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A New Hybrid Iterative Method for Solving Fixed Points Problems for a Finite Family of Multivalued Strictly Pseudo-Contractive Mappings and Convex Minimization Problems in Real Hilbert Spaces

Thierno M. M. Sow

Abstract
In this paper, we investigate the problem of finding a common solution to fixed point problem involving a finite family of multivalued strictly pseudo-contractive mappings and convex minimization problem in the framework of Hilbert spaces. Inspired by the proximal point algorithm and general iterative method, a new iterative method for solving the problem is introduced. Strong convergence theorem of the proposed method is established without any compactness assumption. Our scheme generalize and extend some of the existing results in the literature.

Keywords: *Fixed points problems, Convex minimization problem, Set-valued operators, Iterative methods*

AMS Subject Classification (2020): *Primary: 47H09; Secondary: 49J20; 49J40*

1. Introduction

Let H be a real Hilbert space with the inner product $\langle ., .\rangle$ and norm $\|.\|$ respectively. Let K be a nonempty closed convex subset of H. Consider the following convex minimization problem: find $x \in K$ such that

$$g(x) = \min_{y \in K} g(y),$$

where $g : H \to (-\infty, +\infty)$ be a proper convex and lower semi-continuous. The set of all minimizers of g on K is denoted by $\text{argmin}_{y\in K}\, g(y)$. In 1970, Martinet [21] introduced and studied the proximal point algorithm (PPA) for solving optimization problems. Thereafter the likes of Rockafellar [29], find a solution of the constrained convex minimization problem in the frame work of Hilbert space by using PPA. Let g be a proper convex and lower

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semi-continuous function on H. The PPA is defined as

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \text{argmin}_{y\in H}\Big[g(y) + \dfrac{1}{2\lambda_n}\|x_n - y\|^2\Big], \end{cases} \tag{1.1}$$

where $\lambda_n > 0$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of g provided $\sum_{n=0}^{\infty} \lambda_n = \infty$. In [12], it was shown that a PPA does not necessarily converges strongly. The fact that a PPA does not necessarily converges strongly have been overcome by researchers in this area by introducing a more general PPA in different spaces to obtain a weak and strong convergence . Over the years, researcher have been able to further extend the convex minimization problems by finding a common element of the set of solutions of various convex minimization problems and the set of fixed points for nonexpansive mappings in Hilbert spaces and Banach spaces (see, *e.g.*, Güler [12], Solodov and Svaiter [31], Kamimura and Takahashi [14], Lehdili and Moudafi [15], Reich, [28], Chidume and Djitte [7, 8] and the references therein).

Let (X, d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

The fixed point theory of multi-valued mappings is much more complicated and harder than the corresponding theory of single-valued mappings. However, some classical fixed point theorems for single-valued mappings have already been extended to multi-valued mappings; (see, for example, Brouwer [4], Kakutani [13], Nash [24, 25], Garcia-Falset et al. [27]). The recent fixed point results for multi-valued mappings can be found Blasi et al. [3], Sow [32], Sene et al. [30], Sow et al. [30] and the references cited therein.

Interest in the study of fixed point theory for multi-valued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as Game Theory and Non-Smooth Differential Equations, Optimization.

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see, *e.g.*, [26]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in D\} = d(x, D),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D respectively. The Pompeiu *Hausdorff metric* on $CB(D)$ is defined by:

$$H(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Big\}$$

for all $A, B \in CB(D)$ (see, Berinde [2]). A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called L- *Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\|, \quad \forall x, y \in D(T). \tag{1.2}$$

When $L \in (0, 1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

A mapping $A : K \to H$ is said to be *k-strongly monotone* if there exists $k \in (0, 1)$ such that for all $x, y \in K$,

$$\langle Ax - Ay, x - y\rangle_H \geq k\|x - y\|^2.$$

A mapping $A : K \to H$ is said to be *strongly positive bounded linear* if there exists a constant $k > 0$ such that

$$\langle Ax, x\rangle_H \geq k\|x\|^2, \quad \forall\, x \in K.$$

Remark 1.1. From the definition of A, we note that strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and k-strongly monotone operator.

Great attention has been paid to single-valued nonexpansive mappings (a special kind of strictly pseudo-contractive mappings) because many nonlinear problems can be reduced to fixed point problems of nonexpansive mappings. Among these iterative methods, the Mann iteration method is the mostfavour fixed point algorithm for nonexpansive mappings since many algorithms can be reducedto Mann iteration. Recall that Mann's iteration process [16] is defined as follows: Let C be a nonempty, closed and convex subset of a Banach space X, Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \end{cases} \tag{1.3}$$

$\{\alpha_n\}$ is a sequence in $(0,1)$. But Mann's iteration process has only weak convergence, even in Hilbert space setting. Therefore, many authors try to modify Mann's iteration to have strong convergence for nonlinear operators (see, e.g., [33], [30]).

In 2009, Yao et al. motivated by the fact that Mann's algorithm method is remarkably useful for finding fixed points of a nonexpansive mapping, they proved the following theorem.

Theorem 1.1. *[37] Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For given $x_0 \in H$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n, \end{cases} \tag{1.4}$$

$\{\beta_n\}$ and $\{\alpha_n\}$ are a real sequences in $(0,1)$ satisfying:

$(i)\ \lim_{n\to\infty} \alpha_n = 0;\quad (ii)\ \sum_{n=0}^{\infty} \alpha_n = \infty.$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.4) converge strongly to fixed point of T.

Recently, iterative methods for single-valued nonexpansive mappings have been applied to solve fixed points problems and variational inequality problems in Hilbert spaces, see, e.g.,[18, 19, 35] and the references therein.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle. \tag{1.5}$$

In [35], Xu proved that the sequence $\{x_n\}$ defined by iterative method below with initial guess $x_0 \in H$ chosen arbitrary:

$$x_{n+1} = \alpha_n b + (I-\alpha_n A)Tx_n, \quad n \geq 0, \tag{1.6}$$

converges strongly to the unique solution of the minimization problem (1.5), where T is a nonexpansive mappings in H and A a strongly positive bounded linear operator. In 2006 Marino and Xu [18] extended Moudafi's results [20] and Xu's results [35] via the following general iteration $x_0 \in H$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)Tx_n,\ n \geq 0, \tag{1.7}$$

where$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, A is bounded linear operator on H and T is a nonexpansive. Under suitable conditions, they proved the sequence $\{x_n\}$ defined by (1.7) converges strongly to the fixed point of T, which is a unique solution of the following variational inequality

$$\langle Ax^* - \gamma f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in F(T).$$

The important class of single-valued k-strictly pseudo-contractive maps on Hilbert spaces was introduced by Browder and Petryshyn [5] as a generalization of the class of nonexpansive mappings.

Definition 1.1. Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called k-strictly pseudo-contractive if there exists $k \in (0,1)$ such that

$$\|Tx - Ty\|^2 \quad \leq \quad \|x-y\|^2 + k\|x-y-(Tx-Ty)\|^2, \qquad \forall x, y \in K. \tag{1.8}$$

It is trivial to see that every nonexpansive map is strictly pseudo-contractive. Motivated by this, Chidume et al. [10] introduced the of multivalued strictly pseudo-contractive mappings in real Hilbert as follows.

Definition 1.2. A multi-valued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be *k-strictly pseudo-contractive*, if there exists $k \in (0, 1)$ such for all $x, y \in D(T)$, we have

$$\Big(H(Tx, Ty)\Big)^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \ \forall\, u \in Tx,\, v \in Ty. \tag{1.9}$$

if $k = 1$ in (1.9), the map T is said to be pseudo-contractive.

Remark 1.2. It is easily seen that any multivalued nonexpansive mapping is k-strictly pseudocontractive for any $k \in (0, 1)$. Moreover the inverse is not true (see,e.g., Sene et al. [30]).

With this definition at hand, many mathematicians proved some strong convergence theorems for approximating fixed points of multivalued k-strictly pseudo-contrcative mappings under some compactness conditions (see, for example, Sene et al. [30], Chidume et .al [10], Sow et al. [34]).

In 2019, A. A. Mebawondu [22] introduced the following iterative method to find a common element of the set of minimizers of a convex function and the set of common fixed points of a finite family of multivalued nonexpansive mappings, proved the following theorem.

Theorem 1.2 (A. A. Mebawondu [22])**.** *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $m \geq 1$ be a fixed number, for $i, 1 \leq i \leq m$, let $T_i : K \to CB(K)$ be a multivalued nonexpansive mappings and $f : K \to (-\infty,\ +\infty)$ be a proper convex and lower semi-continuous function such that $\Gamma := \bigcap_{i=1}^{m} F(T_i) \cap argmin_{y\in K} f(y) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in \bigcap_{i=1}^{m} F(T_i)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_1 \in K$ by:*

$$\begin{cases} y_n = J^f_{\lambda_n} x_n, \\ z_n = {\gamma_n}^0 x_n + \displaystyle\sum_{i=1}^{m} {\gamma_n}^i y^i_n, \quad v^i_n \in T_i u_n \\ x_{n+1} = {\alpha_n}^0 z_n + (1 - {\alpha_n}^0) w_n, \quad w_n \in T_i z_n \end{cases} \tag{1.10}$$

where $= {\alpha_n}^0 \subset (0, 1)$, ${\gamma_n}^0 \subset (0, 1)$ and $\{\lambda_n\} \subset]0, \infty[$ satisfy:
$(i)\ \sum_{n=0}^{\infty} {\alpha_n}^0 = \sum_{n=0}^{\infty} {\gamma_n}^0 = 1,\ (ii)\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Then, the sequence $\{x_n\}$ generated by (3.3) converges weakly to an element of Γ.

In the recent years, the problem of finding a common element of the set of solutions of convex minimization and fixed point problems in real Hilbert spaces have been intensively studied by many authors; see, for example, [10, 16, 18, 34**?** , 35] and the references therein.

In this paper, motivated by above results, the fact that the class of multivalued strictly pseudo-contractive mappings contains those of multivalued nonexpansive and multivalued firmly nonexpansive mappings as subclasses and general proximal point algorithm is remarkably useful for solving most important problems with nonlinear operators, we construct and study an explicit iterative method and prove strong convergence theorems by using a modified general proximal point algorithm for approximating for approximating a common element of the set of minimizers of a convex function and the set of common fixed points of a finite family of multivalued strictly pseudo-contractive mappings in the setting of a real Hilbert space which is a solution of some variational inequalities problems. Our result extends and improves the results of A. A. Mebawondu [22], Yao et al. [37], Marino and Xu [18] Rockafellar [29] and many other authors.

2. Preliminaries

Let us recall the following definitions and results which will be used in the sequel.

Let H be a real Hilbert space. Let $\{x_n\}$ be a sequence in H, and let $x \in H$. Weak convergence of x_n to x is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$. Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K, denoted by P_K assigns to each $x \in H$ the unique $P_K x$ with the property

$$\|x - P_K x\| \le \|y - x\|$$

for all $y \in K$. It is well know that P_K satisfies

$$\langle x - P_K x, y - P_K x\rangle \le 0 \tag{2.1}$$

for all $y \in K$.

Definition 2.1. Let H be a real Hilbert space and $T : D(T) \subset H \to 2^H$ be a multivalued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.

Lemma 2.1 (Demiclosedness Principle, [4]). *Let H be a real Hilbert space, K be a nonempty closed and convex subset of H. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2.2 ([6]). *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequality hold:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3 (Xu, [36]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \ge 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *[17] Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \le t_{n_{i+1}}$ for all $i \ge 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \le n : \ t_k \le t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \le t_{\tau(n)+1}.$$

Lemma 2.5. *[19] Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*
(i) If T is a k-strictly pseudo-contractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \le \frac{1 + k}{1 - k}\|x - y\|.$$

(ii) If T is a k-strictly pseudo-contractive mapping, then the mapping $I - T$ is demiclosed at 0.

Lemma 2.6. *[38] Let H be a real Hilbert space. Let K be a nonempty, closed convex subset of H and $A : K \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0, \ L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\{1, \ \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \le (1 - t\tau)\|x - y\|, \ x, y \in K.$$

Lemma 2.7 (Sene et al. [30]). *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $\beta_i \in]0, 1[, i = 1, \cdots, n$ such that $\sum_{i=1}^{n} \beta_i = 1$. Then,*

$$\Big\|\sum_{i=1}^{n} \beta_i u_i\Big\|^2 = \sum_{i=1}^{n} \beta_i\|u_i\|^2 - \sum_{i<j} \beta_i\lambda_j\|u_i - u_j\|^2 \quad \forall \ u_1, u_2, \cdots, u_n \in \ K. \tag{2.2}$$

Let $g : K \to (-\infty,\ +\infty)$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of g in a real Hilbert space H as follows:

$$J_\lambda^g x = \text{argmin}_{u\in K}\Big[g(u) + \frac{1}{2\lambda}\|x-u\|^2\Big],$$

for all $x \in H$. It was shown in [12] that the set of fixed points of the resolvent associated with g coincides with the set of minimizers of g. Also, the resolvent J_λ^g of g is nonexpansive for all $\lambda > 0$ (see [11]).

Lemma 2.8. *(Miyadera [23]) For any $r > 0$ and $\mu > 0$, the following holds:*

$$J_r^g x = J_\mu^g x(\frac{\mu}{r}x + (1-\frac{\mu}{r})J_r^g x).$$

Lemma 2.9 (Sub-differential inequality, [1])**.** *Let $g : H \to (-\infty,\ +\infty)$ be a proper convex and lower semicontinuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{\lambda}\|J_\lambda^g x - y\|^2 - \frac{1}{\lambda}\|x-y\|^2 + \frac{1}{\lambda}\|x - J_\lambda^g x\|^2 + g(J_\lambda^g x) \le g(y). \tag{2.3}$$

3. Main Results

Throughout this section, we will assume that H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $A : K \to H$ be an α-strongly monotone and L-Lipschitzian operator, $m \ge 1$ be a fixed number, for $i, 1 \le i \le m$, let $T_i : K \to CB(K)$ be a multivalued k_i-strictly pseudo-contractive mapping and $g : K \to (-\infty,\ +\infty)$ be a proper convex and lower semi-continuous function such that $\Gamma := \bigcap_{i=1}^{m} F(T_i) \cap \text{argmin}_{y\in K}\, g(y) \neq \emptyset$.

We consider the following fixed point problem:

Problem 1.

$$\textit{find } x \in K \ \textit{ such that } x \in \bigcap_{i=1}^{m} F(T_i). \tag{3.1}$$

We consider the following convex minimization problem:

Problem 2.

$$\textit{find } x \in K \ \textit{ such that } \ \ g(x) \le g(y),\ \ \forall\ y \in K. \tag{3.2}$$

Remark 3.1. We can observe that x^* solves Problem 3.1 and Problem 3.2 if and only if $x^* \in \Gamma$.

We show the main result of this paper, that is, the strong convergence analysis for Algorithm 1.

Algorithm 1. Step 0. *Take $\{\alpha_n\} \subset (0,1),\ \ \eta > 0$, and $\{\lambda_n\} \subset]0,\infty[$ arbitrarily choose $x_0 \in K$; and let $n := 0$.*
Step 1. *Given $x_n \in K$, compute $x_{n+1} \in K$ as*

$$\begin{cases} u_n = \textit{argmin}_{u\in K}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ y_n = \beta_0 u_n + \displaystyle\sum_{i=1}^{m}\beta_i v_n^i,\ \ v_n^i \in T_i u_n \\ x_{n+1} = P_K(I - \alpha_n\eta A)y_n,\ \ n \ge 0. \end{cases} \tag{3.3}$$

Update $n := n+1$ and go to Step 1.
Where $\beta_0 \in]\mu, 1[,\ \ \mu := \max\Big\{k_i,\ \ i = 1, ..., m\Big\},\ \ \beta_i \in]0,1[$ and $\ \beta_0 + \beta_1 + \cdots + \beta_m = 1$.

Theorem 3.1. *Assume that $\ I - T_i$ is demiclosed at origin and $T_i p = \{p\}$ for all $p \in \Gamma$. Suppose that: $(i) \lim_{n\to\infty} \alpha_n = 0$; $\ \ (ii) 0 < \eta < \dfrac{2\alpha}{L^2}$, and $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \ge 0$ and some λ. Then, the sequences $\{x_n\}$ and $\{u_n\}$ defined by Algorithm 1 converge strongly to $x^* \in \Gamma$, which is a unique solution of the following variational inequality:*

$$\langle Ax^*, x^* - p\rangle \le 0,\ \ \ \forall p \in \Gamma. \tag{3.4}$$

Proof. From the choice of η, properties of P_Γ, and A is strongly monotone, then the variational inequality (3.4) has a unique solution in Γ. Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1\ , \frac{1}{\tau}\}\Big)$ where $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. In what follows, we denote x^* to be the unique solution of (3.4). Now, we prove that the sequences $\{x_n\}$ is bounded. Let $p \in \Gamma$. Then, $g(p) \leq g(u)$ for all $u \in K$ This implies that

$$g(p) + \frac{1}{2\lambda_n}\|p-p\|^2 \leq g(u) + \frac{1}{2\lambda_n}\|u-p\|^2$$

and hence $J^g_{\lambda_n}p = p$ for all $n \geq 0$, where $J^g_{\lambda_n}$ is the Moreau-Yosida resolvent of g in K. We have

$$\|u_n - p\| = \|J^g_{\lambda_n}x_n - p\| \leq \|x_n - p\|, \ \ \forall n \geq 0. \tag{3.5}$$

By Using (3.3) and Lemma 2.7, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \Big\|\beta_0(u_n - p) + \sum_{i=1}^m \beta_i(v_n^i - p)\Big\|^2 \\
&= \beta_0\|u_n - p\|^2 + \sum_{i=1}^m \beta_i\|v_n^i - p\|^2 - \sum_{i=1}^m \beta_0\beta_i\|v_n^i - u_n\|^2 - \sum_{1\leq i<j}^m \beta_i\beta_j\|v_n^i - v_n^j\|^2.
\end{aligned}$$

Using the fact that, for $i = 1, \cdots, m$, $T_ip = \{p\}$, we get

$$\|y_n - p\|^2 \leq \beta_0\|u_n - p\|^2 + \sum_{i=1}^m \beta_i\Big(H(T_iu_n, T_ip)\Big)^2 - \sum_{i=1}^m \beta_0\beta_i\|v_n^i - u_n\|^2 - \sum_{1\leq i<j}^m \beta_i\beta_j\|v_n^i - v_n^j\|^2.$$

Using the fact that, for $i = 1, \cdots, m$, T_i is k_i-strictly pseudo-contractive, we have

$$\begin{aligned}
\|y_n - p\|^2 \leq\ & \beta_0\|u_n - p\|^2 + \sum_{i=1}^m \beta_i\Big(\|u_n - p\|^2 + k_i\|v_n^i - u_n\|^2\Big) - \sum_{i=1}^m \beta_0\beta_i\|v_n^i - u_n\|^2 \\
& - \sum_{1\leq i<j}^m \beta_i\beta_j\|v_n^i - v_n^j\|^2.
\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|u_n - p\|^2 - \sum_{i=1}^m \beta_i(\beta_0 - k_i)\|v_n^i - u_n\|^2. \tag{3.6}$$

Since $\beta_0 \in]\mu, 1[$, we obtain,

$$\|y_n - p\Big\| \leq \|u_n - p\Big\| \leq \|x_n - p\|. \tag{3.7}$$

From (3.3), (3.7) and Lemma 2.6, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|(I - \alpha_n\eta A)y_n - p\| \\
&\leq (1 - \tau\alpha_n)\|x_n - p\| + \alpha_n\|\eta Ap\| \\
&\leq \max\{\|x_n - p\|, \frac{\|\eta Ap\|}{\tau}\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\eta Ap\|}{\tau}\}, \ \ n \geq 0.$$

Hence $\{x_n\}$ is bounded also are $\{u_n)\}$, and $\{y_n\}$.

Consequently, by inequality (3.6) and property of μ, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_K(I-\alpha_n\eta A)y_n-p\|^2\\
&\le \|y_n-p-\alpha_n\eta Ay_n\|^2\\
&= \|y_n-p\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|^2\\
&\le \|u_n-p\|^2-\sum_{i=1}^m\beta_i(\beta_0-k_i)\|v_n^i-u_n\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|\\
&\le \|x_n-p\|^2-\sum_{i=1}^m\beta_i(\beta_0-k_i)\|v_n^i-u_n\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|.
\end{aligned}$$

Thus, for every $i, 1\le i\le m$, we get

$$\sum_{i=1}^m\beta_i(\beta_0-k_i)\|u_n^i-v_n\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|.$$

Since $\{x_n\}$ is bounded, then there exists a constant $B>0$ such that for every $i, 1\le i\le m$,

$$\sum_{i=1}^m\beta_i(\beta_0-k_i)\|v_n^i-u_n\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nB. \tag{3.8}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. Now we divide the rest of the proof into two cases.
Case 1. Assume that there is $n_0\in N$ such that $\{\|x_n-p\|\}$ is decreasing for all $n\ge n_0$. Since $\{\|x_n-p\|\}$ is monotonic and bounded, $\{\|x_n-p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\left[\|x_n-p\|^2-\|x_{n+1}-p\|^2\right]=0. \tag{3.9}$$

It then implies from (3.8) that

$$\lim_{n\to\infty}\sum_{i=1}^m\beta_i(\beta_0-k_i)\|v_n^i-u_n\|^2=0. \tag{3.10}$$

Since $\beta_0\in]\mu,1[$, we have

$$\lim_{n\to\infty}\left\|u_n-v_n^i\right\|^2=0. \tag{3.11}$$

Since $v_n^i\in T_iu_n$, it follows that

$$\lim_{n\to\infty}d(u_n,T_iu_n)=0,\ \forall\ i=1,\cdots,m. \tag{3.12}$$

Let $p\in\Gamma$. Using Lemma 2.9 and since $g(p)\le g(u_n)$, we get

$$\|x_n-u_n\|^2\le\|x_n-p\|^2-\|u_n-p\|^2. \tag{3.13}$$

Therefore, from (3.3), Lemma 2.2 and inequality (3.13), we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(I-\alpha_n\eta A)y_n-p\|^2\\
&\le \|y_n-p-\alpha_n\eta Ay_n\|^2\\
&\le \|y_n-p\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|\\
&\le \|u_n-p\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|^2\\
&\le \|x_n-p\|^2-\|x_n-u_n\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|^2
\end{aligned}$$

and hence

$$\|x_n-u_n\|^2\ \le\ \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\eta\|y_n-p\|\|Ay_n\|+\alpha_n^2\|\eta Ay_n\|^2.$$

Thanks inequality (3.9) and $\alpha_n\to0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.14}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, x^* - x_n\rangle \le 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to ω in K and

$$\limsup_{n\to+\infty}\langle Ax^*, x^* - x_n\rangle = \lim_{j\to+\infty}\langle Ax^*, x^* - x_{n_j}\rangle.$$

From (3.12) and the fact that $I - T_i$ are demiclosed, we obtain $\omega \in \bigcap_{i=1}^{m} F(T_i)$. Using (3.3) and Lemma 2.8 we arrive at

$$\begin{aligned}
\|x_n - J_\lambda^g x_n\| &\le \|u_n - J_\lambda^g x_n\| + \|u_n - x_n\| \\
&\le \|J_{\lambda_n}^g x_n - J_\lambda^g x_n\| + \|u_n - x_n\| \\
&\le \|u_n - x_n\| + \|J_\lambda^g\Big(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}^g x_n + \frac{\lambda}{\lambda_n}x_n\Big) - J_\lambda^g x_n\| \\
&\le \|u_n - x_n\| + \|\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}^g x_n + \frac{\lambda}{\lambda_n}x_n - x_n\| \\
&\le \|u_n - x_n\| + \Big(1 - \frac{\lambda}{\lambda_n}\Big)\|u_n - x_n\| \\
&\le \Big(2 - \frac{\lambda}{\lambda_n}\Big)\|u_n - x_n\|.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|x_n - J_\lambda^g x_n\| = 0. \tag{3.15}$$

Since J_λ^g is single valued and nonexpasive, using (3.15) and Lemma 2.1, then $\omega \in F(J_\lambda^g) = \operatorname{argmin}_{u\in K} g(u)$. Therefore, $\omega \in \Gamma$. On other hand, using the fact that x^* solves (3.4), we then have

$$\begin{aligned}
\limsup_{n\to+\infty}\langle Ax^*, x^* - x_n\rangle &= \lim_{j\to+\infty}\langle Ax^*, x^* - x_{n_j}\rangle \\
&= \langle Ax^*, x^* - \omega\rangle \le 0.
\end{aligned}$$

Finally, we show that $x_n \to x^*$.

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_K(I - \eta\alpha_n A)y_n - x^*\|^2 \\
&\le \langle (I - \eta\alpha_n A)y_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle (I - \eta\alpha_n A)y_n - x^* - \alpha_n\eta Ax^* + \alpha_n\eta Ax^*, x_{n+1} - x^*\rangle \\
&\le \|(I - \alpha_n\eta A)(y_n - x^*)\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n\langle \eta Ax^*, x^* - x_{n+1}\rangle \\
&\le (1 - \alpha_n\tau)\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle \eta Ax^*, x^* - x_{n+1}\rangle \\
&\le (1 - \alpha_n\tau)\|x_n - x^*\|^2 + 2\alpha_n\eta\langle Ax^*, x^* - x_{n+1}\rangle.
\end{aligned}$$

From Lemma 2.3, its follows that $x_n \to x^*$. We can check that all the assumptions of Lemma 2.3 are satisfied. Therefore, we deduce $x_n \to x^*$.

Case 2. Assume that there is not $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $\Omega_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \le n,\ \Omega_k \le \Omega_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Omega_{\tau(n)} \le \Omega_{\tau(n)+1}$ for $n \ge n_0$. From (3.8), we have

$$\sum_{i=1}^{m}\beta_i(\beta_0 - k_i)\Big\|u_{\tau(n)} - v_{\tau(n)}^i\Big\|^2 \le \alpha_{\tau(n)}B.$$

Furthermore, we have

$$\lim_{n\to+\infty}\sum_{i=1}^{m}\beta_i(\beta_0 - k_i)\|u_{\tau(n)} - v_{\tau(n)}^i\Big\|^2 = 0.$$

Since $\beta_0 \in]\mu, 1[$, we have

$$\lim_{n\to\infty} \left\| u_{\tau(n)} - v^i_{\tau(n)} \right\|^2 = 0. \tag{3.16}$$

Since $v^i_{\tau(n)} \in T_i u_{\tau(n)}$, it follows that

$$\lim_{n\to\infty} d\Big(u_{\tau(n)}, T_i u_{\tau(n)}\Big) = 0 \ \forall\ i = 1, \cdots, m. \tag{3.17}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly in K and $\limsup\limits_{n\to+\infty} \langle Ax^*, x^* - x_{\tau(n)}\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \leq \alpha_{\tau(n)}[-\tau\|x_{\tau(n)} - x^*\|^2 + 2\eta\langle Ax^*, x^* - x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq \frac{2\eta}{\tau}\langle Ax^*, x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.4, we conclude that

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim\limits_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Now, we apply Algorithm 1 for solving fixed points problem involving multivalued nonexpansive mappings and convex minimization problem without demiclosedness assumption.

Theorem 3.2. *Let H be a real Hilbert space and K be a nonempty, closed convex cone of H. Let $A : K \to H$ be an α-strongly monotone and L-Lipschitzian operator, $m \geq 1$ be a fixed number, for $i, 1 \leq i \leq m$, let $T_i : K \to CB(K)$ be a multivalued k_i-strictly pseudo-contractive mapping and $g : K \to (-\infty,\ +\infty)$ be a proper convex and lower semi-continuous function such that $\Gamma := \bigcap_{i=1}^{m} F(T_i) \cap argmin_{y\in K}\, g(y) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} u_n = argmin_{u\in K}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ y_n = \beta_0 u_n + \displaystyle\sum_{i=1}^{m} \beta_i v^i_n,\ \ v^i_n \in T_i u_n \\ x_{n+1} = P_K(I - \alpha_n \eta A)y_n,\ \ n \geq 0. \end{cases} \tag{3.18}$$

With conditions $\{\alpha_n\} \subset (0, 1)$ and $\eta > 0$ satisfy:
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\quad (ii)$ $0 < \eta < \dfrac{2\alpha}{L^2}$ and $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\beta_0 \in]\mu, 1[$, $\ \mu := \max\Big\{k_i,\ \ i = 1, ..., m\Big\}$, $\ \beta_i \in]0, 1[$ and $\beta_0 + \beta_1 + \cdots \beta_m = 1$.
(iv) $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Then, the sequences $\{x_n\}$ and $\{u_n\}$ defined by Algorithm 1 converge strongly to $x^* \in \Gamma$, which is a minimizer of g in K as well as it is also a common fixed points of T_i in K.

Proof. Since every multivalued nonexpansive mapping is multivalued strictly pseudo-contractive mapping, then, the proof follows Lemma 2.1 and Theorem 3.1. □

Corollary 3.1. *Let H be a real Hilbert space. Let $m \geq 1$ be a fixed number, for $i, 1 \leq i \leq m$, let $T_i : H \to H$ be a k_i-strictly pseudo-contractive mapping and $g : H \to (-\infty, +\infty)$ be a proper convex and lower semi-continuous function such that $\Gamma := \bigcap_{i=1}^{m} F(T_i) \cap argmin_{y \in B}\, g(y) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in H$ by:*

$$\begin{cases} u_n = argmin_{u \in H}\Big[g(u) + \dfrac{1}{2\lambda_n}\|u - x_n\|^2\Big], \\ y_n = \beta_0 u_n + \displaystyle\sum_{i=1}^{m} \beta_i T_i u_n \\ x_{n+1} = (1 - \alpha_n) y_n, \;\; n \geq 0. \end{cases} \tag{3.19}$$

With conditions $\{\alpha_n\} \subset (0, 1)$ satisfies:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad (ii) \sum_{n=0}^{\infty} \alpha_n = \infty,$

(iii) $\beta_0 \in]\mu, 1[, \quad \mu := \max\Big\{k_i, \;\; i = 1, ..., m\Big\}, \quad \beta_i \in]0, 1[$ *and* $\beta_0 + \beta_1 + \cdots \beta_m = 1.$

(iv) *$\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some λ. Then, the sequences $\{x_n\}$ and $\{u_n\}$ defined by Algorithm 1 converge strongly to $x^* \in \Gamma$.*

Proof. Since every single-valued strictly pseudo-contractive is multivalued strictly pseudo-contractive mapping, then, the proof follows Theorem 3.1. □

4. Conclusion

The problem of finding a common element of the set of fixed points of nonlinear operators and the set of solutions of convex minimization problem has attracted much attention because of its extraordinary utility and broad applicability in many branches of mathematical science and engineering. General terative algorithm and proximal point algorithm are remarkably useful methods for solving most important problems with nonlinear operators. In this article, we introduce and analyze a new iterative algorithm for approximating a common solution of an equilibrium problem, variational inequality problems and fixed point problems with a finite family of multivalued strictly pseudo-contractive mappings without imposing any compactness-type condition on either the operators or the space considered. The results obtained in this paper are important improvements of recent important results in this field.

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SITEM for the Conformable Space-Time Fractional (2+1)-Dimensional Breaking Soliton, Third-Order KdV and Burger's Equations

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Abstract
In the present paper, new analytical solutions for the conformable space-time fractional (2+1)-dimensional breaking soliton, third-order KdV and Burger's equations are obtained by using the simplified $\tan(\frac{\phi(\xi)}{2})$-expansion method (SITEM). Here, fractional derivatives are described in conformable sense. The obtained traveling wave solutions are expressed by the trigonometric, hyperbolic, exponential and rational functions. Simulation of the obtained solutions are given at the end of the paper.

Keywords: *Space-time fractional (2+1)-dimensional breaking soliton equations; Space-time fractional third-order KdV equation; Space-time fractional Burger's equation; Simplified* $\tan(\frac{\phi(\xi)}{2})$*-expansion method (SITEM).*

AMS Subject Classification (2020): *Primary: 35C07, 35C08, 35R11.*

1. Introduction

Nonlinear fractional partial differential equations have significant applications in various fields of science and engineering such as fluid mechanics, mechanics of materials, biology, plasma physics, finance, chemistry, image processing (see, for example, [1–5]). Traveling wave solutions to nonlinear fractional partial differential equations play an important role in the study of nonlinear physical phenomena. The traveling wave solutions of the nonlinear partial differential equations have been investigated by using various method such as exponential rational function method, (G'/G)-expansion method, Exp-function method, extended sinh-Gordon equation expansion method, modified exponential rational function method, Jacobi elliptic equation method (see, for example,[6–10]).

(2+1)-dimensional breaking soliton equations describe the (2 + 1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. (G'/G)-expansion method, extended tanh-function method, improved Riccati equations method, sine-cosine method, improved extended Fan sub-equation method, generalized (G'/G)-expansion method and extended three wave method have been applied to the (2+1)-dimensional breaking soliton equations [11–18]. The space-time fractional (2 + 1)-dimensional breaking soliton equations with modified Riemann-Liouville derivative have been solved by using new fractional Jacobi elliptic equation method,

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new fractional sub-equation method, modified simple equation method, improved fractional sub-equation method, exponential rational function method, new fractional Jacobi elliptic equation method [19–23]. (G'/G)-expansion method has been studied for the space fractional (2 + 1)-dimensional breaking soliton equations with modified Riemann-Liouville derivative [24].

The general projective Riccati equation method, Exp-function method, extended hyperbolic function method and collocation method with the modified exponential cubic B-spline have been applied to the third-order KdV equation [25–28]. Time-fractional generalized third-order KdV equation with modified Riemann-Liouville derivative has been solved by using generalized Kudryashov method [29].

Burger's equation plays a major role in the study of nonlinear waves since it is used as a mathematical model in turbulence problems, in the theory of shock waves, and in continuous stochastic processes [30]. Hopf-Cole transformation and a reproducing kernel function method, a semi-analytical iterative method, $(G'/G, 1/G)$-expansion method have been applied to the Burger's equation [31–33].

In this paper, the conformable space-time fractional (2+1)-dimensional breaking soliton, third-order KdV and Burger's equations have been solved by using the simplified $\tan(\frac{\phi(\xi)}{2})$-expansion method (SITEM). SITEM has been applied to the Kundu-Eckhaus equation only for the parameter $p = 0$ in [34]. In our work, SITEM for the nonzero parameter p has been applied to the space-time fractional some evolution equations with conformable fractional derivative. New analytic solutions for these equations have been reported. Note that space-time fractional (2+1)-dimensional breaking soliton, third-order KdV and Burger's equations including conformable derivatives have not yet been solved.

2. Description of the conformable fractional derivative and its properties

For a function $f : (0, \infty) \to R$, the conformable fractional derivative of f of order $0 < \alpha < 1$ is defined as (see, for example, [35])

$$T_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \tag{2.1}$$

Some important properties of the the conformable fractional derivative are as follows:

$$\begin{aligned} T_t^\alpha (af + bg)(t) &= aT_t^\alpha f(t) + bT_t^\alpha g(t), \ \forall a, b \in R, \\ T_t^\alpha (t^\mu) &= \mu t^{\mu-\alpha}, \\ T_t^\alpha (f(g(t)) &= t^{1-\alpha} g'(t) f'(g(t)). \end{aligned} \tag{2.2}$$

3. Analytic solutions to the conformable space-time fractional (2+1)-dimensional breaking soliton equations

The breaking soliton equations can be used to describe the (2 + 1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave propagating along the x-axis. The $u(x, y, t)$ and $v(x, y, t)$ represent the physical field and some potential, respectively. This equation was studied by Bogoyavenskii [36].

Conformable space-time fractional (2+1)-dimensional breaking soliton equations are given in the following form[23]

$$T_t^\alpha u + T_x^\beta T_x^\beta T_y^\theta u + 4uT_x^\beta v + 4vT_x^\beta u = 0, \tag{3.1}$$

$$T_y^\theta u = T_x^\beta v, \ \ 0 < \alpha \le 1, \ \ 0 < \beta \le 1, \ \ 0 < \theta \le 1. \tag{3.2}$$

Let us consider the following transformation

$$u(x, y, t) = U(\xi), \ \ v(x, y, t) = V(\xi), \ \ \xi = k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{y^\theta}{\theta}, \tag{3.3}$$

where k, m and n are constants. Using the third property in Eq.(2.2), we can compute the following derivatives

$$\begin{aligned}
&T_t^{\alpha}u(x,y,t) = T_t^{\alpha}U(\xi) = t^{1-\alpha}\frac{d\xi}{dt}\frac{dU(\xi)}{d\xi} = kU'(\xi),\\
&T_x^{\beta}u(x,y,t) = T_x^{\beta}U(\xi) = x^{1-\beta}\frac{d\xi}{dx}\frac{dU(\xi)}{d\xi} = mU'(\xi),\\
&T_y^{\theta}u(x,y,t) = T_y^{\theta}U(\xi) = y^{1-\theta}\frac{d\xi}{dy}\frac{dU(\xi)}{d\xi} = nU'(\xi),\\
&T_x^{\beta}T_x^{\beta}T_y^{\theta}u(x,y,t) = nm^2kU'''(\xi),\\
&T_x^{\beta}v(x,y,t) = T_x^{\beta}V(\xi) = x^{1-\beta}\frac{d\xi}{dx}\frac{dV(\xi)}{d\xi} = mV'(\xi).
\end{aligned} \tag{3.4}$$

Substituting Eqs.(3.4) into Eqs.(3.1)-(3.2), we obtain the following differential equations

$$kU' + m^2nU''' + 4mUV' + 4mVU' = 0, \tag{3.5}$$

$$nU' = mV'. \tag{3.6}$$

Integrating of Eqs.(3.5)-(3.6) with zero constant of integration and eliminating V, we have

$$kU + m^2nU'' + 4nU^2 = 0. \tag{3.7}$$

Let us suppose that the solution of Eq.(3.7) can be expressed in the following form

$$U(\xi) = \sum_{k=0}^{N} A_k\Big[p + \tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^k + \sum_{k=1}^{N} B_k\Big[p + \tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-k}. \tag{3.8}$$

Here, $\phi(\xi)$ satisfies the following ordinary differential equation

$$\phi'(\xi) = a\sin(\phi(\xi)) + b\cos(\phi(\xi)) + c, \tag{3.9}$$

and $a,\ b,\ c$, $A_k(0 \le k \le N)$ and $B_k(1 \le k \le N)$ are constants to be determined. The solution of Eq. (3.9) is given as follows:

For $b = c,\ \ a = 0$,

$$\tan(\frac{\phi}{2}) = b\xi + c_1 - p.$$

For $b = c,\ \ a \neq 0$,

$$\tan(\frac{\phi}{2}) = c_1\exp(a\xi) - \frac{b}{a}.$$

For $b \neq c,\ \ \Delta = a^2 + b^2 - c^2 > 0$,

$$\tan(\frac{\phi}{2}) = \frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)} - p.$$

For $b \neq c,\ \ \Delta = a^2 + b^2 - c^2 = 0$,

$$\tan(\frac{\phi}{2}) = \frac{a}{b-c} + \frac{2}{b-c}\frac{c_2}{c_1 + c_2\xi}.$$

For $b \neq c,\ \ \Delta = a^2 + b^2 - c^2 < 0$,

$$\tan(\frac{\phi}{2}) = \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)},$$

where c_1 and c_2 are arbitrary constants, $r_1 = (a + p(b-c) + \sqrt{\Delta})/2$ and $r_2 = (a + p(b-c) - \sqrt{\Delta})/2$.

Substituting Eq.(3.8) into Eq.(3.7) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 2. Therefore, Eq.(3.8) reduces to

$$\begin{aligned} U(\xi) &= A_0 + A_1\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big] + A_2\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^2 \\ &+ B_1\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-1} + B_2\Big[p+\tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-2}. \end{aligned} \tag{3.10}$$

Substituting Eq.(3.10) into Eq.(3.7), collecting all the terms with the same power of $\tan(\frac{\phi}{2})$, we can obtain a set of algebraic equations for the unknowns A_0, A_1,A_2, B_1, B_2, k, m, n:

$$\begin{aligned} &8nA_2^2 + 3nA_2b^2m^2 - 6nA_2bcm^2 + 3nA_2c^2m^2 = 0, \\ &64npA_2^2 + ... \end{aligned}$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions:

Case 1: $A_0 = -\frac{3}{8}(b-c)m^2(-b-c+2ap+bp^2-cp^2), A_1 = 0, A_2 = 0, B_1 = \frac{3}{4}m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3), B_2 = -\frac{3}{8}m^2(-b-c+2ap+bp^2-cp^2)^2, k = -\Delta m^2 n:$
For $b = c$ and $a = 0$,

$$U_1(\xi) = -\frac{3}{2}m^2b^2\Big[b\xi + c_1\Big]^{-2}. \tag{3.11}$$

For $b = c$ and $a \neq 0$,

$$\begin{aligned} U_2(\xi) &= \frac{3}{4}m^2(-2ab+2a^2p)\Big[p+c_1\exp(a\xi)-\frac{b}{a}\Big]^{-1} - \frac{3}{8}m^2(-2b+2ap)^2 \\ &. \quad \Big[p+c_1\exp(a\xi)-\frac{b}{a}\Big]^{-2}. \end{aligned} \tag{3.12}$$

For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_3(\xi) &= -\frac{3}{8}(b-c)m^2(-b-c+2ap+bp^2-cp^2) \\ &+ \frac{3}{4}m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3) \\ &. \quad \Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^{-1} \\ &- \frac{3}{8}m^2(-b-c+2ap+bp^2-cp^2)^2\Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^{-2}. \end{aligned} \tag{3.13}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_4(\xi) &= -\frac{3}{8}(b-c)m^2(-b-c+2ap+bp^2-cp^2) \\ &+ \frac{3}{4}m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3) \\ &. \quad \Big[p+\frac{a}{b-c}+\frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-1} \\ &- \frac{3}{8}m^2(-b-c+2ap+bp^2-cp^2)^2\Big[p+\frac{a}{b-c} \\ &- \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-2}. \end{aligned} \tag{3.14}$$

Here $\xi = -\Delta m^2 n\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{x^\theta}{\theta}$.

Case 2: $A_0 = -\frac{1}{8}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2), A_1 = 0, A_2 = 0, B_1 = \frac{3}{4}m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3), B_2 = -\frac{3}{8}m^2(-b-c+2ap+bp^2-cp^2)^2, k = \Delta m^2 n:$

For $b=c$ and $a=0$,

$$U_5(\xi) = -\frac{3}{2}m^2b^2\Big[b\xi + c_1\Big]^{-2}. \tag{3.15}$$

For $b=c$ and $a\neq 0$,

$$\begin{aligned} U_6(\xi) &= -\frac{1}{4}m^2a^2 + \frac{3}{2}m^2(-ab+a^2p)\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-1} \\ &+ \quad -\frac{3}{2}m^2(-b+ap)^2\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-2}. \end{aligned} \tag{3.16}$$

For $\Delta > 0$ and $b\neq c$,

$$\begin{aligned} U_7(\xi) &= -\frac{1}{8}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2) \\ &+ \quad \frac{3}{4}m^2(-ab - ac + 2a^2p - b^2p + c^2p + 3abp^2 - 3acp^2 + b^2p^3 - 2bcp^3 + c^2p^3) \\ &\cdot \quad \Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^{-1} \\ &- \quad \frac{3}{8}m^2(-b-c+2ap+bp^2-cp^2)^2\Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^{-2}. \end{aligned} \tag{3.17}$$

For $\Delta < 0$ and $b\neq c$,

$$\begin{aligned} U_8(\xi) &= -\frac{1}{8}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2) \\ &+ \quad \frac{3}{4}m^2(-ab - ac + 2a^2p - b^2p + c^2p + 3abp^2 - 3acp^2 + b^2p^3 - 2bcp^3 + c^2p^3) \\ &\cdot \quad \Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-1} \\ &- \quad \frac{3}{8}m^2(-b-c+2ap+bp^2-cp^2)^2\Big[p + \frac{a}{b-c} \\ &+ \quad \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-2}. \end{aligned} \tag{3.18}$$

Here $\xi = \Delta m^2 n\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{x^\theta}{\theta}$.

Case 3: $A_0 = -\frac{3}{8}(b-c)m^2(-b-c+2ap+bp^2-cp^2), A_1 = \frac{3}{4}(b-c)m^2(a+bp-cp), A_2 = -\frac{3}{8}(b-c)^2m^2, B_1 = 0, B_2 = 0, k = -\Delta m^2 n:$

For $\Delta > 0$ and $b\neq c$,

$$\begin{aligned} U_9(\xi) &= -\frac{3}{8}(b-c)m^2(-b-c+2ap+bp^2-cp^2) \\ &+ \quad \frac{3}{2}m^2(a+bp-cp)\Big[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big] \\ &- \quad \frac{3}{2}m^2\Big[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^2. \end{aligned} \tag{3.19}$$

For $\Delta < 0$ and $b\neq c$,

$$\begin{aligned} U_{10}(\xi) &= -\frac{3}{8}(b-c)m^2(-b-c+2ap+bp^2-cp^2) \\ &+ \quad \frac{3}{4}m^2(a+bp-cp)\Big[p(b-c) + a + \sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big] \\ &- \quad \frac{3}{8}m^2\Big[p(b-c) + a + \sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^2. \end{aligned} \tag{3.20}$$

Here $\xi = -\Delta m^2 n\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{x^\theta}{\theta}$.

Case 4: $A_0 = -\frac{1}{8}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2)$, $A_1 = \frac{3}{4}(b-c)m^2(a+bp-cp)$, $A_2 = -\frac{3}{8}(b-c)^2m^2, B_1 = 0, B_2 = 0, k = \Delta m^2 n :$
For $b = c$ and $a \neq 0$,

$$U_{11}(\xi) = -\frac{1}{4}m^2a^2. \tag{3.21}$$

For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_{12}(\xi) &= -\frac{1}{8}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2) \\ &+ \frac{3}{2}m^2(a+bp-cp)\Big[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big] \\ &- \frac{3}{2}m^2\Big[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^2. \end{aligned} \tag{3.22}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_{13}(\xi) &= -\frac{1}{8}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2) \\ &+ \frac{3}{4}m^2(a+bp-cp)\Big[p(b-c) + a + \sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big] \\ &- \frac{3}{8}m^2\Big[p(b-c) + a + \sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^2. \end{aligned} \tag{3.23}$$

Here $\xi = \Delta m^2 n\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta} + n\frac{y^\theta}{\theta}$. Using formula $V(\xi) = \frac{n}{m}U(\xi)$ the unknown function $V(\xi)$ can be computed.

The solutions $u_2(x,y,t)$, $u_3(x,y,t)$ and $u_4(x,y,t)$ of the Eqs.(3.1)-(3.2) are simulated as traveling wave solutions for various values of the physical parameters in Fig.1-Fig.6. Figs.1, 2 show kink waves solutions, Figs.3 and 4 show solitary waves solutions, Figs.5, 6 show periodic waves solutions of Eqs.(3.1)-(3.2). Figs.1 and 2 are 3D and 2D plots of the traveling wave solution $u_2(x,1,t)$ and $u_2(x,1,1)$ in Eq.(3.12)for parameters $\alpha = 0.75,\ \beta = 1,\ \theta = 0.5$, $m = -0.05$, $n = 0.5$, $a = 1,\ b = 5,\ c = 5,\ c_1 = 1,\ c_2 = 2$ and $p = 0.1$. Figs.3 and 4 are 3D and 2D plots of the traveling wave solution $u_3(x,1,t)$ and $u_3(x,1,1)$ in Eq.(3.13) for $\alpha = 0.75,\ \beta = 1,\ \theta = 0.5$, $m = 0.5$, $n = 0.2$, $a = 0.1,\ b = 0.5,\ c = 0.02,\ c_1 = 1,\ c_2 = 1$ and $p = 2$. Figs.5 and 6 are 3D and 2D plots of the traveling wave solution $u_4(x,1,t)$ and $u_4(x,1,1)$ in Eq.(3.14) for $\alpha = 0.5,\ \beta = 1,\ \theta = 0.5$, $m = 0.5$, $n = 0.2$, $a = 0.05,\ b = 0.2,\ c = 0.6,\ c_1 = 1,\ c_2 = 1$ and $p = 1$. Note that the 3D graphs describe the behavior of u in space x and time t at fixed $y = 1$, which represents the change of amplitude and shape for each obtained traveling wave solutions. 2D graphs describe the behavior of u in space x at fixed time $t = 1$ and fixed $y = 1$. All graphics in figures are drawn by the aid of Mathematica 10.

4. Analytic solutions to the conformable space-time fractional Korteweg-de Vries (KdV) equation

Conformable space-time fractional KdV equation is given in the following form[25]

$$T_t^\alpha u + T_x^\beta T_x^\beta T_x^\beta u + 6uT_x^\beta u = 0, 0 < \alpha \leq 1,\ \ 0 < \beta \leq 1. \tag{4.1}$$

Let us consider the following transformation

$$u(x,t) = U(\xi),\ \ \xi = k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}, \tag{4.2}$$

where k, m are constants. Substituting (4.2) into Eq.(4.1) we obtain the following differential equations

$$kU' + m^3U''' + 6mUU' = 0. \tag{4.3}$$

Integrating of Eq.(4.3)with zero constant of integration, we have

$$kU + m^3U'' + 3mU^2 = 0. \tag{4.4}$$

Let us suppose that the solution of Eq.(4.4) can be expressed in the form Eq.(3.8). Substituting Eq.(3.8) into Eq.(4.4) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 2. Therefore, Eq.(3.8) reduces to Eq.(3.10). Substituting Eq.(3.10) into Eq.(4.4), collecting all the terms with the same power of $\tan(\frac{\phi}{2})$, we can obtain a set of algebraic equations for the unknowns A_0, A_1,A_2, B_1,B_2, k, m:

$$\begin{aligned}&6A_2^2m + 3A_2b^2m^3 - 6A_2bcm^3 + 3A_2c^2m^3 = 0,\\ &48pA_2^2m + ...\end{aligned}$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions:

Case 1: $A_0 = -\frac{1}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2), A_1 = 0, A_2 = 0, B_1 = m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3), B_2 = -\frac{1}{2}m^2(-b-c+2ap+bp^2-cp^2)^2, k = -\Delta m^3$:
For $b = c$ and $a = 0$,

$$U_1(\xi) = -2m^2b^2\Big[b\xi + c_1\Big]^{-2}. \tag{4.5}$$

For $b = c$ and $a \neq 0$,

$$U_2(\xi) = 2m^2(a^2p-ab)\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-1} - 2m^2(ap-b)^2\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-2}. \tag{4.6}$$

For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_3(\xi) &= -\frac{1}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)\\ &+ m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3)\\ &\cdot \Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^{-1}\\ &- \frac{1}{2}m^2(-b-c+2ap+bp^2-cp^2)^2\Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^{-2}. \end{aligned} \tag{4.7}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_4(\xi) &= -\frac{1}{2}(b-c)m^2(-b-c+2ap+bp^2-cp^2)\\ &+ m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3)\\ &\cdot \Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-1}\\ &- \frac{1}{2}m^2(-b-c+2ap+bp^2-cp^2)^2\Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-2}. \end{aligned} \tag{4.8}$$

Here $\xi = -\Delta m^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}$.

Case 2: $A_0 = -\frac{1}{6}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2), A_1 = 0, A_2 = 0, B_1 = m^2(-ab-ac+2a^2p-b^2p+c^2p+3abp^2-3acp^2+b^2p^3-2bcp^3+c^2p^3), B_2 = -\frac{1}{2}m^2(-b-c+2ap+bp^2-cp^2)^2, k = \Delta m^3$:
For $b = c$ and $a = 0$,

$$U_5(\xi) = -2m^2b^2\Big[b\xi + c_1\Big]^{-2}. \tag{4.9}$$

For $b = c$ and $a \neq 0$,

$$\begin{aligned} U_6(\xi) &= -\frac{1}{3}m^2a^2 + 2m^2(a^2p-ab)\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-1}\\ &- 2m^2(ap-b)^2\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-2}. \end{aligned} \tag{4.10}$$

For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_7(\xi) &= -\frac{1}{6}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2) \\ &+ m^2(-ab - ac + 2a^2p - b^2p + c^2p + 3abp^2 - 3acp^2 + b^2p^3 - 2bcp^3 + c^2p^3) \\ &\cdot \Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^{-1} \\ &- \frac{1}{2}m^2(-b - c + 2ap + bp^2 - cp^2)^2\Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^{-2}. \end{aligned} \tag{4.11}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_8(\xi) &= -\frac{1}{6}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2) \\ &+ m^2(-ab - ac + 2a^2p - b^2p + c^2p + 3abp^2 - 3acp^2 + b^2p^3 - 2bcp^3 + c^2p^3) \\ &\cdot \Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-1} \\ &- \frac{1}{2}m^2(-b - c + 2ap + bp^2 - cp^2)^2\Big[p + \frac{a}{b-c} \\ &+ \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-2}. \end{aligned} \tag{4.12}$$

Here $\xi = \Delta m^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}$.

Case 3: $A_0 = -\frac{1}{2}(b-c)m^2(-b - c + 2ap + bp^2 - cp^2), A_1 = (b-c)m^2(a + bp - cp), A_2 = -\frac{1}{2}(b-c)^2m^2, B_1 = 0, B_2 = 0, k = -\Delta m^3$:
For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_9(\xi) &= -\frac{1}{2}(b-c)m^2(-b - c + 2ap + bp^2 - cp^2) \\ &+ 2m^2(a + bp - cp)\Big[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big] \\ &- 2m^2\Big[\frac{c_1r_1\exp(r_1\xi) + c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi) + c_2\exp(r_2\xi)}\Big]^2 \end{aligned} \tag{4.13}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_{10}(\xi) &= -\frac{1}{2}(b-c)m^2(-b - c + 2ap + bp^2 - cp^2) \\ &+ (b-c)m^2(a + bp - cp)\Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big] \\ &+ -\frac{1}{2}(b-c)^2m^2\Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^2 \end{aligned} \tag{4.14}$$

Here $\xi = -\Delta m^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}$.

Case 4: $A_0 = -\frac{1}{6}m^2(2a^2 - b^2 + c^2 + 6abp - 6acp + 3b^2p^2 - 6bcp^2 + 3c^2p^2), A_1 = (b-c)m^2(a + bp - cp), A_2 = -\frac{1}{2}(b-c)^2m^2, B_1 = 0, B_2 = 0, k = \Delta m^3$:
For $b = c$ and $a \neq 0$,

$$U_{11}(\xi) = -\frac{1}{3}m^2a^2. \tag{4.15}$$

For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_{12}(\xi) &= -\frac{1}{6}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2) \\ &+ 2m^2(a+bp-cp)\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big] \\ &- 2m^2\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^2. \end{aligned} \tag{4.16}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_{13}(\xi) &= -\frac{1}{6}m^2(2a^2-b^2+c^2+6abp-6acp+3b^2p^2-6bcp^2+3c^2p^2) \\ &+ m^2(a+bp-cp)\Big[p(b-c)+a+\sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big] \\ &- \frac{1}{2}m^2\Big[p(b-c)+a+\sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^2. \end{aligned} \tag{4.17}$$

Here $\xi = \Delta m^3\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}$.

The solution $u_{13}(x,t)$ of the Eq.(4.1) is simulated in Fig.7-Fig.8 as periodic waves solutions. 3D plot of the obtained solution $u_{13}(x,t)$ is given for parameters $\alpha = 0.5,\ \beta = 1, m = 0.5, a = 0.5,\ b = 0.25,\ c = 1,\ c_1 = 1,\ c_2 = 3$ and $p = 1$ in Fig.7. Fig.8 demonstrate the same solution with 2D plot for $-40 \le x \le 40$ at $t = 1$.

5. Analytic solutions to the conformable space-time fractional Burger's equation

Conformable space-time fractional Burger's equation is given in the following form[31]

$$T_t^\alpha u + uT_x^\beta u - T_x^\beta T_x^\beta u = 0, 0 < \alpha \le 1,\ \ 0 < \beta \le 1. \tag{5.1}$$

Let us consider the following transformation

$$u(x,t) = U(\xi),\ \ \xi = k\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}, \tag{5.2}$$

where k, m are constants. Substituting Eq.(5.2) into Eq.(5.1) we obtain the following differential equations

$$kU' + mUU' - m^2U'' = 0. \tag{5.3}$$

Integrating of Eq.(5.3)with zero constant of integration, we have

$$kU + \frac{m}{2}U^2 - m^2U' = 0. \tag{5.4}$$

Let us suppose that the solution of Eq.(5.4) can be expressed in the form Eq.(3.8). Substituting Eq.(3.8) into Eq.(5.4) and then by balancing the highest order derivative term and nonlinear term in result equation, the value of N can be determined as 1. Therefore, Eq.(3.8) reduces to

$$U(\xi) = A_0 + A_1\Big[p + \tan\Big(\frac{\phi(\xi)}{2}\Big)\Big] + B_1\Big[p + \tan\Big(\frac{\phi(\xi)}{2}\Big)\Big]^{-1}. \tag{5.5}$$

Substituting Eq.(5.5) into Eq.(5.4), collecting all the terms with the same power of $\tan(\frac{\phi}{2})$, we can obtain a set of algebraic equations for the unknowns A_0, A_1,B_1, k, m:

$$\begin{aligned} & A_1^2m + A_1bm^2 - A_1cm^2 = 0, \\ & 2A_1k + 2A_0A_1m - 2aA_1m^2 + 4A_1^2mp + 2A_1bm^2p - 2A_1cm^2p = 0, \\ & 2A_0k + A_0^2m + 6A_1^2mp^2 + 2A_1B_1m + 6A_1kp - A_1bm^2 - A_1cm^2 - bB_1m^2 \\ + & B_1cm^2 - 4aA_1m^2p + A_1bm^2p^2 - A_1cm^2p^2 + 6A_0A_1mp = 0, \\ & 2B_1k + 4A_1^2mp^3 + 2A_0B_1m + 4A_0kp + 2aB_1m^2 + 6A_1kp^2 + 2A_0^2mp \\ + & 6A_0A_1mp^2 - 2A_1bm^2p - 2A_1cm^2p - 2aA_1m^2p^2 + 4A_1B_1mp = 0, \\ & B_1^2m + A_0^2mp^2 + A_1^2mp^4 + 2B_1kp + bB_1m^2 + B_1cm^2 + 2A_0kp^2 + 2A_1kp^3 \\ + & 2A_0A_1mp^3 + 2A_1B_1mp^2 - A_1bm^2p^2 - A_1cm^2p^2 + 2A_0B_1mp = 0. \end{aligned}$$

Solving the algebraic equations in the Mathematica, we obtain the following set of solutions:

Case 1: $A_0 = -am \pm \sqrt{m^2\Delta} - mp(b-c), A_1 = 0, B_1 = m(-b-c+2ap+bp^2-cp^2), k = \mp m\sqrt{m^2\Delta}$:
For $b = c$ and $a = 0$,

$$U_1(\xi) = -2bm\Big[b\xi + c_1\Big]^{-1}. \tag{5.6}$$

For $b = c$ and $a \neq 0$,

$$U_{2,3}(\xi) = -am \pm \sqrt{m^2a^2} + 2m(ap-b)\Big[p + c_1\exp(a\xi) - \frac{b}{a}\Big]^{-1}. \tag{5.7}$$

For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_{4,5}(\xi) &= -am \pm \sqrt{m^2\Delta} - mp(b-c) \\ &+ m(-b-c+2ap+bp^2-cp^2)\Big[\frac{2}{b-c}\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]^{-1}. \end{aligned} \tag{5.8}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_{6,7}(\xi) &= -am \pm \sqrt{m^2\Delta} - mp(b-c) \\ &+ m(-b-c+2ap+bp^2-cp^2)\Big[p + \frac{a}{b-c} + \frac{\sqrt{-\Delta}}{b-c}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]^{-1}. \end{aligned} \tag{5.9}$$

Here $\xi = \mp m\sqrt{m^2\Delta}\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}$.

Case 2: $A_0 = am \pm \sqrt{m^2\Delta} + mp(b-c), A_1 = -m(b-c), B_1 = 0, k = \mp m\sqrt{m^2\Delta}$:
For $\Delta > 0$ and $b \neq c$,

$$\begin{aligned} U_{8,9}(\xi) &= am \pm \sqrt{m^2\Delta} + mp(b-c) \\ &- 2m\Big[\frac{c_1r_1\exp(r_1\xi)+c_2r_2\exp(r_2\xi)}{c_1\exp(r_1\xi)+c_2\exp(r_2\xi)}\Big]. \end{aligned} \tag{5.10}$$

For $\Delta < 0$ and $b \neq c$,

$$\begin{aligned} U_{10,11}(\xi) &= am \pm \sqrt{m^2\Delta} + mp(b-c) \\ &- m\Big[p(b-c) + a + \sqrt{-\Delta}\frac{-c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\Big]. \end{aligned} \tag{5.11}$$

Here $\xi = \mp m\sqrt{m^2\Delta}\frac{t^\alpha}{\alpha} + m\frac{x^\beta}{\beta}$.

The solution $u_5(x,t)$ in Eq.(5.8) is simulated in Fig.9-Fig.10. These figures show kink wave solutions. Figs.9 and 10 are 3D and 2D plots of the traveling wave solution $u_5(x,t)$ and $u_5(x,1)$ in Eq.(5.8) for $\alpha = 0.75,\ \beta = 1,\ \theta = 0.5$, $m = 0.5$, $a = 2,\ b = 5,\ c = 2,\ c_1 = 1,\ c_2 = 1$ and $p = 0.2$.

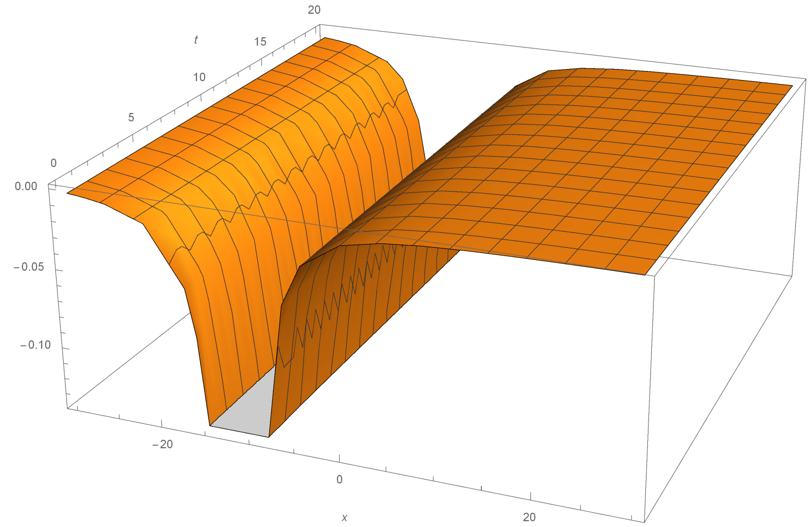

Figure 1. 3D plot of the obtained traveling wave solution $u_2(x, 1, t)$ in Eq.(3.12).

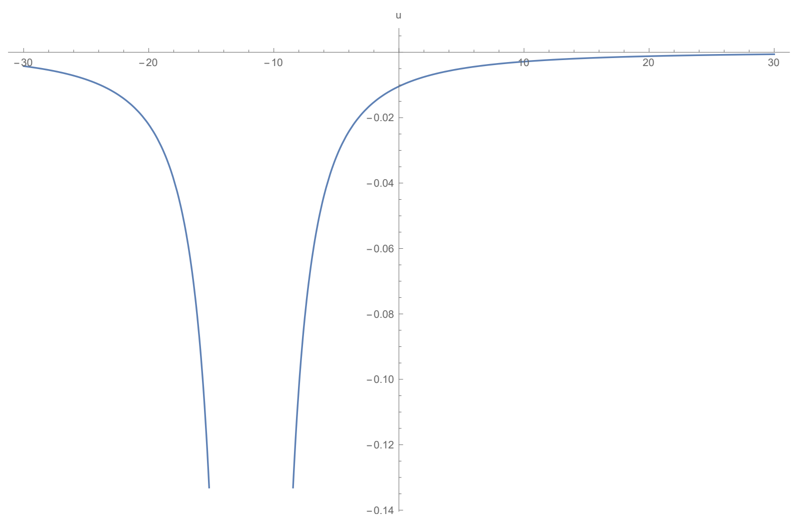

Figure 2. 2D plot of the obtained traveling wave solution $u_2(x, 1, 1)$ in Eq.(3.12).

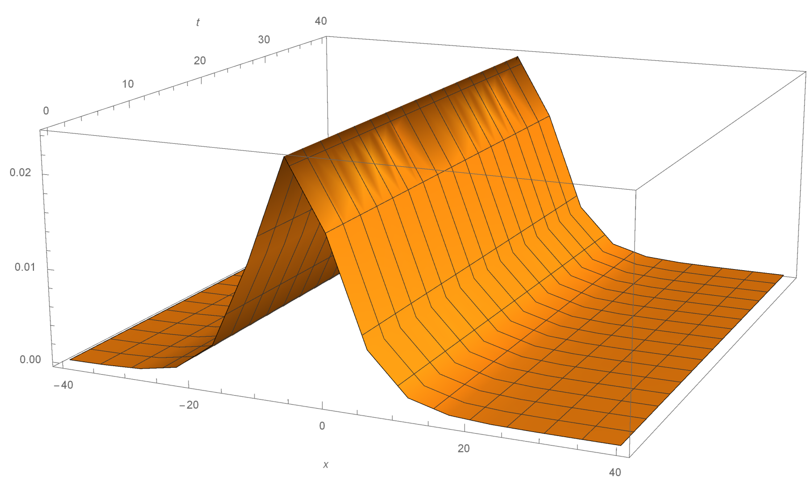

Figure 3. 3D plot of the obtained traveling wave solution $u_3(x, 1, t)$ in Eq.(3.13).

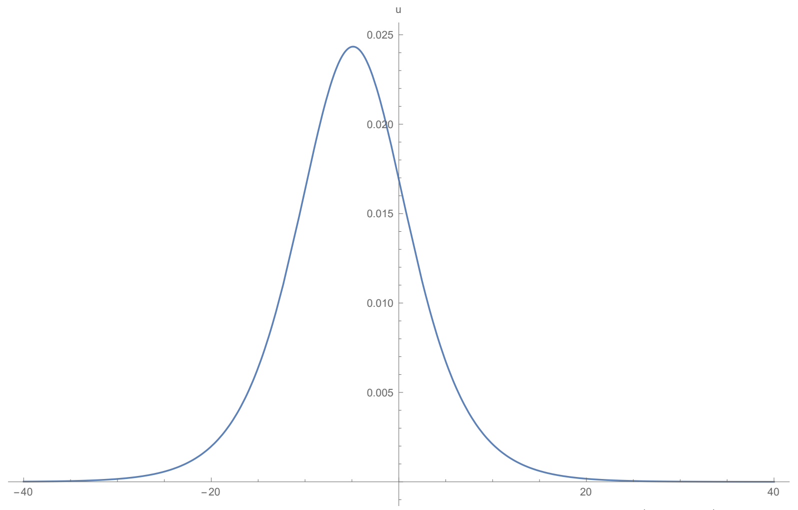

Figure 4. 2D plot of the obtained traveling wave solution $u_3(x, 1, 1)$ in Eq.(3.13).

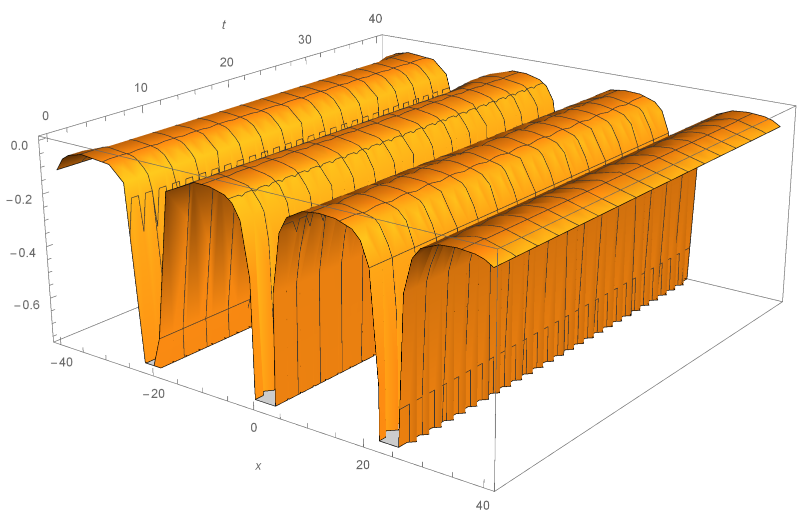

Figure 5. 3D plot of the obtained traveling wave solution $u_4(x, 1, t)$ in Eq.(3.14).

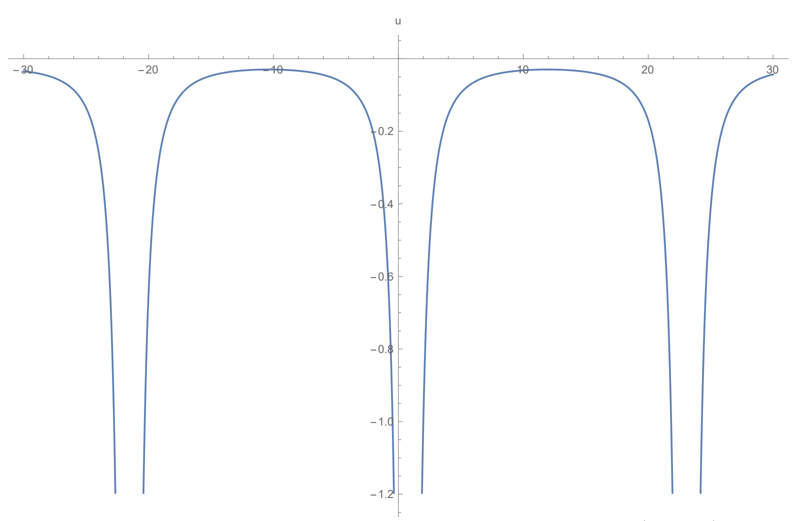

Figure 6. 2D plot of the obtained traveling wave solution $u_4(x, 1, 1)$ in Eq.(3.14).

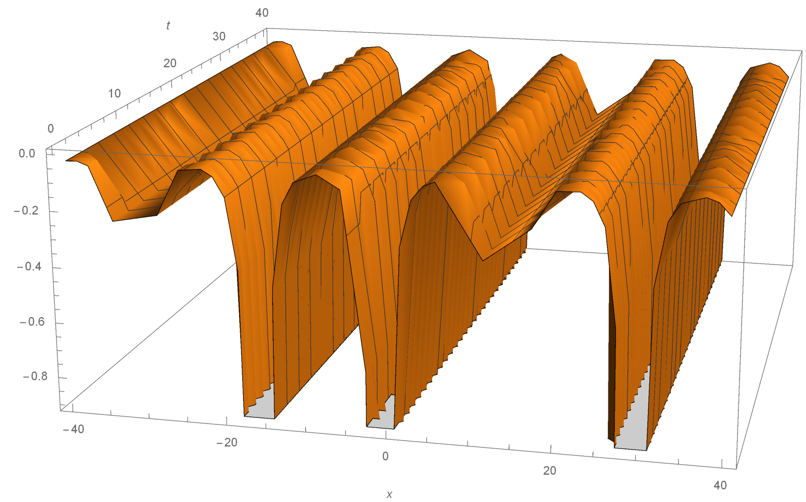

Figure 7. 3D plot of the obtained traveling wave solution $u_{13}(x,t)$ in Eq.(4.17).

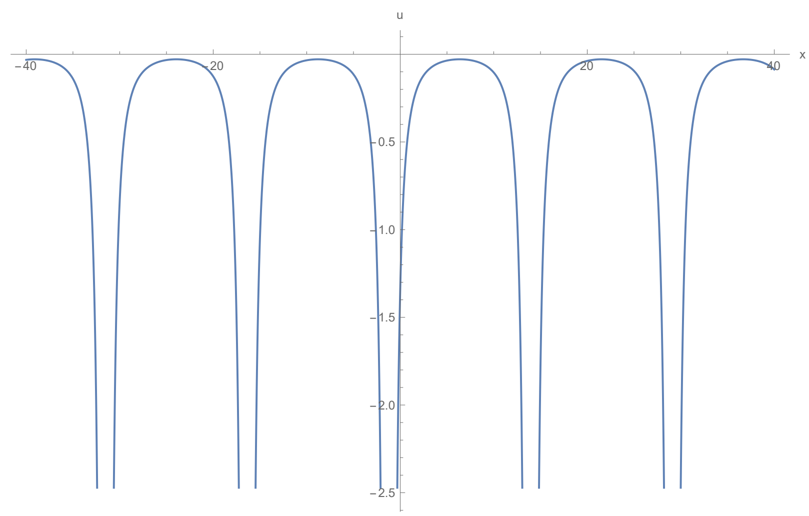

Figure 8. 2D plot of the obtained traveling wave solution $u_{13}(x,1)$ in Eq.(4.17).

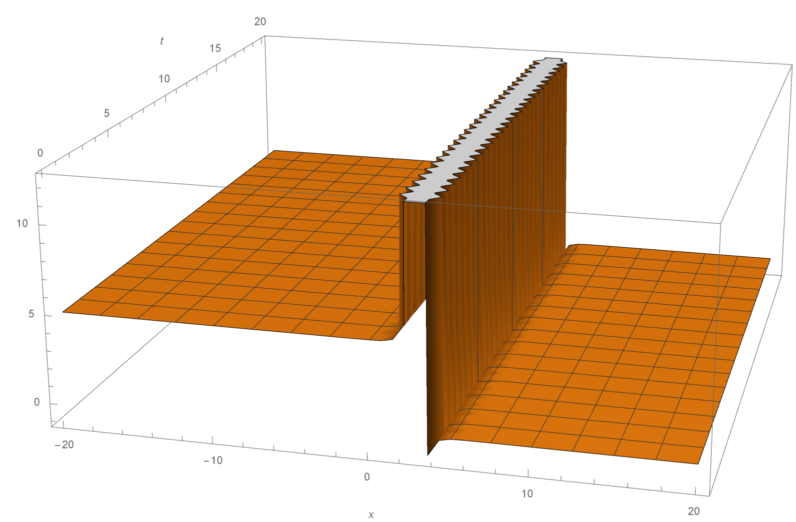

Figure 9. 3D plot of the obtained traveling wave solution $u_5(x,t)$ in Eq.(5.8).

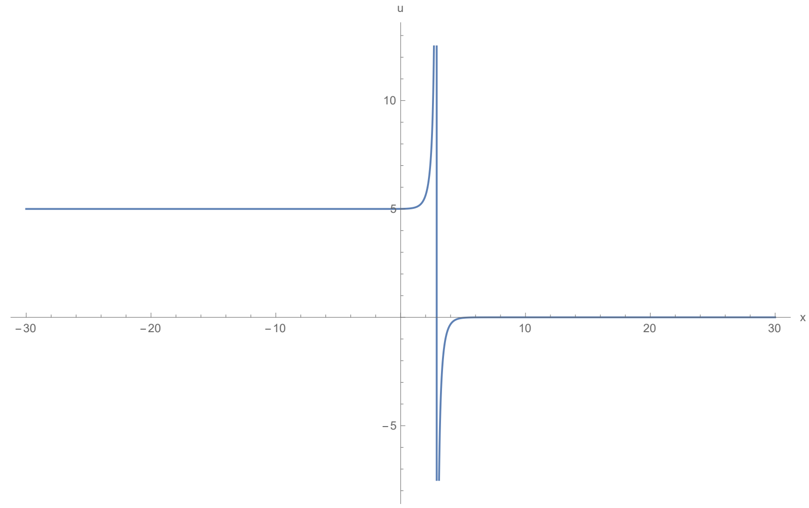

Figure 10. 2D plot of the obtained traveling wave solution $u_5(x, 1)$ in Eq.(5.8).

6. Conclusion

The fundamental goal of the paper has been to construct an approximation to the solution of the conformable space-time fractional (2+1)-dimensional breaking soliton, third-order KdV and Burger's equations by SITEM. The obtained solutions are traveling wave solutions of the conformable space-time fractional (2+1)-dimensional breaking soliton, third-order KdV and Burger's equations. These equations have been converted into its equivalent nonlinear ordinary differential equation by using fractional complex transformation. Solutions of the obtained nonlinear ordinary differential equation have been seek in the form of the summation of the function $p + \tan(\frac{\phi(\xi)}{2})$. Substituting the summation of the function $p + \tan(\frac{\phi(\xi)}{2})$ into the nonlinear ordinary differential equation and equalizing coefficients of the term with the same degree, nonlinear algebraic system is obtained. Solving the nonlinear algebraic system, we have the traveling wave solutions.

There are many types of traveling waves that are of particular interest in solitary wave theory. Three of these types are the solitary waves, the periodic waves and the kink waves. The solitary waves are asymptotically zero at large distances, the periodic waves have periodicity, the kink waves rise or descend from one asymptotic state to another. The 3D and 2D graphics of the obtained solutions have been presented in the paper. Figs.1- 2, Figs.9-10 show kink waves solutions, Figs.5- 6, Figs.7-8 have periodic waves solutions and Figs.3-4 give solitary waves solutions. This method changes the given difficult problems into simple one and solve easily by using MATLAB programming. The obtained solutions are new and have not been reported in former literature. The method can also be applied to other nonlinear fractional partial differential equations.

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Competing interests

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Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Geodesics of Twisted-Sasaki Metric

Abderrahim Zagane

Abstract
The main purpose of the paper is to investigate geodesics on the tangent bundle with respect to the twisted-Sasaki metric. We establish a necessary and sufficient conditions under which a curve be a geodesic respect. Afterward, we also construct some examples of geodesics.

Keywords: *Tangent bundle, Horizontal lift, Vertical lift, Twisted-Sasaki metric, Geodesics.*

AMS Subject Classification (2020): *Primary: 53C22; 53C25 ; Secondary: 53A45; 53C20; 58E20.*

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1. Introduction

The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors such as Sasaki, S. [18], Yano, K. and Ishihara, S. [20], Dombrowski, p. [6], Salimov, A., Gezer, A., and Cengiz, N. [2, 7, 14–16]. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. Musso, E. and Tricerri, F. have introduced the notion of Cheeger-Gromoll metric [13], Jian, W. and Yong, W. have introduced the notion of Rescaled Metric [9], Zagane, A. and Djaa, M. have introduced the notion of Mus-Sasaki metric [12, 21, 22].

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a new metric called twisted-Sasaki metric on the tangent bundle TM. This new natural metric will lead us to interesting results. Afterward we establish a necessary and sufficient conditions under which a curve be a geodesic with respect to the twisted-Sasaki metric.

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$

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defined by :

$$\mathcal{V}_{(x,u)} = Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad a^i \in \mathbb{R}\},$$
$$\mathcal{H}_{(x,u)} = \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}|_{(x,u)}; \quad a^i \in \mathbb{R}\},$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$
$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\}. \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial x^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}, \tag{2.3}$$

$$w^v = (\overline{w}^k + w^i u^j \Gamma_{ij}^k) \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}. \tag{2.4}$$

Lemma 2.1. *[20] Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$, we have following relations*

1. $[X^H, Y^H]_p = [X,Y]_p^H - (R_x(X,Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)_p^V$,
3. $[X^V, Y^V]_p = 0$,

where $p = (x,u) \in TM$.

3. Twisted-Sasaki metric

3.1 Twisted-Sasaki metric

Definition 3.1. Let (M,g) be a Riemannian manifold and $f : M \to [0, +\infty[$ be a positive smooth function on M. On the tangent bundle TM, we define a twisted-Sasaki metric noted g^f by

1 $g^f(X^H, Y^H)_{(x,u)} = g_x(X,Y)$,

2 $g^f(X^H, Y^V)_{(x,u)} = 0$,

3 $g^f(X^V, Y^V)_{(x,u)} = g_x(X,Y) + f(x) g_x(X,u) g_x(Y,u)$,

where $X, Y \in \Gamma(TM)$, $(x,u) \in TM$, f is called twisting function.

Remark 3.1. **1** If $f = 0$ g^f is the Sasaki metric [20],

2 $g^f(X^V, U^V) = \alpha g(X,u)$, $\alpha = 1 + fr^2$ and $r^2 = g(u,u)$,
where $X, U \in \Gamma(TM)$, $U_x = u \in T_xM$ and $(x,u) \in TM$.

In the following, we consider $f \neq 0$, $\alpha = 1 + fr^2$ and $r^2 = g(u,u) = \|u\|^2$ where $\|.\|$ denote the norm with respect to (M,g).

Lemma 3.1. *Let (M,g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function. For all $X, Y \in \Gamma(TM)$, $p = (x,u) \in TM$ and $u \in T_xM$, we have following relations*

1. $X^H(\rho(r^2))_p = 0$,

2. $X^V(\rho(r^2))_p = 2\rho'(r^2)g(X,u)_x$,

3. $X^H(g(Y,u))_p = g(\nabla_X Y, u)_x$,

4. $X^V(g(Y,u)_p = g(X,Y)_x$.

Proof. Locally, if $U : x \in M \to U_x = u = u^i \frac{\partial}{\partial x^i} \in T_x M$ be a local vector field constant on each fiber $T_x M$, then we have

$$\begin{aligned}
1.\quad X^H(\rho(r^2))_p &= \big[X^i \frac{\partial}{\partial x^i}(\rho(r^2)) - \Gamma_{ij}^k X^i y^j \frac{\partial}{\partial y^k}(\rho(r^2))\big]_p \\
&= \big[X^i \rho'(r^2) \frac{\partial}{\partial x^i}(r^2) - \rho'(r^2)\Gamma_{ij}^k X^i y^j \frac{\partial}{\partial y^k}(r^2)\big]_p \\
&= \rho'(r^2)\big[X^i \frac{\partial}{\partial x^i} g_{st} y^s y^t - \Gamma_{ij}^k X^i y^j \frac{\partial}{\partial y^k} g_{st} y^s y^t\big]_p \\
&= \rho'(r^2)\big[X g(U,U)_x - 2(\Gamma_{ij}^k X^i y^j g_{sk} y^s)_p\big] \\
&= \rho'(r^2)[X g(U,U)_x - 2g(U, \nabla_X U)_x] \\
&= 0. \\
2.\quad X^V(\rho(r^2))_p &= [X^i \rho'(r^2) \frac{\partial}{\partial y^i} g_{st} y^s y^t]_p \\
&= 2\rho'(r^2) X^i g_{it} u^t \\
&= 2\rho'(r^2) g(X,u)_x.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Lemma 3.2. *Let (M,g) be a Riemannian manifold, we have the following*

$$\begin{aligned}
1)\ X^H g^f(Y^H, Z^H) &= X g(Y,Z), \\
2)\ X^V g^f(Y^H, Z^H) &= 0, \\
3)\ X^H g^f(Y^V, Z^V) &= g^f((\nabla_X Y)^V, Z^V) + g^f(Y^V, (\nabla_X Z)^V) + X(f)g(Y,u)g(Z,u), \\
4)\ X^V g^f(Y^H, Z^H) &= f\big[g(X,Y)g(Z,u) + g(Y,u)g(X,Z)\big],
\end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$.

Proof. Lemma 3.2 follows from Definition 3.1 and Lemma 3.1. □

3.2 The Levi-Civita connection

We shall calculate the Levi-Civita connection ∇^f of TM with twisted-Sasaki metric g^f. This connection is characterized by the Koszul formula

$$\begin{aligned}
2g^f(\nabla^f_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) &= \widetilde{X} g^f(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y} g^f(\widetilde{Z}, \widetilde{X}) - \widetilde{Z} g^f(\widetilde{X}, \widetilde{Y}) \\
&\quad + g^f(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) + g^f(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) - g^f(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]). \qquad (3.1)
\end{aligned}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(TM)$.

Lemma 3.3. *Let (M,g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric.*

If ∇ (resp ∇^f) denotes the Levi-Civita connection of (M,g) (resp (TM,g^f)), then we have following relations

$$\begin{aligned}
&1)\ g^f(\nabla^f_{X^H}Y^H,Z^H) = g^f\big((\nabla_X Y)^H,Z^H\big),\\
&2)\ g^f(\nabla^f_{X^H}Y^H,Z^V) = -\frac{1}{2}g^f\big((R(X,Y)u)^V,Z^V\big),\\
&3)\ g^f(\nabla^f_{X^H}Y^V,Z^H) = \frac{1}{2}g^f\big((R(u,Y)X)^H,Z^H\big),\\
&4)\ g^f(\nabla^f_{X^H}Y^V,Z^V) = g^f\big((\nabla_X Y)^V,Z^V\big) + \frac{1}{2\alpha}X(f)g(Y,u)g^f(U^V,Z^V),\\
&5)\ g^f(\nabla^f_{X^V}Y^H,Z^H) = \frac{1}{2}g^f\big((R(u,X)Y)^H,Z^H\big),\\
&6)\ g^f(\nabla^f_{X^V}Y^H,Z^V) = \frac{1}{2\alpha}Y(f)g(X,u)g^f(U^V,Z^V),\\
&7)\ g^f(\nabla^f_{X^V}Y^V,Z^H) = \frac{-1}{2}g(X,u)g(Y,u)g^f((grad\, f)^H,Z^H),\\
&8)\ g^f(\nabla^f_{X^V}Y^V,Z^V) = \frac{f}{\alpha}g(X,Y)g^f(U^V,Z^V),
\end{aligned}$$

for all vector fields $X,Y,U\in\Gamma(TM)$, $U_x=u\in T_xM$ and $(x,u)\in TM$, where R denotes the curvature tensor of (M,g).

Proof. The proof of Lemma 3.3 follows directly from Kozul formula (3.1), Lemma 2.1, Definition 3.1 and Lemma 3.2. 1) The statement is obtained as follows

$$\begin{aligned}
2g^f(\nabla^f_{X^H}Y^H,Z^H) =& X^H g^f(Y^H,Z^H) + Y^H g^f(Z^H,X^H) - Z^H g^f(X^H,Y^H)\\
&+ g^f(Z^H,[X^H,Y^H]) + g^f(Y^H,[Z^H,X^H]) - g^f(X^H,[Y^H,Z^H])\\
=& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g^f(Z^H,[X,Y]^H)\\
&+ g^f(Y^H,[Z,X]^H) - g^f(X^H,[Y,Z]^H)\\
=& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y])\\
&+ g(Y,[Z,X]) - g(X,[Y,Z])\\
=& 2g(\nabla_X Y,Z)\\
=& 2g^f((\nabla_X Y)^H,Z^H).
\end{aligned}$$

2) Direct calculations give

$$\begin{aligned}
2g^f(\nabla^f_{X^H}Y^H,Z^V) =& X^H g^f(Y^H,Z^V) + Y^H g^f(Z^V,X^H) - Z^V g^f(X^H,Y^H)\\
&+ g^f(Z^V,[X^H,Y^H]) + g^f(Y^H,[Z^V,X^H]) - g^f(X^H,[Y^H,Z^V])\\
=& g^f(Z^V,[X^H,Y^H])\\
=& -g^f((R(X,Y)u)^V,Z^V).
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

As a direct consequence of Lemma 3.3, we get the following theorem.

Theorem 3.1. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If ∇ (resp ∇^f) denotes the Levi-Civita connection of (M,g) (resp (TM,g^f)), then we have:*

$$\begin{aligned}
1.\ (\nabla^f_{X^H}Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R_x(X,Y)u)^V,\\
2.\ (\nabla^f_{X^H}Y^V)_p &= (\nabla_X Y)^V_p + \frac{1}{2\alpha}X_x(f)g_x(Y,u)U^V_p + \frac{1}{2}(R_x(u,Y)X)^H,\\
3.\ (\nabla^f_{X^V}Y^H)_p &= \frac{1}{2\alpha}Y_x(f)g_x(X,u)U^V_p + \frac{1}{2}(R_x(u,X)Y)^H,\\
4.\ (\nabla^f_{X^V}Y^V)_p &= \frac{-1}{2}g_x(X,u)g_x(Y,u)(grad\, f)^H_p + \frac{f}{\alpha}g_x(X,Y)U^V_p,
\end{aligned}$$

for all vector fields $X,Y,U\in\Gamma(TM)$, $U_x=u\in T_xM$ and $p=(x,u)\in TM$, where R denotes the curvature tensor of (M,g).

4. Geodesics of twisted-Sasaki metric.

Lemma 4.1. *Let (M,g) be a Riemannian manifold. If $X,Y\in\Gamma(TM)$ are vector fields on M and $(x,u)\in TM$ such that $Y_x=u$, then we have*

$$d_xY(X_x)=X^H_{(x,u)}+(\nabla_XY)^V_{(x,u)}.$$

Proof. Let (U,x^i) be a local chart on M in $x\in M$ and $\pi^{-1}(U),x^i,y^j)$ be the induced chart on TM, if $X_x=X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x=Y^i(x)\frac{\partial}{\partial x^i}|_x=u$, then

$$d_xY(X_x)=X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)}+X^i(x)\frac{\partial Y^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,u)}.$$

Thus the horizontal part is given by:

$$\begin{aligned}(d_xY(X_x))^h &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)}-X^i(x)Y^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,u)}\\ &= X^H_{(x,u)},\end{aligned}$$

and the vertical part is given by:

$$\begin{aligned}(d_xY(X_x))^v &= \{X^i(x)\frac{\partial Y^k}{\partial x^i}(x)+X^i(x)Y^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,u)}\\ &= (\nabla_XY)^V_{(x,u)}.\end{aligned}$$

□

Let (M,g) be a Riemannian manifold and $x:I\to M$ be a curve on M. We define a curve $C:I\to TM$ by for all $t\in I, C(t)=(x(t),y(t))$ where $y(t)\in T_{x(t)}M$ i.e. $y(t)$ is a vector field along $x(t)$.

Definition 4.1. ([17, 20]) Let (M,g) be a Riemannian manifold. If $x(t)$ is a curve on M, the curve $C(t)=(x(t),\dot{x}(t))$ is called the natural lift of curve $x(t)$.

Definition 4.2. ([20]) Let (M,g) be a Riemannian manifold and ∇ denotes the Levi-Civita connection of (M,g). A curve $C(t)=(x(t),y(t))$ is said to be a horizontal lift of the cure $x(t)$ if and only if $\nabla_{\dot{x}}y=0$.

Lemma 4.2. *Let (M,g) be a Riemannian manifold and ∇ denotes the Levi-Civita connection of (M,g). If $x(t)$ be a curve on M and $C(t)=(x(t),y(t))$ be a curve on TM, then*

$$\dot{C}=\dot{x}^H+(\nabla_{\dot{x}}y)^V. \tag{4.1}$$

Proof. Locally, if $Y\in\Gamma(TM)$ is a vector field such $Y(x(t))=y(t)$, then we have

$$\dot{C}(t)=dC(t)=dY(x(t)).$$

Using Lemma 4.1, we obtain

$$\dot{C}(t)=dY(x(t))=\dot{x}^H+(\nabla_{\dot{x}}y)^V.$$

□

Theorem 4.1. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If ∇ (resp. ∇^f) denotes the Levi-Civita connection of (M,g) (resp. (TM,g^f)) and $C(t)=(x(t),y(t))$ is the cure on TM such $y(t)$ is a vector field along $x(t)$, then*

$$\begin{aligned}\nabla^f_{\dot{C}}\dot{C} &= (\nabla_{\dot{x}}\dot{x})^H+(R(y,\nabla_{\dot{x}}y)\dot{x})^H-\frac{1}{2}g(\nabla_{\dot{x}}y,y)^2(grad\,f)^H\\ &\quad+(\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V+\frac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y)+f\|\nabla_{\dot{x}}y\|^2\big]y^V.\end{aligned} \tag{4.2}$$

Proof. Using Lemma 4.2, we obtain

$$\begin{aligned}
\nabla^f_{\dot{C}}\dot{C} &= \nabla^f_{[\dot{x}^H + (\nabla_{\dot{x}}y)^V]}[\dot{x}^H + (\nabla_{\dot{x}}y)^V] \\
&= \nabla^f_{\dot{x}^H}\dot{x}^H + \nabla^f_{\dot{x}^H}(\nabla_{\dot{x}}y)^V + \nabla^f_{(\nabla_{\dot{x}}y)^V}\dot{x}^H + \nabla^f_{(\nabla_{\dot{x}}y)^V}(\nabla_{\dot{x}}y)^V \\
&= (\nabla_{\dot{x}}\dot{x})^H - \frac{1}{2}(R(\dot{x},\dot{x})y)^V + (\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V + \frac{1}{2\alpha}\dot{x}(f)g(\nabla_{\dot{x}}y,y)y^V \\
&\quad + \frac{1}{2}(R(y,\nabla_{\dot{x}}y)\dot{x})^H + \frac{1}{2\alpha}\dot{x}(f)g(\nabla_{\dot{x}}y,y)y^V + \frac{1}{2}(R(y,\nabla_{\dot{x}}y)\dot{x})^H \\
&\quad - \frac{1}{2}g(\nabla_{\dot{x}}y,y)g(\nabla_{\dot{x}}y,y)(grad\, f)^H + \frac{f}{\alpha}g(\nabla_{\dot{x}}y,\nabla_{\dot{x}}y)y^V \\
&= (\nabla_{\dot{x}}\dot{x})^H + (R(y,\nabla_{\dot{x}}y)\dot{x})^H - \frac{1}{2}g(\nabla_{\dot{x}}y,y)^2(grad\, f)^H \\
&\quad + (\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V + \frac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y) + f\|\nabla_{\dot{x}}y\|^2\big]y^V.
\end{aligned}$$

□

Theorem 4.2. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. If $C(t)=(x(t),y(t))$ is the cure on (TM,g^f) such $y(t)$ is a vector field along $x(t)$, then $C(t)$ is a geodesic on TM if and only if*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} = \frac{1}{2}g(\nabla_{\dot{x}}y,y)^2 grad\, f - R(y,\nabla_{\dot{x}}y)\dot{x} \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y = -\frac{1}{\alpha}\big[\dot{x}(f)g(\nabla_{\dot{x}}y,y) + f\|\nabla_{\dot{x}}y\|^2\big]y. \end{cases} \tag{4.3}$$

Proof. The statement is a direct consequence of Theorem 4.1 and definition of geodesic. □

Using Theorem 4.2, we deduce following.

Corollary 4.1. *Let (M,g) be a Riemannian manifold and (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. The natural lift $C(t)=(x(t),\dot{x}(t))$ of any geodesic $x(t)$ on (M,g) is a geodesic on (TM,g^f).*

Corollary 4.2. *Let (M,g) be a Riemannian manifold, (TM,g^f) its tangent bundle equipped with the twisted-Sasaki metric. The horizontal lift $C(t)=(x(t),y(t))$ of the curve $x(t)$ is a geodesic on (TM,g^f) if and only if $x(t)$ is a geodesic on (M,g).*

Remark 4.1. Let (M^m,g) be an m-dimensional Riemannian manifold. If $C(t)=(x(t),y(t))$ horizontal lift of the curve $x(t)$, locally we have

$$\begin{aligned}
\nabla_{\dot{x}}y = 0 \quad &\Leftrightarrow \quad \frac{dy^k}{dt} + \Gamma^k_{ij}y^i\frac{dx^j}{dt} = 0 \\
&\Leftrightarrow \quad y'(t) = A(t).y(t),
\end{aligned}$$

where, $A(t)=[a_{kj}]$, $a_{kj} = \sum_{i=1}^{m} -\Gamma^k_{ij}\frac{dx^j}{dt}$.

Remark 4.2.
Using the Remark 4.1, we can construct an infinity of examples of geodesics on (TM,g^f).

Example 4.1. We consider on $\mathbb{R}$ the metric $g=e^x dx^2$.
The Christoffel symbols of the Levi-cita connection associated with g are

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\Big(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\Big) = \frac{1}{2}.$$

1)The geodesics $x(t)$ such that $x(0)=a\in\mathbb{R}, x'(0)=v\in\mathbb{R}$ of g satisfies the equation

$$\frac{d^2x^s}{dt^2} + \sum_{i,j=1}^{n}\frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma^s_{ij} = 0 \Leftrightarrow x'' + \frac{1}{2}(x')^2 = 0.$$

Hence, we get $x'(t) = \dfrac{2v}{2+vt}$ and $x(t) = a + 2\ln(1+\dfrac{vt}{2})$.
Then, the natural lift

$$C_1(t) = (x(t), x'(t)) = \big(a + 2\ln(1+\frac{vt}{2}), \frac{2v}{2+vt}\big)$$

is a geodesic on $T\mathbb{R}$.
2) The curve $C_2(t) = (x(t), y(t))$ such $\nabla_{\dot{x}} y = 0$ satisfies the equation

$$\frac{dy^s}{dt} + y^i \Gamma^s_{ij} \frac{dx^j}{dt} = 0 \Leftrightarrow y' + \frac{1}{2} y x' = 0,$$

after that $y(t) = k.\exp(-\dfrac{v}{2+tv})$, $k \in \mathbb{R}$.
Then, the horizontal lift

$$C_2(t) = (x(t), y(t)) = \big(a + 2\ln(1+\frac{vt}{2}), k.\exp(-\frac{v}{2+tv})\big)$$

is a geodesic on $T\mathbb{R}$.

Corollary 4.3. *Let (M, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric. If f be a constant function, then the curve $C(t) = (x(t), y(t))$ is a geodesic on (TM, g^f) if and only if*

$$\begin{cases} \nabla_{\dot{x}} \dot{x} = -R(y, \nabla_{\dot{x}} y)\dot{x} \\ \\ \nabla_{\dot{x}} \nabla_{\dot{x}} y = -\dfrac{f}{\alpha} \|\nabla_{\dot{x}} y\|^2 y. \end{cases} \tag{4.4}$$

Proof. The statement is a direct consequence of Theorem 4.2. □

Theorem 4.3.
Let (M, g) be a Riemannian manifold, (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric and $x(t)$ be a geodesic on M. If $C(t) = (x(t), y(t))$ is a geodesic on TM such that $\|y(t)\|$ is not a constant, then f is a constant along the curve $x(t)$.

Proof. Let $x(t)$ be a geodesic on M , then $\nabla_{\dot{x}} \dot{x} = 0$. Using the first equation of formula (4.3), we obtain

$$\begin{aligned} g(\nabla_{\dot{x}} \dot{x}, \dot{x}) = 0 \quad &\Rightarrow \quad \frac{1}{2} g(\nabla_{\dot{x}} y, y)^2 g(grad\, f, \dot{x}) - g(R(y, \nabla_{\dot{x}} y)\dot{x}, \dot{x}) = 0 \\ &\Rightarrow \quad \frac{1}{2} g(\nabla_{\dot{x}} y, y)^2 \dot{x}(f) = 0 \\ &\Rightarrow \quad \dot{x}(f) = 0, \end{aligned}$$

as $\|y(t)\|$ is a constant $\Leftrightarrow \dot{x} g(y, y) = 0 \Leftrightarrow g(\nabla_{\dot{x}} y, y) = 0$. □

Corollary 4.4. *Let (M, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric. If $C(t) = (x(t), y(t))$ is the cure on (TM, g^f) such $\|y(t)\|$ is a constant, then the curve $C(t) = (x(t), y(t))$ is a geodesic on (TM, g^f) if and only if*

$$\begin{cases} \nabla_{\dot{x}} \dot{x} = -R(y, \nabla_{\dot{x}} y)\dot{x} \\ \\ \nabla_{\dot{x}} \nabla_{\dot{x}} y = -\dfrac{f}{\alpha} \|\nabla_{\dot{x}} y\|^2 y. \end{cases} \tag{4.5}$$

Proof. The statement is a direct consequence of Theorem 4.2, and we have
$\|y(t)\|$ is a constant $\Leftrightarrow \dot{x} g(y, y) = 0 \Leftrightarrow g(\nabla_{\dot{x}} y, y) = 0$. □

Theorem 4.4. *Let (M, g) be a flat Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric. Then, the cure $C(t) = (x(t), y(t))$ is a geodesic on TM if and only if*

$$\begin{cases} \nabla_{\dot{x}} \dot{x} = \dfrac{1}{2} g(\nabla_{\dot{x}} y, y)^2 grad\, f \\ \\ \nabla_{\dot{x}} \nabla_{\dot{x}} y = -\dfrac{1}{\alpha} \big[\dot{x}(f) g(\nabla_{\dot{x}} y, y) + f \|\nabla_{\dot{x}} y\|^2\big] y. \end{cases} \tag{4.6}$$

Proof. The statement is a direct consequence of Theorem 4.1. □

Corollary 4.5. *Let (M, g) be a flat Riemannian manifold and (TM, g^f) its tangent bundle equipped with the twisted-Sasaki metric. If f is a constant function, then the curve $C(t) = (x(t), y(t))$ is a geodesic on TM implies that $x(t)$ is a geodesic on M.*

Proof. The statement is a direct consequence of Theorem 4.4. □

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Structure Preserving Algorithm for the Logarithm of Symplectic Matrices

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Abstract
The current algorithms use either the full form or the Schur decomposition of the matrix in the inverse scaling and squaring method to compute the matrix logarithm. The inverse scaling and squaring method consists of two main calculations: taking a square root and evaluating the Padé approximants. In this work, we suggest using the structure preserving iteration as an alternative to Denman-Beavers iteration for taking a square root. Numerical experiments show that while using the structure preserving square root iteration in the inverse scaling and squaring method preserves the Hamiltonian structure of matrix logarithm, Denman-Beavers iteration and Schur decomposition cause a structure loss.

Keywords: *Matrix functions; matrix logarithm; symplectic matrix; Hamiltonian matrix; inverse scaling and squaring method*

AMS Subject Classification (2020): *Primary: 00A00 ; Secondary: 00B00; 00C00; 00D00; 00E00; 00F00.*

1. Introduction

The matrix logarithm is not only important for being the inverse function of the matrix exponential, it has also many applications. It has been used by engineers in the continuization process. They compute the logarithm of matrices in converting a discrete process into a continuous one [17, 18]. It has also applications to the stability of differential equations [14, 16]. The growing interest in computing structured matrix functions stems from the fact that predicting and preserving the structure of matrices can help us to explain the results physically and geometrically. The logarithm of structured matrices has applications in the control mechanical systems [4, 15] and in the optometry [6]. Structured matrix logarithm is also used for generalizing Bézier curves to non-Euclidean spaces. Crouch's algorithm, which generalizes De Casteljau algorithm to find polynomial splines on Riemannian manifolds, requires the computation of matrix logarithm when this manifold is a Lie group of matrices [4]. The theory of splines on Lie groups has applications in robotics path planning and air traffic control.

This paper focuses on computing the logarithm of a real symplectic matrix A with the spectrum $\rho(A)$ such that $\rho(A) \cap \mathbb{R}^- = \emptyset$, for which $W = \log A$ is Hamiltonian. In the computation of matrix logarithm, we use the inverse scaling and squaring method proposed by Kenney and Laub [12] and which is based on the relation

$$\log(A) = 2^k \log(A^{1/2^k}).$$

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There are two important calculations in the inverse scaling and squaring method. The first one is taking a square root of a symplectic matrix and the second is the Padé approximation [7]. Moreover, the inverse scaling and squaring method can be applied to A directly or it can be used with the Schur decomposition of A. However, we show that the latter case does not preserve the structure of the symplectic matrices. The aim of this paper is to propose using the structure preserving iteration for the square root in the inverse scaling and squaring method. We analyse this approach in terms of structure error, accuracy and computational cost. Numerical experiments assess the advantage of this approach and suggest using the structure preserving iteration in the inverse scaling and squaring method for the logarithm of symplectic matrices to preserve the Hamiltonian structure.

The paper is organized as follows. Section 2 begins with the definition of symplectic matrices and the matrix logarithm. We also review the inverse scaling and squaring method in this section. In Section 3, we propose our algorithm using the structure preserving square root iteration in the inverse scaling and squaring method. Section 4 presents the numerical findings and analyses our approach in terms of structure error, accuracy and cost. Finally, Section 5 gives a brief summary and critique of the findings .

2. Logarithm of symplectic matrices

2.1 Symplectic matrices

Let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. Consider a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ defined by any nonsingular matrix M, for $x, y \in \mathbb{K}^n$,

$$\langle x, y\rangle_{\mathrm{M}} = \begin{cases} x^T My, & \text{for real or complex bilinear forms,} \\ x^* My, & \text{for sesquilinear forms.} \end{cases}$$

For any matrix $A \in \mathbb{K}^{n\times n}$, there exists a unique operator $A^\star \in \mathbb{K}^{n\times n}$, called the adjoint of A with respect to the scalar product, such that

$$\langle Ax, y\rangle_{\mathrm{M}} = \langle x, A^\star y\rangle_{\mathrm{M}}, \quad \forall x, y \in \mathbb{K}^n.$$

$A^\star$ can be written explicitly

$$A^\star = \begin{cases} M^{-1}A^T M, & \text{for real or complex bilinear forms,} \\ M^{-1}A^* M, & \text{for sesquilinear forms.} \end{cases}$$

Symplectic matrices belong to the automorphism group which is characterized by the adjoint matrix as

$$\mathbb{G} = \{A \in \mathbb{K}^{n\times n} : \ A^\star = A^{-1}\}.$$

So for $M = J$ the matrix $A \in \mathbb{K}^{2n\times 2n}$ is symplectic if $A^T JA = J$. The permutation matrix $J \in \mathbb{R}^{2n\times 2n}$ is given as

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where I_n is the identity matrix of order n.

2.2 Inverse scaling and squaring method

For a given $A \in \mathbb{K}^{n\times n}$ a logarithm of A is any matrix W such that $e^W = A$. We assume that A has no eigenvalues on $\mathbb{R}^-$ so that the existence of a unique principal logarithm is assured as shown in the following theorem.

Theorem 2.1. [7, Thm. 1.31] *Let $A \in \mathbb{K}^{n\times n}$ have no eigenvalues on $\mathbb{R}^-$. There is a unique logarithm W of A all of whose eigenvalues lie in the strip $\{z : -\pi < \mathrm{Im}(z) < \pi\}$. We refer to W as the principal logarithm of A and write $W = \log A$. If A is real, then its principal logarithm is real.*

For $M = J$, a matrix W is called Hamiltonian if $W^\star = -W$, which implies $W^T = JWJ$. The function logarithm maps a symplectic matrix to a Hamiltonian matrix as proved in Theorem 2.2.

Theorem 2.2. [3, Thm. 2.1] *If $A \in \mathbb{K}^{n\times n}$ is a symplectic matrix and $\rho(A) \cap \mathbb{R}^- = \emptyset$, then $\log A = W$ is Hamiltonian.*

Proof. Since A is symplectic it satisfies

$$A^T = -JA^{-1}J = J^{-1}A^{-1}J.$$

Then,

$$\log(A^T) = J^{-1}\log A^{-1} J.$$

As stated in [7, Thm.1.13 and Thm.11.2], $\log(A^T) = (\log A)^T$ and $\log A^{-1} = -\log A$. Therefore,

$$\begin{aligned}(\log A)^T &= -J^{-1}\log A\, J \\ &= J\log A\, J.\end{aligned}$$

Hence, it shows that $\log A$ is Hamiltonian. □

In the computation of matrix logarithm we use the inverse scaling and squaring method. The basic idea of the inverse scaling and squaring method is to take the repetitive square root of A so the result is close to the identity matrix and then use the m-th order Padé approximant r_m. We summarise the method in the following algorithm.

Algorithm 2.3. [12] Given $A \in \mathbb{K}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ this algorithm employs the inverse scaling and squaring method to compute $W = \log A$.

1 Bring A close to an identity matrix by taking k repetitive square root of A
2 Decide the order of $r_m(A^{1/2^k} - I)$ by minimising the cost and maximising the accuracy
3 Approximate $\log(A^{1/2^k})$ by using $r_m(A^{1/2^k} - I) \approx \log(A^{1/2^k})$
4 Rescale to obtain $W \approx 2^k r_m(A^{1/2^k} - I)$

The question we need to deal with is whether the square root function and the Padé approximation preserve the structure or not. After taking the square root of any matrix in the automorphism group it stays in the automorphism group, which is proved in the following theorem.

Theorem 2.4. [13] *Let A be a matrix that has a principal square root $A^{1/2}$. If A is symplectic, then $A^{1/2}$ is symplectic.*

Proof. If A is symplectic, then $A^\star = A^{-1}$. We have the equality

$$(A^\star)^{1/2} = (A^{-1})^{1/2} \Rightarrow (A^{1/2})^\star = (A^{1/2})^{-1}.$$

□

Theorem 2.5. [9, Thm. 6.2] *Let $\mathbb{G}$ be any automorphism group and $A \in \mathbb{G}$. If A has no eigenvalues on R^-, then the iteration*

$$\begin{aligned}Y_{k+1} &= \frac{1}{2}(Y_k + Y_k^{-\star}) \qquad (2.1)\\ &= \frac{1}{2}(Y_k + M^{-1}Y_k^{-T}M)\end{aligned}$$

with starting matrix $Y_1 = \frac{1}{2}(I + A)$, is well defined and Y_k converges quadratically to $A^{1/2}$.

Since a symplectic matrix belongs to an automorphism group $\mathbb{G}$ the advantage of using iteration (2.1) is that it will preserve the symplectic structure and the result will lie in the group to approximately machine precision. With this iteration, we preserve the symplectic structure and when we evaluate the Padé approximant, we obtain the Hamiltonian structure which is proved in the following theorem.

Theorem 2.6. [5] *Let $r_m(X)$ be the diagonal Padé approximants to $\log(I+X)$, $m = 0, 1, \dots$. Let $W = \log A$ and $X = A-I$ with $\rho(X) < 1$. If A is symplectic, then $r_m(A - I)$ is Hamiltonian.*

Proof. We will use the homographic invariance [2, Thm. 1.5.2] under the argument transformations for this proof. Since $f(x) = \log x$ does not have a power series we take $f(x) = \log(1 + x)$. By using the equality $\log(1 + x) = -\log\left(1 + \dfrac{-x}{x+1}\right)$ and [2, Thm. 1.5.2] we get $r_m(x) = -r_m(-x/(x + 1))$.

For the matrix case this formula yields $r_m(X) = -r_m(-X(X + I)^{-1})$. If A is a symplectic matrix, then $A^{-1} = -JA^TJ$. Thus, we can write

$$\begin{aligned} r_m(A - I) &= -r_m(A^{-1} - I) \\ &= -r_m(-JA^TJ - I) \\ &= -r_m(-J(A^T - I)J).\end{aligned}$$

We obtain $r_m(X) = -r_m(-J(X^T)J) = Jr_m(X^T)J = Jr_m(X)^TJ = -J^{-1}r_m(X)^TJ$ which indicates that $r_m(X)$ is Hamiltonian. □

We state in the next theorem that the error in matrix Padé approximation is less than the error in scalar Padé approximation at the norm of the matrix, which is used in the inverse scaling and squaring method to decide the order of Padé approximation.

Theorem 2.7. [11] *For $\|A - I\| < 1$ and any subordinate matrix norm,*

$$\|r_m(A - I) - \log A\| \le |r_m(-\|A - I\|) - \log(1 - \|A - I\|)|. \tag{2.2}$$

Table 1. Maximal values θ_m of $\|A - I\|$ ensure that the bound $\|r_m(A - I) - \log A\|$ does not exceed $u = 2^{-53}$ [7, Table 11.1].

m	1	2	3	4	5	6	7	8	9
θ_m	1.10e-5	1.82e-3	1.62e-2	5.39e-2	1.14e-1	1.87e-1	2.64e-1	3.40e-1	4.11e-1

m	10	11	12	13	14	15	16	32	64
θ_m	4.75e-1	5.31e-1	5.81e-1	6.24e-1	6.62e-1	6.95e-1	7.24e-1	9.17e-1	9.78e-1

The maximal values θ_m of $\|A - I\|$ such that the error bound $\|r_m(A - I) - \log A\|$ does not exceed $u = 2^{-53} \approx 1.1 \times 10^{-16}$ are given in Table 1.

3. Using the structure preserving square root iteration

We adapt the algorithm [7, Alg. 11.10] by using the structure preserving iteration to take a square root. Iteration (2.1) is used to compute the square root of symplectic matrix and it exploits the symplecticity in each iteration. Let it_j be the number of iterations required in each square root. If M was a full matrix, then the operation count would include the inverse of M and the matrix multiplication. However, since $M = J$ is a permutation of diag(± 1) multiplication by J^{-1} is trivial and the cost of each iteration is one matrix inversion per iteration which is $2n^3$ flops. Evaluating the partial fraction form of the Padé approximation with the order m costs $\frac{8}{3}mn^3$ flops. In iteration (2.1) the number of iterations required to take a square root of A typically changes from 16 on the first iterations to 4 for the last few iterations. So the cost of taking a square root of symplectic matrix A at the last few iterations is $8n^3$ flops. It is worth only taking one more square root if it reduces the order of Padé approximation by at least 3. That decrease in the order of Padé approximation can only be obtained when $\|A^{1/2^s} - I\| > \theta_{16}$, where θ_{16} is the value given in Table 1. Taking a square root of A approximately reduces the distance of $A^{1/2^k}$ to the identity matrix by a half. This is easy to see since

$$(I - A^{1/2^{k+1}})(I + A^{1/2^{k+1}}) = I - A^{1/2^k},$$

and $A^{1/2^k} \to I$ as $k \to \infty$, it gives

$$\|I - A^{1/2^{k+1}}\| \approx \frac{1}{2}\|I - A^{1/2^k}\|. \tag{3.1}$$

When $\|A^{1/2^s} - I\| \le \theta_{16}$ is obtained, in order to compare the cost of the Padé approximation and the cost of the square root iteration, we check the inequality

$$\frac{8}{3}(m_1 - m_2)n^3 \le 2n^3 it_j \quad \Rightarrow \quad \frac{4}{3}(m_1 - m_2) \le it_j \tag{3.2}$$

by assuming the same number of iterations is required. In equation (3.2) m_1 and m_2 are the order of Padé approximants before and after the extra square root, respectively. Since the cost of taking 2 more extra square roots exceeds the cost of evaluating the Padé approximant we limit it by taking $p = 2$. That is, only one extra square root is taken if it is required. By using the cost checking (3.2), we present the modified algorithm of the inverse scaling and squaring method using the structure preserving square root iteration (2.1).

Algorithm 3.1. Given a symplectic matrix $A \in \mathbb{K}^{n \times n}$ with no eigenvalues on $\mathbb{R}^-$ this algorithm computes $W = \log A$ by the inverse scaling and squaring method. It uses the constants θ_m given in Table 1 for the Padé approximation and iteration (2.1) to take a square root of A. This algorithm is intended for IEEE double precision arithmetic.

1 $k = 0, p = 0$
2 while true
3 $\tau = \|A - I\|_1$
4   if $\tau < \theta_{16}$
5     $p = p + 1$
6     $m_1 = \min\{i : \tau \le \theta_i, \quad i = 3{:}\,16\}$
7     $m_2 = \min\{i : \tau/2 \le \theta_i, \quad i = 3{:}\,16\}$
8   if $4(m_1 - m_2)/3 \le it_j$ or $p = 2$, $m = m_1$, go to line 13, end
9   end
10   $A \leftarrow A^{1/2}$ by using iteration (2.1)
11   $k = k + 1$
12 end
13 Evaluate $Y = r_m(A - I)$
14 $W = 2^k Y$

Cost: Taking a square root costs $(\sum_{j=1}^{k} it_j)2n^3$ flops where k is the number of square root and evaluating the partial fraction form of the Padé approximation costs about $\frac{8}{3}mn^3$ flops. It is $(\sum_{j=1}^{k} it_j)2n^3 + \frac{8}{3}mn^3$ flops in total.

4. Numerical experiments

4.1 Error measure

The appropriate relative measure of departure from Hamiltonian structure can be computed by [1]

$$\mathrm{err}_H(W) = \frac{\|W^\star + W\|_2}{\|W\|_2}. \tag{4.1}$$

The relative error for the computed logarithm $\widehat{W}$ is given by

$$\mathrm{rel}_{\mathrm{err}}(\widehat{W}) = \frac{\|\widehat{W} - W\|_2}{\|W\|_2}, \tag{4.2}$$

where $W = \log A$ is the "exact" logarithm. In the numerical tests, W is computed at 100 digit precision and we measure the departure from the Hamiltonian structure and the relative error by equations (4.1) and (4.2), respectively.

4.2 Numerical tests

In the numerical experiments, we test using the structure preserving square root iteration (2.1) in the inverse scaling and squaring method in terms of structure loss, accuracy and computational cost. The experiments are carried out in MATLAB R2020b with $u = 1.1 \times 10^{-16}$. We form the full form of symplectic test matrices $A \in \mathbb{R}^{10\times 10}$ by using the function `rand_rsymp` from Jagger's MATLAB Toolbox [10].

Let $\widehat{W}$ represent the computed logarithm and $W = \log A$ represent the "exact" logarithm obtained by using MATLAB's Symbolic Math Toolbox, where A is diagonalized in 100 digit precision as $A = VDV^{-1}$ and computed by $\log A = V \log DV^{-1}$.

In the experiments, the relative error of the logarithm of the symplectic matrices $\mathrm{rel}_{\mathrm{err}}(\widehat{W})$, the structure error $\mathrm{err}_H(W)$ for the Hamiltonian structure and the computational cost are presented. All the results are plotted with the condition number of the symplectic matrices computed by $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$. The legend labels in figures are described as follows:

1. `full_preserve`: $\widehat{W}$ is computed by Algorithm 3.1 using the full form of A with the structure preserving square root iteration (2.1).

2. `full_DB`: $\widehat{W}$ is computed using the full form of A with the scaled product Denman-Beavers iteration [7, Alg. 11.10].

3. `Schur`: $\widehat{W}$ is computed using the Schur decomposition of A [7, Alg. 11.9] with a square root algorithm [7, Alg. 6.3].

Figure 1 provides the comparison of the structure error. We can see that while using the scaled product Denman-Beavers iteration with the full form of the matrix and Schur decomposition cause a structure loss for the ill-conditioned matrices, using Algorithm 3.1 preserves the Hamiltonian structure. We compare the accuracy of the approaches in Figure 2. As shown, while for the well-conditioned matrices, i.e., $\kappa_2(A) \approx 1$ the methods give the good estimate to the "exact" logarithm, we obtain less accurate results for the badly conditioned matrices. Since we obtain almost the same accuracy from three different approaches, we cannot say one is superior to other in terms of accuracy. Figure 3 reveals that using the iterative methods to compute the square root is computationally expensive. However, reducing the matrix to an upper triangular matrix with the Schur decomposition and using a square root algorithm [7, Alg. 6.3] is relatively cheaper than other approaches.

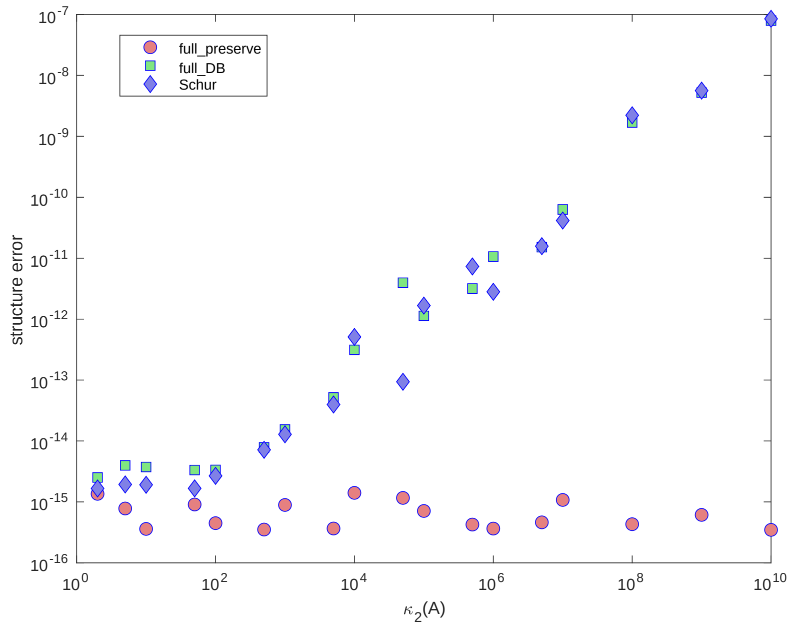

Figure 1. Comparison of the structure error for `full_preserve`, `full_DB` and `Schur`.

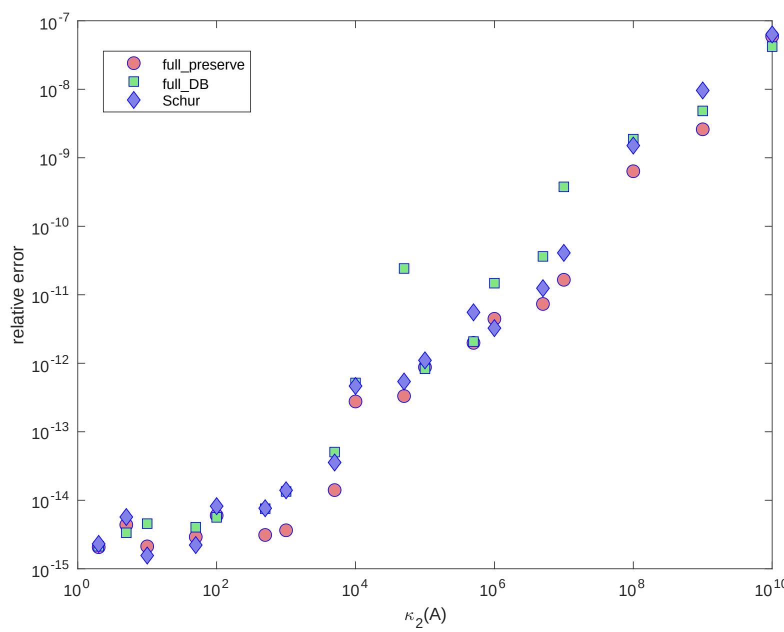

Figure 2. Comparison of the relative error for `full_preserve`, `full_DB` and `Schur`.

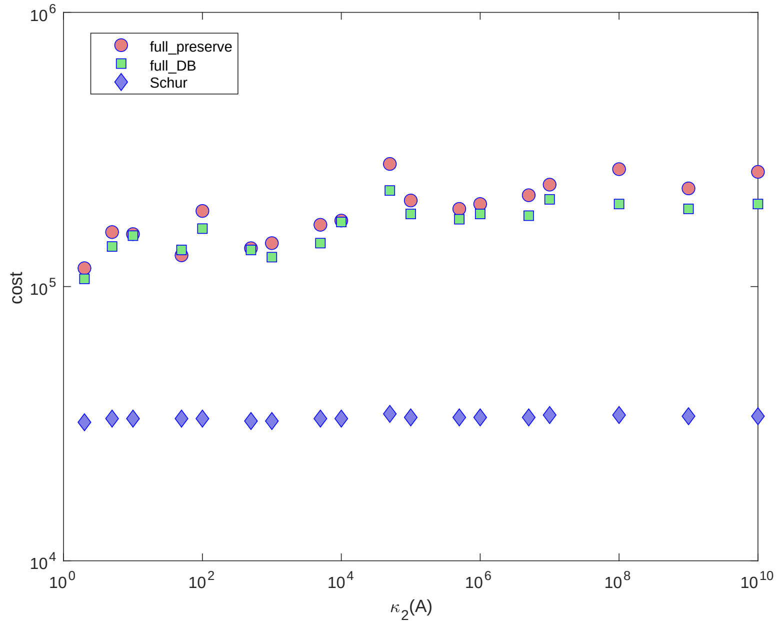

Figure 3. Comparison of the computational cost of `full_preserve`, `full_DB` and `Schur`.

5. Conclusion

We proposed using the structure preserving square root iteration in the inverse scaling and squaring method to compute the logarithm of symplectic matrices and we compared it with the algorithms using either the full form of the matrix with the scaled product Denman-Beavers iteration or the Schur decomposition in terms of the structure loss, the accuracy and the computational cost. The findings show that the best structure is obtained by using the structure preserving square root iteration (2.1) instead of using scaled product Denman-Beavers iteration or the Schur decomposition. While there is not much big difference in their accuracy, however, using the iteration (2.1) with the full form of the symplectic matrices is computationally expensive.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Decay Estimate for the Time-Delayed Fourth-Order Wave Equations

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Abstract
The objective of this article is to analyze the stability of solutions for the following fourth- order nonlinear wave equations with an internal delay term:

$$u_{tt} + \Delta^2 u + u + \sigma_1(t)|u_t(x,t)|^{2m-2}u_t(x,t) + \sigma_2(t)|u_t(x,t-\tau)|^{2m-2}u_t(x,t-\tau) = 0.$$

We obtain appropriate conditions on $\sigma_1(t)$ and $\sigma_2(t)$ for the decay properties of the solutions. The multiplier technique and nonlinear integral inequalities are used in the proof.

Keywords: *Energy decay rate; Fourth order wave; Asymptotic behavior.*

AMS Subject Classification (2020): *Primary: 35B30; Secondary: 35B35; 35G25.*

1. Introduction

In this study, we examine the following initial boundary value problem for the nonlinear fourth-order time-delayed wave equations:

$$\begin{aligned} &u_{tt} + \Delta^2 u + u + \sigma_1(t)|u_t(x,t)|^{2m-2}u_t(x,t) \\ &\qquad + \sigma_2(t)|u_t(x,t-\tau)|^{2m-2}u_t(x,t-\tau) = 0, \text{in} \quad \Omega \times (0,\infty), \quad (1.1) \\ &u(x,t) = \frac{\partial u}{\partial \nu} = 0, \quad \partial\Omega \times (0,\infty), \quad (1.2) \\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \quad (1.3) \\ &u_t(x,t-\tau) = f(x,t-\tau) \quad \text{in} \quad \Omega \times (0,\tau), \quad (1.4) \end{aligned}$$

where $m > 1$ is a constant; σ_1 and σ_2 are positive functions; $\Omega \subset \mathbb{R}^n (n > 4)$ is a bounded domain; $\partial\Omega$ is a smooth boundary of Ω; τ is the time delay and initial function (u_0, u_1, f_0) in a suitable space.
Without the delay term ($\sigma_2 = 0$), the behaviors of the solutions of the fourth-order wave equations have been broadly analyzed in the literature (see [5],[7], [8], [14] and the references therein). Moreover, there are fewer results

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on the stability analysis of the solutions of time- delayed wave equations (see [1], [6], [11], [13] and the references therein). However, there is no detection of the decay rate of the nonlinear fourth-order wave equations with a delay term.
In [2], Benaissa, Benaissa and Messaoudi considered a nonlinear wave equation,

$$u_{tt} - \Delta u + \mu_1 \sigma(t) g_1 (u_t(x,t)) + \mu_2 \sigma(t) g_2 (u_t(x, t - \tau(t))) = 0,$$

where $\tau(t) > 0$ is a time dependent delay term, and μ_1 and μ_2 are positive constants. The existence and decay estimates for the solutions of the initial boundary value problem were proven.
In [3], Benaissa and Messaoudi analyzed the following nonlinear wave equation:

$$u_{tt} - \Delta u + \mu_1 \sigma(t) u_t(x,t) + \mu_2 \sigma(t) u_t(x, t - \tau(t)) + \theta(t) h (\nabla u(x,t)) = 0,$$

and the decay properties of the solutions were determined.
In [12], Ning, Shen and Zhao examined a wave equation of the form

$$u_{tt} + \mathcal{A}u + a(x) [\mu_1 u_t(x,t) + \mu_2 u_t(x, t - \tau)] = 0,$$

where $\mathcal{A}u = -divA(x) = (a_{ij}(x))$ is a symmetric matrix, $a(x)$ is a positive bounded function, and μ_1 and μ_2 are positive constants. The well-posedness of the system and exponential decay of the solutions were established.
In [4], Benaissa, Benguessoum and Messaoudi analyzed the following linear wave equation:

$$u_{tt} - \Delta u + \mu_1(t) u_t(x,t) + \mu_2(t) u_t(x, t - \tau(t)) = 0,$$

under assumptions about $\mu_1(t)$ and $\mu_2(t)$, the existence and decay properties of the solutions of the above equation with the initial boundary values were investigated.
In [9], Li and Chai examined the following damped plate equation:

$$u_{tt} + \mathcal{A}^2 u + b(x) [\mu_1 \beta (u_t(x,t)) + \mu_2 \phi (u_t(x, t - \tau))] = 0,$$

where $\mathcal{A}u = div (A(x) \nabla u)$. The existence of solutions was proven, and the decay rate estimates for the energy were obtained.
The main goal of the present study is to deduce the decay properties of the solutions of the time-delayed fourth-order problem (1.1)-(1.4). To the best of our insight, this problem has not been considered in this respect.
The proof of our principle result is founded on the following Lemma which was demonstrated by Martinez in ([10]).

Lemma 1.1. *([10]) Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non increasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad and \quad \phi(t) \to \infty \quad as \quad t \to \infty. \tag{1.5}$$

Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E(t)^{1+\sigma} \phi^{'}(t) dt \leq \frac{1}{\omega} E(0)^\sigma E(S), \quad 0 \leq S < \infty, \tag{1.6}$$

then $E(t)$ has the following decay properties:

$$if \quad \sigma = 0, \quad then \quad E(t) \leq E(0) e^{1 - \omega\phi(t)}, \forall t \geq 0, \tag{1.7}$$

$$if \quad \sigma > 0, \quad then \quad E(t) \leq E(0) \Big(\frac{1 + \sigma}{1 + \omega\sigma\phi(t)} \Big)^{\frac{1}{\sigma}}, \forall t \geq 0. \tag{1.8}$$

2. asymptotic behavior

In the present section, we aim to constitute a decay property of the solutions of the problem (1.1)-(1.4) using multiplier method and integral inequalities. We use the following variable as in [11].

$$z(x, \rho, t) = u_t(x, t - \tau\rho). \tag{2.1}$$

Hence, we change problem (1.1)-(1.4) to the following problem:

$$u_{tt} + \Delta^2 u + u + \sigma_1(t)|u_t(x,t)|^{2m-2} u_t(x,t) + \sigma_2(t)|z(x,1,t)|^{2m-2} z(x,1,t) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.2}$$

$$\tau z_t + z_\rho = 0 \quad \text{in} \quad \Omega \times (0,1) \times (0,\infty), \tag{2.3}$$

$$u(x,t) = \frac{\partial u}{\partial \nu} = 0 \quad \partial\Omega \times (0,\infty), \tag{2.4}$$

$$z(x,0,t) = u_t(x,t) \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.5}$$

$$z(x,\rho,0) = u_t(x,-\tau\rho) = f(x,-\tau\rho) \quad \text{in} \quad \Omega \times (0,1). \tag{2.6}$$

Lemma 2.1. *Assume that (u, z) is a solution of the new problem (2.2)-(2.6) and $\sigma_1(t)$, $\sigma_2(t)$ satisfy the following properties*
A1: $\sigma_1(t) : \mathbb{R}^+ \to (0,\infty)$ is a non-increasing function on $C^1(\mathbb{R}^+)$ such that

$$|\sigma_1(t)| \le M.$$

A2: $\sigma_2(t) : \mathbb{R}^+ \to (0,\infty)$ is a function on $C^1(\mathbb{R}^+)$ such that

$$|\sigma_2(t)| < M_2 \sigma_1(t),$$

where M and M_2 are positive constants. Then, the positive energy of problem (2.2)-(2.6) satisfies the following inequality:

$$\frac{dE(t)}{dt} \le -\sigma_1(t)(1-\theta_1)\int_\Omega |u_t|^{2m}\,dx - \sigma_1(t)\theta_2 \int_\Omega |z(x,1,t)|^{2m}\,dx,$$

where

$$\theta_1 = \frac{M_2 + \tilde{M}}{2m}, \theta_2 = \frac{\tilde{M} + (2m-1)M_2}{2m},$$

$$(2m-1)M_2 < \tilde{M} < 2m - M_2. \tag{2.7}$$

Proof. By multiplying equation (2.2) by u_t and integrating over Ω we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|u\|^2\right) = -\int_\Omega \sigma_1(t) u_t |u_t|^{2m-1} dx - \int_\Omega \sigma_2(t) u_t |z(x,1,t)|^{2m-2} z(x,1,t) dx. \tag{2.8}$$

Furthermore, multiplying equation (2.3) by function $\gamma_1(t)\,|z(x,\rho,t)|^{2m-2}\, z(x,\rho,t)$ and integrating over $(0,1)\times\Omega$ we derive

$$\frac{d}{dt}\left(\frac{\tau}{2m}\int_\Omega \int_0^1 \gamma_1(t)\,|z(x,\rho,t)|^{2m}\,d\rho dx\right) = \frac{\tau}{2m}\int_\Omega\int_0^1 \gamma_1'(t)\,|z(x,\rho,t)|^{2m}\,d\rho dx - \frac{1}{2m}\int_\Omega \gamma_1(t)\left(|z(x,1,t)|^{2m} - |z(x,0,t)|^{2m}\right)dx, \tag{2.9}$$

where

$$\gamma_1(t) = \tilde{M}\sigma_1(t), \tag{2.10}$$

$$\gamma_1'(t) < 0. \tag{2.11}$$

We define

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|u\|^2 + \frac{\tau}{2m}\int_\Omega \gamma_1(t)\int_0^1 |z(x,\rho,t)|^{2m}\,d\rho dx. \tag{2.12}$$

Hence, by combining equations (2.8) and (2.9), we have

$$\frac{d}{dt}E(t) = -\int_\Omega \sigma_1(t) u_t |u_t|^{2m-1} dx - \int_\Omega \sigma_2(t) u_t |z(x,1,t)|^{2m-1} dx + \frac{\tau}{2m}\int_\Omega\int_0^1 \gamma_1'(t)\,|z(x,\rho,t)|^{2m}\,d\rho dx - \frac{1}{2m}\int_\Omega \gamma_1(t)\left(|z(x,1,t)|^{2m} - |u_t|^{2m}\right)dx,$$

Then, using the definition of $\gamma_1(t)$ (2.10), condition (2.5) and property (2.11) we get

$$\frac{d}{dt}E(t) \leq -\int_\Omega \sigma_1(t)u_t|u_t|^{2m-1}dx - \int_\Omega \sigma_2(t)u_t|z(x,1,t)|^{2m-2}z(x,1,t)dx \\ - \frac{1}{2m}\int_\Omega \tilde{M}\sigma_1(t)\left(|z(x,1,t)|^{2m} - |u_t|^{2m}\right)dx.$$

To estimate the second integral of the above equation, we use Young Inequality to obtain

$$\int_\Omega \sigma_2(t)u_t|z(x,1,t)|^{2m-1}dx \leq \frac{1}{2m}\int_\Omega |\sigma_2(t)|\,|u_t|^{2m}\,dx + \frac{2m-1}{2m}\int_\Omega |\sigma_2(t)|\,|z|^{2m}\,dx. \tag{2.13}$$

Moreover, we have

$$|\sigma_2(t)| < M_2\sigma_1(t). \tag{2.14}$$

Thus, we deduce that inequality

$$\frac{d}{dt}E(t) \leq -\sigma_1(t)\left[1-\theta_1\right]\int_\Omega |u_t|^{2m}\,dx - \sigma_1(t)\theta_2\int_\Omega |z|^{2m}\,dx, \tag{2.15}$$

where

$$\theta_1 = \frac{\tilde{M}+M_2}{2m}, \tag{2.16}$$

and

$$\theta_2 = \frac{\tilde{M}+(2m-1)M_2}{2m}. \tag{2.17}$$

Recalling the property of $\tilde{M}$ (2.7) we have

$$\frac{d}{dt}E(t) \leq 0. \tag{2.18}$$

Hence, the positive energy is non-increasing.

□

Now, we are ready to obtain the decay rate of the solutions of problem (2.2)-(2.6).

Theorem 2.1. *Assume that A1 and A2 hold. Then, there exist positive constants q and ω such that the energy of problem (2.2)-(2.6) satisfies the following property*

$$E(t) \leq E(0)\left(\frac{1+q}{1+\omega q\int_0^t \sigma_1(s)ds}\right)^{\frac{1}{q}}, \quad \forall t > 0,$$

where

$$q > \frac{2m-1}{2},$$

and

$$\omega^{-1} = \frac{2e^{2\tau}}{3}\max\Bigg\{2M, \frac{2}{(q+1)(1-\theta_1)}, \left(\frac{4qe^{2\tau}}{E(0)}\right)^q\left(\frac{1-\theta_1}{2(q+1)}\right)^{q+1}, \frac{qM}{M+1} \\ , \frac{(2m-1)E(0)^{\frac{m-1}{2m-1}}}{2m}\left(\frac{2^{m+2}e^{2\tau}(M^2c_1^2)^m}{m}\right)^{\frac{1}{2m-1}}\left(\frac{1}{1-\theta_1}+\frac{M_2^{\frac{2}{2m-1}}}{\theta_2}\right) \\ , \frac{M\tilde{M}}{2m(q+1)\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)}\Bigg\}.$$

Proof. To establish a decay rate estimate of the positive energy; by multiplying the equation (2.2) by function $\phi'(t)E^q(t)u(x,t)$ and integrating over $(S,T)\times\Omega$ we deduce the following equation,

$$0=\int_S^T\int_\Omega \phi' E^q u\Big[u_{tt}+\Delta^2 u+u+\sigma_1(t)|u_t|^{2m-2}u_t+\sigma_2(t)|z(x,1,t)|^{2m-2}z(x,1,t)\Big]dxdt,$$

where $\phi(t)$ satisfies the hypothesis of Lemma 1.1. Using the boundary conditions, we have

$$\begin{aligned}0=&\int_S^T\int_\Omega\left[\frac{d}{dt}\left(\phi' E^q u u_t\right)-\left(\phi' E^q\right)' u u_t-\phi' E^q u_t^2\right]dxdt\\&+\int_S^T\phi' E^q\int_\Omega(\Delta u)^2dxdt+\int_S^T\phi' E^q\sigma_1(t)\int_\Omega u\left|u_t\right|^{2m-2}u_t dxdt\\&+\int_S^T\phi' E^q\int_\Omega u^2dxdt+\int_S^T\phi' E^q\sigma_2(t)\int_\Omega u\left|z\right|^{2m-2}z(x,1,t)dxdt.\end{aligned}\tag{2.19}$$

Furthermore, by multiplying equation (2.3) by $\phi'(t)E^q(t)\gamma_1(t)e^{-2\tau\rho}\left|z\right|^{2m-2}z(x,\rho,t)$ and integrating over $(S,T)\times\Omega\times(0,1)$, we obtain

$$0=\int_S^T\phi'(t)E^q(t)\int_\Omega\int_0^1\gamma_1(t)e^{-2\tau\rho}\left|z\right|^{2m-2}z(x,\rho,t)\left(\tau z_t+z_\rho\right)d\rho dxdt,$$

with the boundary conditions, we get

$$\begin{aligned}0=&\frac{\tau}{2m}\int_\Omega\int_0^1\phi' E^q\gamma_1 e^{-2\tau\rho}\left|z\right|^{2m}\Big|_S^T d\rho dx+\frac{1}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega e^{-2\tau\rho}\left|z\right|^{2m}\Big|_0^1 dxdt\\&+\frac{\tau}{m}\int_S^T\phi' E^q\gamma_1\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt\\&-\frac{\tau}{2m}\int_S^T\left(\phi' E^q\gamma_1\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt.\end{aligned}\tag{2.20}$$

By taking the sum of equations (2.19) and (2.20), we have

$$\begin{aligned}0=&\int_\Omega\phi' E^q u u_t\Big|_S^T-\int_S^T\int_\Omega\left[\left(\phi' E^q\right)' u u_t-\phi' E^q u_t^2\right]dxdt\\&+\int_S^T\phi' E^q\int_\Omega(\Delta u)^2dxdt+\int_S^T\phi' E^q\int_\Omega u^2dxdt\\&+\int_S^T\phi' E^q\sigma_1(t)\int_\Omega u\left|u_t\right|^{2m-2}u_t dxdt\\&+\int_S^T\phi' E^q\sigma_2(t)\int_\Omega u\left|z\right|^{2m-2}z(x,1,t)dxdt\\&+\frac{\tau}{2m}\int_\Omega\int_0^1\phi' E^q\gamma_1 e^{-2\tau\rho}\left|z\right|^{2m}\Big|_S^T d\rho dx\\&-\frac{\tau}{2m}\int_S^T\left(\phi' E^q\gamma_1\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt\\&+\frac{1}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega e^{-2\tau\rho}\left|z\right|^{2m}\Big|_0^1 dxdt\\&+\frac{\tau}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt.\end{aligned}\tag{2.21}$$

Because of the definition of $E(t)$, we get the following inequality,

$$\begin{aligned}&\frac{\tau}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt>\\&\qquad\int_S^T\phi' E^q e^{-2\tau}\left(2E(t)-\left\|u_t\right\|^2-\left\|\Delta u\right\|^2-\left\|u\right\|^2\right)dt.\end{aligned}$$

By combining the last inequality with equation (2.21), we obtain

$$
\begin{aligned}
2\int_S^T \phi' E^{q+1} e^{-2\tau} dt < & -\left[\phi' E^q \int_\Omega u u_t dx\right]\Big|_S^T + 2\int_S^T \phi' E^q \int_\Omega u_t^2 dx dt \\
& - \int_S^T \phi' E^q \sigma_1(t) \int_\Omega u \left|u_t\right|^{2m-1} dx dt \\
& - \int_S^T \phi' E^q \sigma_2(t) \int_\Omega u \left|z\right|^{2m-2} z(x,1,t) dx dt \\
& + \int_S^T (\phi' E^q)' \int_\Omega u u_t dx dt \\
& - \frac{\tau}{2m} \int_\Omega \int_0^1 \phi' E^q \gamma_1 e^{-2\tau\rho} \left|z\right|^{2m} \Big|_S^T d\rho dx \\
& + \frac{\tau}{2m} \int_S^T (\phi' E^q \gamma_1)' \int_\Omega \int_0^1 e^{-2\tau\rho} \left|z\right|^{2m} d\rho dx dt \\
& - \frac{1}{2m} \int_S^T \phi' E^q \gamma_1 \int_\Omega e^{-2\tau\rho} \left|z\right|^{2m} \Big|_0^1 dx dt.
\end{aligned} \tag{2.22}
$$

By virtue of Young's Inequality, Sobolev inequality, the definition of function $\gamma_1(t)$ (2.10), hypothesis of theorem 2.1 , conclusion of lemma 2.1 and assumption that $\phi'(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function ($0 < |\phi'| < M$) we reach the following inequalities;

$$
\left|\phi' E^q \int_\Omega u u_t dx\right| \le 2ME^{q+1}(t). \tag{2.23}
$$

$$
\left|\phi' E^q \int_\Omega u u_t dx\right|_S^T \le 2ME^{q+1}(S). \tag{2.24}
$$

$$
\left|\int_S^T (\phi' E^q)' \int_\Omega u u_t dx dt\right| \le 2ME^{q+1}(S) + \frac{2Mq}{q+1} E^{q+1}(S). \tag{2.25}
$$

$$
\begin{aligned}
\int_S^T \phi' E^q \sigma_1(t) \int_\Omega u \left|u_t\right|^{2m-1} dx dt \le & \frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m} \int_S^T [-E'(t)] dt \\
& + \frac{\epsilon_1^{2m}}{2m} \int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} c_1 \phi'(t) 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}}\right)^{2m} \left(E^{q+\frac{1}{2}}\right)^{2m} dt.
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
\int_S^T \phi' E^q \sigma_2(t) \int_\Omega u \left|z\right|^{2m-2} z(x,1,t) dx dt \le & \frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m} \int_S^T \left[-E'(t)\right] dt \\
& + \frac{\epsilon_2^{2m}}{2m} \int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} M_2 c_1 \phi' 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}}\right)^{2m} \left(E^{q+\frac{1}{2}}\right)^{2m} dt.
\end{aligned} \tag{2.27}
$$

$$
-\frac{\tau}{2m} \int_\Omega \int_0^1 \phi' E^q \gamma_1 e^{-2\tau\rho} \left|z\right|^{2m} \Big|_S^T d\rho dx \le ME^{q+1}(S). \tag{2.28}
$$

$$
\frac{\tau}{2m} \int_S^T (\phi' E^q \gamma_1)' \int_\Omega \int_0^1 e^{-2\tau\rho} \left|z\right|^{2m} d\rho dx dt \le \frac{Mq}{q+1} E^{q+1}(S). \tag{2.29}
$$

$$
\frac{1}{2m} \int_S^T \phi' E^q \gamma_1 \int_\Omega e^{-2\tau} \left|z(x,1,t)\right|^{2m} dx dt \le \frac{M\tilde{M}}{2m\theta_2(q+1)} E^{q+1}(S). \tag{2.30}
$$

$$
\frac{1}{2m} \int_S^T \phi' E^q \gamma_1 \int_\Omega \left|z(x,0,t)\right|^{2m} dx dt \le \frac{M\tilde{M}}{2m(1-\theta_1)(q+1)} E^{q+1}(S). \tag{2.31}
$$

Based on the estimates (2.23)-(2.31) and equation (2.21),

$$
\begin{aligned}
2\int_S^T \phi' E^{q+1} e^{-2\tau} dt \le\ & 4ME^{q+1}(S) + 2\int_S^T \phi' E^q \int_\Omega u_t^2 dx dt \\
&+ \frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m} \int_S^T [-E'(t)] dt \\
&+ \frac{\epsilon_1^{2m}}{2m} \int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} c_1 \phi' 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}} \right)^{2m} \left(E^{q+\frac{1}{2}} \right)^{2m} dt \\
&+ \frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m} \int_S^T [-E'(t)]\, dt \\
&+ \frac{\epsilon_2^{2m}}{2m} \int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} M_2 c_1 \phi' 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}} \right)^{2m} \left(E^{q+\frac{1}{2}} \right)^{2m} dt \\
&+ \frac{2Mq}{q+1} E^{q+1}(S) + \frac{M\tilde{M}}{2m\theta_2(q+1)} E^{q+1}(S) \\
&+ \frac{M\tilde{M}}{2m(1-\theta_1)(q+1)} E^{q+1}(S),
\end{aligned}
\tag{2.32}
$$

where ϵ_1 and ϵ_2 are positive constants, which will be later selected. Let us define $\phi(t)$ as follows;

$$
\phi(t) = \int_0^t \sigma_1(s) ds,
$$

Dividing region Ω such that $\Omega_1 = \{x; |u_t| \ge 1\}$ and $\Omega_2 = \{x; |u_t| < 1\}$, we get

$$
2\int_S^T \phi' E^q \int_\Omega u_t^2 dx dt = 2\int_S^T \phi' E^q \int_{\Omega_1} u_t^2 dx dt + 2\int_S^T \phi' E^q \int_{\Omega_2} u_t^2 dx dt.
$$

Moreover, using Young's inequality, lemma 2.1 and the definition of $\phi(t)$ we infer the following inequalities

$$
2\int_S^T \phi' E^q \int_{\Omega_1} u_t^2 dx dt \le \frac{2}{(1-\theta_1)(q+1)} E^{q+1}(S), \tag{2.33}
$$

and

$$
\begin{aligned}
2\int_S^T \phi' E^q \int_{\Omega_2} u_t^2 dx dt \le\ & \frac{2\epsilon_3^{\frac{k+1}{2}}}{k+1} \int_S^T [-E'(t)]\, dt \\
&+ \frac{k-1}{(k+1)\epsilon_3^{\frac{k+1}{k-1}}} \left(\frac{2}{1-\theta_1} \right)^{\frac{k+1}{k-1}} \int_S^T \phi' E^{\frac{q(k+1)}{k-1}} dt,
\end{aligned}
\tag{2.34}
$$

where $k > 2m$ and ϵ_3 is a positive constant, which will be later selected. From the estimates (2.33), (2.34) and inequality (2.32), we obtain

$$
\begin{aligned}
2e^{-2\tau}\int_S^T\int_\Omega \phi' E^{q+1}dt \leq & 2ME^{q+1}(S)+\frac{2}{(1-\theta_1)(q+1)}E^{q+1}(S)\\
&+\frac{(k-1)\epsilon_3^{-\frac{k+1}{k-1}}}{k+1}\left(\frac{2}{1-\theta_1}\right)^{\frac{k+1}{k-1}}\int_S^T\phi' E^{\frac{q(k+1)}{k-1}}dt+\frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m}E(S)\\
&+\frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m}E(S)+\frac{\epsilon_1^{2m}}{2m}\int_S^T\left(\frac{\sigma_1^{\frac{1}{2}}c_1\phi' 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}}\right)^{2m}\left(E^{q+\frac{1}{2}}\right)^{2m}dt\\
&+\frac{\epsilon_2^{2m}}{2m}\int_S^T\left(\frac{\sigma_1^{\frac{1}{2m}}M_2c_1\phi' 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}}\right)^{2m}\left(E^{q+\frac{1}{2}}\right)^{2m}dt+\frac{2Mq}{q+1}E^{q+1}(S)\\
&+\frac{2\epsilon_3^{\frac{k+1}{2}}}{k+1}E(S)+\frac{M\tilde{M}}{2m(q+1)}\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)E^{q+1}(S).
\end{aligned}
$$

Selecting $k=2q+1$, we deduce the following inequality,

$$
\begin{aligned}
2e^{-2\tau}\int_S^T\int_\Omega \phi' E^{q+1}dt \leq & 4ME^q(0)E(S)+\frac{2}{(1-\theta_1)(q+1)}E^q(0)E(S)\\
&+\frac{\epsilon_3^{(q+1)}}{q+1}E(S)+\frac{q\epsilon_3^{-\frac{q+1}{q}}}{q+1}\left(\frac{2}{1-\theta_1}\right)^{\frac{q+1}{q}}\int_S^T\phi' E^{q+1}dt\\
&+\frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m}E(S)+\frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m}E(S)\\
&+\frac{M^{2m}\epsilon_1^{2m}}{2m}\left(\frac{c_1 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}}\right)^{2m}E(0)^\theta\int_S^T\phi' E^{q+1}dt\\
&+\frac{\epsilon_2^{2m}}{2m}\left(\frac{M_2c_1 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}}\right)^{2m}E(0)^\theta\int_S^T\phi' E^{q+1}dt+\frac{2Mq}{q+1}E^q(0)E(S)\\
&+\frac{M\tilde{M}}{2m(q+1)}\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)E^q(0)E(S),
\end{aligned}
$$

where $\theta=m(2q+1)-(q+1)$. Selecting $\epsilon_1=\left(\frac{me^{-2\tau}(1-\theta_1)^{2m-1}}{E(0)^\theta 2^{m+2}(M^2c_1^2)^m}\right)^{\frac{1}{2m}}$, $\epsilon_2=\left(\frac{me^{-2\tau}\theta_2^{2m-1}}{E(0)^\theta 2^{m+2}(M_2^2M^2c_1^2)^m}\right)^{\frac{1}{2m}}$ and $\epsilon_3=\frac{1-\theta_1}{2}\left(\frac{(q+1)e^{-2\tau}}{4q}\right)^{\frac{q}{q+1}}$ we get

$$\int_S^T\phi'(t)E^{q+1}(t)dt\leq\frac{1}{\omega}E^q(0)E(S),$$

where

$$
\begin{aligned}
\omega^{-1}=\frac{2e^{2\tau}}{3}\max\Big\{&2M,\frac{2}{(q+1)(1-\theta_1)},\left(\frac{4qe^{2\tau}}{E(0)}\right)^q\left(\frac{1-\theta_1}{2(q+1)}\right)^{q+1},\frac{qM}{M+1}\\
&,\frac{(2m-1)E(0)^{\frac{m-1}{2m-1}}}{2m}\left(\frac{2^{m+2}e^{2\tau}(M^2c_1^2)^m}{m}\right)^{\frac{1}{2m-1}}\left(\frac{1}{1-\theta_1}+\frac{M_2^{\frac{2}{2m-1}}}{\theta_2}\right)\\
&,\frac{M\tilde{M}}{2m(q+1)\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)}\Big\}.
\end{aligned}
$$

and

$$q>\frac{2m-1}{2}.$$

Hence, we deduce the following result from the conclusion of Lemma 1.1

$$E(t) \leq E(0)\left(\frac{1+q}{1+\omega q\int_0^t \sigma_1(s)ds}\right)^{\frac{1}{q}}, \quad \forall t>0.$$

□

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