac{(\tau_2-\tau_1)^2}{16}(\frac{1}{3})^{\frac{1}{p}}$$

$$\times \left[\Big(\frac{1}{3} |\zeta''(\tau_1)|^\ell + \frac{1}{4}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\frac{\tau_1+\tau_2}{2})| - |\zeta''(\tau_1)|^\ell) \Big)^{\frac{1}{\ell}} + \Big(\frac{1}{3} |\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell + \frac{1}{12}\, \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|\zeta''(\tau_2)|^\ell - |\zeta''(\frac{\tau_1+\tau_2}{2})|^\ell) \Big)^{\frac{1}{\ell}} \right].$$

□

Remark 14. *(i) Choosing* $\mathcal{F}^{\aleph}_{\varkappa,\lambda}(\mu-\nu) = \mu - \nu$, *then inequality* (37) *reduce to inequality* (6).

(ii) Under the assumption of Theorem 14, if we take $\aleph = (1, 1, ...)$ *with* $\varkappa = \alpha,\ \lambda = 1$*, we get the following inequality involving classical Mittag–Leffler function*

$$\left|\zeta\Big(\frac{\tau_1+\tau_2}{2}\Big) - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\mu)d\mu\Big)\right| \leq \frac{(\tau_2-\tau_1)^2}{16}(\frac{1}{3})^{\frac{1}{p}} \tag{38}$$
$$\times\left[\Big(\frac{1}{3}|\zeta''(\tau_1)|^{\ell} + \frac{1}{4}E_{\alpha}(|\zeta''(\frac{\tau_1+\tau_2}{2})|^{\ell} - |\zeta''(\tau_1)|^{\ell})\Big)^{\frac{1}{\ell}}\right.$$
$$\left.+\Big(\frac{1}{3}|\zeta''(\frac{\tau_1+\tau_2}{2})|^{\ell} + \frac{1}{12}E_{\alpha}(|\zeta''(\tau_2)|^{\ell} - |\zeta''(\frac{\tau_1+\tau_2}{2})|^{\ell})\Big)^{\frac{1}{\ell}}\right].$$

6. Applications

In this section, we recall the following special means for two positive real numbers τ_1, τ_2 where $\tau_1 < \tau_2$:

(1) The arithmetic mean
$$A = A(\tau_1, \tau_2) = \frac{\tau_1+\tau_2}{2}.$$
(2) The geometric mean
$$G = G(\tau_1, \tau_2) = \sqrt{\tau_1\tau_2}.$$
(3) The harmonic mean
$$H = H(\tau_1, \tau_2) = \frac{2\tau_1\tau_2}{\tau_1+\tau_2}.$$
(4) The p-logarithmic mean
$$L_p = L_p(\tau_1, \tau_2) = \left(\frac{\tau_2^{p+1}-\tau_1^{p+1}}{(p+1)(\tau_2-\tau_1)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\setminus\{0\}.$$
(5) The identric mean
$$I = I(\tau_1, \tau_2) = \frac{1}{e}\left(\frac{\tau_2^{\tau_2}}{\tau_1^{\tau_1}}\right)^{\frac{1}{\tau_2-\tau_1}}.$$
(6) The heronian mean
$$H_{w,s}(\tau_1, \tau_2) = \begin{cases} \Big[\dfrac{\tau_2^s + w(\tau_1\tau_2)^{\frac{s}{2}} + \tau_2^s}{w+2}\Big]^{\frac{1}{s}}, & \text{if } s \neq 0 \\ \sqrt{\tau_1\tau_2}, & \text{if } s = 0. \end{cases}$$

These means have a lot of applications in areas and different type of numerical approximations. However, the following simple relationship are known in the literature.

If we choose $\zeta(\mu) = \mu^s$ for $s \neq 0$ and $x > 0$ in Theorems (11) and (12), as a result we get the following inequalities for means.

Proposition 1. *Let $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 \neq \tau_2$, $\ell \geq 1$ and either $s > 1$ and $(s-1)\ell \geq 1$ or $s < 0$ Then*

$$\left|A(\alpha\tau_1^s, \beta\tau_2^s) + \frac{2-\alpha-\beta}{2}A^s(\tau_1,\tau_2) - L^s(\tau_1,\tau_2)\right| \leq \frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}} \tag{39}$$
$$\times \Bigg[(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)|s\tau_2^{s-1}|^\ell + (4-9\alpha+12\alpha^2-2\alpha^3)$$
$$\times\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|s\tau_1^{s-1}|^\ell - |s\tau_2^{s-1}|^\ell\Big)\Big]^{\frac{1}{\ell}} + (1-2\beta+2\beta^2)^{1-\frac{1}{\ell}}\Big[(6-12\beta+12\beta^2)|s\tau_2^{s-1}|^\ell$$
$$+ (2-3\beta+2\beta^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|s\tau_1^{s-1}|^\ell - |s\tau_2^{s-1}|^\ell)\Big]^{\frac{1}{\ell}}\Bigg].$$

Remark 15. *Under the assumption of Proposition 1, if we take $\aleph = (1, 1, ...)$ with $\varkappa = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function*

$$\left|A(\alpha\tau_1^s, \beta\tau_2^s) + \frac{2-\alpha-\beta}{2}A^s(\tau_1,\tau_2) - L^s(\tau_1,\tau_2)\right| \leq \frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}} \tag{40}$$
$$\times \Bigg[(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)|s\tau_2^{s-1}|^\ell + (4-9\alpha+12\alpha^2-2\alpha^3)$$
$$\times E_\alpha\Big(|s\tau_1^{s-1}|^\ell - |s\tau_2^{s-1}|^\ell\Big)\Big]^{\frac{1}{\ell}} + (1-2\beta+2\beta^2)^{1-\frac{1}{\ell}}\Big[(6-12\beta+12\beta^2)|s\tau_2^{s-1}|^\ell$$
$$+ (2-3\beta+2\beta^3) \times E_\alpha(|s\tau_1^{s-1}|^\ell - |s\tau_2^{s-1}|^\ell)\Big]^{\frac{1}{\ell}}\Bigg].$$

Proposition 2. *Let $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 \neq \tau_2$, $\ell \geq 1$ and either $s > 1$ and $(s-1)\ell \geq 1$ or $s < 0$*

$$\left|A(\alpha\tau_1^s, \beta\tau_2^s) + \frac{2-\alpha-\beta}{2}A^s(\tau_1,\tau_2) - L^s(\tau_1,\tau_2)\right| \leq \frac{(\tau_2-\tau_1)}{4}[\frac{1}{2(\ell+1)(\ell+2)}]^{\frac{1}{\ell}} \tag{41}$$
$$\times \Bigg[\Bigg[\Big(\Big[2(\ell+2)(1-\alpha)^{\ell+1} + 2(\ell+2)\alpha^{\ell+1}\Big]\Big)|s\tau_2^{s-1}|^\ell + \Big[(\ell+3-\alpha)(1-\alpha)^{\ell+1}$$
$$+ (2\ell+4-\alpha)\alpha^{\ell+1}\Big]\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}(|s\tau_1^{s-1}|^\ell - |s\tau_2^{s-1}|^\ell)\Bigg]^{\frac{1}{\ell}} + \Big[\Big(2(\ell+2)(1-\beta)^{\ell+1}$$
$$+ 2(\ell+2)\beta^{q+1}\Big)|s\tau_2^{s-1}|^\ell + \Big(\beta^{\ell+2} + (\ell+1+\beta)(1-\beta)^{\ell+1}\Big)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(|s\tau_1^{s-1}|^\ell - |s\tau_2^{s-1}|^\ell\Big)\Big]^{\frac{1}{\ell}}\Bigg].$$

If we choose $\zeta(\mu) = \ln \mu$ for $\mu > 0$ in theorems (11) and (12), as a result we get the following inequalities for mean.

Proposition 3. *For $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 \neq \tau_2$ and $\ell \geq 1$, we have*

$$\left|\frac{lnG^2(\tau_1^\alpha, \tau_2^\beta)}{2} + \frac{2-\alpha-\beta}{2} lnA(\tau_1,\tau_2) - lnI(\tau_1,\tau_2)\right| \leq \frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}} \Bigg[(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}$$
$$\times \Big[(6-12\alpha+12\alpha^2)(\frac{1}{\tau_2})^\ell + (4-9\alpha+12\alpha^2-2\alpha^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big((\frac{1}{\tau_1})^\ell - (\frac{1}{\tau_2})^\ell\Big)\Big]^{\frac{1}{\ell}} \quad (42)$$
$$+ (1-2\beta+2\beta^2)^{1-\frac{1}{\ell}}\Big[(6-12\beta+12\beta^2)(\frac{1}{\tau_2})^\ell + (2-3\beta+2\beta^3)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big((\frac{1}{\tau_1})^\ell - (\frac{1}{\tau_2})^\ell\Big)\Big]^{\frac{1}{\ell}}\Bigg].$$

Remark 16. *Under the assumption of Proposition 3, if we take $\aleph = (1,1,...)$ with $\varkappa = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function*

$$\left|\frac{lnG^2(\tau_1^\alpha, \tau_2^\beta)}{2} + \frac{2-\alpha-\beta}{2} lnA(\tau_1,\tau_2) - lnI(\tau_1,\tau_2)\right| \leq \frac{\tau_2-\tau_1}{8}(\frac{1}{6})^{\frac{1}{\ell}} \quad (43)$$
$$\times \Bigg[(1-2\alpha+2\alpha^2)^{1-\frac{1}{\ell}}\Big[(6-12\alpha+12\alpha^2)(\frac{1}{\tau_2})^\ell + (4-9\alpha+12\alpha^2-2\alpha^3)E_\alpha\Big((\frac{1}{\tau_1})^\ell - (\frac{1}{\tau_2})^\ell\Big)\Big]^{\frac{1}{\ell}}$$
$$+ (1-2\beta+2\beta^2)^{1-\frac{1}{\ell}}\Big[(6-12\beta+12\beta^2)(\frac{1}{\tau_2})^\ell + (2-3\beta+2\beta^3)E_\alpha\Big((\frac{1}{\tau_1})^\ell - (\frac{1}{\tau_2})^\ell\Big)\Big]^{\frac{1}{\ell}}\Bigg].$$

Proposition 4. *For $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 \neq \tau_2$ and $\ell \geq 1$, we have*

$$\left|\frac{lnG^2(\tau_1^\alpha, \tau_2^\beta)}{2} + \frac{2-\alpha-\beta}{2} lnA(\tau_1,\tau_2) - lnI(\tau_1,\tau_2)\right| \leq \frac{\tau_2-\tau_1}{4}\Big[\frac{1}{2(\ell+1)(\ell+2)}\Big]^{\frac{1}{\ell}} \quad (44)$$
$$\times \Bigg[\Big[\Big(2(\ell+2)(1-\alpha)^{\ell+1} + 2(\ell+2)\alpha^{\ell+1}\Big)(\frac{1}{\tau_2})^\ell + \Big[(q+3-\alpha)(1-\alpha)^{\ell+1} + (2\ell+4-\alpha)\alpha^{\ell+1}\Big]$$
$$\times\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big((\frac{1}{\tau_1})^\ell - (\frac{1}{\tau_2})^\ell\Big)\Big]^{\frac{1}{\ell}} + \Bigg[\Big(2(\ell+2)(1-\beta)^{\ell+1} + 2(\ell+2)\beta^{\ell+1}\Big)\Big(\frac{1}{\tau_2}\Big)^\ell$$
$$+ \Big((q+1+\beta)(1-\beta)^{q+1} + \beta^{q+2}\Big)\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big((\frac{1}{\tau_1})^\ell - (\frac{1}{\tau_2})^\ell\Big)\Big]^{\frac{1}{\ell}}\Bigg].$$

Finally,

Proposition 5. *For $\tau_2 > \tau_1 > 0$, $\tau_1 \neq \tau_2$, $w \geq 0$ and $s \geq 4$ or $0 \neq s < 1$, we have*

$$\left|\frac{H^s_{w,s}(\tau_1,\tau_2)}{H(\tau_1^s,\tau_2^s)} + H^{\frac{s}{2}+1}_{w,(\frac{s}{2}+1)}\Big(\frac{\tau_2}{\tau_1} + \frac{\tau_1}{\tau_2}, 1\Big) - H^s_{w,s}\Big(\frac{L(\tau_1^2,\tau_2^2)}{G^2(\tau_1,\tau_2)}, 1\Big)\right| \quad (45)$$

$$\leq \frac{(\tau_2-\tau_1)A(\tau_1,\tau_2)}{8G^2(\tau_1,\tau_2)}\Bigg[\frac{2|s|}{w+2}\Big(G^{2(s-1)}(\tau_2,\frac{1}{\tau_1})+\frac{w}{2}G^{s-\frac{1}{2}}(\tau_2,\frac{1}{\tau_1})\Big)$$
$$+\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\frac{|s|}{w+2}\Big(G^{2(s-1)}(\tau_1,\frac{1}{\tau_2})+\frac{w}{2}G^{s-\frac{1}{2}}(\tau_1,\frac{1}{\tau_2})\Big)$$
$$-\frac{|s|}{w+2}\Big(G^{2(s-1)}(\tau_2,\frac{1}{\tau_1})+\frac{w}{2}G^{s-\frac{1}{2}}(\tau_2,\frac{1}{\tau_1})\Big)\Big)\Bigg].$$

Proof. Let $\zeta(\mu)=\frac{\mu^s+w\mu^{\frac{s}{2}+1}}{w+2}$ for $\mu>0$ and $s\notin(1,4)$.

$$\zeta'(\mu)=\frac{s}{w+2}\Big(\mu^{s-1}+\frac{w}{2}\mu^{\frac{s}{2}-1}\Big) \tag{46}$$

By Corollary (3) . if follows that

$$\left|\frac{1}{2}\Big[\frac{\zeta(\frac{\tau_2}{\tau_1})+\zeta(\frac{\tau_1}{\tau_2})}{2}+\zeta\Big(\frac{\frac{\tau_2}{\tau_1}+\frac{\tau_1}{\tau_2}}{2}\Big)\Big]-\frac{1}{\frac{\tau_2}{\tau_1}-\frac{\tau_1}{\tau_2}}\int_{\frac{\tau_1}{\tau_2}}^{\frac{\tau_2}{\tau_1}}\zeta(x)dx\right|$$
$$=\Bigg|\frac{1}{2}\Bigg[\frac{1}{2}\Big[\frac{\tau_2^s+w(\tau_1\tau_2)^{\frac{s}{2}}+\tau_1^s}{\tau_1^s(w+2)}+\frac{\tau_1^s+w(\tau_1\tau_2)^{\frac{s}{2}}+\tau_2^s}{\tau_2^s(w+2)}\Big]$$
$$+\frac{(\frac{\tau_2}{\tau_1}+\frac{\tau_1}{\tau_2})^s+w(\frac{\tau_2}{\tau_1}+\frac{\tau_1}{\tau_2})^{\frac{s}{2}}}{w+2}\Bigg]-\frac{1}{w+2}\Bigg[\frac{(\frac{\tau_2}{\tau_1})^{s+1}-(\frac{\tau_1}{\tau_2})^{s+1}}{(s+1)(\frac{\tau_2}{\tau_1}-\frac{\tau_1}{\tau_2})}$$
$$+w\frac{(\frac{\tau_2}{\tau_1})^{\frac{s}{2}+1}-(\frac{\tau_1}{\tau_2})^{\frac{s}{2}+1}}{(\frac{s}{2}+1)(\frac{\tau_2}{\tau_1}-\frac{\tau_1}{\tau_2})}+1\Bigg]\Bigg|$$
$$=\left|\frac{H^s_{w,s}(\tau_1,\tau_2)}{H(\tau_1^s,\tau_2^s)}+H^{\frac{s}{2}+1}_{w,(\frac{s}{2}+1)}(\frac{\tau_2}{\tau_1}+\frac{\tau_1}{\tau_2},1)-H^s_{w,s}\Big(\frac{L(\tau_1^2,\tau_2^2)}{G^2(\tau_1,\tau_2)},1\Big)\right|$$
$$\times\frac{\frac{\tau_2}{\tau_1}-\frac{\tau_1}{\tau_2}}{16}\left[2\left|\zeta'\Big|\frac{\tau_2}{\tau_1}\Big|+\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Big(\Big|\zeta'(\frac{\tau_1}{\tau_2})\Big|-\Big|\zeta'(\frac{\tau_2}{\tau_1})\Big|\Big)\right|\right]$$
$$=\frac{\tau_2^2-\tau_1^2}{16\tau_1\tau_2}\Bigg[2\Big|\frac{s}{w+2}\Big((\frac{\tau_2}{\tau_1})^{s-1}+\frac{w}{2}(\frac{\tau_2}{\tau_1})^{\frac{s}{2}-1}\Big)\Big| \tag{47}$$
$$+\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Bigg(\Big|\frac{s}{w+2}\Big((\frac{\tau_1}{\tau_2})^{s-1}+\frac{w}{2}(\frac{\tau_1}{\tau_2})^{\frac{s}{2}-1}\Big)\Big|-\Big|\frac{s}{w+2}\Big((\frac{\tau_2}{\tau_1})^{s-1}+\frac{w}{2}(\frac{\tau_2}{\tau_1})^{\frac{s}{2}-1}\Big)\Big|\Bigg)\Bigg]$$
$$=\frac{(\tau_2-\tau_1)A(\tau_1,\tau_2)}{8G^2(\tau_1,\tau_2)}\Bigg[\frac{2|s|}{w+2}\Big(G^{2(s-1)}(\tau_2,\frac{1}{\tau_1})+\frac{w}{2}G^{s-\frac{1}{2}}(\tau_2,\frac{1}{\tau_1})\Big) \tag{48}$$
$$+\ \mathcal{F}^{\aleph}_{\varkappa,\lambda}\Bigg(\frac{|s|}{w+2}\Big(G^{2(s-1)}(\tau_1,\frac{1}{\tau_2})+\frac{w}{2}G^{s-\frac{1}{2}}(\tau_1,\frac{1}{\tau_2})\Big)$$

$$- \frac{|s|}{w+2}\left(G^{2(s-1)}(\tau_2, \frac{1}{\tau_1}) + \frac{w}{2}G^{s-\frac{1}{2}}(\tau_2, \frac{1}{\tau_1})\right)\Bigg)\Bigg],$$

obviously (47) and (48) yield (45). □

7. Conclusion

In this paper, we have defined and proved some Hermite-Hadamard and Fejér-type inequalities for generalize convex functions of Riana type. In addition, we find some interesting integral inequalities. All these results are new and amazing in literature. These results of the convex analysis are the basis and argument for many inequalities in pure and applied sciences. One thing to keep in mind, in the field of convex analysis and inequalities if we face problems, generalized notions and conceptions about convex functions are required to obtain pertinent and applicable results. It is high time to find the applications of these inequalities along with efficient numerical methods. We believe that our new results regarding generalize convex function of Raina type will have a very deep research in this fascinating field of inequalities and also in pure and applied sciences. The amazing techniques and wonderful ideas of this paper can be extended on the co-ordinates along with fractional calculus. In the future our goal is that we will continue our research work in this direction furthermore.

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A REVISED GENERALIZED F-TEST FOR TESTING THE EQUALITY OF GROUP MEANS UNDER NON-NORMALITY CAUSED BY SKEWNESS

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Abstract. The non-normality may occur in the data due to several reasons such as the presence of the outlier or skewness. It leads to lose the power and fail control Type I error probability of the tests which are used to test the equality of the group means under heteroscedasticity. To overcome this problem, a revised generalized F-test (RGF) is proposed to test the equality of group means under heteroscedasticity in which non-normality is caused by skewness in this study. An extensive Monte-Carlo simulation study is conducted to investigate and compare the performance of the proposed test with non-parametric alternatives under several values of skewness, and different number of groups. The proposed RGF is the best choice in the high level of skewness for $k = 3, 4, 5$. The Kruskal-Wallis test shows better performance than the others in small and moderate sample sizes for $k = 6$, and 7. It is shown that the proposed RGF test is superior than the non-parametric alternatives in the most of the conditions.

1. Introduction

Classical F-test (CF) is a powerful procedure in testing the equality of group means when the assumptions hold. If one of the assumptions is violated, the power of the CF test is adversely affected. Alexander-Govern (AG), Generalized F (GF), Parametric Bootstrap (PB), and Welch tests are developed when the assumption of variance homogeneity is violated. When the distributional assumption is violated or the distribution of the data is unknown, non-parametric methods may be

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more appropriate than their parametric counterparts. Kruskal-Wallis (KW) test is the non-parametric counterpart of the CF test and does not depend on normality assumption. In the case of unequal variances, Brunner et al. (1997) improved the Brunner-Dette-Munk (BDM) test as the heteroscedastic alternative of KW test.

There are numerous articles about the effect of both non-normality and heteroscedasticity on testing the equality of several group means. To reduce the negative effect of non-normality, researchers have proposed to use the robust estimators rather than the usual maximum likelihood (ML) estimators of the sample mean and variance to obtain better control of Type I error probability under non-normality and variance heterogeneity. Luh and Guo (1999) modified the AG trimmed mean test and Welch trimmed mean test with Johnson's normality transformation to deal with non-normality. Keselman et al. (2002) improved the Welch-James heteroscedastic test. Luh and Guo (2005) adopted a trimmed means method in conjunction with Hall's invertible transformation into AG and Welch test. Cribbie et al. (2007) found that the nonparametric procedure proposed by Brunner et al. (1997) provided good Type I error control. Kulinskaya and Dollinger (2007) modified the Welch test with the robust estimators: Huber (1964) proposed two estimators of location and scale, Hampel's M-estimator of location with scale estimated by the median absolute deviation, and the trimmed mean with scale estimated by the Winsorized standard deviation. Cribbie et al. (2012) proposed the modified PB test based on the trimmed mean. Ochuko et al. (2015) modified the AG test using the one-step m-estimator as central tendency measure to obtain a more powerful test under non-normality. Karagoz and Saracbasi (2016) proposed the robust Brown-Forsythe tests based on median/MAD and median/Q_n to test the equality of Weibull distributed group means in the presence of outliers. Ozdemir et al. (2018) proposed the B_{tj}^2 test based on bootstrapped trimmed means. Yusof et al. (2013) proposed the trimmed F-test combined with the robust scale estimators to overcome the problem of inflating Type I error on testing equality of group means for skewed distributions.

The non-normality may occur in the data due to different reasons such as the presence of outliers or skewness. They may lead to lose power and fail to control Type I error probability of the tests which are used to test the equality of group means under heteroscedasticity. Cavus et al. (2017) modified the GF test to obtain a powerful test for non-normality is caused by outliers. However, it is concluded by the Monte-Carlo simulation study that MGF does not maintain its performance when non-normality caused by skewness. Thus, the primary goal of this article is to propose a revised test for testing the equality of the group means under heteroscedasticity in which non-normality caused by skewness.

The rest of the article is organized as follows: Section 2 provides a literature review about testing the equality of group means. Moreover, the alternative ways in case of assumption violation are described. Section 3 describes the methods that are used to test the equality of group means and the proposed method. An extensive Monte-Carlo simulation study, conducted in Section 4, and Section 5, gives some concluding remarks.

2. Generalized Behrens-Fisher Problem in case of Non-Normality

The linear model underlying the ANOVA F test within the context of a one-way independent group design is given in (1).

$$Y_{ij} = \mu_i + \epsilon_{ij} \tag{1}$$

where Y_{ij} is the dependent variable associated with the ith observation in the jth group for $i = 1, 2, ..., n_i$ and $j = 1, 2, ..., k$. μ_i is the group mean for the ith group, and ϵ_{ij} is the random error component associated with Y_{ij}. The null hypothesis $H_0 : \mu_1 = \mu_2 = ... = \mu_k$ is tested classical F-test assumed that the ϵ_{ij}'s are independent, normally distributed and have an equal variance σ^2 for each group of k. When the normality assumption does not hold, the non-parametric counterparts of the CF test are used. In the following subsections, KW and BDM tests, which are the non-parametric counterparts of CF, are introduced.

2.1. **Bruner-Dette-Munk (BDM) test.** Brunner et al. (1997) proposed the following heterogeneous, rank-based F statistic:

$$T_{BDM} = \frac{N}{tr(M_{11}V)} QMQ' \tag{2}$$

where $M = I - \frac{1}{J}J$, $V = Ndiag(\frac{s_1^2}{n_1}, \frac{s_2^2}{n_2}, ..., \frac{s_j^2}{n_j})$, $s_j^2 = \frac{1}{N^2(n_j-1)} \sum_j R_{ij} - \bar{R}_j^{\,2}$, $Q = \frac{1}{N}(\bar{R}_1 - \frac{1}{2}, \bar{R}_2 - \frac{1}{2}, ..., \bar{R}_j - \frac{1}{2})$ and $\bar{R}_j = \frac{1}{n_j} \sum_{i=1}^{n_j} R_{ij}$.

R_{ij} is the rank of X_{ij}. The null hypothesis is rejected if $T_{BDM} \geq F_{\alpha, v1, v2}$ where $v_1 = \frac{M_{11}[tr(V)]^2}{tr(MVMV)}$, $v2 = \frac{[tr(V)]^2}{tr(V^2\Lambda)}$ and $\Lambda = diag(\frac{1}{n_1-1}, \frac{1}{n_2-1}, ..., \frac{1}{n_j-1})$.

2.2. **Kruskal-Wallis (KW) test.** The well-known non-parametric counterpart of the CF test is the KW test was proposed by Kruskal and Wallis (1952). For this test, the data is ranked so that the smallest of all observations receives a rank of one, and the largest of all observations receives rank N. The test statistic is calculated as

$$T_{KW} = \frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(N+1) \tag{3}$$

is distributed as χ^2_{k-1}, where n_j is the sample size for the jth sample, N is the total sample size, and R_j^2 is the sum of the ranks for the jth sample.

Oshima and Algina (1992) showed that the KW test is sensitive to the presence of unequal variances, particularly when the group sizes are unequal. Fagerland and

Sandvick (2009) indicated that when the observations in each group have different shapes, the KW test may give inaccurate results.

3. Revised Generalized F-Test (RGF)

Weerahandi (1995) proposed the Generalized F-test (GF) using the concept of the generalized p-values to test the equality of several normal distributed group means under heteroscedasticity. Yazici and Cavus (2021) discussed the performance the GF test under various conditions. Cavus et al. (2017) proposed the modified GF (MGF) test by replacing the maximum likelihood estimators of the sample mean and variance with Huber (1964)'s M-estimators to overcome non-normality caused by outlier(s). They conducted a Monte-Carlo simulation study to show the performance of the MGF in terms of power and Type I error probability. It was clearly pointed out that MGF outperforms the alternatives in the case of non-normality caused by outlier(s).

In this paper, the performance of MGF is investigated under heteroscedasticity where non-normality is caused by skewness. However, MGF does not maintain its performance under this case. To solve this problem, MGF is improved by using the Huber (1981) second proposal M-estimators instead of Huber (1964) M-estimators. The difference between these estimators is that the calculation of the scale estimator. Huber (1964) uses the median absolute deviation (MAD) as the scale estimator. However, since the second proposal takes into account the iterative version of MAD, it is used in this study for skewed distributions. The test statistic of the newly proposed test is given in (4).

$$\begin{aligned} T_{RGF} &= \frac{T_N(\bar{X}_1^*, \bar{X}_2^*, ..., \bar{X}_k^*; \sigma_1^{2*}, \sigma_2^{2*}, ..., \sigma_k^{2*})}{T_N(\bar{x}_1, \bar{x}_2, ..., \bar{x}_k; v_1^2/U_1, v_2^2/u_2, ..., v_k^2/U_k)} \\ &= \frac{\sum_{i=1}^k (n_i/\sigma_i^{2*})\bar{X}_i^{2*} - [\sum_{i=1}^k (n_i \bar{X}_i^{2*})/\sigma_i^{2*}] / \sum_{i=1}^k n_i/\sigma_i^{2*}}{\sum_{i=1}^k (n_i U_i/v_i^2)\bar{x}_i^{2*} - [\sum_{i=1}^k (n_i U_i/v_i^2)\bar{x}_i^{2*}]^2 / \sum_{i=1}^k (n_i U_i/v_i^2)} \end{aligned} \tag{4}$$

where $\bar{x}_i^*$ and σ_i^{2*}'s are the mean and scale estimator of Huber (1981) second proposal, respectively. $U_1, U_2, ..., U_k$ are independent random variables with $\chi^2_{n_i-1}, i = 1, 2, ..., k$. Furthermore, $T_N(\bar{X}_1^*, \bar{X}_2^*, ..., \bar{X}_k^*; \sigma_1^{2*}, \sigma_2^{2*}, ..., \sigma_k^{2*})$ $\chi^2_{n_i-1}$ independently of $U_1, U_2, ..., U_k$. The observed value of T_{MGF*} is defined as the value of MGF at $(\bar{X}_1^*, \bar{X}_2^*, ..., \bar{X}_k^*; V_1^2, V_2^2, ..., V_k^2) = (\bar{x}_1^*, \bar{x}_2^*, ..., \bar{x}_k^*; v_1^2, v_2^2, ..., v_k^2)$, the generalized p-value is given in the following equation.

$$p = \#\left(\frac{\chi^2_{k-1}}{T_N(\bar{x}_1^*, \bar{x}_2^*, ..., \bar{x}_k^*; v_1^2/U_1, v_2^2/u_2, ..., v_k^2/U_k} > 1 \right) \tag{5}$$

The p-value of the RGF test is the Monte-Carlo estimate which is computed with 10000 repetitions. It is the rate of the values which provides the condition given in (5).

4. Monte-Carlo Simulation Study

The performance of the RGF test is investigated under heteroscedasticity where non-normality is caused by skewness over the non-parametric counterparts, KW and BDM test. Moreover, MGF is used as the reference test to show the difference between the performance of the RGF. Skewed Exponential Power Distribution (SEPD) is used to simulate the observations from a wide variety of distributions. We included extreme departures from normality as measured by skewness and kurtosis. Zhu and Zilde (2009)'s version of the SEPD is used as follows:

$$f(x \mid \mu, \sigma, t, p) = \begin{cases} \frac{1}{\sigma} K(p)\ exp(-\frac{1}{p}|\frac{x-\mu}{2t\sigma}|^p) & \text{if} \quad x \leq \mu \\ \frac{1}{\sigma} K(p)\ exp(-\frac{1}{p}|\frac{x-\mu}{2(1-t)\sigma}|^p) & \text{if} \quad x > \mu \end{cases} \tag{6}$$

where μ and σ are location and scale parameter, $p \geq 0$ is the kurtosis parameter, $t \in [0,1]$ is skewness parameter and $K(p)$ is the normalization constant, $K(p) = 1/[2p^{1/p}\Gamma(1+1/p)]$. The SEPD corresponds to the Normal distribution when $p = 2$ and $t = 0.5$.

SEPD departures from normality depending on fixed t and p parameters in the simulation study. The Shapiro-Wilk normality test results depend on t and n presented as percentage (%) in Table 1. The results are computed for 10000 samples. SEPD departures from normality when the n increases and t moves away from 0.5 and right-skewed when t is smaller than 0.5, and left-skewed otherwise. A range between 0.1 and 0.9 for t is used to find out the performance of the tests in case of the non-normality caused by skewness.

Beyond the simulation part of this study, power and Type I error probabilities of the tests are calculated with 10000 replications for the nominal size 0.05 concerning the configuration factors which are sample size, effect size, design type, and skewness. Cavus and Yazici (2020) implemented the *doex* (ver.1.2) R package is used to compute the p-values of the RGF and MGF test. The values of Type I error probability and power of the tests are tabulated for both balanced and unbalanced designs and different number of groups ($k = 3, 4, 5, 6, 7$) in the following subsections.

4.1. **The properties of the tests to control the Type I error probability.**

In this section, Type I error probabilities of the tests are given in Tables 2-6 under different scenarios such as heteroscedasticity where non-normality caused by skewness as well as sample size, design type, and the number of the groups (k). According to the Bradley (1978) liberal criterion, a statistical test is considered robust if the empirical Type I error probability is between 0.025 and 0.075 for a nominal level 0.05. The value of the Type I error probability of a test is pointed out with $''*''$ which is in this interval.

It is clearly concluded that MGF has a serious problem controlling the Type I error probability when RGF outperforms others for most of the scenarios. Cavus et al. (2017) showed that the MGF is the superior test for heteroscedasticity and

TABLE 1. The values of skewness and non-normality (%) of the SEPD with fixed n and t parameters

t_i	n_i	skewness	non-normality (%)
0.1	10	0.467	0.159
	30	0.741	0.579
	60	0.827	0.925
0.2	10	0.399	0.121
	30	0.630	0.393
	60	0.705	0.737
0.3	10	0.288	0.083
	30	0.462	0.208
	60	0.523	0.412
0.4	10	0.151	0.058
	30	0.246	0.086
	60	0.278	0.141
0.5	10	0.001	0.046
	30	0.000	0.049
	60	0.000	0.050
0.6	10	-0.154	0.057
	30	-0.252	0.089
	60	-0.280	0.136
0.7	10	-0.289	0.081
	30	-0.470	0.218
	60	-0.523	0.417
0.8	10	-0.398	0.115
	30	-0.632	0.401
	60	-0.706	0.738
0.9	10	-0.467	0.160
	30	-0.739	0.575
	60	-0.828	0.927

non-normality caused by outliers. However, it could not maintain the performance for non-normality caused by skewness in terms of Type I error probability. Since MGF is out of the acceptable interval in almost all cases, it can not be evaluated as a robust test.

Type I error probabilities of the RGF test seem to be very conservative when equal and small sample sizes in case of testing equality of three group means ($k = 3$). However, its Type I error probability is very close to the nominal level for large samples with the high level of skewness. RGF and KW tests outperform others in terms of the controlling Type I error probabilities for $k = 3$. KW test is better than RGF in unbalanced designs when the RGF is better in balanced designs. While

BDM is keeping its performance for all k's, and KW are not keeping in higher values of k.

Results indicated that the RGF (for $k = 3, 4$) and KW (for $k = 3, 4, 5$) tests are fairly robust for deviations from the assumption of normality caused by skewness in the case of small sample sizes. For the higher values of k, none of the procedures considered in this paper control the Type I error probability.

Table 2. Type I error probabilities of the tests for $k = 3$

	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	$n_i = (5, 10, 15)$				$n_i = (10, 10, 10)$			
0.1	0.0549*	0.1428	0.0364*	0.0836	0.0800	0.1672	0.0819	0.0796
0.2	0.0529*	0.1326	0.0341*	0.0773	0.0689*	0.1553	0.0728*	0.0706*
0.3	0.0485*	0.1186	0.0308*	0.0702*	0.0615*	0.1382	0.0712*	0.0697*
0.4	0.0422*	0.1120	0.0304*	0.0629*	0.0536*	0.1238	0.0610*	0.0596*
0.5	0.0398*	0.1070	0.0265*	0.0552*	0.0463*	0.1121	0.0561*	0.0555*
0.6	0.0409*	0.1109	0.0259*	0.0603*	0.0543*	0.1216	0.0612*	0.0606*
0.7	0.0458*	0.1209	0.0299*	0.0694*	0.0617*	0.1372	0.0660*	0.0652*
0.8	0.0501*	0.1337	0.0334*	0.0742*	0.0672*	0.1542	0.0711*	0.0686*
0.9	0.0545*	0.1414	0.0377*	0.0847	0.0782	0.1685	0.0784	0.0759
t_i	$n_i = (15, 30, 45)$				$n_i = (30, 30, 30)$			
0.1	0.0804	0.1078	0.0692*	0.1577	0.0975	0.1274	0.1322	0.1245
0.2	0.0725*	0.1049	0.0572*	0.1322	0.0792	0.1160	0.1116	0.1049
0.3	0.0649*	0.0893	0.0462*	0.0992	0.0696*	0.1013	0.0876	0.0816
0.4	0.0576*	0.0814	0.0310*	0.0710*	0.0593*	0.0831	0.0684*	0.0646*
0.5	0.0526*	0.0778	0.0307*	0.0595*	0.0524*	0.0779	0.0598*	0.0567*
0.6	0.0546*	0.0802	0.0322*	0.0693*	0.0547*	0.0818	0.0668*	0.0626*
0.7	0.0620*	0.0893	0.0421*	0.0944	0.0651*	0.0970	0.0875	0.0815
0.8	0.0703*	0.0978	0.0535*	0.1239	0.0787	0.1150	0.1080	0.1017
0.9	0.0776	0.1025	0.0647*	0.1511	0.0964	0.1292	0.1325	0.1241
t_i	$n_i = (30, 60, 90)$				$n_i = (60, 60, 60)$			
0.1	0.0959	0.1077	0.1198	0.2587	0.1054	0.1184	0.2117	0.2003
0.2	0.0804	0.0931	0.0907	0.1998	0.0969	0.1144	0.1704	0.1610
0.3	0.0672*	0.0800	0.0585*	0.1332	0.0744*	0.0922	0.1170	0.1095
0.4	0.0556*	0.0671*	0.0383*	0.0797	0.0635*	0.0770	0.0783	0.0737*
0.5	0.0553*	0.0659*	0.0319*	0.0643*	0.0581*	0.0686*	0.0644*	0.0603*
0.6	0.0550*	0.0674*	0.0376*	0.0784	0.0620*	0.0760	0.0738*	0.0709*
0.7	0.0666*	0.0781	0.0567*	0.1296	0.0688*	0.0892	0.1141	0.1060
0.8	0.0777	0.0925	0.0877	0.1969	0.0899	0.1056	0.1662	0.1558
0.9	0.0913	0.1028	0.1212	0.2620	0.0970	0.1136	0.2097	0.1990

t: skewness parameter of SEPD, RGF: new proposed test, $(\sigma_1, \sigma_2, \sigma_3) = (0.2, 0.4, 0.6)$

TABLE 3. Type I error probabilities of the tests for $k = 4$

	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i		$n_i = (4, 8, 12, 16)$				$n_i = (10, 10, 10, 10)$		
0.1	0.0691*	0.1810	0.0333*	0.0864	0.0973	0.2272	0.0831	0.0783
0.2	0.0626*	0.1714	0.0318*	0.0784	0.0847	0.2078	0.0761	0.0703*
0.3	0.0529*	0.1475	0.0280*	0.0654*	0.0739*	0.1805	0.0720*	0.0674*
0.4	0.0438*	0.1375	0.0258*	0.0561*	0.0636*	0.1617	0.0640*	0.0605*
0.5	0.0416*	0.1318	0.0244	0.0542*	0.0544*	0.1467	0.0596*	0.0566*
0.6	0.0467*	0.1401	0.0240	0.0592*	0.0627*	0.1585	0.0640*	0.0608*
0.7	0.0540*	0.1599	0.0282*	0.0710*	0.0763	0.1836	0.0739*	0.0692*
0.8	0.0600*	0.1716	0.0300*	0.0785	0.0862	0.2055	0.0787	0.0734*
0.9	0.0664*	0.1863	0.0294*	0.0852	0.0998	0.2282	0.0839	0.0779
t_i		$n_i = (12, 24, 36, 48)$				$n_i = (30, 30, 30, 30)$		
0.1	0.0944	0.1352	0.0707*	0.1850	0.1143	0.1606	0.1470	0.1349
0.2	0.0823	0.1275	0.0521*	0.1466	0.0962	0.1442	0.1281	0.1163
0.3	0.0659*	0.1102	0.0369*	0.1025	0.0798	0.1191	0.0951	0.0854
0.4	0.0544*	0.0894	0.0257*	0.0700*	0.0618*	0.0961	0.0738*	0.0663*
0.5	0.0514*	0.0859	0.0229	0.0573*	0.0578*	0.0905	0.0649*	0.0587*
0.6	0.0602*	0.0967	0.0326*	0.0764	0.0616*	0.0950	0.0698*	0.0634*
0.7	0.0695*	0.1075	0.0419*	0.1029	0.0741*	0.1172	0.0914	0.0824
0.8	0.0800	0.1230	0.0529*	0.1454	0.0940	0.1407	0.1197	0.1083
0.9	0.0943	0.1276	0.0662*	0.1798	0.1134	0.1633	0.1464	0.1342
t_i		$n_i = (24, 48, 72, 96)$				$n_i = (60, 60, 60, 60)$		
0.1	0.1036	0.1181	0.1264	0.3220	0.1250	0.1470	0.2469	0.2215
0.2	0.0883	0.1071	0.0910	0.2403	0.1094	0.1361	0.1985	0.1816
0.3	0.0708*	0.0879	0.0548*	0.1517	0.0856	0.1093	0.1342	0.1208
0.4	0.0580*	0.0752	0.0316*	0.0835	0.0684*	0.0871	0.0854	0.0784
0.5	0.0568*	0.0714*	0.0254*	0.0606*	0.0584*	0.0752	0.0649	0.0577*
0.6	0.0625*	0.0785	0.0341*	0.0841	0.0647*	0.0832	0.0814	0.0739*
0.7	0.0713*	0.0936	0.0585*	0.1504	0.0823	0.1066	0.1270	0.1164
0.8	0.0880	0.1088	0.0911	0.2349	0.1055	0.1301	0.1925	0.1760
0.9	0.1085	0.1228	0.1268	0.3159	0.1209	0.1385	0.2472	0.2269

t: skewness parameter of SEPD, RGF: new proposed test, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.2, 0.4, 0.6, 0.8)$

Several valuable results are obtained. Despite the BDM is the heteroscedastic alternative of the KW test, the property of KW to control Type I error probability is better than BDM for balanced designs and $k = 3$. For higher values of k, the BDM test controls the Type I error probability better than the KW test. KW is not able to control Type I error when the distribution near normal.

4.2. **The results of penalized power of the tests.** Monte-Carlo simulation studies are used to compare the performance of the tests in terms of power and Type I error probability. However, any comparison of the powers is invalid when Type I error probabilities are different. Cavus et al. (2021) proposed the penalized power approach in (7) to compare the power of the tests when Type I error probabilities are different.

$$\gamma = \frac{1-\beta}{\sqrt{1+\left|1-\frac{\alpha_i}{\alpha_0}\right|}} \tag{7}$$

TABLE 4. Type I error probabilities of the tests for $k=5$

	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	$n_i=(5,8,10,12,15)$				$n_i=(10,10,10,10,10)$			
0.1	0.1025	0.2525	0.0435*	0.0924	0.1178	0.2841	0.0866	0.0769
0.2	0.0851	0.2308	0.0397*	0.0810	0.1043	0.2544	0.0802	0.0734*
0.3	0.0730*	0.2021	0.0376*	0.0695*	0.0852	0.2232	0.0714*	0.0655*
0.4	0.0589*	0.1776	0.0324*	0.0605*	0.0720 *	0.1959	0.0682*	0.0615*
0.5	0.0546*	0.1697	0.0299*	0.0557*	0.0629*	0.1772	0.0594*	0.0548*
0.6	0.0596*	0.1832	0.0272*	0.0577*	0.0695*	0.1853	0.0629*	0.0577*
0.7	0.0662*	0.1978	0.0288*	0.0645*	0.0834*	0.2230	0.0706*	0.0650*
0.8	0.0816	0.2267	0.0384*	0.0772	0.0995	0.2464	0.0794	0.0714*
0.9	0.0959	0.2517	0.0415*	0.0885	0.1158	0.2758	0.0839	0.0760
t_i	$n_i=(15,24,30,36,45)$				$n_i=(30,30,30,30,30)$			
0.1	0.1149	0.1662	0.0918	0.1884	0.1288	0.1896	0.1590	0.1415
0.2	0.0965	0.1481	0.0685*	0.1489	0.1077	0.1680	0.1338	0.1191
0.3	0.0833	0.1338	0.0526*	0.1079	0.0894	0.1390	0.1003	0.0888
0.4	0.0681*	0.1064	0.0364*	0.0721*	0.0680*	0.1100	0.0786	0.0699*
0.5	0.0581*	0.0983	0.0327*	0.0587*	0.0599*	0.0994	0.0655*	0.0578*
0.6	0.0586*	0.1006	0.0362*	0.0695*	0.0652*	0.1069	0.0748*	0.0659*
0.7	0.0780	0.1262	0.0538*	0.1064	0.0794	0.1322	0.0941	0.0830
0.8	0.0920	0.1467	0.0716*	0.1488	0.1083	0.1662	0.1326	0.1176
0.9	0.1114	0.1655	0.0909	0.1886	0.1292	0.1889	0.1565	0.1381
t_i	$n_i=(30,48,60,72,90)$				$n_i=(60,60,60,60,60)$			
0.1	0.1242	0.1446	0.1714	0.3440	0.1431	0.1731	0.2819	0.2525
0.2	0.1062	0.1336	0.1256	0.2608	0.1204	0.1545	0.2214	0.1955
0.3	0.0847	0.1088	0.0734*	0.1544	0.0933	0.1201	0.1428	0.1264
0.4	0.0671*	0.0881	0.0454*	0.0862	0.0695*	0.0913	0.0856	0.0725*
0.5	0.0555*	0.0769	0.0364*	0.0606*	0.0615*	0.0784	0.0660*	0.0578*
0.6	0.0668*	0.0870	0.0451*	0.0853	0.0688*	0.0917	0.0859	0.0737*
0.7	0.0819	0.1128	0.0753	0.1556	0.0867	0.1158	0.1382	0.1218
0.8	0.1056	0.1345	0.1268	0.2630	0.1175	0.1500	0.2109	0.1880
0.9	0.1306	0.1455	0.1704	0.3464	0.1368	0.1693	0.2734	0.2451

t: skewness parameter of SEPD, RGF: new proposed test,
$(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5)=(0.2,0.4,0.6,0.8,1.0)$

TABLE 5. Type I error probabilities of the tests for $k = 6$

	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	$n_i = (4, 6, 8, 12, 14, 16)$				$n_i = (10, 10, 10, 10, 10, 10)$			
0.1	0.1038	0.2959	0.0321*	0.0920	0.1419	0.3390	0.0962	0.0848
0.2	0.0922	0.2699	0.0276*	0.0764	0.1190	0.2991	0.0863	0.0772
0.3	0.0726*	0.2406	0.0235	0.0650*	0.0957	0.2566	0.0725*	0.0640*
0.4	0.0616*	0.2202	0.0220	0.0536*	0.0775	0.2255	0.0651*	0.0579*
0.5	0.0574*	0.2034	0.0229	0.0514*	0.0661*	0.2073	0.0596*	0.0530*
0.6	0.0612*	0.2115	0.0237	0.0551*	0.0791	0.2190	0.0630*	0.0575*
0.7	0.0806	0.2376	0.0288*	0.0689*	0.0962	0.2588	0.0730*	0.0656*
0.8	0.0930	0.2680	0.0350*	0.0839	0.1179	0.2916	0.0835	0.0731*
0.9	0.1089	0.2931	0.0355*	0.0957	0.1395	0.3259	0.0926	0.0823
t_i	$n_i = (12, 18, 24, 36, 42, 48)$				$n_i = (30, 30, 30, 30, 30, 30)$			
0.1	0.1244	0.1911	0.0783	0.2213	0.1452	0.2211	0.1732	0.1505
0.2	0.1085	0.1762	0.0617*	0.1704	0.1175	0.189	0.1422	0.1203
0.3	0.0844	0.1473	0.0437*	0.1125	0.0954	0.1551	0.1036	0.0884
0.4	0.0648*	0.1106	0.0283*	0.0709*	0.0744*	0.1212	0.0809	0.0683*
0.5	0.0579*	0.1055	0.0244	0.0540*	0.0636*	0.1083	0.0686*	0.0595*
0.6	0.0614*	0.1145	0.0289*	0.0694*	0.0704*	0.1191	0.0792	0.0663*
0.7	0.0826	0.1411	0.0451*	0.1117	0.0873	0.1479	0.0997	0.0862
0.8	0.1022	0.1677	0.0597*	0.1673	0.1185	0.1874	0.1380	0.1203
0.9	0.1229	0.1907	0.0760	0.2220	0.1446	0.2147	0.1718	0.1472
t_i	$n_i = (24, 36, 48, 72, 84, 96)$				$n_i = (60, 60, 60, 60, 60, 60)$			
0.1	0.1392	0.1681	0.1542	0.3980	0.1608	0.1956	0.3084	0.2722
0.2	0.1161	0.1503	0.1064	0.2932	0.1319	0.1697	0.2356	0.2032
0.3	0.0855	0.1194	0.0616*	0.1764	0.1014	0.1331	0.1534	0.1337
0.4	0.0637*	0.0929	0.0321*	0.0826	0.0723*	0.0950	0.0866	0.0756
0.5	0.0555*	0.0798	0.0245	0.0583*	0.0640*	0.0825	0.0679*	0.0546*
0.6	0.0637*	0.0899	0.0333*	0.0866	0.0699*	0.0947	0.0862	0.0727*
0.7	0.0872	0.1176	0.0621*	0.1761	0.0915	0.1278	0.1443	0.1239
0.8	0.1129	0.1495	0.1046	0.2997	0.1300	0.1668	0.2314	0.2038
0.9	0.1396	0.1685	0.1500	0.3997	0.1584	0.1923	0.3026	0.2680

t: skewness parameter of SEPD, RGF: new proposed test,
$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = (0.2, 0.4, 0.6, 0.8, 1.0, 1.2)$

TABLE 6. Type I error probabilities of the tests for $k = 7$

	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	$n_i = (4, 6, 8, 10, 12, 14, 16)$				$n_i = (10, 10, 10, 10, 10, 10, 10)$			
0.1	0.1218	0.3421	0.0356*	0.0935	0.1597	0.3814	0.0985	0.0840
0.2	0.1077	0.3190	0.0341*	0.0855	0.1349	0.3409	0.0854	0.0765
0.3	0.0860	0.2743	0.0276*	0.0661*	0.1066	0.2914	0.0739*	0.0643*
0.4	0.0704*	0.2507	0.0234	0.0562*	0.0840	0.2552	0.0647*	0.0558*
0.5	0.0670*	0.2281	0.0226	0.0526*	0.0724	0.2327	0.0596*	0.0519*
0.6	0.0699*	0.2348	0.0256*	0.0574*	0.0855	0.2499	0.0661*	0.0586*
0.7	0.0852	0.2710	0.0304*	0.0675*	0.1113	0.2981	0.0747*	0.0654*
0.8	0.1037	0.3079	0.0319*	0.0804	0.1359	0.3441	0.0895	0.0786
0.9	0.1229	0.3459	0.0366*	0.0984	0.1606	0.3750	0.0994	0.0870
t_i	$n_i = (12, 18, 24, 30, 36, 42, 48)$				$n_i = (30, 30, 30, 30, 30, 30, 30)$			
0.1	0.1396	0.2132	0.0849	0.2290	0.1599	0.2429	0.1791	0.1518
0.2	0.1194	0.1920	0.0642*	0.1753	0.1274	0.2097	0.1435	0.1234
0.3	0.0930	0.1635	0.0436*	0.1168	0.1030	0.1698	0.1066	0.0905
0.4	0.0717*	0.1277	0.0306*	0.0696*	0.0779	0.1333	0.0793	0.0668*
0.5	0.0627*	0.1117	0.0244	0.0540*	0.0632*	0.1161	0.0686*	0.0577*
0.6	0.0677*	0.1232	0.0280*	0.0697*	0.0746*	0.1302	0.0782	0.0669*
0.7	0.0880	0.1578	0.0430*	0.1194	0.0960	0.1673	0.1046	0.0882
0.8	0.1132	0.1886	0.0628*	0.1764	0.1293	0.2119	0.1437	0.1213
0.9	0.1358	0.2072	0.0793	0.2276	0.1580	0.2391	0.1820	0.1563
t_i	$n_i = (24, 36, 48, 60, 72, 84, 96)$				$n_i = (60, 60, 60, 60, 60, 60, 60)$			
0.1	0.1600	0.1890	0.1759	0.4405	0.1804	0.2197	0.3376	0.2961
0.2	0.1238	0.1606	0.1185	0.3218	0.1466	0.1934	0.2578	0.2237
0.3	0.0923	0.1312	0.0659*	0.1859	0.1058	0.1477	0.1660	0.1430
0.4	0.0722*	0.1019	0.0370*	0.0898	0.0762	0.1040	0.0907	0.0766
0.5	0.0612*	0.0885	0.0267*	0.0566*	0.0658*	0.0867	0.0682*	0.0537*
0.6	0.0657*	0.0954	0.0369*	0.0873	0.0719*	0.0997	0.0848	0.0704*
0.7	0.0875	0.1260	0.0668*	0.1818	0.0981	0.1396	0.1506	0.1278
0.8	0.1242	0.1621	0.1255	0.3208	0.1418	0.1862	0.2552	0.2217
0.9	0.1569	0.1872	0.1780	0.4419	0.1768	0.2159	0.3295	0.2853

t: skewness parameter of SEPD, RGF: new proposed test,
$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = (0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4)$

where β is Type II error rate, α_i is Type I error of the test and α_0 is the nominal level. Penalized power adjusts the power function with the square root of the percentile deviation between Type I error probability and the nominal level. Thus, penalized power is used to compare the power of the tests in the simulation studies.

Penalized power of the tests are given in Tables 7-11 under different scenarios such as sample size, design type and the number of the groups (k). Indeed a test can be evaluated as an acceptable test when Type I error probability of the test is near to the nominal level and its power as high as possible according to Neyman-Pearson theory. Thus, it is clearly seen that MGF is more powerful than the others. However, MGF can not control Type I error probability so that it can not be considered an acceptable test. The second powerful test is RGF especially for small sample size and k's. Also, it performs better in the unbalanced over the balanced designs. However, its power, like the others, is adversely affected by the increasing number of groups. The power of KW and BDM tests is least affected by skewness. There are cases where the BDM test performs better than the others in terms of power for small sample sizes. The power of the KW test is the lowest one for unbalanced designs with small sample sizes. However, it is seen that KW is more powerful than RGF especially for balanced designs when the sample size is increased.

From the results in Tables 7-11, KW and BDM tests have similar power when the skewness level is high ($t \sim 0.1$ or 0.9) for $k = 3, 4$. However, BDM is more powerful than KW in small sample sizes when the distribution is nearly normal. ($t \sim 0.5$). Moreover, BDM is more powerful than KW for unbalanced designs for $k = 5, 6, 7$ whereas KW is more powerful for balanced designs.

5. Conclusions

There are many modifications proposed to the parametric tests instead of using non-parametric tests in case of non-normality and heteroscedasticity. Cavus et al. (2017) proposed a modification to the GF test under non-normality caused by outliers replacing ML estimators of the sample mean and variance with robust estimators. It was shown that the MGF test is powerful than the alternatives in a Monte-Carlo simulation study. Also, the effectiveness of MGF was illustrated by an example of analyzing regional export data by Cavus et al. (2018a). However, it does not maintain the performance under non-normality caused by skewness.

In this study, MGF is revised by using the advantage of the iterative formed scale estimator of Huber (1981) second proposal for non-normality caused by skewness. The performance of the newly proposed RGF test is investigated in an extensive Monte-Carlo simulation study in terms of Type I error probability and penalized power of the test. According to the results which are given in the previous subsections, RGF is superior when non-normality caused by skewness, especially for lower k's and unbalanced designs. It is more powerful than KW and a better test to control Type I error probability, while the BDM test may be useful for higher k's. However, BDM can not control Type I error probability in case of high values of skewness.

As a result, we suggest researchers use RGF test for $k = 3, 4, 5$. Because of the decreasing performance of RGF for $k = 6, 7$, the KW test may be more appropriate to use for the high level of skewness in small and moderate sample sizes-unbalanced designs. However, BDM test should be used for slightly skewed normal distributions ($t \sim 0.5$) in small and moderate sample sizes-balanced designs. It is concluded and advised that RGF is the best choice in the high level of skewness because of its better performance to control Type I error probability.

TABLE 7. Penalized power of the tests for $k=3$

		RGF	MGF	KW	BDM	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	Δ_i		$n_i=(5,10,15)$				$n_i=(10,10,10)$				$n_i=(15,30,45)$		
0.1	0.2	0,1078	0,1212	0,0546	0,0784	0,0732	0,0961	0,0770	0,0775	0,2828	0,2602	0,1409	0,1181
	0.3	0,2151	0,2012	0,1440	0,1579	0,1191	0,1390	0,1433	0,1450	0,5746	0,5071	0,4342	0,3054
	0.5	0,6112	0,4292	0,5148	0,4479	0,3554	0,3140	0,4344	0,4437	0,7869	0,6795	0,8408	0,5567
0.2	0.2	0,1101	0,1270	0,0596	0,0853	0,0760	0,0975	0,0827	0,0821	0,3001	0,2722	0,1724	0,1440
	0.3	0,2199	0,2076	0,1418	0,1660	0,1315	0,1439	0,1580	0,1599	0,6036	0,5146	0,4959	0,3530
	0.5	0,6086	0,4415	0,5015	0,4772	0,3718	0,3106	0,4469	0,4563	0,8275	0,6880	0,9248	0,6079
0.3	0.2	0,1088	0,1265	0,0673	0,0964	0,0793	0,1024	0,0898	0,0913	0,3232	0,3025	0,2107	0,1934
	0.3	0,2172	0,2093	0,1494	0,1858	0,1383	0,1545	0,1683	0,1713	0,6443	0,5633	0,5561	0,4424
	0.5	0,6036	0,4558	0,4821	0,5064	0,3978	0,3376	0,4604	0,4670	0,8740	0,7459	0,9498	0,7024
0.4	0.2	0,1029	0,1307	0,0739	0,1177	0,0898	0,1113	0,1099	0,1106	0,3486	0,3227	0,2206	0,2714
	0.3	0,2080	0,2189	0,1648	0,2120	0,1626	0,1736	0,2015	0,2046	0,6872	0,5971	0,5327	0,5753
	0.5	0,5752	0,4668	0,4870	0,5590	0,4379	0,3603	0,5032	0,5115	0,9282	0,7812	0,8409	0,8321
0.5	0.2	0,1165	0,1527	0,0848	0,1487	0,1019	0,1298	0,1301	0,1312	0,3798	0,3415	0,2684	0,3711
	0.3	0,2286	0,2447	0,1856	0,2711	0,1853	0,2017	0,2379	0,2398	0,7289	0,6154	0,5744	0,6886
	0.5	0,5585	0,4774	0,4861	0,6327	0,4450	0,3800	0,5394	0,5416	0,9707	0,7983	0,8424	0,9125
0.6	0.2	0,1299	0,1606	0,0996	0,1670	0,1217	0,1412	0,1457	0,1462	0,4017	0,3566	0,3380	0,4261
	0.3	0,2483	0,2546	0,2070	0,3078	0,2055	0,2097	0,2553	0,2552	0,7215	0,6091	0,6276	0,6889
	0.5	0,5863	0,4746	0,4975	0,6385	0,4712	0,3758	0,5302	0,5301	0,9512	0,7844	0,8512	0,8455
0.7	0.2	0,1514	0,1613	0,1240	0,1949	0,1394	0,1583	0,1668	0,1673	0,4104	0,3655	0,4431	0,4346
	0.3	0,2793	0,2539	0,2363	0,3244	0,2265	0,2203	0,2726	0,2727	0,6904	0,5848	0,7196	0,6246
	0.5	0,6116	0,4540	0,5216	0,6132	0,4623	0,3684	0,5228	0,5224	0,8932	0,7415	0,9226	0,7256
0.8	0.2	0,1794	0,1751	0,1468	0,2203	0,1528	0,1643	0,1781	0,1800	0,4112	0,3597	0,5213	0,4311
	0.3	0,3221	0,2582	0,2726	0,3577	0,2358	0,2215	0,2791	0,2817	0,6701	0,5663	0,7920	0,5678
	0.5	0,6523	0,4347	0,5488	0,6101	0,4525	0,3524	0,5070	0,5119	0,8386	0,7094	0,9605	0,6339
0.9	0.2	0,1874	0,1783	0,1659	0,2259	0,1558	0,1686	0,1816	0,1812	0,4152	0,3633	0,5083	0,4100
	0.3	0,3205	0,2572	0,2971	0,3462	0,2356	0,2281	0,2776	0,2792	0,6470	0,5543	0,7420	0,5296
	0.5	0,6278	0,4243	0,5708	0,5787	0,4273	0,3468	0,4927	0,4964	0,7974	0,6921	0,8743	0,5741
t_i	Δ_i		$n_i=(30,30,30)$				$n_i=(30,60,90)$				$n_i=(60,60,60)$		
0.1	0.2	0,1744	0,1673	0,1162	0,1153	0,4705	0,4429	0,2178	0,1703	0,3568	0,3304	0,1674	0,1666
	0.3	0,4156	0,3687	0,2942	0,2985	0,6958	0,6556	0,5432	0,3779	0,6275	0,5873	0,3832	0,3902
	0.5	0,7082	0,6200	0,5991	0,6170	0,7221	0,6814	0,6460	0,4396	0,6888	0,6498	0,4858	0,4994
0.2	0.2	0,1950	0,1756	0,1372	0,1371	0,5118	0,4793	0,2750	0,2101	0,3675	0,3356	0,1992	0,1997
	0.3	0,4558	0,3891	0,3289	0,3340	0,7612	0,7066	0,6321	0,4355	0,6489	0,5957	0,4363	0,4436
	0.5	0,7822	0,6477	0,6517	0,6709	0,7886	0,7328	0,7425	0,5003	0,7183	0,6611	0,5417	0,5572
0.3	0.2	0,2187	0,1987	0,1727	0,1735	0,5665	0,5278	0,4010	0,2968	0,4163	0,3844	0,2719	0,2739
	0.3	0,4862	0,4190	0,4031	0,4122	0,8324	0,7645	0,8149	0,5547	0,7371	0,6656	0,5436	0,5587
	0.5	0,8305	0,6891	0,7337	0,7599	0,8626	0,7906	0,9245	0,6127	0,8198	0,7364	0,6536	0,6756
0.4	0.2	0,2408	0,2313	0,2298	0,2300	0,6386	0,5896	0,4658	0,4627	0,4735	0,4432	0,3933	0,3978
	0.3	0,5293	0,4699	0,4959	0,5043	0,9144	0,8318	0,8204	0,7395	0,7976	0,7270	0,6957	0,7123
	0.5	0,8946	0,7561	0,8331	0,8558	0,9483	0,8632	0,9002	0,7921	0,8874	0,8058	0,7991	0,8237
0.5	0.2	0,2909	0,2722	0,3125	0,3125	0,6706	0,6261	0,5352	0,6229	0,5321	0,4955	0,5297	0,5383
	0.3	0,5771	0,5041	0,5880	0,5956	0,9216	0,8413	0,8106	0,8495	0,8418	0,7760	0,8062	0,8299
	0.5	0,9481	0,7754	0,8934	0,9163	0,9509	0,8710	0,8569	0,8818	0,9275	0,8536	0,8811	0,9106
0.6	0.2	0,3207	0,2985	0,3648	0,3650	0,7084	0,6465	0,6586	0,6539	0,5500	0,5091	0,5824	0,5846
	0.3	0,5910	0,5102	0,6086	0,6193	0,9285	0,8374	0,8672	0,7871	0,8186	0,7379	0,7745	0,7870
	0.5	0,9264	0,7557	0,8458	0,8721	0,9535	0,8613	0,8951	0,7986	0,8978	0,8109	0,8229	0,8396
0.7	0.2	0,3271	0,2971	0,3665	0,3705	0,6722	0,6204	0,7653	0,5538	0,5509	0,4886	0,5171	0,5301
	0.3	0,5714	0,4799	0,5665	0,5787	0,8457	0,7780	0,9212	0,6158	0,7867	0,6836	0,6359	0,6571
	0.5	0,8450	0,6898	0,7398	0,7652	0,8665	0,8001	0,9391	0,6211	0,8523	0,7485	0,6618	0,6865
0.8	0.2	0,3238	0,2913	0,3680	0,3702	0,6472	0,5802	0,6587	0,4695	0,5162	0,4704	0,4616	0,4718
	0.3	0,5377	0,4530	0,5348	0,5434	0,7857	0,7165	0,7462	0,5015	0,6912	0,6314	0,5329	0,5476
	0.5	0,7695	0,6292	0,6688	0,6876	0,8022	0,7352	0,7551	0,5039	0,7455	0,6874	0,5483	0,5662
0.9	0.2	0,3038	0,2845	0,3549	0,3565	0,6033	0,5514	0,5738	0,4158	0,5098	0,4587	0,4276	0,4345
	0.3	0,4946	0,4320	0,4976	0,5058	0,7261	0,6793	0,6360	0,4357	0,6681	0,6112	0,4777	0,4888
	0.5	0,6954	0,5957	0,6046	0,6233	0,7400	0,6974	0,6423	0,4369	0,7178	0,6632	0,4882	0,5012

t: skewness parameter of SEPD, Δ: effect size, RGF: new proposed test, $(\sigma_1,\sigma_2,\sigma_3)=(0.2,0.4,0.6)$

Table 8. Penalized power of the tests for $k = 4$

		RGF	MGF	KW	BDM	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	Δ_i	$n_i = (4, 8, 12, 16)$				$n_i = (10, 10, 10, 10)$				$n_i = (12, 24, 36, 48)$			
0.1	0.2	0,0789	0,1112	0,0360	0,0566	0,0418	0,1696	0,0654	0,0560	0,1556	0,1550	0,0737	0,0734
	0.3	0,1257	0,1476	0,0743	0,0873	0,0541	0,1913	0,0967	0,0809	0,3288	0,2937	0,2099	0,1545
	0.5	0,3244	0,2792	0,2677	0,2377	0,1116	0,2931	0,2309	0,1991	0,6802	0,5720	0,7035	0,4335
0.2	0.2	0,0758	0,1075	0,0372	0,0628	0,0417	0,1645	0,0713	0,0571	0,1559	0,1574	0,0907	0,0829
	0.3	0,1223	0,1420	0,0740	0,0977	0,0543	0,1894	0,1012	0,0813	0,3495	0,3011	0,2643	0,1800
	0.5	0,3201	0,2785	0,2618	0,2500	0,1112	0,3036	0,2410	0,1973	0,7257	0,5876	0,8117	0,4874
0.3	0.2	0,0760	0,1072	0,0405	0,0701	0,0439	0,1553	0,0696	0,0557	0,1798	0,1701	0,0993	0,1106
	0.3	0,1262	0,1524	0,0785	0,1110	0,0569	0,1853	0,1063	0,0846	0,3829	0,3298	0,2656	0,2354
	0.5	0,3430	0,2950	0,2575	0,2861	0,1163	0,3105	0,2556	0,2048	0,8042	0,6255	0,7529	0,5960
0.4	0.2	0,0768	0,1151	0,0479	0,0808	0,0420	0,1596	0,0782	0,0584	0,2055	0,2003	0,1124	0,1560
	0.3	0,1292	0,1609	0,0918	0,1312	0,0591	0,2012	0,1216	0,0918	0,4439	0,3836	0,2925	0,3447
	0.5	0,3375	0,3079	0,2605	0,3397	0,1350	0,3437	0,2900	0,2227	0,8784	0,6939	0,7043	0,7407
0.5	0.2	0,0794	0,1208	0,0566	0,0980	0,0473	0,1726	0,0912	0,0659	0,2332	0,2171	0,1495	0,2281
	0.3	0,1331	0,1680	0,0995	0,1623	0,0673	0,2244	0,1410	0,1044	0,4726	0,4019	0,3310	0,4555
	0.5	0,3372	0,3157	0,2754	0,3800	0,1481	0,3686	0,3107	0,2300	0,9048	0,7051	0,7116	0,8521
0.6	0.2	0,0887	0,1234	0,0599	0,1127	0,0561	0,1877	0,1054	0,0800	0,2413	0,2259	0,2043	0,2718
	0.3	0,1515	0,1793	0,1097	0,1807	0,0807	0,2376	0,1652	0,1232	0,4608	0,3930	0,4213	0,4791
	0.5	0,3775	0,3110	0,2948	0,4145	0,1636	0,3804	0,3294	0,2514	0,8339	0,6595	0,7803	0,7601
0.7	0.2	0,1091	0,1371	0,0774	0,1299	0,0651	0,2043	0,1217	0,0969	0,2467	0,2330	0,2792	0,3014
	0.3	0,1840	0,1835	0,1402	0,2105	0,0917	0,2566	0,1790	0,1445	0,4601	0,3939	0,5225	0,4767
	0.5	0,4022	0,3057	0,3373	0,4228	0,1675	0,3878	0,3417	0,2787	0,7793	0,6246	0,8579	0,6689
0.8	0.2	0,1170	0,1392	0,0886	0,1476	0,0719	0,2347	0,1316	0,1082	0,2639	0,2386	0,3562	0,3114
	0.3	0,1886	0,1851	0,1601	0,2263	0,1009	0,2826	0,1915	0,1584	0,4530	0,3797	0,6002	0,4430
	0.5	0,3986	0,3021	0,3594	0,4362	0,1709	0,4016	0,3480	0,2909	0,7267	0,5845	0,9108	0,5666
0.9	0.2	0,1246	0,1444	0,1000	0,1623	0,0801	0,2463	0,1421	0,1198	0,2642	0,2461	0,3574	0,3131
	0.3	0,1949	0,1914	0,1752	0,2451	0,1074	0,2930	0,2046	0,1747	0,4338	0,3835	0,5760	0,4194
	0.5	0,4006	0,3022	0,3831	0,4416	0,1798	0,4091	0,3596	0,3084	0,6756	0,5769	0,8235	0,5146
t_i	Δ_i	$n_i = (30, 30, 30, 30)$				$n_i = (24, 48, 72, 96)$				$n_i = (60, 60, 60, 60)$			
0.1	0.2	0,0913	0,1152	0,0884	0,0940	0,2682	0,2588	0,1112	0,0960	0,1624	0,1265	0,1153	0,1597
	0.3	0,1707	0,1950	0,1671	0,1816	0,5437	0,5083	0,3317	0,2218	0,3511	0,2651	0,2313	0,3297
	0.5	0,4392	0,4653	0,4532	0,5160	0,6943	0,6503	0,6245	0,3913	0,5782	0,4458	0,4583	0,6784
0.2	0.2	0,0934	0,1245	0,0963	0,0969	0,2882	0,2748	0,1426	0,1133	0,1683	0,1468	0,1259	0,1570
	0.3	0,1767	0,2099	0,1827	0,1853	0,5903	0,5395	0,4047	0,2624	0,3572	0,2981	0,2688	0,3492
	0.5	0,4584	0,4964	0,4891	0,5152	0,7523	0,6831	0,7364	0,4534	0,5979	0,4955	0,5061	0,6809
0.3	0.2	0,0959	0,1399	0,1184	0,1008	0,3314	0,3171	0,2174	0,1560	0,1840	0,1743	0,1553	0,1598
	0.3	0,1955	0,2510	0,2409	0,2111	0,6641	0,6017	0,5724	0,3569	0,4034	0,3712	0,3504	0,3728
	0.5	0,4888	0,5677	0,5788	0,5261	0,8404	0,7541	0,9502	0,5708	0,6638	0,6000	0,6244	0,6833
0.4	0.2	0,1160	0,1691	0,1528	0,1174	0,3892	0,3599	0,2420	0,2546	0,2221	0,2410	0,2316	0,1926
	0.3	0,2289	0,2962	0,3059	0,2384	0,7446	0,6631	0,5780	0,5530	0,4640	0,4830	0,4819	0,4141
	0.5	0,5463	0,6409	0,6786	0,5452	0,9279	0,8151	0,8519	0,7717	0,7454	0,7519	0,7801	0,6899
0.5	0.2	0,1292	0,1906	0,1922	0,1410	0,4299	0,3977	0,3196	0,4162	0,2671	0,3144	0,3506	0,2537
	0.3	0,2460	0,3344	0,3679	0,2698	0,7677	0,6895	0,6245	0,7341	0,5219	0,5788	0,6425	0,4769
	0.5	0,5639	0,6823	0,7500	0,5657	0,9369	0,8357	0,8170	0,9073	0,7962	0,8575	0,9154	0,6937
0.6	0.2	0,1477	0,2133	0,2447	0,1829	0,4384	0,4110	0,4340	0,4713	0,2859	0,3146	0,4090	0,3376
	0.3	0,2709	0,3592	0,4201	0,3216	0,7504	0,6734	0,7313	0,6904	0,5245	0,5386	0,6334	0,5346
	0.5	0,5577	0,6683	0,7520	0,5889	0,8930	0,7971	0,8698	0,7709	0,7549	0,7620	0,8125	0,6971
0.7	0.2	0,1596	0,2183	0,2647	0,2244	0,4561	0,4049	0,5728	0,4337	0,2882	0,2810	0,3934	0,4088
	0.3	0,2709	0,3427	0,4203	0,3677	0,7245	0,6290	0,8328	0,5479	0,4787	0,4440	0,5477	0,5815
	0.5	0,5091	0,5886	0,6777	0,6067	0,8357	0,7285	0,9238	0,5765	0,6690	0,6100	0,6487	0,6990
0.8	0.2	0,1669	0,2201	0,2734	0,2663	0,4521	0,4048	0,5307	0,3904	0,2857	0,2434	0,3644	0,4673
	0.3	0,2717	0,3242	0,4054	0,4067	0,6678	0,5913	0,6958	0,4489	0,4558	0,3728	0,4737	0,6184
	0.5	0,4701	0,5186	0,6031	0,6187	0,7527	0,6760	0,7405	0,4613	0,6053	0,4955	0,5294	0,7013
0.9	0.2	0,1695	0,2042	0,2687	0,2936	0,4298	0,3914	0,4908	0,3553	0,3050	0,2288	0,3507	0,5139
	0.3	0,2634	0,2970	0,3806	0,4234	0,6104	0,5618	0,5992	0,3914	0,4566	0,3391	0,4290	0,6387
	0.5	0,4451	0,4728	0,5500	0,6289	0,6780	0,6367	0,6277	0,3978	0,5897	0,4386	0,4675	0,7036

t: skewness parameter of SEPD, Δ: effect size, RGF: new proposed test, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (0.2, 0.4, 0.6, 0.8, 1.0)$

TABLE 9. Penalized power of the tests for $k = 5$

		RGF	MGF	KW	BDM	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	Δ_i	$n_i = (5, 8, 10, 12, 15)$				$n_i = (10, 10, 10, 10, 10)$				$n_i = (15, 24, 30, 36, 45)$			
0.1	0.2	0,0717	0,1137	0,0342	0,0537	0,0449	0,2050	0,0656	0,0518	0,1053	0,1140	0,0567	0,0756
	0.3	0,0912	0,1282	0,0547	0,0681	0,0490	0,2147	0,0800	0,0638	0,1909	0,1833	0,1204	0,1112
	0.5	0,1710	0,1830	0,1594	0,1449	0,0740	0,2626	0,1521	0,1253	0,4701	0,4027	0,4151	0,2958
0.2	0.2	0,0719	0,1118	0,0368	0,0574	0,0450	0,1958	0,0662	0,0516	0,1083	0,1161	0,0677	0,0727
	0.3	0,0970	0,1273	0,0570	0,0751	0,0504	0,2081	0,0817	0,0649	0,2000	0,1909	0,1435	0,1200
	0.5	0,1815	0,1892	0,1677	0,1644	0,0763	0,2678	0,1543	0,1243	0,5023	0,4231	0,4755	0,3278
0.3	0.2	0,0698	0,1062	0,0406	0,0626	0,0405	0,1831	0,0671	0,0486	0,1133	0,1176	0,0778	0,0806
	0.3	0,0889	0,1246	0,0641	0,0806	0,0475	0,2018	0,0860	0,0643	0,2117	0,1965	0,1769	0,1458
	0.5	0,1889	0,1938	0,1704	0,1790	0,0775	0,2775	0,1761	0,1352	0,5349	0,4458	0,5715	0,4067
0.4	0.2	0,0774	0,1070	0,0461	0,0695	0,0395	0,1726	0,0692	0,0505	0,1244	0,1307	0,0825	0,1053
	0.3	0,1095	0,1307	0,0722	0,1013	0,0456	0,1976	0,0986	0,0723	0,2386	0,2246	0,1935	0,2079
	0.5	0,2147	0,2088	0,1813	0,2140	0,0814	0,2771	0,1919	0,1451	0,6012	0,4991	0,5639	0,5460
0.5	0.2	0,0769	0,1174	0,0465	0,0792	0,0425	0,1839	0,0795	0,0560	0,1501	0,1521	0,1114	0,1593
	0.3	0,1097	0,1462	0,0760	0,1163	0,0543	0,2134	0,1087	0,0767	0,2803	0,2545	0,2355	0,2915
	0.5	0,2390	0,2255	0,1980	0,2568	0,0921	0,3072	0,2193	0,1537	0,6514	0,5216	0,5935	0,6665
0.6	0.2	0,0854	0,1253	0,0530	0,0920	0,0476	0,1965	0,0868	0,0618	0,1724	0,1700	0,1569	0,2058
	0.3	0,1215	0,1515	0,0883	0,1376	0,0603	0,2291	0,1182	0,0857	0,3135	0,2783	0,3090	0,3545
	0.5	0,2502	0,2317	0,2148	0,2958	0,0999	0,3124	0,2223	0,1618	0,6684	0,5251	0,6565	0,6675
0.7	0.2	0,0911	0,1268	0,0637	0,1092	0,0527	0,2262	0,1033	0,0745	0,1729	0,1725	0,2314	0,2247
	0.3	0,1319	0,1531	0,1068	0,1601	0,0690	0,2562	0,1382	0,1041	0,3015	0,2697	0,4041	0,3519
	0.5	0,2628	0,2312	0,2455	0,3231	0,1107	0,3371	0,2433	0,1877	0,5978	0,4811	0,7585	0,5771
0.8	0.2	0,0996	0,1377	0,0798	0,1239	0,0617	0,2381	0,1088	0,0864	0,1868	0,1795	0,2477	0,2364
	0.3	0,1478	0,1666	0,1398	0,1857	0,0778	0,2720	0,1483	0,1162	0,3079	0,2739	0,4081	0,3484
	0.5	0,2622	0,2359	0,2912	0,3279	0,1204	0,3527	0,2546	0,2032	0,5674	0,4532	0,6933	0,5120
0.9	0.2	0,1138	0,1438	0,1043	0,1354	0,0691	0,2594	0,1139	0,0919	0,1994	0,1889	0,2541	0,2481
	0.3	0,1522	0,1687	0,1583	0,1900	0,0852	0,2932	0,1616	0,1298	0,3058	0,2698	0,3987	0,3413
	0.5	0,2604	0,2308	0,3140	0,3246	0,1282	0,3739	0,2659	0,2193	0,5294	0,4294	0,6329	0,4670
t_i	Δ_i	$n_i = (30, 30, 30, 30, 30)$				$n_i = (30, 48, 60, 72, 90)$				$n_i = (60, 60, 60, 60, 60)$			
0.1	0.2	0,0727	0,1108	0,0778	0,0819	0,1580	0,1491	0,0756	0,0885	0,1078	0,0908	0,0964	0,1353
	0.3	0,1077	0,1450	0,1151	0,1260	0,3209	0,3004	0,1725	0,1469	0,2051	0,1619	0,1555	0,2297
	0.5	0,2620	0,2979	0,2928	0,3302	0,6171	0,5706	0,4823	0,3405	0,4669	0,3641	0,3586	0,5564
0.2	0.2	0,0700	0,1127	0,0800	0,0759	0,1641	0,1610	0,0869	0,0898	0,1057	0,1025	0,0968	0,1208
	0.3	0,1097	0,1543	0,1266	0,1250	0,3472	0,3208	0,2138	0,1677	0,2078	0,1825	0,1738	0,2261
	0.5	0,2717	0,3291	0,3232	0,3374	0,6627	0,5917	0,5639	0,3908	0,4870	0,4078	0,4051	0,5530
0.3	0.2	0,0684	0,1133	0,0881	0,0744	0,1783	0,1760	0,1192	0,1052	0,1145	0,1205	0,1137	0,1151
	0.3	0,1123	0,1709	0,1466	0,1270	0,3908	0,3598	0,3054	0,2256	0,2346	0,2311	0,2218	0,2324
	0.5	0,2933	0,3838	0,3854	0,3469	0,7399	0,6543	0,7481	0,5152	0,5461	0,5068	0,5160	0,5682
0.4	0.2	0,0762	0,1305	0,1030	0,0778	0,2098	0,2016	0,1701	0,1581	0,1328	0,1596	0,1614	0,1240
	0.3	0,1317	0,2088	0,1944	0,1502	0,4589	0,4189	0,4195	0,3450	0,2774	0,3071	0,3250	0,2604
	0.5	0,3367	0,4403	0,4660	0,3741	0,8297	0,7255	0,8882	0,7103	0,6304	0,6553	0,7097	0,5950
0.5	0.2	0,0902	0,1552	0,1397	0,0990	0,2633	0,2450	0,2171	0,2598	0,1672	0,2072	0,2381	0,1648
	0.3	0,1533	0,2397	0,2537	0,1805	0,5465	0,4809	0,4800	0,5248	0,3282	0,3786	0,4470	0,3224
	0.5	0,3691	0,4971	0,5577	0,4097	0,9110	0,7760	0,8462	0,8738	0,6932	0,7579	0,8380	0,6293
0.6	0.2	0,1011	0,1658	0,1723	0,1280	0,2843	0,2665	0,3388	0,3323	0,1791	0,2142	0,2960	0,2343
	0.3	0,1726	0,2610	0,2874	0,2174	0,5269	0,4812	0,6220	0,5496	0,3370	0,3761	0,4939	0,4018
	0.5	0,3730	0,4827	0,5684	0,4470	0,8338	0,7304	0,9263	0,7489	0,6491	0,6741	0,7713	0,6562
0.7	0.2	0,1128	0,1857	0,1975	0,1643	0,2999	0,2727	0,3946	0,3427	0,2016	0,2116	0,3110	0,3217
	0.3	0,1795	0,2643	0,3182	0,2700	0,5201	0,4491	0,6203	0,4664	0,3408	0,3294	0,4559	0,4848
	0.5	0,3608	0,4584	0,5553	0,4906	0,7534	0,6411	0,8004	0,5612	0,5836	0,5357	0,6167	0,6771
0.8	0.2	0,1219	0,1787	0,2119	0,2111	0,3055	0,2806	0,3750	0,3150	0,2080	0,1897	0,3054	0,3967
	0.3	0,1886	0,2512	0,3112	0,3157	0,4862	0,4278	0,5225	0,3920	0,3302	0,2853	0,4110	0,5456
	0.5	0,3400	0,4093	0,4896	0,5162	0,6710	0,5906	0,6227	0,4341	0,5267	0,4399	0,5031	0,6866
0.9	0.2	0,1306	0,1822	0,2249	0,2428	0,2976	0,2787	0,3623	0,3052	0,2215	0,1820	0,3035	0,4541
	0.3	0,1904	0,2433	0,3179	0,3504	0,4534	0,4189	0,4743	0,3549	0,3366	0,2651	0,3785	0,5801
	0.5	0,3353	0,3831	0,4766	0,5443	0,6046	0,5680	0,5387	0,3790	0,4987	0,3855	0,4443	0,6925

t: skewness parameter of SEPD, Δ: effect size, RGF: new proposed test, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (0.2, 0.4, 0.6, 0.8, 1.0)$

TABLE 10. Penalized power of the tests for $k=6$

		RGF	MGF	KW	BDM	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	Δ_i	$n_i=(5,8,10,12,14,16)$				$n_i=(10,10,10,10,10,10)$				$n_i=(15,24,30,36,42,48)$			
0.1	0.2	0,0736	0,1227	0,0256	0,0538	0,0499	0,2314	0,0647	0,0523	0,0975	0,1075	0,0534	0,0760
	0.3	0,0874	0,1320	0,0359	0,0600	0,0520	0,2362	0,0726	0,0602	0,1406	0,1427	0,0909	0,0907
	0.5	0,1321	0,1667	0,0838	0,1037	0,0644	0,2584	0,1086	0,0911	0,3320	0,2906	0,2926	0,1959
0.2	0.2	0,0717	0,1175	0,0241	0,0579	0,0457	0,2146	0,0581	0,0448	0,0929	0,1110	0,0576	0,0719
	0.3	0,0825	0,1268	0,0357	0,0665	0,0482	0,2218	0,0676	0,0530	0,1458	0,1486	0,1048	0,0952
	0.5	0,1320	0,1648	0,0861	0,1115	0,0615	0,2574	0,1120	0,0912	0,3495	0,3015	0,3291	0,2199
0.3	0.2	0,0692	0,1141	0,0260	0,0539	0,0403	0,2018	0,0626	0,0440	0,0944	0,1070	0,0581	0,0721
	0.3	0,0878	0,1247	0,0346	0,0675	0,0426	0,2174	0,0753	0,0543	0,1513	0,1599	0,1116	0,1047
	0.5	0,1459	0,1660	0,0920	0,1251	0,0595	0,2609	0,1225	0,0921	0,3933	0,3310	0,3734	0,2840
0.4	0.2	0,0674	0,1152	0,0263	0,0599	0,0375	0,1949	0,0639	0,0444	0,1040	0,1212	0,0568	0,0857
	0.3	0,0834	0,1291	0,0412	0,0761	0,0427	0,2096	0,0782	0,0543	0,1845	0,1869	0,1177	0,1505
	0.5	0,1497	0,1770	0,0967	0,1538	0,0638	0,2626	0,1364	0,0969	0,4529	0,3884	0,3587	0,3890
0.5	0.2	0,0739	0,1189	0,0337	0,0726	0,0376	0,2083	0,0734	0,0495	0,1162	0,1296	0,0634	0,1191
	0.3	0,0928	0,1334	0,0515	0,0994	0,0467	0,2286	0,0923	0,0631	0,1996	0,2009	0,1366	0,2132
	0.5	0,1723	0,1821	0,1137	0,1817	0,0687	0,2953	0,1634	0,1103	0,4946	0,4064	0,3951	0,5240
0.6	0.2	0,0772	0,1184	0,0387	0,0817	0,0435	0,2165	0,0821	0,0567	0,1345	0,1422	0,1016	0,1603
	0.3	0,1011	0,1361	0,0604	0,1122	0,0523	0,2402	0,1050	0,0735	0,2287	0,2181	0,1914	0,2686
	0.5	0,1815	0,1879	0,1332	0,2098	0,0800	0,3098	0,1740	0,1236	0,5127	0,4135	0,4746	0,5561
0.7	0.2	0,0868	0,1272	0,0513	0,0973	0,0535	0,2476	0,0932	0,0694	0,1386	0,1491	0,1588	0,1905
	0.3	0,1125	0,1473	0,0749	0,1320	0,0629	0,2701	0,1170	0,0882	0,2291	0,2172	0,2787	0,2857
	0.5	0,1864	0,1979	0,1555	0,2351	0,0908	0,3357	0,1922	0,1461	0,4748	0,3905	0,6144	0,5041
0.8	0.2	0,0952	0,1382	0,0568	0,1075	0,0612	0,2585	0,1027	0,0788	0,1582	0,1597	0,2052	0,2091
	0.3	0,1192	0,1557	0,0863	0,1431	0,0716	0,2807	0,1287	0,1008	0,2370	0,2244	0,3375	0,2924
	0.5	0,1945	0,2024	0,1873	0,2477	0,0981	0,3408	0,2015	0,1614	0,4539	0,3708	0,6393	0,4464
0.9	0.2	0,0995	0,1446	0,0681	0,1174	0,0678	0,2719	0,1066	0,0882	0,1649	0,1644	0,2182	0,2206
	0.3	0,1264	0,1611	0,1023	0,1553	0,0773	0,2923	0,1336	0,1116	0,2472	0,2257	0,3480	0,2949
	0.5	0,2000	0,2084	0,2094	0,2580	0,1026	0,3479	0,2059	0,1735	0,4409	0,3637	0,6057	0,4070
t_i	Δ_i	$n_i=(30,30,30,30,30,30)$				$n_i=(30,48,60,72,84,96)$				$n_i=(60,60,60,60,60,60)$			
0.1	0.2	0,0684	0,1115	0,0799	0,0853	0,1254	0,1243	0,0620	0,0936	0,0938	0,0864	0,1008	0,1457
	0.3	0,0834	0,1273	0,0963	0,1035	0,2267	0,2197	0,1279	0,1216	0,1442	0,1227	0,1309	0,1981
	0.5	0,1618	0,2135	0,1997	0,2260	0,5181	0,4718	0,4060	0,2633	0,3349	0,2673	0,2729	0,4271
0.2	0.2	0,0619	0,1079	0,0759	0,0710	0,1230	0,1231	0,0631	0,0832	0,0930	0,0956	0,0998	0,1233
	0.3	0,0798	0,1300	0,1004	0,0977	0,2425	0,2299	0,1533	0,1283	0,1481	0,1381	0,1424	0,1824
	0.5	0,1678	0,2336	0,2176	0,2202	0,5612	0,4995	0,4892	0,3005	0,3512	0,3016	0,3089	0,4203
0.3	0.2	0,0595	0,1130	0,0787	0,0663	0,1323	0,1346	0,0827	0,0818	0,0900	0,1020	0,1010	0,1027
	0.3	0,0793	0,1444	0,1137	0,0959	0,2784	0,2539	0,2067	0,1482	0,1565	0,1625	0,1646	0,1704
	0.5	0,1784	0,2677	0,2623	0,2301	0,6448	0,5573	0,6588	0,3937	0,3907	0,3737	0,3902	0,4316
0.4	0.2	0,0624	0,1173	0,0890	0,0634	0,1565	0,1514	0,0953	0,1150	0,0953	0,1248	0,1264	0,0948
	0.3	0,0906	0,1564	0,1366	0,0983	0,3253	0,2942	0,2394	0,2368	0,1817	0,2161	0,2272	0,1850
	0.5	0,2104	0,3140	0,3263	0,2547	0,7522	0,6302	0,6649	0,6115	0,4691	0,5126	0,5476	0,4585
0.5	0.2	0,0696	0,1354	0,1173	0,0798	0,1910	0,1865	0,1279	0,1895	0,1171	0,1526	0,1780	0,1175
	0.3	0,1084	0,1904	0,1876	0,1301	0,3759	0,3454	0,2868	0,3808	0,2150	0,2664	0,3284	0,2239
	0.5	0,2430	0,3645	0,4024	0,2915	0,8239	0,6896	0,6850	0,8038	0,5196	0,5911	0,6986	0,5013
0.6	0.2	0,0828	0,1421	0,1416	0,1048	0,2093	0,2048	0,2073	0,2625	0,1264	0,1644	0,2203	0,1698
	0.3	0,1218	0,2080	0,2216	0,1652	0,4056	0,3659	0,4165	0,4431	0,2283	0,2742	0,3727	0,2966
	0.5	0,2559	0,3719	0,4410	0,3393	0,7823	0,6597	0,7794	0,7079	0,5069	0,5445	0,6771	0,5613
0.7	0.2	0,0941	0,1653	0,1718	0,1434	0,2235	0,2126	0,3164	0,2799	0,1479	0,1662	0,2521	0,2577
	0.3	0,1376	0,2231	0,2497	0,2120	0,3933	0,3528	0,5483	0,3951	0,2432	0,2536	0,3625	0,3827
	0.5	0,2643	0,3746	0,4497	0,3975	0,6820	0,5861	0,8431	0,5170	0,4630	0,4407	0,5548	0,6061
0.8	0.2	0,1068	0,1723	0,1871	0,1844	0,2356	0,2187	0,3222	0,2733	0,1640	0,1564	0,2547	0,3393
	0.3	0,1499	0,2232	0,2620	0,2642	0,3827	0,3363	0,4877	0,3462	0,2470	0,2232	0,3386	0,4599
	0.5	0,2618	0,3429	0,4207	0,4394	0,6113	0,5257	0,6661	0,4022	0,4226	0,3572	0,4557	0,6384
0.9	0.2	0,1156	0,1730	0,1964	0,2158	0,2426	0,2230	0,3180	0,2717	0,1739	0,1500	0,2577	0,3964
	0.3	0,1543	0,2159	0,2653	0,2978	0,3743	0,3346	0,4491	0,3200	0,2489	0,2047	0,3245	0,5093
	0.5	0,2604	0,3231	0,4101	0,4739	0,5580	0,4979	0,5628	0,3504	0,4069	0,3181	0,4066	0,6568

t: skewness parameter of SEPD, Δ: effect size, RGF: new proposed test, $(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)=(0.2,0.4,0.6,0.8,1.0,1.2)$

TABLE 11. Penalized power of the tests for $k = 7$

		RGF	MGF	KW	BDM	RGF	MGF	KW	BDM	RGF	MGF	KW	BDM
t_i	Δ_i	$n_i = (4, 6, 8, 10, 12, 14, 16)$				$n_i = (10, 10, 10, 10, 10, 10, 10)$				$n_i = (12, 18, 24, 30, 36, 42, 48)$			
0.1	0.2	0,0787	0,1301	0,0310	0,0572	0,0518	0,2572	0,0668	0,0519	0,0908	0,1110	0,0494	0,0828
	0.3	0,0853	0,1346	0,0374	0,0635	0,0531	0,2605	0,0704	0,0550	0,1196	0,1303	0,0712	0,0896
	0.5	0,1138	0,1564	0,0754	0,0905	0,0608	0,2784	0,0937	0,0765	0,2299	0,2192	0,1899	0,1465
0.2	0.2	0,0782	0,1282	0,0289	0,0595	0,0469	0,2502	0,0595	0,0450	0,0824	0,1047	0,0466	0,0680
	0.3	0,0838	0,1348	0,0382	0,0647	0,0481	0,2539	0,0627	0,0468	0,1101	0,1277	0,0711	0,0812
	0.5	0,1131	0,1545	0,0704	0,0918	0,0559	0,2762	0,0878	0,0677	0,2403	0,2297	0,2148	0,1618
0.3	0.2	0,0711	0,1190	0,0264	0,0567	0,0422	0,2305	0,0628	0,0444	0,0812	0,1021	0,0435	0,0669
	0.3	0,0793	0,1263	0,0312	0,0630	0,0432	0,2356	0,0691	0,0501	0,1128	0,1327	0,0778	0,0874
	0.5	0,1136	0,1549	0,0635	0,0971	0,0543	0,2617	0,0989	0,0716	0,2593	0,2431	0,2431	0,1945
0.4	0.2	0,0656	0,1181	0,0247	0,0566	0,0371	0,2178	0,0604	0,0396	0,0802	0,1010	0,0429	0,0729
	0.3	0,0760	0,1275	0,0352	0,0687	0,0400	0,2270	0,0689	0,0464	0,1281	0,1424	0,0803	0,1136
	0.5	0,1209	0,1571	0,0748	0,1177	0,0513	0,2637	0,1111	0,0752	0,3053	0,2791	0,2462	0,2719
0.5	0.2	0,0657	0,1156	0,0275	0,0608	0,0398	0,2204	0,0718	0,0458	0,0850	0,1087	0,0488	0,0906
	0.3	0,0791	0,1259	0,0391	0,0817	0,0439	0,2359	0,0848	0,0550	0,1390	0,1550	0,0984	0,1543
	0.5	0,1253	0,1597	0,0855	0,1334	0,0587	0,2874	0,1294	0,0839	0,3447	0,3055	0,2819	0,3791
0.6	0.2	0,0702	0,1191	0,0315	0,0732	0,0432	0,2367	0,0761	0,0533	0,1095	0,1236	0,0738	0,1318
	0.3	0,0866	0,1319	0,0495	0,0963	0,0497	0,2529	0,0944	0,0670	0,1659	0,1721	0,1408	0,2034
	0.5	0,1421	0,1706	0,1020	0,1613	0,0669	0,3018	0,1458	0,1024	0,3575	0,3106	0,3482	0,4184
0.7	0.2	0,0827	0,1266	0,0424	0,0885	0,0523	0,2657	0,0913	0,0655	0,1194	0,1330	0,1309	0,1534
	0.3	0,0956	0,1386	0,0602	0,1140	0,0591	0,2854	0,1105	0,0783	0,1800	0,1772	0,2124	0,2198
	0.5	0,1482	0,1740	0,1163	0,1813	0,0779	0,3258	0,1644	0,1218	0,3505	0,2987	0,4591	0,3927
0.8	0.2	0,0907	0,1391	0,0542	0,0992	0,0591	0,2788	0,0959	0,0744	0,1346	0,1456	0,1645	0,1797
	0.3	0,1100	0,1521	0,0739	0,1268	0,0679	0,2952	0,1147	0,0901	0,1918	0,1869	0,2655	0,2418
	0.5	0,1564	0,1830	0,1354	0,2005	0,0865	0,3338	0,1653	0,1332	0,3492	0,2980	0,5125	0,3786
0.9	0.2	0,0996	0,1453	0,0627	0,1101	0,0690	0,2909	0,1028	0,0843	0,1488	0,1540	0,1837	0,1924
	0.3	0,1165	0,1572	0,0890	0,1344	0,0764	0,3074	0,1227	0,1034	0,2029	0,1954	0,2759	0,2507
	0.5	0,1642	0,1849	0,1599	0,2066	0,0950	0,3444	0,1728	0,1481	0,3442	0,2981	0,4955	0,3616
t_i	Δ_i	$n_i = (30, 30, 30, 30, 30, 30, 30)$				$n_i = (24, 36, 48, 60, 72, 84, 96)$				$n_i = (60, 60, 60, 60, 60, 60, 60)$			
0.1	0.2	0,0718	0,1268	0,0860	0,0894	0,1048	0,1122	0,0597	0,1003	0,0860	0,0830	0,1039	0,1512
	0.3	0,0827	0,1384	0,0980	0,1041	0,1654	0,1628	0,0952	0,1124	0,1154	0,1039	0,1181	0,1790
	0.5	0,1311	0,1847	0,1676	0,1844	0,3950	0,3655	0,2831	0,2038	0,2312	0,1933	0,2101	0,3366
0.2	0.2	0,0665	0,1238	0,0801	0,0757	0,1010	0,1128	0,0569	0,0869	0,0768	0,0855	0,0911	0,1161
	0.3	0,0772	0,1381	0,0973	0,0938	0,1748	0,1744	0,1043	0,1081	0,1067	0,1095	0,1137	0,1502
	0.5	0,1287	0,2019	0,1750	0,1759	0,4344	0,3900	0,3425	0,2256	0,2347	0,2144	0,2268	0,3159
0.3	0.2	0,0621	0,1244	0,0837	0,0676	0,1048	0,1133	0,0653	0,0785	0,0742	0,0903	0,0857	0,0875
	0.3	0,0765	0,1441	0,1073	0,0891	0,1915	0,1877	0,1420	0,1174	0,1092	0,1206	0,1229	0,1287
	0.5	0,1365	0,2237	0,2046	0,1772	0,4921	0,4326	0,4733	0,2940	0,2569	0,2650	0,2773	0,3063
0.4	0.2	0,0582	0,1220	0,0879	0,0601	0,1105	0,1159	0,0697	0,0848	0,0778	0,1034	0,1007	0,0752
	0.3	0,0766	0,1513	0,1260	0,0919	0,2164	0,2073	0,1652	0,1637	0,1267	0,1657	0,1661	0,1298
	0.5	0,1536	0,2606	0,2583	0,1945	0,5690	0,4969	0,5354	0,4592	0,3212	0,3708	0,4051	0,3335
0.5	0.2	0,0642	0,1303	0,1071	0,0694	0,1379	0,1385	0,0887	0,1442	0,0923	0,1276	0,1453	0,0932
	0.3	0,0849	0,1670	0,1519	0,1039	0,2688	0,2488	0,2085	0,2833	0,1551	0,2046	0,2474	0,1625
	0.5	0,1713	0,2951	0,3098	0,2151	0,6463	0,5553	0,5710	0,6773	0,3770	0,4524	0,5666	0,3906
0.6	0.2	0,0725	0,1431	0,1238	0,0882	0,1598	0,1633	0,1599	0,2093	0,1028	0,1453	0,1927	0,1451
	0.3	0,0963	0,1847	0,1821	0,1334	0,3057	0,2844	0,3320	0,3542	0,1741	0,2261	0,3073	0,2360
	0.5	0,1883	0,3071	0,3474	0,2642	0,6557	0,5518	0,7011	0,6345	0,3830	0,4366	0,5722	0,4608
0.7	0.2	0,0811	0,1604	0,1461	0,1181	0,1858	0,1838	0,2549	0,2424	0,1238	0,1498	0,2273	0,2288
	0.3	0,1091	0,1980	0,2062	0,1742	0,3154	0,2861	0,4281	0,3405	0,1903	0,2165	0,3119	0,3285
	0.5	0,1898	0,3028	0,3545	0,3084	0,5963	0,5031	0,7442	0,4809	0,3647	0,3691	0,4893	0,5344
0.8	0.2	0,0934	0,1660	0,1620	0,1604	0,1961	0,1944	0,2589	0,2501	0,1431	0,1443	0,2309	0,3130
	0.3	0,1187	0,1982	0,2166	0,2143	0,3093	0,2798	0,3888	0,3092	0,2010	0,1913	0,2955	0,4124
	0.5	0,1960	0,2849	0,3479	0,3564	0,5242	0,4555	0,5764	0,3808	0,3429	0,3029	0,4082	0,5924
0.9	0.2	0,1054	0,1604	0,1708	0,1868	0,2119	0,1987	0,2707	0,2462	0,1565	0,1412	0,2386	0,3773
	0.3	0,1327	0,1926	0,2231	0,2489	0,3045	0,2802	0,3696	0,2896	0,2151	0,1864	0,2961	0,4739
	0.5	0,2021	0,2707	0,3378	0,3944	0,4829	0,4303	0,4982	0,3282	0,3395	0,2752	0,3799	0,6265

t: skewness parameter of SEPD, Δ: effect size, RGF: new proposed test, $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = (0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4)$

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ON BIVARIATE EXTENSION OF THE UNIVARIATE TRANSMUTED DISTRIBUTION FAMILY

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ABSTRACT. The aim of this study is to examine the bivariate transmuted distributions in the literature and to propose alternative distribution. The method is based on mixing distributions of pairs of order statistics of a sample of size two. Some of proposed distributions allow both negative and positive Pearson correlations with admissible range between pairs of random variates. The results of the study gain importance in terms of eliminating or completing the missing aspects of the bivariate transmuted distributions existing in the literature.

1. INTRODUCTION

Quadratic Rank Transmutation Mapping (QRTM) method proposed by Shaw and Buckley (2009) [20], which is one of the most popular methods of constructing univariate distribution. In multivariate data modelling, instead of investigating the independence between random variables, sometimes one can desire to construct a multivariate distribution of highly correlated random variables. For this purpose, it is important to extend the QRTM technique to two-dimensional distributions both in terms of providing flexibility and making alternative distribution suggestions to the previous works.

Some authors have worked with the bivariate (or multivariate) extension of the univariate transmuted distributions. The bivariate and multivariate extensions of the univariate-transmuted family are first introduced by Bourguignon et al. (2016) [7]. They obtained continuous bivariate and multivariate distributions having transmuted univariate marginals. Merovci et al. (2016) [11] introduced the bivariate version of univariate generalized transmuted G distribution. Alizadeh et

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al. (2017) [1] introduced the bivariate version of univariate generalized transmuted distribution. Unlike the works of Bourguignon et al. (2016) [7] and Merovci et al. (2016) [11], Alizadeh et al. (2017) [1] considered the base distribution as an exponentiated distribution. Merovci et al. (2017) [12] proposed the bivariate extension of the univariate Exponentiated-Transmuted G (ET-G) distribution family. Rezaei et al. (2017) [16] introduced a bivariate version of the generalized exponentiated distribution (GEC-G) by using failure probabilities of nested series and parallel systems. Bakouch et al. (2017) [4] proposed a bivariate version of the transmuted general (T-G) family of distributions. Although Bakouch et al. (2017) [4] use of similar techniques, unlike other authors' works, they proposed the Bivariate T-G distribution by considering different baseline distribution. Unlike the works mentioned above, Sarabia et al. [18] suggests a bivariate distribution with transmuted conditionals.

According to Table 1 of Shaw and Buckley (2009) [20], the essence of the univariate distributions derived from the QRTM is based on minimum and maximum order statistics, therefore this technique may not be easy to apply for bivariate or multivariate distributions. Since a bivariate distribution is constructed over the quadrants, the complement of the bivariate distribution is different from the univariate case.

Some relevant references about construction bivariate and multivariate distributions based on the pairs of order statistics are in [3], [8], [14] and [13]. According to work of Dolati and Úbeda-Flores (2009) [8], a bivariate distribution family was introduced by assuming negative dependence [6] and [21].

Since we inspire by the works of [8] and [13] in the present study, let's give these studies in more detail as follows: Dolati and Úbeda-Flores (2009) [8] introduced two transformations, based on the choice of pairs of order statistics of the marginal distributions. [13] defined that the bivariate distribution having the transmuted marginals, both by examining the univariate QRTM technique and inspired by [8].

In this study, we first examined the bivariate extensions that exist in the literature, and secondly, we discussed the new distribution proposals in the light of the studies of [8] and [13]. The findings obtained as a result of the study show that the proposed distributions can be used as an alternative to the extensions in the literature without any restrictions on the base distribution.

Accordingly, the study is organized as follows: The material and method section will be presented in four parts, these are as follows: Brief introduction of univariate QRTM given by Shaw and Buckley (2009) [20]. The extension of the QRTM technique to the bivariate case proposed by [7] will be considered, and we will discuss by Theorem 1 under which conditions this extension will be a bivariate distribution. Based on the studies of [8] and [13], the introduction of the technique to be used for new distribution proposals will be detailed.

In section 3, new distribution suggestions will be made, and Spearman's rank correlation coefficient will be calculated based on a specially selected distribution as base distribution.

The last part of the study includes the comparison of the proposed distributions with the studies available in the literature, a discussion of the advantages and disadvantages.

2. Material and Method

The quadratic rank transmutation mapping (QRTM) proposed by Shaw and Buckley (2009) [20] is given as $u \longrightarrow u+\lambda u(1-u)$, where $u \in [0,1]$ and $\lambda \in [-1,1]$. We come up with an idea inspired by Shaw and Buckley (2009) [20] as follows:

Let X_1 and X_2 be two independent and identically distributed random variables. Then once recall the distributions of order statistics associated with sample size of 2:

$$\Pr\left(X_{1:2} \leq x\right) = 1 - \Pr\left(X_{1:2} > x\right) = 1 - \left(1 - F\left(x\right)\right)^2 = 2F\left(x\right) - F^2\left(x\right),$$

$$\Pr\left(X_{2:2} \leq x\right) = F(x)^2.$$

Now, a new random variable T is defined by mixing the above order statistics as follows:

$$T = \begin{cases} X_{1:2}, & \text{with probability } \pi_1 \\ X_{2:2}, & \text{with probability } \pi_2, \end{cases}$$

where $\pi_1 + \pi_2 = 1$. Then the distribution of T is as follows:

$$\begin{aligned} \Pr\left(T \leq t\right) &= \pi_1\left(2F\left(t\right) - F^2\left(t\right)\right) + \pi_2\left(F^2\left(t\right)\right) \\ &= \left(2\pi_1\right)F\left(t\right) + \left(\pi_2 - \pi_1\right)F^2\left(t\right). \end{aligned}$$

By letting $\pi_2 = 1 - \pi_1$ then we have

$$\Pr\left(T \leq t\right) = 2\pi_1 F\left(t\right) + \left(1 - 2\pi_1\right)F^2\left(t\right).$$

Since $\pi_1 \in [0,1]$, appropriate parametrization for π_1 can be taken into account as $2\pi_1 = 1+\lambda$. New parameter is in the interval $[-1,1]$. Accordingly, latter probability is as follows:

$$\begin{aligned} \Pr\left(T \leq t\right) &= \left(1+\lambda\right)F\left(t\right) - \lambda\, F^2\left(t\right) \\ &= F\left(t\right) + \lambda F\left(t\right)\left(1 - F\left(t\right)\right), \end{aligned}$$

with $\lambda \in [-1,1]$. As can be seen immediately, if $F\left(t\right) = u$ is taken into account, the above expression is the quadratic rank transmutation proposed in [20]. Here, $F(t)$ is named as "base distribution" and $\Pr\left(T \leq t\right) = G(t)$ is named as "transmuted distribution".

Based on this, bivariate extension of quadratic transmuted distribution proposal of [7] draws our attention in particular. They proposed quadratic rank transmuted bivariate distribution as

$$H\left(x,y\right)=\left(1+\lambda\right)F\left(x,y\right)-\lambda F(x,y)^2, \tag{1}$$

where $\lambda \in [-1,1]$. Marginals of this model are univariate transmuted distributions respectively as $\left(1+\lambda\right)F_x\left(x\right)-\lambda F_x(x)^2$ and $\left(1+\lambda\right)F_y\left(y\right)-\lambda F_y(y)^2$.

Furthermore, just like as in the univariate case, a bivariate distribution should be obtained with the eq. (1) for the extreme values of λ. Such that for $\lambda=-1$, eq. (1) reduces to give $F(x,y)^2$ which is a cumulative distribution function (cdf), and for $\lambda=1$, eq. (1) gives $2F\left(x,y\right)-F(x,y)^2$ which is not a cdf.

Furthermore, $2F\left(x,y\right)-F(x,y)^2$ indicates the probability that is

$$Pr(\{X_1\leq\ x,Y_1\leq y\}\cup\{X_2\leq\ x,Y_2\leq y\}),$$

where $(X_1,\ Y_1)$ and $(X_2,\ Y_2)$ are independent and identically distributed copulas from F. Therefore, there are some issues to overcome for the case of positive values of the transmutation parameter λ. To overcome these issues, we have the following theorem.

Theorem 1. *Let $F\left(x,y\right)$ be a continuous distribution function and $H(x,y)$ be a differentiable on $\Re^2$ where $h\left(x,y\right)$ denotes $\frac{\partial^2 H(x,y)}{\partial x\partial y}$. Then $H\left(x,y\right)=\left(1+\lambda\right)F\left(x,y\right)-\lambda F(x,y)^2$ is a continuous distribution function, if the following conditions hold:*
(i) $-1\leq\lambda\leq 0$,
(ii) $0<\lambda\leq\frac{1}{3}$ and $F\left(x,y\right)$ belongs to Positively Dependent Class.

Proof. Multivariate distribution function must satisfy ((P1)-(P3)) properties (see, Barlow and Proschan, 1975, Chapter 5 [5]). The properties (P1) and (P2) are obviously hold. We prove only (P3).

(i)
(P3) $\frac{\partial^2 H(x,y)}{\partial x\partial y}\geq 0$. For the simplicity, let $f_{xy}=f\left(x,y\right)$, $F_{xy}=F\left(x,y\right)$, and $h_{xy}=\frac{\partial^2 H(x,y)}{\partial x\partial y}$. Then

$$h_{xy}=f_{xy}\left[1+\lambda-2\lambda F_{xy}\right]-2\lambda\frac{\partial F_{xy}}{\partial x}\frac{\partial F_{xy}}{\partial y} \tag{2}$$

where $\frac{\partial F_{xy}}{\partial x}$ and $\frac{\partial F_{xy}}{\partial y}$ are respectively the probabilities of $Pr\left(Y\leq y,\ X=x\right)$ and $Pr\left(X\leq x,\ Y=y\right)$. Obviously, from the eq.(2), for $\lambda\in\ [-1,0]$, $h_{xy}\geq 0$.

(ii)
(P3) Under the assumption of the positive dependence of F, according to [15] and 2, by noting that positive dependence implies $f_{xy}F_{xy}\geq\frac{\partial F_{xy}}{\partial x}\frac{\partial F_{xy}}{\partial y}$. Hence, by the eq.(2), we have

$$\begin{aligned} h_{xy} &\geq f_{xy}\left[1+\lambda-4\lambda F_{xy}\right] \\ &\geq f_{xy}[1-3\lambda]. \end{aligned}$$

By considering $\lambda \leq 1/3$, the latter expression in square brackets is non negative. This completes the proof. □

The transition from univariate case to bivariate or multivariate cases is not so easy. While in univariate case the real line is the complement of $Pr(X \leq x)$, at least in the bivariate case the complement of $Pr\,(X \leq x,\ Y \leq y)$ is on the quadrants.

In response to the above discussions, we consider the work of Dolati and Úbeda-Flores (2009) [8]. They introduced a method for constructing bivariate distribution by using order statistics.

Let $(X_1,\ Y_1)$ and $(X_2,\ Y_2)$ be two independent random vectors with common distribution function $F(x,y)$. Note that, $F(x,y)$ belongs to the distribution family $\mathcal{F}\,(F_x, F_y)$ where F_x and F_y denote respectively marginals of X and Y. Let $X_{(1)}$, $X_{(2)}$ and $Y_{(1)}, Y_{(2)}$ be their corresponding order statistics. According to [8], consider the four probabilities as follows:

$$Pr\left(X_{(1)} \leq x, Y_{(2)} \leq y\right) = F_{xy}\left(2F_y - F_{xy}\right), \tag{3}$$

$$Pr\left(X_{(2)} \leq x, Y_{(1)} \leq y\right) = F_{xy}\left(2F_x - F_{xy}\right), \tag{4}$$

$$Pr\left(X_{(1)} \leq x, Y_{(1)} \leq y\right) = 2F_xF_y + 2F_{xy}\bar{F}_{xy} - F_{xy}^2 \tag{5}$$

and

$$Pr\left(X_{(2)} \leq x, Y_{(2)} \leq y\right) = F_{xy}^2, \tag{6}$$

where $\bar{F}_{xy}$ denotes survival function of (X,Y) *i.e.*, $Pr\,(X > x, Y > y)$. Dolati and Úbeda-Flores (2009) [8] proposed two new distributions. First is a mixture of (3) and (4) with mixing probability $\frac{1}{2}$. In other words,

$$(Z_1, Z_2) = \begin{cases} (X_{(1)}, Y_{(2)}), & \text{with probability } \frac{1}{2} \\ (X_{(2)}, Y_{(1)}), & \text{with probability } \frac{1}{2}. \end{cases}$$

Then the distribution of $(Z_1,\ Z_2)$ is given by

$$H_1\,(x,y) = F_{xy}\left[1 - \bar{F}_{xy}\right]. \tag{7}$$

Second is a mixture of (5) and (6) with mixing probability $\frac{1}{2}$. In other words, If we consider the random vector

$$(T_1, T_2) = \begin{cases} (X_{(1)}, Y_{(1)}), & \text{with probability } \frac{1}{2} \\ (X_{(2)}, Y_{(2)}), & \text{with probability } \frac{1}{2}, \end{cases}$$

the distribution function of $(T_1,\ T_2)$ is given by

$$H_2\left(x,y\right) = F_x F_y + F_{xy}\bar{F}_{xy}. \tag{8}$$

Note that, H_1 and H_2 both belong to $\mathcal{F}\left(F_x, F_y\right)$. Accordingly, a transmuted bivariate distribution was introduced by considering the eq. (5) and eq. (6) in [13] as follows: If we consider the random vector

$$\left(T_1^*, T_2^*\right) = \begin{cases} \left(X_{(1)}, Y_{(1)}\right), \text{ with probability } \frac{1+\lambda}{2} \\ \left(X_{(2)}, Y_{(2)}\right), \text{ with probability } \frac{1-\lambda}{2}, \end{cases}$$

for $\lambda \in [-1,1]$, the distribution function of $(T_1^*,\ T_2^*)$ is given by

$$H_3\left(x,y\right) = \left(1+\lambda\right)\left[F_x F_y + F_{xy}\bar{F}_{xy}\right] - \lambda F_{xy}^2. \tag{9}$$

Note that, H_3 does not belong to $\mathcal{F}\left(F_x, F_y\right)$. Marginals are represented by univariate transmuted distribution as in (1). With descriptions so far, it can easily be said that the genesis of the eq. (9) is one of the bivariate extension of QRTM.

Next section, we try to propose three alternative distributions. The first and second are proposed in the light of the works of [8] and [13]. The third proposal is based on QRTM technique of [20].

3. The Research Findings and Discussion

3.1. **Some Alternative Methods for Constructing Bivariate Distribution Family.** First proposal is as follows: We obtain by mixing H_1 in (7) and H_2 in (8) with respective mixing probabilities $\frac{1+\lambda}{2}$ and $\frac{1-\lambda}{2}$, then we have

$$H_4\left(x,y\right) = \frac{\left(1-\lambda\right)}{2} F_x F_y + \frac{\left(1+\lambda\right)}{2} F_{xy} - \lambda F_{xy}\bar{F}_{xy}, \tag{10}$$

where $\lambda \in\ [-1,1]$. Note that, $H_4 \in F\left(F_x, F_y\right)$. In fact, Dolati and Úbeda-Flores (2009) [8] proposed two different distribution families and made their conclusions on these two families. Therefore, H_4 will be similar to their proposed families.

Second proposal is as follows: Let $(X_1,\ Y_1)$ and (X_2, Y_2) be two independent random vectors with different bivariate distributions $F_x F_y$ and F_{xy}. Then we consider the random pair (V_1, V_2) as follows:

$$\left(V_1, V_2\right) = \begin{cases} \left(min\{X_1, X_2\}, min\{Y_1, Y_2\}\right), \text{ with probability } \frac{1+\lambda}{2} \\ \left(max\{X_1, X_2\}, max\{Y_1, Y_2\}\right), \text{ with probability } \frac{1-\lambda}{2}, \end{cases}$$

where $-1 \le \lambda \le\ 1$. Then the distribution of (V_1, V_2) is given by

$$H_5\left(x,y\right) = \left(1+\lambda\right)F_x F_y + \frac{\left(1+\lambda\right)}{2}\left[F_x F_y \bar{F}_{xy} + F_{xy}\bar{F}_x \bar{F}_y\right] - \lambda F_{xy} F_x F_y. \tag{11}$$

Here, marginal distributions respectively are $H_{5x} = (1+\lambda) F_x - \lambda F_x^2$ and $H_{5y} = (1+\lambda) F_y - \lambda F_y^2$. Note that, H_5 does not belong to $\mathcal{F}(F_x, F_y)$ and its marginals are represented by univariate transmuted distribution as in H_3 given by the eq. (9).

The genesis of the third proposal is based on the QRTM technique of [20], and Rüschendorf's Method of [17]. Then we define a function on the unit square as

$$g(u,v) = uv + k(u,v), \tag{12}$$

where $k(u,v) = 0$ at the endpoints of the unit square, with $\frac{-\partial^2 k(u,v)}{\partial u \partial v} \leq 1$. Specially, by choosing $k(u,v) = \lambda uv(1-u)(1-v)$, with $u = F_x$ and $v = F_y$ the eq. (12) indicates well-known bivariate distribution which is called as Farlie-Gumbel-Morgenstern (FGM) distribution (see Farlie, 1960, [9] and Gumbel, 1960, [10]). Note also that, both $g(u,1)$ and $g(1,v)$ are indicate the QRTM technique of [20]. In the light of these works, we consider

$$k(u,v) = \lambda uv - \lambda C_{uv}^2 + \lambda(1+\lambda) C_{uv} \bar{C}_{uv},$$

where C_{uv} is a bivariate copula, and $\bar{C}_{uv}$ is a survival copula. Accordingly, the third proposal is introduced as

$$H_6(x,y) = (1+\lambda) F_x F_y - \lambda F_{xy}^2 + \lambda(1+\lambda) F_{xy} \bar{F}_{xy}, \tag{13}$$

with the marginal distributions respectively are $(1+\lambda) F_x - \lambda F_x^2$ and $(1+\lambda) F_y - \lambda F_y^2$. The eq. (13) can be seen as another mixture of the eq. (5) and eq. (6). In other words, for $\lambda = -1$, eq. (13) reduces to eq. (6), and for $\lambda = 1$, the eq. (13) reduces to eq.(5). Note that, H_6 does not belong to $\mathcal{F}(F_x, F_y)$ and its marginals are represented by univariate transmuted distribution. Unlike H_3 and H_5 alternatives, H_6 contains the independence class. In other words, for $\lambda = 0$, H_6 produces $F_x F_y$ which is the member of $\mathcal{F}\left((1+\lambda) F_x - \lambda F_x^2, (1+\lambda) F_y - \lambda F_y^2\right)$.

Thus, we just propose H_4, H_5 and H_6 as alternative bivariate distributions. By the next section, considering the special choice of F_{xy}, we will make a comparison amongst to works of [[8], [13], [6], [21]], according to their Spearman's rank correlation coefficients.

3.2. **Spearman's Rho Measures for the New Families of Bivariate Distributions.** For $F \in \mathcal{F}(F_x, F_y)$, Spearman's rho can be expressed as

$$\rho_s = 12 \int_R \int_R \{F_{xy} - F_x F_y\} \, dF_y dF_x \tag{14}$$

(see, [19]).

FGM distribution, a well-known bivariate distribution, will be considered, and calculations of Spearman's rho will be made according to this base distribution. The Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions are given by $F_{xy} = F_x F_y \left[1 + \theta \, \bar{F}_x \bar{F}_y\right]$, for $\theta \in [-1, 1]$. Note that, $\rho_s = \frac{\theta}{3}$ (see [9] and [10]).

The calculated coefficient of Spearman's rho for H_4 (in eq. (10)) can be obtained by

$$\begin{aligned}\rho_s^{H_4} &= \left(\frac{1+\lambda}{2}\right)\rho_s^F - 12\lambda\int_R\int_R F_{xy}\bar{F}_{xy}dF_ydF_x \\ &= \frac{1}{6}\theta - \frac{1}{75}\lambda\theta^2 - \frac{1}{3}\lambda.\end{aligned}$$

In fact, this result overlaps with the result of [8] (see, end of Section 3). Because we created a mixture distribution by H_4 of their proposed distributions. According to their reported results, $\rho_s^{H_4}$ attains minimum value as $-\frac{77}{150} \cong -.513$ at $(\theta = -1,\ \lambda = 1)$ and has a maximum value as as $\frac{77}{150} \cong .513$ at $(\theta = 1,\ \lambda = -1)$.

The calculated coefficient of Spearman's rho for H_5 (in eq. (11)) can be obtained by

$$\rho_s^{H_5} = \frac{1}{3} + \frac{1}{12}\theta + \frac{11}{300}\lambda^2\theta - \frac{1}{3}\lambda^2.$$

$\rho_s^{H_5}$ has a minimum value as $-.12$ at $(\theta = -1,\ \lambda = \pm 1)$ and has a maximum value as $\frac{5}{12} \cong .417$ at $(\theta = 1,\ \lambda = 0)$.

The calculated coefficient of Spearman's rho for H_6 (in eq. (13)) can be obtained by

$$\rho_s^{H_6} = \frac{\theta\lambda^2\left(\lambda^2 + 2\theta + 35\right)}{150}.$$

$\rho_s^{H_6}$ has a minimum value as $\frac{-34}{150} \cong -.227$ at $(\theta = -1,\ \lambda = \pm 1)$ and has a maximum value as $\frac{38}{150} \cong .253$ at $(\theta = 1,\ \ \lambda = \pm 1)$.

For the comparison, we calculate coefficient of Spearman's rho for H_3 (in eq. (9)) proposed by [13] as follows:

$$\rho_s^{H_3} = \frac{1}{3} + \frac{1}{6}\theta + \frac{1}{75}\theta^2 + \frac{11}{150}\lambda^2\theta - \frac{1}{3}\lambda^2.$$

$\rho_s^{H_3}$ has a minimum value as $\frac{-34}{150} \cong -.227$ at $(\theta = -1,\ \lambda = \pm 1)$ and has a maximum value as $\frac{77}{150} \cong .513$ at $(\theta = 1,\ \lambda = 0)$.

We can see that the lower bound of $\rho_s^{H_3}$ is the same of $\rho_s^{H_6}$, and the upper bound is the same as $\rho_s^{H_4}$. Furthermore, in [6] it was reported that for $\theta \in [-1, 0]$, $\rho_s \in [-\frac{1}{4}, \frac{1}{3}]$ and in [21] it was reported that for $\theta \in [-1, 0]$, $\rho_s \in [-\frac{1}{3}, \frac{1}{3}]$.

The upper bounds of Spearman rho for H_3, H_4 and H_5 yield a wider range than FGM does. According to the comparison for the lower bound, H_4 offers wider range than FGM does.

4. Results

We make an in-depth review of the work which is the first of the bivariate extensions of QRTM available in the literature. A discussion is made that this first extension is a bivariate distribution under which conditions. The result of Theorem 1 shows that this bivariate extension cannot be a bivariate distribution function in some values of transmutation parameter λ. Therefore, other extensions existing in literature derived from or similar to this extension proposal may also need to be explored in detail.

To succeed to extend the univariate QRTM to the bivariate case, three new bivariate distribution families have been proposed as an alternative to the bivariate distribution families obtained using bivariate order statistics. We have shown that H_4 has a wider range of dependence measure than FGM has when the base distribution is the specially selected bivariate FGM distribution family is taken. Furthermore, H_5 has also a successful range according to the positive dependence measure.

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A VARIANT OF THE PROOF OF VAN DER WAERDEN'S THEOREM BY FURSTENBERG

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Abstract. Let R be a commutative ring with identity. In this paper, for a given monotone decreasing positive sequence and an increasing sequence of subsets of R, we will define a metric on R using them. Then, we will use this kind of metric to obtain a variant of the proof of Van der Waerden's theorem by Furstenberg [3].

1. Introduction

In 1927, Van der Waerden published a famous theorem [5], which states that if the set of positive integers is divided into finitely many classes, then at least one of these classes contains arbitrarily long arithmetic progressions. This theorem was proved by several different methods. The ergodic theoretic method was established by Furstenberg [3] to prove Van der Waerden's theorem. Notion of a dynamical system plays a fundamental role in the proof. A dynamical system is defined as a pair (X, T), where X being a compact metric space and T is a continuous map (homeomorphism) from X into itself. To obtain number theoretic results, a particular kind of a dynamical system, a symbolic system, is preferable. Let $\Lambda = \{a_1, a_2, ..., a_n\}$ be a finite set and form the space $\Omega = \Lambda^{\mathbb{Z}}$ the set of all sequences with entries from Λ:

$$x \in \Omega \Leftrightarrow x = \{..., x_{-1}, x_0, x_1, ...\}.$$

Observe that Ω can be made a compact metric space, with the following metric:

$$d(x, y) = \inf \left\{ \frac{1}{k+1} : x_i = y_i \text{ for } |i| < k \right\} \text{ for } x, y \in \Omega.$$

When T is the shift homeomorphism, that is to say $Tx_n = x_{n+1}$, from Ω to itself, the pair (Ω, T) is called a symbolic dynamical system.

We shall be interested in defining a new symbolic dynamical system in order to

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prove Van der Waerden's Theorem. For this purpose, we will consider Λ not only as a finite set but also as a ring like $\mathbb{Z}_n$, and define a metric using the notion of filtration.
For a comprehensive treatment of the topological results and the ergodic theoretic method described here, the reader is refered to [2] and [4], respectively.

2. Main Results

Let R be a commutative ring with identity and

$$F : F_1 = \{0\} \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

be a chain of subsets of R such that $R = \bigcup_{i=1}^{\infty} F_i$. The chain F is also called a *filtration* for convenience. For a given monotone decreasing sequence $x := (x_n)$ on $(0, \infty)$, let's start by defining the function $d_{x,F} : R \times R \longrightarrow [0, \infty)$ as follows:

$$d_{x,F}(s,t) = \begin{cases} x_m & \text{if } A_{s,t} \text{ has an upper bound and } m = \max A_{s,t} \\ 0 & \text{otherwise} \end{cases}$$

where $A_{s,t} = \{n \in \mathbb{N} : \forall z \in F_n \; sz = tz\}$ and $s, t \in R$.

Proposition 1. *$d_{x,F}$ is a metric on R.*

Proof. (i) It is clear that $d_{x,F}(s,t) \geq 0$ for all $s, t \in R$.

(ii) If $d_{x,F}(s,t) = 0$, then $A_{s,t}$ has no upper bound and $1_R \in \bigcup_{i=1}^{\infty} F_i$ implies $1_R \in F_j$ for some j. It follows from $s1_R = t1_R$ that $s = t$.

(iii) It is easy to see that $d_{x,F}$ is a symmetric function by the definition.

(iv) Let $d_{x,F}(s,t) = x_m$. Then $sz = tz$ holds for all $z \in F_m$ and there exists an element $y_n \in F_n$ such that $sy_n \neq ty_n$ for $n > m$. For a given arbitrary $r \in R$, suppose $d_{x,F}(s,r) = x_l < x_m$ and $d_{x,F}(r,t) = x_k < x_m$. Then for each $w \in F_n$, where $n = \min\{l, k\} > m$, we have $sw = rw = tw$ which in turn implies that s and t are equal on F_n. This contradicts with $d_{x,F}(s,t) = x_m$. Therefore, $d_{x,F}(s,t) \leq d_{x,F}(s,r) + d_{x,F}(r,t)$ holds for all $s, r, t \in R$.

□

Remark 1. *Suppose that R is an integral domain. For $s, t \in R, s \neq t$, when $sz = tz$ holds, then z must be 0_R due to the fact that $(s-t)z = 0_R$ and R has no nonzero zero divisors. In this case, for an arbitrary filtration F:*

$$F_1 = \{0\} \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots \text{ such that } R = \bigcup_{i=1}^{\infty} F_i,$$

$$d_{x,F}(s,t) = \begin{cases} x_1 & \text{if } s \neq t \\ 0 & \text{if } s = t \;. \end{cases}$$

In other words, $d_{x,F}$ must be the discrete metric on integral domains.

Remark 2. *(i) If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $d_{x,F} \leq d_{y,F}$.*

(ii) If F and G are two filtration such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$, then $d_{x,F} \leq d_{x,G}$ holds.

Example 1. *For the commutative ring $\mathbb{Z}_6$ and the given filtration*

$$F : F_1 = \{0\} \subseteq \{0,2\} \subseteq \{0,2,4\} \subseteq \mathbb{Z}_6,$$

the distance table for a given arbitrary monotone decreasing and positive sequence $x := (x_n)$ as follows;

$d_{x,F}$	*0*	*1*	*2*	*3*	*4*	*5*
0	0	x_1	x_1	x_3	x_1	x_1
1	x_1	0	x_1	x_1	x_3	x_1
2	x_1	x_1	0	x_1	x_1	x_3
3	x_3	x_1	x_1	0	x_1	x_1
4	x_1	x_3	x_1	x_1	0	x_1
5	x_1	x_1	x_3	x_1	x_1	0

Theorem 1. *Suppose that R is a commutative ring with identity and τ is a topology on R. Then τ is the discrete topology if and only if there exist a monotone decreasing and positive sequence $x := (x_n)$ and a filtration F such that $\tau = \tau_{d_{x,F}}$ where $\tau_{d_{x,F}}$ is the metric topology induced by $d_{x,F}$.*

Proof. Let τ be the discrete topology on R. If we choose F to be $F_1 = \{0\} \subseteq F_2 = R$ and x to be an arbitrary monotone decreasing sequence on $(0, \infty)$, then we get the following metric

$$d_{x,F}(s,t) = \begin{cases} x_1 & \text{if } s \neq t \\ 0 & \text{if } s = t \end{cases}$$

on R. As $d_{x,F}$ is a discrete metric, we obtain that $\tau = \tau_{d_{x,F}}$.
Conversely, for a given topology $\tau_{d_{x,F}}$ on R, we will show that $\tau_{d_{x,F}}$ is the discrete topology. Since $1_R \in R = \bigcup\limits_{i=1}^{\infty} F_i$, there is some $k \geq 1$ such that $1_R \in F_k$. It follows from $1_R \in F_k$ that $m = \max A_{s,t} < k$ for all $s, t \in R, s \neq t$. Therefore, if we choose ε from the interval $(0, x_k)$, we get that the open ball of radius ε centred at s is a singleton, it means that

$$B(s, \varepsilon) = \{r \in R : d_{x,F}(s,r) < \varepsilon\} = \{s\}.$$

Hence, $\tau_{d_{x,F}}$ is the discrete topology on R. □

It was shown that using the metric $d_{x,F}$ on a commutative ring with identity, we cannot go beyond discrete topology. Therefore, it is necessary to change the definition to get effective results.

Let R be a commutative ring with identity and

$$F : F_1 = \{0\} \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

be a chain of subsets of R which do not contain any units. For a given monotone decreasing sequence $x := (x_n)$ on $(0, \infty)$ that converges to zero, let us define $d_{x,F} : R \times R \longrightarrow [0, \infty)$ as follows:

$$d_{x,F}(s,t) = \begin{cases} x_m & \text{if } A_{s,t} \text{ has an upper bound and } m = \max A_{s,t} \\ 0 & \text{otherwise} \end{cases}$$

where $A_{s,t} = \{n \in \mathbb{N} : \forall z \in F_n \; sz = tz\}$ and $s, t \in R$.

This time $d_{x,F}$ is a pseudometric on R. For this reason, define the space $R^* = R/_{\sim}$ of equivalence classes by setting

$$s \sim t \Leftrightarrow d_{x,F}(s,t) = 0$$

to obtain a metric.

Proposition 2. *$d^*_{x,F}([s],[t]) = d_{x,F}(s,t)$ is a metric on R^*.*

Example 2. *(i) Since $\mathbb{Z}_2$ has no zero divisors, it is easy to see $d^*_{x,F} = d_{x,F}$ for $\mathbb{Z}_2^{\mathbb{N}}$ and the filtration*

$$F : F_0 = \{\theta\} \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

where $F_n = \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n-times} \times \{0\} \times \cdots$ for each $n \in \mathbb{N}$ and $\theta = (0, 0, \dots)$.

(ii) For the commutative ring $\mathbb{Z}_6^{\mathbb{N}}$ and the filtration

$$F : F_0 = \{\theta\} \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

where $F_n = \underbrace{\{0,2\} \times \cdots \times \{0,2\}}_{n-times} \times \{0\} \times \cdots$ for each $n \in \mathbb{N}$,

we have $[(1,1,...)] = [(4,4,...)]$ since $d_{x,F}((1,1,...),(4,4,...)) = 0$ holds.

Proposition 3. *The operations induced by $+$ and $\cdot$ are continuous on the metric space $\left(\left(\mathbb{Z}_n^{\mathbb{N}}\right)^*, d^*_{x,F}\right)$.*

Proof. (i) $+ : \left(\mathbb{Z}_n^{\mathbb{N}}\right)^* \times \left(\mathbb{Z}_n^{\mathbb{N}}\right)^* \to \left(\mathbb{Z}_n^{\mathbb{N}}\right)^*$, $([a],[b]) \mapsto \left[(a_n + b_n)_{n \in \mathbb{N}}\right]$

Suppose that (a_n, b_n) converges to (a, b). Since $d_{x,F}(a_n, a) + d_{x,F}(b_n, b) \to 0$, for each $\varepsilon > 0$ there exists a natural number m such that for every natural number $n > m$, we have $d_{x,F}(a_n, a) + d_{x,F}(b_n, b) \leq x_k < \varepsilon$ where x_k is the greatest term of the sequence satisfying the inequality. Therefore, we get $a_n z = az$ and $b_n z = bz$ for all $z \in F_k$ which imply $(a_n + b_n) z = (a + b) z$ and $d_{x,F}(a_n + b_n, a + b) \leq x_k < \varepsilon$. Hence, $d^*_{x,F}([a_n] + [b_n], [a] + [b]) \to 0$ and $+$ is continuous.

(ii) $\cdot : \left(\mathbb{Z}_n^{\mathbb{N}}\right)^* \times \left(\mathbb{Z}_n^{\mathbb{N}}\right)^* \to \left(\mathbb{Z}_n^{\mathbb{N}}\right)^*$, $([a],[b]) \mapsto \left[(a_n . b_n)_{n \in \mathbb{N}}\right]$

First, we show the compatibility of multiplication; for any $s, s_1, t, t_1 \in \left(\mathbb{Z}_n^{\mathbb{N}}\right)^*$,

$$s \sim s_1, t \sim t_1 \Rightarrow st \sim s_1 t_1.$$

Suppose that $s \sim s_1, t \sim t_1$. For any $z \in \bigcup_{n=1}^{\infty} F_n$, we have

$$tz = t_1 z \Leftrightarrow (t - t_1)z = 0$$

since $t \sim t_1$. Combining the commutative property of multiplication and $s \sim s_1$ that we obtain

$$s((t - t_1)z) = 0 \Leftrightarrow (st)z = s(t_1 z) = t_1(sz) = (t_1 s_1)z = (s_1 t_1)z.$$

Hence, $st \sim s_1 t_1$.
Now, we prove the continuity of multiplication. Assume that (a_n, b_n) converges to (a, b). Since $d_{x,F}(a_n, a) + d_{x,F}(b_n, b) \to 0$, for each $\varepsilon > 0$ there exists a natural number m such that for every natural number $n > m$, we have $d_{x,F}(a_n, a) + d_{x,F}(b_n, b) \leq x_k < \varepsilon$ where x_k is the greatest term of the sequence satisfying the inequality. Therefore, we get $a_n z = az$ and $b_n z = bz$ for all $z \in F_k$. It follows that we have

$$(a_n b_n)z = a_n(b_n z) = a_n(bz) = (a_n z)b = (az)b = (ab)z$$

for all $z \in F_k$ by the commutative property of multiplication. Thus,

$$d_{x,F}(a_n b_n, ab) \leq x_k < \varepsilon.$$

Hence, $d^*_{x,F}([a_n][b_n], [a][b]) \to 0$ and $\cdot$ is continuous.

□

We have to work with $\mathbb{Z}_n^{\mathbb{Z}}$ instead of $\mathbb{Z}_n^{\mathbb{N}}$ if we want to say that the shift map T on the compact metric space is a homeomorphism. Therefore, let us define a compact metric space on the commutative ring $\mathbb{Z}_n^{\mathbb{Z}}$ using the notion of filtration.

(i) Let n be a prime number.
The metric space $\left(\mathbb{Z}_n^{\mathbb{Z}}, d_{x,F}\right)$ is generated by the filtration

$$F : F_0 = \{\theta\} \subseteq F_1 \subseteq \cdots \subseteq F_m \subseteq \cdots$$

where

$$F_m = \cdots \times \{0\}_{-m} \times (\mathbb{Z}_n)_{-m+1} \times \cdots \times (\mathbb{Z}_n)_{m-1} \times \{0\}_m \times \cdots$$

for each $m \in \mathbb{N}$ and $\theta = (\ldots, 0, 0, \ldots)$.
Since

$$B(s, \varepsilon) = \left\{y \in \mathbb{Z}_2^{\mathbb{Z}} : d_{x,F}(s, y) < \varepsilon\right\} = \\ \cdots \times (\mathbb{Z}_2)_{-m} \times \{s_{-m+1}\} \times \cdots \times \{s_{m-1}\} \times (\mathbb{Z}_2)_m \times \cdots$$

for $s \in \mathbb{Z}_2^{\mathbb{Z}}$ and $\varepsilon > 0$, where $m \in \mathbb{N}$ minimum such that $x_m < \varepsilon$, the metric space $\left(\mathbb{Z}_2^{\mathbb{Z}}, d_{x,F}\right)$ has the same topology as $\left(\{0,1\}^{\mathbb{Z}}, d\right)$ does,

$$\text{where } d(x, y) = \inf\left\{\frac{1}{k+1} : x_i = y_i \text{ for } |i| < k\right\}$$

$x, y \in \{0,1\}^{\mathbb{Z}}$ defined by Furstenberg [3].

(ii) Let n be a composite number.
The metric space $\left(\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*, d_{x,F}^*\right)$ is generated by the filtration
$$F : F_0 = \{\theta\} \subseteq F_1 \subseteq \cdots \subseteq F_m \subseteq \cdots$$
where
$$F_m = \cdots \times \{0\}_{-m} \times \{0,a\}_{-m+1} \times \cdots \times \{0,a\}_{m-1} \times \{0\}_m \times \cdots$$
for each $m \in \mathbb{N}$, $\theta = (\ldots, 0, 0, \ldots)$ and a is the smallest prime divisor of n.

Proposition 4. *The metric space* $\left(\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*, d_{x,F}^*\right)$ *defined above is compact.*

Proof. It is easy to prove that $\left(\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*, d_{x,F}^*\right)$ is a sequentially compact space since $\mathbb{Z}_n$ is finite set. It is sufficient to show that any sequence $(a_n)_{n\in\mathbb{N}}$ of members of $\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*$ has a subsequence converging to an element of $\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*$.
Let $a_n := [(c_{n,m})_{m\in\mathbb{Z}}]$ be a sequence in $\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*$. Take $\alpha_0 \in \mathbb{Z}_n$ such that $I(0) = \{n \in \mathbb{N} : c_{n,0} = \alpha_0\}$ is infinite. Let $s(0)$ be the minimum element (or any) of $I(0)$. Now, take $\alpha_1 \in \mathbb{Z}_n$ such that $I(1) = \{n \in I(0) : c_{n,1} = \alpha_1\}$ is infinite. Let $s(1)$ be any element of $I(1)$ such that $s(1) > s(0)$. Then, take $\alpha_{-1} \in \mathbb{Z}_n$ such that $I(-1) = \{n \in I(1) : c_{n,-1} = \alpha_{-1}\}$ is infinite and choose $s(-1)$ to be any element of $I(-1)$ such that $s(-1) > s(1)$. Continuing in this way, we obtain an element $a := [(\alpha_m)_{m\in\mathbb{Z}}]$ of $\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*$. Besides, $(a_{s(n)})_{n\in\mathbb{Z}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$ that converges to a. Consequently, the metric space $\left(\left(\mathbb{Z}_n^{\mathbb{Z}}\right)^*, d_{x,F}^*\right)$ is compact since it is known that a sequentially compact metric space is also compact. $\square$

Definition 1. *[3] Let (X,T) be a dynamical system. A point $x \in X$ is a recurrent point of (X,T) if for some sequence $n_k \to \infty$, $T^{n_k}x \to x$.*

Let us recall the Birkhoff Multiple Recurrence Theorem. G. D. Birkhoff showed that if X is a compact topological space and T is a continuous map from X to itself, then X has a recurrent point [1] which is called the Birkhoff Recurence Theorem. The following theorem, due to Furstenberg [3], generalizes the Birkhoff Recurrence Theorem since it guarantees the existence of a point which is simultaneously recurrent for $T, T^2, ..., T^n \;\; n \geq 1$.

Theorem 2 (Birkhoff Multiple Recurrence)**.** *[3] Let X be a compact metric space, and let $T : X \to X$ be a continuous map. Then for any integer $r \geq 1$, there exists a point $x \in X$ and a sequence $n_k \to \infty$ with $T^{n_k}x \to x, T^{2n_k}x \to x, ..., T^{rn_k}x \to x$.*

Definition 2. *An arithmetic progression of length l is a sequence of integers of the form*
$$a, a+d, a+2d, ..., a+(l-1)d$$
where $d \neq 0$.

Now, we give a variant of the proof of Van der Waerden's theorem by Furstenberg [3].

Theorem 3 (Van der Waerden)**.** *Let* $\mathbb{Z} = \bigcup_{i=0}^{r-1} C_i$ *be a partition of the integers into* r *subsets. Then one of the sets* C_j *contains an arithmetic progression of length* l*.*

Proof. Let $\mathbb{Z} = \bigcup_{i=0}^{r-1} C_i$ be a partition of the integers.

Case 1 Suppose r is a composite number. Let us define $\xi \in \mathbb{Z}_r^{\mathbb{Z}}$ which corresponds to a partition of $\mathbb{Z}$ with r sets. In other words,

$$\mathbb{Z} = \cup C_i \text{ where } C_i = \{n : \xi_n = i\}$$

represents an equivanlance class partition of $\mathbb{Z}$ in $\Omega := \left(\left(\mathbb{Z}_r^{\mathbb{Z}}\right)^*, d^*_{x,F}\right)$. Let $X \subseteq \Omega$,

$$X = \overline{\{T^n [\xi], n \in \mathbb{Z}\}}$$

be the closure of the set of all translates of $[\xi]$. According to the Birkhoff Multiple Recurrence theorem, there exists $[\beta] \in X$ and an $n > 0$ with the points $[\beta], T^n [\beta], ..., T^{ln} [\beta]$ within distance less than x_1 of one another. We know that for two elements $[\alpha]$ and $[\gamma]$ in Ω, $d^*_{x,F}([\alpha], [\gamma]) < x_1$ implies that one of

$$\alpha_0 = \gamma_0,\ \alpha_0 = \gamma_0 + q,\ ...\ ,\alpha_0 = \gamma_0 + (a-1)\, q$$

is satisfied in $\mathbb{Z}_r$ by virtue of the definition of $d^*_{x,F}$ where $r = aq$ and a is the smallest prime divisor of r. It follows that

$$\begin{gathered}\beta_0 = \beta_{k_{11}\cdot n} = \beta_{k_{12}\cdot n} = \cdots = \beta_{k_{1S_1}\cdot n}\\ \beta_0 + q = \beta_{k_{21}\cdot n} = \beta_{k_{22}\cdot n} = \cdots = \beta_{k_{2S_2}\cdot n}\\ \vdots\\ \beta_0 + (a-1)\, q = \beta_{k_{a1}\cdot n} = \beta_{k_{a2}\cdot n} = \cdots = \beta_{k_{aS_a}\cdot n}\end{gathered}$$

such that $\{1, 2, ..., l\} = \bigcup_{i=1}^{a} \{k_{i1}, ..., k_{iS_i}\}$.
Since $[\beta] \in X = \overline{\{T^j [\xi], j \in \mathbb{Z}\}}$, we can find m so that

$$\begin{gathered}\xi_m = \xi_{m+k_{11}\cdot n} = \xi_{m+k_{12}\cdot n} = \cdots = \xi_{m+k_{1S_1}\cdot n}\\ \xi_m + q = \xi_{m+k_{21}\cdot n} = \xi_{m+k_{22}\cdot n} = \cdots = \xi_{m+k_{2S_2}\cdot n}\\ \vdots\\ \xi_m + (a-1)\, q = \xi_{m+k_{a1}\cdot n} = \xi_{m+k_{a2}\cdot n} = \cdots = \xi_{m+k_{aS_a}\cdot n}.\end{gathered}$$

Then one of $C_i \cup C_{i+q} \cup ... \cup C_{i+(a-1)q}$, where $i \in \{0, 1, ..., q-1\}$, contains an arithmetic progression of length l.
Now, let us consider the partition

$$\mathbb{Z}=\bigcup_{i=0}^{ar-1} D_i \text{ where } C_i=D_i\cup D_{i+r}\cup ...\cup D_{i+(a-1)r} \text{ and } i\in\{0,1,...,r-1\}.$$

If we apply the above method for the partition

$$\mathbb{Z}=\bigcup_{i=0}^{ar-1} D_i \text{ and } \Omega:=\left(\left(\mathbb{Z}_{ar}^{\mathbb{Z}}\right)^*, d_{x,F}^*\right),$$

we obtain that one of $C_i=D_i\cup D_{i+r}\cup ...\cup D_{i+(a-1)r}$, where $i\in\{0,1,...,r-1\}$, contains an arithmetic progression of length l.

Case 2 Let r be a prime number. Now, let us arrange the partition $\mathbb{Z}=\bigcup_{i=0}^{r-1} C_i$ in order to turn into the first case. If we take $r'=2r$ and

$$\mathbb{Z}=\bigcup_{i=0}^{r'-1} E_i \quad \text{where } C_i=E_i\cup E_{i+r} \quad \text{for } i\in\{0,1,...,r-1\},$$

it follows from Case 1 that one of E_i contains an arithmetic progression of length l. □

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THE EFFECT OF SEMI PERFORATED DUCT ON RING SOURCED ACOUSTIC DIFFRACTION

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Abstract. An analytical solution is obtained for the diffraction problem. In an infinite cylindrical duct, the sound waves are emanating by a ring source. The duct is rigid for $z < l$ and perforated for $z > l$. The mixed boundary value problem is defined by a Wiener Hopf equation, by using the Fourier transform technique. Then the numerical solution is obtained. The influence of the parameters of the problem on the diffraction phenomenon is displayed graphically. The present study can be used as a model for different applications. Reducing noise in exhaust systems, ventilation systems are some of these applications.

1. Introduction

Radiation or diffraction of sound waves is an essential problem which has been extensively studied in the literature so far. The duct and pipe structures are commonly used in many industrial devices to control the unwanted and harmful noise, such as exhaust systems, ventilation systems, aircraft jet and modern turbofan engines. For this reason, it is essential to investigate more accurate mathematical models for such engineering problems.

The radiation of sound waves from a semi-infinite rigid duct was first discussed by [1]. In their study, by using the Wiener-Hopf technique, the solution was obtained analytically [2]. Covering the pipe/duct walls with an absorbing lining is an efficient method that has proven beneficial in noise reduction [3–5]. Another method of reducing unwanted sound is to provide additional sound absorption by using perforated structures. The phenomenon of perforated structures has been investigated by various authors [6–10]. This idea is essential because perforated structures provide some possibilities for investigating of sound diffraction.

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The goal of the present study is to consider the diffraction of sound waves emanating from a ring source by an infinite circular cylindrical duct. Duct walls are assumed to be infinitely thin and rigid for $z < l$, perforated for $z > l$. The ring source is located out of the duct and can be moved along the duct axis, but never go beyond the duct exit. This problem is a generalization of a previous work by the author [11] who considered the similar geometry in the case where the numerical results is absent. This geometry can be considered as a model of an acoustic waveguide for noise reduction. Due to the ring source, the total field have angular symmetry which makes the problem simpler than the asymmetric case. This mixed boundary value problem is investigated rigorously through the Wiener-Hopf technique. By applying the Fourier transform, we obtain a Wiener-Hopf equation. After application of usual decomposition and factorisation procedures the solution of the Wiener-Hopf equation is obtained. Then, numerical solution is obtained approximately for various values of the parameters such as the radii of the duct and ring source, frequency, perforated part, etc. The effect of values of the parameters on the diffraction phenomenon is presented graphically.

Validation of graphical results is obtained with unperforated (open) case. When the perforated part is absent, the present study is compared with the study of [12] and the results are found to be in good agreement.

2. Analysis

2.1. **Formulation of the Problem.** We consider the diffraction of sound waves by an infinite cylindrical duct. Infinitely thin duct walls are assumed. The duct is $\{r = a, z \in \mathbb{R}\}$ illuminated by a ring source located at $\{r = b > a, z = -c, c > 0\}$ (see Fig. 1). The part $z < l$ of the inner cylinder is hard walled while the part $z > l$ is perforated. From the installation of the problem, the ring source and the geometry is symmetrical. Therefore, the total field will be independent of azimuth ϕ everywhere in coordinate system (r, ϕ, z). The velocity potential ψ will be used to obtain acoustic pressure p and velocity v via $p = -\rho_0(\partial/\partial t)\psi$ and $\overrightarrow{v} = grad\psi$, where ρ_0 is the density of the undisturbed medium.

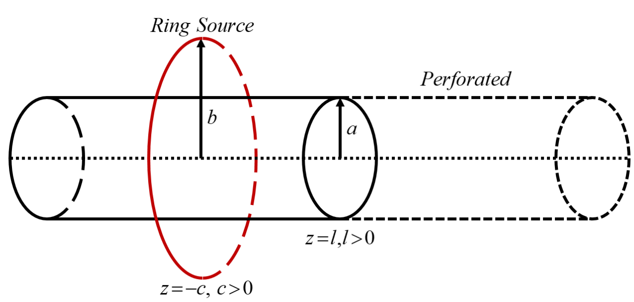

Figure 1. Geometry of the current problem.

For analytical convenience, it is suitable to write the total field as follows

$$\psi^T(r,z) = \begin{cases} \psi_1(r,z); & r > b \\ \psi_2(r,z); & a < r < b \\ \psi_3(r,z); & r < a \end{cases} \tag{1}$$

Time dependence is assumed to be $e^{-i\omega t}$ and suppressed throughout this work where $\omega = 2\pi f$ is the angular frequency and f is the frequency.

2.2. **Derivation of the Wiener-Hopf System.** The unknown fields $\psi_1(r,z), \psi_2(r,z)$ and $\psi_3(r,z)$ satisfy the wave equation for $z \in (-\infty, \infty)$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]\psi_j(r,z) = 0,\ \ j = 1,2,3 \tag{2}$$

with wave number $k = \omega/c_0$ where c_0 is the speed of the sound. By taking Fourier transform of these three equations we obtain the following integral representations

$$\psi_1(r,z) = \frac{k}{2\pi}\int_L A(\alpha)\, H_0^{(1)}(\lambda k r)\, e^{-i\alpha k z} d\alpha \tag{3}$$

$$\psi_2(r,z) = \frac{k}{2\pi}\int_L \left[B(\alpha)\, J_0(\lambda k r) + C(\alpha)\, Y_0(\lambda k r)\right] e^{-i\alpha k z} d\alpha \tag{4}$$

$$\psi_3(r,z) = \frac{k}{2\pi}\int_L D(\alpha)\, J_0(\lambda k r)\, e^{-i\alpha k z} d\alpha \tag{5}$$

where L is a suitable inverse Fourier transform integration contour in the complex α-plane [13]. J_0 and Y_0 are the Bessel and Neumann functions of order zero, respectively. $H_0^{(1)} = J_0 + iY_0$ is the Hankel function of the first type. $\lambda = \sqrt{1-\alpha^2}$ is square root function. The unknown coefficients $A(\alpha), B(\alpha), C(\alpha)$ and $D(\alpha)$, which are introduced in the solution of potential function, are to be determined by applying the following boundary conditions and continuity relations at $r = a$ and $r = b$.

$$\frac{\partial}{\partial r}\psi_2(a,z) = 0,\ \ z < l \tag{6}$$

$$\frac{\partial}{\partial r}\psi_3(a,z) = 0,\ \ z < l \tag{7}$$

$$\frac{\partial}{\partial r}\psi_2(a,z) = \frac{\partial}{\partial r}\psi_3(a,z),\ \ z > l \tag{8}$$

$$\psi_2(a,z) = \psi_3(a,z) + i\frac{\zeta_p}{k}\frac{\partial}{\partial r}\psi_3(a,z),\ \ z > l \tag{9}$$

ζ_p which is given by [14], is the nondimensional specific acoustic impedance, describing the acoustic properties of the perforated cylinder.

$$\zeta_p = \left[0.006 - ik\left(t_w + 0.75 d_h\right)\right]/\sigma \tag{10}$$

where σ is the porosity, d_h is the perforate hole diameter and t_w is the screen thickness. By using the ring source definition, given as

$$\frac{\partial}{\partial r}\psi_1(b,z)-\frac{\partial}{\partial r}\psi_2(b,z)=\delta(z+c)\,,\quad -\infty<z<\infty \tag{11}$$

$$\psi_1(b,z)=\psi_2(b,z),\quad -\infty<z<\infty \tag{12}$$

where δ is dirac delta function. Applying the boundary conditions on $r=a$ for equations (6), (7) and (8) and taking Fourier transforms gives

$$[D(\alpha)-B(\alpha)]\,J_1(\lambda ka)=C(\alpha)\,Y_1(\lambda ka) \tag{13}$$

similarly for equation (7)

$$-D(\alpha)\,\lambda kJ_1(\lambda ka)=e^{i\alpha kl}\Phi^+(\alpha) \tag{14}$$

continuity of pressure at $r=a$ for (9) yields

$$[B(\alpha)-D(\alpha)]\,J_0(\lambda ka)+C(\alpha)\,Y_0(\lambda ka)=e^{i\alpha kl}\Phi^-(\alpha)+i\frac{\zeta_p}{k}e^{i\alpha kl}\Phi^+(\alpha) \tag{15}$$

where $\Phi^+(\alpha)$ and $\Phi^-(\alpha)$ are a function analytic at the upper ($\operatorname{Im}\alpha>0$ or $\operatorname{Im}\alpha=0$ and $\operatorname{Re}\alpha>0$) and lower ($\operatorname{Im}\alpha<0$ or $\operatorname{Im}\alpha=0$ and $\operatorname{Re}\alpha<0$) half plane respectively and defined as

$$\Phi^+(\alpha)=\int_l^\infty\frac{\partial}{\partial r}\psi_3(a,z)\,e^{i\alpha k(z-l)}dz \tag{16}$$

$$\Phi^-(\alpha)=\int_{-\infty}^l[\psi_2(a,z)-\psi_3(a,z)]\,e^{i\alpha k(z-l)}dz \tag{17}$$

The spectral coefficient $D(\alpha)$ can be found easily from (14) while $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are related to each other by the definition of the ring source given in (11,12). By using the boundary conditions on $r=b$, one obtains

$$\lambda kA(\alpha)\,H_1^{(1)}(\lambda kb)=\lambda kB(\alpha)\,J_1(\lambda kb)+\lambda kC(\alpha)\,Y_1(\lambda kb)-e^{-i\alpha kc} \tag{18}$$

$$A(\alpha)\,H_0^{(1)}(\lambda kb)=B(\alpha)\,J_0(\lambda kb)+C(\alpha)\,Y_0(\lambda kb) \tag{19}$$

where $H_1^{(1)}=J_1+iY_1$. From (18) and (19), we get

$$B(\alpha)=A(\alpha)+e^{-i\alpha kc}\frac{\pi b}{2}Y_0(\lambda kb) \tag{20}$$

$$C(\alpha)=iA(\alpha)-e^{-i\alpha kc}\frac{\pi b}{2}J_0(\lambda kb) \tag{21}$$

(13) and (14) allows us to express the coefficients $A(\alpha)$ and $D(\alpha)$ in terms of the analytic function $\Phi^+(u)$

$$A(\alpha) = -\frac{e^{i\alpha kl}\Phi^+(\alpha)}{\lambda k H_1^{(1)}(\lambda ka)} + \frac{\pi b}{2}\frac{e^{-i\alpha kc}}{H_1^{(1)}(\lambda ka)}\left[J_0(\lambda kb)\,Y_1(\lambda ka) - Y_0(\lambda kb)\,J_1(\lambda ka)\right] \tag{22}$$

$$D(\alpha) = -\frac{e^{i\alpha kl}\Phi^+(\alpha)}{\lambda k J_1(\lambda ka)} \tag{23}$$

The substitution of $B(\alpha), C(\alpha)$ and $D(\alpha)$ given by (20), (21), (23) into (15) yields

$$\Phi^+(\alpha)M(\alpha) = \Phi^-(\alpha) + \frac{b}{a}e^{-i\alpha k(c+l)}\frac{H_0^{(1)}(\lambda kb)}{\lambda k H_1^{(1)}(\lambda ka)} \tag{24}$$

where $M(\alpha)$ is kernel function to be factorized

$$M(\alpha) = \frac{J_0(\lambda ka)}{\lambda k J_1(\lambda ka)} - \frac{H_0^{(1)}(\lambda ka)}{\lambda k H_1^{(1)}(\lambda ka)} - i\frac{\zeta_p}{k} \tag{25}$$

2.3. **Solution of the Wiener-Hopf Equation.** Consider the Wiener-Hopf equation in (24) and rearrange it using (25) in the following form

$$\Phi^+(\alpha)M^+(\alpha) = \Phi^-(\alpha)\,M^-(\alpha) + \frac{b}{a}e^{-i\alpha k(c+l)}M^-(\alpha)\,\frac{H_0^{(1)}(\lambda kb)}{\lambda k H_1^{(1)}(\lambda ka)} \tag{26}$$

Here, $M^-(\alpha)$ and $M^+(\alpha)$ are analytic functions in the lower and upper half planes, respectively, From the Wiener-Hopf factorization of $M(\alpha)$ as [15], one gets

$$M(\alpha) = \frac{M^+(\alpha)}{M^-(\alpha)} \tag{27}$$

Now consider (26), by using the classical decomposition procedure for complex term, one gets

$$\Phi^+(\alpha)M^+(\alpha) = \Phi^-(\alpha)\,M^-(\alpha) + I^+(\alpha) + I^-(\alpha) \tag{28}$$

Decomposing $I(\alpha)$ we obtain split functions $I^+(\alpha)$ and $I^-(\alpha)$ which are regular in the upper and lower half planes, respectively [16].

$$I(\alpha) = \frac{b}{a}e^{-i\alpha k(c+l)}M^-(\alpha)\,\frac{H_0^{(1)}(\lambda kb)}{\lambda k H_1^{(1)}(\lambda ka)} = I^+(\alpha) + I^-(\alpha) \tag{29}$$

The Wiener-Hopf equation in (28), yields

$$\Phi^+(\alpha)M^+(\alpha) - I^+(\alpha) = \Phi^-(\alpha)\,M^-(\alpha) + I^-(\alpha) \tag{30}$$

Now both sides of (30) are analytical functions on upper and lower regions, one can obtains the Wiener-Hopf solution

$$\Phi^+(\alpha) = I^+(\alpha)\,/M^+(\alpha) \tag{31}$$

and

$$\Phi^{-}(\alpha) = -I^{-}(\alpha)/M^{-}(\alpha) \tag{32}$$

3. Far Field

The total field in $r > b$ can be evaluated from (3)

$$\psi_1(r,z) = \frac{k}{2\pi}\int\limits_L A(\alpha) H_0^{(1)}(\lambda kr) e^{-i\alpha kz} d\alpha \tag{33}$$

We may write the total field as follows

$$\psi_1(r,z) = \psi_d(r,z) + \psi_i(r,z) + \psi_r(r,z) \tag{34}$$

where

$$\psi_d(r,z) = -\frac{k}{2\pi}\int\limits_L \frac{\Phi^{+}(\alpha)}{\lambda k H_1^{(1)}(\lambda ka)} H_0^{(1)}(\lambda kr) e^{-i\alpha k(z-l)} d\alpha \tag{35}$$

$$\psi_i(r,z) + \psi_r(r,z) = \frac{kb}{4}\int\limits_L \frac{J_0(\lambda kb) Y_1(\lambda ka) - Y_0(\lambda kb) J_1(\lambda ka)}{H_1^{(1)}(\lambda ka)} \times H_0^{(1)}(\lambda kr) e^{-i\alpha k(z+c)} d\alpha \tag{36}$$

Replacing $H_0^{(1)}(\lambda kr)$ by its asymptotic expressions valid for $kr \gg 1$.

$$H_0^{(1)}(\lambda kr) \sim \sqrt{\frac{2}{\pi\lambda kr}} e^{i(\lambda kr - \pi/4)} \tag{37}$$

and applying the saddle point technique [17], we get

$$\psi_1(r,z) = \psi_d(R_1,\theta_1) + \psi_i(R_2,\theta_2) + \psi_r(R_2,\theta_2) \tag{38}$$

where

$$\psi_d(R_1,\theta_1) = \frac{i}{\pi}\frac{\Phi^{+}(-\cos\theta_1)}{\sin\theta_1 H_1^{(1)}(ka\sin\theta_1)}\frac{e^{ikR_1}}{kR_1} \tag{39}$$

$$\begin{aligned}\psi_i(R_2,\theta_2) &+ \psi_r(R_2,\theta_2) \\ &= \frac{kb}{2i}\frac{J_0(kb\sin\theta_2) Y_1(ka\sin\theta_2) - Y_0(kb\sin\theta_2) J_1(ka\sin\theta_2)}{H_1^{(1)}(ka\sin\theta_1)}\frac{e^{ikR_2}}{kR_2}\end{aligned} \tag{40}$$

Here R_1, θ_1 and R_2, θ_2 are the spherical coordinates

$$r = R_1\sin\theta_1,\ z - l = R_1\cos\theta_1 \tag{41}$$

and

$$r = R_2\sin\theta_2,\ z + c = R_2\cos\theta_2 \tag{42}$$

4. Numerical Results

In order to show the effects of the parameters like the radii of the duct and the ring source and the perforated part on the diffracted field, some results displaying a changing of the sound pressure level with different values are presented in this section. Figures are plotted for Sound Pressure Level (SPL), described by

$$SPL = 20\log_{10}\left|\frac{p}{2\cdot 10^{-5}}\right|$$

Far field values are plotted 46 m from the duct edge. Parameter values of the perforated part are taken from the study of [14].

First, variations of sound pressure level for different values of radius (a) are presented in Figure 2 for $f = 1500$ Hz, $b = 0.075$ m, $l = 0.010$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$. In the graph, it is seen that as the value of (a) decreases, the sound pressure level decreases. In Figure 3, similar analysis is

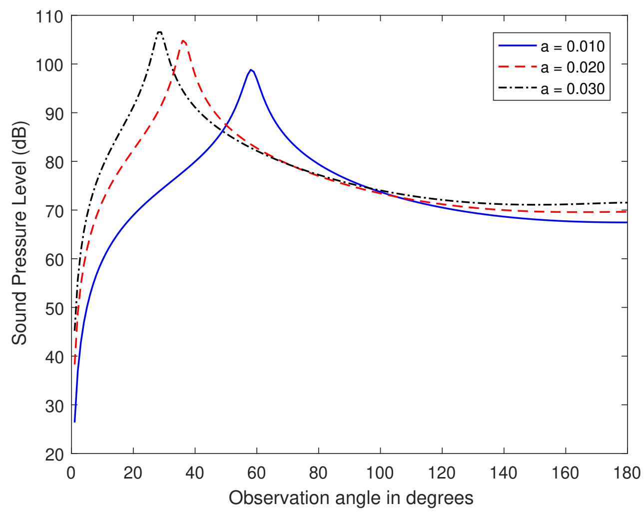

Figure 2. SPL versus the observation angle for different values of the duct radius (a) with $f = 1500$ Hz, $b = 0.075$ m, $l = 0.010$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$.

also carried for different values of ring source radius (b) for $f = 1500$ Hz, $a = 0.010$ m, $l = 0.010$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$. Sound pressure level decreases with decreasing value of ring source radius like in Figure 2.

Figures 4 and 5 display the same effect to the sound pressure level for different values of l and c. For Figure 4, the parameter values are $f = 1500$ Hz, $a = 0.010$

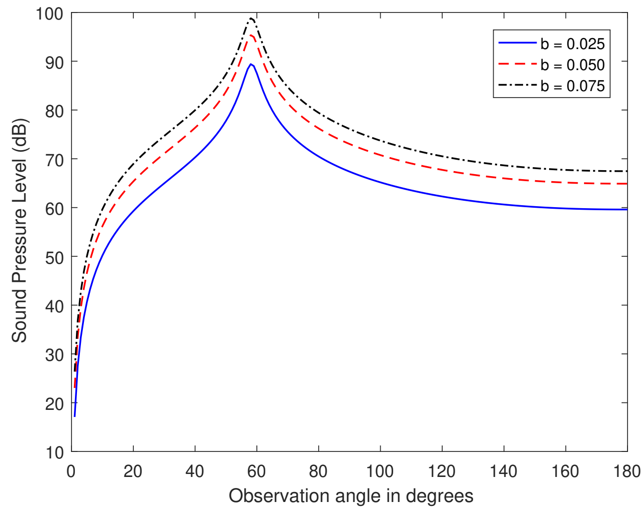

FIGURE 3. SPL versus the observation angle for different values of the ring source radius (b) with $f = 1500$ Hz, $a = 0.010$ m, $l = 0.010$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$.

m, $b = 0.0510$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$ while for Figure 5 the parameter values are $f = 1500$ Hz, $a = 0.010$ m, $b = 0.050$ m, $l = 0.010$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$. Sound pressure level decreases with increasing values of l and c, as expected.

From Figure 6, one can see the effect of the frequency on the sound pressure level. This graph is plotted for $a = 0.010$ m, $b = 0.025$ m, $l = 0.010$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$. The similar effect is observed like in Figure 2.

In Figure 7, the effect of specific acoustic impedance (ζ_p) on sound pressure level is first studied for three different acoustic impedances and compared to open case. This graph is plotted for non dimensional parameters which values are $ka = 1$, $kb = 10$, $kl = 10$, $kc = 6$. It is seen that existing of perforated part makes contribution to the reduction of sound pressure level and when imaginary coefficient of acoustic impedance decreases the pressure level decreases. It should be noted that, due to Equation (10), the real part of ζ_p remains constant and the imaginary part changes for different values of frequency.

In order to show the accuracy of the numerical results obtained in this study, ζ_p is taken zero and the results are compared with the study of [12] for semi-infinite rigid duct. Figure 8 shows that the results are consistent, that is, the results obtained

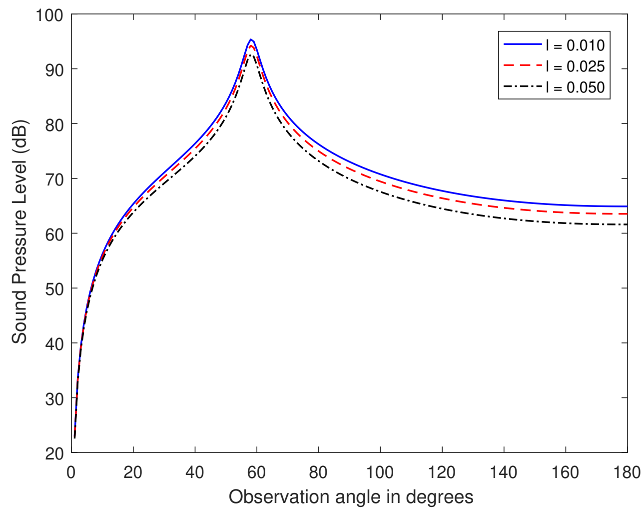

FIGURE 4. SPL versus the observation angle for different values of the duct extension (l) with $f = 1500$ Hz, $a = 0.010$ m, $b = 0.0510$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$.

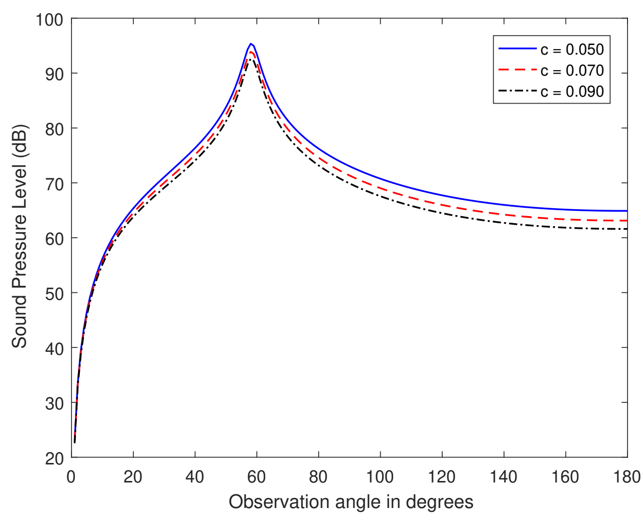

FIGURE 5. SPL versus the observation angle for different values of the ring source axes (c) with $f = 1500$ Hz, $a = 0.010$ m, $b = 0.050$ m, $l = 0.010$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$.

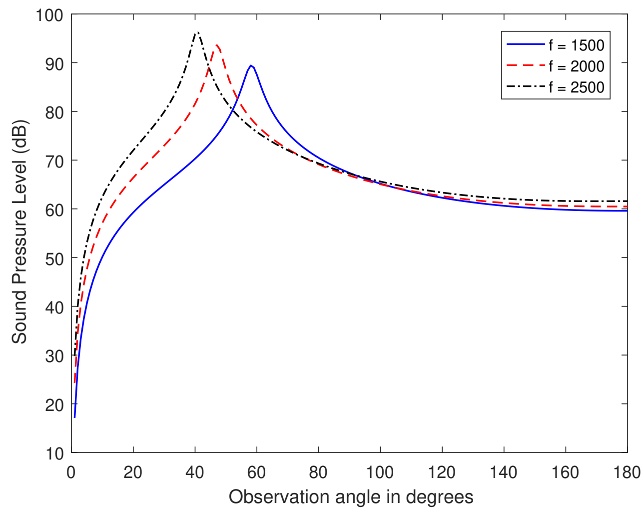

FIGURE 6. SPL versus the observation angle for different values of the frequency (f) with $a = 0.010$ m, $b = 0.025$ m, $l = 0.010$ m, $c = 0.050$ m, $t_w = 0.00081$ m, $d_h = 0.0249$ m, $\sigma = 0.057$.

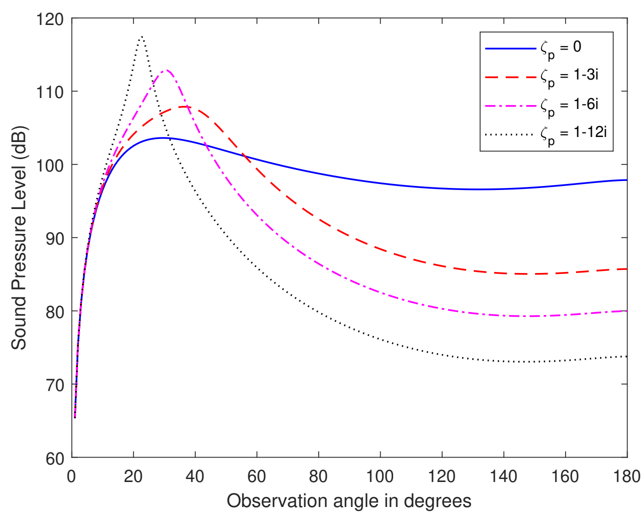

FIGURE 7. SPL versus the observation angle for open perforated duct with $ka = 1$, $kb = 10$, $kl = 10$, $kc = 6$.

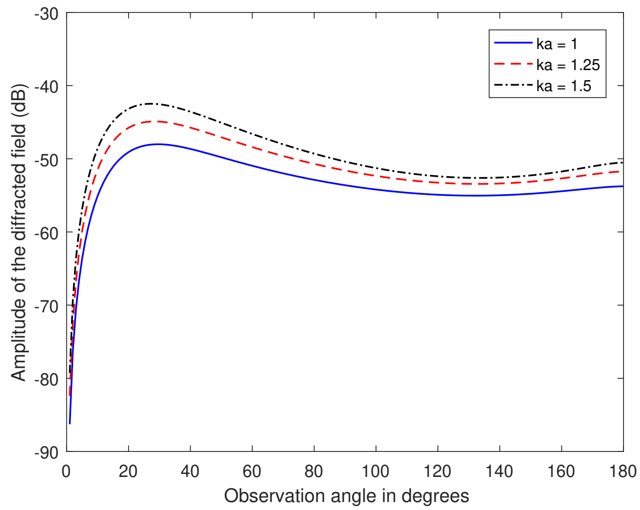

FIGURE 8. Comparison of the diffracted field with the study of [12] for semi infinite rigid duct.

in this study are correct. Notice that for Figure 8, the diffracted field graphic is produced for $20\log_{10}|\psi_d(R_1,\theta_1)|$.

5. Conclusion

In this study, diffraction of sound waves emanating from a ring source by an infinite cylindrical duct which is rigid for $z < l$ and perforated for $z > l$ has been investigated by using the Fourier transform technique in conjunction with the Wiener Hopf technique. The problem is modelled two dimensional due to symmetry of the geometry and of the ring source. A solution is derived by solving the Wiener Hopf equation. To a better understanding the effect of the parameters of the problem such as the radii of the duct and ring source and perforated part on the sound pressure level, graphics are presented for some specific values of the problem. It has been observed that the sound pressure level decreases as the values of the frequency (f), the duct radius (a) and the ring source radius (b) decreases. On the contrary, it is observed that the sound pressure level increases as the values of the duct extension (l) and the ring source axes (c) decreases. The effect of the perforated duct on the sound pressure level is more effective. While a few decibels change in sound pressure level is observed for other parameters of the problem, the variation from the perforated duct is more significant. It is found that presence of the perforated part reduced the sound pressure level when compared with the open part situation. The results are also compared with the study of [12] $(\zeta_p = 0)$ and it is observed that the agreement is perfect.

Declaration of Competing Interests The author has no competing interests to declare.

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COMPARISON OF DIFFERENT ESTIMATION METHODS FOR THE INVERSE WEIGHTED LINDLEY DISTRIBUTION

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Abstract. In this paper, different estimation methods are considered for the parameters of the inverse weighted Lindley (IWL) distribution introduced by Ramos et al.(2019). Parameters of the IWL are estimated by the method of maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CVM) and Anderson Darling (AD). The performances of the estimators are compared using Monte Carlo simulation study via bias, mean square error and deficiency (Def) criteria. Finally, a real data set is analyzed for illustrative purposes.

1. Introduction

Lindley distribution presented by Lindley [7] is an important distribution in statistics and many applied areas because of its flexible mathematical properties. Furthermore, Lindley distribution is more preferable than the exponential distribution in many cases (see [5]). Different generalizations are considered in the literature such as given in [15], [1], [3] to add more flexibility to Lindley distribution. Weighted distributions can extend and provide more flexibility to standard distributions (see [11]). Two-parameter weighted Lindley (WL) distribution is introduced by Ghitany et al. [4]. Mazucheli et al. [3] study on the finite sample properties of the parameters of the WL distribution using four methods. Wang and Wang [14] propose bias-corrected maximum likelihood (bias-corrected ML) estimators for the parameters of the WL distribution. Ramos and Louzada [13] introduce three parameters generalized weighted Lindley distribution. Ramos et al. [12] propose the inverse weighted Lindley (IWL) distribution. The IWL distribution is a component of two mixture model with upside-down bathtub hazard rate function. The IWL distribution is flexible to model data sets in the presence of heterogeneity (see [12]).

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For example, if we are interested in life time of products in a group, it can be considered that the group is heterogeneous. Since the observed failure times of products could be different. In this case, the IWL distribution can be appropriate to describe the heterogeneity in the data.

The IWL distribution is specified by the probability density function (pdf)

$$f(t) = \frac{\lambda^{\phi+1}}{(\phi+\lambda)\Gamma(\phi)} t^{-\phi-1}\left(1+\frac{1}{t}\right)e^{-\lambda t^{-1}}, \tag{1}$$

for all $t > 0$, $\phi > 0$ and $\lambda > 0$ where $\Gamma(\phi)$ is the gamma function which is computed by $\Gamma(\phi) = \int_0^\infty e^{-x}x^{\phi-1}dx$ is the gamma function. The corresponding cumulative distribution function (cdf) is given by

$$F(t) = \frac{\Gamma(\phi, \lambda t^{-1})(\lambda+\phi) + (\lambda t^{-1})^{\phi} e^{-\lambda t^{-1}}}{(\lambda+\phi)\Gamma(\phi)} \tag{2}$$

where $\Gamma(x, y) = \int_x^\infty w^{y-1}e^{-x}dw$ is the upper incomplete gamma. The survival function and hazard function of the IWL distribution are defined as follows

$$S(t) = \frac{\gamma(\phi, \lambda t^{-1})(\lambda+\phi) - (\lambda t^{-1})^{\phi} e^{-\lambda t^{-1}}}{(\lambda+\phi)\Gamma(\phi)}, \tag{3}$$

$$h(t) = \frac{\lambda^{\phi+1} t^{-\phi-1}(1+t^{-1})e^{-\lambda t^{-1}}}{\gamma(\phi, \lambda t^{-1})(\lambda+\phi) - (\lambda t^{-1})^{\phi} e^{-\lambda t^{-1}}}, \tag{4}$$

respectively. Here $\gamma(y, x) = \int_0^x w^{y-1}e^{-w}dw$ is the lower incomplete gamma function. Hazard function plots of the IWL distribution for some selected values of parameters (ϕ, λ) are presented in Figure 1.

We refer to [12] for the further details about the IWL distribution.

Ramos et al. [12] present the ML and Bias-corrected ML estimators for the parameters of the IWL distribution for both complete and censored data and examine the efficienct of bias correction via Monte Carlo simulation.

To the best of our knowledge, parameters of the IWL distribution have not been estimated using different methods, namely, least square (LS), weighted least squares (WLS), Cramér-von Mises (CVM) and Anderson Darling (AD) methods.

In this paper, we propose ML, LS, WLS, CVM and AD estimators for parameters of the IWL distribution. CVM and AD estimators are in the class of minimum distance estimators which are based on minimizing distance between the estimated and empirical cdf with respect to the parameters of interest. Minimum distance estimators are also called as goodness of fit estimators. See [2] and [8] for the further details of goodness of fit estimators. We carry out Monte Carlo simulation study in order to compare performances of the proposed estimators in terms of bias, mean squared error (MSE) and deficiency (Def) criteria.

The rest of paper is organized as follows. Brief descriptions of ML, LS, WLS, CVM and AD methods are given in Section 2. In Section 3, an extensive Monte

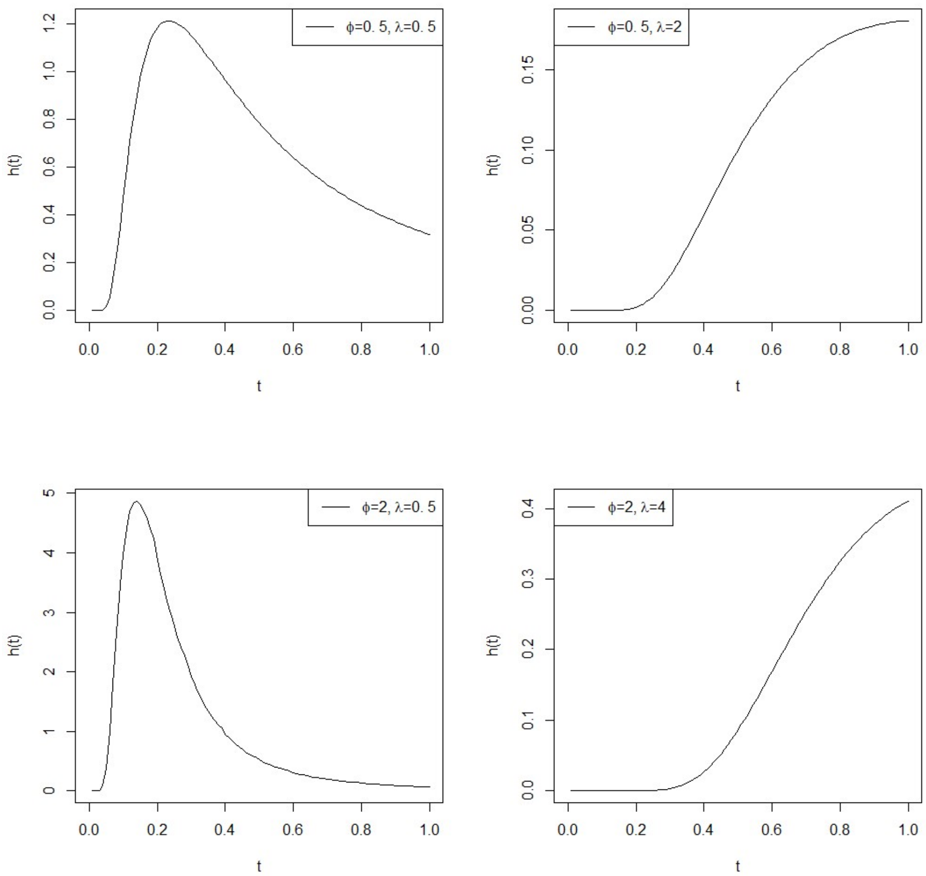

Figure 1. Hazard function plots of the IWL distribution for some selected values of parameters (ϕ, λ).

Carlo simulation study is carried out to compare the performances of the estimators for parameters of the IWL distribution. In Section 4, we give real data application to illustrate the implementation of the proposed methodology. In the final section, the concluding remarks are given.

2. Estimation methods

In this section, we give a brief information of the estimation methods, called as ML, LS, WLS, CVM and AD used to estimate parameters of the IWL in this study.

2.1. **Maximum likelihood estimators.** Let $T_1, T_2, ..., T_n$ be a random sample from the IWL(ϕ, λ) distribution. Then, the log-likelihood function (l) of the observed sample is

$$l = n(\phi+1)\log\lambda - n\log(\lambda+\phi) - n\log\Gamma(\phi) - \lambda\sum_{i=1}^{n}\frac{1}{t_i} - (\phi+1)\sum_{i=1}^{n}\log(t_i). \quad (5)$$

The ML estimators of the parameters ϕ and λ are obtained from the following likelihood equations:

$$\frac{\partial l}{\partial \phi} = n\log(\lambda) - \sum_{i=1}^{n}\log(t_i) - \frac{n}{\lambda+\phi} - n\psi(\phi) = 0 \quad (6)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n(\phi+1)}{\lambda} - \sum_{i=1}^{n}\frac{1}{t_i} - \frac{n}{\lambda+\phi} = 0 \quad (7)$$

where $\psi(k) = \frac{\partial}{\partial k}\log\Gamma(k) = \frac{\Gamma'(k)}{\Gamma(k)}$ is the digamma function. The ML estimate of λ is obtained from equation (7) as

$$\hat{\lambda}_{ML} = \frac{-\hat{\phi}_{ML}(\xi(t)-1) + \sqrt{(\hat{\phi}_{ML}(\xi(t)-1))^2 + 4\xi(t)(\hat{\phi}_{ML}^2 + \hat{\phi}_{ML})}}{2\xi(t)} \quad (8)$$

where $\xi(t) = \sum_{i=1}^{n}(nt_i)^{-1}$. It is obvious that (6) cannot be solved explicitly. Therefore, for computing the ML estimator of ϕ, numerical methods should be performed. See [12] for more details about the ML estimators of the parameters of the IWL distribution.

2.2. **Least Squares Estimation Method.** Let $x_{(i)}$, $i = 1, 2, ..., n$ be the order statistics of a random sample from the IWL distribution. Since $F(x_{(i)})$ behaves as the i-th order statistic of a sample size from $U(0,1)$, expected value and variance of $F(x_{(i)})$ are given as follows:

$$E[F(x_{(i)})] = \frac{i}{n+1} \quad \text{and} \quad Var[F(x_{(i)})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \quad (9)$$

respectively. The LS estimators of the parameters of the IWL distribution are obtained by minimizing the following function with respect to the parameters ϕ and λ.

$$S = \sum_{i=1}^{n}\left(F(x_{(i)}) - \frac{i}{n+1}\right)^2. \quad (10)$$

Here $F(.)$ is the cdf of the IWL given in (2). LS estimators of ϕ and λ are obtained by solving following equations:

$$\frac{\partial S}{\partial \phi} = \sum_{i=1}^{n}\left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1}\right)\Lambda_1(x_{(i)};\phi,\lambda) = 0,$$

$$\frac{\partial S}{\partial \lambda} = \sum_{i=1}^{n} \left(F(x_{(i)}; \phi, \lambda) - \frac{i}{n+1} \right) \Lambda_2(x_{(i)}; \phi, \lambda) = 0, \tag{11}$$

where

$$\begin{aligned}
\Lambda_1(x_{(i)}; \phi, \lambda) &= \frac{\left(\Gamma(\phi,\lambda t^{-1})+\gamma_1(\phi+\lambda)+(\lambda t^{-1})^{\phi}\ln(\lambda t^{-1})e^{-\lambda t^{-1}}\right)\left((\lambda+\phi)\Gamma(\phi)\right)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2} \\
&\quad - \frac{\left(\Gamma(\phi,\lambda t^{-1})(\lambda+\phi)+(\lambda t^{-1})^{\phi}e^{-\lambda t^{-1}}\right)\left(\lambda\gamma_3+\Gamma(\phi)+\phi\gamma_3\right)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2}, \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\Lambda_2(x_{(i)}; \phi, \lambda) &= \frac{\left(\Gamma(\phi,\lambda t^{-1})+\gamma_2(\lambda+\phi)+t^{-1}e^{-\lambda t^{-1}}(\phi(\lambda t^{-1})^{\phi-1}-(\lambda t^{-1})^{\phi})\right)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2} \\
&\quad \times \left((\lambda+\phi)\Gamma(\phi)\right) - \frac{\left(\Gamma(\phi,\lambda t^{-1})(\lambda+\phi)+(\lambda t^{-1})^{\phi}e^{-\lambda t^{-1}}\right)\Gamma(\phi)}{\left((\lambda+\phi)\Gamma(\phi)\right)^2}, \qquad (13)
\end{aligned}$$

respectively. It is obvious that, since equations given in (11) include nonlinear functions, numerical methods should be performed to obtain LS estimators of ϕ and λ.

2.3. **Weighted Least Squares Estimators.** The WLS estimators of the parameters ϕ and λ are obtained by minimizing the following function:

$$S_w = \sum_{i=1}^{n} w_i \left(F(x_{(i)}) - \frac{i}{n+1} \right)^2 \tag{14}$$

where w_i denotes the weight and computed by

$$w_i = \frac{1}{\mathrm{Var}(F(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i-1)}, \quad i = 1, 2, ..., n.$$

The WLS estimators of ϕ and λ are obtained by solving the following nonlinear equations:

$$\begin{aligned}\frac{\partial S_w}{\partial \phi} &= \sum_{i=1}^{n} w_i \left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1} \right) \Lambda_1(x_{(i)};\phi,\lambda) = 0, \\ \frac{\partial S_w}{\partial \lambda} &= \sum_{i=1}^{n} w_i \left(F(x_{(i)};\phi,\lambda) - \frac{i}{n+1} \right) \Lambda_2(x_{(i)};\phi,\lambda) = 0, \end{aligned} \tag{15}$$

respectively. Here Λ_1 and Λ_2 are given in (13). It is clear that WLS estimators should also be obtained using numerical methods, since equations given in (15) cannot be solved explicitly.

2.4. **Cramér-von Mises estimators.** CVM estimators of the parameters of the IWL distribution are obtained by minimizing the following equation with respect to the parameters ϕ and λ.

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_{(i)},\phi,\lambda) - \frac{2i-1}{2n} \right)^2 \tag{16}$$

To obtain the CVM estimators of the parameters, we have to solve the following equations by using numerical methods.

$$\begin{aligned}\frac{\partial CVM}{\partial \phi} &= \sum_{i=1}^{n} \left(F(x_{(i)};\phi,\lambda) - \frac{2i-1}{2n} \right) \Lambda_1(x_{(i)};\phi,\lambda) = 0, \\ \frac{\partial CVM}{\partial \lambda} &= \sum_{i=1}^{n} \left(F(x_{(i)};\phi,\lambda) - \frac{2i-1}{2n} \right) \Lambda_2(x_{(i)};\phi,\lambda) = 0. \end{aligned} \tag{17}$$

Here, Λ_1 and Λ_2 are given in (13).

2.5. **Anderson Darling estimators.** The AD estimators of ϕ and λ are obtained by minimizing the following equation with respect to the parameters of interest.

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log \left[F(x_{(i)}) \left(1 - F(x_{(j)}) \right) \right] \right\}, \tag{18}$$

where $j = n - i + 1$. The AD estimators of ϕ and λ are obtained by solving the nonlinear equations

$$\begin{aligned}\frac{\partial A}{\partial \phi} &= \sum_{i=1}^{n} (2i-1) \left[\frac{\Lambda_1(x_{(i)},\phi,\lambda)}{F(x_{(i)},\phi,\lambda)} - \frac{\Lambda_1(x_{(j)},\phi,\lambda)}{F(x_{(j)},\phi,\lambda)} \right] = 0 \\ \frac{\partial A}{\partial \lambda} &= \sum_{i=1}^{n} (2i-1) \left[\frac{\Lambda_2(x_{(i)},\phi,\lambda)}{F(x_{(i)},\phi,\lambda)} - \frac{\Lambda_2(x_{(j)},\phi,\lambda)}{F(x_{(j)},\phi,\lambda)} \right] = 0, \end{aligned} \tag{19}$$

respectively. Here, Λ_1 and Λ_2 are given in (13). Nonlinear equations given in (19) can be solved by using numerical methods.

3. Simulation study

In this section, we conduct a Monte-Carlo simulation study to compare the performance of the different estimation methods discussed in the previous section. The bias, MSE and Def criteria are used in the comparisons. The bias and MSE are respectively formulated as follows:

$$\text{Bias}(\hat{\theta}) = E(\theta - \hat{\theta}) \quad \text{and} \quad \text{MSE}(\hat{\theta}) = E(\theta - \hat{\theta})^2$$

where $\theta = (\phi, \lambda)$. The mathematical expression of the Def criterion used in this study to compare joint efficiencies of the parameters is given as

$$\text{Def} = \text{MSE}(\hat{\phi}) + \text{MSE}(\hat{\lambda}),$$

see [6] for the further details on DEF. In simulation study, we generate random data from the IWL distribution using the algorithm given by Ramos et al. [12]. The simulation study is performed considering the values: $(\phi, \lambda) = (0.5, 0.5)$, $(0.5, 2)$, $(2, 0.5)$, $(2, 4)$ and $n = (20, 50, 100, 200, 500)$.

For all the numerical computations, we use the R statistical software environment. The ML, LS, WLS, CVM and AD estimators of the parameters are obtained by using "optim" function. Simulation results are given in Table 1-Table 4.

It is observed from Table 1 and Table 2 that the ML estimators of ϕ and λ have the smallest bias for all sample sizes. The ML estimator is also the most efficient one for both ϕ and λ parameters with the smallest MSE values for all cases. The AD estimators of ϕ and λ outperform LS, WLS and CVM estimators in terms of bias and MSE criteria. Overall, the ML estimators of parameters of the IWL distribution is the best estimator in terms of Def criterion. It is followed by AD estimators.

It is observed from Table 3 that the ML estimators of ϕ and λ perform better than the others in terms of bias and MSE criterion in most cases. However AD estimators of ϕ and λ outperform the ML, LS, WLS and CVM estimators in terms of both bias and MSE criteria, when $n = 20$. According to Def, the AD estimator has the best performance for $n = 20$. Otherwise the ML estimator can be preferred.

It is observed from Table 4 that the ML estimators of ϕ and λ have the smallest bias and MSE values in most cases. On the other hand, the bias values of all estimators are close to each other. The AD is the best for $n = 20$ and followed by WLS and LS estimators.

The simulation results show that ML has the best performance with the lowest deficiency almost in all cases. However, AD has a little bit smaller deficiency than the ML when $n = 20$, $\phi = 2$ and $\lambda = 4$. Also, ML has higher deficiency than LS, WLS and AD when $n = 20$, $\phi = 2$ and $\lambda = 0.5$.

Overall, we suggest using the ML methodology for estimating the parameters of the IWL distribution because of its superior performance. Also for the small sample size, the AD estimators can be preferred. It can be also said that CVM estimators of ϕ and λ demonstrate the weakest performance for all cases.

Table 1. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 0.5$, $\lambda = 0.5$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	-0.1784	0.0019	-1.5279	0.3212	0.3231
	LS	-0.2345	0.0028	-2.3822	0.4274	0.4302
	WLS	-0.2291	0.0025	-2.7214	0.3867	0.3892
	CVM	-0.2608	0.0031	-2.5854	0.6440	0.6471
	AD	-0.2096	0.0025	-1.9553	0.3501	0.3526
50	ML	-0.1784	0.0018	-1.4903	0.3138	0.3157
	LS	-0.2341	0.0026	-2.3477	0.4117	0.4144
	WLS	-0.2291	0.0023	-2.6822	0.3645	0.3668
	CVM	-0.2519	0.0029	-2.4238	0.6248	0.6277
	AD	-0.2073	0.0022	-1.9289	0.3421	0.3443
100	ML	-0.1669	0.0017	-1.4630	0.3103	0.3120
	LS	-0.2314	0.0025	-2.3501	0.4013	0.4038
	WLS	-0.2240	0.0023	-2.5933	0.3528	0.3551
	CVM	-0.2497	0.0028	-2.3878	0.6209	0.6237
	AD	-0.2072	0.0021	-1.9823	0.3419	0.3441
200	ML	-0.1518	0.0015	-1.4334	0.3067	0.3082
	LS	-0.2294	0.0023	-2.3326	0.4002	0.4025
	WLS	-0.2233	0.0022	-2.4987	0.3312	0.3334
	CVM	-0.2474	0.0026	-2.3512	0.6076	0.6102
	AD	-0.2022	0.0021	-1.9663	0.3353	0.3374
500	ML	-0.1364	0.0011	-1.4280	0.2952	0.2964
	LS	-0.2234	0.0020	-2.3293	0.3982	0.4002
	WLS	-0.2212	0.0021	-2.4574	0.3166	0.3187
	CVM	-0.2469	0.0024	-2.3367	0.5825	0.5849
	AD	-0.2019	0.0019	-1.9356	0.3285	0.3304

4. Application

In this section, we analyse a real data set taken from the literature to show the implementation of the proposed methods. The data set in Table 5 consist of the failure stresses (in GPa) of 65 single carbon fiber of length 50mm. This data set is taken from Mazucheli et al. [9] in which weighted Lindley (WL) distribution is used.

To fit the IWL distribution to the data set, we use Q-Q plot technique which is one of the well-known and widely used graphical techniques. It is observed from Figure (2) that IWL distribution provides good fit to model the failure stresses data set.

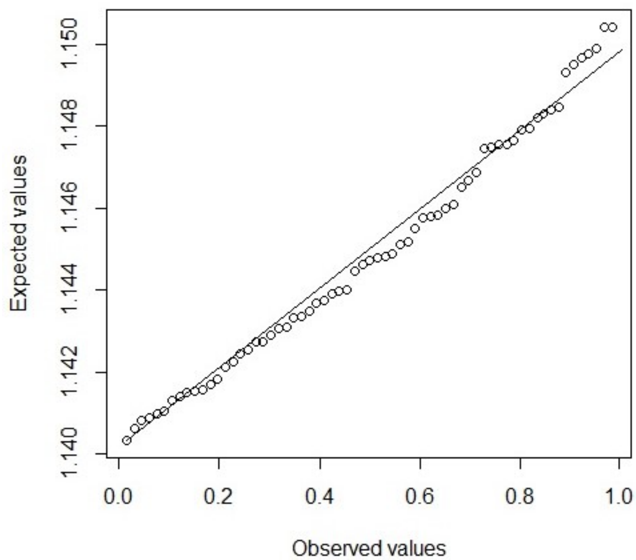

FIGURE 2. IWL QQ plot for the failure stresses data set.

In this study, we use Kolmogorov-Simirnov (KS) test which is a well-known goodness of fit test to test whether the IWL distribution is appropriate for the data.

Furthermore, to identify the parameter estimation methods providing a better fit to the data set, we use Akaike information criterion (AIC), Bayesian information criterion (BCI), the root mean square error (RMSE) and coefficient of determination (R^2) criteria.

We present the estimates of the IWL parameters, AIC, BIC, RMSE, R^2 and p-values obtained from Kolmogrov-Smirnov test are given in Table 6 for the failure stresses data set.

Acording to the results of the KS test given in Table 6, it can be concluded that the IWL distribution with the ML, LS, WLS, CVM and AD estimators of ϕ and λ works quite well to fit the failure stresses data set. However, It is clear from Table 6 that the ML is more desirable according to p-values for the IWL distribution.

It is also obvious from Table 6 that the ML estimates is the most appropriate model among the others. They are followed by the AD estimates. Since it is known that the model having the lowest AIC, the lowest BIC, the lowest RMSE and the highest R^2 value among the models provides better fitting to the data.

Table 2. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 0.5$, $\lambda = 2$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	-0.0884	0.0022	0.2936	0.1503	0.1541
	LS	-0.1597	0.0056	0.6346	0.1749	0.1805
	WLS	-0.1227	0.0038	0.3956	0.2276	0.2298
	CVM	-0.1989	0.0044	0.5366	0.2196	0.2240
	AD	-0.1346	0.0029	0.2968	0.1589	0.1617
50	ML	-0.0863	0.0021	0.2930	0.1485	0.1519
	LS	-0.1582	0.0051	0.6312	0.1702	0.1753
	WLS	-0.1223	0.0034	0.3956	0.2208	0.2229
	CVM	-0.1972	0.0041	0.5226	0.2112	0.2153
	AD	-0.1340	0.0026	0.2913	0.1429	0.1455
100	ML	-0.0855	0.0019	0.2857	0.1376	0.1396
	LS	-0.1578	0.0048	0.5947	0.1673	0.1720
	WLS	-0.1219	0.0030	0.3346	0.2189	0.2220
	CVM	-0.1906	0.0039	0.4985	0.2098	0.2138
	AD	-0.1324	0.0024	0.2791	0.1320	0.1344
200	ML	-0.0846	0.0018	0.2680	0.1296	0.1314
	LS	-0.1566	0.0046	0.5747	0.1573	0.1618
	WLS	-0.1187	0.0027	0.3298	0.2056	0.2083
	CVM	-0.1893	0.0037	0.4757	0.1945	0.1982
	AD	-0.1310	0.0022	0.2587	0.1256	0.1279
500	ML	-0.0838	0.0016	0.2297	0.1172	0.1188
	LS	-0.1487	0.0043	0.5493	0.1494	0.1536
	WLS	-0.1174	0.0025	0.3086	0.1986	0.2011
	CVM	-0.1876	0.0035	0.4328	0.1942	0.1977
	AD	-0.1306	0.0020	0.2328	0.1128	0.1148

5. Conclusion

In this paper, we focus different estimation methods of the unknown parameters of the IWL distribution. We consider ML, LS and WLS as classical methods and CVM and AD as minimum distance methods. As far as we know, LS, WLS, AD and CVM methods have not been used for estimating the parameters of the IWL distribution previously. We compare the performance of the estimators via Monte Carlo simulation study in terms of bias, MSE and Def criteria. The results of simulation study show that among the mentioned estimators, ML has the best performance in most of the cases. Also, it can be concluded that ML is followed by AD especially for small sample sizes. Overall, we suggest using ML methodology

TABLE 3. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 2$, $\lambda = 0.5$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	0.2831	0.0273	-0.7123	0.1688	0.1961
	LS	0.2457	0.0236	-0.6542	0.1583	0.1819
	WLS	0.2396	0.0253	-0.6288	0.1221	0.1474
	CVM	0.3139	0.0295	-0.7245	0.1747	0.2042
	AD	0.1946	0.0217	-0.6125	0.1174	0.1391
50	ML	0.1912	0.0207	-0.6073	0.1049	0.1256
	LS	0.2231	0.0225	-0.6456	0.1466	0.1691
	WLS	0.2065	0.0219	-0.6207	0.1207	0.1426
	CVM	0.3056	0.0278	-0.7098	0.1653	0.1931
	AD	0.1915	0.0212	-0.6098	0.1122	0.1334
100	ML	0.1905	0.0199	-0.5877	0.0972	0.1171
	LS	0.2178	0.0217	-0.6325	0.1352	0.1570
	WLS	0.1976	0.0205	-0.6140	0.1195	0.1400
	CVM	0.2945	0.0266	-0.6947	0.1573	0.1838
	AD	0.1911	0.0201	-0.5927	0.1002	0.1203
200	ML	0.1877	0.0188	-0.5614	0.0954	0.1141
	LS	0.2046	0.0202	-0.6245	0.1294	0.1496
	WLS	0.1912	0.0197	-0.6076	0.1124	0.1321
	CVM	0.2818	0.0242	-0.6544	0.1407	0.1648
	AD	0.1893	0.0193	-0.5706	0.0998	0.1191
500	ML	0.1763	0.0164	-0.5533	0.0826	0.0990
	LS	0.1932	0.0192	-0.6126	0.1122	0.1314
	WLS	0.1846	0.0187	-0.5973	0.1042	0.1229
	CVM	0.2666	0.0211	-0.6286	0.1376	0.1588
	AD	0.1786	0.0176	-0.5683	0.0919	0.1095

to obtain estimators of the IWL distribution. AD gives relatively good results and it is also preferable.

Declaration of Competing Interests The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Table 4. Simulated biases, MSEs and Def values of the ML, LS, WLS, CVM and AD estimators for $\phi = 2$, $\lambda = 4$.

		ϕ		λ		
n	Method	Bias	MSE	Bias	MSE	Def
20	ML	0.1105	0.0106	2.5729	0.3218	0.3324
	LS	0.1127	0.0134	2.6473	0.3457	0.3591
	WLS	0.1110	0.0108	2.6126	0.3462	0.3571
	CVM	0.1312	0.0153	2.7390	0.3959	0.4112
	AD	0.1057	0.0097	2.4957	0.2562	0.2659
50	ML	0.1026	0.0092	2.4559	0.2452	0.2544
	LS	0.1103	0.0128	2.5927	0.3419	0.3547
	WLS	0.1098	0.0095	2.5919	0.3404	0.3499
	CVM	0.1276	0.0146	2.6514	0.3727	0.3872
	AD	0.1033	0.0095	2.4627	0.2496	0.2590
100	ML	0.0956	0.0087	2.4227	0.2383	0.2470
	LS	0.1097	0.0117	2.5569	0.3293	0.3411
	WLS	0.1024	0.0093	2.5224	0.3221	0.3314
	CVM	0.1222	0.0123	2.6007	0.3656	0.3779
	AD	0.1002	0.0090	2.4316	0.2392	0.2483
200	ML	0.0899	0.0083	2.3723	0.2251	0.2334
	LS	0.0977	0.0107	2.4928	0.3118	0.3226
	WLS	0.0965	0.0089	2.4791	0.3076	0.3165
	CVM	0.1152	0.0115	2.5817	0.3422	0.3537
	AD	0.0926	0.0087	2.3917	0.2286	0.2373
500	ML	0.0823	0.0083	2.3357	0.2119	0.2202
	LS	0.0943	0.0107	2.4129	0.3066	0.3173
	WLS	0.0931	0.0089	2.4057	0.2915	0.3004
	CVM	0.1016	0.0115	2.5517	0.3166	0.3282
	AD	0.0893	0.0087	2.3620	0.2148	0.2235

Table 5. The failure stresses (in GPa) of 65 single carbon fibers of length 50 mm.

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.7646	1.807	1.812	1.840	1.852
1.852	1.862	1.864	1.931	1.952	1.974	2.019	2.051	2.055	2.058	2.088	2.125	2.162
2.171	2.172	2.18	2.194	2.211	2.270	2.272	2.280	2.299	2.308	2.335	2.349	2.356
2.386	2.390	2.410	2.430	2.431	2.458	2.471	2.497	2.514	2.558	2.577	2.593	2.601
2.604	2.620	2.633	2.670	2.682	2.699	2.705	2.735	2.785	3.020	3.042	3.116	3.174

TABLE 6. Estimates of the parameters, AIC, BIC, RMSE, R^2 and D values for failure stress data set.

Method	$\hat{\phi}$	$\hat{\lambda}$	AIC	BIC	$RMSE$	R^2	p-value
ML	1.6499	3.3788	250.2429	254.5917	0.1307	0.6023	0.8919
LS	1.6361	4.8553	255.3739	259.7227	0.1393	0.5834	0.8608
WLS	1.6435	4.8799	255.2037	259.5525	0.1393	0.5830	0.8208
CVM	1.6363	4.8553	255.3570	259.7057	0.1393	0.5834	0.8301
AD	1.6196	4.5039	252.1279	256.4767	0.1350	0.5956	0.8624

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A GENERALIZATION OF PURELY EXTENDING MODULES RELATIVE TO A TORSION THEORY

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Abstract. In this work we introduce a new concept, namely, purely τ_s-extending modules (rings) which is torsion-theoretic analogues of extending modules and then we extend many results from extending modules to this new concept. For instance, we show that for any ring R with unit, ${}_RR$ is purely τ_s-extending if and only if every cyclic τ-nonsingular R-module is flat. Also, we make a classification for the direct sums of the rings to be purely τ_s-extending.

1. Introduction

Injective modules have been intensively studied in the 1960s and 1970s in module theory and more generally in algebra. As a generalization of injective modules, extending modules (CS), that is every closed submodule is a direct summand, have been studied widely in last three decades. In general setting, Chatters and Hajarnavis [7], Harmancı and Smith [23], Kamal and Muller [24] and their schools can be mentioned involving studies of extending modules.

Recently, torsion-theoretic analogues of extending modules has been studied on many results and concepts, such primarily studies as, Asgari and Haghany [4], Berktaş, Doğruöz and Tarhan [6], Crivei [11], Çeken and Alkan [12], Doğruöz [13]. Clark [8] defined a module M is *purely extending* if every submodule of M is essential in a pure submodule of M, equivalently every closed submodule of M is pure in M. A submodule K of a module M is *essential (in M)* if $N \cap K \neq 0$ for every non-zero submodule K of M. A submodule K of a module M is *closed (in M)* if K has no proper essential extension in M, i.e., whenever L is a submodule of M such that K is essential in L, then $K = L$. Al-Bahrani [1] generalized purely extending modules as a purely y-extending module using

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s-closed submodules which was defined by Goodearl [21] such as a submodule N of a module M is *s-closed* in M if M/N is nonsingular. So a module M is called *purely y-extending* if every s-closed submodule of M is pure in M. In fact, Al-Bahrani [1] belike misused the terminology of s-closed submodules. They used the term y-closed (purely y-extending) instead of s-closed (purely s-extending) respectively. In this study, we use s-closed submodule and purely s-extending module instead of y-closed submodule and purely y-extending module in the sense of Al-Bahrani [1].

We use the concept 'purity' in the sense of Cohn [10] (as in [8]) which implies definition of Anderson and Fuller [3], that is, a submodule N of an R-module M is called *pure submodule* in M in case $IN = N \cap IM$ for each finitely generated right ideal I of the ring R (see also [26]). In the present paper we introduce purely τ_s-extending modules and then we extend many results from [1], [8] and [21] to this new concept.

For instance, we show that:

Theorem 1: Let R be a τ-torsion ring and M be an R-module. Let $E(M)$ be an injective hull of M. Then M is a purely τ_s-extending module if and only if $A \cap M$ is pure in M for every direct summand A of $E(M)$ such that the submodule $A \cap M$ is τ_s-closed in M.

Proposition 5: Let R be a ring with identity. Then ${}_RR$ is purely τ_s-extending if and only if every cyclic τ-nonsingular R-module is flat.

and

Theorem 6: Let R be a commutative domain and every essential ideal of R is τ-dense in R. Then the following properties are equivalent:

(1): R is a semi-hereditary ring.
(2): $R \oplus R$ is an extending module.
(3): $R \oplus R$ is a purely extending module.
(4): $R \oplus R$ is a purely s-extending module.
(5): $R \oplus R$ is a purely τ_s-extending module.
(6): for each $n \in \mathbb{N}$, $\bigoplus_n R$ is an extending module.
(7): for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely extending module.
(8): for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely s-extending module.
(9): for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely τ_s-extending module.

which is a torsion-theoretic analogue of [8, Proposition 1.6].

Throughout the work R will be an associative ring with identity and all R-modules will be unitary left R-modules unless otherwise stated. *R-Mod* will be the category of unitary left R-modules, and all modules and module homomorphisms will belong to *R-Mod*. By a *class* $\mathcal{X}$ of R-modules we mean a collection of R-modules containing the zero module and closed under isomorphism, i.e., any module which is isomorphic to some module in $\mathcal{X}$ also belongs to $\mathcal{X}$. If a submodule N of a module M belongs to $\mathcal{X}$ class, then N is called *$\mathcal{X}$-submodule* of M. The class of $\mathcal{X}$ closed under extension by short exact sequence we mean for a short exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$$

of R-modules A, B, C, if A and C are bought belong to the class of $\mathcal{X}$, then B is also belongs to $\mathcal{X}$ class.

Let $\tau := (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on R-Mod. The modules in $\mathcal{T}$ are called τ-*torsion* modules and the modules in $\mathcal{F}$ are called τ-*torsion-free* modules. Let $M \in R$-Mod. Then the τ-*torsion submodule* of M, denoted by $\tau(M)$, is defined to be the sum of all τ-torsion submodules of M. Thus $\tau(M)$ is the unique largest τ-torsion submodule of M and $\tau(M/\tau(M)) = 0$ for an R-module M. Also the module M is τ-torsion (resp. τ-torsion-free) if and only if $\tau(M) = M$ (resp. $\tau(M) = 0$). In our study, we mean a ring R is τ-torsion if ${}_RR$ is τ-torsion.

Let M be an R-module. A submodule N of M is called τ-*dense* in M if M/N is τ-torsion. A submodule N of M is called τ-*essential* in M denoted by ($N \leq_{\tau_e} M$) if N is essential in M and M/N is τ-torsion (see [19], originally defined by Tsai in 1965 [29]). Define the set $Z_\tau(M) = \{m \in M \mid Ann(m) \leq_{\tau_e} R\}$. Here $Z_\tau(M)$ is called the τ-*singular submodule* of M. Then the module M is called τ-*singular* if $Z_\tau(M) = M$ and τ-*nonsingular* if $Z_\tau(M) = 0$ ([20]). We mean $Z(M)$ the singular submodule of a module M which is consists of singular elements of M, i.e., elements annihilated by essential left ideals. The module M is singular (resp. nonsingular) if $Z(M) = M$ (resp. $Z(M) = 0$). For the singular and nonsingular notions (see also [21], [22]). If a ring R is τ-torsion, then every left ideal I of R is τ-dense in it, i.e., R/I is τ-torsion in the sense of [19]. Therefore, clearly $Z_\tau(M) = Z(M)$ over a τ-torsion ring R.

For elementary, additional and unexplained terminology the reader is referred to [3] or [30] for module and ring theory, [19] and [28] for torsion theory, [15] for extending modules and [26] for homological algebra.

2. Purely τ_s-Extending Modules

Definition 1. Let M be an R-module and N be a submodule of M. We call N is a τ_s-*closed submodule* of M if the factor module M/N is τ-nonsingular and it is denoted by $N \leq_{\tau_s c} M$.

Definition 2. Let M be an R-module. We call M is a *purely* τ_s-*extending module* if every τ_s-closed submodule of M is pure in M.

Lemma 1. *Let R be a τ-torsion ring. Then every τ_s-closed submodule of a module M is closed in M.*

Proof. Let N be a τ_s-closed submodule of M. Then the factor module M/N is τ-nonsingular i.e., $Z_\tau(M/N) = 0$. Since R is τ-torsion, clearly $Z_\tau(M/N) = Z(M/N)$. Assume N is not closed in M. Then there exists a submodule K of M such that K contains N as an essential submodule. So the factor module K/N

is singular [21]. Hence $Z(K/N) = K/N$. On the other hand, $Z(K/N) = 0$ since $Z(K/N)$ is a submodule of $Z(M/N)$. Hence K/N is nonsingular. But since K/N is singular, it must be zero (i.e $K/N = 0$). Therefore, $N = K$ and so N is closed submodule of M. □

Corollary 1. *Let R be a τ-torsion ring. Then every purely extending R-module is purely τ_s-extending.*

Proof. Let M be a purely extending module and N be a τ_s-closed submodule of M. Since R is τ-torsion N is closed in M by Lemma 1. From [8, Lemma 1.1] every closed submodule of M is pure in M. So N is pure in M. Therefore M is purely τ_s-extending module. □

As in general extending module theory we have some of the fundamental properties of purely τ_s-extending modules as follows:

Lemma 2. *Let $M = M_1 \oplus M_2$ be a purely τ_s-extending module. Then M_1 and M_2 are also purely τ_s-extending modules i.e., any direct summand of a purely τ_s-extending module is purely τ_s-extending.*

Proof. Let $M = M_1 \oplus M_2$ be a purely τ_s-extending module and let N_1 be a τ_s-closed submodule of M_1. Then $Z_\tau(M_1/N_1) = 0$. For the proof we want to show that N_1 is pure in M_1. First let us show that N_1 is τ_s-closed in M i.e., (M/N_1) is τ-nonsingular.

Assume M/N_1 is not τ-nonsingular module. Thus $Z_\tau(M/N_1) \neq 0$. Then there exists an element $N_1 \neq m + N_1 \in M/N_1$ such that $Ann(m + N_1) \leq_{\tau_e} R$. On the other hand, since $m \in M = M_1 \oplus M_2$, there exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2$ and this writing unique. Thus

$$\begin{aligned} Ann(m + N_1) &= Ann((m_1 + m_2) + N_1) = Ann(m_1 + N_1 + m_2 + N_1) \\ &= Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \end{aligned}$$

(see [3, Proposition 2.16]). In addition, since $Ann(m + N_1) \leq_{\tau_e} R$, we have $Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \leq_{\tau_e} R$. Since $Ann(m_1 + N_1) \cap Ann(m_2 + N_1) \subseteq Ann(m_1 + N_1) \subseteq R$, we have $Ann(m_1 + N_1) \leq_{\tau_e} R$. But this contradicts with $Z_\tau(M/N_1) \neq 0$. Hence $Z_\tau(M/N_1) = 0$ i.e., N_1 is a τ_s-closed submodule of M. By the hypothesis N_1 is pure in M since M is purely τ_s-extending module. By [17, Proposition 1.2 (2)] N_1 is pure in M_1. Thus M_1 is purely τ_s-extending module. Similarly it can be shown that M_2 is also purely τ_s-extending module. □

Corollary 2. *Let $M = \bigoplus_{i \in I} M_i$ be a purely τ_s-extending module where I is a finite index set. Then for every $i \in I$, M_i is purely τ_s-extending.*

Proof. It is clear from Lemma 2. □

Lemma 3. *Let C be an R-module. Then C is a τ-nonsingular module if and only if $Hom_R(A, C) = 0$ for every τ-singular R-module A.*

Proof. Let $f : A \longrightarrow C$ be an R-module homomorphism where C is a τ-nonsingular module and A is a τ-singular R-module. Then $f(A) = f(Z_\tau(A))$. We show $f(Z_\tau(A)) \leq Z_\tau(C)$. If $x \in f(Z_\tau(A))$ then there is an element $a \in Z_\tau(A)$ such that $x = f(a)$. So $Ann(a) \leq_{\tau_e} R$. If $r \in Ann(a)$, then $rx = rf(a) = f(ra) = 0$ i.e., $r \in Ann(x)$. Since $Ann(a) \leq Ann(x) \leq R$, we have $Ann(x) \leq_{\tau_e} R$ i.e., $x \in Z_\tau(C)$. By the hypothesis, since $Z_\tau(C) = 0$, $f = 0$ and thus $Hom_R(A, C) = 0$.

For the converse let $Hom_R(A, C) = 0$ for every τ-nonsingular R-module A. Specially $Hom_R(Z_\tau(C), C) = 0$. So the inclusion map $Z_\tau(C) \longrightarrow C$ is zero. Hence $Z_\tau(C) = 0$ and so C is τ-nonsingular module. □

Lemma 4. *The class of τ-nonsingular modules is closed under extensions by short exact sequences.*

Proof. Let C and A be τ-nonsingular modules and consider the following short exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$$

For every τ-singular R-module M, using Lemma 3, we have $Hom_R(M, C) = 0$ and $Hom_R(M, A) = 0$. Then the following short exact sequence

$$0 \longrightarrow Hom_R(M, C) \longrightarrow Hom_R(M, B) \longrightarrow Hom_R(M, A) \longrightarrow 0$$

yields $Hom_R(M, B) = 0$. Again by Lemma 3 the R-module B must be τ-nonsingular. □

Next we can show τ_s-closed submodules have transitivity property.

Lemma 5. *Let M be an R-module and let K and N be submodules of M such that $K \leqslant N$. If K is τ_s-closed submodule of N and N is τ_s-closed submodule of M, then K is τ_s-closed submodule of M.*

Proof. Since K is τ_s-closed submodule of N and N is τ_s-closed submodule of M, $Z_\tau(N/K) = 0$ and $Z_\tau(M/N) = 0$. We must show that $Z_\tau(M/K) = 0$. Consider the following short exact sequence

$$0 \longrightarrow N/K \longrightarrow M/K \longrightarrow M/N \longrightarrow 0$$

By Lemma 4, the class of τ-nonsingular modules are closed under extensions by short exact sequences. Since N/K and M/N are both τ-nonsingular, M/K is τ-nonsingular. Hence $Z_\tau(M/K) = 0$. Thus K is τ_s-closed submodule of M. □

Now we have some basic properties as follows.

Lemma 6. *Any τ_s-closed submodule of a purely τ_s-extending module is purely τ_s-extending.*

Proof. Let M be a purely τ_s-extending module and let N be a τ_s-closed submodule of M. Then M/N is τ-nonsingular. Let K be a τ_s-closed submodule of N. Then by Lemma 5, K is a τ_s-closed submodule of M. Since M is purely τ_s-extending module, K is pure in M. By [17, Proposition 1.2 (2)], K is pure in N. So N is purely τ_s-extending module. □

There exist submodules K, L of a module M such that K and L both closed submodules of M but $K \cap L$ is not closed in K, L or M (see [21, Example 1.6]). But we have the following in our case.

Proposition 1. *Let M be an R-module and N, K be τ_s-closed submodules of M. Then $N \cap K$ is a τ_s-closed submodule of M.*

Proof. Let M be an R-module and N, K be τ_s-closed submodules of M. Then M/K and M/N are τ-nonsingular, i.e., $Z_\tau(M/N) = 0$ and $Z_\tau(M/K) = 0$. Assume $Z_\tau(M/(N \cap K)) \neq 0$. Then there is a $(N \cap K) \neq \bar{m} \in M/(N \cap K)$ such that $Ann(\bar{m}) \leq_{\tau_e} R$. Now for $\bar{m} = m + (N \cap K)$, $m \notin N \cap K$. On the other hand for $m \in M$, choose the elements $\hat{m} = m + N \in M/N$ and $\tilde{m} = m + K \in M/K$. Then we have $Ann(\bar{m}) \subseteq Ann(\hat{m})$ and $Ann(\bar{m}) \subseteq Ann(\tilde{m})$. Indeed, now let $0 \neq r \in Ann(\bar{m})$. Then $r\bar{m} = 0$ and so $rm + (N \cap K) = N \cap K$. Hence $rm \in N \cap K$. So we have $rm \in N$ and $rm \in K$. Thus $rm + N = N$ and $rm + K = K$, i.e. $r\hat{m} = 0$ and $r\tilde{m} = 0$. Consequently $r \in Ann(\hat{m})$ and $r \in Ann(\tilde{m})$. Hence $Ann(\bar{m}) \subseteq Ann(\hat{m})$ and $Ann(\bar{m}) \subseteq Ann(\tilde{m})$. On the other hand, since $Ann(\bar{m}) \leq_{\tau_e} R$ we have $Ann(\hat{m}) \leq_{\tau_e} R$ and $Ann(\tilde{m}) \leq_{\tau_e} R$. Then by hypothesis $Z_\tau(M/N) = 0$ and $Z_\tau(M/K) = 0$, we have $m \in N$ and $m \in K$ and so $m \in N \cap K$. Hence $\bar{m} = m + (N \cap K) = N \cap K$. This is a contradiction. Thus $Z_\tau(M/(N \cap K)) = 0$. Therefore, $N \cap K$ is a τ_s-closed submodule of M. □

Corollary 3. *Any intersection of τ_s-closed submodules is also τ_s-closed.*

Proof. It is an evident result of Proposition 1. □

Lemma 7. *Let M be an R-module and let K, L be submodules of M such that $K \leqslant L$. If L is a τ_s-closed submodule of M, then L/K is a τ_s-closed submodule of M/K.*

Proof. Let L be a τ_s-closed submodule of M. Then $Z_\tau(M/L) = 0$. On the other hand, $(M/K)/(L/K) \cong M/L$ and since τ-nonsingular modules are closed under isomorphisms, $Z_\tau((M/K)/(L/K)) = 0$. Hence L/K is τ_s-closed in M/K. □

Lemma 8. *Let M be an R-module and let K, L be submodules of M such that $K \leq L$. If the submodule L/K is τ_s-closed in M/K, then L is a τ_s-closed submodule of M.*

Proof. Since L/K is a τ_s-closed submodule of M/K, $Z_\tau((M/K)/(L/K)) = 0$. Since $(M/K)/(L/K) \cong M/L$ and τ-nonsingular modules are closed under isomorphisms, $Z_\tau(M/L) = 0$. Hence L is a τ_s-closed submodule of M. □

Proposition 2. *Let M be a purely τ_s-extending R-module and N be a τ_s-closed submodule of M. Then the factor module M/N is purely τ_s-extending.*

Proof. Let M be a purely τ_s-extending R-module and N be a τ_s-closed submodule of M. By the definition of purely τ_s-extending module, N is pure in M. For $N \leq K \leq M$ let K/N be τ_s-closed in M/N. Now $(M/N)/(K/N) \simeq M/K$ and since the τ-nonsingular modules are closed under isomorphisms, $Z_\tau(M/K) = 0$. So K is τ_s-closed submodule of M. Since M is purely τ_s-extending, K is pure in M. By [17, Proposition 1.2 (3)] K/N is pure in M/N. Thus M/N is purely τ_s-extending. □

Let M be an R-module. For an arbitrary submodule N of M by Zorn's Lemma there is a submodule K of M maximal with respect to N is essential in K. The submodule K is called *closure* of N in M ([27]). See also [14] for torsion theoretic version of closures.

Now we give another generalization of closures relative to a torsion theory as follows:

Definition 3. Let M be an R-module and let N be a submodule of M. The smallest τ_s-closed submodule K of M which is containing N is called τ_s-*closure* of N in M. The τ_s-closure of N is denoted by $N^{-\tau_s}$.

Lemma 9. *Every submodule N of an R-module M has a τ_s-closure in M.*

Proof. Let M be an R-module and N be a submodule of M. Now define the set $\mathcal{S} = \{K \leq M \mid N \subseteq K \text{ and } K \leq_{\tau_s c} M\}$. Since $Z_\tau(M/M) = 0$, M is τ_s-closed in M and so $M \in \mathcal{S}$. Then $\mathcal{S}$ is non-empty. Let $\mathcal{C}$ be a chain in $\mathcal{S}$. Take $C = \bigcap_{K_i \in \mathcal{C}} K_i$. By Corollary 3 C is a τ_s-closed submodule of M. Then $C \in \mathcal{S}$. By Zorn's Lemma there is a minimal element in $\mathcal{S}$. If we call this element such as H then H is τ_s-closure of N in M. Thus every submodule N of M has a τ_s-closure in M. □

Proposition 3. *An R-module M is a purely τ_s-extending if and only if the τ_s-closure of N (i.e., $N^{-\tau_s}$) is pure in M for every submodule N of M.*

Proof. Let M be a purely τ_s-extending module. Then every τ_s-closed submodule of M is pure in M. By Zorn's Lemma every submodule N of M has a τ_s-closure in M. By the definition of τ_s-closure, the submodule $N^{-\tau_s}$ is τ_s-closed in M and by the hypothesis the submodule $N^{-\tau_s}$ is pure in M.

Conversely, let K be a τ_s-closed submodule in M. By the definition of τ_s-closure, $K^{-\tau_s} = K$. By the hypothesis $K^{-\tau_s}$ i.e. K is a pure submodule in M. Then any τ_s-closed submodule of M is pure in M. Thus M is a purely τ_s-extending module. □

Theorem 1. *Let R be a τ-torsion ring, let M be an R-module and $E(M)$ be the injective hull of M. Then, M is a purely τ_s-extending module if and only if $A \cap M$ is pure in M for every direct summand A of $E(M)$ such that the submodule $A \cap M$ is τ_s-closed in M.*

Proof. Let R be a τ-torsion ring, M be an R-module, $E(M)$ be the injective hull of M and M be a purely τ_s-extending module. Then for every direct summand A of $E(M)$ such that $A \cap M$ is a τ_s-closed submodule of M it is clear that $A \cap M$ is pure in M.

Conversely, let A be a τ_s-closed submodule of M and let B be a complement of A in M. Then $A \oplus B$ is essential in M [21, Proposition 1.3]. Now it is clear that $A \oplus B$ is essential in $E(M)$. Hence $E(A) \oplus E(B) = E(A \oplus B) = E(M)$ [22]. Since $A = A \cap M \leq_e E(A) \cap M$, $(E(A) \cap M)/A$ is singular (see [21]). Moreover, since R is τ-torsion ring $(E(A) \cap M)/A$ is τ-singular. On the other hand since $(E(A) \cap M)/A \leq M/A$ and A is τ_s-closed submodule of M, M/A is τ-nonsingular and thus $(E(A) \cap M)/A$ is τ-nonsingular. Therefore, $(E(A) \cap M)/A = 0$ and so $E(A) \cap M = A$. Since A is τ_s-closed in M, $E(A) \cap M$ is also τ_s-closed in M. Since $E(A)$ is a direct summand of $E(M)$ by the hypothesis $E(A) \cap M$ is a pure submodule of M. Hence A is pure in M. Thus M is a purely τ_s-extending module. □

Theorem 2. *Let R be a τ-torsion ring, let M be an R-module and let $E(M)$ be the injective hull of M. Assume $A + M$ be a flat module for every direct summand A of $E(M)$ with $A \cap M$ is τ_s-closed submodule of M. Then M is a purely τ_s-extending module.*

Proof. Let A be a direct summand of $E(M)$ such that $A \cap M$ is τ_s-closed in M. Consider the following short exact sequences of R-modules

$$0 \longrightarrow A \cap M \xrightarrow{i_1} M \xrightarrow{f_1} M/(A \cap M) \longrightarrow 0$$

and

$$0 \longrightarrow A \xrightarrow{i_2} A + M \xrightarrow{f_2} (A + M)/A \longrightarrow 0$$

where i_1, i_2 are inclusion maps and f_1, f_2 are natural epimorphisms. Since A is a direct summand of $E(M)$, there is a submodule A' of $E(M)$ such that $E(M) = A \oplus A'$. Thus A is also a direct summand of $A + M$ such as $A + M = (A + M) \cap E(M) = (A + M) \cap (A \oplus A') = A \oplus ((A + M) \cap A')$. Here $((A + M) \cap A')$ is flat as a direct summand of a flat module $A + M$. Since $(A + M)/A \cong ((A + M) \cap A')$, $(A + M)/A$ is flat. On the other hand, the factor module $M/(A \cap M)$ is again flat since $M/(A \cap M) \cong (A + M)/A$. By [17, Theorem 1.7] $A \cap M$ is pure in M. Hence by Theorem 1, M is a purely τ_s-extending module. □

3. Purely τ_s-Extending Rings

If the ring R is purely τ_s-extending as an R-module over itself then R is called *purely τ_s-extending.*

A (von Neumann) regular ring R as an R-module over itself, i.e., ${}_RR$ can be given an example of purely τ_s-extending ring since every left ideal is pure in it by [17, Theorem 2.1].

Fieldhouse in [17] generalizing (von Neumann) regular ring and define, for any ring R, an R-module M is called (von Neumann) *regular* if all its submodules are pure in M.

Therefore, since all (left) R-modules over a (von Neumann) regular ring is regular by [17, Theorem 3.1], thus all R-modules over a (von Neumann) regular ring R is purely τ_s-extending. Also any regular module over any ring R can be given as an example of purely τ_s-extending modules.

3.1. Multiplication Modules.

Let R be a commutative ring and M be an R-module. For every submodule N of M if there exists an ideal I of R such that $N = IM$, then M is called a *multiplication module.* For every submodule N of M let us define

$$(N : M) = \{r \in R \mid rM \subseteq N\}.$$

Then M is an multiplication R-module if and only if $N = (N : M)M$ ([5]).

Definition 4. *[9]* Let M be an R-module and N be a submodule of M. If

$$N = Hom(M, N)N = \Sigma\{\varphi(N) \mid \varphi : M \to N\}$$

then N is called *an idempotent submodule* of M. If every submodule of M is idempotent, then M is called a *fully idempotent module*.

Theorem 3. *[16, Teorem 2.11] Let M be a multiplication R-module and $M = M_1 \oplus M_2$, is a direct sum of fully idempotent submodules M_1 and M_2. Then M is a fully idempotent module.*

Lemma 10. *[16, Lemma 2.13] Let M be a fully idempotent R-module, N be a submodule of M and I be an ideal of R. Then $N \cap MI = NI$, i.e., N is pure in M.*

Now we can give the following teorem by using fully idempotent submodules:

Theorem 4. *Let R be a commutative ring and let $M = M_1 \oplus M_2$ be a multiplication R-module with fully idempotent submodules M_1, M_2 of M. Then M is a purely τ_s-extending module.*

Proof. Let M be a multiplication R-module and N be a τ_s-closed submodule of M. By Teorem 3 M is fully idempotent R-module and by Lemma 10 the τ_s-closed submodule N of M is pure in M. Hence M is purely τ_s-extending. □

Now we can give a characterization of a purely τ_s-extending R-module with a ring as follows:

Proposition 4. *Let R be a commutative ring and let M be a faithful multiplication R-module. If ${}_RR$ is purely τ_s-extending module then M is also purely τ_s-extending module.*

Proof. Let N be a τ_s-closed submodule of M. Since M is multiplication R-module, we can write $N = (N : M)M$. Claim: $(N : M)$ is τ_s-closed submodule in ${}_RR$. Assume $(N : M)$ is not τ_s-closed in R. Then $R/(N : M)$ is not τ-nonsingular that is, $Z_\tau(R/(N : M)) \neq 0$. Then there exists at least one non-zero element $\bar{r}$ of $R/(N : M)$ such that $Ann(r + (N : M))$ is τ-essential in R. So $\bar{r} = r + (N : M) \neq (N : M)$. Then there is an element $0 \neq m_0 \in M$ such that $rm_0 \notin N$. Now $Ann(r + (N : M)) \subseteq Ann(rm_0 + N)$. If $s \in Ann(r + (N : M))$, then $sr + (N : M) = (N : M)$. Hence we have $sr \in (N : M)$ so it is easy to check that $(sr)M \subseteq N$ (*). Let us show that $s \in Ann(rm_0+N)$. Now $s(rm_0+N) = srm_0+N$ but since $(sr)M \subseteq N$ and by (*) for $m_0 \in M$, $srm_0 \in N$, i.e., $srm_0 + N = N$. So $s \in Ann(rm_0 + N)$. Hence we have $Ann(r + (N : M)) \subseteq Ann(rm_0 + N)$. On the other hand, since N is τ_s-closed in M it is clear that M/N τ-nonsingular. So $rm_0 + N = N$ but it contradicts with $rm_0 \notin N$. Hence $(N : M)$ must be τ_s-closed in R. Moreover since ${}_RR$ is purely τ_s-extending, $(N : M)$ is pure in R, i.e., $I(N : M) = IR \cap (N : M)$ for every finitely generated ideal I of R. Thus $I(N : M) = IR \cap (N : M) = I \cap (N : M)$. Therefore, by $N = (N : M)M$ we write $IN = I(N : M)M = (I \cap (N : M))M$. On the other hand, the equality $(I \cap (N : M))M = IM \cap (N : M)M$ holds since R is a commutative ring and M is a faithful multiplication R-module by applying [2, Proposition 1.6 (i)].

Now for the finitely generated ideal I of R, we have
$IN = I(N : M)M = (I \cap (N : M))M = IM \cap (N : M)M = IM \cap N$ ([5]). Therefore, the τ_s-closed submodule N of M is pure in M. Hence M is a purely τ_s-extending module. $\square$

Remark 1. *[26, Proposition 3.46] Let R be an arbitrary ring. The left R-module R is a flat left R-module.*

In the sequel we use the flat ring in the sense of Rotman [26, Proposition 3.46], i.e the ring R is flat if ${}_RR$ is flat.

Proposition 5. *Let R be an arbitrary ring. Then ${}_RR$ is purely τ_s-extending if and only if every cyclic τ-nonsingular R-module is flat.*

Proof. Let ${}_RR$ be a purely τ_s-extending module. Let $M = Ra$ be a cyclic τ-nonsingular R-module which is generated by a. Define the map $f : R \to M$ with $f(r) = ra$. Clearly f is an epimorphism and $\mathrm{Ker}(f) = Ann(a)$. So $R/\mathrm{Ker}(f) = R/Ann(a) \cong Ra$. Moreover, since Ra is a τ-nonsingular module and the class of τ- nonsingular modules is closed under isomorphisms $R/Ann(a)$ is τ-nonsingular. Hence $Ann(a)$ is τ_s-closed in R. By the hypothesis $Ann(a)$ is pure in R. Since R is flat and $Ann(a)$ is pure in R, $R/Ann(a)$ is flat by [3, Lemma 19.18]. Therefore, Ra is flat.

Conversely, let K be a τ_s-closed ideal of R. Then R/K is τ-nonsingular. By the hypothesis R/K is flat as a left R-module. Thus by [3, Lemma 19.18], K is pure in R. Thus ${}_RR$ is a purely τ_s-extending. $\square$

Theorem 5. *Let R be a ring. Then $R \oplus R$ is purely τ_s-extending if and only if every τ-nonsingular 2-generated R-module is flat.*

Proof. Let $M = Rm_1 + Rm_2$ be a τ-nonsingular R-module. Define the map $f : R \oplus R \to M$ with $f(r_1, r_2) = r_1m_1 + r_2m_2$. Now it is clear that f is an epimorphism. Hence $(R \oplus R)/\mathrm{Ker}(f) \cong M$. Since $(R \oplus R)/\mathrm{Ker}(f)$ is τ-nonsingular, $\mathrm{Ker}(f)$ is a τ_s-closed submodule of $R \oplus R$. By the hypothesis $\mathrm{Ker}(f)$ is pure in $R \oplus R$. Since R is is flat as an R-module, $R \oplus R$ is flat ([21]). Thus by [17, Proposition 1.3 (3)], we have the R-module M is flat.

For the converse, let C be a τ_s-closed submodule of $R \oplus R$. Then $(R \oplus R)/C$ is τ-nonsingular. On the other hand, since $R \oplus R$ is a 2-generated R-module, $(R \oplus R)/C$ is also a 2-generated τ-nonsingular R-module. By the hypothesis $(R \oplus R)/C$ is flat. Then by [17, Theorem 1.7] we get C is pure in $R \oplus R$. Thus $R \oplus R$ is purely τ_s-extending. $\square$

Corollary 4. *Let R be a ring and I be a finite index set. Then $\oplus_I R$ is purely τ_s-extending if and only if every τ-nonsingular I-generated R-module is flat.*

3.2. **Semi-hereditary Rings.** Let R be a ring with unit element. If every left (right) ideal of R is projective then R is called a left (right) *hereditary ring* . If every finitely generated left (right) ideal of R is projective then R is called a left (right) *semi-hereditary ring* ([28]). A module M over a commutative domain R is said to be *torsion-free* if for $m \in M$ and $r \in R$, $rm = 0 \Rightarrow r = 0$ or $m = 0$ [25].

Now we can give the following generalized characterization of purely τ_s-extending modules.

Theorem 6. *Let R be a commutative domain and every essential ideal of R is τ-dense in R. Then the following properties are equivalent:*

- **(1):** *R is a semi-hereditary ring.*
- **(2):** *$R \oplus R$ is an extending module.*
- **(3):** *$R \oplus R$ is a purely extending module.*
- **(4):** *$R \oplus R$ is a purely s-extending module.*
- **(5):** *$R \oplus R$ is a purely τ_s-extending module.*
- **(6):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is an extending module.*
- **(7):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely extending module.*
- **(8):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely s-extending module.*
- **(9):** *for each $n \in \mathbb{N}$, $\bigoplus_n R$ is a purely τ_s-extending module.*

Proof. The equivalence of (1), (2) and (6) are given in [15, Corollary 12.10].

In addition the equivalence of (1), (2), (3), (6) and (7) are given in [8, Proposition 1.6].

(3) $\Leftrightarrow$ (4). Every s-closed submodule of a module M is closed in M. But converse is true if M is nonsingular [21, Proposition 2.4]. Here since R is commutative domain, R is nonsingular. Therefore, the notion of closed submodule and s-closed submodule coincide. Thus the proof is clear by [8, Lemma 1.1] in fact, Lemma 1.1 is originally given by Fuchs [18].

(7) $\Leftrightarrow$ (8). It can be easily checked be like (3) $\Leftrightarrow$ (4).

(5) $\Rightarrow$ (4). Let K be a s-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is nonsingular. Since any nonsingular module is τ-nonsingular. $(R \oplus R)/K$ is a τ-nonsingular. By the hypothesis K is pure in $R \oplus R$. Hence $R \oplus R$ is a purely s-extending module.

The implication of (9) $\Rightarrow$ (8) is a generalization of (5) $\Rightarrow$ (4).

(1) $\Rightarrow$ (5). Let K be a τ_s-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is τ-nonsingular. Claim that $(R \oplus R)/K$ is torsion-free R-module. For this fact, let us assume $\overline{m}.r = \overline{0}$ and $r \neq 0$ for $\overline{m} \in (R \oplus R)/K$ and $r \in R$. Here $0 \neq r \in Ann(\overline{m})$. Thus $Ann(\overline{m}) \neq 0$. Since also R is a commutative domain, then all non-zero ideals of R are essential [25, 7.6]. Thus $Ann(\overline{m})$ is essential ideal in R. By hypothesis of the theorem, $Ann(\overline{m})$ is τ-dense in R. Thus $Ann(\overline{m}) \leq_{\tau_e} R$ and so, $\overline{m} \in Z_\tau((R \oplus R)/K)$. In this case, $\overline{m} = 0$ since $(R \oplus R)/K$ is τ-nonsingular. Therefore $(R \oplus R)/K$ is torsion-free. Thus applying [25, Collary 2.31] $(R \oplus R)/K$ is projective since $(R \oplus R)/K$ is 2-generated over the Prüfer domain R. So $(R \oplus R)/K$ is flat by [26, Proposition 3.46]. Thus K is pure in $R \oplus R$ by [17, Proposition 1.3]. Hence $R \oplus R$ is a purely τ_s-extending module

(1) $\Rightarrow$ (9) is also similar to (1) $\Rightarrow$ (5). This completes the proof.

In fact, the proof can be also completed by the following implications.

(4) $\Rightarrow$ (5). Let K be a τ_s-closed submodule of $R \oplus R$. Then $(R \oplus R)/K$ is τ-nonsingular, i.e., $Z_\tau((R \oplus R)/K) = 0$. By assumption, since R is a ring with essential ideal of R is τ-dense in it, τ-nonsingular and nonsingular modules are coincide. Therefore $(R \oplus R)/K$ is nonsingular and so K is s-closed in $R \oplus R$. By hypothesis, K is pure in $R \oplus R$. Therefore, $R \oplus R$ is purely τ_s-extending module.

(8) $\Rightarrow$ (9) is also similar to (4) $\Rightarrow$ (5). $\square$

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AN EXTENSION OF TRAPEZOID INEQUALITY TO THE COMPLEX INTEGRAL

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Abstract. In this paper we extend the trapezoid inequality to the complex integral by providing upper bounds for the quantity

$$\left|(v-u)\,f(u)+(w-v)\,f(w)-\int_{\gamma}f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

1. Introduction

Inequalities providing upper bounds for the quantity

$$\left|(t-a)\,f(a)+(b-t)\,f(b)-\int_a^b f(s)\,ds\right|,\qquad t\in[a,b] \tag{1}$$

are known in the literature as *generalized trapezoid inequalities* and it has been shown in [2] that

$$\begin{aligned}&\left|(t-a)\,f(a)+(b-t)\,f(b)-\int_a^b f(s)\,ds\right| \qquad (2)\\ &\leq\left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\bigvee_a^b(f)\end{aligned}$$

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for any $t \in [a,b]$, provided that f is of bounded variation on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

If f is *absolutely continuous* on $[a,b]$, then (see [1, p. 93])

$$\left|(t-a) f(a)+(b-t) f(b)-\int_a^b f(s)\, ds\right| \tag{3}$$

$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\left\|f'\right\|_{\infty} & \text{if } f' \in L_{\infty}[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1+1/q}\left\|f'\right\|_p & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right](b-a)\left\|f'\right\|_1 & \end{cases}$$

for any $t \in [a,b]$. The constants $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{(q+1)^{1/q}}$ are the best possible.

Finally, for *convex functions* $f:[a,b] \to \mathbb{R}$, we have [4]

$$\begin{aligned} &\frac{1}{2}\left[(b-t)^2 f'_+(t)-(t-a)^2 f'_-(t)\right] \\ &\qquad \leq (b-t) f(b)+(t-a) f(a)-\int_a^b f(s)\, ds \\ &\qquad\qquad \leq \frac{1}{2}\left[(b-t)^2 f'_-(b)-(t-a)^2 f'_-(a)\right] \end{aligned} \tag{4}$$

for any $t \in (a,b)$, provided that $f'_-(b)$ and $f'_+(a)$ are finite. As above, the second inequality also holds for $t=a$ and $t=b$ and the constant $\frac{1}{2}$ is the best possible on both sides of (4).

For other recent results on the trapezoid inequality, see [3], [7], [8], [9] and [11].

In order to extend this result for the complex integral, we need some preparations as follows.

Suppose γ is a *smooth path* parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a)=u$ and $z(b)=w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w}=\gamma$ as

$$\int_\gamma f(z)\, dz=\int_{\gamma_{u,w}} f(z)\, dz:=\int_a^b f(z(t))\, z'(t)\, dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are *piecewise smooth*. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\, dz:=\int_{\gamma_{u,v}} f(z)\, dz+\int_{\gamma_{v,w}} f(z)\, dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{5}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{6}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In this paper we extend the trapezoid inequality to the complex integral, by providing upper bounds for the quantity

$$\left|(v-u)\,f(u) + (w-v)\,f(w) - \int_\gamma f(z)\,dz\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Trapezoid Type Inequalities

We have the following result for functions of complex variable:

Theorem 1. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a, b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\begin{aligned}\left|(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz\right| \\ \leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v|\, |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v|\, |dz| \\ \leq \|f'\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v|\, |dz|, \end{aligned} \tag{7}$$

and

$$\begin{aligned}\left|(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz\right| \\ \leq \|f'\|_{\gamma_{u,v};1} \max_{z\in\gamma_{u,v}} |z-v| + \|f'\|_{\gamma_{v,w};1} \max_{z\in\gamma_{v,w}} |z-v| \\ \leq \|f'\|_{\gamma_{u,w};1} \max_{z\in\gamma_{u,w}} |z-v|. \end{aligned} \tag{8}$$

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}\left|(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz\right| \\ \leq \|f'\|_{\gamma_{u,v};p} \left(\int_{\gamma_{u,v}} |z-v|^q\, |dz|\right)^{1/q} + \|f'\|_{\gamma_{v,w};p} \left(\int_{\gamma_{v,w}} |z-v|^q\, |dz|\right)^{1/q} \\ \leq \|f'\|_{\gamma_{u,w};p} \left(\int_{\gamma_{u,w}} |z-v|^q\, |dz|\right)^{1/q}. \end{aligned} \tag{9}$$

Proof. Using the integration by parts formula (5) twice we have

$$\int_{\gamma_{u,v}} (z-v) f'(z)\, dz = (v-u) f(u) - \int_{\gamma_{u,v}} f(z)\, dz$$

and

$$\int_{\gamma_{v,w}} (z-v) f'(z)\, dz = (w-v) f(w) - \int_{\gamma_{v,w}} f(z)\, dz.$$

If we add these two equalities, we get the following equality of interest

$$(v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz$$

$$= \int_{\gamma_{u,v}} (z-v) f'(z) \, dz + \int_{\gamma_{v,w}} (z-v) f'(z) \, dz = \int_{\gamma} (z-v) f'(z) \, dz \quad (10)$$

with the above assumptions for u, v and w on γ.

Using the properties of modulus and the triangle inequality for the complex integral we have

$$\begin{aligned}
&\left| (v-u) f(u) + (w-v) f(w) - \int_{\gamma} f(z) \, dz \right| \\
&\quad = \left| \int_{\gamma_{u,v}} (z-v) f'(z) \, dz + \int_{\gamma_{v,w}} (z-v) f'(z) \, dz \right| \\
&\quad \le \left| \int_{\gamma_{u,v}} (z-v) f'(z) \, dz \right| + \left| \int_{\gamma_{v,w}} (z-v) f'(z) \, dz \right| \\
&\quad \le \int_{\gamma_{u,v}} |z-v| \, |f'(z)| \, |dz| + \int_{\gamma_{v,w}} |z-v| \, |f'(z)| \, |dz| \\
&\le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v| \, |dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v| \, |dz| \le \|f'\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v| \, |dz| ,
\end{aligned}$$

which proves the inequality (7).

We also have

$$\begin{aligned}
&\int_{\gamma_{u,v}} |z-v| \, |f'(z)| \, |dz| + \int_{\gamma_{v,w}} |z-v| \, |f'(z)| \, |dz| \\
&\le \max_{z \in \gamma_{u,v}} |z-v| \int_{\gamma_{u,v}} |f'(z)| \, |dz| + \max_{z \in \gamma_{v,w}} |z-v| \int_{\gamma_{v,w}} |f'(z)| \, |dz| \\
&\le \max \left\{ \max_{z \in \gamma_{u,v}} |z-v| , \max_{z \in \gamma_{v,w}} |z-v| \right\} \\
&\times \left(\int_{\gamma_{u,v}} |f'(z)| \, |dz| + \int_{\gamma_{v,w}} |f'(z)| \, |dz| \right) = \max_{z \in \gamma_{u,w}} |z-v| \int_{\gamma_{u,w}} |f'(z)| \, |dz| ,
\end{aligned}$$

which proves the inequality (8).

If p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's weighted integral inequality we have

$$\begin{aligned}
&\int_{\gamma_{u,v}} |z-v| \, |f'(z)| \, |dz| + \int_{\gamma_{v,w}} |z-v| \, |f'(z)| \, |dz| \\
&\quad \le \left(\int_{\gamma_{u,v}} |z-v|^q \, |dz| \right)^{1/q} \left(\int_{\gamma_{u,v}} |f'(z)|^p \, |dz| \right)^{1/p} \\
&\qquad + \left(\int_{\gamma_{v,w}} |z-v|^q \, |dz| \right)^{1/q} \left(\int_{\gamma_{v,w}} |f'(z)|^p \, |dz| \right)^{1/p} =: B.
\end{aligned}$$

By the elementary inequality

$$ab + cd \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q},$$

where a, b, c, $d \geq 0$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we also have

$$\begin{aligned} B &\leq \left(\int_{\gamma_{u,v}} |z - v|^q |dz| + \int_{\gamma_{v,w}} |z - v|^q |dz| \right)^{1/q} \\ &\quad \times \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz| + \int_{\gamma_{v,w}} |f'(z)|^p |dz| \right)^{1/p} \\ &= \left(\int_{\gamma_{u,w}} |z - v|^q |dz| \right)^{1/q} \left(\int_{\gamma_{u,w}} |f'(z)|^p |dz| \right)^{1/p}, \end{aligned}$$

which prove the desired result (9). □

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_\gamma f(z)\, dz$ as $\int_u^w f(z)\, dz$.

Using the p-norms defined in the introduction for the segments, namely

$$\|h\|_{[u,w];\infty} = \sup_{z \in [u,w]} |h(z)|$$

and

$$\|h\|_{[u,w];p} = \left(\int_u^w |h(z)|^p |dz| \right)^{1/p} \text{ for } p \geq 1,$$

we can state the following particular case as well:

Corollary 1. *Let f be holomorphic in G, an open domain and suppose $[u, w] \subset G$ is a segment connecting two distinct points u and w in G and $v \in [u, w]$. Then for $v = (1 - s) u + sw$ with $s \in [0, 1]$, we have*

$$\begin{aligned} &\left| (v - u) f(u) + (w - v) f(w) - \int_u^w f(z)\, dz \right| \\ &\qquad \leq \frac{1}{2} |w - u|^2 \left[s^2 \|f'\|_{\gamma_{u,v};\infty} + (1 - s)^2 \|f'\|_{\gamma_{v,w};\infty} \right] \\ &\qquad\qquad \leq |w - u|^2 \left[\frac{1}{4} + \left(s - \frac{1}{2} \right)^2 \right] \|f'\|_{[u,w];\infty}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\left| (v - u) f(u) + (w - v) f(w) - \int_u^w f(z)\, dz \right| \\ &\qquad \leq |w - u| \left\{ s \|f'\|_{[u,v];1} + (1 - s) \|f'\|_{[v,w];1} \right\} \end{aligned}$$

$$\leq |w-u| \left(\frac{1}{2} + \left| s - \frac{1}{2} \right| \right) \|f'\|_{[u,w];1} . \quad (12)$$

If $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1,$ *then*

$$\left| (v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz \right|$$
$$\leq \frac{1}{(q+1)^{1/q}} |w-u|^{1+1/q} \left[s^{1+1/q} \|f'\|_{[u,v];p} + (1-s)^{1+1/q} \|f'\|_{[v,w];p} \right]$$
$$\leq \frac{1}{(q+1)^{1/q}} |w-u|^{1+1/q} \left[s^{q+1} + (1-s)^{q+1} \right]^{1/q} \|f'\|_{[u,w];p} . \quad (13)$$

Proof. Observe that if the segment $[u,w]$ is parametrized by $z(t) = (1-t)u + tw$, then $z'(t) = w-u$

$$\int_u^v |z-v|\,|dz| = |w-u| \int_0^s |(1-t)u + tw - (1-s)u - sw|\, dt$$
$$= |w-u|^2 \int_0^s (s-t)\, dt = \frac{1}{2} |w-u|^2 s^2$$

and

$$\int_v^w |z-v|\,|dz| = |w-u| \int_s^1 |(1-t)u + tw - (1-s)u - sw|\, dt$$
$$= |w-u|^2 \int_s^1 (t-s)\, dt = \frac{1}{2} |w-u|^2 (1-s)^2 .$$

Using the inequality (7) we get

$$\left| (v-u) f(u) + (w-v) f(w) - \int_\gamma f(z)\, dz \right|$$
$$\leq \frac{1}{2} |w-u|^2 s^2 \|f'\|_{\gamma_{u,v};\infty} + \frac{1}{2} |w-u|^2 (1-s)^2 \|f'\|_{\gamma_{v,w};\infty}$$
$$\leq \frac{1}{2} |w-u|^2 \left[s^2 + (1-s)^2 \right] \|f'\|_{\gamma_{u,w};\infty} = |w-u|^2 \left[\frac{1}{4} + \left(s - \frac{1}{2} \right)^2 \right] \|f'\|_{[u,w];\infty} ,$$

which proves (11).

Also

$$\max_{z \in \gamma_{u,v}} |z-v| = \max_{t \in [0,s]} |(1-t)u + tw - (1-s)u - sw| = |w-u|\, s$$

and

$$\max_{z \in \gamma_{v,w}} |z-v| = \max_{t \in [s,1]} \{ |w-u| (1-t) \} = |w-u| (1-s) ,$$

then by (8)

$$
\begin{aligned}
&\left|(v-u) f(u)+(w-v) f(w)-\int_{\gamma} f(z) d z\right| \\
&\qquad \leq|w-u|\left\{s\left\|f^{\prime}\right\|_{[u, v] ; 1}+(1-s)\left\|f^{\prime}\right\|_{[v, w] ; 1}\right\} \\
&\quad \leq|w-u| \max \{s, 1-s\}\left\|f^{\prime}\right\|_{[u, w] ; 1}=|w-u|\left(\frac{1}{2}+\left|s-\frac{1}{2}\right|\right)\left\|f^{\prime}\right\|_{[u, w] ; 1},
\end{aligned}
$$

which proves (12).

Finally, since

$$
\begin{aligned}
\int_{u}^{v}|z-v|^{q}|d z| & =|w-u| \int_{0}^{s}|(1-t) u+t w-(1-s) u-s w|^{q} d t \\
& =|w-u|^{q+1} \int_{0}^{s}(s-t)^{q} d t=\frac{1}{q+1} s^{q+1}|w-u|^{q+1}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{v}^{w}|z-v|^{q}|d z| & =|w-u| \int_{s}^{1}|(1-t) u+t w-(1-s) u-s w|^{q} d t \\
& =|w-u|^{q+1} \int_{s}^{1}(t-s)^{q} d t=\frac{1}{q+1}(1-s)^{q+1}|w-u|^{q+1},
\end{aligned}
$$

hence by (9) we get (13). □

Remark 1. *Let f be holomorphic in G, an open domain and suppose $[u,w] \subset G$ is a segment connecting two distinct points u and w in G. Then*

$$
\begin{aligned}
&\left|\frac{f(u)+f(w)}{2}(w-u)-\int_{u}^{w} f(z) d z\right| \\
&\quad \leq \frac{1}{8}|w-u|^{2}\left[\left\|f^{\prime}\right\|_{\gamma_{u, \frac{u+w}{2}} ; \infty}+\left\|f^{\prime}\right\|_{\gamma_{\frac{u+w}{2}, w} ; \infty}\right] \leq \frac{1}{4}|w-u|^{2}\left\|f^{\prime}\right\|_{[u, w] ; \infty}, \quad (14)
\end{aligned}
$$

and

$$
\left|\frac{f(u)+f(w)}{2}(w-u)-\int_{u}^{w} f(z) d z\right| \leq \frac{1}{2}|w-u|\left\|f^{\prime}\right\|_{[u, w] ; 1}. \quad (15)
$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$
\begin{aligned}
&\left|\frac{f(u)+f(w)}{2}(w-u)-\int_{u}^{w} f(z) d z\right| \\
&\quad \leq \frac{1}{2^{1+1/q}(q+1)^{1/q}}|w-u|^{1+1/q}\left[\left\|f^{\prime}\right\|_{\left[u, \frac{u+w}{2}\right] ; p}+\left\|f^{\prime}\right\|_{\left[\frac{u+w}{2}, w\right] ; p}\right] \\
&\qquad \leq \frac{1}{2(q+1)^{1/q}}|w-u|^{1+1/q}\left\|f^{\prime}\right\|_{[u, w] ; p}. \quad (16)
\end{aligned}
$$

Suppose that $\gamma \subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$. If $v=z(x)$ with $x \in (a,b)$, then $\gamma_{u,w}=\gamma_{u,v} \cup \gamma_{v,w}$.

If we consider $f(z) = \exp(z)$ with $z \in \mathbb{C}$, then

$$\int_{\gamma_{u,w}} \exp(z)\, dz = \exp(w) - \exp(u),$$

$$|\exp(z)| = |\exp(\mathrm{Re}(z) + i\,\mathrm{Im}(z))| = \exp(\mathrm{Re}(z))$$

and by Theorem 1 we have

$$\begin{aligned} &|(v-u)\exp u + (w-v)\exp w - \exp(w) + \exp(u)| \\ &\quad \le \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-v|\,|dz| \\ &\quad + \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-v|\,|dz| \\ &\quad \le \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{u,w};\infty} \int_{\gamma_{u,w}} |z-v|\,|dz|, \end{aligned} \tag{17}$$

and

$$\begin{aligned} &|(v-u)\exp u + (w-v)\exp w - \exp(w) + \exp(u)| \\ &\quad \le \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{u,v};1} \max_{z\in\gamma_{u,v}} |z-v| + \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{v,w};1} \max_{z\in\gamma_{v,w}} |z-v| \\ &\quad \le \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{u,w};1} \max_{z\in\gamma_{u,w}} |z-v|. \end{aligned} \tag{18}$$

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &|(v-u)\exp u + (w-v)\exp w - \exp(w) + \exp(u)| \\ &\quad \le \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{u,v};p} \left(\int_{\gamma_{u,v}} |z-v|^q\,|dz|\right)^{1/q} \\ &\quad + \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{v,w};p} \left(\int_{\gamma_{v,w}} |z-v|^q\,|dz|\right)^{1/q} \\ &\quad \le \|\exp(\mathrm{Re}(\cdot))\|_{\gamma_{u,w};p} \left(\int_{\gamma_{u,w}} |z-v|^q\,|dz|\right)^{1/q}. \end{aligned} \tag{19}$$

With the same assumption of the path γ and if we consider $f(z) = z^n$ with $n \ge 1$, then

$$\int_{\gamma} z^n dz = \frac{w^{n+1} - u^{n+1}}{n+1}$$

and by Theorem 1 we get, by denoting $\ell(z) = z$, $z \in \mathbb{C}$, that

$$\left|(v-u)\,u^n + (w-v)\,w^n - \frac{w^{n+1} - u^{n+1}}{n+1}\right|$$

$$\leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};\infty}\int_{\gamma_{u,v}}|z-v|\,|dz|+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};\infty}\int_{\gamma_{v,w}}|z-v|\,|dz|\right]$$
$$\leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};\infty}\int_{\gamma_{u,w}}|z-v|\,|dz|, \quad (20)$$

and

$$\left|(v-u)\,u^{n}+(w-v)\,w^{n}-\frac{w^{n+1}-u^{n+1}}{n+1}\right|$$
$$\leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};1}\max_{z\in\gamma_{u,v}}|z-v|+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};1}\max_{z\in\gamma_{v,w}}|z-v|\right]$$
$$\leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};1}\max_{z\in\gamma_{u,w}}|z-v|. \quad (21)$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|(v-u)\,u^{n}+(w-v)\,w^{n}-\frac{w^{n+1}-u^{n+1}}{n+1}\right|$$
$$\leq n\left[\left\|\ell^{n-1}\right\|_{\gamma_{u,v};p}\left(\int_{\gamma_{u,v}}|z-v|^{q}\,|dz|\right)^{1/q}+\left\|\ell^{n-1}\right\|_{\gamma_{v,w};p}\left(\int_{\gamma_{v,w}}|z-v|^{q}\,|dz|\right)^{1/q}\right]$$
$$\leq n\left\|\ell^{n-1}\right\|_{\gamma_{u,w};p}\left(\int_{\gamma_{u,w}}|z-v|^{q}\,|dz|\right)^{1/q}, \quad (22)$$

where $\gamma\subset G$ is a smooth path from $z(a)=u$ to $z(b)=w$ and $v=z(x)$ with $x\in(a,b)$.

3. Examples For Circular Paths

Let $[a,b]\subseteq[0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t)=R\exp(it)=R(\cos t+i\sin t),\ t\in[a,b].$$

If $[a,b]=[0,\pi]$ then we get a half circle while for $[a,b]=[0,2\pi]$ we get the full circle.

Since

$$\left|e^{is}-e^{it}\right|^{2}=\left|e^{is}\right|^{2}-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^{2}$$
$$=2-2\cos(s-t)=4\sin^{2}\left(\frac{s-t}{2}\right)$$

for any $t,\ s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r} \quad (23)$$

for any t, $s \in \mathbb{R}$ and $r > 0$. In particular,

$$\left|e^{is} - e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any t, $s \in \mathbb{R}$.

For t, $x \in [a,b] \subseteq [0,2\pi]$ we then have

$$\left|e^{ix} - e^{it}\right| = 2\left|\sin\left(\frac{x-t}{2}\right)\right|.$$

If $u = R\exp(ia)$, $v = R\exp(ix)$ and $w = R\exp(ib)$ then

$$\begin{aligned} v - u &= R\left[\exp(ix) - \exp(ia)\right] = R\left[\cos x + i\sin x - \cos a - i\sin a\right] \\ &= R\left[\cos x - \cos a + i\left(\sin x - \sin a\right)\right]. \end{aligned}$$

Since

$$\cos x - \cos a = -2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right)$$

and

$$\sin x - \sin a = 2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{a+x}{2}\right),$$

hence

$$\begin{aligned} v - u &= R\left[-2\sin\left(\frac{a+x}{2}\right)\sin\left(\frac{x-a}{2}\right) + 2i\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{a+x}{2}\right)\right] \\ &= 2R\sin\left(\frac{x-a}{2}\right)\left[-\sin\left(\frac{a+x}{2}\right) + i\cos\left(\frac{a+x}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{x-a}{2}\right)\left[\cos\left(\frac{a+x}{2}\right) + i\sin\left(\frac{a+x}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]. \end{aligned}$$

Similarly,

$$w - v = 2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]$$

for $a \le x \le b$.

Moreover,

$$z - v = 2Ri\sin\left(\frac{t-x}{2}\right)\exp\left[\left(\frac{t+b}{2}\right)i\right]$$

and

$$|z - v| = \left|2Ri\sin\left(\frac{t-x}{2}\right)\exp\left[\left(\frac{t+b}{2}\right)i\right]\right| = 2R\left|\sin\left(\frac{t-x}{2}\right)\right|$$

for $a \le x$, $t \le b$.

We also have

$$z'(t) = Ri\exp(it) \;\text{ and }\; |z'(t)| = R$$

for $t \in [a,b]$.

Proposition 1. *Let f be holomorphic in G, on open domain and suppose $\gamma_{[a,b],R} \subset G$ with $[a,b] \subseteq [0,2\pi]$ and $R>0$. If $x \in [a,b]$, then*

$$\begin{aligned}
&\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f(R\exp(ia))\right.\\
&\qquad+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f(R\exp(ib))\\
&\qquad\left.-\frac{1}{2}\int_a^b f(R\exp(it))\exp(it)\,dt\right|\\
&\le 4R\left[\|f'(R\exp(i\cdot))\|_{[a,x],\infty}\sin^2\left(\frac{x-a}{4}\right)\right.\\
&\qquad\left.+\|f'(R\exp(i\cdot))\|_{[x,b],\infty}\sin^2\left(\frac{b-x}{4}\right)\right]\\
&\le 4R\|f'(R\exp(i\cdot))\|_{[a,b],\infty}\left[\sin^2\left(\frac{x-a}{4}\right)+\sin^2\left(\frac{b-x}{4}\right)\right]. \qquad (24)
\end{aligned}$$

Proof. We write the inequality (7) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ to get

$$\begin{aligned}
&\left|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f(R\exp(ia))\right.\\
&\qquad+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f(R\exp(ib))\\
&\qquad\left.-Ri\int_a^b f(R\exp(it))\exp(it)\,dt\right|\\
&\le 2R^2\|f'(R\exp(i\cdot))\|_{[a,x],\infty}\int_a^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt\\
&\qquad+2R^2\|f'(R\exp(i\cdot))\|_{[x,b],\infty}\int_x^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f(R\exp(ia))\right.\\
&\qquad+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f(R\exp(ib))\\
&\qquad\left.-\frac{1}{2}\int_a^b f(R\exp(it))\exp(it)\,dt\right|
\end{aligned}$$

$$\le R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}\int_a^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt$$
$$+R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\int_x^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt \quad (25)$$

for $x\in[a,b]$.

Observe that

$$\int_a^x\left|\sin\left(\frac{t-x}{2}\right)\right|dt=\int_a^x\sin\left(\frac{x-t}{2}\right)dt=2-2\cos\left(\frac{x-a}{2}\right)$$
$$=4\sin^2\left(\frac{x-a}{4}\right)$$

and

$$\int_x^b\left|\sin\left(\frac{t-x}{2}\right)\right|dt=\int_x^b\sin\left(\frac{t-x}{2}\right)dt=2-2\cos\left(\frac{b-t}{2}\right)$$
$$=4\sin^2\left(\frac{b-x}{4}\right),$$

which by (25) produce the desired result (24). □

Corollary 2. *With the assumptions of Proposition 1 we have*

$$\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right)\right.$$
$$+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)$$
$$\left.-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|$$
$$\le 4R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],\infty}+\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],\infty}\right]\sin^2\left(\frac{b-a}{8}\right)$$
$$\le 8R\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],\infty}\sin^2\left(\frac{b-a}{8}\right). \quad (26)$$

Remark 2. *The case of semi-circle, namely $a=0$ and $b=\pi$ in (24) gives the inequality*

$$\left|\sin\left(\frac{x}{2}\right)\exp\left[\left(\frac{x}{2}\right)i\right]f\left(R\right)+i\cos\left(\frac{x}{2}\right)\exp\left[\left(\frac{x}{2}\right)i\right]f\left(-R\right)\right.$$
$$\left.-\frac{1}{2}\int_0^\pi f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right|$$
$$\le 4R\left[\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,x],\infty}\sin^2\left(\frac{x}{4}\right)\right.$$

$$
\begin{aligned}
&+ \left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,\pi],\infty} \sin^2\left(\frac{\pi-x}{4}\right)\Bigg] \\
&\leq 4R\left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\pi],\infty}\left[\sin^2\left(\frac{x}{4}\right)+\sin^2\left(\frac{\pi-x}{4}\right)\right], \quad (27)
\end{aligned}
$$

for $x\in[0,\pi]$.

Since

$$
\sin^2\left(\frac{\pi}{8}\right)=\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}=\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{2-\sqrt{2}}{4},
$$

then by taking $x=\frac{\pi}{2}$ in (27), we get

$$
\begin{aligned}
&\left|\frac{1+i}{2}f\left(R\right)+\frac{-1+i}{2}f\left(-R\right)-\frac{1}{2}\int_0^\pi f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\right| \\
&\leq\left(2-\sqrt{2}\right)\left[\left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\frac{\pi}{2}],\infty}+\left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[\frac{\pi}{2},\pi],\infty}\right] \\
&\leq 2\left(2-\sqrt{2}\right)\left\| f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[0,\pi],\infty}. \quad (28)
\end{aligned}
$$

Further, we have the following result as well:

Proposition 2. *With the assumptions of Proposition 1 we have*

$$
\begin{aligned}
&\Bigg|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right) \\
&+\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right) \\
&-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg| \\
&\leq R\left[\max_{t\in[a,x]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^x\left|f'\left(R\exp\left(it\right)\right)\right|dt\right. \\
&\left.+\max_{t\in[x,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_x^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right] \\
&\leq R\max_{t\in[a,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt. \quad (29)
\end{aligned}
$$

Proof. We write the inequality (8) for $\gamma_{[a,b],R}$ and $x\in[a,b]$ to get

$$
\begin{aligned}
&\Bigg|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right) \\
&+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg| \\
&\le 2R^2\left[\max_{t\in[a,x]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^x\left|f'\left(R\exp\left(it\right)\right)\right|dt\right. \\
&\left.+\max_{t\in[x,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_x^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right] \\
&\le 2R^2\max_{t\in[a,b]}\left|\sin\left(\frac{t-x}{2}\right)\right|\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt,
\end{aligned}
$$

which is equivalent to (29). □

In particular, we have:

Corollary 3. *With the assumptions of Proposition 1 we have*

$$
\begin{aligned}
&\Bigg|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right) \\
&\qquad+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right) \\
&-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg|\le R\sin\left(\frac{b-a}{4}\right)\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt. \quad (30)
\end{aligned}
$$

Proof. If we take in (29) $x=\frac{a+b}{2}$, then we get

$$
\begin{aligned}
&\Bigg|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f\left(R\exp\left(ia\right)\right) \\
&+\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f\left(R\exp\left(ib\right)\right)-\frac{1}{2}\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg| \\
&\le R\left[\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_a^{\frac{a+b}{2}}\left|f'\left(R\exp\left(it\right)\right)\right|dt\right. \\
&\left.+\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_{\frac{a+b}{2}}^b\left|f'\left(R\exp\left(it\right)\right)\right|dt\right] \\
&\le R\max_{t\in[a,b]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|\int_a^b\left|f'\left(R\exp\left(it\right)\right)\right|dt. \quad (31)
\end{aligned}
$$

Since the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ have a length less than π, then

$$
\max_{t\in\left[a,\frac{a+b}{2}\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|=\max_{t\in\left[\frac{a+b}{2},b\right]}\left|\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\right|=\sin\left(\frac{b-a}{4}\right)
$$

and by (31) we get (30). □

The case of p-norms is as follows:

Proposition 3. *With the assumptions of Proposition 1 and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\begin{aligned}
&\left|\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f(R\exp(ia))\right. \\
&\qquad + \sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f(R\exp(ib)) \\
&\qquad\qquad \left. -\frac{1}{2}\int_a^b f(R\exp(it))\exp(it)\,dt\right| \qquad (32)
\end{aligned}$$

$$\begin{aligned}
&\le R\left(\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt\right)^{1/q}\|f'(R\exp(i\cdot))\|_{[a,x],p} \\
&\qquad + R\left(\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right)^{1/q}\|f'(R\exp(i\cdot))\|_{[x,b],p} \\
&\le R\left[\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt + \int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right]^{1/q}\|f'(R\exp(i\cdot))\|_{[a,b],p}.
\end{aligned}$$

In particular, for $x = \frac{a+b}{2}$ *we get*

$$\begin{aligned}
&\left|\sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{3a+b}{4}\right)i\right]f(R\exp(ia))\right. \\
&\qquad + \sin\left(\frac{b-a}{4}\right)\exp\left[\left(\frac{a+3b}{4}\right)i\right]f(R\exp(ib)) \\
&\qquad\qquad \left. -\frac{1}{2}\int_a^b f(R\exp(it))\exp(it)\,dt\right| \qquad (33)
\end{aligned}$$

$$\begin{aligned}
&\le R\left(\int_a^{\frac{a+b}{2}} \sin^q\left(\frac{\frac{a+b}{2}-t}{2}\right)dt\right)^{1/q}\|f'(R\exp(i\cdot))\|_{\left[a,\frac{a+b}{2}\right],p} \\
&+ R\left(\int_{\frac{a+b}{2}}^b \sin^q\left(\frac{t-\frac{a+b}{2}}{2}\right)dt\right)^{1/q}\|f'(R\exp(i\cdot))\|_{\left[\frac{a+b}{2},b\right],p} \\
&\le R\left[\int_a^b \sin^q\left(\left|\frac{t-\frac{a+b}{2}}{2}\right|\right)dt\right]^{1/q}\|f'(R\exp(i\cdot))\|_{[a,b],p}.
\end{aligned}$$

Proof. By making use of the inequality (9) for $\gamma_{[a,b],R}$ and $x \in [a,b]$ we get

$$\Bigg|2Ri\sin\left(\frac{x-a}{2}\right)\exp\left[\left(\frac{a+x}{2}\right)i\right]f\left(R\exp\left(ia\right)\right)$$
$$+2Ri\sin\left(\frac{b-x}{2}\right)\exp\left[\left(\frac{x+b}{2}\right)i\right]f\left(R\exp\left(ib\right)\right)-Ri\int_a^b f\left(R\exp\left(it\right)\right)\exp\left(it\right)dt\Bigg|$$
$$\le 2R^2\left(\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,x],p}$$
$$+2R^2\left(\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right)^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[x,b],p}$$
$$\le 2R^2\left[\int_a^x \sin^q\left(\frac{x-t}{2}\right)dt+\int_x^b \sin^q\left(\frac{t-x}{2}\right)dt\right]^{1/q}\left\|f'\left(R\exp\left(i\cdot\right)\right)\right\|_{[a,b],p},$$

which proves the desired result (32). □

The interested reader may consider for examples some fundamental complex functions such as $f(z) = z^n$ with n a natural number, $f(z) = \exp(z)$ or f a trigonometric or a hyperbolic complex function. The details are omitted.

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