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Contents

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MORE ON SEMI QUOTIENT MAPPINGS AND SPACES

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Abstract. In our paper, semi quotient mappings and spaces properties are developed by the change of topology where the notion of semi quotient topology built the interest. Results of this article describe the more interest in our work with the contribution of extremally disconnected concept where the quotient space J/N with this topology $s\tau_Q$ has surprisingly moved to an $s-$topological group.

1. Introduction

A mathematical discipline assembling the topology and group is called the topological group [8, 12]. This discipline has very significant applications in almost all branches of natural sciences. In our arrangement operations of multiplicity and inverse on the continuity and its general forms will be discussed. The study of this weaker form of continuity with topological groups started in 1990s. Twenty-thirty years ago more interesting results relating to the discipline discussed in literature. In 2014, Bosan and Moiz [2] and [5] explored the notion of quasi $s-$topological groups, and quasi irresolute topological groups. We studied the concept of Levine [7] about topological spaces on semi open sets. Different mathematicians like Crossley et.al. [4] studied semi topological properties and Bohn [1] studied semi topological groups. Moreover, Siab et.al. [14] studied irresolute topological groups by using irresolute mappings. In continuation to these concepts Bosan et.al. [3, 5] studied classes of $s-$topological groups and $S-$topological groups. In 2016, Noreen et.al. introduced and defined semi quotient topology which is the generalization of quotient topology for spaces and groups [10]. The motivation behind this work was to study the quotient topology by weakening the open set conditions and also explored semi quotient mappings stronger than semi continuous mappings, and then consider semi quotient spaces and groups [10].

We need also some basic information on a quotient group as: If J is group and H invariant subgroup, consider the collection Q of all left cosets

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$$aH \text{ of } H,\ a \in J.$$

Define a multiplication in Q as follows: For

$$aH,\ bH \in Q,\ \text{we put } aH.bH = abH.$$

It is easy to verify that under this multiplication Q is a group with $eH = H$ as the identity and $a^{\leftarrow}H$ as the inverse of aH in Q. This group is called the quotient or factor group of J by H and it is denoted by J/H. H is invariant subgroup of J. The mapping

$$p : J \to J/H$$

defined by $p(x) = xH$, for each $x \in J$ is called a natural projection.

In this paper, significant results with counter examples have also been proved. We have also used semi homeomorphism [6] and S-homeomorphism [13] changing a discipline to an other and support the discipline.

2. PRELIMINARIES

A mapping $\zeta : K \to M$ between topological spaces K and M is called:

- pre-semi-open [4] if for every semi-open set A of K, the set $\zeta(A)$ is semi-open in M;
- s-open (s-closed) if for every semi-open (semi-closed) set A of K, the set $\zeta(A)$ is open (closed) in M;
- continuous if for each open set $V \subset M$ the set $\zeta^{\leftarrow}(V)$ is open in K.
- semi-continuous [7] (resp. irresolute [4]) if for each open (resp. semi-open) set $V \subset M$ the set $\zeta^{\leftarrow}(V)$ is semi-open in K. Equivalently, the mapping ζ is semi-continuous (irresolute) if for each $x \in K$ and for each open (semi-open) neighbourhood V of $\zeta(x)$, there exists a semi-open neighbourhood U of x such that $\zeta(U) \subset V$;
- semi-homeomorphism [4, 6] if ζ is bijective, irresolute and pre-semi-open;
- S-homeomorphism [3] if ζ is bijective, semi-continuous and pre-semi-open".
- $S-$isomorphism if it is an algebraic isomorphism and topologically an $S-$homeomorphism,
- semi-isomorphism if it is an algebraic isomorphism and topologically a semi-homeomorphism.

Definition 2.1. [1] *An s-topological group is a group $(J, *)$ with a topology τ such that for each $x, y \in J$ and each neighbourhood W of $x * y^{\leftarrow}$ there are semi open neighbourhoods U of x and V of y such that*

$$\mathrm{U} * \mathrm{V}^{\leftarrow} \subset \mathrm{W}.$$

Definition 2.2. [14] *A triple* $(\mathrm{J}, *, \tau)$ *is an irresolute* topological group *with a group* $(\mathrm{J}, *)$ *and a topology τ such that for each* $\mathrm{x},\ \mathrm{y} \in \mathrm{J}$ *and for each semi open neighbourhood* W *of* $\mathrm{x} * \mathrm{y}^{\leftarrow}$, *there exist semi-open neighbourhoods* U *of* x *and* V *of* y *such that*

$$\mathrm{U} * \mathrm{V}^{\leftarrow} \subset \mathrm{W}.$$

3. SEMI QUOTIENT MAPPINGS

Definition 3.1. *A mapping*

$$\phi : \mathrm{K} \to \mathrm{M},$$

where K and M are spaces is semi quotient provided a subset E of M is open in M if and only if $\phi^{\leftarrow}(E)$ is semi open in K.

The differences in mappings of semi quotient, semi continuous and the quotient are illustrated below:

Example 3.1. *Let*

$$\mathrm{K} = \mathrm{M} = \{1, 2, 3\}$$

and let

$$\tau_{\mathrm{K}} = \{\emptyset, \{1\}, \{1, 2\}, \mathrm{K}\}$$

and

$$\tau_{\mathrm{M}} = \{\emptyset, \{1, 2\}, \mathrm{M}\}$$

be topologies on K and M respectively.
Suppose

$$\phi : \mathrm{K} \to \mathrm{M}$$

is a mapping defined by

$$\phi(\alpha) = \alpha, \ \ \alpha \in \mathrm{K}.$$

Since

$$\tau_{\mathrm{M}} \subset \tau_{\mathrm{K}},$$

this mapping ϕ is semi continuous but not semi quotient because $\{1\}$ is not open in M, where as $\phi^{\leftarrow}(\{1\})$ is semi open in K.

Example 3.2. *Let*

$$\mathrm{K} = \{1, 2, 3, 4\}, \ \ \mathrm{M} = \{\mathrm{a}, \mathrm{b}\},$$

$$\tau_{\mathrm{K}} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \mathrm{K}\}, \ \ \tau_{\mathrm{M}} = \{\emptyset, \{\mathrm{b}\}, \mathrm{M}\}.$$

Suppose also

$$\phi : \mathrm{K} \to \mathrm{M} \text{ by}; \ \ \phi(4) = \phi(3) = \phi(2) = \mathrm{b}; \ \ \phi(1) = \mathrm{a}.$$

The mapping ϕ is neither continuous nor quotient but it is semi quotient because $\{b\}$ is open in M but $\phi^{\leftarrow}(\{b\}) = \{2, 3, 4\}$ is not open in K, that is it is not continuous. On the other hand it is semi quotient because the proper subset $\{b\} \in \tau_M, \phi^{\leftarrow}(\{b\}) = \{2, 3, 4\}$ is semi open in K.

Construction: [10] Suppose K is a topological space and M is a set.
Suppose

$$\phi : K \to M \text{ is a mapping}$$

and

$$s\tau_Q := \{E \subset M : \phi^{\leftarrow}(E) \in \mathsf{SO}(K)\},$$

called the semi quotient generalized topology. But $s\tau_Q$ may not be a topology on M [13]. It appears that if K is extremally disconnected, then the intersection of two semi open sets is semi open [9]. Obviously the form $s\tau_Q$ is the finer(stronger) than the topology σ on M:

$$\phi : K \to (M, \sigma)$$

is semi continuous. In deed,

$$\phi : K \to (M, s\tau_Q)$$

is a quotient mapping [11] in the present. In our coming example, we will see the relation between the spaces (M, σ) and $(M, s\tau_Q)$.

Now the special case, suppose ρ is an equivalence relation on K. Suppose

$$p : K \to K/\rho$$

is a projection from K to the set $K/\rho : \forall\ \alpha$ in K, p maps α to $\rho(\alpha)$. The generalized topology $s\tau_Q$ on K/ρ, where K is extremally disconnected with, the mapping p by forced semi continuous, is semi quotient. This important construction can be applied to topologized groups. We see the interesting example below [10] where a quotient topology and the semi quotient topology generated by the same mapping are different.

Example 3.3. *Set*

$$\mathrm{K} = \{1, 2, 3, 4, 5\}$$

with topology

$$\tau = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},$$
$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}, K\}$$

and the collection of semi open sets is

$$SO(K) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\},$$
$$\{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,2,3\}, \{1,2,4\},$$
$$\{1,2,5\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{1,3,4\},$$
$$\{2,3,4\}, \{1,2,3,4\}, \{1,2,4,5\}, \{1,2,3,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}.$$

We define a relation N on K by $\alpha N \gamma \Longleftrightarrow \alpha + \gamma$ is even. Then,

$$N = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$$

forms an equivalence relation, and

$$\mathrm{K/N} = \{\mathrm{N}(1), \mathrm{N}(2)\} = \{\{1,3,5\}, \{2,4\}\}.$$

Suppose

$$\mathrm{p} : \mathrm{K} \to \mathrm{K/N}$$

is a canonical projection. So,

$$\mathrm{p}^{\leftarrow}(\mathrm{N}(1)) = \{1,3,5\} \in \mathsf{SO}(\mathrm{K}),$$

and

$$\mathrm{p}^{\leftarrow}(\mathrm{N}(2)) = \{2,4\} \in \mathsf{SO}(\mathrm{K}),$$

so that

$$s\tau_Q = \{\emptyset, K/N, \{N(1)\}, \{N(2)\}\}$$

is the semi quotient topology on K/N. But, quotient topology on K/N is

$$\tau_Q = \{\emptyset, K/N, \{N(2)\}\}.$$

because here

$$\mathrm{p}^{\leftarrow}(\mathrm{N}(1)) = \{1,3,5\}$$

is not open in K.

Lemma 3.1. [1] *If* $(J, *, \tau)$ *is an* $s-$*topological group, then* A *is semi open in* J *if and only if* $A^{\leftarrow}$ *is semi open in* J*; and if* A *is semi open in* J*, and* $B \subset J$*, then* $A * B$ *and* $B * A$ *are semi open in* J*.*

Lemma 3.2. [14] *If* $(J, *, \tau)$ *is an irresolute topological group, then* A *is semi open in* J *if and only if* $A^{\leftarrow}$ *is semi open in* J*; and if* A *is semi open in* J*, and* $B \subset J$*, then* $A * B$ *and* $B * A$ *are semi open in* J*.*

4. SEMI QUOTIENT MAPPINGS AND SPACES PROPERTIES

In the present sight we will use the concept of $s\tau_Q$ discussed in the previous section establishing some properties.

Theorem 4.1. *If* L *is a closed subgroup of an extremally disconnected irresolute topological group* $(J, *, \tau)$*, and* $n_1 \in J$*, then* λ_{n_1} *is a semi-isomorphism and also*

$$p \circ \ell_{n_1} = \lambda_{n_1} \circ p.$$

Proof. The mappings $p : J \to J/L$ and $\lambda_{n_1} : J/L \to J/L$ are defined by $p(x) = x*L$ and $\lambda_{n_1}(x * L) = n_1 * x * L$ respectively. We see the properties of λ_{n_1} as under:
λ_{n_1} is well defined
Let $x * L = y * L$. This implies $n_1 * x * L = n_1 * y * L$. This implies that $\lambda_{n_1}(x * L) = \lambda_{n_1}(y * L)$.
λ_{n_1} is injective
Let $\lambda_{n_1}(x * L) = \lambda_{n_1}(y * L)$. This implies that $n_1 * x * L = n_1 * y * L$. By left cancelation law, we get $x * L = y * L$.
λ_{n_1} is surjective
For every $n_1 * x * L$ in the range of λ_{n_1}, there exists $x * L$ in the domain of λ_{n_1} such that $\lambda_{n_1}(x * L) = n_1 * x * L$.
λ_{n_1} is homomorphism
Since $p(x*y) = x*y*L = x*L*y*L = p(x)*p(y)$. Therefore, λ_{n_1} is homomorphism.
We have to show that

$$p \circ \ell_{n_1} = \lambda_{n_1} \circ p.$$

In fact, $\forall\ \alpha \in J$ we get,

$$(p \circ \ell_{n_1})(\alpha) = p(n_1 * \alpha) = (n_1 * \alpha) * L = n_1 * (\alpha * L) = \lambda_{n_1}(p(\alpha)) = (\lambda_{n_1} \circ p)(\alpha).$$

Now remaining is to show that λ_{n_1} is pre semi open and irresolute. It is evident from the followings. Suppose

$$\alpha * L \in J/L.$$

For every semi open set U of e_J,

$$p(\alpha * U * L)$$

is a semi open set of

$$\alpha * L \text{ in } J/L.$$

In the same strategy,

$$p(n_1 * \alpha * U * L)$$

is a semi open set of

$$n_1 * \alpha * L$$

in J/L. □

Theorem 4.2. *If $(J, *, \tau_J)$ is an $s-$ topological group, then $(J/N, *, s\tau_Q)$ is extremally disconnected $s-$topological group, where N is an invariant subgroup of J .*

Proof. Since $(J, *, \tau_J)$ is an $s-$ topological group, then by definition for every open neighbourhood W of $x * y^{\leftarrow}$, there exist semi open neighbourhoods U of x and V of y such that $U * V \subset W$. By Lemma 3.1, $U * N$ and $V * N$ are respective semi open neighbourhoods of $x * N$ and $y * N$. By using hypothesis, $U * V * N \subset W * N$. This implies that $(U * N) * (V * N) \subset W * N$. Hence $(J/N, *, s\tau_Q)$ is extremally disconnected $s-$topological group. □

Theorem 4.3. *Let N be an invariant subgroup and $\zeta : J/N \to H/N$ be an $S-$isomorphism of quasi $s-$topological groups. If ζ is semi continuous at the neutral element of the domain, then it is semi continuous at the domain.*

Proof. Let $x*N \in J/N$ and $W*N$ be an open neighbourhood of $y*N = \zeta(x*N)$ in H/N. Then by semi continuity of left translation in H/N, there exists a semi open neighbourhood $V*N$ of the neutral element N of H/N such that (i) $\ell_{y*N}(V*N) = y*N*V*N \subset W*N$. By the hypothesis, ζ is semi continuous at $e_J*N = N$ implies (ii) $\zeta(U*N) \subset N*N$, for some semi open neighbourhood $U*N$ of e_J*N. Also $\ell_{x*N} : J/N \to J/N$ is an $s-$open mapping, the set $\ell_{x*N}(U*N) = x*N*U*N$ is semi open in J/N. Hence $\zeta(x*N*U*N) = y*N*(U*N) \subset y*N*V*N \subset W*N$ (by ii and i). Thus ζ is semi continuous at J/N. □

Theorem 4.4. *Let $(J/N, *, s\tau_J)$ be an extremally disconnected $s-$topological group and $(H/N, *, s\tau_H)$ be an extremally disconnected quasi $s-$topological group with N an invariant subgroup. If $\zeta : J/N \to H/N$ is $S-$isomorphism with $\zeta(x^{\leftarrow}) = (\zeta(x))^{\leftarrow}$, then H/N is also extremally disconnected $s-$topological group.*

Proof. Let $W * N = O_{h_1*N*h_2^{\leftarrow}*N}$ be an open neighbourhood of $h_1 * N * h_2^{\leftarrow} * N$, where $h_1, h_2 \in H$. Then by semi continuity of ζ, $\zeta^{\leftarrow}(W*N) = \zeta^{\leftarrow}(O_{h_1*N*h_2^{\leftarrow}*N})$ is semi open neighbourhood in J/N. Also ζ is bijective, $\zeta(g_1) = h_1$ and $\zeta(g_2) = h_2$, where $g_1, g_2 \in J$. This implies $g_1 = \zeta^{\leftarrow}(h_1), g_2 = \zeta^{\leftarrow}(h_2)$. Since J/N is extremally disconnected $s-$topological group, there are semi open neighbourhoods M_{g_1*N} and M_{g_2*N} such that $M_{g_1*N} * M_{g_2^{\leftarrow}*N} \subset \zeta^{\leftarrow}(W*N)$. This gives that $\zeta(M_{g_1*N} * M_{g_2^{\leftarrow}*N}) \subset W*N$. By homomorphism of ζ, we get $\zeta(M_{g_1*N})*\zeta(M_{g_2^{\leftarrow}*N}) \subset W*N$. Since ζ is $s-$open, then $\zeta(M_{g_1*N})$ and $\zeta(M_{g_2*N})$ are semi open neighbourhoods in H/N. This gives that $(W_{h_1*N}) * (W_{h_2^{\leftarrow}*N}) \subset W*N$, where $\zeta(W_{h_1*N}) = W_{h_1*N}$ and $\zeta(M_{g_2*N}) = W_{h_2*N}$. That is, $(W_{h_1*N}) * (W_{h_2*N})^{\leftarrow} \subset O_{h_1*N*h_2^{\leftarrow}*N}$. □

Theorem 4.5. *If L is a closed invariant subgroup of an extremally disconnected $s-$topological group $(J, *, \tau)$, and if $\zeta : J/L \to H/L$ is $S-$isomorphism with $\zeta(x^{\leftarrow}) = (\zeta(x))^{\leftarrow}$, where $(J/L, *, s\tau_J)$ is an extremally disconnected $s-$topological group and $(H/L, *, s\tau_H)$ is an extremally disconnected quasi $s-$topological group, then $(J/L, *, s\tau_Q)$ is an extremally disconnected $s-$topological group.*

Proof. Suppose the left translations

$$l_g : J \to J \text{ and } l_{p(g)} : J/L \to J/L$$

by $g \in J$, and $p(g) \in J/L$, and the inverse mappings i and i' respectively. $\forall\ \alpha \in J$, we get

$$(p \circ l_g)(\alpha) = g * \alpha * L = (g * L) * (\alpha * L)(= m(g * L, \alpha * L)) = (l_{p(g)} \circ p)(\alpha)$$

and

$$(p \circ i)(\alpha) = \alpha^{-1} * L = (i' \circ p)(\alpha).$$

We see following commutative figures:

$$\begin{array}{ccccc} & & l_g & & \\ J & & \longrightarrow & & J \\ \downarrow & p & & p & \downarrow \\ J/L & & \longrightarrow & & J/L \\ & & l_{p(g)} & & \end{array}$$

$$\begin{array}{ccccc} & & i & & \\ J & & \longrightarrow & & J \\ \downarrow & p & & p & \downarrow \\ J/L & & \longrightarrow & & J/L \\ & & i' & & \end{array}$$

Here p is semi continuous, and the left translation

$$l_g : J \to J$$

$l_g(\alpha) = g * \alpha, \forall \alpha \in J$.
Claim: left translation l_g and the inverse mapping i are irresolute.
If W is a semi open of $g*\alpha$ from range J, $g^{-1}*W$ is a semi open of domain element α satisfying

$$l_g(g^{-1} * W) = W.$$

Suppose W is a semi open in $i(\alpha)$. By Lemma 3.2 [14], W^{-1} is a semi open set of the domain element α satisfying

$$i(W^{-1}) = W.$$

This gives that the left translation $l_{p(g)}$ and the inverse mapping i' must be semi continuous. This gives that J/L is an extremally disconnected quasi $s-$topological group. Thus by Theorem 4.4, we have the required result. □

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THE PATHWAY INTEGRAL OPERATOR INVOLVING EXTENSION OF K-BESSEL-MAITLAND FUNCTION

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Abstract. In the present paper, we establish generalized extension of k-Bessel-Maitland function involving pathway integral operator. We obtain certain composition formulas with pathway fractional integral operators. Further more, Some interesting special cases involving Bessel functions, generalized Bessel functions, generalized Mittag-Leffer functions, generalized k-Mittag-Leffer functions are deduced.

1. Introduction

The study of special functions play an important role in Mathematics, Physics, Chemistry, Biology, Engineering and applied Sciences. It has a wide application of almost all branches of Science and technology. The Bessel-Maitland function [10, 28] is denoted by $J_\nu^\mu(z)$ and is defined as follows:

$$\mathbf{J}_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(n\mu+\nu+1)}. \tag{1.1}$$

The theory of Bessel functions is intimately connected with the theory of certain types of differential equations. A detail account of applications of Bessel functions are given in the book of Watson [27].

Now, Singh *et al.* [25] introduced and investigate of the following generalization of Bessel-Maitland function as follows:

$$\mathbf{J}_{\nu,\tau}^{\mu,q}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_{qn}(-z)^n}{n!\Gamma(n\mu+\nu+1)}, \tag{1.2}$$

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where $\mu, \nu, \tau \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\tau) \geq 0$, and $q \in (0,1) \bigcup \mathbb{N}$ and $(\tau)_{qn} = \frac{\Gamma(\tau+qn)}{\Gamma(\tau)}$ denotes the generalized Pochhammer symbol (see Rainville [21]).

Furthermore, Ghayasuddin *et al.* [7] investigate a new extension of Bessel-Maitland function as follows:

$$\mathbf{J}_{\nu,\tau,\zeta}^{\mu,q,p}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_{qn}(-z)^n}{\Gamma(n\mu+\nu+1)(\zeta)_{pn}}, \tag{1.3}$$

where $\mu, \nu, \tau, \zeta \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\tau) \geq 0, \Re(\zeta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

Recently, Khan *et al.* [9] consider a new generalized Bessel-Maitland function which is defined as:

$$\mathbf{J}_{\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\tau)_{qn}(-z)^n}{\Gamma(n\beta+\alpha+1)(\zeta)_{pn}(\nu)_{n\sigma}}, \tag{1.4}$$

where $\alpha, \beta, \mu, \rho, \nu, \tau, \zeta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0, \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\tau) \geq 0, \Re(\zeta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

In this paper, we consider a new extension of generalized k-Bessel-Maitland function which is defined as:

$$\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-z)^n}{\Gamma_k(n\beta+\alpha+1)(\delta)_{pn,k}(\nu)_{n\sigma,k}}, \tag{1.5}$$

where $k, \alpha, \beta, \mu, \rho, \nu, \tau, \zeta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0, \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

1.1. **Relation with Mittag-Leffler function.**

(1) If we put α by $\alpha - 1$ in (1.5), we get the following result

$$\mathbf{J}_{k,\alpha-1,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(-x) = E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(x), \tag{1.6}$$

where $E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(x)$ is the Mittag-Leffler function defined by Khan and Ahmad [8].

(2) If we put $\mu = \nu = \sigma = \rho = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}_{\alpha-1,\beta,1,1,\delta,p}^{1,1,\gamma,q}(-x) = E_{\alpha,\beta,p}^{\zeta,\tau,q}(x), \tag{1.7}$$

where $E_{\alpha,\beta,p}^{\zeta,\tau,q}(x)$ is the Mittag-Leffler function defined by Salim and Faraz [23].

(3) If we put $\mu = \nu = \sigma = \rho = \zeta = p = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}_{k,\alpha-1,\beta,1,1,1,1}^{1,1,\zeta,q}(-x) = E_{k,\alpha,\beta}^{\tau,q}(x), \tag{1.8}$$

where $E_{k,\alpha,\beta}^{\tau,q}(x)$ is the k-Mittag-Leffler function defined by Chand *et al.* [4].

(4) If we put $\mu = \nu = \sigma = \rho = \zeta = p = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}^{1,1,\gamma,q}_{\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau,q}_{\alpha,\beta}(x), \tag{1.9}$$

where $E^{\tau,q}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Shukla and Prajapati [26].

(5) If we put $\mu = \nu = \sigma = \rho = \zeta = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}^{1,1,\tau,\zeta}_{\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau,\zeta}_{\alpha,\beta}(x), \tag{1.10}$$

where $E^{\tau,q}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Salim [24].

(6) If we put $\mu = \nu = \sigma = \rho = \zeta = p = q = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}^{1,1,\tau}_{k,\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau}_{k,\alpha,\beta}(x), \tag{1.11}$$

where $E^{\tau}_{k,\alpha,\beta}(x)$ is the k-Mittag-Leffler function defined by Dorrego and Cerutti [6].

(7) If we put $\mu = \nu = \sigma = \rho = \zeta = p = q = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}^{1,1,\tau}_{\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau}_{\alpha,\beta}(x), \tag{1.12}$$

where $E^{\tau}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Prabhakar [22].

(8) If we put $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}^{1,1,1}_{\alpha-1,\beta,1,1,1,1}(-x) = E_{\alpha,\beta}(x), \tag{1.13}$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Wiman [28].

(9) If we put $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1, \alpha = 0$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}^{1,1,1}_{0,\beta,1,1,1,1}(-x) = E_{\beta}(x), \tag{1.14}$$

where $E_{\beta}(x)$ is the Mittag-Leffler function defined by Mittag-Leffler [16].

We investigate some special cases of the generalized Bessel Maitland function (1.3) by particular values to the parameters $\mu, \nu, \delta, \gamma, p, q$.

Now, we recall the classical Beta function denoted by $B(a, b)$ and is defined as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, (\Re(a) > 0, \Re(b) > 0). \tag{1.15}$$

(see [21], and also see [10]). The integral representation of the k-Gamma function is given as:

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}) = \int_0^\infty e^{\frac{-t^k}{k}} t^{z-1}dt, \tag{1.16}$$

$k \in \mathbb{R}, z \in \mathbb{C}$,

and k-Beta function is defined as:

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, x>0, y>0. \tag{1.17}$$

The generalized Wright function represented as follows [29, 30, 31]:

$${}_p\Psi_q\left[\begin{array}{cc}(\alpha_1,A_1),...,(\alpha_p,A_p); & \\ & z \\ (\beta_1,B_1),...,(\beta_p,B_p); & \end{array}\right] = {}_p\Psi_q\left((\alpha_j,A_j)_{1,p};(\beta_j,B_j)_{1,q};z\right)$$

$$= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+nA_1)...,\Gamma(\alpha_p+nA_p)}{\Gamma(\beta_1+nB_1)...,\Gamma(\beta_p+nB_p)}\frac{z^n}{n!}. \tag{1.18}$$

In 1961, MacRobert [11] investigate the following interesting result which is given below:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[at+b(1-t)]^{-\alpha-\beta}dt = \frac{1}{a^\alpha b^\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.19}$$

where a and b are non zero constants such that the expression $at+b(1-t)$, for $0 \le t \le 1$, is non zero, provided $\Re(\alpha)>0, \Re(\beta)>0$.

In this paper, we further apply the following useful result which is given below:

$$\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}[at+b(1-t)]^{\frac{-\alpha-\beta}{k}}dt = \frac{1}{a^{\frac{\alpha}{k}}b^{\frac{\beta}{k}}}\frac{k\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}, \tag{1.20}$$

where a and b are non zero constants such that the expression $at+b(1-t)$, for $0 \le t \le 1$, is non zero, provided $\Re(\alpha)>0, \Re(\beta)>0$.
It is easy to see that for $k=1$ the equation (1.20) reduces to known result (1.19).

Recently, by using the pathway idea of Mathai [13] and developed further by Mathai and Haubold [14, 15], Nair [17], we introduce a pathway fractional integral operator which is given below.

Suppose $f(x) \in L(a,b), \eta \in \mathbb{C}, \Re(\eta)>0, a>0$ and the pathway parameter $\alpha<1$ as (cf. [2]), then

$$(P_{0+}^{(\eta,\alpha)}f)(x) = x^\eta \int_0^{[\frac{x}{a(1-\alpha)}]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt. \tag{1.21}$$

For a real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^\delta\right]^{\frac{\beta}{(1-\alpha)}}, \tag{1.22}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and $\gamma > 0$, where c is the normalizing constant and α is called the pathway parameter,

$$c = \frac{1}{2} \frac{\delta\left(a(1-\alpha)\right)^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{(1-\alpha)} + 1\right)}{\Gamma(\frac{\gamma}{\delta})\Gamma\left(\frac{\beta}{(1-\alpha)} + 1\right)}, for\ \alpha < 1 \tag{1.23}$$

$$= \frac{1}{2} \frac{\delta\left(a(1-\alpha)\right)^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\beta}{(1-\alpha)}\right)}{\Gamma(\frac{\gamma}{\delta})\Gamma\left(\frac{\beta}{(1-\alpha)} - \frac{\gamma}{\delta}\right)}, for \frac{1}{1-\alpha} - \frac{\gamma}{\delta} > 0,\ \alpha > 1 \tag{1.24}$$

$$= \frac{1}{2} \frac{(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})},\ \alpha \to 1. \tag{1.25}$$

For $\alpha < 1$, it is a finite range density with $\left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and (1.21) remains in the extended generalized type-1 beta family. The Pathway density in (1.21), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f's. [2]. For $\alpha > 1$,

$$f(x) = c|x|^{\gamma-1}\left[1 + a(1-\alpha)|x|^{\delta}\right]^{-\frac{\beta}{1-\alpha}}, \tag{1.26}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0$ and $\alpha > 0$ which is extended generalized type-2 modal for real x. It includes the type-2 beta density, the F density, the student-t density, the cauchy density and many more. For instance, $\alpha > 1$, writing $(1-\alpha) = -(\alpha - 1)$ gives:

$$(P_{0+}^{(\eta,\alpha)} f)(x) = x^{\eta} \int_{0}^{\left[\frac{x}{-a(1-\alpha)}\right]} \left[1 + \frac{a(\alpha-1)t}{x}\right]^{-\frac{\eta}{(\alpha-1)}} f(t)dt. \tag{1.27}$$

For more basic details about pathway integral operator, one may refer [1, 2, 18, 19, 20].

2. Main Results

The pathway integral operator of k-Bessel-Maitland function is given in the following theorems.

Theorem 2.1. *Let* $k \in \mathcal{R}, \alpha, \beta, \tau, \zeta, \mu, \nu, \rho, \sigma \in \mathcal{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, p, q > 0$ *and* $q \leq \Re(\alpha) + p, \eta \in C, \Re(\frac{\eta}{1-\xi}) > -1, \lambda > 1, w > R.$

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right). \tag{2.1}$$

Proof. On taking L.H.S. of Theorem 2.1, and then expanding the definition of generalized k-Bessel-Maitland function $\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})$, by using (1.18) we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}J_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x)$$

$$= x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\lambda)}\right]} t^{\frac{\beta}{k}-1}\left[1-\frac{a(1-\lambda)t}{x}\right]^{\frac{\eta}{(1-\lambda)}} J^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}(wt^{\frac{\alpha}{k}})dt,$$

$$= x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\lambda)}\right]} t^{\frac{\beta}{k}-1}\left[1-\frac{a(1-\lambda)t}{x}\right]^{\frac{\eta}{(1-\lambda)}} \sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-wt^{\frac{\alpha}{k}})^{n}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}dt,$$

Interchanging the integration and summation under the suitable convergence condition, we obtain

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^{n}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\int_{0}^{\left[\frac{x}{a(1-\lambda)}\right]} t^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}\left[1-\frac{a(1-\lambda)t}{x}\right]^{\frac{\eta}{(1-\lambda)}} dt,$$

Now, interchanging the inner integral by beta function formula (1.12), we get

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^{n}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\int_{0}^{1} u^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}(1-u)^{\frac{\eta}{(1-\lambda)}}\left(\frac{x}{a(1-\lambda)}\right)$$

$$\times\left(\frac{x}{a(1-\lambda)}\right)^{\frac{\beta}{k}+\frac{n\alpha}{k}-1} du,$$

again applying the Beta function formula, we have

$$= \frac{x^{\eta+\frac{\beta}{k}}}{(a(1-\lambda))^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^{n}x^{\frac{n\alpha}{k}}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\frac{\Gamma(\frac{\eta}{(1-\lambda)}+1)\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(\frac{\eta}{(1-\lambda)}+\frac{\beta}{k}+\frac{n\alpha}{k}+1)}\frac{1}{(a(1-\lambda))^{\frac{n\alpha}{k}}}.$$

Now, using the result,

$$\Gamma_k(\lambda) = k^{\frac{\lambda}{k}-1}\Gamma(\frac{\lambda}{k}), \tag{2.2}$$

we get,

$$= \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^{n}x^{\frac{n\alpha}{k}}}{k^{\frac{n\beta+\alpha+1}{k}-1}\Gamma(\frac{n\beta+\alpha+1}{k})(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\frac{\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(\frac{\eta}{(1-\lambda)}+\frac{\beta}{k}+\frac{n\alpha}{k}+1)}\frac{1}{(a(1-\lambda))^{\frac{n\alpha}{k}}},$$

$$= \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),\nu,\sigma,\zeta,p}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right),$$

which is our desired result (2.1).
Thus, the proof of Theorem 2.1 is complete. □

Corollary 2.2. *If we put $\tau = q = 1, \nu = \sigma = p = 1$ in Theorem 2.1, then we get the result corresponding result of Nisar et al.* [19] *as:*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,1,1,\zeta,1}^{\mu,\rho,1,1}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),1,1,\zeta,1}^{\mu,\rho,1,1}\left(-w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right). \tag{2.3}$$

Corollary 2.3. *If we put $\tau = q = 1, \nu = \sigma = p = \zeta = k = 1$ in Theorem 2.1, then we obtain the corresponding result of Nair* [17] *as:*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}\mathbf{J}_{1,\alpha,\beta,1,1,1,1}^{\mu,\rho,1,1}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}\mathbf{J}_{1,\alpha,\beta+1(\frac{\eta}{1-\lambda}),1,1,1,1}^{\mu,\rho,1,1}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right). \tag{2.4}$$

Theorem 2.4. *Let $k \in \mathcal{R}, \alpha, \beta, \tau, \zeta, \mu, \nu, \rho, \sigma \in \mathcal{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, p, q > 0$ and $q \leq \Re(\alpha) + p, \eta \in C, \Re(\frac{\eta}{1-\xi}) > -1, \lambda > 1, w > R$.*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1}),\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{2.5}$$

Proof. On taking L.H.S of (2.5) and applying the definition (1.5) and (1.24), we obtain

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x)$$

$$= x^{\eta}\int_0^{[\frac{x}{-a(1-\lambda)}]} t^{\frac{\beta}{k}-1}\left[1+\frac{a(\lambda-1)t}{x}\right]^{\frac{\eta}{-(\lambda-1)}} J_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})dt,$$

$$= x^{\eta}\int_0^{[\frac{x}{-a(1-\lambda)}]} t^{\frac{\beta}{k}-1}\left[1+\frac{a(\lambda-1)t}{x}\right]^{\frac{\eta}{-(\lambda-1)}} \sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-wt^{\frac{\alpha}{k}})^n}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}dt.$$

Interchanging the integration and summation under the suitable convergence condition, we obtain

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\int_0^{[\frac{x}{-a(1-\lambda)}]} t^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}\left[1+\frac{a(\lambda-1)t}{x}\right]^{\frac{\eta}{-(\lambda-1)}} dt.$$

Now, interchanging the inner integral by beta function formula, we get

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\int_0^1 u^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}(1-u)^{\frac{\eta}{(1-\lambda)}}\left(\frac{x}{a(1-\lambda)}\right)$$

$$\times\left(\frac{x}{a(1-\lambda)}\right)^{\frac{\beta}{k}+\frac{n\alpha}{k}-1} du,$$

again applying the beta function formula, we have

$$= \frac{x^{\eta+\frac{\beta}{k}}}{(-a(1-\lambda)^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n x^{\frac{n\alpha}{k}}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \frac{\Gamma(1-\frac{\upsilon}{(\lambda-1)})\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(1-\frac{\upsilon}{(\lambda-1)}+\frac{\beta}{k}+\frac{n\alpha}{k})} \frac{1}{(-a(\lambda-1))^{\frac{n\alpha}{k}}}.$$

Now, using the result,

$$\Gamma_k(\lambda) = k^{\frac{\lambda}{k}-1}\Gamma(\frac{\lambda}{k}), \tag{2.6}$$

we obtain,

$$= \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}+1}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n x^{\frac{n\alpha}{k}}}{k^{\frac{n\beta+\alpha+1}{k}-1}\Gamma(\frac{n\beta+\alpha+1}{k})(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \frac{\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(1-\frac{\eta}{(1-\lambda)}+\frac{\beta}{k}+\frac{n\alpha}{k})} \frac{1}{(-a(1-\lambda))^{\frac{n\alpha}{k}}},$$

$$= \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}} k^{\frac{\alpha+1}{k}-1}} \mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1}),\nu,\sigma,\zeta,p}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right),$$

which is our desired result (2.5). □

Corollary 2.5. *If we put $\tau = q = 1, \nu = \sigma = p = 1$ in Theorem 2.4, then it reduces to the corresponding result of* [16]*:*

$$P^{(\eta,\lambda)}_{0+}\left[t^{\frac{\beta}{k}-1}\mathbf{J}^{\mu,\rho,1,1}_{k,\alpha,\beta,1,1,\zeta,1}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}+1}k^{\frac{\alpha+1}{k}-1}} \mathbf{J}^{\mu,\rho,1,1}_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{1-\lambda}),1,1,\zeta,1}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{2.7}$$

Corollary 2.6. *If we put $\tau = q = 1, \nu = \sigma = p = \zeta = k = 1$ in Theorem 2.4, then it reduces to the following result of Nair* [17].

$$P^{(\eta,\lambda)}_{0+}\left[t^{\beta-1}\mathbf{J}^{\mu,\rho,1,1}_{1,\alpha,\beta,1,1,1,1}(wt^{\alpha})\right](x) = \frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(1-\lambda)})}{(-a(1-\lambda))^{\beta+1}} \mathbf{J}^{\mu,\rho,1,1}_{1,\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1}),1,1,1,1}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right). \tag{2.8}$$

Theorem 2.7. *Let $k \in \mathcal{R}, \alpha, \beta, \upsilon, \zeta, \mu, \nu, \rho, \sigma, \lambda, \tau \in \mathcal{C}, \Re(\alpha) > -1, \Re(\beta) > 0, \Re(\upsilon) > 0, \Re(\zeta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(\lambda) > 0, \Re(\tau) > 0, p, q > 0$ and $q \leq \Re(\alpha) + p$.*

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}\mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}} dt$$
$$= \frac{\Gamma_k(\zeta)\Gamma_k(\mu)}{\Gamma_k(\tau)\Gamma_k(\mu)a^{\upsilon}b^{\lambda}} \sum_{s=0}^{\infty} \frac{\Gamma_k(\mu+s\rho)\Gamma_k(\gamma+sq)(-2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(\zeta+ps)\Gamma_k(\nu+s\sigma)\Gamma} \frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)}. \tag{2.9}$$

Proof. On taking L.H.S. of Theorem 2.7, using the definition of generalized k-Bessel-Maitland function (1.5) and (1.17), we obtain

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\tau-\xi}{k}}\mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}}dt,$$

$$=\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}\sum_{s=0}^{\infty}\frac{(\mu)_{\rho s,k}(\tau)_{qs,k}}{\Gamma_k(s\beta+\alpha+1)(\zeta)_{ps,k}(\nu)_{s\sigma,k}}\frac{(-2)^{\frac{s}{k}}(ab)^{\frac{s}{k}}t^{\frac{s}{k}}(1-t)^{\frac{s}{k}}}{(at+b(1-t))^{\frac{2s}{k}}}dt,$$

$$=\sum_{s=0}^{\infty}\frac{(\mu)_{\rho s,k}(\tau)_{qs,k}}{\Gamma_k(s\beta+\alpha+1)(\zeta)_{ps,k}(\nu)_{s\sigma,k}}(-2)^{\frac{s}{k}}(ab)^{\frac{s}{k}}\int_0^1 t^{\frac{\upsilon+s}{k}-1}(1-t)^{\frac{\xi+s}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi-2s}{k}}dt,$$

by using the integral (1.17), we obtain

$$=\sum_{s=0}^{\infty}\frac{(\mu)_{\rho s,k}(\tau)_{qs,k}}{\Gamma_k(s\beta+\alpha+1)(\zeta)_{ps,k}(\nu)_{s\sigma,k}}\frac{(-2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{a^{\frac{\tau}{k}}b^{\frac{\lambda}{k}}}\frac{k\Gamma_k(\tau+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)},$$

$$=\frac{\Gamma_k(\zeta)\Gamma_k(\mu)}{\Gamma_k(\tau)\Gamma_k(\mu)a^{\upsilon}b^{\lambda}}\sum_{s=0}^{\infty}\frac{\Gamma_k(\mu+s\rho)\Gamma_k(\tau+sq)(-2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(\zeta+ps)\Gamma_k(\nu+s\sigma)\Gamma}\frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)},$$

we derive required result.
Thus, the proof of Theorem 2.7 is established. □

3. Special Case

In this section, we establish the following potentially useful integral operators involving generalized k-Beta type functions as special cases of our main results:

(1) If we let α by $\alpha-1$ in Theorem 2.1, and then by using (1.6), we get:

$$P^{(\eta,\lambda)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}(wt^{\alpha})\right](x)=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha}{k-1}}}E^{\mu,\rho,\tau,q}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),\nu,\sigma,\zeta,p}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right) \tag{3.1}$$

(2) If we let $\mu=\nu=\sigma=\rho=k=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.7), we obtain:

$$P^{(\eta,\lambda)}_{0+}\left[t^{\beta-1}E^{\zeta,\tau,q}_{\alpha,\beta,p}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\frac{\beta}{k}}}E^{\zeta,\tau,q}_{\alpha,\beta+1(\frac{\eta}{1-\lambda}),p}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.2}$$

(3) If we let $\mu=\nu=\sigma=\rho=\zeta=p=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.8), we obtain

$$P^{(\eta,\lambda)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\tau,q}_{k,\alpha,\beta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha}{k-1}}}E^{1,1,\tau,q}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),1,1,1,1}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right) \tag{3.3}$$

(4) If we let $\mu=\nu=\sigma=\rho=\zeta=p=k=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.9), we attain:

$$P^{(\eta,\lambda)}_{0+}\left[t^{\beta-1}E^{\tau,q}_{\alpha,\beta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E^{\tau,q}_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.4}$$

(5) If we let $\mu=\nu=\sigma=\rho=q=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then bu using (1.10), we get

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau,\zeta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}^{\tau,\zeta}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.5}$$

(6) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.11), we attain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta}^{\tau}(wt^{\alpha})\right](x)=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha}{k-1}}}E_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda})}^{\tau}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right) \tag{3.6}$$

(7) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=k=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.12), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}^{\gamma}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.7}$$

(8) If we let $\mu=\nu=\sigma=\rho=\zeta=\tau=p=q=k=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.13), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{1}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}^{1}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.8}$$

(9) If we let $\mu=\nu=\sigma=\rho=\zeta=\tau=p=q=k=1,\alpha=0$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.14), we find:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\beta}^{1}(w)\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\beta+1(\frac{\eta}{1-\lambda})}^{1}\left(w(\frac{x}{a(1-\lambda)})\right) \tag{3.9}$$

(10) If we let α by $\alpha-1$ in Theorem 2.4, and then by using (1.6), we get:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x)=\frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}E_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1}),\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{3.10}$$

(11) If we let $\mu=\nu=\sigma=\rho=k=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.7), we get:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta,p}^{\tau,\zeta,q}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1}),p}^{\zeta,\gamma,q}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.11}$$

(12) If we let $\mu=\nu=\sigma=\rho=\zeta=p=1$ and replacing α by $\alpha-1$ in Theorem 2.7, and then by using (1.8), we get:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta}^{\tau,q}(wt^{\frac{\alpha}{k}})\right](x)=\frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}E_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1})}^{\gamma,q}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{3.12}$$

(13) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=k=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.9), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau,q}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1})}^{\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.13}$$

(14) If we let $\mu=\nu=\sigma=\rho=q=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then bu using (1.10), we get

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau,\zeta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1})}^{\tau,\zeta}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.14}$$

(15) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.11), we attain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta}^{\tau}(wt^{\frac{\alpha}{k}})\right](x)=\frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}E_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1})}^{\tau}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{3.15}$$

(16) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=k=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.12), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{,\alpha,\beta}^{\tau,q}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1})}^{\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.16}$$

(17) If we let $\mu=\nu=\sigma=\rho=\zeta=\tau=p=q=k=1,\alpha=0$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.13), we find:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\beta}(w)\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\beta+1(1-\frac{\eta}{\lambda-1})}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.17}$$

(18) If we let α by $\alpha-1$ in Theorem 2.7, and then by using (**??**), we get:

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}}dt$$

$$=\frac{\Gamma_k(\zeta)\Gamma_k(\mu)}{\Gamma_k(\tau)\Gamma_k(\mu)a^{\tau}b^{\lambda}}\sum_{s=0}^{\infty}\frac{\Gamma_k(\mu+s\rho)\Gamma_k(\gamma+sq)(2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(\zeta+ps)\Gamma_k(\nu+s\sigma)\Gamma}\frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)} \tag{3.18}$$

(19) If we let $\mu=\nu=\sigma=\rho=k=1$ and replacing α by $\alpha-1$ in Theorem 2.7, and then by using (1.7), we get:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi}E_{\alpha,\beta,p}^{\tau,\zeta,q}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]dt={}_4\Psi_3\left[\begin{array}{cc}(\tau,q),(\upsilon,1),(\lambda,1),(1,1); & \\ & -2ab\\ (\alpha,\beta),(\zeta,p),(\upsilon+\lambda,2),; & \end{array}\right]. \tag{3.19}$$

(20) If we let $\mu=\nu=\sigma=\rho=\zeta=p=1$ and replacing α by $\alpha-1$ in Theorem 2.7, and then by using (1.8), we get:

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}E_{k,\alpha,\beta}^{\tau,q}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}}dt$$

$$= \frac{1}{\Gamma_k(\tau)a^{\upsilon}b^{\lambda}} \sum_{s=0}^{\infty} \frac{\Gamma_k(1+s)\Gamma_k(\tau+sq)(2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(1+s)\Gamma_k(1+s)\Gamma} \frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)}. \quad (3.20)$$

(21) If we let $\mu = \nu = \sigma = \rho = \zeta = p = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7,and then by using (1.9), we attain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}^{\tau,q}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (\tau,q),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.21)$$

(22) If we let $\mu = \nu = \sigma = \rho = q = p = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7,and then by using (1.10), we attain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}^{\tau,\zeta}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_3 \left[\begin{array}{cc} (\tau,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),(\zeta,1),; & \end{array}\right]. \quad (3.22)$$

(23) If we let $\mu = \nu = \sigma = \rho = \zeta = p = q = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.11), we attain:

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}} E_{k,\alpha,\beta}^{\tau}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}} dt$$

$$= \frac{1}{\Gamma_k(\tau)a^{\upsilon}b^{\lambda}} \sum_{s=0}^{\infty} \frac{\Gamma_k(1+s)\Gamma_k(\upsilon+s)(2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(1+s)\Gamma_k(1+s)\Gamma} \frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)}. \quad (3.23)$$

(24) If we let $\mu = \nu = \sigma = \rho = \zeta = p = q = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.12), we obtain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}^{\tau}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (\tau,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.24)$$

(25) If we let $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.13) we obtain

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (1,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.25)$$

(26) If we let $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1, \alpha = 0$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.14) we obtain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\beta}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (1,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (0,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.26)$$

4. **Conclusion**

In the present article, we derive a new generalization of k-Beseel Maitland function and obtain the fractional calculus formula for the same. We also define and study a new fractional integral operators, which contain the extended Bessel Maitland function. If $k = 0$, then all the results of extended Bessel Maitland function will lead to the well-known results of Bessel Maitland function (see [9]).

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SOME CHARACTERIZATIONS ON GEODESIC, ASYMPTOTIC AND SLANT HELICAL TRAJECTORIES ACCORDING TO PAFORS

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Abstract. In this paper, we study the geodesic, asymptotic, and slant helical trajectories according to PAFORS in three-dimensional Euclidean space and give some characterizations on them. Also, we explain how we determine the helix axis for slant helical trajectories (according to PAFORS). Moreover, we develop a method that enables us to find the slant helical trajectory (if exists) lying on a given implicit surface which accepts a given fixed unit direction as an axis and a given angle as the constant angle. This method also gives information when the desired slant helical trajectory does not exist. The results obtained here involve some differential and partial differential equations or they are based on these equations. The aforementioned results are new contributions to the field and they may be useful in some specific applications of particle kinematics and differential geometry.

1. Introduction and Preliminaries

Despite its long history, the theory of surfaces is still an issue of interest in 3-dimensional Euclidean space. Darboux frame, which is constructed on a surface, is an important part of this theory. It exists at all the points of a curve on a regular surface [1]. The success of developing this frame belongs to French mathematician J. G. Darboux. From the discovery of this frame until now, many researchers have presented lots of interesting studies on the theory of surfaces by using this frame. Some of these studies can be found in [2–7]. One of the newest of them is the study [8] presented by Özen and Tosun. They introduced PAFORS (positional adapted frame on the regular surface) for the trajectories with non-vanishing angular momentum in this study.

Let the Euclidean 3-space E^3 be taken into account with the standard scalar product $\langle \mathbf{N}, \mathbf{P} \rangle = n_1p_1 + n_2p_2 + n_3p_3$ where $\mathbf{N} = (n_1,\, n_2,\, n_3),\ \mathbf{P} = (p_1,\, p_2,\, p_3)$ are any vectors in E^3. The norm of $\mathbf{N}$ is given as $\|\mathbf{N}\| = \sqrt{\langle \mathbf{N},\, \mathbf{N} \rangle}$. If a differentiable

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curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ satisfies $\left\| \frac{d\alpha}{ds} \right\| = 1$ for all $s \in I$, it is called a unit speed curve. In that case, s is said to be arc-length parameter of α. A differentiable curve is called a regular curve if its derivative does not equal zero along the curve. An arbitrary regular curve can be reparameterized by the arc-length of itself [9]. Throughout the paper, the differentiation with respect to the arc-length parameter s will be shown with a dash.

Assume that a point particle moves along the trajectory $\alpha : I \subset R \to M \subset E^3$ which is a unit speed curve and lies on a regular surface M. In this case, there exists Darboux frame denoted by $\{\mathbf{T}, \mathbf{Y}, \mathbf{U}\}$ along the trajectory $\alpha = \alpha(s)$. Here, $\mathbf{T}$ is the unit tangent vector of α, $\mathbf{U}$ is the unit normal vector of M restricted to α and $\mathbf{Y}$ is the unit vector given by $\mathbf{Y} = \mathbf{U} \times \mathbf{T}$. The derivative formulas of Darboux frame are given by:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{Y}' \\ \mathbf{U}' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{Y} \\ \mathbf{U} \end{pmatrix}.$$

The functions τ_g, k_g and k_n are called geodesic torsion, geodesic curvature and normal curvature of the curve α, respectively [1, 2]. The following relationships based on these functions are well known [1]:

(1) $\alpha = \alpha(s)$ is an asymptotic curve if and only if $k_n = 0$,
(2) $\alpha = \alpha(s)$ is a geodesic curve if and only if $k_g = 0$.

Another thing that can be of importance is the angular momentum vector of the aforementioned particle about the origin. This vector has an important place in particle kinematics. It is calculated as the vector product of the position vector and linear momentum vector of the particle. It always lies on the instantaneous plane $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$. Let us assume that this vector never equals zero during the motion of the aforementioned particle. This assumption ensures that the functions $\langle \alpha(s), \mathbf{Y}(s) \rangle$ and $\langle \alpha(s), \mathbf{U}(s) \rangle$ are not zero at the same time along the trajectory $\alpha = \alpha(s)$. That is, we can say that the tangent line of $\alpha = \alpha(s)$ at any point does not pass through the origin. In this case, there exists PAFORS denoted by $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ along the trajectory $\alpha = \alpha(s)$. As mentioned earlier, this frame has been recently introduced in the study [8]. PAFORS contains a lot of information about the position vector of the moving particle. Also, it enables us to study together the kinematics of the moving particle on surface, the differential geometry of the trajectory and the differential geometry of the surface. So, it is expected that PAFORS will enable more convenient observation environment for the researchers studying on modern robotics in the future. In the system $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$, $\mathbf{T}(s)$ is the unit tangent vector of the trajectory and it is the common base vector of this frame with the Darboux frame. Consider the vector whose starting point is the foot of the perpendicular (from O to instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$) and the endpoint is the foot of the perpendicular (from O to instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$). The equivalent of this vector at the point $\alpha(s)$ determines the third base vector $\mathbf{H}(s)$ of PAFORS. Therefore, $\mathbf{H}(s)$ is calculated as follows (see [8] for more details):

$$\mathbf{H}(s) = \frac{\langle -\alpha(s), \mathbf{Y}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{Y}(s) + \frac{\langle \alpha(s), \mathbf{U}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{U}(s).$$

On the other hand, the second base vector $\mathbf{G}(s)$ is obtained by vector product $\mathbf{H}(s) \wedge \mathbf{T}(s)$ as in the following:

$$\mathbf{G}(s) = \frac{\langle \alpha(s), \mathbf{U}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}}\mathbf{Y}(s) + \frac{\langle \alpha(s), \mathbf{Y}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}}\mathbf{U}(s).$$

Since tangent vector $\mathbf{T}(s)$ is mutual in both PAFORS and Darboux frame, the vectors $\mathbf{Y}(s)$, $\mathbf{U}(s)$, $\mathbf{G}(s)$ and $\mathbf{H}(s)$ lie on the same plane. Thus, there is a relation between PAFORS and Darboux frame as follows:

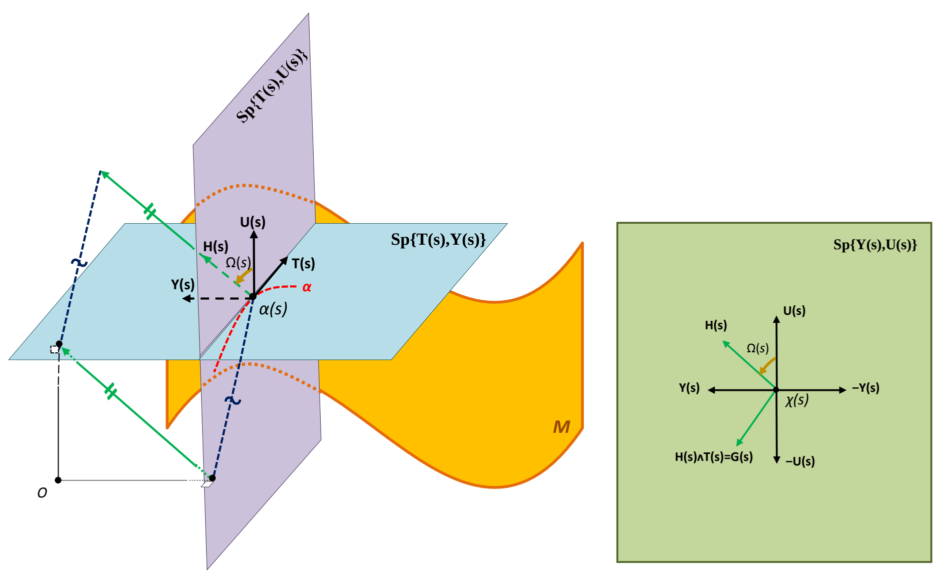
$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{H}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega(s) & -\sin\Omega(s) \\ 0 & \sin\Omega(s) & \cos\Omega(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{U}(s)$ and $\mathbf{H}(s)$ (or likewise $\mathbf{Y}(s)$ and $\mathbf{G}(s)$) which is positively oriented from $\mathbf{U}(s)$ to $\mathbf{H}(s)$ (or likewise from $\mathbf{Y}(s)$ to $\mathbf{G}(s)$)(see Figure 1). Additionally, the derivative formulas of PAFORS are given by

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{G}'(s) \\ \mathbf{H}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{H}(s) \end{pmatrix}$$

where

$$\begin{aligned} k_1(s) &= k_g(s)\cos\Omega(s) - k_n(s)\sin\Omega(s) \\ k_2(s) &= k_g(s)\sin\Omega(s) + k_n(s)\cos\Omega(s) \\ k_3(s) &= \tau_g(s) - \Omega'(s). \end{aligned} \tag{1.1}$$



(A) (B)

Figure 1. An illustration for PAFORS

Since

$$\begin{aligned} \sin\Omega(s) &= \frac{-\langle\alpha(s),\, \mathbf{Y}(s)\rangle}{\sqrt{\langle\alpha(s),\, \mathbf{Y}(s)\rangle^2 + \langle\alpha(s),\, \mathbf{U}(s)\rangle^2}} \\ \cos\Omega(s) &= \frac{\langle\alpha(s),\, \mathbf{U}(s)\rangle}{\sqrt{\langle\alpha(s),\, \mathbf{Y}(s)\rangle^2 + \langle\alpha(s),\, \mathbf{U}(s)\rangle^2}} \qquad (1.2) \\ \tan\Omega(s) &= -\frac{\langle\alpha(s),\, \mathbf{Y}(s)\rangle}{\langle\alpha(s),\, \mathbf{U}(s)\rangle}, \end{aligned}$$

the rotation angle $\Omega(s)$ is determined as follows:

$$\Omega(s) = \begin{cases} \arctan\left(-\frac{\langle\alpha(s),\mathbf{Y}(s)\rangle}{\langle\alpha(s),\mathbf{U}(s)\rangle}\right) \;\; if \;\; \langle\alpha(s),\mathbf{U}(s)\rangle > 0 \\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{Y}(s)\rangle}{\langle\alpha(s),\mathbf{U}(s)\rangle}\right) + \pi \;\; if \;\; \langle\alpha(s),\mathbf{U}(s)\rangle < 0 \\ -\frac{\pi}{2} \;\; if \;\; \langle\alpha(s),\mathbf{U}(s)\rangle = 0\;, \;\; \langle\alpha(s),\mathbf{Y}(s)\rangle > 0 \\ \frac{\pi}{2} \;\; if \;\; \langle\alpha(s),\mathbf{U}(s)\rangle = 0\;, \;\; \langle\alpha(s),\mathbf{Y}(s)\rangle < 0. \end{cases}$$

Any element of the set $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s), k_1(s), k_2(s), k_3(s)\}$ is called as PAFORS apparatus of the curve $\alpha = \alpha\,(s)$ [8].

This paper is organized as follows. In Section 2, we consider the geodesic and asymptotic trajectories according to PAFORS in three-dimensional Euclidean space and give some corollaries for the special cases of these trajectories. In Section 3, we study the slant helical trajectories according to PAFORS and give a method to investigate the existing or not existing of the desired slant helical trajectory on a given implicit surface.

2. Some Characterizations on Geodesic and Asymptotic Trajectories

In the remaining sections, we continue to consider any moving point particle on a regular surface M satisfying the aforesaid assumption and to denote the unit speed parameterization of the trajectory by $\alpha = \alpha(s)$. Also, we will show the parameter interval of the trajectory $\alpha = \alpha(s)$ with I.

Lemma 2.1. *$\alpha = \alpha\,(s)$ is an asymptotic curve if and only if ${k_1}^2 + {k_2}^2 = {k_g}^2$.*

Proof. From the first and second equations in (1.1),

$${k_1}^2 + {k_2}^2 = {k_g}^2 + {k_n}^2$$

can be easily written. Due to the this equality, the remaining part of the proof is obvious. □

Considering above, we give the following lemma without proof.

Lemma 2.2. *$\alpha = \alpha\,(s)$ is a geodesic curve if and only if ${k_1}^2 + {k_2}^2 = {k_n}^2$.*

In the light of the equations (1.1) and (1.2), the following two remarks can be easily given.

Remark 1. *Let $\alpha = \alpha(s)$ be an asymptotic curve on M with $k_g \neq 0$ (The reason why we take k_g as nonzero is to avoid $k_1 = k_2 = 0$). In that case*

$$k_1(s) = k_g(s)\frac{\langle \alpha(s), \mathbf{U}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}} \tag{2.1}$$

$$k_2(s) = -k_g(s)\frac{\langle \alpha(s), \mathbf{Y}(s)\rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s)\rangle^2 + \langle \alpha(s), \mathbf{U}(s)\rangle^2}} \tag{2.2}$$

can be immediately written. By keeping $k_g \neq 0$ and the assumption concerned with the angular momentum in mind, it is very easy to say that $k_1(s)$ and $k_2(s)$ do not equal to zero simultaneously and they verify the followings

$$k_1(s) = 0 \quad \Leftrightarrow \quad \langle \alpha(s), \mathbf{U}(s)\rangle = 0$$
$$k_2(s) = 0 \quad \Leftrightarrow \quad \langle \alpha(s), \mathbf{Y}(s)\rangle = 0.$$

As a result of the proposition $k_1(s) = 0 \Leftrightarrow \langle \alpha(s), \mathbf{U}(s)\rangle = 0$, we can write the equality $k_1(s)\langle \alpha(s), \mathbf{Y}(s)\rangle + k_2(s)\langle \alpha(s), \mathbf{U}(s)\rangle = 0$ when $k_1(s) = 0$. Considering above and using the equations (2.1), (2.2) we have

$$\frac{k_2(s)}{k_1(s)} = -\frac{\langle \alpha(s),\, \mathbf{Y}(s)\rangle}{\langle \alpha(s),\, \mathbf{U}(s)\rangle}$$

where $k_1(s) \neq 0$. So, we can conclude that

$$k_1(s)\langle \alpha(s),\, \mathbf{Y}(s)\rangle + k_2(s)\langle \alpha(s),\, \mathbf{U}(s)\rangle = 0 \tag{2.3}$$

is satisfied along the trajectory $\alpha = \alpha(s)$.

Remark 2. *Let $\alpha = \alpha(s)$ be a geodesic curve on M with $k_n \neq 0$ (The reason why we take k_n as nonzero is to avoid $k_1 = k_2 = 0$). In that case*

$$k_1(s) = k_n(s)\frac{\langle \alpha(s),\, \mathbf{Y}(s)\rangle}{\sqrt{\langle \alpha(s),\, \mathbf{Y}(s)\rangle^2 + \langle \alpha(s),\, \mathbf{U}(s)\rangle^2}} \tag{2.4}$$

$$k_2(s) = k_n(s)\frac{\langle \alpha(s),\, \mathbf{U}(s)\rangle}{\sqrt{\langle \alpha(s),\, \mathbf{Y}(s)\rangle^2 + \langle \alpha(s),\, \mathbf{U}(s)\rangle^2}} \tag{2.5}$$

can be easily written. Similarly, we can easily say that $k_1(s)$ and $k_2(s)$ do not equal to zero simultaneously and they verify the followings

$$k_1(s) = 0 \quad \Leftrightarrow \quad \langle \alpha(s), \mathbf{Y}(s)\rangle = 0$$
$$k_2(s) = 0 \quad \Leftrightarrow \quad \langle \alpha(s), \mathbf{U}(s)\rangle = 0$$

by keeping the assumption concerned with the angular momentum and $k_n \neq 0$ in mind. As a result of the proposition $k_2(s) = 0 \Leftrightarrow \langle \alpha(s), \mathbf{U}(s)\rangle = 0$, we can write $k_2(s)\langle \alpha(s), \mathbf{Y}(s)\rangle - k_1(s)\langle \alpha(s), \mathbf{U}(s)\rangle = 0$ when $k_2(s) = 0$. Taking into consideration above and using the equations (2.4), (2.5) we have

$$\frac{k_1(s)}{k_2(s)} = \frac{\langle \alpha(s),\, \mathbf{Y}(s)\rangle}{\langle \alpha(s),\, \mathbf{U}(s)\rangle}$$

where $k_2(s) \neq 0$. Therefore, we can conclude that

$$k_2(s)\langle \alpha(s),\, \mathbf{Y}(s)\rangle - k_1(s)\langle \alpha(s),\, \mathbf{U}(s)\rangle = 0 \tag{2.6}$$

is satisfied along the trajectory $\alpha = \alpha(s)$.

Theorem 2.3. *Let $\alpha = \alpha(s)$ be an asymptotic curve on M with $k_g \neq 0$. In that case, $\alpha = \alpha(s)$ is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ iff $k_2 = 0$.*

Proof. Assume that the asymptotic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. Then, $\langle \alpha(s), \mathbf{Y}(s)\rangle = 0$ for all the values s of parameter. Considering the equation (2.3), we obtain

$$k_2(s)\langle \alpha(s), \mathbf{U}(s)\rangle = 0 \tag{2.7}$$

for all s. Because of the non-vanishing angular momentum, $\langle \alpha(s), \mathbf{U}(s)\rangle$ never vanishes along α. The first part of the proof is completed by using this information in (2.7).

Conversely, suppose that $k_2 = 0$. From the equation (2.3), we have

$$\forall s \in I, \quad k_1(s)\langle \alpha(s),\, \mathbf{Y}(s)\rangle = 0.$$

Due to Remark 1, we know that $k_1(s)$ and $k_2(s)$ can not equal to zero simultaneously along α. Thus, we can conclude that $k_1(s)$ never vanishes. This gives us the following

$$\forall s \in I, \quad \langle \alpha(s), \mathbf{Y}(s)\rangle = 0$$

which means that the asymptotic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. □

Theorem 2.4. *Let $\alpha = \alpha(s)$ be an asymptotic curve on M with $k_g \neq 0$. In that case, $\alpha = \alpha(s)$ is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ iff $k_1 = 0$.*

Proof. Assume that the asymptotic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. In this case, $\langle \alpha(s), \mathbf{U}(s)\rangle = 0$ for all the values s of the arc-length parameter. Taking into account of the equation (2.3), we have

$$k_1(s)\langle \alpha(s),\, \mathbf{Y}(s)\rangle = 0$$

for all s. Similarly above, $\langle \alpha(s), \mathbf{Y}(s)\rangle$ never vanishes along α thanks to the non-vanishing angular momentum. Then

$$\forall s \in I, \quad k_1(s) = 0$$

can be concluded and this finishes the first part of the proof.

Conversely, assume that $k_1 = 0$. In that case, from the equation (2.3)

$$\forall s \in I, \quad k_2(s)\langle \alpha(s),\, \mathbf{U}(s)\rangle = 0$$

can be written. Because $k_1(s)$ and $k_2(s)$ do not equal to zero simultaneously along $\alpha = \alpha(s)$, we can say

$$\forall s \in I, \quad k_2(s) \neq 0.$$

This yields the following

$$\forall s \in I, \quad \langle \alpha(s), \mathbf{U}(s)\rangle = 0$$

which means that the asymptotic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. □

Theorem 2.5. *Let $\alpha = \alpha(s)$ be an asymptotic curve on M with $k_g \neq 0$. Then the following properties hold:*

(1) *If the position vector of* $\alpha = \alpha(s)$ *always lies on the instantaneous plane* $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$, *differential equation* $k_1k_2' - k_2k_1' + (\Omega' + k_3)(k_1^2 + k_2^2) = 0$ *is satisfied along* α.

(2) *If the differential equation* $k_1k_2' - k_2k_1' + (\Omega' + k_3)(k_1^2 + k_2^2) = 0$ *is satisfied along* α *with* $k_1, k_2 \neq 0$, *then the position vector of* $\alpha = \alpha(s)$ *always lies on the instantaneous plane* $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$.

Proof. One can easily find the equation

$$\begin{aligned} 0 = & -\left(k_1^2(s) + k_2^2(s)\right)\cos\Omega(s)\,\langle\alpha(s),\,\mathbf{T}(s)\rangle \\ & + \left(-k_2(s)(k_3(s) + \Omega'(s)) + k_1'(s)\right)\langle\alpha(s),\,\mathbf{Y}(s)\rangle \\ & + \left(k_1(s)(k_3(s) + \Omega'(s)) + k_2'(s)\right)\langle\alpha(s),\,\mathbf{U}(s)\rangle \end{aligned} \tag{2.8}$$

by differentiating (2.3) and considering the relations between the PAFORS apparatus and Darboux apparatus. If the necessary operations are applied to the equations (2.3) and (2.8) side by side,

$$\begin{aligned} 0 = & \; k_2(s)\left(k_1^2(s) + k_2^2(s)\right)\cos\Omega(s)\,\langle\alpha(s), \mathbf{T}(s)\rangle \\ & + \left(k_1(s)k_2'(s) - k_2(s)k_1'(s) + (\Omega'(s) + k_3(s))(k_1^2(s) + k_2^2(s))\right)\langle\alpha(s), \mathbf{Y}(s)\rangle \end{aligned} \tag{2.9}$$

can be obtained. Now we can investigate the items:

(1) Assume that the position vector of $\alpha = \alpha(s)$ always lies on the instantaneous plane $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$. Then,

$$\langle\alpha(s),\,\mathbf{T}(s)\rangle = 0 \tag{2.10}$$

for all the values s. Taking into consideration (2.9), we get

$$\left(k_1(s)k_2'(s) - k_2(s)k_1'(s) + (\Omega'(s) + k_3(s))(k_1^2(s) + k_2^2(s))\right)\langle\alpha(s), \mathbf{Y}(s)\rangle = 0 \tag{2.11}$$

for all s. Also, differentiating the equation (2.10) gives us the following:

$$k_g(s)\,\langle\alpha(s),\,\mathbf{Y}(s)\rangle + k_n(s)\,\langle\alpha(s),\,\mathbf{U}(s)\rangle = -1.$$

Since $k_n = 0$ for the asymptotic curve $\alpha = \alpha(s)$,

$$\forall s \in I, \quad \langle\alpha(s),\,\mathbf{Y}(s)\rangle = -\frac{1}{k_g(s)} \neq 0 \tag{2.12}$$

can be written. Therefore,

$$k_1k_2' - k_2k_1' + (\Omega' + k_3)(k_1^2 + k_2^2) = 0$$

is obtained from (2.11) and (2.12).

(2) Let the differential equation $k_1k_2' - k_2k_1' + (\Omega' + k_3)(k_1^2 + k_2^2) = 0$ be satisfied along α with $k_1, k_2 \neq 0$. In the light of (2.9), we can write

$$\forall s \in I, \quad k_2(s)\left(k_1^2(s) + k_2^2(s)\right)\cos\Omega(s)\,\langle\alpha(s),\,\mathbf{T}(s)\rangle = 0.$$

Because $k_1, k_2 \neq 0$, we get

$$\forall s \in I, \quad \langle\alpha(s),\,\mathbf{T}(s)\rangle = 0$$

which means that the position vector of the asymptotic curve α always lies on the instantaneous plane $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$ (Notice that $\cos\Omega(s) = \frac{k_1(s)}{k_g(s)}$ under these conditions). □

Theorem 2.6. *Let $\alpha = \alpha(s)$ be a geodesic curve on M with $k_n \neq 0$. In that case, $\alpha = \alpha(s)$ is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ iff $k_1 = 0$.*

Proof. Assume that the geodesic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. Then, $\langle \alpha(s), \mathbf{Y}(s)\rangle = 0$ for all the values s. Taking into account of (2.6), we get

$$-k_1(s)\langle \alpha(s), \mathbf{U}(s)\rangle = 0 \tag{2.13}$$

for all s. Because of the non-vanishing angular momentum, $\langle \alpha(s), \mathbf{U}(s)\rangle$ never vanishes along α. Substituting this into (2.13) completes the first part of the proof.

Conversely, assume that $k_1 = 0$. From (2.6), we obtain

$$\forall s \in I, \quad k_2(s)\langle \alpha(s), \mathbf{Y}(s)\rangle = 0.$$

Due to Remark 2, we know that $k_1(s)$ and $k_2(s)$ can not equal to zero simultaneously along α. Thus, we can conclude that $k_2(s)$ never vanishes. This yields

$$\forall s \in I, \quad \langle \alpha(s), \mathbf{Y}(s)\rangle = 0$$

which means that the geodesic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. □

Theorem 2.7. *Let $\alpha = \alpha(s)$ be a geodesic curve on M with $k_n \neq 0$. In that case, $\alpha = \alpha(s)$ is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ iff $k_2 = 0$.*

Proof. Suppose that the geodesic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. Then, $\langle \alpha(s), \mathbf{U}(s)\rangle = 0$ for all the values s. Considering (2.6), we find

$$k_2(s)\langle \alpha(s), \mathbf{Y}(s)\rangle = 0$$

for all s. Similarly previous proof, we can say that $\langle \alpha(s), \mathbf{Y}(s)\rangle$ never vanishes along α in the light of non-vanishing angular momentum. Then

$$\forall s \in I, \quad k_2(s) = 0$$

can be concluded.

Conversely, assume that $k_2 = 0$. In that case, from (2.6)

$$\forall s \in I, \quad -k_1(s)\langle \alpha(s), \mathbf{U}(s)\rangle = 0$$

can be written. Because $k_1(s)$ and $k_2(s)$ do not equal to zero simultaneously along α, we find

$$\forall s \in I, \quad k_1(s) \neq 0.$$

This yields the following

$$\forall s \in I, \quad \langle \alpha(s), \mathbf{U}(s)\rangle = 0$$

which means that the geodesic curve α is a curve whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. □

Theorem 2.8. *Let $\alpha = \alpha(s)$ be a geodesic curve on M with $k_n \neq 0$. Then the following properties hold:*

(1) *If the position vector of $\alpha = \alpha(s)$ always lies on the instantaneous plane $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$, differential equation $k_2{k_1}' - k_1{k_2}' - (k_3 + \Omega')({k_1}^2 + {k_2}^2) = 0$ is satisfied along α.*

(2) *If the differential equation* $k_2{k_1}'-k_1{k_2}'-(k_3+\Omega')({k_1}^2+{k_2}^2)=0$ *is satisfied along* α *with* $k_1,k_2\neq 0$*, then the position vector of* $\alpha=\alpha(s)$ *always lies on the instantaneous plane* $Sp\{\mathbf{Y}(s),\mathbf{U}(s)\}$.

Proof. One can easily find the equation

$$\begin{aligned} 0 \;=\; & -\left({k_1}^2(s)+{k_2}^2(s)\right)\sin\Omega(s)\,\langle\alpha(s),\,\mathbf{T}(s)\rangle \\ & +\left(k_1(s)(k_3(s)+\Omega'(s))+{k_2}'(s)\right)\langle\alpha(s),\,\mathbf{Y}(s)\rangle \\ & +\left(k_2(s)(k_3(s)+\Omega'(s))-{k_1}'(s)\right)\langle\alpha(s),\,\mathbf{U}(s)\rangle \end{aligned} \tag{2.14}$$

by differentiating (2.6) and taking into consideration the relations between the PAFORS apparatus and Darboux apparatus. If the necessary operations are applied to the equations (2.6) and (2.14) side by side, the equation

$$\begin{aligned} 0 \;=\; & k_2(s)\left({k_1}^2(s)+{k_2}^2(s)\right)\sin\Omega(s)\,\langle\alpha(s),\mathbf{T}(s)\rangle \\ & +\left(k_2(s){k_1}'(s)-k_1(s){k_2}'(s)-(k_3(s)+\Omega'(s))({k_1}^2(s)+{k_2}^2(s))\right)\langle\alpha(s),\mathbf{U}(s)\rangle \end{aligned} \tag{2.15}$$

can be obtained. Now we can investigate the items:

(1) Assume that the position vector of $\alpha=\alpha(s)$ always lies on the instantaneous plane $Sp\{\mathbf{Y}(s),\mathbf{U}(s)\}$. Then,

$$\langle\alpha(s),\,\mathbf{T}(s)\rangle=0 \tag{2.16}$$

for all the values s. Considering the equation (2.15), we get

$$\left(k_2(s){k_1}'(s)-k_1(s){k_2}'(s)-(k_3(s)+\Omega'(s))({k_1}^2(s)+{k_2}^2(s))\right)\langle\alpha(s),\,\mathbf{U}(s)\rangle=0 \tag{2.17}$$

for all s. Also, differentiating (2.16) yields the equation

$$k_g(s)\,\langle\alpha(s),\,\mathbf{Y}(s)\rangle+k_n(s)\,\langle\alpha(s),\,\mathbf{U}(s)\rangle=-1.$$

Because $k_g=0$ for the geodesic curve $\alpha=\alpha(s)$,

$$\forall s\in I,\quad \langle\alpha(s),\,\mathbf{U}(s)\rangle=-\frac{1}{k_n(s)}\neq 0 \tag{2.18}$$

can be written. Thus,

$$k_2{k_1}'-k_1{k_2}'-(k_3+\Omega')({k_1}^2+{k_2}^2)=0$$

is obtained from (2.17) and (2.18).

(2) Let the differential equation $k_2{k_1}'-k_1{k_2}'-(k_3+\Omega')({k_1}^2+{k_2}^2)=0$ be satisfied along α with $k_1,k_2\neq 0$. In the light of (2.15), we can write

$$\forall s\in I,\quad k_2(s)\left({k_1}^2(s)+{k_2}^2(s)\right)\sin\Omega(s)\,\langle\alpha(s),\,\mathbf{T}(s)\rangle=0.$$

Because $k_1,k_2\neq 0$, we find

$$\forall s\in I,\quad \langle\alpha(s),\,\mathbf{T}(s)\rangle=0$$

which means that the position vector of the geodesic curve α always lies on the instantaneous plane $Sp\{\mathbf{Y}(s),\mathbf{U}(s)\}$ (Notice that $\sin\Omega(s)=-\frac{k_1(s)}{k_n(s)}$ under these conditions). □

3. Slant Helical Trajectories According to PAFORS

Slant helix was expressed in [10] as a curve whose principal normal vector makes a constant angle with a fixed direction in E^3. Until now, several kinds of slant helices have been defined and studied in the literature. There can be found some of them in [11–14]. In this section, we take into consideration the slant helical trajectories according to PAFORS and discuss some special cases of them. Also, we give a method to investigate the existing or not existing of the desired slant helical trajectory on a given implicit surface. Note that similar steps and approaches in [15] and [16] will be followed in this section.

As mentioned earlier, we continue to consider any moving point particle on a regular surface M satisfying the aforesaid assumption and to denote the unit speed parameterization of the trajectory by $\alpha = \alpha(s)$.

Firstly, we define $\mathbf{G}-$PAFORS spherical image of the trajectory. We consider this spherical image since it has an important place for the characterization of our slant helical trajectories. The remaining PAFORS spherical images can be topic of a different study.

Definition 3.1. *If we move the vector field $\mathbf{G}$ of $\alpha = \alpha(s)$ to the center O of the unit sphere S^2, we find a curve which $\mathbf{G}(s)$ draws on S^2. We call this curve $\mathbf{G}-$PAFORS spherical image of $\alpha = \alpha(s)$ and show it with ξ_G.*

For $\mathbf{G}-$PAFORS spherical image of $\alpha = \alpha(s)$, we can write

$$\xi_G(s) = \mathbf{G}(s).$$

If this equation is differentiated with respect to s, we get

$$\begin{aligned}
\xi'_G(s) &= -k_1(s)\mathbf{T}(s) + k_3(s)\mathbf{H}(s) \\
\xi''_G(s) &= \left[-k_1'(s) - k_2(s)k_3(s)\right]\mathbf{T}(s) - \left[k_1^2(s) + k_3^2(s)\right]\mathbf{G}(s) \\
&\quad + \left[-k_1(s)k_2(s) + k_3'(s)\right]\mathbf{H}(s) \\
\xi'''_G(s) &= \left[-k_1''(s) - k_2'(s)k_3(s) - 2k_3'(s)k_2(s) + k_1(s)\left(k_1^2(s) + k_2^2(s) + k_3^2(s)\right)\right]\mathbf{T}(s) \\
&\quad -3\left[k_1(s)k_1'(s) + k_3(s)k_3'(s)\right]\mathbf{G}(s) \\
&\quad + \left[k_3''(s) - k_2'(s)k_1(s) - 2k_1'(s)k_2(s) - k_3(s)\left(k_1^2(s) + k_2^2(s) + k_3^2(s)\right)\right]\mathbf{H}(s).
\end{aligned}$$

These equations yield the curvature κ_G and the torsion τ_G of ξ_G as in the following:

$$\kappa_G(s) = \frac{\|\xi'_G(s) \wedge \xi''_G(s)\|}{\|\xi'_G(s)\|^3} = \sqrt{1 + (\zeta_G(s))^2} \tag{3.1}$$

$$\tau_G(s) = \frac{\langle \xi'_G(s) \wedge \xi''_G(s),\ \xi'''_G(s)\rangle}{\|\xi'_G(s) \wedge \xi''_G(s)\|^2} = \frac{\zeta'_G(s)}{\left(1 + (\zeta_G(s))^2\right)\left(k_1^2(s) + k_3^2(s)\right)^{11/2}} \tag{3.2}$$

where

$$\zeta_G(s) = \left(\frac{k_3'k_1 - k_1'k_3 - k_2(k_1^2 + k_3^2)}{\left(k_1^2 + k_3^2\right)^{3/2}}\right)(s). \tag{3.3}$$

Definition 3.2. $\alpha = \alpha(s)$ *is called a slant helical trajectory (according to PAFORS) if the vector field $\mathbf{G}$ of $\alpha = \alpha(s)$ makes a constant angle with a fixed direction.*

If $\alpha = \alpha(s)$ is a slant helical trajectory according to PAFORS, then there exist a constant angle β and a fixed unit vector $\mathbf{g}$ which satisfy

$$\langle \mathbf{G}(s), \mathbf{g}\rangle = \cos\beta$$

for all s.

Theorem 3.1. *Assume that* $(k_1(s), k_3(s)) \neq (0, 0)$. *Then,* α *is a slant helical trajectory according to PAFORS iff the function in* (3.3) *is a constant function.*

Proof. Let $\alpha = \alpha(s)$ with $(k_1(s), k_3(s)) \neq (0, 0)$ be slant helical trajectory according to PAFORS in E^3. In this case, from Definition 3.2, $\mathbf{G}$ makes a constant angle with a fixed direction. Therefore, $\mathbf{G}-$PAFORS spherical image ξ_G of $\alpha = \alpha(s)$ is part of a circle. Thus, it has constant curvature and zero torsion. Using this information in (3.1) and (3.2), we can immediately conclude $\zeta_G(s) = constant$.

Conversely, suppose that $\zeta_G(s) = constant$. In this case, it is not difficult to see that $\kappa_G(s) = constant$ and $\tau_G = 0$. Therefore, $\mathbf{G}-$PAFORS spherical image ξ_G of $\alpha = \alpha(s)$ is part of a circle. So, $\mathbf{G}$ makes a constant angle with a fixed direction and we can finish the proof. □

Corollary 3.2. *Let* $\alpha = \alpha(s)$ *be an asymptotic curve on* M *with* $k_g, k_1 \neq 0$. *Assume that its position vector always lies on the instantaneous plane* $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. *Then,* α *is a slant helical trajectory according to PAFORS iff*

$$\left(\frac{{k_1}^2}{({k_1}^2 + {k_3}^2)^{3/2}}\left(\frac{k_3}{k_1}\right)'\right)(s)$$

is a constant function.

Proof. Where $k_g, k_1 \neq 0$, let the asymptotic curve $\alpha = \alpha(s)$, whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$, be a slant helical trajectory according to PAFORS. From Theorem 2.3, we can write $k_2 = 0$. If we use this in (3.3), we get

$$\zeta_G(s) = \left(\frac{{k_1}^2}{({k_1}^2 + {k_3}^2)^{3/2}}\left(\frac{k_3}{k_1}\right)'\right)(s).$$

In that case, Theorem 3.1 finishes the first part of the proof. Similarly, one can easily complete the other part of the proof. □

Corollary 3.3. *Let* $\alpha = \alpha(s)$ *be an asymptotic curve on* M *with non-zero* k_g *and non-zero* k_3. *Assume that its position vector always lies on the instantaneous plane* $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. *Then,* α *is a slant helical trajectory according to PAFORS iff*

$$\left(\frac{k_2}{k_3}\right)(s)$$

is a constant function.

Proof. Where $k_g, k_3 \neq 0$, let the asymptotic curve $\alpha = \alpha(s)$, whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$, be a slant helical trajectory according to PAFORS. From Theorem 2.4, we can write $k_1 = 0$. If we use this in (3.3), we find

$$\zeta_G(s) = \left(-\frac{k_2}{k_3}\right)(s).$$

In this case, Theorem 3.1 finishes the first part of the proof. Similarly, the other part of the proof is easily completed. So, we omit it. □

Corollary 3.4. *Let $\alpha = \alpha(s)$ be a geodesic curve on M with non-zero k_n and non-zero k_1. Assume that its position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. Then, α is a slant helical trajectory according to PAFORS iff*

$$\left(\frac{{k_1}^2}{({k_1}^2 + {k_3}^2)^{3/2}}\left(\frac{k_3}{k_1}\right)'\right)(s)$$

is a constant function.

Proof. Where $k_n, k_1 \neq 0$, let the geodesic curve $\alpha = \alpha(s)$, whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$, be a slant helical trajectory according to PAFORS. From Theorem 2.7, we can write $k_2 = 0$. If $k_2 = 0$ is substituted in (3.3), we find

$$\zeta_G(s) = \left(\frac{{k_1}^2}{({k_1}^2 + {k_3}^2)^{3/2}}\left(\frac{k_3}{k_1}\right)'\right)(s).$$

Then, the first part of the proof is easily finished considering Theorem 3.1. Similarly, one can easily complete the other part of the proof. □

Corollary 3.5. *Let $\alpha = \alpha(s)$ be a geodesic curve on M with non-zero k_n and non-zero k_3. Assume that its position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. Then, α is a slant helical trajectory according to PAFORS iff*

$$\left(\frac{k_2}{k_3}\right)(s)$$

is a constant function.

Proof. Where $k_n, k_3 \neq 0$, let the geodesic curve $\alpha = \alpha(s)$, whose position vector always lies on the instantaneous plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$, be a slant helical trajectory according to PAFORS. From Theorem 2.6, we can write $k_1 = 0$. If $k_1 = 0$ is substituted in (3.3), we obtain

$$\zeta_G(s) = \left(-\frac{k_2}{k_3}\right)(s).$$

In that case, Theorem 3.1 finishes the first part of the proof. Similarly, the other part of the proof is immediately completed. So, we omit it. □

3.1. **Determination of the helix axis for slant helical trajectories.**

In this subsection, we will discuss on the determination of the helix axis for the slant helical trajectories. Let $\alpha = \alpha(s)$ be a slant helical trajectory according to PAFORS. In that case, there exist a constant angle β and a fixed unit vector $\mathbf{g}$ which satisfy $\langle \mathbf{G}, \mathbf{g}\rangle = \cos\beta = \lambda_2$ where $\mathbf{g} = \lambda_1\mathbf{T} + \lambda_2\mathbf{G} + \lambda_3\mathbf{H}$. With the aid of differentiation with respect to s, we obtain

$$\langle -k_1\mathbf{T} + k_3\mathbf{H},\, \mathbf{g}\rangle = 0. \tag{3.4}$$

Now, let us differentiate the vector $\mathbf{g}$. In that case,

$$({\lambda_1}' - \lambda_2 k_1 - \lambda_3 k_2)\mathbf{T} + (\lambda_1 k_1 - \lambda_3 k_3)\mathbf{G} + ({\lambda_3}' + \lambda_1 k_2 + \lambda_2 k_3)\mathbf{H} = 0$$

is found. This gives us the equation system

$$\begin{aligned} {\lambda_1}' - \lambda_2 k_1 - \lambda_3 k_2 &= 0 \\ \lambda_1 k_1 - \lambda_3 k_3 &= 0 \\ {\lambda_3}' + \lambda_1 k_2 + \lambda_2 k_3 &= 0. \end{aligned} \tag{3.5}$$

Let us solve this system. Firstly, we must emphasize that we will follow similar steps given in [15] and [16] to find the solution of this system. If the equality $\lambda_1 = \frac{k_3}{k_1}\lambda_3\,(k_1(s) \neq 0)$ is written in the equations $(3.5)_1$, $(3.5)_3$ and some necessary operations are applied to these two new equations, we find the differential equation

$$\left(1+\left(\frac{k_3}{k_1}\right)^2\right){\lambda_3}' + \frac{k_3}{k_1}\left(\frac{k_3}{k_1}\right)'\lambda_3 = 0.$$

General solution of this equation can be easily obtained as $\lambda_3 = \mu\frac{k_1}{\sqrt{{k_1}^2+{k_3}^2}}$ where μ is the constant of integration. In that case, it is very easy to find $\lambda_1 = \mu\frac{k_3}{\sqrt{{k_1}^2+{k_3}^2}}$ from the relation $\lambda_1 = \frac{k_3}{k_1}\lambda_3$. Because the vector $\mathbf{g} = \mu\frac{k_3}{\sqrt{{k_1}^2+{k_3}^2}}\mathbf{T} + \cos\beta\mathbf{G} + \mu\frac{k_1}{\sqrt{{k_1}^2+{k_3}^2}}\mathbf{H}$ is taken as a unit vector, the integration constant is derived as $\mu = \pm\sin\beta$. Therefore, $\mathbf{g} = \pm\frac{k_3}{\sqrt{{k_1}^2+{k_3}^2}}\sin\beta\mathbf{T}+\cos\beta\mathbf{G}\pm\frac{k_1}{\sqrt{{k_1}^2+{k_3}^2}}\sin\beta\mathbf{H}$ can be immediately written. Finally, the constant angle β must be determined. Differentiating (3.4) with respect to s,

$$\left\langle(-{k_1}' - k_2k_3)\mathbf{T} + (-{k_1}^2 - {k_3}^2)\mathbf{G} + ({k_3}' - k_1k_2)\mathbf{H},\ \mathbf{g}\right\rangle = 0$$

is found. Thus, we get

$$\pm\sin\beta\left(\frac{k_1{k_3}' - k_3{k_1}' - k_2{k_1}^2 - k_2{k_3}^2}{\sqrt{{k_1}^2+{k_3}^2}}\right) - \cos\beta\left({k_1}^2+{k_3}^2\right) = 0.$$

This gives us the following:

$$\tan\beta = \pm\frac{\left({k_1}^2+{k_3}^2\right)^{3/2}}{k_1{k_3}' - k_3{k_1}' - k_2{k_1}^2 - k_2{k_3}^2}.$$

By means of the above information, one can easily find β and determine the fixed direction generated by the constant vector $\mathbf{g}$ for the slant helical trajectory according to PAFORS.

3.2. **Calculating a slant helical trajectory on a given implicit surface.**
In Euclidean 3-space, suppose that the regular surface M is given in implicit form by $f(x_1, x_2, x_3) = 0$. Let us try to generate a slant helical trajectory lying on M. That is, our aim is to give a method which enables us to find the slant helical trajectory $\alpha(s) = (x_1(s),\ x_2(s),\ x_3(s))$ according to PAFORS (if exists) lying on M which accepts a given fixed unit direction $\mathbf{g} = (a, b, c)$ as an axis and a given angle β as the constant angle (Note that s is the arc-length parameter of α). As it is clear that the unit speed curve $\alpha(s) = (x_1(s),\ x_2(s),\ x_3(s))$ on M satisfies

$$\left\langle\nabla f,\ \frac{d\alpha}{ds}\right\rangle = f_{x_1}\frac{dx_1}{ds} + f_{x_2}\frac{dx_2}{ds} + f_{x_3}\frac{dx_3}{ds} = 0 \tag{3.6}$$

and

$$\left\|\frac{d\alpha}{ds}\right\| = \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2 = 1 \tag{3.7}$$

where $\nabla f = \left(\frac{\partial f}{\partial x_1},\ \frac{\partial f}{\partial x_2},\ \frac{\partial f}{\partial x_3}\right) = (f_{x_1},\ f_{x_2},\ f_{x_3})$ and $\frac{d\alpha}{ds} = \left(\frac{dx_1}{ds},\ \frac{dx_2}{ds},\ \frac{dx_3}{ds}\right)$. We want to obtain $x_1(s)$, $x_2(s)$ and $x_3(s)$ to determine $\alpha(s)$.

Let ($\mathbf{T}$, $\mathbf{G}$, $\mathbf{H}$) and ($\mathbf{T}$, $\mathbf{Y}$, $\mathbf{U}$) denote PAFORS and Darboux frame for the trajectory $\alpha(s) = (x_1(s), x_2(s), x_3(s))$, respectively. From the theory of curves and surfaces we know that $\mathbf{U} = \frac{\nabla f}{\|\nabla f\|}$ and $\mathbf{Y} = \frac{\nabla f}{\|\nabla f\|} \wedge \mathbf{T}$ can be written easily. In view of the relation of PAFORS and Darboux frame, we get

$$\begin{aligned}
\mathbf{G} &= \cos\Omega\mathbf{Y} - \sin\Omega\mathbf{U} \\
&= \cos\Omega\left(\frac{\nabla f}{\|\nabla f\|} \wedge \mathbf{T}\right) - \sin\Omega\frac{\nabla f}{\|\nabla f\|} \\
&= \frac{1}{\|\nabla f\|}\cos\Omega\,(\nabla f \wedge \mathbf{T}) - \frac{1}{\|\nabla f\|}\sin\Omega\nabla f \\
&= \frac{1}{\|\nabla f\|}\cos\Omega\left(f_{x_2}\frac{dx_3}{ds} - f_{x_3}\frac{dx_2}{ds},\, f_{x_3}\frac{dx_1}{ds} - f_{x_1}\frac{dx_3}{ds},\, f_{x_1}\frac{dx_2}{ds} - f_{x_2}\frac{dx_1}{ds}\right) \\
&\quad -\frac{1}{\|\nabla f\|}\sin\Omega\,(f_{x_1},\, f_{x_2},\, f_{x_3}) \\
&= \frac{1}{\|\nabla f\|}\begin{pmatrix} f_{x_2}\cos\Omega\frac{dx_3}{ds} - f_{x_3}\cos\Omega\frac{dx_2}{ds} - f_{x_1}\sin\Omega, \\ f_{x_3}\cos\Omega\frac{dx_1}{ds} - f_{x_1}\cos\Omega\frac{dx_3}{ds} - f_{x_2}\sin\Omega, \\ f_{x_1}\cos\Omega\frac{dx_2}{ds} - f_{x_2}\cos\Omega\frac{dx_1}{ds} - f_{x_3}\sin\Omega \end{pmatrix}.
\end{aligned}$$

If this last equation is considered in $\langle \mathbf{G}, \mathbf{g} \rangle = \cos\beta$,

$$\begin{aligned}
\|\nabla f\|\cos\beta &= \left(af_{x_2}\cos\Omega\frac{dx_3}{ds} - af_{x_3}\cos\Omega\frac{dx_2}{ds} - af_{x_1}\sin\Omega\right) \\
&\quad + \left(bf_{x_3}\cos\Omega\frac{dx_1}{ds} - bf_{x_1}\cos\Omega\frac{dx_3}{ds} - bf_{x_2}\sin\Omega\right) \\
&\quad + \left(cf_{x_1}\cos\Omega\frac{dx_2}{ds} - cf_{x_2}\cos\Omega\frac{dx_1}{ds} - cf_{x_3}\sin\Omega\right)
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla f\|\cos\beta + (af_{x_1} + bf_{x_2} + cf_{x_3})\sin\Omega &= (bf_{x_3}\cos\Omega - cf_{x_2}\cos\Omega)\frac{dx_1}{ds} \\
&\quad + (cf_{x_1}\cos\Omega - af_{x_3}\cos\Omega)\frac{dx_2}{ds} \\
&\quad + (af_{x_2}\cos\Omega - bf_{x_1}\cos\Omega)\frac{dx_3}{ds}
\end{aligned}$$

can be written. By applying necessary operations to this last equation and (3.6) side by side, we obtain

$$\frac{dx_1}{ds} = \frac{1}{\lambda}\left[\begin{array}{l} f_{x_2}\left(\|\nabla f\|\cos\beta + (af_{x_1} + bf_{x_2} + cf_{x_3})\sin\Omega\right) \\ +\left(b\cos\Omega f_{x_1}f_{x_2} - a\cos\Omega {f_{x_2}}^2 + c\cos\Omega f_{x_1}f_{x_3} - a\cos\Omega {f_{x_3}}^2\right)\frac{dx_3}{ds} \end{array}\right] \tag{3.8}$$

and

$$\frac{dx_2}{ds} = \frac{1}{\lambda}\left[\begin{array}{l} -f_{x_1}\left(\|\nabla f\|\cos\beta + (af_{x_1} + bf_{x_2} + cf_{x_3})\sin\Omega\right) \\ +\left(a\cos\Omega f_{x_1}f_{x_2} - b\cos\Omega {f_{x_1}}^2 + c\cos\Omega f_{x_2}f_{x_3} - b\cos\Omega {f_{x_3}}^2\right)\frac{dx_3}{ds} \end{array}\right] \tag{3.9}$$

where $\lambda = f_{x_2}(b\cos\Omega f_{x_3} - c\cos\Omega f_{x_2}) + f_{x_1}(a\cos\Omega f_{x_3} - c\cos\Omega f_{x_1}) \neq 0$. Substituting the equations (3.8) and (3.9) into the equation (3.7) yields the quadratic equation with respect to $\frac{dx_3}{ds}$ as

$$\left(B^2+D^2+1\right)\left(\frac{dx_3}{ds}\right)^2+(2AB+2CD)\frac{dx_3}{ds}+\left(A^2+C^2-1\right)=0 \qquad (3.10)$$

where

$$\begin{aligned} A &= \frac{f_{x_2}\left(\|\nabla f\|\cos\beta+(af_{x_1}+bf_{x_2}+cf_{x_3})\sin\Omega\right)}{\lambda} \\ B &= \frac{b\cos\Omega f_{x_1}f_{x_2}-a\cos\Omega {f_{x_2}}^2+c\cos\Omega f_{x_1}f_{x_3}-a\cos\Omega {f_{x_3}}^2}{\lambda} \\ C &= \frac{-f_{x_1}\left(\|\nabla f\|\cos\beta+(af_{x_1}+bf_{x_2}+cf_{x_3})\sin\Omega\right)}{\lambda} \\ D &= \frac{a\cos\Omega f_{x_1}f_{x_2}-b\cos\Omega {f_{x_1}}^2+c\cos\Omega f_{x_2}f_{x_3}-b\cos\Omega {f_{x_3}}^2}{\lambda}. \end{aligned}$$

From the equation (3.10), we get

$$\frac{dx_3}{ds}=\frac{(-2AB-2CD)\pm\sqrt{(2AB+2CD)^2-4\left(B^2+D^2+1\right)\left(A^2+C^2-1\right)}}{2\left(B^2+D^2+1\right)}. \qquad (3.11)$$

If the equation (3.11) is substituted into the equations (3.8) and (3.9), an explicit 1st order ordinary differential equation system is found. Consequently, together with the initial point

$$\begin{cases} x_1(0)=x_1^* \\ x_2(0)=x_2^* \\ x_3(0)=x_3^* \end{cases}$$

we get an initial value problem. The solution of this problem yields the desired slant helical trajectory (according to PAFORS) on M.

Remark 3. *Considering the obtained results, we can say followings:*

(1) *If* $(2AB+2CD)^2-4\left(B^2+D^2+1\right)\left(A^2+C^2-1\right)<0$ *at* $(x_1^*,\, x_2^*,\, x_3^*)$, *in that case any slant helical trajectory (according to PAFORS) on* M *with the given fixed direction* $\mathbf{g}$ *and angle* β *does not exist.*

(2) *If* $(2AB+2CD)^2-4\left(B^2+D^2+1\right)\left(A^2+C^2-1\right)=0$ *at* $(x_1^*,\, x_2^*,\, x_3^*)$, *in that case we have only one slant helical trajectory (according to PAFORS) on* M *which passes through the initial point and accepts the given fixed unit direction* $\mathbf{g}=(a,b,c)$ *as an axis and the given angle* β *as the constant angle.*

(3) *If* $(2AB+2CD)^2-4\left(B^2+D^2+1\right)\left(A^2+C^2-1\right)>0$ *at* $(x_1^*,\, x_2^*,\, x_3^*)$, *in that case we have two slant helical trajectories (according to PAFORS) on* M *which pass through the initial point and accept the given fixed unit direction* $\mathbf{g}=(a,b,c)$ *as an axis and the given angle* β *as the constant angle.*

4. Conclusion

For a particle moving on a regular surface of E^3, there is a very close relationship between the kinematics of the particle, the differential geometry of the surface, and the differential geometry of the trajectory. As a result of this relationship, moving frames have been used as very useful tools to investigate the concepts of various studies in differential geometry and particle kinematics. PAFORS (Positional Adapted Frame on Regular Surface) has been recently developed for the trajectories having non-vanishing angular momentum by using the own position vector of the moving particle in [8]. It is expected that PAFORS will be widely preferred to discuss many special topics in particle kinematics and differential geometry. The present paper can be seen as the first step of these future studies.

In this paper, the geodesic, asymptotic, and slant helical trajectories are studied according to PAFORS in three-dimensional Euclidean space and some characterizations are given for them. Furthermore, we explain how we determine the helix axis for slant helical trajectories (according to PAFORS). Finally, we give a method to find the slant helical trajectory (if exists) lying on a given implicit surface which accepts a given fixed unit direction as an axis and a given angle as the constant angle.

In the future study, we plan to give MATLAB examples on the use of MATLAB command ODE45 to investigate the specific applications of the aforementioned method related to slant helical trajectories.

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THE CAUCHY PROBLEM OF A PERIODIC KAWAHARA EQUATION IN ANALYTIC GEVREY SPACES

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Abstract. The Cauchy problem for the Kawahara equation with data in analytic Gevrey spaces on the circle is considered and its local well-posedness in these spaces is proved. Using Bourgain-Gevrey type analytic spaces and appropriate bilinear estimates, it is shown that local in time wellposedness holds when the initial data belong to an analytic Gevrey spaces of order σ. Moreover, the solution is not necessarily G^{σ} in time. However, it belongs to $G^{5\sigma}$ near zero for every x on the circle.

1. Introduction, related results and position problem

The shallow water equations describes the flow below a pressure surface in a fluid. They are PDEs of hyperbolic type (or parabolic if we consider viscous shear). For $x \in \mathbb{T},\ t \in \mathbb{R}$, we denote by $u = u(x,t)$. When we write (1.1), we mean the equation number i from the problem (1.1) subjected with the initial data $u(x,0) = u_0(x)$. We consider

$$\begin{cases} \partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u + \mu\partial_x(u^2) = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

here the parameters $\alpha \neq 0$, β and γ are real numbers and μ is a complex number . To outline our contributions, we will extend the results in [2] and [23] , where the solution was obtained in $X_{s,b}$ to the analytic Gevrey-Bourgain spaces $X_{\sigma,\delta,s,b}$ with also regularity in time.

So, from the mathematical point of view, it is important to study the well-posedness and time regularity for the shallow water equations which happens in the water waves with surface tension, in which the Bond number takes on the critical value (See [3], [5], [6], [8]).
Recently, Y. Jia and Z. Huo [2] considered a Cauchy problems (1.1), the authors obtained local well-posedness for data in $H^s(\mathbb{R})$ with $s > -7/4$ for $\partial_x u^2$.

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Motivated by all the above papers, we investigate the well-posedness of (1.1) in Analytic Gevry spaces to extend results in [23]. The second novelty located in the study of Gevrey's temporal regularity for the unique solution, inspired and motivated by [3] and [5] on the temporal regularity of solutions to KdV-type equations with analytical data of Gevrey.

We begin by presenting some ideas to get the well-posedness, we are working mainly on the integral equivalent formulation of (1.1) as

$$u = S(t)u_0 - \int_0^t S(t-t')\partial_x u^2(t')dt', \tag{1.2}$$

where the unit operator related to the corresponding linear equation is

$$S(t) = \mathcal{F}_x^{-1} e^{-it(\alpha\xi^5 - \beta\xi^3 + \gamma\xi)} \mathcal{F}_x. \tag{1.3}$$

Let us define the phase function as follows

$$\phi(\xi) = \alpha\xi^5 - \beta\xi^3 + \gamma\xi, \tag{1.4}$$

We define the needed spaces beginning by the spaces of analytic Gevrey functions that contain our initial data. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let

$$G^{\sigma,\delta,s}(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}); \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{T})} = \sum_{k\in\mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 < \infty \right\}, \tag{1.5}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

For $\delta = 0$, the space $G^{\sigma,\delta,s}(\mathbb{T})$ coincides with the standard Sobolev space $H^s(\mathbb{T})$.

We then define the analytic Gevrey -Bourgain spaces related to Kawahara equation. The completion of the Schwartz class $S(\mathbb{T}\times\mathbb{R})$ is given by $X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})(resp. Y_{\sigma,\delta,s,b})$, for $s, b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, subjected to the norm

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} \langle \tau + \phi(k) \rangle^{2b} \mid \widehat{u}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}}. \tag{1.6}$$

$$\|u\|_{Y_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \left(\int e^{\delta|k|^{1/\sigma}} \langle k \rangle^{s} \langle \tau + \phi(k) \rangle^{b} \mid \widehat{u}(k,\tau) \mid d\tau \right)^2 \right)^{\frac{1}{2}}. \tag{1.7}$$

In addition, let

$$Z_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b} \cap Y_{\sigma,\delta,s,b-1/2}$$

be the Banach space endowed with the norm

$$\|u\|_{Z_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} + \|u\|_{Y_{\sigma,\delta,s,b-1/2}(\mathbb{T}\times\mathbb{R})}. \tag{1.8}$$

For $\delta = 0$, the space $Z_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b} \cap Y_{\sigma,\delta,s,b-1/2}$ coincides with the standard Bourgain type space $Z_{s,b} = X_{s,b} \cap Y_{s,b-1/2}$.

We organize this paper as follows. In Section 2, our main results regarding the well-posedness (Theorem 2.1) and regularity (Theorem 2.2) in the analytic Gevrey-Bourgain spaces for (1.1) are stated. In Section 3, all Theorems by deriving the bilinear estimates are proved in details.

2. Main results

Theorem 2.1. *Let $s > 0, \sigma \geqslant 1, \delta > 0$ and $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$. Then for some real number $b > \frac{1}{2}$, which is near enough to $\frac{1}{2}$, and a constant $T > 0$, such that (1.1) admits a unique local solution $u \in C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right) \cap Z_{\sigma,\delta,s,\frac{1}{2}}$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is Lipschitz continuous from $G^{\sigma,\delta,s}(\mathbb{T})$ to $C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right)$.*

Our next goal is to study Gevrey's temporal regularity of the unique solution obtained in Theorem 2.1. A periodic function $f(x)$ is the Gevrey class of order σ, if there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbb{T}} \mid \partial_x^l f(x) \mid \leq C^{l+1}(l!)^\sigma \quad l = 0, 1, 2, \ldots.$$

Theorem 2.2. *Let $s > 0$, $\sigma \geqslant 1, \delta > 0$ and $\beta = \gamma = \mu = \alpha = 1$. If $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, then the solution $u \in C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right)$ given by Theorem 2.1 belongs to the Gevrey class $G^{5\sigma}$ in time variable.*

3. Proof of main Theorems

We are going to prepare the prove of our main theorems, let us beginning by the embedded result in the next lemma, which is useful for Theorem 2.1.

Lemma 3.1. *Let $s \in \mathbb{R}$, $\sigma \geqslant 1$ and $\delta > 0$, we have*

$$Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R}) \hookrightarrow C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right).$$

3.1. **Existence of solution.** Taking the Fourier transform with respect to x of the Cauchy problems (1.1), after an ordinary calculation, we get

$$u = S(t)u_0 - \int_0^t S(t-t')\partial_x u^2(t')dt',$$

we localize it t by using a cut-off function, satisfying $\psi \in C_0^\infty(\mathbb{R})$, with $\psi = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\mathrm{supp}\psi \subset [-1, 1]$. We consider the operator Φu given by

$$\Phi(u) = \psi(t)S(t)u_0 - \psi(t)\int_0^t S(t-t')\partial_x u^2(t')dt', \tag{3.1}$$

We now estimate the fist part in the right hand side of (3.1).

Lemma 3.2. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geqslant 1$, for some constant $C > 0$, we have*

$$\|\psi(t)S(t)u_0\|_{Z_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} \leq C\ \|u_0\|_{G^{\sigma,\delta,s}(\mathbb{T})}, \tag{3.2}$$

for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$.

Proof. Define the operator A defined by

$$\widehat{Au}^x(k,t) = e^{\delta|k|^{1/\sigma}}\widehat{u}^x(k,t), \tag{3.3}$$

for $\delta = 0$ can be found in Lemma 2.1 of [23] . These inequalities clearly remain valid for $\delta > 0$, as one merely has to replace u_0 by Au_0 in these results. □

We estimate the second part in right hand side of (3.1).

Lemma 3.3. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geqslant 1$, then for some constant $C > 0$, we have*

$$\|\psi(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C\|F\|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}. \tag{3.4}$$

Proof. Define $U = \psi_T(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'$ and using the operator A.

$$\begin{aligned}\widehat{AU}^x(k,t) &= \psi(t)\int_0^t \left(e^{-i(t-t')\phi(k)}\right) e^{\delta|k|^{1/\sigma}}\widehat{F}^x(k,t')\mathrm{d}t', \\ &= \psi(t)\int_0^t \ \widehat{[S(t-t')(AF)]}^x(k,t')\mathrm{d}t'.\end{aligned}$$

Thus,

$$\| U \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} = \| AU \|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} = \|\psi(t)\int_0^t S(t-t')AF(x,t')dt'\|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}.$$

Using Lemma 2.1. in [23], we have

$$\|\psi(t)\int_0^t S(t-t')AF(x,t')dt'\|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C\|AF\|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} = C\|F\|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}.$$

□

In order to treat the different nonlinear terms, we will see several lemmas. Here the bilinear estimate is given in the next lemma.

Lemma 3.4. *If $s > 0$, let $\sigma \geqslant 1$, $\delta > 0$. Then*

$$\| \partial_x(u_1u_2) \|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leqslant C \| u_1 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| u_2 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} . \tag{3.5}$$

Proof. We observe, by considering the operator A in (3.3), that

$$\begin{aligned} e^{\delta|k|^{1/\sigma}}\widehat{u_1u_2} &= (2\pi)^{-2}e^{\delta|k|^{1/\sigma}}\widehat{u_1} * \widehat{u_2} \\ &\leqslant (2\pi)^{-2}\int_{\mathbb{R}^2} e^{\delta|k-\eta|^{1/\sigma}}\widehat{u_1}(k-\eta,\tau-\rho)e^{\delta|\eta|^{1/\sigma}}\widehat{u_2}(\eta,\rho)d\eta d\rho \qquad (3.6) \\ &= \widehat{Au_1Au_2}, \end{aligned}$$

since $\delta \mid k \mid^{1/\sigma} \leqslant \delta \mid k-\eta \mid^{1/\sigma} + \delta \mid \eta \mid^{1/\sigma}, \quad \forall\sigma \geqslant 1$. Then

$$\begin{aligned} \| \partial_x(u_1u_2) \|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} &= \| \partial_x(A(u_1u_2)) \|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\ &\leq \| \partial_x(Au_1Au_2) \|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} . \end{aligned}$$

Now, by using Lemma 2.3. of [23], there exists $C > 0$ such that

$$\begin{aligned} \| \partial_x(Au_1Au_2) \|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} &\leqslant C \| Au_1 \|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| Au_2 \|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\ &= C \| u_1 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| u_2 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} . \end{aligned}$$

□

We are now ready to estimate all the terms in (3.1) by using the bilinear estimates in the above lemmas.

Lemma 3.5. *Let $s > 0$, and $\sigma \geq 1$, $\delta > 0$. Then, for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, with some constant $C > 0$, we have*

$$\| \Phi(u) \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C \left(\| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{T})} + \| u \|^2_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \right), \tag{3.7}$$

and

$$\| \Phi(u) - \Phi(v) \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C \| u - v \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| u + v \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}, \tag{3.8}$$

for all $u, v \in Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})$

Proof. To prove estimate (3.7), we follow

$$\begin{aligned}
\| \Phi(u) \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} &\leq \|\psi_T(t)S(t)u_0\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\
&+ \|\psi_T(t)\int_0^t S(t-t')\partial_x u^2(t')dt'\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\
&\leq C \| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{T})} + C \| \partial_x u^2 \|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\
&\leq C \| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{T})} + C \| u \|^2_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} .
\end{aligned}$$

For the estimate (3.8), we observe that

$$\Phi(u) - \Phi(v) = \psi_T(t) \int_0^t S(t-t') \left(\partial_x u^2 - \partial_x v^2\right)(x,t')dt',$$

where $\omega = \partial_x u^2 - \partial_x v^2$ is now given by

$$\omega = \partial_x(u^2 - v^2) = \partial_x[(u+v)(u-v)],$$

Thus, from the previous results, we obtain (3.8). □

We will show that the map Φ is a contraction on the ball $\mathbb{B}(0,r)$ to $\mathbb{B}(0,r)$. where u_0 satisfies the smallness condition $\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{T})} \leq \frac{1}{18C^2}$ and $r = \frac{1}{6C}$

Lemma 3.6. *Let $s > 0$ and $\sigma \geq 1$. Then, for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, such that the map $\Phi : \mathbb{B}(0,r) \to \mathbb{B}(0,r)$ is a contraction, where $\mathbb{B}(0,r)$ is given by*

$$\mathbb{B}(0,r) = \{u \in Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R}); \|u\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq r\}.$$

Proof. To prove Lemma3.6 we need to use Lemma3.5.

□

This completes the prove of existence Theorem2.1.

3.2. **Continuous dependence of the initial data.** To prove continuous dependence of the initial data in $Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})$ we will prove the following.

Lemma 3.7. *Let $s > 0$ and $\sigma \geq 1$, $\delta > 0$. Then for all $u_0, v_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, if u and v are two solutions to (1.1) corresponding to initial data u_0 and v_0. We have*

$$\|u - v\|_{C([0,T],G^{\sigma,\delta,s}(\mathbb{T}))} \leq 2C_0 C \|u_0 - v_0\|_{G^{\sigma,\delta,s}(\mathbb{T})}. \tag{3.9}$$

Proof. To prove Lemma3.7 we need to use Lemma3.1.

□

This completes the prove of Theorem 2.1.

3.3. **Time regularity.**

Lemma 3.8. *(Proposition 3.1,* [7] *) Let* $s > 0$*, and let* $\delta > 0$*,* $\sigma \geq 1$*,* $u \in C\left([0,T]; G^{\sigma,\delta,s}(\mathbb{T})\right)$ *be the solution of (1.1). Then* $u \in G^{\sigma}$ *in* $x, \forall t \in [0,T]$*, i.e., for some* $C > 0$*, we have*

$$|\partial_x^l u| \leqslant C^{l+1}(l!)^{\sigma}, l \in \{0,1,...\}, \quad \forall x \in \mathbb{T},\ t \in [0,T]. \tag{3.10}$$

In this section, we shall prove the time regularity of the solution as stated in Theorem 2.1 on the circle. The proof on the line is analogous.

Let us consider as in [3], for $\epsilon > 0$, the sequences

$$m_q = \frac{c(q!)^{\sigma}}{(q+1)^2}, (q = 0,1,2,...), \tag{3.11}$$

and

$$M_q = \epsilon^{1-q} m_q, \epsilon > 0 \ \ and \ (q = 1,2,3,...), \tag{3.12}$$

where c will be chosen (see [1]) so that the next inequality holds

$$\sum_{0 \leq l \leq k} \binom{k}{l} m_l m_{k-l} \leqslant m_k. \tag{3.13}$$

Removing the endpoints 0 and k in the left hand side of (3.13) and using the sequence M_q, we obtain

$$\sum_{0 < l < k} \binom{k}{l} M_l M_{k-l} \leq M_k, \forall \epsilon > 0. \tag{3.14}$$

Next, one can check that for any $\epsilon > 0$ the sequence M_q satisfies the following inequality

$$M_j \leqslant \epsilon M_{j+1},\ for\ j \geq 2. \tag{3.15}$$

Also, one can check that for a given $C > 1$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ we have

$$C^{j+1} j! \leqslant M_j,\ for\ j \geq 2. \tag{3.16}$$

By the definition of M_1 and M_2 in (3.12), we have for $j = 1$, that

$$M_1 = a\epsilon M_2, \quad where \quad a = \frac{9}{4(2!)^{\sigma}},$$

for some $C > 0$. Also, we define the following constants

$$M_0 = \frac{c}{8}\ and\ M = max\{2, \frac{8C}{c}, \frac{4C^2}{c}\}. \tag{3.17}$$

The next lemma is the main idea for the proof of Theorem 2.2.

Lemma 3.9. *Let* u *be the solution of (1.1) satisfying (3.10), then there exists* $\epsilon_0 > 0$ *such that for any* $0 < \epsilon \leqslant \epsilon_0$ *we have*

$$|\partial_t^j \partial_x^l u| \leqslant M^{2j+1} M_{l+5j}, j \in \{0,1,2,...\}, l \in \{0,1,2,...\}, \tag{3.18}$$

for all $x \in \mathbb{T},\ t \in [0,T]$.

Proof. (Of Lemma 3.9)
We will prove (3.18) by induction. Let $j = 0$, for $l = 0$, it follows from (3.10) and the definition of M in (3.17) that

$$|u| \leq C \leq M M_0, \quad \forall x \in \mathbb{T},\ t \in [0,T].$$

Similarly, for $l = 1$, we have

$$|\partial_x u| \leq C^2 \leqslant MM_1, \ \forall x \in \mathbb{T}, \ t \in [0, T].$$

By (3.10) and (3.16), for $l \geq 2$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$, we have

$$|\partial_x^l u| \leq C^{l+1}(l!)^\sigma \leq M_l \leq MM_l, \ \forall x \in \mathbb{T}, \ t \in [0, T].$$

This completes the proof of (3.18) for $j = 0$ and $l \in \{0, 1, ...\}$.
Next, we will assume that (3.18) is true for $0 \leq q \leq j$ and $l \in \{0, 1, ...\}$ and we will prove it for $q = j + 1$ and $l \in \{0, 1, ...\}$.
We begin by noting that

$$|\partial_t^{j+1}\partial_x^l u| = |\partial_t^j\partial_x^l(\partial_t u)| \quad \leqslant |\partial_t^j\partial_x^{l+5}u| + |\partial_t^j\partial_x^{l+3}u| + |\partial_t^j\partial_x^{l+1}u| + |\partial_t^j\partial_x^l(\partial_x u^2)|.$$

Using the induction hypotheses and the condition $M > 2$, we estimate the second term $\partial_t^j\partial_x^{l+5}u$, $\partial_t^j\partial_x^{l+3}u$ and $\partial_t^j\partial_x^{l+1}u$ as follows

$$\begin{aligned} |\partial_t^j\partial_x^{l+5}u| &\leqslant M^{2j+1}M_{l+5+5j} = M^{-2}M^{2(j+1)+1}M_{l+5(j+1)}, \\ &\leqslant \frac{1}{4}M^{2(j+1)+1}M_{l+5(j+1)}, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} |\partial_t^j\partial_x^{l+3}u| &\leqslant M^{2j+1}M_{l+3+5j} = M^{-2}M^{2(j+1)+1}M_{l+5j+3)}, \\ &\leqslant \frac{\epsilon^2}{4}M^{2(j+1)+1}M_{l+5(j+1)}, \end{aligned} \tag{3.20}$$

and

$$|\partial_t^j\partial_x^{l+1}u| \quad \leqslant M^{2j+1}M_{l+1+5j} \leqslant \frac{\epsilon^4}{4}M^{2(j+1)+1}M_{l+5(j+1)}. \tag{3.21}$$

All this estimates are taken for the linear terms. For the nonlinear term $(\partial_x u^2)$, using Leibniz's rule twice and the induction hypothesis, we have a different cases. We need the next results.

Lemma 3.10. *([3]) Given $n, k \in \{0, 1, 2, ...\}$ we have*

$$\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q}L_{(n-p)+5(k-q)}L_{p+5q} \leqslant \sum_{r=1}^{m}\binom{m}{r}L_rL_{m-r}, \tag{3.22}$$

where $L_j, j = 0, 1, ..., m$ positive real numbers with $m = n + 5k$

$$\begin{aligned} |\partial_t^j\partial_x^{l+1}(u^2)| &\leqslant \sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}|\partial_t^{j-q}\partial_x^{l+1-p}u||\partial_t^q\partial_x^p u|, \\ &\leqslant \sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M^{2(j-q)+1}M_{l+1-p+5(j-q)}M^{2q+1}M_{p+5q}, \\ &= M^{2(j+1)}\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M_{l+1-p+5(j-q)}M_{p+5q}. \end{aligned}$$

Next, using Lemma 3.10, with $n = l+1, k = j, L_j = M_j, m = l+1+5j$, we obtain

$$\begin{aligned}
&\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M_{l+1-p+5(j-q)}M_{p+5q},\\
&\leqslant \sum_{r=1}^{m}\binom{m}{r}L_rL_{m-r} \le (M_0+\epsilon)M_m,\\
&= (M_0+\epsilon)M_{l+5j+1},
\end{aligned}$$

then

$$\begin{aligned}
|\partial_t^j\partial_x^{l+1}(u^2)| &\leqslant M^{2(j+1)}(M_0+\epsilon)M_{l+5j+1},\\
&\leqslant M^{-2}M^{2(j+1)+1}\epsilon^4(M_0+\epsilon)M_{l+5(j+1)},\\
&\leqslant \frac{\epsilon^4(M_0+\epsilon)}{4}M^{2(j+1)+1}M_{l+5(j+1)}.
\end{aligned}$$

Noting that in the last inequality we have used the fact that $l+5j+1 \ge 2$, since we are assuming that either $j \ne 0$ or $l \ne 0$.

Now, choosing $\epsilon \le \epsilon_0 = \left(\frac{1}{(M_0+\epsilon)}\right)^{\frac{1}{4}} < 1$ to obtain

$$\epsilon^4(M_0+\epsilon) \le \epsilon^4(M_0+1) \le (M_0+1)\left(\frac{1}{(M_0+1)}\right) = 1.$$

Hence,

$$|\partial_t^j\partial_x^{l+1}(u^2)| \le \frac{1}{4}M^{2(j+1)+1}M_{l+5(j+1)}. \tag{3.23}$$

Which completes the proof. □

Proof. (Of Theorem 2.2) By Lemma 3.9, we have

$$|\partial_t^j\partial_x^l u| \leqslant M^{2j+1}M_{l+5j},\ j \in \{0,1,2,...\},\ l \in \{0,1,2,...\},$$

where

$$M_q = \epsilon^{1-q}\frac{c(q!)^\sigma}{(q+1)^2},\ q = 1,2,....$$

Applying this inequality for $j \in \{1,2,...\}$ and $l = 0$ gives

$$\begin{aligned}
|\partial_t^j u| &\leqslant M^{2j+1}M_{5j} = MM^{2j}\epsilon^{1-5j}\frac{c((5j)!)^\sigma}{(5j+1)^2},\\
&\leqslant M\epsilon c\left(\frac{M^2}{\epsilon^5}\right)^j((5j)!)^\sigma,\\
&\leqslant L_0L^j((5j)!)^\sigma,\\
&\leqslant L_0L^jA^{5\sigma j}((j!)^5)^\sigma,\\
&\leqslant A_0^{j+1}(j!)^{5\sigma},
\end{aligned} \tag{3.24}$$

where $L_0 = M\epsilon c,\ L = \dfrac{M^2}{\epsilon^5}$ since $(5j)! \leqslant A^{5j}(j!)^5$ for $A > 0$ and $A_0 = max\{L_0, LA^{5\sigma}\}$.
We also have from (3.18) for $l = j = 0$, that

$$|u| \leq MM_0 = M\frac{c}{8}, \quad \forall x \in \mathbb{T},\ t \in [0,T]. \tag{3.25}$$

Setting $C = max\{M\frac{c}{8}, A_0\}$, it follows from (3.24) and (3.25) that for $j \in \{0,1,2,...\}$, we have

$$|\partial_t^j u| \leqslant C^{j+1}(j!)^{5\sigma}, \quad \forall x \in \mathbb{T},\ t \in [0,T].$$

Hence, $u \in G^{5\sigma}$ in t.
Which completes the proof of Theorem 2.2. □

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UNBOUNDED PERTURBATION TO EVOLUTION PROBLEMS WITH TIME-DEPENDENT SUBDIFFERENTIAL OPERATORS

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Abstract. In this paper, we consider a nonlinear evolution inclusion governed by the subdifferential of a proper convex lower semicontinuous function in a separable Hilbert space. The right-hand side contains a set-valued perturbation with nonempty closed convex and not necessary bounded values. The existence of absolutely continuous solution is stated under different assumptions on the perturbation.

1. Introduction

Nonlinear evolution equations with subdifferential operators plays an important role in the theory of differential inclusions and have been widely investigated by many authors (see [1], [3], [5], [11], [15], [14], [17], [18], [19], [20], [21]), [22], [25]. Such problems appear often in problems of optimal control theory, mechanics and differential games, see for instance [9], [10], [12], [23]. In this work, we prove some existence results for evolution problems governed by subdifferential operator of the form

$$(\mathcal{P})\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + G(t,x(t)) \text{ a.e. } t\in[0,T]; \\ x(0)=x_0\in dom\,\varphi(0,\cdot), \end{cases}$$

in a separable Hilbert space, where φ is a proper convex lower semicontinuous function, $\partial\varphi(\cdot,\cdot)$ is the subdifferntial of φ and $G(\cdot,\cdot)$ is a set-valued mapping with convex closed nonempty values playing the role of a perturbation to the problem. For the unperturbed problem, that is when $G\equiv 0$, the existence and uniqueness of solution have been obtained under various assumptions by many authors, see for instance ([10], [11], [16], [23]). In [16], the author introduced an assumption expressed in terms of the conjugate function $\varphi^*(t,\cdot)$ of the convex function $\varphi(t,\cdot)$, namely, there exists a Lipschitz function $k:\ H\to\mathbf{R}_+$ and an absolutely continuous function $a:\ [0,T]\to\mathbf{R}$ with $\dot{a}\in L^2_{\mathbf{R}}([0,T])$ such that, for all $x\in H$ and $s,t\in[0,T]$,

$$\varphi^*(t,x)\le\varphi^*(s,x)+k(x)\mid a(t)-a(s)\mid .$$

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Some extensions, dealing with set-valued or single-valued perturbations, have been obtained under in general a compactness assumption on the subdifferential ([13], [17]) or on the perturbation [19]. The authors in [19] proved the existence of an absolutely continuous solution with set-valued perturbation satisfying the linear growth condition

$$G(t,x) \subset \beta(t)(1+ \| x \|)K \text{ for all } t \in [T_0, T] \text{ and } x \in H,$$

for some compact subset K and some non-negative function $\beta(\cdot) \in L^2_{\mathbf{R}}([T_0,T])$, $(T_0 \geq 0)$. In the particular case of the so-called sweeping process, i.e., for $\varphi(t,x)$ taken as the indicator function of a closed moving set $C(t,x)$, [13] established the existence of solution with prox-regular sets $C(t,x)$ and $G(\cdot,\cdot)$ with unnecessary bounded closed convex values. For other results, we refer to [6], [24] and the references therein. The main purpose in this paper is to study, in the setting of infinite dimensional Hilbert space H, the perturbed problem $(\mathcal{P})$, and to show how the approach from [13] can be adapted to yield the existence of solutions for $(\mathcal{P})$ with unbounded perturbation, under various assumptions. The paper is organized as follows. In section 2, we give some preliminaries and we recall some results which will be used in the paper. In section 3, we establish the existence theorem for the considered problem $(\mathcal{P})$ for a globally upper hemicontinuous perturbation, then we extend the result obtained in $[0,T]$ to the whole interval $\mathbf{R}_+$. Finally, we weaken the result by taking the perturbation G measurable in the time t and upper semicontinuous in the state x.

2. Preliminaries

Throughout the paper, H is a separable Hilbert space whose inner product is denoted by $\langle\cdot,\cdot\rangle$ and the associated norm by $\|\cdot\|$ and $[0,T]$ is an interval of $\mathbf{R}$. We will denote by $\mathbf{B}$ the closed unit ball of H, $\mathcal{P}_c(H)$ the family of all nonempty closed sets of H and $\mathcal{P}_{cc}(H)$ (resp. $\mathcal{P}_{ck}(H)$) the set of nonempty closed (resp. compact) convex subsets of H.
Let $\varphi : H \to \mathbf{R}\cup\{+\infty\}$ be an extended real-valued lower semicontinuous function, which is proper in the sense that its effective domain $dom\,\varphi$ defined by $dom\ \varphi := \{x \in H \ :\ \varphi(x) < +\infty\}$ is nonempty and, as usual, its Fenchel conjugate is defined by $\varphi^*(v) := \sup_{x\in H}[\langle v,x\rangle - \varphi(x)]$. The subdifferential $\partial\varphi(x)$ of φ at $x \in dom\ \varphi$ is

$$\partial\varphi(x) = \{v \in H :\ \langle v, y-x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in dom\ \varphi\}$$

and its effective domain is $dom\ \partial\varphi = \{x \in\ H :\ \partial\varphi(x) \neq \emptyset\}$. It is well known that if φ is a proper lower semicontinuous convex function, then its subdifferential operator $\partial\varphi$ is a maximal monotone operator and then satisfies the closure property. The function φ is said to be inf-ball compact if for every $r > 0$, the set $\{x \in H :\ \varphi(x) \leq r\}$ is ball-compact, i.e., its intersection with any closed ball in H is compact.
For any subset C of H, $\overline{co}\,C$ stands for the closed convex hull of C and $\sigma(\cdot,C)$ represents the support function of C, that is, for all $\xi \in H$, $\sigma(\xi,C) = \sup_{x\in C}\langle\xi,x\rangle$. We denote by $Proj(\cdot,C)$ the metric projection mapping onto the closed set C, defined by $Proj(x,C) := \{v \in C : d(x,C) = \|v-x\|\}$. A set-valued mapping $G : E \to \mathcal{P}_c(H)$ from a Hausdorff topological space E into subsets of H is said to be upper semicontinuous if, for any open subset $V \subset H$, the set $\{x \in E : G(x) \subset V\}$ is open in E. G is said to be scalarly upper semicontinuous or upper hemicontinuous if, for any $y \in\ H$, the real-valued function $x \mapsto\ \sigma(y, G(x))$ is upper semicontinuous. For

more details concerning the properties of maximal monotone operators in Hilbert space, we refer to [2] and [4]. Basic facts of convex analysis and set-valued mappings can be found in [8]. Let us recall the following result due to [19].

Proposition 2.1. *Let* $\varphi : [T_0, T] \times H \to \mathbf{R}_+ \cup \{+\infty\}$ *be such that:*

- (H_1) *for each* $t \in [T_0, T]$, $\varphi(t, \cdot)$ *is proper convex lower semicontinuous;*
- (H_2) *there exist a* ρ*-Lipschitzean function* $k : H \to \mathbf{R}_+$ *and an absolutely continuous function* $a : [T_0, T] \to \mathbf{R}$*, with a non-negative derivative* $\dot{a} \in L^2_{\mathbf{R}}([T_0, T])$*, such that*

$$\varphi^*(t, x) \leq \varphi^*(s, x) + k(x) \mid a(t) - a(s) \mid$$

for every $(t, s, x) \in [T_0, T] \times [T_0, T] \times H$.

If $h \in L^2_H([T_0, T])$ *and* $x_0 \in dom\, \varphi(T_0, \cdot)$*, then the problem*

$$(\mathcal{P}_h) \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + h(t) \text{ a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in dom\, \varphi(T_0, \cdot) \end{cases}$$

admits a unique absolutely continuous solution $x(\cdot)$ *that satisfies*

$$\int_{T_0}^{T} \| \dot{x}(t) \|^2 \, dt \leq 2c_0 \int_{T_0}^{T} \dot{a}^2(t) dt + \sigma \int_{T_0}^{T} \|h(t)\|^2 dt + c_1$$

with $c_0 = \frac{1}{2}\big(k^2(0) + 3(\rho + 1)^2\big)$, $\sigma = k^2(0) + 3(\rho + 1)^2 + 4$, *and*

$$c_1 = 2\big(T - T_0 + \varphi(T_0, x(T_0)) - \varphi(T, x(T))\big)$$

and for $T_0 \leq t_1 \leq t_2 \leq T$

$$|\varphi(t_2, x(t_2)) - \varphi(t_1, x(t_1))| \leq$$

$$\int_{t_1}^{t_2} \big(k(0) + (\rho + 1) \| \dot{x}(t) + h(t) \| \big)\big(\dot{a}(t) + |h|(t)\big) dt + \int_{t_1}^{t_2} \| \dot{x}(t) + h(t) \|^2 \, dt.$$

We close this section with a set-valued version of Scorza-Dragoni theorem due to [7], Corollary 2.2.

Corollary 2.2. *Let* $I = [T_0, T]$ *and* λ *the Lebesgue measure on* I*, with* σ*-algebra* $\mathcal{L}(I)$*. Let* X *be a Polish space and* Y *be a compact convex metrizable subset of a Hausdorff locally convex space. Let* $G : I \times X \to \mathcal{P}_{ck}(Y)$ *be a multifunction that satisfies the following hypotheses:*

- (i) $\forall t \in I$, *Graph* G_t *is closed in* $X \times Y$*;*
- (ii) $\forall x \in X$, *the multifunction* $t \mapsto G(t, x)$ *admits a measurable selection.*

Then, there exists a measurable multifunction $G_0 : I \times X \to \mathcal{P}_{ck}(Y) \cup \{\emptyset\}$*, which has the following properties:*

1. *there is a* λ*-null set* N *such that* $G_0(t, x) \subset G(t, x)$, $\forall t \notin N$, $\forall x \in X$*;*
2. *if* $u : I \to X$ *and* $v : I \to Y$ *are* $\mathcal{L}(I)$*-measurable functions with* $v(t) \in G(t, u(t))$ *a.e., then* $v(t) \in G_0(t, u(t))$ *a.e.;*
3. *for every* $\varepsilon > 0$*, there is a compact subset* $J_\varepsilon \subset I$ *such that* $\lambda(I \setminus J_\varepsilon) < \varepsilon$*, the graph of the restriction* $G_0/J_\varepsilon \times X$ *is closed and* $\emptyset \neq G_0(t, x) \subset G(t, x)$, $\forall (t, x) \in J_\varepsilon \times X$.

3. The Main Results

Now we are able to proved our first result for the problem $(\mathcal{P})$ with unbounded perturbation. In the development, we will use some ideas from [13] and [19].

Theorem 3.1. *Assume that* $\varphi : [0,T]\times H \to \mathbf{R}_+ \cup \{+\infty\}$ *satisfies* $(H_1),(H_2)$ *and*

(H_3) *φ is inf-ball compact for every $t \in [0,T]$.*

Let $G : [0,T]\times H \to \mathcal{P}_{cc}(H)$ *be such that*

(H_4) *G is upper hemicontinuous with respect to both variables;*

(H_5) *for any $(t,x) \in [0,T]\times H$, the mapping $Proj(0,G(t,x))$ is measurable on $[0,T]$ and there exist some real $\alpha > 0$ such that for all $(t,x)\in[0,T]\times H$,*

$$\|Proj(0,G(t,x))\| = d(0,G(t,x)) \le \alpha.$$

Then, for any $x_0 \in dom\,\varphi(0,\cdot)$ the problem $(\mathcal{P})$ has at least one absolutely continuous solution, satisfying $\int_0^T \|\dot{x}(t)\|^2 dt \le c$, where $c = 2c_0\int_0^T \dot{a}^2(t)dt + \sigma\alpha^2 T + 2\big(T + \varphi(0,x_0)\big)$ and $c_0 = \frac{1}{2}\big(k^2(0)+3(\rho+1)^2\big)$.

Proof. For each $(t,x)\in[0,T]\times H$, denote by $g(t,x)$ the element of minimal norm of the closed convex set $G(t,x)$ of H, that is, $g(t,x) = Proj(0,G(t,x))$. First, we shall construct a sequence of absolutely continuous mappings $(x_n(\cdot))_n$. Define, for every $n \ge 1$, the classical partition of $[0,T]$: for each $0 \le k \le n$, $t_k^n = k\dfrac{T}{n}$. Put $x(t_0^n) = x_0$, and choose y_0^n the element of minimal norm of $G(t_0^n, x_0)$, by (H_5) one has

$$\| y_0^n \| \le \alpha, \tag{3.1}$$

and consider the following differential inclusion on the interval $[t_0^n, t_1^n]$:

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + y_0^n \ \text{ for a.e. } t\in[t_0^n,t_1^n], \\ x(t_0^n) = x_0 \in dom\ \varphi(t_0^n,\cdot), \end{cases}$$

by (3.1) observe that the map $t\mapsto y_0^n$ is in $L_H^2([t_0^n,t_1^n])$, then, by Proposition 2.1 the last problem has a unique absolutely continuous solution that we denote by $x_0^n : [t_0^n,t_1^n]\to H$.

Likewise, for each $k\in\{0,...,n-1\}$ we can construct a finite sequence of absolutely continuous mappings $x_k^n(\cdot) : [t_k^n, t_{k+1}^n] \to H$ such that

$$\begin{cases} -\dot{x}_k^n(t) \in \partial\varphi(t,x_k^n(t)) + y_k^n \ \text{ a.e. } t\in[t_k^n,t_{k+1}^n], \\ x_k^n(t_k^n) = x_{k-1}^n(t_k^n) \in dom\varphi(t_k^n,\cdot). \end{cases} \tag{3.2}$$

where $y_k^n = Proj(0, G(t_k^n, x_{k-1}^n(t_k^n)))$. Recall that, in view of Proposition 2.1, the following inequality holds true in each subinterval $[t_k^n, t_{k+1}^n]$ for any $k\in\{0,...,n-1\}$

$$\begin{aligned} \int_{t_k{}^n}^{t_{k+1}^n} \| \dot{x}_k^n(t) \|^2\, dt &\le 2c_0 \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t)dt + \sigma\int_{t_k{}^n}^{t_{k+1}^n} \| y_k^n \|^2\, dt + c_k \\ &\le 2c_0 \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t)dt + \sigma\int_{t_k{}^n}^{t_{k+1}^n} \alpha^2 dt + c_k, \end{aligned} \tag{3.3}$$

with $c_0 = \frac{1}{2}(k^2(0)+3(\rho+1)^2)$, $\sigma = k^2(0)+3(\rho+1)^2+4$ and

$$c_k = 2[(t_{k+1}^n - t_k^n) + \varphi(t_k^n, x_k^n(t_k^n)) - \varphi(t_{k+1}^n, x_{k+1}^n(t_{k+1}^n))].$$

Now, define x_n and g_n from $[0,T]$ to H by

$$x_n(t) = x_k^n(t) \ \text{ if } \ t\in[t_k^n, t_{k+1}^n[;\ x_n(T) = x_{n-1}^n(T),$$

$$g_n(t) = y_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[;\ g_n(T) = y_{n-1}^n$$

for any $k \in \{0, ..., n-1\}$. Clearly, $x_n(\cdot)$ is absolutely continuous on $[0,T]$. Consider the mapping $\delta_n : [0,T] \to [0,T]$ such that for any $k \in \{0, ..., n-1\}$

$$\delta_n(t) = t_k^n \text{ if } t \in [t_k^n, t_{k+1}^n[;\ \delta_n(T) = T$$

then, observe that for each $t \in [0,T]$, $|\delta_n(t) - t| \leq |t_{k+1}^n - t_k^n| = \dfrac{T}{n}$, so $\delta_n(t) \to t$. Thus, for each $n \geq 1$, we have the following:

(i) $g_n(t) \in G(\delta_n(t), x_n(\delta_n(t)))$, $\forall t \in [0,T]$, $\forall x \in H$;
(ii) $\forall t \in [0,T]:$ $\|g_n(t)\| \leq \alpha$;
(iii) $-\dot{x}_n(t) \in \partial\varphi(t, x_n(t)) + g_n(\delta_n(t))$ a.e. $t \in [0,T]$, $x_n(0) = x_0$.

Further, from (3.3) we have:

$$\sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} \|\dot{x}_n(t)\|^2 dt \leq 2c_0 \sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} \dot{a}^2(t) dt + \sigma\alpha^2 \sum_{k=0}^{n-1} \int_{t_k{}^n}^{t_{k+1}^n} dt + \sum_{k=0}^{n-1} c_k,$$

equivalently

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \leq 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 \int_0^T dt + c_n \leq 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 T + c_n,$$

with $c_n = 2(T + \varphi(0, x_0) - \varphi(T, x_n(T)))$, because φ is non-negative, putting $c' = 2(T + \varphi(0, x_0))$, we may write

$$\int_0^T \|\dot{x}_n(t)\|^2 dt \leq 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 T + c',$$

then $\int_0^T \|\dot{x}_n(t)\|^2 dt \leq c$, where $c = 2c_0 \int_0^T \dot{a}^2(t) dt + \sigma\alpha^2 T + c'$, so

$$\sup_{n \in \mathbf{N}} \int_0^T \|\dot{x}_n(t)\|^2 dt \leq c \tag{3.4}$$

and thus $L = \sup_{n \in \mathbf{N}} \|\dot{x}_n(t)\|_{L_H^2([0,T])} < +\infty$.

Now, let us prove the uniform convergence of some subsequence of $x_n(\cdot)$ to some absolutely continuous mapping $x(\cdot)$. Using the Cauchy-Schwarz inequality and (3.4) for all $s \in [0,T]$ we obtain

$$\| x_n(s) - x_n(0) \|^2 = \| x_n(s) - x_0 \|^2 \leq s \int_0^s \| \dot{x}_n(t) \|^2 \, dt \leq Tc$$

and hence

$$\| x_n(s) \|^2 \leq 2 \| x_0 \|^2 + 2 \| x_n(s) - x_0 \|^2 \leq 2 \| x_0 \|^2 + 2Tc.$$

Consequently, for each n, we get $\| x_n(\cdot) \|_\infty^2 \leq 2 \| x_0 \|^2 + 2Tc$. Then

$$\| x_n(\cdot) \|_\infty \leq M, \tag{3.5}$$

where $M = (2 \| x_0 \|^2 + 2Tc)^{\frac{1}{2}}$. Therefore

$$\| x_n(t) - x_n(s) \| = \| \int_s^t \dot{x}_n(\tau) d\tau \| \leq (t-s)^{\frac{1}{2}} \Big(\int_s^t \| \dot{x}_n(\tau) \|^2 \, d\tau \Big)^{\frac{1}{2}} \leq (t-s)^{\frac{1}{2}} L,$$

so along with (3.5), the set $\{(x_n(\cdot))_n\}$ is bounded and equicontinuous in $C_H([0,T])$, recall that, in view of Proposition 2.1, for any fixed $t \in [0,T]$ and any n, one has

$$|\varphi(t, x_n(t)) - \varphi(0, x(0))| \leq \sup_{n \in \mathbf{N}} \int_0^t (k(0) + (\rho + 1) \| \dot{x}_n(t) + \alpha \|)(\dot{a}(t) + \alpha) dt$$

$$+ \sup_{n\in \mathbf{N}} \int_0^t \| \dot{x}_n(t) + \alpha \|^2 \, dt < +\infty,$$

since φ is inf-ball compact by assumption, the set $\{x_n(t);\ n \in \ \mathbf{N}\}$ is relatively compact in H, so by Ascoli's theorem, we can extract a subsequence of $(x_n(\cdot))_n$ that converges uniformly on $[0, T]$ to some map $x(\cdot) \in C_H([0, T])$. From (3.4), $(\dot{x}_n)_n$ is bounded in $L^2_H([0, T])$, we may then extract a subsequence from the latter subsequence converging weakly in $L^2_H([0, T])$ to some map $v(\cdot)$. The equality $x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s)ds$ for all $t \in [0, T]$ then yields $x(t) = x(0) + \int_0^t v(s)ds$ for all $t \in [0, T]$ and hence the map $x(\cdot)$ is absolutely continuous on $[0, T]$ with $\dot{x}(\cdot) = v(\cdot)$ a.e.

Finally, we show now that $x(\cdot)$ is a solution of $(\mathcal{P})$ on $[0, T]$. Define the step mapping $z_n(t) = g_n(\delta_n(t))$ for all $t \in [0, T]$, one has for almost all $t \in [0, T]$, $-\dot{x}_n(t) \in \partial\, \varphi(t, x_n(t)) + z_n(t)$ with

$$z_n(t) \in \ G(\delta_n(t), x_n(\delta_n(t))). \tag{3.6}$$

Since $\| \ g_n(\delta_n(t)) \ \| \leq \alpha$ for all $n \in \mathbf{N}$ and $t \in [0, T]$, we may suppose that the sequence $(z_n(\cdot))_n$ converges weakly in $L^1_H([0, T])$ to a mapping $z(\cdot) \in L^1_H([0, T])$ with $\| \ z(t) \ \| \leq \alpha$ a.e. $t \in [0, T]$. By Mazur's Theorem, there exists

$$\xi_n \in \overline{co}\{z_q, \ \ q \geq n\} \tag{3.7}$$

such that $(\xi_n(\cdot))_n$ converges strongly in $L^1_H([0, T])$ to $z(\cdot)$. Extracting a subsequence if necessary, we may suppose that $(\xi_n(\cdot))_n$ converges a.e. to $z(\cdot)$, then there is a Lebesgue negligible set $S \subset [0, T]$ such that for every $t \in \ [0, T] \setminus S$, on one hand $\xi_n(t) \to z(t)$ strongly in H, and on the other hand the inclusion (3.6) holds true for every integer $n \geq 1$ as well as the inclusion

$$z(t) \in \bigcap_n \overline{co}\{z_q(t), \ \ q \geq n\}.$$

From the inclusion (3.6), for any $n \in \mathbf{N},\ t \in \ [0, T] \setminus S$ and any $y \in H$:

$$\langle y, z_n(t)\rangle \leq \sigma(y, G(\delta_n(t), x_n(\delta_n(t)))), \tag{3.8}$$

further, for each $n \in \mathbf{N}$ and any $t \in \ [0, T] \setminus S$, from (3.7) we have

$$\langle y, \xi_k(t)\rangle \leq \sup_{q\geq n} \langle y, z_q(t)\rangle \quad \forall\, k \geq n, \tag{3.9}$$

taking the limit in (3.9) as $k \to +\infty$ and by (3.8) one obtains

$$\langle y, z(t)\rangle \leq \sup_{q\geq n} \langle y, z_q(t)\rangle \leq \sup_{q\geq n} \sigma(y, G(\delta_q(t), x_q(\delta_q(t)))),$$

which ensures that $\langle y, z(t)\rangle \leq \limsup_{n\to+\infty} \sigma(y, G(\delta_n(t), x_n(\delta_n(t))))$. Since $\sigma(y, G(\cdot, \cdot))$ is upper semicontinuous on $[0, T] \times H$, then for every $t \in \ [0, T] \setminus S$ and every $y \in H$, $\langle y, z(t)\rangle \leq \sigma(y, G(t, x(t))$, then $z(t) \in G(t, x(t))$ a.e. Further, since $(\dot{x}_n(\cdot) + z_n(\cdot))_n$ converges weakly in $L^1_H([0, T])$ to $\dot{x}(\cdot) + z(\cdot)$ and $(x_n(\cdot))_n$ converges strongly in $L^1_H([0, T])$ to $x(\cdot)$ and since the operator $\partial\varphi(t, \cdot)$ satisfies the closure property as the subdifferential of a proper lower semicontinuous function one obtains $\dot{x}(t) + z(t) \in -\partial\varphi(t, x(t))$ a.e., with $z(t) \in G(t, x(t))$ a.e.

Taking the limit in inequality (3.4) and using the preceding convergence, we get $\int_0^T \|\dot{x}(t)\|^2 dt \leq c.$ □

Note that, obviously, Theorem 3.1 yields for any finite interval of the form $[T_k, T_{k+1}]$ for all $k \in \mathbf{N}$. So, the next Corollary proves on the whole interval $\mathbf{R}_+ := [0, +\infty[$ the existence of solution to the above evolution problem.

Corollary 3.2. *Let $\varphi : \mathbf{R}_+ \times H \to \mathbf{R}_+ \cup \{+\infty\}$ and $G : \mathbf{R}_+ \times H \to \mathcal{P}_{cc}(H)$ be such that the following assumptions hold:*

- (H_1') *the function $x \mapsto \varphi(t,x)$ is proper convex lower semicontinuous, for each $t \in \mathbf{R}_+$.*
- (H_2') *there exist a ρ-Lipschitzean function $k : H \to \mathbf{R}_+$ and an absolutely continuous function $a : \mathbf{R}_+ \to \mathbf{R}$, with a non-negative derivative $\dot{a} \in L^2_{\mathbf{R}}(\mathbf{R}_+)$, such that*

$$\varphi^*(t,x) \leq \varphi^*(s,x) + k(x) \mid a(t) - a(s) \mid$$

  for every $(t,s,x) \in \mathbf{R}_+ \times \mathbf{R}_+ \times H$,
- (H_3') *φ is inf-ball compact for every $t \in \mathbf{R}_+$,*
- (H_4') *G is upper hemicontinuous with respect to both variables,*
- (H_5') *there exists a non-negative function $\alpha(\cdot) \in L^\infty_{loc}(\mathbf{R}_+)$ such that $d(0, G(t,x)) \leq \alpha(t)$ for all $t \in \mathbf{R}_+$ and $x \in H$.*

Then, for any $x_0 \in dom\ \varphi(0,\cdot)$, there exists a mapping $x : \mathbf{R}_+ \to H$ which is locally absolutely continuous on $\mathbf{R}_+$ and satisfies

$$(\mathcal{P}_1) \begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + G(t,x(t)) \ \ a.e. \ \ t \in \mathbf{R}_+, \\ \qquad x(0) = x_0 \in dom\ \varphi(0,\cdot). \end{cases}$$

Proof. We follow the idea of the proof of Theorem 4 in [13]. We consider the partition of $\mathbf{R}_+$ by the points $T_n = n$ for all $n \in \mathbf{N}$. It will suffice to apply Theorem 3.1 in an appropriate way on each interval $[T_n, T_{n+1}]$. By Theorem 3.1, there exists a absolutely continuous solution $x_0 : [T_0, T_1] \to H$ of the differential inclusion

$$-\dot{x}_0(t) \in \partial\varphi(t, x_0(t)) + G(t, x_0(t)),\ t \in [T_0, T_1];\ x_0(T_0) = x_0 \in dom\varphi(T_0, \cdot).$$

Likewise, for each $i \in \{0, \cdots, n-1\}$ we construct an absolutely continuous mapping $x_i : [T_i, T_{i+1}] \to H$ such that

$$\begin{cases} -\dot{x}_i(t) \in \partial\varphi(t, x_i(t)) + G(t, x_i(t)) \ \text{a.e.} \ \ t \in [T_i, T_{i+1}], \\ \qquad x_i(T_i) = x_{i-1}(T_i) \in dom\varphi(T_i, \cdot). \end{cases} \tag{3.10}$$

Taking $x : \mathbf{R}_+ \to H$ defined by $x(t) := x_n(t)$ for all $t \in [T_n, T_{n+1}[$ and $n \in \mathbf{N}$, it is readily seen that x is locally absolutely continuous solution of $(\mathcal{P}_1)$ on $\mathbf{R}_+$. □

In the next theorem, we weaken the hypothesis on G by taking G having a measurable selection with respect to the first variable and upper hemicontinuous on H.

Theorem 3.3. *Under the assumptions of Theorem 3.1 on φ, let $G : [0,T] \times H \to \mathcal{P}_{cc}(H)$ be such that:*

- (a) *for all $t \in [0,T]$, $G(t,\cdot)$ is upper hemicontinuous on H,*
- (b) *for any $x \in H$, $G(\cdot, x)$ has a λ-measurable selection,*
- (c) *for some compact convex subset $K \subset \mathbf{B}$ and some real number $\gamma > 0$, for all $(t,x) \in [0,T] \times H$, one has $G(t,x) \subset \gamma(1+ \parallel x \parallel)K$.*

Then, for any $x_0 \in dom\varphi(0,\cdot)$ the Cauchy problem $(\mathcal{P})$ admits at least one absolutely continuous solution, more precisely, there exist an absolutely continuous mapping $x(\cdot) : [0,T] \to H$ and an integrable mapping $g : [0,T] \to H$ such that $x(0) = x_0$, $x(t) \in dom\varphi(t, x(t))$, for all $t \in [0,T]$, and for almost every $t \in [0,T]$, $g(t) \in G(t, x(t))$ and $-\dot{x}(t) - g(t) \in \partial\varphi(t, x(t))$.

Proof. Choose some positive numbers α, R such that $\alpha = \gamma(1 + R)$ and $R = \sqrt{2}(\|x_0\|^2 + Tc)^{\frac{1}{2}}$, where c is as in Theorem 3.1, and fix a continuous function $\psi : \mathbf{R}_+ \to [0,1]$ such that

$$\psi(\tau) = \begin{cases} 1 & \text{if } \tau \leq R, \\ 0 & \text{if } \tau \geq R+1. \end{cases} \tag{3.11}$$

Let us consider the compact convex metric space $Y := \gamma(1+R)K$, which is a Borel subset of H, and let us define a set-valued mapping $\widehat{G} : [0,T] \times H \to \mathcal{P}_{ck}(Y)$ by

$$\widehat{G}(t,x) := \psi(\|x\|)G(t,x),$$

obviously, $\widehat{G}(\cdot, x)$ has a measurable selection for all $x \in H$ and for each $t \in [0,T]$, the graph of $\widehat{G}(t,\cdot)$ is closed in $H \times Y$, therefore, in view of Corollary 2.2, there exists a measurable set-valued mapping $G_0 : [0,T] \times H \to \mathcal{P}_{ck}(Y) \cup \{\emptyset\}$ such that:

(i) there is a λ-negligible set $N \subset [0,T]$, such that

$$G_0(t,x) \subset \widehat{G}(t,x) \text{ for all } t \notin N \text{ and for all } x \in H; \tag{3.12}$$

(ii) for every $n \geq 1$, there is a compact subset $J_n \subset [0,T]$ such that $\lambda([0,T] \setminus J_n) < \frac{1}{n}$, the graph of the restriction $G_0/J_n \times H$ is closed and $\emptyset \neq G_0(t,x) \subset \widehat{G}(t,x)$, $\forall (t,x) \in J_n \times H$;

further, (ii) implies that there exists an increasing sequence $(J_n)_{n\geq 1}$ of compact subsets of $[0,T]$ such that, for each $n \geq 1$, $G_0/J_n \times H$ is upper semicontinuous with convex compact values. So, by the set-valued version of Dugundji's extension theorem, for each $n \geq 1$, there exists some upper semicontinuous extension $\overline{G}_n$ of $G_0/J_n \times H$ to $[0,T] \times H$ satisfying

$$\overline{G}_n(t,x) \subset \gamma(1+ \| x \|)K, \text{ for all } (t,x) \in [0,T] \times H$$

and $\overline{G}_n(t,x) = G_0(t,x)$ on $J_n \times H$. Further, $d(0, \overline{G}_n(t,x)) \leq \alpha$, $\forall (t,x) \in J_n \times H$. Due to Theorem 3.1 , for each $n \geq 1$, there exists an absolutely continuous map $x_n(\cdot) : [0,T] \to H$ and an integrable map $g_n : [0,T] \to H$ such that $x_n(0) = x_0$ and for almost all $t \in [0,T]$, $-\dot{x}_n(t) \in \partial\varphi(t, x_n(t)) + g_n(t)$, and

$$g_n(t) \in \overline{G}_n(t, x_n(t)), \tag{3.13}$$

with

$$\| g_n(t) \| \leq \alpha \tag{3.14}$$

and

$$\sup_{n\in\mathbf{N}} \int_0^T \| \dot{x}_n(t) \|^2 \, dt \leq c, \tag{3.15}$$

and thus $L = \sup\limits_{n\in\mathbf{N}} \| \dot{x}_n \|_{L^2_H([0,T]} < +\infty$. As in the proof of Theorem 3.1, using the Cauchy-Schwarz inequality and by (3.15) we obtain, for each n,

$$\| x_n(\cdot) \|_\infty \leq \alpha. \tag{3.16}$$

Therefore, for all $s, t \in [0,T]$ one has

$$\| x_n(t) - x_n(s) \| \leq (t-s)^{\frac{1}{2}} \left(\int_0^T \| \dot{x}_n(\tau) \|^2 d\tau \right)^{\frac{1}{2}} \leq (t-s)^{\frac{1}{2}} L,$$

so along with (3.16), the set $\{x_n(\cdot), n \in \mathbf{N}\}$ is bounded and equicontinuous in $\mathrm{C}_H([0,T])$. Recall that, in view of Proposition 2.1, for any fixed $t \in [0,T]$ and any n, one has

$$|\varphi(t, x_n(t)) - \varphi(0, x(0))| < +\infty.$$

So, since φ is inf-ball compact, the set $\{x_n(t), n \in \mathbf{N}\}$ is relatively compact in H. By Ascoli's Theorem, we can extract a subsequence of $(x_n(\cdot))_n$ that converges uniformly on $[0,T]$ to some continuous map $x(\cdot) \in \mathrm{C}_H([0,T])$, that is

$$x_n(\cdot) \to x(\cdot) \quad \text{strongly in} \quad L^2_H([0,T]). \tag{3.17}$$

By (3.15), the sequence $(\dot{x}_n)_n$ is bounded in $L^2_H([0,T])$, we may then extract a subsequence converging weakly in $L^2_H([0,T])$ to some map $v(\cdot)$. The equality

$$x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s) ds, \quad \text{for all } t \in [0,T],$$

then yields

$$x(t) = x(0) + \int_0^t v(s) ds \quad \text{for all } t \in [0,T]$$

and hence the map $x(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{x}(\cdot) = v(\cdot)$ for almost all $t \in [0,T]$ and

$$\dot{x}_n(\cdot) \to \dot{x}(\cdot) \quad \text{weakly in} \quad L^2_H([0,T]). \tag{3.18}$$

Due to (3.14), we may also suppose that, for some map $g(\cdot) \in L^2_H([0,T])$, one has

$$g_n(\cdot) \to g(\cdot) \quad \text{weakly in} \quad L^2_H([0,T]). \tag{3.19}$$

Taking (3.17), (3.18) and (3.19) into account, as in the proof of Theorem 3.1 we have, via the closure property of the subdifferential operator $\partial\varphi(t, \cdot)$ for almost all $t \in [0,T]$ the required inclusion, that is,

$$\dot{x}(t) + g(t) \in -\partial\varphi(t, x(t)) \quad \text{a.e. } t \in [0,T]. \tag{3.20}$$

It remains to prove that $g(t) \in G(t, x(t))$ for almost every $t \in [0,T]$.
Due to (3.19), by Mazur theorem, there exists a sequence $(\xi_n(\cdot))_n$ in $L^1_H([0,T])$ such that

$$\xi_n(\cdot) \in co\{g_q(\cdot), q \geq n\} \quad \text{for all} \quad n \geq 1, \tag{3.21}$$

which converges strongly in $L^1_H([0,T])$ to $g(\cdot)$. Thus, extracting a subsequence if necessary we may suppose that $\xi_n(t) \to g(t)$ for almost every $t \in [0,T]$. So, this along with (3.21), implies that, for some negligible subset $N_1 \subset [0,T]$,

$$g(t) \in \bigcap_n \overline{co}\{g_q(t), q \geq n\} \quad \text{for all} \ \ t \in [0,T] \setminus N_1. \tag{3.22}$$

Taking (3.13) into account, we may also suppose that, for all $n \geq 1$ and for all $t \in [0,T] \setminus N_1$,

$$g_n(t) \in \overline{G}_n(t, x_n(t)). \tag{3.23}$$

Consider the λ-negligible subset $N_2 = ([0,T] \setminus \cup_n J_n) \cup N \cup N_1$, we are going to prove that $g(t) \in G(t, x(t))$ for all $t \in [0,T] \setminus N_2$. Fix any $\tau \in [0,T] \setminus N_2$, from (3.22) and (3.23), it follows that, for any $y \in H$,

$$\langle y, g(\tau) \rangle \leq \limsup_n \sigma(y, \overline{G}_n(\tau, x_n(\tau))). \tag{3.24}$$

On the other hand, by definition of N_2, there exists an integer $p(\tau)$ such that $\tau \in J_{p(\tau)} \setminus N$ and $(J_n)_n$ being increasing, one has $\tau \in J_n$ for all $n \geq p(\tau)$. Consequently, for all $n \geq p(\tau)$,

$$\overline{G}_n(\tau, x_n(\tau)) = G_0(\tau, x_n(\tau)) \subset \widehat{G}(\tau, x_n(\tau)). \tag{3.25}$$

The inclusion coming from (3.12). Note that, by (3.16) one has, for all $n \geq 1$ and for almost all $t \in [0,T]$,

$$\| x_n(t) \| \leq R,$$

and hence by (3.11), for all $n \geq 1$,

$$\widehat{G}(\tau, x_n(\tau)) = G(\tau, x_n(\tau)). \tag{3.26}$$

Therefore, due to (3.24), (3.25) and (3.26) and the fact that $G(\tau, \cdot)$ is scalarly upper semicontinuous, we have

$$\langle y, g(\tau) \rangle \leq \sigma(y, G(\tau, x(\tau))),$$

this being true for any $y \in H$, and $G(\tau, x(\tau))$ being closed and convex, it results that $g(t) \in G(t, x(t))$. Since the latter is satisfied for any $\tau \in [0,T] \setminus N_2$, one has $g(t) \in G(t, x(t))$ a.e. $t \in [0,T]$. This, along with (3.20), proves that $x(\cdot)$ is a solution of $(\mathcal{P})$. □

4. Application

Let φ be the indicator function of a nonempty closed convex moving set $C(t)$, that is, $\varphi(t, x) = I_{C(t)}(x) = 0$ if $x \in C(t)$ and $+\infty$ otherwise. It is well-known that $\partial I_{C(t)}(x) = N_{C(t)}(x)$ the normal cone to C(t) at x. Then problem $(\mathcal{P})$ becomes

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t) + G(t, x(t)) \text{ a.e. } t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases}$$

Problems of this form are known as "sweeping process" and arise in elastoplasticity, contact dynamics, friction dynamics, and granular material (see Moreau [15]). The sweeping process model is also of great interest in nonsmooth mechanics, convex optimization, mathematical economics and more recently in the modeling and simulation of switched electrical circuits as well as the modeling of crowd motion. As an example, let consider dynamics that correspond to an electrical circuit containing nonsmooth devices like diodes. A diode is a device that constitutes a rectifier which permits the easy flow of charges in one direction and restrains the flow in the opposite direction. The ideal model diode is a simple switch.The problem is the following:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{LC} x_1 - \frac{R}{L} x_2 + \frac{1}{L} + \frac{1}{L} u \\ y = -x_2 \text{ and } y_L \in \partial \Phi(y); \end{cases}$$

where $R > 0$ is a resistor, $L > 0$ an inductor, $C > 0$ a capacitor, u is the voltage supply, $x_1(t)$ is the time integral of the current across the capacitance, $x_2(t) = i(t)$ is the current across the circuit, y_L is the voltage of the diode and Φ is the electrical superpotential of the diode. Setting $\varphi = \Phi \circ C$, we get $\partial\varphi(x) = B\partial\Phi(B^t x)$. Therefore, the dynamic of the system is of the form $-\dot{x}(t) \in Ax(t) + \partial\varphi(x(t))$, where

$$A = \begin{pmatrix} 0 & 0 \\ \frac{-1}{LC} & \frac{-R}{L} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

If we suppose that the diode is ideal, then its superpotential and subdifferential are respectively given by $\Phi(x) = I_{\mathbf{R}_+}(x)$ and $\partial\Phi(x) = N_{\mathbf{R}_+}(x) = \begin{cases} \emptyset & \text{if } x < 0 \\ ]-\infty, 0] & \text{if } x = 0 \\ 0 & \text{if } x > 0. \end{cases}$

5. Conclusion

We have established existence results for nonlinear evolution inclusions which are driven by time dependent subdifferential operators, by using a specific and adapted discretization, with technical nuances, in both convex analysis and nonsmooth analysis. We generalize the results when the perturbation, that is, the external forces applied on the system, is with convex but not necessary bounded values. In a forthcoming work, we deal with a nonconvex perturbation by the relaxation (convexification) approach.

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