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Topological Bihyperbolic Modules

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Abstract
The aim of this article is introducing and researching hyperbolic modules, bihyperbolic modules, topological hyperbolic modules and topological bihyperbolic modules. In this regard, we define balanced, convex and absorbing sets in hyperbolic and bihyperbolic modules. In particular, we investigate convex sets in hyperbolic numbers set (it is a hyperbolic module over itself) by considering the isomorphic relation of this set with $2-$dimensional Minkowski space. Moreover, bihyperbolic numbers set is a bihyperbolic module over itself, too. So, we define convex sets in this module by considering hypersurfaces of $4-$dimensional semi Euclidean space that are isomorphic to some subsets of bihyperbolic numbers set. We also study the interior and closure of some special sets and neighbourhoods of the unit element of the module in the introduced topological bihyperbolic modules. In the light of obtained results, new relationships are presented for idempotent representations in topological bihyperbolic modules.

Keywords: Bihyperbolic numbers, hyperbolic numbers, topological bihyperbolic modules, topological modules
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1. Introduction

J. Cockle introduced commutative quaternions as Tessarine numbers in [10, 11, 12]. Besides C. Segre studied these numbers by denominating them bicomplex numbers [3]. Afterwards, G. B. Price comprehensively analyzed bicomplex numbers, functions defined by bicomplex power series, derivatives, integrals, holomorphic functions and also their generalizations to higher dimensions [7]. Actually, the system of bicomplex numbers (Tessarine numbers) is a special case of the commutative fourcomplex numbers system that was generalized by F. Catoni et al. in [6]. The set of generalized commutative quaternions is defined as

$$\{q|\ q = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z;\ t,x,y,z \in \mathbb{R}\}$$

where $\mathrm{i}^2 = \mathrm{k}^2 = \alpha$, $\mathrm{j}^2 = 1$, $\mathrm{ij} = \mathrm{ji} = \mathrm{k}$. A generalized commutative quaternion is called an elliptic, parabolic or hyperbolic commutative quaternion, respectively; provided that $\alpha < 0$, $\alpha = 0$ or $\alpha > 0$. In the case of $\alpha = -1$, the elliptic quaternions corresponds to bicomplex numbers. However, the case of $\alpha = 1$ has not been handled as well as the bicomplex case. In the meantime, the commutative quaternions and their higher versions were considered by S. Olariu and in the case of $\alpha = 1$, a commutative quaternion was called hyperbolic fourcomplex number in [20]. Recently, the set of zeros of polynomials of hyperbolic fourcomplex numbers were studied and these numbers were denominated bihyperbolic numbers since they can be written as a pair of hyperbolic numbers [1].

On the other hand, the hyperbolic fourcomplex numbers are used in digital signal processing and these numbers are called multi-hyperbolic numbers [4]. Also, multi-hyperbolic numbers are a generalization of the hyperbolic fourcomplex numbers, since multi-hyperbolic numbers include the hyperbolic fourcomplex numbers.

Apart from all these, detailed surveys on the algebraic [13], geometric and topological [14], and combinatorial properties [8, 9] of bihyperbolic numbers were given. However, bihyperbolic modules and topological bihyperbolic modules have not investigated yet.

The real or complex vector space, topological vector space and balanced, convex and absorbing sets in these spaces are known very well in the literature [2, 21]. These concepts are thought again with the discovery of the quaternions and especially commutative quaternions. For instance, the bicomplex modules are introduced with the discovery of bicomplex numbers. The set of bicomplex numbers is a commutative ring. Hence, the researches on modules over this ring are accelerated with new results on commutative algebra [5, 16]. Also, topological bicomplex modules are presented and balanced, convex and absorbing sets are investigated in these modules [17, 18].

As its known, the set of hyperbolic numbers is a subalgebra of the algebra of bicomplex numbers and the system of hyperbolic numbers is an active studying area in several disciplines. Besides, hyperbolic module and convex set in this module partially are studied in [15]. In connection with these, we introduce hyperbolic modules, bihyperbolic modules, topological hyperbolic modules and topological bihyperbolic modules. Also, we give new results on these subjects by using the idempotent representations of bihyperbolic numbers which were analyzed in detail [13, 14].

2. Preliminaries

Definition 2.1. *The set of bihyperbolic numbers is defined as*

$$H_2 = \{\,\zeta \mid\ \zeta = z_1 + j_2 z_2,\ \ z_1, z_2 \in H\,(j_1)\}$$

where j_1, j_2 are hyperbolic units satisfying $j_1 j_2 = j_2 j_1 = j_3$, $j_s^2 = 1$, $j_s \neq \pm 1$ for $s = 1,2,3$ and $H\,(j_1) = \{\, z \mid\ z = x + j_1 y :\ x, y \in \mathbb{R}\}$ is the set of hyperbolic numbers based on hyperbolic unit j_1 [13].

Definition 2.2. *The set of multi-hyperbolic numbers is given by*

$$H_n = \left\{ A + j_n B \,\middle|\, A, B \in H_{n-1},\ j_n^2 = 1,\ j_n \neq \pm 1 \right\}$$

for $n \in \mathbb{Z}^+$.

The set H_0 is the real numbers set and the set H_1 is the hyperbolic numbers set corresponding $H\,(\mathrm{j}_1)$ in the previous definition. In the rest of the article, the notion H will be used for the hyperbolic numbers set based on the hyperbolic unit j_1.

The space, null, and time cones of $z_0 \in H$ are defined as

$$SH\,(z_0) = \left\{ z \in H \mid\ (z - z_0)\,\overline{(z - z_0)} > 0 \text{ or } z = z_0 \right\},$$

$$NH\,(z_0) = \left\{ z \in H \mid\ (z - z_0)\,\overline{(z - z_0)} = 0 \right\},$$

and

$$TH\,(z_0) = \left\{ z \in H \mid\ (z - z_0)\,\overline{(z - z_0)} < 0 \text{ or } z = z_0 \right\},$$

respectively [14].

Although the sets H and H_2 are commutative rings with unity according to the addition and multiplication operations, they do not have field structure algebraically since they have non-invertible elements according to multiplication operation.

There are especially non-invertible elements such as

$$e_{1\mathrm{j}_s} = \frac{1 + \mathrm{j}_s}{2} \text{ and } e_{2\mathrm{j}_s} = \frac{1 - \mathrm{j}_s}{2} \text{ for } s = 1,2,3.$$

These numbers are hyperbolic numbers with the hyperbolic units j_s and they are called idempotent elements because of $\left(e_{1\mathrm{j}_s}\right)^n = e_{1\mathrm{j}_s}$ and $\left(e_{2\mathrm{j}_s}\right)^n = e_{2\mathrm{j}_s}$ for $n \in \mathbb{Z}^+$ [13]. Every element of H_2 can be written as a linear decomposition of the set

$\{e_{1,\mathrm{j}_s}, e_{2,\mathrm{j}_s}\}$ in three different ways which are $\zeta = \zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s}$ for $\zeta \in H_2$ with $s = 1,2,3$. The coefficients of the linear decompositions of a bihyperbolic number are bihyperbolic numbers for $s = 1$ and hyperbolic numbers based on the hyperbolic unit j_1 for $s = 2,3$. These representations are given for $s = 1,2$ in [13] and for $s = 3$ in [6]. More details about the idempotent representations of bihyperbolic numbers can be found in [13, 14].

There is another idempotent representation of bihyperbolic numbers in the literature. Briefly, a bihyperbolic number $\zeta = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3$ can be written as $\zeta = w_1 i_1 + w_2 i_2 + w_3 i_3 + w_4 i_4$ where i_1, i_2, i_3 and i_4 are bihyperbolic components such that $i_1 = \frac{1+\mathrm{j}_1+\mathrm{j}_2+\mathrm{j}_3}{4}$, $i_2 = \frac{1-\mathrm{j}_1+\mathrm{j}_2-\mathrm{j}_3}{4}$, $i_3 = \frac{1+\mathrm{j}_1-\mathrm{j}_2-\mathrm{j}_3}{4}$, $i_4 = \frac{1-\mathrm{j}_1-\mathrm{j}_2+\mathrm{j}_3}{4}$ and $w_1 = x_0 + x_1 + x_2 + x_3$, $w_2 = x_0 - x_1 + x_2 - x_3$, $w_3 = x_0 + x_1 - x_2 - x_3$ and $w_4 = x_0 - x_1 - x_2 + x_3$ where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ [20]. Hence, a partial order is defined on the real vector space H_2 by using this representation in [13]. It defines as $\zeta \le \varphi$ for $\zeta, \varphi \in H_2$ if and only if $w_k \le \tilde{w}_k$ where $\zeta = w_k i_k$ and $\varphi = \tilde{w}_k i_k$ for $k = 1,2,3,4$ [13]. Moreover, positive bihyperbolic numbers set is given with this partial order such that $H_2^+ = \{\zeta \mid \zeta = w_k i_k, w_k \ge 0\}$ [13]. Also, positive hyperbolic numbers are known in the literature such that $H^+ = \{z \mid z = x + \mathrm{j}_1 y = (x+y) e_{1,\mathrm{j}_1} + (x-y) e_{2,\mathrm{j}_1},\ x+y \ge 0,\ x-y \ge 0\}$ [5].

On the other hand, a bihyperbolic number $\zeta = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3$ has three conjugates such that $\overline{\zeta}^{\mathrm{j}_1} = x_0 + \mathrm{j}_1 x_1 - \mathrm{j}_2 x_2 - \mathrm{j}_3 x_3$, $\overline{\zeta}^{\mathrm{j}_2} = x_0 - \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 - \mathrm{j}_3 x_3$ and $\overline{\zeta}^{\mathrm{j}_3} = x_0 - \mathrm{j}_1 x_1 - \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3$ [6]. Considering these conjugates, the hyperbolic valued modulus is introduced [9]. It is defined as $|\zeta|_{\mathrm{j}_s} = \sqrt{\left|\zeta \overline{\zeta}^{\mathrm{j}_s}\right|}$ for $s = 1,2,3$ and named j_s−modulus of ζ. Also, by taking $x_0 x_1 - x_2 x_3 = 0$, $x_0 x_2 - x_1 x_3 = 0$ and $x_0 x_3 - x_1 x_2 = 0$, three different hypersurfaces of H_2 are defined such that

$$M_1 = \{x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3 \mid x_0 x_1 - x_2 x_3 = 0\},$$

$$M_2 = \{x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3 \mid x_0 x_2 - x_1 x_3 = 0\}$$

and

$$M_3 = \{x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3 \mid x_0 x_3 - x_1 x_2 = 0\}.$$

The modulus of ζ is given by

$$|\zeta|_{\mathrm{j}_1} = \sqrt{|x_0{}^2 + x_1{}^2 - x_2{}^2 - x_3{}^2|},$$

$$|\zeta|_{\mathrm{j}_2} = \sqrt{|x_0{}^2 - x_1{}^2 + x_2{}^2 - x_3{}^2|}$$

and

$$|\zeta|_{\mathrm{j}_3} = \sqrt{|x_0{}^2 - x_1{}^2 - x_2{}^2 + x_3{}^2|}$$

in M_1, M_2 and M_3, respectively [13]. The cones of a bihyperbolic number $\zeta_0 \in M_k \subseteq H_2$ are classified as

$$SM_k(\zeta_0) = \left\{\zeta \in M_k \mid (\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_k} > 0 \text{ or } \zeta = \zeta_0\right\},$$

$$NM_k(\zeta_0) = \left\{\zeta \in M_k \mid (\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_k} = 0\right\},$$

$$TM_k(\zeta_0) = \left\{\zeta \in M_k \mid (\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_k} < 0 \text{ or } \zeta = \zeta_0\right\}$$

and they are called space cone, null cone, and time cone for $k = 1,2,3$, respectively [14].

Definition 2.3. *Let X be a vector space over a field F (real or complex numbers set) and $\varnothing \ne A \subseteq X$ be a subset. If $\lambda x \in A$ or $\lambda A \subseteq A$ where $\lambda A := \{\lambda x \mid x \in A\}$ for every $x \in A$ and every $\lambda \in F$ with $|\lambda| \le 1$, then A is balanced (circled) set [19].*

Definition 2.4. *Let X be a vector space over the real numbers field $\mathbb{R}$ and $\varnothing \ne A \subseteq X$. A is convex if the line segment connecting x and y is included in A for all $x, y \in A$. This means that $(1-t)x + ty \in A$ for $0 \le t \le 1$ [19].*

Definition 2.5. *Let X be a vector space over a field F (real or complex numbers set) and $\varnothing \ne A \subseteq X$. A is absorbing set, if some real number $\lambda > 0$ for all $x \in X$, $x \in \mu A$ for all scalars $\mu \in F$ that is $|\mu| \ge \lambda$ where $\mu A := \{\mu a \mid a \in A\}$ [19].*

3. Topological Hyperbolic Modules

Definition 3.1. *Let $(X,\oplus)$ be a commutative group. If the operations*

$$\oplus : X\times X\to X \qquad \text{and} \qquad \odot\ : H\times X\to X$$
$$(u,v)\to u+v \qquad\qquad (z,u)\to z\odot u$$

satisfy the properties

$$(z_1z_2)\odot u = z_1\odot(z_2\odot u),$$
$$(z_1+z_2)\odot u = (z_1\odot u)\oplus(z_2\odot u),$$
$$z_1\odot(u\oplus v) = (z_1\odot u)\oplus(z_1\odot v),$$
$$1_H\odot u = u,\ \ (1_H = 1+j_1 0 = 1)$$

for every $z_1,z_2\in H$ and every $u,v\in X$, then $(X,H,\oplus,\odot,+,\cdot)$ is called $H-$module. Later on, $z\odot u$ will be denoted by zu.

Example 3.2. *Hyperbolic numbers set H, bihyperbolic numbers set H_2 and multi-hyperbolic numbers set H_n for $n\in\mathbb{Z}^+$ are $H-$modules.*

Remark 3.3. *Real numbers set $\mathbb{R}$ is not $H-$module because of $H\times\mathbb{R}\to H$.*

Since hyperbolic numbers set H includes the isotropic numbers, the unit balls in H can be classified into three types. So, let us define a new three types of balanced sets by considering three different cases for each hyperbolic number $\lambda=\lambda_1+\mathrm{j}_1\lambda_2\in H$ satisfying $|\lambda|_H=\sqrt{\left|\lambda\overline{\lambda}\right|}=\sqrt{\left|\lambda_1^2-\lambda_2^2\right|}\le 1$.

Definition 3.4. *Let X be a $H-$module, $\varnothing\neq B\subseteq X$ and $\lambda=\lambda_1+j_1\lambda_2\in H$.*

i) B is called $SH-$balanced set if $\lambda B\subseteq B$ for every $\lambda\in SH(O)$ such that $\lambda_1^2-\lambda_2^2\le 1$,

ii) B is called $NH-$balanced set if $\lambda B\subseteq B$ for every $\lambda\in NH(O)$ that is $\lambda_1^2-\lambda_2^2=0$,

iii) B is called $TH-$balanced set if $\lambda B\subseteq B$ for every $\lambda\in TH(O)$ such that $-1\le\lambda_1^2-\lambda_2^2$.

Here, $SH(O)$, $NH(O)$ and $TH(O)$ denotes the space cone, the light cone and the time cone of H at the origin, respectively.

Example 3.5. *The subsets $SH(O)$ and $TH(O)$ in $H-$module H are $SH-$balanced sets. But, they are not $NH-$balanced set and $TH-$balanced set. Also, the subset $NH(O)\subseteq H$ is TH,NH and $SH-$balanced set.*

The partial order on the real vector space H_2 was introduced in [13]. The definition of $H-$convex set is given in [15] by using such an order as follows: Let X be a $H-$module and $\varnothing\neq B\subseteq X$. If $\lambda x+(1-\lambda)y\in B$ for every $x,y\in B$ and $\lambda\in H^+$ with $0\le\lambda\le 1$, then B is called $H-$convex set. Nevertheless, here we investigate especially the $H-$module H. Eventually, three different definitions of convex sets which are geometrically meaningful will be given in $H-$module H for the first time as follows.

Definition 3.6. *Let B be a non-empty subset of $H-$module H. For all $x,y\in B$ and all $\lambda\in\mathbb{R}$ with $0\le\lambda\le 1$,*

i) B is called $SH-$convex set if $y\in SH(x)$ and $\lambda x+(1-\lambda)y\in B$,

ii) B is called $NH-$convex set if $y\in NH(x)$ and $\lambda x+(1-\lambda)y\in B$,

iii) B is called $TH-$convex set if $y\in TH(x)$ and $\lambda x+(1-\lambda)y\in B$.

This definition indicates that the classical definition of the convexity is valid for the convexity of a subset of the hyperbolic numbers set. However, three different convexity types are needed depending on whether the line segments connecting all two different elements of the set belong to either the space cone, the light cone or the time cone.

Definition 3.7. *Let X be a $H-$module and $\varnothing\neq B\subseteq X$. For all $x\in X$,*

i) B is called $SH-$absorbing set if there is a non-negative real number λ such that $x\in\mu B$ for all $\mu\in SH(O)\subseteq H$ with $|\mu|_H\ge\lambda$, *ii) B is called $TH-$absorbing set if there is a non-negative real number λ such that $x \in \mu B$ for all $\mu \in TH(O) \subseteq H$ with $|\mu|_H \geq \lambda$.*

Definition 3.8. *Let X be a $H-$module and τ is a Hausdorff topology on X. If the operations*

$$+ : X \times X \to X$$

$$\cdot \ : H \times X \to X$$

are continuous, then the pair (X, τ) is called a topological hyperbolic module or topological $H-$module.

4. Topological Bihyperbolic Modules

Since $(H_2, +, \cdot)$ is a commutative ring with unity, we can construct a module structure over this ring. For instance, the bihyperbolic numbers set H_2 or the multi-hyperbolic numbers set H_n for $n \in \{2,3,4,\ldots\}$ are H_2-modules.

Let X be an arbitrary H_2-module with the classical addition and multiplication operations. The idempotent representations of the elements of X are given correlatively the elements of H_2 in the following theorem.

Theorem 4.1. *Let X be a H_2-module. Then $X = e_{1j_s}X + e_{2j_s}X$ for $s = 1,2,3$.*

Proof. Let $x \in X$. Then $e_{1\mathrm{j}_s} + e_{2\mathrm{j}_s} = 1$ for $e_{1\mathrm{j}_s}, e_{2\mathrm{j}_s} \in H(\mathrm{j}_s) \subseteq H_2$ and $s = 1,2,3$. Hence, the element x can be written as

$$x = \left(e_{1\mathrm{j}_s} + e_{2\mathrm{j}_s}\right)x = e_{1\mathrm{j}_s}x + e_{2\mathrm{j}_s}x.$$

Since each element of X can be written as above, it can be generalized to the whole set. □

Here if we write $e_{1\mathrm{j}_s}X = X_{1\mathrm{j}_s}$ and $e_{2\mathrm{j}_s}X = X_{2\mathrm{j}_s}$, then $X = X_{1\mathrm{j}_s} + X_{2\mathrm{j}_s}$.

Corollary 4.2. *Let X be a H_2-module. Then, there are $e_{1j_s}X = e_{1j_s}X_{1j_s}$ and $e_{2j_s}X = e_{2j_s}X_{2j_s}$ equations for $s = 1,2,3$.*

Proof. Let $e_{1\mathrm{j}_s}X = X_{1\mathrm{j}_s}$. Then multiplying both sides of this equation from left by $e_{1\mathrm{j}_s}$ gives us $e_{1\mathrm{j}_s}\left(e_{1\mathrm{j}_s}X\right) = e_{1\mathrm{j}_s}X_{1\mathrm{j}_s}$. Hence $e_{1\mathrm{j}_s}X = e_{1\mathrm{j}_s}X_{1\mathrm{j}_s}$, since $e_{1\mathrm{j}_s}$ and $e_{2\mathrm{j}_s}$ are the idempotent elements. Similarly, we can write $e_{2\mathrm{j}_s}\left(e_{2\mathrm{j}_s}X\right) = e_{2\mathrm{j}_s}X_{2\mathrm{j}_s}$ whenever $e_{2\mathrm{j}_s}X = X_{2\mathrm{j}_s}$. So, $e_{2\mathrm{j}_s}X = e_{2\mathrm{j}_s}X_{2\mathrm{j}_s}$ is obtained. □

Corollary 4.3. *Let X be a H_2-module. Then, $X = e_{1j_s}X_{1j_s} + e_{2j_s}X_{2j_s}$ for $s = 1,2,3$.*

Corollary 4.4. *Let X be a H_2-module. Then, X_{1j_s} and X_{2j_s} are H_2-submodules of X for $s = 1,2,3$.*

Proof. Let X be a H_2-module and $X_{1\mathrm{j}_s} \subseteq X$ for $s = 1,2,3$. Moreover, let $t_1, t_2 \in X_{1\mathrm{j}_s}$. There are the elements x and y in X satisfied the equations $t_1 = e_{1\mathrm{j}_s}x$ and $t_2 = e_{1\mathrm{j}_s}y$, since $X_{1\mathrm{j}_s} = e_{1\mathrm{j}_s}X$. $(X, +)$ is a commutative group, since X is a H_2-module. Hence, $x - y \in X$. So, $t_1 - t_2 = e_{1\mathrm{j}_s}x - e_{1\mathrm{j}_s}y = e_{1\mathrm{j}_s}(x-y) \in e_{1\mathrm{j}_s}X = X_{1\mathrm{j}_s}$. On the other hand, let $\zeta \in H_2$ and $t \in X_{1\mathrm{j}_s}$. The product of ζ and t is $\zeta t = \left(\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s} + \zeta_{2\mathrm{j}_s}e_{2\mathrm{j}_s}\right)\left(e_{1\mathrm{j}_s}x\right) = \zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}x$ and $\zeta_{1\mathrm{j}_s}x \in X$ since X a H_2-module. Hence $\zeta t = e_{1\mathrm{j}_s}\zeta_{1\mathrm{j}_s}x \in e_{1\mathrm{j}_s}X = X_{1\mathrm{j}_s}$. Consequently, $X_{1\mathrm{j}_s}$ is a H_2-submodule of the H_2-module X. Similarly, the set $X_{2\mathrm{j}_s}$ is a H_2-submodule of the H_2-module X. □

Especially, the subsets $X_{1\mathrm{j}_s}$ and $X_{2\mathrm{j}_s}$ are $H-$submodules of the H_2-module X for $s = 2,3$ since $\zeta_{1\mathrm{j}_s}, \zeta_{2\mathrm{j}_s} \in H$.

Corollary 4.5. *The subsets $e_{1j_s}H_2$ and $e_{2j_s}H_2$ are H_2-modules for $s = 1,2,3$. Especially, these sets are $H-$modules for $s = 2,3$.*

Definition 4.6. *Let X be a H_2-module. If there is a finite H_2-base such that $\{x_l : l = 1,\ldots,n\} \subseteq X$, then X is a free H_2-module. The free H_2-module X can be written as $X = \left\{ x \mid \ x = \sum\limits_{l=1}^{n} \zeta_l x_l, \ \zeta_l \in H_2, x_l \in X \right\}$.*

Definition 4.7. *Let X be a free H_2-module.*

$$A := \left\{ \tilde{x} \mid \ \tilde{x} = \sum_{l=1}^{n} \zeta_l x_l, \ \zeta_l \in H, x_l \in X \right\} \subseteq X$$

is a free $H-$module depending on the H_2-base of X.

Here, when the elements of any subset A of the free H_2-module X are written as a linear combination of the finite base $\{x_l : l = 1,\dots,n\} \subseteq X$, if the coefficients are bihyperbolic number, then the subset A is a free H_2-module depends on the H_2-base of X.

Example 4.8. *Each element of H_2 can be written as a linear combination of the idempotent elements e_{1j_s} and e_{2j_s} for $s = 1,2,3$ such that $\zeta = \zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s} \in H_2$. Also, the set $\{e_{1j_s}, e_{2j_s}\}$ is linearly independent. Therefore, the subset $\{e_{1j_s}, e_{2j_s}\} \subseteq H_2$ is a base of the H_2. It is known that $\zeta_{1j_s}, \zeta_{2j_s} \in H_2$ for $s = 1$ and $\zeta_{1j_s}, \zeta_{2j_s} \in H$ for $s = 2,3$. So, H_2 is a free H_2-module for $s = 1$. Moreover, H_2 is a free $H-$module according to H_2-base for $s = 2,3$.*

Now, let us give the necessary conditions for any subset of a H_2-module to be balanced, convex or absorbing set. In order to give the conditions specified here, there must be a real-valued norm on the ring in which the module structure is defined. Since there are real-valued norms on the hypersurfaces $M_k \subseteq H_2$ for $k = 1,2,3$, related conditions will be given and theorems will be proved by using the elements of M_k.

Three different balanced (circular) sets, convex sets and two different absorbing (swallowing) sets have emerged on the H_2-module due to the presence of light cone on hypersurfaces $M_k \subseteq H_2$.

Firstly, the following definition of a balanced (circular) set is given by considering the three different conditions for each bihyperbolic number $\zeta \in M_k \subseteq H_2$ satisfying the condition $|\zeta|_{\mathrm{j}_k} = \sqrt{\left|\zeta\overline{\zeta}^{\mathrm{j}_k}\right|} \leq 1$.

Definition 4.9. *Let X be a H_2-module, $\varnothing \neq B \subseteq X$ and $\zeta \in M_k \subseteq H_2$ $(k = 1,2,3)$.*

i) B is called SM_k-balanced set if $\zeta B \subseteq B$ for every $\zeta \in SM_k(O)$ such that $\zeta\overline{\zeta}^{j_k} \leq 1$,

ii) B is called NM_k-balanced set if $\zeta B \subseteq B$ for every $\zeta \in NM_k(O)$ such that $\zeta\overline{\zeta}^{j_k} = 0$,

iii) B is called TM_k-balanced set if $\zeta B \subseteq B$ for every $\zeta \in TM_k(O)$ such that $-1 \leq \zeta\overline{\zeta}^{j_k}$.

Here the sets $SM_k(O)$, $NM_k(O)$ and $TM_k(O)$ are the space cone, the null cone and the time cone at the origin in the hypersurfaces M_k, respectively.

Theorem 4.10. *Let X be a H_2-module and the set B is a SM_k-balanced or TM_k-balanced subset of X for $k = 1,2,3$.*

i) $\zeta B = B$ for every $\zeta \in M_k \subseteq H_2$ such that $|\zeta|_{j_k} = 1$.

ii) $\zeta B = |\zeta|_{j_k} B$ for every $\zeta \in M_k \subseteq H_2$ such that $|\zeta|_{j_k} \neq 0$.

Proof. i) Let $\zeta \in M_k$ such that $|\zeta|_{\mathrm{j}_k} = 1$. Since B is a SM_k-balanced or TM_k-balanced set, $\zeta B \subseteq B$. On the other hand

$$\left|\frac{1}{\zeta}\right|_{\mathrm{j}_k} = \frac{1}{|\zeta|_{\mathrm{j}_k}} = 1.$$

So $\frac{1}{\zeta}B \subseteq B$ and in this way $B \subseteq \zeta B$. Consequently $\zeta B = B$.

ii) Let's take any $\zeta \in M_k$ such that $|\zeta|_{\mathrm{j}_k} \neq 0$. Then

$$\left|\frac{\zeta}{|\zeta|_{\mathrm{j}_k}}\right|_{\mathrm{j}_k} = 1.$$

So,

$$\frac{\zeta}{|\zeta|_{\mathrm{j}_k}} B = B$$

from the condition (i). Hence, we have $\zeta B = |\zeta|_{\mathrm{j}_k} B$.

□

Theorem 4.11. *Let X be a H_2-module and the set B is a SM_k-balanced subset of X for $k = 1,2,3$.*

i) For $s=k=1$, $e_{1j_s}B=B_{1j_s}$ and $e_{2j_s}B=B_{2j_s}$ are SM_k-balanced subsets of H_2-modules $e_{1j_s}X=X_{1j_s}$ and $e_{2j_s}X=X_{2j_s}$, respectively.

ii) For $s,k=2,3$ and $s=k$, $e_{1j_s}B=B_{1j_s}$ and $e_{2j_s}B=B_{2j_s}$ are $SH-$balanced subsets of $H-$modules $e_{1j_s}X=X_{1j_s}$ and $e_{2j_s}X=X_{2j_s}$, respectively.

Proof. i) Let X be a H_2-module and B be a SM_k-balanced subset of X for $k=1$. Therefore, $\zeta x\in B$ for all $x\in B$ and all $\zeta\in SM_k(O)$ such that $\zeta\overline{\zeta}^{\mathrm{j}_k}\leq 1$. Assume that the idempotent representation of ζ is $\zeta=\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}+\zeta_{2\mathrm{j}_s}e_{2\mathrm{j}_s}$ for $s=1$. Since $\zeta\in SM_k(O)$ and $\zeta\overline{\zeta}^{\mathrm{j}_k}\leq 1$, $\zeta_{1\mathrm{j}_s}\in SM_k(O)$ and $\zeta_{1\mathrm{j}_s}\overline{(\zeta_{1\mathrm{j}_s})}^{\mathrm{j}_k}\leq 1$. An element $t\in e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}=e_{1\mathrm{j}_s}B$ is represented by $t=e_{1\mathrm{j}_s}x$ for $x\in B$.

Hence, $\zeta_{1\mathrm{j}_s}t=\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}x=e_{1\mathrm{j}_s}\zeta_{1\mathrm{j}_s}x=e_{1\mathrm{j}_s}\zeta x\in e_{1\mathrm{j}_s}B=e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}$ where $e_{1\mathrm{j}_s}\zeta=e_{1\mathrm{j}_s}\left(\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}+\zeta_{2\mathrm{j}_s}e_{2\mathrm{j}_s}\right)=e_{1\mathrm{j}_s}\zeta_{1\mathrm{j}_s}$. So, the set $e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}$ is SM_k-balanced set of the H_2-module $e_{1\mathrm{j}_s}X_{1\mathrm{j}_s}$. Similarly, the set $e_{2\mathrm{j}_s}B=B_{2\mathrm{j}_s}$ is a SM_k-balanced set of the H_2-module $e_{2\mathrm{j}_s}X=X_{2\mathrm{j}_s}$ for $s=k=1$.

ii) Let X be a H_2-module and B be a SM_k-balanced subset of X for $k=2,3$. Hence, $\zeta x\in B$ for all $x\in B$ and all $\zeta\in SM_k(O)$ such that $\zeta\overline{\zeta}^{\mathrm{j}_k}\leq 1$. The idempotent representation of ζ is $\zeta=\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}+\zeta_{2\mathrm{j}_s}e_{2\mathrm{j}_s}$ for $s=2,3$ and $e_{1\mathrm{j}_s}\zeta=e_{1\mathrm{j}_s}\left(\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}+\zeta_{2\mathrm{j}_s}e_{2\mathrm{j}_s}\right)=e_{1\mathrm{j}_s}\zeta_{1\mathrm{j}_s}$. Moreover, the coefficient $\zeta_{1\mathrm{j}_s}\in H\subseteq H_2$ is $\zeta_{1\mathrm{j}_s}\in SH(O)$ and it provides the inequality $\zeta_{1\mathrm{j}_s}\overline{\zeta_{1\mathrm{j}_s}}\leq 1$ for $s,k=2,3$ $s=k$. An element $t\in e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}=e_{1\mathrm{j}_s}B$ can be written as $t=e_{1\mathrm{j}_s}x$ since $x\in B$. Thus, $\zeta_{1\mathrm{j}_s}t=\zeta_{1\mathrm{j}_s}e_{1\mathrm{j}_s}x=e_{1\mathrm{j}_s}\zeta_{1\mathrm{j}_s}x=e_{1\mathrm{j}_s}\zeta x\in e_{1\mathrm{j}_s}B=e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}$. So, the sets $e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}$ are $SH-$balanced sets of $H-$modules $e_{1\mathrm{j}_s}X_{1\mathrm{j}_s}$ for $s,k=2,3$ and $s=k$. Similarly, the sets $e_{2\mathrm{j}_s}B=B_{2\mathrm{j}_s}$ are $SH-$balanced sets of $H-$modules $e_{2\mathrm{j}_s}X=X_{2\mathrm{j}_s}$ for $s,k=2,3$ $s=k$.

□

Theorem 4.12. *Let X be a H_2-module and B be a NM_k-balanced subset of X for $k=1,2,3$.*

i) For $s=k=1$, $e_{1j_s}B=B_{1j_s}$ and $e_{2j_s}B=B_{2j_s}$ are NM_k-balanced subsets of H_2-modules $e_{1j_s}X=X_{1j_s}$ and $e_{2j_s}X=X_{2j_s}$, respectively.

ii) For $s,k=2,3$ and $s=k$, $e_{1j_s}B=B_{1j_s}$ and $e_{2j_s}B=B_{2j_s}$ are $NH-$balanced subsets of $H-$modules $e_{1j_s}X=X_{1j_s}$ and $e_{2j_s}X=X_{2j_s}$, respectively.

Theorem 4.13. *Let X be a H_2-module and B be a TM_k-balanced subset of X for $k=1,2,3$.*

i) For $s=k=1$, $e_{1j_s}B=B_{1j_s}$ and $e_{2j_s}B=B_{2j_s}$ are TM_k-balanced subsets of H_2-modules $e_{1j_s}X=X_{1j_s}$ and $e_{2j_s}X=X_{2j_s}$, respectively.

ii) For $s,k=2,3$ and $s=k$, $e_{1j_s}B=B_{1j_s}$ and $e_{2j_s}B=B_{2j_s}$ are $TH-$balanced subsets of $H-$modules $e_{1j_s}X=X_{1j_s}$ and $e_{2j_s}X=X_{2j_s}$, respectively.

Theorem 4.14. *Let X be a H_2-module and B be a NM_k-balanced subset of X for $k=1,2,3$. Then $e_{1j_s}B=B_{1j_s}\subseteq B$ and $e_{2j_s}B=B_{2j_s}\subseteq B$ for $s=1,2,3$ and $s\neq k$.*

Proof. Let $x\in B$ and an element $t\in e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}=e_{1\mathrm{j}_s}B$ be given by $t=e_{1\mathrm{j}_s}x$. Since the set B is NM_k-balanced set, $\zeta x\in B$ for all $\zeta\in NM_k(O)$. $e_{1\mathrm{j}_s}\in NM_k(O)$ for $s,k=1,2,3$ and $s\neq k$. Thus, if we choose $\zeta=e_{1\mathrm{j}_s}$, then $e_{1\mathrm{j}_s}B_{1\mathrm{j}_s}\subseteq B$. Similarly, if $\zeta=e_{2\mathrm{j}_s}$ is chosen, $e_{2\mathrm{j}_s}B_{2\mathrm{j}_s}\subseteq B$ for $s,k=1,2,3$ and $s\neq k$. □

The inclusions $e_{1\mathrm{j}_s}B=B_{1\mathrm{j}_s}\subseteq B$ and $e_{2\mathrm{j}_s}B=B_{2\mathrm{j}_s}\subseteq B$ do not exist for a SM_k-balanced or TM_k-balanced subset B of H_2-modules X. Because the idempotent components $e_{1\mathrm{j}_s}$ and $e_{2\mathrm{j}_s}$ are $e_{1\mathrm{j}_s},e_{2\mathrm{j}_s}\notin M_k$ for $s=k$ and $e_{1\mathrm{j}_s},e_{2\mathrm{j}_s}\in NM_k$ for $s\neq k$.

Definition 4.15. *Let X be a H_2-module and $\varnothing\neq B\subseteq X$. B is a H_2-convex set if $\zeta x+(1-\zeta)y\in B$ for all $x,y\in B$ and all $\zeta\in H_2^+$ such that $0\leq\zeta\leq 1$.*

Theorem 4.16. *Let X be a H_2-module and $\varnothing\neq B\subseteq X$ is a H_2-convex subset of X.*

i) The sets $e_{1j_s}B$ and $e_{2j_s}B$ are H_2-convex sets of H_2-modules $e_{1j_s}X$ and $e_{2j_s}X$ for $s=1$, respectively.

ii) The sets $e_{1j_s}B$ and $e_{2j_s}B$ are $H-$convex sets of the $H-$modules $e_{1j_s}X$ and $e_{2j_s}X$ for $s=2,3$, respectively.

iii) *There are the inclusions* $e_{1j_s}B \subseteq B$ *and* $e_{2j_s}B \subseteq B$ *for* $s = 1,2,3$*, if* $\theta \in B$ *where* θ *is the unit element of the* H_2-*module* X.

Proof. i) Let B be a H_2-convex subset of the H_2-module X and $t_1, t_2 \in e_{1j_s}B$ for $s = 1$. There exist $x, y \in B$ such that $t_1 = e_{1j_s}x \in e_{1j_s}B$ and $t_2 = e_{1j_s}y \in e_{1j_s}B$. Consider $\zeta = \zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s} \in H_2^+$ for all $\zeta_{1j_s}, \zeta_{2j_s} \in H_2^+$ such that $\zeta_{1j_s}, \zeta_{2j_s} \in [0,1]$. If $\zeta_{1j_s}, \zeta_{2j_s} \in [0,1]$, then $\zeta \in [0,1]$ [13]. Thus, since the set B is H_2-convex, $\zeta x + (1-\zeta)y \in B$ for $x, y \in B$, $\zeta \in H_2^+$ and $\zeta \in [0,1]$. In that case,

$$\begin{aligned} e_{1j_s}(\zeta x + (1-\zeta)y) &= e_{1j_s}\left(\left(\zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s}\right)x \right.\\ &\quad \left. + \left(1 - \left(\zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s}\right)\right)y\right) \\ &= \zeta_{1j_s}e_{1j_s}x + \left(1-\zeta_{1j_s}\right)e_{1j_s}y \\ &= \zeta_{1j_s}t_1 + \left(1-\zeta_{1j_s}\right)t_2 \in e_{1j_s}B. \end{aligned}$$

From here, the set $e_{1j_s}B$ is a H_2-convex subset of H_2-modules $e_{1j_s}X$. Similarly, it can be proved that the set $e_{2j_s}B$ is H_2-convex subset of H_2-module $e_{2j_s}X$ for $s = 1$.

ii) Let $t_1 = e_{1j_s}x \in e_{1j_s}B$ and $t_2 = e_{1j_s}y \in e_{1j_s}B$ for $x, y \in B$ and $s = 2,3$. $\zeta = \zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s} \in H_2^+$ such that $\zeta_{1j_s}, \zeta_{2j_s} \in H^+$ and $\zeta_{1j_s}, \zeta_{2j_s} \in [0,1]$. Hence, $\zeta \in [0,1]$. Since the set B is H_2-convex set $\zeta x + (1-\zeta)y \in B$. Similarly, we get

$$\begin{aligned} e_{1j_s}(\zeta x + (1-\zeta)y) &= e_{1j_s}\left(\left(\zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s}\right)x \right.\\ &\quad \left. + \left(1 - \left(\zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s}\right)\right)y\right) \\ &= \zeta_{1j_s}e_{1j_s}x + \left(1-\zeta_{1j_s}\right)e_{1j_s}y \\ &= \zeta_{1j_s}t_1 + \left(1-\zeta_{1j_s}\right)t_2 \in e_{1j_s}B. \end{aligned}$$

Hence, the sets $e_{1j_s}B$ for $s = 2,3$ are $H-$convex subsets of $H-$modules $e_{1j_s}X$. Also, it can be proved that the sets $e_{2j_s}B$ are $H-$convex subsets of $H-$modules $e_{2j_s}X$ for $s = 2,3$ in a similar manner.

iii) Let B be a H_2-convex subset of the H_2-module X and $\theta \in B$. $t \in e_{1j_s}B$ for $s = 1,2,3$. There is an element $x \in B$ such that $t = e_{1j_s}x \in e_{1j_s}B$. Considering that $\theta \in B$, since B is H_2-convex subset $e_{1j_s}x + \left(1 - e_{1j_s}\right)\theta = e_{1j_s}x = t \in B$ where $0 \leq e_{1j_s} \leq 1$ and $e_{1j_s} \in H_2^+$ $\left(H^+ \subseteq H_2^+\right)$ for $x, \theta \in B$. Consequently $e_{1j_s}B \subseteq B$ is obtained. Similarly, we deduce $e_{2j_s}B \subseteq B$ for $s = 1,2,3$.

□

Lemma 4.17. *Let* X *be a* H_2-*module and the sets* $\{B_l : \ l \text{ arbitrary}\}$ *be any* H_2-*convex subsets of* X*. Then, the set* $\bigcap_l B_l = B$ *is* H_2-*convex, too.*

Theorem 4.18. *Let* X *be a* H_2-*module and* $\varnothing \neq B \subseteq X$ *be a* H_2-*convex subset. Then,* $B = e_{1j_s}B + e_{2j_s}B$ *for* $s = 1,2,3$.

Proof. Assume that B is a H_2-convex subset of H_2-modules X and take $x \in B$. $e_{1j_s}x \in e_{1j_s}B$ and $e_{2j_s}x \in e_{2j_s}B$ for $s = 1,2,3$. Since $e_{1j_s} + e_{2j_s} = 1$ then

$$x = \left(e_{1j_s} + e_{2j_s}\right)x = e_{1j_s}x + e_{2j_s}x \in e_{1j_s}B + e_{2j_s}B.$$

Thus, $B \subseteq e_{1j_s}B + e_{2j_s}B$. Conversely, let us take $t_1 \in e_{1j_s}B$ and $t_2 \in e_{2j_s}B$ where $t_1 = e_{1j_s}x$ and $t_2 = e_{2j_s}y$ for $x, y \in B$. Since the set B is H_2-convex, $t_1 + t_2 = e_{1j_s}x + e_{2j_s}y = e_{1j_s}x + \left(1 - e_{1j_s}\right)y \in B$ where $e_{1j_s}, e_{2j_s} \in H_2^+$ and $0 \leq e_{1j_s}, e_{2j_s} \leq 1$. Therefore, $e_{1j_s}B + e_{2j_s}B \subseteq B$. This completes the proof.

□

Theorem 4.19. *Let* X *be a* H_2-*module and* $\varnothing \neq B \subseteq X$*. If the sets* $e_{1j_s}B$ *and* $e_{2j_s}B$ *are* H_2-*convex sets for* $s = 1,2,3$ *then the set* $e_{1j_s}B + e_{2j_s}B$ *is a* H_2-*convex subset of* X*, too.*

Proof. Assume that $x, y \in e_{1j_s}B + e_{2j_s}B$ and $\zeta \in H_2^+$ such that $0 \leq \zeta \leq 1$. Then, $x = e_{1j_s}x + e_{2j_s}x$ and $y = e_{1j_s}y + e_{2j_s}y$ where $e_{1j_s}x,\ e_{1j_s}y \in e_{1j_s}B$ and $e_{2j_s}x,\ e_{2j_s}y \in e_{2j_s}B$. The idempotent representation of ζ is $\zeta = \zeta_{1j_s}e_{1j_s} + \zeta_{2j_s}e_{2j_s}$. Hence, $0 \leq \zeta_{1j_s}, \zeta_{2j_s} \leq 1$ and $\zeta_{1j_s}, \zeta_{2j_s} \in H_2^+$ because of $\zeta \in H_2^+$. Since the sets $e_{1j_s}B$ and $e_{2j_s}B$ are H_2-convex, then

$$e_{1j_s}\zeta_{1j_s}x + e_{1j_s}\left(1 - \zeta_{1j_s}\right)y \in e_{1j_s}B,$$

$$e_{2j_s}\zeta_{2j_s}x + e_{2j_s}\left(1 - \zeta_{2j_s}\right)y \in e_{2j_s}B.$$

Therefore,

$$\begin{aligned}\zeta x+(1-\zeta)y &= \zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s}x+\zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s}x+\left(1-\zeta_{1,\mathrm{j}_s}\right)e_{1,\mathrm{j}_s}y+\left(1-\zeta_{2,\mathrm{j}_s}\right)e_{2,\mathrm{j}_s}y\\ &= \zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s}x+\left(1-\zeta_{1,\mathrm{j}_s}\right)e_{1,\mathrm{j}_s}y+\zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s}x+\left(1-\zeta_{2,\mathrm{j}_s}\right)e_{2,\mathrm{j}_s}y\end{aligned}$$

and $[\zeta x+(1-\zeta)y]\in e_{1,\mathrm{j}_s}B+e_{2,\mathrm{j}_s}B$. This proves the assertion.

□

Especially, if we take H_2-modules $X=H_2$, three different convex set definitions which are meaningful geometrically are given for the first time in the following definition.

Definition 4.20. *Let $B\subseteq M_k\subseteq H_2$ be a subset of H_2-module H_2 for $k=1,2,3$. For all $x,y\in B$ and all real numbers $\lambda\in\mathbb{R}$ such that $0\leq\lambda\leq 1$, then*

i) B is called SM_k-convex set if $\lambda x+(1-\lambda)y\in B$ and $y\in SM_k(x)$,

ii) B is called NM_k-convex set if $\lambda x+(1-\lambda)y\in B$ and $y\in NM_k(x)$,

iii) B is called TM_k-convex set if $\lambda x+(1-\lambda)y\in B$ and $y\in TM_k(x)$.

Theorem 4.21. *Let $B\subseteq M_k\subseteq H_2$ be a SM_k-convex subset of H_2-module H_2. The sets $e_{1,j_s}B=e_{1,j_s}B_{1,j_s}$ and $e_{2,j_s}B=e_{2,j_s}B_{2,j_s}$ are, respectively $s,k=1,2,3$,*

i) SM_k-convex subsets of H_2-modules $e_{1,j_s}H_2$ and $e_{2,j_s}H_2$ if $s=k$,

ii) NM_k-convex subsets of H_2-modules $e_{1,j_s}H_2$ and $e_{2,j_s}H_2$ if $s\neq k$.

Proof. i) Let us take $t_1,t_2\in e_{1,\mathrm{j}_s}B_{1,\mathrm{j}_s}$ for $s=k$ $s,k=1,2,3$. There are arbitrary elements $x,y\in B$ such that $t_1=e_{1,\mathrm{j}_s}x$ and $t_2=e_{1,\mathrm{j}_s}y$. Since the set B is a SM_k-convex set, $\lambda x+(1-\lambda)y\in B$ where $y\in SM_k(x)$ and $\lambda\in\mathbb{R}$ such as $0\leq\lambda\leq 1$. Moreover, we find

$$\begin{aligned}e_{1,\mathrm{j}_s}(\lambda x+(1-\lambda)y) &= \lambda e_{1,\mathrm{j}_s}x+(1-\lambda)e_{1,\mathrm{j}_s}y\\ &= \lambda t_1+(1-\lambda)t_2\in e_{1,\mathrm{j}_s}B\\ &= e_{1,\mathrm{j}_s}B_{1,\mathrm{j}_s}.\end{aligned}$$

Also, if $t_1,t_2\in e_{1,\mathrm{j}_s}B_{1,\mathrm{j}_s}$, then $t_1=e_{1,\mathrm{j}_s}t_1$ and $t_2=e_{1,\mathrm{j}_s}t_2$. When $s=k$, if $y\in SM_k(x)$, then $t_2\in SM_k(t_1)$ from [14]. Consequently, the sets $e_{1,\mathrm{j}_s}B$ are SM_k-convex subsets of the H_2-modules $e_{1,\mathrm{j}_s}H_2$. Similarly, it is proven that the sets $e_{2,\mathrm{j}_s}B$ are SM_k-convex subsets of H_2-modules $e_{2,\mathrm{j}_s}H_2$ for $s=k$.

ii) Following a similar way to the first proof and considering that if $y\in SM_k(x)$, then $t_2\in NM_k(t_1)$ for $s\neq k$ from [14], it is proven that the sets $e_{1,\mathrm{j}_s}B$ are NM_k-convex subsets of H_2-modules $e_{1,\mathrm{j}_s}H_2$. Similarly, the sets $e_{2,\mathrm{j}_s}B$ are NM_k-convex subsets of H_2-modules $e_{2,\mathrm{j}_s}H_2$, too.

□

Theorem 4.22. *Let $B\subseteq M_k\subseteq H_2$ be a NM_k-convex subset of H_2-module H_2. The sets $e_{1,j_s}B=e_{1,j_s}B_{1,j_s}$ and $e_{2,j_s}B=e_{2,j_s}B_{2,j_s}$ are NM_k-convex sets of H_2-modules $e_{1,j_s}H_2$ and $e_{2,j_s}H_2$ respectively $s,k=1,2,3$ where $s=k$ or $s\neq k$.*

Theorem 4.23. *Let $B\subseteq M_k\subseteq H_2$ be a TM_k-convex subset of H_2-module H_2. The sets $e_{1,j_s}B=e_{1,j_s}B_{1,j_s}$ and $e_{2,j_s}B=e_{2,j_s}B_{2,j_s}$ are, respectively $s,k=1,2,3$,*

i) TM_k-convex subsets of H_2-modules $e_{1,j_s}H_2$ and $e_{2,j_s}H_2$ if $s=k$,

ii) NM_k-convex subsets of H_2-modules $e_{1,j_s}H_2$ and $e_{2,j_s}H_2$ if $s\neq k$.

Definition 4.24. *Let X be a H_2-module and $\varnothing\neq B\subseteq X$. Some real numbers $\lambda>0$ for all $x\in X$ and for all scalars $\mu\in M_k\subseteq H_2$ such that $|\mu|_{j_k}\geq\lambda$ $(k=1,2,3)$,*

i) B is called SM_k-absorbing set if $x\in\mu B$ and $\mu\in SM_k(O)$,

ii) B is called TM_k-absorbing set if $x\in\mu B$ and $\mu\in TM_k(O)$.

Theorem 4.25. *Let X be a H_2-module and $\varnothing\neq B\subseteq X$. If the subset B is a SM_k-absorbing set $(k=1,2,3)$. Then* *i)* $e_{1,j_s}B = e_{1,j_s}B_{1,j_s}$ *and* $e_{2,j_s}B = e_{2,j_s}B_{2,j_s}$ *are* SM_k-*absorbing sets of* H_2-*modules* $e_{1,j_s}X = X_{1,j_s}$ *and* $e_{2,j_s}X = X_{2,j_s}$ *for* $s = k = 1$*, respectively.*

ii) $e_{1,j_s}B = e_{1,j_s}B_{1,j_s}$ *and* $e_{2,j_s}B = e_{2,j_s}B_{2,j_s}$ *are* $SH-$*absorbing sets of* $H-$*modules* $e_{1,j_s}X = X_{1,j_s}$ *and* $e_{2,j_s}X = X_{2,j_s}$ *for* $s,k = 2,3$ *and* $s = k$*, respectively.*

Proof. i) Let's take $\tilde{x} \in e_{1,j_1}X$ for $s = 1$. There is an element $x \in X$ such that $\tilde{x} = e_{1,j_1}x$. Since B is SM_1-absorbing set for $k = 1$, $x \in \mu B$ for some real numbers $\lambda > 0$ and all scalars $\mu \in SM_1(O)$ such as $|\mu|_{j_1} \geq \lambda$. If we take $\mu = \mu_{1,j_1}e_{1,j_1} + \mu_{2,j_1}e_{2,j_1}$, then

$$\tilde{x} = e_{1,j_1}x \in e_{1,j_1}\mu B = e_{1,j_1}\left(\mu_{1,j_1}e_{1,j_1} + \mu_{2,j_1}e_{2,j_1}\right)B = \mu_{1,j_1}e_{1,j_1}B$$

is obtained. On the other hand, if $\mu \in SM_1(O)$, then $|\mu|_{j_1} = \left|\mu_{1,j_1}\right|_{j_1}$ and hence $\mu_{1,j_1} \in SM_1(O)$ from the [14]. Consequently, $\tilde{x} \in \mu_{1,j_1}e_{1,j_1}B$ for some real numbers $\lambda > 0$ and for all scalars $\mu_{1,j_1} \in SM_1(O)$ such that $\left|\mu_{1,j_1}\right|_{j_1} = |\mu|_{j_1} \geq \lambda$. In that case, the set $e_{1,j_1}B = e_{1,j_1}B_{1,j_1}$ is a SM_1-absorbing subset of H_2-module $e_{1,j_1}X = e_{1,j_1}X_{1,j_1}$.

ii) Consider $\tilde{x} \in e_{1,j_2}X$ for $s = k = 2$ where $\tilde{x} = e_{1,j_2}x$ and $x \in X$. Since B is SM_2-absorbing set for $k = 2$, $x \in \mu B$ for some real numbers $\lambda > 0$ and for all scalars $\mu \in SM_2(O)$ such that $|\mu|_{j_2} \geq \lambda$. Hence

$$\tilde{x} = e_{1,j_2}x \in e_{1,j_2}\mu B = e_{1,j_2}\left(\mu_{1,j_2}e_{1,j_2} + \mu_{2,j_2}e_{2,j_2}\right)B = \mu_{1,j_2}e_{1,j_2}B$$

is obtained where $\mu = \mu_{1,j_2}e_{1,j_2} + \mu_{2,j_2}e_{2,j_2}$. On the other hand, $|\mu|_{j_2} = \left|\mu_{1,j_2}\right|_H$ and $\mu_{1,j_2} \in SH(O)$ from the [14]. Hence, the set $e_{1,j_2}B = e_{1,j_2}B_{1,j_2}$ is $SH-$absorbing set of H_2-modules $e_{1,j_2}X = e_{1,j_2}X_{1,j_2}$. The case $s = k = 3$ can be proved by using the similar way.

□

Theorem 4.26. *Let* X *be a* H_2-*module and* $\varnothing \neq B \subseteq X$*. If the subset* B *is* TM_k-*absorbing set for* $k = 1,2,3$*, then*

i) $e_{1,j_s}B = e_{1,j_s}B_{1,j_s}$ *and* $e_{2,j_s}B = e_{2,j_s}B_{2,j_s}$ *are* TM_k-*absorbing sets of* H_2-*modules* $e_{1,j_s}X = X_{1,j_s}$ *and* $e_{2,j_s}X = X_{2,j_s}$ *for* $s = k = 1$*,*

ii) $e_{1,j_s}B = e_{1,j_s}B_{1,j_s}$ *and* $e_{2,j_s}B = e_{2,j_s}B_{2,j_s}$ *are* $TH-$*absorbing sets of* $H-$*modules* $e_{1,j_s}X = X_{1,j_s}$ *and* $e_{2,j_s}X = X_{2,j_s}$ *for* $s,k = 2,3$ *and* $s = k$*.*

Topological bihyperbolic module which is not previously found in the literature is defined as follows.

Definition 4.27. *Let* X *be a* H_2-*module and* τ *is a Hausdorff topology on* X*. If the operations*

$$\begin{aligned} + &: X \times X \to X \\ \cdot &: H_2 \times X \to X \end{aligned}$$

are continuous, then the pair (X,τ) *is called a topological bihyperbolic module or topological* H_2-*module.*

When the topological vector spaces were introduced in [21], there was a condition such that the single point sets are closed according to the topology on it. The topological vector spaces are Hausdorff space with this condition. But, when the topological vector spaces were introduced in the literature, it was not said that the topology which is corresponding with the topological vector spaces are Hausdorff topology. The reason for this is usually that most of the spaces already provide the Hausdorff property. For instance, the topologies generated by norms on the normed vector space or the topologies generated by metrics are Hausdorff topologies. These structures which are using in the functional analysis frequently appear in the topological vector spaces, too. Although this article has more general structure than these structures, the topology corresponding with H_2-module is taken as Hausdorff topology, unless otherwise stated.

Theorem 4.28. *Let* (X,τ) *be a topological* H_2-*module. The families*

$$\tau_{1,j_s} = \left\{e_{1,j_s}G : \ G \in \tau\right\},$$

$$\tau_{2,j_s} = \left\{e_{2,j_s}G : \ G \in \tau\right\}$$

are Hausdorff topologies on the H_2-*modules* X_{1,j_s} *and* X_{2,j_s} *for* $s = 1,2,3$*, respectively. Especially, they are Hausdorff topologies on* $H-$*modules* X_{1,j_s} *and* X_{2,j_s} *for* $s = 2,3$*, respectively.*

Theorem 4.29. *Let (X,τ) be a topological H_2-module and $\left(X_{i,j_s},\tau_{i,j_s}\right)$ be topological spaces for $s=1,2,3$ and $i=1,2$. Then, the operations*

$$+:X_{i,j_s}\times X_{i,j_s}\rightarrow X_{i,j_s},$$
$$\cdot\ :H_2\times X_{i,j_s}\ \ \rightarrow X_{i,j_s}$$

are continuous.

Especially, the subsets X_{1,j_s} and X_{2,j_s} are $H-$modules of the H_2-modules X, since $\zeta_{1,\mathrm{j}_s},\zeta_{2,\mathrm{j}_s}\in H$ where $\zeta=\zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s}+\zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s}\in H_2$ for $s=2,3$. Hence, the operations

$$+:X_{i,\mathrm{j}_s}\times X_{i,\mathrm{j}_s}\rightarrow X_{i,\mathrm{j}_s}$$
$$\cdot\ :H\times X_{i,\mathrm{j}_s}\ \ \rightarrow X_{i,\mathrm{j}_s}$$

are continuous for $s=2,3$ and $i=1,2$, too.

Corollary 4.30. *Let (X,τ) be a topological H_2-module. The pair $\left(X_{i,j_s},\tau_{i,j_s}\right)$ are topological H_2-modules for $s=1,2,3$ and $i=1,2$. Especially, the pair $\left(X_{i,j_s},\tau_{i,j_s}\right)$ are topological $H-$modules for $s=2,3$ and $i=1,2$, too.*

Theorem 4.31. *Let (X,τ) be a topological H_2-module. If the operation $T_y:X\rightarrow X$ for any $y\in X$ is defined as $T_y(x)=x+y$ for all $x\in X$, then it is a homeomorphism.*

Proof. The operation T_y is continuous by the definition of the topological module and it is bijective by the axioms of the module. Moreover, $T_y^{-1}(x)=T_{-y}(x)=x-y$ and $T_y\circ T_{-y}=T_{-y}\circ T_y=I$ are obtained. Therefore, the operation $T_y^{-1}=T_{-y}$ is also continuous. Consequently, the operation T_y is a homeomorphism. □

Theorem 4.32. *Let (X,τ) be a topological H_2-module. If the operation $M_\zeta:X\rightarrow X$ for any $\zeta\in H_2^*$ is defined as $M_\zeta(x)=\zeta x$ for all $x\in X$, then it is a homeomorphism.*

Proof. The operation M_ζ is continuous by the definition of the topological H_2-module and it is bijective by the axioms of the module. $M_\zeta^{-1}(x)=M_{1/\zeta}(x)=\frac{x}{\zeta}$ for $\zeta\in H_2^*$ and $M_\zeta\circ M_{1/\zeta}=M_{1/\zeta}\circ M_\zeta=I$ are obtained. Hence, the operation $M_\zeta^{-1}=M_{1/\zeta}$ is also continuous. This completes the proof. □

We will investigate the properties of the interiors and the closures of the subsets of the H_2-module X in the following theorems. A° represents the interior of the set A and $\bar{A}$ represents the closure of the set A.

Theorem 4.33. *Let X be a topological H_2-module and $\varnothing\neq B\subseteq X$. Then the followings are satisfied.*

i) $\left(e_{1,j_s}B\right)^\circ=e_{1,j_s}B^\circ$ *and* $\left(e_{2,j_s}B\right)^\circ=e_{2,j_s}B^\circ$ $(s=1,2,3)$.

ii) $\overline{\left(e_{1,j_s}B\right)}=e_{1,j_s}\bar{B}$ *and* $\overline{\left(e_{2,j_s}B\right)}=e_{2,j_s}\bar{B}$ $(s=1,2,3)$.

Proof. i) Let's take $x\in\left(e_{1,\mathrm{j}_s}B\right)^\circ$. There exists an open neighbourhood $G\subseteq X$ such that $x\in e_{1,\mathrm{j}_s}G\subseteq e_{1,\mathrm{j}_s}B$ where $x=e_{1,\mathrm{j}_s}y$ and $y\in G$. Clearly, $y\in G^\circ$. Thus, $x=e_{1,\mathrm{j}_s}y\in e_{1,\mathrm{j}_s}B^\circ$ and $\left(e_{1,\mathrm{j}_s}B\right)^\circ\subseteq e_{1,\mathrm{j}_s}B^\circ$ are obtained. Conversely, let's take $y\in B^\circ$. Hence, $e_{1,\mathrm{j}_s}y\in e_{1,\mathrm{j}_s}B^\circ$. If $y\in B^\circ$, then there is an open neighbourhood $G\subseteq X$ such as $y\in G\subseteq B$. Therefore, $e_{1,\mathrm{j}_s}y\in e_{1,\mathrm{j}_s}G\subseteq e_{1,\mathrm{j}_s}B$. Since G is the open set in X, the set $e_{1,\mathrm{j}_s}G$ is also an open set in $e_{1,\mathrm{j}_s}X$ from Theorem 4.28 , too. Consequently, $e_{1,\mathrm{j}_s}y\in\left(e_{1,\mathrm{j}_s}B\right)^\circ$ and $e_{1,\mathrm{j}_s}B^\circ\subseteq\left(e_{1,\mathrm{j}_s}B\right)^\circ$ are obtained. These two inclusions prove the assertion. Similarly, it can be shown that $\left(e_{\mathrm{j}_s}^2B\right)^\circ=e_{\mathrm{j}_s}^2B^\circ$.

ii) Let's take $x\in\overline{\left(e_{1,\mathrm{j}_s}B\right)}$. There exists a net $\{x_l\}\in e_{1,\mathrm{j}_s}B$ such that $\{x_l\}\rightarrow x$. Moreover, the net $\{y_l\}\in B$ where $\{x_l\}=\left\{e_{1,\mathrm{j}_s}y_l\right\}$ can be taken such as $\{y_l\}\rightarrow y$. Hence, $y\in\bar{B}$. This means that $\{x_l\}=\left\{e_{1,\mathrm{j}_s}y_l\right\}\rightarrow e_{1,\mathrm{j}_s}y$. Since the topological space (X,τ) is Hausdorff, the spaces $\left(e_{1,\mathrm{j}_s}X,\tau_{1,\mathrm{j}_s}\right)$ are Hausdorff, too. So, if there is the limit of a net in the subset $e_{1,\mathrm{j}_s}B\subseteq e_{1,\mathrm{j}_s}X$, it is unique. Therefore, $x=e_{1,\mathrm{j}_s}y\in e_{1,\mathrm{j}_s}\bar{B}$. From here, the inclusion $\overline{\left(e_{1,\mathrm{j}_s}B\right)}\subseteq e_{1,\mathrm{j}_s}\bar{B}$ is obtained. Conversely, take $y\in\bar{B}$. Hence, $e_{1,\mathrm{j}_s}y\in e_{1,\mathrm{j}_s}\bar{B}$. If $y\in\bar{B}$, then there is a net $\{y_l\}\subseteq B$ such that $\{y_l\}\rightarrow y$. Therefore, there exists a net $\left\{e_{1,\mathrm{j}_s}y_l\right\}\subseteq e_{1,\mathrm{j}_s}B$ such as $\left\{e_{1,\mathrm{j}_s}y_l\right\}\rightarrow e_{1,\mathrm{j}_s}y$. So, $e_{1,\mathrm{j}_s}y\in\overline{\left(e_{1,\mathrm{j}_s}B\right)}$ and $e_{1,\mathrm{j}_s}\bar{B}\subseteq\overline{\left(e_{1,\mathrm{j}_s}B\right)}$ are obtained. Similarly, one can prove that $\overline{\left(e_{2,\mathrm{j}_s}B\right)}=e_{2,\mathrm{j}_s}\bar{B}$.

□

Theorem 4.34. *Let X be a topological H_2-module and $\varnothing \neq B \subseteq X$. If B is a H_2-convex subset of X then the following relations are satisfied for $s = 1,2,3$.*

i) $B^{\circ} = e_{1,j_s}B^{\circ} + e_{2,j_s}B^{\circ}$,

ii) $\bar{B} = e_{1,j_s}\bar{B} + e_{2,j_s}\bar{B}$,

iii) B° *is* H_2-*convex,*

iv) $\overline{B}$ *is* H_2-*convex.*

Proof. i) Take into consideration $x \in B^{\circ}$. Then $x = \left(e_{1,j_s} + e_{2,j_s}\right)x = e_{1,j_s}x + e_{2,j_s}x \in e_{1,j_s}B^{\circ} + e_{2,j_s}B^{\circ}$ since $e_{1,j_s} + e_{2,j_s} = 1$. So $B^{\circ} \subseteq e_{1,j_s}B^{\circ} + e_{2,j_s}B^{\circ}$. On the other hand, since B is H_2-convex, $B = e_{1,j_s}B + e_{2,j_s}B$ from Theorem 4.18. Hence, $e_{1,j_s}B^{\circ} + e_{2,j_s}B^{\circ}$ is an open subset of the topological H_2-module X where $e_{1,j_s}B^{\circ} + e_{2,j_s}B^{\circ} \subseteq e_{1,j_s}B + e_{2,j_s}B = B$. But, the largest open set contained in B must be B°. So, $e_{1,j_s}B^{\circ} + e_{2,j_s}B^{\circ} \subseteq B^{\circ}$. This completes the proof.

ii) If $x \in \bar{B}$ is taken, then $x \in e_{1,j_s}\bar{B} + e_{2,j_s}\bar{B}$ and $\bar{B} \subseteq e_{1,j_s}\bar{B} + e_{2,j_s}\bar{B}$ are obtained. Note that in a topological vector space X if $A \subseteq X$ and $B \subseteq X$, then $\bar{A} + \bar{B} \subseteq \overline{A+B}$ [21]. Thus,

$$e_{1,j_s}\bar{B} + e_{2,j_s}\bar{B} = \overline{e_{1,j_s}B} + \overline{e_{2,j_s}B} \subseteq \overline{e_{1,j_s}B + e_{2,j_s}B} = \bar{B}$$

from Theorem 4.33.

iii) Since B is H_2-convex, $\zeta x + (1-\zeta)y \in B$ for all $x, y \in B$ and for all $\zeta \in H_2^+$ such that $0 \leq \zeta \leq 1$. This means that $\zeta x + (1-\zeta)y$ is an element of B when the elements x and y are scanning the set B. So, $\zeta B + (1-\zeta)B \subseteq B$ is obtained. $B^{\circ} = \zeta B^{\circ} + (1-\zeta)B^{\circ} \subseteq B$ since $B^{\circ} \subseteq B$. Assume that $\zeta = 0$. Therefore, $\zeta B^{\circ} + (1-\zeta)B^{\circ} = B^{\circ} \subseteq B^{\circ}$. Now, let's take $\zeta \neq 0$. Since the addition and multiplication with scalar operations are homeomorphisms in X and B° is an open set in X, $\zeta B^{\circ} + (1-\zeta)B^{\circ}$ is an open set, too. But, the largest open set contained in B is B°. So, $\zeta B^{\circ} + (1-\zeta)B^{\circ} \subseteq B^{\circ}$. Consequently, B° is a H_2-convex set.

iv) Let B be a H_2-convex subset of the topological H_2-module X. Let's define an operation

$$\varphi_{\zeta} : X \times X \to X$$

$$(x,y) \to \zeta x + (1-\zeta)y$$

for all $\zeta \in H_2^+$ such that $0 \leq \zeta \leq 1$. Since X is a topological H_2-module, the addition and the multiplication with scalar operations are continuous on X and hence the operation φ_{ζ} is continuous, too. Moreover, since B is H_2-convex, $\varphi_{\zeta}(B \times B) \subseteq B$ for $\zeta \in H_2^+$ such as $0 \leq \zeta \leq 1$. Therefore, $\overline{\varphi_{\zeta}(B \times B)} \subseteq \overline{B}$. So we get $\varphi_{\zeta}\overline{(B \times B)} \subseteq \overline{\varphi_{\zeta}(B \times B)}$ since the operation φ_{ζ} is continuous. Consequently, $\varphi_{\zeta}\left(\overline{B} \times \overline{B}\right) = \varphi_{\zeta}\overline{(B \times B)} \subseteq \overline{B}$. Hence, $\overline{B}$ is a H_2-convex subset of the topological H_2-module X.

□

Theorem 4.35. *Let X be a topological H_2-module and the subset $\varnothing \neq B \subseteq X$ be a SM_k-balanced subset of X for $k = 1,2,3$. Then, the sets $\overline{B}$ and B° are SM_k-balanced sets under the condition $\theta \in B^{\circ}$ where θ is the unit element.*

Proof. Let's take $\zeta \in SM_k(O)$ such that $\zeta\overline{\zeta}^{j_k} \leq 1$. If $\zeta = 0$, then $\zeta\overline{B} = \{\theta\} \subseteq \overline{B}$. We assume that $\zeta \neq 0$. Since $B \subseteq X$ is a SM_k-balanced subset, $\zeta B \subseteq B$. Hence $\overline{\zeta B} \subseteq \overline{B}$. Considering that the multiplication with the scalar operation is a homeomorphism for $\zeta \in H_2^*$ from Theorem 4.32, $\zeta\overline{B} = \overline{\zeta B} \subseteq \overline{B}$ is obtained. Therefore, $\overline{B}$ is a SM_k-balanced set. Assume that $\theta \in B^{\circ}$. First, if $\zeta = 0$, then $\zeta B^{\circ} = \{\theta\} \subseteq B^{\circ}$. Secondly, let's take $\zeta \neq 0$. $\zeta B \subseteq B$ since $B \subseteq X$ is a SM_k-balanced subset. Thus, $(\zeta B)^{\circ} \subseteq B^{\circ}$ and $\zeta B^{\circ} = (\zeta B)^{\circ} \subseteq B^{\circ}$ from Theorem 4.32. Consequently, B° is a SM_k-balanced set. □

Theorem 4.36. *Let X be a topological H_2-module and the subset $\varnothing \neq B \subseteq X$ be a NM_k-balanced subset of X for $k = 1,2,3$. Then $\overline{B}$ is a NM_k-balanced set.*

Proof. Let's take $\zeta \in NM_k(O)$ such that $\zeta\overline{\zeta}^{j_k} = 0$. If $\zeta = 0$, then $\zeta\overline{B} = \{\theta\} \subseteq \overline{B}$. We assume that $\zeta \neq 0$. Since $B \subseteq X$ is a NM_k-balanced subset, $\zeta B \subseteq B$. Hence, $\overline{\zeta B} \subseteq \overline{B}.\zeta\overline{B} \subseteq \overline{\zeta B}$ from Theorem 4.32. Finally, $\zeta\overline{B} \subseteq \overline{\zeta B} \subseteq \overline{B}$ is obtained and so $\overline{B}$ is a NM_k-balanced set. □

The multiplication with scalar operation has inverse only for $\zeta \in H_2^*$. Since the inverse of the multiplication with scalar operation must be continuous so that $\zeta B^\circ \subseteq (\zeta B)^\circ$, B° do not have to be a NM_k-balanced set while the subset B is a NM_k-balanced set.

Theorem 4.37. *Let X be a topological H_2-module and the subset $\varnothing \neq B \subseteq X$ be a TM_k-balanced subset of X for $k=1,2,3$. Then, $\overline{B}$ and B° are TM_k-balanced sets under the condition $\theta \in B^\circ$ where θ is the unit element.*

Theorem 4.38. *Let X be a topological H_2-module. The followings are satisfied for $k=1,2,3$.*

i) All neighbourhoods of the element θ contain a SM_k-absorbing neighbourhood of the element θ in X.

ii) All neighbourhoods of the element θ contain a SM_k-balanced neighbourhood of the element θ in X.

iii) All H_2-convex neighbourhoods of the element θ contain a H_2-convex and SM_k-balanced neighbourhood of the element θ in X.

Proof. i) Let U_θ be any neighbourhood of $\theta \in X$ and V_x be any neighbourhood of $x \in X$. If $\zeta = 0$, then $M_0(x) = \theta$. Since the multiplication with the scalar operation M_ζ is continuous, $M_{A_0}(V_x) \subseteq U_\theta$. Also, there is a neighbourhood of radius $\lambda > 0$ and center $0 \in H_2$ such as $A_0 \subseteq M_k \subseteq H_2$. Therefore, there is a neighbourhood $W_\theta \subseteq U_\theta$ such that $\mu x \in W_\theta$, $|\mu|_{\mathrm{j}_k} \leq \lambda$ and $\mu \in (SM_k(O) \cap A_0)$. Moreover, if we choose $\frac{1}{\lambda} = \delta$, then $\delta > 0$ and $x \in \mu^{-1} W_\theta$ for the scalars μ such as $\left|\mu^{-1}\right|_{\mathrm{j}_k} \geq \delta$. Consequently, W_θ is a SM_k-absorbing subset of X.

ii) Let U_θ be any neighbourhood of the unit element $\theta \in X$. Since $M_0(\theta) = \theta$ and the multiplication with the scalar operation is continuous, there is a neighbourhood of θ such as V_θ and $\mu V_\theta \subseteq U_\theta$ where the elements of the neighbourhood of $0 \in H_2$ with radius $\delta > 0$ are $\mu \in H_2$ and $|\mu|_{\mathrm{j}_k} \leq \delta$. Especially, let's choose $\mu \in SM_k(O)$. If we say $\bigcup\limits_{|\mu|_{\mathrm{j}_k} \leq \delta} \mu V_\theta = A_\theta$, then $\bigcup\limits_{|\mu|_{\mathrm{j}_k} \leq \delta} \mu V_\theta = \theta$ for $\mu = 0$ and $\{\theta\} \subseteq U_\theta$. If $\mu \neq 0$, then A_θ is a neighbourhood of θ and $A_\theta \subseteq U_\theta$. Because the multiplication with the scalar operation is a homeomorphism only for the invertible scalars. On the other hand, take $x \in A_\theta$ and $\zeta \in SM_k(O)$ such that $|\zeta|_{\mathrm{j}_k} \leq 1$. Hence, there is some $y \in V_\theta$ such as $x = \mu y$. We get $\zeta x = \zeta \mu y \in A_\theta$ since $|\zeta\mu|_{\mathrm{j}_k} = |\zeta|_{\mathrm{j}_k} |\mu|_{\mathrm{j}_k} \leq \delta$. So, A_θ is a SM_k-balanced subset of the neighbourhood U_θ.

iii) Let $U_\theta \subseteq X$ be a H_2-convex neighbourhood of $\theta \in X$ and $A = \bigcap\limits_{|\mu|_{\mathrm{j}_k}=1} \mu U_\theta$. There is a SM_k-balanced neighbourhood of θ such that $V_\theta \subseteq U_\theta$ from the previous proposition. Hence, $\mu^{-1} V_\theta = V_\theta$ for $\mu \in SM_k(O)$ such that $|\mu|_{\mathrm{j}_k} = 1$ and $V_\theta \subseteq \mu U_\theta$. Moreover, $V_\theta \subseteq A$. It appears that A is a neighbourhood of θ and $\theta \in A^\circ \subseteq U_\theta$. Now, let's see that the set A° is a H_2-convex and SM_k-balanced subset. Since the images and inverse images of convex sets under linear transformations are convex, the sets μU_θ are H_2-convex for $\mu \in SM_k(O)$ such that $|\mu|_{\mathrm{j}_k} = 1$. Also, the intersection of the H_2-convex sets is H_2-convex. So, the set $A = \bigcap\limits_{|\mu|_{\mathrm{j}_k}=1} \mu U_\theta$ is H_2-convex, too. Hence, the set A° is H_2-convex from Theorem 4.34 (iii). Finally, since μU_θ are H_2-convex sets containing the element θ, $\zeta \mu U_\theta \subseteq \mu U_\theta$ for all $\zeta \in H_2^+$ such that $0 \leq \zeta \leq 1$. On the other hand, $\zeta \lambda A = \bigcap\limits_{|\mu|_{\mathrm{j}_k}=1} \zeta \lambda \mu U_\theta = \bigcap\limits_{|\mu|_{\mathrm{j}_k}=1} \zeta \mu U_\theta \subseteq \bigcap\limits_{|\mu|_{\mathrm{j}_k}=1} \mu U_\theta = A$ for $\lambda \in SM_k(O)$ such that $|\lambda|_{\mathrm{j}_k} = 1$. Hence, the set A is SM_k-balanced. A° is SM_k-balanced according to Theorem 4.35 since $\theta \in A^\circ$.

$\square$

Theorem 4.39. *Let X be a topological H_2-module. Then the following properties are provided for $k=1,2,3$.*

i) All neighbourhoods of the element θ contain a TM_k-absorbing neighbourhood of the element θ in X.

ii) All neighbourhoods of the element θ contain a TM_k-balanced neighbourhood of the element θ in X.

iii) All H_2-convex neighbourhoods of the element θ contain a H_2-convex and TM_k-balanced neighbourhood of the element θ in X.

Since the multiplication with the scalar operation is a homeomorphism only for the scalars which have a multiplicative inverse, the neighbourhood of $\theta \in X$ does not contain NM_k-balanced neighbourhood. Also, a H_2-convex neighbourhood of the element $\theta \in X$ does not contain a NM_k-balanced neighbourhood of the element θ.

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Orthoptic Sets and Quadric Hypersurfaces

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Abstract
Orthoptic curves for the conics are well known. It is the Monge's circle for ellipse and hyperbola, and for parabola it is its directrix. These conics are level sets of quadratic functions in the plane. We consider level sets of quadratic functions in higher dimension, known as quadric hypersurfaces. For these hypersurfaces we present and study their orthoptic sets, which extend the idea of orthoptic curves for conics.

Keywords: Directrix, Monge's circle, orthoptic set, quadric hypersurface
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1. Introduction

In the plane the orthoptic curve is the locus of the points by which pass two perpendicular tangents to the curve, in other words, the locus of the points from which we "see" the curve under a right angle. For the conics in the plane it is related to Monge's work [3].

For ellipse and hyperbola it is called the Monge's circle. Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the Monge's circle is $x^2+y^2=a^2+b^2$, while for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, it is $x^2+y^2=a^2-b^2$, which exists only for $a^2-b^2>0$. For the parabola $y^2=2px$, the orthoptic curve is its directrix $x=-p/2$. See for example [1], [2], [4] for more details.

For these examples in the plane we need two perpendicular tangents to a curve. So the two normal vectors to the tangent planes, which are also normal vectors to the curve, are also orthogonal. One way to consider this locus in higher dimension is to consider a set of tangent planes to the hypersurface such that the set of their normal vectors, to the given tangent planes, form an orthogonal set.

In this paper we consider a natural way to define an orthoptic set associated to a quadric hypersurface. We first present, in Section 2, the surface we are considering and define what we will consider as an orthoptic set. Then some notations are introduced in Section 3. The next two sections contain the presentation and the proofs of our main results. In Section 4 we consider ellipsoid and hyperboloid hypersurfaces. For ellipsoid, the technique in $\mathbb{R}^3$ seems to be due to Monge, as reported in [5] where it is referred to [3]. We present here that it can be extended not only to ellipsoid in $\mathbb{R}^n$, but also to hyperboloid in $\mathbb{R}^n$. Moreover in Section 5 a variant of this technique is also used to determine the orthoptic set for paraboloid hypersurfaces. In the last section, the conclusion, a summary is presented and some questions are raised for future research.

The contribution of this paper is to present results for orthoptic sets, not only for conics in $\mathbb{R}^2$ [4] and quadrics in $\mathbb{R}^3$ [5], but also for quadric hypersurfaces in $\mathbb{R}^n$. Even thought it can be said that the technique for ellipsoid in $\mathbb{R}^3$ can be extended to higher dimension [5], we present this extension not only for ellipsoids, but also for hyperboloids and paraboloids. We will see that it is a nice application of the trace operator of a matrix. Finally, one question remains unanswered. The results say that the orthoptic sets are included in some sets, but are these sets exactly the orthoptic sets. This result is true in $\mathbb{R}^n$ for $n=2,3$, but for $n>3$ it is an open question.

2. Preliminaries

2.1 Quadric hypersurfaces

The two quadratic functions we will study lead to ellipsoid or hyperboloid hypersurface defined by

$$f(x,y) = \sum_{i=1}^{I} \frac{x_i^2}{a_i^2} - \sum_{j=1}^{J} \frac{y_j^2}{b_j^2} = 1,$$

for $(x,y) \in \mathbb{R}^{I+J}$, and to paraboloid surface defined by

$$g(x,y,z) = \sum_{i=1}^{I} \frac{x_i^2}{a_i^2} - \sum_{j=1}^{J} \frac{y_j^2}{b_j^2} - \sum_{k=1}^{K} p_k z_k = 0,$$

for $(x,y,z) \in \mathbb{R}^{I+J+K}$.

2.2 Orthoptic surface

Based on the fact that in the plane each point of the orthoptic curve is associated to two normal vectors to the tangent planes or also to the curve, the next definition is suggested for a generalization in multidimensional Euclidean spaces of the usual orthoptic curve in the plane.

Definition. *Let a hypersurface $\mathscr{S}$ defined by $h(\xi) = 0$ in $\mathbb{R}^L$. The orthoptic set is the set of points common to L tangent planes to $\mathscr{S}$ under the condition that the L normals to the tangent planes form an orthogonal set.*

3. Notations

Let $x = (x_1, \ldots, x_I) \in \mathbb{R}^I$, $y = (y_1, \ldots, y_J) \in \mathbb{R}^J$, $z = (z_1, \ldots, z_K)$ and $p = (p_1, \ldots, p_K) \in \mathbb{R}^K$. Let $N = I + J$ and $M = N + K = I + J + K$. Let us introduce the I'th order diagonal matrix $A = \operatorname{diag}(a_i)$, the J'th order diagonal matrix $B = \operatorname{diag}(b_j)$, and the N'th order diagonal matrix

$$P = \begin{bmatrix} A & O \\ O & \imath B \end{bmatrix},$$

where $\imath$ is the unit complex number such that $\imath^2 = -1$. For any integer $l \in \mathbb{Z}$, we have

$$A^l = \operatorname{diag}(a_i^l) \quad \text{and} \quad B^l = \operatorname{diag}(b_j^l),$$

and also

$$P^l = \begin{bmatrix} A^l & O \\ O & \imath^l B^l \end{bmatrix}.$$

For any (line vector) $q \in \mathbb{R}^L$, q^t will be its (column vector) transpose. So, we can rewrite the quadratic form $f(x,y)$ as

$$f(x,y) = xA^{-2}x^t - yB^{-2}y^t = vP^{-2}v^t = f(v),$$

where $v = (x,y) \in \mathbb{R}^N$, and the quadratic form $g(x,y,z)$ as

$$g(x,y,z) = xA^{-2}x^t - yB^{-2}y^t - 2pz^t = vP^{-2}v^t - 2pz^t = g(w),$$

where $w = (v,z) = (x,y,z) \in \mathbb{R}^M$.

4. Ellipsoid and Hyperboloid hypersurfaces

4.1 Tangent planes

For

$$f(v) = vP^{-2}v^t,$$

a row normal vector to the surface $f(v) = 1$ at a point v_0 of this surface, noted $V(v_0)$, can be taken to be

$$V(v_0) = \frac{1}{2}\nabla f(v_0) = v_0 P^{-2}.$$

The tangent plane to $f(v) = 1$ at v_0 is given by the condition

$$V(v_0)(v - v_0)^t = 0,$$

which gives

$$V(v_0)v^t = V(v_0)v_0^t = v_0 P^{-2} v_0^t = f(v_0) = 1.$$

4.2 Orthoptic set

Let us suppose that there exists a finite sequence of points $\{v_n\}_{n=1}^N$ such that $f(v_n) = 1$ for $n = 1, \ldots, N$, and $\{V(v_n)\}_{n=1}^N$ is an orthogonal set. Let us look for the common point to the N tangent planes to the surface $f(v_n) = 1$ at v_n, that is to say a point $\widetilde{v} = (\widetilde{x}, \widetilde{y})$ such that

$$V(v_n)\widetilde{v}^t = 1$$

for $n = 1, \ldots, N$. We have to solve the linear system

$$\begin{bmatrix} V(v_1) \\ \vdots \\ V(v_N) \end{bmatrix} \widetilde{v}^t = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Using the orthogonality property of the family of normal vectors, we get

$$\begin{bmatrix} V(v_1) \\ \vdots \\ V(v_N) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{V^t(v_1)}{|V(v_1)|^2} & \cdots & \frac{V^t(v_N)}{|V(v_N)|^2} \end{bmatrix}$$

and then

$$\widetilde{v} = \sum_{n=1}^N \frac{1}{|V(v_n)|^2} V(v_n).$$

Again, from the orthogonality condition we get

$$|\widetilde{v}|^2 = \widetilde{v}\widetilde{v}^t = \sum_{n=1}^N \frac{1}{|V(v_n)|^4} V(v_n)V^t(v_n) = \sum_{n=1}^N \frac{1}{|V(v_n)|^2}.$$

Let us look at the inverse. We have

$$I = \begin{bmatrix} V(v_1) \\ \vdots \\ V(v_N) \end{bmatrix} \begin{bmatrix} \frac{V^t(v_1)}{|V(v_1)|^2} & \cdots & \frac{V^t(v_N)}{|V(v_N)|^2} \end{bmatrix}$$

and also

$$\begin{aligned} I &= \begin{bmatrix} \frac{V^t(v_1)}{|V(v_1)|^2} & \cdots & \frac{V^t(v_N)}{|V(v_N)|^2} \end{bmatrix} \begin{bmatrix} V(v_1) \\ \vdots \\ V(v_N) \end{bmatrix} \\ &= \sum_{n=1}^N \frac{1}{|V(v_l)|^2} V^t(v_n)V(v_n) \\ &= \sum_{n=1}^N \frac{1}{|V(v_l)|^2} P^{-2} v_n^t v_n P^{-2}. \end{aligned}$$

Let us observe that

$$P^2 = PIP = \sum_{n=1}^{N} \frac{1}{|V(v_n)|^2} P^{-1} v_n^t v_n P^{-1},$$

and taking the trace on both sides, we get

$$\mathrm{Trace}(P^2) = \sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2,$$

and

$$\begin{aligned}
\mathrm{Trace}(P^2) &= \sum_{n=1}^{N} \frac{1}{|V(v_n)|^2} \mathrm{Trace}(P^{-1} v_n^t v_n P^{-1}) \\
&= \sum_{n=1}^{N} \frac{1}{|V(v_n)|^2} \mathrm{Trace}(v_n P^{-2} v_n^t) \\
&= \sum_{n=1}^{N} \frac{1}{|V(v_n)|^2} f(v_n) \\
&= \sum_{n=1}^{N} \frac{1}{|V(v_n)|^2},
\end{aligned}$$

where we used the fact that $\mathrm{Trace}(HH^t) = \mathrm{Trace}(H^t H)$. So we obtain the result we were looking for.

Theorem 4.1. *Let the hypersurface, ellipsoid or hyperboloid, be defined by*

$$\sum_{i=1}^{I} \frac{x_i^2}{a_i^2} - \sum_{j=1}^{J} \frac{y_j^2}{b_j^2} = 1,$$

in $\mathbb{R}^N$ *where* $N = I + J$. *The orthoptic set of this hypersurface, if it exists, is included in the hypersphere of radius* $\sqrt{\sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2} \geq 0$ *given by*

$$\sum_{i=1}^{I} x_i^2 + \sum_{j=1}^{J} y_j^2 = \sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2.$$

5. Paraboloid hypersurface

5.1 Tangent planes

For

$$g(w) = vP^{-2}v^t - 2pz^t,$$

a row normal vector to the surface $g(w) = 0$ at a point w_0 of this surface, noted $W(w_0)$, can be taken to be

$$W(w_0) = \frac{1}{2} \nabla g(w_0) = (v_0 P^{-2}, -p).$$

The tangent plane to $g(w) = 0$ at w_0 is given by the condition

$$W(w_0)(w - w_0)^t = 0,$$

which gives

$$W(w_0) w^t = W(w_0) w_0^t = v_0 P^{-2} v_0^t - p z_0^t = g(w_0) + p z_0^t = p z_0^t.$$

5.2 Orthoptic set

Let us suppose that there exists a sequence of points $\{w_m\}_{m=1}^M$ such that $g(w_m) = 0$ for $m = 1, \dots, M$, and $\{W(w_m)\}_{m=1}^M$ is an orthogonal sequence. Let us look for the common point to the M tangent planes to the surface $g(w_m) = 0$ at w_m, that is to say a point $\widetilde{w} = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ such that

$$W(w_m)\widetilde{w}^t = pz_m^t$$

for $m = 1, \dots, M$. We have to solve the linear system

$$\begin{bmatrix} W(w_1) \\ \vdots \\ W(w_M) \end{bmatrix} \widetilde{w}^t = \begin{bmatrix} pz_1^t \\ \vdots \\ pz_M^t \end{bmatrix}.$$

Using the orthogonality properties of the family of normal vectors, we get

$$\begin{bmatrix} W(w_1) \\ \vdots \\ W(w_M) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{W^t(w_1)}{|W(w_1)|^2} & \dots & \frac{W^t(w_M)}{|W(w_M)|^2} \end{bmatrix}$$

and then

$$\widetilde{w}^t = \sum_{m=1}^M \frac{1}{|W(w_m)|^2} W^t(w_m) pz_m^t,$$

and so

$$p\widetilde{z}^t = -|p|^2 \sum_{m=1}^M \frac{1}{|W(w_m)|^2} pz_m^t.$$

Let us look at the inverse. We have

$$I = \begin{bmatrix} W(w_1) \\ \vdots \\ W(w_M) \end{bmatrix} \begin{bmatrix} \frac{W^t(w_1)}{|W(w_1)|^2} & \dots & \frac{W^t(w_M)}{|W(w_M)|^2} \end{bmatrix}$$

and also

$$\begin{aligned} I &= \begin{bmatrix} \frac{W^t(w_1)}{|W(w_1)|^2} & \dots & \frac{W^t(w_M)}{|W(w_M)|^2} \end{bmatrix} \begin{bmatrix} W(w_1) \\ \vdots \\ W(w_M) \end{bmatrix} \\ &= \sum_{m=1}^M \frac{1}{|W(w_l)|^2} W^t(w_m) W(w_m) \\ &= \sum_{m=1}^M \frac{1}{|W(w_m)|^2} \begin{bmatrix} P^{-2} v_l^t v_l P^{-2} & P^{-2} v_l^t p \\ p_t v_l P^{-2} & p^t p \end{bmatrix}. \end{aligned}$$

Let us first observe that

$$\begin{aligned} |p|^2 &= \begin{bmatrix} 0 & p \end{bmatrix} I \begin{bmatrix} 0 \\ p^t \end{bmatrix} \\ &= \sum_{m=1}^M \frac{1}{|W(w_m)|^2} \begin{bmatrix} 0 & p \end{bmatrix} \begin{bmatrix} P^{-2} v_l^t v_l P^{-2} & P^{-2} v_l^t p \\ p_t v_l P^{-2} & p^t p \end{bmatrix} \begin{bmatrix} 0 \\ p^t \end{bmatrix} \\ &= \sum_{m=1}^M \frac{1}{|W(w_m)|^2} pp^t pp^t \\ &= \sum_{m=1}^M \frac{1}{|W(w_m)|^2} |p|^4, \end{aligned}$$

so

$$|p|^2 \sum_{m=1}^M \frac{1}{|W(w_m)|^2} = 1.$$

Using any K'th order diagonal matrix $Q = \mathrm{diag}(q_k)$ where $q_k \in \mathbb{R}$ for $k = 1, \ldots, K$, we have

$$\begin{aligned}
\left[\begin{array}{cc} P^2 & 0 \\ 0 & Q^2 \end{array}\right] &= \left[\begin{array}{cc} P & 0 \\ 0 & Q \end{array}\right] I \left[\begin{array}{cc} P & 0 \\ 0 & Q \end{array}\right] \\
&= \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \left[\begin{array}{cc} P^{-1}v_l^t v_l P^{-1} & P^{-1}v_l^t pQ \\ Qp^t v_l P^{-1} & Qp^t pQ \end{array}\right],
\end{aligned}$$

and taking the trace on both sides, we get

$$\mathrm{Trace}(P^2) + \mathrm{Trace}(Q^2) = \sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2 + \sum_{k=1}^{K} q_k^2,$$

and

$$\begin{aligned}
&\sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \mathrm{Trace} \left[\begin{array}{cc} P^{-1}v_l^t v_l P^{-1} & P^{-1}v_l^t pQ \\ Qp^t v_l P^{-1} & Qp^t pQ \end{array}\right] \\
&\quad = \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \left[\mathrm{Trace}(P^{-1}v_m^t v_m P^{-1}) + \mathrm{Trace}(Qp^t pQ)\right] \\
&\quad = \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \left[\mathrm{Trace}(v_m P^{-2} v_m^t) + \mathrm{Trace}(pQ^2 p^t)\right] \\
&\quad = \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \left[v_m P^{-2} v_m^t + pQ^2 p^t\right] \\
&\quad = \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \left[P(w_m) + 2pz_m^t + pQ^2 p^t\right] \\
&\quad = 2\sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} pz_m^t + pQ^2 p^t \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2}.
\end{aligned}$$

For $Q = 0$ we obtain

$$\sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2 = 2\sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} pz_m^t,$$

and for $Q = I$, since $\mathrm{Trace}(Q^2) = \mathrm{Trace}(I) = K$ and $pQ^2p^t = pp^t = |p|^2$, we get

$$\begin{aligned}
\sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2 + K &= 2\sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} pz_m^t + |p|^2 \sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} \\
&= 2\sum_{m=1}^{M} \frac{1}{|W(w_m)|^2} pz_m^t + 1.
\end{aligned}$$

This is possible only for $K = 1$. So we obtain the result we were looking for.

Theorem 5.1. *Let the hypersurface, a paraboloid, defined by*

$$\sum_{i=1}^{I} \frac{x_i^2}{a_i^2} - \sum_{j=1}^{J} \frac{y_j^2}{b_j^2} - \sum_{k=1}^{K} p_k z_k = 0,$$

in $\mathbb{R}^M$ where $M = N + K = I + J + K$.

For $K = 1$, the orthoptic set might exist and, if it exists, is included in the hyperplane

$$z = -\frac{p}{2}\left[\sum_{i=1}^{I} a_i^2 - \sum_{j=1}^{J} b_j^2\right],$$

where we have considered $p > 0$.

For $K > 1$ the orthoptic set does not exist.

Let us observe that the fact that $K = 1$ in this last theorem is not a surprise. Indeed for $K > 1$, since the last K entries of any normal vectors are all equal to 1, it is not possible to find a set of $M = I + J + K$ orthogonal (normal) vectors to the paraboloid as assumed to get the result.

6. Conclusion

We have introduced orthoptic sets for hypersurfaces associated to quadratic forms in $\mathbb{R}^n$. At least one interesting question remains: are the hypersphere in Theorem 4.1 or the hyperplane in Theorem 5.1 exactly the orthoptic surfaces ? In other words, to any point on the given hypersphere or hyperplane does there exists a set of orthogonal normals for which the point is the unique common point to the corresponding set of planes ? As an example, for Theorem 4.1 with $N = 2$ and $I = 1 = J$, if the radius is 0, which means that $a_1 = b_1$, it is not possible to find a set of 2 orthogonal normals, except if we consider that the two asymptotes are tangent at infinity to the hyperbola. So what happens in higher dimension ?

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Risk Assessment of Cognitive and Behavioral Development of Early Childhood Children in Quarantine Days: An AHP Approach

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Abstract
The world is faced with disasters caused by natural or human effects from time to time. The various political, economic, health, and social consequences of these disasters affect people for different periods of time. In natural disasters and especially in epidemic diseases, some measures are taken to protect people from the negative effects of the situation. One of the measures that can be taken is quarantine. The target audience of this study is children aged 5-6 in early childhood. Children of this age group are in the process of gaining skills in expressing their feelings during this period. In addition, the emotional responses of these children can be noticed by a careful observer or even an expert.The aim of the paper is to evaluate the risks of the impacts of quarantine status related to COVID-19 pandemic on cognition and behavior of children staying at home. Risks of the quarantine process in children in early childhood were evaluated using the Pythagorean fuzzy AHP method.

Keywords: COVID-19,early childhood, risk assessment, Pythagorean fuzzy set, analytic hierarchy process, cognitive development, behavioural development
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1. Introduction

Risk is the value determined according to the probability of the damage that dangerous situations can cause. The likelihood and severity of the danger determine the degree of risk. Risk can also be defined as the combination of the probability and violence of the danger, since it has a value determined according to the probability and consequence (severity) of the danger, that is, the potential harm. The risk changes over time. So it is dynamic. Therefore, risk is a manageable phenomenon. Broadly speaking, there are two different approaches to risk: In the first approach, risk means uncertainty. In this case, it can contain both positive and negative consequences. In the second approach, risk means threat/danger. In this case, it contains only negative consequences. It generally has the potential to cause harm. That is, it is dangerous and is often linked to a condition or action that, if left unrestrained, could outcome in undesirable consequences such as illness or injury.

Risk refers to the uncertainty contained in the applied activities. This uncertainty can have positive or negative consequences. The purpose of risk management is to control the consequences of this uncertainty. For this, risk factors must be determined and analysed. Each new unpredictable incident provides valuable experiences for risk executives on how to reply. Corona virus is also no exception, as all other outbreaks are no exception. Based on what is known about the disease so far, some general conclusions can be drawn about how such events should be handled in the future.

Multi-criteria decision making(MCDM) is carried out by modeling the decision process according to the criteria and analyzing it in a way that maximizes the benefit that the decision-maker(DM) will obtain at the end of the continuum. Due to the complexity of the decision-making continuum, the suggestion of a different approach in the literature every day ensures that the MCDM approaches are constantly updated. MCDM approaches consisting of different ad numerous methods have been subjected to different classifications in the literature. Generally, these classes are examined under two groups as MADM(multi-attribute decision making) and MODM(multi-objective decision-making). The AHP (Analytic Hierarchy Process), put forward by Thomas L. Saaty [1], is one of the MADM methods that help the DM. The fact that the criteria can be evaluated analytically by comparison methods without numerical values makes this method more advantageous compared to other methods. This technique speeds up the decision-making process and makes it more systematic. Tuysuz and Kahraman [2] stated that the reliability and accuracy of risks with different dimensions should be evaluated and calculated by taking into account more than one criterion.

Countries or associations generally try to calculate the economic effects of natural disasters first. For example, the European Parliament published a briefing on the economic impacts of the COVID-19 global pandemic in February 2020 [3]. However, people and countries are not only economically affected by disasters. Examples of the sociological effects of COVID-19 can be given from China. The Financial Times reports that courts' demands for divorce have increased dramatically after quarantine in China [4]. An example of the impact of the SARS quarantine on mental health is the study by Hawryluck et al [5]. This study emphasizes that after the virus, the results of the high rate of post-traumatic stress disorder and depression are reached in humans.

In natural disasters and especially in epidemic diseases, some measures are taken to protect people from the negative effects of the situation. One of the measures that can be taken is quarantine. Therefore, Cliff and Smallman-Raynor [6] stated that the quarantine was used to indicate restrictions on the activities of people or animals exposed to infectious diseases during the infectious period. Children, who are members of the society and cannot be isolated from society, should be informed correctly and sufficiently to prevent them from being affected by both the biological effect and the psychological effect of the epidemic. Then, in a study conducted by Lima and Lemos [7] with children, it was emphasized that it was extremely important to inform and raise awareness of children beforehand in order to prevent a pandemic. Because children may face troubles due to the long duration of natural disasters and measures such as quarantine restricting people. Children may face personal losses, collective deaths, and discomfort caused by the diseases caught in natural disasters and outbreaks. These situations can cause adversities such as stress, anxiety, depression, and behavioral disorders in children.

Children's responses to disasters can be examined in three categories: emotion, thought, and behavior. Pfefferbaum et al [8] stated that the behavioral responses of children and adolescents against natural disasters differ from the behavior of adults in the disaster process, however, traces of the reactions of adults to disasters can be seen in the behavior of children. In other words, while children can develop different reactions to disasters than adults, they may show similar responses from time to time. For this reason, it is important to remember that adults should be positive models against children under all conditions.

Children learn a lot of the information they learn through environmental stimuli. Vygotsky [9] states that the interaction of the child with his environment, social relationships, other people, especially adults, play a very important role in cognitive development. The stimuli that it is exposed to in the pandemic process direct the perception of children to the pandemic. In this case, it is clear that children will pay more attention to the pandemic, quarantine, and related stimuli. In the process, the vast majority of stimuli around children, including parents and digital media, lead their perception of COVID-19. If this perception cannot be controlled properly, a false cognition and belief in children will be inevitable.

The most sensitive and vulnerable groups that are affected by the psychological and behavioural effects of disasters are children [10]. In a survey of 1200 social workers published by the BASW (British Association of Social Workers) on March 25, 2020, participating experts stated that they were particularly concerned about children and their parents in the course of the COVID-19 pandemic process [11]. Corona-virus quarantine, which started on 27 January 2020 in Wuhan, China due to the spread of viruses in December 2019, has been shown as the largest quarantine in human history. Schools, workplaces, meetings, social events, and entry-exit to the city have been stopped [12]. In the following days, similar situations in other cities and countries caused this quarantine to be applied in many parts of the world. In a meta-analysis study by Bish and Michie [13], however, it was emphasized that there were some strategies that could be a guide in combating pandemics, and it was emphasized that the confidence of the state was important in combating pandemics.

Gul [14] has integrated the fuzzy analytical hierarchy process (PFAHP) and fuzzy VIKOR (FVIKOR) into the risk assessment process for the field of OHS. Site safety and decoration, repair, and maintenance projects in skyscrapers are of vital importance. Ilkbahar et al [15] using PF Proportional Risk Assessment (PFPRA), PFAHP, and a fuzzy inference system have developed a new integrated approach. In [16], by using Safety and Critical Effect Analysis and PFSs jointly, a new, more exhaustive, and more accurate risk assessment method has been obtained. In [17], the risk assessment of these issues has been examined with the AHP technique. Mahmudova and Jabrailova [18] developed an algorithm to evaluate the functionality of the software using the analytical hierarchy process (AHP) method. An FMEA-based AHP-MOORA integrated approach in Pythagorean fuzzy environment for a pipeline construction project was first developed by Mete [19]. Yucesan and Kahraman [20] used the PFAHP method for risk assessment in hydroelectric power plants. The risk assessment of a hydroelectric power plant project using the TOPSIS method was studied by Zhang et al. [21]. In [22], new convenient foundations of the PFSs method were determined and the validity of these bases was discussed.

In [23], pandemic control measures are discussed on the negative consequences of coronavirus for children. In addition, results regarding the mental health and well-being of children were expressed. Saurabh and Ranjan [24] selected a group of children and adolescents who were quarantined in India as the target audience and examined their quarantine experiences, their adaptation to the quarantine, and the impact of the quarantine on this group. In [25], the psychological effects of quarantine have been investigated by using electronic databases. In this study, results such as trauma, stress symptoms, confusion, and anger were obtained. In addition, it has been stated that the longer the quarantine period, the more negative situations are encountered. Jiao et al [26] worked on the measures recommended to parents and family members to alleviate the fears and concerns of children in the quarantine process. It has been suggested to produce many facilities such as increasing communication, playing games, physical activities, and singing as music therapy in order to eliminate the fears and worries in children. There are similar studies prepared recently ([27], [28], [29], [30], [31]).

The target audience of this study is children aged 5-6 at the end of early childhood. Children of this age group are in the process of gaining skills in expressing their feelings during this period. In addition, the emotional responses of these children can be noticed by a careful observer or even an expert. In addition to those mentioned in the literature, most of the studies related to the effects on the adolescents and children of natural disasters in the World and Turkey focused on the symptoms of "Post Traumatic Stress Disorder" which is one of the psychological effects of disasters [32]. The aim of the work is to evaluate the risks of the impacts of quarantine status related to COVID-19 pandemic on cognition and behavior of children staying at home.

2. Preliminaries

2.1 Pythagorean Fuzzy Sets

Uncertainty is a crucial concept for decision-making problems. It is not easy to make precise decisions in life since each information contains vagueness, uncertainty, imprecision. Fuzzy Set(FS) Theory, Zadeh's [33] pioneering work, proposed a membership function to solve problems such as vagueness, uncertainty, imprecision, and this function took value in the range of [0,1]. FS Theory had solved many problems in practice, but there was no membership function in real life, which only includes acceptances. Rejection is as important as acceptance in real life. Atanassov [34] clarified this problem and posed the Intuitionistic Fuzzy Set(IFS) Theory using the membership function as well as the non-membership function. In IFS, the sum of membership and non-membership grades is 1. This condition is also a limitation for solutions of vagueness, uncertainty, imprecision. Yager [35], [36] has presented a solution to this situation and suggested Pythagorean Fuzzy Sets(PFS). PFS is more comprehensive than IFS because it uses the condition that the sum of the squares of membership and non-membership grades is equal to or less than 1. PFS is also a particular case of the Neutrosophic Set initiated by Smarandache [37].

In this paper, the initial universe, parameters sets will denote U, P, respectively.

The FS has emerged as a generalization of the classical set concept. A function $d_A : U \to [0,1]$ is called FS on U. This indicated by

$$A = \{(u_i, d_A(u_i)) : d_A(u_i) \in [0,1]; \forall u_i \in U\}.$$

Consider the set

$B = \{(u, d_B(u), y_B(u)) : u \in U\}.$

The set B is called an IFS on U, where, $d_B : U \to [0,1]$ and $y_B : U \to [0,1]$ such that $0 \leq d_B(u) + y_B(u) \leq 1$ for any $u \in U$ [34].

$b_B = 1 - d_B(u) - y_B(u)$ is called the degree of indeterminacy.

An PFS C in U is given by

$C = \{(u, d_C(u), y_C(u)) : u \in U\},$

where $d_C : U \to [0,1]$ denotes the degree of membership and $y_C : U \to [0,1]$ denotes the degree of non-membership of the element $u \in \mathscr{U}$ to the set C, respectively, with the condition that $0 \leq [d_C(u)]^2 + [y_C(u)]^2 \leq 1$ [35], [36], [38].

$b_C = \sqrt{1 - [d_C(u)]^2 - [y_C(u)]^2}$ is called the degree of indeterminacy.

Example 2.1. *Let* $U = \{u_1, u_2, u_3\}$ *and* $A(u_1) = (0.8, 0.6), A(u_2) = (0.7, 0.7), A(u_3) = (0.5, 0.6)$ *be three PFNs of* $u_i, (i = 1, 2, 3)$. *Then A is called a PFS with*

$$A = \{(u_1, 0.8, 0.6), (u_2, 0.7, 0.7), (u_3, 0.5, 0.6)\}. \tag{2.1}$$

2.2 PFAHP

One of the techniques that gives the best results in Pythagorean fuzzy AHP. Mohd and Abdullah [39] proposed new method(PFAHP) by integrating PFS into AHP for determination of criteria weight.

Weighted scales for PFAHP method are given in Table 1 [15], where Linguistic terms Certainly Low Importance, Very Low Importance, Low Importance, Below Average Importance, Average Importance, Above Average Importance, High Importance, Very High Importance, Certainly High Importance, Exactly Equal are shown as α, β, γ, δ, ε, η, θ, λ, μ, φ, respectively.

The algorithm of PFAHP as follows:

Step 1. According to experts' evaluations, the pairwise comparison matrix $E = (e_{ik})_{m\times m}$ is created using Table 1.

Step 2. The upper and lower values of the membership and non-membership functions are calculated using Equations 2.2 and 2.3 and the difference matrix $F = (f_{ik})_{m\times m}$ is obtained.

Step 3. The interval multiplicative matrix $G = (g_{ik})_{m\times m}$ is computed using the Equations 2.4 and 2.5.

Step 4. The determinacy value $H = (h_{ik})_{m\times m}$ of the e_{ik} is calculated using the Equation 2.6.

Step 5. The determinacy values and matrix $G = (g_{ik})_{m\times m}$ are multiplied to find the weight matrix before normalization, and the $T = (t_{ik})_{m\times m}$ matrix is constructed using Equation 2.7.

Step 6. The normalized priority weights ω_i are obtained with Equation 2.8.

$$f_{ikI} = d_{ikI}^2 - y_{ikI}^2 \tag{2.2}$$
$$f_{ikU} = d_{ikJ}^2 - y_{ikJ}^2 \tag{2.3}$$
$$g_{ikI} = \sqrt{1000^{f_{ikI}}} \tag{2.4}$$
$$g_{ikJ} = \sqrt{1000^{f_{ikJ}}} \tag{2.5}$$
$$h_{ik} = 1 - \left(d_{ikJ}^2 - d_{ikI}^2\right) - \left(y_{ikJ}^2 - y_{ikI}^2\right) \tag{2.6}$$
$$t_{ik} = \left\{\frac{g_{ikI} + g_{ikJ}}{2}\right\} h_{ik} \tag{2.7}$$
$$\omega_i = \frac{\sum_{k=1}^{m} t_{ik}}{\sum_{i=1}^{m}\sum_{k=1}^{m} t_{ik}} \tag{2.8}$$

Table 1. Weighted scales for the PFAHP

Linguistic terms	PFN equivalents IVPF numbers			
	m_I	m_J	n_I	n_J
α	0.00	0.00	0.90	1.00
β	0.10	0.20	0.80	0.90
γ	0.20	0.35	0.65	0.80
δ	0.35	0.45	0.55	0.65
ε	0.45	0.55	0.45	0.55
η	0.55	0.65	0.35	0.45
θ	0.65	0.80	0.20	0.35
λ	0.80	0.90	0.10	0.20
μ	0.90	1.00	0.00	0.00
φ	0.195	0.195	0.195	0.195

3. COVID-19 Quarantine Implementation

According to identify the criteria to be measured, the cognitive and behavioral status of children should be taken into account when doing risk analysis with respect to their attitudes in quarantine practice. For the weighting procedure, an aggregate of expert opinions consisting of evaluations of Early Childhood experts will be taken. After this stage, the sub-criteria and their weights will be used as entries for the AHP technique to prioritize the objectives and take the final decision. The experts in this study are people working on Early Childhood. Experts cross-check the criteria identified in accordance with the cognitive and behavioral attitudes of these age children and express their evaluations.

The linguistic terms and their numeric labels are:
For Questions to be asked to the child: Yes (1), maybe/some (2), no (3).
For Questions to be asked to parents: too much (1), much (2), some (3), too little (4), none (5).
The survey was prepared to be answered on the internet. Survey questions were asked to children aged 5-6 and their families. The survey includes the following questions:

Questions to be asked to the child:

E1 Do you know Corona-virus?

E2 Does Corona-virus harm people?

E3 Does Corona-virus harm animals?

E4 Can Corona-virus be prevented?

E5 Are you afraid of Corona-virus?

E6 Do you think it's nice not to go to school?

E7 Are you upset that you can't go to school?

E8 Is the obligation to stay home boring?

E9 Can we be protected from Corona-virus by staying at home?

E10 Do you think you can go to school from now on?

Questions to be asked to parents:

P1 Does your child pay more attention to cleaning after Corona-virus?

P2 Has your child's sleep pattern been impaired after Corona-virus?

P3 Have there been changes in your child's nutritional habits after Corona-virus?

Table 2. Classifications of hazards about children's cognition

Current status information(CSI)	Children's COVID-19 knowledge	E1
	The idea of COVID-19 harming people	E2
	The idea of COVID-19 harming animals	E3
	Knowledge of to prevent COVID-19	E4
Affecting children's emotions(ACE)	Children's fear of COVID-19	E5
	Nice not to go to school	E6
	It's sad to not go to school	E7
	The boringness of staying in the compulsory home	E8
Affecting children's thoughts(ACT)	Being protected from COVID-19 by staying at home	E9
	To think that schools can be reopened	E10

P4 Does your child behave anxiously after Corona-virus?

P5 Is your child afraid when a conversation about Corona-virus has passed?

P6 Does your child ask about Corona-virus?

P7 Did your child develop undesirable behaviour after Corona-virus?

P8 Is your child happy because she/he can't go to school?

P9 Has the time your child spent on the Internet after Corona-virus increased?

P10 Has the time your child spent in front of the TV increased after Corona-virus?

The cognitive and behavioral distributions of questions are as follows:

For children's cognition;

C1 Do children know about the current situation? (4 questions)

C2 Does the current situation affect children's emotions? (4 questions)

C3 Does the current situation affect children's thoughts? (2 questions)

For children's behavioral;

B1 Has Corona-virus changed the basic habits of children? (3 questions)

B2 Did behavior change occur in children after quarantine? (5 questions)

B3 Did children's behavior regarding information technologies increase after quarantine? (2 questions)

In this study, from Turkey, 201 children ages 5-6 units and 201 parents were the participants. Opinions of each child and each parent about the questions asked were got. The effect of quarantine on their own cognition in line with the answers given by the children and the effect of the behaviour of their children in line with the observations of the parents have been revealed.

Risk factors were identified as a result of interviews and evaluations with Early Childhood experts. Basic problem and sub-problems related to this problem were created and data were obtained. The evaluations of early childhood experts were obtained for the weights with the acquired data. The risk analysis structure of children's and parents' evaluations is given in Figure 3.1. Cognitive and behavioral risks that can be classified in children are classified in Table 2 and Table 3. In Table 4, Table 5, compromised pairwise comparison tables for CSI and CB are given, respectively. These tables were created according to the evaluations given by the experts by using the values in Table 1. Pythagorean fuzzy numbers are denoted by $<$ *degree of membership*, *degree of non-membership* $>=<\mu_L, \mu_U, \upsilon_L, \upsilon_U>$ in Table 4, Table 5.

For the weighting procedure, the sum of the assessments of the three experts was taken. As a result of expert evaluations, 10 critical criteria for cognitive development, and 10 critical criteria for behavioral development were determined. After this step, in order to identify the priorities of the aims and make final decision, the sub-problems and their weights as PFAHP inputs are studied. Experts are early childhood employees and can compare specified problems, report results, and indicate

Table 3. Classifications of hazards about children's behaviour

Change of basic habits of children(CBHC) in the quarantine period	Change in cleaning habits after COVID-19	P1
	Disruption in sleep pattern after COVID-19	P2
	Change in nutritional habits after COVID-19	P3
Change in behavioural after COVID-19(CB)	Anxiety increase after COVID-19	P4
	The emergence of fear when COVID-19 is spoken	P5
	Asking questions about COVID-19	P6
	Development of undesirable behavior after COVID-19	P7
	The idea that it is good not to go to school	P8
Change in behavior related to Information Technologies(CBIT)	increase in time spent on the internet	P9
	Increase in time spent in front of TV	P10

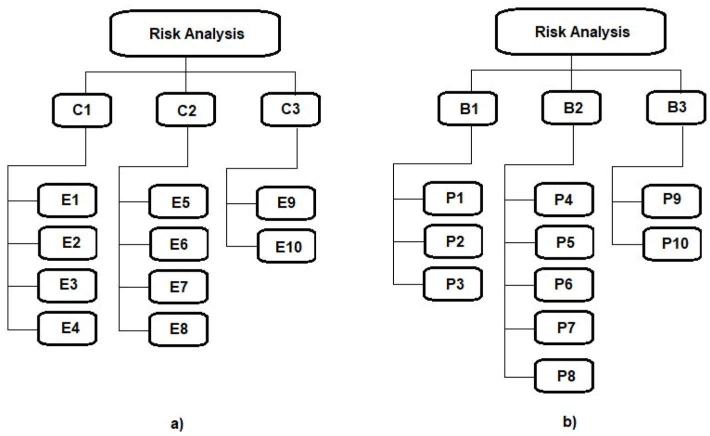

Figure 3.1. Risk analysis a) for children's cognition, b) for children's behavioural

their evaluations. Using pairwise comparison with the PFAHP method, 10 different hazards and associated risks identified for each development situation are weighted. Pairwise comparisons were given by experts for the importance weight of each evaluation criterion. Experts were asked to implement the linguistic variables indicated in Table 1. Here, the linguistic variables are transformed into the corresponding interval-valued PFNs. Since the evaluation degrees of each expert are subjective and will differ from each other, these subjective values are given as compromised pairwise comparison matrices in Table 4 for CSI and Table 5 for CB, respectively. The D matrices and S matrices for CSI and CB are given Tables 6, 7, 8 and 9, respectively. After h_{ik} determinacy values were calculated with Equation 2.6, T matrices (Tables 10 and 11) for CSI and CB were established with Equation 2.7. Further, the importance weights for CSI and CB are indicated in Tables 12 and 13.

Analysis and Discussion

These tables will be calculated in ACT and ACE for the cognitive development category, CBIT and CBC for behavioral development category. Then, the risk factors in each category will be determined. According to the results obtained with the calculated tables, E2 for CSI and P5 for CB were determined as the most important risk factors. The evaluation here will be made for E2 and P5.

Table 4. Linguistic evaluations for CSI

	E1	E2	E3	E4
E1	$<0.195, 0.195, 0.195, 0.195>$	$<0.90, 1.00, 0.00, 0.00>$	$<0.65, 0.80, 0.20, 0.35>$	$<0.80, 0.90, 0.10, 0.20>$
E2	$<0.80, 0.90, 0.10, 0.20>$	$<0.195, 0.195, 0.195, 0.195>$	$<0.54, 0.64, 0.36, 0.46>$	$<0.91, 1.00, 0.05, 0.03>$
E3	$<0.65, 0.80, 0.20, 0.35>$	$<0.81, 0.91, 0.09, 0.13>$	$<0.195, 0.195, 0.195, 0.195>$	$<0.24, 0.33, 0.65, 0.76>$
E4	$<0.90, 1.00, 0.00, 0.00>$	$<0.81, 0.91, 0.09, 0.13>$	$<0.48, 0.59, 0.41, 0.52>$	$<0.195, 0.195, 0.195, 0.195>$

Table 5. Linguistic evaluations for CB

	P4	P5	P6	P7	P8
P4	$< 0.195, 0.195, 0.195, 0.195 >$	$< 0.45, 0.55, 0.45, 0.55 >$	$< 0.91, 1.00, 0.02, 0.01 >$	$< 0.90, 1.00, 0.00, 0.00 >$	$< 0.68, 0.78, 0.22, 0.32 >$
P5	$< 0.90, 1.00, 0.00, 0.00 >$	$< 0.195, 0.195, 0.195, 0.195 >$	$< 0.78, 0.89, 0.11, 0.16 >$	$< 0.92, 1.00, 0.04, 0.02 >$	$< 0.47, 0.59, 0.41, 0.53 >$
P6	$< 0.55, 0.65, 0.35, 0.45 >$	$< 0.68, 0.78, 0.22, 0.32 >$	$< 0.195, 0.195, 0.195, 0.195 >$	$< 0.47, 0.59, 0.41, 0.53 >$	$< 0.24, 0.30, 0.66, 0.76 >$
P7	$< 0.90, 1.00, 0.00, 0.00 >$	$< 0.83, 0.91, 0.09, 0.15 >$	$< 0.66, 0.76, 0.24, 0.30 >$	$< 0.195, 0.195, 0.195, 0.195 >$	$< 0.45, 0.55, 0.45, 0.55 >$
P8	$< 0.10, 0.20, 0.80, 0.90 >$	$< 0.20, 0.35, 0.65, 0.80 >$	$< 0.00, 0.00, 0.90, 1.00 >$	$< 0.19, 0.30, 0.68, 0.81 >$	$< 0.195, 0.195, 0.195, 0.195 >$

Quarantine, which is one of the most important ways to prevent epidemic diseases, requires conscious participation. However, in this process, it is also an important issue to direct the cognition and behaviour of more sensitive and disadvantaged groups such as children. Although the World Health Organization (WHO) states that quarantine increases the capacity of people to control the spread of infectious diseases [40], this may have negative repercussions on people. In addition to the restrictions that may be experienced during the quarantine process, fear, anxiety, etc. related to basic needs and habits can threaten the individual's well-being, especially in terms of mental aspects.

The fact that the stimuli in the environment are intensely related to the virus causes children to learn about the virus. It is possible to be exposed to such an intense flow of information in a short time, to limit life in an instant, to create a perception of danger by talking about unpredictability and death news unnecessarily. According to the results obtained for CSI, the riskiest factor is E2. During the quarantine process, the child is exposed to the flow of information from many sources, from her/his immediate environment to her distant environment. When evaluated within the framework of ecological theory [41], it can be said that sensitivity to interaction between different environments will increase during the quarantine process. The diversity of information reaching the child through family and media-communication technologies reinforces this situation. However, if this information is not suitable for the child's level, misunderstanding and wrong cognition may develop. According to Piaget [42], it is possible that the child who is still in the pre-operational period does not understand the information that contains abstract elements. This situation can cause emotional problems in the child.

Misunderstanding and wrong cognition can disrupt the emotional balance of preschool children. According to the results obtained for CB, the riskiest factor is P5. Piaget [42] stated that newly learned information creates an imbalance in mental processes and that balance will occur with correct experiences. The child may develop fear, anxiety, and panic as a result of the imbalance caused by the information he receives from the environment. However, the exaggerated application of control measures may also increase children's fears.

Gagne [43] stated that learning is a cumulative process. The individual can make sense of the stimuli coming from the environment in her/his mind, associate that information with new situations and use it in solving problems [44], [45]. The beliefs that the Corona-virus harms people, guides the children's other cognitions and behaviours on this issue. In particular, the negative behaviours of one or more of the family members related to the virus also affect the children. Because children imitate adult responses. Even if there are different reasons for children to be affected cognitively, when these and similar triggering factors are combined with the effect of the current period, it is possible to leave permanent problems in children. This situation may negatively affect the healthy preparation of children for adulthood.

Every new experience means new knowledge. Especially children should get the correct information with correct experiences in natural disasters such as epidemics. The information must be coded correctly and transformed into behaviour. For this, administrators should inform the public with correct information and thinking about the psychology of society.

As children model adult reactions, parents should pay attention to their own behaviour and their own discourse in the home. It is also important not to overreact to stimuli received from the media. However, messages sent by the media to children should be filtered. When considered as a whole, it is recommended that parents and adults take a controlled approach without exaggerating their way of interacting with the child. Considering the cognitive and behavioral development of children, parents should not allow children to be exposed to too many news, notifications, and stimuli. However, it is not healthy also to act as if nothing happened or will not happen by moving away from the usual situation.

At this point, as experts [46] have stated, it is important that adults have enough knowledge about the new coronavirus and try to find a balance in order to answer their children's questions well enough without increasing the severity of their anxiety. All possible situations that cause anxiety and fear should be discussed in accordance with the developmental levels of children in this period. Again, the questions of children on these issues should be tried to be answered. The message that children will be safe and that the situation is controllable, especially when necessary precautions are taken, should be given in an age-appropriate manner.

4. Conclusion

The quarantine measures carried out as a result of COVID-19 and the protective / preventive decisions taken in connection with this process are very important for the psychological conditions of early childhood children. Risk assessments related to the negative effects of the cognitive and behavioral development of children in this period have an important effect on decision-making processes. In this study, 10 risk factors for cognitive development and 10 risk factors for behavioral development were

Table 6. Difference matrix for CSI

	E1	**E2**	**E3**	**E4**
E1	$< 0.00, 0.00 >$	$< -0.19, 0.00 >$	$< 0.30, 0.48 >$	$< 0.48, 0.80 >$
E2	$< 0.48, 0.80 >$	$< 0.00, 0.00 >$	$< 0.08, 0.28 >$	$< 0.8272, 0.9975 >$
E3	$< 0.30, 0.48 >$	$< 0.6392, 0.82 >$	$< 0.00, 0.00 >$	$< -0.52, -0.3136 >$
E4	$< 0.81, 1.00 >$	$< 0.6392, 0.82 >$	$< -0.04, 0.18 >$	$< 0.00, 0.00 >$

Table 7. Difference matrix for CB

	P4	**P5**	**P6**	**P7**	**P8**
P4	$< 0.00, 0.00 >$	$< -0.10, -0.10 >$	$< 0.828, 0.9996 >$	$< 0.81, 1.00 >$	$< 0.36, 0.56 >$
P5	$< 0.81, 1.00 >$	$< 0.00, 0.00 >$	$< 0.5828, 0.78 >$	$< 0.846, 0.9984 >$	$< -0.06, 0.18 >$
P6	$< 0.10, 0.30 >$	$< 0.36, 0.56 >$	$< 0.00, 0.00 >$	$< -0.06, 0.18 >$	$< -0.52, -0.3456 >$
P7	$< 0.81, 1.00 >$	$< 0.6664, 0.82 >$	$< 0.3456, 0.52 >$	$< 0.00, 0.00 >$	$< -0.10, 0.10 >$
P8	$< -0.80, -0.60 >$	$< -0.60, -0.30 >$	$< -1.00, -0.81 >$	$< -0.62, -0.3724 >$	$< 0.00, 0.00 >$

Table 8. The interval multiplicative matrix for CSI

	E1	**E2**	**E3**	**E4**
E1	$< 1.00, 1.00 >$	$< 0.52, 1.00 >$	$< 2.81, 5.25 >$	$< 5.25, 15.85 >$
E2	$< 5.25, 15.85 >$	$< 1.00, 1.00 >$	$< 1.32, 2.63 >$	$< 17.41, 31.35 >$
E3	$< 2.82, 5.25 >$	$< 9.42, 17.00 >$	$< 1.00, 1.00 >$	$< 0.17, 0.30 >$
E4	$< 16.40, 31.62 >$	$< 9.1, 17.00 >$	$< 0.79, 1.86 >$	$< 1.00, 1.00 >$

Table 9. The interval multiplicative matrix for CB

	P4	**P5**	**P6**	**P7**	**P8**
P4	$< 1.00, 1.00 >$	$< 0.70, 0.70 >$	$< 17.46, 31.58 >$	$< 16.40, 31.62 >$	$< 3.47, 6.92 >$
P5	$< 16.40, 31.62 >$	$< 1.00, 1.00 >$	$< 7.49, 14.80 >$	$< 18.58, 31.44 >$	$< 0.81, 1.86 >$
P6	$< 2.00, 2.82 >$	$< 3.47, 6.92 >$	$< 1.00, 1.00 >$	$< 0.81, 1.86 >$	$< 0.17, 0.303 >$
P7	$< 16.40, 31.62 >$	$< 10.00, 17.00 >$	$< 3.30, 6.02 >$	$< 1.00, 1.00 >$	$< 0.707, 1.41 >$
P8	$< 0.063, 0.13 >$	$< 0.13, 0.35 >$	$< 0.031, 0.060 >$	$< 0.117, 0.276 >$	$< 1.00, 1.00 >$

Table 10. The weights before normalization for CSI

	E1	**E2**	**E3**	**E4**
E1	1.00	0.152	2.82	8.44
E2	8.44	1.00	1.58	19.80
E3	2.82	10.83	1.00	0.19
E4	4.80	10.70	1.03	1.00

Table 11. The weights before normalization for CB

	P4	**P5**	**P6**	**P7**	**P8**
P4	1.00	0.70	20.35	4.80	4.16
P5	4.80	1.00	8.92	21.26	1.015
P6	1.93	4.16	1.00	1.015	0.20
P7	4.80	11.48	3.87	1.00	1.06
P8	0.08	0.17	0.01	0.15	1.00

Table 12. Importance weights of evaluation for CSI

Criteria	Weight
E1	0.17
E2	0.40
E3	0.20
E4	0.23

Table 13. Importance weights of evaluation for CB

Criteria	Weight
P4	0.31
P5	0.37
P6	0.08
P7	0.22
P8	0.01

determined and evaluated with PFAHP. For this evaluation, the opinions of early childhood experts were taken. Preventive measures have been expressed in order to minimize the most important risk factors identified.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Global Behavior of a System of Second-Order Rational Difference Equations

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Abstract
In this paper, we consider the following system of rational difference equations

$$x_{n+1} = \frac{a+x_n}{b+cy_n+dx_{n-1}},\ y_{n+1} = \frac{\alpha+y_n}{\beta+\gamma x_n+\eta y_{n-1}},\ n = 0,1,2,...$$

where $a,b,c,d,\alpha,\beta,\gamma,\eta \in (0,\infty)$ and the initial values $x_{-1},x_0,y_{-1},y_0 \in (0,\infty)$. Our main aim is to investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system.

Keywords: Equilibrium points, Global behavior, Local stability, Positive solutions, Rate of convergence.
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1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so forth [5, 7, 14, 15]. Recently, there has been a lot of works concerning the global behaviors of positive solutions of rational difference equations and positive solutions of systems of rational difference equations [1, 2, 4, 8, 9, 12, 13]. It is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations and solutions of systems of rational difference equations, although they have very simple forms. One can refer to [1]-[22] and the references cited therein to illustrate this. Therefore, the study of rational difference equations and systems of rational difference equations is worth further consideration.

In [1] M.R.S. Kulenović and M. Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{Ax_ny_n}{1+y_n},\ y_{n+1} = \frac{Bx_ny_n}{1+x_n},\ n = 0,1,2,\ldots,$$

where $A,B \in (0,\infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In [2] S. Kalabusić and M.R.S. Kulenović considered two systems of difference equations

$$x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n},\ y_{n+1} = \frac{\alpha_2+\beta_2 x_n}{y_n},\ n = 0,1,2,\ldots,$$

and

$$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{1+x_n},\ y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{1+y_n},\ n=0,1,2,\ldots,$$

where $\alpha_1,\alpha_2,\beta_2,\gamma_1\in(0,\infty)$ and x_0,y_0 are positive numbers.

In [3], Q. Din et al. investigated behavior of the competitive system of difference equations

$$x_{n+1}=\frac{\alpha_1+\beta_1 x_{n-1}}{a_1+b_1 y_n},\ y_{n+1}=\frac{\alpha_2+\beta_2 y_{n-1}}{a_2+b_2 x_n},\ n=0,1,2,\ldots,$$

where $a_i,b_i,\alpha_i,\beta_i\in(0,\infty)$ for $i\in\{1,2\}$ and initial conditions x_{-1},x_0,y_{-1},y_0 are positive numbers.

In [4], the author investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system

$$x_{n+1}=\frac{ax_n-bx_ny_n}{1+cx_n+dy_n},\ y_{n+1}=\frac{\alpha x_ny_n-\beta y_n}{1+\gamma x_n+\eta y_n},\ n=0,1,2,\ldots,$$

where $a,b,c,d,\alpha,\beta,\gamma,\eta\in(0,\infty)$ and the initial values $(x_0,y_0)\in(0,\infty)$.

Motivated by these above papers, in this paper we will consider the following system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+cy_n+dx_{n-1}},\ y_{n+1}=\frac{\alpha+y_n}{\beta+\gamma x_n+\eta y_{n-1}},\ n=0,1,2,\ldots, \tag{1.1}$$

where $a,b,c,d,\alpha,\beta,\gamma,\eta\in(0,\infty)$ and the initial values $x_{-1},x_0,y_{-1},y_0\in(0,\infty)$. More precisely, we investigate the local asymptotic stability and global stability of equilibrium points, and the rate of convergence of positive solutions of the system (1.1) which converge to its unique positive equilibrium point.

2. Boundedness and persistence

In the first result we will establish the boundedness and persistence of every positive solution of the system (1.1).

Theorem 2.1. *Assume that $b>1$, $d<1$, $\beta>1$ and $\gamma<1$ then every positive solution $\{(x_n,y_n)\}$ of the system (1.1) is bounded and persists.*

Proof. For any positive solution $\{(x_n,y_n)\}$ of the system (1.1), we have

$$x_{n+1}\le\frac{a}{b}+\frac{1}{b}x_n,\ y_{n+1}\le\frac{\alpha}{\beta}+\frac{1}{\beta}y_n,\ n=0,1,2,\ldots. \tag{2.1}$$

Consider the following linear difference equations:

$$u_{n+1}=\frac{a}{b}+\frac{1}{b}u_n,\ v_{n+1}=\frac{\alpha}{\beta}+\frac{1}{\beta}v_n,\ n=0,1,2,\ldots. \tag{2.2}$$

We can see the solutions of (2.2) have the forms

$$u_n=\frac{a}{b-1}+C_1(\frac{1}{b})^n,\ v_n=\frac{\alpha}{\beta-1}+C_2(\frac{1}{\beta})^n,\ n=1,2,\ldots, \tag{2.3}$$

where C_1,C_2 depend on initial conditions u_0,v_0.

Assume that $b>1$ and $\beta>1$ then the sequences $\{u_n\}$ and $\{v_n\}$ are bounded. Suppose that $u_0=x_0$ and $v_0=y_0$ then by comparison we have

$$x_n\le\frac{a}{b-1}=U_1,\ y_n\le\frac{\alpha}{\beta-1}=U_2,\ n=1,2,\ldots. \tag{2.4}$$

Also, from (1.1) and (2.4), we infer

$$\begin{aligned}x_{n+1}&\ge\frac{a}{b+cy_n+dx_{n-1}}\ge\frac{a}{b+cU_2+dx_{n-1}},\\ y_{n+1}&\ge\frac{\alpha}{\beta+\gamma x_n+\eta y_{n-1}}\ge\frac{\alpha}{\beta+\gamma U_1+\eta y_{n-1}},\end{aligned}\quad n=1,2,\ldots. \tag{2.5}$$

Consider the following linear difference equations:

$$s_{n+1} = \delta + ds_{n-1},\ t_{n+1} = \theta + \eta t_n,\ n = 0,1,2,\ldots, \tag{2.6}$$

where $\delta = b + cU_2, \theta = \beta + \gamma U_1$.
We can see the solutions of (2.6) have the forms

$$\begin{aligned} s_n &= \frac{\delta}{1-d} + C_3(\sqrt{d})^n + C_4(-\sqrt{d})^n, \\ t_n &= \frac{\theta}{1-\gamma} + C_5(\sqrt{\gamma})^n + C_6(-\sqrt{\gamma})^n, \end{aligned} \quad n = 1,2,\ldots, \tag{2.7}$$

where C_3, C_4 depend on initial conditions s_{-1}, s_0 and C_5, C_6 depend on initial conditions t_{-1}, t_0.

Assume that $d < 1$ and $\gamma < 1$ then the sequences $\{s_n\}$ and $\{t_n\}$ are bounded. Suppose that $s_{-1} = x_{-1}, s_0 = x_0$ and $t_{-1} = y_{-1}, t_0 = y_0$ then by comparison we have

$$\begin{aligned} x_n &\geq \frac{a}{\delta/(1-d)} = \frac{a(1-d)}{b+cU_2} = \frac{a(1-d)}{b+c\frac{\alpha}{\beta-1}} = \frac{a(1-d)(\beta-1)}{b(\beta-1)+c\alpha} = L_1, \\ y_n &\geq \frac{\alpha}{\theta/(1-\gamma)} = \frac{\alpha(1-\gamma)}{\beta+\gamma U_1} = \frac{\alpha(1-\gamma)}{\beta+\gamma\frac{a}{b-1}} = \frac{\alpha(1-\gamma)(b-1)}{\beta(b-1)+\gamma a} = L_2, \end{aligned} \quad n = 1,2,\ldots. \tag{2.8}$$

From (2.4) and (2.5), we have

$$L_1 \leq x_n \leq U_1,\ L_2 \leq y_n \leq U_2,\ n = 1,2,\ldots. \tag{2.9}$$

Hence, the proof is completed. □

Lemma 2.2. *Let $\{(x_n, y_n)\}$ be a positive solution of the system (1.1). Then, $[L_1, U_1] \times [L_2, U_2]$ is an invariant set for system (1.1).*

Proof. The proof follows by induction. □

3. Global behavior

In the following, we state some main definitions used in this paper.

Let I, J be some intervals of real numbers and let

$$f : I^2 \times J^2 \longrightarrow I \text{ and } g : I^2 \times J^2 \longrightarrow J \tag{3.1}$$

are continuously differentiable functions. Then, for all initial values $(x_{-1}, x_0, y_{-1}, y_0) \in I^2 \times J^2$, the system of difference equations

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}),\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}),\ n = 0,1,2,\ldots, \tag{3.2}$$

has a unique solution $\{(x_n, y_n)\}_{n=1}^{\infty}$.

Definition 3.1. *A point $(\bar{x}, \bar{y})$ is called an equilibrium point of the system (3.2) if*

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}),\ \bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}). \tag{3.3}$$

Definition 3.2. *[3, 5] Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (3.2).*

1. *An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial point $(x_i, y_i), i \in \{-1, 0\}$ if $\sum_{i=-1}^{0} \|(x_i, y_i) - (\bar{x}, \bar{y})\| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$. An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable (the Euclidean norm in $\mathbb{R}^2$ given by $\|(x, y)\| = \sqrt{x^2 + y^2}$ is denoted by $\|.\|$).*

2. *An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\sum_{i=-1}^{0} \|(x_i, y_i) - (\bar{x}, \bar{y})\| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.*

3. *An equilibrium point $(\bar{x},\bar{y})$ is called a global attractor if $(x_n,y_n) \to (\bar{x},\bar{y})$ as $n \to \infty$.*

4. *An equilibrium point $(\bar{x},\bar{y})$ is called an asymptotic global attractor if it is global attractor and stable.*

Definition 3.3. *[3, 5] Let $(\bar{x},\bar{y})$ be an equilibrium point of a map $F = (f, x_n, g, y_n)$, where f and g are continuously differentiable functions at $(\bar{x},\bar{y})$. The linearized system of (3.2) about the equilibrium point $(\bar{x},\bar{y})$ is given by*

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ *and F_J is a Jacobian matrix of the system (3.2) about the equilibrium point $(\bar{x},\bar{y})$.*

In order to corresponding linearized form of system (1.1) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \longrightarrow (f, g, f_1, g_1), \tag{3.4}$$

where $f = x_{n+1}, g = y_{n+1}, f_1 = x_n, g_1 = y_n$. The linearized system of (1.1) about $(\bar{x},\bar{y})$ is given by

$$Y_{n+1} = F_J(\bar{x},\bar{y})Y_n, \tag{3.5}$$

where $Y_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ and the Jacobian matrix of the system (1.1) about the equilibrium point $(\bar{x},\bar{y})$ is given by

$$F_J(\bar{x},\bar{y}) = \begin{pmatrix} \frac{1}{b+c\bar{y}+d\bar{x}} & \frac{-c\bar{x}}{b+c\bar{y}+d\bar{x}} & \frac{-d\bar{x}}{b+c\bar{y}+d\bar{x}} & 0 \\ \frac{-\gamma\bar{y}}{\beta+\gamma\bar{x}+\eta\bar{y}} & \frac{1}{\beta+\gamma\bar{x}+\eta\bar{y}} & 0 & \frac{-\eta\bar{y}}{\beta+\gamma\bar{x}+\eta\bar{y}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{3.6}$$

The following results will be useful in the sequel.

Lemma 3.4. *[3] Assume that $X_{n+1} = F(X_n), n = 0,1,2,\ldots$, is a system of difference equations such that $\bar{X}$ is a fixed point of F. If all eigenvalues of Jacobian matrix F_J about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\bar{X}$ is unstable.*

Lemma 3.5. *[6] Assume that $q_0, q_1, \ldots, q_k$ are real numbers such that*

$$|q_0| + |q_1| + \ldots + |q_k| < 1.$$

Then all roots of the equation

$$\lambda^{k+1} + q_0\lambda^k + \ldots + q_{k-1}\lambda + q_k = 0$$

lie inside the unit disk.

The next theorem will show the existence and uniqueness of positive equilibrium point of the system (1.1).

Theorem 3.6. *Assume that $b > 1, \beta > 1$ and the following conditions are satisfied:*

$$-(c^2\alpha + cd\gamma)U_1^2 - cd(\beta-1)U_1 + d^2\eta < 0, \tag{3.7}$$

and

$$\begin{aligned}&(cd\gamma - d^2\eta)L_1^4 + [(b-1)c\gamma + cd(\beta-1) - 2(b-1)d\eta]L_1^3 \\ &+ [2ad\eta + c^2\alpha + (b-1)c(\beta-1) - ac\gamma - \eta(b-1)^2]L_1^2 \\ &+ [2a(b-1)\eta - ac(\beta-1)]L_1 - a^2\eta > 0,\end{aligned} \tag{3.8}$$

or

$$-(c^2\alpha+cd\gamma)U_1^2-cd(\beta-1)U_1+d^2\eta>0, \tag{3.9}$$

and

$$\begin{aligned}&(cd\gamma-d^2\eta)L_1^4+[(b-1)c\gamma+cd(\beta-1)-2(b-1)d\eta]L_1^3\\&+[2ad\eta+c^2\alpha+(b-1)c(\beta-1)-ac\gamma-\eta(b-1)^2]L_1^2\\&+[2a(b-1)\eta-ac(\beta-1)]L_1-a^2\eta<0,\end{aligned} \tag{3.10}$$

and

$$U_1<\frac{2\sqrt{\alpha\eta}}{\gamma} \tag{3.11}$$

Then there exists unique positive equilibrium point of the system (1.1) *in* $[L_1,U_1]\times[L_2,U_2]$.

Proof. Firstly, we consider the following system of algebraic equations

$$x=\frac{a+x}{b+cy+dx},\ y=\frac{\alpha+y}{\beta+\gamma x+\eta y}. \tag{3.12}$$

From (3.12), it follows that

$$\begin{aligned}y=&\frac{a+x-bx-dx^2}{cx}=\frac{a}{cx}-\frac{d}{c}x-\frac{b-1}{c},\\x=&\frac{\alpha+y-\beta y-\eta y^2}{\gamma y}=\frac{\alpha}{\gamma y}-\frac{\eta}{\gamma}y-\frac{\beta-1}{\gamma}.\end{aligned} \tag{3.13}$$

Set

$$f(x)=\frac{a}{cx}-\frac{d}{c}x-\frac{b-1}{c}, \tag{3.14}$$

and

$$F(x)=\frac{\alpha}{\gamma f(x)}-\frac{\eta}{\gamma}f(x)-\frac{\beta-1}{\gamma}-x. \tag{3.15}$$

We have

$$\begin{aligned}f(U_1)=&\frac{a}{cU_1}-\frac{d}{c}U_1-\frac{b-1}{c}\\=&\frac{a}{c}\frac{b-1}{a}-\frac{d}{c}\frac{a}{b-1}-\frac{b-1}{c}\\=&-\frac{d}{c}\frac{a}{b-1}=-\frac{d}{cU_1},\end{aligned} \tag{3.16}$$

$$\begin{aligned}F(U_1)=&\frac{\alpha}{\gamma f(U_1)}-\frac{\eta}{\gamma}f(U_1)-\frac{\beta-1}{\gamma}-U_1\\=&-\frac{c\alpha U_1}{d\gamma}+\frac{d\eta}{c\gamma U_1}-\frac{\beta-1}{\gamma}-U_1\\=&\frac{-(c^2\alpha+cd\gamma)U_1^2-cd(\beta-1)U_1+d^2\eta}{cd\gamma U_1},\end{aligned} \tag{3.17}$$

$$f(L_1)=\frac{a}{cL_1}-\frac{d}{c}L_1-\frac{b-1}{c}=\frac{-dL_1^2-(b-1)L_1+a}{cL_1}, \tag{3.18}$$

$$
\begin{aligned}
F(L_1) =& \frac{\alpha}{\gamma f(L_1)} - \frac{\eta}{\gamma} f(L_1) - \frac{\beta - 1}{\gamma} - L_1 \\
=& \frac{-\eta[-dL_1^2 - (b-1)L_1 + a]^2 - c\gamma L_1^2[-dL_1^2 - (b-1)L_1 + a]}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
&+ \frac{-c(\beta - 1)L_1[-dL_1^2 - (b-1)L_1 + a] + c^2\alpha L_1^2}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
=& \frac{(cd\gamma - d^2\eta)L_1^4 + [(b-1)c\gamma + cd(\beta - 1) - 2(b-1)d\eta]L_1^3}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
&+ \frac{[2ad\eta + c^2\alpha + (b-1)c(\beta - 1) - ac\gamma - \eta(b-1)^2]L_1^2}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]} \\
&+ \frac{[2a(b-1)\eta - ac(\beta - 1)]L_1 - a^2\eta}{c\gamma L_1[-dL_1^2 - (b-1)L_1 + a]}.
\end{aligned}
\tag{3.19}
$$

From (3.15), we have

$$
F(x) = \frac{\alpha}{\gamma} \cdot \frac{cx}{-dx^2 - (b-1)x + a} - \frac{\eta}{\gamma} \cdot \frac{-dx^2 - (b-1)x + a}{cx} - \frac{\beta - 1}{\gamma} - x. \tag{3.20}
$$

It follows that

$$
\begin{aligned}
F'(x) =& \frac{c\alpha}{\gamma} \cdot \frac{-dx^2 - (b-1)x + a - x(-2dx - b + 1)}{[-dx^2 - (b-1)x + a]^2} \\
&- \frac{\eta}{c\gamma} \cdot \frac{x(-2dx - b + 1) - [-dx^2 - (b-1)x + a]}{x^2} - 1 \\
=& \frac{c\alpha}{\gamma} \cdot \frac{dx^2 + a}{[-dx^2 - (b-1)x + a]^2} + \frac{\eta}{c\gamma} \cdot \frac{dx^2 + a}{x^2} - 1 \\
\geq& 2\frac{\sqrt{\alpha\eta}}{\gamma} \cdot \frac{dx^2 + a}{x[-dx^2 - (b-1)x + a]} - 1 > 2\frac{\sqrt{\alpha\eta}}{\gamma} \cdot \frac{1}{x} - 1.
\end{aligned}
\tag{3.21}
$$

Assume that condition (3.11) is satisfied, then we have $F'(x) > 0$. Hence, $F(x) = 0$ has a unique positive solution in $[L_1, U_1]$. □

Theorem 3.7. *The unique positive equilibrium point $(\bar{x}, \bar{y})$ of system (1.1) is locally asymptotically stable if the following condition holds*

$$
\frac{1 + dU_1}{b + dL_1 + cL_2} + \frac{1 + \eta U_2}{\beta + \gamma L_1 + \eta L_2} + \frac{1 + dU_1 + \eta U_2 + (c\gamma + d\eta)U_1U_2}{(b + dL_1 + cL_2)(\beta + \gamma L_1 + \eta L_2)} < 1. \tag{3.22}
$$

Proof. The characteristic polynomial of Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(\bar{x}, \bar{y})$ is given by

$$
P(\lambda) = \lambda^4 - (A + B)\lambda^3 + (d\bar{x}A + \eta\bar{y}B + AB - c\gamma\bar{x}\bar{y}AB)\lambda^2 - (d\bar{x}AB + \eta\bar{y}AB)\lambda + d\eta\bar{x}\bar{y}AB, \tag{3.23}
$$

where $A = \dfrac{1}{b + d\bar{x} + c\bar{y}}, B = \dfrac{1}{\beta + \gamma\bar{x} + \eta\bar{y}}$.
We have

$$
\begin{aligned}
&|A + B| + |d\bar{x}A + \eta\bar{y}B + AB - c\gamma\bar{x}\bar{y}AB| + |d\bar{x}AB + \eta\bar{y}AB| + |d\eta\bar{x}\bar{y}AB| \\
&< (1 + d\bar{x})A + (1 + \eta\bar{y})B + (1 + d\bar{x} + \eta\bar{y} + c\gamma\bar{x}\bar{y} + d\eta\bar{x}\bar{y})AB \\
&< \frac{1 + dU_1}{b + dL_1 + cL_2} + \frac{1 + \eta U_2}{\beta + \gamma L_1 + \eta L_2} + \frac{1 + dU_1 + \eta U_2 + (c\gamma + d\eta)U_1U_2}{(b + dL_1 + cL_2)(\beta + \gamma L_1 + \eta L_2)} < 1.
\end{aligned}
\tag{3.24}
$$

By using Lemma 3.5, we can see that all the roots of (3.23) satisfy $|\lambda| < 1$, and it follows from Lemma 3.4 that the unique positive equilibrium point $(\bar{x}, \bar{y})$ of the system (1.1) is locally asymptotically stable. Hence, the proof is completed. □

Theorem 3.8. *The unique positive equilibrium point $(\bar{x},\bar{y})$ of system (1.1) is globally asymptotically stable if the following condition holds*

$$a+U_1 < L_1(b+cL_2+dL_1), \alpha+U_2 < L_2(\beta+\gamma L_1+\eta L_2). \tag{3.25}$$

Proof. Arguing as in Theorem 1.1 of [11], we consider the following Lyapunov function:

$$V_n = \bar{x}g(\frac{x_n}{\bar{x}})+\bar{y}g(\frac{y_n}{\bar{y}}), \tag{3.26}$$

where

$$g(x) = x-1-\ln x \geq 0, \forall x>0. \tag{3.27}$$

It is easy to see that V_n is nonnegative function.
Consider

$$\begin{aligned} V_{n+1}-V_n =& \bar{x}(\frac{x_{n+1}}{\bar{x}}-1-\ln\frac{x_{n+1}}{\bar{x}})+\bar{y}(\frac{y_{n+1}}{\bar{y}}-1-\ln\frac{y_{n+1}}{\bar{y}}) \\ &-\bar{x}(\frac{x_n}{\bar{x}}-1-\ln\frac{x_n}{\bar{x}})+\bar{y}(\frac{y_n}{\bar{y}}-1-\ln\frac{y_n}{\bar{y}}) \\ =& \bar{x}(\frac{x_{n+1}-x_n}{\bar{x}}+\ln\frac{x_n}{x_{n+1}})+\bar{y}(\frac{y_{n+1}-y_n}{\bar{y}}+\ln\frac{y_n}{y_{n+1}}). \end{aligned} \tag{3.28}$$

Furthermore, from (3.27) we have

$$\ln\frac{x_n}{x_{n+1}} \leq \frac{x_n}{x_{n+1}}-1, \ln\frac{y_n}{y_{n+1}} \leq \frac{y_n}{y_{n+1}}-1. \tag{3.29}$$

Then, from (3.28) and (3.29) we have

$$\begin{aligned} V_{n+1}-V_n \leq& \bar{x}\left(\frac{x_{n+1}-x_n}{\bar{x}}+\frac{x_n-x_{n+1}}{x_{n+1}}\right)+\bar{y}\left(\frac{y_{n+1}-y_n}{\bar{y}}+\frac{y_n-y_{n+1}}{y_{n+1}}\right) \\ =& \frac{(x_{n+1}-x_n)(x_{n+1}-\bar{x})}{x_{n+1}}+\frac{(y_{n+1}-y_n)(y_{n+1}-\bar{y})}{y_{n+1}} \\ \leq& (U_1-L_1)(1-\frac{\bar{x}}{x_{n+1}})+(U_2-L_2)(1-\frac{\bar{y})}{y_{n+1}}) \\ =& (U_1-L_1)\frac{[a+x_n-\bar{x}(b+cy_n+dx_{n-1})]}{a+x_n} \\ &+(U_2-L_2)\frac{[\alpha+y_n-\bar{y}(\beta+\gamma x_n+\eta y_{n-1})]}{\alpha+y_n} \\ \leq& \frac{(U_1-L_1)[a+U_1-L_1(b+cL_2+dL_1)]}{a+L_1} \\ &+\frac{(U_2-L_2)[\alpha+U_2-L_2(\beta+\gamma L_1+\eta L_2)]}{\alpha+L_2}. \end{aligned} \tag{3.30}$$

By using condition (3.25), we have $V_{n+1}-V_n \leq 0$ for all $n \geq 0$, so that $V_n \geq 0$ is monotonically decreasing sequence. It follows that $\lim_{n\to\infty} V_n$ exists and is nonnegative. Hence, we imply that

$$\lim_{n\to\infty}(V_{n+1}-V_n) = 0. \tag{3.31}$$

Then it follows that $\lim_{n\to\infty} x_{n+1} = \bar{x}$ and $\lim_{n\to\infty} y_{n+1} = \bar{y}$. Furthermore, $V_n \leq V_0$ for all $n \geq 0$, which gives that $(\bar{x},\bar{y}) \in [L_1,U_1]\times[L_2,U_2]$ is uniformly stable. Hence, unique positive equilibrium point $(\bar{x},\bar{y}) \in [L_1,U_1]\times[L_2,U_2]$ of system (1.1) is globally asymptotically stable. ∎

4. Rate of convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of the systems (1.1) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [16] and [17].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n \tag{4.1}$$

where $\mathbf{x}_n$ is a k-dimensional vector, $A \in \mathbb{C}^{k\times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \longrightarrow \mathbb{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty, \tag{4.2}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm; $\|.\|$ also denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|\mathbf{x}\| = \|(x,\ y)\| = \sqrt{x^2 + y^2}. \tag{4.3}$$

Theorem 4.1. *([18]) Assume that condition (4.2) holds. If $\mathbf{x}_n$ is a solution of system (4.1), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 4.2. *([18]) Assume that condition (4.2) holds. If $\mathbf{x}_n$ is a solution of system (4.1), then either $\mathbf{x}_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 4.3. *Assume that $\{(x_n, y_n)\}$ is a positive solution of the system (1.1) such that $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1], \bar{y} \in [L_2, U_2]$. Then the error vector $\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \\ e_{n-1}^1 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \\ x_{n-1} - \bar{x} \\ y_{n-1} - \bar{y} \end{pmatrix}$ of every solution $(x_n, y_n) \neq (\bar{x}, \bar{y})$ of (1.1) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_F(\bar{x}, \bar{y}))| \text{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_F(\bar{x}, \bar{y}))| \text{ for some } i \in \{1,\ 2,\ 3,\ 4\}$$

where $|\lambda_i(J_F(\bar{x}, \bar{y}))|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $(\bar{x}, \bar{y})$.

Proof. Let $\{(x_n, y_n)\}$ be an arbitrary positive solution of the system (1.1) such that $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [L_1, U_1], \bar{y} \in [L_2, U_2]$. Firstly, we will find a system satisfied by the error terms, which are given as

$$\begin{aligned} x_{n+1} - \bar{x} =& \frac{a + x_n}{b + cy_n + dx_{n-1}} - \frac{a + \bar{x}}{b + c\bar{y} + d\bar{x}} \\ =& \frac{1}{(b + cy_n + dx_{n-1})}(x_n - \bar{x}) \\ &- \frac{c(a + \bar{x})}{(b + cy_n + dx_{n-1})(b + c\bar{y} + d\bar{x})}(y_n - \bar{y}) \\ &- \frac{d(a + \bar{x})}{(b + cy_n + dx_{n-1})(b + c\bar{y} + d\bar{x})}(x_{n-1} - \bar{x}), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} y_{n+1}-\bar{y} &= \frac{\alpha+y_n}{\beta+\gamma x_n+\eta y_{n-1}}-\frac{\alpha+\bar{y}}{\beta+\gamma\bar{x}+\eta\bar{y}} \\ &= -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}(x_n-\bar{x}) \\ &\quad+\frac{1}{(\beta+\gamma x_n+\eta y_{n-1})}(y_n-\bar{y}) \\ &\quad-\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}(y_{n-1}-\bar{y}). \end{aligned} \tag{4.5}$$

Let $e_n^1=x_n-\bar{x}$ and $e_n^2=y_n-\bar{y}$, then from (4.4) and (4.5) we have:

$$\begin{aligned} e_{n+1}^1 &= p_n e_n^1+q_n e_n^2+r_n e_{n-1}^1, \\ e_{n+1}^2 &= g_n e_n^1+h_n e_n^2+w_n e_{n-1}^2, \end{aligned}$$

where

$$\begin{aligned} p_n &= \frac{1}{(b+cy_n+dx_{n-1})}, \\ q_n &= -\frac{c(a+\bar{x})}{(b+cy_n+dx_{n-1})(b+c\bar{y}+d\bar{x})}, \\ r_n &= -\frac{d(a+\bar{x})}{(b+cy_n+dx_{n-1})(b+c\bar{y}+d\bar{x})}, \\ g_n &= -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}, \\ h_n &= \frac{1}{(\beta+\gamma x_n+\eta y_{n-1})}, \\ w_n &= -\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma x_n+\eta y_{n-1})(\beta+\gamma\bar{x}+\eta\bar{y})}. \end{aligned}$$

Taking the limmits of $p_n,\ q_n,\ r_n, g_n, h_n$ and w_n as $n\to\infty$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} p_n=\frac{1}{(b+c\bar{y}+d\bar{x})},\ \lim_{n\to\infty} q_n=-\frac{c(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2},\ \lim_{n\to\infty} r_n=-\frac{d(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2}, \\ &\lim_{n\to\infty} g_n=-\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2},\ \lim_{n\to\infty} h_n=\frac{1}{(\beta+\gamma\bar{x}+\eta\bar{y})},\ \lim_{n\to\infty} w_n=-\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2}. \end{aligned}$$

that is

$$\begin{aligned} &p_n=\frac{1}{(b+c\bar{y}+d\bar{x})}+\alpha_n,\ q_n=-\frac{c(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2}+\beta_n,\ r_n=-\frac{d(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2}+\gamma_n, \\ &g_n=-\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2}+\delta_n,\ h_n=\frac{1}{(\beta+\gamma\bar{x}+\eta\bar{y})}+\eta_n,\ w_n=-\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2}+\theta_n. \end{aligned}$$

where $\alpha_n\to 0,\ \beta_n\to 0,\ \gamma_n\to 0, \delta_n\to 0, \eta_n\to 0$ and $\theta_n\to 0$ as $n\to\infty$.

Now, we have system of the form (4.1):

$$\mathbf{e}_{n+1}=(A+B(n))\mathbf{e}_n,$$

where

$$A=\begin{pmatrix} \frac{1}{(b+c\bar{y}+d\bar{x})} & -\frac{c(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2} & -\frac{d(a+\bar{x})}{(b+c\bar{y}+d\bar{x})^2} & 0 \\ -\frac{\gamma(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2} & \frac{1}{(\beta+\gamma\bar{x}+\eta\bar{y})} & 0 & -\frac{\eta(\alpha+\bar{y})}{(\beta+\gamma\bar{x}+\eta\bar{y})^2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B(n) = \begin{pmatrix} \alpha_n & \beta_n & \gamma_n & 0 \\ \delta_n & \eta_n & 0 & \theta_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\|B(n)\| \to 0 \text{ as } n \to \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^1_n \\ e^2_n \end{pmatrix} = A \begin{pmatrix} e^1_n \\ e^2_n \\ e^1_{n-1} \\ e^2_{n-1} \end{pmatrix}.$$

The system is exactly linearized system of (1.1) evaluated at the equilibrium $E = (\bar{x},\ \bar{y})$. Then Theorem 4.1 and Theorem 4.2 imply the result. □

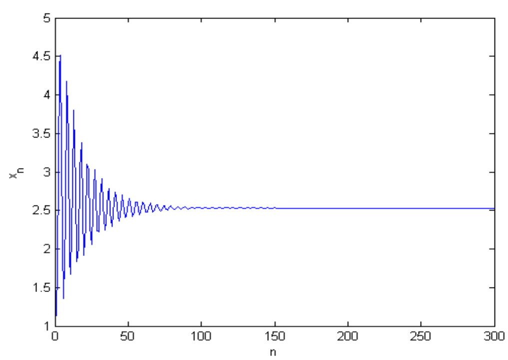
5. Examples

In order to verify our theoretical results and to support our theoretical discussion, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems (1.1). All plots in this section are drawn with Matlab.

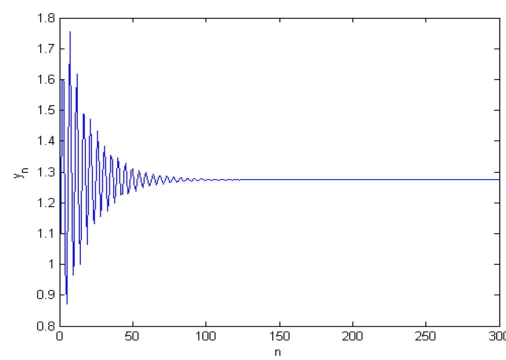
Example 5.1. *Let $a = 3, b = 1.045, c = 0.09, d = 0.8, \alpha = 4, \beta = 1.5, \gamma = 0.69, \eta = 0.7$. The system (1.1) can be written as*

$$x_{n+1} = \frac{3 + x_n}{1.045 + 0.09 y_n + 0.8 x_{n-1}},\ y_{n+1} = \frac{4 + y_n}{1.5 + 0.69 x_n + 0.7 y_{n-1}}, \tag{5.1}$$

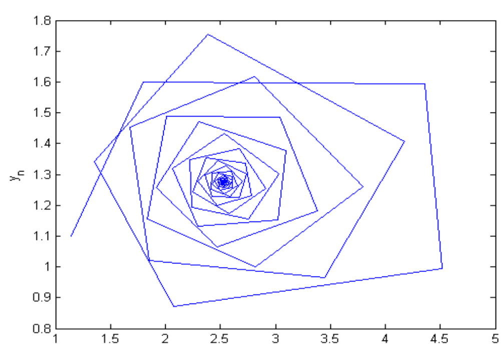
with initial conditions $x_{-1} = 1.14$, $x_0 = 1.8$, $y_{-1} = 1.1$ and $y_0 = 1.6$.



(a) Plot of x_n for the system (5.1)



(b) Plot of y_n for the system (5.1)



(c) An attractor of the system (5.1)

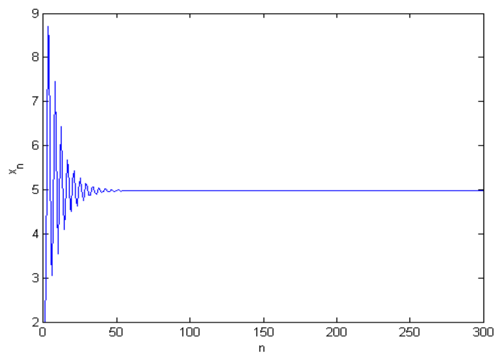
Figure 5.1. Plots for the system (5.1)

In this case, the unique positive equilibrium point of the system (1.1) is global attractor. In Figure 5.1, the plot of x_n is shown in Figure 5.1 (a), the plot of y_n is shown in Figure 5.1 (b), and a phase portrait of the system (5.1) is shown in Figure 5.1 (c).

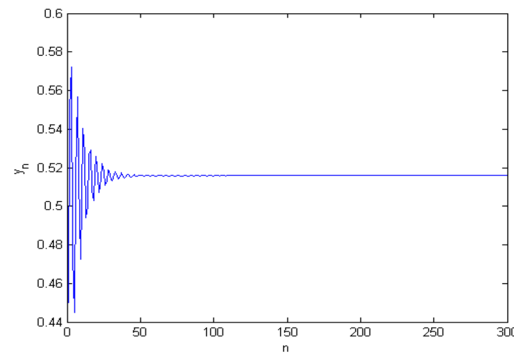
Example 5.2. *Let* $a = 20, b = 1.002, c = 0.07, d = 0.8, \alpha = 0.8, \beta = 2, \gamma = 0.09, \eta = 0.2$. *The system* (1.1) *can be written as*

$$x_{n+1} = \frac{20 + x_n}{1.002 + 0.07y_n + 0.8x_{n-1}}, \; y_{n+1} = \frac{0.8 + y_n}{2 + 0.09x_n + 0.2y_{n-1}}, \tag{5.2}$$

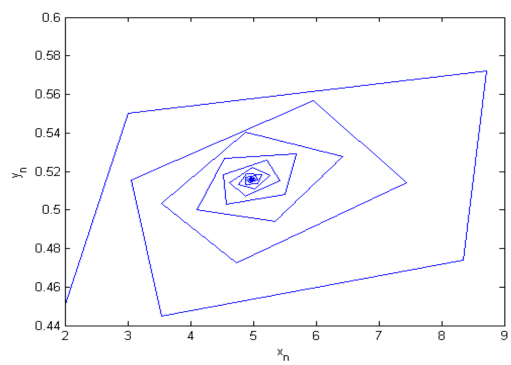
with initial conditions $x_{-1} = 2$, $x_0 = 3$, $y_{-1} = 0.45$ *and* $y_0 = 0.55$.



(a) Plot of x_n for the system (5.2)



(b) Plot of y_n for the system (5.2)



(c) An attractor of the system (5.2)

Figure 5.2. Plots for the system (5.2)

In this case, the unique positive equilibrium point of the system (1.1) is global attractor. In Figure 5.2, the plot of x_n is shown in Figure 5.2 (a), the plot of y_n is shown in Figure 5.2 (b), and a phase portrait of the system (5.2) is shown in Figure 5.2 (c).

6. Conclusion

This work is related to qualitative behavior of the system of second-order rational difference equations. We have investigated the existence and uniqueness of positive equilibrium of system (1.1). Under certain parametric conditions the boundedness and persistence of positive solutions is proved. Moreover, we have shown that unique positive equilibrium point of system (1.1) is locally as well as globally asymptotically stable under certain parametric conditions. Furthermore, the rate of convergence of positive solutions of (1.1) which converge to its unique positive equilibrium point is demonstrated. Finally, numerical examples are established to support our theoretical results.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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A New Pre-Order Relation for Set Optimization using l-difference

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Abstract
A new relation on the subset of the space is defined via ℓ-difference in this work. This is a pre-order relation on the family of nonempty sets. Some relations between this pre-order relation and well-known order relations are investigated. Also, the solution points of a set-valued optimization problem via set and vector approaches are examined.

Keywords: Pre-order relation, Set optimization, Vector approach
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1. Introduction

Optimization problems appear in all parts of our lives. These problems are classified according to the type of objective functions. For example, when the objective function is a set-valued function/mapping/map, the optimization problem is named set-valued optimization problem (shortly, $SVOP$). Recently, $SVOP$ has attracted increasing attention because it has many applications such as finance, control theory, game theory, engineering, statistic, etc.

In the $SVOP$, there are several approaches to solve these optimization problems. Vector and set approaches are the most commonly used types. The first used is the vector approach. In this approach, efficient vectors of the image set of the objective map are investigated. In order to be a solution of a point, the image set of this point has to contain an efficient vector of the image set. The set approach, which is given by Kuroiwa [16, 17], depends on the comparison among values of the objective map. So, an order relation must be used to compare sets in this approach. More information about these approaches and the solution concepts are also available in [4, 6, 8–15, 17, 18], and references therein. In this current investigation, vector and set approaches are considered.

Firstly, Kuroiwa et al. [19] mentioned about set relation based on the ordering cone. Then, they defined six order relations. They gave relationships with each other. By using these order relations, the set optimization approach is constructed by Kuroiwa [16, 17]. Kuroiwa obtained the solutions of $SVOP$ with respect to (shortly, wrt) set approach. Jahn & Ha [6] obtained some new order relations for $SVOP$. Two new partial order relations are defined by Karaman et al. [13] for $SVOP$. There are still sets that can not be compared with these partial and the other order relations. That's why we define a new order relation to compare such sets in this paper.

In order to solve $SVOP$, some methods are used as vectorization, scalarization, directional derivative, subdifferential, embedding space, and so on [1–4, 7–9, 11–15, 21, 23]. The well-known scalarization functions are Gerstewitz, the oriented function of Hiriart-Urruty [5] and generalizations of them. Hernández & Rodríguez-Marín [4] found some optimality conditions for $SVOP$ via derived an extension of Gerstewitz function. Recently, some authors like Khushboo & Lalitha [15], Xu & Li [23], Jiménez et al. [7], Ansari et al. [1] and Chen et al. [2] obtained scalarizations via some extension of the oriented function.

A new relation on the subset of the space is defined via ℓ-difference in this work. We show that this order relation is a pre-order on the family of nonempty sets. Some properties of this pre-order relation are obtained. This pre-order relation is compared with some well-known order relations in the literature. Also, the solutions of $SVOP$ wrt set and vector approaches are examined.

The layout of the study is ordered as follows: The basic definitions and concepts of $SVOP$ are stated and mentioned in section 2. In section 3, a pre-order relation is introduced and some properties are discovered. In section 4, after the solutions concept of $SVOP$ are recalled wrt set approach, the solutions of $SVOP$ are compared according to set and vector approaches.

2. Mathematical Preliminaries

In this study, Y is denoted as a normed space and X is a vector space. Let $K \subset Y$ be given. If $\lambda x \in K$ for all $x \in K$, $\lambda > 0$, then K is called a cone. Assume that cone K is a convex, pointed ($K \cap (-K) = \{0_Y\}$) and closed with the nonempty interior, and Y be ordered by cone K. $\mathscr{P}(Y)$ is denoted the family of proper and nonempty subsets of Y, that is, $\mathscr{P}(Y) := \{A \subset Y : A \neq Y \text{ and } A \text{ is nonempty}\}$. Topological interior and convex hull of any set $A \in \mathscr{P}(Y)$ are indicated by $int(A)$ and $convA$, respectively.

It is denoted that the algebraic sum of A and B by $A+B$, the algebraic difference of A and B by $A-B$, Minkowski (Pontryagin) difference of A and B by $A \dot{-} B := \{x \in Y \mid x+B \subset A\}$ and ℓ-difference of A and B by $A \ominus_\ell B := \{x \in Y \mid x+B \subset A+K\} = (A+K) \dot{-} B$ for any $A, B \in \mathscr{P}(Y)$. Readers can find more information about these in [20–22].

The cone K induces an ordering relations on Y as follow: For $x, x' \in Y$, $x \leq_K x'$ iff $x'-x \in K$, and $x <_K x'$ iff $x'-x \in int(K)$.

Let $A \in \mathscr{P}(Y)$ be a set and $a_0 \in A$. If $A \cap (a_0 - K) = \{a_0\}$ ($A \cap (a_0 + K) = \{a_0\}$), then a_0 is called a minimal (maximal) point of A. The set of all minimal and maximal points of A is indicated by $\min A$ and $\max A$, respectively. Likewise, if $A \cap (a_0 - int(K)) = \emptyset$ ($A \cap (a_0 + int(K)) = \emptyset$), then we say that a_0 is a weak minimal (weak maximal) point of A.

Note that the binary relation $\preceq$ on $\mathscr{S} \subset \mathscr{P}(Y)$ is called a pre-order on $\mathscr{S}$ if $\preceq$ is reflexive and transitive. Also, if pre-order relation $\preceq$ is antisymmetric then the order relation is called a partial order on $\mathscr{S}$.

Definition 2.1. *Let $\mathscr{S} \subset \mathscr{P}(Y)$ and $A, B, C \in \mathscr{S}$ be any sets. The relation $\preceq$ on $\mathscr{S}$ is said to be*

(i) compatible with the addition if $A \preceq B$ implies $A + C \preceq B + C$,

(ii) compatible with positive scalar multiplication if $A \preceq B$ implies $\lambda A \preceq \lambda B$ for all scalars $\lambda > 0$.

Let $F : X \rightrightarrows Y$ be a set-valued function such that $F(x) \in \mathscr{P}(Y)$ for all $x \in X$, and $dom(F) := \{x \in X \mid F(x) \neq \emptyset\}$ be efficient domain set of the set-valued mapping F. $S \subset dom(F)$ be given. Basic $SVOP$ is described by

$$SVOP \begin{cases} \min(\max) F(x) \\ s.t.\ x \in S. \end{cases}$$

We denote the problem by $(v - SVOP)$ when $SVOP$ considers wrt vector approach. Efficient points of the set $F(S) := \bigcup\limits_{x \in S} F(x)$ are investigated to solve $(v - SVOP)$, that is, if $F(x_0)$ contains a minimal (maximal) point of $F(S)$, then $x_0 \in S$ is called a solution of $(v - SVOP)$. In the same way, if $F(x_0)$ contains a weak minimal (weak maximal) point of $F(S)$, then $x_0 \in S$ is entitled a weak solution of $(v - SVOP)$.

In the set approach, a comparison among the values of the set-valued mapping is considered. Namely, efficient sets of $\mathscr{F}(S) := \{F(x) \mid x \in S\}$ are investigated to solve $SVOP$. So, an order relation is needed to solve a $SVOP$ by using the set approach. In the following definition, some order relations are given:

Definition 2.2. *[6, 13, 17] Let $A, B \in \mathscr{P}(Y)$.*

(i) $A \preceq^1 B$ is described by $\forall a \in A,\ \forall b \in B,\ a \leq_K b$,

(ii) $A \preceq^2 B$ is described by $\exists a \in A$ such $\forall b \in B,\ a \leq_K b$,

(iii) $A \preceq^3 B$ is described by $\forall b \in B,\ \exists a \in A$ such $a \leq_K b$,

(iv) $A \preceq^4 B$ is described by $\exists b \in B$, such $\forall a \in A,\ a \leq_K b$,

(v) $A \preceq^5 B$ is described by $\forall a \in A,\ \exists b \in B$ such that $a \leq_K b$,

(vi) $A \preceq^6 B$ is described by $\exists a \in A,\ \exists b \in B$ such that $a \leq_K b$,

(vii) $A \preceq^s B$ is described by $A \preceq^3 B$ and $A \preceq^5 B$,

(viii) $A \preceq^{m_1} B$ *is described by* $(B \dot{-} A) \cap K \neq \emptyset$,

(ix) $A \preceq^{m_2} B$ *is described by* $(A \dot{-} B) \cap (-K) \neq \emptyset$.

It is assumed that $* \in \{1,2,3,4,5,6,s,m_1,m_2\}$ in the rest of the study. In the set approach, the problem is denoted by $(* - SVOP)$ when $SVOP$ considers wrt order relation $\preceq^*$. The efficient set of $\mathscr{F}(S)$ is investigated to solve $(* - SVOP)$. That is, if $F(x_0) \in \mathscr{F}(S)$ is a minimal (resp., maximal) set of $\mathscr{F}(S)$, then x_0 is called a solution of $(* - SVOP)$. Similarly, if $F(x_0) \in \mathscr{F}(S)$ is a weak minimal (resp., weak maximal) set of $\mathscr{F}(S)$, then x_0 is named a weak solution of $(* - SVOP)$.

3. A new Order Relation for Set Approach

In this section, a pre-order relation is derived by using ℓ-difference and some properties of this relation are examined.

Definition 3.1. *Let $A, B \in \mathscr{P}(Y)$. ℓ_1 relation is defined as*

$$A \preceq^{\ell_1} B \; :\Longleftrightarrow (B \ominus_\ell A) \cap K \neq \emptyset.$$

When A and B are taken as singleton, there is a relation between $\preceq^{\ell_1}$ and vector order relation $\leq_K$ on Y as:

$$a \leq_K b \Longrightarrow \{a\} \preceq^{\ell_1} \{b\}$$

for any $A = \{a\}, B = \{b\}$ and $a, b \in Y$.

When two sets don't compare wrt partial order relation $\preceq^{m_1}$, they may be compared wrt order relation $\preceq^{\ell_1}$. For example, when $A = \{(x,0) \in \mathbb{R}^2 \mid 1 \leq x \leq 3\}$ and $B = \{(0,y) \in \mathbb{R}^2 \mid 1 \leq y \leq 3\}$, we have $A \preceq^{\ell_1} B$ and $A \not\preceq^{m_1} B$.

Now, some properties of $\preceq^{\ell_1}$ are presented.

Proposition 3.2. *The order relation $\preceq^{\ell_1}$ has the following properties;*

(i) $\preceq^{\ell_1}$ *is compatible with the addition,*

(ii) $\preceq^{\ell_1}$ *is compatible with the positive scalar multiplication.*

Proof.

(i) Let $A, B, C \in \mathscr{P}(Y)$ and $A \preceq^{\ell_1} B$ be given. Since $A \preceq^{\ell_1} B$, we have $(B \ominus_\ell A) \cap K \neq \emptyset$. That means there exists $\bar{x} \in K$ such that $\bar{x} \in B \ominus_\ell A$. Then, we get $\bar{x} + A \subset B + K$. So, $\bar{x} + A + C \subset B + C + K$, that is $\bar{x} \in (B+C) \ominus_\ell (A+C)$. Therefore,

$$[(B+C) \ominus_\ell (A+C)] \cap K \neq \emptyset.$$

Thus, we obtain $A + C \preceq^{\ell_1} B + C$ that implies $\preceq^{\ell_1}$ is compatible with the addition.

(ii) Let $A \preceq^{\ell_1} B$. We show that $\lambda A \preceq^{\ell_1} \lambda B$ for all scalars $\lambda > 0$. Since $A \preceq^{\ell_1} B$, there exists an $\bar{x} \in K$ such $\bar{x} \in B \ominus_\ell A$, i.e., $\bar{x} + A \subset B + K$. So, we have $\lambda \bar{x} + \lambda A \subset \lambda B + \lambda K = \lambda B + K$ and $\lambda \bar{x} \in K$ because K is cone. Then, we obtain $(\lambda B \ominus_\ell \lambda A) \cap K \neq \emptyset$, i.e., $\lambda A \preceq^{\ell_1} \lambda B$. Hence, $\preceq^{\ell_1}$ is compatible with the positive scalar multiplication.

□

Proposition 3.3. *The order relation $\preceq^{\ell_1}$ has the following properties;*

(i) $\preceq^{\ell_1}$ *is reflexive,*

(ii) $\preceq^{\ell_1}$ *is transitive.*

Proof.

(i) Let $A \in \mathscr{P}(Y)$. Because $0_Y \in A \ominus_\ell A$ and $0_Y \in K$, we have $(A \ominus_\ell A) \cap K \neq \emptyset$. Hence, $A \preceq^{\ell_1} A$.

(ii) Assume that $A \preceq^{\ell_1} B$ and $B \preceq^{\ell_1} C$ for any $A, B, C \in \mathscr{P}(Y)$. We have $(B \ominus_\ell A) \cap K \neq \emptyset$ since $A \preceq^{\ell_1} B$. Then, there exists $x_1 \in K$ such

$$x_1 + A \subset B + K. \tag{3.1}$$

Since $B \preceq^{\ell_1} C$, $(C \ominus_\ell B) \cap K \neq \emptyset$ yields. Then, there exists $x_2 \in K$ such

$$x_2 + B \subset C + K. \tag{3.2}$$

From (3.1) and (3.2) we get $x_1 + x_2 + A \subset x_2 + B + K \subset C + K + K = C + K$. As $x_1 + x_2 + A \subset C + K$ and $x_1 + x_2 \in K$, we obtain $(C \ominus_\ell A) \cap K \neq \emptyset$, i.e., $A \preceq^{\ell_1} C$.

□

Remark 3.4. *The order relation $\preceq^{\ell_1}$ isn't antisymmetric. For example, let $Y=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $A=\{(1,1)\}$ and $B=\{(2,2)\}$ be given. Then, we get $B\ominus_\ell A=\{x\in\mathbb{R}^2 \mid x+A\subset B+K\}=[1,\infty)\times[1,\infty)$ and $A\ominus_\ell B=\{x\in\mathbb{R}^2 \mid x+B\subset A+K\}=[-1,\infty)\times[-1,\infty)$. So, we have $(B\ominus_\ell A)\cap K\neq\emptyset$ and $(A\ominus_\ell B)\cap K\neq\emptyset$, i.e., $A\preceq^{\ell_1} B$ and $B\preceq^{\ell_1} A$. But $A\neq B$. Hence, $\preceq^{\ell_1}$ isn't antisymmetric.*

Corollary 3.5. *The order relation $\preceq^{\ell_1}$ is a pre-order relation on $\mathscr{P}(Y)$.*

Now, a relation between the order relation $\preceq^{\ell_1}$ and order relation $\preceq^{m_1}$ are given.

Proposition 3.6. *Let $A,B\in\mathscr{P}(Y)$. If $A\preceq^{m_1} B$, then $A\preceq^{\ell_1} B$.*

Proof. Let $A\preceq^{m_1} B$, i.e., $(B\dot{-}A)\cap K\neq\emptyset$ be given. There exists an $x\in K$ such $x+A\subset B$. Because K is pointed, we get $x+A\subset B+K$, i.e., $x\in B\ominus_\ell A$. As $x\in K$ and $(B\ominus_\ell A)\cap K\neq\emptyset$, we obtain $A\preceq^{\ell_1} B$ □

Note that $\preceq^{\ell_1}$ doesn't imply $\preceq^{m_1}$. This is presented in the following example.

Example 3.7. *Let $Y=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $A=conv\{(0,0),(1,2)\}$ and $B=conv\{(0,0),(2,1)\}$.*

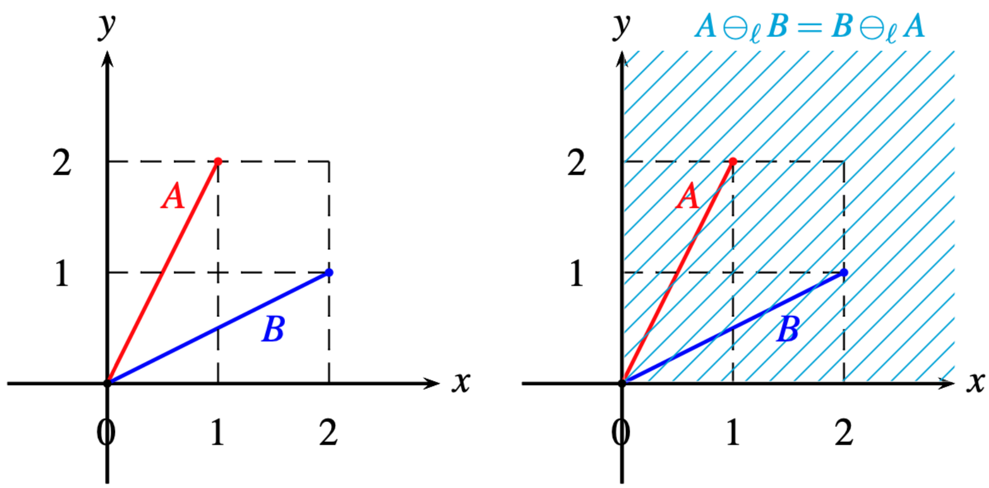

Figure 3.1. $A=conv\{(0,0),(1,2)\}$, $B=conv\{(0,0),(2,1)\}$, $A\ominus_\ell B$ and $B\ominus_\ell A$

As seen in Figure 3.1, since $(B\ominus_\ell A)=K$, we have $(B\ominus_\ell A)\cap K=K$, i.e., $A\preceq^{\ell_1} B$. On the other hand, as $B\dot{-}A=\emptyset$, we get $(B\dot{-}A)\cap K=\emptyset$, i.e., $A\npreceq^{m_1} B$.

In the following definition, strict version of $\preceq^{\ell_1}$ is given.

Definition 3.8. *Let $A,B\in\mathscr{P}(Y)$. The strict ℓ_1 order relation is defined by*

$$A\prec^{\ell_1} B \;:\Longleftrightarrow\; (B\ominus_\ell A)\cap int(K)\neq\emptyset$$

Note that $\prec^{\ell_1}$ implies $\preceq^{\ell_1}$. Namely, if $A\prec^{\ell_1} B$, then $A\preceq^{\ell_1} B$ for all $A,B\in\mathscr{P}(Y)$.

Remark 3.9. *The order relation $\prec^{\ell_1}$ is compatible with not only the addition but also the positive scalar multiplication. Moreover, it is reflexive and transitive. But it isn't antisymmetric.*

One of the most important problems in the set order relations is that some sets can not be compared according to any order relation. Although two sets may not be compared wrt order relation $\preceq^*$, these sets can be compared wrt ℓ_1 order relation. This is illustrated in the accompanying example.

Example 3.10. *Let $K=\mathbb{R}^2_+$, $A=\{(x,y)\in\mathbb{R}^2 \mid 1\leq x\leq 2$ and $3\leq y\leq 4\}$ and $B=\{(x,y)\in\mathbb{R}^2 \mid 3\leq x\leq 4$ and $1\leq y\leq 2\}$.*
As seen Figure 3.2, while $A\npreceq^* B$ we obtain $(B\ominus_\ell A)\cap K\neq\emptyset$, i.e., $A\preceq^{\ell_1} B$.

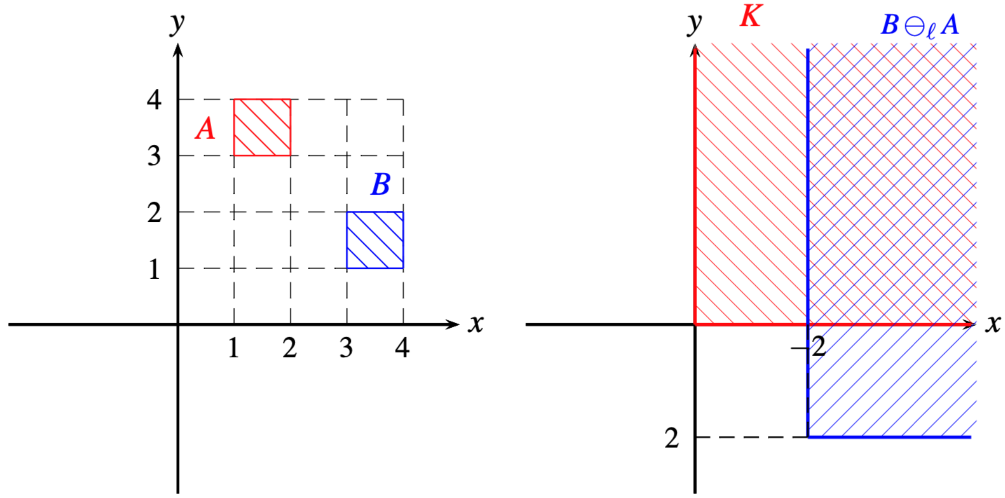

Figure 3.2. $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 \text{ and } 3 \leq y \leq 4\}$, $B = \{(x,y) \in \mathbb{R}^2 \mid 3 \leq x \leq 4 \text{ and } 1 \leq y \leq 2\}$ and $B \ominus_\ell A$

Proposition 3.11. *Let $A, B \in \mathscr{P}(Y)$. Then, the following assertions are satisfied:*

(i) If there exist $a \in A$ and $b \in B$ such that $b \leq_K a$, then $A \preceq^{\ell_1} B$,

(ii) if there exist $a \in A$ and $b \in B$ such that $b <_K a$, then $A \prec^{\ell_1} B$.

Proof. (i) Assume that there exist $a \in A$ and $b \in B$ such that $b \leq_K a$. By contradiction, suppose that $A \npreceq^{l_1} B$. Then, $(B \ominus_\ell A) \cap K = \emptyset$, and we have $k + A \not\subset B + K$ for all $k \in K$. So, $k + a \notin B + K$ for all $k \in K$, $a \in A$. From here, we get $k + a \notin b + K$ for all $k \in K$, $a \in A$ and $b \in B$. Let $k = 0_Y$, then $a \notin b + K$ for all $a \in A$ and $b \in B$. Therefore, for all $a \in A$ and $b \in B$ we get $b \nleq_K a$, which is a contradict.

(ii) This can be proven by similarly to (i).

□

4. Solution Concepts of $SVOP$ According to Set and Vector Approaches

In this part of the study, we obtain relations between the solutions of $(v - SVOP)$ and $(\ell_1 - SVOP)$.

In the following definition, the efficient elements of a family are given wrt pre-order relation $\preceq^{\ell_1}$.

Definition 4.1. *Let $\mathscr{S} \subset \mathscr{P}(Y)$ and $A \in \mathscr{S}$ be given. We call that*

(i) A is an ℓ_1-minimal (ℓ_1-maximal) element of $\mathscr{S}$ iff

$$B \preceq^{\ell_1} A \;\; \text{for some} \;\; B \in \mathscr{S} \Longrightarrow A \preceq^{\ell_1} B \quad (A \preceq^{\ell_1} B \;\; \text{for some} \;\; B \in \mathscr{S} \Longrightarrow B \preceq^{\ell_1} A),$$

(ii) A is a weak ℓ_1-minimal (weak ℓ_1-maximal) element of $\mathscr{S}$ iff

$$B \prec^{\ell_1} A \;\; \text{for some} \;\; B \in \mathscr{S} \Longrightarrow A \prec^{\ell_1} B \quad (A \prec^{\ell_1} B \;\; \text{for some} \;\; B \in \mathscr{S} \Longrightarrow B \prec^{\ell_1} A).$$

If we consider $SVOP$ wrt ℓ_1 order relation, then problem is denoted by

$$(\ell_1 - SVOP) \begin{cases} \min(\max) F(x) \\ s.t. \; x \in S. \end{cases}$$

Let $x_0 \in S$ be given. x_0 is called a solution of $(\ell_1 - SVOP)$ if $F(x_0) \in \mathscr{F}(S)$ is an ℓ_1-minimal (ℓ_1-maximal) set of $\mathscr{F}(S)$. Similarly, x_0 is called a weak solution of $(\ell_1 - SVOP)$ if $F(x_0) \in \mathscr{F}(S)$ is a weak ℓ_1-minimal (weak ℓ_1-maximal) set of $\mathscr{F}(S)$.

The solution of $(\ell_1 - SVOP)$ may not be the solution of $(v - SVOP)$. Now, we will give an example related to this situation.

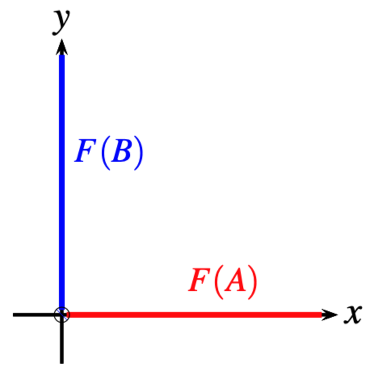

Figure 4.1. $F(A)=\{(x,0)\in\mathbb{R}^2 \mid x>0\}$ and $F(B)=\{(0,y)\in\mathbb{R}^2 \mid y>0\}$

Example 4.2. *Let $Y=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, set-valued map $F:\{A,B\}\rightrightarrows\mathbb{R}^2$ be defined as $F(A)=\{(x,0)\in\mathbb{R}^2 \mid x>0\}$ and $F(B)=\{(0,y)\in\mathbb{R}^2 \mid y>0\}$. Consider the following set-valued optimization problem*

$$SVOP\left\{\begin{array}{l}\min F(x)\\ s.t.\ x\in\{A,B\}.\end{array}\right.$$

As seen in Figure 4.1, $F(A)\cap\min\{F(A)\cup F(B)\}=\emptyset$ and $F(B)\cap\min\{F(A)\cup F(B)\}=\emptyset$. So, A and B pairs of sets are not a solution of $(v-SVOP)$. On the other hand, A and B are solution of (ℓ_1-SVOP) because $F(A)\preceq^{\ell_1}F(B)$ implies $F(B)\preceq^{\ell_1}F(A)$, and $F(B)\preceq^{\ell_1}F(A)$ implies $F(A)\preceq^{\ell_1}F(B)$.

Conversely, the solution of $(v-SVOP)$ may not be the solution of (ℓ_1-SVOP). The following example is related to this situation.

Example 4.3. *Let $Y=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, set-valued map $F:\{1,2\}\rightrightarrows\mathbb{R}^2$ be defined as $F(1)=\{(x,y)\in\mathbb{R}^2 \mid\ x=y \text{ and } x\geq 0\}$ and $F(2)=\{(x,y)\in\mathbb{R}^2 \mid y=-x \text{ and } x\geq 0\}$. Let's consider the following problem*

$$SVOP\left\{\begin{array}{l}\min F(x)\\ s.t.\ x\in\{1,2\}.\end{array}\right.$$

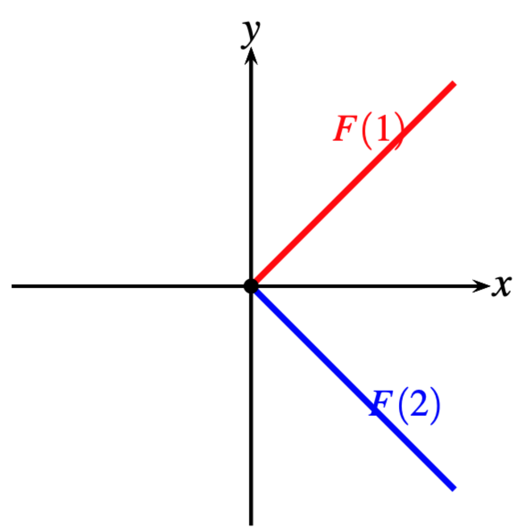

Figure 4.2. $F(1)=\{(x,y)\in\mathbb{R}^2 \mid\ x=y \text{ and } x\geq 0\}$ and $F(2)=\{(x,y)\in\mathbb{R}^2 \mid y=-x \text{ and } x\geq 0\}$

As seen in Figure 4.2, because $\min\{F(1)\cup F(2)\}\cap F(1)\neq\emptyset$ and $\min\{F(1)\cup F(2)\}\cap F(2)\neq\emptyset$, 1 and 2 are solution of $(v-SVOP)$.

Since $(F(2)\ominus_\ell F(1))\cap K\neq\emptyset$, we have $F(1)\preceq^{\ell_1}F(2)$. As $F(1)\ominus_\ell F(2)=\emptyset$, we obtain $(F(1)\ominus_\ell F(2))\cap K=\emptyset$. Hence, we get $F(2)\npreceq^{\ell_1}F(1)$. Because $F(1)\preceq^{\ell_1}F(2)$ doesn't imply $F(2)\preceq^{\ell_1}F(1)$, 2 isn't a solution of (ℓ_1-SVOP). Although 2 is a solution of $(v-SVOP)$, it isn't a solution of (ℓ_1-SVOP).

5. Conclusion

In this study, a new pre-order relation on the family of nonempty sets is introduced, and set-valued optimization problems (ℓ_1-SVOP) are derived. Some optimality conditions can be obtained by using different tools such as vectorization, directional derivative, scalarization, subdifferential etc. for (ℓ_1-SVOP).

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