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Eigenvalue Expansion of Nonsymmetric Linear Compact Operators in Hilbert Space

Ludwig Kohaupt[1]*

Abstract
For a symmetric linear compact resp. symmetric densely defined linear operator with compact inverse, expansion theorems in series of eigenvectors are known. The aim of the present paper is to generalize the known expansion theorems to the case of corresponding operators *without the symmetry property*. For this, we replace the set of orthonormal eigenvectors in the symmetric case by a set of biorthonormal eigenvectors resp. principal vectors in the case of simple eigenvalues resp. general eigenvalues. The results for the operators without the symmetry property are all new. Furthermore, if the operators are symmetric, the generalized results deliver the known expansions. As an application of the results for nonsymmetric operators with simple eigenvalues, we obtain a known expansion in a series of eigenfunctions for a non-selfadjoint Boundary Eigenvalue Problem with ordinary differential operator discussed in a book of Coddington/Levinson. But, we obtain a new result if the eigenvalues are general, that is, not necessarily simple. In addition, for a differential operator of 2nd order with constant coefficients, the eigenfunctions and Green's function are explicitly determined. This result is also new, as far as the author is aware.

Keywords: Applications to non-selfadjoint boundary eigenvalue problems, Densely defined operator with compact inverse, Eigenvalue expansion of nonsymmetric compact operator, Hilbert space, Nonsymmetric compact operator
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[1] *Beuth University of Technology Berlin, Department of Mathematics, Luxemburger Str. 10, 13353 Berlin, Germany, ORCID: 0000-0003-4364-9144*
***Corresponding author**: kohaupt@beuth-hochschule.de
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1. Introduction

The paper is structured as follows.

Section 2 is of preparatory nature and of utmost importance for the subsequent sections; it discusses functions of an operator in a Banach space.

Section 3 is on the expansion of a linear compact operator and of a pertinent projection operator in a series of eigenvectors resp. principal vectors in a Hilbert space.

Section 4 treats densely defined linear operators $T = L$ with compact inverse $G = T^{-1} = L^{-1}$, derives for it expansions in series of eigenvectors resp. of principal vectors and shows that $G_+ = G^*$ not only for simple, but also for general eigenvalues, where $G_+ = L_+^{-1}$ and L_+ is the formal adjoint of L.

In Section 5, applications of the results of Section 4 are made to a non-selfadjoint BEVP taken from [2, Chapter 12], delivering relation [2, Chapter 12, (5.6)]. Here, not only the expansion in a series of eigenfunctions is obtained in the Hilbert space $H = L_2(a,b)$ if the eigenvalues are simple, but also a corresponding expansion in a series of principal functions if the eigenvalues are general.

In Section 6, beyond this, for a differential operator L defined by $Lu(x) = L_{p_0,q_0}u(x) = -u''(x) + p_0u'(x) + q_0u(x)$, $0 \le x \le l$ with real constants p_0 and q_0, the eigenvalues μ_j and pertinent eigenfunctions $\chi_j(x)$ as well as the associated eigenvalues $\overline{\mu}_j = \mu_j$ and eigenfunctions $\psi_j(x)$ of the formally adjoint operator L_+ defined by $L_+v(x) = L_{-p_0,q_0}v(x) = -v''(x) - p_0v'(x) + q_0v(x)$, $0 \le x \le l$ with the biorthonormality property are explicitly determined. Furthermore, the Green's functions $G(x,s) = G(x,s;p_0,q_0)$ pertinent to the operator $L = L_{p_0,q_0}$ as well as the associated Green's function $G_+(x,s) = G^T(x,s) = G(s,x) = G(s,x;-p_0,q_0)$ pertinent to the formally adjoint operator $L_+ = L_{-p_0,q_0}$ are also explicitly determined confirming the general result $G_+ = G^T$ for the linear compact operators G and G_+ defined by the corresponding Green's functions. In Section 7, we compare the present expansion results in abstract Hilbert spaces with known ones. Finally, Section 8 contains the conclusions.

2. Functions of an Operator in a Banach Space

This section contains the basis for the convergence of the studied expansions and is thus of utmost importance for the whole paper.

The method of deriving the expansions for symmetric linear compact operators is no longer applicable when the symmetry property is missing. See, for example the derivation for a symmetric linear compact operator in [14, Theorem 6.4-B, pp.336-337].

A hint what can be done in the nonsymmetric case is found in [2, Chapter 12, 1. Introduction, p.298, first paragraph]. As stated there, an appropriate approach is furnished by the *Cauchy integral method*. There, one can read: "The method ... yields complete information about the convergence of the expansion for any integrable function."

We mention that most theorems of the classical Theory of Functions can be carried over to functions of a complex variable z with values in a complex Banach space.

So, in particular, Cauchy's integral method can be applied to functions with values in a Banach space, that is, in a complete normed space, where the completeness property of the space is essential.

In [2], the special case of the Hilbert function space $H = L_2(a,b)$ is used, that is, a specific complete function space with scalar product.

This is not general enough for our purposes, however. What we need is Cauchy's integral method in a general Banach space. This is treated in the book [6, Chapter I, §5]. However, there Kato assumes that the underlying normed space be finite-dimensional. Then, of course, the space is complete. But, the assumption of finite dimension can be replaced by the completeness of the space since this is the important condition to allow the transition from complex-valued functions of a complex variable to vector-valued functions of a complex variable, as we have already mentioned above. This is done, for instance, in Stummel's paper [13], where Cauchy's integral method is used to show the existence of the resolvent integral for a pair of linear bounded operators $A, B \in B(E,F)$ where E and F are Banach spaces and where it is proven that the completeness property is even not necessary if the operator B is compact.

Here, we study only a single operator $T \in B(E)$, i.e., the pair $(A = T, B = I)$ with the identity operator I in $F = E$ where, for the time being, we assume that the space is complete. In a subsequent paper, we shall investigate whether the completeness property of the space for the series expansion of T can be dropped if T is compact.

For the study of asymptotic expansions for discrete approximations of eigenvalue problems, we refer the reader to [4].

After these preliminary remarks, we turn to functions of an operator in a Banach space as announced in the heading of this section.

We mention that here we use verbatim and almost verbatim passages from [6, Chapter I, §5].

Let $\{0\} \neq E$ be a Banach space over the field $\mathbb{F} = \mathbb{C}$. Whereas in [6, Chapter I] it is supposed that $dim E < \infty$, here we assume that $dim E = \infty$. As already mentioned several times, the following results taken from [6] are valid for $dim E < \infty$ and $dim E = \infty$ if the space is complete.

Let $p(\zeta)$ be the polynomial

$$p(\zeta) = \alpha_0 + \alpha_1\zeta + \cdots \alpha_n\zeta^n,\ \zeta \in \mathbb{C} \tag{2.1}$$

with $\alpha_j \in \mathbb{C}$, $j = 0,1,\cdots,n$. Then the polynomial $p(T) \in B(E)$ is defined by

$$p(T) = \alpha_0 + \alpha_1 T + \cdots \alpha_n T^n,\ \zeta \in \mathbb{C}, \tag{2.2}$$

see [6, Chapter I, §3.3]. Making use of the resolvent

$$R(\zeta) := (T - \zeta)^{-1},\ \zeta \in \mathbb{C}, \tag{2.3}$$

one can now define the function $\phi(T)$ of T for a more general class of functions $\phi(\zeta)$.

Before we do this, we mention that linear compact operators need not have eigenvalues. For example, Volterra integral operators have no eigenvalues. On the other hand, consider a symmetric linear compact operator. Then, such an operator has at least one eigenvalue, and all eigenvalues are real and simple. It may happen that there exits only a finite number of eigenvalues. Further, there is at most a countable set of eigenvalues with the only possible accumulation point zero, and there exists a set of pertinent pairwise orthonormal eigenvectors. Further, it is known that the non-zero elements of the spectrum consist solely of eigenvalues and that, if there is a countable set of eigenvalues, the assocated sequence tends to zero. For all this, see [14, Chapter 6].

Further, according to [5, Theorem 44.1, p.191], one has $\sigma(T)\setminus\{0\} = \sigma_P(T)\setminus\{0\}$ where $\sigma(T)$ is the spectrum of T and $\sigma_P(T)$ the point spectrum consisting of the eigenvalues of T.

Taking this into account, for our general linear compact operator $T \in B(E)$, we suppose that the spectrum $\sigma(T)$ of T has a countable set of non-zero eigenvalues λ_j and that the sequence of eigenvalues tends to zero.

Additionally, we suppose that $0 \notin \sigma(T)$ so that $N(T) = \{0\}$ since without this condition, we cannot obtain relation (2.11) resp.(2.14) below.

Now, suppose that $\phi(\zeta)$ is holomorphic in a domain D of the complex plane containing all the eigenvalues $\lambda_j \neq 0$ of T, and let $C \subset D$ be a simple closed smooth curve with positive direction enclosing all the eigenvalues λ_j in its interior. Then, $\phi(T)$ is defined by the *Dunford-Taylor integral*

$$\phi(T) = -\frac{1}{2\pi i}\int_C \phi(\zeta)R(\zeta)\,d\zeta = -\frac{1}{2\pi i}\int_C \phi(\zeta)(T-\zeta)^{-1}\,d\zeta. \tag{2.4}$$

This is an analogue of the Cauchy integral formula in the Theory of Functions, see [7, Part I, §15, p. 61]. More generally, the curve C may consist of several simple closed rectifiable Jordan curves C_k having positive direction with interiors D'_k such that the union of the D'_k contains all the eigenvalues of T. We note that (2.4) does not depend on C as long as C satisfies these conditions. For the C_k, we can use the circles $C_k = \{z \in \mathbb{C} \mid |z-\lambda_k| = r_k\}$ with sufficiently small radii r_k.

It can be verified that for the polynomial

$$\phi(\zeta) = p(\zeta) = \alpha_0 + \alpha_1\zeta + \cdots \alpha_n\zeta^n,\ \zeta \in \mathbb{C} \tag{2.5}$$

with $\alpha_j \in \mathbb{C}$, $j = 0,1,\cdots,n$, the Dunford-Taylor integral (2.4) is equal to (2.2).

For the special case

$$\phi(\zeta) = p(\zeta) = \zeta, \tag{2.6}$$

we obtain

$$T = -\frac{1}{2\pi i}\int_C TR(\zeta)\,d\zeta = T\left(-\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta\right) = \left(-\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta\right)T. \tag{2.7}$$

Now, we set

$$P := -\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta. \tag{2.8}$$

According to [6, Chapter I, §5, Section 3], P is a continuous projection operator onto the *algebraic eigenspace* $X = P(E) = R(P)$, where $R(P)$ means the range of P. Thus, from (2.7) and (2.8), one obtains

$$T = TP = PT = PTP. \tag{2.9}$$

Now, let the radii r_k be chosen such that

$$C_j \cap C_k = \emptyset,\ j \neq k,\ j,k = 1,2,3,\cdots. \tag{2.10}$$

Then,

$$P = -\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta = \sum_{j=1}^{\infty}\left(-\frac{1}{2\pi i}\int_{C_j} R(\zeta)\,d\zeta\right) = \sum_{j=1}^{\infty} P_j \tag{2.11}$$

with

$$P_j = -\frac{1}{2\pi i}\int_{C_j} R(\zeta)\,d\zeta,\ j = 1,2,3,\cdots. \tag{2.12}$$

At this point, we needed the assumption $0 \notin \sigma(T)$ since otherwise any circle C_0 about $\lambda_0 = 0$ would eventually intersect with the circles C_k for sufficiently large k so that we would not have (2.10) for $j,k \in (1,2,3,\cdots)$. Let J be the sequence

$$J := (1,2,3,\cdots). \tag{2.13}$$

Then, (2.11) can be written as

$$\boxed{P = \sum_{j=1}^{\infty} P_j = \sum_{j\in J} P_j.} \tag{2.14}$$

Because of (2.10), one has

$$P_j P_k = P_k P_j = P_j \delta_{jk},\ j,k \in J. \tag{2.15}$$

Herewith,

$$P_j(E) =: X_j \tag{2.16}$$

is the *algebraic eigenspace of T associated with the eigenvalue* λ_j.

From (2.9), (2.11), and (2.15), we obtain

$$T = PT = TP = PTP = \sum_{j\in J} P_j T = \sum_{j\in J} TP_j = \sum_{j\in J} P_j T P_j, \tag{2.17}$$

and so

$$\begin{aligned} R(T) = T(E) &= (PT)(E) = (TP)(E) = (PTP)(E) \\ &= \sum_{j\in J}(P_j T)(E) = \sum_{j\in J}(TP_j)(E) = \sum_{j\in J}(P_j T P_j)(E). \end{aligned} \tag{2.18}$$

3. Expansion of a Linear Compact Operator and of a Pertinent Projection Operator in Hilbert Space

The aim of the present section is to specify the relation (2.17), i.e.,

$$T = PT = TP = PTP = \sum_{j\in J} P_j T = \sum_{j\in J} TP_j = \sum_{j\in J} P_j T P_j,$$

in more detail. This can best be done in a Hilbert space since, for example, the orthogonal projection Pu of a vector $u \in H$ onto a unit vector $e \in H$ can be written as

$$Pu = (u,e)\,e,$$

that is, by using a scalar product.

In our case, the projection operators P_j are not orthogonal, however. But, the dimension of $R(P_j) = P_j(H)$ is finite-dimensional and represents the *geometric eigenspace* $N_j := N(T-\lambda_j)$ if the eigenvalue λ_j is simple and the *algebraic eigenspace* $X_j := X_{\lambda_j}(T)$ if λ_j is not simple. Now, for finite-dimensional spaces, the author constructed, in earlier work, a set of biorthonormal eigenvectors resp. principal vectors pertinent to a finite-dimensional mapping (usually represented by a matrix with respect to a fixed basis of vectors); here, the mapping is given by $T_j = TP_j = P_j T = P_j T P_j$. Thus, using these biorthonormal sets, it is possible to specify the expressions $T_j u = TP_j u = P_j T u = P_j T P_j u$ for elements $u \in H$ in more detail by using a scalar product. This leads to the desired expansion for Tu. Now, the announced details follow, first for the case of simple eigenvalues, and then for the case of general, not necessarily simple eigenvalues.

3.1 The Case of Simple Eigenvalues

In this subsection, in the case of simple eigenvalues, expansions in a series of eigenvectors are treated; it is organized as follows. First, the conditions for the expansions to hold are stated. Then, the series expansions of Tu as well as of Pu are derived. Finally, the known expansions for a selfadjoint operator $T = A$ are retrieved from the more general result obtained in this subsection.

(i) The Conditions (C1) - (C4)

We assume the following conditions:

(C1) $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product

(C2) $0 \neq T \in B(H)$ is compact (or completely continuous) having countably many simple non-zero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ with $\lim_{k\to\infty} \lambda_k = 0$ pertinent to the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$. Further, $0 \notin \sigma(T)$.

(C3) The eigenvectors of the adjoint T^* of T with the eigenvalues $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \cdots$ are $\psi_1, \psi_2, \psi_3, \cdots$

(C4) $\lambda_i \neq \lambda_j,\, i \neq j,\, i, j = 1, 2, 3 \cdots$

(ii) Series Expansions of Tu as well as of Pu

One has the following theorem.

Theorem 3.1 (Biorthonormality relations for $\lambda_j \neq \lambda_k,\ j \neq k$)

Let the conditions (C1) - (C4) be fulfilled. Then, with appropriate normalization, the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$ and $\psi_1, \psi_2, \psi_3, \cdots$ are orthonormal, that is,

$$(\chi_j, \psi_k) = \delta_{jk},\ j, k \in J. \tag{3.1}$$

Proof: Define the operators

$$P^{(n)} := \sum_{j=1}^{n} P_j \tag{3.2}$$

as well as

$$T^{(n)} := TP^{(n)} = \sum_{j=1}^{n} TP_j. \tag{3.3}$$

Here, $R(T^{(n)}) = (T^{(n)})(H)$ is finite-dimensional with dimension n. From [8, Theorem1], one has

$$(\chi_j, \psi_k) = \delta_{jk},\ j, k = 1, \cdots, n. \tag{3.4}$$

and

$$T^{(n)}\chi_j = \lambda_j \chi_j,\ j = 1, \cdots, n. \tag{3.5}$$

Now, letting $n \to \infty$, relation (3.4) entails (3.1) since $T = \lim_{n\to\infty} T^{(n)}$ according to Section 2. $\diamond$

Furthermore, we obtain the following theorem.

Theorem 3.2 (Expansions of Tu as well as of Pu in a series of eigenvectors)

Let the conditions (C1) - (C4) be fulfilled. Then,

$$Tu = \sum_{j\in J} \lambda_j (u, \psi_j)\chi_j,\ u \in H \tag{3.6}$$

as well as

$$Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H. \tag{3.7}$$

Proof: Let $u \in H$. Then, due to (3.1),

$$P^{(n)}u = \sum_{j=1}^{n} (u, \psi_j)\chi_j \tag{3.8}$$

and thus

$$T^{(n)}u = TP^{(n)} = \sum_{j=1}^{n} \lambda_j (u, \psi_j)\chi_j. \tag{3.9}$$

Now, from Section 2, the limit

$$Pu = \lim_{n\to\infty} P^{(n)}u = \sum_{j=1}^{\infty} (u, \psi_j)\chi_j = \sum_{j\in J} (u, \psi_j)\chi_j \tag{3.10}$$

exists entailing also the existence of the limit

$$Tu = \lim_{n\to\infty} T^{(n)}u = \lim_{n\to\infty} TP^{(n)}u = \sum_{j=1}^{\infty} \lambda_j (u, \psi_j)\chi_j = \sum_{j\in J} \lambda_j (u, \psi_j)\chi_j. \tag{3.11}$$

$\diamond$

Remark: From (3.6) we conclude that

$$\overline{[\chi_1, \chi_2, \chi_3, \cdots]} = T(H) = R(T). \tag{3.12}$$

Further,

$$P : H \mapsto \overline{[\chi_1, \chi_2, \chi_3, \cdots]}. \tag{3.13}$$

$\diamond$

Theorem 3.3
Let the conditions (C1) - (C4) be fulfilled. Then, we obtain

$$u = Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H \tag{3.14}$$

and the projection operator

$$P_0 = I - P : H \mapsto N(T) = \{0\} \iff P_0 = 0. \tag{3.15}$$

Proof: Evidently,

$$u = Pu + (I-P)u,\ u \in H. \tag{3.16}$$

Further,

$$T(P_0 u) = T(I-P)u = Tu - TPu = 0 \tag{3.17}$$

where the last equal sign follows from (2.17). So, $P_0 u \in N(T) = \{0\}$, i.e., $P_0 u = 0,\ u \in H$ or $P_0 = 0$. $\diamond$

If condition (C4) is not fulfilled, one can remedy this by using a biorthonormalization pre-process, as the next lemma shows.

Lemma 3.4
Let the conditions (C1) - (C3) be fulfilled, and let, for instance, $\lambda_{j_1}, \lambda_{j_2}, \cdots, \lambda_{j_p}$ be eigenvalues of T with linearly independent eigenvectors $\chi_{j_1}, \chi_{j_2}, \cdots, \chi_{j_p}$; further, let $\psi_{j_1}, \psi_{j_2}, \cdots, \psi_{j_p}$ be linearly independent eigenvectors pertinent to $\overline{\lambda}_{j_1}, \overline{\lambda}_{j_2}, \cdots, \overline{\lambda}_{j_p}$ of T^*. Then, these eigenvectors can be biorthonormalized such that

$$(\chi_{j_k}, \psi_{j_l}) = \delta_{kl},\ k, l = 1, 2, \cdots, p. \tag{3.18}$$

Proof: See [9, Theorem 3]. $\diamond$

After appropriate application of the biorthonormalization pre-process, condition (C4) is satisfied.

(iii) Special Case of a Selfadjoint Compact Operator $T = A$

If $T = A$ is *selfadjoint and compact* and if there is a countable set of non-zero eigenvalues $\lambda_j,\ j \in J$, then it is known that the relation

$$\lim_{j\to\infty} \lambda_j = 0 \tag{3.19}$$

is fulfilled. Further, the eigenvalues are real, and the pertinent eigenvectors φ_j can be chosen real so that one has

$$\varphi_j = \chi_j = \psi_j,\ j \in J \tag{3.20}$$

meaning that the biorthonormality relations (3.1) turn into the orthonormality relations

$$(\varphi_j, \varphi_k) = \delta_{j,k},\ j,k \in J. \tag{3.21}$$

Thus, if $0 \notin \sigma(A)$, the relations (3.6) and (3.14) turn into the known results

$$Au = \sum_{j \in J} \lambda_j (u, \varphi_j)\varphi_j,\ u \in H \tag{3.22}$$

and

$$u = Pu = \sum_{j \in J} (u, \varphi_j)\varphi_j,\ u \in H. \tag{3.23}$$

3.2 The Case of General Eigenvalues

In this subsection, we do not assume that the eigenvalues of T be simple. Then, we obtain expansions in a series of principal vectors.

This subsection is organized in a similar way as the preceding one.

So, first the conditions for the expansions to hold are stated. Then, the series expansions of Tu as well as of Pu are derived.

(i) The Conditions ($C1'$) - ($C4'$)

In the general case when the eigenvalues need not be simple, we assume the following conditions:

($C1'$) $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

($C2'$) $0 \neq T \in B(H)$ is compact (or completely continuous) having countably many non-zero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ with $\lim_{k\to\infty} \lambda_k = 0$ and the pertinent algebraic eigenspaces $P_j(H) = X_{\lambda_j}(T)$ spanned by the principal vectors $\chi_1^{(j)}, \chi_2^{(j)}, \cdots, \chi_{m_j}^{(j)}$ for $j \in J$, where $\chi_i^{(j)}$ is of stage i. Further, $0 \notin \sigma(T)$.

($C3'$) $\psi_1^{(j)}, \psi_2^{(j)}, \cdots, \psi_{m_j}^{(j)}$ are the principal vectors corresponding to the eigenvalues $\overline{\lambda}_j$, $j \in J$, spanning the algebraic eigenspaces $P_j^*(H) = X_{\overline{\lambda}_j}(T^*)$ for $j \in J$

($C4'$) $\lambda_j \neq \lambda_k,\ j \neq k,\ j,k \in J$

(ii) Series Expansions of Tu as well as of Pu

As a preparation of the expansions in series of principal vectors, we begin with the detailed biorthonormalization process. According to ($C2'$) and ($C3'$), we have

$$\boxed{T\chi_k^{(i)} = \lambda_i \chi_k^{(i)} + \chi_{k-1}^{(i)},\ k = 1,2,\cdots,m_i} \tag{3.24}$$

and

$$\boxed{T^*\psi_l^{(j)} = \overline{\lambda}_j \psi_l^{(j)} + \psi_{l-1}^{(j)},\ l = 1,2,\cdots,m_j.} \tag{3.25}$$

Then, the fact can be used that the principal vectors of stage k are determined only up to a linear combination of principal vectors of stages less than k which was applied in [8] to the chain $\psi_1^{(j)}, \psi_2^{(j)}, \cdots, \psi_{m_j}^{(j)}$ leading to

$$(\chi_k^{(i)}, \psi_l^{(i)}) = 0,\ l \neq m_i - k + 1,\ k = 1,\cdots,m_i \tag{3.26}$$

and

$$(\chi_k^{(i)}, \psi_{m_i-k+1}^{(i)}) \neq 0,\ l = m_i - k + 1,\ k = 1,\cdots,m_i. \tag{3.27}$$

So, with

$$\upsilon_k^{(i)} = \psi_{m_i-k+1}^{(i)}, \tag{3.28}$$

one has

$$(\chi_k^{(i)}, \upsilon_k^{(i)}) \neq 0,\ k = 1, \cdots, m_i. \tag{3.29}$$

Further, according to [8],

$$(\chi_k^{(i)}, \upsilon_l^{(j)}) = 0,\ i \neq j \tag{3.30}$$

$k = 1, \cdots, m_i,\ l = 1, \cdots, m_j$.

Now, replace $\upsilon_k^{(i)}$ in (3,29) by

$$\tilde{\upsilon}_k^{(i)} := \tilde{\psi}_{m_i-k+1} := \beta_{m_i-k+1}^{(i)}\, \psi_{m_i-k+1} = \beta_{m_i-k+1}^{(i)}\, \upsilon_k^{(i)} \tag{3.31}$$

and determine the factor $\beta_{m_i-k+1}^{(i)}$ such that

$$(\chi_k^{(i)}, \tilde{\upsilon}_k^{(i)}) = 1. \tag{3.32}$$

Then,

$$\beta_{m_i-k+1}^{(i)} = 1/\overline{(\chi_k^{(i)}, \upsilon_k^{(j)})} = 1/\overline{(\chi_k^{(i)}, \psi_{m_i-k+1})},\ k = 1, \cdots, m_i \tag{3.33}$$

or

$$\beta_l^{(i)} = 1/\overline{(\chi_{m_i-l+1}^{(i)}, \upsilon_{m_i-l+1}^{(j)})} = 1/\overline{(\chi_{m_i-l+1}^{(i)}, \psi_l)},\ l = 1, \cdots, m_i. \tag{3.34}$$

From (3.31), we obtain

$$\psi_l^{(i)} = \frac{1}{\beta_l^{(i)}}\, \tilde{\psi}_l^{(i)},\ l = 1, \cdots, m_i. \tag{3.35}$$

Inserting this in (3.25) implies

$$T^*(\frac{1}{\beta_l^{(i)}}\, \tilde{\psi}_l^{(j)}) = \overline{\lambda}_j\, (\frac{1}{\beta_l^{(i)}}\, \tilde{\psi}_l^{(j)}) + (\frac{1}{\beta_{l-1}^{(i)}}\, \tilde{\psi}_{l-1}^{(j)}),\ l = 1, 2, \cdots, m_j$$

or

$$\boxed{T^* \tilde{\psi}_l^{(j)} = \overline{\lambda}_j\, \tilde{\psi}_l^{(j)} + \gamma_{l-1}^{(j)}\, \tilde{\psi}_{l-1}^{(j)},\ l = 1, 2, \cdots, m_j} \tag{3.36}$$

with $\beta_0^{(j)} := 1$ and $\tilde{\psi}_0^{(j)} := 0$ as well as

$$\gamma_{l-1}^{(j)} := \beta_l^{(i)}/\beta_{l-1}^{(i)},\ l = 1, 2, \cdots, m_i. \tag{3.37}$$

This means that in the canonical Jordan form of T restricted to the subspace spanned by the principal vectors $\tilde{\psi}_1^{(j)}, \tilde{\psi}_2^{(j)}, \cdots, \tilde{\psi}_{m_j}^{(j)}$, the ones are to be replaced by the $\gamma_{l-1}^{(j)},\ l = 2, \cdots, m_i$.

Due to the above, one has the following lemma.

Lemma 3.5 (Biorthonormality relations for principal vectors)

Let the conditions (C1′) - (C4′) be fulfilled. Then, with the above notations,

$$\boxed{(\chi_k^{(i)}, \tilde{\upsilon}_l^{(j)}) = \delta_{kl}\delta_{ij},} \tag{3.38}$$

$k = 1, \cdots, m_i,\ l = 1, \cdots, m_j,\ i, j \in J$ *with*

$$\tilde{\upsilon}_l^{(j)} = \tilde{\psi}_{m_j-l+1}^{(j)} = \beta_{m_j-l+1}^{(j)}\, \psi_{m_j-l+1}^{(j)} = \beta_{m_j-l+1}^{(j)}\, \upsilon_l^{(j)}, \tag{3.39}$$

$l=1,\cdots,m_j,\ j\in J$ *as well as*

$$\beta^{(j)}_{m_j-l+1}=1/\overline{(\chi^{(j)}_l,\upsilon^{(j)}_l)}=1/\overline{(\chi^{(j)}_l,\psi_{m_j-l+1})}, \tag{3.40}$$

$l=1,\cdots,m_j,\ j\in J.$ ◇

At this point, we mention that

$$(u,v)=(e^{\varphi}u,e^{\varphi}v),\ u,v\in\mathbb{C}^n,\ 0\le\varphi<2\pi$$

which also applies to the pairs of vectors $u=\chi^{(i)}_k, v=\tilde{\upsilon}^{(j)}_l$ in (3.38).

Remark: We note that the matrix

$$((\chi^{(i)}_k,\tilde{\psi}^{(i)}_l))_{k,l=1,\cdots,m_i} \tag{3.41}$$

has the form

$$\begin{bmatrix} & & & 1\\ & & 1 & \\ & \vdots & & \\ 1 & & & \end{bmatrix} \tag{3.42}$$

which is called *cross-diagonal* in [12, p.3] and *anti-diagonal* by other authors. As opposed to this, the matrix $((\chi^{(i)}_k,\tilde{\upsilon}^{(i)}_l))_{k,l=1,\cdots,m_i}$, is equal to the identity matrix and thus *diagonal*. ◇

With Lemma 3.5, we can derive the next theorem that is an analogue to Theorem 3.2.

Theorem 3.6

Let the conditions (C1′) - (C4′) be fulfilled. Then,

$$Tu=\sum_{j\in J}\sum_{k=1}^{m_j}(u,\tilde{\upsilon}^{(j)}_k)[\lambda_j\chi^{(j)}_k+\chi^{(j)}_{k-1}],\ u\in H \tag{3.43}$$

as well as

$$Pu=\sum_{j\in J}\sum_{k=1}^{m_j}(u,\tilde{\upsilon}^{(j)}_k)\chi^{(j)}_k,\ u\in H. \tag{3.44}$$

Proof: Define

$$P^{(n)}u=\sum_{j=1}^{n}P_j=\sum_{j=1}^{n}P_{\lambda_j}(T). \tag{3.45}$$

Since $P^{(n)}(H)$ is finite-dimensional, Lemma 3.4 entails

$$P^{(n)}u=\sum_{j=1}^{n}\sum_{k=1}^{m_j}(u,\tilde{\upsilon}^{(j)}_k)\chi^{(j)}_k,\ u\in H. \tag{3.46}$$

This leads to

$$\begin{aligned} T^{(n)}u: &= TP^{(n)}u=\sum_{j=1}^{n}\sum_{k=1}^{m_j}(u,\tilde{\upsilon}^{(j)}_k)T\chi^{(j)}_k\\ &= \sum_{j=1}^{n}\sum_{k=1}^{m_j}(u,\tilde{\upsilon}^{(j)}_k)[\lambda_j\chi^{(j)}_k+\chi^{(j)}_{k-1}],\ u\in H. \end{aligned} \tag{3.47}$$

From this, it follows, based on Section 2,

$$P=\lim_{n\to\infty}P^{(n)} \tag{3.48}$$

as well as

$$T = \lim_{n\to\infty} T^{(n)} \tag{3.49}$$

in the Banach space $B(H)$. From (3.45) - (3.49), the relations (3.43) and (3.44) follow. ◇

Using (3.44), we obtain the next theorem.

Theorem 3.7

Let the conditions (C1′) - (C4′) be fulfilled. Then,

$$u = Pu = \sum_{j\in J}\sum_{k=1}^{m_j}(u,\tilde{\upsilon}_k^{(j)})\chi_k^{(j)},\; u \in H. \tag{3.50}$$

Proof: The proof is done in the same way as for Theorem 3.3 ◇

Remark: As in the case of simple eigenvalues of T, under the conditions $(C1')$ - $(C4')$ the relation $N(T)=\{0\}$ is equivalent to the property that $\lambda_0 = 0$ is not an eigenvalue of T which, in turn, is equivalent to $\overline{\lambda}_0 = 0$ is not an eigenvalue of T^* or that $N(T^*)=\{0\}$. ◇

Remark: If condition $(C4')$ is not fulfilled, this again can be remedied by a biorthonormalization pre-process described in [9, Theorem 4]. ◇

4. Series Expansions for a Densely Defined Linear Operator with Compact Inverse

The results on linear compact operators in Section 3 can be carried over to densely defined linear operators with compact inverse. The obtained expansions have important applications to BEVPs for ordinary and partial differential equations, where in Section 5, we restrict ourselves to BEVPs for ODEs. Again, it is natural to first handle the case of simple eigenvalues and then the case of general eigenvalues.

4.1 The Case of Simple Eigenvalues

In this subsection, in the case of simple eigenvalues, expansions in series of eigenvectors are treated.

It is structured as follows. We begin with the conditions on the densely defined linear operator L, its formally adjoint operator L_+ and their pertinent compact inverses G and G_+. Then, it is shown that $G_+ = G^*$ where G^* is the adjoint operator of G. Next, the expansions for Gu and Pu in series of eigenvectors are derived.

(i) The Conditions $(C1_d)$ - $(C5_d)$

We assume the following conditions:

$(C1_d)$ $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

$(C2_d)$ $\{0\} \neq H_D$ and H_R are pre-Hilbert spaces with

$$H_D \subset H_R \subset H,\; \overline{H}_D = \overline{H}_R = H$$

and where

$$L : D(L) := H_D \mapsto H_R$$

is a linear operator with the countably many *simple* non-zero eigenvalues $\mu_1,\mu_2,\mu_3,\cdots$ and the property $\lim_{j\to\infty}\mu_j = \infty$ as well as pertinent eigenvectors $\chi_1,\chi_2,\chi_3,\cdots \in H_D$. Further, L possesses a compact inverse

$$G := L^{-1} \in B(H)$$

$(C3_d)$ $\{0\} \neq H_{D,+}$ and H_R are pre-Hilbert spaces with

$$H_{D,+} \subset H_R \subset H,\; \overline{H}_{D,+} = \overline{H}_R = H$$

and where

$$L_+ : D(L_+) := H_{D,+} \mapsto H_R$$

is a linear operator with the countably many *simple* non-zero eigenvalues $\mu_{1,+},\mu_{2,+},\mu_{3,+},\cdots$ and the property $\lim_{j\to\infty}\mu_{j,+} = \infty$ as well as pertinent eigenvectors $\psi_1,\psi_2,\psi_3,\cdots \in H_{D,+}$. Further, L_+ possesses a compact inverse

$$G_+ := L_+^{-1} \in B(H)$$

$(C4_d)$ $(Lu,v) = (u,L_+v),\ u \in H_D,\ v \in H_{D,+}$

$(C5_d)$ $\mu_j \neq \mu_k,\ j \neq k,\ j,k \in J$

We mention that due to the above conditions, $0 \notin \sigma(G)$.

(ii) Series Expansions of Gu and Pu

The first theorem reads as follows.

Theorem 4.1

Let the conditions $(C1_d)$ - $(C5_d)$ be fulfilled. Then,

$$\mu_{j,+} = \overline{\mu}_j,\ j \in J \tag{4.1}$$

and

$$G_+ = G^* \tag{4.2}$$

where $G^* \in B(H)$ is the adjoint operator of G defined by

$$(Gu,v) = (u,G^*u),\ u,v \in H. \tag{4.3}$$

Further, the operator G has the eigenvalues $\lambda_j = 1/\mu_j$ as well as the eigenvectors χ_j, and $G_+ = G^*$ has the eigenvalues $\lambda_{j,+} = \overline{\lambda}_j = 1/\mu_{j,+} = 1/\overline{\mu}_j$ as well as the eigenvectors ψ_j for $j \in J$. In addition, $\lim_{j\to\infty} \lambda_j = 0$.

Proof: Let $\tilde{u},\ \tilde{v} \in H_R$ and

$$u := L^{-1}\tilde{u} = G\tilde{u}$$

as well as

$$v := L_+^{-1}\tilde{v} = G_+\tilde{v}.$$

Then,

$$u \in H_D,\ v \in H_{D,+}.$$

Substituting this in $(C4_d)$ gives

$$(\tilde{u},G_+\tilde{v}) = (G\tilde{u},\tilde{v}),\ \tilde{u},\ \tilde{v} \in H_R$$

or, with new denotations,

$$(u,G_+v) = (Gu,v),\ u,\ v \in H_R,$$

i.e.,

$$(Gu,v) = (u,G_+v),\ u,\ v \in H_R,$$

and thus, because of $\overline{H}_R = H$, also

$$(Gu,v) = (u,G_+v),\ u,\ v \in H.$$

On the other hand,

$$(Gu,v) = (u,G^*v),\ u,\ v \in H$$

and consequently

$$G_+ = G^*.$$

The rest of the proof is obtained in a simple way. ◇

From Theorem 4.1 and the results of Subsection 3.1, we obtain the following corollary.

Corollary 4.2
Let the conditions (C1_d) - (C5_d) be fulfilled. Then,

$$Gu = \sum_{j\in J} \lambda_j (u, \psi_j)\chi_j,\ u \in H, \tag{4.4}$$

$$u = Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H. \tag{4.5}$$

Proof: Because of

$$G\chi_j = \lambda_j \chi_j$$

and

$$G^* \psi_j = G_+ \psi_j = \lambda_{j,+} \psi_j = \overline{\lambda}_j \psi_j,$$

$j \in J$, from Section 3.1 we obtain the relations (4.4) and (4.5). $\diamond$

4.2 The Case of General Eigenvalues

In this subsection, we do not assume that the eigenvalues of L be simple. Then, we obtain expansions in a series of principal vectors.

This subsection is organized in a similar way as the preceding one.

So, first the conditions on the densely defined linear operator L, its formally adjoint operator L_+ and their compact inverses G and G_+ are stated. Next, the expansions of Gu and Pu in series of principal vectors are derived.

(i) The Conditions (C$1'_d$) - (C$5'_d$)
We assume the following conditions:

($C1'_d$) $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

($C2'_d$) $\{0\} \neq H_D$ and H_R are pre-Hilbert spaces with

$$H_D \subset H_R \subset H,\ \overline{H}_D = \overline{H}_R = H$$

and where

$$L : D(L) := H_D \mapsto H_R$$

is a linear operator with the countably many *general* non-zero eigenvalues $\mu_1, \mu_2, \mu_3, \cdots$ and the property $\lim_{j\to\infty} \mu_j = \infty$ as well as pertinent principal vectors $\chi_1^{(j)}, \chi_2^{(j)}, \cdots \chi_{m_j}^{(j)} \in H_D$ $j \in J$, where $\chi_i^{(j)}$ is of stage i. Further, L possesses a compact inverse

$$G := L^{-1} \in B(H)$$

($C3'_d$) $\{0\} \neq H_{D,+}$ and H_R are pre-Hilbert spaces with

$$H_{D,+} \subset H_R \subset H,\ \overline{H}_{D,+} = \overline{H}_R = H$$

and where

$$L_+ : D(L_+) := H_{D,+} \mapsto H_R$$

is a linear operator with the countably many *general* non-zero eigenvalues $\mu_{1,+}, \mu_{2,+}, \mu_{3,+}, \cdots$ and the property $\lim_{j\to\infty} \mu_{j,+} = \infty$ as well as pertinent principal vectors $\psi_1^{(j)}, \psi_2^{(j)}, \cdots \psi_{m_j}^{(j)} \in H_{D,+}$ $j \in J$. Further, L_+ possesses a compact inverse

$$G_+ := L_+^{-1} \in B(H)$$

$(C4'_d)$ $(Lu,v) = (u,L_+v)$, $u \in H_D$, $v \in H_{D,+}$

$(C5'_d)$ $\mu_j \neq \mu_k$, $j \neq k$, $j,k \in J$

Again, we mention that due to the above conditions, $0 \notin \sigma(G)$.

(ii) Series Expansions of Gu and Pu

The next theorem reads as follows.

Theorem 4.3

Let the conditions $(C1'_d)$ - $(C5'_d)$ instead of $(C1_d)$ - $(C5_d)$ be fulfilled. Then, the relations (4.1) - (4.3) as well as $\lim_{j\to\infty}\lambda_j = 0$ *of Theorem 4.1 hold.*

Proof: The proof of Theorem 4.3 is the same as for Theorem 4.1 since it does not depend on the condition that the eigenvalues be simple. $\diamond$

From Theorem 4.3 and the results of Subsection 3.2, we obtain the following corollary.

Corollary 4.4

Let the conditions $(C1'_d)$ - $(C5'_d)$ be fulfilled. Then,

$$Gu = \sum_{j\in J}\sum_{k=1}^{m_j}(u,\tilde{v}_k^{(j)})[\lambda_j\chi_k^{(j)} + \chi_{k-1}^{(j)}],\ u \in H, \tag{4.6}$$

$$u = Pu = \sum_{j\in J}\sum_{k=1}^{m_j}(u,\tilde{v}_k^{(j)})\chi_k^{(j)},\ u \in H. \tag{4.7}$$

$\diamond$

5. Application to a General Non-Selfadjoint BEVP with Ordinary Differential Operator of nth Order

In this section, we apply the results of Section 4 to a general non-selfadjoint BEVP for an ordinary differential operator L of nth order. In doing so, we not only obtain the expansion (1.7) for simple eigenvalues, but also, in addition, those for Pu and Gu in series of eigenfunctions, and further those for general eigenvalues in series of principal functions, which is much more than what is obtained in [2] before.

We mention that this section contains a series of verbatim and almost verbatim passages from [2, Chapter 11].

Now, the details follow.

Let $a \le x \le b$ be a closed bounded interval, and let L be the linear differential operator of nth order with $n \ge 1$ defined by

$$(Lu)(x) := a_n(x)u^{(n)}(x) + a_{n-1}(x)u^{(n-1)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \tag{5.1}$$

where a_k are complex-valued functions of class $C^k[a,b]$ and $a_n(x) \neq 0$ on $[a,b]$. Given any set of $2mn$ complex constants α_{ij}, β_{ij}, $i = 1,2,\cdots,m$, $j = 0,1,\cdots,n-1$, define the m *boundary operators* or *boundary forms* $R_1,\cdots,R_m$ for the functions u on $[a,b]$, for which $u^{(j)}$, $j = 1,2,\cdots,n$ exist at a and b by

$$R_iu := \sum_{j=0}^{n-1}\{\alpha_{ij}u^{(j)}(a) + \beta_{ij}u^{(j)}(b)\} = 0,\ i = 1,2,\cdots,m \tag{5.2}$$

$\Longleftrightarrow$

$$Ru = 0. \tag{5.3}$$

We suppose that R has rank m. Corresponding to any homogeneous boundary value problem (for short: BVP) is a well-defined *"adjoint" problem* (which should better be called *formally adjoint problem*) with the *Lagrange "adjoint operator"* given by

$$\begin{aligned}(L_+v)(x) = (-1)^n(\overline{a}_n(x)v)^{(n)}(x) + (-1)^{n-1}(\overline{a}_{n-1}(x)v)^{(n-1)}(x) + \cdots \\ +(-1)(\overline{a}_1(x)v)'(x) + \overline{a}_0(x)v(x)\end{aligned} \tag{5.4}$$

and a set of adjoint boundary conditions

$$R_+v = 0 \tag{5.5}$$

complementary in a sense to those for the problem pertinent L.

We mention that some authors denote the formally adjoint operator by L^*, see for instance [10]. But, we do not follow this usage since this paper is functional-analysis-oriented and since L^* could be misinterpreted as the adjoint of a densely defined linear operator L, see [1, No.44]. Instead, as in [2], we use a plus sign to denote the formally adjoint operator, here as a subscript instead of a superscript there.

We note that an adjoint boundary condition is not unique, see [2, Theorem 2.1].

Now, we define the pre-Hilbert spaces

$$H_D := D(L) := \{u \in C^n[a,b] \,|\, Ru = 0\} \tag{5.6}$$

and

$$H_{D,+} := D(L_+) := \{v \in C^n[a,b] \,|\, R_+ v = 0\}. \tag{5.7}$$

Then,

$$(Lu, v) = (u, L_+ v),\ u \in H_D,\ v \in H_{D,+}. \tag{5.8}$$

We mention that

$$C_0^\infty[a,b] \subset H_D \subset H_R := C_2[a,b] \subset L_2(a,b) =: H \tag{5.9}$$

where $C_2[a,b]$ is the function space $C[a,b]$ endowed with the norm

$$\|u\|_2 = \left(\int_a^b |u(x)|^2 dx\right)^{\frac{1}{2}}, \tag{5.10}$$

and where the integral is taken in the sense of Riemann which is equal to the Lebesgue integral for $u \in C_2[a,b]$. The space $L_2(a,b)$ is the space of measurable functions such that the above integral (taken in the sense of Lebesgue) is finite.

Corresponding to (5.9), one has

$$C_0^\infty[a,b] \subset H_{D,+} \subset H_R = C_2[a,b] \subset L_2(a,b) = H. \tag{5.11}$$

It is known that

$$\overline{C_0^\infty[a,b]} = L_2(a,b).$$

If R is a boundary form of rank m, the problem

$$\pi_m:\ Lu = 0,\ u \in H_D = D(L) \tag{5.12}$$

is called a *homogeneous BVP of rank m*.

The problem

$$\pi_{2n-m,+}:\ L_+ v = 0,\ v \in H_{D,+} = D(L_+) \tag{5.13}$$

is called the *adjoint BVP*.

One has the following:

π_n and $\pi_{n,+}$ have the same number of independent solutions. see [2, p.293, last line].

The BEVP pertinent to π_n is given by

$$\pi_{n,\mu}:\ Lu = \mu u,\ u \in H_D = D(L) \tag{5.14}$$

and that associated with $\pi_{n,+}$ by

$$\pi_{n,\overline{\mu},+}:\ L_+ v = \overline{\mu} v,\ v \in H_{D,+} = D(L_+). \tag{5.15}$$

Now, let $G(x,s)$ be the Green's function pertinent to the BVP π_n and $G_+(x,s)$ the Green's function associated with $\pi_{n,+}$. Then, the pertinent compact operators $G = L^{-1}$ and $G_+ = L_+^{-1}$ are given by

$$(Gu)(x) = \int_a^b G(x,s)u(s)ds,\ u \in L_2(a,b) \tag{5.16}$$

and

$$(G_+v)(x) = \int_a^b G_+(x,s)v(s)ds,\ v \in L_2(a,b) \tag{5.17}$$

where

$$G_+(x,s) = \overline{G(s,x)},\ x,s \in [a,b], \tag{5.18}$$

see [2, (4.15)] implying for the pertinent operators G in (5.15) and G_+ in (5.17) the relations

$$G_+ = G^*. \tag{5.19}$$

If the conditions $(C1_d)$ - $(C5_d)$ for L in (5.1) and for L_+ in (5.4) are fulfilled, then (5.19) follows also from the abstract results of Section 4, and beyond this, one obtains also the expansions in series of eigenvectors (4.4) and (4.5) in Corollary 4.2 with convergence in the norm $\|\cdot\|_2$, whereas in [2, Chapter 12,(5.6)] only the relation (4.5), i.e.,

$$u = \sum_{j=1}^{\infty}(u,\psi_j)\chi_j,\ u \in H = L_2(a,b)$$

is given.

Beyond this, if the conditions $(C1_d')$ - $(C5_d')$ are fulfilled, then the expansions in series of principal vectors (4.6) and (4.7) are valid in the norm $\|\cdot\|_2$. This case when the eigenvalues are general is not treated in [2] and means a considerable progress in the theory of non-selfadjoint BEVPs.

6. The Case of a Non-Selfadjoint BEVP of 2nd Order

In this section, we further specialize the BEVP discussed in Section 5 by restricting the order of L to $n = 2$ and by employing very simple boundary values. The considered problem is often used as an example in books on Mathematical Physics and is treated there in a special weighted norm. But when it comes to specific examples, the term with the first derivative usually is omitted so that one obtains a selfadjoint problem. Here, we keep this term, and so we get a non-selfadjoint problem of 2nd order.

This section is split up in two subsections.

In Subsection 6.1, the BEVP of 2nd order with real continuous coefficients is established. It goes without saying that the series expansions obtained in Section 5 are valid if the corresponding conditions are fulfilled.

In Subsection 6.2, we further specialize the BEVP of 2nd order to the case when the coefficients are constant. Then, it is possible to explicitly determine the eigenvalues, biorthonormal eigenfunctions, and the Green's functions defining the inverse operators G of L and G_+ of L_+.

6.1 The BEVP of 2nd Order with Real Continuous Coefficients

As a special case of the general differential operator of nth order in Section 5, in this subsection we consider the differential operator of 2nd order

$$Lu(x) := a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x),\ 0 \le x \le l \tag{6.1}$$

with real functions $a_i \in C^i[0,l],\ i = 0,1,2$ and the boundary conditions

$$Ru = 0 \iff u(0) = u(l) = 0, \tag{6.2}$$

cf. e.g., [11, §75, p.362] where $a_2(x) = -1,\ a_1(x) = p(x),\ a_0(x) = q(x),\ l = 1$.

We mention that we have chosen here the interval $[0,l]$ since, in applications to mechanical problems, l means a length.

The formally adjoint operator L_+ reads

$$L_+v(x) := (a_2(x)v)''(x) - (a_1(x)v)'(x) + a_0(x)v(x),\ 0 \le x \le l. \tag{6.3}$$

As adjoint boundary condition, we choose

$$R_+v = 0 \iff v(0) = v(l) = 0 \tag{6.4}$$

so that $R_+ v = Ru = 0$ holds. Here, we have

$$H_D = \{u \in C^2[0,l] \,|\, u(0) = u(l) = 0\} = H_{D,+} \tag{6.5}$$

and

$$H_R = C_2[0,l] \tag{6.6}$$

as well as

$$H = L_2(0,l). \tag{6.7}$$

Herewith,

$$(Lu,v) = (u, L_+ v),\ u \in H_D,\ v \in H_{D,+} \tag{6.8}$$

so that condition $(C4_d)$ is fulfilled.

We further suppose that the differential operator L in (6.1) has a countable set of simple non-zero eigenvalues $\mu_1, \mu_2, \mu_3, \cdots$ with $\lim_{j\to\infty} \mu_j = \infty$. Then, the conditions $(C1_d)$ - $(C5_d)$ are fulfilled, and one has the expansions in series of eigenfunctions (4.4) and (4.5).

6.2 The Special Case of Constant Coefficients

In this subsection, we treat the BEVP of Subsection 6.1 when $a_2(x) = -1,\ a_1(x) = p(x) = p_0,\ a_0(x) = q(x) = q_0$ are constant in the interval $[0,l]$, that is, when $Lu = -u'' + p_0\, u' + q_0\, u$ and thus $L_+ v = -v'' - p_0\, v' + q_0\, v$.

In this special case, it is possible to explicitly determine the eigenvalues μ_j of L resp. μ_j of L_+ and the pertinent eigenfunctions χ_j resp. ψ_j, as the case may be. Further, the Green's functions $G(x,s;p_0,q_0)$ and $G_+(x,s;p_0,q_0)$ defining the inverse compact operators G and $G_+ = G^T$ are explicitly determined. As far as the author is aware, these results have not been obtained, before.

For the sake of brevity, the details of the derivation of these quantities are left to the reader. However, we give some hints for obtaining these results.

(i) The Differential Operators L and L_+ and Pertinent BEVPs

As already announced, in this subsection, we choose constant coefficients in the differential operator L. More precisely, we set

$$a_2(x) = -1,\ a_1(x) = p(x) = p_0,\ a_0(x) = q(x) = q_0 \tag{6.9}$$

with real constants p_0 and q_0 so that

$$(Lu)(x) = -u''(x) + p_0 u'(x) + q_0 u(x),\ 0 \le x \le l \tag{6.10}$$

and

$$(L_+ v)(x) = -v''(x) - p_0 v'(x) + q_0 v(x),\ 0 \le x \le l \tag{6.11}$$

with the same boundary conditions (6.2) and (6.4) as in Subsection 6.1.

We restrict the constant q_0 to $q_0 > 0$.

The pertinent BEVPs read

$$\pi_{2,\mu}:\ Lu = \mu u,\ u \in H_D = D(L) \tag{6.12}$$

and that associated with $\pi_{2,+}$ by

$$\pi_{2,\overline{\mu},+}:\ L_+ v = \overline{\mu} v,\ v \in H_{D,+} = D(L_+). \tag{6.13}$$

(ii) The Eigenvalues and Eigenfunctions

The eigenvalues of L and L_+ are given by

$$\mu = \overline{\mu} = \mu_j = \overline{\mu}_j = \frac{j^2\pi^2}{l^2} + D,\ j \in J \tag{6.14}$$

with the quantity

$$D = D(p_0, q_0) = (\frac{p_0}{2})^2 + q_0 > 0 \tag{6.15}$$

so that

$$\lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \frac{1}{\mu_j} = 0 \tag{6.16}$$

is fulfilled.

The biorthonormal eigenfunctions are found to be

$$\chi_j(x) = \sqrt{\frac{2}{l}} \exp(\frac{p_0}{2}x) \sin j\pi \frac{x}{l},\ 0 \le x \le l,\ j \in J \tag{6.17}$$

and

$$\psi_k(x) = \sqrt{\frac{2}{l}} \exp(-\frac{p_0}{2}x) \sin k\pi \frac{x}{l},\ 0 \le x \le l,\ k \in J \tag{6.18}$$

so that we have

$$(\chi_j, \psi_k) = \int_0^l \chi_j(x)\, \psi_k(x)\, dx = \frac{2}{l} \int_0^l \sin(j\pi \frac{x}{l}) \sin(k\pi \frac{x}{l})\, dx = \delta_{jk},\ j,k \in J. \tag{6.19}$$

Hint: To derive these results, use the ansatz $u(x) = ce^{\kappa x}$ in order to solve the BEVP

$$L_{p_0,q_0} u = \mu u,\ u(0) = u(l) = 0. \tag{6.20}$$

The eigenfunctions $\psi_j(x)$ are obtained from $\chi_j(x)$ by just replacing p_0 by $-p_0$. ◇

(iii) The Green's Function of $L_{p_0,q_0} u = 0,\ u(0) = u(l) = 0$

A set of fundamental solutions of the BVP $L_{p_0,q_0} u = 0,\ u(0) = u(l) = 0$, i.e., when $\mu = 0$, is given by

$$u_1(x) = \exp\left(\frac{p_0}{2}x\right) \sinh \sqrt{D}x,\ 0 \le x \le l, \tag{6.21}$$

$$u_2(x) = \exp\left(\frac{p_0}{2}x\right) \cosh \sqrt{D}x,\ 0 \le x \le l, \tag{6.22}$$

with

$$D = D(p_0, q_0) = (\frac{p_0}{2})^2 + q_0 \tag{6.23}$$

which is also obtained with the ansatz $u(x) = ce^{\kappa x}$ by setting $c = 1$ and taking into account $\mu = 0$ where here D is a discriminant.

Based on these fundamental solutions, we have calculated the Green's functions by the method described in [10, pp.311]. Thus, one gets

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{\sinh\sqrt{D}x\ \sinh\sqrt{D}(l-s)}{\sqrt{D}\sinh\sqrt{D}l} \exp\left(\dfrac{p_0}{2}(x-s)\right),\ 0 \le x \le s \le l, \\[2ex] G_2(x,s) = \dfrac{\sinh\sqrt{D}(l-x)\ \sinh\sqrt{D}s}{\sqrt{D}\sinh\sqrt{D}l} \exp\left(\dfrac{p_0}{2}(x-s)\right)\ 0 \le s \le x \le l. \end{cases} \tag{6.24}$$

For $G_+(x,s)$, we obtain

$$G_+(x,s) = \begin{cases} G_{+,1}(x,s) = \dfrac{\sinh\sqrt{D}(l-x)\ \sinh\sqrt{D}s}{\sqrt{D}\sinh\sqrt{D}l} \exp\left(\dfrac{p_0}{2}(s-x)\right),\ 0 \le x \le s \le l, \\[2ex] G_{+,2}(x,s) = \dfrac{\sinh\sqrt{D}x\ \sinh\sqrt{D}(l-s)}{\sqrt{D}\sinh\sqrt{D}l} \exp\left(\dfrac{p_0}{2}(s-x)\right)\ 0 \le s \le x \le l. \end{cases} \tag{6.25}$$

so that, because of $D = D(p_0,q_0)$,

$$G(x,s) = G(x,s;p_0,q_0) \tag{6.26}$$

and

$$G_+(x,s) = G^T(x,s) = G(s,x) = G(s,x;-p_0,q_0) \tag{6.27}$$

in accordance with the fact that, for the pertinent operators, one has $G_+ = G^T$.

7. Comparison of Present Expansion Results with Known Ones in an Abstract Hilbert Space

The oldest expansion result for compact operators in an abstract Hilbert space being of formal similarity to our results the author found is that in [1, Section 64, pp.172-174]. There, under certain conditions, the expansions of the form

$$h = h_0 + \sum_{j\in J}(h,e_j)e_j, \quad h \in H \tag{7.1}$$

with an element $h_0 \in H_0 := N(T)$ as well as

$$Th = \sum_{j\in J}\mu_j(h,e_j)g_j, \quad h \in H \tag{7.2}$$

can be found. Here, the vectors e_j are the pairwise orthonormal eigenvectors of $A := T^*T$. The associated eigenvalues λ_j can be written in the form

$$\lambda_j = (Ae_j,e_j) = (T^*Te_j,e_j) = (Te_j,Te_j) > 0. \tag{7.3}$$

Therefore, one has $\lambda_j = \mu_j^2$, where $\mu_1 \geq \mu_2 \geq \cdots > 0$.

The vectors g_j are defined by

$$Te_j = \mu_j g_j \tag{7.4}$$

leading to

$$(g_j,g_k) = \delta_{jk}. \tag{7.5}$$

Applying T to (7.1) and using (7.4), we obtain (7.2).

As opposed to this, our result is an expansion in series of eigenvalues and eigenvectors/principal vectors of the compact operator T itself whereas in [1] one has an expansion in series of eigenvalues $\mu_j = \mu_j(T^*T)$ and eigenvectors $e_j = e_j(T^*T)$ of T^*T and the vectors g_j defined in (7.4) that are left singular vectors in the denotation of [3, p.2].

The most recent publication on expansions of a compact operator in an abstract Hilbert space the author has found is [3]. There, it is used that the singular values and singular vectors of T are related to the nonzero eigenvalues and corresponding eigenvectors of T^*T and TT^*. More precisely, one has

$$T\phi_k = \sigma_k\psi_k, \tag{7.6}$$

$$T^*T\phi_k = \sigma_k^2\phi_k, \tag{7.7}$$

$$TT^*\psi_k = \sigma_k^2\psi_k. \tag{7.8}$$

The quantities σ_k are called *singular values*, the vectors ϕ_k are called *right singular vectors* and ψ_k *left singular vectors* in [3, p.2]. Herewith, it is proven that the expansion

$$T = \sum_{k=1}^{\infty}\sigma_k\psi_k\otimes\phi_k \tag{7.9}$$

is valid in $B(H)$. The difference to the present paper is that, in [3], the expansion **is not** in eigenvalues and eigenvectors/principal vectors of the compact operator T itself.

8. Conclusions

In this paper it is shown that expansions in series of eigenvectors valid for symmetric linear compact operators and symmetric densely defined linear operators with compact inverse can be carried over to corresponding nonsymmetric operators where, in the case of general eigenvalues, the expansions are in series of principal vectors. These results are all new and mean a considerable progress in the Spectral Analysis of Nonsymmetric Linear Compact Operators in a Hilbert Space. The expansions discussed in Section 7 are not in series of eigenvectors resp. principal vectors and thus are different from ours. Further, in Natural Sciences and Engineering, expansions in series of eigenvectors and principal vectors are of particular importance. Our results are applicable to general non-selfadjoint BEVPs pertinent to an ordinary differential operator of nth order and deliver even there new results when the eigenvalues are general, that is, not necessarily simple. In a special example of a differential operator of 2nd order with constant coefficients, the eigenvalues, eigenfunctions and the Green's functions are explicitly determined which also seems to be new, as far as the author is aware.

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Reachability Results in Plane Trees

Albert Oloo Nyariaro[1], Isaac Owino Okoth[2]*

Abstract
In this paper, we obtain closed formulas for the number of reachable vertices in labelled plane trees by paths lengths, sinks, leaf sinks, first children, left most path, non-first children, and non-leaves. Our counting objects are plane trees having their edges oriented from a vertex of lower label towards a vertex of higher label. For each statistic, we obtain the average number of reachable vertices. Moreover, we obtain a counting formula for the number of plane trees on n vertices such that exactly $k \leq n$ are reachable from the root.

Keywords: Plane trees, Reachability, Leaf, Sink, First child
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[1] *Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya, ORCID: 0000-0002-2997-0241*
[2] *Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya, ORCID: 0000-0003-4503-4733*
***Corresponding author**: ookoth@maseno.ac.ke
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1. Introduction

Plane trees (or ordered trees) have been studied extensively in the literature. These trees on n vertices are counted by the $(n-1)^{th}$ Catalan number

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

So if the vertices are labelled with labels $1, 2, \ldots, n$ then the total number of these trees is $n!C_{n-1}$. A vertex j is *reachable* from a vertex i if there is a sequence of oriented edges (paths) from vertex i to vertex j, and a path is of length ℓ if there are ℓ edges on the path. In this context, *degree* of a vertex is the number of edges that come out of a vertex if the edges are oriented away from the root. A vertex in which there is no edge that is oriented away from it is called a *sink* whereas a *leaf sink* is a vertex with only one edge oriented towards it but no edge oriented away from it. The vertices with the same parent are called *siblings*. Since the siblings are linearly ordered, they are always drawn in a left-to-right pattern where the leftmost sibling is referred to as *first child*. At a given level ℓ, the left most child is the *eldest child*. A *left most path* refers to a sequence of edges joining eldest children at each level in a plane tree. In this work, we examine the number of reachable vertices from a given root i. We also determine a formula for the number of labelled ordered trees on n vertices such that exactly k vertices are reachable from the root. Plane trees considered here have their edges oriented from a vertex of lower label towards a vertex of higher label. This orientation was introduced in [1]. Equivalent results for t-ary trees have been obtained by the present authors in [3]. We will now refer to these plane trees simply as *trees*. In Section 2, we use path lengths to count the trees. The number of sinks and leaf sinks are the statistics used in Section 3 while in Section 4, we use left most paths and first children. We enumerate trees by non-first children and non-leaves in Section 5. We use mainly generating functions and Lagrange Inversion Formula [5] to prove our results. In most cases, we give asymptotic results as well. Lastly, in Section 6 we give a bijective proof of a formula for the number of trees in which a given number of vertices is reachable from the root.

2. Enumeration by path lengths

In the sequel, we enumerate trees by path lengths. We start by proving the main result of this section.

Theorem 2.1. *The number of trees on n vertices rooted at vertex i such that vertex j of degree d is reachable from the root in ℓ steps is given by*

$$(n-\ell-1)!\frac{2\ell+d}{n+\ell-1}\binom{j-i-1}{\ell-1}\binom{2n-d-3}{n+\ell-2}.$$

Proof. Let $P(x)$ be the generating function for plane trees where x marks the number of non-root vertices. Consider a plane tree rooted at vertex i such that there is a path of length ℓ starting at vertex i and ending at vertex $i+\ell$ of degree d. This path decomposes the tree into left and right plane subtrees upto length ℓ. Thus $(P(x)xP(x))^{\ell}$ is the generating function for the number of the trees with a path of length ℓ starting at i and ending at $i+\ell$. Vertex $i+\ell$ is joined to d other vertices which are connected to other plane trees hence we have $x(xP(x))^{d}$, to represent the vertex and the subtrees of its children. Putting everything together, we obtain $(P(x)xP(x))^{\ell}x(xP(x))^{d}=x^{\ell+d+1}P(x)^{2\ell+d}$ as the generating function of the unlabelled plane tree rooted at vertex i with a path of length ℓ starting at i and ending at vertex $i+\ell$ of degree d. The decomposition is represented by Figure 2.1.

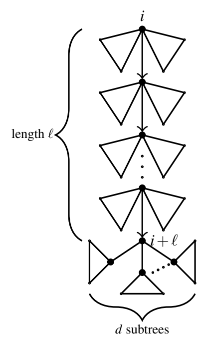

Figure 2.1. Unlabelled plane tree with path length ℓ.

The generating function for unlabelled plane trees is $P(x)=\frac{1}{1-P(x)}$. We let $xP(x)=F(x)$ so that $F(x)=\frac{x}{1-F(x)}$. Applying Lagrange Inversion Formula, we get

$$\begin{aligned}
[x^n]x^{\ell+d+1}P(x)^{2\ell+d} &= [x^n]x^{-\ell+1}F(x)^{2\ell+d}\\
&= [x^{n+\ell-1}]F(x)^{2\ell+d}\\
&= \frac{2\ell+d}{n+\ell-1}[t^{n-\ell-d-1}]\,(1-t)^{-(n+\ell-1)}\\
&= \frac{2\ell+d}{n+\ell-1}[t^{n-\ell-d-1}]\sum_{i\geq 0}\binom{-(n+\ell-1)}{i}(-t)^{i}\\
&= \frac{2\ell+d}{n+\ell-1}[t^{n-\ell-d-1}]\sum_{i\geq 0}\binom{n+\ell+i-2}{i}t^{i}\\
&= \frac{2\ell+d}{n+\ell-1}\binom{2n-d-3}{n-\ell-d-1}.
\end{aligned}$$

This formula counts the number of unlabelled plane trees in which vertex $i+\ell$ of degree d is reachable from the root in ℓ steps. The number of ways of choosing a path of length ℓ from vertex i to vertex j is $\binom{j-i-1}{\ell-1}$. Once the $\ell+1$ vertices on the path are labelled, there are $(n-\ell-1)!$ choices for labelling the remaining vertices. Thus, the number of plane trees in which vertex j of degree d is reachable from root i in ℓ steps is given by

$$(n-\ell-1)!\frac{2\ell+d}{n+\ell-1}\binom{j-i-1}{\ell-1}\binom{2n-d-3}{n+\ell-2}.$$

This completes the proof. $\square$

Setting $\ell = 0$ in the just proved theorem, we get that there are

$$(n-1)!\frac{d}{n-1}\binom{2n-d-3}{n-2} = d(n-2)!\binom{2n-d-3}{n-2}$$

trees in which a root of any label has degree d.

Quite a number of corollaries of Theorem 2.1 follow:

Corollary 2.2. *The total number of trees on n vertices rooted at vertex i such that vertex j is reachable from the root in ℓ steps is given by:*

$$(n-\ell-1)!\frac{2\ell+1}{2n-1}\binom{j-i-1}{\ell-1}\binom{2n-1}{n+\ell}. \tag{2.1}$$

Proof. The result follows by summing over all d in Theorem 2.1. □

By summing over all j in Equation (2.1), we get that

Corollary 2.3. *The number of vertices in trees of order n that are reachable from root i in ℓ steps is given by*

$$(n-\ell-1)!\frac{2\ell+1}{2n-1}\binom{n-i}{\ell}\binom{2n-1}{n+\ell}. \tag{2.2}$$

Moreover, summing over all i in Equation (2.2), we get the total number of reachable vertices:

Corollary 2.4. *There are a total of*

$$(n-\ell-1)!\frac{2\ell+1}{2n-1}\binom{n}{\ell+1}\binom{2n-1}{n+\ell} \tag{2.3}$$

vertices that are reachable from the root in ℓ steps, in trees with n vertices.

Now, summing over all ℓ in Equation (2.3) we obtain

Corollary 2.5. *The total number of vertices in trees on n vertices that are reachable from the root is given by*

$$\frac{n!}{2n-1}\sum_{\ell=0}^{n-1}\frac{2\ell+1}{(\ell+1)!}\binom{2n-1}{n+\ell}.$$

Corollary 2.6. *On average, the number of vertices that are reachable from the root in ℓ steps in a random tree is $\frac{2\ell+1}{(\ell+1)!}$.*

Proof. Dividing the total number of vertices that are reachable from the root in ℓ steps in plane trees (See Equation (2.3)) by the total number of labelled plane trees, we get

$$A = \frac{(n-\ell-1)!\frac{2\ell+1}{2n-1}\binom{n}{\ell+1}\binom{2n-1}{n-\ell-1}}{(n-1)!\binom{2n-2}{n-1}}$$

as the average number of vertices that are reachable in ℓ steps from the root in trees with n vertices. We simplify the average to get

$$A = \frac{2\ell+1}{(l+1)!}\frac{(n-1)!n!}{(n+\ell)!(n-\ell-1)!}.$$

Now, taking limits as $n \to \infty$, we get

$$\begin{aligned}
\lim_{n\to\infty} A &= \frac{2\ell+1}{(\ell+1)!}\lim_{n\to\infty}\frac{(n-1)!n!}{(n+\ell)!(n-\ell-1)!} \\
&= \frac{2\ell+1}{(\ell+1)!}\lim_{n\to\infty}\frac{(n-1)(n-2)(n-3)\cdots(n-\ell)}{(n+\ell)(n+\ell-1)\cdots(n+1)} \\
&= \frac{2\ell+1}{(\ell+1)!}\lim_{n\to\infty}\left(\frac{n^\ell+\cdots}{n^\ell+\cdots}\right) \\
&= \frac{2\ell+1}{(\ell+1)!}.
\end{aligned}$$

Hence the desired result. □

Corollary 2.7. *The number of trees of order n in which there is a path of length ℓ starting at the root and ending at a vertex of degree d is given by*

$$(n-\ell-1)!\frac{2\ell+d}{n+\ell-1}\binom{n}{\ell+1}\binom{2n-d-3}{n+\ell-2}. \tag{2.4}$$

Proof. We obtain the result by summing over all i and j in Theorem 2.1. □

Setting $\ell=0$ in Equation (2.4), we get

$$\frac{n!d}{n-1}\binom{2n-d-3}{n-2},$$

as the formula which counts the number of trees on n vertices such that the root is of degree d. Also setting $\ell=1$ in Equation (2.4), we obtain

$$\frac{(d+2)(n-1)!}{2}\binom{2n-d-3}{n-1}.$$

This formula counts the total number of children of degree d, in all trees of order n.

3. Enumeration by sinks and leaf sinks

In this section, we enumerate trees with respect to sinks and leaf sinks.

Proposition 3.1. *The number of trees of order n in which vertex j, a sink of degree d, is reachable from a root i in ℓ steps is given by*

$$(n-\ell-d-1)!\frac{2\ell+d}{n+\ell-1}\binom{j-i-1}{\ell-1}\binom{j-\ell-1}{d}\binom{2n-d-3}{n+\ell-2}. \tag{3.1}$$

Proof. From the proof of Theorem 2.1, it follows that there are

$$\frac{2\ell+d}{n+\ell-1}\binom{2n-d-3}{n+\ell-2}$$

unlabelled trees with a path of length ℓ starting at a root i and terminating at vertex $i+\ell$ of degree d. Now, consider a path of length ℓ starting at root i and ending at vertex j. There are $\binom{j-i-1}{\ell-1}$ such paths. Since vertex j is a sink of degree d, the labels of the d vertices must be less than j. Thus there are $\binom{j-i-1}{d}$ choices for the labels. Once the $\ell+1$ vertices on the path and the d children of j are labelled, there are $(n-\ell-d-1)!$ choices for the other labels in the tree. Collecting everything, we arrive at the required formula. □

As seen in the previous section, a number of corollaries follow. We obtain the following result by summing over all j in Equation (3.1).

Corollary 3.2. *The total number of sinks of degree d that are reachable, in ℓ steps, from the root i in trees with n vertices is given by*

$$(n-\ell-d-1)!\frac{2\ell+d}{n+\ell-1}\sum_{j=\ell+i}^{n}\binom{j-i-1}{\ell-1}\binom{j-\ell-1}{d}\binom{2n-d-3}{n+\ell-2}.$$

Moreover, setting $\ell=0$ and $j=i$ in Equation (3.1) we obtain that the total number of trees with n vertices such that root i is a sink of degree d is given by

$$(n-d-1)!\frac{d}{n-1}\binom{i-1}{d}\binom{2n-d-3}{n-2}, \tag{3.2}$$

and also by setting setting $\ell=1$ in Equation (3.1), we get that there are

$$(n-d-2)!\frac{2+d}{n}\binom{j-2}{d}\binom{2n-d-3}{n-1} \tag{3.3}$$

children of the root labelled j having degree d in trees on n vertices.

Summing over all j in Equation (3.3) we obtain the number of children of the root, which are also sinks of degree d, in trees of order n. By summing over all i in Equation (3.2), we obtain

Corollary 3.3. *The total number of trees of order n with root sinks of degree d is given by:*

$$(n-d-1)!\frac{d}{n-1}\binom{n}{d+1}\binom{2n-d-3}{n-2}. \tag{3.4}$$

Next, we find an asymptotic result:

Corollary 3.4. *On average there are*

$$\frac{d}{2^{d+1}(d+1)!} \tag{3.5}$$

root sinks of degree d in a random tree.

Proof. Diving the total number of labelled plane trees of order n with root sinks of degree d (Equation (3.4)), by the total number of labelled plane trees we get,

$$\frac{(n-d-1)!\frac{d}{n-1}\binom{n}{d+1}\binom{2n-d-3}{n-2}}{(n-1)!\binom{2n-2}{n-1}},$$

as the average number of root sinks of degree d in a random plane tree on n vertices. Simplifying the average and tending n to infinity, we obtain the required result. ☐

Setting $d=0$ in Equation (3.5) we get that the average number of root sinks of degree 0 is zero. This implies that there is no leaf sink which is also a root. For the remainder of this section, we enumerate the trees by leaf sinks.

Proposition 3.5. *The total number of trees of order n in which vertex j, a leaf sink, is reachable from a root i in ℓ steps is given by the formula,*

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{j-i-1}{\ell-1}\binom{2n-2}{n+\ell-1}. \tag{3.6}$$

Proof. The result follows by setting $d=0$ in Proposition 3.1. However, to show the decomposition we will construct the proof. Let $P(x)$ to be the generating function for plane trees where x is marking a non-root vertex. Consider a plane tree rooted at vertex i such that there is a path of length ℓ starting at vertex i and terminating at a vertex of label $i+\ell$ which is also a leaf sink. The path decomposes the tree into left and right plane subtrees upto length ℓ. See Figure 3.1.

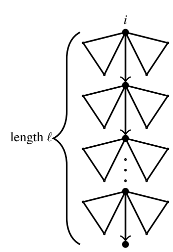

Figure 3.1. Unlabelled plane tree with path length ℓ with vertex $i+\ell$ as a leaf sink.

Vertex $i+\ell$ is not connected to any other tree thus vertex is represented by x in the generating function. So we have $x(P(x)xP(x))^\ell = x(xP(x)^2)^\ell$ as the generating function of the unlabelled trees rooted at vertex i with a path of length ℓ starting at the root and ending at a leaf sink $i+\ell$. The generating function for the number of unlabelled plane trees satisfies $P(x)=\frac{1}{1-xP(x)}$. We set $xP(x)=F(x)$ so that $F(x)=\frac{x}{1-F(x)}$. By Lagrange Inversion Formula, we obtain

$$\begin{aligned}
[x^n]x(xP(x)^2)^\ell = [x^{n+\ell-1}]F(x)^{2\ell} &= \frac{2\ell}{n+\ell-1}[t^{n-\ell-1}]\left((1-t)^{-(n+\ell-1)}\right) \\
&= \frac{2\ell}{n+\ell-1}[t^{n-\ell-1}]\sum_{i\geq 0}\binom{-(n+\ell-1)}{i}(-t)^i
\end{aligned}$$

which reduces to

$$[x^n]x(xP(x)^2)^\ell = \frac{\ell}{n-1}\binom{2n-2}{n+\ell-1}.$$

This formula counts the number of unlabelled plane trees with a path of length ℓ starting at a root i and ending at a leaf sink $i+\ell$. Consider a path of length ℓ starting at vertex i and ending at vertex j. There are $\binom{j-i-1}{\ell-1}$ possible paths. Once the $\ell+1$ vertices on the path have been labelled, there are $(n-\ell-1)!$ ways of labelling the remaining vertices. Therefore, the total number of labelled plane trees of order n in which vertex j is a leaf sink reachable from vertex i in ℓ steps is

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{j-i-1}{\ell-1}\binom{2n-2}{n+\ell-1}.$$

This completes the proof. □

We obtain the following result by summing over all j in Equation (3.6).

Corollary 3.6. *The total number of leaf sinks that are reachable, in ℓ steps, from root i in trees with n vertices is given by*

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{n-i}{\ell}\binom{2n-2}{n+\ell-1}. \tag{3.7}$$

Corollary 3.7. *There are*

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{n}{\ell+1}\binom{2n-2}{n+\ell-1}, \tag{3.8}$$

leaf sinks at step ℓ that are reachable from the root in trees with n vertices.

Proof. The result is evident by summing over all i in Equation (3.7). □

The formula below follows by summing over all ℓ in Equation (3.8).

Corollary 3.8. *The formula for the number of leaf sinks in trees of order n that are reachable from the root is*

$$\frac{n!}{n-1}\sum_{\ell=0}^{n-1}\frac{\ell}{(\ell+1)!}\binom{2n-2}{n+\ell-1}.$$

Corollary 3.9. *The average number of leaf sinks that are reachable from the root in ℓ steps in a random tree is*

$$\frac{\ell}{(\ell+1)!}.$$

Proof. The result follows by dividing the total number of leaf sinks that are reachable from the root in a labelled plane tree, i.e Equation (3.8), by the total number of labelled plane trees, and tending $n\to\infty$. □

4. Enumeration by left most paths and first children

In this section, we continue our investigation of reachable vertices but now according to lengths of left most paths and first children. We begin by left most paths. Recall that a *left most path* refers to a path that joins the eldest children at each level in a plane tree.

Proposition 4.1. *The number of trees of order n in which there is a left most path of length ℓ from a root i to a vertex j is given by*

$$(n-\ell-1)!\frac{\ell+1}{n}\binom{j-i-1}{\ell-1}\binom{2n-\ell-2}{n-\ell-1}. \tag{4.1}$$

Proof. Let $P(x)$ be the generating function for plane trees. Here again, x marks vertices in unrooted plane trees. Figure 4.1 gives the decomposition of these trees by left most path.

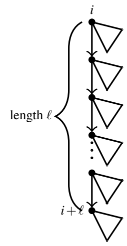

Figure 4.1. Unlabelled plane tree with left most path of length ℓ.

The decomposition shows that $(xP(x))^{\ell+1} = x^{\ell+1}P(x)^{\ell+1}$ is the generating function for the number of unlabelled trees in which there is a left most path of length ℓ. It now remains to extract the coefficient of x^n in the generating function $x^{\ell+1}P(x)^{\ell+1}$. We set $xP(x) = F(x)$ so that $F(x) = \frac{x}{1-F(x)}$ and by Lagrange Inversion Formula we get

$$\begin{aligned}
[x^n](xP(x))^{\ell+1} &= [x^n]F(x)^{\ell+1} \\
&= \frac{\ell+1}{n}[t^{n-\ell-1}](1-t)^{-n} \\
&= \frac{\ell+1}{n}[t^{n-\ell-1}]\sum_{k\geq 0}\binom{-n}{k}(-t)^k \\
&= \frac{\ell+1}{n}[t^{n-\ell-1}]\sum_{k\geq 0}\binom{n-1+k}{k}t^k \\
&= \frac{\ell+1}{n}\binom{2n-\ell-2}{n-\ell-1}.
\end{aligned}$$

There are $\binom{j-i-1}{\ell-1}$ choices for paths of length ℓ between vertices i and j. After the $\ell+1$ vertices on the path have been labelled, by choice of paths, the remaining vertices are labelled in $(n-\ell-1)!$ ways. Therefore, we find that the number of trees of order n in which there is a left most path of length ℓ is given by

$$(n-\ell-1)!\frac{\ell+1}{n}\binom{j-i-1}{\ell-1}\binom{2n-\ell-2}{n-\ell-1}.$$

Thus the proof. $\square$

By summing over all j in Equation (4.1) we find that there are

$$(n-\ell-1)!\frac{\ell+1}{n}\binom{n-i}{\ell}\binom{2n-\ell-2}{n-\ell-1} \tag{4.2}$$

trees on n vertices in which there is a left most path of length ℓ from root i. Also, summing over all i in Equation (4.2), we obtain the formula for the number of trees of order n in which there is a left most path of length ℓ from the root as

$$(n-\ell-1)!\frac{\ell+1}{n}\binom{n}{\ell+1}\binom{2n-\ell-2}{n-\ell-1}. \tag{4.3}$$

Setting $\ell = 0$ in Equation (4.3) we rediscover the formula for the number of labelled plane trees, that is $n!C_{n-1}$ where C_n is the n^{th} Catalan number.

Corollary 4.2. *The average number of eldest children at length ℓ from the root in a random tree is $\frac{1}{\ell!2^{\ell}}$.*

Proof. As before, we divide the total number of labelled plane trees of order n in which there is a left most path of length ℓ, Equation (4.3), by the total number of labelled plane trees. We then simplify the resultant and tend $n \to \infty$. $\square$

We now switch our attention to leaf sinks and left most paths.

Proposition 4.3. *The number of trees of order n rooted at vertex i in which there is a left most path of length ℓ such that the final vertex j is a leaf sink is given by*

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{j-i-1}{\ell-1}\binom{2n-\ell-3}{n-\ell-1}. \tag{4.4}$$

Proof. Let $P(x)$ to be the generating function for plane trees where x is marking a non-root vertex. Consider a plane tree rooted at vertex i such that there is a left most path of length ℓ starting at vertex i and ending at vertex $i+\ell$ which is also a leaf sink. The path decomposes the tree into right plane subtrees upto length ℓ. See Figure 4.2.

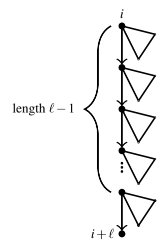

Figure 4.2. Unlabelled plane tree with left most path of length ℓ and the final vertex is a leaf sink.

Vertex $i+\ell$ is not connected to any other subtree. So we have $x(xP(x))^\ell = x^{\ell+1}P(x)^\ell$ as the generating function of the unlabelled plane tree rooted at vertex i with a left most path of length ℓ starting at the root and ending at leaf sink $i+\ell$. We set $xP(x) = F(x)$ and apply Lagrange Inversion Formula, to get

$$\begin{aligned}
[x^n]x^{\ell+1}P(x)^\ell = [x^n]xF(x)^\ell &= [x^{n-1}]F(x)^\ell \\
&= \frac{\ell}{n-\ell}[t^{n-\ell-1}](1-t)^{-(n-1)} \\
&= \frac{\ell}{n-\ell}[t^{n-\ell-1}]\sum_{k\geq 0}\binom{-(n-1)}{k}(-t)^k \\
&= \frac{\ell}{n-1}\binom{2n-\ell-3}{n-\ell-1},
\end{aligned}$$

as the formula for the number of unlabelled plane trees with a left most path of length ℓ starting at a root and ending at a leaf sink. Consider a path of length ℓ starting at vertex i and ending at vertex j. There are $\binom{j-i-1}{\ell-1}$ possible paths. Once the $\ell+1$ vertices on the path have been labelled, there are $(n-\ell-1)!$ ways of labelling the remaining vertices. Therefore, putting everything together we obtain the desired formula. $\square$

By summing over all j in Equation (4.4) we get the total number of trees on n vertices in which there is a left most path of length ℓ from root i and a final vertex is a leaf sink as

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{n-i}{\ell}\binom{2n-\ell-3}{n-\ell-1}. \tag{4.5}$$

Also, by summing over all i in Equation (4.5) we get that the total number of trees on n vertices in which there is a left most path of length ℓ from the root and the final vertex is a leaf sink as

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{n}{\ell+1}\binom{2n-\ell-3}{n-\ell-1}. \tag{4.6}$$

Moreover, if we sum over all ℓ in Equation (4.6) and then simplify, we obtain the formula for number of trees of order n in which there is a left most path starting from the root and the ending vertex is a leaf sink:

$$(n-2)!\sum_{\ell=0}^{n-1}\frac{\ell}{(\ell+1)!}\binom{2n-\ell-3}{n-\ell-1}.$$

Corollary 4.4. *On average, there are $\frac{1}{(\ell+1)!2^{\ell+1}}$ eldest children which are also leaf sinks in random tree.*

Proof. We divide the total number of labelled plane trees in which there is a left most path of length ℓ from the root and the final vertex is a leaf sink (See Equation (4.6)), by the total number of labelled plane trees to obtain

$$\frac{(n-\ell-1)!\frac{\ell}{n-1}\binom{n}{\ell+1}\binom{2n-\ell-3}{n-\ell-1}}{(n-1)!\binom{2n-2}{n-1}}$$

as the average number of labelled plane trees in which there is a left most path and a final vertex is a leaf sink. We then simplify and tend $n \to \infty$ to obtain the desired result. □

Recall that in ordered trees, the children (or siblings) are linearly ordered and are drawn in a left-to-right pattern where the left most child is called the *first child* to the parent. Enumerating the trees by first children we find that,

Proposition 4.5. *The number of trees of order n with vertex i as a root and vertex j as a first child reachable from root in ℓ steps is given by*

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{j-i-1}{\ell-1}\binom{2n-2}{n+\ell-1}.$$

Proof. Let $P(x)$ be the generating function for the plane trees where x represents non-root vertices. Consider a plane tree rooted at vertex i such that there is a path of length ℓ starting at vertex i and terminating at vertex $i+\ell$ which is a first child. The path decomposes the tree into left and right plane subtrees upto vertex $\ell-1$. See Figure 4.3.

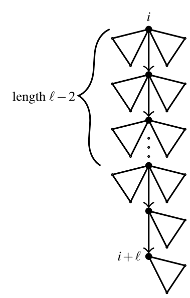

Figure 4.3. Unlabelled plane tree of order n with first child at length ℓ.

Since vertex $i+\ell$, which is the $(\ell+1)^{th}$ vertex, is a first child it's parent has no left subtree. Vertex $i+\ell$ can either have children or not. Thus the decomposition gives $(x(P(x)^2)^{\ell-1}xP(x)xP(x) = x^{\ell+1}P(x)^{2\ell}$ as the generating function for unlabelled plane trees rooted at vertex i such that there is a path of length ℓ starting at the root and ending at a first child $i+\ell$.

Since $P(x) = 1/(1-xP(x))$, we set $xP(x) = F(x)$ and apply Lagrange Inversion Formula, to obtain

$$\begin{aligned}
[x^n]\left(x^{\ell+1}P(x)^{2\ell}\right) &= [x^{n+\ell-1}]F(x)^{2\ell} \\
&= \frac{2\ell}{n+\ell-1}[t^{n-\ell-1}](1-t)^{-(n+\ell-1)} \\
&= \frac{2\ell}{n+\ell-1}[t^{n-\ell-1}]\sum_{i\geq 0}\binom{n+\ell+i-2}{i}t^i \\
&= \frac{2\ell}{n+\ell-1}\binom{2n-3}{n+\ell-2} \\
&= \frac{\ell}{n-1}\binom{2n-2}{n+\ell-1},
\end{aligned}$$

as the formula for unlabelled plane trees with a path of length ℓ starting at a root and terminating at a first child. The formula then follows by choosing the choices of paths between i and j, and labelling the vertices which are not on the path. □

By summing over all j in Equation (4.6), we see that the number of first children that are reachable from root i, at length ℓ, in trees on n vertices is given by

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{n-i}{\ell}\binom{2n-2}{n+\ell-1}. \tag{4.7}$$

In addition, we have: The number of first children at level ℓ in trees of order n that are reachable from the root is given by

$$(n-\ell-1)!\frac{\ell}{n-1}\binom{n}{\ell+1}\binom{2n-2}{n+\ell-1}. \tag{4.8}$$

This is arrived at by summing over all i in Equation (4.7). Also, by summing over all ℓ in Equation (4.8) and simplifying, we obtain the formula for the total number of first children that are reachable from the root in trees of order n:

$$\frac{n!}{n-1}\sum_{\ell=0}^{n-1}\frac{\ell}{n-1}\binom{2n-2}{n+\ell-1}.$$

Corollary 4.6. *The average number of first children that are reachable from the root in ℓ steps in a random tree is $\frac{\ell}{(\ell+1)!}$.*

Proof. By dividing the total number of first children in a labelled plane tree that are reachable from the root (See Equation (4.8)), by the total number of labelled plane trees we obtain,

$$\frac{(n-\ell-1)!\frac{\ell}{n-1}\binom{n}{\ell+1}\binom{2n-2}{n+\ell-1}}{(n-1)!\binom{2n-2}{n-1}}$$

as the average number of first children in a labelled plane tree of order n that are reachable from the root in ℓ steps. We tend $n\to\infty$ to obtain the required result. □

Remark 4.7. Unlabelled plane trees with a path of length ℓ starting at a root i and terminating at a leaf sink j has similar generating function as unlabelled trees with a path of length ℓ starting at a root i and terminating at a first child j. Therefore, they pose similar results if we sum over i, j and ℓ. Asymptotic results are also the same.

Remark 4.8. If the terminal vertex j is a first child which is also a leaf, then the generating function for trees with root i such that there is a path of length ℓ from i to j is given as $(xP(x)^2)^{\ell-1}xP(x)x$. Thus there are

$$\frac{2\ell-1}{n+\ell-1}\binom{2n-3}{n+\ell-2}$$

such trees on n vertices.

5. Enumeration by non-first children and non-leaves

In plane trees, any vertex (child) which is not leftmost child of the parent vertex is called a *non-first child*. A vertex which is not a leaf is a *non leaf*. In this section, we enumerate trees with respect to number of non-first children as well as number of non leaves.

Proposition 5.1. *The number of trees on n vertices rooted at vertex i and having vertex j as a non-first child which is reachable from i, at length ℓ, is given by*

$$(n-\ell-1)!\frac{\ell+1}{n-1}\binom{j-i-1}{\ell-1}\binom{2n-2}{n+\ell}. \tag{5.1}$$

Proof. We obtain the generating function by considering a plane tree rooted at vertex i with a path of length ℓ starting at i and terminating at a non-first child $i+\ell$. The path decomposes the tree into left and right plane subtrees as shown in Figure 5.1.

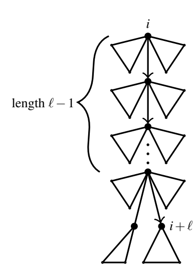

Figure 5.1. Unlabelled plane tree with non-first children at length ℓ.

The decomposition gives $x^{\ell+2}P(x)^{2\ell+2}$ as the generating function for unlabelled plane trees, with a path of length ℓ starting at a root and terminating at a non-first child. The generating function for unlabelled plane trees $P(x) = \frac{1}{1-xP(x)}$. We set $xP(x) = F(x)$ and use Lagrange Inversion formula to obtain

$$\begin{aligned}[x^n]x^{\ell+2}P(x)^{2\ell+2} = [x^{n+\ell}]F(x)^{2\ell+2} &= \frac{2\ell+2}{n+\ell}[t^{n-\ell-2}]\left((1-t)^{-1(n+\ell)}\right)\\ &= \frac{2\ell+2}{n+\ell}[t^{n-\ell-2}]\sum_{i\geq 0}\binom{n+\ell+i-1}{i}t^i\\ &= \frac{\ell+1}{n-1}\binom{2n-2}{n+\ell},\end{aligned}$$

as the formula for the number of non-first children that are reachable at length ℓ from the root in trees of order n. Upon considering the choices for labels on and not on the path, we obtain the desired formula. □

By summing over all j in Equation (5.1), we get that the total number of non-first children at level ℓ in trees of order n that are reachable from vertex i is

$$(n-\ell-1)!\frac{\ell+1}{n-1}\binom{n-i}{\ell}\binom{2n-2}{n+\ell}. \tag{5.2}$$

Also, by summing over all i in Equation (5.2), we obtain the following result: The total number of non-first children at level ℓ that are reachable from the root in a tree on n vertices is given by

$$(n-\ell-1)!\frac{\ell+1}{n-1}\binom{n}{\ell+1}\binom{2n-2}{n+\ell}. \tag{5.3}$$

Moreover, the total number of non-first children in trees of order n that are reachable from the root is given by

$$\frac{n!}{n-1}\sum_{\ell=0}^{n-1}\frac{1}{\ell!}\binom{2n-2}{n+\ell}.$$

This formula is arrived at by summing over all ℓ in Equation (5.3), and simplifying. In a similar fashion as before we have:

Corollary 5.2. *The average number of non-first children that are reachable at length ℓ from the root in a random tree is given by $\frac{1}{\ell!}$.*

For the remainder of this section, we enumerate non-leaf sinks.

Proposition 5.3. *The number of trees of order n with vertex i as a root and vertex j as non-leaf, which is reachable from the root at length ℓ is given by*

$$(n-\ell-1)!\frac{\ell+1}{n-1}\binom{j-i-1}{\ell-1}\binom{2n-2}{n+\ell}. \tag{5.4}$$

Proof. Consider a tree rooted at vertex i with a path of length ℓ from the root to vertex $i+\ell$ which is non-leaf. This path decomposes the tree into left and right subtrees upto step ℓ. Moreover, since vertex $i+\ell$ is a non-leaf, there must be a subtree of $i+\ell$ which may be empty and a subtree, rooted at a child of $i+\ell$. This subtree may also be empty. The decomposition is therefore given by Figure 5.2.

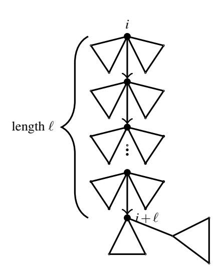

Figure 5.2. Unlabelled plane tree with a non-leaf vertex at length ℓ.

The generating function for the trees is thus $(xP(x)^2)^\ell xP(x)xP(x) = x^{\ell+2}P(x)^{2\ell+2}$, where $P(x)$ is the generating function for unlabelled plane trees with x marking non-root vertices. The required result therefore follows by applying Lagrange Inversion formula, upon setting $xP(x) = F(x)$, and giving the number of choices for the labels on the path and those that are not on the path. □

Remark 5.4. The number of trees with a non-leaf vertex j which is reachable at length ℓ from a root i have similar generating functions (though different decompositions) as for the case of non-first children. The results in the case of non-first children therefore hold for non-leaf vertices.

6. Enumeration by exact number of vertices

The number of exact vertices that are reachable from a given vertex has been studied for the case of labelled ordinary trees. Quite a number of results were obtained by Okoth in his PhD thesis, [2]. Similarly, Seo and Shin [4] established a formula for rooted Cayley trees of order n in which there is a maximal increasing subtree of order k. In this section, we obtain a formula for the number of ordered trees in which a given number of vertices are reachable from the root.

Theorem 6.1. *The total number of trees of order n such that exactly k vertices are reachable from the root is given by*

$$O_{n,k} = \sum_{k \le m+1 \le n} \binom{n}{m+1} z_{m,k-1} \frac{m-k+1}{n-k} (n-k)^{(n-m-1)} \tag{6.1}$$

for $0 \le k < n$, $O_{n,n} = (2n-3)!!$, *where* $n^{(r)} = n(n+1)(n+2)\cdots(n+r-1)$ *is a rising factorial and* $z_{m,k}$ *is the number of ordered trees on* $m+1$ *vertices with additional* $(m-k)$ *decreasing leaves attached to an increasing tree with k edges.*

A subtree of a rooted tree is said to be *increasing* if the labels in the subtree are increasing as one moves away from the root. A *maximal increasing subtree* of a v-rooted tree is an increasing subtree rooted at v and having the highest number of vertices. Seo and Shin [4], showed that Equation (6.1) gives the number of ordered trees on $[n]$ with its maximal decreasing subtree having k vertices. Now, orienting the edges of the ordered tree with n vertices from vertices of lower label towards vertices of higher label, we obtain an ordered tree in which exactly k are reachable from the root if its maximal increasing subtree has k vertices. This proves Theorem 6.1.

Corollary 6.2. *The number of trees of order n having exactly* $k \ge 2$ *vertices reachable from root* 1 *is given by*

$$(2k-3)O_{n-1,k-1} \tag{6.2}$$

where $O_{n,k}$ *is given by Equation* (6.1).

Proof. Consider an ordered tree P of order $n-1$ such that the vertices are labelled $2,3,\cdots,n$. Let the root be of label v_1. Moreover, let the number of vertices that are reachable from v_1 be exactly $k-1$. We follow the following steps in obtaining trees in which exactly k vertices are reachable from root of label 1:
Step 1: Let P_0 be the maximal increasing subtree having vertex set $\{v_1, v_2, \ldots, v_{k-1}\}$ where $v_i < v_{i+1}$ for all i. In P, delete all the edges in P_0 to obtain non-single vertex subtrees $P_1, P_2, \cdots, P_m$.

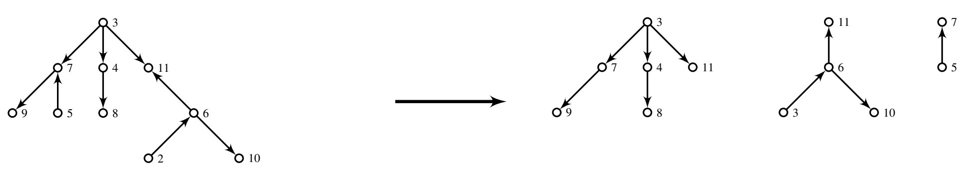

Figure 6.1. Diagram showing Step 1 in the proof of Corollary 6.2

Step 2: Relabel the vertices of the maximal increasing subtree P_0 with the vertex v_1 now as 1, v_2 as v_1, v_3 as v_2 and so on. The maximal increasing tree P_0 still has $k-1$ vertices. There are $2k-3$ positions in the new maximal increasing subtree rooted at 1 to attach vertex v_{k-1}. For each maximal increasing subtree previously rooted at v_1 we obtain $2k-3$ new subtrees rooted at vertex 1 with k reachable vertices.

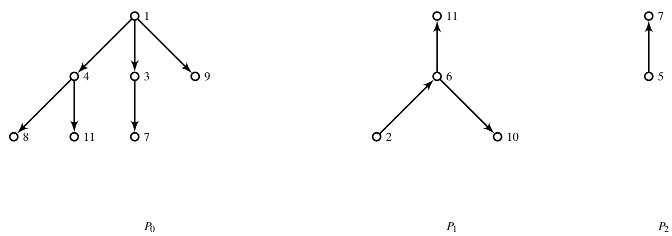

Figure 6.2. Diagram showing Step 2 in the proof of Corollary 6.2

Step 3: Identify vertex v_i in the subtrees $P_1, P_2, \ldots, P_{k-1}$ with vertex v_i in the new maximal increasing subtree, for all $i \in \{1, \ldots, k-1\}$. for which v_i occurs in one of the P_j.

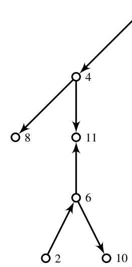

Figure 6.3. Diagram showing Step 3 in the proof of Corollary 6.2

Theorem 6.1, gives the total number of ordered labelled trees of order n such that exactly k vertices are reachable from the root. Now, for maximal increasing subtrees with $k-1$ vertices we substitute k and n in the Equation (6.1) above with $k-1$ and $n-1$ respectively to obtain the formula for the number of labelled trees of order n having exactly k vertices reachable from vertex 1 as

$$(2k-3)O_{n-1,k-1}.$$

Thus the proof is complete. □

Corollary 6.3. *There are*

$$(2n-2i+1)!!(n-i+1)(n+1)(n+2)\cdots(n+i-2)$$

trees on n vertices such that exactly $n-i+1$ reachable vertices from root i.

Proof. There are $(2n-2i+1)!!$ recursive trees on $n-i+1$ vertices (See Lemma 2 in [4]). Since there are $n-(n-i+1) = i-1$ vertices which are not reachable from vertex i, then all the $i-1$ vertices have labels less than i. The number of ways of adding the $i-1$ vertices to recursive tree successively is given by $(n-i+1)(n+1)(n+2)\cdots(n+i-2)$(See Lemma 2 in [4]). Therefore the total number of trees on n vertices with exactly $n-i-1$ vertices reachable from vertex i is given by

$$(2n-2i+1)!!(n-i+1)(n+1)(n+2)\cdots(n+i-2).$$

Hence the desired formula. □

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Noncrossing and Plane Tree-Like Structures

Isaac Owino Okoth[1]*

Abstract
Mathematical trees are connected graphs without cycles, loops and multiple edges. Various trees such as Cayley trees, plane trees, binary trees, d-ary trees, noncrossing trees among others have been studied extensively. Tree-like structures such as Husimi graphs and cacti are graphs which posses the conditions for trees if, instead of vertices, we consider their blocks. In this paper, we use generating functions and bijections to find formulas for the number of noncrossing Husimi graphs, noncrossing cacti and noncrossing oriented cacti. We extend the work to obtain formulas for the number of bicoloured noncrossing Husimi graphs, bicoloured noncrossing cacti and bicoloured noncrossing oriented cacti. Finally, we enumerate plane Husimi graphs, plane cacti and plane oriented cacti according to number of blocks, block types and leaves.

Keywords: Noncrossing tree, Plane tree, Tree-like structure, Husimi graph, Cactus, Oriented cactus, Enumeration
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[1] *Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya, ORCID: 0000-0003-4503-4733*
***Corresponding author**: ookoth@maseno.ac.ke
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1. Introduction

Husimi graph is a connected graph whose blocks are complete graphs. These graphs were introduced and enumerated by Japanese physicist Kodi Husimi in [5]. If the blocks of a connected graph are polygons then the graph is called a *cactus*. Cacti were introduced by Harary and Uhlenbeck in [4] where they appeared as *Husimi trees*. In 1996, Collin Springer [12] introduced and enumerated *oriented cacti.* These are connected graphs whose blocks are oriented cycles. Formulas counting these tree-like structures as well as their coloured counterparts, i.e. structures coloured with the property that blocks of the same colour are not incident to one another, have been obtained. See [1, 3–5, 7, 8, 10, 12] for details. In this paper, we enumerate their noncrossing and plane counterparts. The *degree* of a vertex in a tree-like structure is the number of blocks that are incident to it.

This paper is organized as follows: In Section 2, we enumerate noncrossing Husimi graphs, cacti and oriented cacti by block type and number of blocks. A bijection between these structures and certain polygon dissections is also presented here. Noncrossing tree-like structures whose blocks are coloured using two colours such that no blocks of the same colour are incident to one another are enumerated in Section 3. Lastly in Section 4, we enumerate plane tree-like structures according to block sizes, block types and number of leaves. Some of the results presented here were part of the author's PhD thesis [10].

2. Noncrossing tree-like structures

In this section, we obtain equivalent results for Husimi graphs, cacti and oriented cacti whose blocks do not cross. We shall call these structures as *noncrossing Husimi graphs*, *noncrossing cacti* and noncrossing oriented cacti respectively. The simplest of the noncrossing structures is a *noncrossing tree*. This is a tree drawn in the plane with vertices on the boundary of a circle such that the edges do not cross inside the circle. Marc Noy [9] showed that the number of noncrossing trees on n labelled vertices is given by

$$\frac{1}{2n-1}\binom{3n-3}{n-1}.$$

This result was later generalised to connected graphs by Flajolet and Noy [2]. Before we embark on the enumeration of noncrossing Husimi graphs, let us review the notion of *butterfly decomposition* of noncrossing trees that was introduced in [2]. A *butterfly* is an ordered pair of trees that share a root. If a vertex v in a tree has degree d, then the tree can be decomposed into d butterflies hanging from v.

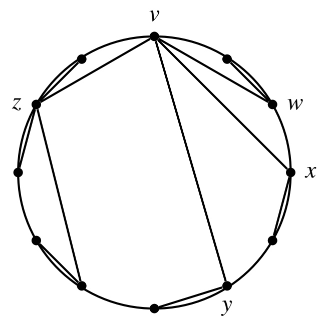

Figure 2.1. Noncrossing tree

In Figure 2.1, there are 4 butterflies rooted at w, x, y and z. The aforementioned authors showed that if $T(x)$ is the generating function for trees and $B(x)$ is the generating function for butterflies then we have the following equations:

$$T(x) = \frac{x}{1-B} \text{ and } B(x) = \frac{T^2}{x}.$$

Theorem 2.1. *Let $(n_2, n_3, \ldots)$ be a sequence of non-negative integers satisfying the condition that $n = \sum_{j\geq 2}(j-1)n_j + 1$. The number $NHG_n(n_2, n_3, \ldots)$ of noncrossing Husimi graphs on $[n]$ having n_j blocks of size j is given by*

$$NHG_n(n_2, n_3, \ldots) = \frac{(2n+k-2)!}{(2n-1)!\prod_{j\geq 2} n_j!} \tag{2.1}$$

where $k = \sum_{j\geq 2} n_j$.

Proof. Let $F(x)$ be the generating function for noncrossing Husimi graphs. Let y_i mark the number of vertices in each block. Adopting the butterfly decomposition of noncrossing trees to noncrossing Husimi graphs, we have that

$$F(x) = \frac{x}{1-\sum_{i\geq 1} y_{i+1}B^i}$$

and

$$B(x) = \frac{F^2}{x}$$

where $B(x)$ is the generating function for butterflies.

Therefore the generating function $F(x)$ satisfies

$$F(x) = \frac{x}{1-\sum\limits_{i\geq 1} y_{i+1}\left(\frac{F^2}{x}\right)^i}.$$

Thus for $G = \frac{F}{\sqrt{x}}$ we have

$$G(x) = \frac{\sqrt{x}}{1-\sum\limits_{i\geq 1} y_{i+1}G^{2i}}.$$

By the Lagrange Inversion Formula, we obtain

$$
\begin{aligned}
[x^n]F(x) = [x^{n-\frac{1}{2}}]G(x) &= \frac{1}{2n-1}[t^{2n-2}]\left(1-\sum_{i\geq 1} y_{i+1}t^{2i}\right)^{-(2n-1)} \\
&= \frac{1}{2n-1}[t^{2n-2}]\sum_{k\geq 0}\binom{-(2n-1)}{k}\left(-\sum_{i\geq 1} y_{i+1}t^{2i}\right)^{k} \\
&= \frac{1}{2n-1}[t^{2n-2}]\sum_{k\geq 0}\binom{2n+k-2}{k}\left(\sum_{i\geq 1} y_{i+1}t^{2i}\right)^{k} \\
&= \frac{1}{2n-1}\sum_{k\geq 0}\binom{2n+k-2}{k}\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}}\frac{k!y_2^{n_2}y_3^{n_3}\ldots}{n_2!n_3!\cdots}. \qquad (2.2)
\end{aligned}
$$

Therefore,

$$NHG_n(n_2,n_3,\ldots) = \frac{1}{2n-1}\binom{2n+k-2}{k}\frac{k!}{\prod_{j\geq 2} n_j!}.$$

□

Corollary 2.2. *The number of noncrossing Husimi graphs on $n \geq 2$ vertices is given by*

$$\frac{1}{n-1}\sum_{k=1}^{n-1}\binom{2n+k-2}{k-1}\binom{n-1}{k}.$$

Proof. We need to show that the number of noncrossing Husimi graphs on n vertices with k blocks is given by the generalised Narayana number,

$$\frac{1}{n-1}\binom{2n+k-2}{k-1}\binom{n-1}{k}. \qquad (2.3)$$

Let $[[n,k]]$ denote the set of all types of partitions of $[n]$ of length k. Since

$$\sum_{P\in[[n-1,k]]}\frac{k!}{n_2!n_3!\cdots} = \binom{n-2}{k-1},$$

the result follows from Equation (2.2). □

The formula (2.3) appears in [11] and [14] as the number of dissections of a convex polygon on $2n$ vertices with $k-1$ noncrossing diagonals such that the number of edges enclosing each interior region is even. We now construct a bijection between the set of these dissections and the noncrossing Husimi graphs.

Lemma 2.3. *There is a bijection between the set of dissections of a convex polygon on $2n$ vertices with $k-1$ noncrossing diagonals such that the number of edges enclosing each interior region is divisible by two and the set of noncrossing Husimi graphs on n vertices with k blocks.*

Proof. Consider a convex polygon on $2n$ vertices such that the vertices are labelled in clockwise direction as $1,1',2,2',\ldots,n,n'$. Let the number of noncrossing diagonal edges be $k-1$ and the number of edges of each interior region be divisible by 2. There are k such regions. Create an edge between any two vertices of label $1,2,\ldots,n$ that are in the same region. A vertex which is incident to more than one region is considered to belong to all the incident regions. The resultant graph is a noncrossing Husimi graph on n vertices with k blocks. See Figure 2.2 for an example. The process can easily be reversed. □

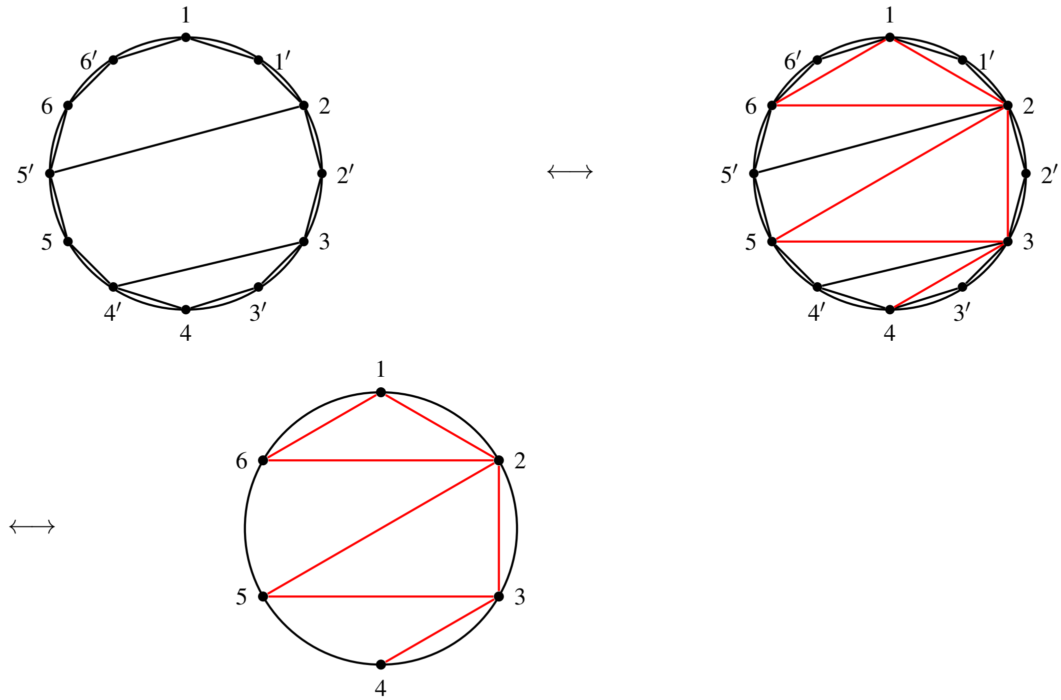

Figure 2.2. Diagram showing the bijection in the proof of Lemma 2.3.

We obtain further corollaries of Theorem 2.1.

Corollary 2.4. *Let $(n_2,n_3,\ldots)$ be a sequence of non-negative integers satisfying the condition that $n=\sum_{j\geq 2}(j-1)n_j+1$. The number $\mathrm{NC}_n(n_2,n_3,\ldots)$ of noncrossing cacti on $[n]$ having n_j blocks of size j is given by*

$$\mathrm{NC}_n(n_2,n_3,\ldots)=\frac{(2n+k-2)!}{(2n-1)!\prod_{j\geq 2}n_j!}, \tag{2.4}$$

where $k=\sum_{j\geq 2}n_j$.

Proof. In the noncrossing setting, there is only one way to turn a complete graph into a cycle thus the required equation follows from Equation (2.1) i.e.,

$$NC_n(n_2,n_3,\ldots)=NHG_n(n_2,n_3,\ldots).$$

□

Corollary 2.5. *The number of noncrossing cacti on $[n]$, where $n\geq 2$, is*

$$\frac{1}{n-1}\sum_{k=1}^{n-1}\binom{2n+k-2}{k-1}\binom{n-1}{k}.$$

Proof. We obtain the formula by summing over all possibilities of $n_2,n_3,\ldots$ and k as in the proof of Corollary 2.2. □

Corollary 2.6. *Let $(n_2,n_3,\ldots)$ be a sequence of non-negative integers satisfying the condition that $n=\sum_{j\geq 2}(j-1)n_j+1$. The number $\mathrm{NOC}_n(n_2,n_3,\ldots)$ of noncrossing oriented cacti on $[n]$ having n_j blocks of size j is given by*

$$\mathrm{NOC}_n(n_2,n_3,\ldots)=\frac{(2n+k-2)!2^{k-n_2}}{(2n-1)!\prod_{j\geq 2}n_j!},$$

where $k=\sum_{j\geq 2}n_j$.

Proof. Since any polygon of size ≥ 3 has 2 orientations, we have

$$NOC_n(n_2,n_3,\ldots)=2^{k-n_2}\cdot NC_n(n_2,n_3,\ldots).$$

The formula thus follows from Equation (2.4). □

Corollary 2.7. *The number of noncrossing oriented cacti on* $[n]$*, where* $n \geq 2$*, is*

$$\sum_{k\geq 0} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{(2n+k-2)!2^{k-n_2}}{(2n-1)!\prod_{j\geq 2} n_j!}.$$

3. Bicoloured noncrossing tree-like structures

In the next proposition, we obtain a formula for the number of noncrossing Husimi graphs on n labelled vertices such that the degrees of the vertices are less than or equal to 2. This will make 2-colouring possible. Recall, from Section 1, that the *degree* of a vertex v in a Husimi graph is the number of blocks that are incident to it.

Proposition 3.1. *Let* $NHG_{n,2}(n_2,n_3,\ldots)$ *be the number of noncrossing Husimi graphs on* $[n]$ *having* n_i *blocks of size* i *such that* $\sum_{i\geq 2}(i-1)n_i+1=n$ *and all the vertices have degree less than or equal to* 2*. Then*

$$NHG_{n,2}(n_2,n_3,\ldots) = \frac{n!2^{k-1}}{(n-k+1)!\prod_{j\geq 2} n_j!} \tag{3.1}$$

where $k=\sum_{j\geq 2} n_j$.

Proof. Let $F(x)$ be the generating function for 2-colourable noncrossing Husimi graphs with root degree 1 (or 0). Let y_i mark blocks of size i. Since each vertex in the block is to have degree less than or equal to two, the generating function satisfies

$$F(x) = x(1+\sum_{i\geq 1} y_{i+1}(2F-x)^i). \tag{3.2}$$

The butterflies of these graphs must be rooted at vertices of degree 1 (or consists of a single vertex). We subtract x to cater for cases in which a butterfly consists of a single vertex.

Setting $G=2F-x$ in Equation (3.2) we obtain

$$G = x(1+2\sum_{i\geq 1} y_{i+1}G^i).$$

G is the generating function for 2-coloured Husimi graphs with root degree 1 (in the case of a single vertex, there are no blocks, thus nothing to be coloured; otherwise there are precisely two colourings). When $y_2=y_3=\cdots=1$, then we obtain the generating function for the large Schröder numbers.

Now, for arbitrary root degree, root degree 2 Husimi graphs are obtained by merging two root degree 1 Husimi graphs. We subtract F for double counting root degree 1 Husimi graphs. The generating function is thus

$$H(x) = \frac{F^2}{x} - F = \frac{G^2}{4x} - \frac{x}{4}.$$

This implies that

$$[x^n]H = \frac{1}{4}[x^{n+1}]G^2.$$

By the Lagrange Inversion Formula, we have

$$\begin{aligned}
\frac{1}{4}[x^{n+1}]G^2 &= \frac{1}{2(n+1)}[t^{n-1}]\left(1+2\sum_{i\geq 1} y_{i+1}t^i\right)^{n+1} \\
&= \frac{1}{2(n+1)}[t^{n-1}]\sum_{k\geq 0}\binom{n+1}{k}\left(2\sum_{i\geq 1} y_{i+1}t^i\right)^k \\
&= \frac{1}{2(n+1)}\sum_{k\geq 0} 2^k\binom{n+1}{k} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\cdots}
\end{aligned}$$

Therefore,

$$NHG_{n,2}(n_2,n_3,\ldots) = \frac{2^{k-1}}{n+1}\binom{n+1}{k}\cdot\frac{k!}{n_2!n_3!\cdots}. \tag{3.3}$$

□

Corollary 3.2. *There are $n \cdot 2^{n-3}$ noncrossing trees on $n \geq 2$ vertices such that all the vertices have degree less than or equal to 2.*

Proof. The result follows from Equation (3.3) by taking $(n_2, n_3, \ldots) = (n-1, 0, \ldots)$ so that $k = n-1$.

Observe that these trees are also noncrossing paths. The corollary thus follows by a simple counting argument as well: first choose a root (in n ways), then 2 choices for each step. $\square$

Corollary 3.3. *Let $NHG_{n,2}$ be the number of noncrossing Husimi graphs on $[n]$ in which all the vertices have degree at most 2. Then*

$$NHG_{n,2} = \frac{1}{n-1} \sum_{k=1}^{n-1} 2^{k-1} \binom{n}{k-1} \binom{n-1}{k}. \tag{3.4}$$

Proof. To prove Formula (3.4), we need to show that the number of noncrossing Husimi graphs on n vertices with k blocks in which each vertex has degree ≤ 2 is given by

$$\frac{2^{k-1}}{n-1} \binom{n}{k-1} \binom{n-1}{k}.$$

Since

$$\sum_{P \in [[n-1,k]]} \frac{k!}{n_2! n_3! \cdots} = \binom{n-2}{k-1},$$

the result follows from Equation (3.3). $\square$

Lemma 3.4. *The number of bicoloured noncrossing Husimi graphs on $[n]$ having n_i blocks of size i such that $\sum_{i \geq 2} (i-1) n_i + 1 = n$ is equal to*

$$\frac{n! 2^k}{(n-k+1)! \prod_{j \geq 2} n_j!}$$

where $k = \sum_{j \geq 2} n_j$.

Proof. Consider a noncrossing Husimi graph on $[n]$ having n_i blocks of size i such that $\sum_{i \geq 2} (i-1) n_i + 1 = n$ and with vertices having degree less than or equal to 2. Let b be a block in the graph. There are two choices for colouring block b and one choice for the remaining blocks. The result thus follows from Equation (3.1). $\square$

Corollary 3.5. *The number of bicoloured noncrossing Husimi graphs on n vertices is given by*

$$\frac{1}{n-1} \sum_{k=1}^{n-1} 2^k \binom{n}{k-1} \binom{n-1}{k}. \tag{3.5}$$

We obtain the following special case by setting $k = n-1$ in Equation (3.5).

Corollary 3.6. *There are $n \cdot 2^{n-2}$ bicoloured noncrossing trees on $n \geq 2$ labelled vertices.*

Corollary 3.7. *The number of bicoloured noncrossing cacti on $[n]$ having n_i cycles of size i such that $\sum_{i \geq 2} (i-1) n_i + 1 = n$ is equal to*

$$\frac{n! 2^k}{(n-k+1)! \prod_{j \geq 2} n_j!},$$

where $k = \sum_{j \geq 2} n_j$.

Corollary 3.8. *The number of bicoloured noncrossing cacti on $[n]$, where $n \geq 2$, is*

$$\frac{1}{n-1} \sum_{k=1}^{n-1} 2^k \binom{n}{k-1} \binom{n-1}{k}.$$

Corollary 3.9. *The number of bicoloured noncrossing oriented cacti on $[n]$ having n_i cycles of size i such that $\sum_{i\geq 2}(i-1)n_i+1 = n$ is equal to*

$$\frac{n!2^{2k-n_2}}{(n-k+1)!\prod_{j\geq 2} n_j!},$$

where $k = \sum_{j\geq 2} n_j$.

Corollary 3.10. *The number of bicoloured noncrossing oriented cacti on $[n]$, for $n \geq 2$, is*

$$\sum_{k\geq 0} \sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{n!2^{2k-n_2}}{(n-k+1)!\prod_{j\geq 2} n_j!}.$$

4. Plane tree-like structures

A *plane Husimi graph* (resp. *plane cactus*) is a Husimi graph (resp. cactus) drawn on the plane such that its blocks are ordered (see, Figure 4.1 for plane cactus).

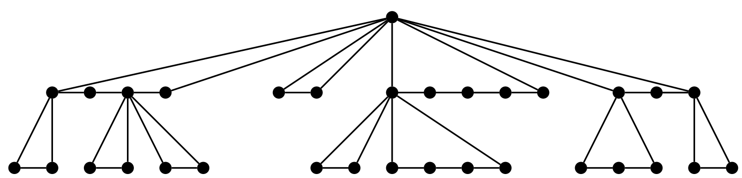

Figure 4.1. Plane cactus on 32 vertices.

In this section, we shall call the number of blocks coming out of a vertex as the *degree* of that vertex. A *leaf* is a non-root vertex which is incident to exactly one block. A non-leaf vertex is referred to as *internal* vertex.

Theorem 4.1. *Let $(n_2,n_3,\ldots)$ be a sequence of non-negative integers satisfying the coherence condition: $n = \sum_{j\geq 2}(j-1)n_j+1$. The number $PHG_n(n_2,n_3,\ldots)$ of plane Husimi graphs on n vertices having n_j blocks of size j is given by*

$$PHG_n(n_2,n_3,\ldots) = \frac{(n+k-1)!}{n!\prod_{j\geq 2} n_j!}, \tag{4.1}$$

where $k = \sum_{j\geq 2} n_j$.

Proof. Let $P(x)$ be the generating function for plane Husimi graphs. Let y_i mark the number of vertices in each block. Then we have

$$P(x) = \frac{x}{1-\sum_{i\geq 1} y_{i+1}P^i}.$$

By the Lagrange Inversion Formula [13], we obtain

$$\begin{aligned}
[x^n]\,P(x) &= \frac{1}{n}[t^{n-1}]\left(1-\sum_{i\geq 1} y_{i+1}t^i\right)^{-n} \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{-n}{k}\left(-\sum_{i\geq 1} y_{i+1}t^i\right)^{k} \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq 0}\binom{n+k-1}{k}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{k} \\
&= \frac{1}{n}\sum_{k\geq 0}\binom{n+k-1}{k}\sum_{\substack{n_2+n_3+\cdots=k \\ n_2+2n_3+\cdots=n-1}} \frac{k!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\cdots}.
\end{aligned} \tag{4.2}$$

Therefore,

$$PHG_n(n_2,n_3,\ldots) = \frac{1}{n}\binom{n+k-1}{k}\frac{k!}{\prod_{j\geq 2} n_j!}.$$

This completes the proof. □

In the proof of the following corollary, we get a formula for the number of plane Husimi graphs with a given number of blocks.

Corollary 4.2. *The number of plane Husimi graphs on $n \geq 2$ vertices is given by*

$$\frac{1}{n}\sum_{k=1}^{n-1}\binom{n+k-1}{k}\binom{n-2}{k-1}.$$

Proof. We need to show that the number of plane Husimi graphs on n vertices with k blocks is given by,

$$\frac{1}{n}\binom{n+k-1}{k}\binom{n-2}{k-1}. \tag{4.3}$$

Let $P(n,k)$ denote the set of all types of partitions of $\{1,2,\ldots,n\}$ of length k. Since

$$\sum_{P\in P(n-1,k)}\frac{k!}{n_2!n_3!\cdots} = \binom{n-2}{k-1}, \tag{4.4}$$

then the formula follows from Equation (4.2). □

Setting $k = n-1$ in Equation (4.3), we recover the formula for plane trees on n vertices. Similarly, setting $n = dn+1$, $n_{d+1} = n$ and $n_i = 0$ for all $i \neq d+1$, in Equation (4.1), we rediscover the formula

$$\frac{1}{dn+1}\binom{(d+1)n}{n}$$

for the number of d-tuplet trees on $dn+1$ vertices obtained in [6]. Here, if $d = 1$ we get the number of plane trees on $n+1$ vertices.

Corollary 4.3. *Let $(n_2,n_3,\ldots)$ be a sequence of non-negative integers satisfying the condition that $n = \sum_{j\geq 2}(j-1)n_j+1$. The number $\mathrm{PC}_n(n_2,n_3,\ldots)$ of plane cacti on n nodes and having n_j blocks of size j is given by*

$$\mathrm{PC}_n(n_2,n_3,\ldots) = \frac{(n+k-1)!}{n!\prod_{j\geq 2} n_j!}, \tag{4.5}$$

where $k = \sum_{j\geq 2} n_j$.

Proof. Since there is only one way to turn a complete graph into a cycle, the required equation follows from Equation (4.1) i.e.,

$$PC_n(n_2,n_3,\ldots) = PHG_n(n_2,n_3,\ldots).$$

□

Corollary 4.4. *The number of plane cacti on n nodes, where $n \geq 2$, is*

$$\frac{1}{n}\sum_{k=1}^{n-1}\binom{n+k-1}{k}\binom{n-2}{k-1}.$$

Proof. We obtain the formula by summing over all possibilities of $n_2,n_3,\ldots$ and k as in the proof of Corollary 4.2. □

Corollary 4.5. *Let $(n_2,n_3,\ldots)$ be a sequence of non-negative integers satisfying the condition that $n = \sum_{j\geq 2}(j-1)n_j+1$. The number $\mathrm{POC}_n(n_2,n_3,\ldots)$ of plane oriented cacti on n vertices and having n_j blocks of size j is given by*

$$\mathrm{POC}_n(n_2,n_3,\ldots) = \frac{(n+k-1)!2^{k-n_2}}{n!\prod_{j\geq 2} n_j!},$$

where $k = \sum_{j\geq 2} n_j$.

Proof. Since any polygon of size ≥ 3 has 2 orientations, we have

$$POC_n(n_2,n_3,\ldots) = 2^{k-n_2}\cdot PC_n(n_2,n_3,\ldots).$$

The result follows from Equation (4.5). $\square$

Corollary 4.6. *The number of plane oriented cacti on n vertices, where $n \geq 2$, is*

$$\sum_{k\geq 0}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}} \frac{(n+k-1)!2^{k-n_2}}{n!\prod_{j\geq 2} n_j!}.$$

For the rest of this paper, we are interested in the number of plane tree-like structures with a given number of leaves.

Theorem 4.7. *Let $(n_2,n_3,\ldots)$ be a sequence of non-negative integers satisfying the coherence condition: $n=\sum_{j\geq 2}(j-1)n_j+1$. The number of plane Husimi graphs on n vertices with ℓ leaves and having n_j blocks of size j is given by*

$$\frac{1}{n}\binom{n}{\ell}\binom{k-1}{n-\ell-1}\frac{k!}{\prod_{j\geq 2} n_j!} \tag{4.6}$$

where $k=\sum_{j\geq 2} n_j$.

Proof. Let $F(x,u)$ be the bivariate generating function for the number of plane Husimi graphs such that x and u are marking vertices and leaves respectively. Again y_i will mark the number of vertices in each block.

Now,

$$F(x,u) = xu + \frac{x}{1-\sum_{i\geq 1} y_{i+1}F(x,u)^i} - x.$$

For convenience, let $w=F(x,u)$ so that $w = x\left(u+\frac{\sum_{i\geq 1} y_{i+1}w^i}{1-\sum_{i\geq 1} y_{i+1}w^i}\right)$. We extract the coefficients of x^n and u^ℓ in the generating function.

$$\begin{aligned}
[x^nu^\ell]F(x,u) = [x^nu^\ell]w &= \frac{1}{n}[u^\ell t^{n-1}]\left(u+\frac{\sum_{i\geq 1} y_{i+1}t^i}{1-\sum_{i\geq 1} y_{i+1}t^i}\right)^n\\
&= \frac{1}{n}[u^\ell t^{n-1}]\sum_{j=0}^{n}\binom{n}{j}u^j\left(\frac{\sum_{i\geq 1} y_{i+1}t^i}{1-\sum_{i\geq 1} y_{i+1}t^i}\right)^{n-j}\\
&= \frac{1}{n}\binom{n}{\ell}[t^{n-1}]\left(\frac{\sum_{i\geq 1} y_{i+1}t^i}{1-\sum_{i\geq 1} y_{i+1}t^i}\right)^{n-\ell}\\
&= \frac{1}{n}\binom{n}{\ell}[t^{n-1}]\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{n-\ell}\left(1-\sum_{i\geq 1} y_{i+1}t^i\right)^{-(n-\ell)}\\
&= \frac{1}{n}\binom{n}{\ell}[t^{n-1}]\sum_{j=0}^{n-\ell}\binom{-(n-\ell)}{j}\left(-\sum_{i\geq 1} y_{i+1}t^i\right)^{j}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{n-\ell}\\
&= \frac{1}{n}\binom{n}{\ell}[t^{n-1}]\sum_{j=0}^{n-\ell}\binom{n-\ell+j-1}{j}\left(\sum_{i\geq 1} y_{i+1}t^i\right)^{n-\ell+j}.
\end{aligned}$$

Let $k=n-\ell+j$ so that

$$[x^nu^\ell]F(x,u) = \frac{1}{n}\binom{n}{\ell}\sum_{k=n-\ell}^{2n-2\ell}\binom{k-1}{n-\ell-1}\sum_{\substack{n_2+n_3+\cdots=k\\ n_2+2n_3+\cdots=n-1}}\frac{k!y_2^{n_2}y_3^{n_3}\cdots}{n_2!n_3!\cdots}.$$

This completes the proof. $\square$

From Equation (4.6) and summing over all n_j as in Equation (4.4), it follows that there are

$$\frac{1}{n}\binom{n}{\ell}\binom{k-1}{n-\ell-1}\binom{n-2}{k-1} \tag{4.7}$$

plane Husimi graphs on n vertices with k blocks and having exactly ℓ leaves. Setting $k=n-1$, we rediscover the famous Narayana number for the number of plane trees with a given number of leaves. Summing over all ℓ, making use of Vandermonde convolution, we obtain Equation (4.3) for the number of plane Husimi graphs on n vertices.

The expected number of leaves in plane Husimi graphs on n vertices with k blocks is

$$\sum_{\ell=1}^{n-1}\frac{\ell}{n}\binom{n}{\ell}\binom{k-1}{n-\ell-1}\binom{n-2}{k-1}=\binom{n+k-2}{k}\binom{n-2}{k-1}$$

and upon division by Equation (4.3), we get that on average there are $(n^2-n)/(n+k-1)$ leaves in the aforementioned plane graphs.

Setting $r=n-\ell$ in Equation (4.7), we obtain the following result.

Corollary 4.8. *There are*

$$\frac{1}{n}\binom{n}{r}\binom{k-1}{r-1}\binom{n-2}{k-1}$$

plane Husimi graphs on n vertices with k blocks and having r internal vertices.

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Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Contact Hamiltonian Description of 1D Frictional Systems

Furkan Semih Dündar[1]*, Gülhan Ayar[2]

Abstract
In this paper, we consider contact Hamiltonian description of 1D frictional dynamics with no conserved force. Friction forces that are monomials of velocity, and the sum of two monomials are considered. For that purpose, quite general forms of contact Hamiltonians are taken into account. We conjecture that it is impossible to give a contact Hamiltonian description of dissipative systems where the friction force is not in the form considered in this paper.

Keywords: Contact geometry, Contact Hamiltonian mechanics, Friction force
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[1] *Department of Mechanical Engineering, Faculty of Technology, Amasya University, Amasya, Turkey, ORCID: 0000-0001-5184-5749*
[2] *Department of Mathematics, Kamil Özdağ Faculty of Science, Karamanoğlu Mehmetbey University, Karaman, Turkey, ORCID: 0000-0002-1018-4590*
***Corresponding author**: furkan.dundar@amasya.edu.tr
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1. Introduction and Preliminaries

We already know that the fundamentals of many applications used in physics go through mathematical calculations. Since the theory of manifolds is used as configuration space in both mathematics and physics, the differential geometric methods used in the theory of manifolds are very important.

The geometry of the contact manifolds is done with the help of odd-dimensional manifolds. The fact that contact geometry can be applied in odd-dimensional manifolds has earned a prominent place in physics as well as differential geometry. Both contact and symplectic manifolds have found application in classical mechanics. In line with these studies, contact geometry was found to be under many physical phenomena and related to many other mathematical structures. Andrew Mclnerney's "First Steps in Differential Geometry" (Ref. [1]) is an important resource for the history of contact geometry and its significance in physics.

Lie [2] was the first to study contact structures systematically. Contact structures were considered in Gibbs' study of thermodynamics [3], Huygens' theory of light, geometric optics and Hamiltonian dynamics [4, 5].

Although the study of mathematical methods in classical mechanics dates back to old times [6], the issue of expressing Hamiltonian dynamics with contact equations is quite new. If we mention some works on Hamiltonian systems with contact equations, in 2016 contact Hamiltonian mechanics have been introduced by Bravetti and et al. [7]. In that paper, authors have focused on the major features of standard symplectic Hamiltonian dynamics and they have showed that all of them can be generalized to the contact case. Later, in Liu's work, the connections between the notions of Hamiltonian system, contact Hamiltonian system and nonholonomic system from the perspective of differential equations and dynamical systems have been described [8]. Also in [9], Dündar has provided a simple contact Hamiltonian description of a system with exponentially vanishing (or zero) potential under a friction term that is quadratic in velocity.

In the light of these previous studies with this present paper, we have provided for contact Hamiltonian description of 1D frictional dynamics with no conserved force. In this way, we have applied contact geometric methods in systems with frictional force (where friction force is not linearly dependent on velocity or where it is a polynomial of velocity). Friction forces that are monomials of velocity, and the sum of two monomials are considered. For that purpose, quite general forms of contact Hamiltonians are taken into account. Furthermore, we have given a conjecture that it is impossible to give a contact Hamiltonian description dissipative systems where the friction force is not in the form considered.

In this article we consider a 1D frictional system. The independent variables are q, p, S. The contact 1-form is given as follows: $\eta = dS + pdq$ [7]. It is easy to check that this expression satisfies the nondegeneracy condition $\eta \wedge d\eta = dS \wedge dp \wedge dq \neq 0$. Moreover, the readers may recognize the pdq term in the contact form as presymplectic potential.

In order to define contact system, we need a contact Hamiltonian. Contact Hamiltonian is a function of positions, momenta and an extra variable S as opposed to usual Hamiltonian function which is a function of positions and momenta. The extra variable, S, helps one describe dissipative systems. Now, we give a basic definition that we will use throughout this study. Let H be a contact Hamiltonian, depending on three variables: q, p, S. The equations of motion are then as follows [7]:

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}, \\
\dot{p} &= -\frac{\partial H}{\partial q} - p\frac{\partial H}{\partial S}, \\
\dot{S} &= p\frac{\partial H}{\partial p} - H.
\end{aligned}
$$

In this paper, we will investigate various forms of contact Hamiltonians to account for friction terms with no potential function. The organization of the paper is as follows: In Section 2 we consider a friction term that is a monomial of $\dot{q}$, in Section 3 we handle the case where the friction term is a sum of two monomials of $\dot{q}$, in Section 4 we give an applciation of Section 3 to a friction term that has linear and quadratic dependence on $\dot{q}$, and finally in Section 5 we conclude the paper by also giving a conjecture.

2. Friction term that is a monomial of $\dot{q}$

The goal of this Section is to find a contact Hamiltonian that will yield a friction term which is a monomial of $\dot{q}$, that is an equation of motion as seen in Equation (2.4). We consider a contact Hamiltonian of the following form:

$$
H = \frac{p^2}{2m} + \lambda p S^a.
$$

The case $a = 1$ gives a quadratic dependence on $\dot{q}$ for the friction term, which is investigated in Ref. [9]. As an ansatz, we let $p = \alpha m\dot{q}$ and $S = S(\dot{q})$. The contact equations of motion are as follows:

$$
\begin{aligned}
\dot{q} &= \frac{p}{m} + \lambda S^a, && (2.1)\\
\dot{p} &= -a\lambda p^2 S^{a-1}, && (2.2)\\
\dot{S} &= \frac{p^2}{2m}. && (2.3)
\end{aligned}
$$

Using the ansatz for p in Equation (2.1) gives us $\lambda S^a = (1-\alpha)\dot{q}$. We want our contact Hamiltonian to produce the following equation of motion:

$$
m\ddot{q} + \gamma\dot{q}^n = 0. \tag{2.4}
$$

Let $S'(\dot{q}) = \partial_{\dot{q}} S(\dot{q})$. Using Equation (2.3) yields:

$$
\ddot{q}S' = \frac{1}{2}m\alpha^2\dot{q}^2,
$$

Use $\ddot{q} = -(\gamma/m)\dot{q}^n$

$$S' = -\frac{1}{2}\frac{m^2\alpha^2}{\gamma}\dot{q}^{2-n},$$
$$S = -\frac{1}{2}\frac{m^2\alpha^2}{\gamma}\frac{\dot{q}^{3-n}}{3-n}$$

We omit the integration constant. With an extra integration constant, the form we found here would not match $\lambda S^a = (1-\alpha)\dot{q}$. Let us use the result we found so far in Equation (2.2) and obtain:

$$\begin{aligned}\alpha m\ddot{q} &= -a\lambda p^2 S^{a-1},\\ &= -a(\alpha m\dot{q})^2\frac{\lambda S^a}{S},\\ &= 2a\gamma(1-\alpha)(3-n)\dot{q}^n,\end{aligned}$$

Use $\ddot{q} = -(\gamma/m)\dot{q}^n$

$$2a = \frac{\alpha}{1-\alpha}\frac{1}{n-3}$$

Choose $\alpha = 2$

$$a = \frac{1}{3-n}$$

We finally obtain λ in terms of m, n, γ using the expression for S and $\lambda S^a = (1-\alpha)\dot{q}$:

$$\lambda = -\left(\frac{2m^2}{\gamma}\frac{1}{n-3}\right)^{\frac{1}{n-3}}.$$

The contact Hamiltonian is as follows:

$$H = \frac{p^2}{2m} - \left(\frac{2m^2}{\gamma}\frac{1}{n-3}\right)^{\frac{1}{n-3}} pS^{1/(3-n)}.$$

This contact Hamiltonian gives us the following equation of motion:

$$m\ddot{q} + \gamma\dot{q}^n = 0,$$

for $n = 2$ and $n > 3$.

3. Friction term that is the sum of two monomials of $\dot{q}$

Our aim in this Section is to find a contact Hamiltonian that will yield a friction term that is a sum of two monomials of $\dot{q}$, that is, an equation of motion of the form $m\ddot{q} + \gamma_A\dot{q}^{n_A} + \gamma_B\dot{q}^{n_B} = 0$. However we will soon see that the only allowed combination is Equation (3.1). Let us consider the following contact Hamiltonian:

$$H = \frac{p^2}{2m} + \sum_k \lambda_k p^{b_k} S^{a_k},$$

where k runs over natural numbers (or any other countable set). The extra terms includes all analytic functions of p, S as well as other type functions with singularities. One can also absorb the first term into the sum, so this form is very general. This type of contact Hamiltonian, though seems quite general, can only model the following type of a differential equation:

$$m\ddot{q} + \gamma_A\dot{q}^{n_A} + \gamma_B\dot{q}^{n_A+1} = 0. \tag{3.1}$$

Let us write down the equations of motion for q, p, S:

$$\dot{q} = \frac{p}{m} + \sum_k b_k \lambda_k p^{b_k - 1} S^{a_k}, \tag{3.2}$$

$$\dot{p} = -\sum_k a_k \lambda_k p^{b_k + 1} S^{a_k - 1}, \tag{3.3}$$

$$\dot{S} = \frac{p^2}{2m} + \sum_k (b_k - 1) \lambda_k p^{b_k} S^{a_k}. \tag{3.4}$$

As an ansatz, let us write $p = \alpha m \dot{q}$ for some constant α. Then the first equation becomes:

$$\dot{q} = \alpha \dot{q} + \sum_k b_k \lambda_k (\alpha m \dot{q})^{b_k - 1} S^{a_k}.$$

Then let $S = \beta \dot{q}^c$ for some constants β, c. We obtain:

$$(1 - \alpha)\dot{q} = \sum_{b_k \neq 0} b_k \lambda_k (\alpha m \dot{q})^{b_k - 1} (\beta \dot{q}^c)^{a_k}.$$

So we obtain the following condition by equating the powers of $\dot{q}$:

$$c a_k + b_k = 2, \quad \text{if } b_k \neq 0, \tag{3.5}$$

and the remaining equation is the following:

$$1 - \alpha = \sum_{b_k \neq 0} b_k \lambda_k (\alpha m)^{b_k - 1} \beta^{a_k}. \tag{3.6}$$

The Equation (3.3) becomes:

$$\begin{aligned} \alpha m \ddot{q} &= -\sum_k a_k \lambda_k (\alpha m \dot{q})^{b_k + 1} (\beta \dot{q}^c)^{a_k - 1}, \\ &= -\sum_{b_k = 0} a_k \lambda_k \alpha m \beta^{a_k - 1} \dot{q}^{1 + c(a_k - 1)} \\ &\quad - \sum_{b_k \neq 0} a_k \lambda_k (\alpha m)^{b_k + 1} \beta^{a_k - 1} \dot{q}^{3 - c}. \end{aligned}$$

Hence we obtain:

$$\begin{aligned} m \ddot{q} = &-\sum_{b_k = 0} a_k \lambda_k m \beta^{a_k - 1} \dot{q}^{1 + c(a_k - 1)} \\ &- \sum_{b_k \neq 0} a_k \lambda_k \frac{(\alpha m)^{b_k + 1}}{\alpha} \beta^{a_k - 1} \dot{q}^{3 - c}. \end{aligned} \tag{3.7}$$

3.1 When $c \neq 0$

In this Subsection we suppose $c \neq 0$.[1] Then Equation (3.4) yields:

$$\beta c \dot{q}^{c-1} \ddot{q} = \frac{1}{2} m \alpha^2 \dot{q}^2 + \sum_k (b_k - 1) \lambda_k (\alpha m \dot{q})^{b_k} (\beta \dot{q}^c)^{a_k}.$$

[1] The case where $c = 0$ is investigated in Subsection 3.2 and causes a vanishing Hamiltonian, but included in this article for completeness.

From this, we obtain:

$$m\ddot{q} = \frac{1}{2}\frac{\alpha^2 m^2}{\beta c}\dot{q}^{3-c} + \sum_k (b_k - 1)\lambda_k (\alpha m)^{b_k}\frac{m}{c}\beta^{a_k-1}\dot{q}^{ca_k+b_k+1-c}.$$

When collected as a sum over $b_k = 0$ and $b_k \neq 0$ we obtain:

$$m\ddot{q} = \frac{1}{2}\frac{\alpha^2 m^2}{\beta c}\dot{q}^{3-c} - \sum_{b_k=0}\lambda_k\frac{m}{c}\beta^{a_k-1}\dot{q}^{1+c_k(a_k-1)} + \sum_{b_k\neq 0}(b_k-1)\lambda_k\frac{(\alpha m)^{b_k}m}{c}\beta^{a_k-1}\dot{q}^{3-c}. \tag{3.8}$$

We now have two equations of motion $m\ddot{q}$. They need to be consistent with each other. So we equate (and suppose $1 + c(a_k - 1) \neq 3 - c$ or $ca_k \neq 2$.) Equation (3.7) and Equation (3.8):

$$\sum_{b_k=0}\lambda_k m\left(a_k - \frac{1}{c}\right)\beta^{a_k-1}\dot{q}^{1+c(a_k-1)} = 0, \tag{3.9}$$

from which we obtain:

$$a_k = 1/c, \quad \text{if } b_k = 0, \tag{3.10}$$

and

$$\frac{1}{2}\frac{\alpha^2 m^2}{\beta c} + \sum_{b_k\neq 0}(b_k-1)\lambda_k\frac{(\alpha m)^{b_k}m}{c}\beta^{a_k-1} = -\sum_{b_k\neq 0}a_k\lambda_k\frac{(\alpha m)^{b_k+1}}{\alpha}\beta^{a_k-1},$$

which yields

$$-\frac{1}{2} = \sum_{b_k\neq 0}\lambda_k(\alpha m)^{b_k-2}m\beta^{a_k}.$$

Since c is a constant, we see that we can only obtain two powers of $\dot{q}$ in the equation of motion. The first is a power of $3-c$ (when $b_k \neq 0$) and the second is a power of $1 + c(a_k - 1) = 2 - c$ (when $b_k = 0$). As a result, it is sufficient to consider two types of variables $(b_k, a_k) \in \{(0, a_A), (b_B, a_B)\}$. In this case the contact Hamiltonian is as follows:

$$H = \frac{p^2}{2m} + \lambda_A S^{a_A} + \lambda_B p^{b_B} S^{a_B}.$$

We equate Equation (3.7) to $-\gamma_A\dot{q}^{n_A} - \gamma_B\dot{q}^{n_B}$ and obtain:

$$-\sum_{b_k=0}a_k\lambda_k m\beta^{a_k-1}\dot{q}^{1+c(a_k-1)} - \sum_{b_k\neq 0}a_k\lambda_k\frac{(\alpha m)^{b_k+1}}{\alpha}\beta^{a_k-1}\dot{q}^{3-c} = -\gamma_A\dot{q}^{n_A} - \gamma_B\dot{q}^{n_B}$$

Putting the relations between the constants we found and their values we get:

$$n_A = 2 - c, \tag{3.11}$$
$$n_B = 3 - c, \tag{3.12}$$
$$\lambda_A = \frac{c\gamma_A}{m}\beta^{1-1/c}, \tag{3.13}$$
$$\lambda_B = \frac{\gamma_B}{a_B m}\frac{\beta^{1-a_B}}{(\alpha m)^{b_B}}. \tag{3.14}$$

Moreoever for λ_B we have one more equation, namely Equation (3.6), which gives us the following constraint:

$$\lambda_B = \frac{1-\alpha}{b_B}\frac{(\alpha m)^{1-b_B}}{\beta^{a_B}} \tag{3.15}$$

By equating Equation (3.14) and Equation (3.15) we obtain:

$$\beta = \frac{m^2\alpha(1-\alpha)}{\gamma_B}\frac{a_B}{b_B} = \frac{m^2\alpha(1-\alpha)}{\gamma_B}\frac{a_B}{2-ca_B}.$$

All in all we have the following relations:

$$\begin{aligned}
a_A &= \frac{1}{2-n_A} = \frac{1}{3-n_B},\\
n_B &= n_A + 1,\\
n_A &= 2 - \frac{1}{a_A},\\
\lambda_A &= \frac{\gamma_A}{ma_A}\beta^{1-a_A},\\
\lambda_B &= \frac{\gamma_B}{ma_B}\frac{\beta^{1-a_B}}{(\alpha m)^{b_B}}.
\end{aligned}$$

Only λ_A can vanish. By letting $\lambda_A = 0$ one can model a system where frictional force is proportional to a monomial of $\dot{q}$.

3.2 When $c = 0$

In this Subsection we consider the case $c = 0$. Under this situation we have $S = \beta$. Hence $\dot{S} = 0$. Equation (3.5) yields the following:

$$\text{if } b_k \neq 0, b_k = 2.$$

When used in Equation (3.4) we obtain the following two equations:

$$\sum_{b_k=0} \lambda_k \beta^{a_k} = 0, \tag{3.16}$$

$$\sum_{b_k=2} \lambda_k \beta^{a_k} = -\frac{1}{2m} \tag{3.17}$$

Since there are two options, we can restrict the set of (b_k, a_k) to two values: $(b_k, a_k) \in \{(0, a_D), (2, a_E)\}$. Equation (3.16) gives us $\lambda_D = 0$. So the contact Hamiltonian is of the following form:

$$H = \frac{p^2}{2m} + \lambda_E p^2 S^{a_E},$$

with $\lambda_E = -\beta^{-a_E}/(2m)$ obtained from Equation (3.17). Let us write the equation of motion derived by $\dot{p}$ (Equation (3.7)):

$$\begin{aligned}
m\ddot{q} &= -\sum_{b_k=0} a_k\lambda_k m\beta^{a_k-1}\dot{q}^{1+c(a_k-1)} - \sum_{b_k\neq 0} a_k\lambda_k \frac{(\alpha m)^{b_k+1}}{\alpha}\beta^{a_k-1}\dot{q}^{3-c},\\
&= -a_D\lambda_D m\beta^{a_D-1}\dot{q} - a_E\lambda_E\frac{(\alpha m)^3}{\alpha}\beta^{a_E-1}\dot{q}^3
\end{aligned}$$

First term vanishes since $\lambda_D = 0$. The second term is proportional to the derivative of $\lambda_E\beta^{a_E} = 1/(2m)$ (see Equation (3.17)) with respect to β and is thus zero.

$$= 0.$$

Finally we obtain the dynamics of a free particle in 1D. As a result, this case is not interesting and does not cause frictional dynamics to appear.

4. Friction term that has linear and quadratic dependence on $\dot{q}$

In this Section, we give an application of Section 3 to the case where there are linear and quadratic dependencies of the friction force on speed ($\dot{q}$). The equation of motion is the following:

$$m\ddot{q} + \gamma_A\dot{q} + \gamma_B\dot{q}^2 = 0.$$

The contact Hamiltonian is the following:

$$H = \frac{p^2}{2m} + \lambda_A S^{a_A} + \lambda_B p^{b_B} S^{a_B}.$$

Using the results of Section 3 we obtain:

$$\begin{aligned} n_A &= 1, \\ n_B &= 2, \\ a_A &= 1, \end{aligned}$$

Let us choose $a_B = b_B = 1$ and $\alpha = 2$

$$\begin{aligned} a_B &= 1, \\ b_B &= 1, \\ \alpha &= 2, \end{aligned}$$

Then we obtain:

$$\begin{aligned} \lambda_A &= \frac{\gamma_A}{m}, \\ \lambda_B &= \frac{\gamma_B}{2m^2}. \end{aligned}$$

So the contact Hamiltonian is the following:

$$H = \frac{p^2}{2m} + \frac{\gamma_A}{m}S + \frac{\gamma_B}{2m^2}pS,$$

and it yields the following equation of motion:

$$m\ddot{q} + \gamma_A\dot{q} + \gamma_B\dot{q}^2 = 0.$$

This is an important step, because in classical mechanics when the speed is low the friction is linear in velocity and when the speed is high the frictional force is quadratic in velocity due to effect of turbulence.

5. Conclusion

In this paper, we mainly focused on contact Hamiltonian description 1D frictional systems. The contact Hamiltonians of the form $H = p^2/2m + \lambda pS^a$ can describe a situation where friction force is a monomial of $\dot{q}$:

$$m\ddot{q} + \gamma\dot{q}^n = 0, \tag{5.1}$$

for $n = 2$ and $n > 3$. The case for $n = 1$ is given in Ref. [7] and the contact Hamiltonian for that case is $H = p^2/2m + V(q) + \gamma S$ and it is the only contact Hamiltonian found so far to include an arbitrary potential. An exponentially decreasing potential in the case of quadratic dependence on velocity of the friction term is found in Ref. [9].

On the other hand, what we found is that a quite general contact Hamiltonian of the form $H = p^2/2m + \sum_k \lambda_k p^{b_k} S^{a_k}$ (which includes all analytic functions of p, S) can at most describe a dissipative system in the following form:

$$m\ddot{q} + \gamma_A \dot{q}^{n_A} + \gamma_B \dot{q}^{n_A+1} = 0. \tag{5.2}$$

In order to solve the contact equations of motion, we considered p, S to be functions of $\dot{q}$ and used two different ansatzes for this purpose. We conjecture that it is impossible to model a dissipative system with no potential that is not of the form appearing in Equation (5.1) or Equation (5.2).

We also have given the contact Hamiltonian description of the following equation of motion:

$$m\ddot{q} + \gamma_A \dot{q} + \gamma_B \dot{q}^2 = 0.$$

This form of frictional force is the most prevalent in nature. When the speed is low the linear term is dominant, and when the speed is high the quadratic term becomes dominant due to turbulence.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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GBS Operators of Bivariate Durrmeyer Operators on Simplex

Harun Çiçek[1]*, Aydın İzgi[2],Mehmet Ayhan[3]

Abstract
We define GBS operators of Durrmeyer operators for bivariate functions on simplex and we give their approximations and rate of their approximations for B-continuous and B-differentiable functions. We show that the GBS type the operators of new Durrmeyer have better approximation than the new operators.

Keywords: Bivariate functions, Bögel continuous, Bögel differentiable, Durrmeyer operators on simplex, GBS-operators, Rate of approximation.
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[1] *Department of Mathematics, Faculty of Science and Arts, Harran University, Şanlıurfa, Turkey, ORCID: https://orcid.org/0000-0003-3018-3015*
[2] *Department of Mathematics, Faculty of Science and Arts, Harran University, Şanlıurfa, Turkey, ORCID: https://orcid.org/0000-0003-3715-8621*
[3] *Department of Mathematics, Faculty of Science and Arts, Harran University, Şanlıurfa, Turkey, ORCID: https://orcid.org/0000-0002-8904-9489*
***Corresponding author**: haruncicek@harran.edu.tr
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1. Preliminaries

Polynomial approach and the classical approximation theory constitute a basic research area in applied mathematics. The development of the approximation theory played an important role in the numerical solution of partial differential equations, data processing sciences, and many other disciplines. For example, it is widely used in geometric modeling in the aerospace and automotive industries to calculate approximate values with basic functions. Work in this field goes back to the 18th century and still continues as a powerful tool in scientific calculations. Furthermore, it is used in civil engineering projects to analyze the energy efficiency and earthquake resistance data of different types of buildings in thermography calculations and earthquake engineering. The purpose of the approximation theory is to provide an approach between function spaces. In this context, the best approximation uses a linear positive operator. An operator that brings a function of positive value in one function space to another function of positive value in another function space is called a positive operator; whereas the operators that are both positive and linear are called linear positive operators. We will introduce a generalization of Bernstein operators that form the basis of linear positive operators. This new generalization to be defined will be a better version of Bernstein operators that contribute to all of the above mentioned fields of study. In this way, it is aimed to have a better approach. Before introducing the operator, if we need to talk about previous studies. Weierstrass, who laid the foundations of the approach with a linear positive operator, said in 1885, that each continuous function as an element of C [a, b] was a sequence that could be approached with a polynomial in the same closed range, but he did not specify the properties of these sequences. In 1912 Bernstein, proved that the sequences in the Weierstrass theorem were the polynomials referred to by his name and exposed them as follows:

$$B_n(h;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} h\left(\frac{k}{n}\right).$$

The modified Bernstein polynomials,

$$D_n(f;x)=(n+1)\sum_{k=0}^{n}\varphi_n^k(x)\left(\int_0^1\varphi_n^k(t)f(t)dt\right),(n\geq 1),$$

where $\varphi_n^k(x)=\begin{pmatrix} n \\ k \end{pmatrix}x^k(1-x)^{n-k}$, $(0\leq x\leq 1)$, were introduced by Durrmeyer [1] and Derriennic [2] gave some results on approximation of function f on [0, 1] by (1)

In [2], it is shown that,for $m\in\mathbb{N}$

$$D_n(t^m;x)=\frac{(n+1)!}{(n+m+1)!}\sum_{r=0}^{m}\begin{pmatrix} m \\ r \end{pmatrix}\frac{m!}{r!}\frac{n!}{(n-r)!}x^r.$$

Denoting by $\Delta=\{(x,y):x\geq 0,\,y\geq 0 \text{ and } x+y\leq 1\}$. Singh [3]defined new class of positive linear operators of order n on Δ by

$$S_n(f;x,y)=\frac{(n+2)!}{n!}\sum_{k=0}^{n}\sum_{j=0}^{n-k}P_{n,k,j}(x,y)\int\int_{\Delta}P_{n,k,j}(u,v)f(u,v)dudv \tag{1.1}$$

where $P_{n,k,j}(x,y)=\begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} n-k \\ j \end{pmatrix}x^ky^j(1-x-y)^{n-k-j}$. Singh proved some results on approximation of function f on Δ by (1.1).

Define $e_i:=e_i(x)=x^i$, $E_i:=E_i(u,x)=(u-x)^i$, $e_{ij}:=e_{ij}(x,y)=x^iy^j$ and $E_{ij}:=E_{ij}(u,v;x,y)=(u-x)^i(v-y)^j$.

Lemma 1.1: ([3])

$$S_n(u^pv^q;x,y)=\frac{(n+2)!}{(n+p+q+2)!}\sum_{r=0}^{p}\sum_{l=0}^{q}\begin{pmatrix} p \\ r \end{pmatrix}\begin{pmatrix} q \\ l \end{pmatrix}\frac{p!q!}{r!l!}x^ry^l$$

In particular,

$$S_n(e_{00};x,y)=1,$$

$$S_n(e_{10};x,y)=\frac{nx+1}{n+3},$$

$$S_n(e_{01};x,y)=\frac{ny+1}{n+3}, \tag{1.2}$$

$$S_n(e_{20};x,y)=\frac{n(n-1)x^2+4nx+2}{(n+3)(n+4)}, \tag{1.3}$$

$$S_n(e_{02};x,y)=\frac{n(n-1)x^2+4nx+2}{(n+3)(n+4)}, \tag{1.4}$$

$$S_n(e_{30};x,y)=\frac{n(n-1)(n-2)x^3+9n(n-1)x^2+18nx+6}{(n+3)(n+4)(n+5)},$$

$$S_n(e_{03};x,y)=\frac{n(n-1)(n-2)y^3+9n(n-1)y^2+18ny+6}{(n+3)(n+4)(n+5)},$$

$$S_n(e_{40};x,y)=\frac{\alpha_n(3)x^4+16\alpha_n(2)x^3+72\alpha_n(1)x+24}{(n+3)(n+4)(n+5)(n+6)},$$

$$S_n(e_{04};x,y)=\frac{\alpha_n(3)y^4+16\alpha_n(2)y^3+72\alpha_n(1)y+24}{(n+3)(n+4)(n+5)(n+6)}.$$

$$S_n(E_{20};x,y) = \frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)},$$
$$S_n(E_{02};x,y) = \frac{2[1+(n-4)y-2(n-6)y^2]}{(n+3)(n+4)},$$

$$S_n(E_{40};x,y)=\frac{12[a_nx^4-2b_nx^3+a_nx^2+6(n-2)x+2]}{(n+3)(n+4)(n+5)(n+6)},$$

and

$$S_n(E_{04};x,y)=\frac{12[a_ny^4-2b_ny^3+a_ny^2+6(n-2)y+2]}{(n+3)(n+4)(n+5)(n+6)}.$$

where $\alpha_n(p)=\dfrac{n!}{(n-p-1)!}$, $a_n=(n^2-31n+30)$, and $b_n=(n^2-28n+20)$. For all $x\in[0,1]$ we have,

$$S_n(E_{20};x,y)\leq\begin{cases}\frac{1}{15} &, \quad n=6\\ \frac{4}{5} &, \quad n<6\\ \frac{n+6}{(n+3)(n+4)} &, \quad n>6\end{cases}$$

The situation for $S_n(E_{02};x,y)$ is the same (11). It is easy to see that

$$\frac{24}{(n+3)(n+4)}<\frac{12}{5(n+3)}$$

for all $n>6$.

And also for all $x\in[0,1]$ we have,

$$S_n(E_{40};x,y)\leq\begin{cases}\frac{15}{32} &, \quad n\leq 30\\ \frac{24}{(n+3)(n+4)} &, \quad n>30\end{cases}$$

The situation for $S_n(E_{04};x,y)$ is the same (12). It is easy to see that

$$\frac{24}{(n+3)(n+4)}<\frac{12}{17(n+3)}$$

for all $n>30$.

Our aim is to extend the operator (1.1) to case B-continuous (Bögel continuous) functions. The term ”B-continous” first was introduced by K. Bögel ([4], [5]). And then we shall present a GBS (Generalized Bögel Sum) operator of (1.1) and some approximation of properties of this operator. The term GBS(Generalized Boolean Sum) operators were introduced by Dobrescu and Matei [7]. The analogous of the well-known Korovkin theorem for approximation of B-continuous functions using GBS operators was given by C.Badea, I. Badea and H. Gonska [8]. The analogous of first modulus of continuity for bivariate B-continuous functions which is named ”mixed modulus of smoothness” was introduced by I. Badea[9]. (see Also H. H. Gonska[10] , C.Badea and C. Cottin [11]).

We show that the operators (2.1) (GBS type the operators of (1.1)) have better approximation than the operators (1.1) in fügures and numerical values.

Definition 1.1:

a) ([4], [5]) A function f is called a B-Continuous function in $(x_0,y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\left[(x,y),(x_0,y_0)\right] = 0.$$

where $\Delta f\left[(x,y),(x_0,y_0)\right] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ represents the mixed difference of f.

b) ([9]).Let $f \in B_b(X \times Y)$. For any $(\delta_1,\delta_2) \in \mathbb{R}^2_{0,+}$, the mixed modulus of smoothness is the function $\omega_{mixed}(f;\delta_1,\delta_2) : \mathbb{R}^2_{0,+} \to \mathbb{R}$ defined by

$$\omega_{mixed}(f;\delta_1,\delta_2) = \sup\left\{\left|\Delta f\left[(x,y),(u,v)\right]\right| : |u-x| \le \delta_1, |v-y| \le \delta_2\right\} \tag{1.5}$$

where $\mathbb{R}_{0,+} := [0,\infty)$.

c) ([6]) A function f is called a B-Differentiable function in $(x_0,y_0) \in X \times Y$ if the following limit is exist and finite,

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left[(x,y),(x_0,y_0)\right]}{(x-y)(x_0-y_0)}.$$

This B-Differentiable of f in (x_0,y_0) is denoted by $D_B f(x_0,y_0)$.

Let F be the class of all functions $f : X \times Y \to \mathbb{R}$. Then we use subsets of F which are given in the following:

$$B(X \times Y) = \{f \in F \ : f \text{ bounded on } X \times Y\}$$

with usual sup-norm $\|\|_\infty$.

$$B_b(X \times Y) = \{f \in F : \left|\Delta f\left[(x,y),(x_0,y_0)\right]\right| \le K, \text{on } X \times Y, K > 0\}$$

is called $B-$bounded functions class with the norm
$\|\|_b = \sup_{(x,y),(x_0,y_0)\in X\times Y} \left|\Delta f\left[(x,y),(x_0,y_0)\right]\right|.$

$$\begin{aligned} C_b(X \times Y) &= \{f \in F : f \text{ is } B-\text{Continuous on } X \times Y\}, \\ D_b(X \times Y) &= \{f \in F : f \text{ is } B-\text{Differentable on } X \times Y\} \end{aligned}$$

If $f : X \times Y \to \mathbb{R}$ is a continuous function in (x_0,y_0), it is also B-Continuous function in (x_0,y_0). A B-continuous function is not necessarily continuous(in usual sense), but the converse is true.

The approximation theorems for bivariate functions were first given by Volkov in [12] and approximation of the GBS operators of associate with operators of two variables were established by [8].

The term GBS(Generalized Boolean Sum) operators were introduced by Badea and Kottin as the following [11]

Definition 1.2. Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator. The operator $UL : C_b(X \times Y) \to B(X \times Y)$ is defined by

$$(ULf)(x,y) = \left(L\left(f(\bullet,y) + f(x,*) - f(\bullet,*)\right)\right)(x,y)$$

is called the GBS operator associated to the operator L, where "$\bullet$" and "$*$" stand for the first and second variable respectively.

From now on, we write $L(f(u,v);x,y)$ and

$$UL(f(u,v);x,y) = L\left(\left(f(u,y) + f(x,v) - f(u,v)\right);x,y\right)$$

instead of (1.2).

Theorem 1.1. ([8].) Let a,b,c,d be real numbers satisfying the inequalities $a < b,\ c < d$ and let $(T_{n,m})$ $(n,m \in \mathbb{Z}^+)$ be a sequence of bivariate linear positive operators, applying $C([a,b] \times [c,d])$ into itself. Suppose the following relations hold for any $(x,y) \in [a,b] \times [c,d]$.

$i)\, T_{n,m}(e_{00};x,y) = 1,$

$ii) T_{n,m}(e_{10};x,y) = x + u_{n,m}(x,y),$

$iii) T_{n,m}(e_{01};x,y) = y + v_{n,m}(x,y),$

$iv) T_{n,m}(e_{20}+e_{02};x,y) = x^2 + y^2 + w_{n,m}(x,y)$

If each of the sequence of $u_{n,m}(x,y), v_{n,m}(x,y)$ and $w_{n,m}(x,y)$ converges to zero uniformly as $n \to \infty$, $m \to \infty$, then the sequence $(T^*_{n,m})$ $(n,m \in \mathbb{Z}^+)$ converges to f uniformly on $[a,b] \times [c,d]$, where $T^*_{n,m}$ represent the GBS operator associate with $T_{n,m}$

Theorem 1.2 ([12]). Let $T : C([a,b] \times [c,d]) \to C([a,b] \times [c,d])$ be a linear positive operator and T^* the GBS operator associate with T.

Then, for any $f \in C([a,b] \times [c,d])$, $(x,y) \in [a,b] \times [c,d]$ and $\delta_1, \delta_2 > 0$, the following holds

$$\begin{aligned} |T^*(f(u,v);x,y) - f(x,y)| &\leq |f(x,y)|\,|1 - T(e_{00};x,y)| + \left\{ |T(e_{00};x,y)| + \frac{1}{\delta_1}\sqrt{T(E_{20};x,y)} + \frac{1}{\delta_2}\sqrt{T(E_{02};x,y)} \right. \\ &\left. + \frac{1}{\delta_1 \delta_2}\sqrt{T(E_{20};x,y)\,T(E_{02};x,y)} \right\} \omega_{mixed}(\delta_1, \delta_2). \end{aligned} \tag{1.6}$$

Theorem 1.3 ([4], [5] , [6]). Let $f : [u,v] \times [x,y] \to \mathbb{R}$ be a function. If f is $B-$differentiable on $[u,v] \times [x,y]$, there exist $(x_0, y_0) \in (u,v) \times (x,y)$ such that

$$\Delta f\left[(u,v),(x,y)\right] = (u-x)(v-y) D_B f(x_0, y_0).$$

Theorem 1.4([13]).Let $T : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UT : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x,y) \in X \times Y$ and $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned} |UT(f(u,v);x,y) - f(x,y)| &\leq |f(x,y)|\,|1 - T(e_{00};x,y)| + 3\,\|D_B f\|_\infty \sqrt{T(E_{20};x,y)\,T(E_{02};x,y)} \\ &+ \left[\sqrt{T(E_{20};x,y)\,T(E_{02};x,y)} + \frac{1}{\delta_1}\sqrt{T(E_{40};x,y)\,T(E_{02};x,y)} \right. \\ &\left. + \frac{1}{\delta_2}\sqrt{T(E_{20};x,y)\,T(E_{04};x,y)} + \frac{1}{\delta_1\delta_2} T(E_{20};x,y)\,T(E_{02};x,y) \right] \omega_{mixed}(D_B f,\, \delta_1, \delta_2). \end{aligned}$$

2. Representation of bivariate GBS operator of Durrmeyer operator

For any $f \in C_b(\Delta)$,($C_b(\Delta)$ is the class of all B-continuous functions on Δ) ,representation of bivariate GBS operator of Durrmeyer operator is

$$F_n(f;x,y) = \frac{(n+2)!}{n!} \sum_{k=0}^{n} \sum_{j=0}^{n-k} P_{n,k,j}(x,\,y) \int\int_{\Delta} P_{n,k,j}(u,\,v)\,[f(u,y) + f(x,v) - f(u,v)]\,du dv. \tag{2.1}$$

It is easy to see $F_n(f;x,y)$ is linear positive operator. Taking into account the relations (1.3), (1.4) and applying Theorem 1.1 we obtain the following theorem.

Theorem 2. 1: The sequence $(F_n)_{n\in\mathbb{N}}$ converges to any $f \in C_b(\Delta)$ uniformly.

It easy tı see the following relation:

$$F_n(u^i v^j;x,y) = x^i y^j \text{ for all } i,\, j = 0, 1, 2, ...$$

That means there is no approximation for any usual continuous function f on $[0,1]$. Mean $F_n(f;x,y) = f$ for all $f \in C(\Delta)$.

Theorem 2. 2 : If $f \in C_b(\Delta)$, then for any $(x,y) \in \Delta$, the following relation holds for all $n > 6$:

$$|F_n(f(u,v);x,y) - f(x,y)| \leq \frac{11}{2}\,\omega_{mixed}(f; \sqrt{\frac{1}{n+3}}, \sqrt{\frac{1}{n+3}}).$$

Proof : Applying Theorem 1.2 , Lemma 1.1 and (11^*) we have

$$
\begin{aligned}
|F_n(f(u,v);x,y)-f(x,y)| &\leq \left(\frac{1}{\delta_1}\sqrt{S_n(E_{20};x,y)}+\frac{1}{\delta_2}\sqrt{S_n(E_{02};x,y)}+\frac{1}{\delta_1\delta_2}\sqrt{S_n(E_{20};x,y)S_n(E_{02};x,y)}\right)\omega_{mixed}(f;\delta_1,\delta_2)\\
&\leq \left(\frac{1}{\delta_1}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}+\frac{1}{\delta_2}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\right.\\
&\quad\left.+\frac{1}{\delta_1\delta_2}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}.\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\right)\omega_{mixed}(f;\delta_1,\delta_2)\\
&\leq \left(\frac{1}{\delta_1}\sqrt{\frac{12}{5}}\sqrt{\frac{1}{n+3}}+\frac{1}{\delta_2}\sqrt{\frac{12}{5}}\sqrt{\frac{1}{n+3}}+\frac{1}{\delta_1\delta_2}\frac{12}{5}\sqrt{\frac{1}{n+3}}\sqrt{\frac{1}{n+3}}\right)\omega_{mixed}(f;\delta_1,\delta_2)
\end{aligned}
$$

If we choose $\delta_1=\sqrt{\frac{1}{n+3}}$ and $\delta_2=\sqrt{\frac{1}{n+3}}$, we get

$$|F_n(f(u,v);x,y)-f(x,y)|\leq\left(2\sqrt{\frac{12}{5}}+\frac{12}{5}\right).\omega_{mixed}(f;\delta_1,\delta_2)$$

and consider $2\sqrt{\frac{12}{5}}+\frac{12}{5}<\frac{11}{2}$, then the proof is beeing comlated.

Theorem 2. 3 : If $f\in D_B(\mathbb{R}^2_{0,+})$ with $D_Bf\in B(\mathbb{R}^2_{0,+})$, then for any $(x,y)\in\Delta$ and $n>30$,

$$|F_n(f(u,v);x,y)-f(x,y)| \leq \frac{36\,\|D_Bf\|_\infty}{5(n+3)}+\frac{17}{10}.\,\omega_{mixed}(D_Bf;\sqrt{\frac{1}{n+3}},\sqrt{\frac{1}{n+3}}).$$

Proof: Applying Theorem 1.4 and Lemma 1.1 (11^* and 12^*) ,we have

$$
\begin{aligned}
|F_n(f(u,v);x,y)-f(x,y)| &\leq 3\,\|D_Bf\|_\infty\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}.\\
&\quad\times\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}.\\
&\quad+\left[\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}\right.\\
&\quad+\frac{1}{\delta_1}\sqrt{\frac{12[a_nx^4-2b_nx^3+a_nx^2+6(n-2)x+2]}{(n+3)(n+4)(n+5)(n+6)}}\sqrt{\frac{2[1+(n-4)y-2(n-6)y^2]}{(n+3)(n+4)}}\\
&\quad+\frac{1}{\delta_2}\sqrt{\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}}.\sqrt{\frac{12[a_ny^4-2b_ny^3+a_ny^2+6(n-2)y+2]}{(n+3)(n+4)(n+5)(n+6)}}\\
&\quad\left.+\frac{1}{\delta_1\delta_2}.\frac{2[1+(n-4)x-2(n-6)x^2]}{(n+3)(n+4)}\,\frac{2[1+(n-4)y-2(n-6)y^2]}{(n+3)(n+4)}\right]\omega_{mixed}(D_Bf,\,\delta_1,\delta_2).
\end{aligned}
$$

$$
\begin{aligned}
|F_n(f(u,v);x,y)-f(x,y)| &\leq \frac{36\,\|D_Bf\|_\infty}{5(n+3)}+\left[\frac{12}{5(n+3)}+\right.\\
&\quad+\frac{1}{\delta_1}\sqrt{\frac{12}{17(n+3)}\,\frac{12}{5(n+3)}}+\frac{1}{\delta_2}\sqrt{\frac{12}{17(n+3)}\,\frac{12}{5(n+3)}}\\
&\quad\left.+\frac{1}{\delta_1\delta_2}.\frac{12}{5(n+3)}\,\frac{12}{5(n+3)}\right]\omega_{mixed}(D_Bf,\,\delta_1,\delta_2)
\end{aligned}
$$

If we choose $\delta_1=\sqrt{\frac{1}{n+3}}$ and $\delta_2=\sqrt{\frac{1}{n+3}}$ and take into account that $\frac{1}{n+3}<\frac{1}{33}$ for $n>30$, then we get desired result.

3. Conclusion

As a result, it is seen that the operator defined in GBS format takes a better approximation. In order to more visibly show that this approximation is better, a numerical value table with the margin of error and a graph can be drawn at different points.

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