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Rank Approach for Equality Relations of BLUPs in Linear Mixed Model and Its Transformed Model

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Abstract

A linear mixed model (LMM) $\mathscr{M} : \mathbf{y} = \mathbf{X}\beta + \mathbf{Zu} + \varepsilon$ with general assumptions and its transformed model $\mathscr{T} : \mathbf{Ty} = \mathbf{TX}\beta + \mathbf{TZu} + \mathbf{T}\varepsilon$ are considered. This work concerns the comparison problem of predictors under $\mathscr{M}$ and $\mathscr{T}$. Our aim is to establish equality relations between the best linear unbiased predictors (BLUPs) of unknown vectors under two LMMs $\mathscr{M}$ and $\mathscr{T}$ through their covariance matrices by using various rank formulas of block matrices and elementary matrix operations.

1. Introduction

Throughout this note, the symbol $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices. $\mathbf{A}'$, $\mathbf{A}^+$, $r(\mathbf{A})$ and $\mathscr{C}(\mathbf{A})$ stand for the transpose, the Moore–Penrose generalized inverse, the rank, and the column space of $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ refers the $m\times m$ identity matrix. Furthermore, $\mathbf{E_A} = \mathbf{A}^\perp = \mathbf{I}_m - \mathbf{AA}^+$ represents the orthogonal projector for $\mathbf{A} \in \mathbb{R}^{m\times n}$.
A linear mixed model (LMM), formulated by

$$\mathscr{M} : \mathbf{y} = \mathbf{X}\beta + \mathbf{Zu} + \varepsilon, \tag{1.1}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X} \in \mathbb{R}^{n\times k}$ and $\mathbf{Z} \in \mathbb{R}^{n\times p}$ are known matrices of arbitrary rank, $\beta \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters, $\mathbf{u} \in \mathbb{R}^{p\times 1}$ is a vector of unobservable random effects, and $\varepsilon \in \mathbb{R}^{n\times 1}$ is an unobservable vector of random errors. LMMs include fixed and random effects and supply helpful tools to explain the variability of model parameters affecting response variables. In statistical inferences of analysis requirements, LMMs may need to be transformed. One of the various transformations is the linear transformation of a given model which is obtained by pre-multiplying the model by a given matrix. In such case, for given transformation matrix $\mathbf{T} \in \mathbb{R}^{m\times n}$, transformed model of $\mathscr{M}$ is obtained as follows

$$\mathscr{T} : \mathbf{Ty} = \mathbf{TX}\beta + \mathbf{TZu} + \mathbf{T}\varepsilon. \tag{1.2}$$

We consider the following vector including all unknown vectors under the models $\mathscr{M}$ and $\mathscr{T}$ to establish simultaneous results on predictors:

$$\phi = \mathbf{K}\beta + \mathbf{Gu} + \mathbf{H}\varepsilon = \mathbf{K}\beta + \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix} \tag{1.3}$$

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for given $\mathbf{K} \in \mathbb{R}^{s\times k}$, $\mathbf{G} \in \mathbb{R}^{s\times p}$, and $\mathbf{H} \in \mathbb{R}^{s\times n}$. We assume the following general assumptions for considered models:

$$\mathrm{E}\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix} = \mathbf{0} \text{ and } \mathrm{D}\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix} = \mathrm{cov}\left\{\begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix}, \begin{bmatrix}\mathbf{u}\\ \varepsilon\end{bmatrix}\right\} = \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix} := \Sigma,$$

where $\Sigma \in \mathbb{R}^{(n+p)\times(n+p)}$ is a positive semi-definite matrix of arbitrary rank and all the elements of Σ are known.
Let $\mathbf{A} = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}$. Then we obtain

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\beta,\ \ \mathrm{D}(\mathbf{y}) = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}' = \mathbf{A}\Sigma\mathbf{A}' := \mathbf{R},$$

$$\mathrm{E}(\phi) = \mathbf{K}\beta,\ \ \mathrm{D}(\phi) = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}' = \mathbf{B}\Sigma\mathbf{B}' := \mathbf{S},$$

$$\mathrm{cov}(\phi, \mathbf{y}) = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}' = \mathbf{B}\Sigma\mathbf{A}' := \mathbf{C}.$$

Further, we assume that $\mathscr{M}$ is consistent, i.e., $\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}$ holds with probability 1 (wp 1), see, e.g., [1]. The consistency of $\mathscr{T}$ is provided with the condition $\mathbf{Ty} \in \mathscr{C}\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix}$ wp 1. It is easy to see that $\mathscr{T}$ is consistent under the consistency of $\mathscr{M}$.
Predictors under original models and their transformed models have different properties. In some cases, due to linear transformation, observable random vectors in transformed models may preserve enough information to predict unknown vectors under original models. For this reason, establishing relationships and comparisons between these models is statistically useful. In prediction problems, covariance matrices of predictors can be used to establish some statistical properties of analysis such as comparison of predictors. Further, some formulas in matrix algebra such as ranks of matrices offer practical ways for simplifying various complicated matrix equations. The matrix rank method based on the fact that $\mathbf{A} = \mathbf{0}$ if and only if $r(\mathbf{A}) = 0$ is one of the useful methods for deriving algebraic and statistical properties of matrix expressions. This study considers the comparison problem of predictors under an LMM and its transformed model under general assumptions. In particular, we establish equality relations between the best linear unbiased predictors (BLUPs) of unknown vectors under $\mathscr{M}$ and $\mathscr{T}$ through their covariance matrices by using various rank formulas for block matrices, the matrix rank method, and elementary matrix operations. We also give some results for certain specific forms of ϕ which correspond to the best linear unbiased estimators (BLUEs) of unknown parameters under $\mathscr{M}$ and $\mathscr{T}$. To derive the results, we use the following situations to establish equalities between two random vectors, see, e.g., [2] and [3]. Let $\mathbf{u}$ be a random vector

(a) If both $\mathrm{E}(\mathbf{F}_1\mathbf{u} - \mathbf{F}_2\mathbf{u}) = \mathbf{0}$ and $\mathrm{D}(\mathbf{F}_1\mathbf{u} - \mathbf{F}_2\mathbf{u}) = \mathbf{0}$ hold, $\mathbf{F}_1\mathbf{u} = \mathbf{F}_2\mathbf{u}$ holds wp 1.
(b) If both $\mathrm{E}(\mathbf{F}_1\mathbf{u}) = \mathrm{E}(\mathbf{F}_2\mathbf{u})$ and $\mathrm{D}(\mathbf{F}_1\mathbf{u}) = \mathrm{D}(\mathbf{F}_2\mathbf{u})$ hold, the expectation and covariance of $\mathbf{F}_1\mathbf{u}$ and $\mathbf{F}_2\mathbf{u}$ are equal, respectively.

Further, we use the following formulas for ranks of block matrices to establish the results in this study. They are given in the following lemma; see [4] and [5].

Lemma 1.1. *Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{N} \in \mathbb{R}^{m\times k}$, $\mathbf{P} \in \mathbb{R}^{l\times n}$, *and* $\mathbf{Q} \in \mathbb{R}^{l\times k}$. *Then,*

$$r\begin{bmatrix}\mathbf{M}, & \mathbf{N}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{E_M N}) = r(\mathbf{N}) + r(\mathbf{E_N M}),$$

$$r\begin{bmatrix}\mathbf{M}\\ \mathbf{P}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{P E_{M'}}) = r(\mathbf{P}) + r(\mathbf{M E_{P'}}),$$

$$r\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{0}\end{bmatrix} = r(\mathbf{N}) + r(\mathbf{P}) + r(\mathbf{E_N M E_{P'}}), \tag{1.4}$$

$$r\begin{bmatrix}\mathbf{MM}' & \mathbf{N}\\ \mathbf{N}' & \mathbf{0}\end{bmatrix} = r\begin{bmatrix}\mathbf{M}, & \mathbf{N}\end{bmatrix} + r(\mathbf{N}),$$

$$r\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{Q}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{Q} - \mathbf{PM}^{+}\mathbf{N}) \ \ \textit{if}\ \mathscr{C}(\mathbf{N}) \subseteq \mathscr{C}(\mathbf{M}) \ \ \textit{and}\ \ \mathscr{C}(\mathbf{P}') \subseteq \mathscr{C}(\mathbf{M}'), \tag{1.5}$$

Statistical inference of LMMs is an important part in the data analysis, and some previous and recent studies on relations between predictors under these models can be found in, e.g., [6]-[19], among others. Searching relationships between a linear model and its transformed model is one of the essential issues in linear regression analysis. For transformation approaches of linear models, we may refer [2], [20]-[28].

2. Notes on BLUPs in LMMs

To obtain some results of the BLUPs under models $\mathscr{M}$ and $\mathscr{T}$, we need some fundamental facts on BLUPs under LMMs. In this section, we review the predictability conditions and then we give the fundamental BLUP equations and related properties under $\mathscr{M}$ and $\mathscr{T}$.
The predictability requirement of vector ϕ in (1.3) under $\mathscr{M}$ is described as holding the inclusion $\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}')$. This requirement also corresponds to the estimability of vector $\mathbf{K}\beta$ under $\mathscr{M}$; see, e.g., [29]. For transformed model $\mathscr{T}$, the predictability requirement of vector ϕ is $\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}'\mathbf{T}')$. It's obvious that the predictability of ϕ under $\mathscr{T}$ shows predictability of ϕ under $\mathscr{M}$.
Let ϕ predictable under $\mathscr{M}$. If there exists $\mathbf{Ly}$ such that

$$\mathrm{D}(\mathbf{Ly}-\phi) = \min \text{ subject to } \mathrm{E}(\mathbf{Ly}-\phi) = \mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{Ly}$ is defined to be the BLUP of ϕ and is denoted by $\mathbf{Ly} = \mathrm{BLUP}_{\mathscr{M}}(\phi) = \mathrm{BLUP}_{\mathscr{M}}(\mathbf{K}\beta+\mathbf{Gu}+\mathbf{H}\varepsilon)$, is originated from [30]. If $\mathbf{G}=\mathbf{0}$ and $\mathbf{H}=\mathbf{0}$, $\mathbf{Ly}$ corresponds the BLUE of $\mathbf{K}\beta$, denoted by $\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta)$, under $\mathscr{M}$.
We have the following comprehensive result for the algebraic expressions of the BLUPs of ϕ and also properties of the BLUPs; as a detailed study for linear random effects models see [3].

Lemma 2.1. *Let $\mathscr{T}$ be as given in (1.2) and let ϕ in (1.3) be predictable under $\mathscr{T}$. In this case,*

$$\mathrm{E}(\mathbf{L}_t\mathbf{Ty}-\phi) = \mathbf{0} \;\; and \;\; \mathrm{D}(\mathbf{L}_t\mathbf{Ty}-\phi) = min \Leftrightarrow \mathbf{L}_t\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = \left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]. \tag{2.1}$$

The equation in (2.1) is called the fundamental BLUP *equation and*

$$\mathrm{BLUP}_{\mathscr{T}}(\phi) = \mathbf{L}_t\mathbf{Ty} = \left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{T} + \mathbf{U}_t\mathbf{W}_t^{\perp}\mathbf{T}\right)\mathbf{y}, \tag{2.2}$$

where $\mathbf{U}_t \in \mathbb{R}^{s\times m}$ is arbitrary and $\mathbf{W}_t = \left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right]$. In particular,

(a) $\mathbf{L}_t$ is unique $\Leftrightarrow r\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = m$.
(b) $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ is unique wp 1 $\Leftrightarrow \mathscr{T}$ is consistent.
(c) The rank of matrix $\mathbf{W}_t$ satisfies $r\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = r\left[\mathbf{TX}, \quad \mathbf{TRT}'\right]$.
(d) $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ satisfies

$$\mathrm{D}[\mathrm{BLUP}_{\mathscr{T}}(\phi)] = \left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TRT}'\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\right)',$$

$$\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{T}}(\phi)] = \left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)\Sigma\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)'. \tag{2.3}$$

Let ϕ in (1.3) be predictable under $\mathscr{M}$. By setting $\mathbf{T}=\mathbf{I}_n$ in Lemma 2.1, we obtain the following well-known results on BLUP of ϕ under $\mathscr{M}$. We may also refer [31] and for deriving the BLUPs under linear random-effects models see, [17].

$$\mathrm{BLUP}_{\mathscr{M}}(\phi) = \mathbf{Ly} = \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+} + \mathbf{UW}^{\perp}\right)\mathbf{y}, \tag{2.4}$$

$$\mathrm{D}[\mathrm{BLUP}_{\mathscr{M}}(\phi)] = \left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{R}\left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\right)',$$

$$\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] = \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{A}-\mathbf{B}\right)\Sigma\left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{A}-\mathbf{B}\right)', \tag{2.5}$$

where $\mathbf{U} \in \mathbb{R}^{s\times n}$ is arbitrary and $\mathbf{W} = \left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right]$. Further, we can write the following results.

(a) $\mathbf{L}$ in (2.4) is unique $\Leftrightarrow r\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = n$.
(b) $\mathrm{BLUP}_{\mathscr{M}}(\phi)$ is unique wp 1 $\Leftrightarrow \mathscr{M}$ is consistent.
(c) The rank of matrix $\mathbf{W}$ satisfies $r\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = r\left[\mathbf{X}, \quad \mathbf{R}\right]$.

3. Equality relations of BLUPs in LMMs

In this section, we establish equality relations between BLUPs of ϕ under $\mathscr{M}$ and $\mathscr{T}$ through their covariance matrices by using block matrices' rank formulas and elementary matrix operations. Related conclusions are also given for some special forms of ϕ. Equality relations between covariance matrices of BLUPs of ϕ under the models, which is obtained in the following results, correspond to the equality situations given in Section 1, respectively, by combining the following result:

$$\mathrm{E}[\mathrm{BLUP}_{\mathscr{M}}(\phi)] = \mathrm{E}[\mathrm{BLUP}_{\mathscr{T}}(\phi)] = \mathbf{K}\beta.$$

Theorem 3.1. *Let ϕ in (1.3) be predictable under $\mathscr{T}$ in (1.2) (also predictable under $\mathscr{M}$ in (1.1)). Let* $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\phi)$ *be as given in (2.2) and (2.4), respectively. Then,*

$$\mathrm{BLUP}_{\mathscr{M}}(\phi) = \mathrm{BLUP}_{\mathscr{T}}(\phi) \textit{ wp 1}$$
$$\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C} & -\mathbf{CT'} & \mathbf{K} & -\mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}) + r(\mathbf{TX}).$$

Proof. Note from (2.2) and (2.4) that

$$\begin{aligned} r(\mathrm{D}[\mathrm{BLUP}_{\mathscr{M}}(\phi) - \mathrm{BLUP}_{\mathscr{T}}(\phi)]) &= r\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}\mathbf{W}^{+}\mathbf{R} - \begin{bmatrix}\mathbf{K}, & \mathbf{CT'}(\mathbf{TX})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{TR}\right) \\ &= r\left(\left[\begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}, \quad \begin{bmatrix}\mathbf{K}, & \mathbf{CT'}(\mathbf{TX})^{\perp}\end{bmatrix}\right]\begin{bmatrix}\mathbf{W} & \mathbf{0} \\ \mathbf{0} & -\mathbf{W}_t\end{bmatrix}^{+}\begin{bmatrix}\mathbf{R} \\ \mathbf{TR}\end{bmatrix}\right), \end{aligned} \tag{3.1}$$

where $\mathbf{W}_t = \begin{bmatrix}\mathbf{TX}, & \mathbf{TRT'}(\mathbf{TX})^{\perp}\end{bmatrix}$ and $\mathbf{W} = \begin{bmatrix}\mathbf{X}, & \mathbf{RX}^{\perp}\end{bmatrix}$. We can apply (1.5) to (3.1) since $\mathscr{C}(\mathbf{TR}) = \mathscr{C}(\mathbf{TRT'}) \subseteq \mathscr{C}(\mathbf{W}_t)$, $\mathscr{C}(\mathbf{R}) \subseteq \mathscr{C}(\mathbf{W})$, $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CT'}(\mathbf{TX})^{\perp}\end{bmatrix}'\right) \subseteq \mathscr{C}(\mathbf{W}_t')$, and $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{CX}^{\perp}\end{bmatrix}'\right) \subseteq \mathscr{C}(\mathbf{W}')$ hold. Then, by simplifying Lemma 1.1, and congruence operations, (3.1) is equivalently written as

$$r\begin{bmatrix} \mathbf{X} & \mathbf{RX}^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} & -\mathbf{TX} & -\mathbf{TRT'}(\mathbf{TX})^{\perp} & \mathbf{TR} \\ \mathbf{K} & \mathbf{CX}^{\perp} & \mathbf{K} & \mathbf{CT'}(\mathbf{TX})^{\perp} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix}\mathbf{X}, & \mathbf{RX}^{\perp}\end{bmatrix} - r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT'}(\mathbf{TX})^{\perp}\end{bmatrix}$$

$$= r\begin{bmatrix} \mathbf{X} & \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} & -\mathbf{TX} & -\mathbf{TRT'} & \mathbf{TR} \\ \mathbf{K} & \mathbf{C} & \mathbf{K} & \mathbf{CT'} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X'T'} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} - r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT'}\end{bmatrix} - r(\mathbf{X}) - r(\mathbf{TX})$$

$$= r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C} & -\mathbf{CT'} & \mathbf{K} & -\mathbf{K} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} - r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} - r(\mathbf{X}) - r(\mathbf{TX}). \tag{3.2}$$

The required result is seen from (3.2) by using the matrix rank method. □

Corollary 3.2. *Let models $\mathscr{M}$ and $\mathscr{T}$ be as given in (1.1) and (1.2), respectively.*

(a) Assume that $\mathbf{K}\beta$ is estimable under $\mathscr{T}$ (also estimable under $\mathscr{M}$). Then

$$\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta) = \mathrm{BLUE}_{\mathscr{T}}(\mathbf{K}\beta) \textit{ wp 1}$$
$$\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K} & -\mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}) + r(\mathbf{TX}).$$

(b) $\mathbf{X}\beta$ is estimable under $\mathscr{T}$ $\Leftrightarrow r(\mathbf{TX}) = r(\mathbf{X})$ (also note that $\mathbf{X}\beta$ is always estimable under $\mathscr{M}$). Then

$$\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\beta) = \mathrm{BLUE}_{\mathscr{T}}(\mathbf{X}\beta) \textit{ wp 1} \Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{R} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{TX} & \mathbf{TR} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}).$$

Theorem 3.3. *Let ϕ in (1.3) be predictable under $\mathscr{T}$ in (1.2) (also predictable under $\mathscr{M}$ in (1.1)). Let* $\mathrm{BLUP}_{\mathscr{T}}(\phi)$ *and* $\mathrm{BLUP}_{\mathscr{M}}(\phi)$ *be as given in (2.2) and (2.4), respectively. Then*

$$\mathrm{D}[\phi - \mathrm{BLUP}_{\mathscr{T}}(\phi)] = \mathrm{D}[\phi - \mathrm{BLUP}_{\mathscr{M}}(\phi)]$$
$$\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{C'} \\ \mathbf{0} & \mathbf{TRT'} & \mathbf{0} & \mathbf{TX} & \mathbf{TC'} \\ \mathbf{X'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K'} \\ \mathbf{0} & \mathbf{X'T'} & \mathbf{0} & \mathbf{0} & \mathbf{K'} \\ -\mathbf{C} & \mathbf{CT'} & -\mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r(\mathbf{TX}) + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}).$$

Proof. By using (2.3) and (1.5), we obtain

$$
\begin{aligned}
& r(\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{T}}(\phi)]) \\
&= r\left(\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\left(\left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)\Sigma\left(\left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}-\mathbf{B}\right)'\right) \\
&= r\begin{bmatrix} \Sigma & \Sigma\left(\left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}\right)'-\Sigma\mathbf{B}' \\ \left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TA}\Sigma-\mathbf{B}\Sigma & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] \end{bmatrix}-r(\Sigma) \\
&= r\left(\begin{bmatrix} \Sigma & -\Sigma\mathbf{B}' \\ -\mathbf{B}\Sigma & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] \end{bmatrix}+\begin{bmatrix} \Sigma\mathbf{A}'\mathbf{T}' & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right] \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{W}_t \\ \mathbf{W}_t' & \mathbf{0} \end{bmatrix}^{+}\right. \\
&\qquad \left.\times\begin{bmatrix} \mathbf{TA}\Sigma & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]' \end{bmatrix}\right)-r(\Sigma),
\end{aligned}
\tag{3.3}
$$

where $\mathbf{W}_t=\left[\mathbf{TX},\quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right]$. We can apply (1.5) to (3.3) since

$$
\mathscr{C}(\mathbf{TA}\Sigma)=\mathscr{C}(\mathbf{TRT}')\subseteq\mathscr{C}(\mathbf{W}_t) \quad \textit{and} \quad \mathscr{C}\left(\left[\mathbf{K},\quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]'\right)\subseteq\mathscr{C}(\mathbf{W}_t').
$$

Then (3.3) is equivalently written as

$$
\begin{aligned}
& r\begin{bmatrix} \mathbf{0} & -\mathbf{TX} & -\mathbf{TRT}'(\mathbf{TX})^{\perp} & \mathbf{TA}\Sigma & \mathbf{0} \\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -(\mathbf{TX})^{\perp}\mathbf{TRT}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{TX})^{\perp}\mathbf{TC}' \\ \Sigma\mathbf{A}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \Sigma & -\Sigma\mathbf{B}' \\ \mathbf{0} & \mathbf{K} & \mathbf{CT}'(\mathbf{TX})^{\perp} & -\mathbf{B}\Sigma & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] \end{bmatrix}-r(\Sigma)-2r\left[\mathbf{TX},\quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] \\
&= r\begin{bmatrix} -\mathbf{TRT}' & -\mathbf{TX} & -\mathbf{TRT}'(\mathbf{TX})^{\perp} & \mathbf{TC}' \\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -(\mathbf{TX})^{\perp}\mathbf{TRT}' & \mathbf{0} & \mathbf{0} & (\mathbf{TX})^{\perp}\mathbf{TC}' \\ \mathbf{CT}' & \mathbf{K} & \mathbf{CT}'(\mathbf{TX})^{\perp} & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\mathbf{S} \end{bmatrix}-2r\left[\mathbf{TX},\quad \mathbf{TRT}'\right] \\
&= r\begin{bmatrix} -\mathbf{TRT}' & -\mathbf{TX} & \mathbf{TC}' \\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{K}' \\ \mathbf{CT}' & \mathbf{K} & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]-\mathbf{S} \end{bmatrix}-2r\left[\mathbf{TX},\quad \mathbf{TRT}'\right]+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}] \\
&= r\left(\begin{bmatrix} \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ \mathbf{CT}' & \mathbf{S} & \mathbf{K} \\ \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix}-\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\right)+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}]-2r\left[\mathbf{TX},\quad \mathbf{TRT}'\right].
\end{aligned}
\tag{3.4}
$$

We can apply (1.5) to (3.4) after setting the expression of $\mathrm{D}[\phi-\mathrm{BLUP}_{\mathscr{M}}(\phi)]$ given in (2.5). In this case, in a similar way to obtaining (3.3), (3.4) is equivalently written as

$$
\begin{aligned}
& r\left(\begin{bmatrix} \Sigma & \mathbf{0} & -\Sigma\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ -\mathbf{B}\Sigma & \mathbf{CT}' & \mathbf{S} & \mathbf{K} \\ \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix}+\begin{bmatrix} \Sigma\mathbf{A}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left[\mathbf{K},\quad \mathbf{CX}^{\perp}\right] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{W} \\ \mathbf{W}' & \mathbf{0} \end{bmatrix}^{+}\begin{bmatrix} \mathbf{A}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[\mathbf{K},\quad \mathbf{CX}^{\perp}\right]' & \mathbf{0} \end{bmatrix}\right) \\
&\quad -2r\left[\mathbf{TX},\quad \mathbf{TRT}'\right]+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}]-r(\Sigma),
\end{aligned}
\tag{3.5}
$$

where $\mathbf{W}=\left[\mathbf{X},\quad \mathbf{RX}^{\perp}\right]$. We can reapply (1.5) to (3.5) since $\mathscr{C}(\mathbf{A}\Sigma)=\mathscr{C}(\mathbf{R})\subseteq\mathscr{C}(\mathbf{W})$ and $\mathscr{C}\left(\left[\mathbf{K},\quad \mathbf{CX}^{\perp}\right]'\right)\subseteq\mathscr{C}(\mathbf{W}')$. Then from Lemma 1.1, and some congruence operations, (3.5) is equivalently written as

$$
\begin{aligned}
& r\begin{bmatrix} \mathbf{0} & -\mathbf{X} & -\mathbf{RX}^{\perp} & \mathbf{A}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0} \\ -\mathbf{X}^{\perp}\mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\mathbf{C}' & \mathbf{0} \\ \Sigma\mathbf{A}' & \mathbf{0} & \mathbf{0} & \Sigma & \mathbf{0} & -\Sigma\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ \mathbf{0} & \mathbf{K} & \mathbf{CX}^{\perp} & -\mathbf{B}\Sigma & \mathbf{CT}' & \mathbf{S} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix}+r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}]-2r\left[\mathbf{TX},\quad \mathbf{TRT}'\right] \\
&\quad -r(\Sigma)-2r\left[\mathbf{X},\quad \mathbf{RX}^{\perp}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= r\begin{bmatrix} -\mathbf{R} & -\mathbf{X} & -\mathbf{R}\mathbf{X}^{\perp} & \mathbf{0} & \mathbf{C}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0} \\ -\mathbf{X}^{\perp}\mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\mathbf{C}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ \mathbf{C} & \mathbf{K} & \mathbf{C}\mathbf{X}^{\perp} & \mathbf{CT}' & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix} + r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}] - 2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} \\
&\quad - 2r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= r\begin{bmatrix} -\mathbf{R} & -\mathbf{X} & \mathbf{0} & \mathbf{C}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{TRT}' & \mathbf{TC}' & \mathbf{TX} \\ \mathbf{C} & \mathbf{K} & \mathbf{CT}' & \mathbf{0} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{K}' & \mathbf{0} \end{bmatrix} + r[(\mathbf{TX})^{\perp}\mathbf{TRT}'(\mathbf{TX})^{\perp}] + r(\mathbf{X}^{\perp}\mathbf{R}\mathbf{X}^{\perp}) - 2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} \\
&\quad - 2r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{C}' \\ \mathbf{0} & \mathbf{TRT}' & \mathbf{0} & \mathbf{TX} & \mathbf{TC}' \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -\mathbf{C} & \mathbf{CT}' & -\mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix} + r\begin{bmatrix} \mathbf{TRT}' & \mathbf{TX} \\ \mathbf{X}'\mathbf{T}' & \mathbf{0} \end{bmatrix} - 2r(\mathbf{TX}) + r\begin{bmatrix} \mathbf{R} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - 2r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} \\
&\quad - 2r\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} - 2r(\mathbf{X}).
\end{aligned} \tag{3.6}
$$

The required result is seen from (3.6) by using (1.4) and the matrix rank method. □

Corollary 3.4. *Let models $\mathscr{M}$ and $\mathscr{T}$ be as given in* (1.1) *and* (1.2)*, respectively.*

(a) *Assume that $\mathbf{K}\beta$ is estimable under $\mathscr{T}$ (also estimable under $\mathscr{M}$). Then*

$$
\begin{aligned}
&\mathrm{D}[\mathrm{BLUE}_{\mathscr{T}}(\mathbf{K}\beta)] = \mathrm{D}[\mathrm{BLUE}_{\mathscr{M}}(\mathbf{K}\beta)] \\
&\Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{TRT}' & \mathbf{0} & \mathbf{TX} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{0} & -\mathbf{K} & \mathbf{K} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r(\mathbf{TX}) + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix} + r(\mathbf{X}).
\end{aligned}
$$

(b) *$\mathbf{X}\beta$ is estimable under $\mathscr{T}$ $\Leftrightarrow r(\mathbf{TX}) = r(\mathbf{X})$ (also note that $\mathbf{X}\beta$ is always estimable under $\mathscr{M}$). Then*

$$
\mathrm{D}[\mathrm{BLUE}_{\mathscr{T}}(\mathbf{X}\beta)] = \mathrm{D}[\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}\beta)] \Leftrightarrow r\begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & \mathbf{TRT}' & \mathbf{TX} \\ \mathbf{X}' & -\mathbf{X}'\mathbf{T}' & \mathbf{0} \end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix} + r\begin{bmatrix}\mathbf{TX}, & \mathbf{TR}\end{bmatrix}.
$$

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Some Approximation Results on $\lambda-$Szász-Mirakjan-Kantorovich Operators

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Abstract

In this article, we purpose to obtain several approximation properties of Szász-Mirakjan-Kantorovich operators with shape parameter $\lambda \in [-1,1]$. We compute some preliminaries results such as moments and central moments for these operators. Next, we derive the Korovkin type convergence theorem, estimate the degree of convergence with respect to the moduli of continuity, for the functions belong to Lipschitz-type class and Peetre's K-functional, respectively. Further, we investigate Voronovskaya type asymptotic theorem and give the comparison of the convergence of these newly defined operators to the certain functions with some graphics.

1. Introduction

In [1, 2], Szász and Mirakjan defined and introduced the following polynomials

$$S_m(\mu;y) = \sum_{j=0}^{\infty} \mu\left(\frac{j}{m}\right) s_{m,j}(y), \tag{1.1}$$

where $y \geq 0$, $m \in \mathbb{N}$, $\mu \in C[0,\infty)$ and Szász-Mirakjan basis functions $s_{m,j}(y)$ are given as below:

$$s_{m,j}(y) = e^{-my}\frac{(my)^j}{j!}.$$

A Kantorovich variant of (1.1) operators is presented by Ditzian and Totik [3] as follows:

$$K_m(\mu;y) = m\sum_{j=0}^{\infty} s_{m,j}(y) \int_{\frac{j}{m}}^{\frac{j+1}{m}} \mu(t)\,dt,\ y \geq 0. \tag{1.2}$$

Various approximation features of (1.1) and (1.2) operators have been introduced by many authors. More details on these directions, we refer the readers to [4]-[12].
Very recently, Qi et al. [13] defined a new generalization of $\lambda-$Szász-Mirakjan operators with shape parameter $\lambda \in [-1,1]$, as below:

$$S_{m,\lambda}(\mu;y) = \sum_{j=0}^{\infty} \mu\left(\frac{j}{m}\right) \widetilde{s}_{m,j}(\lambda;y),$$

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where Szász-Mirakjan bases functions $\widetilde{s}_{m,j}(\lambda;y)$ with shape parameter $\lambda \in [-1,1]$:

$$\widetilde{s}_{m,0}(\lambda;y) = s_{m,0}(y) - \frac{\lambda}{m+1} s_{m+1,1}(y);$$

$$\widetilde{s}_{m,i}(\lambda;y) = s_{m,i}(y) + \lambda \left(\frac{m-2i+1}{m^2-1} s_{m+1,i}(y) \;-\frac{m-2i-1}{m^2-1} s_{m+1,i+1}(y) \right), \; (i = 1,2,...,\infty, \; y \in [0,\infty)). \tag{1.3}$$

They studied several theorems such as Korovkin approximation, local approximation, Lipschitz type convergence, Voronovskaja and Grüss-Voronovskaja type for these new form operators. In the literature, recently several researchers have obtained some approximation results for various linear positive operators with shape parameter $\lambda \in [-1,1]$, one can refer to [14]-[23].
Now, motivated by all above mentioned works, we propose the Kantorovich kind of $\lambda-$Szász-Mirakjan operators as follows:

$$R_{m,\lambda}(\mu;y) = m \sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;y) \int_{\frac{j}{m}}^{\frac{j+1}{m}} \mu(t)\,dt, \quad y \in [0,\infty), \tag{1.4}$$

where $\widetilde{s}_{m,j}(\lambda;y)$ $(j = 0,1,..\infty)$ given in (1.3) and $\lambda \in [-1,1]$.
The structure of this work is organized as follows: In section 2, we compute some moments and central moments. In section 3, we establish Korovkin type approximation theorem and discuss the order of convergence in terms of the usual moduli of continuity, for the function belongs to Lipschitz-type class and Peetre's K-functional, respectively. In section 4, we derive a Voronovskaya type asymptotic theorem. In the final section, we show the comparison of the convergence of operators (1.4) to the certain functions for the different values of m and λ. We also compare the convergence of operators (1.2) and (1.4) to the certain function to see the behaviour of λ parameter.

2. Preliminaries

Lemma 2.1. *[13]. For the $\lambda-$Szász-Mirakjan operators $S_{m,\lambda}(\mu;y)$ following expressions are satisfied:*

$$S_{m,\lambda}(1;y) = 1;$$

$$S_{m,\lambda}(t;y) = y + \left[\frac{1 - e^{-(m+1)y} - 2y}{m(m-1)} \right] \lambda;$$

$$S_{m,\lambda}(t^2;y) = y^2 + \frac{y}{m} + \left[\frac{2y + e^{-(m+1)y} - 1 - 4(m+1)y^2}{m^2(m-1)} \right] \lambda;$$

$$S_{m,\lambda}(t^3;y) = y^3 + \frac{3y^2}{m} + \frac{y}{m^2} + \left[\frac{1 - e^{-(m+1)y} - 2y + 3(m-3)(m+1)y^2 - 6(m+1)y^3}{m^3(m-1)} \right] \lambda;$$

$$S_{m,\lambda}(t^4;y) = y^4 + \frac{6y^3}{m} + \frac{7y^2}{m^2} + \frac{y}{m^3} + \left[\frac{e^{-(m+1)y} - 1 + 2my + 2(3m-11)(m+1)y^2 + 4(m-8)(m+1)^2y^3 - 8(m+1)^3y^4}{m^4(m-1)} \right] \lambda.$$

Lemma 2.2. *Let the operators $R_{m,\lambda}$ be defined by (1.4). Then, we have*

$$R_{m,\lambda}(1;y) = 1; \tag{2.1}$$

$$R_{m,\lambda}(t;y) = y + \frac{1}{2m} + \left[\frac{1 - e^{-(m+1)y} - 2y}{m(m-1)} \right] \lambda; \tag{2.2}$$

$$R_{m,\lambda}(t^2;y) = y^2 + \frac{2y}{m} + \frac{1}{3m^2} + \left[\frac{-4(m+1)y^2}{m^2(m-1)} \right] \lambda; \tag{2.3}$$

$$R_{m,\lambda}(t^3;y) = y^3 + \frac{9y^2}{2m} + \frac{7y}{2m^2} + \frac{1}{4m^3} + \left[\frac{3(m-5)(m+1)y^2 - y - 6(m+1)y^3 + \frac{1}{2} - \frac{1}{2}e^{-(m+1)y}}{m^3(m-1)} \right] \lambda; \tag{2.4}$$

$$R_{m,\lambda}(t^4;y) = y^4 + \frac{8y^3}{m} + \frac{15y^2}{m^2} + \frac{6y}{m^3} + \frac{1}{5m^4} + \left[\frac{2(m-1)y + 12(m-4)(m+1)y^2 + 4(m-11)(m+1)^2y^3 - 8(m+1)^3y^4}{m^4(m-1)} \right] \lambda. \tag{2.5}$$

Proof. Taking Özger et al. [24] in to account and using (1.4), it is easy to see $\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;y)=1$, hence we get (2.1). Now, with the help of Lemma 2.1, we will compute expressions (2.2) and (2.3).

$$\begin{aligned}
R_{m,\lambda}(t;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}} t\,dt = \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{2j+1}{2m}\\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j}{m}+\frac{1}{2m}\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\\
&= S_{m,\lambda}(t;y)+\frac{1}{2m}=y+\frac{1}{2m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda,
\end{aligned}$$

$$\begin{aligned}
R_{m,\lambda}(t^2;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}} t^2dt = \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{3j^2+3j+1}{3m^2}\\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j^2}{m^2}+\frac{1}{m}\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j}{m}+\frac{1}{3m^2}\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\\
&= S_{m,\lambda}(t^2;y)+\frac{1}{m}S_{m,\lambda}(t;y)+\frac{1}{3m^2}=y^2+\frac{2y}{m}+\frac{1}{3m^2}+\left[\frac{-4(m+1)y^2}{m^2(m-1)}\right]\lambda.
\end{aligned}$$

Analogously, taking into consideration Lemma 2.1, hence we can arrive expressions (2.4) and (2.5) by simple computation, thus we omitted details. □

Corollary 2.3. *Let $y\in[0,\infty)$, $m>1$ and $\lambda\in[-1,1]$. As a consequence of Lemma 2.2, we obtain the following relations:*

$$\begin{aligned}
(i)\ R_{m,\lambda}(t-y;y) &= \frac{1}{2m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda\\
&\leq \frac{m+1+2e^{-(m+1)y}+4y}{2m(m-1)} := \beta_m(y);
\end{aligned}$$

$$\begin{aligned}
(ii)\ R_{m,\lambda}((t-y)^2;y) &= \frac{y}{m}+\frac{1}{3m^2}+\left[\frac{2(e^{-(m+1)y}-1)y}{m(m-1)}-\frac{4y^2}{m^2(m-1)}\right]\lambda\\
&\leq \frac{y}{m}+\frac{1}{3m^2}+\frac{2(e^{-(m+1)y}+1)y}{m(m-1)}+\frac{4y^2}{m^2(m-1)} := \gamma_m(y);
\end{aligned}$$

$$\begin{aligned}
(iii)\ R_{m,\lambda}((t-y)^4;y) = \frac{3y^2}{m^2}+\frac{5y}{m^3}+\frac{1}{5m^4}+\Bigg(&\frac{2(me^{-(m+1)y}-1)y}{m^4(m-1)}+\frac{4(3m^2-8m-12)y^2}{m^4(m-1)}\\
&-\frac{4(3m^3+3m^2-6m-11)+4m^3e^{-(m+1)y})y^3}{m^4(m-1)}-\frac{8y^4}{m^4(m-1)}\Bigg)\lambda.
\end{aligned}$$

Lemma 2.4. *Let $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, the following expressions holds true:*

$$\begin{aligned}
(i)\ \lim_{m\to\infty} mR_{m,\lambda}(t-y;y) &= \frac{1}{2};\\
(ii)\ \lim_{m\to\infty} mR_{m,\lambda}((t-y)^2;y) &= y;\\
(iii)\ \lim_{m\to\infty} m^2R_{m,\lambda}((t-y)^4;y) &= 3y^2.
\end{aligned}$$

3. Direct theorems of $R_{m,\lambda}$

In the next theorem, we introduce a Korovkin type approximation theorem. As it is known, the space $C[0,\infty)$ denotes the all continuous and bounded functions on $[0,\infty)$ and it is equipped with the sup-norm for a function μ as follows:

$$\|\mu\|_{[0,\infty)}=\sup_{y\in[0,\infty)}|\mu(y)|.$$

Theorem 3.1. *Let $\mu \in C[0,\infty)$, then $R_{m,\lambda}(\mu;y)$ converge uniformly to μ on $[0,\infty)$.*

Proof. According to the Bohman-Korovkin theorem [25], it is sufficient to verify

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|R_{m,\lambda}(t^s;y)-y^s\right|=0,\ \text{for } s=0,1,2.$$

Using (2.1), for $s=0$, it can be seen that above expression is clear.
For $s=1$, in view of (2.2), we have

$$\begin{aligned}\lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|R_{m,\lambda}(t;y)-y\right| &= \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|\frac{1}{2m}+\left(\frac{1-e^{-(m+1)y}}{m(m-1)}-\frac{2y}{m(m-1)}\right)\lambda\right| \\ &\leq \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left(\frac{m+1+2e^{-(m+1)y}+4y}{2m(m-1)}\right)=0.\end{aligned}$$

Similarly, by (2.3), one has

$$\begin{aligned}\lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|R_{m,\lambda}(t^2;y)-y^2\right| &= \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|\frac{2y}{m}+\frac{1}{3m^2}+\left(\frac{-4(m+1)}{m^2(m-1)}y^2\right)\lambda\right| \\ &\leq \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left(\frac{2y}{m}+\frac{1}{3m^2}+\frac{4(m+1)}{m^2(m-1)}y^2\right)=0.\end{aligned}$$

Hence, we get the required sequel. □

Further, we discuss the order of convergence in connection with the usual moduli of continuity, for the function belong to Lipschitz type continuous and Peetre's K-functional. The Peetre's K-functional is defined by

$$K_2(\mu,\eta)=\inf_{\nu\in C^2[0,\infty)}\left\{\|\mu-\nu\|+\eta\left\|\nu''\right\|\right\},$$

where $\eta>0$ and $C^2[0,\infty)=\{\nu\in C[0,\infty):\nu',\nu''\in C[0,\infty)\}$.
Taking into account [26], there exist an absolute constant $C>0$ such that

$$K_2(\mu;\eta)\leq C\omega_2(\mu;\sqrt{\eta}),\qquad \eta>0, \tag{3.1}$$

where

$$\omega_2(\mu;\eta)=\sup_{0<\alpha\leq\eta}\ \sup_{y\in[0,\infty)}|\mu(y+2\alpha)-2\mu(y+\alpha)+\mu(y)|,$$

is the second order modulus of smoothness of the function $\mu\in C[0,\infty)$. Further, by

$$\omega(\mu;\eta):=\sup_{0<\alpha\leq\eta}\ \sup_{y\in[0,\infty)}|\mu(y+\alpha)-\mu(y)|,$$

we denote the usual moduli of continuity of $\mu\in C[0,\infty)$. Since $\eta>0$, $\omega(\mu;\eta)$ has some useful properties see details: [27]. Also, we give an element of Lipschitz continuous function with $Lip_L(\zeta)$, where $L>0$ and $0<\zeta\leq 1$. If the expression below:

$$|\mu(t)-\mu(y)|\leq L|t-y|^{\zeta},\qquad (t,y\in\mathbb{R}),$$

holds, then one can say a function μ is belong to $Lip_L(\zeta)$.

Theorem 3.2. *Let $\mu\in C[0,\infty)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, we have following inequality verify*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right|\leq 2\omega(\mu;\sqrt{\gamma_m(y)}),$$

where $\gamma_m(y)$ given as in Corollary 2.3.

Proof. Using the well-known property of moduli of continuity $|\mu(t)-\mu(y)|\leq\left(1+\frac{|t-y|}{\delta}\right)\omega(\mu;\delta)$ and after operating $R_{m,\lambda}(.;y)$, it becomes

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right|\leq\left(1+\frac{1}{\delta}R_{m,\lambda}(|t-y|;y)\right)\omega(\mu;\delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality and from Corollary 2.3, we get

$$\begin{aligned}\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq \left(1+\frac{1}{\delta}\sqrt{R_{m,\lambda}((t-y)^2;y)}\right)\omega(\mu;\delta) \\ &\leq \left(1+\frac{1}{\delta}\sqrt{\gamma_m(y)}\right)\omega(\mu;\delta).\end{aligned}$$

Taking $\delta=\sqrt{\gamma_m(y)}$, hence we obtain the proof of Theorem 3.2. □

Theorem 3.3. *Let $\mu \in Lip_L(\zeta)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$. Then, we obtain*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq L(\gamma_m(y))^{\frac{\zeta}{2}}.$$

Proof. By the linearity and monotonicity of the operators (1.4), it follows

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq R_{m,\lambda}(|\mu(t)-\mu(y)|;y) \leq m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}|\mu(t)-\mu(y)|\,dt \leq Lm\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}|t-y|^{\zeta}\,dt.$$

Utilizing the Hölder's inequality with $p_1=\frac{2}{\zeta}$ and $p_2=\frac{2}{2-\zeta}$ and in view of Corollary 2.3 and Lemma 2.2, we arrive

$$\begin{aligned}
\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq L\left\{m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}(t-y)^2\,dt\right\}^{\frac{\zeta}{2}}\left\{\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\right\}^{\frac{2-\zeta}{2}}\\
&= L\left\{R_{m,\lambda}((t-y)^2;y)\right\}^{\frac{\zeta}{2}}\left\{R_{m,\lambda}(1;y)\right\}^{\frac{2-\zeta}{2}} \leq L(\gamma_m(y))^{\frac{\zeta}{2}}.
\end{aligned}$$

Thus, we get the proof of this theorem. □

Theorem 3.4. *For all $\mu \in C[0,\infty)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$, the following inequality holds:*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq C\omega_2(\mu;\frac{1}{2}\sqrt{\gamma_m(y)+(\beta_m(y))^2}+\omega(\mu;\beta_m(y)),$$

where $C>0$ is a constant, $\beta_m(y)$, $\gamma_m(y)$ defined as in Corollary 2.3.

Proof. Let $\mu \in C[0,\infty)$. We denote $\alpha_{m,\lambda}(y) := y+\frac{1}{2m}+\left[\frac{1-2y-e^{-(m+1)y}}{m(m-1)}\right]\lambda$, it is obvious that $\alpha_{m,\lambda}(y) \in [0,\infty)$ for sufficently large m. We define the following auxiliary operators:

$$\widehat{R}_{m,\lambda}(\mu;y)=R_{m,\lambda}(\mu;y)-\mu(\alpha_{m,\lambda}(y))+\mu(y). \tag{3.2}$$

In view of (2.1) and (2.2), it follows that

$$\widehat{R}_{m,\lambda}(t-y;y)=0.$$

By Taylor's formula, one has

$$\xi(t)=\xi(y)+(t-y)\xi'(y)+\int_y^t(t-u)\xi''(u)du, \quad (\xi \in C^2[0,\infty)). \tag{3.3}$$

After operating $\widehat{R}_{m,\lambda}(.;y)$ to (3.3), yields

$$\begin{aligned}
\widehat{R}_{m,\lambda}(\xi;y)-\xi(y) &= \widehat{R}_{m,\lambda}((t-y)\xi'(y);y)+\widehat{R}_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)\\
&= \xi'(y)\widehat{R}_{m,\lambda}(t-y;y)+R_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)-\int_y^{\alpha_{m,\lambda}(y)}(\alpha_{m,\lambda}(y)-u)\xi''(u)du\\
&= R_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)-\int_y^{\alpha_{m,\lambda}(y)}(\alpha_{m,\lambda}(y)-u)\xi''(u)du.
\end{aligned}$$

Taking Lemma 2.2 and (3.2) into the account, we get

$$\begin{aligned}
\left|\widehat{R}_{m,\lambda}(\xi;y)-\xi(y)\right| &\leq \left|R_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)\right|+\left|\int_y^{\alpha_{m,\lambda}(y)}(\alpha_{m,\lambda}(y)-u)\xi''(u)du\right|\\
&\leq R_{m,\lambda}(\left|\int_y^t(t-u)\right||\xi''(u)||du|;y)+\int_y^{\alpha_{m,\lambda}(y)}|\alpha_{m,\lambda}(y)-u|\,|\xi''(u)|\,|du|\\
&\leq \|\xi''\|\left\{R_{m,\lambda}((t-y)^2;y)+\left(\alpha_{m,\lambda}(y)-y\right)^2\right\} \leq \left\{\gamma_m(y)+(\beta_m(y))^2\right\}\|\xi''\|.
\end{aligned}$$

Also from (2.1), (2.2) and (3.2), it deduce the following

$$\left|\widehat{R}_{m,\lambda}(\mu;y)\right| \leq \left|R_{m,\lambda}(\mu;y)\right| + 2\|\mu\| \leq \|\mu\| R_{m,\lambda}(1;y) + 2\|\mu\| \leq 3\|\mu\|. \tag{3.4}$$

On the other hand, by (3.3) and (3.4) imply

$$\begin{aligned} \left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq \left|\widehat{R}_{m,\lambda}(\mu-\xi;y)-(\mu-\xi)(y)\right| + \left|\widehat{R}_{m,\lambda}(\xi;y)-\xi(y)\right| + \left|\mu(y)-\mu(\alpha_{m,\lambda}(y))\right| \\ &\leq 4\|\mu-\xi\| + \left\{\gamma_m(y)+(\beta_m(y))^2\right\}\|\xi''\| + \omega(\mu;\beta_m(y)). \end{aligned}$$

On account of this, if we take the infimum on the right hand side over all $\xi \in C^2[0,\infty)$ and by (3.1), we arrive

$$\begin{aligned} \left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq 4K_2\Big(\mu;\frac{\left\{\gamma_m(y)+(\beta_m(y))^2\right\}}{4}\Big) + \omega(\mu;\beta_{m,\lambda}(y)) \\ &\leq C\omega_2\Big(\mu;\frac{1}{2}\sqrt{\gamma_m(y)+(\beta_m(y))^2}\Big) + \omega(\mu;\beta_m(y)). \end{aligned}$$

Hence, we obtain the proof of this theorem. □

Theorem 3.5. *If $\mu \in C^1[0,\infty) := \{\mu : \mu'$ is continuous and bounded on $[0,\infty)\}$, then for all $y \in [0,\infty)$ and $\lambda \in [-1,1]$, we arrive*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq \beta_m(y)\left|\mu'(y)\right| + 2\sqrt{\gamma_m(y)}\,\omega(\mu';\sqrt{\gamma_m(y)}),$$

where $\beta_m(y)$, $\gamma_m(y)$ defined as in Corollary 2.3.

Proof. Let $\mu \in C^1[0,\infty)$. For any $y,t \in [0,\infty)$, we get

$$\mu(t)-\mu(y) = \mu'(y)(t-y) + \int_y^t (\mu'(u)-\mu'(y))du.$$

After operating $R_{m,\lambda}(.;y)$ to the both sides of above expression, it gives

$$R_{m,\lambda}(\mu(t)-\mu(y);y) = \mu'(y)R_{m,\lambda}(t-y;y) + R_{m,\lambda}\Big(\int_y^t (\mu'(u)-\mu'(y))du;y\Big).$$

Taking into consideration the following well-known property

$$|\mu(u)-\mu(y)| \leq \left(1+\frac{|u-y|}{\delta}\right)\omega(\mu;\delta), \quad \delta>0,$$

then

$$\left|\int_y^t \left|\mu'(u)-\mu'(y)\right|du\right| \leq \left(\frac{(t-y)^2}{\delta} + |t-y|\right)\omega(\mu';\delta).$$

Hence,

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq \left|R_{m,\lambda}(t-y;y)\right|\left|\mu'(y)\right| + \left[\frac{R_{m,\lambda}((t-y)^2;y)}{\delta} + R_{m,\lambda}(|t-y|;y)\right]\omega(\mu';\delta).$$

Applying Cauchy-Bunyakovsky-Schwarz inequality on the right hand side of foregoing inequality and taking into consideration Corollary 2.3, we find

$$\begin{aligned} \left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq \left|R_{m,\lambda}(t-y;y)\right|\left|\mu'(y)\right| + \omega(\mu';\delta)\left[\frac{\sqrt{R_{m,\lambda}((t-y)^2;y)}}{\delta}+1\right]\sqrt{R_{m,\lambda}((t-y)^2;y)} \\ &\leq \beta_m(y)\left|\mu'(y)\right| + \omega(\mu';\delta)\left[\frac{\sqrt{\gamma_m(y)}}{\delta}+1\right]\sqrt{\gamma_m(y)}. \end{aligned}$$

By taking $\delta = \sqrt{\gamma_m(y)}$, the required result is obtained. □

4. Voronovskaya type asymptotic theorem

Theorem 4.1. *Let $\mu \in C[0,\infty)$ such that $\mu', \mu'' \in C[0,\infty)$ and $\lambda \in [-1,1]$, then we have for any $y \in [0,\infty)$ that*

$$\lim_{m\to\infty} m\left[R_{m,\lambda}(\mu;y)-\mu(y)\right] = \frac{\mu'(y)+y\mu''(y)}{2}.$$

Proof. Suppose that $y \in [0,\infty)$ and $\mu', \mu'' \in C[0,\infty)$. From Taylor's formula, one has

$$\mu(t) = \mu(y)+(t-y)\mu'(y)+\frac{1}{2}(t-y)^2\mu''(y)+(t-y)^2\phi(t;y). \tag{4.1}$$

In (4.1), $\phi(t;y)$ is a Peano of the remainder term and by the fact that $\phi(.;y) \in C[0,\infty)$, we have $\lim_{t\to y}\phi(t;y) = 0$.
After operating $R_{m,\lambda}(.;y)$ to (4.1), hence

$$R_{m,\lambda}(\mu;y)-\mu(y) = R_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}R_{m,\lambda}((t-y)^2;y)\mu''(y)+R_{m,\lambda}((t-y)^2\phi(t;y);y).$$

If we take the limit of the both sides of above relation as $m\to\infty$, then

$$\lim_{m\to\infty} m(R_{m,\lambda}(\mu;y)-\mu(y) = \lim_{m\to\infty} m\left(R_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}R_{m,\lambda}((t-y)^2;y)\mu''(y)+R_{m,\lambda}((t-y)^2\phi(t;y);y)\right). \tag{4.2}$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality to the last term on the right hand side of the above expression, it becomes

$$\lim_{m\to\infty} mR_{m,\lambda}((t-y)^2\phi(t;y);y) \leq \sqrt{\lim_{m\to\infty} R_{m,\lambda}(\phi^2(t;y);y)}\sqrt{\lim_{m\to\infty} m^2R_{m,\lambda}((t-y)^4;y)}.$$

It is observed that as $\phi(t;y) \in C[0,\infty)$, thus by Theorem 3.1, $\lim_{t\to y}\phi(t;y) = 0$. It follows that

$$\lim_{m\to\infty} R_{m,\lambda}(\phi^2(t;y);y) = \phi^2(y;y) = 0. \tag{4.3}$$

If we combine (4.2)-(4.3) and in view of Lemma 2.4 (iii), we arrive

$$\lim_{m\to\infty} mR_{m,\lambda}((t-y)^2\phi(t;y);y) = 0.$$

Thus, we obtain the desired sequel as follows:

$$\lim_{m\to\infty} m\left[R_{m,\lambda}(\mu;y)-\mu(y)\right] = \frac{\mu'(y)+y\mu''(y)}{2}.$$

□

5. Graphical analysis

In this section, we give some graphics to see the convergence of operators (1.4) to the certain functions. Also, we compare the convergence of our newly defined operators (1.4) with the operators (1.2) with the different values of m and λ.
In Figure 5.1, for $\lambda = 0.5$ and $m = 10, 40, 70$ respectively, we demonstrate the convergence of operators (1.4) to $\mu(y) = e^y$. In Figure 5.2, for $\lambda = 0.9$ and $m = 10, 40, 70$ respectively, we show the convergence of operators (1.4) to $\mu(y) = cos(\pi y)$. In Figure 5.3, we denote with LKMS:= λ-Szász-Mirakjan-Kantorovich operators defined by (1.4) and KMS:= Szász-Mirakjan-Kantorovich operators defined by (1.2). We compare the convergence of operators (1.4) with (1.2) for $\lambda = 0.5$, $m = 10$ to $\mu(y) = e^y$. We can conclude from Figure 5.1 and Figure 5.2 that, as the values of m increases than the convergence of operators (1.4) to the functions becomes better. Moreover, in Figure 5.3 it can be seen that for $\lambda = 0.5$ and $m = 10$ operators (1.4) have better approximation than operators (1.2).

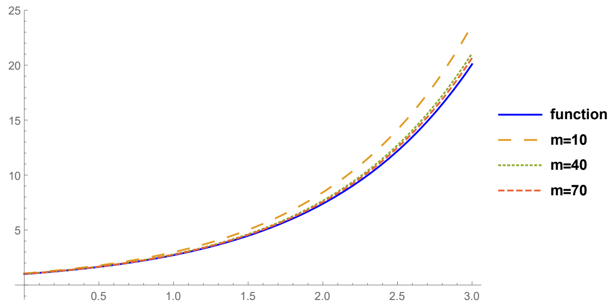

Figure 5.1: The convergence of $R_{m,\lambda}(\mu;y)$ to $\mu(y) = e^y$ for $\lambda = 0.5$ and $m = 10, 40, 70$.

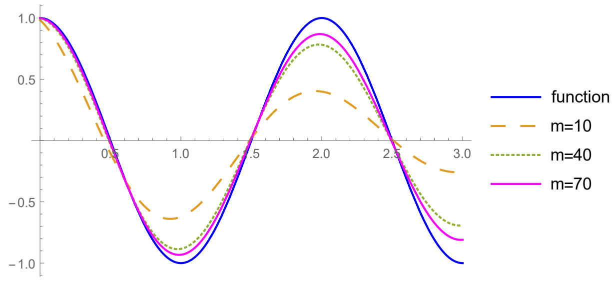

Figure 5.2: The convergence of $R_{m,\lambda}(\mu;y)$ to $\mu(y) = cos(\pi y)$ for $\lambda = 0.9$ and $m = 10, 40, 70$.

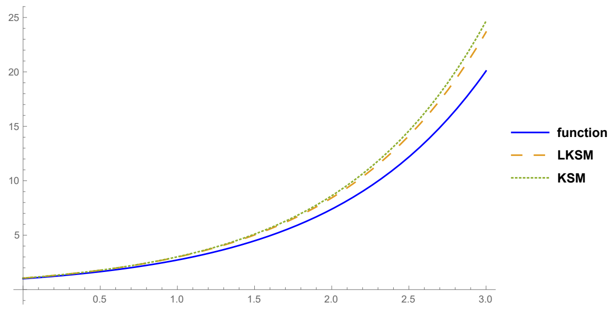

Figure 5.3: The convergence of $R_{m,\lambda}(\mu;y)$ and $K_m(\mu;y)$ to $\mu(y) = e^y$ for $\lambda = 0.5$ and $m = 10$.

6. Conclusion

In the present paper, we introduced Szász-Mirakjan-Kantorovich operators based on shape parameter $\lambda \in [-1,1]$. The importance of parameter λ, give us more flexibility in modeling. We derived a Korovkin type convergence theorem, estimated the degree of convergence in terms of the moduli of continuity, for the functions belong to Lipschitz class and Peetre's K-functional, respectively. We also discussed Voronovskaya type asymptotic theorem. Moreover, we gave the comparison of the convergence of our newly constructed operators (1.4) to the certain functions with some graphics and also we compared the convergence of (1.4) between (1.2). As future works, we will consider the Stancu, Durrmeyer and Baskakov type λ-Szász-Mirakjan operators.

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Author's contributions

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Some Fixed Point Theorems on b-θ-metric spaces via b-simulation Functions

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Abstract

We introduce the concept of b-θ-metric space as a generalization of θ-metric space and investigate some of its properties. Then, we establish a fixed point theorem in b-θ-metric spaces via b-simulation functions. Thus, we deduce Banach type fixed point in such spaces. Also, we discuss some fixed point results in relation to existing ones.

1. Introduction

Fixed point theory plays a fundamental role in various fields of mathematics, engineering and applied science. A basic result in fixed point theory is the Banach contraction principle which is an important tool for solving nonlinear analysis' problems. This result has been generalized and extended in various generalized metric spaces.

Many authors have generalized metric spaces in several ways. Bakhtin [1] introduced the concept of b-metric space, which is a generalized form of metric space (see also [2]). Since then, several authors have many fixed point results for single- valued and multi- valued operators in b- metric spaces (see [2]-[4]).

Khojasteh et al. [5] introduced θ-metric space by using a more generalized inequality instead of triangle inequality. They are inspired by fuzzy metric spaces, which are generalizations of metric spaces. Then they proved Banach and Caristi type fixed point in θ-metric spaces.

Khojasteh et al. [6] introduced $\mathscr{Z}$-contraction as a new type of nonlinear contractions via simulation function which is useful to express a family of contractivity conditions. After then Chanda and Dey [7] obtained some fixed point results on θ-metric spaces by using simulation functions. Also, Demma et al.[8] deduced several related results in fixed point theory in b-metric space via b-simulation functions.

In this paper, we defined b-θ-metric space as a generalization of b-metric space with the help of $\mathscr{B}$-action and studied its fundamental properties. Also, we compare it to both b-metric and θ-metric space. Then we obtain a fixed point result in b-θ-metric spaces by using b-simulation functions. So we get the Banach contraction principle in such spaces. Finally, we give some fixed point results regarding existing ones in b-metric spaces and θ-metric spaces.

2. Preliminaries

Definition 2.1. *[1, 2] Let W be a nonempty set and $b \geq 1$ be a given real number. A function $d : W \times W \to [0,\infty)$ is a b-metric on W iff it satisfies the following conditions for all $\omega, \varpi, \rho \in W$.*

$(b1)$ $d(\omega,\varpi) = 0$ *iff* $\omega = \varpi$.

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$(b2)$ $d(\omega,\varpi)=d(\varpi,\omega)$.

$(b3)$ $d(\omega,\varpi)\leq b[d(\omega,\rho)+d(\rho,\varpi)]$.
Then, the pair (W,d) *is called a b-metric space.*

Definition 2.2. *[5] Let* $\theta:[0,\infty)\times[0,\infty)\to[0,\infty)$ *be a continuous mapping with respect to each variable.* θ *is called an* $\mathscr{B}$*-action iff it satisfies the following conditions:*

$(B1)$ $\theta(0,0)=0$ *and* $\theta(\omega,\varpi)=\theta(\varpi,\omega)$ *for all* $\omega,\varpi\geq 0$,

$(B2)$ $\theta(\omega,\varpi)<\theta(\rho,\nu)$ *if* $\omega<\rho$ *and* $\varpi\leq\nu$ *or* $\omega\leq\rho$ *and* $\varpi<\nu$.

$(B3)$ *For each* $r\in Im(\theta)-\{0\}$ *and for each* $\omega\in(0,r]$, *there exists* $\varpi\in(0,r]$ *such that* $\theta(\omega,\varpi)=r$, *where* $Im(\theta)=\{\theta(\omega,\varpi):\omega>0,\varpi\geq 0\}$.

$(B4)$ $\theta(\omega,0)\leq\omega$ *for all* $\omega>0$.
The set of all $\mathscr{B}$*-actions is denoted by* $\mathscr{M}$.

Definition 2.3. *[5] Let* W *be a nonempty set. A mapping* $d_\theta:W\times W\to[0,\infty)$ *is called a* θ*-metric on* W *with respect to* $\mathscr{B}$*-action* $\theta\in\mathscr{M}$ *if* d_θ *satisfies the following conditions:*

$(\theta1)$ $d_\theta(\omega,\varpi)=0$ *iff* $\omega=\varpi$,

$(\theta2)$ $d_\theta(\omega,\varpi)=d_\theta(\varpi,\omega)$,

$(\theta3)$ $d_\theta(\omega,\varpi)\leq\theta(d_\theta(\omega,\rho),d_\theta(\rho,\varpi))$ *for all* $\omega,\varpi,\rho\in W$.
Then, the pair (W,d_θ) *is called a* θ*-metric space.*

Definition 2.4. *[8] Let* (W,d) *be a b-metric space. A b-simulation function is a function* $\varsigma:[0,\infty)\times[0,\infty)\to\mathbb{R}$ *satisfying the following conditions:*

$(\varsigma1)$ $\varsigma(\omega,\varpi)<\varpi-\omega$ *for all* $\omega,\varpi>0$.

$(\varsigma2)$ *If* $\{\omega_n\},\{\varpi_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$0<\lim_{n\to\infty}\omega_n\leq\lim_{n\to\infty}\inf\varpi_n\leq\lim_{n\to\infty}\sup\varpi_n\leq b\lim_{n\to\infty}\omega_n<\infty$$

then

$$\lim_{n\to\infty}\sup\varsigma(b\omega_n,\varpi_n)<0.$$

3. Main results

Definition 3.1. *Let* W *be a nonempty set and* $b\geq 1$ *be a given real number. A mapping* $d_\theta^b:W\times W\to[0,\infty)$ *is called a b-*θ*-metric on* W *with respect to* $\mathscr{B}$*-action* $\theta\in\mathscr{M}$ *if it satisfies the following properties for each* $\omega,\varpi,\rho\in W$.

$(b\theta1)$ $d_\theta^b(\omega,\varpi)=0$ *iff* $\omega=\varpi$.

$(b\theta2)$ $d_\theta^b(\omega,\varpi)=d_\theta^b(\varpi,\omega)$.

$(b\theta3)$ $d_\theta^b(\omega,\varpi)\leq b\theta(d_\theta^b(\omega,\rho),d_\theta^b(\rho,\varpi))$.
Then, the pair (W,d_θ^b) *is called b-*θ*-metric space.*

Remark 3.2. *Every* θ*-metric space is b-*θ*-metric space and the concept of b-*θ*-metric space coincides with the concept of* θ*-metric space when* $b=1$.

Example 3.3. *Let* $W=\{\omega,\varpi,\nu\}$ *and* $d_\theta^b:W\times W\to[0,\infty)$ *be defined by*

$$d_\theta^b(\omega,\varpi)=d_\theta^b(\varpi,\omega)=d_\theta^b(\omega,\nu)=d_\theta^b(\nu,\omega)=1$$
$$d_\theta^b(\varpi,\nu)=d_\theta^b(\nu,\varpi)=2,\ d_\theta^b(\omega,\omega)=d_\theta^b(\varpi,\varpi)=d_\theta^b(\nu,\nu)=0.$$

Suppose that $\theta(u,\rho)=\frac{1}{2}(u+\rho)$. *Then,* (W,d_θ^b) *is b-*θ*-metric space with* $b=2$ *but it is not* θ*-metric space since* $d_\theta^b(\varpi,\nu)>\theta(d_\theta^b(\varpi,\omega),d_\theta^b(\omega,\nu))$.

Remark 3.4. *The concept of b-*θ*-metric space coincides with the concept of b-metric space when* $\theta(u,\rho)=u+\rho$. *Every b-*θ*-metric space is b-metric space when* $\theta(u,\rho)=k(u+\rho),k\in(0,1]$.

Example 3.5. *Let* $W=\{\omega,\varpi,\nu\}$ *and* $d_\theta^b:W\times W\to[0,\infty)$ *be defined by*

$$d_\theta^b(\nu,\omega)=d_\theta^b(\omega,\nu)=d_\theta^b(\varpi,\nu)=d_\theta^b(\nu,\varpi)=1$$
$$d_\theta^b(\omega,\varpi)=d_\theta^b(\varpi,\omega)=3,\ d_\theta^b(\omega,\omega)=d_\theta^b(\varpi,\varpi)=d_\theta^b(\nu,\nu)=0.$$

Suppose that $\theta(u,\rho)=\frac{u\rho}{1+u\rho}$. *Then,* (W,d_θ^b) *is b-metric space with* $b=\frac{3}{2}$ *but it is not b-*θ*-metric space.*

Definition 3.6. *Let* (W,d_θ^b) *be a b-*θ*-metric space. We define the open ball with center* ω *and radius* $r>0$ *by*

$$B_{d_\theta^b}(\omega, r) = \{\varpi \in W : d_\theta^b(\omega, \varpi) < r\}$$

Example 3.7. *$W = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and let $d_\theta^b : W \times W \to [0,\infty)$ be defined by*

$$d_\theta^b(0,1) = d_\theta^b(1,0) = 2,$$

$$d_\theta^b(1, \tfrac{1}{n}) = d_\theta^b(\tfrac{1}{n}, 1) = \tfrac{1}{n} \text{ if } n \geq 2,$$

$$d_\theta^b(0, \tfrac{1}{n}) = d_\theta^b(\tfrac{1}{n}, 0) = 3 \text{ if } n \geq 2,$$

$$d_\theta^b(\tfrac{1}{n}, \tfrac{1}{m}) = d_\theta^b(\tfrac{1}{m}, \tfrac{1}{n}) = \tfrac{1}{n} + \tfrac{1}{m} \text{ if } m, n \geq 2,$$

$$d_\theta^b(m,n) = 0 \text{ iff } m = n.$$

Suppose that $\theta(u,\rho) = u + \rho + u\rho$. Then, (W, d_θ^b) is a b-θ-metric space with $b = 2$. $B_{d_\theta^b}(0,3) = \{0,1\}$ and there is no open ball with center 1 contained in $B_{d_\theta^b}(0,3)$. Thus, $B_{d_\theta^b}(0,3)$ is not open.

Definition 3.8. *Let (W, d_θ^b) be a b-θ-metric space. Then, a sequence $\{\varpi_n\}$ in W is said to be*

1. *convergent iff there exists $\varpi \in W$ such that $d_\theta^b(\varpi_n, \varpi) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} \varpi_n = \varpi$,*
2. *Cauchy iff $d_\theta^b(\varpi_n, \varpi_m) \to 0$ as $n, m \to \infty$.*

Definition 3.9. *The b-θ-metric space (W, d_θ^b) is complete if every Cauchy sequence in W converges to $\varpi \in W$.*

Proposition 3.10. *If (W, d_θ^b) is a b-θ-metric space, then the following hold:*

1. *The limit of a convergent sequence is unique.*
2. *Each convergent sequence is a Cauchy sequence.*

Proof.

1. Suppose that $\lim_{n\to\infty} \varpi_n = \varpi$ and $\lim_{n\to\infty} \varpi_n = \omega$. We claim that $\varpi = \omega$. Since $\lim_{n\to\infty} \varpi_n = \varpi$ and $\lim_{n\to\infty} \varpi_n = \omega$, then $d_\theta^b(\varpi_n, \varpi) \to 0$ and $d_\theta^b(\varpi_n, \omega) \to 0$ as $n \to \infty$. From $(b\theta 3)$, we have
$$0 \leq d_\theta^b(\varpi, \omega) \leq b\theta(d_\theta^b(\varpi_n, \varpi), d_\theta^b(\varpi_n, \omega)).$$
Letting $n \to \infty$ in the above inequality, using the continuity of θ, we get $d_\theta^b(\varpi, \omega) = 0$. Thus, $\varpi = \omega$.
2. It is obvious.

□

Example 3.11. *Let $W = \mathbb{N} \cup \{\infty\}$ and let $d_\theta^b : W \times W \to [0,\infty)$ be defined by*

$$d_\theta^b(\varpi, \omega) = \begin{cases} 5 & \text{if } \varpi, \omega \in \mathbb{N} (\varpi \neq \omega), \\ 2 & \text{if one of } \varpi, \omega \in \mathbb{N} \text{ and the other is } \infty, \\ 0 & \text{if } \varpi = \omega. \end{cases}$$

Suppose that $\theta(u,\rho) = \sqrt{u^2 + \rho^2}$. Then, (W, d_θ^b) is a b-θ-metric space with $b = 2$. Let $\varpi_n = 5n$ for each $n \in \mathbb{N}$. Then, $d_\theta^b(5n, 2) \to 5$ as $n \to \infty$. But $d_\theta^b(\infty, 2) \to 2$ since $\varpi_n \to \infty$. Thus, it is not continuous.

4. Fixed point results

Let $W \neq \emptyset$ and T be a self mapping on W. Let $\varpi_0 \in W$ and $\varpi_n = T\varpi_{n-1}$ for all $n \in \mathbb{N}$. Then, $\{\varpi_n\}$ is called a Picard sequence of initial point at ϖ_0 and $Fix(T) = \{\varpi \in W : \varpi = T\varpi\}$ is the set of fixed points of T.

Theorem 4.1. *Let (W, d_θ^b) be a complete b-θ-metric space and let $T : W \to W$ be a mapping. Suppose that there exists a b-simulation function ς such that*

$$\varsigma(bd_\theta^b(T\varpi, T\rho), d_\theta^b(\varpi, \rho)) \geq 0 \text{ for all } \varpi, \rho \in W.$$

Then, T has a unique fixed point.

Proof. Let $\{\varpi_n\}$ be a sequence of Picard with initial point $\varpi_0 \in W$. Suppose that $\varpi_n \neq \varpi_{n-1}$ for all $n \in \mathbb{N}$. We prove this theorem in 4 cases.

Case 1: We claim that $\lim_{n\to\infty} d_\theta^b(\varpi_{n-1},\varpi_n)=0$.
By the hypotheses and using $(\varsigma 1)$, respectively, we have

$$\begin{aligned} 0 &\leq \varsigma(bd_\theta^b(\varpi_n,\varpi_{n+1}),d_\theta^b(\varpi_{n-1},\varpi_n)) \\ &< d_\theta^b(\varpi_{n-1},\varpi_n)-bd_\theta^b(\varpi_n,\varpi_{n+1}). \end{aligned}$$

Thus, for all $n\in\mathbb{N}$, we get

$$bd_\theta^b(\varpi_n,\varpi_{n+1})<d_\theta^b(\varpi_{n-1},\varpi_n).$$

That is, $\{d_\theta^b(\varpi_{n-1},\varpi_n)\}$ is a decreasing sequence of positive real numbers. Hence, there exists $r\geq 0$ such that $\lim_{n\to\infty} d_\theta^b(\varpi_{n-1},\varpi_n)=r$. Assume $r>0$. From $(\varsigma 2)$ for $\nu_n=d_\theta^b(\varpi_n,\varpi_{n+1}),\omega_n=d_\theta^b(\varpi_{n-1},\varpi_n)$, we obtain

$$0 \quad \leq \lim_{n\to\infty}\sup\varsigma(bd_\theta^b(\varpi_n,\varpi_{n+1}),d_\theta^b(\varpi_{n-1},\varpi_n))<0.$$

This is a contradiction. Thus, $r=0$. That is $\lim_{n\to\infty} d_\theta^b(\varpi_{n-1},\varpi_n)=0$.

Case 2: Our aim is to show that $\{\varpi_n\}$ is a bounded sequence.
Suppose that ϖ_n is not a bounded sequence. Then, there exists a subsequence $\{\varpi_{n(k)}\}$ of $\{\varpi_n\}$ such that $n(1)=1$ and $n(k+1)$ is the minimum integer for each $k\in\mathbb{N}$ such that

$$d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})>1 \text{ and } d_\theta^b(\varpi_m,\varpi_{n(k)})\leq 1 \text{ for } n(k)\leq m\leq n(k+1)-1.$$

Thus, by using $(b\theta 3)$, we have

$$\begin{aligned} 1<d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)}) &\leq b\theta(d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k+1)-1}),d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)})) \\ &\leq b\theta(d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k+1)-1}),1). \end{aligned}$$

By taking the limit from two sides of above inequality, we get

$$1<\lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})\leq b.$$

From Case 1 and $(b\theta 3)$, we have

$$\begin{aligned} bd_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)}) &\leq d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1}) \\ &\leq b\theta(d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)}),d_\theta^b(\varpi_{n(k)},\varpi_{n(k)-1})) \\ &\leq b\theta(1,d_\theta^b(\varpi_{n(k)},\varpi_{n(k)-1})). \end{aligned}$$

Again by taking the limit from two sides of above inequality, we obtain

$$b<\lim_{k\to\infty} bd_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})\leq \lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1})\leq b$$

Thus,

$$\lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1})=b \text{ and } \lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})=1.$$

Now, by $(\varsigma 2)$, with $\nu_k=d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})$ and $\omega_k=d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1})$, we get

$$0\leq \lim_{k\to\infty}\varsigma(b\nu_k,\omega_k)<0.$$

This is a contradiction. Hence, $\{\varpi_n\}$ is a bounded sequence.

Case 3: We will show that $\{\varpi_n\}$ is a Cauchy sequence.
Let $M_n=\sup\{d_\theta^b(\varpi_i,\varpi_j):i,j\geq n \text{ and } n\in\mathbb{N}\}$. From Case 2, for each $n\in\mathbb{N}$, $M_n<\infty$. Here, M_n is a positive decreasing sequence. So, there exists $M\geq 0$ such that $\lim_{n\to\infty} M_n=M$.
Assume that $M>0$. For $k\in\mathbb{N}$, there exist $n(k),m(k)\in\mathbb{N}$ such that $m(k)>n(k)\geq k$ and

$$M_k-\frac{1}{k}<d_\theta^b(\varpi_{m(k)},\varpi_{n(k)})\leq M_k.$$

After taking the limit in the above inequality, we have

$$\lim_{k\to\infty} d_\theta^b(\varpi_{m(k)},\varpi_{n(k)})=M.$$

From Case 1 and the definition of M_n, we obtain

$$bd_\theta^b(\varpi_{m(k)},\varpi_{n(k)}) \leq d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1}) \leq M_{k-1}.$$

Again, by taking the limit, we find

$$bM \leq \lim_{k\to\infty}\inf d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1}) \leq \lim_{k\to\infty}\sup d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1}) \leq M.$$

If $b>1$, then $M=0$. If $b=1$, from $(\varsigma 2)$ with $\nu_k = d_\theta^b(\varpi_{m(k)},\varpi_{n(k)})$ and $\omega_k = d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1})$, we obtain

$$0 \leq \lim_{k\to\infty}\sup \varsigma(b\nu_k,\omega_k) < 0.$$

This is a contradiction. Thus, $M=0$. This implies that $\{\varpi_n\}$ is a Cauchy sequence.

Case 4: Since (W,d_θ^b) is a complete b-θ-metric space and $\{\varpi_n\}$ is a Cauchy sequence from Case 3, there exists $\rho\in W$ such that $\lim_{n\to\infty}\varpi_n=\rho$. We must show that $\rho\in Fix(T)$. From Case 1,

$$bd_\theta^b(T\varpi_n,T\rho) \leq d_\theta^b(\varpi_n,\rho) \text{ for all } n\in\mathbb{N}.$$

Thus,

$$\begin{aligned} 0 \leq d_\theta^b(\rho,T\rho) &\leq b\theta(d_\theta^b(\rho,\varpi_{n+1}),d_\theta^b(\varpi_{n+1},T\rho)) \\ &< b\theta(d_\theta^b(\rho,\varpi_{n+1}),\frac{1}{b}d_\theta^b(\varpi_n,\rho)). \end{aligned}$$

By taking the limit from two sides of above inequality, we get $d_\theta^b(\rho,T\rho)=0$ since $\lim_{n\to\infty}\varpi_n=\rho$. Therefore, $\rho=T\rho$.

Finally, we must show that the uniqueness of fixed point. Assume that there exists $w\in W$ such that $w=Tw$ and $w\neq\rho$. By Case 1, we get

$$0 \leq bd_\theta^b(Tw,T\rho) \leq d_\theta^b(w,\rho).$$

This implies that $b\leq 1$. This is a contradiction with our assumption. Hence, T has a unique fixed point. □

Corollary 4.2. *Let (W,d_θ^b) be a complete b-θ-metric space and $T:W\to W$ be a mapping satisfies the following inequality*

$$bd_\theta^b(T\omega,T\varpi) \leq \alpha d_\theta^b(\omega,\varpi)$$

for each $\omega,\varpi\in W$, where $\alpha\in[0,1)$. Then, T has a unique fixed point.

Proof. It follows from Theorem 4.1 if we take b-simulation function as $\varsigma(\nu,\rho)=\alpha\rho-\nu$ for all $\nu,\rho\geq 0$. □

Remark 4.3. *Let (W,d_θ^b) be a complete b-θ-metric space.*

1. *Theorem 3.4 in [8] is obtained from Theorem 4.1 by taking $\theta(\nu,\rho)=\nu+\rho$.*
2. *Theorem 3.3 in [7] is obtained from Theorem 4.1 by taking $b=1$.*

Now, we illustrate the validity of fixed point result in Theorem 4.1 by the following examples.

Example 4.4. *Let $W=[0,\infty)$ and $d_\theta^b: W\times W\to[0,\infty)$ be defined by $d_\theta^b(\omega,\varpi)) = |\ \omega-\varpi\ |^3$. Also, we take $\theta(\nu,\rho)=\nu+\rho+\nu\rho$. Then, (W,d_θ^b) is a complete b-θ-metric space with $b=4$. Define a mapping $T:W\to W$ by $T\omega=\frac{\omega}{a}$ for all $\omega\in W$ and $a>0, a\neq 1$. From Theorem 4.1, T has a unique fixed point $u=0$ for b-simulation function $\varsigma(\nu,\rho)=\lambda\rho-\nu$ where $\lambda\geq\frac{4}{a^3}$ for all $\nu,\rho\in[0,\infty)$, since*

$$\begin{aligned} \varsigma(4d_\theta^b(T\omega,T\varpi),d_\theta^b(\omega,\varpi)) &= \lambda d_\theta^b(\omega,\varpi)-4d_\theta^b(T\omega,T\varpi) \\ &= \lambda(|\ \omega-\varpi\ |^3)-4(|\ T\omega-T\varpi\ |^3) \\ &= \lambda(|\ \omega-\varpi\ |^3)-4(|\ \frac{\omega}{a}-\frac{\varpi}{a}\ |^3) \\ &= (\lambda-\frac{4}{a^3})(|\ \omega-\varpi\ |^3) \\ &\geq 0. \end{aligned}$$

Example 4.5. *Let $W=[0,1]$ and $d_\theta^b : W\times W \to [0,\infty)$ be defined by $d_\theta^b(\omega,\varpi) = \mid \omega-\varpi \mid^2$. Also, we take $\theta(\nu,\rho)=\sqrt{\nu^2+\rho^2}$. Then, (W,d_θ^b) is a complete b-θ-metric space with $b=2\sqrt{2}$. Define a mapping $T:[0,1]\to[0,1]$ by $T\omega = \frac{\omega}{\sqrt{2}}+a$ for all $\omega\in W$ and $a<\frac{\sqrt{2}-1}{\sqrt{2}}$. From Theorem 4.1, T has a unique fixed point $u=\frac{\sqrt{2}a}{\sqrt{2}-1}$ for b-simulation function $\varsigma(\nu,\rho)=\lambda\rho-\nu$ where $\lambda\geq\sqrt{2}$ for all $\nu,\rho\in[0,\infty)$, since*

$$
\begin{aligned}
\varsigma(2\sqrt{2}d_\theta^b(T\omega,T\varpi),d_\theta^b(\omega,\varpi)) &= \lambda d_\theta^b(\omega,\varpi)-2\sqrt{2}d_\theta^b(T\omega,T\varpi)\\
&= \lambda(\mid \omega-\varpi\mid^2)-2\sqrt{2}(\mid T\omega-T\varpi\mid^2)\\
&= \lambda(\mid \omega-\varpi\mid^2)-2\sqrt{2}(\mid \frac{\omega}{\sqrt{2}}-\frac{\varpi}{\sqrt{2}}\mid^2)\\
&= (\lambda-\frac{2\sqrt{2}}{2})(\mid \omega-\varpi\mid^2)\\
&\geq 0.
\end{aligned}
$$

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Dynamic Behavior of Euler-Maclaurin Methods for Differential Equations with Piecewise Constant Arguments of Advanced and Retarded Type

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Abstract

The paper deals with three dynamic properties of the numerical solution for differential equations with piecewise constant arguments of advanced and retarded type: oscillation, stability and convergence. The Euler-Maclaurin methods are used to discretize the equations. According to the characteristic theory of the difference equation, the oscillation and stability conditions of the numerical solution are obtained. It is proved that the convergence order of numerical method is $2n+2$. Furthermore, the relationship between stability and oscillation is discussed for analytic solution and numerical solution, respectively. Finally, several numerical examples confirm the corresponding conclusions.

1. Introduction

As a special type of delay differential equations [1]- [4], differential equations with piecewise constant argument [5]- [9] (abbreviated as EPCA) has some characteristics of continuous and discrete dynamic system, so it has important value in practical application such as population biology [10], neural networks [11, 12], predator-prey model [13], epidemiology [14] and so on. In recent years, the comprehensive exploration of EPCA has become a scientific issue widely concerned by scholars in various fields. Because of the complexity of this kind of equation in structure, it is difficult to solve it accurately. Therefore, it is necessary to study the numerical solution of EPCA, and then clarify the applicability of numerical method in EPCA.

In the study of differential equations with piecewise constant arguments, much research has been focused on the properties of numerical solution of EPCA. Gao [15] considered numerical oscillation of the Runge-Kutta method for EPCA of mixed type. In [16], convergence and stability of stochastic EPCA in split-step theta method was considered. The stability of the Runge-Kutta method for nonlinear neutral EPCA was studied in [17]. Wang and Yao [18] studied the stability and oscillation of a kind of functional differential equation. Liang et al. [19] considered numerical stability of system $u'(t) = Lu(t) + Mu([t])$ with matrix coefficients in the case of 2-norm. Different from previous studies, this paper mainly considers the numerical oscillation, stability and convergence of Euler-Maclaurin methods for forward EPCA with advanced and retarded type, and gives some new conclusions.

Consider the following equation:

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t+1]), x(0) = c_0, \tag{1.1}$$

where $[\cdot]$ designates the greatest-integer function.

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Denote

$$b_0(t)=e^{at}+a^{-1}a_0(e^{at}-1), b_1(t)=a^{-1}a_1(e^{at}-1), \lambda=b_0(1)/(1-b_1).$$

Theorem 1.1. *[20] Eq. (1.1) has on a unique solution*

$$x(t)=(b_0(\{t\})+\lambda b_1(\{t\}))\lambda^{[t]}c_0, \tag{1.2}$$

where $\{t\}$ is the fractional part of t, if $b_1(1)\neq 1$.

In particular, the solution of Eq. (1.1) is

$$x(t)=\left(1+\frac{a_0+a_1}{1-a_1}\{t\}\right)\left(\frac{1+a_0}{1-a_1}\right)^{[t]}c_0,$$

for $a=0$.

Theorem 1.2. *[20] The solution $x=0$ of Eq. (1.1) is stable (asymptotically stable) as $t\to+\infty$, if and only if*

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a(e^a+1)}{e^a-1}\right)\geq 0. \tag{1.3}$$

Theorem 1.3. *[20] In each internal $(n,n+1)$, the solution of Eq. (1.1) with the condition $x(0)=c_0\neq 0$ has exact roots*

$$t_n=n+\frac{1}{a}ln\frac{a_0+a_1e^a}{a+a_0+a_1}$$

if

$$\left(a_0+\frac{ae^a}{e^a-1}\right)\left(a_1-\frac{a}{e^a-1}\right)>0. \tag{1.4}$$

If (1.4) is not satisfied and $a_0\neq -ae^a/(e^a-1), c_0\neq 0$, then solution (1.2) has no zero in $[0,+\infty)$.

2. Numerical oscillation and non-oscillation

2.1. Euler-Maclaurin methods and convergence

Firstly, we introduce Bernoulli's numbers and Bernoulli's polynomials as follows:

$$\frac{z}{e^z-1}=\sum_{j=0}^{\infty}\frac{B_j}{j!}z^j, |z|<2\pi,$$

$$\frac{ze^{xz}}{e^z-1}=\sum_{j=0}^{\infty}\frac{B_j(x)}{j!}z^j, |z|<2\pi,$$

where B_j and $B_j(x), j=0,1,2\cdots$ are called Bernoulli's number and the jth-order Bernoulli's polynomial, respectively.

Lemma 2.1. *[21] B_j and $B_j(x)$ have the following several properties:*
(I) $B_0=1, B_1=-\frac{1}{2}, B_{2j}=2(-1)^{j+1}(2j)!\sum_{k=1}^{\infty}(2k\pi)^{-2j}, B_{2j+1}=0, j\geq 1,$
(II) $B_0(x)=1, B_1(x)=x-\frac{1}{2}, B_2(x)=x^2-x+\frac{1}{6}, B_k(x)=\sum_{j=0}^{k}\binom{k}{j}B_jx^{k-j}.$

Lemma 2.2. *[22] Suppose that $f(x)$ has $2n+3$rd continuous derivative on $[t_i,t_{i+1}]$, then we have*

$$\left|\int_{t_i}^{t_{i+1}}f(t)dt-\frac{h}{2}\left[f(t_{i+1})+f(t_i)\right]+\sum_{j=1}^{n}\frac{B_{2j}h^{2j}}{(2j)!}\left[f^{(2j-1)}(t_{i+1})-f^{(2j-1)}(t_i)\right]\right|=O\left(h^{2n+3}\right). \tag{2.1}$$

Let $h=\frac{1}{m}$ be a given step-size and t_i be defined by $t_i=ih, i=0,1,2\cdots$, then let $i=km+l, l=0,1,2,\cdots,m-1$. The derivative $x^{(j)}(t)$ exists in every interval $[k,k+1)$. We suppose

$$f(t)=x'(t)=ax(t)+a_0x([t])+a_1x([t+1])$$

for all $j=0,1,2\cdots$, then we have

$$\begin{aligned} f'(t)&=x''(t)=ax'(t)=a^2x(t)+aa_0x([t])+aa_1x([t+1]),\\ f^{(j)}(t)&=x^{(j+1)}(t)=a^{j+1}x(t)+a^ja_0x([t])+a^ja_1x([t+1]).\end{aligned} \tag{2.2}$$

Apply (2.2) to (2.1), we get

$$x_{i+1} = x_i + \frac{ha}{2}(x_{i+1}+x_i) + ha_0 x_{km} + ha_1 x_{(k+1)m} - \sum_{j=1}^{n} \frac{B_{2j}(ah)^{2j}}{(2j)!}(x_{i+1}-x_i). \tag{2.3}$$

Since $i = km+l, l = 0,1,2,\cdots,m-1$, (2.3) can be expressed as:

$$x_{(k+1)m} = \frac{1+a_0}{1-a_1} x_{km}, \tag{2.4}$$

$$x_{km+l+1} = (1+(l+1)ha_0)x_{km} + (l+1)ha_1 x_{(k+1)m}, \tag{2.5}$$

for $a = 0$, and

$$x_{(k+1)m} = \frac{R(z)^m + \frac{a_0}{a}(R(z)^m - 1)}{1 - \frac{a_1}{a}(R(z)^m - 1)} x_{km}, \tag{2.6}$$

$$x_{km+l+1} = \left(R(z)^{l+1} + \frac{a_0}{a}\left(R(z)^{l+1} - 1\right)\right) x_{km} + \frac{a_1}{a}\left(R(z)^{l+1} - 1\right) x_{(k+1)m}, \tag{2.7}$$

for $a \neq 0$, where $l = 0,1,\cdots,m-2$, $z = ah$, $\phi(z) = 1 - \frac{z}{2} + \sum_{j=1}^{n} \frac{B_{2j}z^{2j}}{(2j)!}$ and $R(z) = 1 + \frac{z}{\phi(z)}$ is the stability function of the Euler-Maclaurin methods.

Theorem 2.3. *For every given $n \in N$, the Euler-Maclaurin method is of order $2n+2$.*

Proof. Let $km \le i < (k+1)m - 1$, then by Lemma 2.2 and $f(t) = x'(t)$, we get

$$\begin{aligned} x(t_{i+1}) - x(t_i) = \int_{t_i}^{t_{i+1}} x'(t)dt =& \frac{ha}{2}[x(t_{i+1}) + x(t_i)] + ha_0 x(k) + ha_1 x(k+1) \\ &- \sum_{j=1}^{n} \frac{B_{2j}(ah)^{2j}}{(2j)!}[x(t_{i+1}) - x(t_i)] + O\left(h^{2n+3}\right). \end{aligned}$$

Let $i = (k+1)m - 1$, then for any $0 < \varepsilon < h$, we have

$$\begin{aligned} x(t_{i+1}-\varepsilon) - x(t_i) = \int_{t_i}^{t_{i+1}-\varepsilon} x'(t)dt =& \frac{ha}{2}[x(t_{i+1}-\varepsilon) + x(t_i)] + ha_0 x(k) + ha_1 x(k+1) \\ &- \sum_{j=1}^{n} \frac{B_{2j}(ah)^{2j}}{(2j)!}[x(t_{i+1}-\varepsilon) + x(t_i)] + O\left(h^{2n+3}\right). \end{aligned} \tag{2.8}$$

Let $\varepsilon \to 0^+$ in (2.8), (2.7) holds true for $i = (k+1)m - 1$. Suppose

$$(x(t_{i+1}) - x_{i+1})\left(1 + \frac{ha}{2} + \sum_{j=1}^{n} \frac{B_{2j}(ha)^{2j}}{(2j)!}\right) = O\left(h^{2n+3}\right),$$

then from (2.4)-(2.7) we obtain

$$(x(t_{i+1}) - x_{i+1})\left(1 + \frac{ha}{2} + \sum_{j=1}^{n} \frac{B_{2j}(ha)^{2j}}{(2j)!}\right) = O\left(h^{2n+3}\right),$$

the proof is complete. □

2.2. Oscillation analysis

Theorem 2.4. *If $\{x_n\}$ and $\{x_{km}\}$ are given by (2.5), (2.7) and (2.4), (2.6), respectively, then $\{x_n\}$ is non-oscillatory if and only if $\{x_{km}\}$ is non-oscillatory.*

Proof. The necessity is obvious for $a \neq 0$. Sufficiency: if $\{x_{km}\}$ is non-oscillatory, without loss of generality, we assume that $\{x_{km}\}$ is an eventually negative solution of (2.6), that is, there exists a $k_0 \in R$ such that $x_{km} < 0$ for $k > k_0$. In order to prove $x_{km+l} < 0$ for all $k > k_0 + 1$ and $l = 0,1,\cdots,m-1$, we suppose $a_0 < 0, a_1 < 0$. If $a > 0$, then $1 < R(z) < \infty$ and $R(z)^{-m} \le R(z)^{-l}$, therefore from (2.7) we have

$$\begin{aligned} R(z)^{-l} x_{km+l} &= \left(1 + \frac{a_0}{a}\left(1 - R(z)^{-l}\right)\right) x_{km} + \frac{a_1}{a}\left(1 - R(z)^{-l}\right) x_{(k+1)m} \\ &\le \left(1 + \frac{a_0}{a}\left(1 - R(z)^{-m}\right)\right) x_{km} + \frac{a_1}{a}\left(1 - R(z)^{-m}\right) x_{(k+1)m} \\ &= R(z)^{-m} x_{(k+1)m} < 0. \end{aligned}$$

So $x_{km+l} < 0$. The case of $a < 0$ and $a = 0$ can be studied in the same way. The proof is complete. □

By Theorem 2.4, we can get the following theorem.

Theorem 2.5. *The following propositions are equivalent:*
(I) $\{x_n\}$ *is oscillatory ;*
(II) $\{x_{km}\}$ *is oscillatory ;*
(III) The two cases hold
(i) $a_0 < -\frac{aR(z)^m}{R(z)^m-1}$ *and* $a_1 < \frac{a}{R(z)^m-1}$,
(ii) $a_0 > -\frac{aR(z)^m}{R(z)^m-1}$ *and* $a_1 > \frac{a}{R(z)^m-1}$,
for $a \neq 0$, *and*
(i) $a_0 < -1$ *and* $a_1 < 1$,
(ii) $a_0 > -1$ *and* $a_1 > 1$,
for $a = 0$.

Proof. According to Theorem 2.4, the equivalence of (I) and (II) is obvious, then we prove that (II) and (III) are equivalent. $\{x_n\}$ is oscillatory for $a \neq 0$ if and only if the corresponding characteristic equation has no positive roots, which is equivalent to

$$\frac{R(z)^m + \frac{a_0}{a}\left(R(z)^m - 1\right)}{1 - \frac{a_1}{a}\left(R(z)^m - 1\right)} < 0,$$

so we have

$$R(z)^m + \tfrac{a_0}{a}\left(R(z)^m - 1\right) < 0 \text{ and } 1 - \tfrac{a_1}{a}\left(R(z)^m - 1\right) > 0$$

or

$$R(z)^m + \tfrac{a_0}{a}\left(R(z)^m - 1\right) > 0 \text{ and } 1 - \tfrac{a_1}{a}\left(R(z)^m - 1\right) < 0,$$

that is

$$a_0 < -\tfrac{aR(z)^m}{R(z)^m-1} \text{ and } a_1 < \tfrac{a}{R(z)^m-1}$$

or

$$a_0 > -\tfrac{aR(z)^m}{R(z)^m-1} \text{ and } a_1 > \tfrac{a}{R(z)^m-1}.$$

In the same way, $\lambda = \frac{1+a_0}{1-a_1}$ for $a = 0$. The proof is complete. □

From Theorem 1.3, we have the following corollary.

Corollary 2.6. *If any of the following conditions holds true:*
(I) When $a \neq 0$,
(i) $a_0 < -\frac{ae^a}{e^a-1}$ *and* $a_1 < \frac{a}{e^a-1}$,
(ii) $a_0 > -\frac{ae^a}{e^a-1}$ *and* $a_1 > \frac{a}{e^a-1}$,
(II) When $a = 0$,
(i) $a_0 < -1$ *and* $a_1 < 1$,
(ii) $a_0 > -1$ *and* $a_1 > 1$,
then every solution of Eq. (1.1) is oscillatory.

Lemma 2.7. *[21] If* $|z| \leq 1$, *then we have* $\phi(z) \geq \frac{1}{2}\cdot$ *for* $z > 0$ *and* $\phi(z) \geq 1$ *for* $z \leq 0$.

Lemma 2.8. *[21] If* $|z| \leq 1$, *then*
(I) $\phi(z) \leq \frac{z}{e^z-1}$, *n is even ;*
(II) $\phi(z) \geq \frac{z}{e^z-1}$, *n is odd.*

Theorem 2.9. *If* $a \neq 0$, *then the Euler-Maclaurin methods preserve the oscillation of Eq. (1.1) if and only if n is even.*

Proof. According to Theorem 2.5 and Corollary 2.6, we can get the Euler-Maclaurin methods preserve the oscillation of (1.1) if and only if

$$\tfrac{ae^a}{e^a-1} \leq -\tfrac{aR(z)^m}{R(z)^m-1} \text{ or } \tfrac{a}{e^a-1} \geq \tfrac{a}{R(z)^m-1}$$

holds true. If $a > 0$, we have

$$\tfrac{e^a}{e^a-1} \geq \tfrac{R(z)^m}{R(z)^m-1} \text{ or } e^a \leq R(z)^m.$$

Since the function $y = \frac{x}{x-1}$ is decreasing, so
$$e^a \le R(z)^m.$$
Therefore,
$$\phi(z) \le \frac{z}{e^z - 1}.$$
From Lemma 2.8, n is even. The case of $a < 0$ can be proved in the same way. □

Theorem 2.10. *If $a \neq 0$, then the Euler-Maclaurin methods preserve the non-oscillation of (1.1) if and only if n is odd.*

From Theorem 2.5 and Corollary 2.6, we can get this proof.

Theorem 2.11. *When $a = 0$, the Euler-Maclaurin methods preserve the oscillation and non-oscillation of (1.1) for any $n \in N$.*

3. Relationship between stability and oscillation

From Theorem 1.2, we have the following corollary.

Corollary 3.1. *The analytic solution of Eq. (1.1) is asymptotically stable as $t \to +\infty$, if and only if*
$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\left(e^a+1\right)}{e^a-1}\right) > 0$$
for $a \neq 0$, and
$$(a_0+a_1)(a_1-a_0-2) > 0$$
for $a = 0$.

Theorem 3.2. *The numerical solution of Eq. (1.1) is asymptotically stable ($x_n \to 0$ as $n \to \infty$) if and only if*
$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\left(R(z)^m+1\right)}{R(z)^m-1}\right) > 0$$
for $a \neq 0$, and
$$(a_0+a_1)(a_1-a_0-2) > 0$$
for $a = 0$.

Proof. According to (2.3) and (2.5), it is well known that $x_n \to 0$ as $n \to \infty$ if and only if $|\hat{\lambda}| < 1$, where
$$\hat{\lambda} = \frac{R(z)^m + \frac{a_0}{a}\left(R(z)^m - 1\right)}{1 - \frac{a_1}{a}\left(R(z)^m - 1\right)}$$
for $a \neq 0$, and
$$\hat{\lambda} = \frac{1+a_0}{1-a_1}$$
for $a = 0$. So we have
$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\left(R(z)^m+1\right)}{R(z)^m-1}\right) > 0$$
for $a \neq 0$, and
$$(a_0+a_1)(a_1-a_0-2) > 0$$
for $a = 0$. This completes the proof. □

According to Corollary 2.6 and Corollary 3.1, we get the conclusion for the analytic solution.

Theorem 3.3. *When $a \neq 0$, the analytic solution of Eq. (1.1) is*

(A1) non-oscillatory and asymptotically stable if
$$(a+a_0+a_1)\left(a_1 - a_0 - \tfrac{a(e^a+1)}{e^a-1}\right) > 0,\ a_0 < -\tfrac{ae^a}{e^a-1} \text{ and } a_1 \ge \tfrac{a}{e^a-1}$$
or
$$(a+a_0+a_1)\left(a_1 - a_0 - \tfrac{a(e^a+1)}{e^a-1}\right) > 0,\ a_0 > -\tfrac{ae^a}{e^a-1} \text{ and } a_1 \le \tfrac{a}{e^a-1}$$
holds true.

(A2) non-oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\tfrac{ae^a}{e^a-1}\ and\ a_1\geq\tfrac{a}{e^a-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\tfrac{ae^a}{e^a-1}\ and\ a_1\leq\tfrac{a}{e^a-1}$$

holds true.

(A3) oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\tfrac{ae^a}{e^a-1}\ and\ a_1<\tfrac{a}{e^a-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\tfrac{ae^a}{e^a-1}\ and\ a_1>\tfrac{a}{e^a-1}$$

holds true.

(A4) oscillatory and asymptotically stable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)>0,\ a_0<-\tfrac{ae^a}{e^a-1}\ and\ a_1<\tfrac{a}{e^a-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)>0,\ a_0>-\tfrac{ae^a}{e^a-1}\ and\ a_1>\tfrac{a}{e^a-1}$$

holds true.

According to Theorem 2.5 and Theorem 3.2, we get the corresponding conclusion for the numerical solution.

Theorem 3.4. *When $a\neq 0$, the numerical solution of (1.1) is*

(B1) non-oscillatory and asymptoticallystable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0<-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1\geq\tfrac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0>-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1\leq\tfrac{a}{R(z)^m-1}$$

holds true.

(B2) non-oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1\geq\tfrac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1\leq\tfrac{a}{R(z)^m-1}$$

holds true.

(B3) oscillatory and unstable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0<-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1<\tfrac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(e^a+1)}{e^a-1}\right)\leq 0,\ a_0>-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1>\tfrac{a}{R(z)^m-1}$$

holds true.

(B4) oscillatory and asymptotically stable if

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0<-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1<\tfrac{a}{R(z)^m-1}$$

or

$$(a+a_0+a_1)\left(a_1-a_0-\tfrac{a(R(z)^m+1)}{R(z)^m-1}\right)>0,\ a_0>-\tfrac{aR(z)^m}{R(z)^m-1}\ and\ a_1>\tfrac{a}{R(z)^m-1}$$

holds true.

Theorem 3.5. *When $a=0$, the analytic solution and numerical solution of Eq. (1.1) are both*

(C1) non-oscillatory and asymptotically stable if

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0<-1\ and\ a_1\geq 1$$

or

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0>-1 \text{ and } a_1\leq 1.$$

(C2) *non-oscillatory and unstable if*

$$(a_0+a_1)(a_1-a_0-2)\leq 0,\ a_0<-1 \text{ and } a_1\geq 1$$

or

$$(a_0+a_1)(a_1-a_0-2)\leq 0,\ a_0>-1 \text{ and } a_1\leq 1.$$

(C3) *oscillatory and unstable if*

$$(a_0+a_1)(a_1-a_0-2)\leq 0,\ a_0<-1 \text{ and } a_1<1$$

or

$$(a_0+a_1)(a_1-a_0-2)\leq 0,\ a_0>-1 \text{ and } a_1>1.$$

(C4) *oscillatory and asymptotically stable if*

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0<-1 \text{ and } a_1<1$$

or

$$(a_0+a_1)(a_1-a_0-2)>0,\ a_0>-1 \text{ and } a_1>1.$$

4. Numerical examples

Consider the following equations

$$x'(t)=-x(t)-2x([t])+5x([t+1]),\quad x(0)=1, \tag{4.1}$$

$$x'(t)=x(t)+4x([t])-3x([t+1]),\quad x(0)=1, \tag{4.2}$$

$$x'(t)=x(t)+x([t])+2x([t+1]),\quad x(0)=1, \tag{4.3}$$

$$x'(t)=-2x(t)-3x([t])-2x([t+1]),\quad x(0)=1. \tag{4.4}$$

From Theorem 1.1, the analytic solution of Eq. (4.1) is $x(10)\approx 1.51037040806E-4$ at $t=10$. We listed the absolute errors (AE) and the relative errors (RE) at $n=2$ and $t=10$ and the ratio of the errors of the case $m=20$ over that of $m=40$. We can see from Table 1 that the Euler-Maclaurin methods is of order 6 when $n=2$. The Euler-Maclaurin methods have good convergence for this kind of equations.
Further, from (4.1) we know that the coefficients are $a=-1, a_0=-2, a_1=5$, then

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a\left(e^a+1\right)}{e^a-1}\right)\approx 9.6721>0, a_0<-\frac{ae^a}{e^a-1}\approx -0.5820 \text{ and } a_1\geq \frac{a}{e^a-1}\approx 1.5820,$$

so (A1) in Theorem 3.3 holds true. On the other hand, let $m=50, n=3$, we have

$$z=ha=\frac{a}{m}=-0.02, B_{2j}=2.3404\times 10^{-8}, \phi(z)=1.0100, R(z)=1+\frac{z}{\phi(z)}=0.9802.$$

Table 1: The errors of the Euler-Maclaurin methods $(n=2)$

	AE	RE
$m=2$	$3.0083E-10$	$1.9918E-6$
$m=3$	$2.6198E-11$	$1.7345E-7$
$m=5$	$1.2172E-12$	$8.0591E-9$
$m=10$	$1.8986E-14$	$1.2570E-10$
$m=20$	$2.9751E-16$	$1.9697E-12$
$m=40$	$4.0115E-18$	$2.6560E-14$
ratio	74.16	74.16

Because

$$\phi(z) \geq \frac{z}{e^z - 1} \approx 1.0100,$$

then we obtain

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\left(R(z)^m+1\right)}{R(z)^m-1}\right) \approx 9.6721 > 0, a_0 < -\frac{aR(z)^m}{R(z)^m-1} \approx -0.5820 \text{ and } a_1 \geq \frac{a}{R(z)^m-1} \approx 1.5820,$$

so (B1) in Theorem 3.4 holds true.

From Figure 4.1 we can see that the analytic solution and the numerical solution of (4.1) are asymptotically stable and non-oscillatory, which is agreement with Theorems 3.3 (A1) and 3.4 (B1).

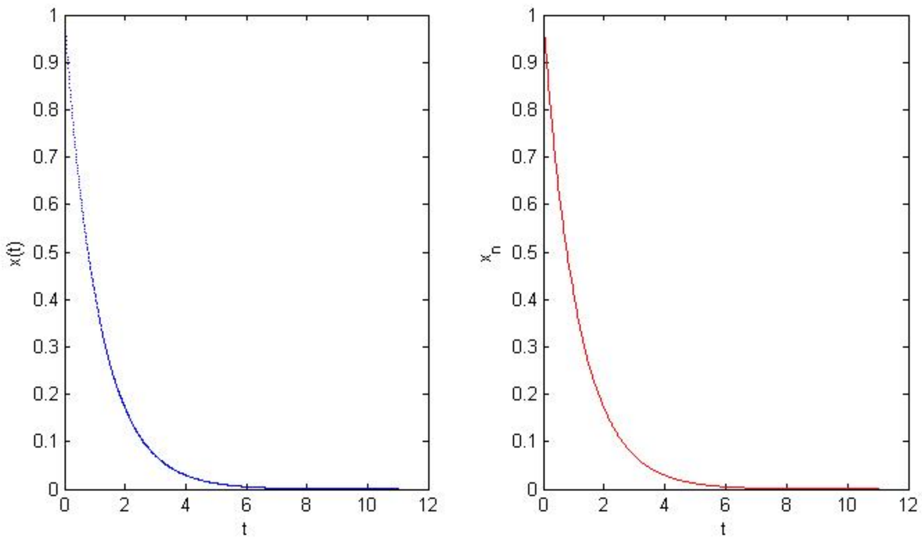

Figure 4.1: The analytic solution (left) and the numerical solution (right, $n = 3$) of (4.1).

From (4.2) we know that the coefficients are $a = 1, a_0 = 4, a_1 = -3$, then

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\left(e^a+1\right)}{e^a-1}\right) \approx -18.3279 \leq 0, a_0 > -\frac{ae^a}{e^a-1} \approx -1.5820 \text{ and } a_1 \leq \frac{a}{e^a-1} \approx 0.5820,$$

so (A2) in Theorem 3.3 holds true. On the other hand, let $m = 50, n = 3$, we have

$$z = ha = \frac{a}{m} = 0.02, B_{2j} = 2.3404 \times 10^{-8}, \phi(z) = 0.9900, R(z) = 1 + \frac{z}{\phi(z)} = 1.0202.$$

Because

$$\phi(z) \geq \frac{z}{e^z - 1} \approx 0.9900,$$

then we obtain

$$(a+a_0+a_1)\left(a_1 - a_0 - \frac{a\left(R(z)^m+1\right)}{R(z)^m-1}\right) \approx -18.3279 \leq 0, a_0 > -\frac{aR(z)^m}{R(z)^m-1} \approx -1.5820 \text{ and } a_1 \leq \frac{a}{R(z)^m-1} \approx 0.5820,$$

so (B2) in Theorem 3.4 holds true.

From Figure 4.2 we can see that the analytic solution and the numerical solution of (4.2) are unstable and non-oscillatory, which is agreement with Theorems 3.3 (A2) and 3.4 (B2).

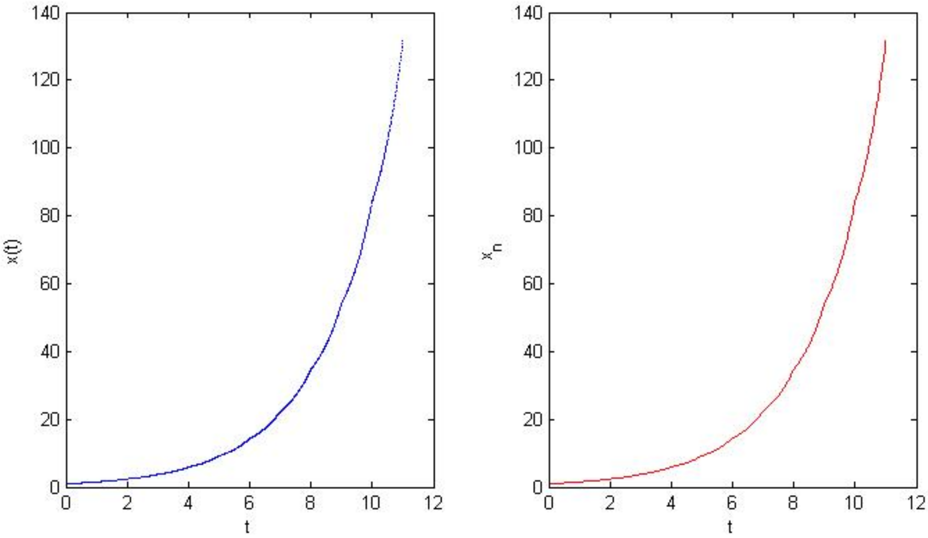

Figure 4.2: The analytic solution (left) and the numerical solution (right, $n = 3$) of (4.2).

From (4.3) we know that the coefficients are $a = 1, a_0 = 1, a_1 = 2$, then

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a\left(e^a+1\right)}{e^a-1}\right) \approx -4.6558 \leq 0, a_0 > -\frac{ae^a}{e^a-1} \approx -1.5820 \text{ and } a_1 > \frac{a}{e^a-1} \approx 0.5820,$$

so (A3) in Theorem 3.3 holds true. On the other hand, let $m = 50, n = 4$, we have

$$z = ha = \frac{a}{m} = 0.02, B_{2j} = 2.3404 \times 10^{-8}, \phi(z) = 0.9900, R(z) = 1 + \frac{z}{\phi(z)} = 1.0202.$$

Because

$$\phi(z) \geq \frac{z}{e^z-1} \approx 0.9900,$$

then we obtain

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a\left(R(z)^m+1\right)}{R(z)^m-1}\right) \approx -4.6558 \leq 0, a_0 > -\frac{aR(z)^m}{R(z)^m-1} \approx -1.5820 \text{ and } a_1 > \frac{a}{R(z)^m-1} \approx 0.5820,$$

so (B3) in Theorem 3.4 holds true.
From Figure 4.3 we can see that the analytic solution and the numerical solution of (4.3) are unstable and oscillatory, which is agreement with Theorems 3.3 (A3) and 3.4 (B3).

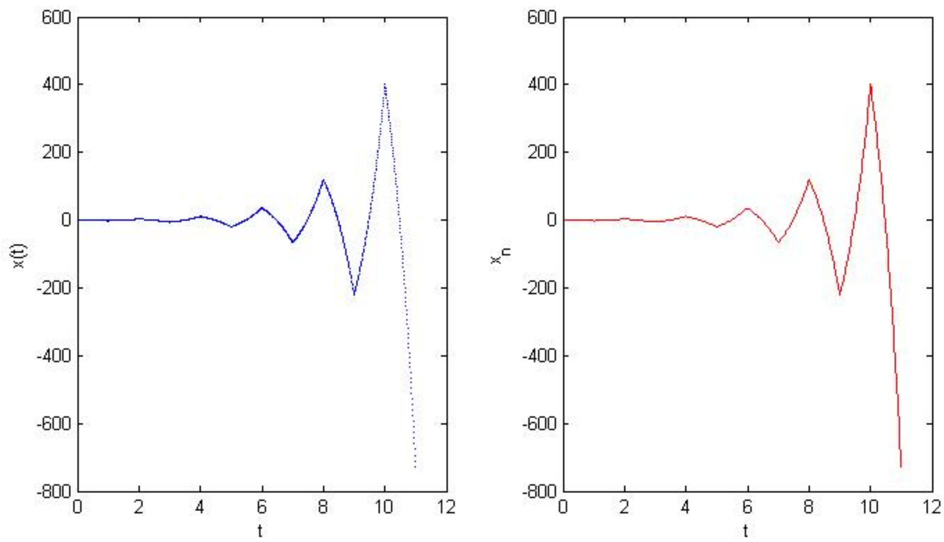

Figure 4.3: The analytic solution (left) and the numerical solution (right, $n = 4$) of (4.3).

From (4.4) we know that the coefficients are $a=-2, a_0=-3, a_1=-2$, then

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a\,(e^a+1)}{e^a-1}\right)\approx 11.3825>0, a_0<-\frac{ae^a}{e^a-1}\approx -0.3130 \text{ and } a_1<\frac{a}{e^a-1}\approx 2.3130,$$

so (A4) in Theorem 3.3 holds true. On the other hand, let $m=50, n=4$, we have

$$z=ha=\frac{a}{m}=-0.04, B_{2j}=2.3404\times 10^{-8}, \phi(z)=1.0201, R(z)=1+\frac{z}{\phi(z)}=0.9608.$$

Because

$$\phi(z)\geq \frac{z}{e^z-1}\approx 1.0201,$$

then we obtain

$$(a+a_0+a_1)\left(a_1-a_0-\frac{a\,(R(z)^m+1)}{R(z)^m-1}\right)\approx 11.3825>0, a_0<-\frac{aR(z)^m}{R(z)^m-1}\approx -0.3130 \text{ and } a_1<\frac{a}{R(z)^m-1}\approx 2.3130,$$

so (B4) in Theorem 3.4 holds true.

From Figure 4.4 we can see that the analytic solution and the numerical solution of (4.4) are asymptotically stable and oscillatory, which is agreement with Theorems 3.3 (A4) and 3.4 (B4).

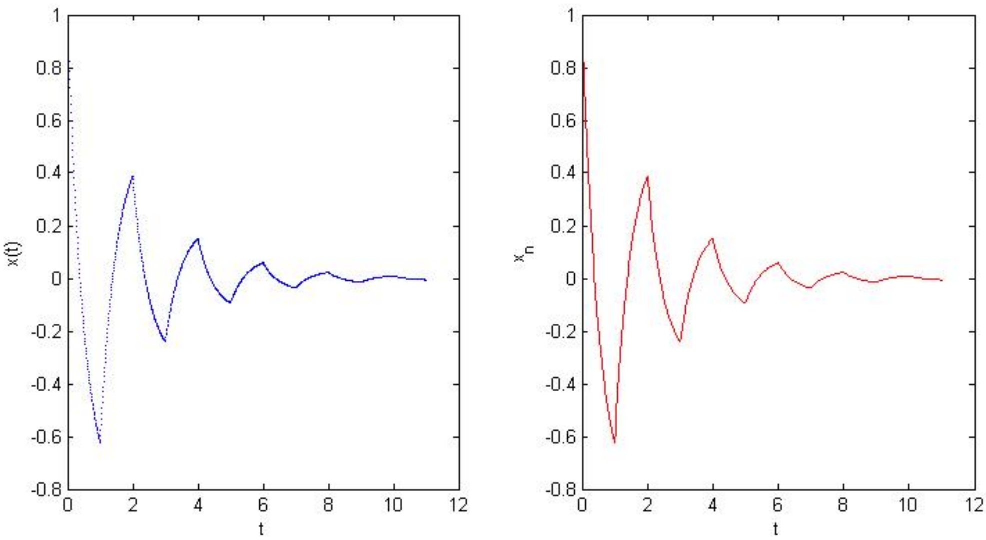

Figure 4.4: The analytic solution (left) and the numerical solution (right, $n=4$) of (4.4).

In particular, when $a=0$, Eq. (4.1) becomes

$$x'(t)=-2x([t])+5x([t+1]),\quad x(0)=1, \tag{4.5}$$

that is, $a_0=-2, a_1=5$, so we have

$$(a_0+a_1)\,(a_1-a_0-2)=15>0,\quad a_0<-1 \text{ and } a_1\geq 1,$$

so (C1) in Theorem 3.5 holds true.

From Figure 4.5 we also see that the analytic solution and the numerical solution of (4.5) are asymptotically stable and non-oscillatory, which is agreement with Theorem 3.5 (C1).

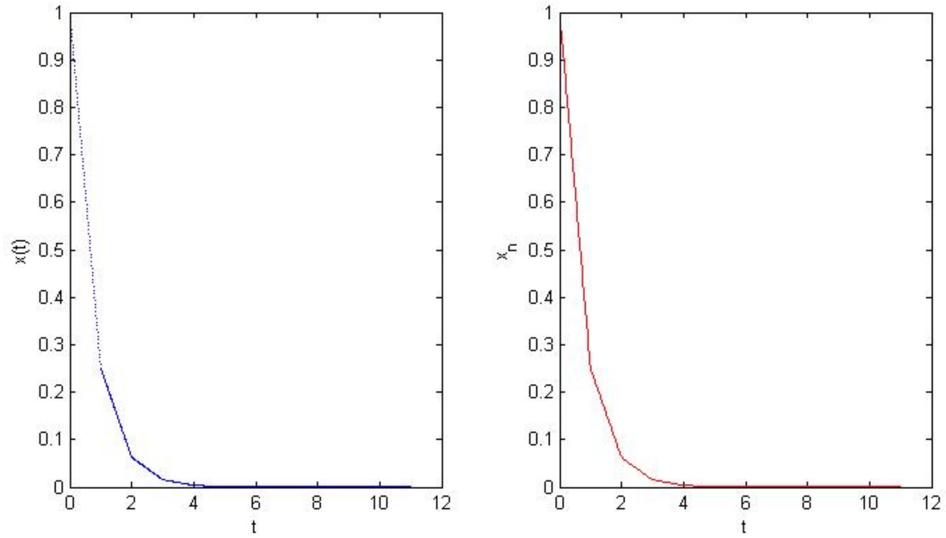

Figure 4.5: The analytic solution (left) and the numerical solution (right) of (4.5).

When $a=0$, Eq. (4.2) becomes

$$x'(t)=4x([t])-3x([t+1]), \quad x(0)=1, \tag{4.6}$$

that is, $a_0=4, a_1=-3$, so we have

$$(a_0+a_1)(a_1-a_0-2)=-9\le 0, \quad a_0>-1 \text{ and } a_1\le 1,$$

so (C2) in Theorem 3.5 holds true.
From Figure 4.6 we also see that the analytic solution and the numerical solution of (4.6) are unstable and non-oscillatory, which is agreement with Theorem 3.5 (C2).

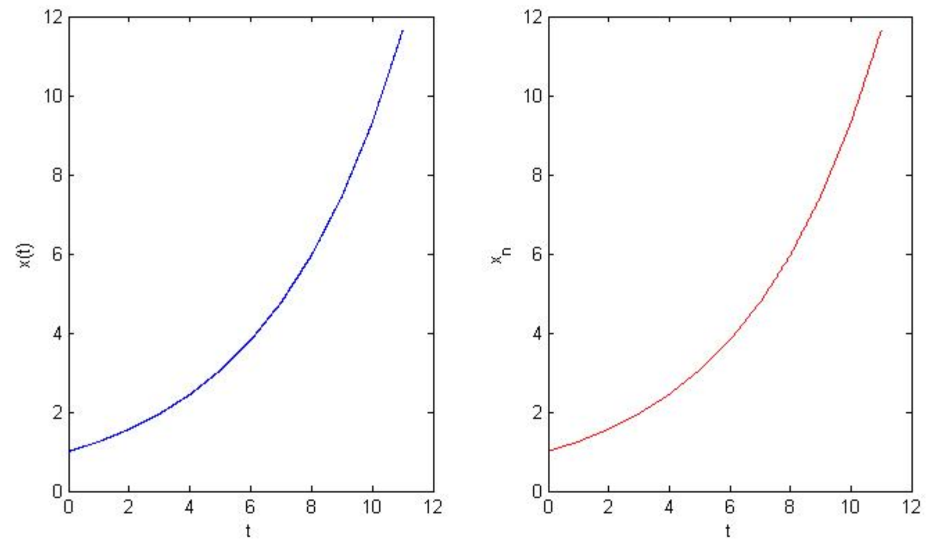

Figure 4.6: The analytic solution (left) and the numerical solution (right) of (4.6).

When $a=0$, Eq. (4.3) becomes

$$x'(t)=x([t])+2x([t+1]), \quad x(0)=1, \tag{4.7}$$

that is, $a_0=1, a_1=2$, so we have

$$(a_0+a_1)(a_1-a_0-2)=-3\le 0, \quad a_0>-1 \text{ and } a_1>1,$$

so (C3) in Theorem 3.5 holds true.
From Figure 4.7 we also see that the analytic solution and the numerical solution of (4.7) are unstable and oscillatory, which is agreement with Theorem 3.5 (C3).

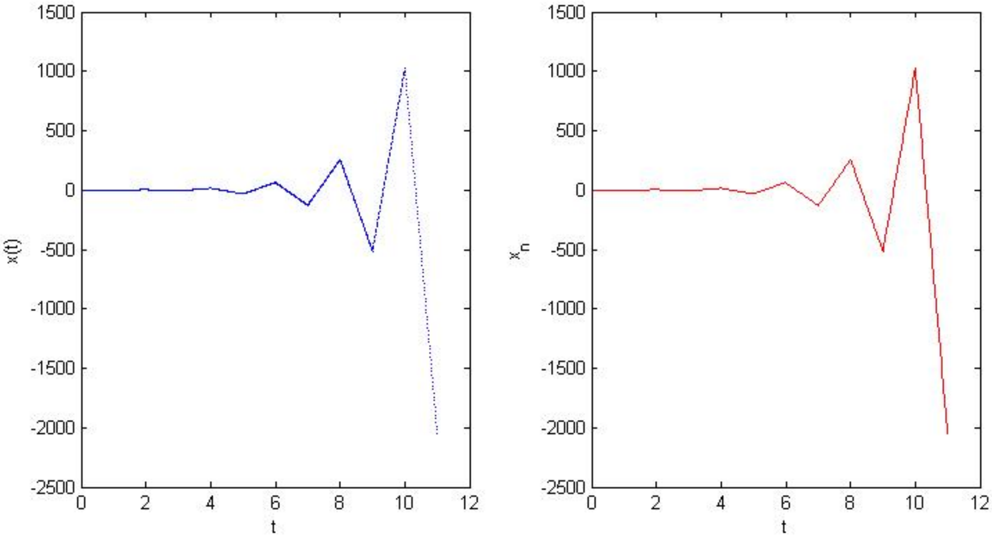

Figure 4.7: The analytic solution (left) and the numerical solution (right) of (4.7).

When $a = 0$, Eq. (4.4) becomes

$$x'(t) = -3x([t]) - 2x([t+1]), \quad x(0) = 1, \tag{4.8}$$

that is, $a_0 = -3, a_1 = -2$, so we have

$$(a_0 + a_1)(a_1 - a_0 - 2) = 5 > 0, \quad a_0 < -1 \text{ and } a_1 < 1,$$

so (C4) in Theorem 3.5 holds true.
From Figure 4.8 we also see that the analytic solution and the numerical solution of (4.8) are asymptotically stable and oscillatory, which is agreement with Theorem 3.5 (C4).

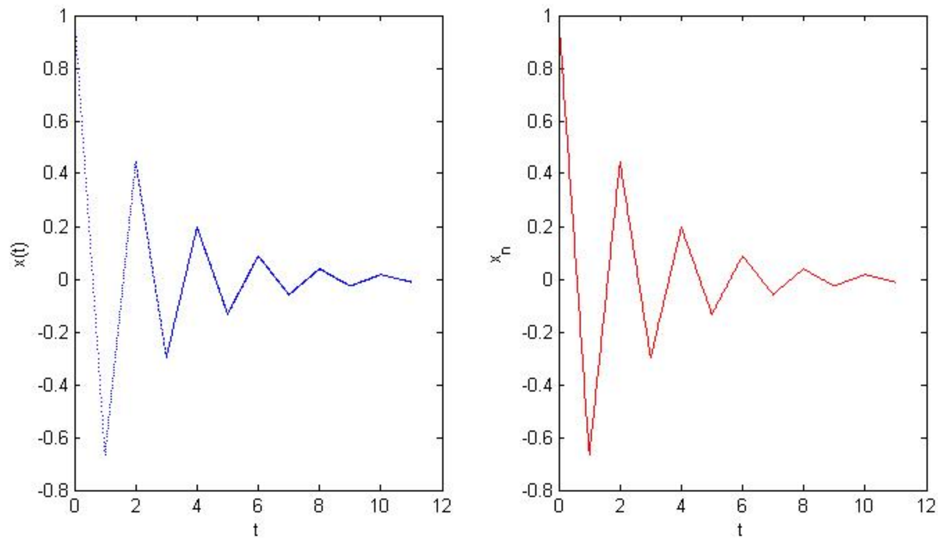

Figure 4.8: The analytic solution (left) and the numerical solution (right) of (4.8).

5. Conclusion

In this paper, the Euler-Maclaurin methods are applied to discrete differential equations with piecewise constant arguments of advanced and retarded type. We obtained the stability, oscillation conditions and convergence order of numerical methods. The type of Euler-Maclaurin methods for solving differential equations with piecewise constant arguments is extended and the results of corresponding literature are generalized. In the future, we will consider the application of the numerical method to the multi-dimensional and fractional cases.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Appendix A

The following code is the Matlab detail of Table 1.

```
%n = 2
syms d;
a = -1;
a0 = -2;
a1 = 5;
x0 = 1;
t = 10;
m = 2;
h = 1/m;
z = h*a;
R1 = symsum(1/d^2,1,inf);R1 = double(R1);
R2 = symsum(1/d^4,1,inf);R2 = double(R2);
A = 1-z/2+R1*z^2/(2*pi^2)-R2*z^4/(8*pi^4);
R = 1+z/A;
k1 = (R^m+(a0/a)*(R^m-1))/(1-(a1/a)*(R^m-1));
x = zeros(1,11);
x(1) = x0;
for k = 1:10
    x(k+1) = k1*x(k)
end
b0 = (exp(a)+(exp(a)-1)*(a0/a))/(1-(a1/a)*(exp(a)-1));
X = b0^10;
AE = abs(x(11)-X)
RE = abs(AE/X)
```

Appendix B

The following code is the Matlab detail of Figure 4.1.

```
a = -1;
a0 = -2;
a1 = 5;
x0 = 1;
%t = 10;
m = 50;
h = 1/m;
z = h*a;
for j = 1:3
    for k = 1:10
        B = 2*(-1)^(j+1)*factorial(2*j)*sum((2*k*pi)^(-2*j));
        A = 1-z/2+sum((B*z^(2*j))/factorial(2*j));
    end
end
R = 1+z/A;
k1 = (R^m+(a0/a)*(R^m-1))/(1-(a1/a)*(R^m-1));
x = zeros(1,12*m);
%x(0) = x0;
x(m) = x0;
t = zeros(1,11*m+1);
for k = 1:11
    x(m*(k+1)) = k1*x(m*k);
    for l = 0:m-2
        k2 = R^(l+1)+(a0/a)*(R^(l+1)-1);
        k3 = (a1/a)*(R^(l+1)-1);
        x(k*m+l+1) = k2*x(k*m)+k3*x((k+1)*m);
    end
end
y = x(m:end);
for i = 0:11*m
```

```
    t(i+1) = i/m;
end
subplot(1,2,2)
plot(t,y,'r-')
xlabel('t');
ylabel('x_n');
hold on;
for   n = 0 : 10
    for   t = n : 0.01 : n+1
        z = ((exp(a*(t-n)) + (a0/a)*(exp(a*(t-n))-1)) + (exp(a) + (a0/a)*(exp(a)-1))/(1-(a1/a)*(exp(a)-
1))*(a1/a)*(exp(a*(t-n))-1))*((exp(a) + (a0/a)*(exp(a)-1))/(1-(a1/a)*(exp(a)-1)))^n;
        subplot(1,2,1)
        plot(t,z,'b-.')
        hold on
    end
end
hold off
xlabel('t');
ylabel('x(t)');
```

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Numerical Algorithm for Solving General Linear Elliptic Quaternionic Matrix Equations

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Abstract

In this study, we develop a general method to solve the general linear elliptic quaternionic matrix equations by means of real representation of elliptic quaternion matrices. A pseudo-code for our approach that provides the solution of the linear elliptic quaternionic matrix equations is expressed. Moreover, we apply this method to the well-known Slyvester matrix equations and Kalman Yakubovich matrix equations over the elliptic quaternion algebra.

1. Introduction and Preliminaries

Real quaternions are a four-dimensional number system that was first expressed by Hamilton in 1843, based on the idea of generalizing complex numbers [1]. Hamilton first tried to describe the 3-dimensional number system as follows:

$$q = q_0 + q_1 i + q_2 j$$

where $q_0, q_1, q_2 \in \mathbb{R}$ and $i^2 = j^2 = -1$. However, he saw that this number system does not provide the closure property under multiplication. In this way, Hamilton saw that there could not be a system similar to any 3-dimensional complex number system and defined the 4-dimensional number system is known as the real quaternion in the following way:

$$\mathbb{K} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \ q_0, q_1, q_2, q_3 \in \mathbb{R} \text{ and } i, j, k \notin \mathbb{R}\} \tag{1.1}$$

such that

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ ik = -ki = -j, \ jk = -kj = i. \tag{1.2}$$

There are many applications of real quaternion algebra in different fields of the scientific world. The main areas are kinematics, mechanics, quantum physics, chemistry, image-signal restoration, and game development. For this reason, there are many studies related to real quaternions in literature [2]-[6].

On the other hand, Segre defined commutative quaternions in 1892 [7]. One of the most essential properties of a commutative quaternion is that it meets the commutative property of multiplication. The commutative quaternion algebra is a significant factor in fields such as Hopfield neural networks, digital signal, and image processing [8]-[11]. Therefore, commutative

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quaternion algebra theory has been increasingly important in recent years.

Elliptic quaternions are the generalized form of commutative quaternions. The set of elliptic quaternions is a commutative ring under a commutative law and combination law of a four-dimensional Clifford algebra. Moreover, this set contains non-trivial idempotents, nilpotent elements, and zero-divisors [8, 12, 13].
The set of elliptic quaternions with basic elements $1, i, j$ and k is represented as

$$\mathbb{H}_\alpha = \{a = a_0 + a_1 i + a_2 j + a_3 k : \ a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R} \text{ and } i, j, k \notin \mathbb{R}\} \tag{1.3}$$

which satisfy the equalities $i^2 = k^2 = \alpha,\ j^2 = 1,\ ij = ji = k,\ jk = kj = i,\ ki = ik = \alpha j,\ \alpha < 0,\ \alpha \in \mathbb{R}$, [8]. Addition of any two elliptic quaternions $a = a_0 + a_1 i + a_2 j + a_3 k,\ b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{H}_\alpha$ is given by $a + b = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$. Scalar multiplication of a elliptic quaternion $a \in \mathbb{H}_\alpha$ with a scalar $\lambda \in \mathbb{R}$ is expressed as $\lambda a = \lambda (a_0 + a_1 i + a_2 j + a_3 k) = \lambda a_0 + \lambda a_1 i + \lambda a_2 j + \lambda a_3 k$. In addition, the operation of the quaternionic multiplication of two elliptic quaternions $a, b \in \mathbb{H}_\alpha$ is expressed as

$$\begin{aligned} ab = &(a_0 b_0 + \alpha a_1 b_1 + a_2 b_2 + \alpha a_3 b_3) + (a_1 b_0 + a_0 b_1 + a_3 b_2 + a_2 b_3)i \\ &+ (a_0 b_2 + a_2 b_0 + \alpha a_1 b_3 + \alpha a_3 b_1)j + (a_3 b_0 + a_0 b_3 + a_1 b_2 + a_2 b_1)k. \end{aligned} \tag{1.4}$$

On the other hand, we know that the elliptic quaternion $a \in \mathbb{H}_\alpha$ has three types of the conjugate: ${}^1\overline{a} = a_0 - a_1 i + a_2 j - a_3 k$, ${}^2\overline{a} = a_0 - a_1 i - a_2 j + a_3 k$ and ${}^3\overline{a} = a_0 + a_1 i - a_2 j - a_3 k$. Additionally, the norm of the elliptic quaternion $a \in \mathbb{H}_\alpha$ is

$$\|a\| = \sqrt[4]{a\,({}^1\overline{a})\,({}^2\overline{a})\,({}^3\overline{a})} = \sqrt[4]{\left[(a_0 + a_2)^2 - \alpha(a_1 + a_3)^2\right]\ \left[(a_0 - a_2)^2 - \alpha(a_1 - a_3)^2\right]}. \tag{1.5}$$

If $a \in \mathbb{H}_\alpha$ and $\|a\| \neq 0$ then there exists multiplicative inverse of the elliptic quaternion a. So, multiplicative inverse of the elliptic quaternion a is $a^{-1} = \frac{({}^1\overline{a})({}^2\overline{a})({}^3\overline{a})}{\|a\|^4}$ [8, 12].

For

$$\mathbb{H}'_\alpha = \left\{ \begin{pmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \in \mathbb{R}^{4\times 4} : a_0,\ a_1,\ a_2, a_3 \in \mathbb{R} \right\}, \tag{1.6}$$

$\mathbb{H}_\alpha$ is algebraically isomorphic to the matrix algebra $\mathbb{H}'_\alpha$ through the bijective map

$$\phi : \mathbb{H}_\alpha \to \mathbb{H}'_\alpha,\ \ \phi_a = \begin{pmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}. \tag{1.7}$$

Thus, every elliptic quaternion $a \in \mathbb{H}_\alpha$ has a real matrix representation

$$\phi_a = \begin{pmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \tag{1.8}$$

in $\mathbb{H}'_\alpha$ [8].

Theorem 1.1. *([8, 12]). For $a, b \in \mathbb{H}_\alpha$ and $\lambda \in \mathbb{R}$, the following identities are satisfied:*

1. $a = b \Leftrightarrow \phi_a = \phi_b$,
2. $\phi_{(a+b)} = \phi_a + \phi_b$,
3. $\phi_{(ab)} = \phi_a \phi_b$,
4. $\phi_{(\phi_{(a)} b)} = \phi_a \phi_b$,
5. $\phi_{(\lambda a)} = \lambda \phi_a$,
6. $trace(\phi_a) = a + {}^1\overline{a} + {}^2\overline{a} + {}^3\overline{a}$,
7. $\|a\|^4 = |\det(\phi_a)|$.

Let's denote by $\mathbb{H}_\alpha^{m\times n}$ which is the set of all $m\times n$ type matrices with elliptic quaternion entries. $\mathbb{H}_\alpha^{m\times n}$ with the ordinary matrix summation and multiplication is a ring with identity. The conjugates of elliptic quaternion matrix $A=(a_{ij})\in\mathbb{H}_\alpha^{m\times n}$ which has three types of conjugate are given the following as:

$$^{1}\overline{A}=\left(^{1}\overline{a_{ij}}\right)\in\mathbb{H}_\alpha^{m\times n},\ ^{2}\overline{A}=\left(^{2}\overline{a_{ij}}\right)\in\mathbb{H}_\alpha^{m\times n}\ \textit{and}\ ^{3}\overline{A}=\left(^{3}\overline{a_{ij}}\right)\in\mathbb{H}_\alpha^{m\times n}.$$

Also, elliptic quaternion matrix $A=(a_{ij})\in\mathbb{H}_\alpha^{m\times n}$ can be expressed as $A=A_0+A_1i+A_2j+A_3k$ where $A_0,\ A_1,\ A_2,\ A_3\in\mathbb{R}^{m\times n}$. Then, $^{1}\overline{A}=A_0-A_1i+A_2j-A_3k$, $^{2}\overline{A}=A_0-A_1i-A_2j+A_3k$ and $^{3}\overline{A}=A_0+A_1i-A_2j-A_3k$. A matrix $A^T\in\mathbb{H}_\alpha^{n\times m}$ is transpose of $A\in\mathbb{H}_\alpha^{m\times n}$. Also $A^{*_s}=\left(^{s}\overline{A}\right)^T\in\mathbb{H}_\alpha^{m\times n}$, $s=1,2,3$, is called conjugate transpose with respect to the s^{th} conjugate of $A\in\mathbb{H}_\alpha^{m\times n}$, [12].

Theorem 1.2. *([12]) Let's assume that A and B are elliptic quaternion matrices of appropriate sizes. Then the following expressions are provided:*

1. $\left(^{s}\overline{A}\right)^T={}^{s}\overline{\left(A^T\right)}$,
2. $(AB)^{*_s}=B^{*_s}A^{*_s}$,
3. $(AB)^T=B^TA^T$,
4. $^{s}\overline{(AB)}=\left(^{s}\overline{A}\right)\left(^{s}\overline{B}\right)$,
5. *If* A^{-1} *and* B^{-1} *exist then* $(AB)^{-1}=B^{-1}A^{-1}$,
6. *If* A^{-1} *exists* $(A^{*_s})^{-1}=\left(A^{-1}\right)^{*_s}$,
7. $\left(^{s}\overline{A}\right)^{-1}={}^{s}\overline{\left(A^{-1}\right)}$.

For any elliptic quaternion matrix $A=A_0+A_1i+A_2j+A_3k\in\mathbb{H}_\alpha^{m\times n}$, the real representation Φ_A of the elliptic quaternion matrix A were given in [13] as follows,

$$\Phi_A=\begin{pmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3\\ A_1 & A_0 & A_3 & A_2\\ A_2 & \alpha A_3 & A_0 & \alpha A_1\\ A_3 & A_2 & A_1 & A_0\end{pmatrix}\in\mathbb{R}^{4m\times 4n}$$

in here $A_0,A_1,A_2,A_3\in\mathbb{R}^{m\times n}$, $\alpha\in\mathbb{R}$ and $\alpha<0$.

Theorem 1.3. *([13]) Let* $A,\ B\in\mathbb{H}^{m\times n}$, $C\in\mathbb{H}_\alpha^{n\times p}$ *and* $\lambda\in\mathbb{R}$ *be given. In that case, following identities for the elliptic quaternion matrix are satisfied:*

1. $A=B\Leftrightarrow\Phi_A=\Phi_B$, $\Phi_{A+B}=\Phi_A+\Phi_B$,
2. $\Phi_{AC}=\Phi_A\Phi_C$, $\Phi_{\lambda A}=\lambda\Phi_A$,
3. $A=\frac{1}{2-2\alpha}E_{4m}\Phi_A\left(^{1}\overline{E}_{4n}\right)^T$ *where* $E_{4t}=\begin{pmatrix}I_t & iI_t & jI_t & kI_t\end{pmatrix}\in\mathbb{H}^{t\times 4t}$,
4. *If A is a nonsingular matrix of size m, then*

$$\Phi_{A^{-1}}=\Phi_A^{-1},\quad A^{-1}=\frac{1}{2-2\alpha}E_{4m}\Phi_A^{-1}\left(^{1}\overline{E}_{4n}\right)^T,$$

5. $\Phi_{A^-}=\Phi_A^-$, $A^-=\frac{1}{2-2\alpha}E_{4m}\Phi_A^-\left(^{1}\overline{E}_{4n}\right)^T$ *are generalized inverse of* Φ_A *and* A, *respectively,*
6. $\Phi_A=R_{4m}^{-1}\Phi_AR_{4n}$, $\Phi_A=S_{4m}^{-1}\Phi_AS_{4n}$ *and* $\Phi_A=T_{4m}^{-1}\Phi_AT_{4n}$ *where*

$$R_{4t}=\begin{pmatrix}0 & \alpha I_t & 0 & 0\\ I_t & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha I_t\\ 0 & 0 & I_t & 0\end{pmatrix},\ S_{4t}=\begin{pmatrix}0 & 0 & I_t & 0\\ 0 & 0 & 0 & I_t\\ I_t & 0 & 0 & 0\\ 0 & I_t & 0 & 0\end{pmatrix},\ T_{4t}=\begin{pmatrix}0 & 0 & 0 & \alpha I_t\\ 0 & 0 & I_t & 0\\ 0 & \alpha I_t & 0 & 0\\ I_t & 0 & 0 & 0\end{pmatrix}.$$

2. On solutions of general linear elliptic quaternionic matrix equations

In this section, we study the solutions of the equations

$$A_1XB_1+\cdots+A_lXB_l=C \tag{2.1}$$

by means of the real representations of elliptic quaternion matrices, where $A_s\in\mathbb{H}_\alpha^{m\times n}$, $B_s\in\mathbb{H}_\alpha^{p\times q}$, $C\in\mathbb{H}_\alpha^{m\times q}$ and $s=1,2,3,\dots,l$.

Theorem 2.1. *The elliptic quaternionic matrix equation given by (2.1) has a solution X if and only if the real matrix equation*

$$\Phi_{A_1} Y \Phi_{B_1} + \cdots + \Phi_{A_l} Y \Phi_{B_l} = \Phi_C \tag{2.2}$$

has a solution $Y \in \mathbb{R}^{4n\times 4p}$*, in which case, if* $Y \in \mathbb{R}^{4n\times 4p}$ *is a solution of the real matrix equation (2.2), then the matrix*

$$X = \frac{1}{2-2\alpha} E_{4n} Y' \left({}^{1}\overline{E}_{4p} \right)^{T} \tag{2.3}$$

is a solution of (2.1) where

$$Y' = \frac{1}{4}\left(Y + R_{4n} Y R_{4p}^{-1} + S_{4n} Y S_{4p}^{-1} + T_{4n} Y T_{4p}^{-1} \right) \tag{2.4}$$

and

$$E_{4t} = \begin{pmatrix} I_t & iI_t & jI_t & kI_t \end{pmatrix} \in \mathbb{H}_{\alpha}^{t\times 4t},\ t = n, p.$$

Proof. Suppose that the real matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix},\ Y_{uv} \in \mathbb{R}^{n\times p},\ u,v = 1,2,3,4 \tag{2.5}$$

is a solution to the equation (2.2), then, we say that the matrix given in (2.3) is a solution to equation (2.1). According to Theorem 1.3, we get

$$\Phi_{A_s} = R_{4m}^{-1}\Phi_{A_s}R_{4n},\ \ \Phi_{B_s} = R_{4p}^{-1}\Phi_{B_s}R_{4q} \text{ and } \ \Phi_C = R_{4m}^{-1}\Phi_C R_{4q},$$

$$\Phi_{A_s} = S_{4m}^{-1}\Phi_{A_s}S_{4n},\ \ \Phi_{B_s} = S_{4p}^{-1}\Phi_{B_s}S_{4q} \text{ and } \ \Phi_C = S_{4m}^{-1}\Phi_C S_{4q},$$

$$\Phi_{A_s} = T_{4m}^{-1}\Phi_{A_s}T_{4n},\ \ \Phi_{B_s} = T_{4p}^{-1}\Phi_{B_s}T_{4q} \text{ and } \ \Phi_C = T_{4m}^{-1}\Phi_C T_{4q}.$$

where $s = 1,2,3,\ldots,l$. Substituting them into (2.2), respectively, and simplifying the corresponding equation, we have three equations as follows,

$$\begin{aligned} &\Phi_{A_1}\left(R_{4n}YR_{4p}^{-1}\right)\Phi_{B_1} + \cdots + \Phi_{A_l}\left(R_{4n}YR_{4p}^{-1}\right)\Phi_{B_l} = (\Phi_C)\,, \\ &\Phi_{A_1}\left(S_{4n}YS_{4p}^{-1}\right)\Phi_{B_1} + \cdots + \Phi_{A_l}\left(S_{4n}YS_{4p}^{-1}\right)\Phi_{B_l} = (\Phi_C)\,, \\ &\Phi_{A_1}\left(T_{4n}YT_{4p}^{-1}\right)\Phi_{B_1} + \cdots + \Phi_{A_l}\left(T_{4n}YT_{4p}^{-1}\right)\Phi_{B_l} = (\Phi_C)\,. \end{aligned} \tag{2.6}$$

This equation express that if Y is a solution of the equation given by (2.2), then $R_{4n}YR_{4p}^{-1}$, $S_{4n}YS_{4p}^{-1}$ and $T_{4n}YT_{4p}^{-1}$ are also solutions of the real matrix equation defined by (2.2). Thus the undermentioned real matrix:

$$Y' = \frac{1}{4}\left(Y + R_{4n} Y R_{4p}^{-1} + S_{4n} Y S_{4p}^{-1} + T_{4n} Y T_{4p}^{-1} \right) \tag{2.7}$$

is a solution to (2.2). By substituting (2.5) in (2.7) and making necessary simplifications, it can easily be written by

$$Y' = \begin{pmatrix} Z_0 & \alpha Z_1 & Z_2 & \alpha Z_3 \\ Z_1 & Z_0 & Z_3 & Z_2 \\ Z_2 & \alpha Z_3 & Z_0 & \alpha Z_1 \\ Z_3 & Z_2 & Z_1 & Z_0 \end{pmatrix}$$

where

$$
\begin{aligned}
Z_0 &= \tfrac{1}{4}\left(Y_{11}+Y_{22}+Y_{33}+Y_{44}\right), \ Z_1 = \tfrac{1}{4}\left(\tfrac{Y_{12}}{\alpha}+Y_{21}+\tfrac{Y_{34}}{\alpha}+Y_{43}\right), \\
Z_2 &= \tfrac{1}{4}\left(Y_{13}+Y_{24}+Y_{31}+Y_{42}\right), \ Z_3 = \tfrac{1}{4}\left(\tfrac{Y_{14}}{\alpha}+Y_{23}+\tfrac{Y_{32}}{\alpha}+Y_{41}\right).
\end{aligned} \tag{2.8}
$$

Thus, we get $\Phi_X = Y'$. From Theorem 1.3, we obtain

$$
X = \frac{1}{2-2\alpha}\left(I_n\, iI_n\, jI_n\, kI_n\right)Y'\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix} = Z_0+Z_1 i+Z_2 j+Z_3 k. \tag{2.9}
$$

Moreover, since $\Phi_X = Y'$ the elliptic quaternionic matrix equation given in (2.1) has a solution if and only if the real matrix equation given in (2.2) has a solution.

□

3. Numerical algorithm

Considering the discussions in the previous section, now, we provide numerical Algorithm for solving general linear elliptic quaternionic matrix equation

$$
A_1XB_1+\cdots+A_lXB_l = C
$$

where $A_s \in \mathbb{H}_\alpha^{m\times n}$, $B_s \in \mathbb{H}_\alpha^{p\times q}$, $C \in \mathbb{H}_\alpha^{m\times q}$ and $s = 1,2,3,...,l$.

Algorithm 1 Numerical Algorithm for Solving General Linear Elliptic Quaternionic Matrix Equations

1: **Begin**
2: Input $A_s \in \mathbb{H}_\alpha^{m\times n}$, $B_s \in \mathbb{H}_\alpha^{p\times q}$ and $C \in \mathbb{H}_\alpha^{m\times q}$ where $1 \le s \le l$.
3: Form Φ_{A_s}, Φ_{B_s} and Φ_C.
4: Compute Y and $Y' = \frac{1}{4}\left(Y + R_{4n}YR_{4p}^{-1} + S_{4n}YS_{4p}^{-1} + T_{4n}YT_{4p}^{-1}\right)$.
5: Calculate $X = \frac{1}{2-2\alpha}\left(I_n \;\; iI_n \;\; jI_n \;\; kI_n\right)Y'\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix}$.
6: **End**

4. Numerical examples

For $l = 2$, the special case of (2.1) is given by

$$
A_1XB_1+A_2XB_2 = C \tag{4.1}
$$

where $A_1, A_2 \in \mathbb{H}_\alpha^{m\times n}$, $B_1, B_2 \in \mathbb{H}_\alpha^{p\times q}$ and $C \in \mathbb{H}_\alpha^{m\times q}$. If $B_1 = I_p$, $A_2 = -I_n$, $m = n$, $p = q$ are taken in (4.1), we have elliptic quaternionic Sylvester matrix equation $AX - XB = C$. Similarly, $A_1 = I_n$, $B_1 = I_p$, $m = n$, $p = q$, $A_2 = -A$ and $B_2 = B$ are taken in (4.1) we have elliptic quaternionic Kalman-Yakubovich matrix equation $X - AXB = C$.

In the literature, the equations $AX - XB = C$ and $X - AXB = C$ are known as the Sylvester matrix equation and the Kalman-Yakubovich matrix equation, respectively. These equations play an important role in control theory, signal processing, filtering, image restoration, decoupling techniques for ordinary and partial differential equations, and block-diagonalization of matrices, [14]-[18]. In this section, we obtain the solutions of the given elliptic quaternionic matrix equations $AX - XB = C$ and $X - AXB = C$ according to our Algorithm.

Note that all computations in the rest of the paper are performed on an Intel i7-3630QM@2.40 GHz/16GB computer using MATHEMATICA 9 software.

Let's take $\alpha = -2$ specifically to solve the elliptic quaternionic Kalman Yakubovich matrix equation

$$
X - \begin{pmatrix} 1+k & i \\ j-k & 1-j \end{pmatrix} X \begin{pmatrix} j & 1+2i \\ k & i+j \end{pmatrix} = \begin{pmatrix} 3+i+3j+k & 2+2i+7j+k \\ 5+2i-6j+k & -7-2i-2j+8k \end{pmatrix}.
$$

Real representation of given equation is

$$Y-\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & -2 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} Y \begin{pmatrix} 0 & 1 & 0 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 3 & 4 & -2 & 6 & 1 & 7 & 0 & -2 \\ 7 & -12 & 0 & 6 & -7 & 3 & -4 & -18 \\ 1 & -3 & 3 & 4 & 0 & 1 & 1 & 7 \\ 0 & -3 & 7 & -12 & 2 & 9 & -7 & 3 \\ 1 & 7 & 0 & -2 & 3 & 4 & -2 & 6 \\ -7 & 3 & -4 & -18 & 7 & -12 & 0 & 6 \\ 0 & 1 & 1 & 7 & 1 & -3 & 3 & 4 \\ 2 & 9 & -7 & 3 & 0 & -3 & 7 & -12 \end{pmatrix}.$$

If we solve this equation, we have

$$Y=\begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2 & -4 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$X=\frac{1}{24}\begin{pmatrix} I_2 & iI_2 & jI_2 & kI_2 \end{pmatrix}\left(Y+R_8YR_8^{-1}+S_8YS_8^{-1}+T_8YT_8^{-1}\right)\begin{pmatrix} I_2 \\ -iI_2 \\ jI_2 \\ -kI_2 \end{pmatrix}$$

$$=\begin{pmatrix} 1-i & j \\ 1+k & j+2k \end{pmatrix}.$$

Similarly, let's take $\alpha=-5$ specifically to solve the elliptic quaternionic Sylvester matrix equation

$$\begin{pmatrix} 1+i & i+3j+2k \\ 3k & 2 \end{pmatrix}X-X\begin{pmatrix} i & j+2k \\ 5+i & 2-3j \end{pmatrix}=\begin{pmatrix} -46+13i-19j+k & -19+6i-35j+15k \\ 25-22i-8j+7k & 48-6i+21k \end{pmatrix}.$$

The solution of real representation of given elliptic quaternionic Sylvester matrix equation is

$$Y=\begin{pmatrix} 1 & 2 & -5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -25 & 1 & 3 & -20 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 4 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 & 1 & 2 & -5 & 0 \\ 1 & 3 & -20 & 0 & 0 & 0 & 0 & -25 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 4 & 0 & 1 & 3 & 0 & 5 & 0 & 0 \end{pmatrix}.$$

Thus, we get

$$X=\frac{1}{24}\begin{pmatrix} I_2 & iI_2 & jI_2 & kI_2 \end{pmatrix}\left(Y+R_8YR_8^{-1}+S_8YS_8^{-1}+T_8YT_8^{-1}\right)\begin{pmatrix} I_2 \\ -iI_2 \\ jI_2 \\ -kI_2 \end{pmatrix}$$

$$=\begin{pmatrix} 1+i+j & 2 \\ j+4k & 5i+3j \end{pmatrix}.$$

5. Conclusion

In this study, we established the solution of general linear elliptic quaternionic matrix equations with the help of the real representation of elliptic quaternion matrices and expressed an Algorithm for the solutions of these equations. In addition, we investigated solutions of elliptic quaternionic Sylvester and Kalman Yakubovich matrix equations, which are essential applications in various areas of science. Actually, general linear matrix equations over the complex field form a special class of general linear elliptic quaternionic matrix equations. Thus, the obtained results extend, generalize and complement the scope of general linear matrix equations known in the literature.

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Some New Traveling Wave Solutions of Nonlinear Fluid Models via the MSE Method

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Abstract

In this study, some new exact wave solutions of nonlinear partial differential equations are investigated by the modified simple equation method. This method is applied to the $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation and the $(3+1)$-dimensional Jimbo-Miwa equation. Our applications reveal how to use the proposed method to solve nonlinear partial differential equations with the balance number equal to two. Consequently, some new exact traveling wave solutions of these equations are achieved, and types of waves are determined. To verify our results and draw the graphs of the solutions, we use the Mathematica package program.

1. Introduction

Nonlinear partial differential equations (NPDEs) have proved to be precious instruments for the modelling of physical phenomena, and have been the focus of many researchers due to their extensive use in several areas such as mathematical physics, biology, nonlinear optics, fluid mechanics, ocean engineering, chemical physics, plasma physics etc. [1]-[6]. Thus, it has gained great importance in the literature to examine the solutions of these equations to explain the nonlinear complex processes in nature. However, exact solutions of equations in the nonlinear form are not always obtained by classical methods. In recent times, many useful methods and techniques such as the modified simple equation (MSE) method [7], the improved $\tan(\varphi/2)$-expansion method [8], the extended rational sine-cosine method [9], the $(G'/G, 1/G)$-expansion method [10], the improved F-expansion method [11], the modified exp $(-\phi(\varepsilon))$-expansion method [12], the first integral method [13], the (G'/G)-expansion method [14] etc. have been enhanced to find traveling wave solutions. In this paper, we propose the MSE method, which is a remarkable and useful method for finding various solutions of NPDEs. This method converts NPDEs into nonlinear ordinary differential equations (NODEs) with wave transformation. Also, the advantage of the proposed method is that the general solution form is defined as the sum of the finite series and an unknown function in this solution form is determined according to the solution of a system of algebraic equations obtained from the main equation. Compared to other methods in the literature such as $(G'/G, 1/G)$-expansion method, the sine-cosine method, the improved F-expansion method, the (G'/G)-expansion method, etc., the MSE method does not require symbolic computational software programs to solve algebraic equation systems. In addition, the unknown function in this method is not depend on a pre-defined function or a solution of the ODE, and the obtained exact solutions have arbitrary coefficients. Thus, the traveling wave solutions can be obtained in a new and extensible form. We observe that this method is highly systematic, understandable and applicable. We perform the MSE method to NPDEs, namely, the $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation [15] and the $(3+1)$-dimensional Jimbo-Miwa equation [16]. The CBS equation is a frequently used model in fluid dynamics that describes and explains situations such as fusion, annihilation and fission of complex waves [17]. The Jimbo-Miwa equation is

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used in fluid mechanics to define some specific $(3+1)$-dimensional nonlinear waves, and this equation is the second equation in the notable Kadomtsev–Petviashvili hierarchy of integrable systems [18]. As a result, new exact solutions of the equations are obtained and their graphs are drawn to observe the physical behaviors of these solutions. The article is concerted in the following: In Sec. 2, we summarize the illustration of the MSE method. In Sec. 3, applications of the MSE method are given. In Sec. 4, we draw graphs of wave solutions and physical explanations. Sec. 5 includes the conclusion.

2. The modified simple equation method

In this section, we present the major steps of the MSE method [7]:
Consider the NPDE in the following:

$$G\left(u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},...\right)=0, \tag{2.1}$$

where G is a polynomial of $u\left(x,y,t\right)$ and its several partial derivatives.
Step 1. We use the traveling wave transformation

$$u\left(x,y,t\right)=u\left(\Upsilon\right),\;\;\Upsilon=x+y-\Theta t, \tag{2.2}$$

to reduce (2.1) into the succeeding NODE:

$$R\left(u,u',u'',u''',...\right)=0, \tag{2.3}$$

where R is a polynomial in $u\left(\Upsilon\right)$ and its all derivatives with respect to Υ.
Step 2. Suppose that the solution of (2.3) can be expressed in the form,

$$u\left(\Upsilon\right)=\sum_{k=0}^{N}A_k\left[\frac{\phi'\left(\Upsilon\right)}{\phi\left(\Upsilon\right)}\right]^k, \tag{2.4}$$

where A_k are arbitrary constants $\left(A_N\neq 0\right)$ and $\phi\left(\Upsilon\right)$ is an unknown function to be calculated.
Step 3. We determine balancing number N in (2.4) by considering the homogeneous balance between the highest order nonlinear terms and the highest order derivatives occurred in (2.3).
Step 4. We replace (2.4) and its derivatives into (2.3). Hereby, we have a polynomial of $\phi\left(\Upsilon\right)$. Then, we equalize all the coefficients of $\phi^{-i}\left(\Upsilon\right)$ $\left(i=0,1,2\dots\right)$ to zero in this polynomial. This operation gives a system of equations to obtain A_k and $\phi\left(\Upsilon\right)$. Thus, we achieve the exact solution of (2.1).

3. Applications

In this section, the MSE method is applied to nonlinear equations which express some special physical phenomena and wave solutions of these equations are obtained.

3.1. $(2+1)$-dimensional Calogera-Bogoyavlenskii-Schiff (CBS) equation

This equation was examined by Schiff and Bogoyavlenskii in varied ways. Bogoyavlenskii used the modified Lax formalism, while Schiff obtained the similar equation by reducing the self-dual Yang-Mills equation [19]. This equation has various forms for different coefficients. Also, many studies in the literature obtain different solution types of this equation [17], [20]-[23]. The $(2+1)$-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation is as follows [15]:

$$u_{xxxy}+2u_yu_{xx}+4u_xu_{xy}+u_{xt}=0, \tag{3.1}$$

where x, y represent the position of the wave and t represents the time. Applying the wave transformation in (2.2) to (3.1), integrating once respect to Υ and considering the integration constant as zero, we attain nonlinear ODE in the following form:

$$u'''+3\left(u'\right)^2-\Theta u'=0. \tag{3.2}$$

Now, using the transformation $u'(\Upsilon)=v\left(\Upsilon\right)$, (3.2) reduces to

$$v''+3v^2-\Theta v=0. \tag{3.3}$$

Balancing v'' and v^2 in (3.3), we find $N=2$. Consequently, (2.4) turns into the following form:

$$v\left(\Upsilon\right)=A_0+A_1\left(\frac{\phi'\left(\Upsilon\right)}{\phi\left(\Upsilon\right)}\right)+A_2\left(\frac{\phi'\left(\Upsilon\right)}{\phi\left(\Upsilon\right)}\right)^2. \tag{3.4}$$

Substituting (3.4) and its derivatives into (3.3), and setting all the coefficients with the same power of $\phi^{-i}(\Upsilon)$, we attain a system as follows:

$$
\begin{aligned}
(\phi)^0 &: 3A_0^2 - \Theta A_0 = 0, \\
(\phi)^{-1} &: 6A_0A_1\phi'(\Upsilon) - \Theta A_1\phi'(\Upsilon) + A_1\phi'''(\Upsilon) = 0, \qquad (3.5) \\
(\phi)^{-2} &: 3A_1^2(\phi'(\Upsilon))^2 + 6A_0A_2(\phi'(\Upsilon))^2 - \Theta A_2(\phi'(\Upsilon))^2 - 3A_1\phi'(\Upsilon)\phi''(\Upsilon) \\
&\quad +2A_2\phi'''(\Upsilon)\phi'(\Upsilon) + 2A_2(\phi''(\Upsilon))^2 = 0, \qquad (3.6) \\
(\phi)^{-3} &: 6A_1A_2(\phi'(\Upsilon))^3 + 2A_1(\phi'(\Upsilon))^3 - 10A_2(\phi'(\Upsilon))^2\phi''(\Upsilon) = 0, \qquad (3.7) \\
(\phi)^{-4} &: 3A_2^2(\phi'(\Upsilon))^4 + 6A_2(\phi'(\Upsilon))^4 = 0.
\end{aligned}
$$

Case 1: $A_0 = 0$, $A_1 \neq 0$, $A_2 = -2$ and $\phi'(\Upsilon) \neq 0$. In this case, by using (3.5) and (3.7), we obtain $\phi'(\Upsilon) = 2\frac{c_1}{A_1}e^{\frac{2\Theta}{A_1}\Upsilon}$ and $\phi(\Upsilon) = \frac{c_1}{\Theta}e^{\frac{2\Theta}{A_1}\Upsilon} + c_2$. Here and throughout the paper, c_1 and c_2 are arbitrary constants of integration. Then, we use these equations and (3.6), we achieve $A_1 = \pm 2\sqrt{\Theta}$. Inserting A_0, A_1, A_2, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.4), we deduce the exact solution of (3.1) as follows:

$$
v(\Upsilon) = \pm 2\sqrt{\Theta}\left(\frac{\pm\frac{c_1}{\sqrt{\Theta}}e^{\pm\sqrt{\Theta}\Upsilon}}{\frac{c_1}{\Theta}e^{\pm\sqrt{\Theta}\Upsilon} + c_2}\right) - 2\left(\frac{\pm\frac{c_1}{\sqrt{\Theta}}e^{\pm\sqrt{\Theta}\Upsilon}}{\frac{c_1}{\Theta}e^{\pm\sqrt{\Theta}\Upsilon} + c_2}\right)^2,
$$

where $\Upsilon = x + y - \Theta t$.
Now, by using hyperbolic function features, we obtain the wave solutions when $c_1 = \Theta$ and $c_2 = 1$ as:

$$
v_{1,2}(x,y,t) = \Theta\left(1 + \tanh\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right) - \frac{\Theta}{2}\left(1 + \tanh\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right)^2,
$$

$$
u_{1,2}(x,y,t) = \sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \qquad (3.8)
$$

When $c_1 = \Theta$, $c_2 = -1$ as:

$$
v_{3,4}(x,y,t) = \Theta\left(1 + \coth\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right) - \frac{\Theta}{2}\left(1 + \coth\left(\frac{\pm\sqrt{\Theta}}{2}(x+y-\Theta t)\right)\right)^2,
$$

$$
u_{3,4}(x,y,t) = \sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \qquad (3.9)
$$

Case 2: $A_0 = \frac{\Theta}{3}$, $A_1 \neq 0$, $A_2 = -2$ and $\phi'(\Upsilon) \neq 0$. By using (3.5) and (3.7), we obtain $\phi'(\Upsilon) = 2\frac{c_1}{A_1}e^{\frac{-2\Theta}{A_1}\Upsilon}$ and $\phi(\Upsilon) = c_2 - \frac{c_1}{\Theta}e^{\frac{-2\Theta}{A_1}\Upsilon}$. Considering these equations and (3.6), we have $A_1 = \pm 2i\sqrt{\Theta}$. Now, inserting A_0, A_1, A_2, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.4), the exact solution of (3.1) follows as:

$$
v(\Upsilon) = \frac{\Theta}{3} \pm 2i\sqrt{\Theta}\left(\frac{\frac{\pm\sqrt{c_1}}{i\sqrt{\Theta}}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}{c_2 - \frac{c_1}{\Theta}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}\right) - 2\left(\frac{\frac{\pm\sqrt{c_1}}{i\sqrt{\Theta}}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}{c_2 - \frac{c_1}{\Theta}e^{\mp\frac{\sqrt{\Theta}}{i}\Upsilon}}\right)^2.
$$

where $\Upsilon = x + y - \Theta t$.
Hence, by using hyperbolic function features, we achieve the wave solutions for $c_1 = -\Theta$ and $c_2 = 1$ as:

$$
v_{5,6}(x,y,t) = \frac{\Theta}{3} - \Theta\left(1 + \tanh\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right) + \frac{\Theta}{2}\left(1 + \tanh\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right)^2,
$$

$$
u_{5,6}(x,y,t) = \frac{\Theta(x+y-\Theta t)}{3} - \sqrt{\Theta}\tan\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \qquad (3.10)
$$

For $c_1 = -\Theta$ and $c_2 = -1$ as:

$$v_{7,8}(x,y,t) = \frac{\Theta}{3} - \Theta\left(1+\coth\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right) + \frac{\Theta}{2}\left(1+\coth\left(\frac{\mp\sqrt{\Theta}}{2i}(x+y-\Theta t)\right)\right)^2,$$

$$u_{7,8}(x,y,t) = \frac{\Theta(x+y-\Theta t)}{3} + \sqrt{\Theta}\cot\left(\frac{\sqrt{\Theta}}{2}(x+y-\Theta t)\right). \tag{3.11}$$

3.2. (3+1)-dimensional Jimbo-Miwa equation

This equation appears in many areas of science, such as geochemistry, fluid mechanics, optical fiber, astrophysics, plasma physics, chemical kinematics and solid state physics [24]. Furthermore, there are many studies in the literature investigating the different forms of solutions for this equation [18], [25]-[28].
The $(3+1)$-dimensional Jimbo-Miwa equation is as follows [16]:

$$\begin{aligned} u_{xxxy} + 6u_x u_y + 3uv_{xx} + 3u_{xx}v + 3u_{yt} - 3u_{zz} &= 0, \\ u_y &= v_x, \end{aligned} \tag{3.12}$$

where x,y,z represent the position of the wave and t represents the time. Using the wave transformation in the following:

$$u(x,y,z,t) = u(\Upsilon), \quad v(x,y,z,t) = v(\Upsilon), \quad \Upsilon = x+y+z-\Theta t,$$

and three times integrating with respect to Υ, considering the integration constants as zero, (3.12) converts to nonlinear ODE:

$$u'' + 3u^2 - 3(\Theta+1)u = 0. \tag{3.13}$$

Balancing u'' and u^2 in (3.13), we get $N=2$. Therefore, (2.4) turns into the following form:

$$u(\Upsilon) = A_0 + A_1\left(\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right) + A_2\left(\frac{\phi'(\Upsilon)}{\phi(\Upsilon)}\right)^2. \tag{3.14}$$

Substituting (3.14) and its derivatives into (3.13), and editing all the coefficients with the same power of $\phi^{-i}(\Upsilon)$, we obtain a system as follows:

$$(\phi)^0 : 3A_0^2 - 3(\Theta+1)A_0 = 0,$$

$$(\phi)^{-1} : 6A_0A_1\phi'(\Upsilon) - 3(\Theta+1)A_1\phi'(\Upsilon) + A_1\phi'''(\Upsilon) = 0, \tag{3.15}$$

$$\begin{aligned} (\phi)^{-2} : 3A_1^2(\phi'(\Upsilon))^2 + 6A_0A_2(\phi'(\Upsilon))^2 - 3(\Theta+1)A_2(\phi'(\Upsilon))^2 \\ -3A_1\phi'(\Upsilon)\phi''(\Upsilon) + 2A_2\phi'''(\Upsilon)\phi'(\Upsilon) + 2A_2(\phi''(\Upsilon))^2 = 0, \end{aligned} \tag{3.16}$$

$$(\phi)^{-3} : 6A_1A_2(\phi'(\Upsilon))^3 + 2A_1(\phi'(\Upsilon))^3 - 10A_2(\phi'(\Upsilon))^2\phi''(\Upsilon) = 0, \tag{3.17}$$

$$(\phi)^{-4} : 3A_2^2(\phi'(\Upsilon))^4 + 6A_2(\phi'(\Upsilon))^4 = 0.$$

Case 1: $A_0 = 0$, $A_1 \neq 0$, $A_2 = -2$ and $\phi'(\Upsilon) \neq 0$. From (3.15) and (3.17), we get $\phi'(\Upsilon) = 2\frac{c_1}{A_1}e^{\frac{6(\Theta+1)}{A_1}\Upsilon}$ and $\phi(\Upsilon) = \frac{c_1}{3(\Theta+1)}e^{\frac{6(\Theta+1)}{A_1}\Upsilon} + c_2$. Then, by these equations and (3.16), we deduce $A_1 = \pm 2\sqrt{3(\Theta+1)}$. Substituting A_0, A_1, A_2, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.14) we have the exact solution of (3.12) as in the following:

$$u(\Upsilon) = \pm 2\sqrt{3(\Theta+1)}\left(\frac{\pm\frac{c_1}{\sqrt{3(\Theta+1)}}e^{\pm\sqrt{3(\Theta+1)}\Upsilon}}{\frac{c_1}{3(\Theta+1)}e^{\pm\sqrt{3(\Theta+1)}\Upsilon} + c_2}\right) - 2\left(\frac{\pm\frac{c_1}{\sqrt{3(\Theta+1)}}e^{\pm\sqrt{3(\Theta+1)}\Upsilon}}{\frac{c_1}{3(\Theta+1)}e^{\pm\sqrt{3(\Theta+1)}\Upsilon} + c_2}\right)^2.$$

where $\Upsilon = x+y+z-\Theta t$.
Hence, by using hyperbolic function properties, we get the wave solutions when $c_1 = 3(\Theta+1)$ and $c_2 = 1$ as:

$$\begin{aligned} u_{1,2}(x,y,z,t) = 3(\Theta+1)\left(1+\tanh\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right) \\ -\frac{3(\Theta+1)}{2}\left(1+\tanh\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right)^2. \end{aligned} \tag{3.18}$$

When $c_1 = 3(\Theta+1)$ and $c_2 = -1$ as:

$$u_{3,4}(x,y,z,t) = 3(\Theta+1)\left(1+\coth\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right) \\ -\frac{3(\Theta+1)}{2}\left(1+\coth\left(\frac{\pm\sqrt{3(\Theta+1)}}{2}(x+y+z-\Theta t)\right)\right)^2. \tag{3.19}$$

Case 2: $A_0 = \Theta+1$, $A_1 \neq 0$, $A_2 = -2$ and $\phi'(\Upsilon) \neq 0$. Taking (3.15) and (3.17) into account, we get $\phi'(\Upsilon) = 2\frac{c_1}{A_1}e^{\frac{-6(\Theta+1)}{A_1}\Upsilon}$ and $\phi(\Upsilon) = c_2 - \frac{c_1}{3(\Theta+1)}e^{\frac{-6(\Theta+1)}{A_1}\Upsilon}$. From these equations and (3.16), we have $A_1 = \pm 2i\sqrt{3(\Theta+1)}$. Substituting A_0, A_1, A_2, $\phi(\Upsilon)$ and $\phi'(\Upsilon)$ into (3.14), we get the exact solutions of (3.12) as follows:

$$u(\Upsilon) = (\Theta+1) \pm 2i\sqrt{3(\Theta+1)}\left(\frac{\frac{\pm c_1}{i\sqrt{3(\Theta+1)}}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}}{\frac{-c_1}{3(\Theta+1)}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}+c_2}\right) - 2\left(\frac{\frac{\pm c_1}{i\sqrt{3(\Theta+1)}}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}}{\frac{-c_1}{3(\Theta+1)}e^{\frac{\mp 3\sqrt{\Theta+1}}{i\sqrt{3}}\Upsilon}+c_2}\right)^2.$$

where $\Upsilon = x+y+z-\Theta t$.
Then, by using hyperbolic function properties, the wave solutions are obtained for $c_1 = -3(\Theta+1)$ and $c_2 = 1$ as:

$$u_{5,6}(x,y,z,t) = (\Theta+1) - 3(\Theta+1)\left(1+\tanh\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right) \\ -\frac{3(\Theta+1)}{2}\left(1+\tanh\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right)^2. \tag{3.20}$$

For $c_1 = -3(\Theta+1)$ and $c_2 = -1$ as:

$$u_{7,8}(x,y,z,t) = (\Theta+1) - 3(\Theta+1)\left(1+\coth\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right) \\ -\frac{3(\Theta+1)}{2}\left(1+\coth\left(\frac{\mp\sqrt{3(\Theta+1)}}{2i}(x+y+z-\Theta t)\right)\right)^2. \tag{3.21}$$

Moreover, the values of $v(x,y,z,t)$ can be easily calculated according to the $u_y = v_x$.

Consequently, the set of exact solutions for the CBS and the Jimbo-Miwa equations can be expanded by selecting more varied arbitrary constants c_1 and c_2.

4. Physical explanation and graphs

This part shows physical behaviour of the achieved exact wave solutions of the CBS and the Jimbo-Miwa equations. The MSE method is implemented to both equations and the new traveling wave solutions are obtained in (3.8), (3.9), (3.10), (3.11) and (3.18), (3.19), (3.20), (3.21), respectively. These results are drawn with proper values in different types of graphs and intervals such as 3D $(-8 \leq x,t \leq 8)$, 2D $(-8 \leq x \leq 8)$ and contour graph $(0 \leq x,t \leq 10)$. Other independent variables y and z are used with appropriate values in the solution graphs.

4.1. Graphs of solutions for the CBS equation:

Fig.4.1-(a), (b), (c), (d) demonstrate (3.8) $u_{1,2}(x,y,t)$, (3.9) $u_{3,4}(x,y,t)$ for $\Theta = 1.39$, and (3.10) $u_{5,6}(x,y,t)$, (3.11) $u_{7,8}(x,y,t)$ for $\Theta = 1.5$, respectively. Fig.4.2-(a)-(b) represent (3.8) $u_{1,2}(x,y,t)$ and (3.9) $u_{3,4}(x,y,t)$ for $\Theta = 1.39$, $t = 1$ and $y = 0$. Also, Fig.4.2-(c)-(d) show (3.10) $u_{5,6}(x,y,t)$ and (3.11) $u_{7,8}(x,y,t)$ for $\Theta = 1.5$, $t = 1$ and $y = 0$.

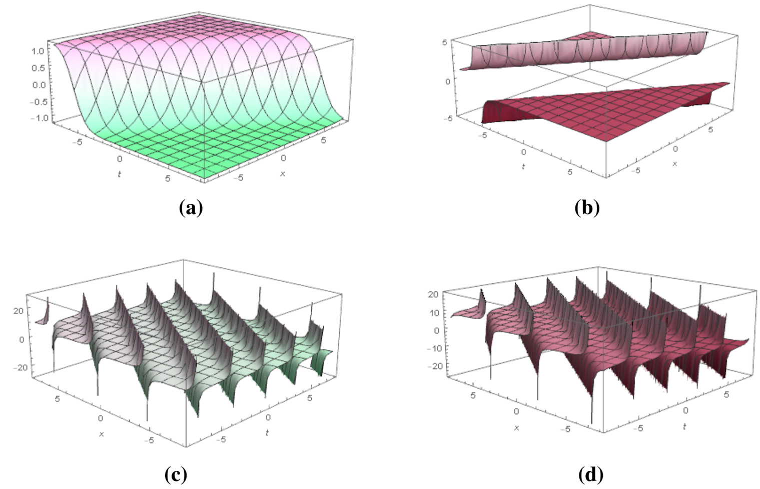


Figure 4.1: 3D-graphs.

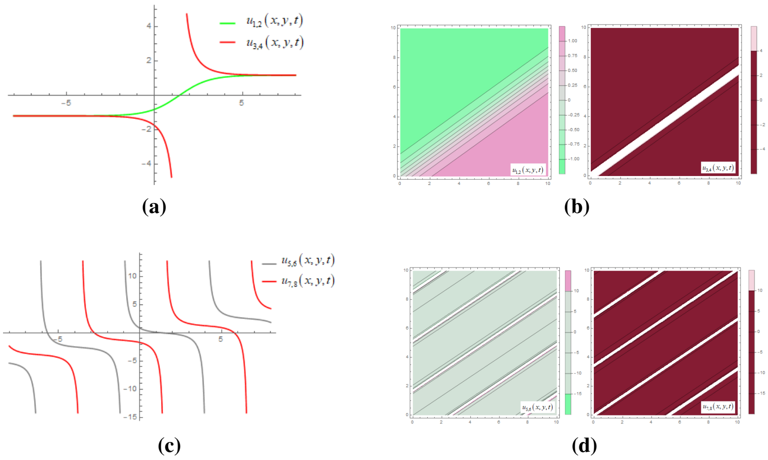


Figure 4.2: (a)-(c) 2D-graphs. (b)-(d) Contour graphs.

4.2. Graphs of solutions for the Jimbo-Miwa equation:

Fig.4.3-(a), (b), (c), (d) indicate (3.18) $u_1(x,y,z,t)$, (3.19) $u_3(x,y,z,t)$ for $\Theta = 1.2$, and (3.20) $u_5(x,y,z,t)$, (3.21) $u_7(x,y,z,t)$ for $\Theta = 1.5$, respectively. Fig.4.4-(a)-(b) express (3.18) $u_1(x,y,z,t)$ and (3.19) $u_3(x,y,z,t)$ for $\Theta = 1.2$, $t = 1$, $y = 0$ and $z = 0$. Further, Fig.4.4-(c)-(d) represent (3.20) $u_5(x,y,z,t)$ and (3.21) $u_7(x,y,z,t)$ for $\Theta = 1.5$, $t = 1$, $y = 0$ and $z = 0$.

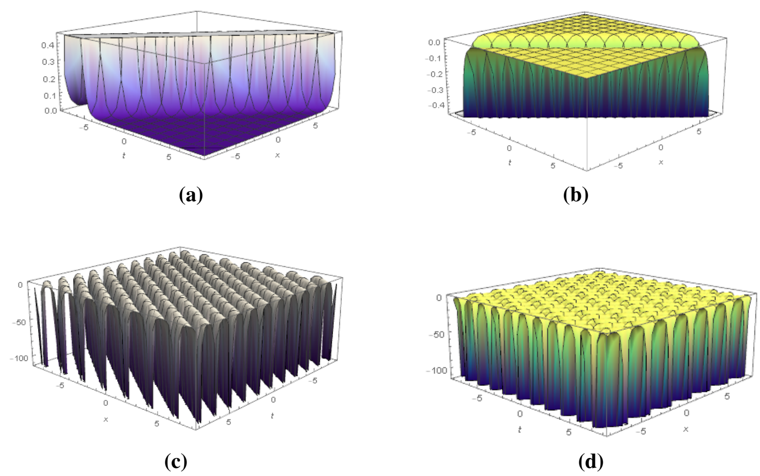
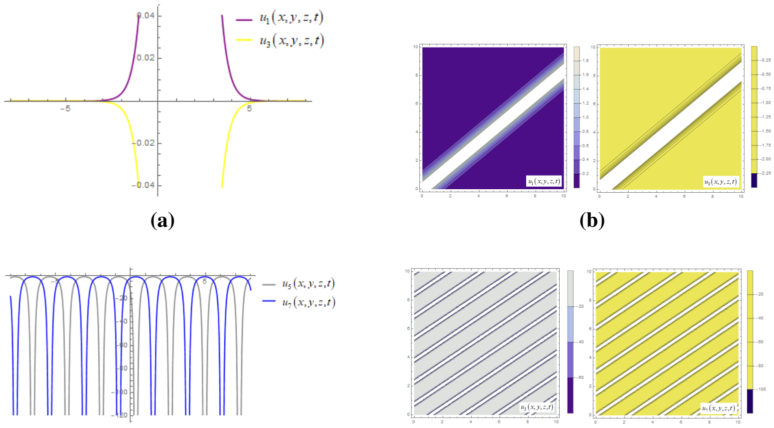


Figure 4.3: 3D-graphs.



(a) **(b)** **(c)** **(d)**

Figure 4.4: (a)-(c) 2D-graphs. (b)-(d) Contour graphs.

As a consequence, we have achieved some new wave solutions of equations (3.1) and (3.12) in hyperbolic and trigonometric forms. The graphs show that the resulting solitary wave solutions have several shapes, such as periodic and kink forms with respect to the wave speed Θ.

5. Conclusion

We have implemented the MSE method to attain some new exact solutions of the $(2+1)$-dimensional CBS equation and the $(3+1)$-dimensional Jimbo-Miwa equation. The correctness of the solutions has been demonstrated using the Mathematica package program. The graphics of the solutions have been plotted according to the appropriate values. The features of the MSE method allow us to obtain new traveling wave solutions to explain some complex physical phenomena. Consequently, our results show that the proposed method is practical, straightforward and effective for finding solutions to physics and engineering models. In our future studies, this effective and useful method will be applied to some other nonlinear equations involving integer and fractional derivatives expressing different complex phenomena.

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Competing interests

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Author's contributions

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Special Ruled Surface in de-Sitter 3-Space

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Abstract

In this paper, timelike base curve and spacelike main geodesic with the timelike ruled surface are studied, which is a special class of ruled surface in de-Sitter space S_1^3. A ruled surface in the de-Sitter space S_1^3 is obtained by moving a geodesic along a curve. So we will call these surfaces in the de-Sitter space as the geodesic ruled surface. Developable ruled surface, striction point, striction curve, dispersion parameter, and orthogonal trajectory concepts are investigated for the obtained geodesic ruled surface.

1. Introduction

The de-Sitter space is a model for physical events, and many physical phenomena can be explained by these models. Therefore, the surface varieties in de-Sitter space are very important. The surface types in different spaces guide the areas related to our daily life such as architecture and geometric design and therefore, the ruled surfaces in de-Sitter space are of great importance. It can be seen during history via the Euclidean motif in BC first, then spherical motif in the medieval and hyperbolic motif in the modern times in the architectures. In the future, architectural structures and geometric designs using de-Sitter lines will enter our daily lives. There is more than one causal character for surfaces, curves, and lines of de-Sitter space due to the structure of de-Sitter space. Since the surface of de-Sitter space can be considered as spacelike and timelike, then also curves and lines of de-Sitter space can be considered as spacelike and timelike.

Let $U \subset \mathbb{R}^2$ be an open subset, and let $x : U \to S_1^3$ be an embedding. If the vector subspace $\tilde{U}$ which generated by $\{x_{u_1}, x_{u_2}\}$ contains at least a timelike vector field then x is called **timelike surface** in S_1^3,i.e., the normal on the surface is a spacelike vector. In [1], Turgut and Hacısalihoglu studied timelike ruled surfaces in the Minkowski-3 space. They showed that these surfaces are obtained by a timelike straight line moving along a spacelike curve. A ruled surface is a surface generated by a straight line l moving along a curve α [1]. The various positions of the generating line l are called the rulings of the surface. Similarly, they studied spacelike ruled surfaces in the Minkowski-3 space [2]. Sabuncuoğlu studied generalized ruled surfaces in Euclidean $n-$space E^n and showed that the necessary and sufficient condition for the $n-$dimensional ruled surface to be a minimal surface is that the curves perpendicular to the rectangular space are asymptotic curves [3]. Later, Mert introduced spacelike ruled surfaces in the hyperboloid model of hyperbolic 3-space in Minkowski space, and using the properties of hyperbolic space, she investigated the properties of these type ruled surfaces [4].

Let $x : M \longrightarrow \mathbb{R}_1^4$ be an immersion of a surface M into $\mathbb{R}_1^4$. We say that x is timelike (*resp.* spacelike, lightlike) if the induced metric on M via x is Lorentzian (*resp. Riemannian, degenerated*). If $\langle x,x\rangle = 1$, then x is an immersion of S_1^3 [5]. Since geodesic which is lines of de-Sitter space on a ruled surface can be obtained by moving of curves in space, a sort of ruled surface can be captured up to causal characters of the base curve and main geodesic. In this paper, we investigate timelike ruled surfaces which have a base curve as timelike and main geodesic as spacelike in de-Sitter space S_1^3. A ruled surface is a surface obtained by a geodesic d_s^α moving along a curve α . Therefore, such surfaces may also be called geodesic ruled

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surfaces. Thus, the geodesic ruled surface has a parameterization in S_1^3 as follows

$$\varphi(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,Z(s)$$

where α is called the base curve and Z is called the director vector of d_s^{α}. If the tangent plane is constant along with a fixed ruling, then the ruled surface is called a developable geodesic ruled surface.

2. Preliminaries

For basic notions and properties of the Lorentz-Minkowski space from the viewpoint of Lorentz geometry, see [6]. Let R_1^4 be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4\ .$$

Then, R_1^4 is called Lorentzian 4- space or 4-dimensional Minkowski space. The Lorentzian norm (length) of x is defined to be

$$\|x\| = |\langle x,x\rangle|^{\frac{1}{2}}\,.$$

If $\left(x_1^i, x_2^i, x_3^i, x_5^i\right)$ is the coordinate of x_i with respect to canonical basis $\{e_1, e_2, e_3, e_4\}$ of R_1^4, then the lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix}.$$

One can easily see that

$$\langle x_1 \times x_2 \times x_3, x_4\rangle = \det(x_1, x_2, x_3, x_4)\,.$$

Given a vector $v \in R_1^4$ and a real number c, the hyperplane with pseudonormal v is defined by

$$HP(v,c) = \left\{x \in \mathbb{R}_1^4 \,|\, \langle x,v\rangle = c\right\}$$

We say that $HP(v,c)$ is a spacelike hyperplane, timelike hyperplane or lightlike hyperplane if v is timelike, spacelike or lightlike, respectively. We have the following three types of pseudo-spheres in R_1^4 :

$$\begin{aligned} &\text{Hyperbolic-3 space} &&: H^3(-1) = \left\{x \in R_1^4 \,|\, \langle x,x\rangle = -1, x_0 \geq 1\right\}, \\ &\text{de Sitter 3- space} &&: S_1^3 = \left\{x \in R_1^4 \,|\, \langle x,x\rangle = 1\right\}, \\ &\text{(open) lightcone} &&: LC^* = \left\{x \in R_1^4 \backslash \{0\} \,|\, \langle x,x\rangle = 0, x_0 > 0\right\}. \end{aligned}$$

We also define the lightcone $3-$sphere

$$S_+^3 = \{x = (x_1, x_2, x_3, x_4) \,|\, \langle x,x\rangle = 0, x_1 = 1\}\,.$$

A hypersurface given by the intersection of S_1^3 with a spacelike (resp.timelike) hyperplane is called an elliptic hyperquadric (resp. hyperbolic hyperquadric). If $c \neq 0$ and $HP(v,c)$ are lightlike, then $HP(v,c) \cap S_1^3$ is a de Sitter horosphere.
In the point of view of Kasedou [7], we construct the extrinsic differential geometry of curves in S_1^3. Since S_1^3 is a Riemannian manifold, the regular curve $\gamma : I \to S_1^3$ is given by the arclength parameter.

Theorem 2.1. *i) If $\gamma : I \to S_1^3$ is a spacelike curve with unit speed, then Frenet-Serret type formulae are obtained*

$$\begin{cases} \gamma'(s) &= t(s) \\ t'(s) &= \kappa_d(s)\,n(s) - \gamma(s) \\ n'(s) &= -\kappa_d(s)\,t(s) - \tau_d(s)\,e(s) \\ e'(s) &= -\tau_d(s)\,n(s) \end{cases}$$

where

$$\kappa_d(s) = \left\|t'(s) + \gamma(s)\right\|$$

and

$$\tau_d(s) = -\frac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{(\kappa_d(s))^2},$$

in [8].

ii) If $\gamma : I \to S_1^3$ is a timelike curve with unit speed, then Frenet-Serret type formulae are obtained

$$\begin{cases} \gamma'(s) & = t(s) \\ t'(s) & = \kappa_d(s)\, n(s) + \gamma(s) \\ n'(s) & = -\kappa_d(s)\, t(s) + \tau_d(s)\, e(s) \\ e'(s) & = -\tau_d(s)\, n(s) \end{cases}$$

where

$$\kappa_d(s) = \left\| t'(s) - \gamma(s) \right\|$$

and

$$\tau_d(s) = -\frac{\det\left(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s)\right)}{\left(\kappa_d(s)\right)^2},$$

in [8].

It is easily seen that $\kappa_d(s) = 0$ if and only if there exists a lightlike vector c such that $\gamma(s) - c$ is a geodesic.
Now we give extrinsic differential geometry on surfaces in S_1^3 due to Kasedou [7].
Let $U \subset \mathbb{R}^2$ is an open subset, and $x : U \to S_1^3$ is a regular surface $M = x(U)$. Since M is a timelike surface, there is

$$e(u) = \frac{x(u) \wedge x_{u_1}(u) \wedge x_{u_2}(u)}{\| x(u) \wedge x_{u_1}(u) \wedge x_{u_2}(u) \|}$$

such that

$$\langle e, x \rangle \equiv \langle e, x_{u_i} \rangle \equiv 0, \langle e, e \rangle = 1.$$

Thus there is de Sitter Gauss image of x which is defined by mapping $E : U \to S_1^3$,

$$E(u) = e(u).$$

The lightcone Gauss image of x is defined by map $L^{\pm} : U \to LC^*$,

$$L^{\pm}(u) = x(u) \pm e(u).$$

The derivative $dx(u_0)$ can be identified by the mapping 1_{T_pM} on the tangent space T_pM. Therefore, we have

$$dL^{\pm}(u_0) = 1_{T_pM} \pm dE(u_0).$$

The linear transformations

$$S_p^{\pm} := -dL^{\pm}(u_0) : T_pM \to T_pM$$

and

$$A_p := -dE(u_0) : T_pM \to T_pM$$

are called the hyperbolic shape operator and de Sitter shape operator of M at $p = x(u_o)$, respectively.
Let $\bar{K}_i^{\pm}(p)$ and $K_i(p), (i = 1, 2)$ be the eigenvalues of $S_p^{\pm}$ and A_p. Since

$$S_p^{\pm} = -1_{T_pM} \pm A_p\ ,$$

$S_p^{\pm}$ and A_p have the same eigenvectors and relations

$$\bar{K}_i^{\pm}(p) = -1 \pm K_i(p).$$

$\bar{K}_i^{\pm}(p)$ and $K_i(p)$, $(i = 1, 2)$ are called hyperbolic and de Sitter principal curvatures of M at p, respectively.
Let $\gamma(s)$ be a unit speed curve on M, with $p = \gamma(u_1(s_0), u_2(s_0))$. We consider the hyperbolic curvature vector

$$k(s) = t'(s) - \gamma(s)$$

and the de Sitter normal curvature

$$K_n^{\pm}(s_0) = \left\langle k(s_0), L^{\pm}(u_1(s_0), u_2(s_0)) \right\rangle = \left\langle t'(s_0), L^{\pm}(u_1(s_0), u_2(s_0)) \right\rangle + 1$$

of $p = \gamma(u_1(s_0), u_2(s_0))$. The de Sitter normal curvature depends only on the point p and the unit tangent vector of M at p. The hyperbolic normal curvature of $\gamma(s)$ is defined to be

$$\bar{K}_n^{\pm}(s) = K_n^{\pm}(s) - 1.$$

The extrinsic (de Sitter) Gauss curvature and mean curvature of M at p is given by

$$K_e(u_0) = \det A_p = K_1(p) K_2(p)$$

and

$$K_d(u_0) = \frac{1}{2} TraceA_p = \frac{K_1(p) + K_2(p)}{2}.$$

3. $TS-$geodesic ruled surface in de-Sitter 3-space

Now let's investigate the timelike ruled surfaces that its base curve is a timelike curve and its direction geodesic is a spacelike geodesic in the de-Sitter space S_1^3. Hereinafter, in terms of brevity, we call the $TS-$geodesic ruled surfaces the geodesic ruled surfaces whose base curve is timelike and the direction geodesic is spacelike.
Let α be a differentiable timelike curve with the unit speed in de-Sitter space S_1^3, then it is defined by

$$\alpha : I \to S_1^3 \subset R_1^4 \ , \ \alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s)) \ , \forall s \in I$$

where $\{0\} \subset I \subset \mathbb{R}$. In here

$$\langle \alpha(s), \alpha(s) \rangle = 1$$

and since α base curve is a timelike curve, we have

$$\left\langle \alpha'(s), \alpha'(s) \right\rangle = -1.$$

Let's assume that

$$\langle \alpha(s), Z(s) \rangle = 0, \forall s \in I$$

where

$$Z : I \to S_1^3, Z(s) = (z_1(s), z_2(s), z_3(s), z_4(s))$$

and

$$\langle Z(s), Z(s) \rangle = 1.$$

Then, a geodesic d_s^α in de-Sitter space S_1^3 has a parametrization

$$d_s^\alpha : \mathbb{R} \to S_1^3, d_s^\alpha(t) = (\cos t)\,\alpha(s) + (\sin t)\,Z(s)$$

where $\alpha(s)$ is a initial point and $Z(s)$ is the direction vector of d_s^α [6]. Here frenet components of base curve $\alpha(s)$ are $\{T_\alpha, N_\alpha, B_\alpha, \kappa_d, \tau_d\}$. Let T_d be tangent of geodesic d_s^α at the point $\alpha(s)$ and assume that T_d and T_α are linearly independent for all $s \in I$. Then, we obtain $(I \times \mathbb{R}, \varphi)$ parametrized by $\varphi : I \times \mathbb{R} \to S_1^3$

$$\varphi(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,Z(s).$$

This $(I \times \mathbb{R}, \varphi)$ surface is called a geodesic ruled surface which is produced by the geodesic d_s^α. Let us denote this geodesic ruled surface with M. Then we can give the following definition.

Definition 3.1. *The surface obtained by moving a given d_s^α spacelike geodesic along a given α timelike curve is called the $TS-$geodesic ruled surface in the de-Sitter space S_1^3, where d_s^α is the* ***direction geodesic of the $TS-$geodesic ruled surface*** *and the α curve is called the* ***base curve of $TS-$geodesic ruled surface.***

Let us find the orthonormal base of tangent space $\chi(M)$ of geodesic ruled surface M along the timelike curve α. If T is a unit tangent vector of timelike curve α and Z is the unit director vector of spacelike geodesic d_s^{α}, then we can choose spacelike vector field such that

$$Y=\tilde{T}_d+\left\langle\tilde{T}_d,T\right\rangle T$$

that is orthogonal to T in this plane, where

$$\tilde{T}_d=\frac{T_d}{\|T_d\|}$$

is the unit tangent of geodesic d_s^{α} and

$$T_d=(\cos t)\,T_{\alpha(s)}+(\sin t)\,T_{Z(s)}.$$

Also, if we take

$$X=\frac{Y}{\|Y\|},$$

then

$$\|X\|=1,\langle X,T\rangle=0\text{ and }\langle T,T\rangle=-1.$$

Thus, $\{X,T\}$ are the orthonormal vectors of $\chi(M)$. Also,

$$\xi=\varphi\times T\times X$$

is the normal vector of $TS-$geodesic ruled surface M in de-Sitter space S_1^3, that is

$$\begin{aligned}\xi &\in\chi^{\perp}(M)\\ \chi\left(S_1^3\right) &=Sp\{X,T\}\oplus Sp\{\xi\}\\ \chi\left(R_1^4\right) &=Sp\{X,T\}\oplus Sp\{\xi,\varphi\}\end{aligned}$$

In this case, system $\{\varphi,T,X,\xi\}$ is the orthonormal base of M.
Now let investigate the alteration of this system along the timelike curve α. The Levi-Civita connection of R_1^4,S_1^3, and M is denoted $\overline{\overline{D}},\bar{D}$, and D, respectively. Then we have the Gauss formulas [9]

$$\begin{cases}\overline{\overline{D}}_XY=\bar{D}_XY-\langle X,Y\rangle\alpha & ,\tilde{A}(X)=\overline{\overline{D}}_X\alpha=I(X)\\ \bar{D}_XY=D_XY-\langle A(X),Y\rangle\xi & ,A(X)=\bar{D}_X\xi.\end{cases}$$

In de-Sitter space S_1^3, let's derive the $\{T,X,\xi\}$ orthonormal frame along timelike curve α. In this case, we get the system in S_1^3

$$\begin{cases}\bar{D}_TT &=aX+b\xi\\ \bar{D}_TX &=aT+c\xi\\ \bar{D}_T\xi &=bT-cX\end{cases}$$

The matrix representation of this system is

$$\begin{bmatrix}\bar{D}_TT\\ \bar{D}_TX\\ \bar{D}_T\xi\end{bmatrix}=\begin{bmatrix}0&a&b\\ a&0&c\\ b&-c&0\end{bmatrix}\begin{bmatrix}T\\ X\\ \xi\end{bmatrix},$$

where

$$a=\left\langle\bar{D}_TT,X\right\rangle,b=\left\langle\bar{D}_TT,\xi\right\rangle\text{ and }c=\left\langle\bar{D}_TX,\xi\right\rangle$$

Now, in R_1^4, let's derive the $\{\varphi,T,X,\xi\}$ orthonormal frame along timelike curve α. In this case, we get the system

$$\begin{cases}\overline{\overline{D}}_T\varphi &=(\cos t+a\sin t)\,T+(c\sin t)\,\xi\\ \overline{\overline{D}}_TT &=\varphi+aX+c\xi\\ \overline{\overline{D}}_TX &=aT+c\xi\\ \overline{\overline{D}}_T\xi &=bT-cX\end{cases}\tag{3.1}$$

in R_1^4. System 3.1 have the for matrix form

$$\begin{bmatrix} \overline{\overline{D}}_T\varphi \\ \overline{\overline{D}}_T T \\ \overline{\overline{D}}_T X \\ \overline{\overline{D}}_T \xi \end{bmatrix} = \begin{bmatrix} 0 & \cos t + a\sin t & 0 & c\sin t \\ 1 & 0 & a & b \\ 0 & a & 0 & c \\ 0 & b & -c & 0 \end{bmatrix} \begin{bmatrix} \varphi \\ T \\ X \\ \xi \end{bmatrix}$$

For ruled surface M that is given by parametrization

$$\varphi : I \times \mathbb{R} \to S_1^3, \varphi(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,X(s)$$

$$\begin{cases} E &= \langle \varphi_s, \varphi_s \rangle = -(\cos t + a\sin t)^2 + c^2\sin^2 t \\ F &= \langle \varphi_s, \varphi_t \rangle = 0 \\ G &= \langle \varphi_t, \varphi_t \rangle = 1, \end{cases}$$

where

$$\langle \xi, \xi \rangle = F^2 - EG = -E.$$

Since ξ is the spacelike vector that is

$$\langle \xi, \xi \rangle > 0,$$

then the geodesic ruled surface is a timelike surface and

$$E < 0.$$

Let us the denote domain of t by

$$J = \{t \mid E = E(t) < 0\}.$$

$$\varphi_{t_o} : I \times \{t_0\} \to M \ , \ \varphi_{t_0}(s,t_0) = (\cos t_0)\,\alpha(s) + (\sin t_0)\,X(s)$$

determines a curve of $TS-$geodesic ruled surface M where t is constant in its domain. The tangent vector field of this curve is

$$A = (\cos t_0 + a\sin t_0)\,T(s) + c(\sin t_0)\,\xi(s).$$

Since

$$\langle A, A \rangle = E$$

and

$$E < 0,$$

then A is a timelike vector. Thus φ_{t_0} curve is a timelike curve and also

$$\langle X, A \rangle = 0$$

Remark 3.2. *Since the stereographic projection is a conformal map, using stereographic projection, the following example can be provided from [10].*

Example 3.3. *Let us take $TS-$geodesic ruled surface M in de-Sitter space S_1^3 given by parametrization*

$$\varphi : I \times \mathbb{R} \to S_1^3 \ , \ \varphi(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,X(s)\ .$$

In here, if

$$\alpha(s) = (\sinh s, 0, \cosh s, 0)$$

and

$$X(s) = \left(-\cosh s, \sqrt{2}, \sinh s, 0\right)$$

are chosen, then $\varphi(s,t)$ is $TS-$geodesic ruled surface in de-Sitter space S_1^3.

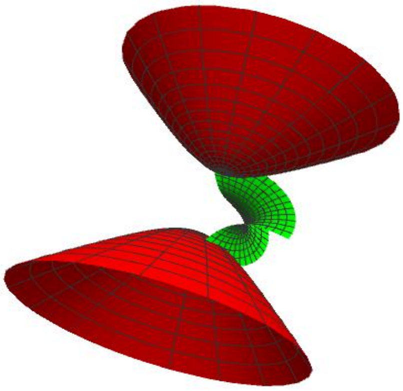

Figure 3.1: Timelike Geodesic Ruled Surface in de-Sitter 3-Space

4. Developable timelike geodesic ruled surfaces

Definition 4.1. *If the tangent planes of a $TS-$geodesic ruled surface in S_1^3 are the same along its main geodesics, then this timelike ruled surface is called a **developable timelike geodesic ruled surface.***

Theorem 4.2. *Let M be timelike ruled surface whose are base curve as timelike and main geodesic as spacelike in de-Sitter space S_1^3. Then the tangent planes are the same along the main geodesic if and only if $c = 0$.*

Proof. Let M be a $TS-$geodesic ruled surface in de-Sitter space S_1^3, and suppose that tangent planes of this ruled surface are the same along with one of its main geodesics. We consider the tangent vector field

$$A = (\cos t_0 + a\sin t_0)\,T\,(s) + c\,(\sin t_0)\,\xi\,(s)$$

of curve $\varphi_{t_0} : I \times \{t_0\} \to M$ which is at $t_0 \in I$. Since φ_{t_0} is the parameter curve of M, the vector A is in the tangent plane of the surface M. Hence

$$c = 0.$$

Conversely, assume that

$$c = 0.$$

In this case, since

$$A = (\cos t_0 + a\sin t_0)\,T\,(s)$$

and

$$T_{\varphi(t_0,s)}M = sp\,\{T, X\} = sp\,\{T, A\}\,.$$

This means that the tangent planes are the same along with one of its main geodesics. □

Corollary 4.3. *The $TS-$geodesic ruled surface M in de-Sitter space S_1^3 is a developable surface if and only if $c = 0$.*

Corollary 4.4. *For $TS-$geodesic ruled surface M in de-Sitter space S_1^3,*

$$b = -\det\left(T, X, \varphi, \overline{\overline{D}}_T T\right) \text{ and } c = -\det\left(T, X, \varphi, \overline{\overline{D}}_T X\right)$$

Example 4.5. *The surface of example-1 above is an example of a developable ruled surface in de-Sitter space S_1^3. Really, for timelike geodesic ruled surface M in de-Sitter space S_1^3 given by parametrization*

$$\varphi : I \times \mathbb{R} \to S_1^3\,,\ \varphi\,(s,t) = (\cos t)\,\alpha\,(s) + (\sin t)\,X\,(s)$$

if

$$\alpha\,(s) = (\sinh s, 0, \cosh s, 0)$$

and

$$X(s)=\left(-\cosh s,\sqrt{2},\sinh s,0\right)$$

are chosen, then

$$c=-\det\left(T,X,\varphi,\overline{\overline{D}}_T X\right)=\begin{vmatrix} \cosh s & -\sin\left(\frac{s}{\sqrt{2}}\right) & \cos\left(\frac{s}{\sqrt{2}}\right) & 0\\ -\cosh s & -\sin s & \cos s & 0\\ \cos t\sinh s-\sin t\cosh s & \sqrt{2}\sin t & \cos t\cosh s+\sin t\sinh s & 0\\ -\sinh s & 0 & \cosh s & 0 \end{vmatrix}.$$

Therefore, it is clear that

$$c=0.$$

5. A striction point and position vector of a striction point

Definition 5.1. *Let $TS-$geodesic ruled surface be given in de-Sitter space S_1^3. If there exists a common perpendicular of two neighbors the main geodesic of timelike geodesic ruled surface the foot of this perpendicular on principal geodesic is called* ***striction point.***

Definition 5.2. *When the main geodesic of $TS-$geodesic ruled surface in de-Sitter space S_1^3 creates the timelike geodesic ruled surface through the base curve, the geometrical place of the striction points of the ruled surface is called the* ***striction curve*** *of M.*

If w be the distance between the striction point of the timelike geodesic ruled surface and base curve, then position vector $\bar{\alpha}(s)$ can be defined by

$$\bar{\alpha}(s,w)=(\cos w)\,\alpha(s)+(\sin w)\,X(s)$$

where $\alpha(s)$ is the position vector of the timelike base curve and $X(s)$ is the direction vector of the spacelike main geodesic. The parameter w can be written as the combination of the position vector of the base curve and direction vector of the timelike geodesic ruled surface. Let the first two of three neighbor geodesic of the timelike ruled surface be

$$d_s^{\alpha}=(\cos t)\,\alpha(s)+(\sin t)\,X(s)$$

and

$$d_{s+\Delta s}^{\alpha}=(\cos t)\,\alpha(s+\Delta s)+(\sin t)\,X(s+\Delta s)$$

where $X(s)$ and $X(s)+\bar{D}_{T(s)}X(s)$ are the direction vectors of these main geodesic, respectively. Also let P,P' and Q,Q' be the feet on the main geodesic of the common perpendicular of the neighbor geodesic. Thus P and Q are two different striction points. The direction of common perpendicular first two main geodesics are linearly dependent to the vector

$$\alpha(s)\times X(s)\times\left[X(s)+\bar{D}_{T(s)}X(s)\right].$$

Therefore

$$\alpha(s)\times X(s)\times\left[X(s)+\bar{D}_{T(s)}X(s)\right]=\alpha(s)\times X(s)\times\bar{D}_{T(s)}X(s).$$

The vector $\overrightarrow{PQ}$ coincides with the vector $\overrightarrow{PP'}$ in the limiting position, and $\overrightarrow{PQ}$ will be the tangent vector of the striction curve. Since

$$\left\langle X(s),\overrightarrow{PQ}\right\rangle=0 \text{ and } \left\langle X(s)+\bar{D}_{T(s)}X(s),\overrightarrow{PQ}\right\rangle=0$$

we obtain

$$\left\langle \bar{D}_{T(s)}X(s),\overrightarrow{PQ}\right\rangle=0.$$

Thus we get

$$\left\langle \bar{D}_{T(s)}X(s),\bar{D}_{T(s)}\bar{\alpha}(s)\right\rangle=0. \tag{5.1}$$

On the other hand, since

$$\bar{D}_{T(s)}\bar{\alpha}(s)=\overline{\overline{D}}_{T(s)}\bar{\alpha}(s)+\left\langle T(s),\bar{\alpha}(s)\right\rangle\bar{\alpha}(s)$$

we obtain

$$\bar{D}_{T(s)}\bar{\alpha}(s) = \overline{\overline{D}}_{T(s)}\bar{\alpha}(s).$$

Consequently, from 5.1, we have

$$\left\langle \overline{\overline{D}}_{T(s)}X(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s) \right\rangle = 0$$

and then

$$\frac{\sin w}{\cos w} = \frac{a}{-a^2 + c^2},$$

that is

$$w = \arctan\left(\frac{a}{-a^2 + c^2}\right)$$

and

$$\cos w = \frac{-a^2 + c^2}{\sqrt{a^2 + (-a^2 + c^2)^2}}, \ \sin w = \frac{a}{\sqrt{a^2 + (-a^2 + c^2)^2}}.$$

So, the position vector of the striction curve is

$$\bar{\alpha}(s) = \left(\frac{-a^2 + c^2}{\sqrt{a^2 + (-a^2 + c^2)^2}}\right)\alpha(s) + \left(\frac{a}{\sqrt{a^2 + (-a^2 + c^2)^2}}\right)X(s). \tag{5.2}$$

Theorem 5.3. *The distance between the striction point of the timelike geodesic ruled surface and base curve is constant, that is*

$$w = \arctan\left(\frac{a}{-a^2 + c^2}\right).$$

Proof. Since

$$\langle X(s), PQ \rangle = 0,$$

we obtain

$$\left\langle X(s), \bar{D}_{T(s)}\bar{\alpha}(s) \right\rangle = 0$$

and

$$\bar{D}_{T(s)}\bar{\alpha}(s) = \overline{\overline{D}}_{T(s)}\bar{\alpha}(s).$$

Thus

$$\left\langle X(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s) \right\rangle = 0$$

and

$$(\cos w)\frac{dw}{ds} = 0,$$

which implies that

$$\frac{dw}{ds} = 0$$

and so, w is constant. □

Theorem 5.4. *The striction curve of an undevelopable $TS-$geodesic ruled surface in de-Sitter space S_1^3 is independent from choosing base curve.*

Proof. Let us denote two $TS-$geodesic ruled surface in de-Sitter space S_1^3 by

$$\begin{aligned}\varphi(t,s) &= (\cos t)\,\alpha(s) + (\sin t)\,X(s)\\ \varphi(t,s) &= (\cos t)\,\beta(s) + (\sin t)\,X(s),\end{aligned}$$

where α and β are two different base curves of the timelike geodesic ruled surface in S_1^3. Then the striction curves of timelike geodesic ruled surface are

$$\begin{aligned}\bar{\alpha}(s) &= \left(\frac{-a^2+c^2}{\sqrt{a^2+\left(-a^2+c^2\right)^2}}\right)\alpha(s) + \left(\frac{a}{-\sqrt{a^2+\left(-a^2+c^2\right)^2}}\right)(\cos t)\,X(s)\\ \bar{\beta}(s) &= \left(\frac{-a^2+c^2}{\sqrt{a^2+\left(-a^2+c^2\right)^2}}\right)\beta(s) + \left(\frac{a}{-\sqrt{a^2+\left(-a^2+c^2\right)^2}}\right)X(s)\end{aligned}$$

If we subtract $\bar{\beta}(s)$ from $\bar{\alpha}(s)$ and use 5.2, we obtain

$$\bar{\alpha}(s) - \bar{\beta}(s) = 0$$

which gives up the proof. □

Theorem 5.5. *Let M be undevelopable $TS-$geodesic ruled surface in de-Sitter space S_1^3. The point $\varphi(s,v_0)$ is striction point on the main geodesic which passes through $\alpha(s)$ point if and only if $\bar{D}_{T(s)}X(s)$ is a normal vector of the tangent plane on $\varphi(s,v_0)$ point.*

Proof. Let M be undevelopable $TS-$geodesic ruled surface in de-Sitter space S_1^3. Suppose that $\bar{D}_{T(s)}X(s)$ is a normal vector of the tangent plane on $\varphi(s,v_0)$ point. Since the tangent vector field of $\varphi_{v_0} : I \times \{v_0\} \to M$ given by

$$A = (\cos v_0 + a\sin v_0)\,T(s) + c(\sin v_0)\,\xi(s),$$

then

$$\left\langle \bar{D}_{T(s)}X(s), A\right\rangle = 0.$$

Thus, we obtain

$$\frac{\sin v_o}{\cos v_0} = \frac{a}{-a^2+c^2}.$$

Therefore $\varphi(s,v_0)$ is a striction point of M.
Conversely, suppose that $\varphi(s,v_0)$ is a striction point with main geodesic passing through the point $\alpha(s)$. Thus, we have

$$\left\langle \bar{D}_{T(s)}X(s), X(s)\right\rangle = 0,$$

$$\left\langle \bar{D}_{T(s)}X(s), A\right\rangle = -a(\cos v_0 + a\sin v_0) + c^2\sin v_0.$$

Since $\varphi(s,v_0)$ is striction point, then we get

$$-a(\cos v_0 + a\sin v_0) + c^2\sin v_0 = 0.$$

Hence, we obtain

$$\left\langle \bar{D}_{T(s)}X(s), A\right\rangle = 0.$$

So, $\bar{D}_{T(s)}X(s)$ is a normal vector of tangent plane at $\varphi(s,v_0)$. □

Remark 5.6. *Let $\bar{D}_{T(s)}X(s)$ be a normal vector of the tangent plane on the striction point. From the equality, we conclude that*

$$\left\langle \bar{D}_{T(s)}X(s), \bar{D}_{T(s)}X(s)\right\rangle = -a^2+c^2,$$

i) If $-a^2+c^2>0$, then $\bar{D}_{T(s)}X(s)$ is a spacelike normal vector field.

ii) If $-a^2+c^2<0$, then $\bar{D}_{T(s)}X(s)$ is a timelike normal vector field.

Theorem 5.7. *Let M be undevelopable $TS-$geodesic ruled surface in de-Sitter space S_1^3. The striction curve $\bar{\alpha}(s)$ has the form.*

i) If $-a^2+c^2>0$, then the striction curve $\bar{\alpha}(s)$ is a timelike curve.

ii) If $-a^2+c^2<0$, then the striction curve $\bar{\alpha}(s)$ is a spacelike curve.

Proof. We need to show that the tangent vector field of striction curve $\bar{\alpha}$ is a spacelike vector field or timelike vector field. It is clear that

$$\left\langle \overline{\overline{D}}_{T(s)}\bar{\alpha}(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s)\right\rangle = \frac{-c^2}{-a^2+c^2}\cos^2 w,$$

where

$$\overline{\overline{D}}_{T(s)}\bar{\alpha}(s) = (\cos w)\overline{\overline{D}}_{T(s)}\alpha(s) + \frac{a}{-a^2+c^2}(\cos w)\overline{\overline{D}}_{T(s)}X(s).$$

If

$$-a^2+c^2>0,$$

that is

$$\left\langle \overline{\overline{D}}_{T(s)}\bar{\alpha}(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s)\right\rangle < 0,$$

then $\bar{\alpha}(s)$ is timelike curve and similarly, if

$$-a^2+c^2<0,$$

that is

$$\left\langle \overline{\overline{D}}_{T(s)}\bar{\alpha}(s), \overline{\overline{D}}_{T(s)}\bar{\alpha}(s)\right\rangle > 0,$$

then $\bar{\alpha}(s)$ is spacelike curve. □

6. Dispersion parameter

Let the base curve of a $TS-$geodesic ruled surface M be the striction curve in de-Sitter space S_1^3. Then, the distance from the striction point to the base curve is

$$w = \arctan\left(\frac{a}{-a^2+c^2}\right) = 0.$$

Hence, we have

$$a=0$$

and since

$$\bar{D}_{T(s)}X(s) = aT(s) + c\xi(s),$$

the vector field $\bar{D}_{T(s)}X(s)$ and normal of surface $\xi(s)$ are linearly independent. Therefore, there exists $\lambda\in\mathbb{R}$ for the equality

$$\xi(s) = \lambda\bar{D}_{T(s)}X(s).$$

On the other hand, since

$$\xi(s) = \varphi\times X\times T$$

and

$$\varphi = (\cos t)\alpha(s) + (\sin t)X(s),$$

we have

$$\xi(s) = (\cos t)[\alpha(s)\times X(s)\times T(s)].$$

Therefore, we have

$$\lambda \bar{D}_{T(s)} X(s) = (\cos t)\left[\alpha(s) \times X(s) \times T(s)\right].$$

If we take the scalar product with $\bar{D}_{T(s)} X(s)$ of both sides of the above equality, then we have

$$\lambda = (\cos t)\frac{\det\left(\alpha(s), T(s), X(s), \bar{D}_{T(s)} X(s)\right)}{\left\langle \bar{D}_{T(s)} X(s), \bar{D}_{T(s)} X(s)\right\rangle},$$

where λ is called a **dispersion parameter of $TS-$geodesic ruled surface in de-Sitter space S_1^3.**

Example 6.1. *The surface of example-1 above is an example of a developable ruled surface in de-Sitter space S_1^3. It is clear that for timelike geodesic ruled surface M in de-Sitter space S_1^3 given by parametrization*

$$\varphi : I \times \mathbb{R} \to S_1^3\,,\ \varphi(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,X(s)$$

if

$$\alpha(s) = (\sinh s, 0 \cosh s, 0)$$

and

$$X(s) = \left(-\cosh s, \sqrt{2}, \sinh s, 0\right)$$

are chosen, then we can derive

$$\det\left(T, X, \varphi, \bar{\bar{\bar{\Delta}}}_T X\right) = 0.$$

Therefore

$$\lambda = (\cos t)\frac{\det\left(\alpha(s), T(s), X(s), \bar{\Delta}_{T(s)} X(s)\right)}{\left\langle \bar{\Delta}_{T(s)} X(s), \bar{\Delta}_{T(s)} X(s)\right\rangle} = 0.$$

Theorem 6.2. *The $TS-$geodesic ruled surface M in de-Sitter space S_1^3 is developable if and only if the dispersion parameter of M is zero.*

Proof. From Corollary-1 and Corollary-2, we get

$$c = -\det\left(T(s), X(s), \alpha(s), \bar{\bar{\bar{D}}}_{T(s)} X(s)\right) = 0.$$

It is clear from the definition of the dispersion parameter that

$$\lambda = (\cos t)\frac{\det\left(\alpha(s), T(s), X(s), \bar{D}_{T(s)} X(s)\right)}{\left\langle \bar{D}_{T(s)} X(s), \bar{D}_{T(s)} X(s)\right\rangle} = 0.$$

□

Definition 6.3. *If there exists a curve that cuts vertically each main geodesic of the $TS-$geodesic ruled surface in de-Sitter space S_1^3, then this curve is called* ***orthogonal trajectory of $TS-$geodesic ruled surface in de-Sitter space S_1^3.***

Theorem 6.4. *Let M be a $TS-$geodesic ruled surface in de-Sitter space S_1^3. There is only one orthogonal trajectory which passes through every point of M.*

Proof. Let M be a $TS-$geodesic ruled surface given by the parametrization $\varphi : I \times J \to S_1^3 \subset R_1^4$,

$$\varphi(s,t) = (\cos t)\,\alpha(s) + (\sin t)\,Z(s).$$

Then, the orthogonal trajectory of M is $\beta : \tilde{I} \subset I \to M$,

$$\beta(s) = [\cos f(s)]\,\alpha(s) + [\sin f(s)]\,Z(s).$$

Since

$$\left\langle \bar{D}_{T(s)} \beta(s), Z(s)\right\rangle = 0,$$

we get

$$f(s) = -\int \left\langle \overline{\overline{D}}_{T(s)} \alpha(s), Z(s) \right\rangle ds + h,$$

where $\langle Z(s), Z(s) \rangle = 1$. If we take

$$F(s) = -\int \left\langle \overline{\overline{D}}_{T(s)} \alpha(s), Z(s) \right\rangle ds,$$

we get

$$f(s) = F(s) + h.$$

Since h is chosen arbitrary, there are a lot of curves that satisfy the condition

$$\left\langle \bar{D}_{T(s)} \beta(s), Z(s) \right\rangle = 0.$$

Let us now find $s \in \mathbb{R}$ such that

$$P_0 = [\cos(F(s) + h)]\, \alpha(s) + [\sin(F(s) + h)]\, Z(s).$$

This leads to

$$[\cos f(s)]\, \alpha(s) + [\sin f(s)]\, Z(s) = [\cos v_0]\, \alpha(s_0) + [\sin v_0]\, Z(s_0)$$

So,

$$\alpha(s_0) = \alpha(s), v_0 = f(s).$$

If we choose interval I such that α is one to one, then we get

$$s = s_0.$$

Thus,

$$h = f(s_0) - F(s_0).$$

Consequently, there exists only one orthogonal trajectory passing through the point P_0. Therefore, $\tilde{I}$ must be equal to I. □

Theorem 6.5. *Let M be undevelopable $TS-$geodesic ruled surface in de-Sitter space S_1^3. The shortest distance along the orthogonal trajectory between of any two main geodesics of M is the distance measured along curve $\varphi_t : I \to M$ corresponding to*

$$t = \frac{1}{2} \arctan\left(\frac{2a}{1 - a^2 + c^2}\right).$$

Proof. Let us take two geodesics passing through points $\alpha(s_1)$ and $\alpha(s_2)$ where $s_1, s_2 \in I$ and $s_1 < s_2$. Also, let us denote distance obtained along orthogonal trajector t =constant between these lines by $d(t)$. Then,

$$d(t) = \int_{s_1}^{s_2} \|A\|\, ds = \sqrt{-(\cos t + a \sin t)^2 + c^2 \sin^2 t}\, (s_2 - s_1),$$

where

$$A = (\cos t + a \sin t)\, T(s) + c(\sin t)\, \xi(s).$$

If $d'(t) = 0$, then $d(t)$ takes minimum value. Hence we get

$$t = \frac{1}{2} \arctan\left(\frac{2a}{1 - a^2 + c^2}\right).$$

□

Theorem 6.6. *Let M be $TS-$geodesic ruled surface in de-Sitter space S_1^3. The geodesic of M is both asymptotic and geodesic curves.*

Proof. Let X be a tangent vector field of a geodesic of a $TS-$geodesic ruled surface M. Since every geodesic in ruled surface M, it is a geodesic S_1^3. Thus we get

$$\bar{D}_X X = 0.$$

From [9] , we also get

$$\bar{D}_X X = D_X X - \langle S(X), X\rangle \xi.$$

Thus

$$D_X X = \langle S(X), X\rangle \xi.$$

Therefore

$$D_X X \in \chi(M) \text{ and } \langle S(X), X\rangle \xi \in \chi^{\perp}(M).$$

Since the metric on M is nondegenerate, we get

$$\chi\left(S_1^3\right) = \chi(M) \oplus \chi^{\perp}(M) \text{ and } \chi(M) \cap \chi^{\perp}(M) = \{0\}.$$

Thus

$$D_X X = 0 \text{ and } \langle S(X), X\rangle = 0.$$

The proof is completed. □

Theorem 6.7. *Let M be $TS-$geodesic ruled surface in de-Sitter space S_1^3. Then*

$$K(p) \geq 0 \textit{ for all } p \in M$$

where K is the Gauss curvature function of M.

Proof. Let X be the tangent vector field of the main geodesic at point $p \in M$ and take the orthonormal basis $\{X, Y\}$ of $\chi(M)$. Since M is a timelike ruled surface, X ,Y are timelike and spacelike vector fields, respectively. The Weingarten operator S of M can be written

$$\begin{array}{ll} S(X) & = -\langle S(X), X\rangle X + \langle S(X), Y\rangle Y \\ S(Y) & = -\langle S(Y), X\rangle X + \langle S(Y), Y\rangle Y \end{array}.$$

In this case, the matrix

$$S = \begin{bmatrix} -\langle S(X), X\rangle & \langle S(X), Y\rangle \\ -\langle S(Y), X\rangle & \langle S(Y), Y\rangle \end{bmatrix}$$

is corresponding to Weingarten operator S. On the other hand, the Weingarten operator S is selfadjoint,

$$\langle S(Y), X\rangle = \langle Y, S(X)\rangle .$$

Also, by Theorem 6.6, we conclude

$$\langle S(X), X\rangle = 0, \langle S(Y), Y\rangle = 0.$$

Hence, from the definition of Gauss curvature, we get

$$K = \det S = \langle S(X), Y\rangle^2.$$

The proof is completed. □

Theorem 6.8. *Let M be a $TS-$geodesic ruled surface in de-Sitter space S_1^3. Then*

$$\begin{cases} \varphi \times T \times X & = \xi \\ T \times X \times \xi & = -\varphi \\ \xi \times \varphi \times T & = -X \\ X \times \xi \times \varphi & = -T, \end{cases}$$

where T is a unit tangent vector of base curve, φ is the position vector of M, X is unit tangent vector field of the main geodesic of M and ξ is unit normal vector field of M.

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Tubular Surfaces Around a Null Curve and Its Spherical Images

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Abstract

In this study, we define tubular surfaces whose center curves are null curves and their spherical images in Minkowski $3-$space. Firstly, we give the interior properties of the surfaces and calculate their invariant curvatures. Then, we obtain some special characterizations for the parameter curves of the surfaces. Finally, we demonstrate the theory via example and give their visualizations with the help of Mathematica.

1. Introduction

Minkowski space is defined as the basic model of quantum physics. Many notions in Euclidean space are different in this space. There are three spheres such as de Sitter $2-$space, hyperbolic $2-$space, and lightcone. Moreover, curves are divided into three groups due to the casual characters of their tangent vectors in the Minkowski space. An arbitrary curve is called as a spacelike curve, a timelike curve or a null (lightlike) curve, if its tangent vector is a spacelike vector, a timelike vector or a null (lightlike) vector, respectively. Similarly, a surface is called a timelike, spacelike, or lightlike surface if its normal vector lies on the de Sitter $2-$space, hyperbolic $2-$space, or null cone, respectively. Null curves have different properties than spacelike and timelike curves. So, the author [1] has defined Cartan frame as the most useful frame, and he used this frame to study null curves. Also, studies in the differential geometry are examined in two classes as null and non-null structures (see [2]- [4]).

A canal surface is defined as an envelope of one parameter family of spheres centered by a space curve. A tubular surface is a canal surface with constant radius. Many authors have studied on the characaterizations of tubular and canal surfaces [5]- [10]. The authors [4] have studied some characterizatons of the tubular surfaces generated by non-null curves in Minkowski $3-$space. Blaga [11] has presented a new approach to the tubular surfaces and provided CAD applications. Arslan *et.al.* [12] have obtained a medical application of the tube surfaces. In [13], they have examined a new type of the canal surface.

A tubular surface is one of the fundamental objects in geometric modelling. It appears in many application areas such as the networks of blood vessels and the neurons in medicine, hose systems, surface modeling in CAGD and CAD/CAM systems. On the other hand, null curves are important curves in general relativity. The surfaces produced by these curves provide good models for the study of different horizon types. In this study, we indicate the tubular surface around a null curve since they are generated by parabolas. To find geodesics on tubular surfaces are important to found the shortest distances between two points on a surface. Asymptotic curve on a surface whose osculating plane at each point coincides with the tangent plane to the surface at that point. Therefore, we have obtained characterizations of these curves on the surface. Also, we have examined the singular points of the tubular surface and the condition of the tubular surface being a Weingarten surface. Finally, we have investigated the tubular surfaces formed by spherical images of the null curve.

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2. Preliminaries

The standard metric of the Minkowski 3−space is

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

where x_i and y_j $(i,j=1,2,3)$ are shown the coefficients of the vectors x and y, respectively [14]. Since $\langle,\rangle$ is an indefinite metric, recall that a vector $u \in E_1^3$ has three categories: if $\langle u,u\rangle > 0$ or $u=0$ it is a spacelike vector, if $\langle u,u\rangle < 0$ it is a timelike vector and if $\langle u,u\rangle = 0$ and $u \neq 0$ it is a null (lightlike) vector. Also, an arbitrary curve α in E_1^3 is called as spacelike, timelike or null (lightlike) curves according to casual character of the tangent vector. Cartan [1] has defined a frame $\{L(s)=\alpha'(s), N(s), W(s)\}$ similar to the Frenet frame for a null curve $\alpha(s)$, called Cartan frame satisfying

$$\langle L,L\rangle = \langle N,N\rangle = 0, \langle L,N\rangle = 1,$$
$$\langle W,L\rangle = \langle W,N\rangle = 0, \langle W,W\rangle = 1,$$

with $L\times N = W, W\times L = L$ and $N\times W = N$. The Cartan equations are

$$\begin{aligned} L'(s) &= k_1(s)W(s), \\ N'(s) &= k_2(s)W(s), \\ W'(s) &= -k_2(s)L(s) - k_1(s)N(s), \end{aligned}$$

where $k_1(s) = \langle \alpha'', \alpha''\rangle^{1/2}$ and $k_2(s) = \langle N'(s), W(s)\rangle$ are Cartan curvature functions [15].
For investigate the interior geometry of the parametric surface $X(s,\theta)$ at the point $X(s_0,\theta_0)$, we use the first fundamental form. The coefficients of the first fundamental form are calculated as $e=\langle X_s,X_s\rangle$, $f=\langle X_s,X_\theta\rangle$, $g=\langle X_\theta,X_\theta\rangle$. The Gauss map of the surface $X(s,\theta)$ is U such that $\{X_s,X_\theta,U\}$ is an orthogonal frame along the surface. Let ε be a sign function of the Gauss map U, this is used to determine the causal character of the surface. If $\varepsilon=1$ or $\varepsilon=-1$, then the surface is the timelike surface or the spacelike surface, respectively. The coefficients of the second fundamental form are $\ell = \langle X_{ss},U\rangle$, $m=\langle X_{s\theta},U\rangle$, $n=\langle X_{\theta\theta},U\rangle$. The invariant curvatures K and H of the surface are calculated as:

$$K := \frac{\varepsilon\left(\ell n - m^2\right)}{eg-f^2} \text{ and } H := \frac{\varepsilon\left(en-2fm+g\ell\right)}{2\left(eg-f^2\right)} \tag{2.1}$$

where K, H are called as Gaussian curvature and mean curvature of the surface, respectively. A surface in Minkowski 3−space is called as linear Weingarten surface if its invariant curvatures is satisfied the equation $2aH+bK=c$, where a,b,c are real numbers and $(a,b,c)\neq(0,0,0)$ [16].
The parametric equation of the canal surface is given by

$$X(s,\theta) = \alpha(s) + r(s)(\cos\theta N(s) + \sin\theta B(s))$$

where $N(s)$ and $B(s)$ are the Frenet normal vectors of the spine curve α. There are three kind of the tubular surface with respect to causal characters of the non-null curves in Minkowski 3−space [4].

3. Tubular surface around a null curve

In this section, we will analyze the properties of the tubular surface whose center curve is a null curve α and characterize some special curves on this surface.
Lopez [14] defined that the orbit of a point lies in the null plane is a parabola. The parabola in the null plane play the same role as the circle in Euclidean ambient. In [17], the authors is defined the tubular surface around the null curve $\alpha(s)$ as follows:

$$X(s,\theta) = \alpha(s) + \theta N(s) + \theta^2 W(s)$$

where $N(s)$ and $W(s)$ are the Cartan frame vectors of the null curve α and the parameter θ is characterized the parabola lies on the null plane spanned by the vectors $N(s)$ and $W(s)$. The coefficients of the first fundamental form are given by

$$\begin{aligned} e &= -2\theta^2 k_1(s)(1-\theta^2 k_2(s)) + \theta^2 k_2^2(s), \\ f &= 1+\theta^2 k_2(s), \quad g = 4\theta^2. \end{aligned} \tag{3.1}$$

The Gauss map U of the tubular surface $X(s,\theta)$ is calculated as

$$U = \frac{1}{\sqrt{A}}\{-2\theta(1-\theta^2 k_2(s))L(s) - (2\theta^3 k_1(s) + \theta k_2(s))N(s) + (1-\theta^2 k_2(s))W(s)\}$$

and also, the coefficients of the second fundamental form are

$$\begin{aligned} \ell &= \tfrac{1}{\sqrt{A}}\{2\theta^5(k_1k_2'-k_1'k_2)+2\theta^4k_1k_2^2+2\theta^3k_1'+\theta^2(k_2^3-k_1k_2)+\theta k_2'+k_1\}, \\ m &= \tfrac{1}{\sqrt{A}}\{\theta^2k_2^2(s)+4\theta^2k_1(s)+k_2(s)\}, \quad n=\tfrac{2(1-\theta^2k_2(s))}{\sqrt{A}}, \end{aligned} \tag{3.2}$$

where $A=\varepsilon(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)$ and the derivatives are taken by the parameter s.

Proposition 3.1. *The tubular surface $X(s,\theta)$ generated by the null curve α is a regular surface if and only if it satisfies the following conditions $\theta^2\neq\frac{1}{k_2(s)}$ and $\theta^2\neq\frac{\left(-3k_2(s)\pm\sqrt{9k_2^2(s)-32k_1(s)}\right)}{16k_1(s)}$ for $k_2^2(s)\geq\frac{32k_1(s)}{9}$.*

Proof. The condition $eg-f^2\neq 0$ provide for every regular surface at the point (s,θ). By using the equation (3.1), we obtain $-(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)\neq 0$ for the surface $X(s,\theta)$ and this equation gives us the desired conditions. □

Remark 3.2. *The tubular surface $X(s,\theta)$ has singular points at (s_0,θ_0) if and only if the equation $(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)=0$ is satisfied for the points (s_0,θ_0).*

Using the equation (2.1), the invariant curvatures of the surface $X(s,\theta)$ can be computed as

$$K=-\frac{\varepsilon}{A^2}\left\{\begin{array}{c} (1-\theta^2k_2)(2\theta^3k_1+\theta k_2)(2k_2'\theta^2-2k_2^2\theta-16k_1k_2\theta^3) \\ +(1-\theta^2k_2)^2(4\theta^3k_1'+2k_1+2k_2'\theta-16k_1^2\theta^4-8k_1k_2\theta^2-k_2^2) \\ -4k_2^2\theta^2(2\theta^3k_1+\theta k_2)^2 \end{array}\right\}$$

and

$$H=-\frac{\varepsilon}{A^{3/2}}\left\{\begin{array}{c} (1-\theta^2k_2)(-4\theta^2k_1-k_2+4\theta^5k_1'-2\theta^4k_1k_2+2\theta^3k_2') \\ +(2\theta^3k_1+\theta k_2)(-2\theta k_2+2\theta^4k_2') \end{array}\right\}$$

where $A=\varepsilon(1-\theta^2k_2(s))(8\theta^4k_1(s)+3\theta^2k_2(s)+1)$.

Theorem 3.3. *The $s-$parameter curves of $X(s,\theta)$ are the geodesic curves if and only if the condition is satisfied*

$$k_2'(s)(k_2(s)+\theta^2k_1(s))-k_1'(s)(1-\theta^2k_2(s))=0$$

in terms of the Cartan curvatures of the null curve α.

Proof. If the normal vector of the surface and second derivative of a curve lying on the surface are linearly dependent, then the curve is called the geodesic curve of the surface [8]. Based on this definition, we obtain the following system of equations for the $s-$parameter curves on the regular tubular surface $X(s,\theta)$:

$$\left\{\begin{array}{c} (1-\theta^2k_2)[-k_2'\theta^2+k_2(2\theta^3k_1+\theta k_2)-2\theta k_1(1-\theta^2k_2)]=0, \\ (1-\theta^2k_2)(2\theta^3k_1^2-\theta^2k_1')+(2\theta^3k_1+\theta k_2)(-\theta^2k_1k_2+\theta k_2')=0, \\ (1-\theta^2k_2)(-2\theta^3k_1'-2\theta^2k_1k_2)+(2\theta^3k_1+\theta k_2)(\theta^2k_2'+\theta k_2^2)=0. \end{array}\right. \tag{3.3}$$

Since $X(s,\theta)$ is the regular surface, we have $-k_2'\theta^2+k_2(2\theta^3k_1+\theta k_2)-2\theta k_1(1-\theta^2k_2)=0$. If this equation is solved together with the last two equations in equation (3.3), we get

$$k_2'(s)(k_2(s)+\theta^2k_1(s))-k_1'(s)(1-\theta^2k_2(s))=0.$$

□

Corollary 3.4. *The $s-$parameter curves of the surface with the Cartan curvatures $k_1(s)=\frac{1}{1-(as^n+b)\theta^2}(\frac{a^2s^{2n}}{2}+abs^n+c)$ and $k_2(s)=as^n+b,(n\geq 1$ and a,b,c*

are constants$)$ are the geodesic curves on the tubular surface.

Proof. If we consider $k_2(s)=as^n+b$ for the constants a, b and substituting this equation into the equation $k_2'(s)(k_2(s)+\theta^2k_1(s))-k_1'(s)(1-\theta^2k_2(s))=0$, then the following differential equation is obtained

$$k_1'(s)-\frac{ans^{n-1}\theta^2}{1-(as^n+b)\theta^2}k_1(s)=\frac{ans^{n-1}(as^n+b)}{1-(as^n+b)\theta^2}.$$

From solution of the ODE according to function $k_1(s)$, the first Cartan curvature is found as $k_1(s)=\frac{1}{1-(as^n+b)\theta^2}(\frac{a^2s^{2n}}{2}+abs^n+c)$. □

Theorem 3.5. *The $s-$parameter curves of $X(s,\theta)$ are the asymptotic curves if and only if the condition is fulfilled the following equation*

$$2\theta^5(k_1k_2'-k_1'k_2)+2\theta^4k_1k_2^2+2\theta^3k_1'+\theta^2(k_2^3-k_1k_2)+\theta k_2'+k_1=0$$

where $k_1(s)$ and $k_2(s)$ are the Cartan curvatures of the curve $\alpha(s)$.

Proof. If the normal vector of the surface is tangent to second derivative of a curve lying on the surface, this curve is called as the asymptotic curve of the surface, that is $\ell=0$ [8]. The desired result is obtained from the expression of $\ell=0$ in the equation (3.2) for the $s-$parameter curves on the regular tubular surface $X(s,\theta)$. □

Theorem 3.6. *The $\theta-$parameter curves of the regular surface $X(s,\theta)$ are neither geodesic curve nor asymptotic curve.*

Proof. For $\theta-$ parameter curves to be geodesic, it must provide the condition $U\times X_{\theta\theta}=0$. From this condition, we obtain

$$\frac{1}{\sqrt{A}}\{-4\theta(1-\theta^2k_2(s))L(s)+2(2\theta^3k_1(s)+\theta k_2(s))N(s)\}=0.$$

Since $1-\theta^2k_2(s)=0$ conflicts with the regularity condition of the surface $X(s,\theta)$, the $\theta-$ parameter curves cannot be geodesic curves. If the $\theta-$ parameter curves are to be asymptotic curve, then the coefficient of the second fundamental form n in equation (3.2) must be equal to zero. This condition conflicts with the regularity condition of the surface $X(s,\theta)$. So, the $\theta-$ parameter curves cannot be an asymptotic curve. □

4. Tubular surfaces around the spherical images of the null curve

In this section, we introduce tubular surfaces formed by spherical images of the null curve α. First, we will give definitions of the spheres in the Minkowski $3-$space. There are three kinds of spheres in E_1^3: de Sitter $2-$space, hyperbolic $2-$space, and lightlike cone. These are respectively:

$$S_1^2=\left\{p\in E_1^3\,|\ \langle p,p\rangle=1\right\},H_0^2=\left\{p\in E_1^3\,|\ \langle p,p\rangle=-1\right\}$$
$$\text{and } Q^2=\left\{p\in E_1^3\,|\ \langle p,p\rangle=0\right\}.$$

Now, we will give the definitions of the spherical images of the null curve. The null Cartan vector field L of the curve α is located at the center of the lightcone, the geometric location of this vector with respect to each point s indicates a curve on the lightcone Q^2, which is called the spherical (L) image of the curve α. In this definition, the spherical (N) image of the curve is defined by taking the null Cartan vector N instead of L. The spherical (W) image of the null curve is defined by the geometric location of the spacelike vector W on the de Sitter $2-$space S_1^2.
Note: Unless stated otherwise, the parameter θ given for each surface is different from each other.

4.1. Tubular surface around the spherical (L) and (N) images of the null curve

Let the spine curves of the tubular surfaces is respectively the spherical (L) and (N) images of the null curve α, that is, $\beta_i(s_i)=i(s)$ where the function s_i is the arc length parameter of the (i) image curve and $s_i=\int_0^s k_j(s)ds$ where indices are respectively $i=L$, $j=1$ and $i=N$, $j=2$. In [18], the author defined the Darboux frame $\left\{\beta_i(s_i),t_i(s_i)=\frac{d\beta}{ds_i},y_i(s_i)=\beta_i(s_i)\times t_i(s_i),\varkappa_i(s_i)\right\}$ of the spacelike curve β_i on the lightcone Q^2. The Darboux frame apparatus are calculated as follows:

$$\beta_L(s_L)=L(s),\ t_L(s_L)=W(s),\ y_L(s_L)=N(s)\text{ and }\varkappa_L(s_L)=-\tfrac{k_2(s)}{k_1(s)},$$

and

$$\beta_N(s_N)=N(s),\ t_N(s_N)=W(s),\ y_N(s_N)=L(s)\text{ and }\varkappa_N(s_N)=-\tfrac{k_1(s)}{k_2(s)}.$$

The Darbox equations are given by

$$\begin{aligned}\beta_i{}'(s_i)&=t_i(s_i),\\ t_i{}'(s_i)&=\varkappa_i(s_i)\beta_i(s_i)-y_i(s_i),\\ y_i{}'(s_i)&=-\varkappa_i(s_i)t_i(s_i),\end{aligned}$$

where $y_i\times t_i=y_i$, $\beta_i\times y_i=t_i$ and $t_i\times\beta_i=\beta_i$. The **spacelike** tubular surface around the curve β is

$$\mathscr{X}(s_i,\theta)=(1+\theta)\beta_i(s_i)+\theta^2y_i(s_i)$$

with the Gauss map $\mathscr{U}=\frac{1}{2\sqrt{|\theta|}}(\beta_i(s_i)-2\theta y_i(s_i))$. The coefficients of the first and second fundamental forms of $\mathscr{X}(s,\theta)$ are found by

$$\begin{gathered}\mathscr{E}=(1+\theta-\theta^2\varkappa_i(s_i))^2,\quad \mathscr{F}=0,\quad \mathscr{G}=4\theta,\\ \mathscr{L}=-\tfrac{(1+\theta-\theta^2\varkappa_i(s_i))(1+2\theta\varkappa_i(s_i))}{2\sqrt{|\theta|}},\quad \mathscr{M}=0,\quad \mathscr{N}=\tfrac{1}{\sqrt{|\theta|}}.\end{gathered}\tag{4.1}$$

Proposition 4.1. *The tubular surface $\mathscr{X}(s_i,\theta)$ is a regular surface if and only if it has the condition $\theta \neq \frac{1\mp\sqrt{1+4\varkappa_i(s_i)}}{2\varkappa_i(s_i)}$.*

Proof. The condition $\mathscr{E}\mathscr{G}-\mathscr{F}^2\neq 0$ must be provided for a regular surface. By using the equation (4.1), we obtain $1+\theta-\theta^2\varkappa_i(s_i)\neq 0$ for the surface $\mathscr{X}(s_i,\theta)$ and the desired condition is obtained from the solution of this equation with respect to θ. □

Remark 4.2. *The surface $\mathscr{X}(s_i,\theta)$ has the singular points at the points $(s_0,\theta_0=\frac{1\mp\sqrt{1+4\varkappa_i(s_0)}}{2\varkappa_i(s_0)}$.*

From equation (2.1) and $\varepsilon=-1$, the curvatures of the surface $\mathscr{X}(s_i,\theta)$ are calculated as follows:

$$K=-\frac{(1+2\theta\varkappa_i(s_i))}{8\theta^2(1+\theta-\theta^2\varkappa_i(s_i))} \ \ and \ \ H=-2\theta^{3/2}\left(\frac{1}{16\theta^3}+K\right).$$

Theorem 4.3. *The $s-$parameter curves of $\mathscr{X}(s_i,\theta)$ are the geodesic curves if and only if the Darboux curvature $\varkappa_i(s_i)=1/2\theta$ is a constant, this means that the image curve (i) is a planar curve.*

Proof. The $s-$parameter curves on the regular tubular surface $\mathscr{X}(s_i,\theta)$ are the geodesic curves if and only if $\mathscr{X}_{s_is_i}\times\mathscr{U}=0$. From the last equation, we obtain

$$\frac{1}{2|\ \theta\ |^{1/2}}(1-2\theta\varkappa_i)(1+\theta-\theta^2\varkappa_i)=0.$$

Since the surface is the regular, then $1+\theta-\theta^2\varkappa_i\neq 0$. So the curvature $\varkappa_i(s_i)=1/2\theta$ is obtained as a constant. □

Theorem 4.4. *The $s-$parameter curves of $\mathscr{X}(s_i,\theta)$ are the asymptotic curves if and only if the image curve (i) is a planar curve.*

Proof. The $s-$parameter curves on the regular tubular surface $\mathscr{X}(s_i,\theta)$ are the asymptotic curves if and only if $\langle\mathscr{X}_{s_is_i},\mathscr{U}\rangle=0$. From here, we get $\varkappa_i(s_i)=-1/2\theta$. Since the parameter θ is a constant for the $s-$parameter curves, the curvature $\varkappa_i(s_i)$ is a constant. □

Theorem 4.5. *The $\theta-$parameter curves on the regular surface $X(s,\theta)$ are neither geodesic curve nor asymptotic curve.*

Proof. For $\theta-$ parameter curves, $\mathscr{X}_{\theta\theta}\times\mathscr{U}\neq 0$ and $\mathscr{N}\neq 0$ are satisfied, so the $\theta-$ parameter curves cannot be a geodesic curve and an asymptotic curve. □

Since the proofs of the theorems and propositions involving the properties of the tubular surfaces consisting of $W-$ image curve of the null curve are similar to the proofs given above, the following theorems and propositions will be given without proof.

4.2. Tubular surface around the spherical (W) image of the null curve

Let γ be the (W) image curve of the null curve α. In this subsection, the spine curve of the surface $\mathbf{X}(s_W,\theta)$ will take as the curve γ, that is, $\gamma(s_W)=W(s)$ where the function $s_W=\int_0^s\sqrt{2\mid k_1(s)k_2(s)\mid}ds$ is the arc length parameter of the (W) image curve. The Darboux frame apparatus are given by

$$\begin{aligned}
\gamma(s_W) &= W(s),\\
t_W(s_W) &= -\frac{1}{\sqrt{2\mid k_1(s)k_2(s)\mid}}(k_2(s)L(s)+k_1(s)N(s)),\\
y_W(s_W) &= \frac{1}{\sqrt{2\mid k_1(s)k_2(s)\mid}}(-k_2(s)L(s)+k_1(s)N(s)),\\
\varkappa_W(s_W) &= \frac{k_1'(s)k_2(s)-k_1(s)k_2'(s)}{(2\mid k_1(s)k_2(s)\mid)^{3/2}}.
\end{aligned}$$

There are two cases here: $k_1(s)k_2(s)\neq 0$ and $k_1(s)k_2(s)=0$.
Case 1: $\mathbf{k_1(s)k_2(s)\neq 0}$. We will examine this situation as two sub-cases.
Case 1.1: If $\mathbf{k_1(s)k_2(s)>0}$, then the curve γ is a spacelike curve on de Sitter $2-$space S_1^2. In [14], the Darboux equations are

$$\begin{aligned}
\gamma'(s_W) &= t_W(s_W),\\
t_W{}'(s_W) &= -\gamma(s_W)+\varkappa_W(s_W)y_W(s_W),\\
y_W{}'(s_W) &= \varkappa_W(s_W)t_W(s_W),
\end{aligned}$$

where $y_W\times t_W=\gamma$, $\gamma\times y_W=t_W$ and $t_W\times\gamma=-y_W$. The **timelike** tubular surface around the curve γ is

$$\mathbf{X}(s_W,\theta)=(1+r\cosh\theta)\gamma(s_W)-r\sinh\theta y_W(s_W)$$

with the Gauss map $\mathbf{U} = \cosh\theta\gamma(s_W) - \sinh\theta y_W(s_W)$. The coefficients of the first and second fundamental forms of $\mathbf{X}(s,\theta)$ are found by

$$\mathbf{E} = (1 + r\cosh\theta - r\varkappa_W(s_W)\sinh\theta)^2, \quad \mathbf{F} = 0, \quad \mathbf{G} = -r^2,$$

$$\mathbf{L} = (\varkappa_W(s_W)\sinh\theta - \cosh\theta)(1 + r\cosh\theta - r\varkappa_W(s_W)\sinh\theta), \quad \mathbf{M} = 0, \quad \mathbf{N} = r.$$

Proposition 4.6. *The tubular surface $\mathbf{X}(s_W,\theta)$ is a regular surface if and only if it has the condition $1 + r\cosh\theta - r\varkappa_W(s_W)\sinh\theta \neq 0$.*

Remark 4.7. *The surface $\mathbf{X}(s_W,\theta)$ has the singular points satisfying the equation $1 + r\cosh\theta - r\varkappa_W(s_W)\sinh\theta = 0$.*

From equation (2.1) and $\varepsilon = 1$, the curvatures of the surface $\mathbf{X}(s_W,\theta)$ are calculated as follows:

$$\mathbf{K} = \frac{(\cosh\theta - \varkappa_W(s_W)\sinh\theta)}{r(1 + r\cosh\theta - r\varkappa_W(s_W)\sinh\theta)} \quad \text{and} \quad \mathbf{H} = -\frac{1}{2}\left(\frac{1}{r} + rK\right).$$

Remark 4.8. *Since the surface $\mathbf{X}(s_W,\theta)$ has the condition $2\mathbf{H} + r\mathbf{K} = -\frac{1}{r}$, the surface $\mathbf{X}(s_W,\theta)$ is a linear Weingarten surface.*

Theorem 4.9. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the geodesic curves if and only if the Darboux curvature $\varkappa_W(s_W) = \tanh\theta$ is a constant.*

Theorem 4.10. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the asymptotic curves if and only if the curvature $\varkappa_W$ is a constant and equal to* $\coth\theta$.

Theorem 4.11. *The $\theta-$parameter curves on the regular surface $\mathbf{X}(s_W,\theta)$ are always a geodesic curve and cannot be an asymptotic curve.*

Case 1.2: If $\mathbf{k_1(s)k_2(s) < 0}$, then the curve γ is a timelike curve on de Sitter $2-$space S_1^2. In [14], the Darboux equations are

$$\begin{aligned} \gamma'(s_W) &= t_W(s_W), \\ t_W{}'(s_W) &= \gamma(s_W) + \varkappa_W(s_W)y_W(s_W), \\ y_W{}'(s_W) &= \varkappa_W(s_W)t_W(s_W), \end{aligned}$$

where $y_W \times t_W = -\gamma$, $\gamma \times y_W = t_W$ and $t_W \times \gamma = -y_W$. The timelike tubular surface around the curve γ is

$$\mathbf{X}(s_W,\theta) = (1 + r\cos\theta)\gamma(s_W) + r\sin\theta y_W(s_W)$$

with the spacelike Gauss map $\mathbf{U} = \cos\theta\gamma(s_W) + \sin\theta y_W(s_W)$. The coefficients of the first and second fundamental forms of $\mathbf{X}(s,\theta)$ are found by

$$\mathbf{E} = -(1 + r\cos\theta + r\varkappa_W(s_W)\sin\theta)^2, \quad \mathbf{F} = 0, \quad \mathbf{G} = r^2,$$

$$\mathbf{L} = (\cos\theta + \varkappa_W(s_W)\sin\theta)(1 + r\cos\theta + r\varkappa_W(s_W)\sin\theta), \quad \mathbf{M} = 0, \quad \mathbf{N} = -r.$$

Proposition 4.12. *The tubular surface $\mathbf{X}(s_W,\theta)$ is a regular surface if and only if it has the condition $1 + r\cos\theta + r\varkappa_W(s_W)\sin\theta \neq 0$.*

Remark 4.13. *The surface $\mathbf{X}(s_W,\theta)$ has the singular points satisfying the equation $1 + r\cos\theta + r\varkappa_W(s_W)\sin\theta = 0$.*

The invariant curvatures of the surface $\mathbf{X}(s_W,\theta)$ are calculated as follows:

$$\mathbf{K} = \frac{(\cos\theta + \varkappa_W(s_W)\sin\theta)}{r(1 + r\cos\theta + r\varkappa_W(s_W)\sin\theta)} \quad \text{and} \quad \mathbf{H} = -\frac{1}{2}\left(\frac{1}{r} + r\mathbf{K}\right).$$

Since the surface $\mathbf{X}(s_W,\theta)$ has the condition $2\mathbf{H} + r\mathbf{K} = -\frac{1}{r}$, the surface $\mathbf{X}(s_W,\theta)$ is a linear Weingarten surface.

Theorem 4.14. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the geodesic curves if and only if the Darboux curvature $\varkappa_W(s_W) = \tan\theta$ is a constant.*

Theorem 4.15. *The $s-$parameter curves of $\mathbf{X}(s_W,\theta)$ are the asymptotic curves if and only if the curvature of the (W) image curve is a constant.*

Theorem 4.16. *The $\theta-$parameter curves on the regular surface $\mathbf{X}(s_W,\theta)$ are always the geodesic curves and they cannot be the asymptotic curves.*

Case 2: $\mathbf{k_1(s)k_2(s) = 0}$. If $k_1(s) = 0$, the null curve α is a planar line. So, we will examine the case of $k_2(s) = 0$. For $k_1(s) \neq 0$ and $k_2(s) = 0$, the curve α is called a generalized null cubic curve in [15] and it is given by

$$\alpha(s) = \left(\frac{1}{\sqrt{2}} \left((s + \frac{\phi(s)}{2} \right), \frac{1}{\sqrt{2}} \left(s - \frac{\phi(s)}{2} \right), \psi(s) \right)$$

where $\phi\prime(s) = (\psi\prime(s))^2$. The third Cartan vector of the curve α is $W(s) = \left(\frac{\psi'(s)}{\sqrt{2}}, -\frac{\psi'(s)}{\sqrt{2}}, 1 \right)$. To find the tubular surface around the (W) image curve of the generalized null cubic curve, we calculate the Cartan frame of the (W) image curve as follows: $\overline{L} = \left(\frac{\psi''(s)}{\sqrt{2}}, -\frac{\psi''(s)}{\sqrt{2}}, 0 \right), \overline{N}(s) = \left(-\frac{1}{\sqrt{2}\psi''(s)}, -\frac{1}{\sqrt{2}\psi''(s)}, 0 \right), \overline{W} = (0,0,1)$.
From these vectors, we obtain $\psi''(s) = \sqrt{2}a$, where a is a constant. The surface consisting of the (W) image curve of the generalized null cubic curve α can be written as

$$\mathbf{X}(s_W, \theta) = W(s) + \theta\overline{N}(s) + \theta^2\overline{W}(s).$$

This tubular surface is degenerated to a plane.

5. Visualization

In this section, we give the tubular surfaces whose center curves are a null curve α and its spherical images. Then, we calculate the some special curves on these surfaces and find the singular points of them. Also, we visualize the our calculations with Mathematica.
Let $\alpha = \alpha(s)$ be a null curve is defined by

$$\alpha(s) = \left(s, \frac{1}{5}\sin(5s+4) + 1, -\frac{1}{5}\cos(5s+4) - 1 \right)$$

with the Cartan frame apparatus

$$\begin{aligned} L(s) &= (1, \cos(5s+4), \sin(5s+4)), \\ N(s) &= \tfrac{1}{2}(-1, \cos(5s+4), \sin(5s+4)), \\ W(s) &= (0, -\sin(5s+4), \cos(5s+4)), \end{aligned}$$

$k_1(s) = 5$ and $k_2(s) = 5/2$. The parametric form of the tubular surface $X(s,\theta)$ is given as follows

$$X(s,\theta) = \alpha(s) + \theta N(s) + \theta^2 W(s),$$

$$\begin{aligned} X(s,\theta) = \Bigg(& s - \frac{\theta}{2}, 1 + \frac{\theta}{2}\cos(5s+4) + \frac{1}{5}\sin(5s+4) - \theta^2\sin(5s+4), \\ & -1 + \frac{\theta}{2}\sin(5s+4) - \frac{1}{5}\cos(5s+4) + \theta^2\cos(5s+4) \Bigg). \end{aligned}$$

We calculate the singular points on the surface $X(s,\theta)$, then we obtain two curves consisting of singular points in Figure (5.1). Since the Cartan curvatures of the curve α are constants, the $s-$parameter curves in Figure (5.2) of the tubular surface $X(s,\theta)$

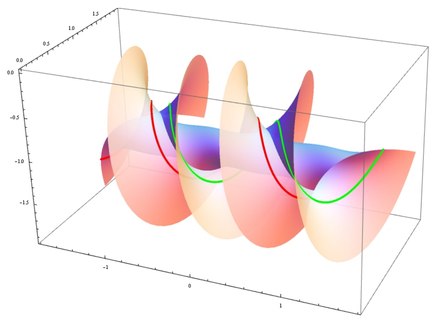

Figure 5.1: Tubular surface generated by the null curve α for $s \in [-2\pi/5, 2\pi/5]$, $\theta \in [-1,1]$ and its singular curves for $\theta = \sqrt{\frac{2}{5}}$ (red), $\theta = -\sqrt{\frac{2}{5}}$ (green).

are always the geodesic curves. Since the condition given in the Theorem (3.5) is not satisfied in this example, the $s-$parameter curves are not the asymptotic curves. Also, we have shown in Theorem (3.6) that the $\theta-$parameter curves in Figure (5.3) are neither geodesic curves nor asymptotic curves.

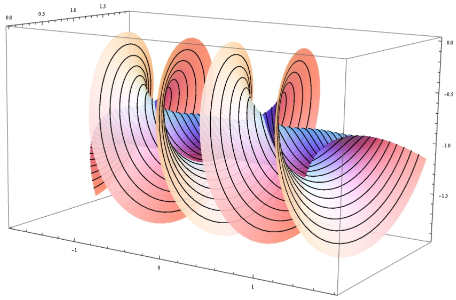

Figure 5.2: Geodesic $s-$parameter curves on the tubular surface $X(s,\theta)$

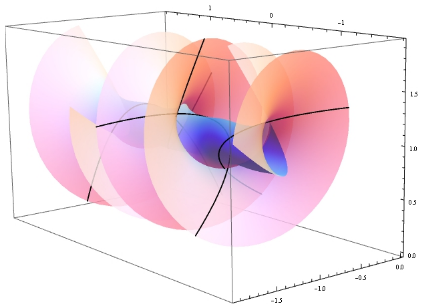

Figure 5.3: The $\theta-$parameter curves on the tubular surface $X(s,\theta)$

The tubular surface around (L) image curve is given by

$$\mathscr{X}(s_L,\theta)=(1+\theta)L(s)+\theta^2N(s),$$

$$\mathscr{X}(s_L,\theta)=\left(1+\theta-\frac{\theta^2}{2},\left(1+\theta+\frac{\theta^2}{2}\right)\cos(5s+4),\left(1+\theta+\frac{\theta^2}{2}\right)\sin(5s+4)\right)$$

where $s_L=5s$. The tubular surface $\mathscr{X}(s_L,\theta)$ has no singular points. The $s-$parameter curves are the geodesic curves for $\theta=-1$ in Figure (5.4) (red) and are the asymptotic curves for $\theta=1$ in Figure (5.4) (green). The $\theta-$parameter curves on the tubular surface $\mathscr{X}(s_L,\theta)$ are shown in Figure (5.5).

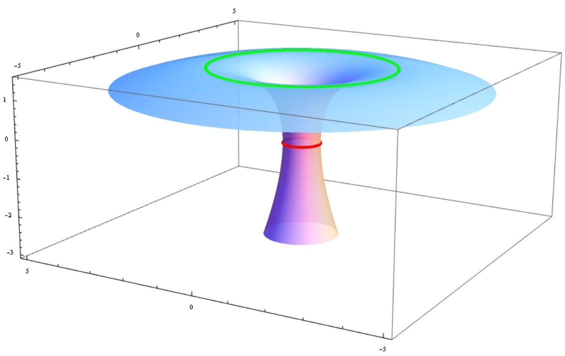

Figure 5.4: Geodesic $s-$parameter curves on $\mathscr{X}(s_L,\theta)$ for $\theta=-1$ (red) and $\theta=1$ (green).

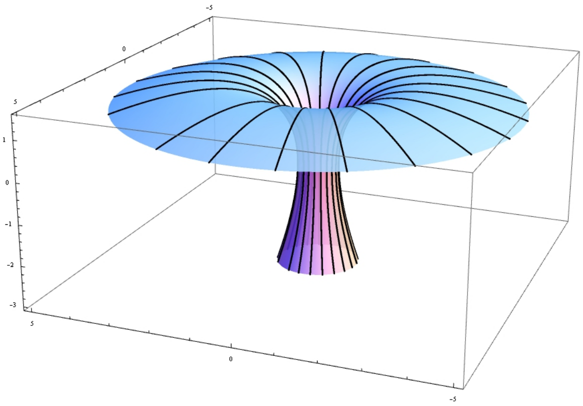

Figure 5.5: The $\theta-$parameter curves on $\mathscr{X}(s_L,\theta)$.

The tubular surface around (N) image curve is given by

$$\mathscr{X}(s_N,\theta)=(1+\theta)N(s)+\theta^2L(s),$$

$$\mathscr{X}(s_N,\theta)=\left(-\frac{1}{2}-\frac{\theta}{2}+\theta^2,\left(\frac{1}{2}+\frac{\theta}{2}+\theta^2\right)\cos(5s+4),\left(\frac{1}{2}+\frac{\theta}{2}+\theta^2\right)\sin(5s+4)\right)$$

where $s_N=5s/2$. The tubular surface $\mathscr{X}(s_N,\theta)$ has no singular points. The $s-$parameter curves are the geodesic curves for $\theta=-1/4$ in Figure (5.6) (red) and the asymptotic curves for $\theta=1/4$ in Figure (5.6) (green). The $\theta-$parameter curves on the tubular surface $\mathscr{X}(s_L,\theta)$ are shown in Figure (5.7).

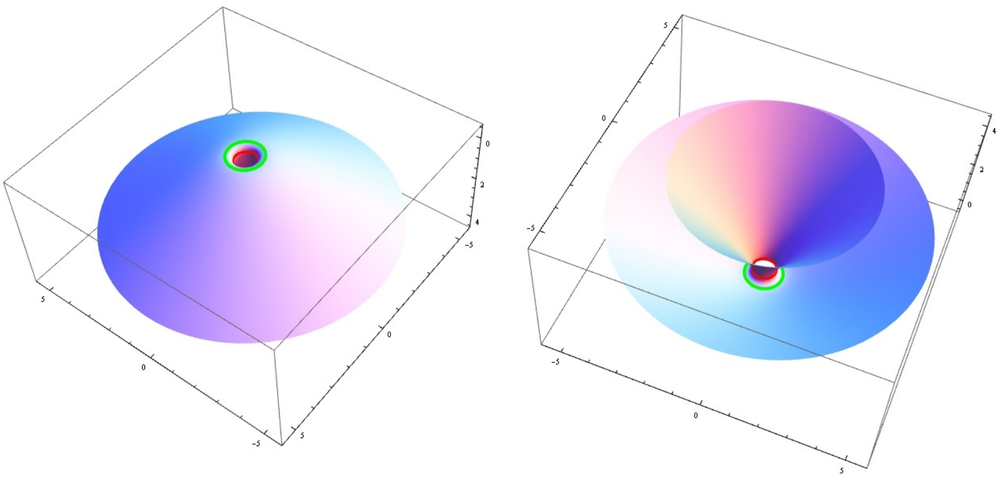

Figure 5.6: Geodesic $s-$parameter curves on $\mathscr{X}(s_N,\theta)$ for $\theta=-0.25$ (red) and $\theta=0.25$ (green).

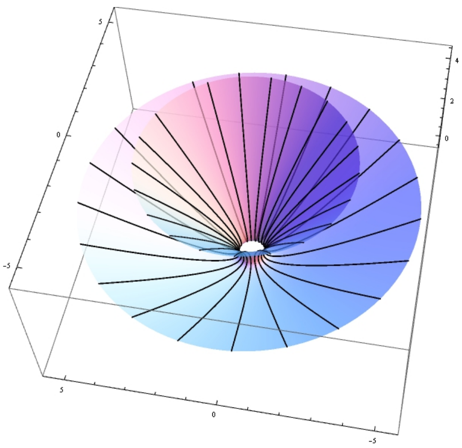

Figure 5.7: The $\theta-$parameter curves on $\mathscr{X}(s_N,\theta)$.

Since $\mathbf{k_1(s)k_2(s)} > \mathbf{0}$, the tubular surface generated by (W) image curve is

$$\mathbf{X}(s_W,\theta) = (1+r\cosh\theta)W(s) - r\sinh\theta\left(-\frac{1}{2}L(s)+N(s)\right),$$

$$\mathbf{X}(s_W,\theta) = (r\sinh\theta, -(1+r\cosh\theta)\sin(5s+4), (1+r\cosh\theta)\cos(5s+4))$$

where $s_W = 5s$. Since $\varkappa_W(s_W) = 0$ and $r > 0$, the equation in Remark (4.7) has no real root. So, there is no singular points on the tubular surface $\mathbf{X}(s_W,\theta)$. Some special curves on $\mathbf{X}(s_W,\theta)$ are shown in Figure (5.8).

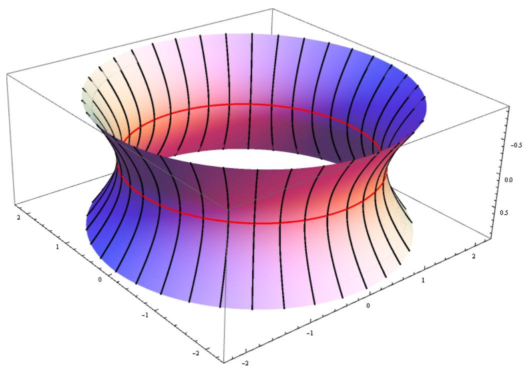

Figure 5.8: Geodesic $s-$parameter curve for $\theta = 0$ (red) and geodesic $\theta-$parameter curves (black) for $r = 0.8$ on the surface $\mathbf{X}(s_W,\theta)$.

6. Conclusion

This study is important in terms of finding tubular surface formed by the null curve and its image curves on the Minkowski spheres. Their singular points are characterized in terms of Cartan frame and Darboux frame apparatus. It is also noteworthy that to use the Darboux frame instead of the Frenet frame, this is provided an opportunity to examine the expressions in their simplest form.

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Author's contributions

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