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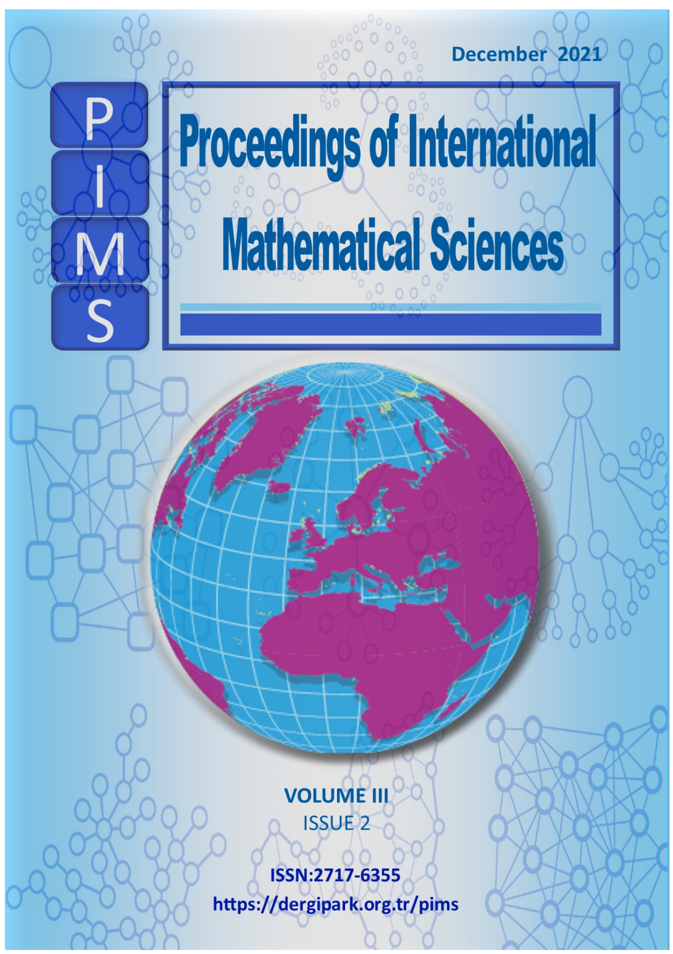

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A UNIFORMLY STABLE SOLVABILITY OF NLBVP FOR PARAMETERIZED ODE

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Abstract. Nonlocal boundary value problem of the first kind for an ordinary linear second order differential equation with positive parameter at the highest derivative is considered. The existence and uniqueness, as well as, a uniformly stable estimate of classical solution is established under accurate condition on coefficients and location of nonlocal data carriers of multipoint boundary value condition. An essentiality of the revealed condition is confirmed by ill-posed problem examples.

1. Introduction

The article of A.N. Tikhonov [1] gave the reason for a wide range study in the field of parameterized differential equations. The joint paper of A.V. Bitsadze and A.A. Samarskii [2] motivated a lot of research in the field of differential problems which are identifiable as nonlocal boundary problems.

In our paper, we consider nonlocal boundary value problem (NLBVP) of the first kind[1] for ordinary differential equation (ODE)

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = -f(x), \ \ 0 < x < 1$$

with a positive parameter $\varepsilon > 0$. Herein, for an unknown solution, we consider the nonlocal boundary value condition (NLBVC) which is given by linear combination of the values in boundary and interior points of $[0, 1]$. Our task is to study the question of a uniformly stable solvability of such NLBVP in respect of the classical solution from $C^2(0, 1) \cap C[0, 1]$.

In [3], for Sturm-Liouville operator the NLBVP of the first kind

$$[k(x)u']' - q(x)u = -f(x), \ 0 < x < 1, \ \ u(0) = 0, \ u(1) = \sum_{i=1}^{n} \alpha_i u(\xi_i)$$

was researched for $k(x) \in C^1[0, 1]$, $f(x)$, $q(x) \in C[0, 1]$, $k(x) \geq m_0 > 0$, $q(x) \geq 0$. The existence, uniqueness and a priori estimate of classical solution was established

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[1]The term "NLBVP of the first kind" was introduced by V.A. Il'in and E.I. Moiseev in [3].

for the case if all coefficients $\alpha_i,\ i=1,...,n$ have the same sign and satisfy the condition

$$-\infty < \sum_{i=1}^{n} \alpha_i \le 1.$$

For the same problem, but under the condition that $\alpha_i,\ i=1,...,n$ have an arbitrary sign and satisfy the condition

$$\sum_{i=1}^{n} \frac{(\alpha_i + |\alpha_i|)}{2} \int_0^{\xi_i} \frac{1}{k(\tau)} d\tau < \int_0^1 \frac{1}{k(\tau)} d\tau,$$

the existence and uniqueness of classical solution was proved in [4].

In [5], it was proved that singularly perturbed NLBVP of the first kind

$$-\varepsilon^2 y''(x) + g(x)y(x) = h(x),\ \ 0 < x < 1,\quad y(0)=0,\ \hat{\ell}y = d,$$

has a unique solution if and only if the solution of Dirichlet problem

$$\varepsilon^2 u''(x) - g(x)u(x) = 0,\ \ u(0)=0,\ u(1)=1$$

satisfies the condition $\hat{\ell}u \neq 0$, where $g(x) \ge K^2 > 0$, $K \in \boldsymbol{R}$, $\hat{\ell}y \equiv y(1) - \sum\limits_{i=1}^{m} c_i y(s_i)$, $s_i \in (0,1)$.

In [6], the existence, uniqueness and a priori estimate of classical solution

$$||u||_{W_2^2[0,1]} \le C||f||_{L_2[0,1]}$$

were proved for NLBVP

$$\begin{cases} [k(x)u'(x)]' + r(x)u'(x) - q(x)u(x) = -f(x),\ \ 0 < x < 1, \\ u(0)=0,\ \ u(1) = \alpha u(\zeta) - \beta u(\eta), \end{cases}$$

where $k(x) \in C^1[0,1]$, $f(x)$, $r(x)$ and $q(x) \in C[0,1]$, $k(x) \ge m_0 > 0$, $|r(x)| < \mu$, $q(x) \ge 0$, $x \in [0,1]$, $\mu < m_0$, $\zeta \in (0,1)$, $\eta \in (0,1)$, in addition, $\alpha > 0$, $\beta > 0$, $-\infty < \alpha - \beta \le 1$ if $\zeta < \eta$, $\alpha \le 1$ if $\eta < \zeta$.

In [7], the existence, uniqueness and a priori estimate of classical solution were proved for NLBVP with double-side NLBVC of the first kind

$$\begin{cases} [k(x)u'(x)]' + r(x)u'(x) - q(x)u(x) = -f(x),\ \ 0 < x < 1, \\ u(0) = \alpha_0 u(\zeta_0) - \beta_0 u(\eta_0), u(1) = \alpha_1 u(\zeta_1) - \beta_1 u(\eta_1) \end{cases}$$

where $k(x) \in C^1[0,1]$, $f(x),\ r(x),\ q(x) \in C[0,1]$, $k(x) \ge m_0 > 0$, $q(x) \ge 0$, $x \in [0,1]$, $\zeta_i \in (0,1)$, $\eta_i \in (0,1)$, $i=0,1$, $\max\{\zeta_0,\eta_0\} < \min\{\zeta_1,\eta_1\}$, in addition, $\alpha_i > 0$, $\beta_i > 0$, $i=0,1$, $S_0 \le 1$, $S_1 \le 1$, $S_0 + S_1 < 2$, herewith $S_0 = \alpha_0 - \beta_0$ for $\eta_0 \le \zeta_0$, $S_0 = \alpha_0$ for $\zeta_0 < \eta_0$, $S_1 = \alpha_1 - \beta_1$ for $\zeta_1 \le \eta_1$, $S_1 = \alpha_1$ for $\eta_1 < \zeta_1$.

In [8, p. 68-72], a uniformly stable solvability was reported for parameterized NLBVP

$$\begin{cases} -\varepsilon u''(x) + b(x)u(x) = f(x),\ \ 0 < x < 1, \\ u(0) - \alpha u(\zeta) = \phi_0,\ \ u(1) - \beta u(\eta) = \phi_1 \end{cases}$$

where $\varepsilon > 0$, $b(x),\ f(x) \in C[0,1]$, $b(x) \ge b^* > 0$, $0 < \zeta < \eta < 1$, $-\infty < \alpha < 1$, $-\infty < \beta < 1$, $\alpha\beta \neq 0$, $\varphi_i = const$, $i=0,1$.

In [9], the solution of NLBVP, which was formulated in [5], was constructed by using the truncated orthogonal series and corresponding solution of the reduced problem.

In [10], under the condition that classical solution of the Dirichlet problem

$$-\varepsilon w'' + a(x)w' + b(x)w = 0,\ 0 < x < 1,\ w(0) = 0,\ w(1) = 1$$

satisfies the inequality $w(1) - \sum\limits_{i=1}^{m-2} c_i w(s_i) \neq 0$, the behaviour of exact solution was analized for NLBVP

$$-\varepsilon u'' + a(x)u' + b(x)u = f(x),\ 0 < x < 1,\ u(0) = A,\ u(1) = \sum_{i=1}^{m-2} c_i u(s_i) + B,$$

where $0 < \varepsilon << 1$, $a(x) \geq \alpha > 0$, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions on $[0,1]$, $s_i \in (0,1)$, $i = 1,2,...,m-2$.

In summary, it is natural that NLBVP's solvability, as well as, the behaviour of its classical solution depends on coefficients, their signs, values, and, at least, data carriers location of given nonlocal condition. It is the reason why the aim of our paper is to reveal explicit condition of a uniform solvability for parameterized linear second order ODE with abstract double-side nonlocal condition of the first kind. In general, naturally that the information on a uniform solvability of differential problem is also actual for its numerical interpretation.

Additionally, sufficiently detailed overview on NLBVP for ODE is enclosed in [3, 4, 6, 7, 13], the survey on boundary value problems respectively parameterized ODE is represented by [14].

2. Differential problem

We consider the NLBVP

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = -f(x), \quad 0 < x < 1, \tag{2.1}$$

$$\ell_0(u) \equiv u(0) - \sum_{k=1}^{m_0} \alpha_k u(\zeta_k) = \varphi_0, \quad \ell_1(u) \equiv u(1) - \sum_{l=1}^{m_1} \beta_l u(\eta_l) = \varphi_1, \tag{2.2}$$

where $\varepsilon > 0$ is a parameter, $a(x)$, $b(x)$, $f(x) \in C[0,1]$, $m_i \geq 2$, $i = 0,1$, $\varphi_i \in R$, $i = 0,1$, $\zeta_k \in (0,1)$, $k = 1,...,m_0$, $\eta_l \in (0,1)$, $l = 1,...,m_1$ are so that

$$0 < \zeta_1 < \zeta_2 < ... < \zeta_{m_0} < \eta_1 < \eta_2 < ... < \eta_{m_1} < 1, \tag{2.3}$$

in addition, $\alpha_k \in R$, $k = 1,...,m_0$, $\beta_l \in R$, $l = 1,...,m_1$ are nonzero coefficients. Next condition is denoted by $\boldsymbol{A}$:

- if all α_k are not of the same sign, then $\alpha_k > 0$ only for $k = 1,...,m_\iota$, or $\alpha_k > 0$ only for $k = m_\iota + 1,...,m_0$, where m_ι is some natural number, $1 \leq m_\iota < m_0$;
- if all β_l are not of the same sign, then $\beta_l > 0$ only for $l = 1,...,m_\kappa$, or $\beta_l > 0$ only for $l = m_\kappa + 1,...,m_1$, where m_κ is some natural number, $1 \leq m_\kappa < m_1$.

Further, we will use the designations:

$$\alpha = \sum_{k=1}^{m_0} \alpha_k, \quad \alpha^+ = \sum_{k=1}^{m_0} \frac{\alpha_k + |\alpha_k|}{2}, \quad \alpha^- = \sum_{k=1}^{m_0} \frac{\alpha_k - |\alpha_k|}{2},$$

$$\beta = \sum_{l=1}^{m_1} \beta_l, \quad \beta^+ = \sum_{l=1}^{m_1} \frac{\beta_l + |\beta_l|}{2}, \quad \beta^- = \sum_{l=1}^{m_1} \frac{\beta_l - |\beta_l|}{2},$$

$$S_0 = \begin{cases} \alpha^+ + \alpha^-, & \text{if } \alpha_{m_\iota} < 0,\ \alpha_{m_\iota+1} > 0, \\ \alpha^+, & \text{if } \alpha_{m_\iota} > 0,\ \alpha_{m_\iota+1} < 0, \\ \alpha, & \text{if } \alpha_k,\ k = 1,...,m_0 \text{ have the same sign,} \end{cases}$$

$$S_1 = \begin{cases} \beta^+ + \beta^-, \text{ if } \beta_{m_\kappa} > 0, \ \beta_{m_{\kappa+1}} < 0, \\ \beta^+, \text{ if } \beta_{m_\kappa} < 0, \ \beta_{m_{\kappa+1}} > 0, \\ \beta, \text{ if } \beta_l, \ l = 1, ..., m_1 \text{ have the same sign.} \end{cases}$$

Definition. The function $u(x)$ is a classical solution of NLBVP (2.1)-(2.2) if it belongs to $C^2(0,1) \cap C[0,1]$, satisfies the equation (2.1) and NLBVC (2.2).

Let each one of NLBVC (2.2) encloses different sign coefficients. Let us suppose that classical solution $u(x)$ of NLBVP (2.1)-(2.2) exists. Then, in view of the mean value (MV) property [3, p. 1198-1199], by analogy with [13, p. 39], this classical solution satisfies some reduced NLBVC

$$u(0) - \alpha^+ u(\zeta^+) - \alpha^- u(\zeta^-) = \varphi_0, \quad u(1) - \beta^+ u(\eta^+) - \beta^- u(\eta^-) = \varphi_1, \qquad (2.4)$$

where $\zeta^+ \in [\zeta_1, \zeta_{m_0}]$, $\zeta^- \in [\zeta_1, \zeta_{m_0}]$, $\eta^+ \in [\eta_1, \eta_{m_1}]$, $\eta^- \in [\eta_1, \eta_{m_1}]$ and, therefore, $u(x)$ is classical solution of NLBVP (2.1),(2.4) too[2]. In respect of (2.4), we denote

$$\bar{\ell}_0(u) \equiv u(0) - \alpha^+ u(\zeta^+) - \alpha^- u(\zeta^-), \quad \bar{\ell}_1(u) \equiv u(1) - \beta^+ u(\eta^+) - \beta^- u(\eta^-). \quad (2.5)$$

Hence, for S_i, $i = 0, 1$, we have

$$S_0 = \begin{cases} \alpha^+ + \alpha^-, \text{ if } \zeta^- < \zeta^+, \\ \alpha^+, \text{ if } \zeta^+ < \zeta^-, \\ \alpha, \text{ if all } \alpha_k, \ k = 1, ..., m_0 \text{ have the same sign,} \end{cases} \qquad (2.6)$$

$$S_1 = \begin{cases} \beta^+ + \beta^-, \text{ if } \eta^+ < \eta^-, \\ \beta^+, \text{ if } \eta^- < \eta^+, \\ \beta, \text{ if all } \beta_l, \ l = 1, ..., m_1 \text{ have the same sign.} \end{cases} \qquad (2.7)$$

Additionally, in view of (2.3) and $\boldsymbol{A}$, we have

$$\zeta^- \neq \zeta^+, \ \ \eta^- \neq \eta^+, \ \ \max\{\zeta^-, \zeta^+\} < \min\{\eta^-, \eta^+\}. \qquad (2.8)$$

Our first result is

Lemma 2.1. *Let $S_i \leq 1$, $\varphi_i \neq 0$, $i = 0, 1$. If $u(x)$ is classical solution of the problem (2.1),(2.4), then $v(x) = u(x) + \varphi_0 q_0(x) + \varphi_1 q_1(x)$ is classical solution of the problem*

$$Lv(x) = -f_1(x), \ 0 < x < 1, \quad \bar{\ell}_0(v) = 0, \ \bar{\ell}_1(v) = 0 \qquad (2.9)$$

for $f_1(x) = f(x) - \varphi_0 Lq_0(x) - \varphi_1 Lq_1(x)$, where $q_i(x)$, $i = 0, 1$ are some cubic polinoms .
Let $S_i \leq 1$, $i = 0, 1$. Let only one of φ_0, φ_1 be nonzero, i.e., $\varphi_{i_*} \neq 0$, $i_* \in \{0, 1\}$. If $u(x)$ is classical solution of (2.1),(2.4), then $v(x) = u(x) + \varphi_{i_*} q_{i_*}(x)$ is classical solution of the problem (2.9) for $f_1(x) = f(x) - \varphi_{i_*} Lq_{i_*}(x)$, where $q_{i_*}(x)$ is some cubic polinom.

Proof. Assume that $q_0(x)$, $q_1(x) \in C^2(0,1)$ are an arbitrary functions. Then it is obvious that

$$Lv(x) = -[f(x) - \varphi_0 Lq_0(x) - \varphi_1 Lq_1(x)] = -f_1(x)$$

[2]Thus we will say: "the problem (2.1),(2.2) is reduciable to (2.1),(2.4)", or, for example, "condition (2.2) is redused to (2.4)", or "condition (2.2) is reduciable to (2.4)", or "(2.4) is reduced nonlocal condition" and etc..

i.e., $v(x)$ satisfies the differential equation (2.9). Let us construct some polinomials $q_0(x)$ and $q_1(x)$, so that the function $v(x)$ will satisfy NLBVC (2.9). Put $\varphi_i \neq 0$, $i = 0, 1$. We look for the functions

$$q_0(x) = c_0(1-x)(\eta^+ - x)(\eta^- - x), \tag{2.10}$$

$$q_1(x) = c_1 x(\zeta^+ - x)(\zeta^- - x), \tag{2.11}$$

where an unknown constants c_0 and c_1 have to be defined. Since

$$\overline{\ell}_0(q_1) = 0, \quad \overline{\ell}_1(q_0) = 0, \tag{2.12}$$

then, in view of (2.5) and (2.4),

$$\overline{\ell}_0(v) = \varphi_0 + \varphi_0\overline{\ell}_0(q_0) + \varphi_1\overline{\ell}_0(q_1) = \varphi_0[1 + \overline{\ell}_0(q_0)], \tag{2.13}$$

$$\overline{\ell}_1(v) = \varphi_1 + \varphi_0\overline{\ell}_1(q_0) + \varphi_1\overline{\ell}_1(q_1) = \varphi_1[1 + \overline{\ell}_1(q_1)]. \tag{2.14}$$

Since $v(x)$ has to satify (2.9), then, in view of (2.13) and (2.14), the equalities

$$1 + \overline{\ell}_0(q_0) = 0, \quad 1 + \overline{\ell}_1(q_1) = 0 \tag{2.15}$$

have to be satisfied for $q_i(x)$, $i = 0, 1$. Hence, we have

$$c_0 = -(E_0)^{-1}, \tag{2.16}$$

$$c_1 = -(D_0)^{-1}, \tag{2.17}$$

for

$$E_0 = \eta^+\eta^- - \alpha^+(1-\zeta^+)(\eta^+-\zeta^+)(\eta^--\zeta^+) - \alpha^-(1-\zeta^-)(\eta^+-\zeta^-)(\eta^--\zeta^-), \tag{2.18}$$

$$D_0 = (1-\zeta^+)(1-\zeta^-) - \beta^+\eta^+(\zeta^+-\eta^+)(\zeta^--\eta^+) - \beta^-\eta^-(\zeta^+-\eta^-)(\zeta^--\eta^-), \tag{2.19}$$

where $E_0 \neq 0$, $D_0 \neq 0$ since $E_0 > 0$, $D_0 > 0$. Indeed, in view of (2.8), from (2.18) and (2.19), correspondingly, we get

$$E_0 > \begin{cases} (1 - [\alpha^+ + \alpha^-])(1-\zeta^-)(\eta^+ - \zeta^-)(\eta^- - \zeta^-), & \text{if } \zeta^- < \zeta^+, \\ (1-\alpha^+)(1-\zeta^-)(\eta^+ - \zeta^-)(\eta^- - \zeta^-), & \text{if } \zeta^+ < \zeta^-, \end{cases} \tag{2.20}$$

$$D_0 > \begin{cases} (1 - [\beta^+ + \beta^-])\eta^-(\eta^- - \zeta^+)(\eta^- - \zeta^-), & \text{if } \eta^+ < \eta^-, \\ (1-\beta^+)\eta^-(\eta^- - \zeta^+)(\eta^- - \zeta^-), & \text{if } \eta^- < \eta^+. \end{cases} \tag{2.21}$$

Then from (2.20) and (2.21), correspondingly, in view of (2.6) and (2.7), we have

$$E_0 > \begin{cases} (1-S_0)(1-\zeta^-)(\eta^+ - \zeta^-)(\eta^- - \zeta^-), & \text{if } \zeta^- < \zeta^+,\ 0 < S_0 \leq 1, \\ (1-\zeta^-)(\eta^+ - \zeta^-)(\eta^- - \zeta^-), & \text{if } \zeta^- < \zeta^+,\ -\infty < S_0 \leq 0, \\ (1-S_0)(1-\zeta^-)(\eta^+ - \zeta^-)(\eta^- - \zeta^-), & \text{if } \zeta^+ < \zeta^-,\ 0 \leq S_0 \leq 1, \end{cases}$$

$$D_0 > \begin{cases} (1-S_1)\eta^-(\eta^- - \zeta^+)(\eta^- - \zeta^-), & \text{if } \eta^+ < \eta^-,\ 0 < S_1 \leq 1, \\ \eta^-(\eta^- - \zeta^+)(\eta^- - \zeta^-), & \text{if } \eta^+ < \eta^-,\ -\infty < S_1 \leq 0, \\ (1-S_1)\eta^-(\eta^- - \zeta^+)(\eta^- - \zeta^-), & \text{if } \eta^- < \eta^+,\ 0 \leq S_1 \leq 1. \end{cases}$$

Hence, $E_0 > 0$, $D_0 > 0$, therefore, $E_0 \neq 0$, $D_0 \neq 0$. Thus, in view of (2.16) and (2.17), the polinomials (2.10) and (2.11) are defined. In view of (2.12) and (2.15), $v(x)$ satisfies nonlocal conditions of (2.9). Since $u(x) \in C^2(0,1) \cap C[0,1]$, then $v(x) \in C^2(0,1) \cap C[0,1]$ too, therefore, $v(x)$ is classical solution of NLBVP (2.9).

By similar way, it is easy to prove the second statement for the case if one of two data φ_i, $i = 0, 1$ is zero, but another one is nonzero. Lemma 2.1 is proved. □

Let all coefficients $\alpha_k,\ \ k = 1, ..., m_0$ have the same sign and all coefficients $\beta_l,\ \ l = 1, ..., m_1$ have the same sign, (the signs of α_k and β_l can be different). Then, by analogy with (2.4), the classical solution of NLBVP (2.1),(2.2) satisfies the condition

$$u(0) - \alpha u(\zeta) = \varphi_0, \quad u(1) - \beta u(\eta) = \varphi_1,$$

for some $\zeta \in [\zeta_1, \zeta_{m_0}],\ \eta \in [\eta_1, \eta_{m_1}]$, so that $\zeta < \eta$ in view of (3).

Corollary 2.2. *Let $S_i \le 1,\ \varphi_i \ne 0,\ i = 0, 1$. If $u(x)$ is some classical solution of the problem*

$$Lu(x) = -f(x),\ 0 < x < 1,\ \ u(0) - \alpha u(\zeta) = \varphi_0,\ u(1) - \beta u(\eta) = \varphi_1$$

for $\zeta < \eta$, then $v(x) = u(x) + \varphi_0 q_0(x) + \varphi_1 q_1(x)$ is classical solution of the problem

$$Lv(x) = -f_1(x),\ 0 < x < 1,\ \ v(0) - \alpha v(\zeta) = 0,\ v(1) - \beta v(\eta) = 0$$

for $f_1(x) = f(x) - \varphi_0 Lq_0(x) - \varphi_1 Lq_1(x)$, $q_0(x) = c_0(1-x)(\eta - x)$, $q_1(x) = c_1 x(\zeta - x)$, $c_0 = -[\eta - \alpha(1-\zeta)(\eta - \zeta)]^{-1}$, $c_1 = -[(1-\zeta) - \beta\eta(\eta - \zeta)]^{-1}$.

Proof. This is provable by analogy with Lemma 2.1. Corollary 2.2 is proved. □

Corollary 2.3. *The statement of Corollary 2.2 is true for the case if all coefficients $\alpha_k,\ \ k = 1, ..., m_0$ have the same sign, but there are different sign coefficients among $\beta_l,\ \ l = 1, ..., m_1$ (or vice versa).*

Proof. This is provable by analogy with Lemma 2.1. Corollary 2.3 is proved. □

3. A uniform stability estimate

Here, we establish a uniformly stable estimate. Our basic result is

Theorem 3.1. *Let $a(x) \ge a_0 > 0$, $b(x) \ge b_0 \ge 0$, $x \in [0, 1]$. Let conditions (2.3), **A** hold. If $S_0 \le 1$, $S_1 \le 1$ and, in addition, $S_1 < 1$ if $b_0 = 0$, then a uniformly stable estimate*

$$|u(x)| \le C(|\varphi_0| + |\varphi_1| + \max_{0 \le y \le 1} |f(y)|), \quad 0 \le x \le 1 \tag{3.1}$$

holds for classical solution of NLBVP (2.1),(2.2).

Proof. Let $u(x)$ be some classical solution of NLBVP (2.1),(2.2). Since (2.2) is reduciable to (2.4), then $u(x)$ is classical solution of NLBVP (2.1),(2.4). In view of Lemma 2.1, the function $v(x) = u(x) + \varphi_0 q_0(x) + \varphi_1 q_1(x)$ is classical solution of NLBVP (2.9). Assume that a uniformly stable estimate holds for $v(x)$, i.e.,

$$|v(x)| \le C_1 \max_{0 \le y \le 1} |f_1(y)|, \quad 0 \le x \le 1 \tag{3.2}$$

for some independent of ε constant C_1, where $f_1(x) = f(x) - \varphi_0 Lq_0(x) - \varphi_1 Lq_1(x)$. Then, by virtue of the triangle inequality,

$$|u(x)| \le C_1 \max_{0 \le y \le 1} |f_1(y)| + |\varphi_0| \max_{0 \le y \le 1} |q_0(y)| + |\varphi_1| \max_{0 \le y \le 1} |q_1(y)|,\ \ 0 \le x \le 1,$$

so that

$$|u(x)| \le C_1 \max_{0 \le y \le 1} |f(y)| + C_2|\varphi_0| + C_3|\varphi_1|,\ \ 0 \le x \le 1.$$

Thus, if (3.2) is true, then (3.1) is also true for the constant $C = \max\{C_1, C_2, C_3\}$. So, to prove (3.1) it will be sufficient to obtain the estimate (3.2) for the solution of NLBVP (2.9). Further, to establish (3.2) we will consider three subcases:

first - all coefficients $\alpha_k,\ k = 1, ..., m_0$ have the same sign and all coefficients β_l, $l = 1, ..., m_1$ have the same sign (the signs of α_k and β_l can be different);
second - each one of two nonlocal conditions (2.2) encloses different sign coefficients;
third - one condition of (2.2) has the same sign coefficients, but another one encloses different sign coefficients.
Subcase 1. Put $\alpha_k,\ k = 1, ..., m_0$ have the same sign, β_l, $l = 1, ..., m_1$ have the same sign (the signs of α_k and β_l can be different). Our task is to prove the estimate (3.2) for classical solution of NLBVP (2.9). By virtue of MV property [3, p. 1198-1199] in respect of NLBVC (2.9), we get

$$v(0) = \alpha v(\zeta), \quad v(1) = \beta v(\eta) \tag{3.3}$$

for some $\zeta \in [\zeta_1, \zeta_{m_0}]$, $\eta \in [\eta_1, \eta_{m_1}]$. If $\alpha < 0$, then, in view of Bolzano theorem, $v(x_0) = 0$ at some point $x_0 \in (0, \zeta)$, i.e., $v(x)$ satisfies boundary value condition (BVC) of the first kind at x_0. If $\alpha > 0$, then $\omega_0(0) = \omega_0(\zeta)$ for the function

$$\omega_0(x) = v(x)\frac{(\alpha - 1)x + \zeta}{\zeta}. \tag{3.4}$$

By virtue of Rolle's theorem, $\omega_0'(x_0) = 0$ at some point $x_0 \in (0, \zeta)$. Hence,

$$v'(x_0) - h_0 v(x_0) = 0, \qquad h_0 = \frac{1 - \alpha}{\zeta - x_0(1 - \alpha)}, \tag{3.5}$$

so that $h_0 \geq 0$ since our theorem condition requires the bound $S_0 \leq 1$. It means that $v(x)$ satisfies BVC of the third kind at x_0 if $0 < \alpha < 1$, or of the second kind if $\alpha = 1$.

Similarly, for $\beta < 0$ we obtain BVC of the first kind $v(x_1) = 0$ at some point $x_1 \in (\eta, 1)$, as well as, for $\beta > 0$ we get BVC of the third kind if $0 < \beta < 1$, or of the second kind if $\beta = 1$ at some point $x_1 \in (\eta, 1)$, i.e.,

$$v'(x_1) + h_1 v(x_1) = 0, \qquad h_1 = \frac{1 - \beta}{\beta(1 - x_1) + x_1 - \eta}, \tag{3.6}$$

so that $h_1 \geq 0$ since the theorem condition requires the bound $S_1 \leq 1$. Note that to get (3.6) we use the function

$$\omega_1(x) = v(x)\frac{\beta(x - 1) + \eta - x}{\eta - 1} \tag{3.7}$$

and corresponding equalities $\omega_1(1) = \omega_1(\eta)$, $\omega_1'(x_1) = 0$.

In summary, we revealed that on some interval $[x_0, x_1]$ the function $v(x)$ satisfies the boundary value problem (BVP)

$$Lv(x) = -f_1(x),\ x_0 < x < x_1, \quad \delta_0 v'(x_0) - h_0 v(x_0) = 0,\ \ \delta_1 v'(x_1) + h_1 v(x_1) = 0,$$

where $\delta_0 = 1$ if $\alpha > 0$ and $\delta_1 = 1$ if $\beta > 0$, in addition, $\delta_0 = 0$, $h_0 = 1$ if $\alpha < 0$, and $\delta_1 = 0$, $h_1 = 1$ if $\beta < 0$. Hence, in view of the variable replacement

$$t = (x_1 - x_0)^{-1}(x - x_0), \tag{3.8}$$

we get that the function $\tilde{v}(t) = v(x(t))$ satisfies the BVP

$$\begin{cases} \tilde{L}\tilde{v}(t) \equiv \varepsilon\tilde{v}''(t) + \tilde{a}(t)\tilde{v}'(t) - \tilde{b}(t)\tilde{v}(t) = -\tilde{f}_1(t), \quad 0 < t < 1, \\ \tilde{h}_0\tilde{v}(0) - \delta_0\tilde{v}'(0) = 0, \quad \tilde{h}_1\tilde{v}(1) + \delta_1\tilde{v}'(1) = 0, \end{cases} \tag{3.9}$$

where

$$\tilde{a}(t) = (x_1 - x_0)a(x(t)),\ \ \tilde{b}(t) = (x_1 - x_0)^2 b(x(t)),\ \ \tilde{f}_1(t) = (x_1 - x_0)^2 f_1(x(t)),$$

$$x(t) = (x_1 - x_0)t + x_0,\ \ \tilde{a}(t) \geq (\eta_1 - \zeta_{m_0})a_0,\ \ \tilde{b}(t) \geq (\eta_1 - \zeta_{m_0})^2 b_0,\ \ 0 \leq t \leq 1,$$

in addition, $\tilde{h}_0$, δ_0, $\tilde{h}_1$, δ_1 are defined by the specification

$$\begin{aligned} &\delta_0 = 0,\ \tilde{h}_0 = 1 \quad \text{for } \alpha < 0,\\ &\delta_0 = 1,\ \tilde{h}_0 = (x_1 - x_0)h_0 \quad \text{for } 0 < \alpha \leq 1,\\ &\delta_1 = 0,\ \tilde{h}_1 = 1 \quad \text{for } \beta < 0,\\ &\delta_1 = 1,\ \tilde{h}_1 = (x_1 - x_0)h_1 \quad \text{for } 0 < \beta \leq 1, \end{aligned}$$

herewith, $\tilde{h}_1 + (\eta_1 - \zeta_{m_0})^2 b_0 > 0$ since the theorem condition requires $S_1 < 1$ for $b_0 = 0$ (it means that $0 < \beta < 1$ for $b_0 = 0$) and, therefore, we have $\tilde{h}_1 > 0$. Further, for classical solution of BVP (3.9), by virtue of [12, p. 100-103], we get a uniform on ε stability estimate

$$|\tilde{v}(t)| \leq C_4 \max_{0 \leq y \leq 1} |\tilde{L}\tilde{v}(y)|,\ \ 0 \leq t \leq 1, \tag{3.10}$$

therefore, in view of the variable replacement,

$$|v(x)| \leq C_4 \max_{x_0 \leq y \leq x_1} |f_1(y)|,\quad x_0 \leq x \leq x_1, \tag{3.11}$$

where C_4 is an ε-independent constant. Since $\zeta \in (x_0, x_1)$ and $\eta \in (x_0, x_1)$, then

$$|v(\zeta)| \leq C_4 \max_{x_0 \leq x \leq x_1} |f_1(x)|,\quad |v(\eta)| \leq C_4 \max_{x_0 \leq x \leq x_1} |f_1(x)|.$$

Hence, in view of NLBVC (2.9),

$$|v(0)| \leq C_5 \max_{0 \leq x \leq 1} |f_1(x)|,\quad |v(1)| \leq C_5 \max_{0 \leq x \leq 1} |f_1(x)|, \tag{3.12}$$

where $C_5 = C_4 \max\{|\alpha|, |\beta|\}$. Now, in view of (3.12), we interpret the solution of NLBVP (2.9) as classical solution of Dirichlet problem

$$Lv(x) = -f_1(x),\ 0 < x < 1,\quad v(0) = \gamma_0,\ \ v(1) = \gamma_1, \tag{3.13}$$

where

$$|\gamma_i| \leq C_5 \max_{0 \leq x \leq 1} |f_1(x)|,\ \ i = 0, 1. \tag{3.14}$$

Then, by virtue of [12, p. 100-103], we obtain a unifromly stable estimate

$$|v(x)| \leq C_6 \big(|\gamma_0| + |\gamma_1| + \max_{0 \leq y \leq 1} |Lv(y)|\big),\ \ 0 \leq x \leq 1, \tag{3.15}$$

where C_6 is some ε-independent constant. In view of (3.14), the estimate (3.2) is true. Therefore, a uniform on ε stability estimate (3.1) is proved.

Subcase 2. Put that each one of two conditions (2.2) encloses different sign coefficients. Then $u(x)$ satisfies some reduced condition (2.4). We will prove the estimate (3.2). Further, we admit that $v(0) \neq 0$ and $v(1) \neq 0$, since for the case if $v(0) = 0$ or $v(1) = 0$ the estimate (3.2) is provable by the same approach which we use here.

Firstly, assume that $\zeta^- < \zeta^+$ in respect of (2.4).

a) If $\operatorname{sign}[v(0)v(\zeta^-)] \neq 1$ or $\operatorname{sign}[v(0)v(\zeta^+)] \neq 1$, then there is some point x_0, $x_0 \in (0, \zeta^-)$ or $x_0 \in (0, \zeta^+)$ correspondingly, so that $v(x_0) = 0$.

b) If $\operatorname{sign}[v(0)v(\zeta^-)] = 1$ and $\operatorname{sign}[v(0)v(\zeta^+)] = 1$, then, by virtue of MV propety[3, p. 1198-1199] in respect of the first nonlocal condition (2.4), we have

$$(1 + |\alpha^-|)v(\zeta_0) = \alpha^+ v(\zeta^+)$$

for some $\zeta_0 \in [0, \zeta^-]$, herewith $\zeta_0 < \zeta^+$. Then, in view of the condition $S_0 \leq 1$,

$$v(\zeta_0) = \alpha_0 v(\zeta^+),\quad \alpha_0 = \frac{\alpha^+}{1 + |\alpha^-|},\quad 0 < \alpha_0 \leq 1. \tag{3.16}$$

Hence, by using the function

$$\hat{w}_0(x) = v(x)\frac{(\alpha_0 - 1)x + \zeta^+ - \alpha_0\zeta_0}{\zeta^+ - \zeta_0},$$

we get $\hat{w}_0(\zeta_0) = \hat{w}_0(\zeta^+)$, so that, by virtue of Rolle's theorem, $\hat{w}'(x_0) = 0$ at some point $x_0 \in (\zeta_0, \zeta^+)$, and, therefore,

$$v'(x_0) - h_0 v(x_0) = 0, \qquad h_0 = \frac{1-\alpha_0}{(\alpha_0 - 1)x_0 + \zeta^+ - \alpha_0\zeta_0}, \quad h_0 \geq 0. \tag{3.17}$$

Now, assume that $\zeta^+ < \zeta^-$ in respect of (2.4).
a) If $\text{sign}[v(0)v(\zeta^+)] \neq 1$ or $\text{sign}[v(0)v(\zeta^-)] \neq 1$, then there is some point x_0, $x_0 \in (0, \zeta^+)$ or $x_0 \in (0, \zeta^-)$ correspondingly, so that $v(x_0) = 0$.
b) If $\text{sign}[v(0)v(\zeta^+)] = 1$ and $\text{sign}[v(0)v(\zeta^-)] = 1$, then there is some value $\tilde{\alpha}_0$, $0 < \tilde{\alpha}_0 < \alpha^+$, so that, in view of the condition $S_0 \leq 1$,

$$v(0) = \tilde{\alpha}_0 v(\zeta^+), \qquad 0 < \tilde{\alpha}_0 < 1. \tag{3.18}$$

Hence, by using

$$\tilde{\omega}_0(x) = v(x)\frac{(\tilde{\alpha}_0 - 1)x + \zeta^+}{\zeta^+},$$

we get $\tilde{\omega}_0(0) = \tilde{\omega}_0(\zeta^+)$, then $\tilde{\omega}'(x_0) = 0$ at some point x_0, $x_0 \in (0, \zeta^+)$, and, therefore,

$$v'(x_0) - h_0 v(x_0) = 0, \qquad h_0 = \frac{1-\tilde{\alpha}_0}{\zeta^+ - x_0(1-\tilde{\alpha}_0)}, \quad h_0 > 0. \tag{3.19}$$

In summary, we revealed that at some point x_0 the solution of (2.9) satisfies one of the left-side BVC:

$\ell_{x_0,1}(v) \equiv v(x_0) = 0,\ \ 0 < x_0 < \zeta^- < \zeta^+,$
$\ell_{x_0,2}(v) \equiv v(x_0) = 0,\ \ 0 < x_0 < \zeta^+,\ \ 0 < \zeta^- < \zeta^+,$
$\ell_{x_0,3}(v) \equiv v'(x_0) - h_0 v(x_0) = 0,\ \ h_0 \geq 0,\ \ 0 < x_0 < \zeta^+,\ \ 0 < \zeta^- < \zeta^+,$
$\ell_{x_0,4}(v) \equiv v(x_0) = 0,\ \ 0 < x_0 < \zeta^+ < \zeta^-,$
$\ell_{x_0,5}(v) \equiv v(x_0) = 0,\ \ 0 < x_0 < \zeta^-,\ \ 0 < \zeta^+ < \zeta^-,$
$\ell_{x_0,6}(v) \equiv v'(x_0) - h_0 v(x_0) = 0,\ \ h_0 > 0,\ \ 0 < x_0 < \zeta^+ < \zeta^-.$

By similar way, we reveal that at some point x_1 the solution of (2.9) satisfies one of the right-side BVC:

$\ell_{x_1,1}(v) \equiv v(x_1) = 0,\ \ \eta^+ < \eta^- < x_1 < 1,$
$\ell_{x_1,2}(v) \equiv v(x_1) = 0,\ \ \eta^+ < x_1 < 1,\ \ \eta^+ < \eta^- < 1,$
$\ell_{x_1,3}(v) \equiv v'(x_1) + h_1 v(x_1) = 0,\ \ h_1 \geq 0,\ \ \eta^+ < x_1 < 1,\ \ \eta^+ < \eta^- < 1,$
$\ell_{x_1,4}(v) \equiv v(x_1) = 0,\ \ \eta^- < \eta^+ < x_1 < 1,$
$\ell_{x_1,5}(v) \equiv v(x_1) = 0,\ \ \eta^- < x_1 < 1,\ \ \eta^- < \eta^+ < 1,$
$\ell_{x_1,6}(v) \equiv v'(x_1) + h_1 v(x_1) = 0,\ \ h_1 > 0,\ \ \eta^- < \eta^+ < x_1 < 1,$

where similarly (3.16)-(3.17), by using

$$\hat{w}_1(x) = v(x)\frac{(\beta_0 - 1)x + \eta^+ - \beta_0\eta_0}{\eta^+ - \eta_0},$$

we define

$$h_1 = \frac{1-\beta_0}{\beta_0(\eta_0 - x_1) + x_1 - \eta^+}, \quad \beta_0 = \frac{\beta^+}{1+|\beta^-|},$$

for $\ell_{x_1,3}(v)=0$, as well as, similarly (3.18)-(3.19), by using

$$\tilde{\omega}_1(x)=v(x)\frac{(\tilde{\beta}_0-1)x+\eta^+-\tilde{\beta}_0}{\eta^+-1}$$

for the case if $\operatorname{sign}[v(1)v(\eta^+)]=\operatorname{sign}[v(1)v(\eta^-)]=1$, we define

$$h_1=\frac{1-\tilde{\beta}_0}{\tilde{\beta}_0(1-x_1)+x_1-\eta^+}$$

for $\ell_{x_1,6}(v)=0$. Here, $\tilde{\beta}_0$ is an appropriate value, so that $v(1)=\tilde{\beta}_0 v(\eta^+)$, herewith $0<\tilde{\beta}_0<\beta^+$, so $0<\tilde{\beta}_0<1$ since $S_1\le 1$ in view of theorem condition.

Further, let $v(x)$ satisfies some pair of BVC $\ell_{x_0,i}(v)=0$, $\ell_{x_1,j}(v)=0$, $i=1,...,6$, $j=1,...,6$.

2.1. Assume, that $\zeta^-,\zeta^+,\eta^-,\eta^+\in(x_0,x_1)$. Note, it is always fulfiled for any pair $\ell_{x_0,i}(v)=0$, $\ell_{x_1,j}(v)=0$, $i=1,4,6$, $j=1,4,6$. Similarly to the Subcase 1, by virtue of (3.8), we obtain the BVP (3.9)

$$\tilde{L}\tilde{v}(t)=-\tilde{f}_1(t),\ \ 0<t<1,\quad \tilde{h}_0\tilde{v}(0)-\delta_0\tilde{v}'(0)=0,\quad \tilde{h}_1\tilde{v}(1)+\delta_1\tilde{v}'(1)=0,$$

where

$$\begin{aligned}
&\delta_0=0,\ \tilde{h}_0=1 \quad \text{for conditions } \ell_{x_0,i}(v)=0,\ i=1,2,4,5,\\
&\delta_0=1,\ \tilde{h}_0=(x_1-x_0)h_0 \quad \text{for conditions } \ell_{x_0,i}(v)=0,\ i=3,6,\\
&\delta_1=0,\ \tilde{h}_1=1 \quad \text{for conditions } \ell_{x_1,j}(v)=0,\ j=1,2,4,5,\\
&\delta_1=1,\ \tilde{h}_1=(x_1-x_0)h_1 \quad \text{for conditions } \ell_{x_1,j}(v)=0,\ j=3,6,
\end{aligned}$$

herewith, $\tilde{h}_1+(\eta_1-\zeta_{m_0})^2b_0>0$ for $\ell_{x_1,j}(v)=0$, $j=1,2,4,5,6$ since $\tilde{h}_1>0$, in addition, $\tilde{h}_1+(\eta_1-\zeta_{m_0})^2b_0>0$ for $\ell_{x_1,3}(v)=0$ in view of the theorem requirement $S_1<1$. Then the estimate (3.10) holds for $\tilde{v}(x)$. Since (3.10) results in (3.11), then, in view of $\zeta^-,\zeta^+,\eta^-,\eta^+\in(x_0,x_1)$, we get

$$|v(\zeta^-)|\le C_4\max_{x_0\le x\le x_1}|f_1(x)|,\qquad |v(\zeta^+)|\le C_4\max_{x_0\le x\le x_1}|f_1(x)|, \tag{3.20}$$

$$|v(\eta^-)|\le C_4\max_{x_0\le x\le x_1}|f_1(x)|,\qquad |v(\eta^+)|\le C_4\max_{x_0\le x\le x_1}|f_1(x)|. \tag{3.21}$$

In view of (3.20) and (3.21), the estimate (3.12) follows from NLBVC (2.9) for $C_5=C_4\max\{|\alpha^-|+|\alpha^+|,\ |\beta^-|+|\beta^+|\}$. Therefore, we can interpret the solution of NLBVP (2.9) as classical solution of the Dirichlet's problem (3.13) and, by virtue of [12, p. 100-103], state that a unifromly stable estimate (3.15) holds. Hence, in view of (3.14), we get the estimate (3.11). At least, the validity of (3.11) is sufficient to confirm that (3.1) is true.

2.2. Assume, that only one of two points ζ^- or ζ^+ belongs to (x_0,x_1), as well as, only one of two points η^- or η^+ belongs to (x_0,x_1) (it is available in respect of any pair of the conditions $\ell_{x_0,i}(v)=0$, $\ell_{x_1,j}(v)=0$, $i=2,3,5$, $j=2,3,5$). Then, since (3.10)-(3.11) holds, then one of two estimates (3.20) holds, as well as, one of two estimates (3.21) holds too. Thus, the NLBVP (2.9) is reduciable to the problem

$$Lv(x)=-f_1(x),\ 0<x<1,\ v(0)=\alpha^*v(\zeta^*)+\varphi_0^*,\ v(1)=\beta^*v(\eta^*)+\varphi_1^*, \tag{3.22}$$

where

$$|\varphi_0^*|\le C_5\max_{0\le x\le 1}|f_1(x)|,\quad |\varphi_1^*|\le C_5\max_{0\le x\le 1}|f_1(x)|, \tag{3.23}$$

$C_5 = C_4 \max\{|\alpha^-| + |\alpha^+|,\ |\beta^-| + |\beta^+|\}$, the pair α^* and ζ^* is performed by α^- and ζ^-, or by α^+ and ζ^+ correspondingly, the pair β^* and η^* is performed by β^+ and η^+, or by β^- and η^-. So, in view of the theorem condition, $-\infty < \alpha^* \le 1$, $-\infty < \beta^* \le 1$, $\beta^* < 1$ if $b_0 = 0$. Now, firstly by virtue of Lemma 2.1 in respect of the problem (3.22), and then, by reasoning similar Section 1, we obtain the analogy of (3.1) for the problem (3.22), i.e.,

$$|v(x)| \le C(|\varphi_0^*| + |\varphi_1^*| + \max_{0\le y\le 1} |f_1(y)|), \quad 0 \le x \le 1. \tag{3.24}$$

Then, in view of (3.23), the estimate (3.24) results in (3.2), and, therefore, the estimate (3.1) is true.

2.3. Assume, that three of four points ζ^-, ζ^+, η^-, η^+ belong to (x_0, x_1). Then, by combined reasoning of 2.1-2.2, one can prove that the estimate (3.1) is true.

Subcase 3. Assume, that one of two conditions (2.2) encloses the coefficients of the same sign, but another one encloses different sign coefficients. Then (3.1) can be proved by virtue of combined approach of Subcases 1-2. Theorem 3.1 is proved. □

4. The existence and uniqueness

Firstly, we prove

Lemma 4.1. *Let (2.3) and the condition* $\boldsymbol{A}$ *are fulfiled,* $S_i \le 1$, $\varphi_i \ne 0$, $i = 0, 1$. *If* $u(x)$ *is some classical solution of NLBVP (2.1),(2.2), then*

$$v(x) = u(x) + \varphi_0 q_0(x) + \varphi_1 q_1(x)$$

is classical solution of the problem

$$Lv(x) = -f_1(x),\ 0 < x < 1, \quad \ell_0(v) = 0,\ \ \ell_1(v) = 0 \tag{4.1}$$

for $f_1(x) = f(x) - \varphi_0 Lq_0(x) - \varphi_1 Lq_1(x)$, *where* $q_0(x) = c_0(1-x)\prod_{l=1}^{m_1}(\eta_l - x)$, $q_1(x) = c_1 x \prod_{k=1}^{m_0}(x - \zeta_k)$, *herewith an appropriate constant* $c_i \in \boldsymbol{R}$, $c_i \ne 0$, $i = 0, 1$.

If $v(x)$ *is some classical solution of (4.1), then* $u(x) = v(x) - \varphi_0 q_0(x) - \varphi_1 q_1(x)$ *is classical solution of NLBVP (2.1),(2.2) for* $f(x) = f_1(x) + \varphi_0 Lq_0(x) + \varphi_1 Lq_1(x)$.

Let only one of φ_i, $i \in \{0, 1\}$ *be nonzero, put* $\varphi_{i_*} \ne 0$, $i_* \in \{0, 1\}$. *If* $u(x)$ *is some classical solution of NLBVP (2.1),(2.2), then* $v(x) = u(x) + \varphi_{i_*} q_{i_*}(x)$ *is classical solution of the problem (4.1) for the function* $f_1(x) = f(x) - \varphi_{i_*} Lq_{i_*}(x)$. *Vice versa, if* $v(x)$ *is some classical solution of (4.1), then* $u(x) = v(x) - \varphi_{i_*} q_*(x)$ *is classical solution of NLBVP (2.1),(2.2) for* $f(x) = f_1(x) + \varphi_{i_*} Lq_{i_*}(x)$.

Proof. Put $\varphi_i \ne 0$, $i = 0, 1$. Obviously that $Lv(x) = -f_1(x)$, $0 < x < 1$ for $q_i(x)$ and any nonzero c_i, $i = 0, 1$. Let us find c_i, $i = 0, 1$, so that the $v(x)$ will satisfy NLBVC (4.1). Note,

$$\ell_0(q_1) = 0,\ \ell_1(q_0) = 0. \tag{4.2}$$

Since $v(x)$ has to satisfy the NLBVC (4.1), then the expressions

$$l_0(v) = \varphi_0[1 + \ell_0(q_0)] = 0, \quad l_1(v) = \varphi_1[1 + \ell_1(q_1)] = 0 \tag{4.3}$$

have to be true, therefore,

$$1 + \ell_0(q_0) = 0, \quad 1 + \ell_1(q_1) = 0 \tag{4.4}$$

have to be true too. Hence,

$$c_0 = -(E_0)^{-1}, \quad c_1 = -(D_0)^{-1}, \tag{4.5}$$

where

$$E_0 = \prod_{l=1}^{m_1} \eta_l - \sum_{k=1}^{m_0} \alpha_k (1-\zeta_k) \prod_{l=1}^{m_1} (\eta_l - \zeta_k), \tag{4.6}$$

$$D_0 = \prod_{k=1}^{m_0} (1-\zeta_k) - \sum_{l=1}^{m_1} \beta_l \eta_l \prod_{k=1}^{m_0} (\eta_l - \zeta_k), \tag{4.7}$$

herewith $E_0 \neq 0$, $D_0 \neq 0$, moreover, $E_0 > 0$, $D_0 > 0$. Actually, by virtue of the condition $S_0 \leq 1$ in respect of (4.6), we have

$$E_0 > \begin{cases} \prod\limits_{l=1}^{m_1} \eta_l > 0, \text{ if } \alpha_k < 0, \ k = 1, ..., m_0, \\ (1-\alpha) \prod\limits_{l=1}^{m_1} \eta_l \geq 0, \text{ if } \alpha_k > 0, \ k = 1, ..., m_0, \\ (1-S_0) \prod\limits_{l=1}^{m_1} \eta_l \geq 0, \text{ if all } \alpha_k, \ k = 1, ..., m_0 \text{ have not the same sign,} \end{cases}$$

where, in view of (2.6), $S_0 = \alpha^+$ for $\zeta^+ < \zeta^-$, $S_0 = \alpha^- + \alpha^+$ for $\zeta^- < \zeta^+$. Indeed, it is clear for the case if all coefficients α_k, $k = 1, ..., m_0$ have the same sign, as well as, for the case if $\zeta^+ < \zeta^-$. Let us confirm that $E_0 > 0$ for the case if all α_k, $k = 1, ..., m_0$ have not the same sign and $\zeta^- < \zeta^+$. In view of (4.6),

$$E_0 > \prod_{l=1}^{m_1} \eta_l - (1-\zeta_{m_\iota})\alpha^- \prod_{l=1}^{m_1} (\eta_l - \zeta_{m_\iota}) - (1-\zeta_{m_\iota+1})\alpha^+ \prod_{l=1}^{m_1} (\eta_l - \zeta_{m_\iota+1}),$$

then

$$E_0 > \prod_{l=1}^{m_1} \eta_l - (1-\zeta_{m_\iota})(\alpha^- + \alpha^+) \prod_{l=1}^{m_1} (\eta_l - \zeta_{m_\iota}).$$

Hence, $E_0 > \prod\limits_{l=1}^{m_1} \eta_l > 0$ for $-\infty < S_0 \leq 0$, $E_0 > [1 - (\alpha^- + \alpha^+)] \prod\limits_{l=1}^{m_1} \eta_l \geq 0$ for $0 < S_0 \leq 1$, where $S_0 = \alpha^- + \alpha^+$. Thus, we proved finally that $E_0 > 0$, then $E_0 \neq 0$, therefore, the constant c_0 is definable by the first formula (4.5). Similarly, by virtue of the condition $S_1 \leq 1$ for (4.7), it is easy to confirm that $D_0 > 0$ and prove that the constant c_1 is definable by the second formula of (4.5).

Let us prove second statement of lemma. Obviously,

$$Lu(x) = Lv(x) - \sum_{i=1}^{2} \varphi_i Lq_i(x) = -f_1(x) - \sum_{i=1}^{2} \varphi_i Lq_i(x) = -f(x), \ \ 0 < x < 1,$$

in addition, since $v(x)$ satisfies NLBVC (4.1), then, in view of (4.2)-(4.5), we get

$$\ell_0(u) = \ell_0(v) - \varphi_0 \ell_0(q_0) - \varphi_1 \ell_0(q_1) = -\varphi_0 \ell_0(q_0) = \varphi_0,$$

$$\ell_1(u) = \ell_1(v) - \varphi_0 \ell_1(q_0) - \varphi_1 \ell_1(q_1) = -\varphi_1 \ell_1(q_1) = \varphi_1.$$

To finish this proof, by the same way as in above it is easy to confirm, that the third statement of lemma is true. Lemma 4.1 is proved. □

Theorem 4.2. *Let $a(x) \geq a_0 > 0$, $b(x) \geq b_0 \geq 0$ for $x \in [0,1]$. Let (2.3) and the condition **A** are fulfilled. If $S_i \leq 1$, $i = 0, 1$ and, in addition, $S_1 < 1$ if $b_0 = 0$, then classical solution of NLBVP (2.1),(2.2) exists and is a unique.*

Proof. Because all conditions of Theorem 3.1 are fulfiled, then the uniqueness of classical solution follows from stabilty estimate (3.1).

In view of Lemma 4.1, to prove the existence, it is sufficient to establish that classical solution of the differential problem (4.1) exists. The problem (4.1) is equivalent to the differential problem

$$[k(x)v']' - q(x)v = -\tilde{f}_1(x), \;\; 0 < x < 1, \;\;\; \ell_0(v) = 0, \;\; \ell_1(v) = 0, \tag{4.8}$$

where

$$k(x) = \exp\Big(\frac{1}{\varepsilon}\int\limits_0^x a(t)dt\Big), \;\; q(x) = b(x)k(x), \;\;\; \tilde{f}_1(x) = f_1(x)k(x),$$

therefore, it will be sufficient to prove that classical solution of (4.8) exists. To prove it let us use the fact that for any continuous function $F(x), \; x \in [0,1]$ the differential problem

$$[k(x)v']' - v/k(x) = F(x), \;\; 0 < x < 1, \;\;\; \ell_0 v = 0, \;\; \ell_1 v = 0 \tag{4.9}$$

has the solution

$$v(x) = A\sinh(P(x)) + B\cosh(P(x)) + \int\limits_0^x \sinh\big(P(x) - P(t)\big)F(t)dt, \tag{4.10}$$

where

$$P(x) = \int\limits_0^x \big(k(\tau)\big)^{-1} d\tau, \tag{4.11}$$

$$A = -\Big[\ell_1(s_P)\Big]^{-1}\Big\{B\ell_1(c_P) + \ell_1\Big(\int\limits_0^x \sinh\big(P(x) - P(t)\big)F(t)dt\Big)\Big\},$$

$$B = -\Big[\ell_0(c_P) - \ell_0(s_P)\frac{\ell_1(c_P)}{\ell_1(s_P)}\Big]^{-1}\Big\{\ell_0\Big(\int\limits_0^x \sinh\big(P(x) - P(t)\big)F(t)dt\Big)$$

$$-\frac{\ell_0(s_P)}{\ell_1(s_P)}\ell_1\Big(\int\limits_0^x \sinh\big(P(x) - P(t)\big)F(t)dt\Big)\Big\},$$

herewith for the convenience we use next designation

$$s_P(x) = \sinh(P(x)), \;\; c_P(x) = \cosh(P(x)). \tag{4.12}$$

Actually, written in the square brackets expression is nonzero for A and B. Indeed, since the function $P(x)$ is nonnegative and strictly increasing in $[0,1]$, then

$$\ell_1(s_P) > \begin{cases} s_P(1) > 0, \;\; \text{if } \beta_l < 0, \;\; l = 1, ..., m_1, \\ s_P(1) - \beta s_P(\eta_{m_1}) > 0, \;\; \text{if } \beta_l > 0, \;\; l = 1, ..., m_1. \end{cases} \tag{4.13}$$

If all coefficients $\beta_l, \; l = 1, ..., m_1$ have not the same sign, then, in view of the MV property [3, p. 1198-1199],

$$\ell_1(s_P) = s_P(1) - \beta^+ s_P(\tilde{\eta}^+) - \beta^- s_P(\tilde{\eta}^-), \tag{4.14}$$

where $\tilde{\eta}^+,\ \tilde{\eta}^- \in [\eta_1, \eta_{m_1}]$.[3] In view of (4.14), if $\tilde{\eta}^- < \tilde{\eta}^+$, then

$$\ell_1(s_P) > s_P(1) - \beta^+ s_P(\eta_{m_1}) > 0.$$

If $\tilde{\eta}^+ < \tilde{\eta}^-$, then $\beta^+ + \beta^- = S_1$, then, in view of (4.14), we get

$$\ell_1(s_P) > s_P(1) - S_1 s_P(\eta_{m_\kappa}) > 0.$$

In summary, we proved that

$$\ell_1(s_P) > 0. \tag{4.15}$$

Now, let us prove that the expression inside square brackets for B is nonzero too. Let us denote this expression by G, i.e.,

$$G = \ell_0(c_P) - \ell_0(s_P) \times \frac{\ell_1(c_P)}{\ell_1(s_P)}.$$

Additionally, we denote

$$\varPhi(x) = c_P(x) - s_P(x) \times \frac{\ell_1(c_P)}{\ell_1(s_P)}.$$

Since

$$G = 1 - \sum_{k=1}^{m_0} \alpha_k c_P(\zeta_k) + \sum_{k=1}^{m_0} \alpha_k s_P(\zeta_k) \times \frac{\ell_1(c_P)}{\ell_1(s_P)},$$

then, by virtue of MV property, there are some points[4] $\tilde{\zeta},\ \tilde{\zeta}^+,\ \tilde{\zeta}^- \in [\zeta_1, \zeta_{m_0}]$, so that

$$G = \begin{cases} 1 - \alpha\varPhi(\tilde{\zeta}), \quad \text{if} \quad \alpha_k, \ \ k = 1, ..., m_0 \ \text{ have the same sign,} \\ 1 - \alpha^+\varPhi(\tilde{\zeta}^+) - \alpha^-\varPhi(\tilde{\zeta}^-), \ \ \text{if all} \ \ \alpha_k, k = 1, ..., m_0 \ \text{ have not} \\ \text{the same sign.} \end{cases} \tag{4.16}$$

Further, in respect of $\varPhi(x)$, $x \in [0, \zeta_{m_0}]$, we have

$$\varPhi(x) = c_P(x) - s_P(x) \times \frac{c_P(1) - \sum\limits_{l=1}^{m_1} \beta_l c_P(\eta_l)}{s_P(1) - \sum\limits_{l=1}^{m_1} \beta_l s_P(\eta_l)}$$

$$= \Big[\sinh\big(P(1) - P(x)\big) - \sum_{l=1}^{m_1} \beta_l \sinh\big(P(\eta_l) - P(x)\big)\Big] \times \frac{1}{\ell_1(s_P)}.$$

Hence, by virtue of MV property [3, p. 1198-1199], for $x \in [0, \zeta_{m_0}]$ we obtain that

$$\varPhi(x) = \frac{\sinh\big(P(1) - P(x)\big) - \beta \sinh\big(P(\hat{\eta}) - P(x)\big)}{\ell_1(s_P)}$$

for some $\hat{\eta} \in [\eta_1, \eta_{m_1}]$ if β_l, $l = 1, ..., m_1$ have the same sign. However, if all β_l, $l = 1, ..., m_1$ have not the same sign[5], then

$$\varPhi(x) = \frac{\sinh\big(P(1) - P(x)\big) - \beta^+ \sinh\big(P(\hat{\eta}^+) - P(x)\big) - \beta^- \sinh\big(P(\hat{\eta}^-) - P(x)\big)}{\ell_1(s_P)}$$

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[3] Note, $\tilde{\eta}^+ < \tilde{\eta}^-$ if $\eta^+ < \eta^-$, or, alternatively, $\tilde{\eta}^- < \tilde{\eta}^+$ if $\eta^- < \eta^+$, where η^+ and η^- are some designated points respectively NLBVC (2.4).

[4] Note, $\tilde{\zeta}^+ < \tilde{\zeta}^-$ if $\zeta^+ < \zeta^-$, or, alternatively, $\tilde{\zeta}^- < \tilde{\zeta}^+$ if $\zeta^- < \zeta^+$, where ζ^+ and ζ^- are some designated points respectively NLBVC (2.4).

[5] Note, $\hat{\eta}^+ < \hat{\eta}^-$ if $\eta^+ < \eta^-$, or, alternatively $\hat{\eta}^- < \hat{\eta}^+$ if $\eta^- < \eta^+$, where η^+ and η^- are defined in respect of NLBVC (2.4).

for some $\hat{\eta}^+ \in [\eta_1, \eta_{m_1}]$, $\hat{\eta}^- \in [\eta_1, \eta_{m_1}]$. Hence, in view of (4.15), for $x \in [0, \zeta_{m_0}]$ we have:

$$\Phi(x) > \frac{\sinh(P(1) - P(x))}{\ell_1(s_P)}, \text{ if } \beta < 0; \tag{4.17}$$

$$\Phi(x) > \frac{\sinh(P(1) - P(x)) - \beta \sinh(P(\hat{\eta}) - P(x))}{\ell_1(s_P)}, \text{ if } 0 < \beta \le 1; \tag{4.18}$$

$$\Phi(x) > \frac{\sinh(P(1) - P(x)) - S_1 \sinh(P(\hat{\eta}^-) - P(x))}{\ell_1(s_P)}, \text{ if } \hat{\eta}^+ < \hat{\eta}^-, \tag{4.19}$$

herewith $S_1 = \beta^+ + \beta^-$;

$$\Phi(x) > \frac{\sinh(P(1) - P(x)) - S_1 \sinh(P(\hat{\eta}^+) - P(x))}{\ell_1(s_P)}, \text{ if } \hat{\eta}^- < \hat{\eta}^+, \tag{4.20}$$

herewith $S_1 = \beta^+$. Since $S_1 \le 1$, then for $x \in [0, \zeta_{m_0}]$, in view of (4.17)-(4.20), we obtain the inequality

$$\Phi(x) > 0. \tag{4.21}$$

Further, for the case if β_l, $l = 1, ..., m_1$ have the same sign we get

$$\Phi'(x) = -\frac{\sinh\big(P(1) - P(x)\big) - \beta \sinh\big(P(\hat{\eta}) - P(x)\big)}{k(x)\ell_1(s_P)},$$

for the case if all β_l, $l = 1, ..., m_1$ are not of the same sign we get

$$\Phi'(x) = -\frac{\sinh\big(P(1) - P(x)\big) - \beta^+ \sinh\big(P(\hat{\eta}^+) - P(x)\big) - \beta^- \sinh\big(P(\hat{\eta}^-) - P(x)\big)}{k(x)\ell_1(s_P)}.$$

Hence,

$$\Phi'(x) = -\frac{\Phi(x)}{k(x)}.$$

Now, in view of (4.21), for $x \in [0, \zeta_{m_0}]$ we have

$$\Phi'(x) < 0. \tag{4.22}$$

Then $\Phi(x)$ is strictly decreasing positive function in $[0, \zeta_{m_0}]$, in addition, in view of (4.11), $\Phi(0) = 1$, therefore, $0 < \Phi(x) < 1$ for $x \in (0, \zeta_{m_0}]$. Hence, in view of (4.16)-(4.20), we get

$$G > \begin{cases} 1 - (\alpha^+ + \alpha^-)\Phi(\tilde{\zeta}^-), \text{ if } \tilde{\zeta}^- < \tilde{\zeta}^+, \\ 1 - \alpha^+\Phi(\tilde{\zeta}^+), \text{ if } \tilde{\zeta}^+ < \tilde{\zeta}^-, \\ 1 - \Phi(\zeta_1), \text{ if } \alpha_k, \ k = 1, ..., m_0 \text{ have the same sign.} \end{cases} \tag{4.23}$$

Since $S_0 \le 1$, then

$$G > 0. \tag{4.24}$$

Thus, in view of (4.15) and (4.24), the coefficients A and B are uniquely definable in respect of the formula (4.10), i.e., the function $v(x)$ is the solution of the differential problem (4.9). Further, by substituting

$$F(x) = \big[q(x) - 1/k(x)\big]v(x) - \tilde{f}_1(x) \tag{4.25}$$

into the equation (4.9), we obtain that the problem (4.9) is equivivalent to the Fredholm's integral equation of the second kind

$$v(x) = \int_0^1 K(x,t)v(t)dt + \hat{f}(x), \tag{4.26}$$

where

$$\begin{cases} K(x,t) = K_0(x,t) + \sum\limits_{k=1}^{m_0}[Z_k(x,t) + \tilde{Z}_k(x,t)] \\ \qquad + \sum\limits_{l=1}^{m_1}[H_l(x,t) + \tilde{H}_l(x,t)] + \sum\limits_{i=1}^{3} R_i(x,t), \end{cases} \tag{4.27}$$

herewith:

$$K_0(x,t) = \begin{cases} \sinh\big(P(x) - P(t)\big)\big[q(t) - 1/k(t)\big], & \text{if } (x,t) \in Q_x, \\ 0, & \text{if } (x,t) \in \overline{Q}_x \end{cases}$$

for $Q_x = \{0 \le x \le 1,\ 0 \le t \le x\}$, $\overline{Q}_x = \{0 \le x \le 1,\ \ x \le t \le 1\}$;

$$Z_k(x,t) = \begin{cases} G^{-1}\alpha_k c_P(x)\sinh(P(\zeta_k) - P(t))[q(t) - 1/k(t)], & \text{if } (x,t) \in Q_{\zeta_k}, \\ 0, & \text{if } (x,t) \in \overline{Q}_{\zeta_k} \end{cases}$$

and

$$\tilde{Z}_k(x,t) = -\frac{\ell_1(c_P)Z_k(x,t)}{\ell_1(s_P)}\tanh(P(x))$$

for $Q_{\zeta_k} = \{0 \le x \le 1,\ 0 \le t \le \zeta_k\}$, $\overline{Q}_{\zeta_k} = \{0 \le x \le 1,\ \zeta_k \le t \le 1\}$, $k = 1, ..., m_0$;

$$H_l(x,t) = \begin{cases} [\ell_1(s_P)G]^{-1}\beta_l\ell_0(s_P)c_P(x)\sinh(P(\eta_l) - P(t))[q(t) - 1/k(t)], \text{ if } (x,t) \in Q_{\eta_l}, \\ 0, \quad \text{if } (x,t) \in \overline{Q}_{\eta_l} \end{cases}$$

and

$$\tilde{H}_l(x,t) = -\frac{GH_l(x,t)}{\ell_1(s_P)}\tanh(P(x))$$

for $Q_{\eta_l} = \{0 \le x \le 1,\ 0 \le t \le \eta_l\}$, $\overline{Q}_{\eta_l} = \{0 \le x \le 1,\ \eta_l \le t \le 1\}$, $l = 1, ..., m_1$;
in addition, for $(x,t) \in Q_1$, $Q_1 = \{0 \le x \le 1,\ 0 \le t \le 1\}$

$$R_1(x,t) = \frac{\ell_0(s_P)c_P(x)}{G\ell_1(s_P)}\sinh(P(1) - P(t))[q(t) - 1/k(t)],$$

$$R_2(x,t) = -\frac{\ell_1(c_P)R_1(x,t)}{\ell_1(s_P)}\tanh(P(x))$$

and

$$R_3 = -\frac{s_P(x)}{\ell_1(s_P)}\sinh\big(P(1) - P(t)\big)[q(t) - 1/k(t)];$$

at least for $x \in [0,1]$

$$\begin{cases} \hat{f}(x) = -G^{-1}c_P(x)T_2 + [\ell_1(s_P)]^{-1}s_P(x)T_1 \\ \qquad -[G\ell_1(s_P)]^{-1}\ell_1(c_P)s_P(x)c_P(x)T_2 - \int\limits_0^x \sinh\big(P(x) - P(t)\big)\tilde{f}_1(t)dt, \end{cases} \tag{4.28}$$

where

$$T_1 = \ell_1\Big(\int\limits_0^x \sinh\big(P(x) - P(t)\big)\tilde{f}_1(t)dt\Big),$$

$$T_2 = \frac{\ell_0(s_P)}{\ell_1(s_P)}T_1 - \ell_0\Big(\int\limits_0^x \sinh\big(P(x) - P(t)\big)\tilde{f}_1(t)dt\Big).$$

Since the summands of (4.27) are continuous functions in $[0,1]\times[0,1]$, then the sum $K(x,t)$ is also the continuous function in $[0,1]\times[0,1]$. Therefore, the Fredholm's alternative holds for the integral equation (4.26) in respect of the Hilbert space $L_2(0,1)$. Because $\tilde{f}_1(x) = f_1(x)k(x)$, $f_1(x) \in C[0,1]$, then, in view of the formula for $\hat{f}(x)$, we have $\hat{f}(x) \in C[0,1]$. Since $K(x,t) \in C([0,1]\times[0,1])$, $\hat{f}(x) \in C[0,1]$, then, belonged to $L_2(0,1)$ solution of the integral equation (4.26) belongs to $C[0,1]$ actually. Then for $v(x) \in C[0,1]$, $k(x) \in C^1([0,1]$ and $q(x) \in C[0,1]$, the integral $\int\limits_0^1 K(x,t)v(t)dt$, as the function of x, belongs to $C^2[0,1]$. In addition, from the formula for $\hat{f}(x)$ it follows that $\hat{f}(x) \in C^2[0,1]$ since $\tilde{f}_1(x) \in C[0,1]$. In summary, any solution from $L_2(0,1)$ of the integral equation (4.26) belongs to $C^2[0,1]$. Then it is sufficient to prove that (4.26) has only the trivial solution if $\hat{f}(x) \equiv 0$ on $[0,1]$.

Put $\hat{f}(x) \equiv 0$ on $[0,1]$ for the integral equation (4.26). Then $\tilde{f}_1(x) \equiv 0$ on $[0,1]$. Indeed, since $\hat{f}(0) = 0$, then, in view of (4.12) and (4.28), $T_2 = 0$. Therefore,

$$\hat{f}(x) = \frac{s_P(x)}{\ell_1(s_P)}T_1 - \int\limits_0^x \sinh\big(P(x)-P(t)\big)\tilde{f}_1(t)dt.$$

Hence,

$$\hat{f}'(x) = \frac{c_P(x)}{k(x)\ell_1(s_P)}T_1 - \frac{1}{k(x)}\int\limits_0^x \cosh\big(P(x)-P(t)\big)\tilde{f}_1(t)dt,$$

herewith, since we put $\hat{f}(x) \equiv 0$, then $\hat{f}'(x) \equiv 0$ on $[0,1]$. Since $\hat{f}'(0) = 0$, then $T_1 = 0$, therefore,

$$\int\limits_0^x \cosh\big(P(x)-P(t)\big)\tilde{f}_1(t)dt \equiv 0$$

on $[0,1]$. Hence, similarly [13, p. 46], we obtain that $\tilde{f}_1(x) \equiv 0$ on $[0,1]$. Since $\tilde{f}_1(x) = f_1(x)k(x)$, then $f_1(x) \equiv 0$ on $[0,1]$. Since $f_1(x) \equiv 0$, then, in view of Theorem 3.1, the NLBVP (4.1) has only trivial solution $v(x) \equiv 0$. Hence, because the problem (4.1) is equivalent to the differential problem (4.8), (4.8) is equivalent to the differential problem (4.9) for the defined by (4.25) function $F(x)$ and (4.9) is equivivalent to the integral equation (4.26), then (4.26) has only trivial solution if $\hat{f}(x) \equiv 0$ on $[0,1]$.

Thus we proved that the solution $v(x)$ of the integral equation (4.26) exits and belongs to $C^2[0,1]$, then, in view of the equivalence, $v(x)$ is classical solution of NLBVP (4.1) at the same time. By virtue of Lemma 4.1, since classical solution $v(x)$ of NLBVP (4.1) exits, then classical solution $u(x)$ of NLBVP (2.1),(2.2) exists too. Theorem 4.2 is proved. □

5. Ill-posed statement examples

Next examples show that stated for NLBVP (2.1)-(2.2) condition on S_i, $i = 0,1$ is essential for well-posedness of the problem.

Example 1. The problem

$$\varepsilon u''(x) + a(x)u'(x) = 0,\ 0 < x < 1,\ \ u(0) = u(\zeta),\ u(1) = u(\eta)$$

is ill-posed, it has infinite number of solutions $u(x) = const$. It shows the essentiality of condition $S_1 < 1$ for the case if $b_0 = 0$.

Example 2. The problem

$$\varepsilon u''(x) + a(x)u'(x) = 0,\ 0 < x < 1,\ \ u(0) = u(\zeta),\ u(1) = u(\eta) + 1$$

is ill-posed, it has no solution for $0 < \zeta < \eta < 1$. Indeed, assume that some solution of the problem exists, then $u'(\xi) = 0$ at some point $\xi \in (0, \zeta)$, therefore, $u'(x) \equiv 0$ on $[\xi, 1]$, then $u(x) = const$ on $[0, 1]$, so that it conflicts with the condition $u(1) = u(\eta) + 1$. It shows the essentiality of condition $S_1 < 1$ for the case $b_0 = 0$.
Example 3. The problem

$$\varepsilon u''(x) + u'(x) = 1,\ 0 < x < 1,\ \ u(0) = 0,\ u(1) = u(\eta)$$

is ill-posed, it has the unstable on parameter solution

$$u(x) = -(1 - \eta)[\exp(-x/\varepsilon) - 1][\exp(-1/\varepsilon) - \exp(-\eta/\varepsilon)]^{-1} + x,$$

$u(x) \to -\infty$ at each nonzero point x if $\varepsilon \to 0$. It shows the essentiality of condition $S_1 < 1$ for the case if $b_0 = 0$.
Example 4. The problem

$$\varepsilon u''(x) + au'(x) - bu(x) = 0,\ 0 < x < 1,\ \ u(0) = 0,\ u(1) = \beta u(\eta) \tag{5.1}$$

is ill-posed, in general, for each $\beta > 1$, $\eta \in (0, \beta^{-1})$, $a = const > 0$, $b = const \geq 0$. It always has infinite number of solutions for some parameter value $\varepsilon = \varepsilon^*$. This fact shows the essentiality of condition $S_1 \leq 1$, $i = 0, 1$. Let us confirm the ill-posedness.
(i) If $b = 0$, then for an arbitrary constant C the function

$$u(x) = C\exp(-ax/\varepsilon) - C \tag{5.2}$$

satisfies the equation (5.1) and the condition $u(0) = 0$. By choosing $C \neq 0$ and substituting (5.2) into nonlocal condition (5.1) we get

$$\beta = \frac{1 - e^{-a/\varepsilon}}{1 - e^{-a\eta/\varepsilon}}. \tag{5.3}$$

Note, the equality (5.3) is impossible for $-\infty < \beta \leq 1$, $\eta \in (0, 1)$. However, for $\beta > 1$ and for each $\eta \in (0, \beta^{-1})$ the formula (5.3) is true for some value $\varepsilon = \varepsilon^*$. Indeed, for each $\varepsilon > 0$ the function $g(\varepsilon) = (1 - e^{-a/\varepsilon})/(1 - e^{-a\eta/\varepsilon})$ is positive and continuous, $\lim_{\varepsilon\to+0} g(\varepsilon) = 1$, $\lim_{\varepsilon\to+\infty} g(\varepsilon) = \eta^{-1}$, the $g(\varepsilon)$ reachs the value β at some argument ε^* since $1 < \beta < \eta^{-1}$. So, $g(\varepsilon^*) = \beta$, i.e., (5.3) is true for $\varepsilon = \varepsilon^*$, therefore, $u(x)$ is solution of (5.1). Since $C \neq 0$ is an arbitrary constant for (5.2), then (5.1) has infinite number of solutions.
(ii) If $b > 0$, then for an arbitrary constant C the function

$$u(x) = C\exp(\lambda_1 x) - C\exp(\lambda_2 x) \tag{5.4}$$

satisfies the equation (5.1) and the condition $u(0) = 0$ for

$$\lambda_1 = -a/2\varepsilon - \sqrt{(a/2\varepsilon)^2 + b/\varepsilon},\ \ \lambda_2 = -a/2\varepsilon + \sqrt{(a/2\varepsilon)^2 + b/\varepsilon}.$$

By choosing $C \neq 0$ and substituting (5.4) into (5.1) we get

$$\beta = e^{\lambda_1(1-\eta)}\frac{1 - e^{\lambda_2 - \lambda_1}}{1 - e^{\eta(\lambda_2 - \lambda_1)}} = \frac{e^{\lambda_1} - e^{\lambda_2}}{e^{\lambda_1\eta} - e^{\lambda_2\eta}}. \tag{5.5}$$

The equality (5.5) is impossible for $-\infty < \beta \leq 0$. Moreover, (5.5) is impossible for $0 < \beta \leq 1$ too. It follows from the relation $h(1) - \beta h(\eta) \neq 0$ for the function $h(t) = e^{\lambda_1 t} - e^{\lambda_2 t}$, $t \in (0, 1]$ in view of $h(t) < 0$, $h'(t) < 0$. However, for $\beta > 1$, the formula (5.5) is true for some value $\varepsilon = \varepsilon^*$ in respect of each $\eta \in (\beta^{-1}, 1)$. Indeed, since $z(\varepsilon) = h(1)/h(\eta)$ is positive and continuous function in $(0, +\infty)$, $\lim_{\varepsilon\to+0} z(\varepsilon) = +\infty$, $\lim_{\varepsilon\to+\infty} z(\varepsilon) = \eta^{-1}$, then $z(\varepsilon^*) = \beta$ for some $\varepsilon^* \in (0, +\infty)$. So, (5.5) is true for $\varepsilon = \varepsilon^*$, then $u(x)$ is solution of (5.1) for

$\lambda_i = \lambda_i(\varepsilon^*)$. Since $C \neq 0$ is an arbitrary constant for (5.4), then (5.1) has infinite number of solutions.

6. Conclusion

In this article we studied NLBVP of the first kind for linear second order ODE with positive parameter at the highest derivative. We researched the well-posed solvability of the problem in respect of classical solution. Under new and accurate condition on coefficients and nonlocal carriers of NLBVC, we obtained a uniform on parameter stability estimate of classical solution, proved existence and uniqueness. We demonstrated examples of ill-posed problems for some cases if coefficients of NLBVC does not satisfy stated herein well-posedness condition, i.e., we confirmed, in general, that established in our paper condition is essential.

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FACTOR RELATIONS BETWEEN SOME SUMMABILITY METHODS

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Abstract. In the present paper, using the result of Bennett [1] on characterization of factorable matrices, we give necessary and sufficient conditions in order that $\Sigma\lambda_n x_n$ is summable $|R,p_n|_s$ whenever $\Sigma\mu_n x_n$ is summable $|C,0|_k$, and $\Sigma\lambda_n x_n$ is summable $|C,0|_s$ whenever $\Sigma\mu_n x_n$ is summable $|R,p_n|_r$, . where $1<k\leq s<\infty$. Therefore we also extend some known results.

1. Introduction

Consider an infinite series Σx_n with partial sum s_n, and by (σ_n^α), we denote the n-th Cesàro means of order α with $\alpha>-1$ of the sequence (s_n). The series Σx_n is said to be summable $|C,\alpha|_k, k\geq 1$, if $\left(n^{1-1/k}\left(\sigma_n^\alpha-\sigma_{n-1}^\alpha\right)\right)\in\ell_k$ (see [7]), where ℓ_k is the set of all sequences consisting k- absolutely convergent series. Note that the summability $|C,0|_k$ reduces to $\left(n^{1-1/k}x_n\right)\in\ell_k$. Let (p_n) be a sequence of positive real numbers with $P_n=p_0+p_1+\cdots+p_n\to\infty$ as $n\to n$. The sequence-to-sequence transformation

$$u_n=\frac{1}{P_n}\sum_{n=0}^{n}p_ns_n. \tag{1.1}$$

defines the sequence (u_n) of the (R,p_n) Riesz means of the sequence (s_n), generated by the sequence of numbers (p_n). The series Σx_n is said to be summable $|R,p_n|_k, k\geq 1$, if $\left(n^{1-1/k}\left(u_n-u_{n-1}\right)\right)\in\ell_k$ (see [19]).

A summability method Y is said to be include another summability method X, if every series summable by X is also summable by Y. If the methods include each other, then, these methods are called equivalent. Hereof, the inclusion relations of the absolute summability methods of single series were studied by various authors (see, for example, [2-24]).

The following result was established by Bor [2].

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Theorem 1.1. Let $1 < k < \infty$ and

$$\sum_{n=v}^{\infty} \frac{n^{k-1} p_n^k}{P_n^k P_{n-1}} = O\left(\frac{v^{k-1} p_{v-1}^k}{P_{v-1}^k}\right). \tag{1.2}$$

If exists $d > 1$ such that

$$\frac{P_{n+1}}{P_n} \geq d \text{ for all } n \geq 1, \tag{1.3}$$

then, the summability methods $|R, p_n|_k$ and $|C, 0|_k$ are equivalent.

Also, in [16], this result was extended as follows.

Theorem 1.2. Let $1 < k \leq s < \infty$. Then, the necessary and sufficient condition in order that the summability method $|R, p_n|_s$ includes the summability method $|C, 0|_k$ is

$$\left\{\sum_{v=1}^{m} \frac{P_{v-1}^{k^*}}{v}\right\}^{1/k^*} \left\{\sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} p_n}{P_n P_{n-1}}\right)^s\right\}^{1/s} = O(1),$$

where k^* denotes the conjugate of the index $k > 1$, i.e., $1/k + 1/k^* = 1$.

Theorem 1.3. Let $1 < k \leq s < \infty$. Then, the necessary and sufficient condition in order that the summability method $|C, 0|_s$ includes the summability method $|R, p_n|_k$ is

$$\left\{\sum_{v=m-1}^{m} \frac{1}{v} \left|\frac{P_{v-1} P_v}{p_v}\right|^{k^*}\right\}^{1/k^*} \left\{\sum_{n=m}^{m+1} \frac{n^{s-1}}{P_n^s}\right\}^{1/s} = O(1).$$

2. The main Result

This paper gives necessary and sufficient conditions in order that $\Sigma \lambda_n x_n$ is summable $|C, 0|_s$ whenever $\Sigma \mu_n x_n$, is summable $|R, p_n|_k$, and also $\Sigma \lambda_n x_n$ is summable $|R, p_n|_s$ whenever $\Sigma \mu_n x_n$, is summable $|C, 0|_k$, where$1 < r \leq s < \infty$, which generalizes the above results.

A factorable matrix T is defined by

$$t_{nv} = \begin{cases} b_n a_v, 0 \leq v \leq n, \\ 0, \qquad v > n. \end{cases}$$

where (b_n) and (a_n) are sequences of real or complex numbers.

Now we prove the following theorems.

Theorem 2.1. Let $1 < k \leq s < \infty$ and $\lambda = (\lambda_n)$ be a sequence of numbers. Further, let $\mu = (\mu_n)$ be a sequence of non-zero numbers. Then, necessary and sufficient condition in order that $\Sigma \lambda_n x_n$ is summable $|R, p_n|_s$ whenever $\Sigma \mu_n x_n$ is summable $|C, 0|_k$ is

$$\left\{\sum_{v=1}^{m} \frac{P_{v-1}^{k^*}}{v} \left|\frac{\lambda_v}{\mu_v}\right|^{k^*}\right\}^{1/k^*} \left\{\sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} p_n}{P_n P_{n-1}}\right)^s\right\}^{1/s} = O(1). \tag{2.1}$$

Theorem 2.2. Let $1 < k \leq s < \infty$, λ and μ be as in Theorem 2.1. Then, necessary and sufficient condition in order that $\Sigma \lambda_n x_n$ is summable $|C, 0|_s$ whenever $\Sigma \mu_n x_n$ is summable $|R, p_n|_k$ is

$$\left\{\sum_{v=m-1}^{m}\frac{1}{v}\left(\frac{P_{v-1}P_v}{p_v}\right)^{k^*}\right\}^{1/k^*}\left\{\sum_{n=m}^{m+1}\left|\frac{n^{1/s^*}\lambda_n}{P_n\mu_n}\right|^s\right\}^{1/s}=O(1). \tag{2.2}$$

It may be noticed that Theorem 2.1 and Theorem 2.2. are, in the special case $\mu_n = \lambda_n = 1$ for all $n \geq 0$, reduced to Theorem 1.2. and Theorem 1.3, respectively.

Also, if $p_n = 1$ for all $n \geq 0$, then the summability$|R, p_n|_k$ coincides with the summability $|C, 1|_k$. Further, $P_n = n + 1$ and

$$\sum_{n=m}^{\infty}\frac{p_n}{P_{n-1}P_n^s}=\sum_{n=m}^{\infty}\frac{1}{n(n+1)^s}=O\left(\frac{1}{m^s}\right).$$

Hence, the following results is immediately obtained.

Corollary 2.3. Let $1 < k \leq s < \infty$, λ and μ be as in Theorem 2.1. Then, necessary and sufficient condition in order that $\Sigma x_n\lambda_n$ is summable $|C, 1|_s$ whenever $\Sigma x_n\mu_n$ is summable $|C, 0|_k$ is

$$\sum_{v=1}^{m}v^{k^*-1}\left|\frac{\lambda_v}{\mu_v}\right|^{k^*}=O(m^{k^*}).$$

Corollary 2.4. Let $1 < k \leq s < \infty$, λ and μ be as in Theorem 2.1. Then, necessary and sufficient condition in order that $\Sigma x_n\lambda_n$ is summable $|C, 0|_s$ whenever $\Sigma x_n\mu_n$ is summable $|C, 1|_k$ is

$$\sum_{n=m}^{m+1}\frac{1}{P_n^s}\left|\frac{\lambda_n}{\mu_n}\right|^s=O(m^{1-2s-s/k}).$$

Proof of Theorem 2.1. We first note a result of Bennett [1] that a factorable matrix T defines a bounded linear operator $L_T : \ell_k \to \ell_s$ such that $L_T(x) = T(x)$ for all $x \in \ell_k$ if and only if

$$\left(\sum_{v=0}^{m}|a_v|^{k^*}\right)^{1/k^*}\left(\sum_{n=m}^{\infty}|b_n|^s\right)^{1/s}=O(1), \tag{2.3}$$

where $k^*$is the conjugate of indices k. Let σ_n^0 and u_n be Cesàro $(C, 0)$ and Riesz means (R, p_n) of the series $\Sigma\mu_n x_n$ and $\Sigma\lambda_n x_n$, respectively. Then, by (1.1),

$$\sigma_n^0=\sum_{v=0}^{n}\mu_v x_v$$

$$u_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{r=0}^{v}\lambda_r x_r$$

and so $\Delta u_0 = \lambda_0 x_0$,

$$\Delta u_n=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\lambda_v x_v, \text{ for } n \geq 1.$$

Now, say $t'_n = n^{1/s^*}\Delta u_n$ and $\sigma_n^{0\prime} = n^{1/k^*}\mu_n x_n$ for $n \geq 1$. Then, it easily seen that

$$\begin{aligned} t'_n &= \frac{n^{1/s^*}p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v^{1/k^*}\mu_v}\sigma_v^{0\prime} \\ &= \sum_{v=1}^{\infty} t_{nv}\sigma_v^{0\prime} \end{aligned}$$

where the matrix $T = (t_{nv})$ is given by

$$t_{nv} = \begin{cases} \frac{n^{1/s^*}p_nP_{v-1}\lambda_v}{P_nP_{n-1}v^{1/k^*}\mu_v}, & 1 \leq v \leq n, \\ 0, & v > n. \end{cases}$$

This means that $\Sigma x_n\lambda_n$ is summable $|R, p_n|_s$ whenever $\Sigma x_n\mu_n$ is summable $|C, 0|_k$ if and only if $(t'_n) \in \ell_s$ for all $\left(\sigma_n^{0\prime}\right) \in \ell_k$, or, $T : \ell_k \to \ell_s$ is a bounded linear operator. Thus, by applying (2.3) to the matrix T, we have (2.1).

Proof of Theorem 2.2. Let u_n and σ_n^0 be means of Riesz (R, p_n) and Cesàro $(C, 0)$ of the series $\Sigma\mu_n x_n$ and $\Sigma\lambda_n x_n$, respectively. Then, as above, $\Delta\sigma_n^0 = \lambda_n x_n$, and also $\Delta u_0 = \mu_0 x_0$,

$$\Delta u_n = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\mu_v x_v, \text{for } n \geq 1 \tag{2.4}$$

By inversion of (2.4), it can be stated that, for $n \geq 1$,

$$x_n = \frac{1}{\mu_nP_{n-1}}\left(\frac{P_{n-1}P_n}{p_n}\Delta u_n - \frac{P_{n-1}P_{n-2}}{p_{n-1}}\Delta u_{n-1}\right)$$

Say $t'_n = n^{1/k^*}\Delta u_n$ and $\sigma_n^{0\prime} = n^{1/s^*}\lambda_n x_n$ for $n \geq 1$. Then, it can be written that

$$\begin{aligned} \sigma_n^{0\prime} &= \frac{n^{1/s^*}\lambda_n}{\mu_nP_{n-1}}\left(\frac{P_{n-1}P_nt'_n}{n^{1/k^*}p_n} - \frac{P_{n-1}P_{n-2}t'_{n-1}}{(n-1)^{1/k^*}p_{n-1}}\right) \\ &= \sum_{v=1}^{\infty} d_{nv}t'_v \end{aligned}$$

where the matrix $D = (d_{nv})$ is defined by

$$d_{nv} = \begin{cases} \frac{n^{1/s^*}\lambda_n}{\mu_nP_{n-1}}\left(-\frac{P_{n-1}P_{n-2}}{(n-1)^{1/k^*}p_{n-1}}\right), & v = n-1 \\ \frac{n^{1/s^*}\lambda_n}{\mu_nP_{n-1}}\left(\frac{P_{n-1}P_n}{n^{1/k^*}p_n}\right), & v = n \\ 0, & v > n. \end{cases}$$

This gives that $\Sigma x_n\lambda_n$ is summable $|C, 0|_s$ whenever $\Sigma x_n\mu_n$ is summable $|R, p_n|_k$ if and only if $\left(\sigma_n^{0\prime}\right) \in \ell_s$ for all $(t'_n) \in \ell_k$, or, $D : \ell_k \to \ell_s$ is a bounded linear operator. Thus, by applying (2.3) to the matrix D, we get (2.2).

This completes the proof.

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SOME COINCIDENCE BEST PROXIMITY POINT RESULTS IN S-METRIC SPACES

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Abstract. In this paper, we introduce the notions of S-proximal Berinde g-cyclic contraction of two nonself mappings and S-proximal Berinde g-contractions of the first kind and second kind in an S-metric space and prove some coincidence best proximity point theorems for these types of nonself mappings in this space. Also, we give two examples to analyze and support our main results. The results presented here generalize some results in the existing literature.

1. Introduction

Let (X, d) be a metric space and $J : X \to X$ be a self mapping. A fixed point problem is to find a point x in X such that $Jx = x$ or $d(x, Jx) = 0$. In this direction, Banach [1] proved his famous result "Banach contraction principle", which states that "let (X, d) be a complete metric space and $J : X \to X$ be a contraction mapping, then J has a unique fixed point". Later, many authors studied the results dealing with "fixed point" in different spaces (see, e.g., [2]-[7]).

Let (X, d) be a metric space, Y and Z be two nonempty subsets of X and $J : Y \to Z$ be a nonself mapping. A point $x \in Y$ is called a best proximity point of J if $d(x, Jx) = \triangle_{YZ}$ where $\triangle_{YZ} = d(Y, Z) = \inf\{d(x, y) : x \in Y,\, y \in Z\}$. Clearly, if J is a self mapping, then the best proximity point problem reduces to a fixed point problem. In this way, the best proximity point problem can be viewed as a natural generalization of a fixed point problem.

A coincidence best proximity point problem is to find a point x in Y such that $d(gx, Jx) = \triangle_{YZ}$, where g is a self mapping on Y. If g is an identity mapping on Y, then it can be observed that a coincidence best proximity point is essentially a best proximity point. Hence, the coincidence best proximity point problem is an extension of the best proximity point problem. There are several results dealing with proximity point problem in different spaces (see [8]-[12]).

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In 2011, Basha [13] studied and established best proximity point theorems for the proximal contractions of the first kind and second kind, and proximal cyclic contractions in a metric space. More recently, Klanarong and Chaiya [14] presented coincidence best proximity point theorems for the proximal Berinde g-contractions of the first kind and second kind, and proximal Berinde g-cyclic contractions which are more general than the nonself mappings considered in [13].

In 2012, Sedghi et al. [15] introduced the notion of S-metric space and investigated the topology of this space. They also characterized some well-known fixed point results in the context of S-metric space. Later, some authors have published the best proximity point and coincidence best proximity point results on the setting of S-metric space (for details, see [16]-[18]).

Inspired and motivated by the above results, in this paper, we introduce the notions of S-proximal Berinde g-cyclic contractions of two nonself mappings and S-proximal Berinde g-contractions of the first kind and second kind in an S-metric space and establish some coincidence best proximity point theorems for these kinds of nonself mappings in this space. We also give two examples to support our results. The results presented in this paper can be regarded as an extension of corresponding results from a metric space to an S-metric space.

2. Preliminaries and lemmas

In this section, we recall some definitions and lemmas which are needed in the sequel.

The notion of an S-metric space is introduced as a generalization of a metric space as follows.

Definition 2.1. *(see* [15, Definition 2.1]*) Let X be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:*

(S1) $S(x, y, z) \geq 0$;

(S2) $S(x, y, z) = 0$ *if and only if* $x = y = z$;

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

for all $x, y, z, a \in X$. Then the function S is called an S-metric on X, and the pair (X, S) is called an S-metric space.

Some geometric examples for S-metric spaces can be seen in [15].

The following lemma can be considered as the symmetry condition and it will be used in the proofs of some theorems.

Lemma 2.1. *(see* [15, Lemma 2.5]*) Let (X, S) be an S-metric space. Then*

$$S(x, x, y) = S(y, y, x) \quad \textit{for all } x, y \in X.$$

We need the following result which can easily be derived from Definition 2.1 and Lemma 2.1.

Lemma 2.2. *(see* [18, Remark 2.6]*) Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \leq S(x, x, y) + 2S(y, y, z) \quad \textit{for all } x, y, z \in X.$$

Sedghi et al. [15, 19] defined some basic topological concepts in an S-metric space as follow.

Definition 2.2. *(see* [15, Definition 2.6]*) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x, r)$ is defined as follows:*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\}.$$

Definition 2.3. *(see* [15, Definition 2.8 (3)-(5)]*) Let (X, S) be an S-metric space.*

(i) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$,

(ii) A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n, m \to \infty$. That is, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x) < \varepsilon$ for all $n, m \geq n_0$. We write $x_n \to x$ for brevity.

(iii) The S-metric space (X, S) is called complete if every Cauchy sequence in (X, S) is convergent in (X, S).

Definition 2.4. *(see* [19, Corollary 2.4]*) Let X and X' be two S-metric spaces, and let $f : X \to X'$ be a function. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ for any sequence $\{x_n\}$ in X such that $x_n \to x$. We say that f is continuous on X if f is continuous at every point $x \in X$.*

Özgür and Taş [20, 21] defined the concepts of cluster point and closed set in an S-metric space.

Definition 2.5. *Let (X, S) be an S-metric space and $Y \subseteq X$ be any subset.*

(i) (see [20, Definition 4.2]*) A point $x \in X$ is a cluster point of Y if*

$$(B_S(x, r) - \{x\}) \cap Y \neq \emptyset$$

for every $r > 0$. The set of all cluster points of Y is denoted by Y_S'.

(ii) (see [21, Definition 3.3]*) Let (X, S) be an S-metric space and $Y \subseteq X$. The subset Y is called closed if the set of cluster points of Y is contained by Y, that is, $Y_S' \subset Y$.*

Özgür and Taş [21] also defined the concept of sub-S-metric space and gave a property for closed subsets in complete S-metric spaces.

Definition 2.6. *(see* [21, Definition 3.2]*) Let (X, S) be an S-metric space and Y be a nonempty subset of X. Let a function $S_Y : Y \times Y \times Y \to [0, \infty)$ be defined by*

$$S_Y(x, y, z) = S(x, y, z) \quad \text{for all } x, y, z \in Y.$$

Then S_Y is called a reduced S-metric and (Y, S_Y) is called a sub-S-metric space of (X, S).

Proposition 2.3. *(see* [21, Proposition 3.4]*) If (X, S) is a complete S-metric space and Y is a closed set in (X, S), then (Y, S_Y) is complete.*

The relation between a metric and an S-metric was given in [22] as follows.

Lemma 2.4. *(see* [22, Lemma 1.12]*) Let (X, d) be a metric space. Then the following properties are satisfied:*

1) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.

2) $x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).

3) $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d).

4) (X, d) is complete if and only if (X, S_d) is complete.

In [23], the function S_d was called an S-metric generated by d. We know some examples of an S-metric which are not generated by any metric (see [22, 23] for more details).

On the other hand, Gupta [24] claimed that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x,y) = S(x,x,y) + S(y,y,x), \quad \forall x,y \in X. \tag{2.1}$$

However, the function $d_S(x,y)$ defined in (2.1) does not always a metric because the triangle inequality is not satisfied for all elements of X everywhere (see [23] for more details).

Khanpanuk [18] defined the following concepts in an S-metric space.

Definition 2.7. *(see* [18, Definition 3.4]*) Let (X,S) be an S-metric space. A mapping $g : X \to X$ is called an isometry if*

$$S(gx,gy,gz) = S(x,y,z), \quad \forall x,y,z \in X.$$

Clearly, a self mapping which is an isometry is continuous.

Definition 2.8. *(see* [18, Definition 3.5]*) Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ be a mapping and $g : Y \to Y$ be an isometry. The mapping J is said to preserve the isometric distance with respect to g if*

$$S(Jgx, Jgy, Jgz) = S(Jx, Jy, Jz) \quad \forall x,y,z \in Y.$$

Klanarong and Chaiya [14] introduced the following new classes of nonself mappings in a metric space.

Definition 2.9. *(see* [14, Definitions 3.2 and 3.4]*) Let (X,d) be a metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. The mapping J is said to be*

(i) a proximal Berinde g-contraction of the first kind if there exist $\alpha \in [0,1)$ and $L_1 \geq 0$ such that

$$\begin{aligned} d(gu_1, Jx_1) &= d(gu_2, Jx_2) = \triangle_{YZ} \\ &\Longrightarrow \\ d(gu_1, gu_2) &\leq \alpha d(gx_1, gx_2) + L_1 \min\{d(gx_1, gu_2), d(gx_2, gu_1)\} \end{aligned}$$

for all $x_1, x_2, u_1, u_2 \in Y$,

(ii) a proximal Berinde g-contraction of the second kind if there exist $\beta \in [0,1)$ and $L_2 \geq 0$ such that

$$\begin{aligned} d(gu_1, Jx_1) &= d(gu_2, Jx_2) = \triangle_{YZ} \\ &\Longrightarrow \\ d(Jgu_1, Jgu_2) &\leq \beta d(Jgx_1, Jgx_2) + L_2 \min\{d(Jgx_1, Jgu_2), d(Jgx_2, Jgu_1)\} \end{aligned}$$

for all $x_1, x_2, u_1, u_2 \in Y$.

In the case $L_1 = 0$ (or $L_2 = 0$) and $gx = x$ for all $x \in Y$, it is easy to see that a proximal Berinde g-contraction of the first kind (or the second kind) reduces to proximal contraction of the first kind (or the second kind) which was introduced in [13]. But the converse is not true (see [14, Example 3.3]).

Definition 2.10. *(see* [14, Definition 3.5]*) Let (X,d) be a metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z, T : Y \to Z$ and $g : Y \cup Z \to Y \cup Z$*

be mappings. The pair (J,T) is said to be a proximal Berinde g-cyclic contraction if there exist $\gamma \in [0,1)$ and $L \geq 0$ such that

$$\begin{aligned} d(gu_1, Jx_1) &= d(gu_2, Tx_2) = \triangle_{YZ} \\ &\Longrightarrow \\ d(gu_1, gu_2) &\leq \gamma d(gx_1, gx_2) + (1-\gamma)d(Y,Z) + Ld(gx_1, gu_1) \end{aligned}$$

for all $x_1, gu_1 \in Y$ and $x_2, gu_2 \in Z$.

In the case $L = 0$ and $gx = x$ for all $x \in Y \cup Z$, it is easy to see that a proximal Berinde g-cyclic contraction reduces to a proximal cyclic contraction which was introduced in [13].

3. Main Results

Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. We define the following sets:

$$\begin{aligned} \triangle^S_{YZ} &= S(Y,Y,Z) = \inf\{S(x,x,y) : x \in Y, y \in Z\}, \\ Y_0 &= \left\{x \in Y : \text{there exists some } y \in Z \text{ such that } S(x,x,y) = \triangle^S_{YZ}\right\}, \\ Z_0 &= \left\{y \in Z : \text{there exists some } x \in Y \text{ such that } S(x,x,y) = \triangle^S_{YZ}\right\}. \end{aligned}$$

Definition 3.1. *Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. A point $x \in Y$ is said to be a coincidence best proximity point of the pair (g,J) if $S(gx,gx,Jx) = \triangle^S_{YZ}$.*

Note that if g is the identity mapping on Y in Definition 3.1, then the point x is the best proximity point of J.

Definition 3.2. *Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$, $T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ be mappings. An element $(x,y) \in Y \times Z$ is called a coincidence best proximity point of the triple (g,J,T) if $(gx,gy) \in Y \times Z$ and $S(gx,gx,Jx) = S(gy,gy,Ty) = S(x,x,y) = \triangle^S_{YZ}$.*

Note that if g is the identity mapping on $Y \cup Z$ in Definition 3.2, then the point x and y is the best proximity point of J and T, respectively.

Now, we introduce the S-proximal Berinde g-contractions of the first kind and second kind in an S-metric space.

Definition 3.3. *Let (X,S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ be mappings. The mapping J is said to be*

(i) an S-proximal Berinde g-contraction of the first kind if there exist $\alpha \in [0,1)$ and $L_1 \geq 0$ such that

$$\begin{aligned} S(gu_1, gu_1, Jx_1) &= S(gu_2, gu_2, Jx_2) = \triangle^S_{YZ} \\ &\Longrightarrow \\ S(gu_1, gu_1, gu_2) &\leq \alpha S(gx_1, gx_1, gx_2) \\ &\quad + L_1 \min\{S(gx_1, gx_1, gu_2), S(gx_2, gx_2, gu_1)\} \quad (3.1) \end{aligned}$$

for all $x_1, x_2, u_1, u_2 \in Y$,

(ii) an S-proximal Berinde g-contraction of the second kind if there exist $\beta \in [0,1)$ and $L_2 \geq 0$ such that

$$\begin{aligned} S(gu_1, gu_1, Jx_1) &= S(gu_2, gu_2, Jx_2) = \triangle^S_{YZ} \\ &\Longrightarrow \\ S(Jgu_1, Jgu_1, Jgu_2) &\leq \beta S(Jgx_1, Jgx_1, Jgx_2) + L_2 \min\{S(Jgx_1, Jgx_1, Jgu_2), \\ & \quad S(Jgx_2, Jgx_2, Jgu_1)\} \end{aligned} \tag{3.2}$$

for all $x_1, x_2, u_1, u_2 \in Y$.

Now, we define the S-proximal Berinde g-cyclic contraction in an S-metric space.

Definition 3.4. *Let (X, S) be an S-metric space and Y, Z be two nonempty subsets of X. Let $J : Y \to Z, T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ be mappings. The pair (J, T) is said to be an S-proximal Berinde g-cyclic contraction if there exist $\gamma \in [0,1)$ and $L \geq 0$ such that*

$$\begin{aligned} S(gu_1, gu_1, Jx_1) &= S(gu_2, gu_2, Tx_2) = \triangle^S_{YZ} \\ &\Longrightarrow \\ S(gu_1, gu_1, gu_2) &\leq \gamma S(gx_1, gx_1, gx_2) + (1-\gamma)\triangle^S_{YZ} + LS(gx_1, gx_1, gu_1) \end{aligned}$$

for all $x_1, gu_1 \in Y$ and $x_2, gu_2 \in Z$.

Next, we give the following coincidence best proximity point result in an S-metric space.

Theorem 3.1. *Let (X, S) be a complete S-metric space and Y, Z be two nonempty closed subsets of X. Let $J : Y \to Z$, $T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ satisfy the following conditions:*

(i) J and T are S-proximal Berinde g-contractions of the first kind, i.e., there exist $\alpha, \beta \in [0,1)$ and $L_1, L_2 \geq 0$ such that J and T satisfy the condition (3.1), respectively;

(ii) $J(Y_0) \subseteq Z_0$ and $T(Z_0) \subseteq Y_0$;

(iii) g is an isometry with $\emptyset \neq Y_0 \subseteq g(Y_0)$ and $Z_0 \subseteq g(Z_0)$.

(iv) The pair (J, T) is an S-proximal Berinde g-cyclic contraction.

Then, there exists a point $x \in Y$ and there exists a point $y \in Z$ such that

$$S(gx, gx, Jx) = S(gy, gy, Ty) = S(x, x, y) = \triangle^S_{YZ}. \tag{3.3}$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \text{for all } n \in \mathbb{N}$$

converges to the element x, and for any fixed $y_0 \in Z_0$, the sequence $\{y_n\}$ defined by

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \text{for all } n \in \mathbb{N}$$

converges to the element y. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the there exists unique element x and there exists unique element y which satisfy the equation (3.3).

Proof. Let $x_0 \in Y_0$ be given. Since $J(Y_0) \subseteq Z_0$, $Jx_0 \in Z_0$. Hence there is $z_1 \in Y$ such that $S(z_1, z_1, Jx_0) = \triangle^S_{YZ}$ which implies that $z_1 \in Y_0$. As $Y_0 \subseteq g(Y_0)$, there exists $x_1 \in Y_0$ such that $gx_1 = z_1$, so $S(gx_1, gx_1, Jx_0) = S(z_1, z_1, Jx_0) = \triangle^S_{YZ}$. In a similar way, there is $x_2 \in Y_0$ such that $S(gx_2, gx_2, Jx_1) = \triangle^S_{YZ}$. Inductively, we can construct a sequence $\{x_n\}$ in Y_0 such that

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}.$$

Since J is an S-proximal Berinde g-contraction of the first kind, for $x_{n-1}, x_n, x_{n+1} \in Y_0$, $S(gx_n, gx_n, Jx_{n-1}) = S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}$ implies that

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &\leq \alpha S(gx_{n-1}, gx_{n-1}, gx_n) \\ &\quad + L_1 \min\{S(gx_{n-1}, gx_{n-1}, gx_{n+1}), S(gx_n, gx_n, gx_n)\} \\ &= \alpha S(gx_{n-1}, gx_{n-1}, gx_n) \end{aligned}$$

for all $n \in \mathbb{N}$. It follows from g being an isometry that

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(gx_n, gx_n, gx_{n+1}) \\ &\leq \alpha S(gx_{n-1}, gx_{n-1}, gx_n) \\ &\leq \alpha^2 S(gx_{n-2}, gx_{n-2}, gx_{n-1}) \\ &\ \ \vdots \\ &\leq \alpha^n S(gx_0, gx_0, gx_1) \\ &= \alpha^n S(x_0, x_0, x_1) \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$. Since $\alpha \in [0, 1)$, then we have

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

For positive integers m and n with $m > n$, it follows that

$$\begin{aligned} & S(x_n, x_n, x_m) \\ \leq\ & 2S(x_{m-1}, x_{m-1}, x_m) + S(x_n, x_n, x_{m-1}) \\ \leq\ & 2S(x_{m-1}, x_{m-1}, x_m) + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + S(x_n, x_n, x_{m-2}) \\ \leq\ & 2S(x_{m-1}, x_{m-1}, x_m) + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + ... + S(x_n, x_n, x_{n+1}). \end{aligned}$$

Now, for $m = n + r; r \geq 1$ and (3.4), we obtain

$$S(x_n, x_n, x_{n+r}) \leq 2\alpha^{n+r-1} S(x_0, x_0, x_1) + 2\alpha^{n+r-2} S(x_0, x_0, x_1) + ... + \alpha^n S(x_0, x_0, x_1).$$

By taking limit as $n \to \infty$, we deduce

$$\lim_{n\to\infty} S(x_n, x_n, x_m) = 0.$$

That is, $\{x_n\}$ is a Cauchy sequence in Y. Since (Y, S_Y) is a complete S-metric space, so there exists $x \in Y$ such that $x_n \to x$ as $n \to \infty$. Similarly, since $T(Z_0) \subseteq Y_0$ and $Z_0 \subseteq g(Z_0)$, there exists a sequence $\{y_n\}$ in Z_0 such that

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N},$$

and which converges to some element $y \in Z$. Since the pair (J, T) is an S-proximal Berinde g-cyclic contraction and

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ} = S(gy_{n+1}, gy_{n+1}, Ty_n), \quad \forall n \in \mathbb{N},$$

there exist $\gamma \in [0, 1)$ and $L \geq 0$ such that

$$S(gx_{n+1}, gx_{n+1}, gy_{n+1}) \leq \gamma S(gx_n, gx_n, gy_n) + (1 - \gamma)\triangle^S_{YZ} + LS(gx_n, gx_n, gx_{n+1}).$$

It implies that

$$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq \gamma S(x_n, x_n, y_n) + (1 - \gamma)\triangle^S_{YZ} + LS(x_n, x_n, x_{n+1}).$$

Taking limit as $n \to \infty$, we have

$$S(x,x,y) \le \gamma S(x,x,y) + (1-\gamma)\triangle^S_{YZ} + LS(x,x,x)$$

yields that

$$S(x,x,y) \le \triangle^S_{YZ}.$$

Then $S(x,x,y) = \triangle^S_{YZ}$, that is, $x \in Y_0$ and $y \in Z_0$. Since $J(Y_0) \subseteq Z_0$ and $T(Z_0) \subseteq Y_0$, then $Jx \in Z_0$ and $Ty \in Y_0$. Hence there exists $w \in Y_0$ and $z \in Z_0$ such that

$$S(gw,gw,Jx) = \triangle^S_{YZ} = S(gv,gv,Ty)$$

since $Y_0 \subseteq g(Y_0)$ and $Z_0 \subseteq g(Z_0)$. Since J is an S-proximal Berinde g-contraction of the first kind and

$$S(gw,gw,Jx) = S(gx_{n+1},gx_{n+1},Jx_n) = \triangle^S_{YZ},$$

we obtain that

$$\begin{aligned} S(gw,gw,gx_{n+1}) &\le \alpha S(gx,gx,gx_n) + L_1 \min\{S(gx,gx,gx_{n+1}), S(gx_n,gx_n,gw)\} \\ &= \alpha S(gx,gx,gx_n) + L_1 S(gx,gx,gx_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$, by the continuity of g, we get $S(gw,gw,gx) = 0$, and so, $gx = gw$. It is implies that

$$S(gx,gx,Jx) = \triangle^S_{YZ}.$$

Similarly, it is easy to verify that $S(gy,gy,Ty) = \triangle^S_{YZ}$. Thus, we can conclude that

$$S(gx,gx,Jx) = S(gy,gy,Ty) = S(x,x,y) = \triangle^S_{YZ}$$

Therefore, the pair (x,y) is a coincidence best proximity point of the triple (g,J,T). Next, we will show that the pair (x,y) is unique. Suppose that $\alpha+L_1<1, \beta+L_2<1$ and there exists $x \ne x^* \in Y$ such that

$$S(gx^*,gx^*,Jx^*) = \triangle^S_{YZ}.$$

Since J is an S-proximal Berinde g-contraction of the first kind, it follows that

$$\begin{aligned} S(gx,gx,gx^*) &\le \alpha S(gx,gx,gx^*) + L_1 \min\{S(gx,gx,gx^*), S(gx^*,gx^*,gx)\} \\ &= (\alpha + L_1)\, S(gx,gx,gx^*). \end{aligned}$$

Since $\alpha + L_1 < 1$, then we have $S(gx,gx,gx^*) = 0$. It follows that $x = x^*$, which implies that there exists a unique $x \in Y$ such that $S(gx,gx,Jx) = \triangle^S_{YZ}$. Similarly, we can show that there exists a unique $y \in Z$ such that $S(gy,gy,Ty) = \triangle^S_{YZ}$. Therefore, the pair (x,y) is the unique coincidence best proximity point of the triple (g,J,T).

Now, we give an example to illustrate Theorem 3.1.

Example 3.1. *Let $(\mathbb{R}^2,d)$ be the Euclidean metric space. Define*

$$S(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}.$$

Then $(\mathbb{R}^2,S)$ is an S-metric space. Let $Y = \{(0,y); -1 \le y \le 1\}$ and $Z = \{(1,y); -1 \le y \le 1\}$. Then $\triangle^S_{YZ} = 1$, $Y_0 = Y$ and $Z_0 = Z$. Define the mappings $J : Y \to Z, T : Z \to Y$ and $g : Y \cup Z \to Y \cup Z$ by

$$J(0,y) = \left(1, \frac{y}{2}\right), T(1,y) = \left(0, \frac{y}{2}\right) \text{ and } g(x,y) = (x,-y).$$

Clearly, $Y_0 = g(Y_0)$, $Z_0 = g(Z_0), J(Y_0) = \left\{\left(1, \frac{y}{2}\right); -1 \leq y \leq 1\right\} \subset Z_0$, $T(Z_0) = \left\{\left(0, \frac{y}{2}\right); -1 \leq y \leq 1\right\} \subset Y_0$ *and the mapping* g *is an isometry. Obviously, the mappings* J *and* T *are* S*-proximal Berinde* g*-contractions of the first kind and the pair* (J, T) *is an* S*-proximal Berinde* g*-cyclic contraction. Hence, the all conditions of Theorem 3.1 are satisfied and the element* $\{(0,0),(1,0)\}$ *in* $Y \times Z$ *is the unique coincidence best proximity point of the triple* (g, J, T)*.*

If we take $L_1 = 0, L_2 = 0$ and $L = 0$ in Theorem 3.1, then we obtain the following coincidence best proximity theorem.

Theorem 3.2. *Let* X, Y, Z, Y_0, Z_0, J, T *and* g *satisfy the hypotheses of Theorem 3.1. Then, there exists a unique point* $x \in Y$ *and there exists a uniqe point* $y \in Z$ *such that*

$$S(gx, gx, Jx) = S(gy, gy, Ty) = S(x, x, y) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$*, the sequence* $\{x_n\}$ *defined by*

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element x*, and for any fixed* $y_0 \in Z_0$*, the sequence* $\{y_n\}$ *defined by*

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element y*.*

If we take $gx = x$ for all $x \in Y \cup Z$ in Theorem 3.1, then we immediately obtain the following theorem.

Theorem 3.3. *Let* X, Y, Z, Y_0, Z_0, J *and* T *satisfy the hypotheses of Theorem 3.1. Then, there exists a point* $x \in Y$ *and there exists a point* $y \in Z$ *such that*

$$S(x, x, Jx) = S(y, y, Ty) = S(x, x, y) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$*, the sequence* $\{x_n\}$ *defined by*

$$S(x_{n+1}, x_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element x*, and for any fixed* $y_0 \in Z_0$*, the sequence* $\{y_n\}$ *defined by*

$$S(y_{n+1}, y_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element y*. In addition, if* $\alpha + L_1 < 1$ *and* $\beta + L_2 < 1$*, then the point* x *and* y *is the unique best proximity point of* J *and* T*, respectively.*

If we suppose that J and T are continuous mappings instead of the condition (iv) in Theorem 3.1, then we obtain the following theorem.

Theorem 3.4. *Let* (X, S) *be a complete* S*-metric space and* Y*,* Z *be two nonempty closed subsets of* X*. Let* $J : Y \to Z$*,* $T : Z \to Y$ *and* $g : Y \cup Z \to Y \cup Z$ *satisfy the following conditions:*

(i) J *and* T *are* S*-proximal Berinde* g*-contractions of the first kind, i.e., there exist* $\alpha, \beta \in [0, 1)$ *and* $L_1, L_2 \geq 0$ *such that* J *and* T *satisfy the condition (3.1), respectively;*

(ii) J *and* T *are continuous mappings such that* $J(Y_0) \subseteq Z_0$ *and* $T(Z_0) \subseteq Y_0$*;*

(iii) g *is an isometry with* $\emptyset \neq Y_0 \subseteq g(Y_0)$ *and* $Z_0 \subseteq g(Z_0)$*.*

Then, there exists a point $x \in Y$ *and there exists a point* $y \in Z$ *such that*

$$S(gx, gx, Jx) = S(gy, gy, Ty) = \triangle^S_{YZ}. \tag{3.5}$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element x, and for any fixed $y_0 \in Z_0$, the sequence $\{y_n\}$ defined by

$$S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \textit{for all } n \in \mathbb{N}$$

converges to the element y. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the there exists unique element x and there exists unique element y which satisfy the equation (3.5).

Proof. By the proof of Theorem 3.1, we get that the sequences $\{x_n\}$ in Y_0 and $\{y_n\}$ in Z_0 such that

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = S(gy_{n+1}, gy_{n+1}, Ty_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

converge to some elements $x \in Y$ and $y \in Z$, respectively. Since J, T and g are continuous mappings, then we have that $Jx_n \to Jx, Ty_n \to Ty$ and $gx_{n+1} \to gx, gy_{n+1} \to gy$. Taking limit in (3.6) as $n \to \infty$, we conclude that

$$S(gx, gx, Jx) = S(gy, gy, Ty) = \triangle^S_{YZ}.$$

The proof of uniqueness of the elements x and y follows as in Theorem 3.1.

Next, we establish a coincidence best proximity point result for an S-proximal Berinde g-contraction of the first kind and second kind in an S-metric space.

Theorem 3.5. *Let (X, S) be a complete S-metric space and Y, Z be two nonempty closed subsets of X. Let $J : Y \to Z$ and $g : Y \to Y$ satisfy the following conditions:*

(i) J is an S-proximal Berinde g-contraction of the first kind and second kind, i.e., there exist $\alpha, \beta \in [0, 1)$ and $L_1, L_2 \geq 0$ such that J satisfies the conditions (3.1) and (3.2), respectively;

(ii) J preserves the isometric distance with respect to g and $J(Y_0) \subseteq Z_0$;

(iii) g is an isometry with $\emptyset \neq Y_0 \subseteq g(Y_0)$.

Then, there exists a point $x \in Y$ such that

$$S(gx, gx, Jx) = \triangle^S_{YZ}. \tag{3.7}$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

converges to the element x. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the there exists unique element x which satisfy the equation (3.7).

Proof. Following similar arguments to those given in proof of Theorem 3.1, we deduce that the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

is convergent to some $x \in Y$. Since J is an S-proximal Berinde g-contraction of the second kind and preserves the isometric distance with respect to g, then we have

$$\begin{aligned}
& S(Jx_n, Jx_n, Jx_{n+1}) \\
= \; & S(Jgx_n, Jgx_n, Jgx_{n+1}) \\
\leq \; & \beta S(Jgx_{n-1}, Jgx_{n-1}, Jgx_n) \\
& + L_2 \min\{S(Jgx_{n-1}, Jgx_{n-1}, Jgx_{n+1}), S(Jgx_n, Jgx_n, Jgx_n)\} \\
= \; & \beta S(Jgx_{n-1}, Jgx_{n-1}, Jgx_n) \\
= \; & \beta S(Jx_{n-1}, Jx_{n-1}, Jx_n).
\end{aligned}$$

Similarly, in the proof of Theorem 3.1, we can show that $\{Jx_n\}$ is a Cauchy sequence and converges to some element $y \in Z$. Therefore we can conclude that

$$S(gx, gx, y) = \lim_{n\to\infty} S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ},$$

that is, $gx \in Y_0$. Since $Y_0 \subseteq g(Y_0)$, there exists $z \in Y_0$ such that $gx = gz$ and so $S(gx, gx, gz) = 0$. By the fact that g is an isometry, we get $S(x, x, z) = S(gx, gx, gz) = 0$. Hence $x = z \in Y_0$ and so $Jx \in J(Y_0) \subseteq Z_0$. Then there exists $u \in Y_0$ such that

$$S(gu, gu, Jx) = \triangle^S_{YZ}. \tag{3.8}$$

It follows from J being an S-proximal Berinde g-contraction of the first kind that

$$\begin{aligned} S(gu, gu, gx_{n+1}) &\leq \alpha S(gx, gx, gx_n) + L_1 \min\{S(gx, gx, gx_{n+1}), S(gx_n, gx_n, gu)\} \\ &\leq \alpha S(gx, gx, gx_n) + L_1 S(gx, gx, gx_{n+1}) \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in (3.9), we conclude that $gu = gx$. Therefore, from (3.8), we have

$$S(gx, gx, Jx) = \triangle^S_{YZ},$$

that is, x is a coincidence best proximity point of the pair (g, J). The proof of uniqueness of the element x follows as in Theorem 3.1.

The following example illustrates the preceding coincidence best proximity point theorem.

Example 3.2. *Let $X = \mathbb{R}$ and $S(x, y, z) = \max\{|x-y|, |y-z|, |z-x|\}$. Then $(\mathbb{R}, S)$ is an S-metric space. Let $Y = [-2, 2]$ and $Z = \{-3\} \cup [3, 4]$. Then $\triangle^S_{YZ} = 1$, $Y_0 = \{-2, 2\}$ and $Z_0 = \{-3, 3\}$. Define the mappings $J : Y \to Z$ and $g : Y \to Y$ by*

$$Jx = \begin{cases} 3, & \text{if } x \text{ is rational} \\ 4, & \text{otherwise} \end{cases} \quad \text{and} \quad gx = -x.$$

Clearly, $Y_0 = g(Y_0), J(Y_0) = \{3\} \subset Z_0$ and the mapping g is an isometry. Obviously, the mapping J preserves the isometric distance with respect to g and it is an S-proximal Berinde g-contraction of the first kind and second kind. Thus, the all conditions of Theorem 3.5 are fulfilled and the element -2 in Y is the unique coincidence best proximity point of the pair (g, J).

If we take $L_1 = 0$ and $L_2 = 0$ in Theorem 3.5, then we obtain the following coincidence best proximity theorem.

Theorem 3.6. *Let X, Y, Z, Y_0, Z_0, J and g satisfy the hypotheses of Theorem 3.5. Then, there exists a unique point $x \in Y$ such that*

$$S(gx, gx, Jx) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(gx_{n+1}, gx_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

converges to the element x.

If we take $gx = x$ for all $x \in Y$ in Theorem 3.5, then we get the following theorem.

Theorem 3.7. *Let X, Y, Z, Y_0, Z_0 and J satisfy the hypotheses of Theorem 3.5. Then, there exists a point $x \in Y$ such that*

$$S(x, x, Jx) = \triangle^S_{YZ}.$$

Moreover, for any fixed $x_0 \in Y_0$, the sequence $\{x_n\}$ defined by

$$S(x_{n+1}, x_{n+1}, Jx_n) = \triangle^S_{YZ}, \quad \forall n \in \mathbb{N}$$

converges to the element x. In addition, if $\alpha + L_1 < 1$ and $\beta + L_2 < 1$, then the element x is the unique best proximity point of J.

Remark. *Since both the proximal contraction of the first kind and the proximal Berinde g-contraction of the first kind are special cases of the S-proximal Berinde g-contraction of the first kind, Theorems 3.1-3.3 generalize the corresponding results for both the proximal contraction and the proximal Berinde g-contraction of the first kind. Same is the case for Theorems 3.5-3.7 dealing with the S-proximal Berinde g-contraction of the first kind and second kind.*

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ON MULTISET MINIMAL STRUCTURE TOPOLOGICAL SPACE

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Abstract. In this article we have established the concept of multi-continuity in minimal structure spaces (in short $\mathscr{M}$ space) and the notion of product minimal space in Multiset topological space. Continuity between $\mathscr{M}$-space, generalized Multiset topology and Multiset ideal topological spaces. We have investigated some basic properties of $\mathscr{M}$–continuity in Multiset topological space, such as composition of $\mathscr{M}$–continuous functions, product of $\mathscr{M}$–continuous functions in product Multiset topological space etc.

1. Introduction

Cantor's set is not enough for representing the all kind of situations of our real world. In Cantor's set theory, repetition of elements is not allowed. however, there are many situations where repetition of elements plays a vital role. This led the introduction of the theory of the notion of Multisets, which was first studied by Blizard [1] in the year 1989. Thus, a Multiset is a collection of elements in which certain elements may occur more than once and number of times an element occurs is called its multiplicity.

In this article our aim is to study the properties of continuous function on Multiset minimal space and Multiset generalized topological spaces. Minimal stricture space is the minimum restriction for the topology by containing empty set and whole set. Many authors have studied in the direction of Multiset ideal and generalized Multiset topological spaces. This work aim is relating the Multiset ideal, Multiset filter and generalized Multiset topological space

In 1991 Bilzard[2] designed the Multiset theory and further developed the Multiset theory in 1989. Many researchers have defined the Multiset topological spaces; one may refer to New axioms in topological spaces[4], Separation axioms on Multiset topological Space [5], Relations and functions in Multiset context [6]. There after different properties of Multiset topological space, such as compactness studied by Mahanta and Samanta [9], Multiset quasicoincidence studied by Shravan and

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Tripathy Multiset quasiconcidence between Multisets, continuous function on Multisets, generalized closed Multiset [12, 13, 14, 15]. Multiset mixed topological space between two Multisets studied by Tripathy and Das [16].
In this article we have established many results between Multiset minimal stricture space and Multiset ideal topological space.
We define different types of continuous function between two multiset generalized topological space, Multiset ideal and generalized topological space. By the results of this article we can study the ideal structure and generalized topological structure at a time, and we can can find the similarity between the topological structures.
This paper is organized in the following way:
Section-1 is the introduction part. In this section, author focused on the previous work and back ground of the research. In the Section-2, author provides some preliminary results and definitions for the article which is necessary for the work. Section-3 is the main section in which we established main results of this article. In Section-4, we analyzed the continuous function from Multiset minimal stricture to generalized Multiset topology and Multiset ideal topological spaces. Section-5 is the conclusion section in which the future plane and the application field have been analyzed by the author.

2. Preliminaries and Definitions

In this section, we provide some basic definitions and notations those will be used throughout this article.

A Multiset (M-set) with domain set X, in which no element occurs more than m times is denoted by $[X]^m$. The count function C_m on X represents the repetition of an element, denoted by $C_m(x)$, for $x \in X$. When $C_m(x) = 1$, for all $x \in X$, then the Multiset becomes a Cantor's set.

Thought the articles we shall use the definition of Multiset mixed topological space (Shravan and Tripathy [13]) and ultra-Separation Axioms in Generalized Topological Space (Powar and Rajak [10]) for the union, intersection, compliment, support set, empty set, equality of M-sets, partial whole sub-M-sets etc.

Definition 2.1. A domain X is defined as a set of elements from which M-sets are constructed. The M-set space $[X]^m$ is the set of all M-set whose elements are in X such that no element in the M-set occurs more than m times. The set $[X]^\infty$ is the set of all M-sets over a domain X such that there is no limit on the number of occurrences of an element in an M-set.
Let $P, N \in [X]^m$. Then, the following relations between M-sets are defined:

(1) P is a sub-M-set of N denoted by $P \subseteq N$, if $C_P(x) \leq C_N(x)$ for all $x \in X$.

(2) $P = N$ if $P \subseteq N$ and $N \subseteq P$.

(3) P is a proper sub-M-set of N denoted by $P \subset N$, if $C_P(x) \leq C_N(x)$, for all $x \in X$ and there exists at least one element $x \in X$ such that $C_P(x) < C_N(x)$.

(4) $Q = P \cup N$, if $C_Q(x) = max\{C_P(x), C_N(x)\}$, for all $x \in X$.

(5) $Q = P \cap N$, if $C_Q(x) = min\{C_P(x), C_N(x)\}$, for all $x \in X$.

(6) Addition of P and N is also a new M-set $Q = P \bigoplus N$ such that $C_Q(x) = min\{C_P(x) + C_N(x), m\}$, for all $x \in X$.

(7) Subtraction of P and N results is also an M-set $Q = P \ominus N$ such that $C_Q(x) = max\{C_P(x) - C_N(x), 0\}$, for all $x \in X$, where $\bigoplus$ and $\ominus$ represent M-set addition and M-set subtraction, respectively.

(8) An M-set P is empty if $C_P(x) = 0$, for all $x \in X$.

(9) The support set of P denoted by P^* is a subset of X and $P^* = \{x \in X : C_P(x) > 0\}$; that is, P^* is an ordinary set and it is also called root set.

(10) The cardinality of an M-set P drawn from a set X is $Card(P) = \sum_{x \in X} C_P(x)$.

(11) P and N are said to be equivalent if and only if $Card(P) = Card(N)$.

Definition 2.2. Let $M \in [X]^m$ and $N \subseteq M$. Then, the complement N^c of N in $[X]^m$ is an element of $[X]^m$ such that $N^c = M - N$.

Definition 2.3. Let $M \subseteq [X]^m$ and $P^*(M)$. Then τ is called a Multiset topology of M if satisfies the following properties,
1. The whole M-set M and the empty M-set $\emptyset$ are in τ.
2. The M-set union of elements of any sub-collection of τ is in τ.
3. The M-set intersection of the elements of any finite sub-collection of τ is in τ.
Then, the pair (M, τ) is called an Multiset topological space (M-topological space). The elements of τ are called open M-sets. The complement of an open M-set in (M, τ) is said to be closed M-set.

Definition 2.4. Given a sub-M-set N of M-topological space. Then, the interior of N is denoted by $int(N)$ and is defined as the M-set union of all open M-sets contained in N, i.e., $C_{int(N)}(x) = max\{C_G(x) : G \subseteq N\}$.

Definition 2.5. Given a sub-M-set A of an M-topological space (M, τ). Then, the closure of A is defined as the M-set intersection of all closed M-sets containing A and is denoted by $cl(A)$, i.e., $C_{cl(A)}(x) = min\{C_K(x) : A \subseteq K\}$.

Definition 2.6. Let (M, τ) be an M-topological space, and M_1 is a sub-M-set of M. The collection $\tau_N = \{U^{'} = U \cap M_1 : U \in \tau\}$ is an M-topology on M_1, called the subspace M-topology. With this M-topology, M_1 is called a subspace of M and its open M-sets consisting of all M-set intersections of open M-sets of M with M_1.

Definition 2.7. A non-empty collection I of sub-M-sets of a non-empty M-set M is said to be an M-set ideal on M, if it satisfies the following conditions:

(i). $N_1 \in I$ and $N_2 \subseteq N_1$ with $C_{N_2}(x) \leq C_{N_1}(x)$, for all $x \in X \rightarrow N_2 \in I$.
(ii). $N_1 \in I, N_2 \in I \rightarrow N_1 \cup N_2 \in I$. The M-set ideal is abbreviated as M-ideal. The triplet (M, τ, I) is called Multiset ideal topological space with the ideal I and Multiset topology τ.

Definition 2.8. Let $[X]^w$ be a space of M-sets. A Multipoint is a M-set M in X such that $C_M(x) = \begin{cases} k, for x \in M; \\ 0, otherwise, \end{cases}$
where $k \in \{1, 2, 3, ...w\}$ and $C_M(x)$ is the multiplicity of x in X.

A multipoint, denoted by $\{k/x\}$ is a subset of a M-set M or $\{k/x\} \in M \; if \; 0 \leq k \leq C_M(x)$ and Singleton sub-M-set if $k = C_M(x)$, for all $x \in X$.

Let (M, τ) be a M-topological space and I be an M-ideal on M. Let N be a sub-M-set of M. Then, the local function denoted by $N^*(I, \tau)$ is defined by, $N^*(I, \tau) = \{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_I(x_j), I \in I$, for all $U \in N_q(m_i/x_i)$ and at least one $x_j \in X\}$, where $N_q(m_i/x_i)$ is the set of $q - nbhd$ of m_i/x_i. We will write $N^*(I)$ or N^* in place of $N^*(I, \tau)$.

Definition 2.9. Let M be any non-empty M-set and τ be the collection of subsets of the M-set M. the pair (M, τ) is said to be a generalized M-set topological space if the following property holds
1. $M, \emptyset \in \tau$.
2. If $H, G \in \tau$ then $H \cap G \in \tau$.
3. If $u_{i\in\Lambda} \in \tau$ then $\cup_{i\in\Lambda} u_i \in \tau$.

Note 2.1. The generalized M-set topological space is the generalized form of M-set topology. Sometimes, we denote generalized M-set topology by $(M(N), \tau)$, where $\cup_{i\in\Lambda} u_i = N$.

3. Main Results.

In this section we established a topology from the M-set minimal structure, and study different properties on multi-continuity between the different types of M-set topological space and M-set minimal space.

Definition 3.1. A family $\mathscr{M} \subseteq 2^X(\mathscr{P}(X)$ is said to be M-set minimal structure on X if $\emptyset, X \in \mathscr{M}$.

In this case $(M, \mathscr{M})$ is called a M-set minimal space. Throughout this paper $(X, \mathscr{M})$ means M-set minimal space. The M-set minimal space is abbreviated as M-minimal space.

Definition 3.2. A M-minimal space is called an M-set minimal topological space if it satisfy the properties of finite intersection and arbitrary union property.

Example 3.1. Let M be a nonempty M-set on $[X]^w$. Then, the filter $\mathscr{F}$ and the ideal do not form a M-set minimal structure on X. Since, $\mathscr{F}$ does not contain emptyset $\emptyset$ and ideal does not contain the whole set M.

Definition 3.3. A set $A \in P^*(X)$ is said to be a $\mathscr{M}$–open set if for $A \in \mathscr{M}$, $B \in P^*(X)$ is a $\mathscr{M}$–closed set if $M \ominus B \in \mathscr{M}$. We get
$\mathscr{M} - int(A) = \cup\{u : u \subseteq A, u \in \mathscr{M}\}$.
$\mathscr{M} - cl(A) = \cap\{F : A \subseteq F, M \ominus F \in \mathscr{M}\}$.

In view of above definition we formulate the following proposition:

Proposition 3.1.
1. $\mathscr{M} - int(A)$ is the largest $\mathscr{M}$-open M-set contained in A.
2. $\mathscr{M} - cl(A)$ is the smallest $\mathscr{M}$-closed M-set containing A.

Definition 3.4. Let $(M, \mathscr{M})$ be a M-set $\mathscr{M}$-space on $[X]^w$, and N be a sub-M-set of M. We define the following
(i) A $\mathscr{M}$-semi-open M-set if $N \subseteq Cl(int(N))$ with $C_N(x) \leq C_{cl(int(N))}(x)$, for all $x \in X$;
(ii) A $\mathscr{M}$-semi-closed M-set if $int(cl(N)) \subseteq N$ with $C_{int(cl(N))}(x) \leq C_N(x)$, for all $x \in X$;
(iii) A $\mathscr{M}$-semi-pre-open M-set if $N \subseteq Cl(int(cl(N)))$ with $C_N(x) \leq C_{cl(int(cl(N)))}(x)$, for all $x \in X$;
(iv) A $\mathscr{M}$-semi-pre closed M-set if $int(cl(int(N))) \subseteq N$ with $C_{int(cl(int(N)))}(x) \leq C_N(x)$, for all $x \in X$;
(v) A $\mathscr{M}$-pre-open M-set if $N \subseteq int(cl(N))$ with $C_N(x) \leq C_{int(cl(N)))}(x)$, for all $x \in X$.

Theorem 3.1. Let $(M, \mathscr{M})$ be a M-minimal space. Then, for $N, K \in P^*(X)$,

1. $\mathscr{M} - int(N) \subseteq N$ and $\mathscr{M} - int(N) = N$ iff N is an $\mathscr{M}$–open M-set.

2. $\mathscr{M} - cl(N) \subseteq N$ iff N is an $\mathscr{M}$–closed M-set.

3. $\mathscr{M} - int(N) \subseteq \mathscr{M} - int(K)$ and $\mathscr{M} - cl(N) \subseteq \mathscr{M} - cl(K)$ if $N \subseteq K$.

4. $\mathscr{M} - int(N \cap K) \subseteq (\mathscr{M} - int(N)) \cap (\mathscr{M} - int(K))$ and $(\mathscr{M} - int(N)) \cup (\mathscr{M} - int(K)) \cup \mathscr{M} - int(N \cup K)$.

5. $\mathscr{M} - cl(N \cup K) \subseteq (\mathscr{M} - cl(N)) \cup (\mathscr{M} - cl(K))$ and $\mathscr{M} - cl(N \cap K) \subseteq (\mathscr{M} - cl(N)) \cup (\mathscr{M} - cl(K))$.

6. $\mathscr{M} - int(\mathscr{M} - int(N)) = \mathscr{M} - int(N)$ and $\mathscr{M} - cl(\mathscr{M} - cl(N)) = \mathscr{M} - cl(N)$.

7. $x \in \mathscr{M} - cl(N)$ if and only if every $\mathscr{M}$–open M-set U containing x such that $U \cap N \neq \emptyset$.

8. $(M \ominus \mathscr{M} - cl(N)) = \mathscr{M} - int(M \ominus N)$ and $(M \ominus \mathscr{M} - int(N)) = \mathscr{M} - Cl(M \ominus N)$.

Proof (1) $\mathscr{M} - int(N)$ is the largest open M-set contained in N. So the proof is clear. Now, let us consider N is $\mathscr{M}$-open M-set. Then, $N \subseteq \mathscr{M} - int(N)$. Therefore, $N = \mathscr{M} - int(N)$.
Let us consider $N = \mathscr{M} - int(N)$. Again $\mathscr{M} - int(N)$ is the largest open M-set contained in N. Hence, N is $\mathscr{M}$-open M-set.

(2). Let $\mathscr{M} - cl(N)$ be a small closed M-set containing N. Therefore, $N \subseteq \mathscr{M} - cl(N)$. Again, N is $\mathscr{M}$-closed M-set. This implies, the closure of all multipoint of N contained in N with $C_{cl(N)}(x) \leq C_N(x)$, for all $x \in N$. Hence the theorem proved.

(3) From the definition we have $\mathscr{M} - int(N) \subseteq N$ and $\mathscr{M} - int(K) \subseteq K$ for any M-set $N, K \subseteq M$. Given that $N \subseteq K$ this implies, $\mathscr{M} - int(N) \subseteq N \subseteq K$ So, $\mathscr{M} - int(N) \subseteq K$. By the definition of M-set interior point set is the largest open M-set of the M-set contained in the M-set. Hence, $\mathscr{M} - int(K)$ is the largest open M-set in K. Again, $\mathscr{M} - int(N)$ is an open M-set contained in K. So the only possible case $\mathscr{M} - int(N) \subseteq \mathscr{M} - int(K)$.

(4) For any two M-sets N and K, we have $N \cap K \subseteq N$ and $N \cap K \subseteq K$. Using the above results we have, $\mathscr{M} - int(N \cap K) \subseteq \mathscr{M} - int(N)$ and $\mathscr{M} - int(N \cap K) \subseteq \mathscr{M} - int(K)$. This implies, $\mathscr{M} - int(N \cap K) \subseteq (\mathscr{M} - int(N)) \cap (\mathscr{M} - int(K))$.
Again, for any two M-set N and K, we have $N \subseteq N \cup K$ and $K \subseteq N \cup K$. This implies, $\mathscr{M} - int(N) \subseteq \mathscr{M} - int(N \cup K)$ and $\mathscr{M} - int(K) \subseteq \mathscr{M} - int(N \cup K)$. Hence, $(\mathscr{M} - int(N)) \cup (\mathscr{M} - int(K)) \subseteq \mathscr{M} - int(N \cup K)$.

(5) We have, $N \subseteq (m - cl(N))$, $K \subseteq (\mathscr{M} - cl(K))$. Therefore, $N \cup K \subseteq (\mathscr{M} - cl(N)) \cup (\mathscr{M} - cl(K))$ and so $\mathscr{M} - cl(N \cup K) \subseteq \mathscr{M} - cl((\mathscr{M} - cl(N)) \cup (\mathscr{M} - cl(K))) = (\mathscr{M} - cl(N)) \cup \mathscr{M} - cl(K))$.
Hence, $m - cl(N \cup K) \subseteq (m - cl(N)) \cup \mathscr{M} - cl(K))$ as union of two $\mathscr{M}$-closed M-sets is also a $\mathscr{M}$-closed M-set and $\mathscr{M}$-closure of a $\mathscr{M}$-closed M-set is also a $\mathscr{M}$-closed M-set.
Similarly, for the $\mathscr{M} - cl(N \cap K) \subseteq (\mathscr{M} - cl(N)) \cap (\mathscr{M} - cl(K))$.

For (6), (7) and (8), the results are holds using the definition and above proofs.

Lemma 3.1. A sub-M-set N in a M-set minimal space $(M, \mathscr{M})$ is said to be $\mathscr{M}$-semi-pre-closed M-set if and only if $N = spcl(N)$.

Definition 3.5. A sub-M-set N of a M-set topological space (M, τ) is called as
(1) M-set generalized semi-closed (briefly mgs-closed) set if $msCl(N) \subseteq U$ whenever $N \subseteq U$ and U is open M-set in generalized $\mathscr{M}$- space.
(2) M-set generalized minimal semi-pre closed (briefly $\mathscr{M} - mgsp$-closed) set if $spcl(N) \subseteq U$ whenever $N \subseteq U$ and U is $\mathscr{M}$-open M-set in generalized M space.

Definition 3.6. Let (M_1, τ_1) and (M_2, τ_2) be two M-set minimal spaces. Then, a function $f : M_1 \to M_2$ is called as

(i) multi-semi-continuous if the inverse image of each $\mathscr{M}$-open M-set of M_2 is $\mathscr{M}$-semi-open M-set in M_1.

(ii) multi minimal semi pre-continuous if the inverse image of each m-open M-set of M_2 is $m-$semi-pre open M-set in M_1.

(iii) multiset minimal g-continuous if the inverse image of each m-open M-set of M_2 is $\mathscr{M} - mg$-open M-set in M_1.

(iv) multiset minimal gp-continuous if the inverse image of each m-open M-set of M_2 is $\mathscr{M} - mgp$-open M-set in M_1.

(v) multiset minimal gsp-continuous if the inverse image of each m-open M-set of M_2 is $\mathscr{M} - gsp$-open M-set in M_1.

(vi) multiset minimal pre-continuous if the inverse image of each m-open M-set of M_2 is m-pre open M-set in M_1.

Definition 3.7. Let (M, I, τ) be an M-set ideal topological space on $[X]^w$. A subset A of M-set ideal space is said to be pre-I-open M-set if $A \subseteq int(cl(A^*))$ with $C_A(x) \leq C_{int(cl(A^*))}(x)$ for all $x \in X$. The compliment of pre-I-open M-set is called pre-I-closed M-set.

Definition 3.8. Let (M, I, τ) be an M-set ideal topological space in $[X]^W$. A sub M-set A of M is called a semi-I-open M-set if $A \subseteq cl(int(A^*))$ with $C_A(x) \leq C_{cl(int(A^*))}(x)$, for all $x \in X$. The complement of a semi-open M-set is called a $semi - I - closed$ M-set.

Proposition 3.2. Let (M_1, τ_1) and (M_2, τ_2) be two M-set minimal spaces on M. A function $f : M_1 \to M_2$ be a bijective mapping and $\{U_i : i \in \Delta\}$ be a family of $\mathscr{M}$-open M-sets of M_2. Then, we have
(i) $f^{-1}(\cup_{i\in\Delta} U_i) = \cup_{i\in\Delta} f^{-1}(U_i)$.
(ii) $f^{-1}(\cap_{i\in\Delta} U_i) = \cap_{i\in\Delta} f^{-1}(U_i)$.

Lemma 3.2. Every M-set topological space is a M-set minimal space, but the converse is not necessary.

Lemma 3.2. Every M-set minimal space is a generalized M-set minimal space, but the converse is not necessary.

4. Properties of $\mathscr{M} - msgp$ Generalized Multiset and Ideal-Continuous Functions

We state the following two results without proof, which follow on using standard theory.

Theorem 4.1. Let $f : (M_1, \tau_1) \to (M_2, \tau_2)$ and $g : (M_2, \tau_2) \to (M_3, \tau_3)$ be any two maps. Then, $g \circ f$ is $\mathscr{M}$-continuous if g is $\mathscr{M}$-continuous and f is $\mathscr{M}$-continuous.
Proof: The proof is so easy, so omitted.

Proposition 4.1.. Let (M_1, τ) be M-set minimal space, and $M_2 \subseteq M_1$. Then, $(M_2, \tau \cap M_2)$ will be an M-set minimal structure. Further, for $M_2 \subseteq M_1$, $(M_2, \tau \cap M_2)$ is a weaker M-set minimal structure space.

Definition 4.1. Let (M_1, τ_1) and (M_2, τ_2) be two M-set minimal spaces. A function $f : M_1 \to M_2$ is called $\mathscr{M}mgsp$-continuous if $f^{-1}(N)$ if $\mathscr{M} - mgsp$-closed in M_1 for every closed M-set N of M_2.

Lemma 4.1.

1. Every g-continuous function is $\mathscr{M} - mgsp$-continuous function.

2. Every $\mathscr{M}-$ pre-continuous function is generalized semi-continuous function.

Theorem 4.2. Let (M_1, τ_1) be a M-set minimal space, and (M_2, τ_2) be a generalized M-set topology. If the bijective function $f : M_1 \to M_2$ is $\mathscr{M}$-semi-pre continuous and $\mathscr{M}$-open M-set, then f is $\mathscr{M}-$ pre-continious.
Proof: Let N be $\mathscr{M}$-closed in M_2 and let $f^{-1}(N) \subseteq K$, where K is $\mathscr{M}$-open set in M_1. Clearly, $N \subseteq f(K)$. Since $f(K)$ is open M-set in M_2 as f is open, and as N is $\mathscr{M}$-closed in M_2, then $\mathscr{M} - spCl(N) \subseteq f(K)$ and thus $f^{-1}(spCl(N)) \subseteq K$. Since f is bijective and $\mathscr{M} - mspCl(N)$ is a $\mathscr{M}$ semi-pre closed M-set, then $f^{-1}(spCl(N))$ is $\mathscr{M}$-semi-pre closed M-set in M_1. Thus, $mspCl(f^{-1}(N)) \subseteq spCl(f^{-1}(mspCl(N)))$ $= f^{-1}(mspCl(N)) \subseteq K$. So, $f^{-1}(N)$ is $mgsp$-closed set and f is $\mathscr{M}-$-pre-continious.

Theorem 4.3. Let $f : M_1 \to M_2$ be a pre-$\mathscr{M} - mgsp$-continuous and $g : M_2 \to M_3$ is $\mathscr{M}$-semi-pre-continuous. Then, their composition $g \circ f$ is $\mathscr{M}mgsp$-continuous.
Proof: The proof is straight forward, so omitted.

Definition 4.2. A function $f : M_1 \to M_2$ is called strongly $\mathscr{M} - mgsp$-continuous if the inverse image of every $\mathscr{M}$-open M-set of M_2 is $\mathscr{M}$-open M-set in M_1.

Lemma 4.2. every strongly $\mathscr{M} - mgsp$-continuous function is an $\mathscr{M}$-continuous.

Definition 4.3. A function $f : M_1 \to M_2$ is called strongly $\mathscr{M}$-continuous if the inverse image of every sub-M-set in M_2 is $\mathscr{M} - cl - open$ in M_1.

On the basis of the above definition we give the following results:

Theorem 4.4. If the function $f : M_1 \to M_2$ is strongly $\mathscr{M}$-continuous, then f is strongly $\mathscr{M} - mgsp$-continuous.

Theorem 4.5 If $f : M_1 \to M_2$ be an $m - I - gn$-continuous function from an M-set ideal topological space to another M-set minimal space, then the following are equivalent.

1. for every local function A^* in M_1 there exist an M-set local function $f(A^*)$ in M_2.

2. For the M-set ideal I there exist an ideal $f(I)$ in M_2.

3. For every $mgsp$-closed (Open) set in M_2 there exist an multiset-semi-pre-ideal closed ($\mathscr{M}$-open) set in M_1.

Theorem 4.6. If the function $f : M_1 \to M_2$ be $m - I$ -continuous and the function $g : M_2 \to M_3$ be $\mathscr{M}$-continuous, then $g \circ f : M_1 \to M_3$ is $\mathscr{M}$-continuous.

Proof: Let N be an $\mathscr{M}$-open M-set in M_3. Since, g is $\mathscr{M}$-continuous, so $g^{-1}(N)$ is an $m - I$-semi-pre-open set in M_2. Again, f is an $m - I$ -continuous. Hence, $f^{-1}(g^{-1}(N))$ is an $m - I$-open M-set in M_1. But, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$. So $g \circ f$ is an $\mathscr{M}$-continuous.

5. Conclusion

In this article, we introduced the notion of M-set minimal space and M-set minimal topological spaces. Besides, we defined different types of M-set minimal continuous function between two M-set minimal space. M-set minimal continuous functions can help to study the structure of M-set minimal spaces. The common property between two M-set minimal structure can be analze by the continious functions. Many interesting results can established between M-set ideal, generalized M-set topological and M-set minimal structure spaces. It is hoped that, in future many new investigations can be done in this direction.

Conflict of Interest: The authors declare that there is no conflict of interest.

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THE HASSE-MINKOWSKI THEOREM AND LEGENDRE'S THEOREM FOR QUADRATIC FORMS IN TWO AND THREE VARIABLES

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Abstract. Determining the solvability of equations has been an extended and fundamental study in Mathematics. The local-global principle states two objects are equivalent globally if and only if they are equivalent locally at all places. By applying this principle, the Hasse - Minkowski theorem is able to identify the existence of rational solutions of an equation. This paper explores the applications of the Hasse-Minkowski theorem to homogeneous quadratic forms in two and three variables. After providing some of the necessary proofs and definitions, we have been able to introduce some complete computer programs implementing the Hasse-Minkowski theorems and Legendre theorem with some supporting functions like the Eratosthenes sieve.

Reasons for Retraction. Our paper was hugely inspired by Dr. Hohner's master thesis, "The Hasse-Minkowski Theorem in Two and Three Variables." More than half the length of our paper is our original programming implementation of various theorems, like the Hasse-Minkowski theorem and Legendre's theorem, and many supporting concepts, along with the algorithm analysis. We also shorten many proofs from Dr. Hohner's paper by either providing an alternative shorter version or summarizing them. We credit him in section 1 on the binary and ternary quadratic form and the bibliography. However, the location of the credit section 1 was supposed to be before section 1, and this is a formatting mistake. Even though we made an effort to credit Dr. Hohner's work, it could still be insufficient. We think it would be best to retract the paper for those listed reasons.

1. Binary and ternary quadratic form

What follows has been inspired by *The Hasse-Minkowski Theorem in Two and Three Variables* by Hoehner, S [1].
A quadratic form is a polynomial with all the terms of degree two. The 2-variable quadratic form, which is also called binary form, has the following general form:

$$q(x, y) = ax^2 + bxy + cy^2. \tag{1.1}$$

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Similarly, the 3-variable quadratic form is called the ternary form and has the general form of:

$$q(x, y, z) = ax^2 + bxy + cy^2 + dyz + ez^2 + fxz. \tag{1.2}$$

Theorem 1.1. *Every quadratic form q in n variables over a field of characteristic not equal to 2 is equivalent to a diagonal form:*

$$q(x) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2. \tag{1.3}$$

Since the general form is equivalent to diagonal form, we only need to consider the diagonal form to determine the integral solvability. Hence, we just need to look at the equations of form $q(x, y) = ax^2 + by^2$ for the binary case and $q(x, y) = ax^2 + by^2 + cz^2$, where a, b and c are integers.

Consider the binary diagonal form. If we have any rational coefficient, by the homogeneity of the equation $g(x, y) = 0$, we could clear the denominators to obtain an equation with integral coefficients. We also claim that the greatest common divisor of a and b is 1. Given that $gcd(a, b) = g$ and $g > 1$, we could divide $ax^2 + by^2 = 0$ by g to get $q(x, y) = \frac{a}{g}x^2 + \frac{b}{g}y^2$ and obtain $gcd(\frac{a}{g}, \frac{b}{g}) = 1$.

Also, we assume that a and b are square-free. If a is not square-free, $a = a's^2$, where a' is an integer. Then, we have $a = ax^2 + by^2 = a'(sx)^2 + by^2 = 0$. We could repeat the same process to clear all the squares from a and b which eventually leads to square-free coefficients.

Finally, we claim that $ab < 0$. If $ab = 0$, either one or both of the coefficients is 0 and we could not obtain a non-trivial solution. And, if $ab > 0$, the equation $f(x, y) = ax^2 + by^2$ will not have any solution since it would be either negative or positive.

Similarly, following the same reasoning, we get pairwise relatively prime, square-free coefficients for ternary form.

2. MODULAR ARITHMETIC

Definition 2.1. *An integer is called a quadratic residue modulo n if there exists an integer x such that*

$$x^2 \equiv q \ (mod\ n). \tag{2.1}$$

Due to the Legendre symbol, we could speed up the process of determining if a number is a quadratic residue modulo an odd prime. The Legendre symbol is defined as below.

Definition 2.2. *The Legendre symbol is a function of a and p, where p is an odd prime, defined as:*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \textit{if a is a quadratic residue modulo p and } a \not\equiv 0 \pmod{p}, \\ -1 & \textit{if a is a non-quadratic residue modulo p}, \\ 0 & \textit{if } a \equiv 0 \pmod{p}. \end{cases} \tag{2.2}$$

In addition, the Legendre symbol has the following properties:

(1) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

(2) If $a \equiv b \pmod{p}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$
(3) $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$
(4) $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$
(5) $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{1}{4}(p-1)\cdot(q-1)}$.

For the proof of above Legendre symbol properties, see pages 99, 100 and 102 in [3].
Furthermore, if an odd integer n has the prime factorization of $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and any integer a, we have a generalization of the Legendre symbol called the Jacobi symbol, stating that:

$$\left(\frac{a}{1}\right) = 1 \tag{2.3}$$

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1}\left(\frac{a}{p_2}\right)^{\alpha_2} \cdots \left(\frac{a}{p_k}\right)^{\alpha_k}. \tag{2.4}$$

Similar to the Legendre symbol, the Jacobi symbol also has some properties that we use to prove the Hasse-Minkowski theorem:
(1) $\left(\frac{a_1 a_2}{n}\right) = \left(\frac{a_1}{n}\right)\left(\frac{a_2}{n}\right)$
(2) If $a_1 \equiv a_2 \pmod{n}$, then $\left(\frac{a_1}{n}\right) = \left(\frac{a_2}{n}\right)$
(3) $\left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}}$
(4) $\left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}}$
(5) If $gcd(a, n) = 1$, then $\left(\frac{a}{n}\right)\left(\frac{n}{a}\right) = (-1)^{\frac{1}{4}(a-1)\cdot(n-1)}$

3. THE HASSE-MINKOWSKI THEOREM FOR BINARY FORMS

In order to prove the Hasse-Minkowski theorem for binary forms, we need the following theorems.

Theorem 3.1. *The Chinese Remainder Theorem. Suppose n_i are pairwise coprime and a_1, a_2,…,a_k is any sequence of integers, then there exists an integer x such that:*

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned} \tag{3.1}$$

and the solution x is unique modulo n, where $n = \prod_{i=1}^{k} n_i$.

Theorem 3.2. *Suppose a is an integer, b is a natural number, and let $b = \prod_{i=1}^{n} p_i^{\varepsilon_i}$ be the prime factorization of b. Then a is a quadratic residue modulo b if and only if a is a quadratic residue modulo $p_i^{\varepsilon_i}$ for $i = 1, \dots, n$.*

Proof for Theorem 3.2. Suppose a is a quadratic residue modulo b. We then have $a \equiv x^2 \pmod{b}$ for some integer x. Since $p_i^{\varepsilon_i} \mid b$, we also have $a \equiv x^2 \pmod{p_i^{\varepsilon_i}}$.
To prove the order direction, if a is a quadratic residue modulo $p_i^{\varepsilon_i}$, we have $a \equiv x^2$

(mod $p_i^{\varepsilon_i}$), if $j \neq k$, $gcd(p_j^{\varepsilon_j}, p_k^{\varepsilon_k})$. Thus, we could apply the Chinese Remainder Theorem to the congruences $x \equiv x_i \pmod{p_i^{\varepsilon_i}}$ where $i = 1, \ldots, n$. Obtaining $x^2 \equiv x_i^2 \equiv a \pmod{p_i^{\varepsilon_i}}$ from the Chinese Remainder theorem, we thus have $x^2 \equiv a$ (mod $\prod_{i=1}^{n} p_i^{\varepsilon_i}$) or a is a quadratic residue modulo b.

Theorem 3.3. *Dirichlet's Theorem on Arithmetic Progressions. For any two positive coprime integers a and d, there are infinitely many primes of the form $a + nd$, where n is also a positive integer*

Theorem 3.4. *The congruence $x^2 \equiv a$ (mod p) is solvable for every prime p if and only if $a = b^2$ for some $b \in \mathbb{Z}$.*

Proof for Theorem 3.4. Suppose $a = b^2$ for some b, we have $x^2 \equiv a \equiv b^2$ (mod p). Therefore, for all prime p, we have a solution $x \equiv b \pmod{p}$.
To prove the other direction, we try to prove an equivalent statement "if $a \neq b^2$ for some b, a is not a quadratic residue modulo for every prime p."
Suppose a is a positive non-square. Then, if $a = 2$, we could just choose $p = 5$ and apply property 4 from the Legendre symbol to get $\left(\frac{2}{5}\right) = (-1)^{\frac{5^2-1}{8}} = -1$. Otherwise, a could be factored into $p_1 p_2 \ldots p_k$ for $p_1, \ldots, p_k$ prime. Also, a has an odd prime divisor p_k. Now we choose a prime such that $p \equiv 1 \pmod{8}$, $p \equiv 1 \pmod{p_i}$ for $i = 1, 2, \ldots, k-1$ and $p \equiv a \pmod{p_k}$. Such a prime number p exists according to Theorem 3.3. Then, since p_k is not a quadratic residue modulo p, a is not a quadratic non residue modulo p. Thus, we have proved Theorem 3.4 for the case where a is positive.
If a number is negative, it is not a square. We present all negative numbers in the form of $-a$ where a is a positive integer. Let p be a prime number and apply property 1 from the Legendre symbol to get $\left(\frac{-a}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{a}{p}\right)$. We then apply property 3 to obtain $\left(\frac{-1}{p}\right)\left(\frac{a}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{a}{p}\right)$. If a is a square, we can choose $p = 3$ to get $(-1)^{\frac{3-1}{2}}\left(\frac{a}{p}\right) = (-1) \cdot 1 = -1$. If a is a non square, we choose $p = 5$ to obtain $(-1)^{\frac{5-1}{2}}\left(\frac{a}{p}\right) = 1 \cdot (-1) = -1$.

Theorem 3.5. *The Hasse-Minkowski Theorem 1. Let a and b be nonzero, square-free, relatively prime integers of opposite signs. If for each prime p the congruence $ax^2 + by^2 \equiv 0$ (mod p) has a solution in integers (x, y) both not divisible by p, then $ax^2 + by^2 = 0$ has a nontrivial integral solution.*

Consider the first case where $p \nmid ab$, we claim that $gcd(x, p) = 1$. We can prove this statement by using contradiction. Suppose $gcd(x, p) > 1$, then we have $p \mid x$. Hence, $ax^2 + by^2 \equiv by^2 \equiv 0 \pmod{p}$. Also, we could see that either $p \mid b$ or $p \mid y$. Since we assume that $p \nmid ab$, we have $p \mid y$. Now that we have $p \mid x$ and $p \mid y$, this contradicts our assumption that the solution (x, y) to nontrivial modulo p, establishing our claim that $gcd(x, p) = 1$. Now, from $ax^2 + by^2 \equiv 0 \pmod{p}$, we have $ax^2 \equiv -by^2 \pmod{p}$ and by multiplying the congruence on both sides by $-b$, we obtain $-bax^2 \equiv (by)^2 \pmod{p}$. Since $gcd(x, p) = 1$, we could divide

$-bax^2 \equiv (by)^2$ by x^2 to obtain $-ba \equiv (\frac{by}{x})^2$. Thus, $-ba$ is a quadratic residue modulo p for all $p \nmid ab$. Now, assume $p \mid ab$. We have $-ab \equiv 0^2 \pmod{p}$, therefore $-ab$ is a quadratic residue modulo p for all $p \mid ab$.
Thus, $-ba$ is a quadratic residue modulo for all primes p. According to Theorem 3.4, we have $-ba = d^2$ for some integer d. Plugging the pair of integer (b, d) into $f(x, y)$, we obtain $f(b, d) = ab^2 + bd^2 = ab^2 + b(-ab) = 0$. Hence, we have found a nontrivial integral solution to equation $f(x, y) = 0$.

4. The Hasse-Minkowski Theorem for Ternary Forms

Theorem 4.1. *Legendre's Theorem. Suppose a, b, c are non-zero square-free, pairwise relatively prime integers not all of the same sign. Then the equation $ax^2 + by^2 + cz^2 = 0$ has a non-trivial solution if and only if the following conditions are satisfied: (i) $-bc$ is a quadratic residue modulo $|a|$, (ii) $-ab$ is a quadratic residue modulo $|c|$, and (iii) $-ac$ is a quadratic residue modulo $|b|$.*

Definition 4.1. *Let (x_0, y_0, z_0) be a nontrivial integral solution to the congruence $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$, and at most one of x_0, y_0, z_0 is divisible by p, then we call (x_0, y_0, z_0) a p-focused solution.*

Theorem 4.2. *Hasse-Minkowski 2. Let a, b, c be nonzero, square-free, pairwise relatively prime integers not all the same sign. If for each odd prime $p \mid abc$ the congruence $ax^2 + by^2 + cz^2 \equiv 0 \pmod{m}$ has a p-focused solution in integers (x, y, z), then $ax^2 + by^2 + cz^2 = 0$ has a nontrivial integral solution.*

Proof for theorem 4.2. Let p be an odd prime, $p \mid a$ and $f(x, y, z) \equiv 0 \pmod{p}$ has a p-focused solution. According to Theorem 3.2, to prove $-bc$ is a quadratic residue modulo $|a|$, it suffices to show $-bc$ is a quadratic residue modulo p for all $p \mid a$.
Suppose (x_0, y_0, z_0) is a p-focused solution to the congruence. Since $p \mid a$, we have $by_0^2 + cz_0^2 \equiv 0 \pmod{p}$. If $p = 2$ or $p \mid bc$, we have $-bc \equiv 0 \pmod{p}$ and it is a quadratic residue modulo p. If $p \nmid bc$, we obtain $\gcd(b, p) = \gcd(c, p) = 1$. We also know that at most one of x_0, y_0, z_0 is divisible by p. First, suppose p doesn't divide x_0, y_0 or z_0. We have

$$-by_0^2 \equiv cz_0^2 \pmod{p}. \tag{4.1}$$

Divide both sides by z_0^2 to get

$$-b(y_0 z_0^{-1})^2 \equiv c \pmod{p}. \tag{4.2}$$

Multiply both sides by $-b$ to obtain

$$-bc \equiv (by_0 z_0^{-1})^2 \pmod{p}. \tag{4.3}$$

Now suppose p divides exactly one of x_0, y_0, z_0. In the case where $p \mid x_0$, we are done. Suppose $p \mid y_0$ and $p \nmid z_0$, we have

$$cz_0^2 \equiv 0 \pmod{p}. \tag{4.4}$$

Divide both sides by z_0 to get

$$c \equiv 0 \pmod{p}. \tag{4.5}$$

Multiply both sides by $-b$ to obtain

$$-bc \equiv 0 \pmod{p}. \tag{4.6}$$

So, we have $-bc$ a quadratic modulo p. Hence, $-bc$ is a quadratic residue modulo p. The case where $p \mid z_0$ and $p \nmid y_0$ could be proved using a similar procedure. Since the congruence $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$ has a p-focused solution for all $p \mid a$, we have $-bc$ a quadratic residue modulo $|a|$. Similarly, we can determine that $-ac$ is a quadratic residue modulo $|b|$ and $-ab$ is a quadratic modulo c.
We do not need to consider the case where p is even or $p = 2$ since $-bc$, $-ac$, $-ad$ are either odd and even. Thus, they are congruent to 0 or 1 modulo 2 and both 0 and 1 are squares.
Finally, we need to show that if $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$ has a p-focused solution for all odd $p \mid abc$, then $f(x, y, z) = 0$ has a nontrivial integral solution. Since $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$ has a p-focused solution for all odd $p \mid abc$, it has a p-focused solution for all odd $p \mid a$, $p \mid b$ and $p \mid c$. We can also determine that $-bc$ is a quadratic residue modulo $|a|$, $-ac$ is a quadratic residue modulo $|b|$, $-ab$ is a quadratic residue modulo $|c|$. Hence, according to the Legendre's Theorem, the equation $ax^2 + by^2 + cz^2 = 0$ has a nontrivial integral solution.

5. HASSE-MINKOWSKI AND LEGENDRE THEOREM IMPLEMENTATION

Let $f(x, y, z) = ax^2 + by^2 + cz^2$. Obviously, since checking whether a congruence $f(x, y, z) \equiv 0 \pmod{p}$ has a p-focused solution is a tedious task in real life, especially when abc has a lot of prime factors or when a, b, c are large, we could write a computer program to check it.

Eratosthenes Sieve

Eratosthenes Sieve is an old algorithm used to rapidly identify all the primes to a certain limit. The program first gets the integers a, b and c from the keyboard. Then, it creates the Eratosthenes sieve of primes that are odd and divide abc. The code below is the modified Eratosthenes sieve function written in C++.
The parameter *upperBound* is the maximum number which we would check if it is a prime number. The program always calls the function with *upperBound* $= abc$. Then, we create a bitset, a data structure that stores bits, named *flag*. Suppose i is a number from 2 to *upperBound*, given that $flag[i] = 1$, then i is prime, and vice versa. Next, we reset our bitset which would set all the value of *flag* to 1. Our first loop iterates from 2 to $upperBound$ and for every number, if *flag[i]* $= 1$. Next, we process the second loop that iterates every multiple of that prime number to *upperBound*. For every multiples of that prime, we set the corresponding $flag$ value to 0 since the multiple of a prime can not be a prime. After the second loop, we would append our prime to a vector named *primes* to store it.

Function. $sieve(upperBound)$

Pseudocode

Input. *upperBound*, the maximum number to check if it is a prime number.
Determine. Every prime less than or equal to $upperBound + 1$.

(1) $primes \leftarrow$ an empty dynamic array, $flag \leftarrow$ an bitset
(2) $upperBound \leftarrow |upperBound|$
(3) for $i \leftarrow 0$ to 1000009
(4) $\quad flag_i \leftarrow 1$

(5) for $i \leftarrow 2$ to $upperBound + 1$
(6) if $flag_i = 1$
(7) $j \leftarrow 2i$
(8) while $j <= sievesize$
(9) $flag_j \leftarrow 0$
(10) if $i \neq 2$ and $flag_i \equiv 0 \pmod{upperBound}$
(11) append i to $primes$

C++ Implementation

```
bitset<10000010> flag;
vector<int> primes;
int a, b, c;

void sieve(long upperBound) {
    upperBound = abs(upperBound);
    flag.set();
    flag[0] = flag[1] = 0;
    for (long long i = 2; i <= upperBound; i++)
        if (flag[i]) {
            for (long long j = i * i; j <= upperBound; j += i) flag[j] = 0;
            if(i != 2 && upperBound % i == 0) primes.push_back((int)i);
        }
}
```

The Hasse-Minkowski Theorem 2

Suppose p is a prime that divides abc. To check for p-focused solution, we write a boolean method, *pFocusedCheck*, with parameter $primes$, the prime to check. *pFocusedCheck* has three loops that create every combination of x, y, z, where x, y, z are integer and less than $primes$. For every combination, if it is a $primes$-focused solution we immediately return true. After it finishes three loops, we would haven't found a $primes$-focused solution, thus return false.

Function. *pFocusedCheck*($prime$)

Pseudocode

Input. $primes$, the prime number to look for a $primes$-focused solution to the congruence.

Output. Return true if there is a $primes$-focused solution, otherwise returns false.

(1) $x \leftarrow$ an int, $y \leftarrow$ an int, $z \leftarrow$ an int
(2) for $x \leftarrow 0$ to $prime - 1$
(3) for $y \leftarrow 0$ to $prime - 1$
(4) for $z \leftarrow 0$ to $prime - 1$
(5) if $ax^2 + by^2 + cz^2 \equiv 0 \pmod{primes}$ and at most one of x, y, z is divisible by $primes$.
(6) return true
(7) return false

C++ Implementation

```
bool pFocusedCheck(int prime){
    int x, y, z;
```

```
    for(x = 0; x < prime; ++x){
        for(y = 0; y < prime; ++y){
            for(z = 0; z < prime; ++z){
                if((((a * (x * x))  + (b * (y * y)) + (c * (z * z))) % prime == 0)
                   && (((x % prime) == 0) + ((y % prime) == 0) + ((z % prime) == 0) <= 1)){
                    return true;
                }
            }
        }
    }
    return false;
}
```

Then, we create a function named $HasseMinkowski2Check$ that loops through the *sieve* vector to check whether the congruence $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$ has a p-focused solution. The function returns true if the congruence has a p-focused solution to every p, otherwise, returns false.

Function. $HasseMinkowski2Check()$

Pseudocode

Output. Returns true if for every p, the congruence $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$ has a p-focused solution, otherwise, returns false.

(1) for every prime in *primes*
(2)   if not *pFocusedCheck*(prime)
(3)   return false
(4) return true

C++ Implementation

```
bool HasseMinkowski2Check(){
    for(int i = 0; i < primes.size(); ++i){
        if(!pFocusedCheck(primes[i])){
            return false;
        };
    }
    return true;
}
```

Legendre's Theorem.

Initially, we want to implement the Legendre's symbol. We define *LegendreSymbol* function with two parameters, *toCheck* and *modulo*. The function returns 0 if *toCheck* $\equiv 0$ (mod *modulo*) and returns 1 if there exists an x such that $x^2 \equiv$ *toCheck* (mod *modulo*), elsewise returns -1.

First, if *toCheck* $\equiv 0$ (mod *modulo*), the function immediately returns 0. Next, if *toCheck* is negative, applying property 1 and 3 of the Legendre symbol, we can calculate $\left(\frac{-1}{p}\right)$ and save the result to a variable named *offset*. Otherwise, *offset* is set as 1. We, then, apply property 2 of the Legendre symbol to make *toCheck* less than *modulo*. Now, we make a loop that iterates from 1 to *modulo* $- 1$. If there exists a number i in that range such that $i^2 \equiv$ *toCheck* (mod *modulo*), we return $1\cdot$*offset*. Otherwise, after finishing the loop, we return $-1\cdot$*offset*

Function. *LegendreSymbol()*
Pseudocode
Input. *toCheck*, the number to check if it is a quadratic residue
modulo, the modulo
Output. Returns 0 if *toCheck* $\equiv 0 \pmod{modulo}$ and returns 1 if *toCheck* is a quadratic residue modulo *modulo*, elsewise returns -1.

(1) if $toCheck \equiv 0 \pmod{modulo}$
(2) return 0
(3) if $toCheck < 0$
(4) $offset \leftarrow -1^{\frac{modulo-1}{2}}$
(5) else $offset \leftarrow 1$
(6) $toCheck \leftarrow |toCheck|$
(7) while $toCheck > modulo$
(8) $toCheck \leftarrow toCheck \bmod modulo$
(9) for $i \leftarrow 1$ to $modulo$
(10) if $i^2 \equiv toCheck \pmod{modulo}$
(11) return $1 \cdot offset$
(12) return $-1 \cdot offset$

C++ Implementation

```
int LegendreSymbol(int toCheck, int modulo){
    if(toCheck % modulo == 0) return 0;
    int offset = (toCheck < 0) ?  (int)(pow(-1, (modulo - 1) / 2)) : 1;
    toCheck = aflag(toCheck);
    while (toCheck > modulo){
        toCheck %= modulo;
    }

    for(int i = 1; i < modulo; ++i){
        if((i * i) % modulo == toCheck) return 1 * offset;
    }
    return -1 * offset;
}
```

Next, we only need to to write the Legendre theorem function. We will name it *LegendreCheck*.
Function. *LegendreCheck()*
Pseudocode
Output. return true if $-bc$ is a quadratic residue modulo $|a|$, $-ab$ is a quadratic residue modulo $|c|$ and $-ac$ is a quadratic residue modulo $|b|$. Otherwise, return false.

(1) bool *ans*
(2) temp $\leftarrow LegendreSymbol(-b * c, abs(a))$
(3) $ans \leftarrow temp = 0$ or $temp = 1$
(4) temp $\leftarrow LegendreSymbol(-a * b, abs(c))$
(5) $ans \leftarrow (temp = 0$ or $temp = 1)$ and ans
(6) temp $\leftarrow LegendreSymbol(-a * c, abs(b))$
(7) $ans \leftarrow temp = 0$ or $temp = 1$ and ans

(8) reuturn *ans*

C++ Implementation

```
bool LegendreCheck(){
    int temp = LegendreSymbol(-b * c, abs(a));
    bool ans = (temp == 0 || temp == 1);
    temp =  LegendreSymbol(-a * b, abs(c));
    ans &= (temp == 0 || temp == 1);
    temp = LegendreSymbol(-a * c, abs(b));
    return (ans & ((temp == 0 || temp == 1)));
}
```

Sample Program Run

We now add a few print functions to the code and try running two inputs in order to test our program.

Input 1

Input.

```
a = 1
b = 1
c = -3
```

Output.

```
Legendre Theorem Check
-bc is  a quadratic residue modulo |a|
-ab is not a quadratic residue modulo |c|
-ac is  a quadratic residue modulo |b|
There is no nontrivial integral solution to f(x, y, z) = 0

Hasse-Minkowski Theorem Check
There is no 3-focused solution
There is no nontrivial integral solution to f(x, y, z) = 0
```

Input 2

Input.

```
a = -7
b = 15
c = 13
```

Output.

```
Legendre Theorem Check
-bc is a quadratic residue modulo |a|
-ab is a quadratic residue modulo |c|
-ac is a quadratic residue modulo |b|
There are nontrivial integral solutions to f(x, y, z) = 0

Hasse-Minkowski Theorem Check
There is a 3-focused solution: x = 1, y = 0, z = 1
There is a 5-focused solution: x = 1, y = 0, z = 2
There is a 7-focused solution: x = 0, y = 1, z = 1
There is a 13-focused solution: x = 1, y = 6, z = 0
```

```
There are nontrivial integral solutions to f(x, y, z) = 0
```

We can see that in both cases the result is the same as the Legendre theorem and the Hasse-Minkowski theorem. We can also modify the program or add more functions depending on the task we intend to apply them to.

6. Conclusion

We have proved two Hasse-Minkowski theorems which facilitate the problem of determining the integral solvability of quadratic forms. After the Hasse-Minkowski theorem, in the binary form, we could find a prime p which $f(x, y) \equiv 0 \pmod{p}$ does not have a solution (x, y) both not divisible by p to show that $f(x, y) = 0$ does not have nontrivial integral solutions. In the ternary form, the Hasse-Minkowski theorem reduces the problem to determining if there is a p-focused solution to the congruence $f(x, y, z) = 0 \pmod{p}$, which p is finite. The crux of this paper is the introduction of a complete program implementing the Hasse-Minkowski theorems and Legendre theorem with some supporting functions like the Eratosthenes sieve and the Legendre symbol.

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