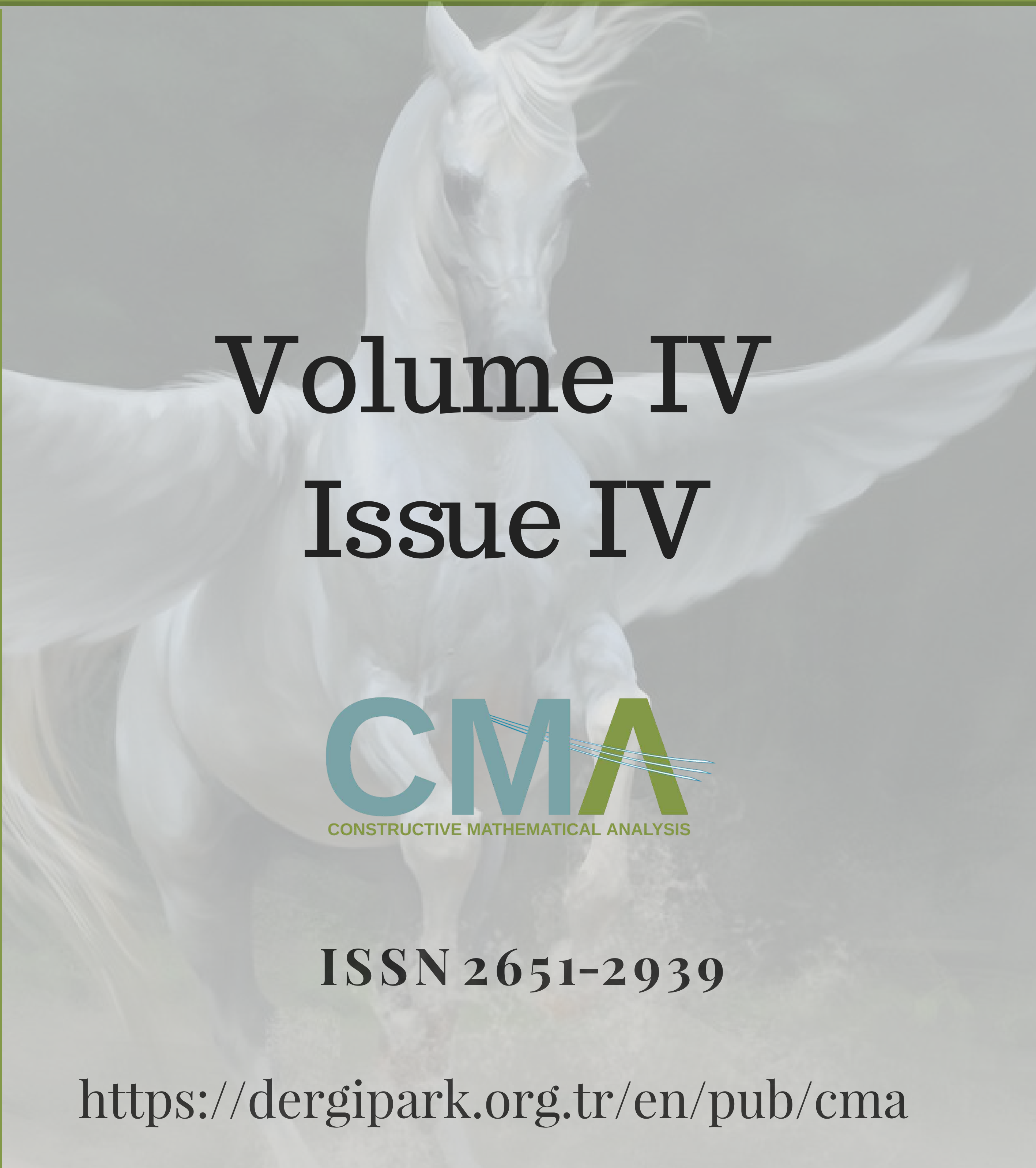
CONSTRUCTIVE MATHEMATICAL ANALYSIS



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Research Article

Continuous prime systems satisfying $N(x) = c(x-1)+1$

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ABSTRACT. Hilberdink showed that a continuous prime system for which there exists a constant A such that the function $N(x) - Ax$ is periodic satisfies $N(x) = c(x-1)+1$. He further showed that there exists a constant $c_0 > 2$, such that there exists a continuous prime system of this form if and only if $c \leq c_0$. Here, we determine c_0 numerically to be $1.25479 \cdot 10^{19} \pm 2 \cdot 10^{14}$. To do so we compute a representation for a twisted exponential function as a sum over the roots of the Riemann zeta function. We then give explicit bounds for the error obtained when restricting the occurring sum to a finite number of zeros. .

Keywords: Beurling primes, explicit formulae, continuous prime systems, Riemann zeta function.

2020 Mathematics Subject Classification: 11N80, 11Y60, 30A10, 33E20, 65E05.

1. INTRODUCTION AND RESULTS

Let S be the space of right-continuous functions $f : \mathbb{R} \to \mathbb{R}$ of bounded local variation, for which $f(x) = 0$ for $x < 1$. Let S^+ be the subset consisting of non-decreasing functions. For functions $f, g \in S$ define the Mellin-Stieltjes convolution $f * g$ by means of the equation

$$(f * g)(x) = \int_{1^-}^{x} f(x/t)dg(t),$$

and the convolution exponential $\exp_* g$ as

$$\exp_* g = \sum_{n=0}^{\infty} \frac{g^{*n}}{n!},$$

where g^{*n} denotes n-fold iterated convolution. For $\pi \in S^+$ define $\Pi(x) = \sum_{k \geq 1} \frac{1}{k}\pi(x^{1/k})$ and $N = \exp_* \Pi$. If the sum defining Π converges for all x, then we call the pair (Π, N) a continuous prime system with prime counting function π. Note that if $\pi(x)$ denotes the number of ordinary primes below x, we obtain $N(x) = \lfloor x \rfloor$, and $\Pi(x)$ is the weighted number of prime powers below x introduced by Riemann. If more generally $\pi(x)$ is a step function with integral jumps, then $N(x)$ is the counting function of an arithmetic semigroup in the sense of Knopfmacher ([5]).

Starting with the work of Beurling, there has been ongoing interest in continuous prime systems. Hilberdink ([4]) showed that if there is some c, such that $N(x) - cx$ is periodic and

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continuously differentiable, then $N(x) = c(x-1)+1$. This led him to ask, for which c such a number system exists. Define the holomorphic function f as

$$f(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}(e^{z/n} - 1) = \sum_{k=1}^{\infty} \frac{z^k}{k!\zeta(k+1)}.$$

He then proved the following.

Theorem 1.1. *There exists a continuous prime system satisfying $N(x) = c(x-1)+1$ if and only if $f(x) \geq f((1-c)x)$ for all $x \geq 0$. Moreover, there exists some $c_0 > 2$ such that there exists such a prime system if and only if $c \leq c_0$.*

Here, we determine c_0 numerically. Clearly, the existence of c_0 is equivalent to the statement that $f(x)$ is positive for some $x < 0$. Hilberdink proved the existence of such an x using Landau's ineffective criterion on the continuation of Dirichlet series with non-negative coefficients, therefore his proof does not yield any bound on c_0.
We prove the following.

Theorem 1.2. *The constant c_0 from Theorem 1.1 satisfies*

$$\left|c_0 - 1.25479 \cdot 10^{19}\right| \leq 2 \cdot 10^{14}.$$

2. Asymptotic estimates for f

In the sequel θ denotes a complex number of modulus ≤ 1, which may be different in all equations and may depend on all occurring parameters. As in the case of Landau symbols, equations containing θ may only be read from left to right, e.g. we have $\theta = 2\theta$, but not $2\theta = \theta$. The following is a version of Stirling's formula with an explicit error term, derived by Boyd ([3]).

Lemma 2.1. *For $|\arg z| \leq \frac{\pi}{2}$ we have*

$$\Gamma(z) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \left(1 + \theta\frac{1+\sqrt{2}}{2\pi^2|z|}\right).$$

We can now come to the main result of this section. We denote the non-trivial roots of ζ by ρ, and the imaginary part of ρ by γ.

Lemma 2.2. *Let $T \geq 100$ be a real number such that all roots of ζ in the rectangle $0 \leq \sigma \leq 1$, $|t| \leq T$ are simple with real part $\frac{1}{2}$, and that ζ has no root with imaginary part T. Put*

$$\delta = \min_{-2\leq\sigma\leq 2} |\zeta(\sigma + iT)|.$$

Then we have for real $x > e^2$ the estimate

$$f(-x) = \frac{1}{x^2\zeta'(-1)} + \frac{1}{\sqrt{x}} \sum_{|\gamma|<T} \frac{\Gamma(1-\rho)}{\zeta'(\rho)} x^{i\gamma} + \theta\frac{15.18}{x^{5/2}} + \theta(0.85\log x + 0.88\delta^{-1})T^2 e^{-\pi T/2} \tag{2.1}$$

Proof. From the Mellin transform

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)x^{-s}ds = e^{-x} - 1,$$

we deduce

$$f(-x) = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\Gamma(1-s)}{\zeta(s)} x^{s-1} ds.$$

We shift the path of integration to the path going from $1+\frac{1}{\log x}-i\infty$ to $1+\frac{1}{\log x}-iT$, then to $-\frac{3}{2}-iT$, to $-\frac{3}{2}+iT$, further to $1+\frac{1}{\log x}+iT$, and finally to $1+\frac{1}{\log x}+i\infty$. Doing so we encounter one singularity at $s=-1$ with residuum $\frac{1}{x^2\zeta'(-1)}$, and one singularity with residuum $\frac{\Gamma(1-\rho)}{\zeta'(\rho)}x^{1/2+i\gamma}$ for each non-trivial root ρ in the rectangle $0\leq\sigma\leq 1$, $|t|<T$. Note that the pole of ζ at 1 and the pole of Γ at 0 cancel each other. The integral over the new path will be bounded from above. We have

$$\left|\int_{-\frac{3}{2}-iT}^{-\frac{3}{2}+iT} \frac{\Gamma(1-s)}{\zeta(s)} x^{s-1} ds\right| \leq \frac{1}{x^{5/2}}\int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty}\left|\frac{\Gamma(1-s)}{\zeta(s)}\right| ds,$$

and since Γ decreases rapidly along every line parallel to the imaginary axis, the last integral can easily be evaluated numerically to be ≤ 95.32. On the line $\Re\, s = 1+\frac{1}{\log x}$, we have

$$\frac{1}{|\zeta(s)|} < \zeta(1+\frac{1}{\log x}) < 1+\int_1^\infty \frac{dt}{t^{1+1/\log x}} = 1+\log x,$$

thus

$$\left|\int_{1+\frac{1}{\log x}+iT}^{1+\frac{1}{\log x}+i\infty} \frac{\Gamma(1-s)}{\zeta(s)} x^{s-1} ds\right| \leq e(1+\log x)\int_{1+\frac{1}{\log x}+iT}^{1+\frac{1}{\log x}+i\infty} |\Gamma(1-s)|\, ds,$$

and from Lemma 2.1, we obtain that for $x\geq e^2$ the right hand side is bounded above by

$$e(1+\log x)\int_T^\infty (t+1)e^{-\pi t/2}\, dt = e(1+\log x)(\frac{2}{\pi}T+\frac{4+2\pi}{\pi^2})e^{-\pi T/2}.$$

Finally, we have

$$\left|\int_{-\frac{3}{2}+iT}^{1+\frac{1}{\log x}+iT} \frac{\Gamma(1-s)}{\zeta(s)} x^{s-1} ds\right| \leq \delta^{-1}(T+1)e^{-\pi T/2}\int_{-\frac{3}{2}}^{1+\frac{1}{\log x}} x^{\sigma-1} d\sigma \leq e(T+1)e^{-\pi T/2}\delta^{-1}.$$

We conclude that the modulus of the integral over the new path is bounded above by

$$\begin{aligned}&\frac{95.32}{x^{5/2}}+(1+\log x)(3.462T+5.665)e^{-\pi T/2}+2e(T+1)\delta^{-1}e^{\pi T/2}\\ \leq&\frac{95.32}{x^{5/2}}+(5.491\delta^{-1}+5.279\log x)Te^{-\pi T/2},\end{aligned}$$

where we used the bounds $1+\log x\leq\frac{3}{2}\log x$ and $T\geq 100$. Taking the factor $\frac{1}{2\pi}$ into account our claim follows. □

Note that even if we assume RH and the simplicity of all roots, we cannot get an explicit formula depending only on x and T, since it might be that $\zeta'(\rho)$ could be very close to 0. However, as in the explicit formula for $\sum_{n\leq x}\mu(n)$, we do get an explicit formula valid for all suitable values of T. We refer the reader to [6, section 14.27] for details.

Lemma 2.3. *We have $f(-x)<0$ for $0<x<2.5\cdot 10^6$, and $f(-x)<9.2\cdot 10^{-13}$ for all $x>0$.*

Proof. We claim that in the range $7 < x < 2.5\cdot 10^6$ the first negative summand in (2.1) dominates the other terms. We put $T = 100$. A straightforward computation yields $\delta \geq 1.19$, together with $\zeta'(-1) = -0.165421\ldots$ we obtain

$$f(-x) \leq -\frac{6.045}{x^2} + \frac{15.18}{x^{5/2}} + \frac{1}{\sqrt{x}} \sum_{|\gamma|\leq 100} \left|\frac{\Gamma(1-\rho)}{\zeta'(\rho)}\right| + (0.85\log x + 0.74)\cdot 6.05\cdot 10^{-65}$$
$$\leq -\frac{6.045}{x^2} + \frac{15.18}{x^{5/2}} + \frac{1.44\cdot 10^{-9}}{\sqrt{x}} + (5.15\log x + 4.48)\cdot 10^{-65}.$$

From this, we conclude $f(-x) < 0$ for $7 < x < 2.5\cdot 10^6$ as well as $f(-x) < 9.2\cdot 10^{-13}$ for $2.5\cdot 10^6 < x < e^{10^{50}}$. If x is very big we use estimates for the summatory function of the Möbius function. We have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}(e^{-x/n} - 1) \leq \sum_{n=1}^{\infty} m(n) \left|e^{-x/n} - e^{-x/(n+1)}\right|,$$

where $m(x) = \left|\sum_{n\leq x} \frac{\mu(n)}{n}\right|$. Bordellés ([2]) has shown that $m(x) \leq \frac{546}{\log^2 x}$ for $x > 1$, hence for $x > e^{24}$ we get

$$|f(-x)| \leq e^{-x} + \sum_{n=2}^{\infty} \frac{546}{\log^2 n} e^{-x/n} \left|e^{-x/(n(n+1))} - 1\right|$$
$$\leq e^{-x} + \sum_{n=2}^{\infty} \frac{546}{\log^2 n} e^{-x/n} \min\left(\frac{2x}{n(n+1)}, 1\right)$$
$$\leq x e^{-x^{1/3}} + \frac{1092}{\log^2 x^{2/3}} \sum_{n\geq x^{2/3}} \frac{1}{n^2}$$
$$\leq e^{24-e^8} + \frac{4.27}{x^{2/3}},$$

which is sufficiently small for $x > 10^{19}$. In the range $\frac{1}{2} \leq x \leq 7$, we can compute f with high precision using its Taylor series. We have

$$|f''(-x)| = \left|\sum_{n=1}^{\infty} \frac{\mu(n)e^{-x/n}}{n^3}\right| \leq \sum_{n=1}^{\infty} \frac{e^{-x/n}}{n^3} \leq e^{-x} + \int_0^{\infty} \frac{e^{-x/t}}{t^3}\,dt = e^{-x} + \frac{1}{x^2}.$$

Thus for a given x_0, we compute $f(x_0)$ and $f'(x_0)$, estimate $f''(x_0)$, and obtain an interval for which f is negative. Finally in the range $0 < x \leq \frac{1}{2}$, we have

$$f'(-x) = \sum_{k=1}^{\infty} \frac{(-x)^{k-1}}{(k-1)!\zeta(k+1)} \geq \frac{1}{\zeta(2)} - \frac{x}{\zeta(3)} > 0$$

together with $f(0) = 0$, we conclude that $f(-x) < 0$ in $0 < x \leq \frac{1}{2}$ as well. Hence the lemma is proven for all $x > 0$. □

3. Computation of c_0

The problem of computing c_0 is equivalent to finding the infimum of all c, such that there exists some $y > 0$ with $f(-y) \geq f(\frac{y}{c-1})$. Since $f(x)$ is increasing for $x \geq 0$, the right hand side is decreasing with c, hence our problem is equivalent to minimizing $\frac{x}{y}$ subject to the relations $x, y > 0$, $f(x) = f(-y)$.

By Lemma 2.3, we have $f(-y) < 9.2 \cdot 10^{-13}$. As $f(x) \geq \frac{x}{\zeta(2)}$ for $x \geq 0$, the equation $f(-y) = f(x)$ implies $x < 2 \cdot 10^{-12}$. Together with $f(-y) < 0$ for $y < 2.5 \cdot 10^6$, we obtain $\frac{x}{y} < 5 \cdot 10^{18}$ for all $x, y > 0$ satisfying $f(x) = f(-y)$. This crude lower bound is surprisingly close to the actual value for c.

For two positive real numbers y_1, y_2, we say that y_1 is better than y_2, if $f(-y_1) > 0$, and either $f(-y_2) \leq 0$ or for the real numbers $x_1, x_2 > 0$ defined by the equation $f(-y_i) = f(x_i)$ we have $\frac{x_1}{y_1} < \frac{x_2}{y_2}$. Clearly if y_1 is better than y_2, then y_2 cannot solve our optimization problem. We first show that in this way the range of y can be restricted to a bounded interval.

Lemma 3.4. *Suppose that $x_1 > 0$ satisfies $f(-x_1) > 0$. Then x_1 is better than all x_2 satisfying $x_2 > \frac{9.2 \cdot 10^{-13}}{f(-x_1)} x_1$.*

Proof. Suppose that $x_2 > x_1$, and that x_1 is not better than x_2. Let y_1, y_2 be given by the equations $f(-x_i) = f(y_i)$. We then have $y_2 > y_1$, and since f is convex in $x \geq 0$, we conclude that $\frac{f(y_2)}{y_2} > \frac{f(y_1)}{y_1}$, thus $\frac{f(-x_2)}{x_2} > \frac{f(-x_1)}{x_1}$. Our claim now follows from Lemma 2.3. □

We now apply Lemma 2.2 with $T = 100$ and neglect all roots except $\frac{1}{2} + i\gamma_1$, where $\gamma_1 = 14.13\ldots$ to find

$$\begin{aligned} f(-x) &= \frac{1}{x^2 \zeta'(-1)} + \frac{2}{\sqrt{x}} \Re \frac{x^{i\gamma_1} \Gamma(\frac{1}{2} - i\gamma_1)}{\zeta'(\rho_1)} \\ &\quad + \theta \left(\frac{4 \cdot 10^{-14}}{x^{1/2}} + \frac{15.18}{x^{5/2}} + (5.15 \log x + 4.48) \cdot 10^{-65} \right) \\ &= \frac{1}{x^2 \zeta'(-1)} + \frac{10^{-14}}{\sqrt{x}} \Re \left(x^{i\gamma_1} (-14102 + 143259i + 5\theta) \right) \\ &\quad + \theta \left(\frac{15.18}{x^{5/2}} + (5.15 \log x + 4.48) \cdot 10^{-65} \right) \\ &= \frac{1}{x^2 \zeta'(-1)} + \frac{10^{-14}}{\sqrt{x}} \Re \left(x^{i\gamma_1} (-14102 + 143259i + 21\theta) \right), \end{aligned}$$

provided that $2.5 \cdot 10^6 \leq x \leq 10^{50}$. Putting $s = \log(-x)$, we obtain

$$f(-e^s) = \frac{e^{-2s}}{\zeta'(-1)} + (143951 + 22\theta) \cdot 10^{-14} e^{-s/2} \cos(\gamma_1 s + 1.66892). \tag{3.2}$$

In particular we obtain $f(-e^{15}) > 2.3 \cdot 10^{-13}$, thus, using Lemma 3.4, e^{15} is better than all x satisfying $x > 4 \cdot e^{15}$. In particular we only have to consider values of x, for which the approximation (3.2) is valid. Considering the power series for f we find that for $x \in [0, 10^{20}]$ with $f(-x) > 0$ the unique value y with $f(y) = f(-x)$ satisfies $y < \zeta(2) f(-x)$, as well as

$$f(y) < \frac{y}{\zeta(2)} + e^y - 1 - y < \frac{y}{\zeta(2)} + y^2,$$

thus

$$y > \zeta(2) f(-x) - (\zeta(2) f(-x))^2 > \left(1 - \frac{3 \cdot 10^{-9}}{\sqrt{x}}\right) \zeta(2) f(-x),$$

and therefore $y = \left(1 + \frac{3 \cdot 10^{-9}\theta}{\sqrt{x}}\right) \zeta(2) f(-x)$. We conclude that in the relevant range the function to be minimized is

$$\left(1 + \frac{4 \cdot 10^{-9}\theta}{\sqrt{x}}\right) \frac{x}{f(-x)},$$

subject to the condition $f(-x) > 0$. Since this condition in particular implies that the first, negative, summand in (3.2) is of smaller absolute value than the second, we obtain that we have to minimize the inverse of

$$\frac{e^{-3s}}{\zeta'(-1)} + (143951 + 24\theta) \cdot 10^{-14} e^{-3s/2} \cos(\gamma_1 s + 1.66892)$$

subject to the condition that this expression is positive, that is, we have to find the largest local maximum of this function. The first positive local maximum of this function occurs at $s = 14.99$ with a value of $7.01 \cdot 10^{-20}$, the second at 15.44 with a value of $7.97 \cdot 10^{-20}$, the third at 15.88 with a value $5.26 \cdot 10^{-20}$. All further local maxima are much smaller. The precision is sufficient to guarantee that the maximum is attained in the interval $[15.43, 15.45]$ and has a value in the interval $[7.9 \cdot 10^{-20}, 8 \cdot 10^{-20}]$.

We can now refine our computation by using the latter bound to improve the error in (2.1). We put $T = 100$ in Lemma 2.2 and get

$$\begin{aligned} f(-x) &= \frac{1}{x^2\zeta'(-1)} + \frac{2}{\sqrt{x}} \sum_{j=1}^{29} \frac{\Gamma(\frac{1}{2} - i\gamma_j)}{\zeta'(\frac{1}{2} + i\gamma_j)} \Re x^{i\gamma_j} + 1.69 \cdot 10^{-16}\theta \\ &= \frac{1}{x^2\zeta'(-1)} + \frac{2}{\sqrt{x}} \left| \frac{\Gamma(\frac{1}{2} - i\gamma_j)}{\zeta'(\frac{1}{2} + i\gamma_j)} \right| \cos\left(\gamma_1 \log x + \arg \frac{\Gamma(\frac{1}{2} - i\gamma_j)}{\zeta'(\frac{1}{2} + i\gamma_j)} \right) + 1.75 \cdot 10^{-16}\theta \end{aligned}$$

for $e^{15.43} < x < e^{15.45}$. From this, we find that the maximum of $\frac{f(-x)}{x}$ is attained in $\log x = 15.4382 + \theta 0.0001$ and has a value $(796947 + \theta) \cdot 10^{-25}$, and the value of c_0 is $(1.25479 + 0.000002\theta) \cdot 10^{19}$. The proof of Theorem 1.2 is complete.

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Research Article

Matrix valued positive definite kernels related to the generalized Aitken's integral for Gaussians

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ABSTRACT. We introduce a method to construct general multivariate positive definite kernels on a nonempty set X that employs a prescribed bounded completely monotone function and special multivariate functions on X. The method is consistent with a generalized version of Aitken's integral formula for Gaussians. In the case in which X is a cartesian product, the method produces nonseparable positive definite kernels that may be useful in multivariate interpolation. In addition, it can be interpreted as an abstract multivariate version of the well-established Gneiting's model for constructing space-time covariances commonly highly cited in the literature. Many parametric models discussed in statistics can be interpreted as particular cases of the method.

Keywords: Positive definite kernels, conditionally negative definite functions, Aitken's integral, Schur exponential, Oppenheim's inequality, Gneiting's model.

2020 Mathematics Subject Classification: 42A82, 47A56.

1. INTRODUCTION

Let X be a nonempty set and write $M_q(\mathbb{C})$ to denote the set of all $q\times q$ matrices with complex entries. A kernel $K=[K_{m,n}]_{m,n=1}^q : X\times X\to M_q(\mathbb{C})$ is *positive definite* if for every positive integer N at most the cardinality of X and distinct points $x_1,\ldots,x_N$ in X, the block matrix $[[K_{m,n}(x_\mu,x_\nu)]_{\mu,\nu=1}^N]_{m,n=1}^q$ of order Nq is positive semi-definite, that is,

$$\sum_{\mu,\nu=1}^{N} c_\mu^* K(x_\mu,x_\nu)c_\nu=\sum_{m,n=1}^{q}\sum_{\mu,\nu=1}^{N}\overline{c_\mu^m}c_\nu^n K_{m,n}(x_\mu,x_\nu)\geq 0, \tag{1.1}$$

whenever $c_1,\ldots,c_N$ are column vectors in $\mathbb{C}^q$ and $c_\mu=[c_\mu^1\ \ldots\ c_\mu^q]^\intercal$. The star notation refers to conjugate transposition of column vectors in $\mathbb{C}^q$. If the matrices $[[K_{m,n}(x_\mu,x_\nu)]_{\mu,\nu=1}^N]_{m,n=1}^q$ are all positive definite, that is, the inequalities in (1.1) are strict when at least one of the vectors c_μ is nonzero, then the positive definite kernel K is termed *strictly positive definite* on X. The two classes of kernels introduced above will be denoted by $PD_q(X)$ and $SPD_q(X)$, respectively. Kernels in these classes correspond to the standard positive definite kernels studied in [4] when we set $q=1$ and identify $M_q(\mathbb{C})$ with $\mathbb{C}$. The importance of matrix valued positive definite kernels in their various formats may be ratified in the references [2, 18, 19, 25].

Examples of kernels in $PD_q(X)$ and $SPD_q(X)$ can be easily constructed. If A is a positive semi-definite matrix in $M_q(\mathbb{C})$, then the constant kernel

$$K(x,x')=A,\quad x,x'\in X,$$

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belongs to $PD_q(X)$. If $f_1, \ldots, f_q$ are kernels in $PD_1(X)$, then the kernel K given by the formula

$$K(x, x') = \text{Diag}(f_1(x, x'), \ldots, f_q(x, x')), \quad x, x' \in X,$$

belongs to $PD_q(X)$. Further, if all the f_m belong to $SPD_1(X)$, then K belongs to $SPD_q(X)$. Moving the other way around, if K is a kernel in $PD_q(X)$ and $c \in \mathbb{C}^q$, then

$$f(x, x') = c^* K(x, x') c, \quad x, x' \in X,$$

defines a function in $PD_1(X)$. If $c \neq 0$ and K belongs to $SPD_q(X)$, then f actually belongs to $SPD_1(X)$.

The purpose of this paper is to introduce methods to construct abstract matrix-valued mappings with the additional requirement of positive definiteness and strict positive definiteness. In many cases, the methods yield easy to handle and flexible models, once it encompasses common models found in geophysical sciences, including probabilistic weather forecasting, data assimilation, statistical analysis of climate model output, etc. when one makes the right choice for X and set a metric structure in it.

The method itself will be based on bounded completely monotone functions and special matrix valued functions attached to the notion of conditional negative definiteness. Recall that the *complete monotonicity* of a function $f : (0, \infty) \to \mathbb{R}$ is characterized by two properties: f is $C^\infty(0, \infty)$ and $(-1)^n f^{(n)}(t) \geq 0$ for $n = 0, 1, \ldots$ and $t \in (0, \infty)$. Throughout the paper, we will not distinguish between a bounded completely monotone function and its unique continuous extension to $[0, \infty)$.

A kernel $K = [K_{m,n}]_{m,n=1}^q : X \times X \to M_q(\mathbb{C})$ is *conditionally negative definite* if it is Hermitian and the block matrices $[[K_{m,n}(x_\mu, x_\nu)]_{\mu,\nu=1}^N]_{m,n=1}^q$ are of *negative type*, that is, the quadratic forms (1.1) are ≤ 0 whenever the vectors c_μ satisfy $\sum_{\mu=1}^N c_\mu = 0$. The conditionally negative definite kernel K is *strictly conditionally negative definite* if the matrices $[[K_{m,n}(x_\mu, x_\nu)]_{\mu,\nu=1}^N]_{m,n=1}^q$ are of *strict negative type* for $N \geq 2$, that is, the quadratic forms are negative whenever $N \geq 2$ and at least one c_μ is nonzero. These two classes of kernels will be denoted by $CND_q(X)$ and $SCND_q(X)$, respectively. Examples of kernels in $CND_1(X)$ and $SCND_1(X)$ can be found in [4] while connections between the classes $PD_1(X)$ and $CND_1(X)$ are described in [3, 4, 10]. As for examples in the classes $CND_q(X)$ and $SCND_q(X)$, one may employ these connections and imitate the procedures adopted for producing kernels in $PD_q(X)$ and $SPD_q(X)$ previously mentioned.

All the major results we intend to prove here will be based on a generalization of Aitken's integral formula for computing Gaussians: if A is a positive definite matrix in $M_q(\mathbb{R})$ (the subset of $M_q(\mathbb{C})$ formed by matrices with real entries only) and b is a vector in $\mathbb{R}^q$, then

$$\int_{\mathbb{R}^q} e^{-u^\intercal A u + i\, b^\intercal u} du = \frac{\pi^{q/2}}{\sqrt{\det\, A}} e^{-b^\intercal (4A)^{-1} b}.$$

Aitken's integral itself corresponds to the formula above in the case $b = 0$. A proof for the generalized Aitken's integral formula can be reached by mimicking the proof of Aitken's integral in [24, p. 340] but an independent proof is available in [15]. This reference also contains univariate results that may be considered as versions of some of the results to be described here.

Before we proceed to the outline of the paper, it is worth mentioning that if X is actually a cartesian product of sets, the method to be presented here lead to nonseparable kernels, i.e., kernels on two variables which are not mixed up, a desirable property in applications. Meanwhile, in some specific cases, the method will upgrade to a generalization of the well established Gneiting's contribution in [7] on the construction of kernels in $PD_1(\mathbb{R}^q \times \mathbb{R}^d)$. Gneiting's classical result is as follows: for a bounded completely monotone function $\phi : (0,\infty) \to \mathbb{R}$ and a positive valued function f with a completely monotone derivative, it asserts that the formula

$$G_r((x,y),(x',y')) = \frac{1}{f(\|y-y'\|^2)^r}\phi\left(\frac{\|x-x'\|^2}{f(\|y-y'\|^2)}\right), \quad x,x' \in \mathbb{R}^q;\ y,y' \in \mathbb{R}^d, \tag{1.2}$$

defines a kernel G_r in $PD_1(\mathbb{R}^q \times \mathbb{R}^d)$, whenever $r \geq d/2$ and $\|\cdot\|$ denotes the usual norms in both $\mathbb{R}^q$ and $\mathbb{R}^d$. The boundedness of ϕ is required in order to make $\phi(0^+) < \infty$. The references [12, 13, 16, 20] include some extensions and generalizations of this important result along with additional references on the topic.

The paper proceeds as follows. Section 2 begins with the description of two additional notions to be employed in the paper, one for families of vector functions and another for families of matrix functions, along with examples. The first major result of the paper is Theorem 2.4: it describes a method to construct kernels in $PD_p(Y)$ from bounded completely monotone functions, special families of vector functions on Y and special families of matrix functions on Y. Further, it provides a sufficient condition in order that the resulting kernel be in $SPD_p(Y)$. At the end of the section, we discuss some examples and detach a relevant consequence of Theorem 2.4. The main result in Section 3 expands Theorem 2.4 via integration with respect to a convenient measure, a usual procedure adopted in approximation theory and statistics in order to produce new positive definite functions from a family of parameterized positive definite functions. We separate a special simpler version of the theorem in Corollary 3.3. Section 4 describes extensions of Theorems 2.4 and 3.1 that lead to kernels in $PD_p(X \times Y)$. Applications and a multivariate abstract extension of the classical Gneiting's result are described.

2. Positive definiteness on a single set

This section contains the first main contribution in the paper to be made explicit in Theorem 2.4. It provides a method to construct functions in $PD_q(Y)$ using completely monotonic functions via Aitken's formula. A sufficient condition for strict positive definiteness is included. The contribution itself demands two notions for families of functions with domain Y which we now discuss.

For a matrix function G in $CND_q(Y)$ and a vector u from $\mathbb{C}^q$, the kernel

$$(y,y') \in Y \times Y \mapsto u^* G(y,y') u$$

belongs to $CND_1(Y)$. Further, the kernel belongs to the class $SCND_1(Y)$ whenever G belongs to $SCND_q(Y)$ and u is nonzero. Theorem 2.4 will demand a family $\{G_{m,n} : m,n = 1,\ldots,p\}$ for which all the matrix kernels

$$(y,y') \in Y \times Y \mapsto [u^\intercal G_{m,n}(y,y')u]_{m,n=1}^p, \quad u \in \mathbb{R}^q,$$

belong to $CND_p(Y)$. Since this is not easily achievable, the following examples are apposite.

Example 2.1. Define

$$G_{m,n}(y,y') = g_m(y) + g_n(y'), \quad y,y' \in X,$$

where $g_m, g_n : Y \to M_q(\mathbb{C})$ are functions subject to our choice. If $y_1, \ldots, y_N$ are distinct points in Y, $c_1, \ldots, c_N$ are vectors in $\mathbb{C}^p$ such that $\sum_{\mu=1}^N c_\mu = 0$ and $u \in \mathbb{C}^q$, then

$$\sum_{\mu,\nu=1}^{N} c_\mu^* \left[u^\intercal G_{m,n}(y_\mu, y_\nu)u\right]_{m,n=1}^p c_\nu = \sum_{n=1}^{p}\sum_{\nu=1}^{N} c_\nu^n \sum_{\mu=1}^{N}\sum_{m=1}^{p} \overline{c_\mu^m} u^\intercal g_m(y_\mu)u + \sum_{m=1}^{p}\sum_{\mu=1}^{N} \overline{c_\mu^m} \sum_{\nu=1}^{N}\sum_{n=1}^{p} c_\nu^n u^\intercal g_n(y_\nu)u = 0,$$

that is, the matrix function

$$(y, y') \in X \times X \mapsto \left[u^\intercal G_{m,n}(y, y')u\right]_{m,n=1}^p$$

belongs to $CND_p(Y)$.

Example 2.2. Set $G_{m,n} = 0$ when $m \neq n$ and pick each $G_{m,m}$ in the class $CND_q(Y)$. Keeping the c_μ and the y_μ as in Example 2.1, it is easily seen that

$$\sum_{\mu,\nu=1}^{N} c_\mu^* \left[u^\intercal G_{m,n}(y_\mu, y_\nu)u\right]_{m,n=1}^p c_\nu = \sum_{m=1}^{p}\sum_{\mu,\nu=1}^{N} \overline{c_\mu^m} c_\nu^m u^\intercal G_{m,m}(y_\mu, y_\nu)u \leq 0.$$

Thus, the matrix function

$$(y, y') \in X \times X \mapsto \left[u^\intercal G_{m,n}(y, y')u\right]_{m,n=1}^p$$

belongs to $CND_p(Y)$.

Theorem 2.4 will also need special families $\{H_{m,n} : m, n = 1, \ldots, p\}$ of vector functions $H_{m,n} : Y \times Y \to \mathbb{C}^q$. Precisely, it will require families for which all the matrix functions

$$(y, y') \in Y \times Y \mapsto \left[e^{i\, H_{m,n}(y,y')^* u}\right]_{m,n=1}^p, \quad u \in \mathbb{R}^q,$$

belong to $PD_p(Y)$. Again, this is not easy to achieve, reason why a simple example is handy.

Example 2.3. Let us set

$$H_{m,n}(y, y') = h_m(y) - h_n(y'), \quad y, y' \in Y,$$

where $h_m : Y \to \mathbb{R}^q$, $m = 1, \ldots, p$. If $y_1, \ldots, y_N$ are distinct points in Y and $c_1, \ldots, c_N$ are vectors in $\mathbb{C}^p$, then

$$\sum_{\mu,\nu=1}^{N} c_\mu^* \left[e^{i\, H_{m,n}(y_\mu,y_\nu)^\intercal u}\right]_{m,n=1}^p c_\nu = \left|\sum_{\mu=1}^{N}\sum_{m=1}^{p} \overline{c_\mu^m}\, e^{i\, h_m(y_\mu)^\intercal u}\right|^2 \geq 0, \quad u \in \mathbb{R}^q,$$

that is, the kernels

$$(y, y') \in Y \times Y \mapsto \left[e^{i\, H_{m,n}(y,y')^* u}\right]_{m,n=1}^p, \quad u \in \mathbb{R}^q,$$

belong to $PD_p(Y)$.

We observe that if the matrix functions

$$(y, y') \in Y \times Y \mapsto \left[e^{i\, H_{m,n}(y,y')^* u}\right]_{m,n=1}^p, \quad u \in \mathbb{R}^q,$$

belong to $PD_p(Y)$, then each $H_{m,n}$ must be anti-symmetric in the sense that

$$\text{Re}\, H_{m,n}(y, y') = -\text{Re}\, H_{m,n}(y', y), \quad y, y' \in Y.$$

In particular,

$$\operatorname{Re} H_{m,n}(y,y)=0, \quad m,n=1,\ldots,p;\ y\in Y.$$

Finally, some specific properties of Hadamard exponentials will be needed. We recall that if A is a matrix in $M_q(\mathbb{C})$, then its *Hadamard exponential* is the matrix

$$e^{\circ A} := [e^{A_{\mu\nu}}]_{\mu,\nu=1}^{q}.$$

If $A\in M_q(\mathbb{R})$ is symmetric and of negative type, then the Hadamard exponential of $-A$ is positive semi-definite. It is positive definite if and only if

$$A_{\mu\mu}+A_{\nu\nu}<2A_{\mu\nu}, \quad \mu\neq\nu.$$

These facts are proved in Lemma 2.5 in [21] albeit [14] analyzed similar properties earlier. As an obvious consequence, we have that if $A\in M_q(\mathbb{R})$ is of strict negative type, then the Hadamard exponential of $-A$ is positive definite. The recasting of this property for block matrices is as follows: if a real symmetric block matrix $A=[[A_{mn}(\mu\nu)]_{\mu,\nu=1}^{N}]_{m,n=1}^{q}$ is of negative type, then the Hadamard exponential of $-A$ is positive definite if and only if

$$A_{mm}(\mu\mu)+A_{nn}(\nu\nu)<2A_{mn}(\mu\nu), \quad |m-n|+|\mu-\nu|>0. \tag{2.3}$$

Below, we will use the symbol "$\bullet$" to denote the Schur product of two matrices of same size.

Theorem 2.4. *Let ϕ be a bounded and completely monotone function. For each m,n in $\{1,\ldots,p\}$, let $G_{m,n}: Y\times Y\to M_q(\mathbb{R})$ be a matrix function with range containing positive definite matrices only and $H_{m,n}: Y\times Y\to\mathbb{R}^q$ a vector function. Assume the matrix functions*

$$(y,y')\in Y\times Y\mapsto [u^{\intercal}G_{m,n}(y,y')u]_{m,n=1}^{p}, \quad u\in\mathbb{R}^q,$$

belong to $CND_p(Y)$ and that

$$(y,y')\in Y\times Y\mapsto \left[e^{i\,H_{m,n}(y,y')^{\intercal}u}\right]_{m,n=1}^{p}, \quad u\in\mathbb{R}^q,$$

belong to $PD_p(Y)$. The following assertions hold for the kernel $K: Y\times Y\to M_p(\mathbb{R})$ given by the formula

$$K(y,y')=\left[\frac{\phi\left(H_{m,n}(y,y')^{\intercal}G_{m,n}(y,y')^{-1}H_{m,n}(y,y')\right)}{\sqrt{\det G_{m,n}(y,y')}}\right]_{m,n=1}^{p}, \quad y,y'\in Y.$$

(i) K belongs to $PD_p(Y)$.

(ii) If ϕ is not identically 0 and there exists an open subset U of $\mathbb{R}^q\setminus\{0\}$ so that

$$u^{\intercal}[G_{m,m}(y,y)+G_{n,n}(y',y')-2G_{m,n}(y,y')]u<0, \quad (m,y)\neq(n,y');\ u\in U,$$

then K belongs to $SPD_p(Y)$.

Proof. We begin by proving Assertion (i) in the case where ϕ is a constant function, that is, the case in which

$$K(y,y')=\left[\frac{\phi(0)}{\sqrt{\det G_{m,n}(y,y')}}\right]_{m,n=1}^{p}, \quad y,y'\in Y.$$

Since each matrix $G_{m,n}(y,y')$ is positive definite, we may apply Aitken's integral formula to obtain

$$K(y,y')=\frac{\phi(0)}{\pi^{q/2}}\left[\int_{\mathbb{R}^q} e^{-u^{\intercal}G_{m,n}(y,y')u}du\right]_{m,n=1}^{p}, \quad y,y'\in Y. \tag{2.4}$$

If $y_1, \ldots, y_N$ are distinct points in Y and $c_1, \ldots, c_N$ are vectors in $\mathbb{R}^p$, then

$$\sum_{\mu,\nu=1}^{N} c_\mu^\intercal K(y_\mu, y_\nu) c_\nu = \frac{\phi(0)}{\pi^{q/2}} \sum_{\mu,\nu=1}^{N} \sum_{m,n=1}^{p} c_\mu^m c_\nu^n \int_{\mathbb{R}^q} e^{-u^\intercal G_{m,n}(y_\mu, y_\nu) u} du$$
$$= \frac{\phi(0)}{\pi^{q/2}} \int_{\mathbb{R}^q} \sum_{m,n=1}^{p} \sum_{\mu,\nu=1}^{N} c_\mu^m c_\nu^n e^{-u^\intercal G_{m,n}(y_\mu, y_\nu) u} du.$$

One of the assumptions on the $G_{m,n}$ now yields that

$$\sum_{m,n=1}^{p} \sum_{\mu,\nu=1}^{N} c_\mu^m c_\nu^n e^{-u^\intercal G_{m,n}(y_\mu, y_\nu) u} \geq 0, \quad u \in \mathbb{R}^q,$$

and Assertion (i) follows in this case. In the general case, the Bernstein-Widder Theorem ([23, p. 3]) implies that

$$K(y, y') = \left[\frac{1}{\sqrt{\det G_{m,n}(y, y')}} \int_{[0,\infty)} e^{-H_{m,n}(y,y')^\intercal G_{m,n}(y,y')^{-1} H_{m,n}(y,y')\, s} d\sigma(s) \right]_{m,n=1}^{p}$$

for some finite and positive measure σ on $[0, \infty)$. On the other hand, the generalized Aitken's integral formula provides the alternative representation

$$K(y, y') = \left[\frac{1}{\pi^{q/2}} \int_{[0,\infty)} \left(\int_{\mathbb{R}^q} e^{-u^\intercal G_{m,n}(y,y') u} e^{2i\sqrt{s} H_{m,n}(y,y')^\intercal u} du \right) d\sigma(s) \right]_{m,n=1}^{p}.$$

If the y_μ are as before and the c_μ are now complex vectors, the quadratic form

$$Q := \sum_{\mu,\nu=1}^{N} c_\mu^* K(y_\mu, y_\nu) c_\nu$$

becomes

$$Q = \frac{1}{\pi^{q/2}} \sum_{\mu,\nu=1}^{N} \sum_{m,n=1}^{p} \overline{c_\mu^m} c_\nu^n \int_{[0,\infty)} \int_{\mathbb{R}^q} e^{-u^\intercal G_{m,n}(y_\mu, y_\nu) u} e^{i\, 2\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal u} du d\sigma(s)$$
$$= \frac{1}{\pi^{q/2}} \int_{[0,\infty)} \int_{\mathbb{R}^q} \sum_{m,n=1}^{p} \sum_{\mu,\nu=1}^{N} \overline{c_\mu^m} c_\nu^n c_\mu^* e^{-u^\intercal G_{m,n}(y_\mu, y_\nu) u} e^{i\, 2\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal u} du d\sigma(s).$$

The assumption on each $(y, y') \in Y \times Y \mapsto \left[e^{i\, H_{m,n}(y,y')^\intercal u} \right]_{m,n=1}^{p}$ settles the positive semi-definiteness of

$$\left[\left[e^{i\, 2\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal u} \right]_{\mu,\nu=1}^{N} \right]_{m,n=1}^{p} \tag{2.5}$$

while the Schur Product Theorem ratifies the positive semi-definiteness of each Schur product

$$\left[\left[e^{-u^\intercal G_{m,n}(y_\mu, y_\nu) u} \right]_{\mu,\nu=1}^{N} \right]_{m,n=1}^{p} \bullet \left[\left[e^{i\, 2\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal u} \right]_{\mu,\nu=1}^{N} \right]_{m,n=1}^{p}. \tag{2.6}$$

These arguments validate the inequality $Q \geq 0$.
Let us keep the notation used above to prove Assertion (ii). Assume further that the c_μ are not all zero vectors. If there exists an open subset U of $\mathbb{R}^q \setminus \{0\}$ so that

$$u^\intercal [G_{m,m}(y,y) + G_{n,n}(y',y') - 2G_{m,n}(y,y')]u < 0, \quad (m,y) \neq (n,y');\ u \in U,$$

we can infer via (2.3) that the block matrix

$$\left[\left[e^{-u^\intercal G_{m,n}(y_\mu,y_\nu)u}\right]_{\mu,\nu=1}^{N}\right]_{m,n=1}^{p}$$

is positive definite whenever $u \in U$. Thus, if ϕ is constant and not identically 0, then $Q > 0$ by Formula (2.4). If ϕ is nonconstant, first we invoke our assumption on the $H_{m,n}$ in order to see that the diagonal entries in each block matrix (2.5) are all equal to 1. An application of Oppenheim's inequality ([9, p. 509]) shows that the Schur product (2.6) is positive definite for $u \in U$ and $s \geq 0$. In particular,

$$\int_{\mathbb{R}^q} \sum_{m,n=1}^{p} \sum_{\mu,\nu=1}^{N} \overline{c_\mu^m} c_\nu^n c_\mu^* e^{-u^\intercal G_{m,n}(y_\mu,y_\nu)u} e^{i\,2\sqrt{s}H_{m,n}(y_\mu,y_\nu)^\intercal u}\, du > 0, \quad s \geq 0.$$

Since σ is not the zero measure we may go one step further and infer that $Q > 0$. □

Remark 2.5. Theorem 17 in [22] is a very special case of Theorem 2.4-(i).

Next, we present some examples that illustrate our findings.

Example 2.6. For $m = 1, \ldots, p$, let $g_m : Y \to M_q(\mathbb{R})$ be a function with range containing positive definite matrices only and $h_m : Y \to \mathbb{R}^q$ an arbitrary function. Setting $G_{m,n}(y,y') = g_m(y) + g_n(y')$, $y,y' \in Y$ and $H_{m,n}(y,y') = h_m(y) - h_n(y')$, $y,y' \in Y$, the assumptions in Theorem 2.4 are satisfied. Thus, the formula

$$\left[\frac{\phi\left((h_m(y)-h_n(y'))^\intercal (g_m(y)+g_n(y'))^{-1}(h_m(y)-h_n(y'))\right)}{\sqrt{\det[(g_m(y)+g_n(y')]}}\right]_{m,n=1}^{p}, \quad y,y' \in Y,$$

defines a kernel in $PD_p(Y)$ whenever ϕ is bounded completely monotone function. The inequalities in Theorem 2.4-(ii) cannot be matched in this abstract example.

Example 2.7. For $m,n = 1, \ldots, p$, let us set

$$G_{m,n}(y,y') = g_{m,n}(y,y')I_q, \quad y,y' \in Y,$$

where each $g_{m,n}$ is a positive valued kernel on Y and $(y,y') \in Y \times Y \mapsto [g_{m,n}(y,y')]_{m,n=1}^{p}$ belongs to $CND_p(Y)$. Observe that for each m and n,

$$u^\intercal G_{m,n}(y,y')u = \|u\|^2 g_{m,n}(y,y'), \quad u \in \mathbb{R}^q; y,y' \in Y.$$

On the other hand, if $c_1, \ldots, c_N$ are column vectors satisfying $\sum_{\mu=1}^{N} c_\mu = 0$ and $y_1, \ldots, y_N$ belong to Y, then

$$\sum_{m,n=1}^{p} \sum_{\mu,\nu=1}^{N} \overline{c_\mu^m} c_\nu^n u^\intercal G_{m,n}(y_\mu,y_\nu)u = \|u\|^2 \sum_{m,n=1}^{p} \sum_{\mu,\nu=1}^{N} \overline{c_\mu^m} c_\nu^n g_{m,n}(y_\mu,y_\nu) \leq 0,$$

that is, each kernel

$$(y,y') \in Y \times Y \mapsto [u^\intercal G_{m,n}(y,y')u]_{m,n=1}^{p},$$

belongs to $CND_p(Y)$. If the $H_{m,n}$ satisfies the assumptions of Theorem 2.4, then it is promptly seen that the formula

$$K(y,y') = \left[\frac{1}{g_{m,n}(y,y')^{q/2}}\phi\left(\frac{\|H_{m,n}(y,y')\|^2}{g_{m,n}(y,y')}\right)\right]_{m,n=1}^{p}, \quad y,y' \in Y,$$

defines a matrix kernel in $PD_p(Y)$ whenever ϕ is a bounded completely monotone function.

Example 2.8. If we take $H_{m,n}$ as in Example 2.6, then the kernel K in Example 2.7 takes the form

$$K(y,y') = \left[\frac{1}{g_{m,n}(y,y')^{q/2}}\phi\left(\frac{\|h_m(y)-h_n(y')\|^2}{g_{m,n}(y,y')}\right)\right]_{m,n=1}^{p}, \quad y,y' \in Y.$$

This example has a structure that resembles that of Gneiting's model in [7] for the construction of space-time covariances. We can get even closer by setting $g_{m,n} := g$ for all m and n, where $g : Y \to (0,\infty)$ belongs to $CND_1(Y)$, a choice that leads to

$$K(y,y') = \frac{1}{g(y,y')^{q/2}}\left[\phi\left(\frac{\|h_m(y)-h_n(y')\|^2}{g(y,y')}\right)\right]_{m,n=1}^{p}, \quad y,y' \in Y.$$

The setting adopted in both Examples 2.7 and 2.8 is a particular case of that detached in Theorem 2.9 below. Needless to say that the theorem can be interpreted as a multivariate version of the Gneiting's criterion in [7].

Theorem 2.9. *Let ϕ be a bounded and completely monotone function. Let g be a positive valued kernel in $CND_1(Y)$ and for each m,n in $\{1,\ldots,p\}$, define*

$$G_{m,n}(y,y') = g(y,y')I_q, \quad y,y' \in Y.$$

If $H_{m,n} : Y \times Y \to \mathbb{R}^q$ is a vector function such that the matrix functions

$$(y,y') \in Y \times Y \mapsto \left[e^{i\,H_{m,n}(y,y')^\intercal u}\right]_{m,n=1}^{p}, \quad u \in \mathbb{R}^q,$$

belong to $PD_p(Y)$, then the following assertions hold for the kernel $K : Y \times Y \to M_p(\mathbb{R})$ given by the formula

$$K(y,y') = \frac{1}{g(y,y')^{q/2}}\left[\phi\left(\frac{\|H_{m,n}(y,y')\|^2}{g(y,y')}\right)\right]_{m,n=1}^{p}, \quad y,y' \in Y.$$

(i) K belongs to $PD_p(Y)$.
(ii) If ϕ is not identically 0 and $g(y,y)+g(y',y')-2g(y,y')<0$ for $y \neq y'$, then K belongs to $SPD_p(Y)$.

3. An extension of the main result via integration

Here, we extend the results proved in Section 2 by introducing a scale mixture in the formula that defines the positive definite kernels.

Our first contribution here is as follows.

Theorem 3.1. *Let ρ be a nonzero positive measure on $(0,\infty)$ and ϕ a bounded and completely monotone function. For each m,n in $\{1,\ldots,p\}$, let $G_{m,n} : Y \times Y \to M_q(\mathbb{R})$ be a matrix function with range containing positive definite matrices only, $H_{m,n} : Y \times Y \to \mathbb{R}^q$ a vector function and $\{P^s_{m,n}\}_{s>0}$ a family of kernels on Y such that each function $s \in (0,\infty) \mapsto P^s_{m,n}(y,y')$ is ρ-integrable. If the matrix functions*

$$(y,y') \in Y \times Y \mapsto [u^\intercal G_{m,n}(y,y')u]_{m,n=1}^{p}, \quad u \in \mathbb{R}^q,$$

$$(y,y') \in Y \times Y \mapsto \left[e^{i\,H_{m,n}(y,y')^\intercal u}\right]_{m,n=1}^{p}, \quad u \in \mathbb{R}^q,$$

and

$$(y, y') \in Y \times Y \mapsto [P^s_{m,n}(y, y')]^p_{m,n=1}, \quad s > 0,$$

belong to $CND_p(Y)$, $PD_p(Y)$ and $PD_p(Y)$, respectively, then the kernel $K : Y \times Y \to M_p(\mathbb{R})$ given by the formula

$$K(y, y') = \left[\frac{1}{\sqrt{\det G_{m,n}(y, y')}} \times \int_{(0,\infty)} \phi\left(H_{m,n}(y, y')^\intercal G_{m,n}(y, y')^{-1} H_{m,n}(y, y')\, s\right) P^s_{m,n}(y, y') d\rho(s)\right]^p_{m,n=1}$$

belongs to $PD_p(Y)$.

Proof. Let $y_1, \ldots, y_N$ be distinct points in Y, $c_1, \ldots, c_N$ vectors in $\mathbb{C}^p$ and set

$$Q := \sum_{\mu,\nu=1}^N c^*_\mu K(y_\mu, y_\nu) c_\nu.$$

Direct calculation shows that

$$Q = \int_{(0,\infty)} \sum_{m,n=1}^p \sum_{\mu,\nu=1}^N \frac{\phi\left(\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal G_{m,n}(y_\mu, y_\nu)^{-1} \sqrt{s} H_{m,n}(y_\mu, y_\nu)\right)}{\sqrt{\det G_{m,n}(y_\mu, y_\nu)}} P_{m,n}(y_\mu, y_\nu) d\rho(s).$$

As in the proof of Theorem 2.4, the matrix functions

$$(y, y') \in Y \times Y \mapsto \left[e^{i\sqrt{s} H_{m,n}(y, y')^\intercal u}\right]^p_{m,n=1}, \quad u \in \mathbb{R}^q; s > 0,$$

belong to $PD_p(Y)$. However, since the assumptions on the $G_{m,n}$ are the same as those in Theorem 2.4, we can apply Theorem 2.4-(i) in order to see that each matrix

$$\left[\left[\frac{\phi\left(\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal G_{m,n}(y_\mu, y_\nu)^{-1} \sqrt{s} H_{m,n}(y_\mu, y_\nu)\right)}{\sqrt{\det G_{m,n}(y_\mu, y_\nu)}}\right]^N_{\mu,\nu=1}\right]^p_{m,n=1} \tag{3.7}$$

is positive semi-definite. As for the matrices

$$\left[\left[P^s_{m,n}(y_\mu, y_\nu)\right]^N_{\mu,\nu=1}\right]^p_{m,n=1}, \tag{3.8}$$

they are positive semi-definite as well by our assumption on the family $\{P^s_{m,n}\}_{s>0}$. Thus, the Schur Product Theorem implies that

$$\sum_{m,n=1}^p \sum_{\mu,\nu=1}^N \frac{\phi\left(\sqrt{s} H_{m,n}(y_\mu, y_\nu)^\intercal G_{m,n}(y_\mu, y_\nu)^{-1} \sqrt{s} H_{m,n}(y_\mu, y_\nu)\right)}{\sqrt{\det G_{m,n}(y_\mu, y_\nu)}} P_{m,n}(y_\mu, y_\nu) \geq 0, \quad s > 0.$$

Therefore, $Q \geq 0$. □

As for strict positive definiteness, the following consequence of Theorem 3.1 holds.

Theorem 3.2. *Under the setting of Theorem 3.1, if ϕ is not identically zero, then the following additional assertions hold for the kernel K:*

(i) *If there exists an open subset U of $\mathbb{R}^q \setminus \{0\}$ so that*

$$u^\intercal[G_{m,m}(y,y)+G_{n,n}(y',y')-2G_{m,n}(y,y')]u<0, \quad (m,y)\neq(n,y');\ u\in U,$$

and a ρ-measurable subset A of $(0,\infty)$ so that $\rho(A)>0$ and

$$P^s_{m,m}(y,y)>0, \quad m\in\{1,\dots,p\};\ y\in Y;\ s\in A,$$

then K belongs to $SPD_p(Y)$.

(ii) *If there exists a ρ-measurable subset A of $(0,\infty)$ so that $\rho(A)>0$ and*

$$(y,y')\in Y\times Y\mapsto [P^s_{m,n}(y,y')]_{m,n=1}^p\in SPD_p(Y), \quad s\in A,$$

then K belongs to $SPD_p(Y)$.

Proof. Let the x_μ and the c_μ be as in the proof of Theorem 3.1. Further, assume at least one c_μ is nonzero. If the assumptions in (i) hold, then Theorem 2.4-(ii) implies that each matrix (3.7) is positive definite while the diagonal entries in the matrices in (3.8) are all positive for $s\in A$. Therefore, by Oppenheim's inequality, we can assert that

$$\sum_{m,n=1}^p\sum_{\mu,\nu=1}^N \frac{\phi\left(\sqrt{s}H_{m,n}(y_\mu,y_\nu)^\intercal G_{m,n}(y_\mu,y_\nu)^{-1}\sqrt{s}H_{m,n}(y_\mu,y_\nu)\right)}{\sqrt{\det G_{m,n}(y_\mu,y_\nu)}}P_{m,n}(y_\mu,y_\nu)>0, \quad s\in A.$$

Since the measure ρ is nonzero, $Q>0$. If the assumptions in (ii) hold, we may reach the very same conclusion once the diagonal elements in the matrices in (3.7) are given by

$$\frac{\phi(0)}{\sqrt{\det G_{m,m}(y_\mu,y_\mu)}}>0, \quad m=1,\dots,p;\ \mu=1,\dots,q.$$

Indeed, Oppenheim's inequality once again would imply that $Q>0$. □

A specially chosen family $\{G_{m,n}: m,n=1,\dots,p\}$ in Theorem 3.1 leads to the following improved abstract multivariate version of Gneiting's criterion in [7].

Corollary 3.3. *Let $\phi:(0,\infty)\to\mathbb{R}$ be a bounded and completely monotone function. For $m,n=1,2,\dots,p$, set $G_{m,n}(y,y')=g_{m,n}(y,y')I_q$, $y,y'\in Y$, where each $g_{m,n}$ is a positive valued kernel in $CND_1(Y)$, let $H_{m,n}:Y\times Y\to\mathbb{R}^q$ be a vector function and $\{P^s_{m,n}\}_{s>0}$ a family of kernels on Y such that each function $s\in(0,\infty)\mapsto P^s_{m,n}(y,y')$ is ρ-integrable. If the matrix functions*

$$(y,y')\in Y\times Y\mapsto[u^\intercal G_{m,n}(y,y')u]_{m,n=1}^p, \quad u\in\mathbb{R}^q,$$
$$(y,y')\in Y\times Y\mapsto\left[e^{i\,H_{m,n}(y,y')^\intercal u}\right]_{m,n=1}^p, \quad u\in\mathbb{R}^q,$$

and

$$(y,y')\in Y\times Y\mapsto[P^s_{m,n}(y,y')]_{m,n=1}^p, \quad s>0,$$

belong to $CND_p(Y)$, $PD_p(Y)$ and $PD_p(Y)$, respectively, then the kernel $K:Y\times Y\to M_p(\mathbb{R})$ given by the formula

$$K(y,y')=\left[\frac{1}{g_{m,m}(y,y')^{q/2}}\int_0^\infty\phi\left(\frac{\|H_{m,n}(y,y')\|^2s}{g_{m,n}(y,y')}\right)P^s_{m,n}(y,y')d\rho(s)\right]_{m,n=1}^p$$

belongs to $PD_p(Y)$. Further, if ϕ is not identically 0, the following two additional assertions hold:

(i) *If* $g_{m,m}(y,y) + g_{n,n}(y',y') - 2g_{m,n}(y,y') < 0$ *when* $(m,y) \neq (n,y')$ *and there exists a* ρ-*measurable subset* A *of* $(0,\infty)$ *so that* $\rho(A) > 0$ *and*

$$P^s_{m,m}(y,y) > 0, \quad m \in \{1,\ldots,p\};\ y \in Y;\ s \in A,$$

then K *belongs to* $SPD_p(Y)$.

(ii) *If there exists a* ρ-*measurable subset* A *of* $(0,\infty)$ *so that* $\rho(A) > 0$ *and*

$$(y,y') \in Y \times Y \mapsto [P^s_{m,n}(y,y')]^p_{m,n=1}$$

belongs to $SPD_p(Y)$ *for* $s \in A$, *then* K *belongs to* $SPD_p(Y)$.

4. The main results in the case of a product of sets

An easy way to construct kernels in $PD_p(X \times Y)$ is given by the product of a kernel in $PD_p(X)$ with another one in $PD_p(Y)$, a fact that can be ratified via the Schur Product Theorem. The separable kernels produced by this method may be not suitable if one needs strong interactions between X and Y. The main result in this section will provide a version of Theorem 2.4 that leads to kernels in $PD_p(X \times Y)$ and except for very particular cases, the kernels produced by this version will be nonseparable. In particular, the aforementioned interactions are possible. The result explains, from a mathematical point of view, some important practical models adopted in the statistical literature. The proofs will be omitted once they are very similar to those of the theorems proved in Sections 2 and 3.

Theorem 4.1. *Let* $\phi : (0,\infty) \to \mathbb{R}$ *be a bounded and completely monotone function. For each* m,n *in* $\{1,\ldots,p\}$, *let* $G_{m,n} : Y \times Y \to M_q(\mathbb{R})$ *be a matrix function with range containing positive definite matrices only and* $H_{m,n} : X \times X \to \mathbb{R}^q$ *a vector function. If the matrix functions*

$$(y,y') \in Y \times Y \mapsto [u^\intercal G_{m,n}(y,y')u]^p_{m,n=1}, \quad u \in \mathbb{R}^q,$$

belong to $CND_p(Y)$ *and*

$$(x,x') \in X \times X \mapsto \left[e^{i\, H_{m,n}(x,x')^\intercal u}\right]^p_{m,n=1}, \quad u \in \mathbb{R}^q,$$

belong to $PD_p(X)$, *then the kernel* $K : (X \times Y)^2 \to M_p(\mathbb{R})$ *given by*

$$K((x,y),(x',y')) = \left[\frac{\phi\left(H_{m,n}(x,x')^\intercal G_{m,n}(y,y')^{-1} H_{m,n}(x,x')\right)}{\sqrt{\det G_{m,n}(y,y')}}\right]^p_{m,n=1}$$

belongs to $PD_p(X \times Y)$.

In Example 4.2 below, we illustrate Theorem 4.1 in the case $X = \mathbb{R}$ and $Y = S^d$, the unit sphere in $\mathbb{R}^{d+1}$.

Example 4.2. Define $H_{m,n}(x,x') = h_m(x) - h_n(x')$, $x,x' \in \mathbb{R}$, where each $h_m : \mathbb{R} \to \mathbb{R}^q$ is an arbitrary function. If δ denotes the geodesic distance in S^d, set

$$G_{m,n}(y,y') = [m + n + \delta(y,y')]I_q, \quad y,y' \in S^d.$$

It is well known that $(y,y') \in S^d \times S^d \mapsto \delta(y,y')$ belongs to $CND_1(S^d)$ (see Section 4 in [1]). Hence, each $G_{m,n}$ has range containing positive definite matrices only. On the other hand, according to Examples 2.7 and 2.8, each kernel

$$(y,y') \in Y \times S^d \mapsto [u^\intercal G_{m,n}(y,y')u]^p_{m,n=1}, \quad u \in \mathbb{R}^q,$$

belongs to $CND_p(S^d)$. It follows that

$$K((x,y),(x',y')) = \left[\frac{1}{[m+n+\delta(y,y')]^{q/2}}\phi\left(\frac{\|h_m(x)-h_n(x')\|^2}{m+n+\delta(y,y')}\right)\right]_{m,n=1}^p$$

belongs to $PD_p(\mathbb{R}\times S^d)$. The choice

$$h_m(x) = (x,0,\ldots,0)^\intercal, \quad x\in\mathbb{R};\ m=1,\ldots,p,$$

leads to the simpler example

$$K((x,y),(x',y')) = \left[\frac{1}{[m+n+\delta(y,y')]^{q/2}}\phi\left(\frac{(x-x')^2}{m+n+\delta(y,y')}\right)\right]_{m,n=1}^p$$

belonging to $PD_p(\mathbb{R}\times S^d)$.

A version of Theorem 3.1 for kernels acting on the product $X\times Y$ is as follows.

Theorem 4.3. *Let ρ be a nonzero positive measure on $(0,\infty)$ and ϕ a bounded and completely monotone function. For each m,n in $\{1,\ldots,p\}$, let $G_{m,n}: Y\times Y\to M_q(\mathbb{R})$ be a matrix function with range containing positive definite matrices only, $H_{m,n}: X\times X\to\mathbb{R}^q$ vector functions and $\{P^s_{m,n}\}_{s>0}$ a family of kernels on $X\times Y$ such that each function $s\in(0,\infty)\mapsto P^s_{m,n}((x,y),(x',y'))$ is ρ-integrable. If the matrix functions*

$$(y,y')\in Y\times Y\mapsto [u^\intercal G_{m,n}(y,y')u]_{m,n=1}^p, \quad u\in\mathbb{R}^q,$$
$$(x,x')\in X\times X\mapsto \left[e^{i\,H_{m,n}(x,x')^\intercal u}\right]_{m,n=1}^p, \quad u\in\mathbb{R}^q,$$

and

$$((x,y),(x',y'))\in Y\times Y\mapsto [P^s_{m,n}((x,y),(x',y'))]_{m,n=1}^p, \quad s>0,$$

belong to $CND_p(Y)$, $PD_p(Y)$ and $PD_p(X\times Y)$, respectively, then $K=[K_{m,n}]_{m,n=1}^p: (X\times Y)^2\to M_p(\mathbb{R})$ given by the formula

$$K_{m,n}((x,y),(x',y'))$$
$$=\frac{1}{\sqrt{\det G_{m,n}(y,y')}}\int_{(0,\infty)}\phi\left(H_{m,n}(x,x')^\intercal G_{m,n}(y,y')^{-1}H_{m,n}(x,x')\,s\right)P^s_{m,n}((x,y),(x',y'))d\rho(s)$$

belongs to $PD_p(X\times Y)$.

We now move to some specific applications of Theorem 4.3.

Example 4.4. Here, we will employ the formula deduced in Theorem 1.1 in [6]:

$$\mathcal{M}_\nu(r\sqrt{u}) = \frac{r^{2\nu}}{2^{2\nu}\Gamma(\nu)}\int_0^\infty e^{-s\,u}e^{-r^2/4s}s^{-\nu-1}ds, \quad r,u>0,$$

that defines the so-called *Matérn function*. This function is studied in details in [6]. We may apply Theorem 3.1 with $\phi(u)=\exp(-u)$, $u>0$ and $d\rho(s)=e^{-r^2/4s}s^{-1}ds$. If for $x,x'\in X$ and $y,y'\in Y$ we set

$$2v_{m,n}((x,y),(x',y')) := v_m(x,y)+v_n(x',y'),$$

where $v_m: X\times Y\to(0,\infty)$, for all m, and

$$P^s_{m,n}((x,y),(x',y')) := \frac{r^{2v_{m,n}((x,y),(x',y'))}s^{-v_{m,n}((x,y),(x',y'))}}{2^{2v_{m,n}((x,y),(x',y'))}},$$

for $x, x' \in X$ and $y, y' \in Y$, it is easily seen that the kernels

$$((x,y),(x',y')) \in (X\times Y)^2 \mapsto \left[P^s_{m,n}((x,x'),(y,y'))\right]_{m,n=1}^p, \quad s>0,$$

belong to $PD_p(X\times Y)$. If each $s\in(0,\infty)\mapsto s^{-v_{m,n}((x,y),(x',y'))/2}$ is ρ-integrable, Theorem 4.3 implies that the formula

$$K_{m,n}((x,y),(x',y')) = \frac{\Gamma(v_{m,n}((x,y),(x',y')))}{\sqrt{\det G_{m,n}(y,y')}} \times \mathcal{M}_{v_{m,n}((x,y),(x',y'))}(r(H_{m,n}(x,x')^\intercal G_{m,n}(y,y')^{-1}H_{m,n}(x,x'))^{1/2})$$

defines a kernel $K((x,y),(x',y')) = [K_{m,n}((x,y),(x',y'))]_{m,n=1}^p$ that belongs to $PD_p(X\times Y)$, as long as the $G_{m,n}$ and the $H_{m,n}$ satisfy the assumptions of the theorem. We could also modify the $P^s_{m,n}$ by introducing a matrix $[r_{m,n}]_{m,n=1}^p$ with positive entries, by setting

$$P^s_{m,n}((x,y),(x',y')) := \frac{r_{m,n}^{2v_{m,n}((x,y),(x',y'))}s^{-v_{m,n}((x,y),(x',y'))}}{2^{2v_{m,n}((x,y),(x',y'))}}$$

for $x, x' \in X$ and $y, y' \in Y$, as long as the kernels

$$((x,y),(x',y')) \in (X\times Y)^2 \mapsto \left[P^s_{m,n}((x,y),(x',y'))\right]_{m,n=1}^p, \quad s>0,$$

stay in $PD_p(X\times Y)$. In this case, the outcome of Theorem 3.1 would be that the formula

$$K_{m,n}((x,y),(x',y')) = \frac{\Gamma(v_{m,n}((x,y),(x',y')))}{\sqrt{\det G_{m,n}(y,y')}} \times \mathcal{M}_{v_{m,n}((x,y),(x',y'))}(r_{m,n}(H_{m,n}(x,x')^\intercal G_{m,n}(y,y')^{-1}H_{m,n}(x,x'))^{1/2})$$

defines a kernel $K((x,y),(x',y')) = [K_{m,n}((x,y),(x',y'))]_{m,n=1}^p$ in $PD_p(X\times Y)$, if we keep the assumptions on the $G_{m,n}$ and the $H_{m,n}$ required in the theorem. An specific and simple example in the space-time setting can be produced in analogy with Theorem 1 in [5]: set $Y=\mathbb{R}^d$, $X=\mathbb{R}$,

$$G_{m,n}(y,y') = g(\|y-y'\|^2)I_q, \quad y,y'\in\mathbb{R}^d; m,n=1,\ldots,p,$$

where $g:(0,\infty)\to(0,\infty)$ has a completely monotone derivative and

$$H_{m,n}(x,x') = x-x', \quad x,x'\in\mathbb{R}; m,n=1,\ldots,p.$$

Since $(y,y')\in\mathbb{R}^d\times\mathbb{R}^d\mapsto g(\|y-y'\|^2)$ belongs to $CND_1(\mathbb{R}^d)$ by a result of Micchelli ([17]), it follows that the matrix kernels $(y,y')\in Y\times Y\mapsto [u^\intercal G_{m,n}(y,y')u]_{m,n=1}^p$, $u\in\mathbb{R}^q$, belong to $CND_p(\mathbb{R}^d)$. If we put

$$v_{m,n}((x,y),(x',y')) = \frac{v_m+v_n}{2}, \quad x,x'\in\mathbb{R}; y,y'\in Y; m,n=1,\ldots,p,$$

in which each v_m is a positive constant and properly specify $[r_{m,n}]_{m,n=1}^p$, then for $x,x'\in\mathbb{R}$ and $y,y'\in\mathbb{R}^d$ the formula

$$P^s_{m,n}((x,y),(x',y')) := \frac{r_{m,n}^{v_m+v_n}s^{-(v_m+v_n)/2}}{2^{v_m+v_n}}, \quad m,n=1,\ldots,p,$$

defines kernels

$$((x,y),(x',y')) \in (\mathbb{R}\times\mathbb{R}^d)^2 \mapsto [P^s_{m,n}((x,y),(x',y'))]_{m,n=1}^p, \quad s>0,$$

in $PD_p(\mathbb{R}\times\mathbb{R}^d)$. An application of Theorem 4.3 would lead to

$$K_{m,n}((x,y),(x',y')) = \frac{\Gamma((v_m+v_n)/2)}{g(\|y-y'\|^{p/2})}\mathcal{M}_{(v_m+v_n)/2}\left(r_{mn}\frac{\|x-x'\|^2}{g(\|y-y'\|^2)}\right)$$

with $K((x,y),(x',y')) = [K_{m,n}((x,y),(x',y'))]_{m,n=1}^p$ in $PD_p(\mathbb{R}\times\mathbb{R}^d)$. We observe that the factor $\Gamma((v_m+v_n)/2)$ can be eliminated as long as we can specify $[r_{mn}]_{m,n=1}^p$ in such a way that $[r_{m,n}^{v_m+v_n}/\Gamma((v_m+v_n)/2)]_{m,n=1}^p$ is a positive definite matrix. Theorem 1 in [11] is another construction that fits into Theorem 4.3. Details on that will be left to the readers.

Example 4.5. The so-called generalized Cauchy function ([8, p. 337]) is given by

$$\frac{1}{(1+cu^\gamma)^\nu} = \frac{c^{-\nu}}{\Gamma(\nu)}\int_0^\infty e^{-s\,u^\gamma}s^\nu d\rho(s), \quad u\geq 0,$$

where $c>0$, $\nu>1$, $\gamma\in(0,1]$ and $d\rho(s) = s^{-1}\exp(-s/c)$. In order to apply Theorem 4.3, we now set $\phi(u) = e^{-u^\gamma}$, $u>0$ and

$$P^s_{m,n}((x,y),(x',y')) = \left(\frac{s}{c}\right)^{v_m(x,y)+v_n(x',y')}, \quad s>0;\ y,y'\in Y,$$

where $v_m : X\times Y \to (0,\infty)$ is chosen in such a way that each $s\in(0,\infty)\mapsto s^{v_m(x,y)/2}$ is ρ-integrable. The outcome is that

$$K_{m,n}((x,y),(x',y')) = \frac{\Gamma(v_m(x,y)+v_n(x',y'))}{\sqrt{\det G_{m,n}(y,y')}} \times \frac{1}{(1+c(H_{m,n}(x,x')^\intercal G_{m,n}(y,y')^{-1}H_{m,n}(x,x')^\gamma))^{v_m(x,y)+v_n(x',y')}},$$

defines a kernel $K((x,y),(x',y')) = [K_{m,n}((x,y),(x',y'))]_{m,n=1}^p$ in $PD_p(X\times Y)$, if we keep the assumptions on the $G_{m,n}$ and the $H_{m,n}$ required in the theorem. Arguments similar to those developed in the second half of Example 4.4 leads to an example aligned with Theorem 2 in [5].

5. A further extension

As a final remark let us point an improvement that one can make in all the theorems proved in this paper. If for each m and n in $\{1,\ldots,p\}$, $G_{m,n}: Y\times Y\to M_q(\mathbb{R})$ is a matrix function with range containing positive definite matrices only, Theorem 2.4 justifies the following fact: if the matrix kernels

$$(y,y')\in Y\times Y\mapsto [u^\intercal G_{m,n}(y,y')u]_{m,n=1}^p, \quad u\in\mathbb{R}^q,$$

belong to $CND_p(Y)$, then the kernel K given by

$$K(y,y') = \left[\frac{1}{\sqrt{\det G_{m,n}(y,y')}}\right]_{m,n=1}^p, \quad y,y'\in Y,$$

belongs to $PD_p(Y)$. Under the same setting, it follows from the Schur Product Theorem that

$$K_l(y,y') = \left[\frac{1}{[\det G_{m,n}(y,y')]^{l/2}}\right]_{m,n=1}^p, \quad y,y'\in Y,$$

belongs to $PD_p(Y)$ whenever $l\in\{1,2,\ldots\}$. In particular, we can introduce the same power $l/2$ in the assertions of all the theorems proved in the paper.

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Parameterized families of polylog integrals

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ABSTRACT. It is commonly known that integrals containing log-polylog integrands admit representations in terms of special functions such as the Dirichlet eta and Dirichlet beta functions. We investigate two parameterized families of such integrals and in a particular case demonstrate a connection with the Herglotz function. In the process of the investigation, we recover some known Euler sum equalities and discover some new identities.

Keywords: Harmonic number, gamma function, alternating harmonic number, Riemann zeta function, polylogarithm function, polygamma functions, linear Euler sums.

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1. INTRODUCTION, PRELIMINARIES AND NOTATION

In the recent past many books ([5], [20], [35]) have been published whereby the authors describe the connection of the representation of some integrals in terms of Euler sums. Likewise the following papers investigate certain integrals that can be represented by Euler sums [7], [17], [27]. In this paper, we consider two parameterized families of log-polylog integrals that admit solutions dependent on Euler sums, thereby extending the integrals considered by ([3], [6], [14], [23], [36]). We investigate parameterized families of integrals of the type

$$I^{b}_{+,-}(a,p,q,t) = \int_{x} \frac{x^{a}\ln^{p}(x)\,\mathrm{Li}_{t}(x^{bq})}{1 \pm x^{b}}\,dx, \tag{1.1}$$

$$K^{b}_{+,-}(a,p,q,t) = \int_{x} \frac{x^{a}\ln^{p}(x)\,\mathrm{Li}_{t}(-x^{bq})}{1 \pm x^{b}}\,dx,$$

where $a \geq -2,\ b \in \mathbb{R}^{+},\ p \in \mathbb{N}_0,\ q \in \mathbb{N},\ t \in \mathbb{N}_0$ and for the domain of $x \in (0,1)$. Here and elsewhere, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^{+}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers, positive real numbers, integers and positive integers respectively and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^{-} := \mathbb{Z} \backslash \mathbb{N}_0$. In the case $(a,b) = (0,2)$, we also study the integrals

$$J(p,q,t) = \int_{x} \frac{\ln^{p}(x)\,\mathrm{Li}_{t}(x^{2q})}{1 - x^{2}}\,dx, \tag{1.2}$$

$$M(p,q,t) = \int_{x} \frac{\ln^{p}(x)\,\mathrm{Li}_{t}(-x^{2q})}{1 - x^{2}}\,dx$$

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in the positive half line $x \geq 0$. In a particular case of the integral $K_{+}^{b}(a,p,q,t)$, we make a connection with the Herglotz function [15]. Some other related papers dealing with polylog integrals and Euler sums are [4], [9], [24], [25], [26] and the excellent books [18] and [34]. We describe some notation and special functions, to be used in the following, in the analysis of the integrals (1.1) and (1.2). The generalized harmonic number $H_n^{(t)}(\alpha)$ are defined as

$$H_n^{(t)}(\alpha) = \sum_{j=1}^{n} \frac{1}{(j+\alpha)^t}, \alpha \in \mathbb{C}\backslash\{-1,-2,-3,...\}, t \in \mathbb{C}, n \in \mathbb{N}$$

and when $\alpha = 0$, $H_n^{(t)}(0) = H_n^{(t)}$ are ordinary harmonic numbers of order t, an empty sum is designated as $H_0^{(t)} = 0$. For complex values of z, $z \in \mathbb{C}\backslash\{0,-1,-2,-3,...\}$, $\psi(z)$ is the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z)$ is the familiar gamma function, (see, e.g. [33], sections 1.1 and 1.3). We know that for $n \geq 1$, $\psi(n+1) - \psi(1) = H_n$ with $\psi(1) = -\gamma$, where γ is the Euler Mascheroni constant and $\psi(n)$ is the digamma function. The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}.$$

The difference of the polygamma functions and generalized harmonic numbers are connected by the zeta function such that for $z \in \mathbb{C}\backslash\{0,-1,-2,-3,...\}$ we have the identity

$$H_z^{(m+1)} - \frac{(-1)^m}{m!}\psi^{(m)}(z+1) = \zeta(m+1). \tag{1.3}$$

The Dirichlet lambda function $\lambda(z)$,

$$\zeta(z) + \eta(z) = 2\lambda(z) \tag{1.4}$$

connects the zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, with the alternating zeta function $\eta(z)$. It is widely known that integrals of the type (1.1) may be represented by Euler sums and therefore in terms of special functions such as the Dirichlet beta function. The following papers [27], [28] and [29] also examined some integrals in terms of Euler sums. Some examples will be given highlighting specific cases of the integrals, some of which cannot be evaluated by a computer mathematical package such as "Mathematica".

2. Polylog Integrals With Positive Argument

Consider the following.

Theorem 2.1. *Let* $(p,q,t) \in \mathbb{N}_0, q \neq 0, a \geq -2,$ *and denote,*

$$I_{+}(a,p,q,t) = \int_0^1 \frac{x^a \ln^p(x) \operatorname{Li}_t(x^q)}{1+x} dx. \tag{2.5}$$

For an even integer q

$$I_{+}(a,p,q,t) = (-1)^p p! \sum_{n\geq 1} H_n^{(t)} \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}}, \tag{2.6}$$

for an odd integer q

$$I_+\left(a,p,q,t\right)=\left(-1\right)^p p!\sum_{n\geq 1}\left(-1\right)^{n+1}\left(H_n^{(t)}-\frac{1}{2^{t-1}}H_{\left[\frac{n}{2}\right]}^{(t)}\right)\sum_{j=1}^{q}\frac{\left(-1\right)^{j+1}}{\left(qn+j+a\right)^{p+1}}, \tag{2.7}$$

and for $q\in\mathbb{R}^+\setminus\{0\}$

$$I_+\left(a,p,q,t\right)=\frac{\left(-1\right)^p p!}{2^{p+1}}\sum_{n\geq 1}\frac{1}{n^t}\left(H_{\frac{qn+a}{2}}^{(p+1)}-H_{\frac{qn+a-1}{2}}^{(p+1)}\right), \tag{2.8}$$

where $H_n^{(t)}$ *are harmonic numbers of order* t *and* $[z]$ *denotes the greatest integer that is less than or equal to* z.

Proof. The alternating harmonic numbers $A\left(n,t\right)$ of order t are defined by

$$A\left(n,t\right)=\sum_{k=1}^{n}\frac{\left(-1\right)^{k+1}}{k^t},\quad n\in\mathbb{N};t\in\mathbb{C} \tag{2.9}$$

then, see [2],

$$A\left(n,t\right)=\sum_{k=1}^{n}\frac{\left(-1\right)^{k+1}}{k^t}=H_n^{(t)}-\frac{1}{2^{t-1}}H_{\left[\frac{n}{2}\right]}^{(t)}.$$

The Dirichlet eta function

$$\eta\left(t\right)=\lim_{n\to\infty}A\left(n,t\right)=\sum_{n\geq 1}\frac{\left(-1\right)^{n+1}}{n^t}, Re\left(t\right)>0.$$

For $x\in\left(0,1\right)$, a Taylor series expansion gives

$$\mathrm{Li}_t(x^q)=\sum_{n\geq 1}\frac{x^{qn}}{n^t},\ \frac{1}{1+x}=\sum_{n\geq 0}\left(-1\right)^n x^n.$$

By the Cauchy product of two convergent series, then it follows that for q an even integer

$$\frac{x^a\mathrm{Li}_t(x^q)}{1+x}=\sum_{n\geq 1}H_n^{(t)}\sum_{j=1}^{q}\left(-1\right)^{j+1}x^{qn+j+a-1}$$

and therefore, for q an even integer

$$\frac{x^a\ln^p\left(x\right)\mathrm{Li}_t(x^q)}{1+x}=\sum_{n\geq 1}H_n^{(t)}\sum_{j=1}^{q}\left(-1\right)^{j+1}x^{qn+j+a-1}\ln^p\left(x\right).$$

Integrating both sides for $x\in(0,1)$, we have, after reversing the order of summation and integration, which is justified by the uniform convergence theorem

$$\begin{aligned}\int_0^1\frac{x^a\ln^p\left(x\right)\mathrm{Li}_t(x^q)}{1+x}dx&=\sum_{n\geq 1}H_n^{(t)}\sum_{j=0}^{q}\left(-1\right)^{j+1}\int_0^1 x^{qn+j+a-1}\ln^p\left(x\right)dx\\&=\left(-1\right)^p p!\sum_{n\geq 1}H_n^{(t)}\sum_{j=1}^{q}\frac{\left(-1\right)^{j+1}}{\left(qn+j+a\right)^{p+1}}.\end{aligned}$$

For q an odd integer, we have

$$\frac{x^a\ln^p\left(x\right)\mathrm{Li}_t(x^q)}{1+x}=\sum_{n\geq 1}\left(-1\right)^{n+1}A\left(n,t\right)\sum_{j=1}^{q}\left(-1\right)^{j+1}x^{qn+j+a-1}\ln^p\left(x\right)$$

and

$$\int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(x^q)}{1+x}dx = (-1)^p\, p! \sum_{n\geq 1} (-1)^{n+1} \left(H_n^{(t)} - \frac{1}{2^{t-1}} H_{\left[\frac{n}{2}\right]}^{(t)} \right) \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}}.$$

By simple expansion, we have

$$\sum_{n\geq 1} \frac{(-1)^{n+1} H_{\left[\frac{n}{2}\right]}^{(t)}}{(qn+\alpha)^{p+1}} = \sum_{n\geq 1} \frac{H_n^{(t)}}{(q(2n+1)+\alpha)^{p+1}} - \sum_{n\geq 1} \frac{H_n^{(t)}}{(2qn+\alpha)^{p+1}}$$

and therefore we can also express

$$\begin{aligned}\frac{(-1)^p}{p!} \int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(x^q)}{1+x}dx &= \sum_{n\geq 1} (-1)^{n+1} H_n^{(t)} \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(qn+j+a)^{p+1}} + \frac{1}{2^{t-1}} \sum_{n\geq 1} H_n^{(t)} \\ &\times \left(\sum_{j=1}^{q} \frac{(-1)^{j+1}}{(2qn+j+a)^{p+1}} - \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(q(2n+1)+j+a)^{p+1}} \right).\end{aligned}$$

For the representation (2.8), we can write

$$\begin{aligned}\int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(x^q)}{1+x}dx &= (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \sum_{j\geq 0} \frac{(-1)^j}{(qn+j+a+1)^{p+1}} \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(\zeta\left(p+1, \frac{qn+a+1}{2}\right) - \zeta\left(p+1, \frac{qn+a+2}{2}\right) \right) \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(\psi^{(p)}\left(\frac{qn+a+2}{2}\right) - \psi^{(p)}\left(\frac{qn+a+1}{2}\right) \right) \\ &= \frac{(-1)^p\, p!}{2^{p+1}} \sum_{n\geq 1} \frac{1}{n^t} \left(H_{\frac{qn+a}{2}}^{(p+1)} - H_{\frac{qn+a-1}{2}}^{(p+1)} \right)\end{aligned}$$

and the proof is finished. □

The next theorem deals with a related integral similar to (2.5).

Theorem 2.2. *For* $(p,t) \in \mathbb{N}, a \geq -2$, *and for* q *a positive integer, then*

$$I_-(a,p,q,t) = \int_0^1 \frac{x^a \ln^p(x)}{1-x} \mathrm{Li}_t(x^q)dx$$

$$= (-1)^p\, p! \sum_{n\geq 1} H_n^{(t)} \sum_{j=1}^{q} \frac{1}{(qn+j+a)^{p+1}}. \tag{2.10}$$

For $q \in \mathbb{R}^+ \setminus \{0\}$

$$I_-(a,p,q,t) = (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \left(\zeta(p+1) - H_{nq+a}^{(p+1)} \right), \tag{2.11}$$

where $H_{nq+a}^{(p+1)}$ *are shifted harmonic numbers of order* $p+1$.

Proof. A Taylor series expansion of

$$\mathrm{Li}_t(x^q) = \sum_{n\geq 1} \frac{x^{qn}}{n^t} \text{ and } \frac{1}{1-x} = \sum_{j\geq 0} x^j$$

allows us to write

$$\begin{aligned} I_-(a,p,q,t) &= \sum_{n\geq 1} H_n^{(t)} \sum_{j=1}^{q} \int_0^1 x^{qn+j+a-1} \ln^p(x)\, dx \\ &= (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \sum_{j=1}^{q} \frac{1}{(qn+j+a)^{p+1}}. \end{aligned}$$

For the representation (2.11), we notice

$$\begin{aligned} I_-(a,p,q,t) &= \sum_{n\geq 1} \frac{1}{n^t} \sum_{j\geq 0} \frac{(-1)^p\, p!}{(qn+j+a+1)^{p+1}} \\ &= (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \zeta(p+1, qn+a+1) \\ &= (-1)^p\, p! \sum_{n\geq 1} \frac{(-1)^{p+1}}{p! n^t} \psi^{(p)}(qn+a+1). \end{aligned}$$

From the identity (1.3), we obtain the required representation

$$I_-(a,p,q,t) = (-1)^p\, p! \sum_{n\geq 1} \frac{1}{n^t} \left(\zeta(p+1) - H_{nq+a}^{(p+1)}\right).$$

We remark that Coffey [7] obtained solutions of various special cases of $I_-(0,p,1,1)$ in terms of Euler sums. □

Remark 2.1. *For $(p,q) \in \mathbb{N}_0$, we see from (2.10) and (2.11) the remarkable Euler sum identity*

$$\sum_{n\geq 1} H_n^{(t)} \sum_{j=1}^{q} \frac{1}{(qn+j+a)^{p+1}} = \sum_{n\geq 1} \frac{1}{n^t} \left(\zeta(p+1) - H_{nq+a}^{(p+1)}\right). \tag{2.12}$$

Using the notation developed by [13] and generalized by the authors of the paper [1], we define

$$S_{p,q}^{++}(\alpha,\beta) = \sum_{n\geq 1} \frac{H_n^{(p)}(\alpha)}{(n+\beta)^q}, \quad S_{p,q}^{+-}(\alpha,\beta) = \sum_{n\geq 1} \frac{(-1)^{n+1} H_n^{(p)}(\alpha)}{(n+\beta)^q},$$

where

$$\zeta(p,\alpha) = H_n^{(p)}(\alpha) = \sum_{j=1}^{n} \frac{1}{(n+\alpha)^p}, \qquad n \in \mathbb{N}, p \in \mathbb{C}, \alpha \in \mathbb{C}\backslash\mathbb{Z}^-.$$

In the case $\alpha = 0, \beta = 0$, we write $S_{p,q}^{++}(0,0) = S_{p,q}^{++}$ and $S_{p,q}^{+-}(0,0) = S_{p,q}^{+-}$. For $a = 0$, upon rearranging and simplifying we obtain a new Euler identity

$$\sum_{n\geq 1} H_n^{(t)} \sum_{j=1}^{q-1} \frac{1}{(qn+j)^{p+1}} + \sum_{n\geq 1} \frac{H_{qn}^{(p+1)}}{n^t} + \frac{1}{q^{p+1}} S_{t,p+1}^{++} = \zeta(t)\zeta(p+1) + \frac{1}{q^{p+1}} \zeta(t+p+1).$$

If we choose $q=1, a\in\mathbb{R}, a>-1$, *then*

$$S^{++}_{p+1,t}(0,a)+S^{++}_{t,p+1}(a,0)=\zeta(t)\zeta(p+1)+\sum_{n\geq1}\frac{1}{n^t(n+a)^{p+1}},$$

which confirms Theorem 3.1 obtained by [1], *using shuffle (or reciprocity) properties of Euler sums. If we now let* $a=0$, *we recover the well known identity*

$$S^{++}_{p+1,t}+S^{++}_{t,p+1}=\zeta(t)\zeta(p+1)+\zeta(t+p+1).$$

In the special case $p+1=t$,

$$\sum_{n\geq1}H^{(t)}_n\sum_{j=1}^{q-1}\frac{1}{(qn+j)^t}+\sum_{n\geq1}\frac{H^{(t)}_{qn}}{n^t}=\left(1-\frac{1}{q^{t+1}}\right)\zeta^2(t)+\frac{1}{q^{t+1}}\zeta(2t).$$

From (2.12) *with* $q=2, a=0$ *(and renaming* $p+1$ *as* p*), we have*

$$\frac{1}{2^p}S^{++}_{p,t}\left(0,\frac12\right)=\zeta(t)\zeta(p)-\frac{1}{2^p}\zeta(t+p)+\left(\frac{1}{2^{p-1}}-2^{t-1}\right)S^{++}_{p,t}-\frac{1}{2^p}S^{++}_{t,p}+2^{t-1}S^{+-}_{p,t}$$

and when $p=t$, *we can simplify to obtain the new identity*

$$\frac{1}{2^t}S^{++}_{t,t}\left(0,\frac12\right)-2^{t-1}S^{+-}_{t,t}=\zeta(t)\eta(t)-2^{t-2}\eta(2t)+\left(\frac{3}{2^{t+1}}-2^{t-2}\right)\zeta^2(t).$$

In terms of the harmonic numbers at an argument of half integer values we have

$$\frac{1}{2^t}S^{++}_{t,t}\left(0,\frac12\right)-2^{t-1}S^{+-}_{t,t}=2^{t-1}S^{++}_{t,t}-2^{t-1}\lambda(t)\eta(t)-\frac12\sum_{n\geq1}\frac{H^{(t)}_{\frac{n}{2}}}{n^t}-\frac12\sum_{n\geq1}\frac{(-1)^{n+1}H^{(t)}_{\frac{n}{2}}}{n^t},$$

where it has been shown in [29] *that*

$$\sum_{n\geq1}\frac{(-1)^{n+1}H^{(t)}_{\frac{n}{2}}}{n^t}=2^{t-1}\left(\eta(2t)-\eta^2(t)\right)$$

and

$$\sum_{n\geq1}\frac{H^{(t)}_{\frac{n}{2}}}{n^t}=2^{t-1}\left(\eta(2t)-\eta^2(t)\right)+\frac{1}{2^t}\left(\zeta(2t)+\zeta^2(t)\right).$$

Some other log-sine-polylog integrals involving alternating Euler sums have recently been investigated by [17].

Remark 2.2. *For the two cases where* $b\in\mathbb{R}^+, a+1>-b$

$$I^b_+(a,p,q,t)=\int_0^1\frac{x^a\ln^p(x)\operatorname{Li}_t(x^{bq})}{1+x^b}dx=\frac{1}{b^{p+1}}I_+\left(\frac{a+1-b}{b},p,q,t\right) \tag{2.13}$$

$$=\frac{(-1)^{p+1}p!}{(2b)^{p+1}}\sum_{n\geq1}\frac{1}{n^t}\left(H^{(p+1)}_{\frac{qn}{2}+\frac{a+1-2b}{2b}}-H^{(p+1)}_{\frac{qn}{2}+\frac{a+1-b}{2b}}\right)$$

and

$$I^b_-(a,p,q,t)=\int_0^1\frac{x^a\ln^p(x)\operatorname{Li}_t(x^{bq})}{1-x^b}dx=\frac{1}{b^{p+1}}I_-\left(\frac{a+1-b}{b},p,q,t\right). \tag{2.14}$$

The following theorem applies.

Theorem 2.3. *For $p,q,t\in\mathbb{N},\ a=0$ and $b=2$ then*

$$J(p,q,t)=\int_0^\infty \frac{\ln^p(x)\,\mathrm{Li}_t(x^{2q})}{1-x^2}dx=\int_0^\infty f(x;p,q,t)\,dx$$

$$=\int_0^1 \ln^p(\tanh\theta)\,\mathrm{Li}_t(\tanh^{2q}\theta)d\theta \tag{2.15}$$

$$=\left(1+(-1)^{p+t}\right)I_-^2(0,p,q,t)+(-1)^{p+t}\frac{(2\pi i)^t}{t!}\int_0^1\frac{\ln^p(x)}{1-x^2}B\left(t,\frac{\ln\left(x^{2q}\right)}{2\pi i}\right)dx, \tag{2.16}$$

where

$$f(x;p,q,t)=\frac{\ln^p(x)}{1-x^2}\mathrm{Li}_t(x^{2q}),$$

$I_-^2(0,p,q,t)$ is given by (2.14) and $B\left(t,\frac{\ln(x^{2q})}{2\pi i}\right)$ is the Bernoulli polynomial.

Proof. We begin with

$$J(p,q,t)=\int_0^\infty \frac{\ln^p(x)\,\mathrm{Li}_t(x^{2q})}{1-x^2}dx=\int_0^\infty f(x;p,q,t)\,dx$$

and put

$$J(p,q,t)=\int_0^\infty f(x;p,q,t)\,dx=\int_0^1 f(x;p,q,t)\,dx+\int_1^\infty f(x;p,q,t)\,dx.$$

We notice that $f(x;p,q,t)$ is continuous, bounded and differentiable on the interval $x\in(0,1]$, with $\lim\limits_{x\to0^+}f(x;p,q,t)=\lim\limits_{x\to1}f(x;p,q,t)=0$. Now we make the transformation $xy=1$ in the third integral so that

$$\int_0^\infty f(x;p,q,t)\,dx=\int_0^1 f(x;p,q,t)\,dx+(-1)^p\int_0^1\frac{\ln^p(y)}{1-y^2}\mathrm{Li}_t(y^{-2q})dy. \tag{2.17}$$

From Erdělyi et. al. [11], Jonquiěre's relation states

$$\mathrm{Li}_s(z)+e^{i\pi s}\mathrm{Li}_s(\frac{1}{z})=\frac{\left(2\pi e^{i\pi}\right)^s}{\Gamma(s)}\zeta\left(1-s,\frac{\ln z}{2\pi i}\right), \tag{2.18}$$

where $\mathrm{Li}_s(z)$ is a polylogarithm, $i=\sqrt{-1},\ \Gamma(s)$ is the gamma function, $s\in\mathbb{C}$ and $\zeta\left(1-s,\frac{\ln z}{2\pi i}\right)$ is the Hurwitz zeta function and z is not a member of the real interval $[0,1]$. A modified version of (2.18) is given by Crandall [9] as follows. For integer t and $z\in\mathbb{C}$,

$$\mathrm{Li}_t(z)+(-1)^t\,\mathrm{Li}_t(\frac{1}{z})=-\frac{(2\pi i)^t}{t!}B\left(t,\frac{\ln(z)}{2\pi i}\right)-2\pi i\Theta(z)\frac{\ln^{t-1}(z)}{(t-1)!}, \tag{2.19}$$

where $B\left(t, \frac{\ln(z)}{2\pi i}\right)$ is the Bernoulli polynomial (see, e.g. [33], sections 1.7), and $\Theta(z)$ is a time dependent step function

$$\Theta(z) = \begin{cases} 1, \text{ if } Im(z) < 0 \text{ or } \text{ z} \in [1, \infty) \\ 0, \text{ otherwise} \end{cases}.$$

The function $\Theta(z)$ is intended to provide the conventional behavior in the branch when and only when z is in the lower half plane union with the real cut $[1, \infty)$. For convenience, we list

$$B\left(4, \frac{\ln(z)}{2\pi i}\right) = \frac{1}{16\pi^4} \ln^4 z - \frac{i}{4\pi^3} \ln^3 z - \frac{i}{4\pi^2} \ln^2 z - \frac{1}{30},$$

$$B\left(3, \frac{\ln(z)}{2\pi i}\right) = -\frac{i}{4\pi} \ln z + \frac{3}{8\pi^2} \ln^2 z + \frac{i}{8\pi^3} \ln^3 z.$$

Now, we can substitute (2.19) into (2.17), so that

$$\int_0^\infty f(x; p, q, t)\, dx = \left(1 + (-1)^{p+t}\right) \int_0^1 f(x; p, q, t)\, dx + (-1)^{p+t} \frac{(2\pi i)^t}{t!} \int_0^1 \frac{\ln^p(x)}{1 - x^2} B\left(t, \frac{\ln(x^{2q})}{2\pi i}\right) dx.$$

The integral

$$I_-^2(0, p, q, t) = \int_0^1 \frac{\ln^p(x) \operatorname{Li}_t(x^{2q})}{1 - x^2} dx$$

has been evaluated in Theorem 2.1 and therefore

$$J(p, q, t) = \left(1 + (-1)^{p+t}\right) I_-^2(0, p, q, t) + (-1)^{p+t} \frac{(2\pi i)^t}{t!} \int_0^1 \frac{\ln^p(x)}{1 - x^2} B\left(t, \frac{\ln(x^{2q})}{2\pi i}\right) dx$$

and the proof is finished. Note that the integral $I_-^2(0, p, q, t)$ does not contribute to $J(p, q, t)$ in the case when $p + t$ is an odd integer. The third integral in (2.15) is obtained by the substitution $x = \tanh\theta$. □

Remark 2.3. *Utilizing (2.19) we are able to evaluate the related integral, from Theorem 2.2, (or from (2.14))*

$$\int_0^1 \frac{x^a \ln^p(x) \operatorname{Li}_t(x^{-bq})}{1 - x^b} dx = (-1)^{t+1} I_-^b(a, p, q, t) + (-1)^{t+1} \frac{(2\pi i)^t}{t!} \int_0^1 \frac{x^a \ln^p(x) B\left(t, \frac{\ln(x^{bq})}{2\pi i}\right)}{1 - x^b} dx.$$

Some examples follow. First we record here the following result, given in [27], that will be required for the evaluation of some Euler sums.

Theorem 2.4. *Let α be a real number $\alpha \neq -1, -2, -1, \ldots$, and assume that $m \in \mathbb{N} \backslash \{1\}$. Then*

$$\sum_{n \geq 1} \frac{H_n}{(n + \alpha)^m} = \frac{(-1)^m}{(m-1)!} \left[\begin{array}{c} (\psi(\alpha) + \gamma)\, \psi^{(m-1)}(\alpha) \\ -\frac{1}{2}\psi^{(m)}(\alpha) + \sum_{j=1}^{m-2} \binom{m-2}{j} \psi^{(j)}(\alpha)\, \psi^{(m-j-1)}(\alpha) \end{array} \right],$$

where γ is the Euler Mascheroni constant.

Example 2.1.
1. For $a=-1, q=2, p=2m, m\in\mathbb{N}$ and $t+p$ of even weight

$$I_+(-1,2m,2,t)=\int_0^1 \frac{x^{-1}\ln^{2m}(x)\operatorname{Li}_t(x^2)}{1+x}dx$$
$$=\frac{(2m)!}{2^{2m+1}}S^{++}_{t,2m+1}-\frac{(2m)!}{2^{2m+1}}S^{++}_{t,2m+1}\left(0,\frac{1}{2}\right)$$

and can be evaluated explicitly in terms of special functions since we have the known Euler sum relations, $S^{++}_{t,2m+1}$ and $S^{++}_{t,2m+1}\left(0,\frac{1}{2}\right)$ defined in Remark 2.1.
2. For $t=p=q=1$ and $a=-\frac{1}{2}$

$$I_+\left(-\frac{1}{2},1,1,1\right)=\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{1}{4}\right)^2}-\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{3}{4}\right)^2}-\sum_{n\geq1}\frac{(-1)^{n+1}H_n}{\left(n+\frac{1}{2}\right)^2},$$

here, the Euler sums $\sum_{n\geq1}\frac{H_n}{(n+x)^m}$ are evaluated using Theorem 2.4, so that

$$I_+\left(-\frac{1}{2},1,1,1\right)=8G\ln2+8Im\left(\operatorname{Li}_3\left(\frac{1\pm i}{2}\right)\right)-\frac{1}{4}\pi\ln^2 2-\frac{5}{16}\pi^3,$$

where $G=\sum_{n\geq0}\frac{(-1)^n}{(2n+1)^2}$. is Catalan's constant. Sofo and Nimbran [32] have shown that the imaginary part of the trilogarithm:

$$W(3):=Im\left(\operatorname{Li}_3\left(\frac{1\pm i}{2}\right)\right)$$
$$=\sum_{n\geq1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\,n^3}$$
$$=\sum_{n\geq1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right)$$

and Lewin ([16], p.164, 296) has also given

$$Re\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right)=\frac{1}{48}\ln^3 2+\frac{35}{64}\zeta(3)$$

and therefore

$$I_+\left(-\frac{1}{2},1,1,1\right)=8G\ln2+8W(3)-\frac{1}{4}\pi\ln^2 2-\frac{5}{16}\pi^3.$$

3. For $t=q=1, p=2$ and $a=-\frac{1}{2}$

$$I_+\left(-\frac{1}{2},1,1,2\right)=2\sum_{n\geq1}\frac{(-1)^{n+1}H_n}{\left(n+\frac{1}{2}\right)^3}-\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{1}{4}\right)^3}+\frac{1}{4}\sum_{n\geq1}\frac{H_n}{\left(n+\frac{3}{4}\right)^3}$$
$$=\frac{63}{8}\pi\zeta(3)+2\pi^2G+\frac{13}{8}\pi^3\ln2-102\beta(4),$$

where the Dirichlet beta function, $\beta(z)$ or Dirichlet L function is given by, see Finch [12],

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z};\ for\ \ Re(z)>0,$$

where $\beta(2) = G$ *is Catalan's constant. From Remark* 2.2

$$I_-^b\left(b-1,t-1,\frac{1}{2},t\right) = \int_0^1 \frac{x^{b-1}\ln^{t-1}(x)\,\mathrm{Li}_t(x^{b/2})}{1-x^b}dx$$
$$= \frac{(-1)^t(t-1)!}{b^t}\left(2^{-t}\zeta(2t) + \left(1-2^{-t}\right)\zeta^2(t) + 2^{t-1}\left(\eta(2t)-\eta^2(t)\right)\right).$$

4. *For* $a = -\frac{1}{2}, p = 1, q = 1, t = 2$

$$\int_0^1 \frac{x^{-1/2}\ln(x)\,\mathrm{Li}_2(x)}{1-x}dx = 16L(3) - \frac{55}{4}\zeta(4),$$

where, see [13],

$$L(3) = S_{1,3}^{+-} = \frac{11}{4}\zeta(4) - \frac{7}{4}\zeta(3)\ln 2 + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2Li_4\left(\frac{1}{2}\right) \tag{2.20}$$

and

$$\int_0^1 \frac{x^{-1/2}\ln(x)\,\mathrm{Li}_2(\frac{1}{x})}{1-x}dx = \frac{175}{4}\zeta(4) - 16L(3) + i14\pi\zeta(3).$$

5. *For* $b = 1, p = 1, q = 1, t = 2$

$$I_-^2(-2,1,1,2) = \int_0^1 \frac{x^{-2}\ln(x)\,\mathrm{Li}_2(x^2)}{1-x^2}dx = 8\ln 2 + 4L(3) - 4\zeta(2) - \frac{55}{16}\zeta(4).$$

6. *From* (2.13)

$$I_+^b(b-1,t-1,1,t) = \frac{(-1)^t(t-1)!}{2b^t}\left(\eta^2(t) - \zeta(2t)\right).$$

7. *From* (2.16)

$$J(3,2,1) = \int_0^\infty \frac{\ln^3(x)\,\mathrm{Li}_1(x^4)}{1-x^2}dx = \frac{21}{8}\pi^2\zeta(3) + \frac{3}{4}\pi^3 G + \frac{3}{8}\pi^4\ln 2 + 6\pi\beta(4) + i\frac{\pi^5}{16}.$$

In the next section we consider the integral (2.5) with negative polylog argument.

3. Polylog Integrals With Negative Argument

Theorem 3.5. *Let* $(p,q,t) \in \mathbb{N}_0, q \neq 0, a \geq -2,$ *and denote*

$$K_+(a,p,q,t) = \int_0^1 \frac{x^a \ln^p(x)\,\mathrm{Li}_t(-x^q)}{1+x}dx. \tag{3.21}$$

For q an odd integer

$$K_+(a,p,q,t) = (-1)^p p! \sum_{n\geq 1}(-1)^{n+1} H_n^{(t)} \sum_{j=1}^q \frac{(-1)^j}{(qn+j+a)^{p+1}}, \tag{3.22}$$

for q an even integer

$$K_+(a,p,q,t) = (-1)^p p! \sum_{n\geq 1}\left(H_n^{(t)} - \frac{1}{2^{t-1}}H_{[\frac{n}{2}]}^{(t)}\right)\sum_{j=1}^q \frac{(-1)^j}{(qn+j+a)^{p+1}}, \tag{3.23}$$

and for $q \in \mathbb{R}^+ \setminus \{0\}$

$$K_+(a,p,q,t) = \frac{(-1)^p p!}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^n}{n^t} \left(H^{(p+1)}_{\frac{qn+a}{2}} - H^{(p+1)}_{\frac{qn+a-1}{2}} \right), \tag{3.24}$$

where $H_n^{(t)}$ *are harmonic numbers of order* t *and* $[z]$ *denotes the greatest integer that is less than or equal to* z.

Proof. The results (3.22), (3.23) and (3.24) can be proven *Mutatis Mutandis* with respect to Theorem 2.1. □

3.1. **Connection to the Herglotz function.**

From Theorem 3.5, let

$$\Lambda(q) = -K_+(0,0,q,1) = -\int_0^1 \frac{\mathrm{Li}_1(-x^q)}{1+x} dx = \int_0^1 \frac{\ln(1+x^q)}{1+x} dx.$$

In the paper [22], Zagier stated that Henri Cohen ([8], Ex. 60, p. 902-903) showed him the identity

$$\Lambda\left(1+\sqrt{2}\right) = \frac{1}{2}\ln 2\left(\ln 2 + \ln\left(1+\sqrt{2}\right)\right) - \frac{1}{4}\zeta(2).$$

Radchenko and Zagier [22], evaluated many other cases such as $\Lambda\left(\frac{2}{5}\right)$ and $\Lambda\left(4+\sqrt{17}\right)$ and gave the relation

$$\Lambda(q) = F(2q) - 2F(q) + F\left(\frac{q}{2}\right) + \frac{1}{2q}\zeta(2)$$

in terms of the function

$$F(q) = \sum_{n\geq 1} \frac{1}{n}\left(\psi(nq) - \ln(nq)\right), q \in \mathbb{C}\setminus(-\infty, 0].$$

The function $F(q)$ was introduced and studied by Zagier [37] and he obtained some functional equations that $F(q)$ satisfies, namely, for $q \in \mathbb{C}\setminus(-\infty,0]$

$$F(q) - F(q+1) - F\left(\frac{q}{1+q}\right) + F(1) = \mathrm{Li}_2\left(\frac{1}{1+q}\right)$$

and

$$F(q) + F\left(\frac{1}{q}\right) - 2F(1) = \frac{1}{2}\ln^2 q - \frac{(q-1)^2}{q}\zeta(2).$$

A similar function to $F(q)$ was also studied by Herglotz in [15] and therefore Radchenko and Zagier [22] named it the Herglotz function. Herglotz [15] also studied the integral $-K_+(0,0,q,1)$ and found explicit values for $\Lambda\left(4+\sqrt{15}\right)$, $\Lambda\left(6+\sqrt{35}\right)$ and $\Lambda\left(12+\sqrt{143}\right)$. Many other identities of this kind were found by Muzzafar and Williams [19], together with some sufficient conditions on q under which one can evaluate $\Lambda\left(q+\sqrt{q^2-1}\right)$. In Section 6, Radchenko and Zaiger [22] give a systematic account, at special values of quadratic units of these identities and list two tables with specific solutions. Radchenko and Zagier [22] study, among other things, the relation of this function with the Dedekind eta-function, functional equations satisfied by $F(q)$ in connection with Hecke operators, the cohomological aspects of $F(q)$ and its special values at positive rationals and quadratic units. Recently, Dixit et al. [10] extended the study to higher Herglotz functionals. From (3.24) we can see that

$$\Lambda(q) = \frac{1}{2} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(H_{\frac{qn}{2}} - H_{\frac{qn-1}{2}} \right)$$

and from the identity of the multiple argument of polygamma functions,

$$2H_{qn} - 2\ln 2 = H_{\frac{qn}{2}} + H_{\frac{qn}{2}-\frac{1}{2}}$$

implies

$$\begin{aligned}\Lambda(q) &= \ln^2 2 - \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(H_{qn} - H_{\frac{qn}{2}} \right) \\ &= \ln^2 2 - \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(\psi(qn+1) - \psi\left(\frac{qn}{2}+1\right) \right).\end{aligned}$$

In the case $q = 1/2$

$$\Lambda\left(\frac{1}{2}\right) = \frac{1}{4}\ln^2 2 + \frac{1}{8}\zeta(2).$$

Consider the case $q = 2m, m \in \mathbb{N}$, then

$$\Lambda(2m) = \ln^2 2 + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(\psi(mn+1) - \psi(2mn+1) \right)$$

and using the known identities, see [31], for the digamma sums, we can write

$$\begin{aligned}\Lambda(2m) = \int_0^1 \frac{\ln(1+x^{2m})}{1+x} dx =& \frac{1}{2} \sum_{j=0}^{2m-1} \ln^2 \left(2\sin\left(\frac{(2j+1)\pi}{4m} \right) \right) \\ &+ \frac{1-2m^2}{8m}\zeta(2) + \ln^2 2 - \frac{1}{2} \sum_{j=0}^{m-1} \ln^2 \left(2\sin\left(\frac{(2j+1)\pi}{2m} \right) \right),\end{aligned}$$

where, in particular

$$\Lambda(6) = 2\ln^2\left(1+\sqrt{3}\right) - 2\ln 2 \ln\left(1+\sqrt{3}\right) + \frac{5}{4}\ln^2 2 - \frac{17}{24}\zeta(2).$$

From the functional relationship

$$\ln(1+x^q) - \ln\left(1+x^{-q}\right) = q\ln x$$

we can evaluate the related $\Lambda(-q)$ integral

$$\Lambda(-q) = \int_0^1 \frac{\ln(1+x^{-q})}{1+x} dx = \Lambda(q) - \frac{q}{2}\zeta(2),$$

here

$$\Lambda(-6) = 2\ln^2\left(1+\sqrt{3}\right) - 2\ln 2 \ln\left(1+\sqrt{3}\right) + \frac{5}{4}\ln^2 2 + \frac{55}{24}\zeta(2).$$

For the case of q odd, we also have the representation (3.22) and for $q = 3$,

$$\Lambda(3) = \ln 2 \ln 3 - \frac{1}{2}\ln^2 2 - \sum_{n\geq 1} \frac{\cos(\pi n/3)}{2^{n-1}n^2}.$$

Theorem 3.6. *Let* $(p,q,t)\in\mathbb{N}, q\neq 0, a\geq -2,$ *and denote*

$$K_{-}(a,p,q,t)=\int_0^1 \frac{x^a\ln^p(x)\operatorname{Li}_t(-x^q)}{1-x}dx.$$

Then, for $q\in\mathbb{N}$

$$K_{-}(a,p,q,t)=(-1)^p p!\sum_{n\geq 1}\left(H_n^{(t)}-\frac{1}{2^{t-1}}H_{\left[\frac{n}{2}\right]}^{(t)}\right)\sum_{j=1}^{q}\frac{1}{(qn+j+a)^{p+1}},$$

and for $q\in\mathbb{R}^+\backslash\{0\}$

$$K_{-}(a,p,q,t)=(-1)^p p!\zeta(p+1)\eta(t)-(-1)^p p!\sum_{n\geq 1}\frac{(-1)^{n+1}H_{qn+a}^{(p+1)}}{n^t},$$

where $\eta(t)$ *is the Dirichlet eta function, or the alternating zeta function.*

Proof. The proof follows *Mutatis Mutandis* with respect to Theorem 2.1. □

Remark 3.4. *For the two cases where* $b\in\mathbb{R}^+, a+1>-b$

$$K_{+}^{b}(a,p,q,t)=\int_0^1 \frac{x^a\ln^p(x)\operatorname{Li}_t(-x^{bq})}{1+x^b}dx=\frac{1}{b^{p+1}}K_{+}\left(\frac{a+1-b}{b},p,q,t\right)$$

and

$$K_{-}^{b}(a,p,q,t)=\int_0^1 \frac{x^a\ln^p(x)\operatorname{Li}_t(-x^{bq})}{1-x^b}dx=\frac{1}{b^{p+1}}K_{-}\left(\frac{a+1-b}{b},p,q,t\right).$$

In particular

$$K_{-}^{2}(0,p,q,t)=\int_0^1 \frac{\ln^p(x)\operatorname{Li}_t(-x^{2q})}{1-x^2}dx=\frac{1}{2^{p+1}}K_{-}\left(-\frac{1}{2},p,q,t\right) \tag{3.25}$$

$$=\begin{cases}\frac{(-1)^p p!}{2^{p+1}}\sum_{n\geq 1}\frac{(-1)^{n+1}H_{qn-\frac{1}{2}}^{(p+1)}}{n^t}-\frac{(-1)^p p!}{2^{p+1}}\zeta(p+1)\eta(t), \textit{ for } q\in\mathbb{R}^+\\ (-1)^{p+1}p!\sum_{n\geq 1}A(n,t)\sum_{j=1}^{q}\frac{1}{(2qn+2j-1)^{p+1}}, \textit{ for } q\in\mathbb{N}\end{cases}.$$

Now, we provide a theorem for the representation of a special case of the integral (3.25) in the half plane $x\geq 0$.

Theorem 3.7. *For* $b=2, a=0; p,t\in\mathbb{N}, q>0$

$$M(p,q,t)=\int_0^\infty \frac{\ln^p(x)\operatorname{Li}_t(-x^{2q})}{1-x^2}dx=\int_0^\infty g(x;p,q,t)\,dx \tag{3.26}$$

$$=\int_0^1 \ln^p(\tanh\theta)\operatorname{Li}_t(-\tanh^{2q}\theta)d\theta$$

$$= \left(1+(-1)^{p+t}\right) K_{-}^{2}\left(0,p,q,t\right) + 2\sum_{j=0}^{\left[\frac{t}{2}\right]} (2q)^{t-2j}\, p! \begin{pmatrix} p+t-2j \\ p \end{pmatrix} \eta(2j)\lambda\left(p+t+1-2j\right), \tag{3.27}$$

where

$$g\left(x;p,q,t\right) = \frac{\ln^{p}\left(x\right)\mathrm{Li}_{t}(-x^{2q})}{1-x^{2}}, \tag{3.28}$$

$K_{-}^{2}\left(0,p,q,t\right)$ is given by (3.25), $\eta(2j)$ is the Dirichlet eta function, $\lambda\left(\cdot\right)$ is the Dirichlet lambda function (1.4) and $\left[\frac{t}{2}\right]$ is the Floor function.

Proof. Using the same technique as in Theorem 2.3, we arrive at

$$\int_{0}^{\infty} g\left(x;p,q,t\right)dx = \int_{0}^{1} g\left(x;p,q,t\right)dx + (-1)^{p}\int_{0}^{1} \frac{\ln^{p}\left(y\right)}{1-y^{2}}\mathrm{Li}_{t}(-y^{-2q})dy. \tag{3.29}$$

From Lewin ([16], p.299), Jonquiěre's relation states

$$\mathrm{Li}_{s}(-z) + (-1)^{t}\,\mathrm{Li}_{s}(-\frac{1}{z}) = -2\sum_{j=0}^{\left[\frac{t}{2}\right]} \frac{(\ln z)^{t-2j}}{(t-2j)!}\eta(2j) = 2\sum_{j=0}^{\left[\frac{t}{2}\right]} \frac{(\ln z)^{t-2j}}{(t-2j)!}\mathrm{Li}_{2j}(-1), \tag{3.30}$$

where $\mathrm{Li}_{s}(z)$ is a polylogarithm. The relation (3.30) can also be written in terms of Bernoulli numbers so that

$$\mathrm{Li}_{t}(-z) + (-1)^{t}\,\mathrm{Li}_{t}(-\frac{1}{z}) = \frac{1}{t!}\sum_{j=0}^{t} \left(1-2^{1-j}\right)\begin{pmatrix} t \\ j \end{pmatrix} B_{j}\left(2\pi i\right)^{j}\left(\ln z\right)^{t-2j},$$

where B_j are the Bernoulli numbers. Now we can substitute (3.30) into (3.29), so that

$$\int_{0}^{\infty} g\left(x;p,q,t\right)dx = \left(1+(-1)^{p+t}\right)\int_{0}^{1} g\left(x;p,q,t\right)dx + 2\left(-1\right)^{p+t}\sum_{j=0}^{\left[\frac{t}{2}\right]} \frac{(2q)^{t-2j}}{(t-2j)!}\eta(2j)\int_{0}^{1} \frac{\ln^{p+t-2j}\left(x\right)}{1-x^{2}}dx.$$

The integral

$$K_{-}^{2}\left(0,p,q,t\right) = \int_{0}^{1} \frac{\ln^{p}\left(x\right)\mathrm{Li}_{t}(-x^{2q})}{1-x^{2}}dx$$

and

$$\int_{0}^{1} \frac{\ln^{p+t-2j}\left(x\right)}{1-x^{2}}dx = (-1)^{p+t}\left(p+t-2j\right)!\lambda\left(p+t-1-2j\right).$$

Therefore we obtain (3.27) and the proof is finished. Note that the integral $K_{-}^{2}\left(0,p,q,t\right)$ does not contribute to $M\left(p,q,t\right)$ in the case when $p+t$ is an odd integer. The third integral in (3.26) is obtained by the substitution $x = \tanh\theta$. □

Remark 3.5. *It can be noted, from Jonquiěre's relation (3.30) and using the integrals in remark 3.4 that we are able to determine the value of the integrals*

$$\int_0^1 \frac{x^a \ln^p(x) \operatorname{Li}_t(-x^{-bq})}{1-x^b} dx = (-1)^{t+1} K_-^b(a,p,q,t) + 2(-1)^{t+1} \sum_{j=0}^{[\frac{t}{2}]} \frac{\eta(2j)}{(t-2j)!} \int_0^1 \frac{x^a \ln^p(x) \ln^{t-2j}(-x^{bq})}{1-x^b} dx. \tag{3.31}$$

Some examples follow.

Example 3.2.
1. From (3.22) and (3.24) for $q=1, a=0$,

$$\sum_{n\geq 1} \frac{(-1)^n}{(n+1)^{p+1}} H_n^{(t)} = \frac{1}{2^{p+1}} \sum_{n\geq 1} \frac{(-1)^n}{n^t} \left(H_{\frac{n}{2}}^{(p+1)} - H_{\frac{n-1}{2}}^{(p+1)} \right),$$

from the polygamma multiplication formula [30]

$$2^{p+1} H_n^{(p+1)} = 2^{p+1}\eta(p+1) + H_{\frac{n}{2}}^{(p+1)} + H_{\frac{n-1}{2}}^{(p+1)}$$

we can write

$$S_{t,p+1}^{+-} - \eta(p+t+1) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^t} \left(H_n^{(p+1)} - \eta(p+1) - \frac{1}{2^p} H_{\frac{n}{2}}^{(p+1)} \right)$$

and therefore

$$\frac{1}{2^p} \sum_{n\geq 1} \frac{(-1)^{n+1} H_{\frac{n}{2}}^{(p+1)}}{n^t} = \eta(p+t+1) - \eta(p+1)\eta(t) + S_{p+1,t}^{+-} - S_{t,p+1}^{+-}.$$

If $p+1=t$,

$$\sum_{n\geq 1} \frac{(-1)^{n+1} H_{\frac{n}{2}}^{(t)}}{n^t} = 2^{t-1}\left(\eta(2t) - \eta^2(t)\right).$$

2. From Theorem 3.6 with $q=2, a=0$, we have

$$\begin{aligned} &\zeta(p+1)\eta(t) - \sum_{n\geq 1} \frac{(-1)^{n+1} H_{2n}^{(p+1)}}{n^t} \\ &= \frac{1}{2^{p+1}} \left(S_{t,p+1}^{++}\left(0,\frac{1}{2}\right) + S_{t,p+1}^{++}(0,1) \right) \\ &- \frac{1}{2^{1+2p+t}} \left(S_{t,p+1}^{++}\left(0,\frac{1}{4}\right) + S_{t,p+1}^{++}\left(0,\frac{1}{2}\right) + S_{t,p+1}^{++}\left(0,\frac{3}{4}\right) + S_{t,p+1}^{++}(0,1) \right). \end{aligned}$$

Simplifying we obtain the new identity

$$\begin{aligned} \sum_{n\geq 1} \frac{(-1)^{n+1} H_{2n}^{(p+1)}}{n^t} &= \zeta(p+1)\eta(t) + 2^{-1-2p-t}\left(S_{t,p+1}^{++}\left(0,\frac{1}{4}\right) + S_{t,p+1}^{++}\left(0,\frac{3}{4}\right) \right) \\ &+ \left(2^{-1-2p-t} - 2^{-1-p}\right)\left(S_{t,p+1}^{++} - \zeta(p+t+1) + S_{t,p+1}^{++}\left(0,\frac{1}{2}\right) \right). \end{aligned}$$

In particular, when $t=1$ we obtain an analogous identity, (to (5.6) in [1])

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n}^{(p+1)}}{n}=\zeta(p+1)\ln 2+2^{-2-2p}\left(S_{1,p+1}^{++}\left(0,\frac{1}{4}\right)+S_{1,p+1}^{++}\left(0,\frac{3}{4}\right)\right)$$
$$+\left(2^{-2-2p}-2^{-1-p}\right)\left(S_{1,p+1}^{++}-\zeta(p+2)+S_{1,p+1}^{++}\left(0,\frac{1}{2}\right)\right),$$

the expression $S_{1,p+1}^{++}(0,\alpha)=\sum_{n\geq 1}\frac{H_n}{(n+\alpha)^{p+1}}$ can be evaluated by Theorem 2.4.
3. For $a=1,p=4,q=2,t=1$

$$K_{-}(1,4,2,1)=\int_0^1\frac{x\ln^4(x)\,\mathrm{Li}_1(-x^2)}{1-x}dx$$
$$=24\sum_{n\geq 1}\frac{(-1)^{n+1}H_{2n+1}^{(5)}}{n}-24\zeta(5)\ln 2$$
$$=48(1-G)\beta(4)+48G-240+12\pi+24\ln 2+\frac{3}{2}\pi^3$$
$$+\frac{5}{32}\pi^5+\frac{15453}{256}\zeta(6)-\frac{27}{128}\zeta^2(3)-\frac{1581}{64}\zeta(5)\ln 2.$$

4. For $a=-\frac{3}{2},p=0,q=1,t=2$

$$\int_0^1\frac{x^{-3/2}\mathrm{Li}_2(-x)}{1-x}dx=-2\sum_{n\geq 1}\frac{(-1)^{n+1}H_n^{(2)}}{2n-1}=\frac{11}{45}\zeta(3)+\zeta(2)$$
$$+\frac{\pi}{4}\ln^2 2+4\ln 2-2\pi-4G\ln 2-8W(3).$$

5. For $a=0,p=0,q=\frac{1}{2},t=3$

$$\int_0^1\frac{\mathrm{Li}_3(-x^{1/2})}{1+x}dx=-2\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^3}\left(H_{\frac{n}{2}}-\ln 2-H_{\frac{n}{4}}\right)$$
$$=\frac{65}{128}\zeta(4)-\eta(3)\ln 2-\frac{3}{8}L(3).$$

6. For $p=t-1,q>0,t\in\mathbb{N}$

$$M(t-1,q,t)=\int_0^\infty\frac{\ln^{t-1}(x)\,\mathrm{Li}_t(-x^{2q})}{1-x^2}dx$$
$$=2(t-1)!\sum_{j=0}^{\left[\frac{t}{2}\right]}(2q)^{t-2j}\binom{2t-2j-1}{t-1}\eta(2j)\lambda(2t-2j),$$

where $\eta(0) = \frac{1}{2}$.

7. For $p = 2m-1$, *where* $m \in \mathbb{N},\ q = 1, t \in \mathbb{N}$

$$\frac{1}{(2m-1)!}M\left(2m-1,1,t\right) = \frac{1}{(2m-1)!}\int\limits_{0}^{\infty} \frac{\ln^{2m-1}(x)\,\mathrm{Li}_t(-x^2)}{1-x^2}dx$$
$$= \left(\frac{\zeta(2m)}{2^{2m}} + \eta(2m)\right)\eta(t) - \sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}\left(H_{2n}^{(2m)} - \frac{1}{2^{2m}}H_n^{(2m)}\right)$$
$$+ 2\sum_{j=0}^{\left[\frac{t}{2}\right]}(2)^{t-2j}\left(\begin{array}{c} 2m-1+t-2j \\ 2m-1 \end{array}\right)\eta(2j)\lambda\left(2m+t-2j\right).$$

In particular

$$M\left(7,1,1\right) = \frac{427}{64}\pi^7 G + \frac{17}{16}\pi^8\ln 2 + \frac{525}{8}\pi^5\beta\left(4\right) + 630\pi^3\beta\left(6\right) + 5040\pi\beta\left(8\right).$$

8. For $p = t,\ q = \frac{1}{2}, t \in \mathbb{N}$

$$\frac{1}{2}M\left(t,\frac{1}{2},t\right) = \frac{1}{2}\int\limits_{0}^{\infty}\frac{\ln^t(x)\,\mathrm{Li}_t(-x)}{1-x^2}dx = \sum_{j=0}^{\left[\frac{t}{2}\right]} t!\left(\begin{array}{c} 2t-2j \\ t \end{array}\right)\eta(2j)\lambda\left(2t+1-2j\right)$$
$$+ (-1)^t\, t!\left(S_{t+1,t}^{+-} - \frac{1}{2^{t+1}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}H_{\frac{n}{2}}^{(t+1)}\right)$$
$$- (-1)^t\, t!\eta(t)\left(\eta(t+1) + \frac{1}{2^{t+1}}\zeta(t+1)\right)$$

and the Euler sum $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^t}H_{\frac{n}{2}}^{(t+1)}$ *can be explicitly evaluated by the techniques developed in* [26], [28] *and* [29]. *Other authors have also evaluated particular case of these integrals, Coffey* [7] *has evaluated, amongst other results,* $K_{+}\left(0,1,1,2\right)$.

9. For $a = \frac{1}{2}, p = 2, q = 1, t = 2$

$$K_{-}\left(\frac{1}{2},2,1,2\right) = \int\limits_{0}^{1}\frac{\sqrt{x}\ln^2(x)\,\mathrm{Li}_2(-x)}{1-x}dx = 48\pi\beta\left(4\right) + 384 - 128G - 48\pi$$
$$+ 8\zeta\left(2\right) + 2\pi^3 G - 2\pi^3 - 96\ln 2 - 7\zeta\left(2\right)\zeta\left(3\right) - 186\zeta\left(5\right)$$

and

$$\int\limits_{0}^{1}\frac{\sqrt{x}\ln^2(x)\,\mathrm{Li}_2(-\frac{1}{x})}{1-x}dx = 128G - 48\pi\beta\left(4\right) + 48\pi + 8\zeta\left(2\right)$$
$$+ 96\ln 2 - 2\pi^3 G - 7\zeta\left(2\right)\zeta\left(3\right) - 186\zeta\left(5\right).$$

Remark 3.6. *From Theorem 3.7 we can identify a new Euler identity in the case of even weight $p+t$. Consider the case $p=t$, then we can write*

$$2K_{-}^{2}\left(0,t,q,t\right)=M\left(t,q,t\right)-2t!\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(2q\right)^{t-2j}\left(\begin{array}{c}2t-2j\\ t\end{array}\right)\eta(2j)\lambda\left(2t+1-2j\right)$$

from which we extract the Euler identity

$$\sum_{n\geq1}\frac{\left(-1\right)^{n+1}}{n^{t}}\left(H_{2qn}^{(t+1)}-\frac{1}{2^{t+1}}H_{qn}^{(t+1)}\right)=\frac{\left(-1\right)^{t}}{2t!}M\left(t,q,t\right) \tag{3.32}$$
$$+\left(-1\right)^{t}\sum_{j=0}^{\left[\frac{t}{2}\right]}\left(2q\right)^{t-2j}\left(\begin{array}{c}2t-2j\\ t\end{array}\right)\eta(2j)\lambda\left(2t+1-2j\right)+\eta(t)\left(\eta(t+1)+\frac{1}{2^{t+1}}\zeta(t+1)\right).$$

Since for $q=1$ Flajolet and Salvy [13] give explicit values for $S_{t+1,t}^{+-}$, then we can obtain an explicit identity for $\sum_{n\geq1}\frac{(-1)^{n+1}H_{2n}^{(t+1)}}{n^{t}}$. Iterating for values $q=1,2,3...$ allows us to obtain new Euler sum identities for $\sum_{n\geq1}\frac{(-1)^{n+1}H_{2qn}^{(t+1)}}{n^{t}}$, $t\in\mathbb{N}$. Let

$$S_{p,t}^{+-}\left(\alpha,\beta;q\right)=\sum_{n\geq1}\frac{\left(-1\right)^{n+1}H_{qn}^{(p)}\left(\alpha\right)}{\left(n+\beta\right)^{t}},$$

then from (3.32) we offer the following examples.

$$S_{2,1}^{+-}\left(0,0;2\right)=2\zeta\left(3\right)-\frac{1}{2}\pi G-\frac{1}{8}\zeta\left(2\right)\ln2.$$
$$S_{3,2}^{+-}\left(0,0;2\right)=3\pi\beta\left(4\right)+\frac{1}{8}\pi^{3}G-\frac{2997}{256}\zeta\left(5\right)+\frac{3}{32}\zeta\left(2\right)\zeta\left(3\right).$$

$$\begin{aligned}S_{3,2}^{+-}\left(0,0;4\right)=&\frac{\pi^{2}}{512\sqrt{2}}\left(\begin{array}{c}3\pi\left(\psi'\left(\frac{1}{8}\right)+\psi'\left(\frac{3}{8}\right)-\psi'\left(\frac{5}{8}\right)-\psi'\left(\frac{7}{8}\right)\right)\\ -\psi''\left(\frac{1}{8}\right)+\psi''\left(\frac{3}{8}\right)+\psi''\left(\frac{5}{8}\right)-\psi''\left(\frac{7}{8}\right)\end{array}\right)\\ &+\frac{\pi}{512\sqrt{2}}\left(\psi'''\left(\frac{1}{8}\right)+\psi'''\left(\frac{3}{8}\right)-\psi'''\left(\frac{5}{8}\right)-\psi'''\left(\frac{7}{8}\right)\right)\\ &+\frac{1}{8}\left(3\pi\beta\left(4\right)+\frac{1}{8}\pi^{3}G-\frac{2997}{256}\zeta\left(2\right)+\frac{3}{32}\zeta\left(2\right)\zeta\left(3\right)\right)-186\zeta\left(5\right).\end{aligned}$$

In the case where $p=2,q=2,t=3$, we can evaluate the result

$$S_{2,3}^{+-}\left(0,0;2\right)=\frac{1973}{128}\zeta\left(5\right)+\frac{61}{32}\zeta\left(2\right)\zeta\left(3\right)-6\pi\beta\left(4\right).$$

The case $p=t+1,t\in\mathbb{N}$ and $\beta=1$ results in

$$S_{t+1,t}^{+-}\left(0,1;q\right)+S_{t+1,t}^{+-}\left(0,0;q\right)=\frac{\eta\left(2t+1\right)}{q^{t+1}}+\sum_{j=1}^{q-1}\sum_{r=0}^{t-1}q^{r}\left(\begin{array}{c}t+r\\ r\end{array}\right)\frac{\eta\left(t-r\right)}{j^{t+r+1}}$$
$$+\sum_{j=1}^{q-1}\sum_{r=0}^{t}\left(-1\right)^{r}\left(\begin{array}{c}t+r-1\\ r\end{array}\right)\frac{q^{t}}{j^{t+r}}\sum_{n\geq1}\frac{\left(-1\right)^{n+1}}{\left(qn-j\right)^{t+1-r}}. \tag{3.33}$$

The result (3.33) follows from the consideration

$$S_{t+1,t}^{+-}(0,1;q)+S_{t+1,t}^{+-}(0,0;q)=\sum_{n\geq 1}\frac{(-1)^{n+1}}{q^{t+1}n^{t}\left(n-\frac{j}{q}\right)^{t+1}}$$

and by the known decomposition formula, originally due to Euler ([21], p.48, Eq.(9))

$$\frac{1}{n^{t}(n-\alpha)^{t+1}}=\sum_{r=0}^{t-1}(-1)^{t+1}\binom{t+r}{r}\frac{1}{n^{t+r}\alpha^{t+r+1}}$$
$$+\sum_{r=0}^{t}(-1)^{r}\binom{t+r-1}{r}\frac{1}{\alpha^{t+r}(n-\alpha)^{t+1-r}}.$$

The classical identity follows, upon putting $q=1$, in which case

$$S_{t+1,t}^{+-}(0,1;1)+S_{t+1,t}^{+-}(0,0;1)=\eta(2t+1).$$

Concluding Remarks. We have extended the current available knowledge for the representation of Euler sums. Moreover, we have demonstrated two parameterized families of log-polylog families that admit solutions dependent on Euler sums and in a particular case have demonstrated a connection with the Herglotz function. As a result of this line of research we expect further studies in the areas of polylog integrals and generalized Herglotz functions.

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Research Article

van der Corput inequality for real line and Wiener-Wintner theorem for amenable groups

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ABSTRACT. We extend the classical van der Corput inequality to the real line. As a consequence, we obtain a simple proof of the Wiener-Wintner theorem for the $\mathbb{R}$-action which assert that for any family of maps $(T_t)_{t\in\mathbb{R}}$ acting on the Lebesgue measure space $(\Omega, \mathcal{A}, \mu)$, where μ is a probability measure and for any $t \in \mathbb{R}$, T_t is measure-preserving transformation on measure space $(\Omega, \mathcal{A}, \mu)$ with $T_t \circ T_s = T_{t+s}$, for any $t, s \in \mathbb{R}$. Then, for any $f \in L^1(\mu)$, there is a single null set off which $\lim\limits_{T\to+\infty} \frac{1}{T}\int_0^T f(T_t\omega)e^{2i\pi\theta t}dt$ exists for all $\theta \in \mathbb{R}$. We further present the joining proof of the amenable group version of Wiener-Wintner theorem due to Ornstein and Weiss .

Keywords: van der Corput inequality, Wiener-Wintner theorem, joinings, amenable group.

2020 Mathematics Subject Classification: 11K06, 28D05, 47A35.

1. INTRODUCTION

In this paper, using our generalization of van der Corput inequality for the real line, we present a simple proof of Wiener-Wintner theorem for the flows. We further present the joining proof of the amenable groups version of it due to Ornstein and Weiss [13]. This accomplished by applying the Furstenberg joinings machinery. The classical Wiener-Wintner theorem [15] assert the following.

Theorem 1.1. *Let $(\Omega, \mathcal{A}, \mu, T)$ be a dynamical system, where μ is a probability measure. Then, for any f in $L^1(\mu)$, there is a set Ω' of full measure such that for any $\omega \in \Omega'$ the sums*

$$\frac{1}{N}\sum_{0}^{N-1} f(T^n\omega)z^n$$

converge for all z in the unit circle $C = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

The original proof can be found in [15]. Subsequently, Furstenberg in [6] obtain a joining proof of Wiener-Wintner theorem. Later, I. Assani [2], A. Below & V. Losert [3] proved the stronger version of this theorem. This stronger version is due to Bourgain [4]. Theirs proofs is based on the Hellinger integral (known also as affinity principle). In [10], E. Lesigne generalize Wiener-Wintner theorem to the polynomial case. His proof is based on the Furstenberg's joinings technique. Afterwards, in [11], using van der Corput inequality and the spectral theory

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of skew products, he extended the stronger version of polynomial Wiener-Wintner theorem to the case of weak-wixing dynamical systems[1].

In this paper, we extend van der Corput inequality to the continuous time and we give a simple proof of the flow version of Wiener-Wintner theorem. We further present the Ornstein-Weiss's joining of the amenable group version of this fundamental theorem in ergodic theory. The proof is based on Furstenberg's joinings machinery combined with the recent result of E. Lindenstrauss [12].

The plan of the paper is as follows. In Section 2, we state and prove the continuous van der Corput inequality and the flow version of Wiener-Wintner theorem. In section 3, we state and prove the amenable group version of Wiener-Wintner theorem.

2. VAN DER CORPUT FOR REAL LINE

In this section, we state our first main result.

Theorem 2.2 (van der Corput). *Let* $(u(t))_{t\in[0,T]}$ *be an integrable complex valued function and* $S \in (0,T]$. *Then*

$$\left|\int_0^T u(t)dt\right|^2 \leq \frac{S+T}{S^2}\int_0^S\int_0^S\int_0^T u(t+s'-s)\overline{u}(t)dsds'dt.$$

Proof. We start by noticing that we have

$$S\int_0^T u(t)dt = \int_0^{T+S}\int_0^S \tilde{u}(t-s)dsdt,$$

where $\tilde{u}$ stand for

$$\tilde{u}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ u(t) & \text{if } 0 \leq t \leq T, \\ 0 & \text{if not.} \end{cases}$$

Indeed, we have

$$\begin{aligned} \int_0^{T+S}\int_0^S \tilde{u}(t-s)dsdt &= \int_0^S\int_{-s}^{T+S-s} \tilde{u}(t)dtds \\ &= \int_0^S\int_0^T u(t)dtds \\ &= S\int_0^T u(t)dt. \end{aligned}$$

Whence,

$$S^2\left|\int_0^T u(t)dt\right|^2 = \left|\int_0^{T+S}\int_0^S \tilde{u}(t-s)dsdt\right|^2.$$

Now, applying Cauchy-Schwarz inequality, we obtain

$$S^2\left|\int_0^T u(t)dt\right|^2 \leq (T+S)\left(\int_0^{T+S}\left|\int_0^S \tilde{u}(t-s)ds\right|^2 dt\right).$$

[1]Seven year after the first version of this note was written, M. Lacey and E. Terwillegar [9] produce an oscillation proof of the Hilbert version of Wiener-Wintner theorem.

But

$$\int_0^T \left| \int_0^S \tilde{u}(t-s)ds \right|^2 dt = \int_0^T \int_0^S \int_0^S \tilde{u}(t-s)\overline{\tilde{u}}(t-s')dsds'$$
$$= \int_0^T \int_0^S \int_0^S \tilde{u}(t-s)\overline{\tilde{u}}(t-s')dsds'dt$$
$$= \int_0^S \int_0^S \int_0^T \tilde{u}(t+s'-s)\overline{\tilde{u}}(t)dtdsds'.$$

Whence

$$\left| \int_0^T u(t)dt \right|^2 \leq \frac{S+T}{S^2} \int_0^S \int_0^S \int_0^T u(t+s'-s)\overline{u}(t)dsds'dt.$$

This achieve the proof of the theorem. □

Theorem 2.3 (Limit version of continuous van der Corput theorem). *Let $(u(t))_{t\in\mathbb{R}}$ be a bounded complex valued function. Then*

$$\limsup_{T\to\infty} \left| \frac{1}{T} \int_0^T u(t)dt \right|^2$$
$$\leq \limsup_{S\to\infty} \frac{1}{S^2} \int_0^S \int_0^S \limsup_{T\to\infty} \frac{1}{T} \int_0^T u(t+s'-s)\overline{u}(t)dsds'dt.$$

Proof. Straightforward from Theorem 2.2. □

Now, let us state the continuous version of Wiener-Wintner theorem.

Theorem 2.4 (Continuous version of Wiener-Wintner theorem). *Let $(T_t)_{t\in\mathbb{R}}$ be a maps acting on the Lebesgue measure space $(\Omega, \mathcal{A}, \mu)$, where μ is a probability measure and for any $t \in \mathbb{R}$, T_t is measure-preserving transformation on measure space $(\Omega, \mathcal{A}, \mu)$ with $T_t \circ T_s = T_{t+s}$, for any $t, s \in \mathbb{R}$. Then, for any $f \in L^1(\mu)$, there is a single null set off which*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(T_t\omega)e^{2i\pi\theta t}dt$$

exists for all $\theta \in \mathbb{R}$.

We will assume without loss of generality that μ ergodic. Indeed, on can use the ergodic decomposition of μ. So, it is sufficient to prove the following :

Theorem 2.5. *For any f in $L^2(\mu)$, there is a set Ω' of full measure such that the sums*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(T_t\omega)e^{2i\pi\theta t}dt$$

converge to 0 for all θ in $\mathbb{R}$, where $e^{2\pi i\theta} \notin e(T)$ and $\omega \in \Omega'$. $e(T)$ stand for the set of eigenvalue of the Koopman operator $U_T \ : \ g \mapsto g \circ T$.

Before proceeding to the proof of Theorem 2.5, let us notice that it suffices to prove it for a dense class of functions (L^2 functions for instance). Indeed, put

$$R(\omega, f) = \limsup_{T\longrightarrow+\infty} \left| \int_0^T f(T_t(\omega))e^{2\pi it\theta}dt \right|,$$

and assume that g in the dense class for which theorem holds. Then

$$R(\omega, f) = R(\omega, f - g),$$

and

$$\mu\{\omega : R(\omega, f - g) > \epsilon\} \leq \frac{||f - g||_1}{\epsilon}.$$

We thus get by the density of $L^2(\mu)$ in $L^1(\mu)$, that there exist g in $L^2(\mu)$ such that : $||f - g||_1 < \epsilon^2$. Then

$$\mu\{\omega : R(\omega, f - g) > \epsilon\} \leq \epsilon.$$

Since ϵ is arbitrary, we see $R(\omega, f) = 0$ a.e., where the null set excluded is independent of θ.

We start by recalling that by Bochner theorem, for any $f \in L^2(X)$, there exists a unique finite Borel measure σ_f on $\mathbb{R}$ such that

$$\widehat{\sigma_f}(t) = \int_{\mathbb{R}} e^{-it\xi}\, d\sigma_f(\xi) = \langle U_t f, f\rangle = \int_{\Omega} f \circ T_t(\omega) \cdot \overline{f}(\omega)\, d\mu(\omega).$$

σ_f is called the *spectral measure* of f. If f is eigenfunction with eigenfrequency λ, then the spectral measure is the Dirac measure at λ.

We need also the following fundamental results from [1].

Theorem 2.6. *Let $(\Omega, \mathcal{A}, \mu, (T_t)_{t\in\mathbb{R}})$ be an* ergodic *dynamical flow. Then, for any $S > 0$ and all $f, g \in L^2(X)$, for almost all $\omega \in \Omega$, we have*

$$\lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^{\tau} f(T_{t+s}\omega) \cdot g(T_t\omega)\, dt = \int_{\Omega} f \circ T_s \cdot g\, d\mu$$

uniformly for s in the interval $[-S, S]$.

This yields the exact result need it.

Corollary 2.1. *Let $f \in L^2(\mu)$. There exist a full measure subset Ω_f of Ω such that, for any $\omega \in \Omega_f$ and any $s \in \mathbb{R}$, we have*

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} f(T_{t+s}\omega) \cdot \overline{f}(T_t\omega)\, dt = \int_X f \circ T_s \cdot \overline{f}\, d\mu.$$

Proof of Theorem 2.5. Let f in $L^\infty(\mu)$ and $\omega \in \Omega_f$ as in Corollary 2.1, then we have

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} f(T_{t+s}\omega) \cdot \overline{f}(T_t\omega)\, dt = \int_X f \circ T_s \cdot \overline{f}\, d\mu$$
$$\stackrel{\text{def}}{=} < f \circ T_s, f > .$$

Put

$$u(t) = f(T_t\omega)e^{2\pi i t\theta},$$

and apply further van der Corput's inequality (Theorem 2.2) to get

$$\left|\frac{1}{\tau}\int_0^{\tau} f(T_t\omega)e^{2\pi i t\theta}dt\right|^2$$
$$\leq \frac{S+\tau}{\tau S^2}\int_0^S\int_0^S e^{2\pi i(s-s')\theta}\frac{1}{\tau}\int_0^{\tau} f(T_{t+s-s'})\overline{f}(T_t\omega)dtdsds'.$$

We thus deduce that for almost all ω and all $\theta \in \mathbb{R}$, we have

$$\limsup_{\tau\to\infty}\left|\frac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi it\theta}dt\right|^2$$
$$\leq\frac{1}{S^2}\int_0^S\int_0^S e^{2\pi i(s-s')\theta}\left(\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(T_{t+s-s'})\overline{f}(T_t\omega)dt\right)dsds'.$$

This combined with Corollary 2.1 gives

$$\limsup_{\tau\to\infty}\left|\frac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi it\theta}dt\right|^2$$
$$\leq\frac{1}{S^2}\int_0^S\int_0^S\left(\int_{\mathbb{R}} e^{2\pi i(s-s')(\theta-\gamma)}d\sigma_f(s-s')\right)dsds',$$

where σ_f stand for the spectral measure of f. But, since

$$\frac{1}{S^2}\int_0^S\int_0^S e^{2\pi i(s-s')(\theta-\gamma)}dsds'=\left|\frac{1}{S}\int_0^S e^{2\pi is(\theta-\gamma)}ds\right|^2$$

if $\theta\neq\gamma$, we have

$$\lim_{S\to\infty}\frac{1}{S^2}\int_0^S\int_0^S e^{2\pi i(s-s')(\theta-\gamma)}dsds'=0.$$

Whence, if $e^{2\pi i\theta}$ is not a eigenvalue of (T_t), we have

$$\lim_{S\to\infty}\frac{1}{S^2}\int_0^S\int_0^S\left(\int_{\mathbb{R}} e^{2\pi i(s-s')(\theta-\gamma)}d\sigma_f(s-s')\right)dsds'=0.$$

Since all the sums are bounded, we deduce from Lebesgue theorem that for almost all ω, and for all θ in $\mathbb{R}$, where $e^{2\pi i\theta}\notin e(T)$,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(T_t\omega)e^{2\pi it\theta}dt=0,$$

and this finish the proof of the theorem. □

3. JOINING'S PROOF OF WIENER-WINTNER THEOREM FOR ACTION OF AMENABLE GROUP

In this section, we deal with actions on Lebesgue spaces, that is, given a locally compact groupe G and the a Lebesgue space $(X,\mathcal{A},\mu)$, a $G-$*action* is a measurable mapping $G\times X\to X$, $(g,x)\mapsto g.x$, such that for all $g,h\in G$, $g.(h.x)=(gh).x$ and $e.x=x$ for almost all $x\in X$ (where e is the identity in G). Furthermore, $T_g\ :\ x\mapsto g.x$ is measure -preserving for every $g\in G$. We will mainly concerned with G which is amenable group (locally compact second countable) or the subclass of locally compact abelian groups.

We recall that G is an amenable group if for any compact $K\subset G$ and $\delta>0$ there is a compact set $F\subset G$ such that

$$\mathrm{h}_L(F\Delta KF)<\delta\mathrm{h}_L(F),\tag{3.1}$$

where h_L stand for the left Haar measure on G. It is well known that the amenability is equivalent to the existence of Følner sequence (F_n), that is, (F_n) is a sequence of compact subsets of G for which for every compact K and $\delta>0$, for all large enough n we have that F_n satisfy (3.1).

Assume further that (F_n) satisfy the so-called Shulman Condition ,that is, for some $C > 0$ and all n

$$\mathrm{h}_L\left(\bigcup_{k\leq n} F_k^{-1}F_n\right) \leq C.\mathrm{h}_L\left(F_n\right). \tag{3.2}$$

Under this assumptions, E. Lindenstrauss proved that the Birkhoff pointwise ergodic theorem holds, that is, then for any $f \in L^1(\mu)$, there is a G-invariant $f^* \in L^1(\mu)$ such that

$$\lim \frac{1}{\mathrm{h}_L\left(F_n\right)} \int_{F_n} f(g\omega)d\mathrm{h}_L(g) = f^*(\omega) \text{ a.e..}$$

To formulate the G-version of Wiener-Wintner theorem, we replace the group rotations by homomorphisms Θ from G to a finite dimensional unitary group U_d. The canonical action in this case is given by $g.u = \Theta(g).u$, $u \in U_d$ and $g \in G$. The invariant measure is the Haar measure on U_d. In this setting, we formulate the Wiener-Wintner theorem as follows:

Theorem 3.7 (Group version of Wiener-Wintner theorem)**.** *Let G be an amenable group acting on a Lebesgue space $(\Omega, \mathcal{A}, \mu)$ and assume that G satisfy Shulman condition. Let $f \in L^\infty(\mu)$. Then, there is a set Ω_f of full measure such for any $\omega \in \Omega_f$*

$$\frac{1}{h_L\left(F_n\right)} \int_{F_n} f(g\omega)\phi(\Theta(a)u)dh_L(g)$$

converge for all finite dimensional unitary representation Θ of G into U_d (all d), all continuous function ϕ on U_d and all $u \in U_d$. We further have that the limit on the orthocomplement of the space spanned by the finite dimensional invariant subspaces is zero.

Before proceeding to the proof let us recall some important tools.

A *joining* of two actions of the same group $\mathcal{X} = (X, \mathcal{A}, \mu, G)$ and $\mathcal{Y} = (Y, \mathcal{B}, \nu, G)$ is the probability measure λ on $(X \times Y, \mathcal{A} \times \mathcal{B})$ which is invariant under the diagonal action of G $(g.(x, y) = (g.x, g.y))$ and whose marginals on $(\mathcal{A} \times Y)$ and $(X \times \mathcal{B})$ are μ and ν respectively (i.e. if $A \in \mathcal{A}$, $\lambda(A \times Y) = \mu(A)$; and if $B \in \mathcal{B}$, $\lambda(X \times B) = \nu(B)$). The set of joinings is never empty (take $\mu \times \nu$). As we deal with Lebesgue spaces, a joining λ of two ergodic G-actions $\mathcal{X}$ and $\mathcal{Y}$ has the property that there exists a Lebesgue space Ω and the probability ¶ on Ω such that $\lambda = \int \lambda_\omega d¶(\omega)$, where λ_ω is ergodic (this is just the ergodic decomposition of λ, and as the marginals of λ are ergodic a.e., λ_ω is joining). Therefore the set of ergodic joinings is never empty.[2]

Historically, joinings were introduced by H. Furstenberg in his paper [7] on disjointness. In particular, he defined the important notion of disjointness for $\mathbb{Z}$*-action* in the following way : $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ is *disjoint* if the only joining between them is the product joining. In the case of *G-action*, we have the following definition.

Definition 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two actions of the same group G. $\mathcal{X}$ and $\mathcal{Y}$ are disjoint if the only joining between them is the product joining. We denote this disjointness by $\mathcal{X} \perp \mathcal{Y}$.*

In the case of $\mathbb{Z}$-actions, H. Hahn & W. Parry obtain in [8] that if two transformations have mutually singular maximal types, then they are disjoint. But, as for the joinings theory, the spectral theory of $\mathbb{Z}$-actions can be extended to the case of locally abelien $G-$actions. Therefore, we have the following group version of Hahn-Parry theorem.

Theorem 3.8 (Hahn & Parry)**.** *If two G-actions $\mathcal{X}$ and $\mathcal{Y}$ have mutually singular maximal spectral types, then they are disjoint.*

[2] see [5], for instance.

Proof. Let recall that the spectral measure of a function $f \in L^2(X)$ under the operators U_g (defined on $L^2(X)$ by $U_g(f) = f \circ T_g$) is the measure σ_f on $\hat{G}$ (dual group of G, i.e., the set of all continuous characters of G), where its Fourier transform $\overset{\wedge}{\sigma_f}$ is given by $\overset{\wedge}{\sigma_f}(g) = < U_g f, f >$. Now, we follows the proof given in [14]. In $X \times Y$ endowed with a joining measure λ, consider $f_1 \in L^2(X)$ and $f_2 \in L^2(Y)$ and consider H_{f_1} the $L^2(\lambda)$ closure of the linear span of the functions $(U_g(f_1) - \int f_1 d\mu) \times 1_Y, g \in G$. The projection of $1_X \times f_2$ on H_{f_1} will have a spectral measure absolutely continuous with respect to the spectral type of U_g on $L^2(X)$ and thus has to be 0. Therefore $1_X \times f_2 \perp (f_1 - \int f_1) \times 1_Y$, and $\int f_1(x) f_2(y) d\lambda(x,y) = \int f_1 d\mu \int f_2 d\nu$. □

From this theorem, we have the following.

Corollary 3.2. *Let χ_0 be a non trivial character and define the action of G on torus $\mathbb{T}$ by $(g, e^{ix}) \mapsto \chi_0(g)e^{ix}$. Assume that for any $n \in \mathbb{Z}$, the character χ_0^n define on G by $g \mapsto \chi_0(g^n)$ is not eigenvalue of the G-action on $\mathcal{X}$. The G-action on $\mathbb{T}$ and the G-action on $\mathcal{X}$ are disjoint.*

Proof. Let recall that χ_0 is a eigenvalue of G- action if there exists a eigenfunction $f \in L^2(X, \mu)$ such that $f \circ T_g = \chi_0(g)\ f$. We deduce that the spectral measure of f is $||f||_2^2 \delta_{\chi_0}$ (δ_{χ_0} is the Dirac measure on χ_0). Since for any $n \in \mathbb{Z}$, χ_0^n is not eigenvalue of G-action on $\mathcal{X}$, we conclude that the maximal spectral types of this two G-actions are mutually singular. Now apply the Hahn-Parry theorem to complete the proof. □

For the general case of amenable group which satisfy Shulman condition, we have the following lemma from [13].

Lemma 3.1. *Let U be the closure of $\Theta(G)$ in U_d. Then, if the product $(U, \Theta, G) \times (\Omega, \mathbf{A}, \mu, G)$ is ergodic then there is only on G-invariant measure on $U \times \Omega$ that projects onto μ on Ω.*

Proof of Theorem 3.7. We start by assuming without lost of generality that the action on $(\Omega, \mathbf{A}, \mu, G)$ is ergodic and by presenting the proof for the case when G is locally Abelien group. Let $f \in L^\infty(\mu)$ and ϕ continuous function. Then, by the pointwise theorem, there is a set of full measure of ω. Assume that ω is in this subset and let $\chi_0 \in \hat{G}$ such χ_0 is not eigenvalue. Then, the product $(U, \Theta, G) \times (\Omega, \mathbf{A}, \mu, G)$ is ergodic. Moreover, by taking a subsequence (n_k), we can assume that

$$\lim_{k \longrightarrow +\infty} \frac{1}{\mathrm{h}_L(F_{n_k})} \int_{F_{n_k}} f(g\omega)\phi(\Theta(a)u) d\mathrm{h}_L(g) = \lambda(f \otimes \phi).$$

It follows that λ is a joining and by Corollary 3.2, $\lambda = dh \times \mu$. We end the proof by noticing that there is a countable of eigenvalue. The general case follows in the same manner by taking

$$F(\omega) = \int \psi(u) I(u_1 \omega) du,$$

where I is a bounded invariant functions on $U \times \Omega$ and ψ is any positive continuous function on u. Therefore, transforming F by g is the same as transforming ψ by $\Theta(g)$. We thus have that a nonconstant I will give rise to finite dimensional invariant subspaces for G on Ω. Moreover, by taking (U, Θ, G) not in the list of countable representations (U_j, Θ_j, G), the condition of Lemma 3.1 is satisfied and therefore as before the only joining is the product measure, and we are done. □

Question 3.9. *We ask on the possible extension of van der Corput inequality to the locally compact group and its application to produce a direct proof of the group version of Wiener-Wintner theorem.*

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