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Contents

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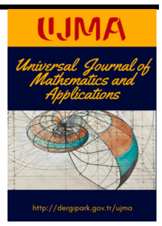

Sesqui-Harmonic Curves in LP-Sasakian Manifolds

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Abstract

In this article, we characterize interpolating sesqui-harmonic spacelike curves in a four-dimensional conformally and quasi-conformally flat and conformally symmetric Lorentzian Para-Sasakian manifold. We give some theorems for these curves.

1. Introduction

Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds and $\sigma:(M_1,g_1)\to(M_2,g_2)$ be a smooth map. The equation

$$\mathbb{L}(\sigma)=\frac{1}{2}\int_{M_1}|\,d\sigma\,|^2\,\vartheta_{g_1}$$

gives the critical points of energy functional The Euler-Lagrange equation of the energy functional gives the harmonic equation defined by vanishing of

$$\tau(\sigma)=trace\nabla d\sigma,$$

where $\tau(\sigma)$ is called the tension field of the map σ.
Biharmonic maps between Riemannian manifolds were studied in [1]. Biharmonic maps between Riemannian manifolds $\psi:(M_1,g_1)\to(M_2,g_2)$ are the critical points of the bienergy functional

$$\mathbb{L}_2(\sigma)=\frac{1}{2}\int_{M_1}|\,\tau(\sigma)\,|^2\,\vartheta_{g_1}.$$

In [2], G.Y. Jiang derived the variations of bienergy formulas and showed that

$$\begin{aligned}\tau_2(\sigma)&=-J^{\sigma}(\tau(\sigma))\\&=-\triangle\tau(\Psi)-traceR^N(d\sigma,\tau(\sigma))d\sigma,\end{aligned}$$

where J^{σ} is the Jacobi operator of σ. The equation $\tau_2(\sigma)=0$ is called biharmonic equation.
Interpolating sesqui-harmonic maps were studied by Branding [3]. The author defined an action functional for maps between Riemannian manifolds that interpolated between the actions for harmonic and biharmonic maps. Ψ is interpolating sesqui-harmonic if it is critical point of $_{\delta_1,\delta_2}(\Psi)$,

$$\mathbb{L}_{\delta_1,\delta_2}(\Psi)=\delta_1\int_{M_1}|d\Psi|^2v_{g_1}+\delta_2\int_{M_1}|\tau(\Psi)|^2v_{g_1}, \tag{1.1}$$

where $\delta_1,\delta_2\in\mathbb{R}$ [3].

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For $\delta_1,\delta_2 \in \mathbb{R}$ the equation

$$\tau_{\delta_1,\delta_2}(\Psi)=\delta_2\tau_2(\Psi)-\delta_1\tau(\Psi)=0, \tag{1.2}$$

is the interpolating sesqui-harmonic map equation [3].
An interpolating sesqui-harmonic map is biminimal if variations of (1.1) that are normal to the image $\Psi(M_1)\subset M_2$ and $\delta_2=1$, $\delta_1>0$ [4]. In a 3-dimensional sphere, interpolating sesqui-harmonic curves were studied in [3]. Interpolating sesqui-harmonic Legendre curves in Sasakian space forms were characterized in [5]. Recently, Yüksel Perktaş et all. introduced biharmonic and biminimal Legendre curves in 3-dimensional f-Kenmotsu manifold [6]. Moreover, spacelike and timelike curves characterized in a four dimensional manifold to be proper biharmonic in [7]. Motivated by the above studies, in this paper, we examine interpolating sesqui-harmonic curves in 4-dimensional LP-Sasakian manifold.

2. Preliminaries

2.1. Lorentzian almost paracontact manifolds

Let M be an n-dimensional differentiable manifold equipped with a structure (ϕ,ζ,η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form on M such that [8]

$$\phi^2=Id+\eta\otimes\zeta \tag{2.1}$$

$$\eta(\zeta)=-1. \tag{2.2}$$

Also, we have

$$\eta\circ\phi=0,\quad \phi\zeta=0,\quad rank(\phi)=n-1.$$

If M admits a Lorentzian metric g, such that

$$g(\phi V,\phi W)=g(V,W)+\eta(V)\eta(W), \tag{2.3}$$

then M is said to admit a *Lorentzian almost paracontact structure* (ϕ,ζ,η,g).
The manifold M endowed with a Lorentzian almost paracontact structure (ϕ,ζ,η,g) is called a L*orentzian almost paracontact manifold* [8,9]. In equations (2.1) and (2.2) if we replace ζ by $-\zeta$, we obtain an almost paracontact structure on M defined by I. Sato [10].
A Lorentzian almost paracontact manifold (M,ϕ,ζ,η,g) is called a *Lorentzian para-Sasakian manifold* [8] if

$$(\nabla_V\phi)W=g(V,W)\zeta+\eta(W)V+2\eta(V)\eta(W)\zeta. \tag{2.4}$$

It is well konown that, conformal curvature tensor $\tilde{C}$ is given by

$$\tilde{C}(V,W)Z \;=\; R(V,W)Z-\frac{1}{n-2}\left\{\ S(W,Z)V-S(V,Z)W+g(W,Z)V-g(V,Z)QW\ \right\}+\left(\frac{r}{(n-1)(n-2)}\right)\{g(W,Z)V-g(V,Z)W\},$$

where S is the Ricci tensor and r is the scalar curvature. If $C=0$, then Lorentzian para-Sasakian manifold is called *conformally flat*.
Also, quasi conformal curvature tensor $\hat{C}$ is defined by

$$\hat{C}(V,W)Z \;=\; \alpha R(V,W)Z-\beta\left\{\ S(W,Z)V-S(V,Z)W+g(W,Z)QV-g(V,Z)QW\ \right\}-\left(\frac{r}{n}\left(\frac{\alpha}{(n-1)}+2\beta\right)\right)\{g(W,Z)V-g(V,Z)W\},$$

where α,β constants such that $\alpha\beta\neq 0$. If $\hat{C}=0$, then Lorentzian para-Sasakian manifold is called quasi conformally flat.
A conformally flat and quasi conformally flat LP-Sasakian manifold M^n $(n>3)$ is of constant curvature 1 and also a LP-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation $R(V,W)\cdot C=0$ holds [11]. For a conformally symmetric Riemannian manifold [12], we have $\nabla C=0$. So, for a conformally symmetric space $R(V,W)\cdot C=0$ satisfies. Therefore a conformally symmetric LP-Sasakian manifold is locally isometric to a Lorentzian unit sphere [11].
In this case, for conformally flat, quasi conformally flat and conformally symmetric LP-Sasakian manifold M, for every $V,W,Z\in TM$ [11], we have

$$R(V,W)Z=g(W,Z)V-g(V,Z)W. \tag{2.5}$$

3. Main results

In this section, we give our main results about interpolating sesqui-harmonic curves in a conformally flat, quasi conformally flat and conformally symmetric LP-Sasakian manifold $\tilde{M}$. From now on, we will consider such a manifold as $\tilde{M}$.

Theorem 3.1. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma: I\to\tilde{M}$ be a curve parametrized by arclength s with $\{t,n,b_1,b_2\}$ orthonormal Frenet frame such that first binormal vector b_1 is timelike. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) γ is a circle with $\rho_1=\sqrt{1-\frac{\delta_1}{\delta_2}}$,
or
ii) γ is a helix with $\rho_1^2-\rho_2^2=1-\frac{\delta_1}{\delta_2}$
where $\frac{\delta_1}{\delta_2}<1$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the first binormal vector b_1 of $\{t,n,b_1,b_2\}$ orthonormal Frenet frame is a timelike vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & \rho_2 & 0 \\ 0 & \rho_2 & 0 & \rho_3 \\ 0 & 0 & \rho_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.1}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve γ [13].
By using (3.1) and equation (2.5), we obtain

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1 \rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1\rho_1')t + (\rho_1'' - \rho_1^3 + \rho_1\rho_2^2)n + (2\rho_1'\rho_2 + \rho_1\rho_2')b_1 + (\rho_1\rho_2\rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

Considering above equations in (1.2), we have

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 + \rho_1\rho_2^2 + \rho_1)\delta_2 \\ -\rho_1\delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

Thus, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1 = const. > 0 \quad \rho_2 = const.$$

$$\rho_1^2 - \rho_2^2 = 1 - \frac{\delta_1}{\delta_2},$$

$$\rho_2\rho_3 = 0.$$

So, we get the proof. □

Theorem 3.2. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t,n,b_1,b_2\}$ orthonormal Frenet frame such that second binormal vector b_2 is timelike. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) γ is a circle with $\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}$,
or
ii) γ is a helix with $\rho_1^2 + \rho_2^2 = 1 - \frac{\delta_1}{\delta_2}$
where $\frac{\delta_1}{\delta_2} < 1$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the vector b_2 of $\{t,n,b_1,b_2\}$ orthonormal Frenet frame is a timelike vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & \rho_2 & 0 \\ 0 & -\rho_2 & 0 & \rho_3 \\ 0 & 0 & \rho_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.2}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve [13].
From (3.2) and (2.5), we get

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1 \rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1\rho_1')t + (\rho_1'' - \rho_1^3 - \rho_1\rho_2^2)n + (2\rho_1'\rho_2 + \rho_1\rho_2')b_1 + (\rho_1\rho_2\rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

Considering above equations in (1.2), we have

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 - \rho_1\rho_2^2 + \rho_1)\delta_2 \\ -\rho_1\delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

In this case, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1 = const. > 0 \quad \rho_2 = const.$$

$$\rho_1^2 + \rho_2^2 = 1 - \frac{\delta_1}{\delta_2},$$

$$\rho_2\rho_3 = 0.$$

This equation proves our assertion. □

Theorem 3.3. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t, n, b_1, b_2\}$ orthonormal Frenet frame such that binormal vector b_1 is null. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) $\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}}$ and
and
ii) $\rho_2 = 0$ or $|ln|\rho_2(s) = -\int \rho_3(s)ds$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the first binormal vector b_1 of $\{t, n, b_1, b_2\}$ orthonormal Frenet frame is a null(lightlike) vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & \rho_2 & 0 & -\rho_3 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.3}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve [13].
By use of (3.3) and equation (2.5), we have

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1\rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1\rho_1')t + (\rho_1'' - \rho_1^3 + \rho_1)n + (2\rho_1'\rho_2 + \rho_1\rho_2')b_1 + (\rho_1\rho_2\rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

In view of (1.2), we arrive at

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 + \rho_1)\delta_2 \\ -\rho_1\delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

Thus, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1\rho_1' = 0$$

$$(\rho_1'' - \rho_1^3 + \rho_1)\delta_2 - \rho_1\delta_1 = 0,$$

$$2\rho_1'\rho_2 + \rho_1\rho_2' + \rho_1\rho_2\rho_3 = 0.$$

If we consider non-geodesic solution, we obtain

$$\rho_1 = \sqrt{1 - \frac{\delta_1}{\delta_2}},$$

$$\rho_2' + \rho_2\rho_3 = 0,$$

where $\frac{\delta_1}{\delta_2} < 1$. □

Theorem 3.4. *Let $\tilde{M}$ be a 4-dimensional LP-Sasakian manifold and $\gamma : I \to \tilde{M}$ be a curve parametrized by arclength s with $\{t, n, b_1, b_2\}$ orthonormal Frenet frame such that normal vector n is timelike. Then γ is a interpolating sesqui-harmonic curve if and only if either*
i) γ is a circle with $\rho_1 = \sqrt{\frac{\delta_1}{\delta_2} - 1}$,
or
ii) γ is a helix with $\rho_1^2 + \rho_2^2 = \frac{\delta_1}{\delta_2} - 1$
where $\frac{\delta_1}{\delta_2} > 1$.

Proof. Let $\tilde{M}$ be a four-dimensional LP-Sasakian manifold and γ be a parametrized curve on $\tilde{M}$. If the normal vector n of $\{t,n,b_1,b_2\}$ orthonormal Frenet frame is a timelike vector, then the Frenet equations of the curve γ given as

$$\begin{bmatrix} \nabla_t t \\ \nabla_t n \\ \nabla_t b_1 \\ \nabla_t b_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho_1 & 0 & 0 \\ \rho_1 & 0 & \rho_2 & 0 \\ 0 & \rho_2 & 0 & \rho_3 \\ 0 & 0 & -\rho_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix} \tag{3.4}$$

where ρ_1, ρ_2, ρ_3 are respectively the first, the second and the third curvature of the curve [13].
By using (3.4) and equation (2.5), we obtain

$$\nabla_t t = \rho_1 n,$$

$$\nabla_t \nabla_t t = -\rho_1^2 t + \rho_1' n + \rho_1 \rho_2 b_1,$$

$$\nabla_t \nabla_t \nabla_t t = -(3\rho_1 \rho_1')t + (\rho_1'' + \rho_1^3 + \rho_1 \rho_2^2 + \rho_1)n + (2\rho_1' \rho_2 + \rho_1 \rho_2')b_1 + (\rho_1 \rho_2 \rho_3)b_2,$$

and

$$R(t, \nabla_t t)t = -\rho_1 n.$$

Considering above equations in (1.2), we have

$$\tau_{\delta_1,\delta_2}(\Psi) = -(3\rho_1\rho_1')\delta_2 t + \left\{ \begin{array}{c} (\rho_1'' - \rho_1^3 + \rho_1 k_2^2 + \rho_1)\delta_2 \\ -\rho_1 \delta_1 \end{array} \right\} n + (2\rho_1'\rho_2 + \rho_1\rho_2')\delta_2 b_1 + (\rho_1\rho_2\rho_3)\delta_2 b_2.$$

Thus, γ is a interpolating sesqui-harmonic curve if and only if

$$\rho_1 = const. > 0 \quad \rho_2 = const.$$

$$\rho_1^2 + \rho_2^2 = \frac{\delta_1}{\delta_2} - 1,$$

$$\rho_2 \rho_3 = 0.$$

So, we get the proof. □

4. Conclusion

In this paper we charaecterized spacelike curves to be Sesqui-harmonic curves in LP-Sasakian manifolds. We gave four theorems about these curves. These theorems showed that if we change the vector fields of the Frenet frame $\{t,n,b_1,b_2\}$, then the equation of Sesqui-harmonic curves change. So, we introduced four different spacelike Sesqui-harmonic curves in this manner.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Ideal Convergent Difference Double Sequence Spaces in Intuitionistic Fuzzy Normed Linear Spaces

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Abstract

In this paper, we introduce difference double sequence spaces $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and $I_2^{0(\mu,\upsilon)}(M,\Delta)$ in the intuitionistic fuzzy normed linear spaces. We also investigate some topological properties of these spaces.

Introduction

Fuzzy set theory firstly defined by Zadeh [39] has been applied many fields of engineering such as in non-linear dynamic systems [10] , in the population dynamics [5], in the quantum physics [27], but also in various fields of mathematics such as in metric and topological spaces [7, 9, 12], in the theory of functions [11, 38] , in the approximation theory [4]. Fuzzy topology plays an essential role in fuzzy theory. It deals with such conditions where the classical theories break down. The intuitionistic fuzzy normed space and intuitionistic fuzzy n-normed space which were investigated in [32] and [36] are the most important improvements in fuzzy topology. In the last years, the concepts of intuitionistic fuzzy I-convergent difference sequence spaces and intuitionistic fuzzy I-convergent difference double sequence spaces have been studied in [21]- [**?**] and [23]- [24], respectively.

The concept of statistical convergence was given by Steinhaus [34] and Fast [8] using the definition of density of the set of natural numbers. Many years later, statistical convergence was discussed by many researchers in the theory of Fourier analysis, ergodic theory and number theory. Some statistical convergence types were studied in [1]- [3] and [29]. As an extended definition of statistical convergence, definition of I-convergence was introduced by Kostyrko, Salat and Wilczynski [26] by using the idea of I of subsets of the set of natural numbers. I-convergence of double sequences $x = (x_{ij})$ has been studied in [30]- [31]. Recently, I- and I^*- convergence of double sequences have been studied by Das et. al [6]. Also, related studies can be found in [13]- [17].

Some new sequence spaces were introduced by means of various matrix transformations in [18], [19], [28] and [35]. Kızmaz [25] defined the difference sequence spaces with the difference matrix as follows:

$$X(\Delta) = \{x = (x_k) \in \omega : \Delta x \in X\}$$

for $X = l_\infty$, c, c_0, where $\Delta x_k = x_k - x_{k+1}$ and Δ denotes the difference matrix $\Delta = (\Delta_{nk})$ defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n \le k \le n+1, \\ 0, & \text{if } 0 \le k < n. \end{cases}$$

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In this paper, we introduce difference double sequence spaces $I_2^{(\mu,\upsilon)}(M,\Delta)$ and $I_2^{0(\mu,\upsilon)}(M,\Delta)$ in the intuitionistic fuzzy normed linear spaces. We also investigate some topological properties of these new spaces.

Basic definitions

In this section, we give some definitions and notations which will be used for this study.

Definition 2.1. ([33]) A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a*1=a$ for all $a\in[0,1]$,
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for each $a,\ b,\ c,d\in[0,1]$.

Definition 2.2. ([33]) A binary operation $\circ:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(i) $\circ$ is associative and commutative,
(ii) $\circ$ is continuous,
(iii) $a\circ 0=a$ for all $a\in[0,1]$,
(iv) $a\circ b\le c\circ d$ whenever $a\le c$ and $b\le d$ for each $a,\ b,\ c,d\in[0,1]$.

Definition 2.3. ([32]) The five-tuple $(X,\mu,\upsilon,*,\circ)$ is said to be intuitionistic fuzzy normed linear space (or shortly IFNLS) is where X is a linear space over a field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, υ are fuzzy sets on $X\times(0,\infty)$, μ denotes the degree of membership and υ denotes the degree of nonmembership of $(x,t)\in X\times(0,\infty)$ satisfying the following conditions for every $x,y\in X$ and $s,t>0$:

(i) $\mu(x,t)+\upsilon(x,t)\le 1$,
(ii) $\mu(x,t)>0$,
(iii) $\mu(x,t)=1$ if and only if $x=0$,
(iv) $\mu(\alpha x,t)=\mu\left(x,\frac{t}{|\alpha|}\right)$ if $\alpha\neq 0$,
(v) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,
(vi) $\mu(x,.):(0,\infty)\to[0,1]$ is continuous,
(vii) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,
(viii) $\upsilon(x,t)<1$,
(ix) $\upsilon(x,t)=0$ if and only if $x=0$,
(x) $\upsilon(\alpha x,t)=\upsilon\left(x,\frac{t}{|\alpha|}\right)$ if $\alpha\neq 0$,
(xi) $\upsilon(x,t)\circ\upsilon(y,s)\ge\upsilon(x+y,s+t)$,
(xii) $\upsilon(x,.):(0,\infty)\to[0,1]$ is continuous,
(xiii) $\lim_{t\to\infty}\upsilon(x,t)=0$ and $\lim_{t\to 0}\upsilon(x,t)=1$.

In this case (μ,υ) is called intuitionistic fuzzy norm.

Example 2.1. ([32]) Let(X, $\|.\|$) be a normed space, and let $a*b=ab$ and $a\circ b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in X$ and every $t>0$, consider

$\mu(x,t):=\frac{t}{t+\|x\|}$ and $\upsilon(x,t):=\frac{\|x\|}{t+\|x\|}$.

Then $(X,\mu,\upsilon,*,\circ)$ is an IFNLS.

Definition 2.4. ([32]) Let $(X,\mu,\upsilon,*,\circ)$ be an IFNLS. For $t>0$, the open ball $B_x(r,t)$ with center $x\in X$ and radius $r\in(0,1)$ is defined as

$$B_x(r,t)=\{y\in X:\mu(x-y,t)>1-r \text{ and } \upsilon(x-y,t)<r\}.$$

Definition 2.5.([26]) If X is a non-empty set, then a family of sets $I\subset P(X)$ is called an ideal in X if and only if

(i) $\emptyset\in I$,
(ii) $A,B\in I$ implies that $A\cup B\in I$, and
(iii) for each $A\in I$ and $B\subset A$ we have $B\in I$,

where $P(X)$ is the power set of X.

Definition 2.6.([26]) If X is a non-empty set, then a non-empty family of sets $F\subset P(X)$ is called a filter on X if and only if

(i) $\emptyset\notin F$,

(ii) $A,B \in F$ implies that $A \cap B \in F$, and
(iii) for each $A \in F$ and $A \subset B$, we have $B \in F$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if it contains $\{\{x\} : x \in X\}$.
A relation between the concepts of an ideal and a filter is given by the following proposition.

Proposition 2.1. ([26]) Let $I \subset P(X)$ be a non-trivial ideal. Then the class $F = F(I) = \{M \subset N : M = X - A,\ for\ some\ A \in I\}$ is a filter on X. $F = F(I)$ is called the filter associated with the ideal I.

Definition 2.7 ([30]) Let I_2 be a non-trivial ideal of $N \times N$ and $(X, \mu, \upsilon, *, \circ)$ be an IFNLS. A double sequence $x = (x_{ij})$ of elements of X is said to be I_2-convergent to $L \in X$ with respect to the intuitionistic fuzzy linear norm (μ, υ) if, for every $\varepsilon > 0$ and $t > 0$, the set

$$\left\{(i,j) \in N \times N : \mu\left(x_{ij} - L, t\right) \leq 1 - \varepsilon \text{ or } \upsilon\left(x_{ij} - L, t\right) \geq \varepsilon\right\} \in I_2.$$

In this case, we write $I_2^{(\mu,\upsilon)} - \lim x = L$.

Definition 2.9. ([20]) An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function M is replaced by $M(x+y) \leq M(x+y) + M(y)$, then this function is called modulus function.

Remark 2.1. ([20]) If M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Main results

In this paper, we introduce a variant of ideal convergent difference double sequence spaces in the intuitionistic fuzzy normed linear spaces. We also investigate some topological properties of these new spaces.
Let w_2 be the space of all double sequences in the intuitionistic fuzzy normed linear spaces. We define the following sequence spaces:

$$I_2{}^{(\mu,\upsilon)}(M,\Delta) =$$

$$\{(x_{ij}) \in w_2 : \left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij} - L, t\right)}{\rho}) \leq 1 - \varepsilon \text{ or } M(\frac{\upsilon\left(\Delta x_{ij} - L, t\right)}{\rho}) \geq \varepsilon\right\} \in I_2\}$$

and

$$I_2^{0(\mu,\upsilon)}(M,\Delta) =$$

$$\{(x_{ij}) \in w_2 : \left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij}, t\right)}{\rho}) \leq 1 - \varepsilon \text{ or } M(\frac{\upsilon\left(\Delta x_{ij}, t\right)}{\rho}) \geq \varepsilon\right\} \in I_2\}.$$

Theorem 3.1. The spaces $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and $I_2^{0(\mu,\upsilon)}(M,\Delta)$ are linear spaces.

Proof. We prove the result for $I_2{}^{(\mu,\upsilon)}(M,\Delta)$. Similarly, it can be proved for $I_2^{0(\mu,\upsilon)}(M,\Delta)$. Let $(x_{ij}), (y_{ij}) \in I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and α, β be scalars. The proof is trivial for $\alpha = 0$ and $\beta = 0$. Let $\alpha \neq 0$ and $\beta \neq 0$. For a given $\varepsilon > 0$, choose $s > 0$ such that $(1-\varepsilon) * (1-\varepsilon) > 1 - s$ and $\varepsilon \circ \varepsilon < s$. Hence, we have

$$A_1 = \left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) \leq 1 - \varepsilon \text{ or } M(\frac{\upsilon\left(\Delta x_{ij} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) \geq \varepsilon\right\} \in I_2,$$

$$A_2 = \left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij} - L_1, \frac{t}{2|\beta|}\right)}{\rho}) \leq 1 - \varepsilon \text{ or } M(\frac{\upsilon\left(\Delta x_{ij} - L_1, \frac{t}{2|\beta|}\right)}{\rho}) \geq \varepsilon\right\} \in I_2,$$

$$A_1^c = \left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) > 1 - \varepsilon \text{ and } M(\frac{\upsilon\left(\Delta x_{ij} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) < \varepsilon\right\} \in F(I_2),$$

and

$$A_2^c = \left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij} - L_1, \frac{t}{2|\beta|}\right)}{\rho}) > 1 - \varepsilon \text{ and } M(\frac{\upsilon\left(\Delta x_{ij} - L_1, \frac{t}{2|\beta|}\right)}{\rho}) < \varepsilon\right\} \in F(I_2).$$

Let define the set $A_3 = A_1 \cup A_2$. Hence $A_3 \in I_2$. It follows that A_3^c is a non-empty set in $F(I_2)$. We will prove that for every $(x_{ij}), (y_{ij}) \in I_2{}^{(\mu,\upsilon)}(M,\Delta)$,

$$A_3^c \subset \left\{(i,j) \in N \times N : M(\tfrac{\mu((\alpha.\Delta x_{ij} + \beta.\Delta y_{ij}) - (\alpha.L_1 + \beta.L_2), t)}{\rho}) > 1 - s\right.$$
$$\left.\text{and } M(\tfrac{\upsilon((\alpha.\Delta x_{ij} + \beta.\Delta y_{ij}) - (\alpha.L_1 + \beta.L_2), t)}{\rho}) < s\right\}.$$

Let $(m,n) \in A_3^c$. In this case,

$$M(\frac{\mu\left(\Delta x_{mn} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) > 1-\varepsilon \quad \text{and } M(\frac{\upsilon\left(\Delta x_{mn} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) < \varepsilon,$$

and

$$M(\frac{\mu\left(\Delta y_{mn} - L_2, \frac{t}{2|\beta|}\right)}{\rho}) > 1-\varepsilon \quad \text{and } M(\frac{\upsilon\left(\Delta y_{mn} - L_2, \frac{t}{2|\beta|}\right)}{\rho}) < \varepsilon \quad .$$

Then

$$M(\frac{\mu((\alpha.\Delta x_{mn} + \beta.\Delta y_{mn}) - (\alpha.L_1 + \beta.L_2), t)}{\rho})$$

$$\geq M(\frac{\mu(\alpha.\Delta x_{mn} - \alpha.L_1, t/2)}{\rho}) * M(\frac{\mu(\beta.\Delta y_{mn} - \beta.L_2, t/2)}{\rho})$$

$$= M(\frac{\mu\left(\Delta x_{mn} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) * M(\frac{\mu\left(\Delta y_{mn} - L_2, \frac{t}{2|\beta|}\right)}{\rho}) > (1-\varepsilon)*(1-\varepsilon) > 1-s$$

and

$$M(\frac{\upsilon((\alpha.\Delta x_{mn} + \beta.\Delta y_{mn}) - (\alpha.L_1 + \beta.L_2), t)}{\rho})$$

$$\leq M(\frac{\upsilon(\alpha.\Delta x_{mn} - \alpha.L_1, t/2)}{\rho}) \circ M(\frac{\upsilon(\beta.\Delta y_{mn} - \beta.L_2, t/2)}{\rho})$$

$$= M(\frac{\upsilon\left(\Delta x_{mn} - L_1, \frac{t}{2|\alpha|}\right)}{\rho}) \circ M(\frac{\upsilon\left(\Delta y_{mn} - L_2, \frac{t}{2|\beta|}\right)}{\rho}) < \varepsilon \circ \varepsilon < s.$$

This proves that

$$A_3^c \subset \left\{(i,j) \in N \times N : M(\tfrac{\mu((\alpha.\Delta x_{ij} + \beta.\Delta y_{ij}) - (\alpha.L_1 + \beta.L_2), t)}{\rho}) > 1-s \right.$$
$$\left. \text{and } M(\tfrac{\upsilon((\alpha.\Delta x_{ij} + \beta.\Delta y_{ij}) - (\alpha.L_1 + \beta.L_2), t)}{\rho}) < s \right\}.$$

Hence $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ is a linear space.

Theorem 3.2. Every closed ball $B_x^c(r,t)(M)$ is an open set in $I_2{}^{(\mu,\upsilon)}(M,\Delta)$.

Proof. Let $B_x(r,t)(M)$ be an open ball with centre $x \in I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and radius $r \in (0,1)$ with respect to t, i.e.

$$B_x(r,t)(M) = \{y \in I_2{}^{(\mu,\upsilon)}(M,\Delta) :$$
$$\left\{(i,j) \in N \times N : M(\frac{\mu\left(\Delta x_{ij} - \Delta y_{ij}, t\right)}{\rho}) \leq 1-r \text{ or } M(\frac{\mu\left(\Delta x_{ij} - \Delta y_{ij}, t\right)}{\rho}) \geq r\right\} \in I_2\}.$$

Let $y \in B_x^c(r,t)(M)$. So $M(\frac{\mu\left(\Delta x - \Delta y, t\right)}{\rho}) > 1-r$ and $M(\frac{\upsilon\left(\Delta x - \Delta y, t\right)}{\rho}) < r$.

Since $M(\frac{\mu\left(\Delta x - \Delta y, t\right)}{\rho}) > 1-r$, there exists $t_0 \in (0,t)$ such that $M(\frac{\mu\left(\Delta x - \Delta y, t_0\right)}{\rho}) > 1-r$ and $M(\frac{\upsilon\left(\Delta x - \Delta y, t_0\right)}{\rho}) < r$.

Let $r_0 = M(\frac{\mu\left(\Delta x - \Delta y, t_0\right)}{\rho})$. Since $r_0 > 1-r$, there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$ and so there exists $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1-s$ and $(1-r_0) \circ (1-r_2) < s$.

Let $r_3 = max\{r_1, r_2\}$. Then $1-s < r_0 * r_1 \leq r_0 * r_3$ and $(1-r_0) \circ (1-r_3) \leq (1-r_0) \circ (1-r_2) < s$.

Consider the closed balls $B_y^c(1-r_3, t-t_0)(M)$ and $B_x^c(r,t)(M)$. We prove that $B_y^c(1-r_3, t-t_0)(M) \subset B_x^c(r,t)(M)$. Let $z \in B_y^c(1-r_3, t-$

$t_0)(M)$. Then $M(\frac{\mu(\Delta y-\Delta z,t-t_0)}{\rho}) > r_3$ and $M(\frac{\upsilon(\Delta y-\Delta z,t-t_0)}{\rho}) < 1-r_3$. Hence

$$M(\frac{\mu(\Delta x-\Delta z,t)}{\rho}) \geq M(\frac{\mu(\Delta x-\Delta y,t_0)}{\rho}) * M(\frac{\mu(\Delta y-\Delta z,t-t_0)}{\rho}) > r_0 * r_3 \geq r_0 * r_1 > 1-s > 1-r,$$

and

$$M(\frac{\upsilon(\Delta x-\Delta z,t)}{\rho}) \leq M(\frac{\upsilon(\Delta x-\Delta y,t_0)}{\rho}) \circ M(\frac{\upsilon(\Delta y-\Delta z,t-t_0)}{\rho})$$
$$< (1-r_0) \circ (1-r_3) < s < r.$$

Thus $z \in B_x^c(r,t)(M)$ and it proves that $B_y^c(1-r_3,t-t_0)(M) \subset B_x^c(r,t)(M)$.

Remark 3.1. It is clear that $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ is an IFNLS. Define

$$\tau_2{}^{(\mu,\upsilon)}(M,\Delta) = \{A \subset I_2{}^{(\mu,\upsilon)}(M,\Delta) :$$
$$\textit{for each}\ x \in A,\ \textit{there exist}\ t > 0\ \textit{and}\ r \in (0,1)\ \textit{such that}\ B_x^c(r,t)(M) \subset A\}.$$

Then $\tau_2{}^{(\mu,\upsilon)}(M,\Delta)$ is a topology on $I_2{}^{(\mu,\upsilon)}(M,\Delta)$.

Theorem 3.3. The topology $\tau_2{}^{(\mu,\upsilon)}(M,\Delta)$ on $I_2^{0(\mu,\upsilon)}(M,\Delta)$ is first countable.

Proof. It is clear that $\{B_x^c(\frac{1}{n},\frac{1}{n})(M) : n \in N\}$ is a local base at $x \in I_2{}^{(\mu,\upsilon)}(M,\Delta)$. Then, the topology $\tau_2{}^{(\mu,\upsilon)}(M,\Delta)$ on $I_2^{0(\mu,\upsilon)}(M,\Delta)$ is first countable.

Theorem 3.4. $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and $I_2^{0(\mu,\upsilon)}(M,\Delta)$ are Hausdorff spaces.

Proof. Let $x,y \in I_2{}^{(\mu,\upsilon)}(M,\Delta)$ such that $x \neq y$. Then $0 < M(\frac{\mu(\Delta x-\Delta y,t)}{\rho}) < 1$ and $0 < M(\frac{\upsilon(\Delta x-\Delta z,t)}{\rho}) < 1$.

Define r_1, r_2 and r such that $r_1 = M(\frac{\mu(\Delta x-\Delta y,t)}{\rho})$, $r_2 = M(\frac{\upsilon(\Delta x-\Delta y,t)}{\rho})$ and $r = max\{r_1, 1-r_2\}$. Then for each $r_0 \in (r,1)$ there exist r_3 and r_4 such that $r_3 * r_4 \geq r_0$ and $(1-r_3) \circ (1-r_4) \leq (1-r_0)$.

Let $r_5 = max\{r_3,(1-r_4)\}$ and consider the closed balls $B_x^c(1-r_5,\frac{t}{2})(M)$ and $B_y^c(1-r_5,\frac{t}{2})(M)$. Then, clearly $B_x^c(1-r_5,\frac{t}{2})(M) \cap B_y^c(1-r_5,\frac{t}{2})(M) = \emptyset$.

Suppose that $z \in B_x^c(1-r_5,\frac{t}{2})(M) \cap B_y^c(1-r_5,\frac{t}{2})(M)$. So,

$$r_1 = M(\frac{\mu(\Delta x-\Delta y,t)}{\rho}) \geq M(\frac{\mu(\Delta x-\Delta z,t/2)}{\rho}) * M(\frac{\mu(\Delta y-\Delta z,t/2)}{\rho})$$
$$\geq r_5 * r_5 \geq r_3 * r_4 \geq r_0 > r \text{ and}$$
$$r_2 = M(\frac{\upsilon(\Delta x-\Delta y,t)}{\rho}) \leq M(\frac{\upsilon(\Delta x-\Delta z,t/2)}{\rho}) \circ M(\frac{\upsilon(\Delta y-\Delta z,t/2)}{\rho})$$
$$\leq (1-r_5) \circ (1-r_5) \leq (1-r_3) \circ (1-r_4) \leq (1-r_0) < 1-r,$$

which is a contradiction. Hence $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ is a Hausdorff space.

Theorem 3.5. Let $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ be an IFNLS, $\tau_2{}^{(\mu,\upsilon)}(M,\Delta)$ be a topology on $I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and (x_{ij}) be a sequence in $I_2{}^{(\mu,\upsilon)}(M,\Delta)$. Then a sequence (x_{ij}) is Δ-convergent to Δx_0 with respect to the intuitionistic fuzzy linear norm (μ,υ) if and only if $M(\frac{\mu(\Delta x_{ij}-\Delta x_0,t)}{\rho}) \longrightarrow 1$ and $M(\frac{\upsilon(\Delta x_{ij}-\Delta x_0,t)}{\rho}) \longrightarrow 0$ as $i,j \longrightarrow \infty$.

Proof. Let $B_{x_0}(r,t)(M)$ be an open ball with centre $x_0 \in I_2{}^{(\mu,\upsilon)}(M,\Delta)$ and radius $r \in (0,1)$ with respect to t, i.e.

$$B_{x_0}(r,t)(M) = \{(x_{ij}) \in I_2{}^{(\mu,\upsilon)}(M,\Delta) :$$
$$\left\{(i,j) \in N \times N : M(\frac{\mu(\Delta x_{ij}-\Delta x_0,t)}{\rho}) \leq 1-r \text{ or } M(\frac{\mu(\Delta x_{ij}-\Delta x_0,t)}{\rho}) \geq r\right\} \in I_2\}.$$

Suppose (x_{ij}) is Δ-convergent to Δx_0 with respect to the intuitionistic fuzzy linear norm (μ,υ). Then for $r \in (0,1)$ and $t > 0$, there exists $k_0 \in N$ such that $(x_{ij}) \in B_{x_0}^c(r,t)(M)$ for all $i,j \geq k_0$. Thus,

$$\left\{(i,j) \in N \times N : M(\frac{\mu(\Delta x_{ij}-\Delta x_0,t)}{\rho}) > 1-r \text{ and } M(\frac{\upsilon(\Delta x_{ij}-\Delta x_0,t)}{\rho}) < r\right\} \in F(I_2).$$

So $1-M(\frac{\mu\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})<r$ and $M(\frac{\upsilon\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})<r$, for all $i,j\geq k_0$. Then $M(\frac{\mu\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})\longrightarrow 1$ and $M(\frac{\upsilon\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})\longrightarrow 0$ as $i,j\longrightarrow\infty$.

Conversely, if for each $t>0$,

$M(\frac{\mu\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})\longrightarrow 1$ and $M(\frac{\upsilon\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})\longrightarrow 0$ as $i,j\longrightarrow\infty$. Then for $r\in(0,1)$, there exists $k_0\in N$ such that $1-M(\frac{\mu\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})<r$ and $M(\frac{\upsilon\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})<r$ for all $i,j\geq k_0$. So, $M(\frac{\mu\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})>1-r$ and $M(\frac{\upsilon\left(\Delta x_{ij}-\Delta x_0,t\right)}{\rho})<r$ for all $i,j\geq k_0$. Hence $(x_{ij})\in B^c_{x_0}(r,t)(M)$ for all $i,j\geq k_0$. This proves that a sequence (x_{ij}) is Δ-convergent to Δx_0 with respect to the intuitionistic fuzzy linear norm (μ,υ).

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Author's contributions

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Fixed Point Theorems for Almost Contractions in Extended b-Metric Spaces

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Abstract

In this paper, we define the concept of almost contraction in extended b-metric spaces. We prove some common fixed point theorems for mappings satisfying almost contractions in extended b-metric spaces. These results extend and generalize the corresponding results given in the literature.

1. Introduction and Preliminaries

Main and earlier result of fixed point theory is the Banach contraction principle which guarantees existence and uniqueness of fixed point. It was proved in complete metric spaces by Banach in 1922. Banach contraction principle was applied as a important method in mathematics and other sciences. Some problems of Mathematics and other sciences didn't solve using Banach contraction principle. Thus, more general fixed point theorems be needed. Some of these theorems were gived in more general spaces of metric spaces, some of them were gived by new contaction mappings which are more general than Banach contraction principle. $b-$metric spaces was introduced by Bakhtin [3] and Czerwik [8] as a generalizations of metric spaces. They proved the contraction mapping principle in b-metric spaces. Recently, Kamran [11] introduced extended b-metric spaces using the idea of b-metric spaces as a new type of generalized metric space and they proved some fixed point theorems on this space. Also some generalized fixed point theorems proved in extended b-metric spaces [1, 2, 6, 7, 9, 10, 12–14].
In this work, we define almost contraction in extended b-metric spaces which was defined in metric spaces by Berinde [4, 5]. And we prove fixed point theorems for mappings satisfying these type contractions.

Definition 1.1. *[11] Let X be a nonempty set and $\theta : X \times X \to [1,\infty)$ be a mapping. A function $d_\theta : X \times X \to [0,\infty)$ is called extended b-metric if it satisfies, for all $x,y,z \in X$*

($d_\theta 1$) $d_\theta(x,y) = 0$ *if and only if* $x = y$,
($d_\theta 2$) $d_\theta(x,y) = d_\theta(y,x)$,
($d_\theta 3$) $d_\theta(x,y) \le \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$.

In this case, the pair (X, d_θ) is called extended b-metric space, in short extended-bMS.

Example 1.2. *[11] Let $X = \{1,2,3\}$ and $\theta : X \times X \to [1,\infty)$, $\theta(x,y) = 1 + x + y$. Define $d_\theta : X \times X \to [0,\infty)$ as*

$$\begin{aligned} d_\theta(x,y) &= 0 \quad \text{for } x = y \\ d_\theta(1,2) &= 80, \quad d_\theta(1,3) = 1000, \quad d_\theta(2,3) = 600. \end{aligned}$$

Then, (X, d_θ) is an extended-bMS.

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Example 1.3. *[1] Let $X=[0,1]$ and $\theta : X\times X\to[1,\infty)$,*

$$\begin{aligned}\theta(x,y) &= \frac{x+y+1}{x+y} \quad \text{for } x,y\in(0,1]\\ \theta(x,y) &= 1, \quad \text{for } x,y=0\end{aligned}$$

Define $d_\theta : X\times X\to[0,\infty)$ as

$$\begin{aligned}d_\theta(x,y) &= \frac{1}{xy} \quad \text{for } x,y\in(0,1],\ x\neq y,\\ d_\theta(x,y) &= 0 \quad \text{for } x,y\in[0,1],\ x=y,\\ d_\theta(x,0) &= \frac{1}{x} \quad \text{for } x\in(0,1].\end{aligned}$$

Then, (X,d_θ) is an extended-bMS.

Definition 1.4. *[11] Let (X,d_θ) be an extended-bMS.*
(i) *A sequence $\{x_n\}$ in X is said to converge to $x\in X$, if for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $d_\theta(x_n,x)<\varepsilon$ for all $n\geq N$. In this case, we write*

$$\lim_{n\to\infty} x_n = x.$$

(ii) *A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $d_\theta(x_n,x_m)<\varepsilon$ for all $n,m\geq N$.*
(iii) *(X,d_θ) is said to be complete if every Cauchy sequence in X is convergent.*

Let (X,d_θ) be extended-*bMS*. If d_θ is continuous, then every convergent sequence has a unique limit.

2. Fixed point theorems

Theorem 2.1. *Let (X,d_θ) be a complete extended-bMS and $f,g : X\to X$ be two self mappings satisfying*

$$d_\theta(fx,gy)\leq \delta M(x,y)+LN(x,y) \tag{2.1}$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n\geq 1$ and

$$M(x,y)=\max\{d_\theta(x,y),d_\theta(x,fx),d_\theta(y,gy)\}$$

$$N(x,y)=\min\{d_\theta(x,fx),d_\theta(y,gy),d_\theta(x,gy),d_\theta(y,fx)\}.$$

Then f and g have a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ in X as $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n\geq 1$. Suppose that there is some $n\geq 1$ such that $x_n=x_{n+1}$. If $n=2k$, then $x_{2k}=x_{2k+1}$ and from (2.1),

$$d_\theta(x_{2k+1},x_{2k+2})=d_\theta(fx_{2k},gx_{2k+1})\leq \delta M(x_{2k},x_{2k+1})+LN(x_{2k},x_{2k+1})$$

where

$$\begin{aligned}M(x_{2k},x_{2k+1}) &= \max\{d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k},fx_{2k}),d_\theta(x_{2k+1},gx_{2k+1})\}\\ &= \max\{d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k+1},x_{2k+2})\}\\ &= \max\{0,0,d_\theta(x_{2k+1},x_{2k+2})\}\end{aligned}$$

and

$$\begin{aligned}N(x_{2k},x_{2k+1}) &= \min\{d_\theta(x_{2k},fx_{2k}),d_\theta(x_{2k+1},gx_{2k+1}),d_\theta(x_{2k},gx_{2k+1}),d_\theta(x_{2k+1},fx_{2k})\}\\ &= \min\{d_\theta(x_{2k},x_{2k+1}),d_\theta(x_{2k+1},x_{2k+2}),d_\theta(x_{2k},x_{2k+2}),d_\theta(x_{2k+1},x_{2k+1})\}\\ &= 0\end{aligned}$$

Thus, we have

$$d_\theta(x_{2k+1},x_{2k+2})\leq \delta d_\theta(x_{2k+1},x_{2k+2})$$

which is a contradiction with $\delta\in[0,1)$. Therefore $x_{2k+1}=x_{2k+2}$. Hence, we have $x_{2k}=x_{2k+1}=x_{2k+2}$. It means that $x_{2k}=fx_{2k}=gx_{2k}$,i.e. x_{2k} is a common fixed point of f and g.
If $n=2k+1$, then using same arguments, it can be shown that x_{2k+1} is a common fixed point of f and g.
Now, suppose $x_n\neq x_{n+1}$ for all $n\geq 1$.

$$d_\theta(x_{2n+1},x_{2n+2})=d_\theta(fx_{2n},gx_{2n+1})\leq \delta M(x_{2n},x_{2n+1})+LN(x_{2n},x_{2n+1}) \tag{2.2}$$

where

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= \max\{d_\theta(x_{2n},x_{2n+1}),d_\theta(x_{2n},fx_{2n}),d_\theta(x_{2n+1},gx_{2n+1})\} \\ &= \max\{d_\theta(x_{2n},x_{2n+1}),d_\theta(x_{2n},x_{2n+1}),d_\theta(x_{2n+1},x_{2n+2})\} \\ &= \max\{d_\theta(x_{2n},x_{2n+1}),d_\theta(x_{2n+1},x_{2n+2})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{2n},x_{2n+1}) &= \min\{d_\theta(x_{2n},fx_{2n}),d_\theta(x_{2n+1},gx_{2n+1}),d_\theta(x_{2n},gx_{2n+1}),d_\theta(x_{2n+1},fx_{2n})\} \\ &= \min\{d_\theta(x_{2n},x_{2n+1}),d_\theta(x_{2n+1},x_{2n+2}),d_\theta(x_{2n},x_{2n+2}),0\} \\ &= 0. \end{aligned}$$

If $M(x_{2n},x_{2n+1}) = d_\theta(x_{2n+1},x_{2n+2})$, then by (2.2)

$$d_\theta(x_{2n+1},x_{2n+2}) \leq \delta d_\theta(x_{2n+1},x_{2n+2})$$

which is a contradiction. Thus $M(x_{2n},x_{2n+1}) = d_\theta(x_{2n},x_{2n+1})$ and from (2.2)

$$d_\theta(x_{2n+1},x_{2n+2}) \leq \delta d_\theta(x_{2n},x_{2n+1}).$$

Similarly it can be proved that

$$d_\theta(x_{2n+3},x_{2n+2}) \leq \delta d_\theta(x_{2n+2},x_{2n+1}).$$

So,

$$d_\theta(x_{n+1},x_n) \leq \delta d_\theta(x_n,x_{n-1}) \leq \delta^n d_\theta(x_1,x_0)$$

for all $n \geq 1$.
We show that $\{x_n\}$ is a Cauchy sequence. For all $p \geq 1$,

$$\begin{aligned} d_\theta(x_n,x_{n+p}) &\leq \theta(x_n,x_{n+p})\left[d_\theta(x_n,x_{n+1})+d_\theta(x_{n+1},x_{n+p})\right] \\ &\leq \theta(x_n,x_{n+p})\left[\delta^n d_\theta(x_0,x_1)+d_\theta(x_{n+1},x_{n+p})\right] \\ &\quad ... \\ &\leq \theta(x_n,x_{n+p})\,\delta^n d_\theta(x_0,x_1)+\theta(x_n,x_{n+p})\,\theta(x_{n+1},x_{n+p})\,\delta^{n+1}d_\theta(x_0,x_1) \\ &\quad +...+\theta(x_n,x_{n+p})\,...\theta(x_{n+p-1},x_{n+p})\,\delta^{n+p-1}d_\theta(x_0,x_1) \\ &= d_\theta(x_0,x_1)\sum_{i=1}^{n+p-1}\delta^i\prod_{j=1}^{i}\theta(x_{n+j},x_{n+p}). \end{aligned}$$

The last inequality is dominated by

$$\sum_{i=1}^{n+p-1}\delta^i\prod_{j=1}^{i}\theta(x_{n+j},x_{n+p}) \leq \sum_{i=1}^{n+p-1}\delta^i\times\prod_{j=1}^{i}\theta(x_j,x_{n+p}).$$

By the ratio test, the series$\sum_{i=1}^{\infty}S_i$ where $S_i = \delta^i\prod_{j=1}^{i}\theta(x_j,x_{n+p})$ converges to some $z \in (0,\infty)$. Indeed, $\lim_{i\to\infty}\frac{S_{i+1}}{S_i} = \lim_{i\to\infty}\delta\theta(x_i,x_{i+p}) < 1$.
Thus, we have $a = \sum_{i=1}^{\infty}\delta^i\prod_{j=1}^{i}\theta(x_j,x_{n+p})$ with the partial sum $a_n = \sum_{i=1}^{n}\delta^i\prod_{j=1}^{i}\theta(x_j,x_{n+p})$.
Hence, for $n \leq 1$, $p \leq 1$ we have

$$d_\theta(x_n,x_{n+p}) \leq \delta^n d_\theta(x_0,x_1)\left[S_{n+p-1}-S_{n-1}\right]. \tag{2.3}$$

Letting $n\to\infty$ in (2.3), we conclude that the sequence $\{x_n\}$ is a Cauchy sequence. By completeness of (X,d_θ), there exists $r \in X$ such that $x_n \to r$ as $n\to\infty$.
Now, we prove that $fr = r$. By $b-$rectangular inequality ,

$$d_\theta(x_{2n+1},gr) = d_\theta(fx_{2n},gr) \leq \delta M(x_{2n},r)+LN(x_{2n},r)$$

where

$$M(x_{2n},r) = \max\{d_\theta(x_{2n},r),d_\theta(x_{2n},x_{2n+1}),d_\theta(r,gr)\} \to d_\theta(r,gr),$$

as $n\to\infty$ and

$$N(x_{2n},r) = \min\{d(x_{2n},x_{2n+1}),d(r,gr),d(x_{2n},gr),d(r,x_{2n+1})\} \to 0.$$

Hence, taking the limit as $n\to\infty$, we obtain

$$d_\theta(r,gr) \leq \delta d_\theta(r,gr)+L.0$$

that is $fr = r$. Hence r is a fixed point of g.
Now, we show $r = fr$. Suppose $r \neq fr$, By (2.1)

$$d_\theta(fr,r) = d_\theta(fr,gr) \leq \delta M(r,r) + LN(r,r)$$

where

$$\begin{aligned} M(r,r) &= \max\{d_\theta(r,r), d_\theta(r,fr), d_\theta(r,gr)\} \\ &= \max\{0,0,d_\theta(r,fr)\} \\ &= d_\theta(r,gr) \end{aligned}$$

and

$$\begin{aligned} N(r,r) &= \min\{d_\theta(r,fr), d_\theta(r,gr), d_\theta(r,gr), d_\theta(r,fr)\} \\ &= 0. \end{aligned}$$

Thus, we have

$$d_\theta(fr,r) \leq \delta d_\theta(fr,r)$$

which is a contradiction. Thus $r = fr$.
Now, we show that uniqueness, Suppose r and t are different common fixed points of f and g. By (2.1),

$$d_\theta(r,t) = d_\theta(fr,gt) \leq \delta M(r,t) + LN(r,t) \tag{2.4}$$

where

$$\begin{aligned} M(r,t) &= \max\{d_\theta(r,t), d_\theta(r,fr), d_\theta(t,gt)\} \\ &= d_\theta(r,t) \end{aligned}$$

and

$$\begin{aligned} N(r,t) &= \min\{d_\theta(r,fr), d_\theta(t,gt), d_\theta(r,gt), d_\theta(t,fr)\} \\ &= 0. \end{aligned}$$

From (2.4)

$$d_\theta(r,t) \leq \delta d_\theta(r,t)$$

So $d_\theta(r,t) = 0$, i.e. $r = t$.

Example 2.2. *Let $X = [0,1]$ and $\theta : X \times X \to [1,\infty)$, $\theta(x,y) = 1 + x + y$. Define $d_\theta : X \times X \to [0,\infty)$ such that $d_\theta(x,y) = (x-y)^2$ with for all $x,y \in X$. Let $f,g : X \to X$ be defined as*

$$f(x) = \frac{x}{2}, \quad g(x) = \frac{3x}{4}.$$

Then, d_θ is complete extended $b-$metric on X. We have

$$d_\theta(fx,gy) = \left(\frac{x}{2} - \frac{3y}{4}\right)^2 \leq \delta M(x,y) + LN(x,y)$$

where

$$M(x,y) = \max\left\{(x-y)^2, \left(\frac{x}{2}\right)^2, \left(\frac{y}{4}\right)^2\right\}$$

$$N(x,y) = \min\left\{\left(\frac{x}{2}\right)^2, \left(\frac{y}{4}\right)^2, \left(x - \frac{3y}{4}\right)^2, \left(y - \frac{x}{2}\right)^2\right\}.$$

with $\delta = \frac{3}{4}$ and $L \geq 0$.

□

If $x = y$,

$$d_\theta(fx,gy) = \left(\frac{x}{2} - \frac{3y}{4}\right)^2 = \left(\frac{x}{4}\right)^2 \leq \frac{3}{4}\left(\frac{x}{2}\right)^2 + L\left(\frac{x}{4}\right)^2.$$

If $x = 0, y \neq 0$

$$d_\theta(fx,gy) = \left(0 - \frac{3y}{4}\right)^2 = \left(\frac{3y}{4}\right)^2 \leq \frac{3}{4}y^2 + L.0.$$

If $y=0, x\neq 0$,

$$d_\theta(fx,gy)=\left(\frac{x}{2}-0\right)^2\leq\frac{3}{4}x^2+L.0.$$

If $x\neq y\neq 0$,

$$d_\theta(fx,gy)=\left(\frac{x}{2}-\frac{3y}{4}\right)^2\leq\frac{3}{4}M(x,y)+LN(x,y)$$

Also, for each $x\in X$ $f^n x=\frac{x}{2^n}$, we have

$$\lim_{n,m\to\infty}\theta(x_n,x_m)=\lim_{n,m\to\infty}\theta\left(\frac{x}{2^n}+\frac{x}{2^m}+1\right)<\frac{4}{3}.$$

Thus all conditions of Theorem 2.1 are satisfied and $x=0$ is a unique fixed point of f and g.

Corollary 2.3. *Let (X,d_θ) be a complete extended-bMS space and $f,g:X\to X$ be self mappings satisfying*

$$d_\theta(fx,gy)\leq\delta d_\theta(x,y)+L\min\{d_\theta(x,fx),d_\theta(y,gy),d_\theta(x,gy),d_\theta(y,fx)\}$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n\geq 1$. Then f and g have a unique fixed point.

Corollary 2.4. *Let (X,d_θ) be a complete extended-bMS space and $f:X\to X$ be a self mapping satisfying*

$$d_\theta(fx,fy)\leq\delta M(x,y)+LN(x,y)$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{n+1}=fx_n$, where

$$M(x,y)=\max\{d_\theta(x,y),d_\theta(x,fx),d_\theta(y,fy)\}$$

$$N(x,y)=\min\{d_\theta(x,fx),d_\theta(y,fy),d_\theta(x,fy),d_\theta(y,fx)\}.$$

Then f has a unique fixed point.

Corollary 2.5. *Let (X,d_θ) be a complete extended-bMS space and $f:X\to X$ be a self mapping satisfying*

$$d_\theta(fx,fy)\leq\delta d_\theta(x,y)+L\min\{d_\theta(x,fx),d_\theta(y,fy),d_\theta(x,fy),d_\theta(y,fx)\}$$

for all $x,y\in X$, where $\delta\in[0,1)$ and $L\geq 0$ such that for each $x_0\in X$, $\lim_{n,m\to\infty}\theta(x_n,x_m)<\frac{1}{\delta}$ with $x_{n+1}=fx_n$. Then f has a unique fixed point.

3. Conclusion

The development of the field of fixed point theory depends on the generalization of the Banach Contraction principle on complete metric spaces. This generalization or extension comes up by either introducing new types of contractions or by working on a more general structured space such as extended b-metric spaces. In this article, we have proven some fixed point theorems for almost contraction in extended b-metric spaces and hence our results generalize many existing results in the literature.

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A Study on f-Rectifying Curves in Euclidean n-Space

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Abstract

A rectifying curve in Euclidean n-space $\mathbb{E}^n$ is defined as an arc-length parametrized curve γ in $\mathbb{E}^n$ such that its position vector always lies in its rectifying space (i.e., the orthogonal complement of its principal normal vector field) in $\mathbb{E}^n$. In this paper, in analogy to this, we introduce the notion of an f-rectifying curve in $\mathbb{E}^n$ as a curve γ in $\mathbb{E}^n$ parametrized by its arc-length s such that its f-position vector field γ_f, defined by $\gamma_f(s) = \int f(s)d\gamma$, always lies in its rectifying space in $\mathbb{E}^n$, where f is a nowhere vanishing real-valued integrable function in parameter s. The main purpose is to characterize and classify such curves in $\mathbb{E}^n$.

1. Introduction

Let $\mathbb{E}^3$ denote the Euclidean 3-space (i.e., the three-dimensional real vector space $\mathbb{R}^3$ endowed with the *standard inner product* $\langle \cdot, \cdot \rangle$). Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit-speed curve (i.e., a curve in $\mathbb{E}^3$ parametrized by *arc length function s*) of class at least $\mathscr{C}^3$ (i.e., possessing continuous derivatives at least up to third order). Needless to mention, I denotes a non-trivial interval in $\mathbb{R}$, i.e., a connected set in $\mathbb{R}$ containing at least two points. We consider the *Frenet apparatus* $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$ for the curve γ which is defined as follows: $T_\gamma = \gamma'$ is the unit *tangent vector field* along γ; N_γ is the unit *principal normal vector field* along γ obtained by normalizing the acceleration vector field T'_γ; $B_\gamma = T_\gamma \times N_\gamma$ is the unit *binormal vector field* along γ and it is the unique vector field along γ orthogonal to both T_γ and N_γ so that the *dynamic Frenet frame* $\{T_\gamma, N_\gamma, B_\gamma\}$ is positive definite along γ having the same orientation as that of $\mathbb{E}^3$; κ_γ is the *curvature* and τ_γ is the *torsion* of γ. If γ is of class at least $\mathscr{C}^5$, then its curvature κ_γ and torsion τ_γ are at least twice differentiable. Moreover, γ reduces to a *tortuous curve* in $\mathbb{E}^3$ if it has nowhere vanishing curvature κ_γ and torsion τ_γ (cf. [1] or [2]).

At each point $\gamma(s)$ on γ, the planes spanned by $\{T_\gamma(s), N_\gamma(s)\}$, $\{T_\gamma(s), B_\gamma(s)\}$ and $\{N_\gamma(s), B_\gamma(s)\}$ are respectively called the *osculating plane*, *rectifying plane* and *normal plane* of γ ([1,2]). It is well known from elementary *Differential Geometry* that a space curve γ lies in a *plane* in $\mathbb{E}^3$ if its position vector field always lies in its osculating planes, and it lies on a *sphere* in $\mathbb{E}^3$ if its position vector field always lies in its normal planes. In this point of view, it is natural to inquire the geometric question: *Does there exist a space curve $\gamma : I \longrightarrow \mathbb{E}^3$ whose position vector field always lies in its rectifying planes?* The existence of such space curves was introduced by B.Y. Chen in his paper [3] and named as *rectifying curves*. Thus, the position vector field of a rectifying curve $\gamma : I \longrightarrow \mathbb{E}^3$ parametrized by arc length function s satisfies the equation

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s)$$

for some smooth functions $\lambda, \mu : I \longrightarrow \mathbb{R}$. In [3], B.Y. Chen explored some characterizations of rectifying curves in $\mathbb{E}^3$ in terms of distance functions, tangential, normal and binormal components of their position vector field and also in terms of ratios of their curvature and torsion. Also, he attempted for a classification of such curves in $\mathbb{E}^3$ based on a sort of dilation applied on unit-speed curves on the unit sphere $\mathbb{S}^2(1)$.

In [4], B.Y. Chen and F. Dillen observed that rectifying curves can be viewed as *centrodes* and *extremal curves* in $\mathbb{E}^3$. Moreover, they found a relation between rectifying curves and centrodes which performs a significant role in defining the curves of constant procession in *Differential Geometry* as well as in *Kinematics* or, in general, *Mechanics*. Thereafter, several characterizations of rectifying curves in

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Euclidean spaces were evolved in [5–8]. Meanwhile, the notion of rectifying curves were extended to several ambient spaces, e.g., 3D sphere $\mathbb{S}^3(r)$ [9], 3D hyperbolic space $\mathbb{H}^3(-r)$ [10], Minkowski 3-space $\mathbb{E}^3_1$ [11,12], Minkowski space-time $\mathbb{E}^4_1$ [13–15]. Furthermore, a new kind of curves were studied in $\mathbb{E}^3$ which generalizes rectifying curves and helices [16]. Also, some characterizations and classification of non-null and null f-rectifying curves (which are a sort of generalization of rectifying curves) were investigated in *Minkowski* 3*-space* $\mathbb{E}^3_1$ [17,18], *Minkowski space-time* $\mathbb{E}^4_1$ [19] and Euclidean 4-space [20].

In section 2, we give requisite preliminaries and then, in section 3, we introduce the notion of f-rectifying curves in $\mathbb{E}^n$. Thereafter, section 4 is devoted to investigate some simple geometric characterizations of f-rectifying curves in $\mathbb{E}^n$. Afterwards, section 5 is dedicated to classify f-rectifying curves in terms of their f-position vectors in $\mathbb{E}^n$. Finally, we conclude our study in section 6. This is how the paper is organised.

2. Preliminaries

The *Euclidean n-space* $\mathbb{E}^n$ is the n-dimensional real vector space $\mathbb{R}^n$ equipped with the *standard inner product* $\langle\cdot,\cdot\rangle$ defined by

$$\langle x,y\rangle := \sum_{i=1}^{n} x_i y_i$$

for all tangent vectors $x=(x_1,x_2,\dots,x_n), y=(y_1,y_2,\dots,y_n)$ to $\mathbb{R}^n$. As usual, the *norm* or *length* of a tangent vector $x=(x_1,x_2,\dots,x_n)$ to $\mathbb{R}^n$ is denoted and defined by

$$\|x\| := \sqrt{\langle x,x\rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}\,.$$

Let $\gamma : J \longrightarrow \mathbb{E}^n$ be an arbitrary differentiable curve parametrized by t and γ' denotes its velocity vector field in $\mathbb{E}^n$. Also, we assume that γ is regular, i.e., its velocity vector field γ' is nowhere vanishing. If we change the parameter t by arc-length function $s : J \longrightarrow I$ based at t_0 given by

$$s(t) = \int_{t_0}^{t} \left\|\gamma'(u)\right\| du$$

such that $\|\gamma'(s)\| = \sqrt{\langle\gamma'(s),\gamma'(s)\rangle} = 1$, i.e., $\langle\gamma'(s),\gamma'(s)\rangle = 1$, then $\gamma : I \longrightarrow \mathbb{E}^n$ is referred to as an *arc-length parametrized* or a *unit-speed* curve in $\mathbb{E}^n$. We may consider that γ is of class at least $\mathscr{C}^4$. Now, let T_γ, N_γ denote respectively the unit *tangent vector field* and the unit *principal normal vector field* of γ and for each $i\in\{1,2,\dots,n-2\}$, let B_{γ_i} denote the unit i-th *binormal vector field* of γ so that $\{T_\gamma,N_\gamma,B_{\gamma_1},B_{\gamma_2},\dots,B_{\gamma_{n-2}}\}$ forms the positive definite *dynamic Frenet frame* along γ having the same orientation as that of $\mathbb{E}^n$. Also, for each $i\in\{1,2,\dots,n-1\}$, let κ_{γ_i} denote the i-th *curvature* of γ. Then the Frenet-Serret formulae for the curve γ are given by ([21,22])

$$\begin{pmatrix} T'_\gamma \\ N'_\gamma \\ B'_{\gamma_1} \\ B'_{\gamma_2} \\ \vdots \\ B'_{\gamma_{n-2}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{\gamma_1} & 0 & 0 & \cdots & 0 & 0 \\ -\kappa_{\gamma_1} & 0 & \kappa_{\gamma_2} & 0 & \cdots & 0 & 0 \\ 0 & -\kappa_{\gamma_2} & 0 & \kappa_{\gamma_3} & \cdots & 0 & 0 \\ 0 & 0 & -\kappa_{\gamma_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \kappa_{\gamma_{n-1}} & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_{\gamma_1} \\ B_{\gamma_2} \\ \vdots \\ B_{\gamma_{n-2}} \end{pmatrix}. \tag{2.1}$$

From the above formulae, it follows that $\kappa_{\gamma_{n-1}} \not\equiv 0$ if and only if the curve γ lies wholly in $\mathbb{E}^n$. This is equivalent to saying that $\kappa_{\gamma_{n-1}} \equiv 0$ if and only if the curve γ lies wholly in a hypersurface in $\mathbb{E}^n$ (cf. [21,22]). We recall that the hypersurface in $\mathbb{E}^n$ defined by

$$\mathbb{S}^{n-1}(1) := \{x\in\mathbb{E}^n : \langle x,x\rangle = 1\}$$

is called the *unit sphere* with centre at the origin in $\mathbb{E}^n$. We also recall that the *rectifying space* of the curve γ in $\mathbb{E}^n$ is the orthogonal complement $N_\gamma{}^\perp$ of its principal normal vector field N_γ in $\mathbb{E}^n$ defined by

$$N_\gamma{}^\perp := \left\{x\in\mathbb{E}^n : \langle x,N_\gamma\rangle = 0\right\}.$$

3. Notion of f-rectifying curves in $\mathbb{E}^n$

Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve (parametrized by arc length s) with Frenet apparatus $\{T_\gamma,N_\gamma,B_{\gamma_1},B_{\gamma_2},\dots,B_{\gamma_{n-2}},\kappa_{\gamma_1},\kappa_{\gamma_2},\dots,\kappa_{\gamma_{n-1}}\}$. As found in [8], γ is a rectifying curve in $\mathbb{E}^n$ if and only if its position vector field always lies in its rectifying space, i.e., if and only if its position vector field satisfies

$$\gamma(s) = \lambda(s)T_\gamma(s) + \sum_{i=1}^{n-2}\mu_i(s)B_{\gamma_i}(s)$$

for some differentiable functions $\lambda,\mu_1,\mu_2,\dots,\mu_{n-2} : I \longrightarrow \mathbb{R}$. Now, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function. Then the *f-position vector field* of γ is denoted by γ_f and is defined by

$$\gamma_f(s) = \int f(s)\,d\gamma.$$

Here, the integral sign is used in this sense that on differentiation of previous equation, one finds

$$\gamma'_f(s) = f(s)T_\gamma(s)$$

so that γ_f is an *integral curve* of the vector field fT_γ along γ in $\mathbb{E}^n$. Using this concept of f-position vector field of a curve in $\mathbb{E}^n$, we define an f-rectifying curve in $\mathbb{E}^n$ as follows:

Definition 3.1. *Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve with Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}, \ldots, B_{\gamma_{n-2}}, \kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}\}$ and $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in arc-length parameter s of γ with at least $(n-2)$-times differentiable primitive function F. Then γ is referred to as an f-rectifying curve in $\mathbb{E}^n$ if its f-position vector field γ_f always lies in its rectifying space in $\mathbb{E}^n$, i.e., if its f-position vector field γ_f satisfies the equation*

$$\gamma_f(s) = \lambda(s)T_\gamma(s) + \sum_{i=1}^{n-2} \mu_i(s)B_{\gamma_i}(s) \tag{3.1}$$

for some differentiable functions $\lambda, \mu_1, \mu_2, \ldots, \mu_{n-2} : I \longrightarrow \mathbb{R}$.

Remark 3.2. *In particular, if the function f is a non-zero constant on I, then, up to isometries (rigid motions) of $\mathbb{E}^n$, an f-rectifying curve $\gamma : I \longrightarrow \mathbb{E}^n$ is congruent to a rectifying curve in $\mathbb{E}^n$ and the study coincides with the same incorporated in [8].*

4. Some geometric characterizations of f-rectifying curves in $\mathbb{E}^n$

In this section, we present some geometrical characterizations of unit-speed f-rectifying curves in $\mathbb{E}^n$ in terms of the norm functions, tangential components, normal components, binormal components of their f-position vector field.

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve (parametrized by arc length s) having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$ and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F. If γ is a f-rectifying curve in $\mathbb{E}^n$, then the following statements are true:*

1. *The norm function $\rho = \|\gamma_f\|$ is given by $\rho(s) = \sqrt{F^2(s) + c^2}$, where c is a non-zero constant.*
2. *The tangential component $\langle \gamma_f, T_\gamma \rangle$ of γ_f is given by $\langle \gamma_f(s), T_\gamma(s) \rangle = F(s)$.*
3. *The normal component $\gamma_f^{N_\gamma}$ of γ_f has a constant length and the norm function ρ is non-constant.*
4. *The first binormal component $\langle \gamma_f, B_{\gamma_1} \rangle$ and the second binormal component $\langle \gamma_f, B_{\gamma_2} \rangle$ of γ_f are respectively given by*

$$\langle \gamma_f(s), B_{\gamma_1}(s) \rangle = \frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s), \quad \langle \gamma_f(s), B_{\gamma_2}(s) \rangle = \frac{1}{\kappa_{\gamma_3}(s)} \frac{d}{ds} \left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right)$$

and for each $i \in \{2, 3, \ldots, n-3\}$, the $(i+1)$-th binormal component $\left\langle \gamma_f, B_{\gamma_{i+1}} \right\rangle$ of γ_f is given by

$$\left\langle \gamma_f(s), B_{\gamma_{i+1}}(s) \right\rangle = \frac{1}{\kappa_{\gamma_{i+2}}(s)} \left[\kappa_{\gamma_{i+1}}(s) \left\langle \gamma_f(s), B_{\gamma_{i-1}}(s) \right\rangle + \langle \gamma_f(s), B_{\gamma_i}(s) \rangle \right].$$

Conversely, if $\gamma : I \longrightarrow \mathbb{E}^n$ is a unit-speed curve having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$, and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F such that any one of the statements (1), (2), (3) or (4) is true, then γ is an f-rectifying curve in $\mathbb{E}^n$.

Proof. First, for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ with at least $(n-2)$-times differentiable primitive function F, let $\gamma : I \longrightarrow \mathbb{E}^n$ be an f-rectifying curve in $\mathbb{E}^n$ having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \ldots, \kappa_{\gamma_{n-1}}$. Then for some differentiable functions $\lambda, \mu_1, \mu_2, \ldots, \mu_{n-2} : I \longrightarrow \mathbb{R}$, the f-position vector field γ_f of γ satisfies

$$\gamma_f(s) = \lambda(s)T_\gamma(s) + \sum_{i=1}^{n-2} \mu_i(s)B_{\gamma_i}(s). \tag{4.1}$$

Differentiating (4.1) and then applying the Frenet-Serret formulae (2.1), we obtain

$$\begin{aligned} f(s)T_\gamma(s) \quad = \quad & \lambda'(s)T_\gamma(s) + \left(\lambda(s)\kappa_{\gamma_1}(s) - \mu_1(s)\kappa_{\gamma_2}(s)\right) N_\gamma(s) + \left(\mu_1'(s) - \mu_2(s)\kappa_{\gamma_3}(s)\right) B_{\gamma_1}(s) \\ & + \sum_{i=2}^{n-3} \Big(\mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s) + \mu_i'(s) - \mu_{i+1}(s)\kappa_{\gamma_{i+2}}(s)\Big) B_{\gamma_i}(s) + \Big(\mu_{n-3}(s)\kappa_{\gamma_{n-1}}(s) + \mu_{n-2}'(s)\Big) B_{\gamma_{n-2}}(s) \end{aligned}$$

which gives the following set of relations

$$\begin{cases} \lambda'(s) = f(s), \\ \lambda(s)\kappa_{\gamma_1}(s) - \mu_1(s)\kappa_{\gamma_2}(s) = 0, \\ \mu_1'(s) - \mu_2(s)\kappa_{\gamma_3}(s) = 0, \\ \mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s) + \mu_i'(s) - \mu_{i+1}(s)\kappa_{\gamma_{i+2}}(s) = 0 \quad \text{for } i \in \{2, 3, \ldots, n-3\}, \\ \mu_{n-3}(s)\kappa_{\gamma_{n-1}}(s) + \mu_{n-2}'(s) = 0. \end{cases} \tag{4.2}$$

From the first $n-1$ relations of (4.2), we find

$$\begin{cases} \lambda(s) = F(s), \\ \mu_1(s) = \dfrac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s), \\ \mu_2(s) = \dfrac{1}{\kappa_{\gamma_3}(s)} \dfrac{d}{ds} \left(\dfrac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right), \\ \mu_{i+1}(s) = \dfrac{1}{\kappa_{\gamma_{i+2}}(s)} \left[\mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s) + \mu_i'(s) \right] \quad \text{for } i \in \{2, 3, \ldots, n-3\}. \end{cases} \tag{4.3}$$

On the other hand, from the last $n-2$ relations of (4.2), we get

$$\mu_1(s)\left(\mu_1'(s)-\mu_2(s)\kappa_{\gamma_3}(s)\right)+\sum_{i=2}^{n-3}\mu_i(s)\left(\mu_{i-1}(s)\kappa_{\gamma_{i+1}}(s)+\mu_i'(s)-\mu_{i+1}(s)\kappa_{\gamma_{i+2}}(s)\right)+\mu_{n-2}(s)\left(\mu_{n-2}'(s)+\mu_{n-3}(s)\kappa_{\gamma_{n-1}}(s)\right)=0$$

which reduces to

$$\sum_{i=1}^{n-2}\mu_i(s)\mu_i'(s)=0. \tag{4.4}$$

Integrating (4.4), we obtain

$$\sum_{i=1}^{n-2}\mu_i^2(s)=c^2, \tag{4.5}$$

where c is an arbitrary non-zero constant. Using (4.1), (4.3) and (4.5), the norm function $\rho=\|\gamma_f\|$ is given by

$$\rho^2(s)=\left\|\gamma_f(s)\right\|^2=\left\langle\gamma_f(s),\gamma_f(s)\right\rangle=F^2(s)+\sum_{i=1}^{n-2}\mu_i^2(s)=F^2(s)+c^2.$$

This proves the statement (1). Again, using (4.1) and (4.3), the tangential component $\left\langle\gamma_f,T_\gamma\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),T_\gamma(s)\right\rangle=\lambda(s)=F(s).$$

This proves the statement (2). Now, for each $s\in I$, $\gamma_f(s)$ can be decomposed as

$$\alpha_f(s)=\nu(s)T_\gamma(s)+\alpha_f^{N_\gamma}(s)$$

for some differentiable function $\nu:I\longrightarrow\mathbb{R}$, where $\gamma_f^{N_\gamma}$ denotes the normal component of γ_f. Thus, in view of (4.1), $\gamma_f^{N_\gamma}$ is given by

$$\gamma_f^{N_\gamma}(s)=\sum_{i=1}^{n-2}\mu_i(s)B_{\gamma_i}(s).$$

Therefore, we have

$$\left\|\gamma_f^{N_\gamma}(s)\right\|=\sqrt{\left\langle\gamma_f^{N_\gamma}(s),\gamma_f^{N_\gamma}(s)\right\rangle}=\sqrt{\sum_{i=1}^{n-2}\mu_i^2(s)}. \tag{4.6}$$

Now, by using (4.5) in (4.6), we find $\|\gamma_f^{N_\gamma}(s)\|=c$. This proves the statement (3). Finally, using (4.1) and (4.3), the first binormal component $\left\langle\gamma_f,B_{\gamma_1}\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),B_{\gamma_1}(s)\right\rangle=\mu_1(s)=\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)}F(s),$$

the second binormal component $\left\langle\gamma_f,B_{\gamma_2}\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),B_{\gamma_2}(s)\right\rangle=\mu_2(s)=\frac{1}{\kappa_{\gamma_3}(s)}\frac{d}{ds}\left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)}F(s)\right)$$

and for each $i\in\{2,3,\dots,n-3\}$, the $(i+1)$-th binormal component $\left\langle\gamma_f,B_{\gamma_{i+1}}\right\rangle$ of γ_f is given by

$$\left\langle\gamma_f(s),B_{\gamma_{i+1}}(s)\right\rangle=\mu_{i+1}(s)=\frac{1}{\kappa_{\gamma_{i+2}}(s)}\left[\kappa_{\gamma_{i+1}}(s)\left\langle\gamma_f(s),B_{\gamma_{i-1}}(s)\right\rangle+\left\langle\gamma_f(s),B_{\gamma_i}(s)\right\rangle\right].$$

Thus the statement (4) is proved.

Conversely, let $\gamma:I\longrightarrow\mathbb{E}^n$ be a unit-speed curve having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1},\kappa_{\gamma_2},\dots,\kappa_{\gamma_{n-1}}$, and $f:I\longrightarrow\mathbb{R}$ be a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F such that the statement (1) or the statement (2) is true. Then, in either case, we must have

$$\left\langle\gamma_f(s),T_\gamma(s)\right\rangle=F(s). \tag{4.7}$$

Differentiating (4.7) and then using the Frenet-Serret formulae (2.1), we finally obtain

$$\left\langle\gamma_f(s),N_\gamma(s)\right\rangle=0.$$

This implies that γ_f lies in the rectifying space of γ and hence γ is an f-rectifying curve in $\mathbb{E}^n$.

Next, we assume that the statement (3) is true. Then $\|\gamma_f^{N_\gamma}\|=$ a constant $=c$, say. Again, the normal component $\gamma_f^{N_\gamma}$ is given by

$$\gamma_f(s)=F(s)T_\gamma(s)+\gamma_f^{N_\gamma}(s)$$

and hence we have

$$\langle \gamma_f(s), \gamma_f(s) \rangle = \langle \gamma_f(s), T_\gamma(s) \rangle^2 + c^2. \tag{4.8}$$

Differentiating (4.8) and then applying the Frenet-Serret formulae (2.1), we obtain

$$\langle \gamma_f(s), N_\gamma(s) \rangle = 0.$$

This proves that γ_f lies in the rectifying space of γ and hence γ is an f-rectifying curve in $\mathbb{E}^n$.

Finally, we assume that the statement (4) is true. Then the first binormal component and the second binormal component of γ_f are respectively given by

$$\langle \gamma_f(s), B_{\gamma_1}(s) \rangle = \frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s), \tag{4.9}$$

$$\langle \gamma_f(s), B_{\gamma_2}(s) \rangle = \frac{1}{\kappa_{\gamma_3}(s)} \frac{d}{ds} \left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right). \tag{4.10}$$

Differentiating (4.9) and by using the Frenet-Serret formulae (2.1), we obtain

$$-\kappa_{\gamma_2}(s) \langle \gamma_f(s), N_\gamma(s) \rangle + \kappa_{\gamma_3}(s) \langle \gamma_f(s), B_{\gamma_2}(s) \rangle = \frac{d}{ds} \left(\frac{\kappa_{\gamma_1}(s)}{\kappa_{\gamma_2}(s)} F(s) \right). \tag{4.11}$$

Combining (4.10) and (4.11), we find

$$\langle \gamma_f(s), N_\gamma(s) \rangle = 0.$$

Consequently, γ_f lies in the rectifying space of γ and hence γ is an f-rectifying curve in $\mathbb{E}^n$. □

5. Classification of f-rectifying curves in $\mathbb{E}^n$

In many papers (e.g., [3], [7], [8], [11] etc.), several interesting results were found primarily attempting towards the classification of rectifying curves which are mostly based on their parametrizations. In this section, we attempt for the same in $\mathbb{E}^n$ and this classification is totally based on the parametrizations of their f-position vector field.

Theorem 5.1. *Let $\gamma : I \longrightarrow \mathbb{E}^n$ be a unit-speed curve (parametrized by arc-length s) having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \dots, \kappa_{\gamma_{n-1}}$ and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function with at least $(n-2)$-times differentiable primitive function F. Then γ is an f-rectifying curve in $\mathbb{E}^n$ if and only if, up to a parametrization, its f-position vector field γ_f is given by*

$$\gamma_f(t) = c \sec \left(t + \arctan \left(\frac{F(s_0)}{c} \right) \right) \beta(t),$$

where c is a positive constant, $s_0 \in I$ and $\beta : J \longrightarrow \mathbb{S}^{n-1}(1)$ is a unit-speed curve having $t : I \longrightarrow J$ as arc length function based at s_0.

Proof. First, for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ with at least $(n-2)$-times differentiable primitive function F, let $\gamma : I \longrightarrow \mathbb{E}^n$ be an f-rectifying curve having nowhere vanishing $n-1$ curvatures $\kappa_{\gamma_1}, \kappa_{\gamma_2}, \dots, \kappa_{\gamma_{n-1}}$. Then by Theorem 4.1, the norm function $\rho = \|\gamma_f\|$is given by

$$\rho(s) = \sqrt{F^2(s) + c^2}, \tag{5.1}$$

where we may choose c as a positive constant. Now, we define a curve $\beta : I \longrightarrow \mathbb{E}^n$ by

$$\beta(s) := \frac{1}{\rho(s)} \gamma_f(s). \tag{5.2}$$

Then we find

$$\langle \beta(s), \beta(s) \rangle = 1. \tag{5.3}$$

Therefore, β is a curve in the unit-sphere $\mathbb{S}^{n-1}(1)$. Differentiating (5.3), we get

$$\langle \beta(s), \beta'(s) \rangle = 0. \tag{5.4}$$

Now, from (5.1) and (5.2), we obtain

$$\gamma_f(s) = \beta(s) \sqrt{F^2(s) + c^2}. \tag{5.5}$$

Again, differentiating (5.5), we obtain

$$f(s) T_\gamma(s) = \beta'(s) \sqrt{F^2(s) + c^2} + \frac{\beta(s) f(s) F(s)}{\sqrt{F^2(s) + c^2}}. \tag{5.6}$$

Using (5.3), (5.4) and (5.6), we obtain

$$\left\langle \beta'(s), \beta'(s) \right\rangle = \frac{c^2 f^2(s)}{\left(F^2(s)+c^2\right)^2}. \tag{5.7}$$

Therefore, we get

$$\left\|\beta'(s)\right\| = \sqrt{\langle \beta'(s), \beta'(s)\rangle} = \frac{c\, f(s)}{F^2(s)+c^2}. \tag{5.8}$$

Now, for some $s_0 \in I$, let $t : I \longrightarrow J$ be arc-length parameter of β given by

$$t = \int_{s_0}^{s} \left\|y'(u)\right\| du. \tag{5.9}$$

Then we have

$$\begin{aligned} t &= \int_{s_0}^{s} \frac{c\, f(u)}{F^2(u)+c^2}\, du \\ \Longrightarrow \quad t &= \arctan\left(\frac{F(s)}{c}\right) - \arctan\left(\frac{F(s_0)}{c}\right) \\ \Longrightarrow \quad s &= F^{-1}\left(c \tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right)\right). \end{aligned} \tag{5.10}$$

Substituting (5.10) in (5.5), we obtain the f-position vector field of γ as follows:

$$\gamma_f(t) = c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right) \beta(t).$$

Conversely, let γ be a unit-speed curve in $\mathbb{E}^n$ such that, up to a parametrization, its f-position vector field γ_f is defined by

$$\gamma_f(t) := c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right) \beta(t), \tag{5.11}$$

where c is a positive constant and $\beta : J \longrightarrow \mathbb{S}^{n-1}(1)$ is a unit-speed curve having $t : I \longrightarrow J$ as arc length function based at s_0. Differentiating (5.11), we obtain

$$\gamma_f{}'(t) = c \sec\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right) \left[\tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right) \beta(t) + 1\right] \beta'(t). \tag{5.12}$$

Since β is a unit-speed curve in the unit-sphere $\mathbb{S}^{n-1}(1)$, we have $\langle \beta'(t), \beta'(t)\rangle = 1$, $\langle \beta(t), \beta(t)\rangle = 1$ and consequently $\langle \beta(t), \beta'(t)\rangle = 0$. Therefore, from (5.11) and (5.12), we have

$$\left\langle \gamma_f(t), \gamma_f(t) \right\rangle = c^2 \sec^2\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right), \tag{5.13}$$

$$\left\langle \gamma_f(t), \gamma_f{}'(t) \right\rangle = c^2 \sec^2\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right) \tan\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right), \tag{5.14}$$

$$\left\langle \gamma_f{}'(t), \gamma_f{}'(t) \right\rangle = c^2 \sec^4\left(t + \arctan\left(\frac{F(s_0)}{c}\right)\right). \tag{5.15}$$

Now, if we put

$$t = \arctan\left(\frac{F(s)}{c}\right) - \arctan\left(\frac{F(s_0)}{c}\right),$$

then s becomes arc length parameter of γ and equations (5.13), (5.14), (5.15) reduce to

$$\left\langle \gamma_f(s), \gamma_f(s) \right\rangle = c^2 \sec^2\left(\frac{F(s)}{c}\right), \tag{5.16}$$

$$\left\langle \gamma_f(s), \gamma_f{}'(s) \right\rangle = c^2 \sec^2\left(\frac{F(s)}{c}\right) \tan\left(\frac{F(s)}{c}\right), \tag{5.17}$$

$$\left\langle \gamma_f{}'(s), \gamma_f{}'(s) \right\rangle = c^2 \sec^4\left(\frac{F(s)}{c}\right). \tag{5.18}$$

Again, the normal component $\gamma_f^{N_\gamma}$ of γ_f is given by

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle = \left\langle \gamma_f(s), \gamma_f(s) \right\rangle - \frac{\left\langle \gamma_f(s), \gamma_f{}'(s) \right\rangle^2}{\left\langle \gamma_f{}'(s), \gamma_f{}'(s) \right\rangle}.$$

Then substituting (5.16), (5.17)) and (5.18) in the previous equation, we obtain

$$\left\langle \gamma_f^{N_\gamma}(s), \gamma_f^{N_\gamma}(s) \right\rangle = \left\| \gamma_f^{N_\gamma}(s) \right\|^2 = c^2.$$

This implies that the normal component $\gamma_f^{N_\gamma}$ of γ_f has a constant length. Also, the norm function $\rho = \|\gamma_f\|$ is given by

$$\rho(s) = \sqrt{\langle \gamma_f(s), \gamma_f(s) \rangle} = c \sec\left(\frac{F(s)}{c}\right)$$

and it is non-constant. Therefore, by applying the Theorem 4.1, we conclude that γ is an f-rectifying curve in $\mathbb{E}^n$. □

6. Conclusion

It goes without saying that f-rectifying curves in Euclidean spaces are a sort of generalizations of rectifying curves therein. In this paper, we presented a study on f-rectifying curves in Euclidean n-space $\mathbb{E}^n$. Predominantly, we explored two main theorems demonstrating some necessary and sufficient conditions for a regular curve to be an f-rectifying curve in $\mathbb{E}^n$. The first theorem portrays some geometric characterizations of f-rectifying curves in $\mathbb{E}^n$ in connection with norm functions, tangential, normal and $n-2$ binormal components of their f-position vector field. Whereas the second theorem classifies such curves based on parametrization of their f-position vector field. Moreover, it yields an important characterization: namely, the f-position vector field of an f-rectifying curve in $\mathbb{E}^n$ is a dilation of a unit-speed curve in the unit $(n-1)$-sphere $\mathbb{S}^{n-1}(1)$ with dilation factor $c \sec\left(t+\arctan\left(\frac{F(s_0)}{c}\right)\right)$ for some constants $c>0$ and s_0. Extensions of such study to other ambient spaces may be considered as problems of interest.

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Refinements of Hermite-Hadamard Type Inequalities for s-Convex Functions with Applications to Special Means

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Abstract

In this paper, we establish some Hermite-Hadamard type inequalities for s-convex functions in the first and second sense. Some applications to special means for real numbers are also given.

1. Introduction

Let $f: I \subset R \to R$be a convex function on the interval I of real numbers and $a,b \in I$ with $a<b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

is known as Hermite-Hadamard's inequality for convex functions [4].
In [13] and [4, pp.278]), the following concept was introduced by Orlicz.
A function $f: R^+ \to R$, where $R^+ = [0,\infty)$, is said to be s-convex in the first sense if:

$$f(\alpha_1 u + \beta_1 v) \leq {\alpha_1}^s f(u) + {\beta_1}^s f(v),$$

for all $u,v \in R^+$, $\alpha_1, \beta_1 \geq 0$and $s \in (0,1]$ with ${\alpha_1}^s + {\beta_1}^s = 1$. The class of s-convex functions in the first sense is usually denoted with K_s^1.
In [9] and [4, pp.288]), Hudzik and Maligranda considered, among others, the class of functions which is s-convex in the second sense. This class is defined in the following way:
$f: [0,\infty) \to R$ is called s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x,y \in [0,\infty), \lambda \in [0,1]$ and for some fixed $s \in (0,1]$. The class of s-convex functions in the second sense is usually denoted with K_s^2.
In [5], S.S. Dragomir and S. Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense:

Theorem 1.1: Suppose that $\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}(M^{1/q}+N^{1/q})(|f'(a)|+|f'(b)|)$ is an s-convex function in the second sense, where $s \in (0,1)$and let $a_1 = |f'(a)|^p, b_1 = 2|f'(b)|^p, a_2 = 2|f'(a)|^p, b_2 = |f'(b)|^p$. If $f \in L_1([a,b])$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

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The constant $k = 1/(s+1)$ is the best possible in the second inequalitiy in (1.1).
In [6], S.S. Dragomir presented the following result:
Theorem 1.2: Let $f : [a,b] \to R$ be a L-Lipschitzian mapping on $[a,b]$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{2}{3} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{6} \right| \leq \frac{5}{36} L (b-a)^2. \tag{1.2}$$

In [7], S.S. Dragomir et al. gave the following result:
Theorem 1.3: Suppose $f : [a,b] \to R$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L_1([a,b])$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{2}{3} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{6} \right| \leq \frac{(b-a)}{3} \|f'\|_1, \tag{1.3}$$

where $\|f'\|_1 = \int_a^b |f'(x)|\,dx$.
Note that the bound of (1.3) for L-Lipschitzian is $\frac{5}{36} L(b-a)$ [7].
In [16], Y. Shuang and F. Qi established the following results:
Theorem 1.4 ([16,Theorem 3.5]): Let $f : R_o = (0,\infty] \to R$ be a differentiable function on R_o, a,$b \in R_o$ with $a < b$ and $f' \in L_1([a,b])$. If $|f'|^q$is (α,m)-convex on $[0, \frac{b}{m}]$ for $(\alpha,m) \in (0,1]^2$ and $q > 1$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{3}{4} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{8} \right|$$

$$\leq \frac{b-a}{4} \left[\frac{(q-1)(3^{\frac{2q-1}{q-1}}+1)}{2^{\frac{2(2q-1)}{q-1}}(2q-1)} \right]^{1-1/q} \left[\frac{1}{\alpha+1} |f'(a)|^q + \frac{m\alpha}{\alpha+1} \left| f'\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} + \left[\frac{1}{\alpha+1} \left| f'\left(\frac{a+b}{2}\right) \right|^q + \frac{m\alpha}{\alpha+1} \left| f'\left(\frac{b}{m}\right) \right|^q \right]^{1/q}$$

Theorem 1.5 ([16,Corollary 3.6]): Under the assumptions of Theorem 1.4, if $\alpha = m = 1$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{3}{4} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{8} \right|$$

$$\leq \frac{(b-a)}{4} \left[\frac{(q-1)(3^{\frac{2q-1}{q-1}}+1)}{2^{\frac{2(2q-1)}{q-1}}(2q-1)} \right]^{1-\frac{1}{q}} \left\{ \left[\frac{|f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q}{2} \right]^{1/q} + \left[\frac{\left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q}{2} \right]^{1/q} \right\}. \tag{1.4}$$

In [17], Y. Shuang et al. gave the following results:
Theorem 1.6 ([17,Theorem 3.2]): Let $f : I \subset R_o \to R$ be a differentiable function on I^o, $a,b \in I$ with $a < b$ and $f' \in L_1([a,b])$. If $|f'|^q$is s-convex function on $[a,b]$ for some fixed $s \in (0,1]$ and $q > 1$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{4}{5} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{10} \right|$$

$$\leq \frac{(b-a)}{4} \left[\frac{(q-1)(4^{\frac{2q-1}{q-1}}+1)}{5^{\frac{(2q-1)}{q-1}}(2q-1)} \right]^{1-\frac{1}{q}} \left\{ \left[\frac{|f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q}{s+1} \right]^{1/q} + \left[\frac{\left| f'\left(\frac{a+b}{2}\right) \right|^q + \left| f'(b) \right|^q}{s+1} \right]^{1/q} \right\}.$$

Theorem 1.7 ([17,Corollary 3.3]): Under the assumptions of Theorem 1.6, for s=1, then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{4}{5} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{10} \right|$$

$$\leq \frac{(b-a)}{4} \left[\frac{(q-1)(4^{\frac{2q-1}{q-1}}+1)}{5^{\frac{(2q-1)}{q-1}}(2q-1)} \right]^{1-\frac{1}{q}} \left\{ \left[\frac{\left| f'(a) \right|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q}{2} \right]^{1/q} + \left[\frac{\left| f'\left(\frac{a+b}{2}\right) \right|^q + \left| f'(b) \right|^q}{2} \right]^{1/q} \right\}. \tag{1.5}$$

In [8], T. Du et al. gave the following results:
Theorem 1.8 [8,Corollary 2.8]): Let $f : I \subset R_o \to R$ be a differentiable function on I^o, where $a,b \in I^o$ such that $0 < a < b$. If $t = k = \frac{1}{2}$, $-1 < s \leq 1$ *and* $m = 1$, the inequality holds for (s,m)-convex functions:

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{b-a}{8^{1-\frac{1}{q}}} \left(\frac{1}{2^{s+2}(s+1)(s+2)} \right)^{\frac{1}{q}}$$

$$\times \left\{ \left[\left| f'(b) \right|^q + \left(s2^{s+1}+1 \right) \left| f'(a) \right|^q \right]^{\frac{1}{q}} + \left[\left| f'(a) \right|^q + \left(s2^{s+1}+1 \right) \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right\}. \tag{1.6}$$

Theorem 1.9 ([8,Corollary 2.5]): Let $f : I \subset R_o \to R$ be a differentiable function on I^o, where $a,b \in I^o$ such that $0 < a < b$. If the mapping $|f'|^{p/(p-1)}$ is (s,m)-convex on $[a,b]$, then we get, for $t = k = \frac{1}{2}$ *and* $m = 1$,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{2(s+1)}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{1+\frac{1}{p}}$$

$$\times\left\{\left[\left|f'(a)\right|^q + \left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[\left|f'\left(\tfrac{a+b}{2}\right)\right|^q + \left|f'(b)\right|^q\right]^{1/q}\right\}. \tag{1.7}$$

In [12], U.S. Kırmacı et al. gave the following result:
Theorem 1.10 ([12,Theorem 3]): Let $f : I \to R,\ I \subset [0,\infty)$ be a differentiable function on I^o such that $f' \in L_1([a,b])$, where $a,b \in I,\ a < b$. If $|f'|^q$is s-convex function on $[a,b]$ for some fixed $s \in (0,1)$ and $q > 1$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{2}$$

$$\times\left\{\left[\left|f'(a)\right|^q + \left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[\left|f'\left(\tfrac{a+b}{2}\right)\right|^q + \left|f'(b)\right|^q\right]^{1/q}\right\}. \tag{1.8}$$

In [10], author gave some inequalities for differentiable convex and concave mappings with applications to special means of real numbers.
The aim of this paper is to establish refinements inequalities of Hermite-Hadamard type for s-convex functions in the second sense.
In the development of pure and applied mathematics, convexity has played a key role. In linear programing, combinatory, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamics and in the theory of relativity, integral inequalities have various applications.
For several recent results concerning integral inequalities for convex, quasi-convex, s-convex and $(\alpha,m)-$convex functions, we refer the reader to [1-18].
Throughout we suppose I is an interval on R and $a,b,c,A,B \in I^0$ with $a \le A \le c \le B \le b$. $(c \ne a,b), p,q \in R$ and $f : I^0 \to R$ is differentiable. (I^0denotes the interior of I.)

2. Main Results

First, we give the following Lemma.
Lemma 2.1 [10]: Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0, $a,b \in I^0$ with $a < b$. If $f' \in L_1([a,b])$, then we have

$$f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= (a-b)\left[\int_0^c (t-A)f'(ta+(1-t)b)dt + \int_c^1 (t-B)f'(ta+(1-t)b)dt\right],$$

where $a,b,c,A,B \in I^0$ *with* $a \le A \le c \le B \le b$.

Proof: Let S:[a,b]$\to R$ be defined by

$$S(t) = \begin{cases} t-A, & t \in [0,c], \\ t-B, & t \in (c,1]. \end{cases}$$

Integrating by parts and using the change of the variable $x = ta+(1-t)b$, we have

$$\int_0^1 S(t)f'(ta+(1-t)b)dt = \int_0^c (t-A)f'(ta+(1-t)b)dt + \int_c^1 (t-B)f'(ta+(1-t)b)dt$$
$$= \frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A - \frac{1}{b-a}\int_a^b f(x)dx\right].$$

Hence we have the conclusion.

Remark 2.2: i). Applying Lemma 2.1 for $c = 1/2$, then we obtain the Lemma 2.1 given by T. Du et al. in [8, (for m=1)].
ii). Applying Lemma 2.1 for $A = \frac{1}{6},\ B = \frac{5}{6}$ *and* $c = 1/2$, then we obtain the Lemma 2.1 given by Qaisar and He in [15,(for m=1)].
iii). Applying Lemma 2.1 for $A = B = c = \frac{1}{2}$, then we get the Lemma 2.1 given by S.S.Dragomir and R.P. Agarwal in [3].
iv). Applying Lemma 2.1 for $A = 0,\ B = 1$ *and* $c = \frac{1}{2}$, then we get the Lemma 2.1 given by author in [11].

In the following theorems, we present generalized integral inequalities via s-convex mappings in the first and second sense.
Theorem 2.3: Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0 and let $p > 1$. If $|f'|^{p/(p-1)}$ is s-convex mapping in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right|$$

$$\leq\left[\frac{A^{p+1}+(c-A)^{p+1}}{p+1}\right]^{1/p}\left(\frac{c^{s+1}|f'(a)|^q+(1-(1-c)^{s+1})|f'(b)|^q}{s+1}\right)^{1/q}+\left[\frac{(B-c)^{p+1}+(1-B)^{p+1}}{p+1}\right]^{1/p}\left(\frac{(1-c^{s+1})|f'(a)|^q+(1-c)^{s+1}|f'(b)|^q}{s+1}\right)^{1/q}. \tag{2.1}$$

Proof: From Lemma 2.1, we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \leq \int_0^c |t-A|\,|f'(ta+(1-t)b)|\,dt+\int_c^1 |t-B|\,|f'(ta+(1-t)b|\,dt. \tag{2.2}$$

Using the Hölder's inequality for $p>1$, we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \tag{2.3}$$
$$\leq\left(\int_0^c |t-A|^p\,dt\right)^{1/p}\left(\int_0^c \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}+\left(\int_c^1 |t-B|^p\,dt\right)^{1/p}\left(\int_c^1 \left|f'(ta+(1-t)b\right|^q dt\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.Since $|f'|^q$ is s- convex mapping in the second sense on $[a,b]$, we obtain

$$\int_0^c \left|f'(ta+(1-t)b)\right|^q dt\leq\int_0^c \left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt=\frac{c^{s+1}|f'(a)|^q+(1-(1-c)^{s+1})|f'(b)|^q}{s+1} \tag{2.4}$$

and

$$\int_c^1 \left|f'(ta+(1-t)b)\right|^q dt\leq\int_c^1 \left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt=\frac{(1-c^{s+1})|f'(a)|^q+(1-c)^{s+1}|f'(b)|^q}{s+1}. \tag{2.5}$$

Where,

$$\int_c^1 t^s dt=\frac{1-c^{s+1}}{s+1},\int_c^1 (1-t)^s dt=\frac{(1-c)^{s+1}}{s+1},\quad \int_0^c (1-t)^s dt=\frac{1-(1-c)^{s+1}}{s+1},\quad \int_0^c t^s dt=\frac{c^{s+1}}{s+1}.$$

Also, we have

$$P_p=\int_0^c |t-A|^p\,dt=\int_0^A (A-t)^p dt+\int_A^c (t-A)^p dt=\frac{A^{p+1}+(c-A)^{p+1}}{p+1}, \tag{2.6}$$

$$M_p=\int_c^1 |t-B|^p\,dt=\int_c^B (B-t)^p dt+\int_B^1 (t-B)^p dt=\frac{(B-c)^{p+1}+(1-B)^{p+1}}{p+1}. \tag{2.7}$$

A combination of (2.3)-(2.7) gives the required inequality (2.1).

Corollary 2.4: Under the assumptions of Theorem 2.3,
i). When $A=0,\ B=1,\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(\frac{a+b}{2})\right|$$
$$\leq\frac{b-a}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}(s+1)^{\frac{1}{q}}}\left[\left[\left(\frac{1}{2^{s+1}}\left|f'(a)\right|^q+\left(1-\frac{1}{2^{s+1}}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]+\left[\left(1-\frac{1}{2^{s+1}}\right)\left|f'(a)\right|^q+\left(\frac{1}{2^{s+1}}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right].$$

Using the fact that

$$\sum_{k=1}^n (a_k+b_k)^s\leq\sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s, \tag{2.8}$$

for $0\leq s<1$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}(s+1)^{\frac{1}{q}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

For $p>1$, then $p+1>2$ and so $\frac{1}{(p+1)^{\frac{1}{p}}}<\frac{1}{2^{\frac{1}{p}}}$ *and also* $\frac{1}{(s+1)^{\frac{1}{q}}}\leq 1,$

for $s\in(0,1)$, $q\in(1,\infty)$. Hence, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{2.4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})(\left|f^{'(a)}\right|+\left|f'(b)\right|).$$

ii). When $A=B=\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{2.4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})(\left|f^{'(a)}\right|+\left|f'(b)\right|).$$

iii). When $A=\frac{1}{4},B=\frac{3}{4},c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{2}(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2})\right|$$
$$\leq\frac{b-a}{4.4^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|\right).$$

iv). When $A=\frac{1}{6},B=\frac{5}{6},c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\leq\frac{(b-a)\left(2^{p+1}+1\right)^{\frac{1}{p}}}{6.(12)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|\right).$$

v). When $A=\frac{1}{8},\ B=\frac{7}{8},c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\right|\leq\frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8.(16)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|\right).$$

vi). When $A=\frac{1}{10},B=\frac{9}{10},c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right|\leq\frac{(b-a)\left(4^{p+1}+1\right)^{\frac{1}{p}}}{10.(20)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|\right).$$

vii). When $A=\frac{1}{12},B=\frac{11}{12},c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{5}{6}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{12}\right|\leq\frac{(b-a)\left(5^{p+1}+1\right)^{\frac{1}{p}}}{12.(24)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|\right).$$

Theorem 2.5: Let $f:I^0\subset R\to R$ be a differentiable mapping on I^0 and let $p>1$. If $|f'|^{p/(p-1)}$ is s-convex mapping in the first sense on$[a,b]$ for some fixed $s\in(0,1]$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \tag{2.9}$$
$$\leq\ {P_p}^{1/p}\left(\frac{c^{s+1}\,|f'(a)|^q+(c(s+1)-c^{s+1})\,|f'(b)|^q}{s+1}\right)^{1/q} \tag{2.10}$$
$$+\ {M_p}^{1/p}\left(\frac{(1-c^{s+1})\,|f'(a)|^q+((1-c)(s+1)-(1-c^{s+1}))\,|f'(b)|^q}{s+1}\right)^{1/q} \tag{2.11}$$

Where P_p *and* M_p are as in (2.6) and (2.7) respectively.

Proof: From Lemma 2.1 and using the Hölder's inequality for $p>1$, we get inequality (2.3). Since $|f'|^q$is s- convex mapping in the first sense on $[a,b]$, we obtain

$$\int_0^c\left|f'(ta+(1-t)b)\right|^q dt\leq\int_0^c\left[t^s\left|f'(a)\right|^q+(1-t^s)\left|f'(b)\right|^q\right]dt=\frac{c^{s+1}\,|f'(a)|^q+(c(s+1)-c^{s+1})\,|f'(b)|^q}{s+1} \tag{2.12}$$

and

$$\int_c^1\left|f'(ta+(1-t)b)\right|^q dt\leq\int_c^1\left[t^s\left|f'(a)\right|^q+(1-t^s)\left|f'(b)\right|^q\right]dt$$

$$=\frac{(1-c^{s+1})\,|f'(a)|^q+((1-c)(s+1)-(1-c^{s+1}))\,|f'(b)|^q}{s+1}, \tag{2.13}$$

where,
$\int_c^1 t^s dt = \frac{1-c^{s+1}}{s+1}$, $\int_c^1 (1-t^s)dt = 1-c-\frac{1-c^{s+1}}{s+1}$, $\int_0^c (1-t^s)dt = c-\frac{c^{s+1}}{s+1}$, $\int_0^c t^s dt = \frac{c^{s+1}}{s+1}$. From (2.3),(2.6),(2.7),(2.12) and (2.13), we deduce required inequality (2.9).

Theorem 2.6: Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0 and let $p > 1$. If $|f'|^{p/(p-1)}$ is s-convex mapping in the second sense on $[a,b]$ for some fixed $s \in (0,1)$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \le P_p{}^{1/p}\left(c\frac{|f'(ca+(1-c)b)|^q+|f'(b)|^q}{s+1}\right)^{1/q} + M_p{}^{1/p}\left((1-c)\frac{|f'(a)|^q+|f'(ca+(1-c)b)|^q}{s+1}\right)^{1/q}. \tag{2.14}$$

Where P_p *and* M_p are as in (2.6) and (2.7) respectively.

Proof: : From Lemma 2.1 and using the Hölder's inequality for $p > 1$, we get inequality (2.3). Let us substitute $x = ta+(1-t)b$ *and* $dx = (a-b)dt$, we get

$$\int_0^c |f'(ta+(1-t)b)|^q dt \le \frac{1}{a-b}\int_b^{ac+(1-c)b} |f'(x)|^q dx = \frac{c}{(a-b)c}\int_b^{ac+(1-c)b} |f'(x)|^q dx$$

and

$$\int_c^1 |f'(ta+(1-t)b)|^q dt \le \frac{1}{a-b}\int_{ac+(1-c)b}^{a} |f'(x)|^q dx = \frac{1-c}{(a-b)(1-c)}\int_{ac+(1-c)b}^{a} |f'(x)|^q dx.$$

Since $|f'|^q$is s-convex mapping in the second sense on $[a,b]$, using the above inequalities and by inequality (1.1), we have

$$\int_0^c |f'(ta+(1-t)b)|^q dt \le c\frac{|f'(ca+(1-c)b)|^q+|f'(b)|^q}{s+1} \tag{2.15}$$

and

$$\int_c^1 |f'(ta+(1-t)b)|^q dt \le (1-c)\frac{|f'(a)|^q+|f'(ca+(1-c)b)|^q}{s+1}. \tag{2.16}$$

From (2.3),(2.6),(2.7),(2.15) and (2.16), we obtain required inequality (2.14).

Corollary 2.7: Under the assumptions of Theorem 2.6, using the inequality (2.8) and since $\frac{1}{(p+1)^{\frac{1}{p}}} < \frac{1}{2^{\frac{1}{p}}}$ *and* $\frac{1}{(s+1)^{\frac{1}{q}}} \le 1$,

i). When $A = 0$, $B = 1$, $c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{b-a}{4.2^{\frac{1}{p}}}\left(2\left|f'^{(\frac{a+b}{2})}\right|+\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

ii). When $A = B = c = 1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{4.2^{\frac{1}{p}}}\left(2\left|f'^{(\frac{a+b}{2})}\right|+\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right). \tag{2.17}$$

iii). When $A = \frac{1}{4}, B = \frac{3}{4}, c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2})\right| \le \frac{b-a}{8.4^{\frac{1}{p}}}\left(2\left|f'^{(\frac{a+b}{2})}\right|+\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

iv). When $A = \frac{1}{6}, B = \frac{5}{6}, c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{2}{3}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{6}\right| \le \frac{(b-a)\left(2^{p+1}+1\right)^{\frac{1}{p}}}{12.6^{\frac{1}{p}}}(2\left|f'^{(\frac{a+b}{2})}\right|+\left|f'^{(a)}\right|++|f'(b)|).$$

v). When $A = \frac{1}{8}, B = \frac{7}{8}, c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{3}{4}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{8}\right| \le \frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{16.8^{\frac{1}{p}}}(2\left|f'^{(\frac{a+b}{2})}\right|+\left|f'^{(a)}\right|+ \quad +|f'(b)|). \tag{2.18}$$

vi). When $A = \frac{1}{10}, B = \frac{9}{10}, c = 1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{4}{5}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{10}\right| \le \frac{(b-a)\left(4^{p+1}+1\right)^{\frac{1}{p}}}{20.10^{\frac{1}{p}}}(2\left|f'^{(\frac{a+b}{2})}\right|+\left|f'^{(a)}\right|+ +|f'(b)|). \tag{2.19}$$

vii). When $A=\frac{1}{12}, B=\frac{11}{12}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{5}{6}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{12}\right|\leq\frac{(b-a)\left(5^{p+1}+1\right)^{\frac{1}{p}}}{24.12^{\frac{1}{p}}}\left(2\left|f'^{\left(\frac{a+b}{2}\right)}\right|+\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

Remark 2.8: The followings are observed that:
i) The inequality (2.18) is a refinement of inequality (1.4) presented by Y. Shuang and F. Qi in [16]
ii) The inequality (2.19)) is a refinement of inequality (1.5) presented by Y. Shuang et al. in [17]
iii) The inequality (2.17) is both a refinement of inequality (1.7) given by T. Du et al. in [8] and the inequality (1.8) given by Kirmaci et al. in [12].
Theorem 2.9: Let $f: I^0\subset R\to R$ be a differentiable mapping on I^0 and let $p>1$. If $|f'|^{p/(p-1)}$ is s-concave mapping on$[a,b]$ for some fixed $s\in(0,1)$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \tag{2.20}$$

$$\leq {P_p}^{1/p}\left(2^{s-1}\left|f'\left(\tfrac{c}{2}a+\tfrac{2-c}{2}b\right)\right|^q\right)^{1/q}+{M_p}^{1/p}\left(2^{s-1}\left|f'\left(\tfrac{1+c}{2}a+\tfrac{1-c}{2}b\right)\right|^q\right)^{1/q}.$$

Where P_p *and* M_p are as in (2.6) and (2.7) respectively.

Proof: : From Lemma 2.1 and using the Hölder's inequality for $p>1$, we get inequality (2.3). Since $|f'|^q$is s-concave mapping on $[a,b]$ and using inequality (1.1), we have

$$\int_0^c\left|f'(ta+(1-t)b)\right|^q dt\leq 2^{s-1}\left|f'\left(\frac{c}{2}a+\frac{2-c}{2}b\right)\right|^q \tag{2.21}$$

and

$$\int_c^1\left|f'(ta+(1-t)b)\right|^q dt\leq 2^{s-1}\left|f'\left(\frac{1+c}{2}a+\frac{1-c}{2}b\right)\right|^q. \tag{2.22}$$

From (2.3),(2.6),(2.7),(2.21) and (2.22), we obtain required inequality (2.20).

Corollary 2.10: Under the assumptions of Theorem 2.9, using the inequality (2.8) and since $2^{(s-1)/q}<1$ *and* $\frac{1}{(p+1)^{\frac{1}{p}}}<\frac{1}{2^{\frac{1}{p}}}$ *for* $s\in(0,1)$ *and* $q\in(1,\infty)$,

i). When $A=0,\ B=1,\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}2^{\frac{s-1}{q}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right]$$

$$\leq\frac{1}{2.4^{\frac{1}{p}}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

ii). When $A=\ B=\ c=\frac{1}{2}$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2.4^{\frac{1}{p}}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

iii). When $A=\frac{1}{4}, B=\frac{3}{4}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2})\right|$$

$$\leq\frac{b-a}{4.4^{\frac{1}{p}}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

iv) When $A=\frac{1}{6}, B=\frac{5}{6}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\leq\frac{(b-a)\left(2^{p+1}+1\right)^{\frac{1}{p}}}{6^{\frac{1}{p}+1}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

v). When $A=\frac{1}{8}, B=\frac{7}{8}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\right|\leq\frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8^{\frac{1}{p}+1}}\left[\left|f'\left(\tfrac{a+3b}{4}\right)\right|+\left|f'\left(\tfrac{3a+b}{4}\right)\right|\right].$$

vi). When $A=\frac{1}{10}, B=\frac{9}{10}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right|\leq\frac{(b-a)\left(4^{p+1}+1\right)^{\frac{1}{p}}}{10^{\frac{1}{p}+1}}\left[\left|f'\left(\tfrac{a+3b}{4}\right)\right|+\left|f'\left(\tfrac{3a+b}{4}\right)\right|\right].$$

vii). When $A=\frac{1}{12}, B=\frac{11}{12}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{5}{6}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{12}\right|\leq\frac{(b-a)\left(5^{p+1}+1\right)^{\frac{1}{p}}}{12^{\frac{1}{p}+1}}\left[\left|f'\left(\tfrac{a+3b}{4}\right)\right|+\left|f'\left(\tfrac{3a+b}{4}\right)\right|\right].$$

Theorem 2.11: Let $f:I^0\subset R\to R$ be a differentiable mapping on I^0, $a,b\in I^0$ with $a<b$ and let $p\geq 1$. If the mapping $|f'|^p$ is s-convex in the first sense on $[a,b]$ for some fixed $s\in(0,1]$, then we have

$$\begin{aligned}&\left|f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\tfrac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq(a-b)\left\{P_2^{1-\frac{1}{p}}\left[T_1\left|f'(a)\right|^p+(P_2-T_1)\left|f'(b)\right|^p\right]^{1/p}+M_2^{1-\frac{1}{p}}\left[N_1\left|f'(a)\right|^p+(M_2-N_1)\left|f'(b)\right|^p\right]^{1/p}\right\},\end{aligned}\tag{2.23}$$

where,

$$\begin{aligned}P_2&=\frac{A^2+(c-A)^2}{2}, T_1=\frac{2}{(s+1)(s+2)}A^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{A}{s+1}\right],\\M_2&=\frac{(B-c)^2+(1-B)^2}{2}, N_1=\frac{2}{(s+1)(s+2)}B^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{B}{s+1}\right]+\frac{1}{s+2}-\frac{B}{s+1}.\end{aligned}$$

Proof: From Lemma 2.1, we get the inequality (2.2). By the power-mean inequality, we obtain

$$\int_0^c|t-A|\left|f'(ta+(1-t)b\right|dt\leq\left(\int_0^c|t-A|\,dt\right)^{1-\frac{1}{p}}\left(\int_0^c|t-A|\left|f'(ta+(1-t)b\right|^p dt\right)^{1/p}\tag{2.24}$$

and

$$\int_c^1|t-B|\left|f'(ta+(1-t)b\right|dt\leq\left(\int_c^1|t-B|\,dt\right)^{1-\frac{1}{p}}\left(\int_c^1|t-B|\left|f'(ta+(1-t)b)\right|^p dt\right)^{1/p}.\tag{2.25}$$

Since $|f'|^p$is s-convex in the first sense, we have

$$\int_0^c|t-A|\left|f'(ta+(1-t)b\right|^p dt\leq\int_0^c|t-A|\left(t^s\left|f'(a)\right|^p+(1-t^s)\left|f'(b)\right|^p\right)dt\tag{2.26}$$

$$\leq T_1\left|f^{'(a)}\right|^p+(P_2-T_1)\left|f'(b)\right|^p$$

and

$$\int_c^1|t-B|\left|f'(ta+(1-t)b\right|^p dt\leq\int_c^1|t-B|\left(t^s\left|f'(a)\right|^p+(1-t^s)\left|f'(b)\right|^p\right)dt\tag{2.27}$$

$$\leq N_1\left|f'(a)\right|^p+(M_2-N_1)\left|f'(b)\right|^p.$$

where,

$$P_2=\int_0^c|t-A|\,dt=\int_0^A(A-t)dt+\int_A^c(t-A)dt=\frac{A^2+(c-A)^2}{2},$$

$$T_1=\int_0^c|t-A|t^s dt=\int_0^A(A-t)t^s dt+\int_A^c(t-A)t^s dt=\frac{2}{(s+1)(s+2)}A^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{A}{s+1}\right],$$

$$M_2=\int_c^1|t-B|\,dt=\int_c^B(B-t)dt+\int_B^1(t-B)dt=\frac{(B-c)^2+(1-B)^2}{2},$$

$$N_1=\int_c^1|t-B|t^s dt=\int_c^B(B-t)t^s dt+\int_B^1(t-B)t^s dt=\frac{2}{(s+1)(s+2)}B^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{B}{s+1}\right]++\frac{1}{s+2}-\frac{B}{s+1}\tag{2.28}$$

and

$$P_2-T_1=\int_0^c |t-A|(1-t)dt=\int_0^A (A-t)(1-t)dt+\int_A^c (t-A)(1-t)dt,$$
$$M_2-N_1=\int_c^1 |t-B|(1-t)dt=\int_c^B (B-t)(1-t)dt+\int_B^1 (t-B)(1-t)dt.$$

A combination of (2.2) and (2.24)-(2.28) gives the required inequality (2.23).

Theorem 2.12: Let $f:I^0\subset R\to R$ be a differentiable mapping on I^0, $a,b\in I^0$ with $a<b$ and let $p\geq 1$. If the mapping $|f'|^p$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then we have

$$\begin{aligned}&\left|f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\tfrac{1}{b-a}\int_a^b f(x)dx\right|\\ &\leq (a-b)\left\{P_2^{1-\frac{1}{p}}\left[T_1|f'(a)|^p+T_2|f'(b)|^p\right]^{1/p}+M_2^{1-\frac{1}{p}}\left[N_1|f'(a)|^p+N_2|f'(b)|^p\right]^{1/p}\right\},\end{aligned} \tag{2.29}$$

where,

$$T_2=\frac{A}{s+1}+\frac{2(1-A)^{s+2}}{(s+1)(s+2)}-(1-c)^{s+1}\left[\frac{c-A}{s+1}+\frac{1-c}{(s+1)(s+2)}\right]-\frac{1}{(s+1)(s+2)},$$

$N_2=\frac{2(1-B)^{s+2}}{(s+1)(s+2)}+(1-c)^{s+1}\left[\frac{B-c}{s+1}-\frac{1-c}{(s+1)(s+2)}\right]$ andP_2,M_2,T_1,N_1 are as in (2.28).

Proof: From Lemma 2.1, we have the inequality (2.2). By the power-mean inequality, we get inequalities (2.24) and (2.25). Since $|f'|^p$is s-convex mapping in the second sense on $[a,b]$, we have

$$\int_0^c |t-A|\left|f'(ta+(1-t)b\right|^p dt\leq\int_0^c |t-A|\left(t^s|f'(a)|^p+(1-t)^s|f'(b)|^p\right)dt\leq T_1|f'(a)|^p+T_2|f'(b)|^p \tag{2.30}$$

and

$$\int_c^1 |t-B|\left|f'(ta+(1-t)b\right|^p dt\leq\int_c^1 |t-B|\left(t^s|f'(a)|^p+(1-t)^s|f'(b)|^p\right)dt\leq N_1|f'(a)|^p+N_2|f'(b)|^p. \tag{2.31}$$

Where,

$$T_2=\int_0^c |t-A|(1-t)^s dt=\int_0^A (A-t)(1-t)^s dt+\int_A^c (t-A)(1-t)^s dt$$
$$=\frac{A}{s+1}+\frac{2(1-A)^{s+2}}{(s+1)(s+2)}-(1-c)^{s+1}\left[\frac{c-A}{s+1}+\frac{1-c}{(s+1)(s+2)}\right]-\frac{1}{(s+1)(s+2)},$$

$$N_2=\int_c^1 |t-B|(1-t)^s dt=\int_c^B (B-t)(1-t)^s dt+\int_B^1 (t-B)(1-t)^s dt$$

$$=\frac{2(1-B)^{s+2}}{(s+1)(s+2)}+(1-c)^{s+1}\left[\frac{B-c}{s+1}-\frac{1-c}{(s+1)(s+2)}\right] \tag{2.32}$$

and T_1,N_1 are as in (2.28). A combination of (2.2),(2.24),(2.25), (2.30),(2.31) and (2.32) gives the required inequality (2.29).

Corollary 2.13: Under the assumptions of Theorem 2.12 and using the inequality (2.8),

i) When $A=0,\ B=1,\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(T_{1'}^{\frac{1}{p}}+N_{1'}^{\frac{1}{p}})\left|f^{'(a)}\right|+(T_{2'}^{\frac{1}{p}}+N_{2'}^{\frac{1}{p}})\left|f^{'(b)}\right|], \tag{2.33}$$

where,

$$T_{1'}=\frac{1}{2^{s+2}(s+2)},\ N_{1'}=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)},\ T_{2'}=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)},\ N_{2'}=\frac{s+1}{2^{s+2}(s+1)(s+2)}.$$

Taking s=1 and p=1 in (2.33) yields

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(\frac{1}{24}^{\frac{1}{p}}+\frac{1}{12}^{\frac{1}{p}})(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|)]$$
$$\leq\frac{b-a}{8}(\left|f^{'(a)}\right|+\left|f^{'(b)}\right|).$$

ii) When $A=\ B=\ c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(T_{1''}^{\frac{1}{p}}+N_{1''}^{\frac{1}{p}})\left|f^{'(a)}\right|+(T_{2''}^{\frac{1}{p}}+N_{2''}^{\frac{1}{p}})\left|f^{'(b)}\right|], \tag{2.34}$$

where,

$$T_{1''}=\frac{1}{2^{s+2}(s+1)(s+2)},\ N_{1''}=\frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)},\ T_{2''}=\frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)},\ N_{2''}=\frac{1}{2^{s+2}(s+1)(s+2)}.$$

Taking s=1 and p=1 in (2.34) yields

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(\frac{1}{48}^{\frac{1}{p}}+\frac{5}{48}^{\frac{1}{p}})(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|)]$$
$$\leq\frac{b-a}{8}\left(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

iii) When $A=\frac{1}{6},\ B=\frac{5}{6},\ c=\frac{1}{2}$ *and* $s=1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\leq(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{p}}(\frac{90}{1296})^{1/p}(\left|f'^{(a)}\right|+\left|f'(b)\right|)$$
$$\leq\frac{5(b-a)}{72}(\left|f'^{(a)}\right|+\left|f'(b)\right|).$$

If $|f'(x)|\leq L$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\leq\frac{5(b-a)}{36}L. \tag{2.35}$$

iv) When $A=\frac{1}{10},\ B=\frac{9}{10},\ c=\frac{1}{2}$ *and* $s=1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{8}{10}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right|\leq(b-a)\left(\frac{17}{200}\right)^{1-\frac{1}{p}}(\frac{17}{200})^{1/p}(\left|f'^{(a)}\right|+\left|f'(b)\right|)$$
$$\leq\frac{17(b-a)}{200}(\left|f'^{(a)}\right|+\left|f'(b)\right|).$$

Remark 2.14: The followings are observed that
i)The inequality (2.34) is a refinement of inequality (1.6) presented by Du et al. in [8]
ii) The inequality (2.35) is a refinement of inequality (1.2) established by Dragomir et al. in [6] and the same as inequality (1.3) presented by Dragomir in [7].

3. Applications To Special Means

We shall consider the means for arbitrary real numbers $\alpha,\beta,\alpha\neq\beta$. We take
$A(\alpha,\beta)=\frac{\alpha+\beta}{2}$, $\alpha,\beta\in$R, (arithmetic mean)
$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}$, n∈Z\{-1,0}, $\alpha,\beta\in$R, $\alpha\neq\beta$, (generalized log-mean)

In [9] and [4, pp 288], the following example is given:
Let $s\in(0,1)$and $a,b,c\in R$. We define function $f:[0,\infty)\to R$ as

$$f(t)=\begin{cases} a, & t=0\\ bt^s+c, & t>0\end{cases}$$

if $b\geq 0$ and $0\leq c\leq a$, then $f\in K_s^2$. Hence, for $a=c=0, b=1$, we have $f(t)=t^s, f:[0,1]\to[0,1], f\in K_s^2$.

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 3.1: Let $a,b\in I^o,\ 0<a<b$ *and* $0<s<1$. Then we have, for all $p>1$
i) $|L_s^s(a,b)-A^s(a,b)|\leq\frac{s(b-a)}{4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})A(|a|^{s-1},|b|^{s-1})$.
ii) $|L_s^s(a,b)-A(a^s,b^s)|\leq\frac{s(b-a)}{4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})A(|a|^{s-1},|b|^{s-1})$.
Proof: The assertions follow from Corollaries 2.4-i and 2.4-ii applied to the mapping $f(x)=x^s, f:[0,1]\to[0,1]$, respectively.
Proposition 3.2: Let $a,b\in I^o,\ 0<a<b$ *and* $0<s<1$. Then we have, for all $p>1$
i)$|L_s^s(a,b)-A^s(a,b)|\leq\frac{s(b-a)}{2^{\frac{1}{p}+1}}(|A(a,b)|^{s-1}+A(|a|^{s-1},|b|^{s-1}))$.
ii) $|L_s^s(a,b)-A(a^s,b^s)|\leq\frac{s(b-a)}{2^{\frac{1}{p}+1}}(|A(a,b)|^{s-1}+A(|a|^{s-1},|b|^{s-1}))$.
iii) $\left|L_s^s(a,b)-\frac{3A^s(a,b)}{4}-\frac{A(a^s,b^s)}{8}\right|\leq\frac{s(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8^{\frac{1}{p}+1}}\left(\left(|A(a,b)|^{s-1}+A\left(|a|^{s-1},|b|^{s-1}\right)\right)\right)$.
Proof: The assertions follow from Corollaries 2.7-i, 2.7-ii and 2.7-v applied to the mapping $f(x)=x^s, f:[0,1]\to[0,1]$, respectively.

Proposition 3.3: Let $a,b \in I^o$, $0 < a < b$ *and* $0 < s < 1$. Then we have, for all $p \geq 1$

i)$|L_s^s(a,b) - A^s(a,b)| \leq \frac{s(b-a)}{4.8^{-\frac{1}{p}}} A(\left(T^{\frac{1}{p}}_{1'+N_{1'}^{\frac{1}{p}}} \right) |a|^{s-1}, \left(T^{\frac{1}{p}}_{2'+N_{2'}^{\frac{1}{p}}} \right) |b|^{s-1})$.

ii) $|L_s^s(a,b) - A(a^s,b^s)| \leq \frac{s(b-a)}{4.8^{-\frac{1}{p}}} A(\left(T_{1''}^{\frac{1}{p}} + N_{1''}^{\frac{1}{p}} \right) |a|^{s-1}, \left(T_{2''}^{\frac{1}{p}} + N_{2''}^{\frac{1}{p}} \right) |b|^{s-1})$.

iii) $\left| L_s^s(a,b) - \frac{2A^s(a,b)}{3} - \frac{A(a^s,b^s)}{6} \right| \leq \frac{5s(b-a)}{36} A\left(|a|^{s-1}, |b|^{s-1} \right)$.

Proof: The assertions follow from Corollaries 2.13-i, 2.13-ii and 2.13-iii applied to the mapping $f(x) = x^s, f : [0,1] \to [0,1]$, respectively.

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