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Projective Synchronization of The Modified Fractional-Order Hyperchaotic Rössler System and Its Application in Secure Communication

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Abstract

In this paper, we propose a new approach to investigate the chaos projective synchronization of the modified fractional-order hyperchaotic Rössler system and its application in secure communication. The proposed communication system consists of four main elements including: modulation, master system, slave system and demodulation. The main idea of this approach is to inject the bounded or unbounded message into one of the parameters of the proposed system using the exponential function. However, the way of injecting the message in the modulation parameter must not remove the hyperchaotic character of the signal sent to the slave system. The slave system adaptively synchronizes with the master system, and the information signal can be recovered. Based on the Lyapunov stability theory, an adaptation laws and adaptive control are designed to achieve projection synchronization of the modified system. Numerical simulations are performed to show the feasibility of the proposed secure communication scheme.

1. Introduction

The concept of using chaos theory for communication systems was essentially inspired by the work of Pecora and Carroll in 1990 [1]. They discovered that two identical chaotic systems with different initial conditions can synchronize if they are properly coupled.
The chaotic transmission is a mode of secure communication that arises from the inclusion of chaos in transmission systems. The main idea of the chaotic transmission is to inject the message into a chaotic signal to hide this information and send it to the receiver system through a public channel. Thus, after the synchronization of the two chaotic systems (transmitter and receiver), the encrypted information is thus recovered at the receiver system. On the other hand, in literature, one often finds the name of the fractional derivation to the generalization of the derivation to an arbitrary order. The concepts of derivation and fractional integration are often associated with the names of Riemann-Liouville, whereas the question about the generalization of these notions is older.
With particular attention from physicists as well as engineers, a remarkable research activity has been devoted to fractional computing. Indeed, it has been found that many real physical systems are better characterized by dynamic models of fractional orders, such as diffusion systems [2], chemical systems [3], electrochemical systems [4], biological systems [5] and viscoelastic systems [6], etc. The use of classical models based on a classical derivation is therefore not appropriate. Chaos synchronization phenomena have been of particular interest in the study of chaotic and hyperchaotic dynamical systems, since they can be applied to large areas of engineering and information science, particularly in secure communication [7], control processing [8] and cryptology [9].
The basic configuration of a synchronization system consists of two chaotic or hyperchaotic systems: a transmitter system and a receiver system. Note that the two previous systems can be identical (with different initial conditions) or completely different. The transmitter system synchronizes the receiver system via one or several signals. In the literature, divers control methods have been applied to achieve synchronization, such as approximated auxiliary system [10], active control [11], adaptative control [12] and fuzy adaptive control [13]. Using

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these methods, several concepts of chaotic and hyperchaotic synchronization have also been extended, such as complete synchronization [14], anti-synchronization [15], generalized synchronization [16], projective synchronization [17] and modified projective synchronization [18]. A great deal of work has been done in recent years, exploiting chaotic and hyperchaotic signals in the context of secure communications. Indeed, their characteristics, sensitivities to the initial conditions, deterministic dynamics, ergodicity and structure complexity, are well adapted to secure transmissions [19–21].
In most of the secure communication systems proposed above, the size of the message must be small enough, otherwise an hyperchaotic system may be asymptotically stable, which may render the retrieval of the transmitted signal unsuccessful. However, in some real applications, various messages to be transmitted can be unbounded.
In [22], X Wu et al. have proposed a new secure communication scheme based on the projective generalized synchronization of a hyperchaotic system, where the signal of the message is bounded or unbounded. However, it should be mentioned that the fundamental results of the previous work apply only to integer-order hyperchaotic systems to the design of the secure communication system. So, it is very interesting to extend them to the general case of fractional order systems and the work in this area is still considered a stimulating research topic.
Motivated by the above considerations, in this paper, we propose a new simple approach to solve both the problem of projective synchronization in the modified fractional-order hyperchaotic Rössler system and that of the transmission security, where the signal of the message is bounded or unbounded.
The current manuscript is organized as follows: In Section 2, we present the system description and some preliminaries. The main result of this paper concerning a new secure communication scheme based on fractional order hyperchaotic system is mainly presented in Section 3. Therefore, in order to achieve this purpose, a modified adaptative control and a parameter update rule are designed. Numerical simulations are presented to show the viability and efficiency of the proposed method in Section 4. Finally, we conclude our paper with a short summary in Section 5.

2. System description and preliminaries

Consider the new hyperchaotic system [23] written by the dynamic equations:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 + x_4, \\ \dot{x}_2 = x_1 + a_1 x_2, \\ \dot{x}_3 = x_1 x_3 - a_3 x_3 + a_2, \\ \dot{x}_4 = a_4 x_1. \end{cases} \tag{2.1}$$

For the parameter values $a_2 = 0.01$, $a_3 = 5$, $a_4 = 0.1$ and $0.16 \leq a_1 \leq 0.19$, the system has large hyperchaotic region. The variation of the three largest Lyapunov exponents for different values of a_1 is given in Figure 2.1.
From the Figure 2.1, one can say that there are two positive lyapunov's exponents, when $0.16 \leq a_1 \leq 0.19$, wich means that the system is

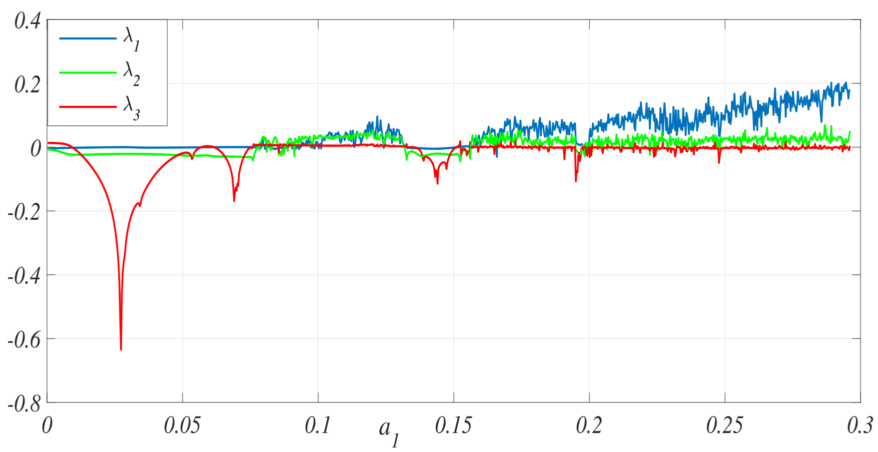

Figure 2.1: The three largest Lyapunov's exponents of system (2.1)

hyperchaotic.
The fractional version of the system (2.1) is governed by:

$$\begin{cases} D^{\alpha_1} x_1 = -x_2 - x_3 + x_4, \\ D^{\alpha_2} x_2 = x_1 + a_1 x_2, \\ D^{\alpha_3} x_3 = x_1 x_3 - a_3 x_3 + a_2, \\ D^{\alpha_4} x_4 = a_4 x_1. \end{cases} \tag{2.2}$$

where $\alpha_i \in]0,1[$, $i = 1,2,3,4$ are fractional-orders, and D^{α} is the Caputo derivative, which is defined as:

$$D^{\alpha} x(t) = J^{n-\alpha} x^{(n)}(t), \quad \alpha \in (0,1), \tag{2.3}$$

were $n = \lceil \alpha \rceil$, i.e., n is the first integer which is not less than α; $x^{(n)}$ is the general n-order derivative and J^{γ} is the γ-order Riemann–Liouville integral operator expressed as follows:

$$J^{\gamma} y = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} y(\tau) d\tau, \tag{2.4}$$

where $\Gamma(.)$ is the gamma function.

Remark 2.1. *The major advantage of the Caputo definition is that the initial conditions for fractional-order differential equations take a similar form as for integer-order differential equations.*

Remark 2.2. *In system (2.1), the fractional-order system is called a commensurate fractional-order system if* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, *otherwise the system is called an incommensurate fractional-order system.*

3. Main results

The Main results of this part is mainly devoted to a new secure communication scheme. This method is based on the projective synchronization (PS) of the modified fractional Rösler system, using the parametric modulation technique. Figure 3.1 describes the proposed hyperchaotic communication scheme based on parametric modulation. The signal of the message to be sent can be bounded or unbounded. The proposed communication system consists of four main elements including: modulation (using exponontial function), master system, slave system and demodulation. Finally, the original message signal transmitted can be successfully recovered by the estimated parameter and the proposed invertible function.

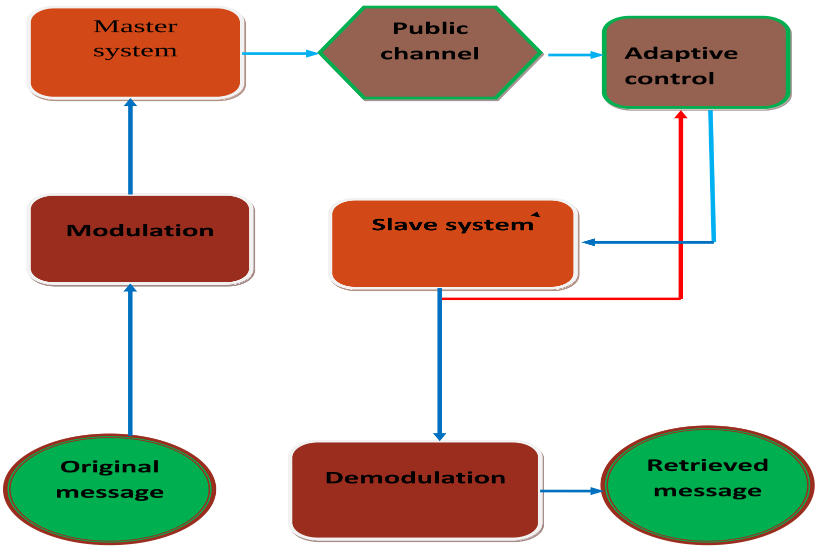

Figure 3.1: Principal diagram of the prposed secure communication.

In the proposed communication system, we plan to modulate in the unknown parameter a_1 of the system(2.2).
Let $m(t)$ be the signal of the message. Now let's define a new unknown parameter $A = A_1(t)$. In order to preserve the hyperchaotic behavior of the transmitter system studied, we propose the following parametric modulation technique:

$$A_1(t) = 0.03e^{-m(t)} + 0.16, \qquad m(t) \geq 0, \tag{3.1}$$

where $e^{(\cdot)}$ is the exponontial function.
Now, we replace the parameter a_1 of the system (2.2) by A_1, we have:

$$\begin{cases} D^{\alpha_1}x_1 = -x_2 - x_3 + x_4, \\ D^{\alpha_2}x_2 = x_1 + A_1x_2, \\ D^{\alpha_3}x_3 = x_1x_3 - a_3x_3 + a_2, \\ D^{\alpha_4}x_4 = a_4x_1, \end{cases} \tag{3.2}$$

where x_1, x_2, x_3, x_4 are chaotic signals that must be transmitted to the receiver system via a public channel. Since $A_1(t) \in [0.16, 0.19]$, the resulting system (3.2) is still hyperchaotic. Then we can take the system (3.2) as the master system.
Consider also the hyperchaotic slave system, which is supposed to be written by:

$$\begin{cases} D^{\alpha_1}y_1 = -y_2 - y_3 + y_4 + u_1, \\ D^{\alpha_2}y_2 = y_1 + \hat{A}_1y_2 + u_2, \\ D^{\alpha_3}y_3 = y_1y_3 - a_3y_3 + a_2 + u_3, \\ D^{\alpha_4}y_4 = a_4y_1 + u_4, \end{cases} \tag{3.3}$$

where $\hat{A}_1$ is the estimated parameter of A_1 and $u_i, i = 1,2,3,4$ are the controls to be determined.
Our main objective is to design a modified adaptive control u_i (for all $i = 1,2,3,4$) and a parameter $\hat{A}_1$ realizing a practical PS between the master system (3.2) and the slave system (3.3) and finally $\hat{A}_1$ converges towards the value A_1.
To quantify this goal, the synchronization error is defined as:

$$e_i = y_i - \theta x_i, \ \ i = 1,2,3,4, \tag{3.4}$$

where θ is a scaling factor defining a proportional relationship between the two synchronized systems.
Therefore, the complete synchronization and anti-synchronization are the special cases of a PS, when θ takes the values $+1$ and -1, respectively.
Let us also define the estimation error as:

$$e_{A_1} = A_1 - \hat{A}_1. \tag{3.5}$$

The error dynamics is easily obtained in the form:

$$D^{\alpha_i} e_i = D^{\alpha_i} y_i - \theta D^{\alpha_i} x_i, \ \ i = 1,2,3,4. \tag{3.6}$$

Inserting(3.2) and (3.3) in (3.6) yields the following:

$$\begin{cases} D^{\alpha_1} e_1 = -e_2 - e_3 + e_4 + u_1, \\ D^{\alpha_2} e_2 = e_1 + \hat{A}_1 e_2 - \theta e_{A_1} x_2 + u_2, \\ D^{\alpha_3} e_3 = -a_3 e_3 + y_1 y_3 - \theta x_1 x_3 + a_2(1-\theta) + u_3, \\ D^{\alpha_4} e_4 = a_4 e_1 + u_4. \end{cases} \tag{3.7}$$

Differentiating (3.5) from t, we have:

$$\dot{e}_{A_1} = -0.03\dot{m}e^{-m} - \dot{\hat{A}}_1 \tag{3.8}$$

On the basis of the previous discussions, we shall state and prove the following result:

Theorem 3.1. *(Main results) If the adaptive control parameter coordinates are selected as:*

$$\begin{cases} u_1 = e_2 + e_3 - e_4 - k_1 D^{\alpha_1 - 1} e_1, \\ u_2 = -e_1 - \hat{A}_1 e_2 + \theta e_{A_1} x_2 - D^{\alpha_2 - 1}(\theta e_{A_1} x_2 + k_2 e_2), \\ u_3 = a_3 e_3 - y_1 y_3 + \theta x_1 x_3 - a_2(1-\theta) - k_3 D^{\alpha_3 - 1} e_3, \\ u_4 = -a_4 e_1 - k_4 D^{\alpha_4 - 1} e_4, \end{cases} \tag{3.9}$$

where k_i, $i = 1,2,3,4$ are positive control gains,
and the update law for the parameter estimate is taken as:

$$\dot{\hat{A}}_1 = -\theta e_2 x_2 - 0.03\dot{m}e^{-m}, \tag{3.10}$$

then the PS between the two identical systems (3.2) and (3.3) is achieved.

Proof. Inserting (3.9) into (3.7), we get the error dynamic system as follows:

$$\begin{cases} D^{\alpha_1} e_1 = -k_1 D^{\alpha_1 - 1} e_1, \\ D^{\alpha_2} e_2 = -D^{\alpha_2 - 1}(\theta e_{A_1} x_2 + k_2 e_2), \\ D^{\alpha_3} e_3 = -k_3 D^{\alpha_3 - 1} e_3, \\ D^{\alpha 4} e_4 = -k_4 D^{\alpha_4 - 1} e_4. \end{cases} \tag{3.11}$$

Consider the Lyapunov function candidate as:

$$V = \frac{1}{2}\left(\sum_{i=1}^{4} e_i^2 + e_{A_1}^2\right). \tag{3.12}$$

Obviously, V is a positive semi-definite function on $\mathbb{R}^5$.
The time derivative of V along the error system (3.11) is:

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{4} e_i \dot{e}_i + e_{A_1}\dot{e}_{A_1} \\ &= \sum_{i=1}^{4} e_i D^{1-\alpha_i}(D^{\alpha_i} e_i) + e_{A_1}\dot{e}_{A_1} \\ &= e_1(-k_1 e_1) - e_2(\theta e_{A_1} x_2 + k_2 e_2) + e_3(-k_3 e_3) + e_4(-k_4 e_4) + e_{A_1}(-0.03\dot{m}\exp(-m) - \dot{\hat{A}}_1) \\ &= -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2 + k_4 e_4^2) + e_{A_1}(-\theta e_2 x_2 - 0.03\dot{m}\exp(-m) - \dot{\hat{A}}_1). \end{aligned} \tag{3.13}$$

Substituting the adaptation law(3.10) in (3.13), we have:

$$\dot{V} = -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2 + k_4 e_4^2), \tag{3.14}$$

which is negative semi-definite on $\mathbb{R}^5$. Therefore, according to Lyapunov stability theory, the synchronization errors e_i, $i = 1,2,3,4$ converge asymptotically to zero, i.e. the PS between the master system (3.2) and the slave system (3.3)is achieved. This completes the proof. □

Remark 3.2. *According to the proposed transformation function (3.1), the recovered signal message should be defined by:*

$$\hat{m}(t) = \ln\left(\frac{0.03}{\hat{A}_1(t) - 0.16}\right). \tag{3.15}$$

Once the synchronization errors e_i, $i = 1,2,3,4$ approaches zero, it means:

$$\hat{A}_1(t) \to A_1(t), \text{ when } t \to \infty. \tag{3.16}$$

Hence, we have:

$$\hat{m}(t) = \ln\left(\frac{0.03}{\hat{A}_1(t) - 0.16}\right) \to m(t) = \ln\left(\frac{0.03}{A_1(t) - 0.16}\right), \text{ when } t \to \infty. \tag{3.17}$$

Therefore, it can be concluded that the message signal can be finally recovered precisely by the identified parameter and the corresponding demodulation method.

4. Numerical simulations

In this section, computer simulations will be provided to verify the feasibility of the proposed communication system. The Adams-Bashforth-Moulton method is used to solve the fractional systems.

4.1. Case of a bounded information signal

Here, the hidden message signal in the slave system is given by:

$$m(t) = 3 - \cos(2t) - 2\cos(3t). \tag{4.1}$$

Obviously, $0 \le m(t) \le 6$. According to the equation (3.1), we can select $A_1(t)$ as follows:

$$A_1(t) = 0.03e^{(-3+\cos(2t)+2\cos(3t))} + 0.16. \tag{4.2}$$

It follows that $A_1(0) = 0.19$.
The initial condition for the adaptation law is given by: $\hat{A}_1(0) = 0.19$.
So the initial condition of the estimation error is given by: $e_{A_1}(0) = 0$.
The initial conditions of the two systems (3.2) and (3.3) are selected respectively as:

$$x_1(0) = -0.02,\ x_2(0) = -0.01,\ x_3(0) = -0.046,\ x_4(0) = 0.02. \tag{4.3}$$

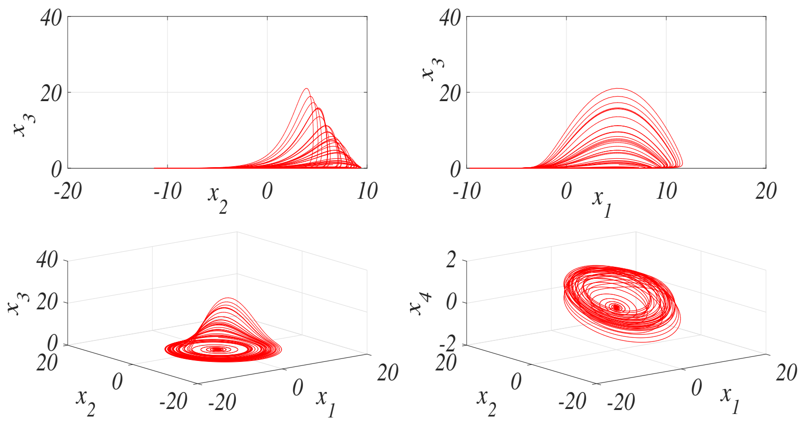

Figure 4.1: Projections of phase portraits of the resulting system (3.2). Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

$$y_1(0) = -0.08,\ y_2(0) = -0.08,\ y_3(0) = 0.128,\ y_4(0) = 0.07 \tag{4.4}$$

The parameter θ is selected randomly as:

$$\theta = 3. \tag{4.5}$$

As a result, the initial system error conditions are given by:

$$e_1(0) = -0.02,\ e_2(0) = -0.05,\ e_3(0) = 0.01,\ e_4(0) = 0.01. \tag{4.6}$$

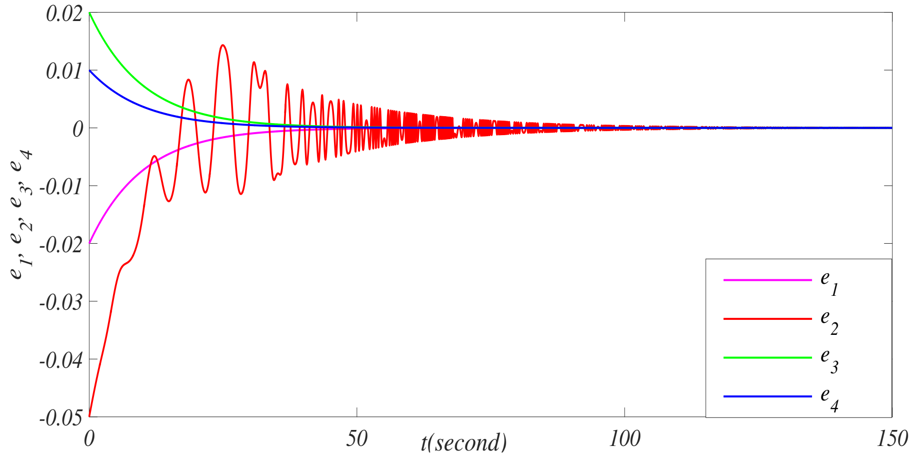

Figure 4.2: Time evolution of the synchronization errors. Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

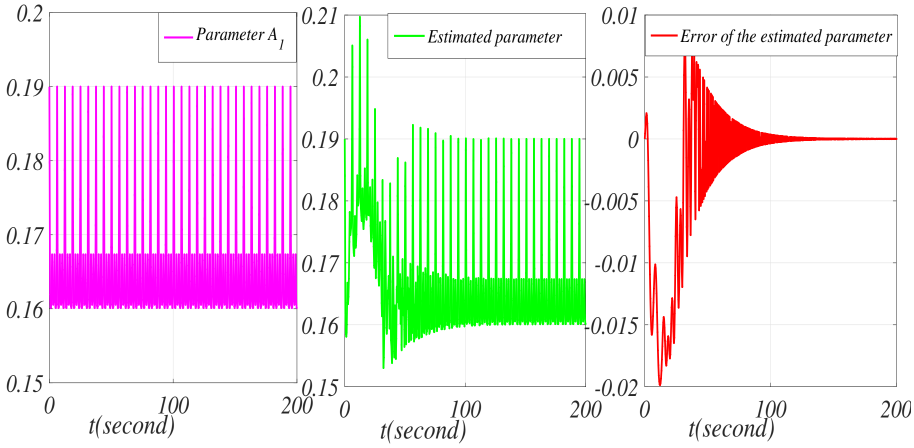

Figure 4.3: Time evolution of the parameter, the estimated parameter and the error of the estimated parameter. Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

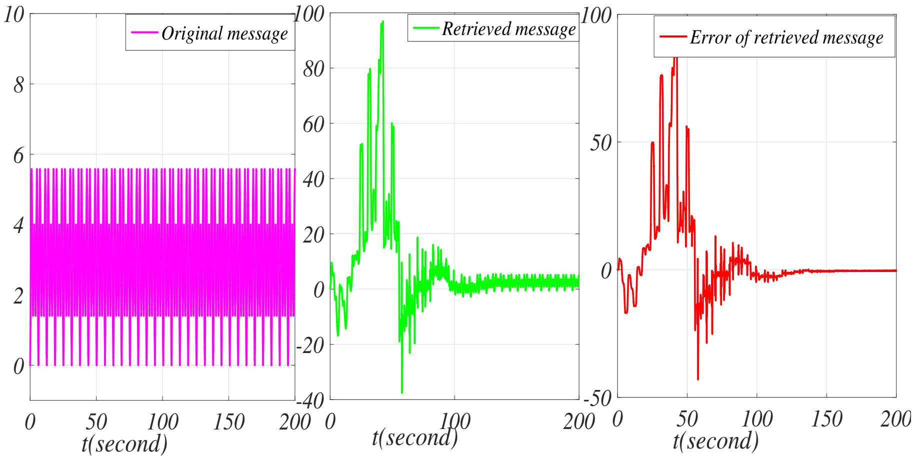

Figure 4.4: Time evolution of the original message, the retrieved message and the error of retrieved message. Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

The gain (design) parameters are chosen as follows:

$$k_1 = k_2 = k_4 = k_4 = 0.1. \tag{4.7}$$

The orders of fractional derivatives are chosen as:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.98, 0.98, 0.97, 0.97). \tag{4.8}$$

Figure 4.1 illustrates the projections of phase portraits of the resulting system (3.2). The simulation results of the proposed communication system are shown in Figures 4.2, 4.3 and 4.4.

Remark 4.1. *From the Figure 4.2, we can easily see that the errors synchronisation e_i, $i = 1,2,3,4$ converge asymptotically towards zero quickly, i.e., the PS between the master system and the slave system is obtained.*
On the other hand, Figure 4.4 describes the original message signal $m(t)$, the recovered message signal $\hat{m}(t)$ and the signal error via the demodulator (3.15).
From these figures, we can easily see that the error of the parameter converges quickly to zero, when $t \geq 100s$, which shows that the reconstructed signal $\hat{m}(t)$ coincides with the original message signal $m(t)$ with good precision, and the goal of secure communication is achieved.

4.2. Case of an unbounded information signal

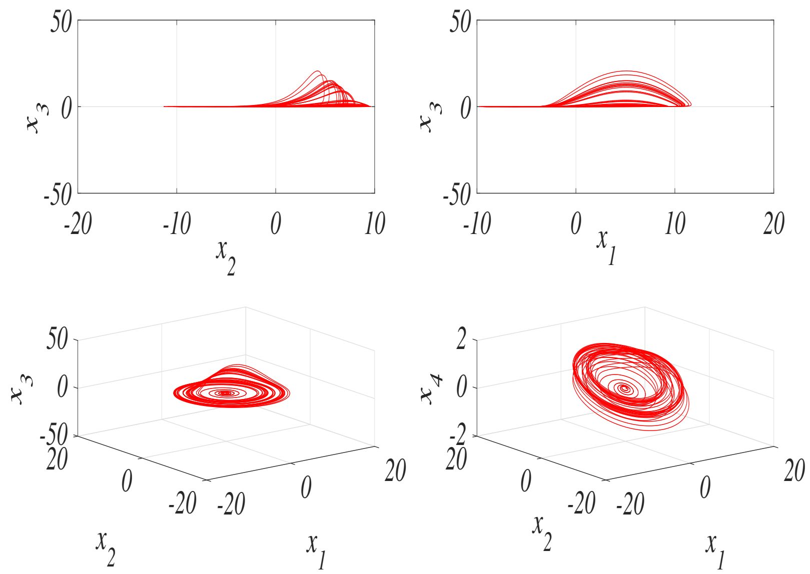

Figure 4.5: Different hyperchaotic attractors of the resulting system (3.2). Case of unbounded information signal: $m(t) = 0.05(t+\sin(t))$

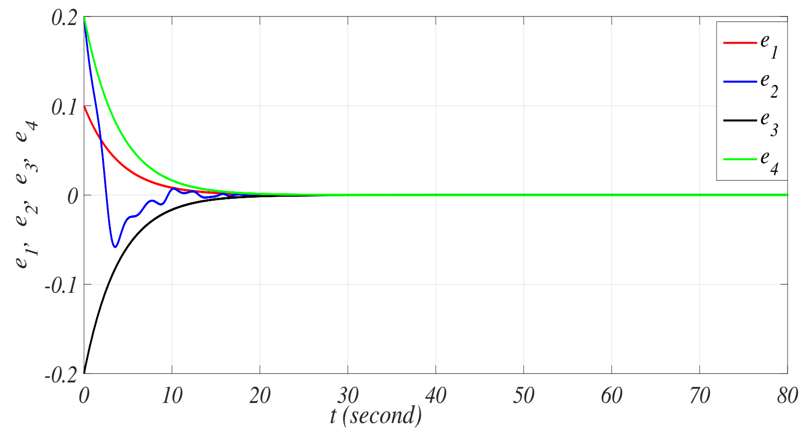

Figure 4.6: Time evolution of the synchronization errors. Case of unbounded information signal: $m(t) = 0.05(t+\sin(t))$

In this case, the message signal is taken as follows:

$$m(t) = 0.05(t + sin(t)), \tag{4.9}$$

According to the equation(3.1), $A_1(t)$ can be obtained as follows:

$$A_1(t) = 0.03e^{(-0.05(t+sin(t)))} + 0.16. \tag{4.10}$$

It follows that $A_1(0) = 0.19$.
The initial condition for the adaptation law is given by: $\hat{A}_1(0) = 0.19$.

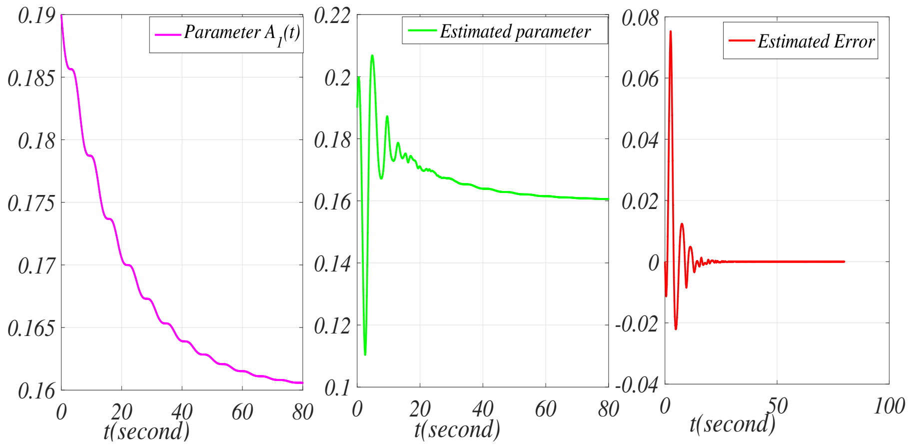

Figure 4.7: Time evolution of the parameter, the estimated parameter and the error of the estimated parameter. Case of unbounded information signal: $m(t) = 0.05(t + \sin(t))$

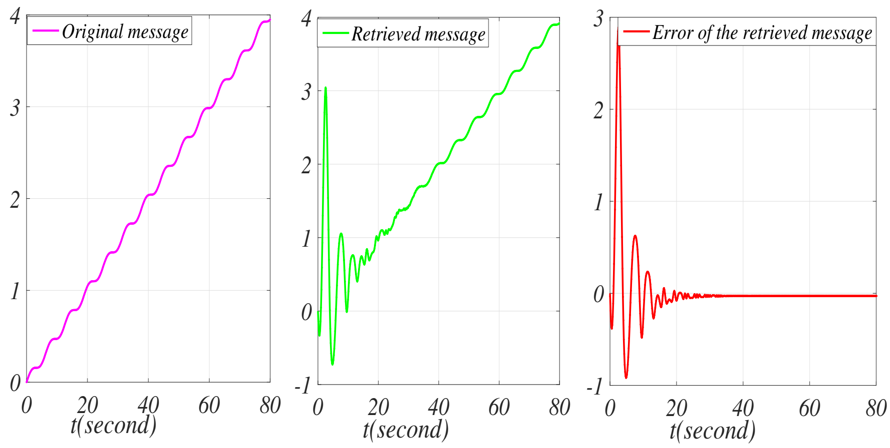

Figure 4.8: Time evolution of the original message, the retrieved message and the error of the retrieved message. Case of unbounded information signal: $m(t) = 0.05(t + \sin(t))$

So the initial condition of the estimation error is given by $e_{A_1}(0) = 0$.
The initial conditions of the two systems (3.2) and (3.3) are selected respectively as:

$$x_1(0) = 0.1,\ x_2(0) = -0.1,\ x_3(0) = -0.2,\ x_4(0) = 0.2. \tag{4.11}$$

$$y_1(0) = 0.3,\ y_2(0) = 0,\ y_3(0) = -0.6,\ y_4(0) = 0.6. \tag{4.12}$$

The scale parameter θ is randomly selected as:

$$\theta = 2. \tag{4.13}$$

Therefore, the initial system error conditions are given by:

$$e_1(0) = 0.1,\ e_2(0) = 0.2,\ e_3(0) = -0.2,\ e_4(0) = 0.2. \tag{4.14}$$

The gain parameters are chosen as follows:

$$k_1 = k_3 = k_4 = 0.25,\ k_2 = 0.5. \tag{4.15}$$

The orders of fractional derivatives are chosen as:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.98, 0.98, 0.98, 0.97). \tag{4.16}$$

The different hyperchaotic attractors of the resulting system (3.2) is shown in Figure 4.5 . The simulation results of the proposed communication system are shown in Figures 4.6, 4.7 and 4.8.

Remark 4.2. *From the Figure 4.6 , its easy to show that all of the synchronization errors e_i $i = 1, 2, 3, 4$, approach to zero quickly. Therefore, the proposed systems are globally synchronized.*
The original message signal $m(t)$, the recovered message signal $\hat{m}(t)$ and the signal error $\hat{m}(t) - m(t)$ are shown in Figure 4.8, which shows that the reconstructed signal $\hat{m}(t)$ coincides with the original message signal $m(t)$ with good precision, and the goal of secure communication is achieved.

5. Conclusion

In the present paper, a new approach for hyperchaotic secure communication method is included by using the parametric modulation technique. Two kinds of secure communication schemes in the case that the hidden message is bounded or unbounded are presented for the possible application in real engineering. We think that we have achieved two important goals. First one, using Lyapunov method, a modified adaptative controller and update law for a parameter estimate are introduced to achieve PS of fractional-order hyperchaotic systems. In particular, the errors system converge to zero quickly, which helps to find the time required. The most important part of this analysis is the proper design of modulation technique so that the message signals in both cases (bounded or unbounded) can be successfully and secretly transmitted via four main elements, namely: modulation, master system, slave system and demodulation. Finally, numerical simulations were provided to verify the effectiveness and feasibility of the proposed secure communication scheme.

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Hermite-Hadamard type Inequalities via p–Harmonic Exponential type Convexity and Applications

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Abstract

In this work, we introduce the idea and concept of p–harmonic exponential type convex functions. We elaborate on the newly introduced idea by examples and some interesting algebraic properties. In addition, we attain the novel version of Hermite–Hadamard type inequality in the mode of the newly introduced definition and on the basis of lemmas, some refinements of the Hermite–Hadamard type inequalities in the support of the newly introduced idea are established. Finally, we investigate and explore some integral inequalities in the form of applications for the arithmetic, geometric, harmonic and logarithmic means. The amazing tools and interesting ideas of this work may inspire and motivate further research in this direction furthermore.

1. Introduction

Theory of convexity present an active and attractive field of research. Many researchers endeavor, attempt and maintain his work on the concept of convexity, extend and generalize its variant forms in different ways using innovative ideas and fruitful techniques. This theory provides us with unified and unique framework to develop and organize highly efficient numerical tools to tackle and solve a wide class of problems that arise in pure and applied mathematics. In recent years, the concept of convexity has been improved, generalized, and extended in many directions. A number of studies have shown that the theory of convex functions has a close relationship with the theory of inequalities.

The integral inequalities are useful and have remarkable importance in optimization theory, functional analysis, physics and statistical theory. In the research area, inequalities have a lot of applications in probability, statistical problems and numerical quadrature formulas [10, 19, 20]. Due to many generalizations and extensions convex analysis and inequalities have become an attractive, interesting and absorbing field for the researchers and for attention reader can refer to [7, 17, 18, 21, 29].

It is well known that the harmonic mean is the special case of power mean. This mean has a lot of applications in different field of sciences which are computer science, geometry, probability, finance, trigonometry, statistics and electric circuit theory. Harmonic mean is the most appropriate measure for rates and ratios because it equalizes the weights of each data point. Harmonic mean is used to define the harmonic convex set. In 2003, first time harmonic convex set was introduced by Shi [27]. Harmonic and p–harmonic convex function was first time introduced and discussed by Anderson et al. [2] and Noor et al. [22] respectively. Nowadays a lot of people are working on exponential type convexity [5, 6]. Dragomir [9] introduced the class of exponential type convexity. After Dragomir, Awan [3] studied and investigated a new class of exponentially convex functions. Kadakal introduced a new definition of exponential type convexity in [16]. The amazing importance and applications of exponential type convexity is used to manipulate for statistical learning, information sciences, data mining, stochastic optimization and sequential prediction [1, 26, 28] and the references therein.
The principal focus and main aim of this note is to explore and define the idea of p–harmonic exponential type convex functions and in the support of these newly introduced functions, we attain its algebraic properties. Some interesting examples with logic are given as well. In addition, we attain the novel version of Hermite–Hadamard inequality in the mode of the newly discussed idea. Furthermore, we explore a new lemma and in order to this lemma, we attain some refinements of Hermite–Hadamard-type inequality in the manner of this newly

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explored definition. Finally, as applications, some new inequalities for the arithmetic, geometric and harmonic means are established. The awe-inspiring concepts and formidable tools of this paper may invigorate and revitalize for additional research in this worthy and absorbing field. Before we start, we need the following necessary known definitions and literature.

2. Preliminaries

In this section we recall some known concepts.

Definition 2.1. *[21] Let $\psi : I \to \mathbb{R}$ be a real valued function. A function ψ is said to be convex, if*

$$\psi(\kappa\sigma_1 + (1-\kappa)\sigma_2) \leq \kappa\psi(\sigma_1) + (1-\kappa)\psi(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\kappa \in [0,1]$.

Definition 2.2. *[15] A function $\psi : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be harmonic convex, if*

$$\psi\left(\frac{\sigma_1\sigma_2}{\kappa\sigma_2 + (1-\kappa)\sigma_1}\right) \leq \kappa\psi(\sigma_1) + (1-\kappa)\psi(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\kappa \in [0,1]$.

For the harmonic convex function, İşcan [15] provided the Hermite–Hadamard type inequality.

Definition 2.3. *[23] A function $\psi : I \to \mathbb{R}$ is said to be p–harmonic convex, if*

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p + (1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \leq \kappa\psi(\sigma_1) + (1-\kappa)\psi(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\kappa \in [0,1]$.

Note that $\kappa = \frac{1}{2}$ in the above Definition 2.3, we get the following inequality

$$\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p + \sigma_2^p}\right]^{\frac{1}{p}}\right) \leq \frac{\psi(\sigma_1) + \psi(\sigma_2)}{2},$$

holds for all $\sigma_1, \sigma_2 \in I$.
The function ψ is called Jensen p–harmonic convex function.
If we put $p = -1$ and $p = 1$, then p–harmonic convex sets and p–harmonic convex functions collapses to classical convex sets, harmonic convex sets and harmonic convex functions respectively.
We organise the paper in following way. Firstly, we will give the idea and its algebraic properties of p–harmonic exponential type convex functions. Secondly, we will derive the new sort of Hermite–Hadamard type and refinements of Hermite–Hadamard type inequalities by using the newly introduced idea. Finally, we will give some applications for means and conclusion.

3. p–harmonic Exponential Type Convex Functions and its Properties

We are going to introduce a new definition called p–harmonic exponential type convex function and will study some of their algebraic properties. Throughout the paper, one thing get in mind p–harmonic exp convex function represents p–harmonic exponential type convex function.

Definition 3.1. *A function $\psi : I \subseteq (0,+\infty) \to [0,+\infty)$ is called p–harmonic exp convex, if*

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p + (1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \leq \left(e^{\kappa} - 1\right)\psi(\sigma_1) + \left(e^{1-\kappa} - 1\right)\psi(\sigma_2),$$

holds for every $\sigma_1, \sigma_2 \in I$, and $\kappa \in [0,1]$.

Remark 3.2. *(i) Taking $p = 1$ in Definition 3.1, we obtain the following new definition about harmonically exp type convex function:*

$$\psi\left(\frac{\sigma_1\sigma_2}{\kappa\sigma_2 + (1-\kappa)\sigma_1}\right) \leq \left(e^{\kappa} - 1\right)\psi(\sigma_1) + \left(e^{1-\kappa} - 1\right)\psi(\sigma_2).$$

(ii) Taking $p = -1$ in Definition 3.1, then we get a definition namely exponential type convex function which is defined by Kadakal et al. [16].

That is the beauty of this newly introduce definition if we put the different values of p, then we obtain new inequalities and also found some results which connect with previous results.

Lemma 3.3. *The following inequalities $e^{\kappa} - 1 \geq \kappa$ and $e^{1-\kappa} - 1 \geq 1 - \kappa$ are hold. If for all $\kappa \in [0,1]$.*

Proof. The rest of the proof is clearly seen. □

Proposition 3.4. *Every p–harmonic convex function is p–harmonic exp convex function.*

Proof. Using the definition of p–harmonic convex function and from the lemma 3.3 , since $\kappa \leq e^{\kappa}-1$ and $1-\kappa \leq e^{1-\kappa}-1$ for all $\kappa \in [0,1]$, we have

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \leq \kappa\psi(\sigma_1)+(1-\kappa)\psi(\sigma_2) \leq \left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2).$$

□

Proposition 3.5. *Every p–harmonic exp convex function is p–harmonic h–convex function with* $h(\kappa)=(e^{\kappa}-1)$.

Proof.

$$\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) \leq \left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2) \leq h(\kappa)\psi(\sigma_1)+h(1-\kappa)\psi(\sigma_2).$$

□

Remark 3.6. (i) *If* $p=1$ *in Proposition 3.5, then as a result we get harmonically convex function, which is introduced by Noor et al. in [25].*
(ii) *If* $p=-1$ *in Proposition 3.5, then as a result we get h–convex function, which is defined by Varošanec et al. [29].*

Now we make and investigate some examples by way of newly introduced definition.

Example 3.7. *If* $\psi(\sigma)=\sigma^{p+1}$, $\forall\sigma\in(0,\infty)$ *is p–harmonic convex function, then according to Proposition 3.4, it is a p–harmonic exp convex function.*

Example 3.8. *If* $\psi(\sigma)=\frac{1}{\sigma^{2p}}$, $\forall\sigma\in\mathbb{R}\setminus\{0\}$ *is p–harmonic convex function, then according to Proposition 3.4, it is a p–harmonic exp convex function.*

Now, we will discuss and investigate some of its algebraic properties.

Theorem 3.9. *Let* $\psi,\varphi:[\sigma_1,\sigma_2]\to\mathbb{R}$. *If* ψ *and* φ *are two p–harmonic exp convex functions, then*
(i) $\psi+\varphi$ *is a p–harmonic exp convex function.*
(ii) *For* $c\in\mathbb{R}(c\geq 0)$, $c\psi$ *is a p–harmonic exp convex function.*

Proof. (i) Let ψ and φ be a p–harmonic exp convex, then

$$\begin{aligned}(\psi+\varphi)\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) &= \psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)+\varphi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\\ &\leq \left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)+\left(e^{\kappa}-1\right)\varphi(\sigma_1)+\left(e^{1-\kappa}-1\right)\varphi(\sigma_2)\\ &= \left(e^{\kappa}-1\right)[\psi(\sigma_1)+\varphi(\sigma_1)]+\left(e^{1-\kappa}-1\right)[\psi(\sigma_2)+\varphi(\sigma_2)]\\ &= \left(e^{\kappa}-1\right)(\psi+\varphi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(\psi+\varphi)(\sigma_2).\end{aligned}$$

(ii) Let ψ be a p–harmonic exp convex, then

$$\begin{aligned}(c\psi)\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) &\leq c\left[\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)\right]\\ &= \left(e^{\kappa}-1\right)c\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)c\psi(\sigma_2)\\ &= \left(e^{\kappa}-1\right)(c\psi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(c\psi)(\sigma_2),\end{aligned}$$

which completes the proof. □

Remark 3.10. (i) *If* $p=1$ *in Theorem 3.9, then as a result we get the* $\psi+\varphi$ *and* $c\psi$ *are harmonic exp convex functions.*
(ii) *If* $p=-1$ *in Theorem 3.9, then as a result we get Theorem 2.1 in [16].*

Theorem 3.11. *Let* $\psi:I=[\sigma_1,\sigma_2]\to J$ *be p–harmonic convex function and* $\varphi:J\to\mathbb{R}$ *is non-decreasing and exp convex function. Then the function* $\varphi\circ\psi:I=[\sigma_1,\sigma_2]\to\mathbb{R}$ *is a p–harmonic exp convex function.*

Proof. $\forall\ \sigma_1,\sigma_2\in I$, and $\kappa\in[0,1]$, we have

$$\begin{aligned}(\varphi\circ\psi)\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right) &= \varphi\left(\psi\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\\ &\leq \varphi\left(\kappa\psi(\sigma_1)+(1-\kappa)\psi(\sigma_2)\right)\\ &\leq \left(e^{\kappa}-1\right)\varphi(\psi(\sigma_1))+\left(e^{1-\kappa}-1\right)\varphi(\psi(\sigma_2))\\ &= \left(e^{\kappa}-1\right)(\varphi\circ\psi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(\varphi\circ\psi)(\sigma_2).\end{aligned}$$

□

Remark 3.12. *(i) In case of being $p=1$, as a result we attain the following new inequality*

$$(\varphi\circ\psi)\left[\frac{\sigma_1\sigma_2}{\kappa\sigma_2+(1-\kappa)\sigma_1}\right]\le\left(e^{\kappa}-1\right)(\varphi\circ\psi)(\sigma_1)+\left(e^{1-\kappa}-1\right)(\varphi\circ\psi)(\sigma_2).$$

(ii) In case of being $p=-1$, then as a result the above Theorem collapses to the Theorem 2.2 in [16].

Theorem 3.13. *Let $0<\sigma_1<\sigma_2$, $\psi_j:[\sigma_1,\sigma_2]\to[0,+\infty)$ be a class of p–harmonic exp convex functions and $\psi(u)=\sup_j\psi_j(u)$. Then ψ is a p–harmonic exp convex function and $U=\{u\in[\sigma_1,\sigma_2]:\psi(u)<+\infty\}$ is an interval.*

Proof. Let $\sigma_1,\sigma_2\in U$ and $\kappa\in[0,1]$, then

$$\begin{aligned}\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)&=\sup_j\psi_j\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\\&\le\left(e^{\kappa}-1\right)\sup_j\psi_j(\sigma_1)+\left(e^{1-\kappa}-1\right)\sup_j\psi_j(\sigma_2)\\&=\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)<+\infty,\end{aligned}$$

which completes the proof. □

Remark 3.14. *In case of being $p=-1$ in Theorem 3.13, as a result we get Theorem 2.3 in [16].*

Theorem 3.15. *If $\psi:[\sigma_1,\sigma_2]\to\mathbb{R}$ is a p–harmonic exp convex then ψ is bounded on $[\sigma_1,\sigma_2]$.*

Proof. Let $x\in[\sigma_1,\sigma_2]$ and $L=\max\{\psi(\sigma_1),\psi(\sigma_2)\}$, then there $\exists\ \kappa\in[0,1]$ such that $x=\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}$. Thus, since $e^{\kappa}\le e$ and $e^{1-\kappa}\le e$, we have

$$\begin{aligned}\psi(x)=\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)&\le\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)\\&\le\left(e^{\kappa}+e^{1-\kappa}-2\right)\cdot L\\&\le 2L[e-1]=M,\end{aligned}$$

The above proof clearly shows that ψ is bounded above from M. For bounded below, the readers using the identical concept as in Theorem (2.4) in [16]. □

Remark 3.16. *In case of being $p=-1$, we obtain Theorem 2.4 in [16].*

4. Hermite–Hadamard type inequality via p–harmonic exponential type convexity

The main object of this section is to investigate and prove a new version of Hermite–Hadamard type inequality using p–harmonic exp convexity.

Theorem 4.1. *Let $\psi:[\sigma_1,\sigma_2]\to[0,+\infty)$ be a p–harmonic exp convex function. If $\psi\in L[\sigma_1,\sigma_2]$, then*

$$\frac{1}{2\left(\sqrt{e}-1\right)}\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)\le\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu\le\left[\psi(\sigma_1)+\psi(\sigma_2)\right](e-2).$$

Proof. Since ψ is a p–harmonic exp convex function, we have

$$\psi\left(\left[\frac{x^py^p}{\kappa y^p+(1-\kappa)x^p}\right]^{\frac{1}{p}}\right)\le\left(e^{\kappa}-1\right)\psi(x)+\left(e^{1-\kappa}-1\right)\psi(y),$$

which lead to

$$\psi\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{\frac{1}{p}}\right)\le\left(\sqrt{e}-1\right)\psi(x)+\left(\sqrt{e}-1\right)\psi(y).$$

Using the change of variables, we get

$$\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)\le\left(\sqrt{e}-1\right)\times\left\{\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\left(\kappa\sigma_2^p+(1-\kappa)\sigma_1^p\right)}\right]^{\frac{1}{p}}\right)+\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\left(\kappa\sigma_1^p+(1-\kappa)\sigma_2^p\right)}\right]^{\frac{1}{p}}\right)\right\}.$$

Integrating the above inequality with respect to κ on $[0,1]$, we obtain

$$\frac{1}{2\left(\sqrt{e}-1\right)}\psi\left(\left[\frac{2\sigma_1^p\sigma_2^p}{\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)\le\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu,$$

which completes the left side inequality.

For the right side inequality, first of all we change the variable of integration by $\nu=\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}$ and using Definition 3.1 for the function ψ, we have

$$\begin{aligned}
\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(\nu)}{\nu^{p+1}}d\nu &= \int_0^1\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)d\kappa\\
&\leq \int_0^1\left[\left(e^{\kappa}-1\right)\psi(\sigma_1)+\left(e^{1-\kappa}-1\right)\psi(\sigma_2)\right]d\kappa\\
&= \psi(\sigma_1)\int_0^1\left(e^{t}-1\right)d\kappa+\psi(\sigma_2)\int_0^1\left(e^{1-\kappa}-1\right)d\kappa\\
&= \left[\psi(\sigma_1)+\psi(\sigma_2)\right](e-2),
\end{aligned}$$

which completes the proof. □

Remark 4.2. *(i) In case of being $p=-1$, then as a result we obtain Theorem 3.1 in [16].*
(ii) In case of being $p=1$, then as a result we obtain Corollary 1 in [11].

5. Refinements of Hermite–Hadamard type inequality via p–harmonic exponential type convexity

In this section, in order to prove our main results regarding on some Hermite–Hadamard type inequalities for p–harmonic exp convex function, we need the following lemmas:

Lemma 5.1. *. Let $\psi: I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$, then*

$$\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\int_0^1\frac{\mu(\kappa)}{A_\kappa^{p+1}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)d\kappa,$$

where $A_\kappa=\left[\kappa\sigma_2^p+(1-\kappa)\sigma_1^p\right]^{\frac{1}{p}}$ and $\mu(\kappa)=(1-2\kappa)$.

Proof. Let

$$I=\frac{\sigma_2^p-\sigma_1^p}{2p\sigma_1^p\sigma_2^p}\int_0^1(1-2\kappa)\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{1+\frac{1}{p}}\psi'\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)$$

Using integration by parts

$$\begin{aligned}
I &= \frac{\sigma_2^p-\sigma_1^p}{2p\sigma_1^p\sigma_2^p}\left\{\left|\frac{-p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}(1-2\kappa)\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)\right|_0^1-\frac{2p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_0^1\psi\left(\left[\frac{\sigma_1^p\sigma_2^p}{\kappa\sigma_2^p+(1-\kappa)\sigma_1^p}\right]^{\frac{1}{p}}\right)d\kappa\right\}\\
&= \frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx.
\end{aligned}$$

□

Lemma 5.2. *[24]. Let $\psi: I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$, then*

$$\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\int_0^1\frac{\mu(\kappa)}{A_\kappa^{p+1}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)d\kappa,$$

where $A_\kappa=\left[\kappa\sigma_2^p+(1-\kappa)\sigma_1^p\right]^{\frac{1}{p}}$ and

$$\mu(\kappa)=\begin{cases}\kappa-\frac{1}{8}, & \text{if } \kappa\in[0,\frac{1}{3})\\ \kappa-\frac{1}{2}, & \text{if } \kappa\in[\frac{1}{3},\frac{2}{3})\\ \kappa-\frac{7}{8}, & \text{if } \kappa\in[\frac{2}{3},1].\end{cases}$$

Theorem 5.3. *Let $\psi : I = [\sigma_1, \sigma_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable function on the I° of I. If $\psi' \in L[\sigma_1, \sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on I, $q \geq 1$, then*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{G_1^{1-\frac{1}{q}}\left[G_2|\psi'(\sigma_1)|^q + G_3|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_1 = \int_0^1 \frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa, \quad G_2 = \int_0^1 \frac{|1-2\kappa|(e^\kappa-1)}{A_\kappa^{1+p}}d\kappa,$$
$$G_3 = \int_0^1 \frac{|1-2\kappa|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}d\kappa.$$

Proof. Using Lemma 5.1, properties of modulus, power mean inequality and p–harmonic exp convexity of the $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\frac{|1-2\kappa|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}d\kappa\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\frac{|1-2\kappa|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa + \int_0^1\frac{|1-2\kappa|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{G_1^{1-\frac{1}{q}}\left[G_2|\psi'(\sigma_1)|^q+G_3|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which completes the proof. □

Corollary 5.4. *Under the assumptions of Theorem 5.3 with $p = -1$, we have the following new result*

$$\begin{aligned}
&\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right| \\
&\leq \frac{(\sigma_2-\sigma_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{8\sqrt{e}-2e-7}{2}\right)\left\{\left[|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 5.5. *Under the assumptions of Theorem 5.3 with $p = 1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^2}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2-\sigma_1)}{2}\left\{G_1'^{1-\frac{1}{q}}\left[G_2'|\psi'(\sigma_1)|^q+G_3'|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_1' = \int_0^1\frac{|1-2t|}{A_\kappa^2}d\kappa, \quad G_2' = \int_0^1\frac{|1-2\kappa|(e^\kappa-1)}{A_\kappa^2}d\kappa,$$
$$G_3' = \int_0^1\frac{|1-2\kappa|(e^{1-\kappa}-1)}{A_\kappa^2}d\kappa.$$

Theorem 5.6. *Let $\psi : I = [\sigma_1, \sigma_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable function on the I° of I. If $\psi' \in L[\sigma_1, \sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on I, $r, q \geq 1$, $\frac{1}{r}+\frac{1}{q} \geq 1$ then*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2} - \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_a^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\times\left\{G_4^{\frac{1}{r}}\left[G_5|\psi'(\sigma_1)|^q+G_6|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_4 = \int_0^1|1-2\kappa|^r d\kappa, \quad G_5 = \int_0^1\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,$$

$$G_6 = \int_0^1\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa.$$

Proof. Using Lemma 5.1, properties of modulus, Hölder's inequality and p–harmonic exp convexity of the $|\psi'|^q$, we have

$$\begin{aligned}\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| &\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\int_0^1\frac{|1-2\kappa|}{A_\kappa^{p+1}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa\\ &\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{\frac{1}{r}}\left(\int_0^1\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\\ &\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{\frac{1}{r}}\right.\\ &\left.\times\left(\int_0^1\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}\right\}\\ &= \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{2p}\left\{G_4^{\frac{1}{r}}\left[G_5|\psi'(\sigma_1)|^q+G_6|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},\end{aligned}$$

which completes the proof. □

Corollary 5.7. *Under the assumptions of Theorem 5.6 with $p=-1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right|\leq\frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^1|1-2\kappa|^r d\kappa\right)^{\frac{1}{r}}(e-2)\left(|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\right)^{\frac{1}{q}}.$$

Corollary 5.8. *Under the assumptions of Theorem 5.6 with $p=1$, we have the following new result*

$$\left|\frac{\psi(\sigma_1)+\psi(\sigma_2)}{2}-\frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^2}dx\right|\leq\frac{\sigma_1\sigma_2(\sigma_2-\sigma_1)}{2}\left\{G_4'^{\frac{1}{r}}\left[G_5'|\psi'(\sigma_1)|^q+G_6'|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\},$$

where

$$G_4'=\int_0^1|1-2\kappa|^r d\kappa,\quad G_5'=\int_0^1\frac{(e^\kappa-1)}{A_\kappa^{2q}}d\kappa,$$

$$G_6'=\int_0^1\frac{(e^{1-\kappa}-1)}{A_\kappa^{2q}}d\kappa.$$

Theorem 5.9. *Let $\psi:I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on I, $q\geq 1$ then*

$$\begin{aligned}&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\ &\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\left\{B_1^{1-\frac{1}{q}}[B_4|\psi'(\sigma_1)|^q+B_5|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right.\\ &\left.+B_2^{1-\frac{1}{q}}[B_6|\psi'(\sigma_1)|^q+B_7|\psi'(\sigma_2)|^q]^{\frac{1}{q}}+B_3^{1-\frac{1}{q}}[B_8|\psi'(\sigma_1)|^q+B_9|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right\},\end{aligned}$$

where

$$B_1=\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{p+1}}d\kappa,\quad B_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{p+1}}d\kappa,\quad B_3=\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{p+1}}d\kappa,$$

$$B_4=\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^\kappa-1)}{A_\kappa^{p+1}}d\kappa,\quad B_5=\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^{1-\kappa}-1)}{A_\kappa^{p+1}}d\kappa,$$

$$B_6=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^\kappa-1)}{A_\kappa^{p+1}}d\kappa,\quad B_7=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^{1-\kappa}-1)}{A_\kappa^{p+1}}d\kappa,$$

$$B_8=\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|(e^\kappa-1)}{A_\kappa^{p+1}}d\kappa,\quad B_9=\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|(e^{1-\kappa}-1)}{A_\kappa^{p+1}}d\kappa.$$

Proof. Using Lemma 5.2, properties of modulus, power mean inequality and p-harmonic exp convexity of the $|\psi'|^q$, we have

$$\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right|$$

$$\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left[\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{2}{3}}^1\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa\right]$$

$$\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left[\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right]\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}$$
$$\times\left[\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]}{A_\kappa^{1+p}}d\kappa\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left[\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa+\int_0^{\frac{1}{3}}\frac{|\kappa-\frac{1}{8}|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|}{A_\kappa^{1+p}}d\kappa\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|\kappa-\frac{1}{2}|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|}{A_t^{1+p}}dt\right)^{1-\frac{1}{q}}\times\left(\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|(e^\kappa-1)}{A_\kappa^{1+p}}|\psi'(\sigma_1)|^q d\kappa+\int_{\frac{2}{3}}^{1}\frac{|\kappa-\frac{7}{8}|(e^{1-\kappa}-1)}{A_\kappa^{1+p}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}\right]$$
$$=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\left\{B_1^{1-\frac{1}{q}}[B_4|\psi'(\sigma_1)|^q+B_5|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right.$$
$$\left.+B_2^{1-\frac{1}{q}}[B_6|\psi'(\sigma_1)|^q+B_7|\psi'(\sigma_2)|^q]^{\frac{1}{q}}+B_3^{1-\frac{1}{q}}[B_8|\psi'(\sigma_1)|^q+B_9|\psi'(\sigma_2)|^q]^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 5.10. *Under the assumptions of Theorem 5.9 with $p=-1$, we have the following new result*

$$\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\psi\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\psi(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right|$$
$$\leq(\sigma_2-\sigma_1)\left\{\left(\frac{17}{576}\right)\left[0.0069|\psi'(\sigma_1)|^q+0.036|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{0.183}{360}\right)\left[|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}+\left(\frac{17}{576}\right)\left[0.036|\psi'(\sigma_1)|^q+0.0069|\psi'(\sigma_2)|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 5.11. *Let $\psi:I=[\sigma_1,\sigma_2]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be differentiable function on the I° of I. If $\psi'\in L[\sigma_1,\sigma_2]$ and $|\psi'|^q$ is a p–harmonic exp convex function on $I, r, q\geq 1$ and $\frac{1}{r}+\frac{1}{q}\geq 1$ then*

$$\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right|$$

$$\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{10}|\psi'(\sigma_1)|^q+B_{11}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{12}|\psi'(\sigma_1)|^q+B_{13}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{14}|f'(a)|^q dt+B_{15}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right\},$$

where

$$B_{10}=\int_0^{\frac{1}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,\quad B_{11}=\int_0^{\frac{1}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa,$$

$$B_{12}=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,\quad B_{13}=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa,$$

$$B_{14}=\int_{\frac{2}{3}}^{1}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}d\kappa,\quad B_{15}=\int_{\frac{2}{3}}^{1}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}d\kappa.$$

Proof. Using Lemma 5.2, properties of modulus, Hölder's inequality and p–harmonic exponential convexity of the $|\psi'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{\sigma_1^p+2\sigma_2^p}\right]^{\frac{1}{p}}\right)+3\psi\left(\left[\frac{3\sigma_1^p\sigma_2^p}{2\sigma_1^p+\sigma_2^p}\right]^{\frac{1}{p}}\right)+\psi(\sigma_2)\right]-\frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\psi(x)}{x^{1+p}}dx\right| \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p} \\
&\times\left[\int_0^{\frac{1}{3}}|\kappa-\frac{1}{8}|\left|\frac{1}{A_\kappa^{1+p}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}|\kappa-\frac{1}{2}|\left|\frac{1}{A_\kappa^{1+p}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa+\int_{\frac{2}{3}}^{1}|\kappa-\frac{7}{8}|\left|\frac{1}{A_\kappa^{1+p}}\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|d\kappa\right] \\
&\leq \frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\left\{\left(\int_0^{\frac{1}{3}}|\kappa-\frac{1}{8}|^r d\kappa\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right. \\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\kappa-\frac{1}{2}|^r d\kappa\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{2}{3}}^{1}|\kappa-\frac{7}{8}|^r d\kappa\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\frac{1}{A_\kappa^{(1+p)q}}\left|\psi'\left(\frac{\sigma_1\sigma_2}{A_\kappa}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right\}\leq\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\int_0^{\frac{1}{3}}|\kappa-\frac{1}{8}|^r d\kappa\right)^{\frac{1}{r}}\right. \\
&\times\left(\int_0^{\frac{1}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\kappa-\frac{1}{2}|^r d\kappa\right)^{\frac{1}{r}} \\
&\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}|\kappa-\frac{7}{8}|^r d\kappa\right)^{\frac{1}{r}} \\
&\left.\times\left(\int_{\frac{2}{3}}^{1}\frac{1}{A_\kappa^{(1+p)q}}\left[(e^\kappa-1)|\psi'(\sigma_1)|^q+(e^{1-\kappa}-1)|\psi'(\sigma_2)|^q\right]d\kappa\right)^{\frac{1}{q}}\right\} \\
&=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_1)|^q d\kappa+\int_0^{\frac{1}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\right. \\
&\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}|\varphi'(\sigma_1)|^q d\kappa+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}} \\
&\left.\times\left(\int_{\frac{2}{3}}^{1}\frac{(e^\kappa-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_1)|^q d\kappa+\int_{\frac{2}{3}}^{1}\frac{(e^{1-\kappa}-1)}{A_\kappa^{(1+p)q}}|\psi'(\sigma_2)|^q d\kappa\right)^{\frac{1}{q}}\right\} \\
&=\frac{\sigma_1\sigma_2(\sigma_2^p-\sigma_1^p)}{p}\times\left\{\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{10}|\psi'(\sigma_1)|^q+B_{11}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{12}|\psi'(\sigma_1)|^q+B_{13}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(B_{14}|\psi'(\sigma_1)|^q d\kappa+B_{15}|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right\},
\end{aligned}$$

which completes the proof. □

Corollary 5.12. *Under the assumptions of Theorem 5.11 with $p=-1$, we have the following new result*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\psi(\sigma_1)+3\psi\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\psi\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\psi(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\psi(x)dx\right| \\
&\leq(\sigma_2-\sigma_1)\left[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(0.0623|\psi'(\sigma_1)|^q+0.4372|\psi'(\sigma_2)|^q)^{\frac{1}{q}}+\left(\frac{1}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}0.2188(|\psi'(\sigma_1)|^q+|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}(0.4372|\psi'(\sigma_1)|^q+0.0623|\psi'(\sigma_2)|^q)^{\frac{1}{q}}\right].
\end{aligned}$$

6. Applications

In this section, we recall the following special means of two positive numbers σ_1,σ_2 with $\sigma_1<\sigma_2$:

(1) The arithmetic mean

$$A=A(\sigma_1,\sigma_2)=\frac{\sigma_1+\sigma_2}{2}.$$

(2) The geometric mean

$$G=G(\sigma_1,\sigma_2)=\sqrt{\sigma_1\sigma_2}.$$

(3) The harmonic mean

$$H=H(\sigma_1,\sigma_2)=\frac{2\sigma_1\sigma_2}{\sigma_1+\sigma_2}.$$

(4) The logarithmic mean

$$L = L(\sigma_1,\sigma_2) = \frac{\sigma_2-\sigma_1}{\ln\sigma_2-\ln\sigma_1}.$$

These means have a lot of applications in areas and different type of numerical approximations. However, the following simple relationship is known in the literature.

$$H(\sigma_1,\sigma_2) \le G(\sigma_1,\sigma_2) \le L(\sigma_1,\sigma_2) \le A(\sigma_1,\sigma_2).$$

Proposition 6.1. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H_p(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2^{1-p}-\sigma_1^{1-p}}{1-p}\right) \le A(\sigma_1,\sigma_2)[2e-4]. \tag{6.1}$$

Proof. Taking $\psi(\sigma)=\sigma$ for $\nu>0$ in Theorem 4.1, then inequality (6.1) is easily captured. □

Proposition 6.2. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H_{2p}^{-1}(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2^{\frac{1}{2}-p}-\sigma_1^{\frac{1}{2}-p}}{\frac{1}{2}-p}\right)^{-1} \le A^{-1}(\sqrt{\sigma}_1,\sqrt{\sigma}_2)[2e-4]. \tag{6.2}$$

Proof. Taking $\psi(\sigma)=\frac{1}{\sqrt{\sigma}}$ for $\sigma>0$ in Theorem 4.1, then inequality (6.2) is easily captured. □

Proposition 6.3. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2-\sigma_1}{L(\sigma_1,\sigma_2)}\right) \le A(\sigma_1^p,\sigma_2^p)[2e-4]. \tag{6.3}$$

Proof. Taking $\psi(\sigma)=\sigma^p$ for $\sigma>0$ in Theorem 4.1, then inequality (6.3) is easily captured. □

Proposition 6.4. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}H_p^2(\sigma_1^p,\sigma_2^p) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\left(\frac{\sigma_2^{2-p}-\sigma_1^{2-p}}{2-p}\right) \le A(\sigma_1^2,\sigma_2^2)[2e-4]. \tag{6.4}$$

Proof. Taking $\psi(\sigma)=\sigma^2$ for $\sigma>0$ in Theorem 4.1, then inequality (6.4) is easily captured. □

Proposition 6.5. *Let $0<\sigma_1<\sigma_2$ and $p\ge 1$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}\ln G(\sigma_1,\sigma_2) \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{-\ln x}{x^{p+1}}dx \le \ln H_p(\sigma_1^p,\sigma_2^p)[2e-4]. \tag{6.5}$$

Proof. Taking $\psi(\sigma)=-\ln\sigma$ for $\sigma>0$ in Theorem 4.1, then inequality (6.5) is easily captured. □

Proposition 6.6. *Let $0<\sigma_1<\sigma_2$. Then we get the following inequality*

$$\frac{1}{2(\sqrt{e}-1)}e^{H(\sigma_1,\sigma_2)} \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{e^x}{x^{p+1}}dx \le A(e^{\sigma_1},e^{\sigma_2})[2e-4]. \tag{6.6}$$

Proof. Taking $\psi(\sigma)=e^\sigma$ for $\sigma>0$ in Theorem 4.1, then inequality (6.6) is easily captured. □

Proposition 6.7. *Let $0<\sigma_1<\sigma_2$. Then we get the following inequality*

$$A(\sin\sigma_1,\sin\sigma_2)[2e-4] \le \frac{p\sigma_1^p\sigma_2^p}{\sigma_2^p-\sigma_1^p}\int_{\sigma_1}^{\sigma_2}\frac{\sin x}{x^{p+1}}dx \le \frac{1}{2(\sqrt{e}-1)}\sin H_p(\sigma_1,\sigma_2). \tag{6.7}$$

Proof. Taking $\psi(\nu)=\sin(-\nu)$ for $\nu\in(0,\frac{\pi}{2})$ in Theorem 4.1, then inequality (6.7) is easily captured. □

Remark 6.8. *The above discussed means are well–known in literature because these means have fruitful importance and magnificent applications in machine learning, probability, statistics and numerical approximation [4, 8]. But we believe that in the future we will try to find the applications of He Chengtian mean (also called as He Chengtian average), which was introduced by the first time a famous ancient Chinese mathematician He Chengtian [12]. This mean was extended to solve nonlinear oscillators and it is called as He's max–min approach (also called as He's max–min method), which was further developed into a frequency–amplitude formulation for nonlinear oscillators [13, 14].*

7. Conclusion

We have introduced and investigated some algebraic properties of a new class of functions namely p–harmonic exp convex. We showed that our new introduced class of function have some nice properties. New version of Hermite–Hadamard type inequality and an integral identity for the differentiable function are obtained. It is the time to find the applications and importance of these inequalities along with efficient numerical tools and methods. The interesting tools and fruitful ideas of this paper can be extended and generalized on the co-ordinates along with fractional calculus. Further, this new concept will be opening new door of investigations toward fractal integration and differentiations in convexity, preinvexity and fractal image processing. We hope the consequences and techniques of this article will energize and inspire the researcher to explore a more interesting sequel in this area.

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Some Abelian, Tauberian and Core Theorems Related to the (V,λ)-Summability

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Abstract

For a non-decreasing sequence of the positive integers tending to infinity $\lambda=(\lambda_m)$ such that $\lambda_{m+1}-\lambda_m\leq 1$, $\lambda_1=1$; (V,λ)-summability defined as the limit of the generalized de la Valée-Pousin of a sequence, [10]. In the present research, we establish some *Tauberian*, *Abelian* and *Core* theorems related to the (V,λ)-summability.

1. Preliminaries

Let $\mathbb{R}$ be the set of the reel numbers and $\mathbb{C}$ be the set of the complex numbers. Let c and ℓ_∞ be the space of all complex valued convergent and bounded sequences, one by one. Let $\lambda=(\lambda_m)$ be a non-decreasing sequence of the positive integers tending to ∞ such that $\lambda_1=1$, $\lambda_{m+1}\leq\lambda_m+1$. A real number sequence $x=(x_n)$ is said to be (V,λ)-summable to the value l if

$$\lim_m t_m(x)=l$$

exists, where

$$t_m(x)=\frac{1}{\lambda_m}\sum_{n\in I_m}x_n,\quad I_m=[m-\lambda_m+1,m].$$

By (V,λ), we mean the set of all (V,λ)-summable sequences, i.e.,

$$(V,\lambda)=\left\{x=(x_n):\lim_m t_m(x)=l\ \text{ for }\ \text{some}\ \ l\in\mathbb{R}\right\}.$$

Also, by $(V,\lambda)_0$ we denote the space of all sequences which (V,λ)-summable to zero. It is clear that in the case $\lambda_m=m$ for all m, (V,λ)-summability reduces to the Cesáro summability, [11]. If $x\in(V,\lambda)$ and $\lim_m t_m(x)=l$, then we have $(V,\lambda)-\lim x=l$.
Let E be a subset of $\mathbb{N}$ (the set of natural numbers). Natural density δ of E given by the following equality:

$$\delta(E)=\lim_n\frac{1}{n}|\{k\leq n:k\in E\}|.$$

The number sequence $x=(x_k)$ is said to be statistically convergent to the number l if for every $\varepsilon>0$, $\delta(\{k:|x_k-l|\geq\varepsilon\})=0$, [7]. In this case, we write: $st-\lim x=l$, where st and st_0 are the sets of all statistically convergent and statistically null sequences, respectively.
For a given non-negative regular matrix A, the number

$$\delta_A(K)=\lim_n\sum_{k\in K}a_{nk}$$

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is said to be the A-density of $K \subseteq \mathbb{N}$, [8]. A sequence $x = (x_k)$ is said to be A-statistically convergent to the number s if for every $\varepsilon > 0$, the set $\{k : |x_k - s| \geq \varepsilon\}$ has A-density zero, [2]. Thus, the following equation is valid: $st_A - \lim x = s$. By $st(A)$ and $st(A)_0$, we respectively show the set of all A-statistically convergent and A-statistically null sequences.
For example, if we choose $E \subset \mathbb{N}$ such as $E = \{n^2 : n = 1,2,3\cdots\}$ then it is easy to see that $\delta(E) = 0$. A real number sequence $x = (x_k)$ is said to be statistically convergence to the number l if for every $\varepsilon > 0$, $\delta\{k : |x_k - l|\} = 0$, [7]. For example, let

$$x_k = \begin{cases} k & , \quad k = n^2 \text{ for all } n = 1,2,3,\cdots \\ \frac{1}{k} & , \quad \text{otherwise.} \end{cases}$$

Then it obvious that $\lim x_k$ does not exist. But since $\delta(E) = \delta(\{n^2 : n = 1,2,3\cdots\}) = 0$, we write $st - \lim x_k = \lim_k \frac{1}{k} = 0$. If (x_k) is statistically convergence to a number l, then we write $st - \lim x = l$. By st and st_0, we denote the set of all statistically convergent and statistically null sequence, respectively. If a sequence is A-statistically convergent to l, then we can write $st_A - \lim x = l$.
Let $x = (x_k)$ be a sequence in $\mathbb{C}$ and R_k be the least convex closed region of complex plane containing $x_k, x_{k+1}, x_{k+2}, \ldots$. The Knopp Core (or $\mathscr{K}$-core) of x is defined by the intersection of all R_k (k=1,2,...), [1, pp.137]. In [12], it is indicate that

$$\mathscr{K} - core(x) = \bigcap_{z\in\mathbb{C}} B_x(z)$$

for any bounded sequence x, where $B_x(z) = \left\{w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z|\right\}$.
In [6], the notion of the statistical core of a complex number sequence introduced by Fridy and Orhan [9] has been extended to the A-statistical core (or st_A-core) and it is shown that for a A-statistically bounded seqeunce x

$$st_A - core(x) = \bigcap_{z\in\mathbb{C}} C_x(z)$$

where $C_x(z) = \left\{w \in \mathbb{C} : |w - z| \leq st_A - \limsup |x_k - z|\right\}$. The inclusion theorems related to the $\mathscr{K}$-core and st_A-core has been worked by many authors [3–5].
Let D be an infinite matrix of complex entries d_{nk} and $x = (x_k)$ be a complex valued sequence. Then $Dx = \{(Dx)_n\}$ is called the transformed sequence of x, if $(Dx)_n = \sum_k d_{nk}x_k$ converges for each n. For two sequence spaces M and N we say that $D \in (M,N)$ if $Dx \in N$ for each $x \in M$. If M and N are equipped with the limits $M - \lim$ and $N - \lim$, respectively, $D \in (M,N)$ and $N - \lim_n (Dx)_n = M - \lim_k x_k$ for all $x \in M$, then we say D regularly transforms M into N and write $D \in (M,N)_{reg}$.
Recently, similar works studied by some authors, see [13–17]. In the present paper, we have proved some *Abelian*, *Tauberian* and *Core* theorems related to the (V,λ)-summability.

2. Tauberian and Abelian Theorems

For any sequence spaces X and Y, an *Abelian* theorem is a theorem such that states the inclusion $X \subset Y$. The *Tauberian* theorem is a one of the form $X \cap Z \subset Y$, where Z is also a sequence space and $Y \subset X$.
Our first result for (V,λ) is an *Abelian* theorem.

Theorem 2.1. *$c_{(C,1)} \subset (V,\lambda)$ if and only if*

$$\liminf_m \frac{m}{\lambda_m} = 1, \tag{2.1}$$

where $c_{(C,1)}$ is the space of all Cesáro summable sequences.

Proof. Let $x \in c_{(C,1)}$ and

$$\lim_m \frac{1}{m}\sum_{n=1}^{m} x_n = l.$$

Then, for any given $\varepsilon > 0$ and enough large m,

$$\left|\frac{1}{m}\sum_{n=1}^{m} x_n - l\right| < \varepsilon.$$

Now, one can write that

$$\begin{aligned}\left|\frac{1}{\lambda_m}\sum_{n\in I_m}(x_n - l)\right| &= \left|\frac{1}{\lambda_m}\sum_{n=1}^{m}(x_n - l) - \frac{1}{\lambda_m}\sum_{n=1}^{m-\lambda_m}(x_n - l)\right| \\ &\leq \frac{m}{\lambda_m}\left|\frac{1}{m}\sum_{n=1}^{m}(x_n - l)\right| + \frac{m-\lambda_m}{\lambda_m}\left|\frac{1}{m-\lambda_m}\sum_{n=1}^{m-\lambda_m}(x_n - l)\right| \\ &\leq \frac{m}{\lambda_m}\varepsilon + \frac{m-\lambda_m}{\lambda_m}\varepsilon \\ &\leq \varepsilon\left(2\frac{m}{\lambda_m} - 1\right).\end{aligned}$$

Therefore, it is clear that $\lim_m t_m(x) = l$ if and only if (2.1) holds. This completes the theorem. □

Since $c \subset c_{(C,1)}$, the following result is obvious.

Corollary 2.2. *If (2.1) holds then* $c \subset (V,\lambda)$.

Theorem 2.3. $(V,\lambda)_0 \cap c_0 \subset (c_0)_{(C,1)}$, *where* $(c_0)_{(C,1)}$ *is the space of all Cesáro summable to zero sequences.*

Proof. Let $x \in (V,\lambda)_0 \cap c_0$. Thus, for any $\varepsilon > 0$ and enough large m,n, $|t_m(x)| \leq \varepsilon/2$ and $|x_n| \leq \varepsilon/2$. Hence, we have

$$\begin{aligned} \left|\frac{1}{m}\sum_{n=1}^{m} x_n\right| &= \left|\frac{1}{m}\sum_{n=1}^{m-\lambda_m} x_n + t_m(x)\right| \\ &\leq \frac{1}{m}\sum_{n=1}^{m-\lambda_m} |x_n| + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2}\left(1 - \frac{\lambda_m}{m} + \frac{2}{m}\right). \end{aligned}$$

Also, since λ_m/m is bounded by 1, the following inequality is true:

$$\left|\frac{1}{m}\sum_{n=1}^{m} x_n\right| \leq \frac{\varepsilon}{m}$$

which gives the result. □

Since $(t_m(x)-l) \in (V,\lambda)_0$ and $(x_n - l) \in c_0$, we have the following outcome which is a *Tauberian* theorem.

Theorem 2.4. $(V,\lambda)\cap c \subset c_{(C,1)}$.

3. Core Theorems

Definition 3.1. *Let R_m be the least closed convex hull containing $t_m, t_{m+1}, t_{m+2}, \ldots$. Then, $\mathscr{K}_\lambda$-core of x is the intersection of all R_m, i.e.,*

$$\mathscr{K}_\lambda - core(x) = \bigcap_{m=1}^{\infty} R_m.$$

In fact, we define $\mathscr{K}_\lambda$-core of x by the $\mathscr{K}$-core of the sequence (t_m). Thus, one may state the following theorem which is an parallel of $\mathscr{K}$-core.
One can prove the following theorem by replacing (t_m) in place of (x_k), which is analogues of theorem given in [12] for Knopp core.

Theorem 3.2. *Let, for any $z \in \mathbb{C}$,*

$$G_x(z) = \left\{w \in \mathbb{C} : |w - z| \leq \limsup_m |t_m(x) - z|\right\}.$$

So, for any $x \in \ell_\infty$,

$$\mathscr{K}_\lambda - core(x) = \bigcap_{z\in\mathbb{C}} G_x(z).$$

At present, we are in a position to construct the inclusion theorems. First of all, we prove several lemmas which will be helpful to the proof of the next theorems.

Lemma 3.3. *Let X be any sequence space. Then, $B \in (X,(V,\lambda))$ if and only if $D \in (X,c)$, where $D = (d_{nk})$ is defined by*

$$d_{nk} = \left\{\frac{1}{\lambda_n}\sum_{j\in I_n} b_{jk},\ (n \in \mathbb{N})\right\}. \tag{3.1}$$

Proof. Let $x \in X$ and take into consideration the equality

$$\frac{1}{\lambda_m}\sum_{j\in I_m}\sum_{k=0}^{n} b_{jk}x_k = \sum_{k=0}^{n}\frac{1}{\lambda_m}\sum_{j\in I_m} b_{jk}x_k;\ (m,n \in \mathbb{N})$$

which yields as $n \longrightarrow \infty$ that

$$\frac{1}{\lambda_m}\sum_{j\in I_m}(Bx)_j = (Dx)_m;\ (m \in \mathbb{N}),$$

where $D = (d_{nk})$ is defined by (3.1).
Thus, it is obvious that $B \in (X,(V,\lambda))$ if and only if $D \in (X,c)$. As a result, the proof is complete. □

For the special cases of the sequence space X, one can state the following lemmas.

Lemma 3.4. *$B \in (c,\ (V,\lambda))_{reg}$ if and only if*

$$\sup_m \sum_k \left| \frac{1}{\lambda_m} \sum_{n\in I_m} b_{nk} \right| < \infty,$$
$$\lim_m \frac{1}{\lambda_m} \sum_{n\in I_m} b_{nk} = 0, \forall k,$$
$$\lim_m \sum_k \frac{1}{\lambda_m} \sum_{n\in I_m} b_{nk} = 1.$$

Following lemma is an analogues of Theorem 3.2 in [4]. One can prove it by same technique. So, we omit the proof.

Lemma 3.5. *$B \in (st(A)\cap \ell_\infty, (V,\lambda))_{reg}$ if and only if $B \in (c,(V,\lambda))_{reg}$ and*

$$\lim_m \sum_{k\in E} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{nk} \Big| = 0 \tag{3.2}$$

for every $E \subset \mathbb{N}$ with $\delta_A(E) = 0$.

By choosing A as Cesáro matrix

$$a_{nk} = \begin{cases} 1/n & , \quad n \geq k \\ 0 & , \quad \text{others.} \end{cases}$$

we get following lemma.

Lemma 3.6. *$B \in (S\cap \ell_\infty, (V,\lambda))_{reg}$ if and only if $B \in (c,(V,\lambda))_{reg}$ and*

$$\lim_m \sum_{k\in E} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{nk} \Big| = 0$$

for every $E \subset \mathbb{N}$ with $\delta(E) = 0$.

Now, we can give the following theorem.

Theorem 3.7. *Let $\|B\| = \sup_n \sum_k |b_{nk}| < \infty$. Then, $\mathscr{K}_\lambda$-core$(Bx) \subseteq \mathscr{K}$-core$(x)$ for all $x \in \ell_\infty$ if and only if $B \in (c,(V,\lambda))_{reg}$ and*

$$\lim_m \sum_k \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{nk} \Big| = 1. \tag{3.3}$$

Proof. (*Necessity*). Let $x \in c$ with $\lim x = l$. Then, $\mathscr{K}$-core$(x) = \{l\}$ which implies that $\mathscr{K}_\lambda$-core$(Bx) \subseteq \{l\}$. Since the assumption $\|B\| < \infty$ implies the boundedness of Bx, $\mathscr{K}_\lambda$-core$(Bx) = \{l\}$ and therefore $(V,\lambda) - \lim Bx = l$. This implies that $B \in (c,(V,\lambda))_{reg}$.
Let's assume that the condition(3.3) is not satisfy. Then we have,

$$\lim_m \sum_k \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{nk} \Big| > 1.$$

The conditions of the Lemma 3.4 give us to choose two strictly increasing sequences $\{k(n_i)\}$ and $\{n_i\}$ $(i = 1,2,\ldots)$ of positive integers such that

$$\sum_{k=0}^{k(n_i-1)} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{n_i,k} \Big| < \frac{1}{4}, \quad \sum_{k=k(n_{i-1})+1}^{k(n_i)} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{n_i,k} \Big| > 1 + \frac{1}{2}$$

and

$$\sum_{k=k(n_i)+1}^{\infty} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{n_i,k} \Big| < \frac{1}{4}.$$

At present, let's define a sequence $x = (x_k)$ by

$$x_k = sgn\Big(\frac{1}{\lambda_m} \sum_{n\in I_m} b_{n_i,k} \Big), \quad k(n_{i-1}) + 1 \leq k < k(n_i),$$

where m is an integer as defined in the choosing $\lambda = \lambda_m$. Then, it is clear that $\mathscr{K}$-core$(x) \subseteq B_x(0)$. Also,

$$\Big| \sum_k \frac{1}{\lambda_m} \sum_{n\in I_m} b_{n_i,k} x_k \Big| \geq \sum_{k=k(n_{i-1})+1}^{k(n_i)} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{n_i,k} \Big| - \sum_{k=0}^{k(n_i-1)} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{n_i,k} \Big| - \sum_{k=k(n_i)+1}^{\infty} \frac{1}{\lambda_m} \Big| \sum_{n\in I_m} b_{n_i,k} \Big| > 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = 1.$$

Since $B \in (c,(V,\lambda))_{reg}$, it follows that (Bx) has a subsequence whose (V,λ)-limit can not be in $B_x(0)$. This is a contradiction with that $\mathscr{K}_\lambda$-core$(Bx) \subseteq \mathscr{K}$-core$(x)$. Hence, the condition (3.3) have to be satisfy.

(*Sufficiency*). Let $w \in \mathscr{K}_\lambda$-core$(Bx)$. So, for any given $z \in \mathbb{C}$, one get

$$\begin{aligned} |w-z| &\le \limsup_m |t_m(Bx)-z| \\ &= \limsup_m \left|z-\sum_k c_{mk}x_k\right| \\ &\le \limsup_m \left|\sum_k c_{mk}(z-x_k)\right| + \limsup_m |z|\left|1-\sum_k c_{mk}\right| \\ &= \limsup_m \left|\sum_k c_{mk}(z-x_k)\right| \end{aligned} \tag{3.4}$$

where

$$c_{mk} = \frac{1}{\lambda_m}\sum_{n\in I_m} b_{nk}.$$

Now, let $\limsup_k |x_k - z| = l$. Subsequently, for any $\varepsilon > 0$, $|x_k - z| \le l+\varepsilon$ whenever $k \ge k_0$. Thus, the following inequality applies:

$$\begin{aligned} \left|\sum_k c_{mk}(z-x_k)\right| &= \left|\sum_{k<k_0} c_{mk}(z-x_k) + \sum_{k\ge k_0} c_{mk}(z-x_k)\right| \\ &\le \sup_k |z-x_k| \sum_{k<k_0} |c_{mk}| + (l+\varepsilon)\sum_{k\ge k_0}|c_{mk}| \\ &\le \sup_k |z-x_k| \sum_{k<k_0} |c_{mk}| + (l+\varepsilon)\sum_k |c_{mk}|. \end{aligned} \tag{3.5}$$

Therefore, applying $\limsup_m$ and combining (3.4) with (3.5), we have

$$|w-z| \le \limsup_m \left|\sum_k c_{mk}(z-x_k)\right| \le l$$

which shows that $w \in \mathscr{K}$-core(x). The proof is completed. □

Theorem 3.8. *Let $\|B\| = \sup_n \sum_k |b_{nk}| < \infty$. Then, $\mathscr{K}_\lambda$-core$(Bx) \subseteq st_A$-core(x) for all $x \in \ell_\infty$ if and only if $B \in (st(A)\cap\ell_\infty,(V,\lambda))_{reg}$ and the condition (3.3) are satisfy.*

Proof. (*Necessity*). By choosing $x \in st(A)\cap\ell_\infty$, as in Theorem 3.7, we immediately have that $B \in (st(A)\cap\ell_\infty,(V,\lambda))_{reg}$.
On the other hand, since st_A-core$(x) \subseteq \mathscr{K}$-core$(x)$ for any sequence x [6], the necessity of the condition (3.3) follows from Theorem 3.7.
(*Sufficiency*). Let we take $w \in \mathscr{K}_\lambda$-core$(Bx)$. So, we have again the condition (3.4). At present, if $st_A - \limsup|x_k - z| = s$, then for any $\varepsilon > 0$, the set E defined by $E = \{k : |x_k - z| > s+\varepsilon\}$ has zero A-density. At present, we get

$$\begin{aligned} \left|\sum_k c_{mk}(z-x_k)\right| &= \left|\sum_{k\in E} c_{mk}(z-x_k) + \sum_{k\notin E} c_{mk}(z-x_k)\right| \\ &\le \sup_k |z-x_k| \sum_{k\in E} |c_{mk}| + (s+\varepsilon)\sum_{k\notin E}|c_{mk}| \\ &\le \sup_k |z-x_k| \sum_{k\in E} |c_{mk}| + (s+\varepsilon)\sum_k |c_{mk}|. \end{aligned}$$

Hence, applying the operator $\limsup_m$ and using the condition (3.3) with (3.2), we can write that

$$\limsup_m \left|\sum_k c_{mk}(z-x_k)\right| \le s+\varepsilon. \tag{3.6}$$

Finally, combining (3.4) with (3.6), we get

$$|w-z| \le st_A - \limsup_k |x_k - z|$$

which shows that $w \in st_A$-core(x). □

As a consequence of Theorem 3.8, we get

Corollary 3.9. *Let $\|B\| = \sup_n \sum_k |b_{nk}| < \infty$. Then, $\mathscr{K}_\lambda$-core$(Bx) \subseteq st$-core(x) for all $x \in \ell_\infty$ if and only if $B \in (st\cap\ell_\infty,(V,\lambda))_{reg}$ and (3.3) holds.*

4. Conclusion

In this paper, we obtained new some *Tauberian*, *Abelian* and *Core* theorems related to the (V,λ)-summability.

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On the Solutions of a Fourth Order Difference Equation

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Abstract

In this paper, we solve and study the global behavior of the well defined solutions of the difference equation

$$x_{n+1}=\frac{x_n x_{n-3}}{Ax_{n-2}+Bx_{n-3}}, \quad n=0,1,...,$$

where $A,B>0$ and the initial values x_{-i}, $i\in\{0,1,2,3\}$ are real numbers.

1. Introduction

In [1], we determined an explicit formula for the solutions of the fourth order difference equation

$$x_{n+1}=\frac{x_n x_{n-2}}{ax_{n-2}+bx_{n-3}},\ n=0,1,...,$$

where a,b are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are real numbers.
In [2] and [8], we determined the forbidden set and introduced an explicit formula for the solutions of each of the two fourth order difference equations (respectively)

$$x_{n+1}=\frac{ax_n x_{n-2}}{-bx_n+cx_{n-3}},\ n=0,1,...,$$

and

$$x_{n+1}=\frac{ax_n x_{n-2}}{bx_n+cx_{n-3}},\ n=0,1,...,$$

where a,b,c are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are real numbers.
In [10], the authors studied the qualitative analysis of the fourth order difference equation

$$x_{n+1}=ax_{n-1}+\frac{bx_{n-1}}{cx_{n-1}-dx_{n-3}},\ n=0,1,...,$$

where a,b,c,d are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are arbitrary real numbers.
In [15], the authors obtained the solutions of the fourth order difference equation

$$x_{n+1}=\frac{x_n x_{n-3}}{x_{n-2}(\pm 1\pm x_n x_{n-3})}, \quad n=0,1,...,$$

where the initial conditions are arbitrary real numbers.
In [24], the author studied the boundedness character of the positive solutions of the fourth order difference equation

$$x_{n+1}=\max\{A,\frac{x_n^p}{x_{n-3}^p}\},\ n=0,1,...,$$

where the parameters A and p are positive real numbers. For more on difference equations (See [3]- [7], [9], [11]- [14], [16]- [23]) and the references therein.

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In this paper, we study the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{Ax_{n-2} + Bx_{n-3}}, \quad n = 0,1,..., \tag{1.1}$$

where $A,B > 0$ and the initial values x_{-i}, $i \in \{0,1,2,3\}$ are real numbers.
The transformation

$$z_n = \frac{x_{n-1}}{x_n}, \text{ with } z_{-2} = \frac{x_{-3}}{x_{-2}}, \ z_{-1} = \frac{x_{-2}}{x_{-1}} \text{ and } z_0 = \frac{x_{-1}}{x_0} \tag{1.2}$$

reduces Equation (1.1) into the difference equation

$$z_{n+1} = \frac{A}{z_{n-2}} + B, \ n = 0,1,.... \tag{1.3}$$

During this paper, we suppose that

$$\theta_j = \frac{\lambda_+^j - \lambda_-^j}{\sqrt{B^2 + 4A}},$$

where $\lambda_- = \frac{B}{2} - \frac{\sqrt{B^2+4A}}{2}$ and $\lambda_+ = \frac{B}{2} + \frac{\sqrt{B^2+4A}}{2}$, $j = 0,1,....$
Let $\mu_l(j) = Ax_l\theta_j + x_{l-1}\theta_{j+1}$, $l \in \{0,-1,-2\}$ and $j = 0,1,....$
We give the following Lemma without proof:

Lemma 1.1. *The following identities are true:*

1. $A\theta_j + B\theta_{j+1} = \theta_{j+2}, \ j = 0,1,....$
2. $A\mu_l(j) + B\mu_l(j+1) = \mu_l(j+2), \ l \in \{0,-1,-2\}$ *and* $j = 0,1,....$

2. Solution of Equation (1.1)

In this section, we shall give two invariant sets and introduce the solution of Equation (1.1).
Consider the sets

$$D_+ = \{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : -\frac{u_0}{(\lambda_+/A)^3} = \frac{u_{-1}}{(\lambda_+/A)^2} = -\frac{u_{-2}}{\lambda_+/A} = u_{-3}\}$$

and

$$D_- = \{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : -\frac{u_0}{(\lambda_-/A)^3} = \frac{u_{-1}}{(\lambda_-/A)^2} = -\frac{u_{-2}}{\lambda_-/A} = u_{-3}\}.$$

Theorem 2.1. *The two sets D_+ and D_- are invariant sets for Equation* (1.1).

Proof. Let $(x_0,x_{-1},x_{-2},x_{-3}) \in D_+$. We show that $(x_n,x_{n-1},x_{n-2},x_{n-3}) \in D_+$ for each $n \in \mathbb{N}$. The proof is by induction on n.
The point $(x_0,x_{-1},x_{-2},x_{-3}) \in D_+$ implies

$$-\frac{x_0}{(\lambda_+/A)^3} = \frac{x_{-1}}{(\lambda_+/A)^2} = -\frac{x_{-2}}{\lambda_+/A} = x_{-3}\}.$$

Now for $n = 1$, we have

$$\begin{aligned} x_1 &= \frac{x_0x_{-3}}{Ax_{-2} + Bx_{-3}} = \frac{-(\lambda_+/A)^2 x_{-2}(A/\lambda_+)x_{-2}}{Ax_{-2} - B(A/\lambda_+)x_{-2}} \\ &= -\frac{1}{A^2}\frac{\lambda_+ x_{-2}}{1 - B/\lambda_+} = -\frac{1}{(A/\lambda_+)^3}x_{-2}. \end{aligned}$$

Then we have

$$-\frac{x_1}{(\lambda_+/A)^3} = \frac{x_0}{(\lambda_+/A)^2} = -\frac{x_{-1}}{\lambda_+/A} = x_{-2}.$$

This implies that $(x_1,x_0,x_{-1},x_{-2}) \in D_+$.
Suppose now that for a certain $n \in \mathbb{N}$, $(x_n,x_{n-1},x_{n-2},x_{n-3}) \in D_+$. That is

$$-\frac{x_n}{(\lambda_+/A)^3} = \frac{x_{n-1}}{(\lambda_+/A)^2} = -\frac{x_{n-2}}{\lambda_+/A} = x_{n-3}.$$

Then

$$\begin{aligned} x_{n+1} &= \frac{x_nx_{n-3}}{Ax_{n-2} + Bx_{n-3}} = \frac{-(\lambda_+/A)^2 x_{n-2}(A/\lambda_+)x_{n-2}}{Ax_{n-2} - B(A/\lambda_+)x_{n-2}} \\ &= -\frac{1}{A^2}\frac{\lambda_+ x_{n-2}}{1 - B/\lambda_+} = -\frac{1}{(A/\lambda_+)^3}x_{n-2}. \end{aligned}$$

Then we have

$$-\frac{x_{n+1}}{(\lambda_+/A)^3} = \frac{x_n}{(\lambda_+/A)^2} = -\frac{x_{n-1}}{\lambda_+/A} = x_{n-2}.$$

This means that the point $(x_{n+1},x_n,x_{n-1},x_{n-2}) \in D_+$. Therefore, D_+ is an invariant set for Equation (1.1).
By similar way, we can show that D_- is an invariant set for Equation (1.1).
This completes the proof. □

Theorem 2.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a well defined solution of Equation (1.1). Then*

$$x_n = \begin{cases} \frac{v}{\mu_{-2}(\frac{n+2}{3})\mu_{-1}(\frac{n-1}{3})\mu_0(\frac{n-1}{3})}, & n = 1, 4, ..., \\ \frac{v}{\mu_{-2}(\frac{n+1}{3})\mu_{-1}(\frac{n+1}{3})\mu_0(\frac{n-2}{3})}, & n = 2, 5, ..., \\ \frac{v}{\mu_{-2}(\frac{n}{3})\mu_{-1}(\frac{n}{3})\mu_0(\frac{n}{3})}, & n = 3, 6, ..., \end{cases} \tag{2.1}$$

where $v = x_0x_{-1}x_{-2}x_{-3}$.

Proof. We can write the given solution (2.1) as

$$x_{3m+1} = \frac{v}{\mu_{-2}(m+1)\mu_{-1}(m)\mu_0(m)},$$

$$x_{3m+2} = \frac{v}{\mu_{-2}(m+1)\mu_{-1}(m+1)\mu_0(m)},$$

and

$$x_{3m+3} = \frac{v}{\mu_{-2}(m+1)\mu_{-1}(m+1)\mu_0(m+1)}.$$

When $m = 0$,

$$\begin{aligned} x_1 &= \frac{v}{\mu_{-2}(1)\mu_{-1}(0)\mu_0(0)} = \frac{v}{(Ax_{-2}+Bx_{-3})x_{-2}x_{-1}} \\ &= \frac{x_0x_{-3}}{Ax_{-2}+Bx_{-3}}, \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{v}{\mu_{-2}(1)\mu_{-1}(1)\mu_0(0)} = \frac{v}{(Ax_{-2}+Bx_{-3})(Ax_{-1}+Bx_{-2})x_{-1}} \\ &= \frac{x_1x_{-2}}{Ax_{-1}+Bx_{-2}}, \end{aligned}$$

and

$$\begin{aligned} x_3 &= \frac{v}{\mu_{-2}(1)\mu_{-1}(1)\mu_0(1)} = \frac{v}{(Ax_{-2}+Bx_{-3})(Ax_{-1}+Bx_{-2})(Ax_0+Bx_{-1})} \\ &= \frac{x_0x_{-3}}{Ax_{-2}+Bx_{-3}} \frac{x_{-2}x_{-1}}{(Ax_{-1}+Bx_{-2})(Ax_0+Bx_{-1})} \\ &= \frac{x_1x_{-2}}{Ax_{-1}+Bx_{-2}} \frac{x_{-1}}{Ax_0+Bx_{-1}} = \frac{x_2x_{-1}}{Ax_0+Bx_{-1}}. \end{aligned}$$

Suppose that the result is true for $m > 0$.
Then

$$\begin{aligned} \frac{x_{3m}x_{3m-3}}{Ax_{3m-2}+Bx_{3m-3}} &= \frac{\frac{v}{\mu_{-2}(m)\mu_{-1}(m)\mu_0(m)} \frac{v}{\mu_{-2}(m-1)\mu_{-1}(m-1)\mu_0(m-1)}}{\frac{Av}{\mu_{-2}(m)\mu_{-1}(m-1)\mu_0(m-1)} + \frac{Bv}{\mu_{-2}(m-1)\mu_{-1}(m-1)\mu_0(m-1)}} \\ &= \frac{\frac{v}{\mu_{-1}(m)\mu_0(m)}}{A\mu_{-2}(m-1)+B\mu_{-2}(m)}. \end{aligned}$$

Using Lemma (1.1), we have

$$A\mu_{-2}(m-1)+B\mu_{-2}(m) = \mu_{-2}(m+1).$$

Then

$$\begin{aligned} &\frac{x_{3m}x_{3m-3}}{Ax_{3m-2}+Bx_{3m-3}} \\ &= \frac{\frac{v}{\mu_{-1}(m)\mu_0(m)}}{A\mu_{-2}(m-1)+B\mu_{-2}(m)} \\ &= \frac{v}{\mu_{-2}(m+1)\mu_{-1}(m)\mu_0(m)} = x_{3m+1}. \end{aligned}$$

Similarly we can show that

$$\frac{x_{3m+1}x_{3m-2}}{Ax_{3m-1}+Bx_{3m-2}} = x_{3m+2} \text{ and } \frac{x_{3m+2}x_{3m-1}}{Ax_{3m}+Bx_{3m-1}} = x_{3m+3}.$$

This completes the proof. □

3. Global behavior of Equation (1.1)

In this section, we introduce the forbidden set and determine the global behavior of Equation (1.1). Clear that, if $x_0 = 0$ and $x_{-1}x_{-2}x_{-3} \neq 0$, then x_4 is undefined. If $x_{-1} = 0$ and $x_0x_{-2}x_{-3} \neq 0$, then x_7 is undefined. If $x_{-2} = 0$ and $x_0x_{-1}x_{-3} \neq 0$, then x_6 is undefined. Finally, if $x_{-3} = 0$ and $x_0x_{-1}x_{-2} \neq 0$, then x_5 is undefined.
The following result provides the forbidden set of Equation (1.1).

Theorem 3.1. *The forbidden set of equation (1.1) as*

$$F = \bigcup_{i=0}^{3}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_{-i} = 0\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_0 = -\frac{u_{-1}}{A}\frac{\theta_{m+1}}{\theta_m}\} \cup$$
$$\bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_{-1} = -\frac{u_{-2}}{A}\frac{\theta_{m+1}}{\theta_m}\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_{-2} = -\frac{u_{-3}}{A}\frac{\theta_{m+1}}{\theta_m}\}.$$

Theorem 3.2. *Assume that $\{x_n\}_{n=-3}^{\infty}$ is a well defined solution of Equation (1.1). Then the following statements are true:*

1. *If $A+B > 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.*
2. *If $A+B < 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ is unbounded.*

Proof. We can write $\theta_j = \lambda_+^j \dfrac{(1-(\frac{\lambda_-}{\lambda_+})^j)}{\sqrt{B^2+4A}}$.

1. If $A+B > 1$, then $\lambda_+ > 1$. That is $\theta_m \to \infty$ as $m \to \infty$. This implies that

$$x_{3m} = \frac{\nu}{\mu_{-2}(m)\mu_{-1}(m)\mu_0(m)} \to 0 \text{ as } m \to \infty.$$

Similarly, we can show that

$$x_{3m+1} \to 0 \text{ as } m \to \infty \text{ and } x_{3m+2} \to 0 \text{ as } m \to \infty.$$

For (2), it is enough to note that $\lambda_+ < 1$ when $A+B < 1$. This completes the proof. □

Theorem 3.3. *Assume that $A+B = 1$, then every well defined solution $\{x_n\}_{n=-3}^{\infty}$ of Equation (1.1) converges to a finite limit.*

Proof. When $A+B = 1$, we have $\lambda_+ = 1$. Then

$$\mu_{-j}(m) = Ax_{-j}\theta_m + x_{-j-1}\theta_{m+1} \to \frac{Ax_{-j}+x_{-j-1}}{\sqrt{B^2+4A}} \text{ as } m \to \infty, \; j \in \{0,1,2\}.$$

This implies that

$$x_{3m} = \frac{\nu}{\mu_{-2}(m)\mu_{-1}(m)\mu_0(m)} \to \frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty.$$

Similarly, we have that

$$x_{3m+1} \to \frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty,$$

and

$$x_{3m+2} \to \frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty.$$

Therefore, the solution $\{x_n\}_{n=-3}^{\infty}$ of Equation (1.1) converges to

$$\frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty.$$

This completes the proof. □

Example (1) Figure 1. shows that, if $A = 0.2$, $B = 0.4$ $(A+B < 1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = -1$ and $x_0 = 3$ is unbounded.
Example (2) Figure 2. shows that, if $A = 1.6$, $B = 0.3$ $(A+B > 1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = -1$ and $x_0 = 3$ converges to zero.
Example (3) Figure 3. shows that, if $A = 0.62$, $B = 0.38$ $(A+B = 1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with $x_{-3} = -1$, $x_{-2} = 1.2$, $x_{-1} = 2.5$ and $x_0 = 1.7$ converges to

$$\frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \simeq 8.666.$$

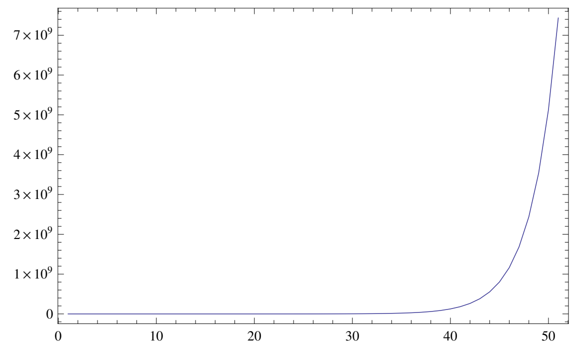

Figure 3.1: $x_{n+1} = \dfrac{x_n x_{n-3}}{0.2x_{n-2} + 0.4x_{n-3}}$

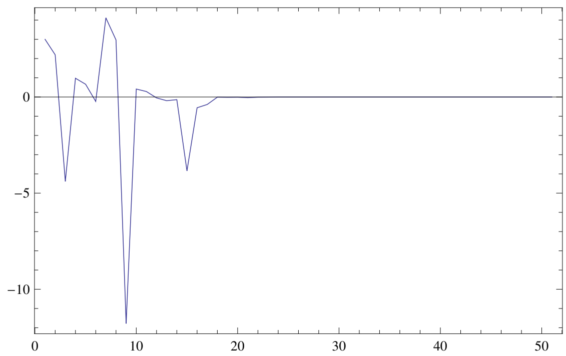

Figure 3.2: $x_{n+1} = \dfrac{x_n x_{n-3}}{1.6x_{n-2} + 0.3x_{n-3}}$

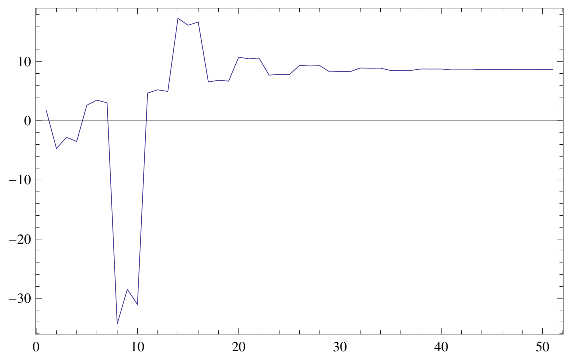

Figure 3.3: $x_{n+1} = \dfrac{x_n x_{n-3}}{0.62x_{n-2} + 0.38x_{n-3}}$

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Blow Up and Exponential Growth to a Petrovsky Equation with Degenerate Damping

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Abstract

This paper deals with the initial boundary value problem of Petrovsky type equation with degenerate damping. Under some appropriate conditions, we study the finite time blow up and exponential growth of solutions with negative initial energy.

1. Introduction

We investigate the following initial boundary value problem:

$$\begin{cases} u_{tt}+\Delta^2 u-\int\limits_0^t \mu(t-s)\Delta^2 u(s)\,ds+|u|^{\upsilon}\partial j(u_t)=|u|^{\gamma-1}u & \text{in } \Omega\times(0,+\infty), \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x) & \text{on } x\in\Omega, \\ u(x,t)=\frac{\partial}{\partial n}u(x,t)=0 & \text{on } x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\partial j(s)$ denotes the sub-differential of $j(s)$ [1], n is the outer normal and Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. The Petrovsky type equations are orginated from the study of beams and plates and so often arise in many branches of physics such as optics, nuclear physics, geophysics and ocean acoustics. Rivera et al. [2] considered

$$u_{tt}+\Delta^2 u-\int_0^t \mu(t-s)\Delta^2 u(s)ds-\gamma\Delta u_{tt}=0,$$

and proved the asymptotic behaviour of solution with the initial and dynamical boundary conditions.
The following problem was studied by Alabau-Boussouira et al. [3]

$$u_{tt}+\Delta^2 u-\int_0^t \mu(t-s)\Delta^2 u(s)ds=f(u). \tag{1.2}$$

The authors studied exponential and polynomial decay results of solutions when the memory μ decay exponentially and polynomially, respectively. Afterwards, Tahamtani ve Shahrouzi [4] investigated the existence of weak solutions for problem (1.2). In addition, the authors proved blow up of solutions with positive and negative initial energy in finite time.
In [5], Li and Gao discussed the following equation

$$u_{tt}+\Delta^2 u-\int_0^t \mu(t-s)\Delta^2 u(s)ds+|u_t|^{p-2}u_t=|u|^{\gamma-2}u. \tag{1.3}$$

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The authors studied blow up result of solutions under suitable conditions of the initial datum and the relaxation function. Furthermore, problem (1.3) has been studied by Liu et al. [6] and the finite time blow-up of solutions with arbitrary high initial energy has been proved. Recently, Liu et al. [7] investigated problem (1.3) with case ($p=2$) and proved blow up of solution with $E(0)\leq M$, M is positive constant. Furthermore, the authors studied blow up of solutions with $E(0)>M$ by applying concavity method.
On the other hand, Messaoudi [8] investigated the following problem

$$u_{tt}+\Delta^2 u+|u_t|^{p-2}u_t=|u|^{\gamma-2}u. \tag{1.4}$$

The author studied an existence result and global solution in case $p\geq\gamma$. Then, blow-up of solutions with negative initial energy and $p<\gamma$ was proved. Then, Chen and Zhou [9] discussed blow up with positive initial energy for (1.4) and showed that the solution blows up in finite time for vanishing initial energy case ($p=2$). Moreover, the problem (1.4) with Δu_t term has been considered by Pişkin and Polat [10] and the authors proved decay estimates of the solution by using Nakao's inequality. Some other studies on Petrovsky equations are [11], [12], [13], [14].
The hyperbolic models with degenerate damping also are of much interest in material science and physics. It particularly appears in physics when the friction is modulated by the strains. There are a lot of studies that has degenerate damping terms, namely $\delta(u)h(u_t)$ here $\delta(u)$ is a positive function and h is nonlinear, (see [15–21]).
Motivated by previous results, we prove several results concerning the blow up and exponential growth of solution for the problem (1.1). It should be noted here that we can say that the study is both quite difficult and interesting and the analysis are more subtle because of the degenerate damping.
The remaining part of this paper is organized as follows: In the next section, we study the nonexistence of solutions. The exponential growth result is presented in Section 3.

2. Preliminaries

Now, we present some preliminary material which will be helpful in the proof of our results. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\|=\|.\|_{L^2(\Omega)}$ and $L^q(\Omega)$ norm $\|.\|_q=\|.\|_{L^q(\Omega)}$.
(A1) $\upsilon, p\geq 0$, $\gamma>1$; $\upsilon\leq\frac{n}{n-2}$, $\gamma+1\leq\frac{2n}{n-2}$ if $n\geq 3$. There exist positive constants c, c_0, c_1 such that for all $s,k\in R$ $j(s):R\to R$ be a C^1 convex real function satisfies

- $j(s)\geq c|s|^{p+1}$,
- $j'(s)$ is single valued and $|j'(s)|\leq c_0|s|^p$,
- $(j'(s)-j'(k))(s-k)\geq c_1|s-k|^{p+1}$.

(A2) $u_0(x)\in H_0^2(\Omega)$, $u_1(x)\in L^2(\Omega)$.
(A3) Assume $\mu(\tau):R^+\to R^+$ satisfies

$$\mu(\tau)\geq 0,\ \mu'(\tau)\leq 0,$$

for all $s\in R^+$ and

$$\int_0^t\mu(\tau)d\tau<1.$$

(A4) $\int_0^t\mu(\tau)d\tau<\frac{\gamma-1}{\gamma+1}$.
We use the following notations

$$l=1-\int_0^t\mu(\tau)d\tau,$$
$$(\mu\diamond\theta)(t)=\int_0^t\mu(t-\tau)\int_\Omega|\theta(t)-\theta(\tau)|dxd\tau.$$

The said solution of (1.1) satisfies the energy identity

$$E(t)+\frac{1}{2}\mu(t)\|\Delta u\|^2-\frac{1}{2}(\mu'\diamond\Delta u)(t)+\int_0^t\int_\Omega|u(\tau)|^{\upsilon}j(u_t)(\tau)dxd\tau=E(0), \tag{2.1}$$

where

$$E(t)=\frac{1}{2}\left[\|u_t\|^2+\left(1-\int_0^t\mu(s)ds\right)\|\Delta u\|^2+(\mu\diamond\Delta u)(t)\right]-\frac{1}{\gamma+1}\|u\|_{\gamma+1}^{\gamma+1} \tag{2.2}$$

and

$$E(0)=\frac{1}{2}\left[\|u_1\|^2+\|\Delta u_0\|^2\right]-\frac{1}{\gamma+1}\|u_0\|_{\gamma+1}^{\gamma+1}. \tag{2.3}$$

Moreover, by computation, we get

$$E(t)\leq E(0). \tag{2.4}$$

3. Blow up

In this section, we shall prove the blow up results of the solutions for problem (1.1).

Theorem 3.1. *Let* $(A1)$-$(A4)$ *hold. Assume further that* $\gamma > \upsilon + p$, $E(0) < 0$ *and* u *be a any solution to (1.1) on the interval* $[0,T]$, *then* T *is necessarily finite, i.e.* u *can't be continued for all* $t > 0$.

Proof. We assume that the solution exists for all time and we arrive to a contradiction. Set

$$H(t) = -E(t). \tag{3.1}$$

By using (2.1), we get

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \frac{1}{2}\mu(t)\|\Delta u\|^2 - \frac{1}{2}(\mu' \diamond \Delta u)(t) + \int_\Omega |u(t)|^\upsilon j(u_t) u_t dx \\ &\geq \int_\Omega |u(t)|^\upsilon j(u_t) u_t dx \\ &\geq c_0 \int_\Omega |u(t)|^\upsilon |u_t|^{p+1} dx. \end{aligned} \tag{3.2}$$

Hence, we find

$$0 < H(0) \leq H(t) \leq \frac{1}{\gamma+1}\|u\|_{\gamma+1}^{\gamma+1}, \quad t \geq 0. \tag{3.3}$$

Define

$$K(t) = H^{1-\rho}(t) + \varepsilon \int_\Omega u u_t dx,$$

where $\rho = \min\left\{\frac{\gamma-p-\upsilon}{p(\gamma+1)}, \frac{\gamma-1}{2(\gamma+1)}\right\}$ and ε is a positive constant.
Taking the derivative of $K(t)$ and using Eq.(1.1), we get

$$\begin{aligned} K'(t) &= (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\Delta u\|^2 + \varepsilon\int_0^t \mu(t-s)\int_\Omega \Delta u(s)\Delta u(t) dxds \\ &\quad - \varepsilon\int_\Omega |u(t)|^\upsilon u(t)\partial j(u_t)(t) dx + \varepsilon\|u\|_{\gamma+1}^{\gamma+1} \\ &= (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\Delta u\|^2 + \varepsilon\int_0^t \mu(s)ds\|\Delta u\|^2 + \varepsilon\int_0^t \mu(t-s)\int_\Omega \Delta u(t)(\Delta u(s) - \Delta u(t)) dxds \\ &\quad - \varepsilon\int_\Omega |u(t)|^\upsilon u(t)\partial j(u_t)(t) dx + \varepsilon\|u\|_{\gamma+1}^{\gamma+1}. \end{aligned} \tag{3.4}$$

By applying Young's inequality to estimate the fifth term of (3.4) as follows

$$\left|\int_0^t \mu(t-s)\int_\Omega \Delta u(t)(\Delta u(s) - \Delta u(t)) dxds\right| \leq \int_0^t \mu(s)ds\|\Delta u\|^2 + \frac{1}{4}(\mu \diamond \Delta u)(t). \tag{3.5}$$

From $(A3)$, since $0 < l \leq 1$. Then it follows from the definition of $H(t)$ that

$$-\|\Delta u\|^2 = \frac{2}{l}H(t) + \frac{1}{l}\|u_t\|^2 + \frac{1}{l}(\mu \diamond \Delta u)(t) - \frac{2}{l(\gamma+1)}\|u\|_{\gamma+1}^{\gamma+1}. \tag{3.6}$$

Combining (3.4)-(3.6), we obtain

$$\begin{aligned} K'(t) &\geq (1-\rho)H^{-\rho}(t)H'(t) + \varepsilon\left(1 + \frac{1}{l}\right)\|u_t\|^2 + \frac{2}{l}H(t) + \left(\frac{1}{l} - \frac{1}{4}\right)(\mu \diamond \Delta u)(t) \\ &\quad - \varepsilon\int_\Omega |u(t)|^\upsilon u(t)\partial j(u_t)(t) dx + \varepsilon\left(1 - \frac{2}{l(\gamma+1)}\right)\|u\|_{\gamma+1}^{\gamma+1}. \end{aligned} \tag{3.7}$$

By assumption $\int_0^t \mu(\tau)d\tau < \frac{\gamma-1}{\gamma+1}$, we have $1 - \frac{2}{l(\gamma+1)} > 0$.
In order to estimate fifth term in (3.7), since $q > \upsilon + p$, from assumption $(A1)$ and thanks to Holder's inequality and Young's inequality, we obtain

$$\begin{aligned} \left|\int_\Omega |u(t)|^\upsilon u(t)\partial j(u_t)(t) dx\right| &\leq \int_\Omega |u(t)|^{\upsilon+1-\frac{\upsilon+p+1}{p+1}} |u(t)|^{\frac{\upsilon+p+1}{p+1}} |u_t(t)|^p dx \\ &\leq C_0\left(\int_\Omega |u(t)|^\upsilon |u_t(t)|^{p+1} dx\right)^{\frac{p}{p+1}}\left(\int_\Omega |u(t)|^{\upsilon+p+1} dx\right)^{\frac{1}{p+1}} \\ &\leq C_0|\Omega|^{\frac{\gamma-\upsilon-p}{\gamma+1}}\left(\int_\Omega |u(t)|^\upsilon |u_t(t)|^{p+1} dx\right)^{\frac{p}{p+1}}\|u(t)\|_{\gamma+1}^{\frac{\upsilon+p+1}{p+1}} \\ &\leq \beta\left(H'(t)\right)^{\frac{p}{p+1}}\|u(t)\|_{\gamma+1}^{\frac{\upsilon+p+1}{p+1}} \\ &\leq \beta\left(\delta^{-\frac{1}{p}}H'(t) + \delta\|u(t)\|_{\gamma+1}^{\upsilon+p+1}\right), \end{aligned} \tag{3.8}$$

where constant $\delta > 0$ is specified later and $\beta = C_0 C_1^{-\frac{p}{p+1}} |\Omega|^{\frac{\gamma-\upsilon-p}{\gamma+1}}$.
Hence, (3.7) becomes

$$K'(t) \geq \left[(1-\rho) H^{-\rho}(t) - \varepsilon\beta\delta^{-\frac{1}{p}}\right] H'(t) + \varepsilon\left(1+\frac{1}{l}\right)\|u_t\|^2 + \varepsilon\frac{2}{l}H(t) + \varepsilon\left(\frac{1}{l}-\frac{1}{4}\right)(\mu\diamond\Delta u)(t)$$
$$+\varepsilon\left(1-\frac{2}{l(\gamma+1)}\right)\|u\|_{\gamma+1}^{\gamma+1} - \varepsilon\beta\delta\|u(t)\|_{\gamma+1}^{\upsilon+p+1}. \tag{3.9}$$

The choice of δ $\left(\text{i.e. } \delta = \frac{1}{\beta}\left(\frac{1}{2}-\frac{1}{l(\gamma+1)}\right)\|u\|_{\gamma+1}^{\gamma-\upsilon-p}\right)$, then

$$\varepsilon\beta\delta\|u(t)\|_{\gamma+1}^{\gamma+p+1} = \varepsilon\left(\frac{1}{2}-\frac{1}{l(\gamma+1)}\right)\|u\|_{\gamma+1}^{\gamma+1}.$$

Furthermore, since $\|u\|_{\gamma+1} \geq [(\gamma+1)H(0)]^{\frac{1}{\gamma+1}}$ by (3.3) and $\upsilon+p-\gamma+p(\gamma+1)\rho \leq 0$, then

$$\begin{aligned}(1-\rho)H^{-\rho}(t) - \varepsilon\beta\delta^{-\frac{1}{p}} &= H^{-\rho}(t)\left[1-\rho-\varepsilon\beta\delta^{-\frac{1}{p}}H^{\rho}(t)\right]\\
&\geq H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{1}{l(\gamma+1)}\right)^{-\frac{1}{p}}(\gamma+1)^{-\rho}\|u\|_{\gamma+1}^{\frac{p+\upsilon-\gamma+p(\gamma+1)\rho}{p}}\right]\\
&\geq H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\left(\frac{1}{2}-\frac{1}{l(\gamma+1)}\right)^{-\frac{1}{p}}(\gamma+1)^{-\rho-\frac{q-p-\upsilon}{p(q+1)}}H(0)_{\gamma+1}^{\rho-\frac{\gamma-\upsilon-p}{p(\gamma+1)}}\right]\\
&\geq H^{-\rho}(t)\left[1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\chi\right],\end{aligned} \tag{3.10}$$

where $\chi = \left(\frac{1}{2}-\frac{1}{l(\gamma+1)}\right)^{-\frac{1}{p}}(\gamma+1)^{\rho-\frac{q-p-\upsilon}{p(q+1)}}H(0)_{\gamma+1}^{\rho-\frac{\gamma-\upsilon-p}{p(\gamma+1)}}$. Now, we choose ε to be sufficiently small such that

$$1-\rho-\varepsilon\beta^{1+\frac{1}{p}}\chi > 0.$$

Then (3.10) and (3.9) yield

$$K'(t) \geq \varepsilon C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{\gamma+1}^{\gamma+1} + (\mu\diamond\Delta u)(t)\right], \tag{3.11}$$

where $C > 0$ is a constant that does not depended on ε. Especially, (3.11) means that $K(t)$ is increasing on $[0,T)$, with

$$K(t) = H^{1-\rho}(t) + \varepsilon\int_\Omega u u_t dx \geq H^{1-\rho}(0) + \varepsilon\int_\Omega u_0 u_1 dx.$$

We also select ε to be sufficiently small such that $K(0) > 0$, thus $K(t) \geq K(0) > 0$ for $t \geq 0$.
Let $\eta = \frac{1}{1-\rho}$. Since $0 < \rho < \frac{1}{2}$, it is evident that $2 > \eta > 1$. By using the following inequality

$$|x+y|^\eta \leq 2^{\eta-1}\left(|x|^\eta + |y|^\eta\right) \quad \text{for} \quad \eta \geq 1,$$

applying Young's inequality, we get

$$\begin{aligned}K^\eta(t) &\leq 2^{\eta-1}\left(H(t) + \varepsilon\|u(t)\|^\eta\|u_t(t)\|^\eta\right)\\
&\leq C\left(H(t) + \|u_t(t)\|^2 + \|u(t)\|_{\gamma+1}^{\frac{1}{\frac{1}{2}-\rho}}\right).\end{aligned} \tag{3.12}$$

By the choice of ρ, we have $\frac{1}{2}-\rho > \frac{1}{\gamma+1}$. Now applying the inequality

$$a^\sigma \leq \left(1+\frac{1}{b}\right)(b+a), \quad a \geq 0, \quad 0 \leq \sigma \leq 1, \quad b > 0,$$

and taking $a = \|u(t)\|_{\gamma+1}^{\gamma+1}$, $\eta = \frac{1}{\left(\frac{1}{2}-\rho\right)(\gamma+1)} < 1$, and $b = H(0)$, we obtain

$$\begin{aligned}\|u(t)\|_{\gamma+1}^{\frac{1}{\frac{1}{2}-\rho}} &\leq \left(1+\frac{1}{H(0)}\right)\left(H(0) + \|u(t)\|_{\gamma+1}^{\gamma+1}\right)\\
&\leq C\left(H(t) + \|u(t)\|_{\gamma+1}^{\gamma+1}\right).\end{aligned} \tag{3.13}$$

Therefore, by combining of (3.12) and (3.13), we obatin

$$\begin{aligned}K^\eta(t) &\leq C\left(H(t) + \|u_t(t)\|^2 + \|u(t)\|_{\gamma+1}^{\gamma+1}\right)\\
&\leq C\left(H(t) + \|u_t(t)\|^2 + \|u(t)\|_{\gamma+1}^{\gamma+1} + (\mu\diamond\Delta u)(t)\right).\end{aligned} \tag{3.14}$$

Thus, (3.11) and (3.14) arrive at

$$K'(t) \geq CK^\eta(t), \quad t \in [0,T]. \tag{3.15}$$

In the end, from (3.15) and $\eta = \frac{1}{1-\rho} > 1$, we see that $K(t) = H^{1-\rho}(t) + \varepsilon\int_\Omega u u_t dx$ blow up in finite time. This completes the proof. □

4. Growth

In this section, we goal to show that the energy grow up as an exponential function as time as goes to infinity.

Theorem 4.1. *Let* $(A1)$*-*$(A4)$ *hold. Assume further that* $\gamma > \upsilon + p$ *and* $E(0) < 0$ *and* u *be a any solution to (1.1) grows exponentially.*

Proof. We define

$$Z(t) = H(t) + \varepsilon \int_\Omega u u_t dx, \tag{4.1}$$

where $H(t) = -E(t)$ and $0 < \varepsilon \leq 1$. By derivating (4.1) and using Eq.(1.1), we have

$$\begin{aligned} Z'(t) &= H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 + \varepsilon \int_0^t \mu(t-s) \int_\Omega \Delta u(s) \Delta u(t) dxds - \varepsilon \int_\Omega |u(t)|^\upsilon u(t) \partial j(u_t)(t) dx + \varepsilon \|u\|_{\gamma+1}^{\gamma+1} \\ &= H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\Delta u\|^2 + \varepsilon \int_0^t \mu(s) ds \|\Delta u\|^2 + \varepsilon \int_0^t \mu(t-s) \int_\Omega \Delta u(t) (\Delta u(s) - \Delta u(t)) dxds \\ &- \varepsilon \int_\Omega |u(t)|^\upsilon u(t) \partial j(u_t)(t) dx + \varepsilon \|u\|_{\gamma+1}^{\gamma+1}. \end{aligned} \tag{4.2}$$

By using (3.5), the assumption $(A3)$ and the definition $H(t)$, we have $0 < l \leq 1$ and

$$Z'(t) \geq H'(t) + \varepsilon \left(1 + \frac{1}{l}\right) \|u_t\|^2 + \frac{2}{l} H(t) + \left(\frac{1}{l} - \frac{1}{4}\right) (\mu \diamond \Delta u)(t) - \varepsilon \int_\Omega |u(t)|^\upsilon u(t) \partial j(u_t)(t) dx + \varepsilon \left(1 - \frac{2}{l(\gamma+1)}\right) \|u\|_{\gamma+1}^{\gamma+1}. \tag{4.3}$$

By the assumption $\int_0^t \mu(\tau) d\tau < \frac{\gamma-1}{\gamma+1}$ and using (3.8), we get

$$\begin{aligned} Z'(t) &\geq \left[1 - \varepsilon \beta \delta^{-\frac{1}{p}}\right] H'(t) + \varepsilon \left(1 + \frac{1}{l}\right) \|u_t\|^2 + \varepsilon \frac{2}{l} H(t) + \varepsilon \left(\frac{1}{l} - \frac{1}{4}\right) (\mu \diamond \Delta u)(t) \\ &+ \varepsilon \left(1 - \frac{2}{l(\gamma+1)}\right) \|u\|_{\gamma+1}^{\gamma+1} - \varepsilon \beta \delta \|u(t)\|_{\gamma+1}^{\upsilon+p+1}. \end{aligned} \tag{4.4}$$

The choice of δ $\left(\text{i.e. } \delta = \frac{1}{\beta} \left(\frac{1}{2} - \frac{1}{l(\gamma+1)}\right) \|u\|_{\gamma+1}^{\gamma-\upsilon-p}\right)$, then

$$\varepsilon \beta \delta \|u(t)\|_{\gamma+1}^{\upsilon+p+1} = \varepsilon \left(\frac{1}{2} - \frac{1}{l(\gamma+1)}\right) \|u\|_{\gamma+1}^{\gamma+1}.$$

Furthermore, since $\|u\|_{\gamma+1} \geq [(\gamma+1) H(0)]^{\frac{1}{\gamma+1}}$ by (3.3) and assumption $\upsilon + p - \gamma \leq 0$, then

$$\begin{aligned} 1 - \varepsilon \beta \delta^{-\frac{1}{p}} &\geq 1 - \varepsilon \beta^{1+\frac{1}{p}} \left(\frac{1}{2} - \frac{1}{l(\gamma+1)}\right)^{-\frac{1}{p}} (\gamma+1)^{-\frac{\gamma-p-\upsilon}{p(\gamma+1)}} H(0)_{\gamma+1}^{-\frac{\gamma-\upsilon-p}{p(\gamma+1)}} \\ &\geq 1 - \varepsilon \beta^{1+\frac{1}{p}} P, \end{aligned}$$

where $P = \left(\frac{1}{2} - \frac{1}{l(\gamma+1)}\right)^{-\frac{1}{p}} (\gamma+1)^{-\frac{\gamma-p-\upsilon}{p(\gamma+1)}} H(0)_{\gamma+1}^{-\frac{\gamma-\upsilon-p}{p(\gamma+1)}}$. Now, we choose ε to be sufficiently small such that

$$1 - \varepsilon \beta^{1+\frac{1}{p}} P > 0.$$

Thus,

$$Z'(t) \geq \varepsilon C \left[H(t) + \|u_t(t)\|^2 + \|u\|_{\gamma+1}^{\gamma+1} + (\mu \diamond \Delta u)(t)\right] \tag{4.5}$$

where $C > 0$ is a constant that does not depended on ε.
Now, applying Young's inequality, and Sobolev Poincare inequality we have

$$\begin{aligned} Z(t) &\leq H(t) + \varepsilon \|u\| \|u_t\| \\ &\leq C \left(H(t) + \|u_t\|^2 + \|u\|^2\right). \end{aligned}$$

Now, in order the estimate $\|u\|^2$ term we apply the inequality $a^l \leq (a+1) \leq (1 + \frac{1}{b})(a+b)$ for $a = \|u\|_{\gamma+1}^{\gamma+1}$, $l = 2/\gamma + 1 < 1$, $b = H(0)$, we have

$$\begin{aligned} \|u\|^2 &\leq C \|u\|_{\gamma+1}^2 \\ &= C \left(\|u\|_{\gamma+1}^{\gamma+1}\right)^{\frac{2}{\gamma+1}} \\ &\leq \left(1 + \frac{1}{H(0)}\right) \left(\|u\|_{\gamma+1}^{\gamma+1} + H(0)\right) \\ &\leq C \left(\|u\|_{\gamma+1}^{\gamma+1} + H(t)\right). \end{aligned} \tag{4.6}$$

Thus,

$$Z(t) \leq C\left[H(t) + \|u_t(t)\|^2 + \|u\|_{\gamma+1}^{\gamma+1} + (\mu \diamond \Delta u)(t)\right]. \tag{4.7}$$

Therefore, (4.5) and (4.7) arrive at

$$Z'(t) \geq \xi Z(t), \quad t \geq 0$$

This completes the proof. □

5. Conclusion

In this work, we obtained the finite time bolw up and growth of solutions for a Petrovsky equation with degenerate damping in a bounded domain. This improves and extends many results in the literature.

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