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An Investigation on The Behaviour of Unbounded Operators in Γ-Hilbert Space

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Keywords:
Γ-Hilbert space, Closed operator, Densely defined operator, Self-adjoint of densely defined operator, Symmetric of densely defined operator.

Abstract — In this paper, we investigate about the behavior of unbounded operators in Γ-Hilbert Space. Here we discussed about the adjoint, self-adjoint, symmetric and other related properties of densely defined operator. We proof some related theorems and corollaries and will show the characterizations of these operators in Γ-Hilbert space.

Subject Classification (2020): 46CXX, 46C05, 46C07,46C15,46C99,47L06.

1. Introduction

Γ-Hilbert space plays an important role in generalization of general linear quadratic control problems in an abstract space [1] which was motivated from the work of L.Debnath and Pitor Mikusinski [8] but there not enough literature found to study about the unbounded operators in Γ-Hilbert space. The definition of Γ-Hilbert space was introduced by Bhattacharya D.K. and T.E. Aman in their paper "Γ-Hilbert space and linear quadratic control problem" in 2003 [9]. Further development was made in 2017 by A.Ghosh, A.Das and T.E. Aman in their research paper [1]. In [6] S.Islam and A.Das discussed about the properties of bounded operators in Γ-Hilbert Space. Boundedness of an operator is a great tool to elaborate Γ-Hilbert Space. We often deal with operators which are not bounded. In this paper, we will briefly discuss the concept, methods and theory of unbounded operators in Γ-Hilbert Space. In this paper, after consulting the main author, we have made some changes in the main definition of Γ-Hilbert space [9].

First, we recall the definitions of Γ-Hilbert Space.

Definition 1.1. Let E be the linear space over the field F and Γ be a semi group with respect to addition. A mapping $\langle .,.,. \rangle : E \times \Gamma \times E \to$ F ($\mathbb{R}$ or $\mathbb{C}$) is called a Γ-Inner product on (E, Γ) if

(i) $\langle .,.,. \rangle$ is linear in first variable and additive in second variable.
(ii) $\langle u, \gamma, v \rangle = \langle v, \gamma, u \rangle \; \forall$ u, v ∈ E and γ ∈ Γ.
(iii) $\langle u, \gamma, u \rangle > 0 \; \forall\, u \neq 0$.
(iv) $\langle u, \gamma, u \rangle = 0$ if at least one of u, γ is zero.

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$[(E, \Gamma), \langle .,.,. \rangle]$ is called a Γ-inner product space over F.

A complete Γ-inner product space is called Γ-Hilbert space.

Using the Γ-inner product, we may define three types of norm in a Γ-Hilbert space, namely (i) γ-norm (ii) $\Gamma_{\inf}$- norm and (iii) Γ-norm.

Definition 1.2. Now if we write $\|u\|_{\gamma}{}^{2} = \langle u, \gamma, u \rangle$, for $u \in H$ and $\gamma \in \Gamma$ then $\|u\|_{\gamma}{}^{2}$ satisfy all the conditions of norm.

Definition 1.3. If we define $\|u\|_{\Gamma_{\inf}} = \inf\{\|u\|_{\gamma} : \gamma \in \Gamma\}$. Clearly $\Gamma_{\inf}$-norm satisfy all the conditions of the norm for $u \in H$.

Definition 1.4. If we write $\|u\|_{\Gamma} = \{\|u\|_{\gamma} : \gamma \in \Gamma\}$ then this norm is called the Γ-norm of the Γ-Hilbert space.

Definition 1.5. Let L be a non-empty subset of a Γ-Hilbert space H_{Γ}. Two elements x and y are said to be γ-orthogonal if their inner product $\langle x, \gamma, y \rangle = 0$. In symbol, we write $x \perp_{\gamma} y$.

2. Basic Concepts

In this section, we briefly discuss about the definition of densely defined operator and the adjoint, self-adjoint , symmetric etc of that operator. Also, related examples and theorem are mentioned in this part.

2.1. Extension of operators

Let S and T be two operators in a vector space E. D_S and D_T are the domains of S and T respectively. If

$$D_S \subset D_T \quad \text{and} \quad Sx = Tx \qquad \text{for every } x \in D_S$$

then T is called an extension of S and we write $S \subset T$.

2.2. Densely defined operator

An operator T defined a linear map T from a subspace of H_{Γ} to H_{Γ} is called an operator in H_{Γ} and the subspace denoted by D_T, is called the domain of T. Now an operator T is defined in a normed space E is called densely defined if its domain D_T is a dense subset of E , that is $\quad$ cl $D_T = E$.

Example 2.2.1. The differential operator $\frac{d}{dx}$ is densely defined in $L^2(\mathbb{R})$, because the subspace of differentiable functions is dense in $L(\mathbb{R})^2$.

Theorem 2.2.2. Let T be a densely defined operator in a Γ-Hilbert space H_{Γ} and let E be the set of all $y \in H_{\Gamma}$ for which $\langle Tx, \gamma, x \rangle$ where $\gamma \in \Gamma$ is a continuous functional on D_T. There exists a unique operator S defined on E such that

$$\langle Tx, \gamma, x \rangle = \langle x, \gamma, Sy \rangle \text{ for all } x \in D_T \text{ and } y \in E .$$

Proof: For any $y \in E$, consider the functional $f_y(x) = \langle Tx, \gamma, x \rangle$ where $\gamma \in \Gamma$. Being continuous on a dense subspace of H_{Γ} , has a unique extension to a continuous functional $\tilde{f}_y$ on H_{Γ}.

By Riesz representation theorem, there exists a unique $Z_y \in \mathrm{H}_\Gamma$ such that $\tilde{f}_y(x) = \langle x, \gamma, Z_y\rangle\ \forall\, x \in \mathrm{H}_\Gamma$. Now if we define $S(y) = Z_y$, then we will have

$$\begin{aligned}\langle Tx, \gamma, x\rangle = f_y(x) &= \tilde{f}_y(x)\\ &= \langle x, \gamma, Z_y\rangle\\ &= \langle x, \gamma, Sy\rangle \text{ for all } x \in \mathrm{D_T}\,,\ \mathrm{y} \in \mathrm{E} \text{ and } \gamma \in \Gamma\,.\end{aligned}$$

Also the linearity of S is obvious.

2.3. Adjoint of densely defined operator

Let T be an operator which is densely defined in a Γ-Hilbert space H_Γ. The adjoint T^* of T is the operator defined on the set of all $y \in \mathrm{H}_\Gamma$ for which $\langle Tx, \gamma, x\rangle$ where $\gamma \in \Gamma$ is a continuous function on $\mathrm{D_T}$ and such that

$$\langle Tx, \gamma, x\rangle = \langle x, \gamma, T^*y\rangle \text{ for all } x \in \mathrm{D_T} \text{ and } y \in D_{T^*}$$

Example 2.3.1. Let $C^1{}_0(\mathbb{R})$ denote the space of all continuously differentiable functions on $\mathbb{R}$. This is also a dense subspace of $L^2(\mathbb{R})$. Now consider the differentiable operator D which defined on $C^1{}_0(\mathbb{R})$. Since

$$\begin{aligned}\langle Dx, \gamma, y\rangle &= \int_{-\infty}^{\infty}\left(\frac{d}{dt}x(t)\right)\gamma\, \overline{y(t)}\ dt\\ &= -\int_{-\infty}^{\infty} x(t)\left(\frac{d}{dt}\,\overline{y(t)}\right)\gamma\, dt \qquad \text{for all} \quad \gamma \in \Gamma\,.\end{aligned}$$

$\therefore \langle Dx, \gamma, y\rangle$ is a continuous functional on $C^1{}_0(\mathbb{R})$.

Moreover,

$$\begin{aligned}\langle Dx, \gamma, y\rangle &= -\int_{-\infty}^{\infty} x(t)\left(\frac{d}{dt}\,\overline{y(t)}\right)\gamma\, dt\,.\\ &= \int_{-\infty}^{\infty} x(t)\ \overline{\left(-\frac{d}{dt}(y(t)\right)}\,\gamma\, dt\,.\end{aligned}$$

Here it is not correct to write $D^* = -D$, since the domain of D^* is not $C^1{}_0(\mathbb{R})$.

2.4. Self –adjoint of densely defined operator

Let T be a densely defined operator in a Γ-Hilbert space H_Γ. Then T is called self-adjoint if $T = T^*$.

Note. $T = T^*$ implies that $D_{T^*} = D_T$ and $T(x) = T^*(x)$ for all $x \in D_T$. If T is a densely defined operator in H_Γ which is bounded then T has a unique extension to a bounded operator in H_Γ. Then the domain of T as well as its adjoint T^*, is the whole space H_Γ . If T is unbounded operators ,then T has an adjoint T^* such that $T(x) = T^*(x)$ whenever $x \in D_T \cap D_{T^*}$, but $D_{T^*} \neq D_T$ and thus T is not self-adjoint.

2.5. Symmetric Operator

We now consider a special kind of operator in Γ-Hilbert space . An operator T which is densely defined in Γ-Hilbert space H_Γ is called symmetric if for all $x, y \in D_T$, we have

$$\langle Tx, \gamma, y\rangle = \langle x, \gamma, Ty\rangle \text{ for all } \gamma \in \Gamma\,.$$

It is clear that if T is symmetric , then $\langle T(x), \gamma, x\rangle \in \mathbb{R}$ for every $x \in D_T$ and $\gamma \in \Gamma$. Also, it follows that a densely defined operator T is symmetric if and only if T^* extends T. If T is symmetric and $\mathrm{D_T} = \mathrm{H}_\Gamma$, then T is in fact a bounded operator on H_Γ. This leads as follows,

Let $E = \{\mathrm{T(x)} : \mathrm{x} \in \mathrm{H}_\Gamma\,, \|\mathrm{x}\|_\gamma \leq 1\}$. Then for a fixed $y \in \mathrm{H}_\Gamma$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}|\langle T(x), \gamma, y\rangle| &= |\langle x, \gamma, T(y)\rangle|\\ &\leq \|x\|\,\|\gamma\|\|T(y)\|\\ &\leq \|T(y)\| \text{ for all } x \in \mathrm{H}_\Gamma \text{ with } \|x\|, \|\gamma\| \leq 1\,.\end{aligned}$$

Also clearly every self-adjoint operator is symmetric.

Example 2.5.1. Suppose we consider an operator $A = \frac{id}{dt}$ with the domain $D_A = \{ f \in L^2([a,b]) :$ f' is continuous and $f(a) = f(b) = 0 \}$.
Now, since for all $\gamma \in \Gamma$, we have

$$\begin{aligned}\langle Af, \gamma, g\rangle &= \int_a^b if'(t)\,\gamma\,\overline{g(t)}\,dt \\ &= \int_a^b f(t)\,\gamma\,\overline{ig'(t)}\,dt \\ &= \langle f, \gamma, Ag\rangle\end{aligned}$$

$\therefore\ \langle Af, \gamma, g\rangle = \langle f, \gamma, Ag\rangle$
for all $f, g \in D_A$, A is symmetric.

$\langle Af, \gamma, g\rangle$ is a continuous functional on D_A for any function g continuously differentiable , no need to satisfying $g(a) = g(b)$.

Consequently , $D_{A^*} \neq D_A$ and A is not self-adjoint.

2.6. Closed Operator

A linear operator $T : E_1 \to E_2$ is said to be closed when the graph $G(T) = \{\langle x, \gamma, Tx\rangle : x \in D_T \text{ and } \gamma \in \Gamma\}$ is a closed subspace of $E_1 \times E_2$ that is

$$x_n \in D_T,\ x_n \to x \text{ and } Tx_n \to y$$

implies $x \in D_T$ and $Tx = y$.

3. Main Results

Theorem 3.1. Let A and B be densely defined operators in a Γ-Hilbert space H_Γ.
(a) If $A \subset B$, then $B^* \subset A^*$.
(b) If D_{B^*} is dense in H_Γ, then $B \subset B^{**}$.

Proof. (a) Let us consider $y \in D_{B^*}$ and $\gamma \in \Gamma$. Then as a function of x, $\langle Bx, \gamma, y\rangle$ is a continuous functional on D_B. Also $\langle Bx, \gamma, y\rangle$ is a continuous functional on D_A since $D_A \subset D_B$.

Now, $Bx = Ax$ for $x \in D_A$, so $\langle Ax, \gamma, y\rangle$ is a continuous functional on D_A. This proves that $y \in D_{A^*}$. Then the equality $A^*y = B^*y$ for $y \in D_{B^*}$ follows from the uniqueness of the adjoint operator.

(b) Let $x \in D_B$. Then for every $y \in D_{B^*}$ and $\gamma \in \Gamma$,we have

$$\langle Bx, \gamma, y\rangle = \langle x, \gamma, B^*y\rangle$$

It can be rewrite as

$$\langle B^*y, \gamma, x\rangle = \langle y, \gamma, Bx\rangle .$$

Since D_{B^*} is dense in H_Γ, B^{**} exists and we have

$\langle B^*y, \gamma, x\rangle = \langle y, \gamma, B^{**}x\rangle$ for all $y \in D_{B^*}$, $x \in D_{B^{**}}$ and $\gamma \in \Gamma$.

Now, by the proof of (a), we can show that $D_B \subset D_{B^{**}}$ and $B(x) = B^{**}(x)$ for any $x \in D_B$. Thus $B \subset B^{**}$.

Theorem 3.2. If T is a one-to-one operator in a Γ-Hilbert space and both T and its inverse T^{-1} are densely defined , then T^* is also one- to-one and $(T^*)^{-1} = (T^{-1})^*$.

Proof. Let $y \in D_{T^*}$. Then for every $x \in D_{T^{-1}}$ and $\gamma \in \Gamma$, we have $T^{-1}x \in D_T$ and hence

$$\begin{aligned}\langle T^{-1}x, \gamma, T^*x\rangle &= \langle TT^{-1}x, \gamma, y\rangle \\ &= \langle x, \gamma, y\rangle .\end{aligned}$$

This follows that $T^*y \in D_{(T^{-1})^*}$.

And also,

$$(T^{-1})^*\, T^*y = (T\, T^{-1})^*\, y = y \tag{3.1}$$

Now we take an arbitrary $y \in D_{(T^{-1})^*}$ .Then for each $x \in D_T$ and $\gamma \in \Gamma$, we have

$$Tx \in D_{T^{-1}} .$$

Hence

$$\langle Tx, \gamma, (T^{-1})^*y\rangle = \langle T^{-1}Tx, \gamma, y\rangle = \langle x, y\rangle \tag{3.2}$$

This shows that $(T^{-1})^*y \in D_{T^*}$. And $T^*(T^{-1})^*y = (T^{-1}T)^*y = y$. Now, from (3.1) and (3.2) it follows that $(T^*)^{-1} = (T^{-1})^*$.

Theorem 3.3. If A, B and AB are densely defined operators in H_Γ , then $B^*\ A^* = (AB)^*$.

Proof. Let $x \in D_{AB}$ and $y \in D_{B^*A^*}$. Since $x \in D_B$ and $A^*y \in D_{B^*}$, it follows that

$$\langle Bx, \gamma, A^*y\rangle = \langle x, \gamma, B^*A^*y\rangle \text{ for all } \gamma \in \Gamma .$$

On the other side, since $Bx \in D_A$ and $y \in D_{A^*}$, we have

$$\langle ABx, \gamma, y\rangle = \langle Bx, \gamma, A^*y\rangle \text{ for all } \gamma \in \Gamma .$$

Hence

$$\langle ABx, \gamma, y\rangle = \langle x, \gamma, B^*A^*y\rangle .$$

Since this holds for all $x \in D_{AB}$,we have $y \in D_{(AB)^*}$ and $(B^*\ A^*)y = (AB)^*y$. This implies, $B^*\ A^* = (AB)^*$.

Theorem 3.4. A densely defined operator T in a Γ-Hilbert space H_Γ is symmetric if and only if $T = T^*$.

Proof: Let us suppose $T = T^*$. Since for all $x \in D_T$ and $y \in D_{T^*}$ we have

$$\langle Tx, \gamma, y\rangle = \langle x, \gamma, T^*y\rangle \text{ where } \gamma \in \Gamma \tag{3.3}$$

Again we have

$$\langle Tx, \gamma, y\rangle = \langle x, \gamma, Ty\rangle \text{ for all } x, y \in D_T \tag{3.4}$$

Thus, T is symmetric. If T is symmetric then combining (3.3) and (3.4) we can conclude $T = T^*$.

Corollary 3.5. If T is a densely defined symmetric operator, then T^* is the maximal symmetric extension of T.

Proof. Let S be a symmetric operator in a Γ-Hilbert space H_Γ such that $T \subset S$. Then by the Theorem 3.3, we have

$$S^* \subset T^* .$$

Hence, $T \subset S \subset S^* \subset T^*$.

Theorem 3.6. If T is closed and invertible, then T^{-1} is closed.

Proof. Let us suppose that graph of T that is $G(T)$ is closed and $G(T) = \{(x, \gamma, Tx): x \in D_T \text{ and } \gamma \in \Gamma\}$. Then obviously

$$G(T^{-1}) = \{(Tx, \gamma, x): x \in D_T \text{ and } \gamma \in \Gamma\} \text{ is closed.}$$

Theorem 3.7. If T is densely defined operator, then T^* is closed.

Proof: If $y_n \in D_{A^*}$, $y_n \to y$ and $A^* y_n \to z$, then for any $x \in D_A$ & $\gamma \in \Gamma$ we have

$$\begin{aligned} \langle Ax, \gamma, y \rangle &= \lim_{n\to\infty} \langle Ax, \gamma, y_n \rangle \\ &= \lim_{n\to\infty} \langle x, \gamma, A^* y_n \rangle \\ &= \langle x, \gamma, z \rangle \end{aligned}$$

Hence, $y \in D_{A^*}$ and $A^* y = z$.

Note. If the given operator A is not closed then is it possible to extend A to a closed operator? Answer to that problem is to use the closure of $G(A)$ in $H_\Gamma \times H_\Gamma$ to define an operator. If closure of $G(A)$ defines an operator, then extension of A is closed.

Theorem 3.8. Every symmetric and densely defined operator in Γ-Hilbert space has a closed symmetric extension.

Proof. Let A be a densely defined, symmetric operator in a Γ-Hilbert space H_Γ. At first, we will show that condition $x_n \in D_A$, $x_n \to 0$, as $Ax_n \to y$ which implies that $y = 0$, is satisfied.

Let $x_n \to 0$ and $Ax_n \to y$. Since A is symmetric then for all $\gamma \in \Gamma$ we have

$$\begin{aligned} \langle y, \gamma, z \rangle &= \lim_{n\to 0} \langle Ax_n, \gamma, z \rangle \\ &= \lim_{n\to 0} \langle x_n, \gamma, Az \rangle \\ &= 0, \quad \text{for any } z \in D_A . \end{aligned}$$

This implies $y = 0$, as D_A is dense in H_Γ.

Now we have that there exists a closed operator B such that $G(B) = \mathrm{Cl}G(A)$ and hence $A \subset B$. We have to prove that B is symmetric. If $x, y \in D_B$, then there exists $x_n, y_n \in D_A$ such that

$$x_n \to x \quad , \quad Ax_n \to Ax$$

and

$$y_n \to y \quad , \quad Bx_n \to Bx .$$

Since A is a symmetric operator , we have

$$\langle Ax_n, \gamma, y_n \rangle = \langle x_n, \gamma, Ay_n \rangle \text{ for all } \gamma \in \Gamma .$$

Then by letting $n \to \infty$, we have

$$\langle Bx, \gamma, y\rangle = \langle x, \gamma, By\rangle.$$

Hence B is symmetric.

Theorem 3.9. Let T be a closed densely defined operator in a Γ-Hilbert space H_Γ. Then

(a) For any $v, w \in \mathrm{H}_\Gamma$, there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that $T(x) + y = v$ and $x - T^*(y) = w$.
(b) For any $w \in \mathrm{H}_\Gamma$, there exist unique $x \in D_{T^*T}$ such that $x + T^*T(x) = w$.

Proof. (a) Consider the Γ-Hilbert space $\mathrm{H}_{\Gamma_1} = \mathrm{H}_\Gamma \times \mathrm{H}_\Gamma$. Since T is closed, $G(T) = \{(x, \gamma, T(x)) : x \in D_T \text{ and } \gamma \in \Gamma\}$ is a closed subspace of H_{Γ_1}. Then by the projection theorem we have

$$\mathrm{H}_{\Gamma_1} = \mathrm{G(T)} + \mathrm{G(T)}^{\perp\gamma},$$

with

$$\mathrm{G(T)} \cap \mathrm{G(T)}^{\perp\gamma} = \{0\}.$$

Now, $(u, y) \in \mathrm{G(T)}^{\perp\gamma}$ if and only if $\langle (x, Tx), \gamma, (u, y)\rangle = 0$ for all $x \in D_T$ and $\gamma \in \Gamma$. This implies, $\langle x, \gamma, u\rangle + \langle T(x), \gamma, y\rangle = 0$. That is $(u, y) \in \mathrm{G(T)}^{\perp\gamma}$ if and only if $\langle T(x), \gamma, y\rangle = \langle x, \gamma, -u\rangle$ for all $x \in D_T$. In other way,

$$(u, y) \in \mathrm{G(T)}^{\perp\gamma} \text{ if and only if } y \in D_{T^*} \text{ and } u = -T^*(y).$$

Since $(w, v) \in \mathrm{H}_\Gamma \times \mathrm{H}_\Gamma$, then there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that

$$(w, \gamma, v) = (x, \gamma, T(x)) + (-T^*(y), \gamma, y) \text{ for all } \gamma \in \Gamma.$$

That is, $w = x - T^*(y)$ and $v = T(x) + y$.

(b) Letting $v = 0$ in (a), then there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that $T(x) + y = 0$ and $x - T^*(y) = w$. Thus $x - T^*(-T(x)) = 0$ implies, $x + T^*T(x) = w$, as desired.

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XMod$_{\mathbf{Lie}}$ Fibred Over Lie Algebras

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Keywords
Crossed modules, Fibration, Lie algebras, Pullback

Abstract — In this work, we showed that the category of crossed modules over Lie algebras is fibred over the category of Lie algebras by illustrating that the forgetful functor is a fibration.

Subject Classification (2020): 18N45, 18G45, 17B66.

1. Introduction

The crossed module is defined by Whitehead on groups in [1]. In his work, algebraic structures that CW-complex has homotopy 2-type were the crossed modules. In the following years, Lichtenbaum, Schlessinger [2] and Gerstenhaber [3] have made different definitions on the crossed modules of Lie algebras. The definition and some of the categorical properties of crossed modules over algebras can be found in [4] Shammu's work. The crossed modules of commutative algebras could be seen in the work of Porter [5].

Lie algebras were first studied by Marius Sophus Lie in 1870s and independently by Wilhelm Killing in the 1880s to create infinitely small transformations. Lie algebras is defined by Hermann Weyl in 1930s. Also, crossed modules of Lie algebras studied by Kassel and Loday [6]. They investigated this notion with computational algebraic structures that are equivalent to simplicial Lie algebras with Moore complex of length 1. For more details [7–9]. Akça and Arvasi examined simplicial Lie algebras and crossed Lie algebras equivalency and applied to Lie crossed squares [10].

To show the fibration feature, the characteristics of the functor defined between the categories should be examined. The first known definition of fibration was described by Heinz Hopf [11] in his article as Hopf Fibration. In this study, the category of crossed modules on Lie algebras is referred to **XMod**$_{\mathbf{Lie}}$ that this structure obtained before.

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2. Preliminary

Definition 2.1. Let A be a commutative ring with identity if the bilinear function

$$[,]: M \times M \longrightarrow M$$

called multiplication satisfies

L1. $[x, x] = 0$

L2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

for $x, y, z \in M$ then M is called a Lie algebra over A with $[,]$. Property $L1$ with bilinearity implies the following two conditions.

L3. $[x, y] = -[y, x]$,

L4. $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

Recall that if $\alpha : N \longrightarrow M$ is a Lie algebra morphism then for all $m_1, m_2 \in M$

$$\alpha([m_1, m_2]) = [\alpha(m_1), \alpha(m_2)]$$

we will denote the category of Lie algebras with **Lie** [12].

Definition 2.2. Let A and N be two Lie k-algebras. If the N-algebra morphism

$$\mu : N \to A$$

with Lie action of A on N given by

$$\begin{gathered} A \times N \to N \\ (a, n) \mapsto a \cdot n \end{gathered}$$

satisfies

$XMod_{Lie}1.$ $\mu(a \cdot n) = [a, \mu(n)]$

$XMod_{Lie}2.$ $n' \cdot \mu(n) = [n', n]$

then the triple (N, A, μ) is called crossed modules of Lie algebras [6].

Definition 2.3. Let (A, N, μ), (A', N', μ') be two crossed modules of Lie algebras. A morphism

$$(f, \phi) : (A, N, \mu) \longrightarrow (A', N', \mu')$$

of crossed modules of Lie algebras consists of pair of Lie algebra morphism $f : N \longrightarrow N$ and $\phi : A \longrightarrow A'$ such that

$$f(n \cdot a) = f(n) \cdot \phi(a)$$

for all $a \in A$ and $n \in N$ and the following diagram

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\mu"'] & N' \arrow[d, "\mu'"] \\ A \arrow[r, "\phi"'] & A' \end{tikzcd}$$

commutes. That is $\mu' f = \phi\mu$. We will denote this category with $\mathbf{XMod_{Lie}}$.

Example 2.4. Let R be a Lie algebra and I be an ideal of R.

$$\begin{aligned} \partial : \quad & I \to R \\ & i \mapsto i \end{aligned}$$

and the action of R on I is given as Lie product

$$\begin{aligned} R \times I &\to I \\ (r,i) &\mapsto [r,i] \end{aligned}$$

CM1.

$$\begin{aligned} \partial(r \cdot i) &= \partial[r,i] \\ &= [r,i] \\ &= [r,\partial i] \end{aligned}$$

CM2.

$$\begin{aligned} (\partial r) \cdot r' &= [\partial r, r'] \\ &= [r,r'] \end{aligned}$$

So, (I,R,∂) is a crossed module of Lie algebras.

3. $\mathbf{XMod_{Lie}}$ Fibred Over Lie Algebras

In this section, we will show that the forgetful functor

$$\theta : \mathbf{XMod_{Lie}} \to \mathbf{Lie}$$

which takes $\mu : M \to N \in Ob(\mathbf{XMod_{Lie}})$ in its base Lie Algebra N is a fibration. That is given a map of Lie algebras, we can obtain the crossed modules of Lie algebras can be constructed via pullback. Furthermore, the functor θ has a left adjoint.

Proposition 3.1. The functor

$$\theta : \mathbf{XMod_{Lie}} \to \mathbf{Lie}$$

is fibred.

Proof.

To prove that θ is fibred we will give the construction with pullback crossed module Lie algebras.

Let $L \to N$ be an object in $\mathbf{XMod_{Lie}}$ and $M \to N$ is a homomorphism of Lie algebras

$$\begin{tikzcd} & L \arrow[d, "\mu"] \\ M \arrow[r, "\sigma"'] & N \end{tikzcd}$$

Define $P = \{(l,m) : \mu(l) = \sigma(m)\} \subset L \times M$ and mappings

$$\begin{array}{ll} i_1 : P \to L & \qquad i_2 : P \to M \\ (l,m) \mapsto l & \qquad (l,m) \mapsto m \end{array}$$

That is we obtain the following diagram.

$$\begin{tikzcd} P \arrow[r, "i_1"] \arrow[d, "i_2"'] & L \arrow[d, "\mu"] \\ M \arrow[r, "\sigma"'] & N \end{tikzcd}$$

It is clear that i_1 and i_2 are Lie algebra morphims. For $(l,m) = p \in P$ we have

$$\begin{aligned} (\sigma \circ i_2)(p) &= (\sigma \circ i_2)(l,m) \\ &= \sigma(i_2(l,m)) \\ &= \sigma(m) \\ &= \mu(l) \\ &= \mu(i_1(l,m)) \\ &= (\mu \circ i_1)(l,m) \\ &= (\mu \circ i_1)(p) \end{aligned}$$

the diagram is commutative.

Claim $\eta : P \to N \in Ob(\mathbf{XMod_{Lie}})$.

For $p = (l,m), p' = (l',m') \in P$ defining the Lie bracket as

$$[(l,m),(l',m')] = ([l,l'],[m,m'])$$

the action is

$$\begin{array}{rcl} N \times P & \to & P \\ (n,p) & \mapsto & n \cdot p = n \cdot (l,m) = (n \cdot l, n \cdot m) \end{array}$$

and η is

$$\begin{array}{rcl} P & \to & N \\ p & \mapsto & \eta(p) = \eta(l,m) = \mu(l) = \sigma(m) \end{array}$$

The conditions are:

XMLie1. For $n \in A$ and $p \in P$

$$\begin{aligned}\eta(n \cdot p) &= \eta(n \cdot (l,m)) \\ &= \eta(n \cdot l, n \cdot m) \\ &= \sigma(n \cdot m) \\ &= [n, \sigma(m)] \\ &= [n, \eta(l,m)] \\ &= [n, \eta(p)]\end{aligned}$$

XMLie2. For $p, p' \in P$

$$\begin{aligned}\eta(p) \cdot p' &= \eta((l,m) \cdot (l',m')) \\ &= \sigma(m) \cdot (l',m') \\ &= (\sigma(m) \cdot l', \sigma(m) \cdot m') \\ &= (\mu(l) \cdot l', \sigma(m) \cdot m') \\ &= ([l,l'],[m,m']) \\ &= [(l,m),(l',m')]\end{aligned}$$

Claim 1: Let $X \in Ob(\mathbf{Lie})$ and $\alpha_1 : X \to L, \alpha_2 : X \to M \in Mor(\mathbf{Lie})$. Then there exists

$$\begin{aligned}h : X &\to P \\ x &\mapsto (\alpha_1(x), \alpha_2(x))\end{aligned}$$

Since $\alpha_1, \alpha_2 \in Mor(\mathbf{Lie})$ we get $h \in Mor(\mathbf{Lie})$. For $x \in X$ we have

$$(i_1 \circ h)_{(x)} = i_1(h(x)) = i_1(\alpha_1(x), \alpha_2(x)) = \alpha_1(x)$$

$$(i_2 \circ h)_{(x)} = i_2(h(x)) = i_2(\alpha_1(x), \alpha_2(x)) = \alpha_2(x)$$

that is the diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "h"] \arrow[drr, bend left, "\alpha_1"] \arrow[ddr, bend right, "\alpha_2"'] & & \\ & P \arrow[r, "i_1"] \arrow[d, "i_2"] & L \\ & M & \end{tikzcd}$$

is commutative.

Claim 2: h is unique.

Let $h' : X \to P \in Mor(\mathbf{Lie})$ defined as $h(x) = (l,m) = p$ for $x \in X$ such that $i_1 \circ h' = \alpha_1$ and $i_2 \circ h' = \alpha_2$.
For $x \in X$ we have

$$(i_1 \circ h')_{(x)} = i_1(h'(x)) = i_1(l,m) = l = \alpha_1(x)$$

$$(i_2 \circ h')_{(x)} = i_2(h'(x)) = i_2(l,m) = m = \alpha_2(x)$$

that is

$$h'(x) = (l,m) = (\alpha_1(x), \alpha_2(x)) = h(x)$$

As a result in the diagram

$$\begin{array}{ccc} P & \xrightarrow{i_1} & L \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ M & \xrightarrow[\sigma]{} & N \end{array}$$

the morphism (i_1, σ) becomes cartesian morphism over $\phi(i_1, \sigma) = \sigma'$ which shows ϕ is a fibration of categories.

Theorem 3.2. The functor $F : \mathbf{XMod_{Lie}} \longrightarrow \mathbf{Lie}$ which is given by $F(L,N,\mu) = N$(base Lie algebra) has a right adjoint.

Proof.
Let $G : \mathbf{Lie} \longrightarrow \mathbf{XMod_{Lie}}$ be defined as $G(M) = (M, M, id)$. Then for any crossed module of Lie algebra say (L,N,μ) and a Lie Algebra. M define the morphism

$$\phi : \mathbf{Lie}(F(L,N,\mu), M) \longrightarrow \mathbf{XMod_{Lie}}((L,N,\mu), G(M))$$

as follows, if $\alpha : N \longrightarrow M$ is a Lie algebra morphism then

$$\phi(\alpha) = (\alpha \circ \mu, \alpha) : (L,N,\mu) \longrightarrow (M,M,id)$$

is a morphism in $\mathbf{XMod_{Lie}}$.

$$\begin{array}{ccc} L & \xrightarrow{\alpha\circ\mu} & M \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle id} \\ N & \xrightarrow[\alpha]{} & M \end{array}$$

i)

$$\begin{aligned} \alpha \circ \mu(n \cdot l) &= \alpha(n,l) \\ &= [\alpha(n), \alpha(l)] \\ &= \alpha(n) \cdot \alpha \circ \mu(l) \end{aligned}$$

ii) The commutativity of this diagram is obvious.

Conversely, for any morphism in $\mathbf{XMod_{Lie}}$ say

$$(\alpha, \beta) : (L,N,\mu) \longrightarrow (M,M,id)$$

Let us define $\psi(\alpha, \beta) = \beta : N \longrightarrow M$ which is a Lie algebra morphism. Then we get $\psi \circ \phi = 1_{\mathbf{Lie}}$ and $\phi \circ \psi = 1_{\mathbf{XMod_{Lie}}}$.

Therefore, ϕ is a bijection.

Now let us show that naturality in (L,N,μ) and M.

Moreover, for any morphism $(\alpha, \beta) : (L,N,\mu) \longrightarrow (P,S,\partial)$ and a Lie algebra morphism $h : A \longrightarrow B$ the following diagrams are commutative.

$$\begin{tikzcd}
\mathbf{Lie}(F(L,N,\mu),A) \arrow[r] \arrow[d] & \mathbf{XMod_{Lie}}((L,N,\mu),G(A)) \arrow[d] \\
\mathbf{Lie}(F(L,N,\mu),B) \arrow[r] & \mathbf{XMod_{Lie}}((L,N,\mu),G(B))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Lie}(F(L,N,\mu),A) \arrow[r] \arrow[d] & \mathbf{XMod_{Lie}}((L,N,\mu),G(A)) \arrow[d] \\
\mathbf{Lie}(F(P,S,\partial),A) \arrow[r] & \mathbf{XMod_{Lie}}((P,S,\partial),G(A))
\end{tikzcd}$$

Thus ϕ is an isomorphism.

4. Conclusion

In this paper we show that the category $\mathbf{XMod_{Lie}}$ is fibred over Lie algebras. Further work can be done by investigating induced structure on $\mathbf{XMod_{Lie}}$ for a cofibration. Then, it is of interest to investigate the functor

$$\psi^* : \mathbf{XMod_{Lie}}/N \to \mathbf{XMod_{Lie}}/M$$

has a right adjoint

$$\psi_* : \mathbf{XMod_{Lie}}/M \to \mathbf{XMod_{Lie}}/N$$

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Some Pair Difference Cordial Graphs

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Keywords
Path,
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Ladder

Abstract — Let $G=(V,E)$ be a (p,q) graph. Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L=\{\pm1,\pm2,\pm3,\cdots,\pm\rho\}$ called the set of labels. Consider a mapping $f: V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $\left|f(u)-f(v)\right|$ such that $\left|\Delta_{f_1}-\Delta_{f_1^c}\right| \le 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of $P_n \odot K_1$,$P_n \odot K_2$,$C_n \odot K_1$,$P_n \odot 2K_1$,$L_n \odot K_1$,$G_n \odot K_1$, where G_n is a gear graph and etc.

Subject Classification (2020): 05C78.

1. Introduction

In this paper we consider only finite, undirected and simple graphs. Cordial labeling was introduced in [1] and more cordial related labeling was studied in [2, 3]. Corona product operations used in several areas of graph theory [4–6]. In [7] the notion of pair difference cordial labeling of a graph was introduced and also in the same article pair difference cordial labeling behaviour of path, cycle, star, ladder have been studied. The pair difference cordial labeling behavior of snake related graph and butterfly graph have been investigated in [8]. In this paper we have study about the pair difference cordiality of some graphs using corona product operations like $P_n \odot K_1$,$P_n \odot K_2$,$C_n \odot K_1$,$P_n \odot 2K_1$,$L_n \odot K_1$,$G_n \odot K_1$, where G_n is a gear graph.

2. Preliminaries

Definition 2.1. [9] Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. The join G_1+G_2 as $G_1 \cup G_2$ together with all the edges joining vertices of V_1 to the vertices of V_2.

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Definition 2.2. [9] The corona graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy G_2, where G_1 is graph of order n.

Definition 2.3. [10] The graph $W_n = C_n + K_1$ is called the wheel graph.

Definition 2.4. [10] The ladder L_n is the product graph $P_n X K_2$ with $2n$ vertices and $3n-2$ edges.Let $V(L_n) = \{a_i, b_i : 1 \leq i \leq n\}$ and $E(L_n) = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1}, b_i b_{i+1} : 1 \leq i \leq n\}$.

Definition 2.5. [5] The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.Let $V(G_n) = \{x, x_i, y_i : 1 \leq i \leq n\}$ and $E(G_n) = \{xx_i : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{y_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_n x_1\}$.

3. Main Results

$P_n \odot K_1$ is pair difference cordial for all values of n [7]. Now we investigate the pair difference cordiality of $P_n \odot K_2$.

Theorem 3.1. $P_n \odot K_2$ is pair difference cordial for all values of n.

Proof.
Let $V(P_n \odot K_2) = \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_2) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_i z_i, x_i y_i, x_i z_i : 1 \leq i \leq n\}$.Clearly $P_n \odot K_2$ has $3n$ vertices and $4n-1$ edges. There are two cases arises.

Case 1.n is even.

First assign the labels $\frac{3n}{2}, \frac{3n-2}{2}, \frac{3n-4}{2}, \cdots, n+1$ to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n}{2}}$ and assign the labels $-\frac{3n}{2}$, $-\frac{3n-2}{2}$, $-\frac{3n-4}{2}$, $\cdots$,$-(n+1)$ to the vertices $x_{\frac{n+2}{2}}, x_{\frac{n+4}{2}}, x_{\frac{n+6}{2}}, \cdots, x_n$. Next assign the labels $1, 3, 5, \cdots, n-1$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(n-1)$ to the vertices $y_{\frac{n+2}{2}}, y_{\frac{n+4}{2}}, y_{\frac{n+6}{2}}, \cdots, y_n$. Now assign the labels $2, 4, 6, \cdots, n$ to the vertices $z_1, z_2, z_3, \cdots, z_{\frac{n}{2}}$ and assign the labels $-2, -4, -6, \cdots, -n$ to the vertices $z_{\frac{n+2}{2}}, z_{\frac{n+4}{2}}, z_{\frac{n+6}{2}}, \cdots, z_n$.

Case 2.n is odd.

Assign the labels $\frac{3n-3}{2}$, $\frac{3n-5}{2}, \frac{3n-7}{2}, \cdots, n$ to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n-1}{2}}$ and assign the labels $-\frac{3n-3}{2}$,$-\frac{3n-5}{2}$, $-\frac{3n-7}{2}$,$\cdots, -n$ to the vertices $x_{\frac{n+1}{2}}$, $x_{\frac{n+3}{2}}$, $x_{\frac{n+5}{2}}$, $\cdots$,x_{n-1}. Now assign the labels $1, 3, 5, \cdots, n-2$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n-1}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(n-2)$ to the vertices $y_{\frac{n+1}{2}}, y_{\frac{n+3}{2}}, y_{\frac{n+5}{2}}, \cdots, y_{n-1}$. Next assign the labels $2, 4, 6, \cdots, n-1$ to the vertices $z_1, z_2, z_3, \cdots, z_{\frac{n-1}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(n-1)$ to the vertices $z_{\frac{n+1}{2}}$, $z_{\frac{n+3}{2}}$, $z_{\frac{n+5}{2}}$,$\cdots$, z_{n-1}. Finally assign the labels $\frac{3n-1}{2}$, $-(\frac{3n-1}{2})$, $\frac{3n-3}{2}$ to the vertices x_n, y_n, z_n. The Table 1 given below establish that this vertex labeling f is a pair difference cordial of $P_n \odot K_2$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$2n-1$	$2n$
n is even	$2n$	$2n-1$

Table 1

$C_n \odot K_1$ is pair difference cordial for all values of $n \geq 3$ [7]. We now investigate the pair difference cordiality of $C_n \odot K_2, n \geq 3$.

Theorem 3.2. $C_n \odot K_2$ is pair difference cordial for all values of $n \geq 3$.

Proof.
Let $V(C_n \odot K_2) = \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(C_n \odot K_2) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{y_i z_i, x_i y_i, x_i z_i : 1 \leq i \leq n\}$. Obviously $C_n \odot K_2$ has $3n$ vertices and $4n$ edges. As in theorem 3.2, Assign the labels to the vertices $x_i, y_i, z_i (1 \leq i \leq n)$ of $C_n \odot K_2$. This vertex labeling f yields that $\Delta_{f_1} = \Delta_{f_1^c} = 2n$.

Theorem 3.3. $P_n \odot 2K_1$ is pair difference cordial for all values of n.

Proof.
Let $V(P_n \odot 2K_1) = \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(P_n \odot 2K_1) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i, x_i z_i : 1 \leq i \leq n\}$. Note that $P_n \odot K_2$ has $3n$ vertices and $3n-1$ edges.There are two cases arises.

Case 1.n is even.

Assign the labels $2, 5, 8, \cdots, \frac{3n-2}{2}$ respectively to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n}{2}}$ and assign the labels $-1, -4, -7, \cdots$, -3n-4$_{\overline{2}}$ to the vertices $x_{\frac{n+2}{2}}, x_{\frac{n+4}{2}}, x_{\frac{n+6}{2}}, \cdots, x_n$ respectively. Next we assign the labels $1, 4, 7, \cdots, \frac{3n-4}{2}$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n}{2}}$ respectively and assign the labels $-2, -5, -8, \cdots, -\frac{3n-2}{2}$ respectively to the vertices $y_{\frac{n+2}{2}}, y_{\frac{n+4}{2}}, y_{\frac{n+6}{2}}, \cdots, y_n$. We now assign the labels $3, 6, 9, \cdots, \frac{3n}{2}$ respectively to the vertices $z_1, z_2, z_3, \cdots, z_{\frac{n}{2}}$ and assign the labels $-3, -6, -9, \cdots, -\frac{3n}{2}$ to the vertices $z_{\frac{n+2}{2}}, z_{\frac{n+4}{2}}, z_{\frac{n+6}{2}}, \cdots, z_n$ respectively.

Case 2.n is odd.

Assign the labels $2, 5, 8, \cdots, \frac{3n-5}{2}$ respectively to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n-1}{2}}$ and assign the labels $-1, -4, -7, \cdots$, -3n-7$_{\overline{2}}$ to the vertices $x_{\frac{n+1}{2}}, x_{\frac{n+3}{2}}, x_{\frac{n+5}{2}}, \cdots, x_{n-1}$ respectively. Next assign the labels $1, 4, 7, \cdots, \frac{3n-7}{2}$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n-1}{2}}$ respectively and assign the labels $-2, -5, -8, \cdots, -\frac{3n-5}{2}$ respectively to the vertices $y_{\frac{n+1}{2}}, y_{\frac{n+3}{2}}, y_{\frac{n+5}{2}}, \cdots, y_{n-1}$. Now assign the labels $3, 6, 9, \cdots, \frac{3n-3}{2}$ respectively to the vertices $z_1, z_2, z_3 \cdots, z_{\frac{n-1}{2}}$ and assign the labels $-3, -6, -9, \cdots, -\frac{3n-3}{2}$ to the vertices $z_{\frac{n+1}{2}}, z_{\frac{n+3}{2}}, z_{\frac{n+5}{2}}, \cdots, z_{n-1}$ respectively. Finally assign the labels $\frac{3n-1}{2}, -\frac{3n-1}{2}, \frac{3n-3}{2}$ to the vertices x_n, y_n, z_n.
The Table 2 given below establish that this vertex labeling f is a pair difference cordial of $P_n \odot 2K_1$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n}{2}$

Table 2

Theorem 3.4. $C_n \odot 2K_1$ is pair difference cordial for all values of $n \geq 3$.

Proof.
Let $V(C_n \odot 2K_1) = \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(P_n \odot 2K_1) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{x_i y_i, x_i z_i : 1 \leq i \leq n\}$. Clearly $C_n \odot 2K_1$ has $3n$ vertices and $3n$ edges. As in theorem 3.4, Assign the labels to the vertices $x_i, y_i, z_i (1 \leq i \leq n)$ of $C_n \odot 2K_1$.
The Table 3 given below establish that this vertex labeling f is a pair difference cordial of $C_n \odot 2K_1$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
n is even	$\frac{3n}{2}$	$\frac{3n}{2}$

Table 3

Theorem 3.5. $L_n \odot K_1$ is pair difference cordial for all values of n.

Proof.
Let $V(L_n \odot K_1) = V(L_n) \cup \{x_i, y_i : 1 \leq i \leq n\}$ and $E(L_n \odot K_1) = E(L_n) \cup \{a_i x_i, b_i y_i : 1 \leq i \leq n\}$. Note that $L_n \odot K_1$ has $4n$ vertices and $4n-2$ edges. Assign the labels $1, 2, 3, \cdots, n$ to the vertices $a_1, a_2, a_3, \cdots, a_n$ and assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $b_1, b_2, b_3, \cdots, b_n$. Next assign the labels $2n, 2n-1, 2n-2, \cdots, n+1$ to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $-2n, -2n+1, -2n+2, \cdots, -n-1$ to the vertices $y_1, y_2, y_3, \cdots, y_n$. This vertex labeling gives that, $\Delta_{f_1} = 2n, \Delta_{f_1^c} = 2n-1$.

Theorem 3.6. $L_n \odot 2K_1$ is pair difference cordial for all values of n.

Proof.
Let $V(L_n \odot 2K_1) = V(L_n) \cup \{x_i, y_i, u_i, v_i : 1 \leq i \leq n\}, E(L_n \odot 2K_1) = E(L_n) \cup \{a_i x_i, a_i u_i, b_i v_i, b_i y_i : 1 \leq i \leq n\}$. Obviously $L_n \odot 2K_1$ has $6n$ vertices and $7n-2$ edges.

Case 1. n is even.

Define the map $f : V(L_n \odot 2K_1) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$ by

$$
\begin{aligned}
f(a_1) &= 2, & f(b_1) &= -2, \\
f(x_1) &= 1, & f(y_1) &= -1, \\
f(u_1) &= 3, & f(v_1) &= -3, \\
f(a_i) &= f(a_{i-1}) + 3, & 2 &\leq i \leq \frac{n}{2}, \\
f(b_i) &= f(b_{i-1}) - 3, & 2 &\leq i \leq n-1, \\
f(x_i) &= f(x_{i-1}) + 3, & 2 &\leq i \leq \frac{n}{2}, \\
f(u_i) &= f(u_{i-1}) + 3, & 2 &\leq i \leq n, \\
f(y_i) &= f(y_{i-1}) - 3, & 2 &\leq i \leq n-1, \\
f(v_i) &= f(v_{i-1}) - 3, & 2 &\leq i \leq n-1,
\end{aligned}
$$

$$
\begin{aligned}
f(a_{\frac{n+2}{2}}) &= f(a_{\frac{n}{2}}) + 2, \\
f(x_{\frac{n+2}{2}}) &= f(x_{\frac{n}{2}}) + 4, \\
f(a_{\frac{n+2i}{2}}) &= f(a_{\frac{n+2}{2}}) + 3i - 3, && 2 \le i \le \frac{n}{2}, \\
f(u_{\frac{n+2i}{2}}) &= f(u_{\frac{n+2}{2}}) + 3i - 3, && 2 \le i \le \frac{n}{2}, \\
f(x_{\frac{n+2i}{2}}) &= f(x_{\frac{n+2}{2}}) + 3i - 3, && 2 \le i \le \frac{n}{2}, \\
f(b_n) &= -f(a_n), \\
f(v_n) &= -f(u_n), \\
f(x_n) &= -f(y_n).
\end{aligned}
$$

Case 2. n is odd.

Define the map $f : V(L_n \odot 2K_1) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$ by

$$
\begin{aligned}
f(a_1) &= 2, && f(b_1) = -2, \\
f(x_1) &= 1, && f(y_1) = -1, \\
f(u_1) &= 3, && f(v_1) = -3, \\
f(a_i) &= f(a_{i-1}) + 3, && 2 \le i \le \frac{n+1}{2}, \\
f(x_i) &= f(x_{i-1}) + 3, && 2 \le i \le \frac{n+1}{2}, \\
f(u_i) &= f(u_{i-1}) + 3, && 1 \le i \le n,
\end{aligned}
$$

$$
\begin{aligned}
f(a_{\frac{n+3}{2}}) &= f(a_{\frac{n+1}{2}}) + 2, \\
f(x_{\frac{n+3}{2}}) &= f(x_{\frac{n+1}{2}}) + 4, \\
f(a_{\frac{n+2i+1}{2}}) &= f(a_{\frac{n+1}{2}}) + 3i - 3, && 2 \le i \le \frac{n-1}{2}, \\
f(u_{\frac{n+2i+1}{2}}) &= f(u_{\frac{n+1}{2}}) + 3i - 3, && 2 \le i \le \frac{n-1}{2}, \\
f(x_{\frac{n+2i+1}{2}}) &= f(x_{\frac{n+1}{2}}) + 3i - 3, && 2 \le i \le \frac{n-1}{2}, \\
f(b_i) &= -f(a_i), && 1 \le i \le n, \\
f(v_i) &= -f(u_i), && 1 \le i \le n, \\
f(x_i) &= -f(y_i), && 1 \le i \le n.
\end{aligned}
$$

The Table 3 given below establish that this vertex labeling f is a pair difference cordial of $L_n \odot 2K_1$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{7n-3}{2}$	$\frac{7n-1}{2}$
n is even	$\frac{7n-2}{2}$	$\frac{7n-2}{2}$

Table 4

Theorem 3.7. $W_n \odot 2K_1$ is pair difference cordial for all values of $n \geq 3$.

Proof.

Let $V(W_n \odot 2K_1) = \{x_i, y_i, z_i : 1 \leq i \leq n\} \cup \{x, w_1, w_2\}$ and $E(W_n \odot 2K_1) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{xx_i, x_i y_i, x_i z_i : 1 \leq i \leq n\} \cup \{x_1 x_n, xw_1, xw_2\}$. Note that $W_n \odot 2K_1$ has $3n+3$ vertices and $4n+2$ edges.

Case 1. n is even.

Define the map $f : V(W_n \odot 2K_1) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+2}{2}\}$ by

$$
\begin{aligned}
f(x_1) &= 2, & f(y_1) &= 1, \\
f(z_1) &= 3, & f(w_1) &= \frac{3n+2}{2}, \\
f(w_2) &= -\frac{3n+2}{2}, & f(x) &= \frac{3n}{2}, \\
f(x_i) &= f(x_{i-1}) + 3, & 2 &\leq i \leq \frac{n}{2}, \\
f(y_i) &= f(y_{i-1}) + 3, & 2 &\leq i \leq \frac{n}{2}, \\
f(z_i) &= f(z_{i-1}) + 3, & 2 &\leq i \leq \frac{n}{2}, \\
f(x_{\frac{n+2i}{2}}) &= -f(x_i), & 1 &\leq i \leq \frac{n}{2}, \\
f(y_{\frac{n+2i}{2}}) &= -f(y_i), & 1 &\leq i \leq \frac{n}{2}, \\
f(z_{\frac{n+2i}{2}}) &= -f(z_i), & 1 &\leq i \leq \frac{n}{2},
\end{aligned}
$$

Case 2. n is odd.

Define the map $f : V(W_n \odot 2K_1) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+3}{2}\}$ by

$$
\begin{aligned}
f(x_1) &= 2, & f(y_1) &= 1, \\
f(z_1) &= 3, & f(w_1) &= -\frac{3n+1}{2}, \\
f(w_2) &= -\frac{3n+3}{2}, & f(x_n) &= \frac{3n+1}{2}, \\
f(y_n) &= \frac{3n-1}{2}, & f(z_n) &= \frac{3n+3}{2}, \\
f(x) &= -\frac{3n-1}{2}, \\
f(x_i) &= f(x_{i-1}) + 3, & 2 &\leq i \leq \frac{n-1}{2}, \\
f(y_i) &= f(y_{i-1}) + 3, & 2 &\leq i \leq \frac{n-1}{2}, \\
f(z_i) &= f(z_{i-1}) + 3, & 2 &\leq i \leq \frac{n-1}{2}, \\
f(x_{\frac{n+2i-1}{2}}) &= -f(x_i), & 1 &\leq i \leq \frac{n-1}{2}, \\
f(y_{\frac{n+2i-1}{2}}) &= -f(y_i), & 1 &\leq i \leq \frac{n-1}{2}, \\
f(z_{\frac{n+2i-1}{2}}) &= -f(z_i), & 1 &\leq i \leq \frac{n-1}{2},
\end{aligned}
$$

In both cases $\Delta_{f_1} = \Delta_{f_1^c} = 2n+1$. Therefore $W_n \odot 2K_1$ is pair difference cordial for all values of $n \geq 3$.

Theorem 3.8. $G_n \odot 2K_1$ is pair difference cordial for all values of $n \geq 3$.

Proof.

Let $V(G_n \odot 2K_1) = V(G_n) \cup \{a_i, a_i', b_i, b_i' : 1 \le i \le n\} \cup \{a, a'\}$ and $V(G_n \odot 2K_1) = E(G_n) \cup \{x_i a_i, x_i a_i', y_i b_i, y_i b_i', : 1 \le i \le n\} \cup \{xa, xa'\}$. Clearly $G_n \odot 2K_1$ has $6n+3$ vertices and $7n+2$ edges. **Case 1.** $n \equiv 0 \pmod 4$.

Define the map $f : V(G_n \odot 2K_1) \to \{\pm 1, \pm 2, \cdots, \pm \frac{6n+2}{2}\}$ by

$$
\begin{aligned}
&f(x_1) = 2, && f(a_1) = 1,\\
&f(a_1') = 3, && f(a) = 3n+1,\\
&f(a') = -(3n+1), && f(x) = 3n,\\
&f(x_i) = f(x_{i-1}) + 3, && 2 \le i \le \frac{3n}{4},\\
&f(a_i) = f(a_{i-1}) + 3, && 2 \le i \le \frac{3n}{4},\\
&f(a_i') = f(a_{i-1}') + 3, && 2 \le i \le \frac{3n}{4},\\
&f(x_{\frac{3n+4}{4}}) = f(x_{\frac{3n}{4}}) + 2, &&\\
&f(a_{\frac{3n+4}{4}}) = f(a_{\frac{3n}{4}}) + 4, &&\\
&f(a'_{\frac{3n+4}{4}}) = f(a'_{\frac{3n}{4}}) + 3, &&\\
&f(x_{\frac{3n+4i}{4}}) = f(x_{\frac{3n+4i-4}{4}}) + 3, && 2 \le i \le \frac{n-4}{4},\\
&f(a_{\frac{3n+4i}{4}}) = f(a_{\frac{3n+4i-4}{4}}) + 3, && 2 \le i \le \frac{n-4}{4},\\
&f(a'_{\frac{3n+4i}{4}}) = f(a'_{\frac{3n+4i-4}{4}}) + 3, && 2 \le i \le \frac{n-4}{4},\\
&f(y_i) = -f(x_i), && 1 \le i \le n,\\
&f(b_i) = -f(a_i), && 1 \le i \le n,\\
&f(b_i') = -f(a_i'), && 1 \le i \le n.
\end{aligned}
$$

Case 2. $n \equiv 1 \pmod 4$.

Define the map $f : V(G_n \odot 2K_1) \to \{\pm 1, \pm 2, \cdots, \pm \frac{6n+3}{2}\}$ by

$$
\begin{aligned}
&f(x_1) = 2, && f(a_1) = 1,\\
&f(a_1') = 3, && f(a) = 3n+1,\\
&f(a') = -(3n+1), && f(x) = 3n,\\
&f(x_i) = f(x_{i-1}) + 3, && 2 \le i \le \frac{3n+1}{4},\\
&f(a_i) = f(a_{i-1}) + 3, && 2 \le i \le \frac{3n+1}{4},\\
&f(a_i') = f(a_{i-1}') + 3, && 2 \le i \le \frac{3n+1}{4},\\
&f(x_{\frac{3n+5}{4}}) = f(x_{\frac{3n+1}{4}}) + 2, &&\\
&f(a_{\frac{3n+5}{4}}) = f(a_{\frac{3n+1}{4}}) + 4, &&\\
&f(a'_{\frac{3n+5}{4}}) = f(a'_{\frac{3n+1}{4}}) + 3, &&
\end{aligned}
$$

$$
\begin{aligned}
f(x_{\frac{3n+4i+1}{4}}) &= f(x_{\frac{3n+4i-3}{4}})+3, && 2\le i\le \frac{n-3}{4},\\
f(a_{\frac{3n+4i+1}{4}}) &= f(a_{\frac{3n+4i-3}{4}})+3, && 2\le i\le \frac{n-3}{4},\\
f(a'_{\frac{3n+4i+1}{4}}) &= f(a'_{\frac{3n+4i-3}{4}})+3, && 2\le i\le \frac{n-3}{4},\\
f(y_i) &= -f(x_i), && 1\le i\le n,\\
f(b_i) &= -f(a_i), && 1\le i\le n,\\
f(b'_i) &= -f(a'_i), && 1\le i\le n.
\end{aligned}
$$

Case 3. $n\equiv 2 \pmod 4$.

Define the map $f: V(G_n\odot 2K_1)\to\{\pm1,\pm2,\cdots,\pm\frac{6n+2}{2}\}$ by

$$
\begin{aligned}
f(x_1) &= 2, && f(a_1)=1,\\
f(a'_1) &= 3, && f(a)=3n+1,\\
f(a') &= -(3n+1), && f(x)=3n,\\
f(x_i) &= f(x_{i-1})+3, && 2\le i\le \frac{3n+2}{4},\\
f(a_i) &= f(a_{i-1})+3, && 2\le i\le \frac{3n+2}{4},\\
f(a'_i) &= f(a'_{i-1})+3, && 2\le i\le \frac{3n+2}{4},\\
f(x_{\frac{3n+6}{4}}) &= f(x_{\frac{3n+2}{4}})+2, &&\\
f(a_{\frac{3n+6}{4}}) &= f(a_{\frac{3n+2}{4}})+4, &&\\
f(a'_{\frac{3n+6}{4}}) &= f(a'_{\frac{3n+2}{4}})+3, &&\\
f(x_{\frac{3n+4i+2}{4}}) &= f(x_{\frac{3n+4i-2}{4}})+3, && 2\le i\le \frac{n-2}{4},\\
f(a_{\frac{3n+4i+2}{4}}) &= f(a_{\frac{3n+4i-2}{4}})+3, && 2\le i\le \frac{n-2}{4},\\
f(a'_{\frac{3n+4i+2}{4}}) &= f(a'_{\frac{3n+4i-2}{4}})+3, && 2\le i\le \frac{n-2}{4},\\
f(y_i) &= -f(x_i), && 1\le i\le n,\\
f(b_i) &= -f(a_i), && 1\le i\le n,\\
f(b'_i) &= -f(a'_i), && 1\le i\le n.
\end{aligned}
$$

Case 4. $n\equiv 3 \pmod 4$.

Define the map $f: V(G_n\odot 2K_1)\to\{\pm1,\pm2,\cdots,\pm\frac{6n+3}{2}\}$ by

$$
\begin{aligned}
f(x_1) &= 2, && f(a_1)=1,\\
f(a'_1) &= 3, && f(a)=3n+1,\\
f(a') &= -(3n+1), && f(x)=3n,
\end{aligned}
$$

$$
\begin{aligned}
f(x_i) &= f(x_{i-1})+3, && 2\le i\le \frac{3n+3}{4},\\
f(a_i) &= f(a_{i-1})+3, && 2\le i\le \frac{3n+3}{4},\\
f(a'_i) &= f(a'_{i-1})+3, && 2\le i\le \frac{3n+3}{4},\\
f(x_{\frac{3n+7}{4}}) &= f(x_{\frac{3n+3}{4}})+2, &&\\
f(a_{\frac{3n+7}{4}}) &= f(a_{\frac{3n+3}{4}})+4, &&\\
f(a'_{\frac{3n+7}{4}}) &= f(a'_{\frac{3n+3}{4}})+3, &&\\
f(x_{\frac{3n+4i+3}{4}}) &= f(x_{\frac{3n+4i-1}{4}})+3, && 2\le i\le \frac{n-1}{4},\\
f(a_{\frac{3n+4i+3}{4}}) &= f(a_{\frac{3n+4i-1}{4}})+3, && 2\le i\le \frac{n-1}{4},\\
f(a'_{\frac{3n+4i+3}{4}}) &= f(a'_{\frac{3n+4i-1}{4}})+3, && 2\le i\le \frac{n-1}{4},\\
f(y_i) &= -f(x_i), && 1\le i\le n,\\
f(b_i) &= -f(a_i), && 1\le i\le n,\\
f(b'_i) &= -f(a'_i), && 1\le i\le n.
\end{aligned}
$$

The Table 4 given below establish that this vertex labeling f is a pair difference cordial of $G_n \odot 2K_1, n\ge 3$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{7n+1}{2}$	$\frac{7n+3}{2}$
n is even	$\frac{7n+2}{2}$	$\frac{7n+2}{2}$

Table 5

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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AVD Proper Edge Coloring of Some Cycle Related Graphs

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Abstract — The adjacent vertex-distinguishing proper edge-coloring is the minimum number of colors required for a proper edge-coloring of G, in which no two adjacent vertices are incident to edges colored with the same set of colors. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G is called the adjacent vertex-distinguishing proper edge-chromatic index. In this paper, we compute adjacent vertex-distinguishing proper edge-chromatic index of Anti-prism, sunflower graph, double sunflower graph, triangular winged prism, rectangular winged prism and Polygonal snake graph.

Subject Classification (2020): 05C15, 05C38.

1. Introduction

The terminology and notations we refer to Bondy and Murthy [4]. Let G be a finite, simple, undirected and connected graph. Let $\Delta(G)$ denote the maximum degree of G. A *proper edge-coloring* σ is a mapping from $E(G)$ to the set of colors such that any two adjacent edges receive distinct colors. For any vertex v of G, let $S_\sigma(v)$ denote the set of the colors of all edges incident to v. A proper edge-coloring σ is said to an *adjacent vertex-distinguishing* (AVD) if $S_\sigma(u) \neq S_\sigma(v)$, for every adjacent vertices u and v. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G, denoted by $\chi'_{as}(G)$, is called the *adjacent vertex-distinguishing proper edge-chromatic index (AVD proper edge-chromatic index)* of G. Thus, $\chi'_{as}(G) \geq \chi'(G)$.

Conjecture 1.1. [11] For any connected graph G ($|V(G)| \geq 6$), there is $\chi'_{as}(G) \leq \Delta(G) + 2$. If H is a subgraph of G, it is interesting that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is not always true.

Let $K_{m,n}$ be the complete bipartite graph, then $\chi'_{as}(K_{2,3}) = 3$ and $K_{2,3} - e$ for any edge, then $\chi'_{as}(K_{2,3} - e) = 4$. Deletion of an edge of a graph may also decrease the coloring number of the graph. Let $n \geq 3$, then $\chi'_{as}(K_{1,n}) = n$ and $\chi'_{as}(K_{1,n} - e) = n - 1$.

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In [11] Zhang et al. proved: if G has n components G_i, $1 \leq i \leq n$, with at least three vertices in each, then $\chi'_{as}(G) = max_{1 \leq i \leq n}\{\chi'_{as}(G_i)\}$. So we consider only connected graphs. For a tree T with $|V(T)| \geq 3$, if any two vertices of maximum degree are non-adjacent, then $\chi'_{as}(T) = \Delta(T)$. If T has two vertices of maximum degree which are adjacent, then $\chi'_{as}(T) = \Delta(T) + 1$. For cycle C_n we have $\chi'_{as}(C_n) = 3$, for $n \equiv 0\ (mod 3)$, otherwise $\chi'_{as}(C_n) = 4$ for $n \not\equiv 0\ (mod 3)$ and $n \neq 5$, $\chi'_{as}(C_n) = 5$, for $n = 5$. For the complete bipartite graph $K_{m,n}$ for $1 \leq m \leq n$, we have $\chi'_{as}(K_{m.n}) = n$ if $m < n$, and $\chi'_{as}(K_{m.n}) = n + 2$ if $m = n \geq 2$. For the complete graph K_n $(n \geq 3)$, we have $\chi'_{as}(K_n) = n$ for $n \equiv 1\ (mod\ 2)$; $\chi'_{as}(K_n) = n + 1$ for $n \equiv 0\ (mod\ 2)$. If G is a graph which has two adjacent maximum degree vertices, then $\chi'_{as}(G) \geq \Delta(G) + 1$. If G is a graph such that the degree of any two adjacent vertices is different, then $\chi'_{as}(G) = \Delta(G)$. In [9] Shiu proved: for $n \geq 3$, we have $\chi'_{as}(F_n) = n$, if $n = 3,4$ and $\chi'_{as}(F_{n-1}) = n - 1$, for $n \geq 5$. For $n \geq 3$, we have $\chi'_{as}(W_n) = 5$, if $n = 3$, and $\chi'_{as}(W_n) = n$, for $n \geq 4$. In [7] Hatami prove that if G is a graph with no isolated edges and maximum degree $\Delta(G) > 10^{20}$, then $\chi'_{as} \leq \Delta + 300$. In [2] Balister et al. proved: if G is a k-chromatic graph with no isolated edges, then $\chi'_{as}(G) \leq \Delta(G) + O(\log k)$. In [1] Axenovich et al. obtained upper bound for adjacent vertex-distinguishing edge-colorings of graphs. In [3] Baril et al. obtained exact values for adjacent vertex-distinguishing edge-coloring of meshes. In [5] Bu et al. finding adjacent vertex-distinguishing edge-colorings of planar graphs with girth at least six. In [6] Chen et al. obtained adjacent vertex-distinguishing proper edge-coloring of planar bipartite graphs with $\Delta = 9,10$ or 11.

In this paper, we compute adjacent vertex-distinguishing edge-chromatic index of Anti- prism, sunflower graph, double sunflower graph, triangular winged prism, rectangular winged prism and Polygonal snake graph.

Observation 1.1. If a connected graph G contains two adjacent vertices of degree $\Delta(G)$, then $\chi'_t(G) \geq \Delta(G) + 1$.

Observation 1.2. If G is a graph such that the degree of any two adjacent vertices is different, then $\chi'_{as}(G) = \Delta(G)$.

2. AVD Proper Edge-chromatic Index of Anti-prism Graph, Sunflower Graph, Double Sunflower Graph, Triangular Winged Prism and Rectangular Winged Prism

In this section, The AVD proper edge-chromatic index of Anti-prism graph, Sunflower graph, Double Sunflower graph, Triangular winged prism and Rectangular winged prism graph will be discussed. We have the following results.

2.1. AVD Proper Edge-chromatic Index of Anti-prism Graph

If $C_n \square K_2, n \geq 3$, is called prism graph, where $\square$ is Cartesian product, and it is denoted by D_n

By an Anti-prism graph of order n denoted by A_n, we mean a graph obtained from a prism graph D_n by adding some crossing edges $x_i y_{(i+1)(mod\ n)}$, $i = 1,2, \dots, n$. [10]

Theorem 2.1. $\chi'_{as}(A_n) = 5,\ for\ n \geq 3$.

Proof. Let $C_n = x_1x_2 \dots x_nx_1$, For $n \geq 4$ and $x_1', x_2', \dots, x_n'$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$ to form A_n. In A_n, for $i \in \{1,2,\dots,n\}$, let $e_i = x_ix_{i+1}$, $e_i' = x_i'x_{i+1}'$, $f_i = x_ix_i'$, $g_i = x_ix_{i+1}'$, where $x_{n+1} = x_1$, $x_{n+1}' = x_1'$.

Define $\sigma : E(A_3) \to \{1,2,3,4,5\}$ as follows: $\sigma(e_1) = 1, \sigma(e_2) = 4, \sigma(e_3) = 5, \sigma(e_1') = 4, \sigma(e_2') = 5, \sigma(e_3') = 1, \sigma(f_1) = \sigma(f_2) = \sigma(f_3) = 3, \sigma(g_1) = \sigma(g_2) = \sigma(g_3) = 2$. Therefore σ is proper-edge coloring. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,2,3,5\}, S_\sigma(x_2) = \{1,2,3,4\}, S_\sigma(x_3) = \{2,3,4,5\}, S_\sigma(x_1') = \{1,2,3,4\}, S_\sigma(x_2') = \{2,3,4,5\}$, $S_\sigma(x_3') = \{1,2,3,5\}$. Hence σ is an AVD proper edge-coloring A_3. By observation 1.1, $\chi'_{as}(A_3) \geq 5$ and so $\chi'_{as}(A_3) = 5$. Define $\sigma : E(A_4) \to \{1,2,3,4,5\}$ as follows: $\sigma(e_1) = \sigma(e_1') = 1, \sigma(e_2) = \sigma(e_2') = 4, \sigma(e_3) = \sigma(e_3') = 5, \sigma(e_4) = \sigma(e_4') = 3, \sigma(f_1) = \sigma(f_2) = \sigma(f_3) = \sigma(f_4) = 2, \sigma(g_1) = 5, \sigma(g_2) = 3, \sigma(g_3) = 1, \sigma(g_4) = 4$. Therefore σ is proper-edge coloring. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,2,3,5\}, S_\sigma(x_2) = \{1,2,3,4\}, S_\sigma(x_3) = \{1,2,4,5\}, S_\sigma(x_4) = \{2,3,4,5\}, S_\sigma(x_1') = \{1,2,3,4\}, S_\sigma(x_2') = \{1,2,4,5\}, S_\sigma(x_3') = \{2,3,4,5\}, S_\sigma(x_4') = \{1,2,3,5\}$. Hence σ is an AVD proper edge-coloring A_5. By observation 1.1, $\chi'_{as}(A_4) \geq 5$ and so $\chi'_{as}(A_4) = 5$. Define $\sigma: E(A_5) \to \{1,2,3,4,5\}$ as follows: $(e_1) = \sigma(e_1') = 2, \sigma(e_2) = 5, \sigma(e_2') = 3, \sigma(e_3) = 2, \sigma(e_3') = 5, \sigma(e_4) = 3, \sigma(e_4') = 2, \sigma(e_5) = \sigma(e_5') = 5, \sigma(f_1) = 3, \sigma(f_2) = 4 = \sigma(f_3), \sigma(f_4) = 1 = \sigma(f_5), \sigma(g_1) = 1 = \sigma(g_2), \sigma(g_3) = 3, \sigma(g_4) = 4 = \sigma(g_5)$. Therefore σ is proper-edge coloring. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,2,3,5\}, S_\sigma(x_2) = \{1,2,4,5\}, S_\sigma(x_3) = \{2,3,4,5\}, S_\sigma(x_4) = \{1,2,3,4\}, S_\sigma(x_5) = \{1,3,4,5\}, S_\sigma(x_1') = \{2,3,4,5\}, S_\sigma(x_2') = \{1,2,3,4\}, S_\sigma(x_3') = \{1,3,4,5\}, S_\sigma(x_4') = \{1,2,3,5\}, S_\sigma(x_5') = \{1,2,4,5\}$. Hence σ is an AVD proper edge-coloring A_5. By observation 1.1, $\chi'_{as}(A_5) \geq 5$ and so $\chi'_{as}(A_5) = 5$.

For $n \geq 6$, since $\Delta(A_n) = 4$, by observation 1.1. $\chi'_{as}(A_n) \geq 5$. To show $\chi'_{as}(A_n) \leq 5$. we consider five cases and in each case, we first define $\sigma : E(A_n) \to \{1,2,3,4,5\}$ as follows:

For $n \equiv 0 \pmod 3$

For $i \in \{1,2,\dots,n\}$,

$$\sigma(e_i) = \begin{cases} 5 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$\sigma(e_i') = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$\sigma(f_i) = 4,\ \sigma(g_i) = 1$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$\text{For } i \in \{1,2,\dots,n\}, S_\sigma(x_i) = \begin{cases} \{1,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,3,4\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$S_\sigma(x_i') = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 1 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e_1')$

For $i \in \{2,3,\dots,n-1\}$, $\sigma(e_i) = \sigma(e_i') = \begin{cases} 4 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 3 = \sigma(e_n')$

$\sigma(f_1) = 4, \sigma(f_2) = 2,$

For $i \in \{3,4,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 5 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 5,$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-2\}$, $\sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \\ 5 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-1}) = 1, \sigma(g_n) = 2.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,3,4,5\}$

For $i \in \{2,3,\dots,n\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

For $i \in \{1,2,\dots,n-1\}$, $S_\sigma(x_i') = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$S_\sigma(x_n') = \{1,3,4,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 2 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e_1')$

For $i \in \{2,3,\dots,n-2\}$, $\sigma(e_i) = \sigma(e_i') = \begin{cases} 4 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 3 = \sigma(e_n')$, $\sigma(e_{n-1}) = 5 = \sigma(e_{n-1}')$

$\sigma(f_1) = \sigma(f_2) = 2$

For $i \in \{3,4,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 5 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 1,$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-2\}$, $\sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \\ 5 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-1}) = 2, \sigma(g_n) = 4$.

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,3,5\}$

For $i \in \{2,3,\dots,n-1\}, S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_n) = \{1,3,4,5\}$

For $i \in \{1,2,\dots,n-2\}, S_\sigma(x'_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$S_\sigma(x'_{n-1}) = \{1,3,4,5\}, S_\sigma(x'_n) = \{1,2,3,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 4 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e'_1)$

For $i \in \{2,3,\dots,n-4\}, \sigma(e_i) = \sigma(e'_i) = \begin{cases} 4 & \text{if } i \text{ is even} \\ 5 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 3 = \sigma(e'_n),\ \sigma(e_{n-1}) = 5 = \sigma(e'_{n-1}),$

$\sigma(e_{n-2}) = 1 = \sigma(e'_{n-2}),\ \sigma(e_{n-3}) = 2 = \sigma(e'_{n-3})$

$\sigma(f_1) = \sigma(f_2) = 2$

For $i \in \{3,4,\dots,n-3\}, \sigma(f_i) = \begin{cases} 2 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 1, \sigma(f_{n-1}) = 4 = \sigma(f_{n-2})$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-4\},\ \sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \\ 2 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-3}) = 5, \sigma(g_{n-2}) = 3, \sigma(g_{n-1}) = 2, \sigma(g_n) = 4$.

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,3,5\},\ S_\sigma(x_2) = \{1,2,3,4\}$

For $i \in \{3,4,\dots,n-3\}, S_\sigma(x_i) = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$S_\sigma(x_n) = \{1,3,4,5\},\ S_\sigma(x_{n-1}) = \{1,2,4,5\},\ S_\sigma(x_{n-2}) = \{1,2,3,4\}$

$S_\sigma(x_1') = \{1,2,3,4\}$

For $i \in \{2,3,\dots,n-4\}, S_\sigma(x_i') = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_{n-3}') = \{1,2,3,4\}, S_\sigma(x_{n-2}') = \{1,2,4,5\}, S_\sigma(x_{n-1}') = \{1,3,4,5\}, S_\sigma(x_n') = \{1,2,3,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 5 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e_1')$

For $i \in \{2,3,\dots,n-4\}, \sigma(e_i) = \sigma(e_i') = \begin{cases} 4 & \text{if } i \text{ is even} \\ 5 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 2 = \sigma(e_n'),\ \sigma(e_{n-1}) = 3 = \sigma(e_{n-1}'),$

$\sigma(e_{n-2}) = 5 = \sigma(e_{n-2}'),\ \sigma(e_{n-3}) = 3 = \sigma(e_{n-3}')$

$\sigma(f_1) = 3, \sigma(f_2) = 2$

For $i \in \{3,4,\dots,n-3\}, \sigma(f_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 5, \sigma(f_{n-1}) = 4, \sigma(f_{n-2}) = 1$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-4\},\ \sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-3}) = 4, \sigma(g_{n-2}) = 2, \sigma(g_{n-1}) = 1, \sigma(g_n) = 4.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,3,5\},\ S_\sigma(x_2) = \{1,2,3,4\}$

For $i \in \{3,4,\dots,n-2\}, S_\sigma(x_i) = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$S_\sigma(x_n) = \{2,3,4,5\},\ S_\sigma(x_{n-1}) = \{1,3,4,5\},\ S_\sigma(x_{n-2}) = \{1,2,3,5\}$

$S_\sigma(x_1') = \{1,2,3,4\}$

For $i \in \{2,3,\dots,n-4\}, S_\sigma(x_i') = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_{n-3}') = \{1,2,3,5\}, S_\sigma(x_{n-2}') = \{1,3,4,5\}, S_\sigma(x_{n-1}') = \{2,3,4,5\}, S_\sigma(x_n') = \{1,2,3,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

2.2. AVD Proper Edge-chromatic Index of Sunflower Graph

By an sun flower graph of order n denoted by SF_n, we mean a graph that is isomorphic to a graph obtained from Anti-prism graph A_n by deleting edges $y_i y_{(i+1)(mod\ n)}$, $i = 1,2,\dots,n$.

Theorem 2.2. $\chi'_{as}(SF_n) = 5,\ for\ n \geq 4$.

Proof. Let $C_n = x_1 x_2 \dots x_n x_1$ For $n \geq 4$ and $x'_1, x'_2, \dots, x'_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$ to form SF_n. In SF_n, for $i \in \{1,2,\dots,n\}$, let $e_i = x_i x_{i+1}$, $f_i = x_i x'_i$, and $g_i = x'_i x_{i+1}$, where $x_{n+1} = x_1$.

Define $\sigma : E(SF_3) \to \{1,2,3,4,5\}$ as follows: $(e_1) = 1, \sigma(e_2) = 2, \sigma(e_3) = 5$, $\sigma(f_1) = \sigma(f_2) = \sigma(f_3) = 3$, $\sigma(g_1) = \sigma(g_2) = \sigma(g_3) = 4$. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,3,4,5\}, S_\sigma(x_2) = \{1,2,3,4\}, S_\sigma(x_3) = \{2,3,4,5\}, S_\sigma(x'_1) = S_\sigma(x'_2) = S_\sigma(x'_3) = \{3,4\}$. Therefore σ is an AVD proper edge-coloring SF_n. Hence, $\chi'_{as}(SF_3) = 5$.

For $n \geq 4$, since $\Delta(SF_n) = 4$, by observation 1.1. $\chi'_{as}(SF_n) \geq 5$. To show $\chi'_{as}(SF_n) \leq 5$. we consider two cases first define $\sigma : E(SF_n) \to \{1,2,3,4,5\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\dots,n\}$

$$\sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 5 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}$$

$\sigma(g_i) = 3$,

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,3\}$

For $i \in \{1,2,3,\dots,n\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,5\} & \text{if } i \text{ is odd} \\ \{1,2,3,4\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(x'_i) = \begin{cases} \{3,5\} & \text{if } i \text{ is odd} \\ \{3,4\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of SF_n. Hence, $\chi'_{as}(SF_n) = 5$

Case 2. If n is odd

For $i \in \{1,2,\dots,n-1\}$, $\sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

$\sigma(e_n) = 5$

$\sigma(f_1) = 4$,

For $i \in \{2,3,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is even} \\ 5 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(f_n) = 4$,

For $i \in \{1,2,\dots,n\}$, $\sigma(g_i) = 3$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,3,4,5\}$

For $i \in \{2,3,\ldots,n-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \text{ is even} \\ \{1,2,3,5\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n) = \{2,3,4,5\}$

$S_\sigma(x'_1) = \{3,4\}$

For $i \in \{2,3,\ldots,n-1\}$, $S_\sigma(x'_i) = \begin{cases} \{3,4\} & \text{if } i \text{ is even} \\ \{3,5\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x'_n) = \{3,4\}$

Therefore σ is an AVD proper edge-coloring of SF_n. Hence, $\chi'_{as}(SF_n) = 5$.

2.3. AVD Proper Edge-chromatic Index of Double Sunflower Graph

By a double sunflower graph of order n denoted by DSF_n, is a graph obtained from the graph SF_n by inserting a new vertex z_i on each edges $x_i x_{i+1}$ and adding edges $y_i z_i$ for each i.

Theorem 2.3. $\chi'_{as}(DSF_n) = 4,\ for\, n \geq 4,$

Proof. Let $C_n = x_1 x_2 \ldots x_n x_1$ For $n \geq 4$ and $x'_1, x'_2, \ldots, x'_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be newly added vertices corresponding to the sub division of each edge of the cycle C_n to form DSF_n. In DSF_n, for $i \in \{1,2,\ldots,n\}$, let $e_i = x_i y_i$, $e'_i = y_i x_{i+1}$ $f_i = x_i x'_i$, $g_i = x'_i x_{i+1}$ and $h_i = x'_i y_i$ where $x_{n+1} = x_1$.

For $n \geq 4$, since $\Delta(DSF_n) = 4$, by observation 1.2. $\chi'_{as}(DSF_n) \geq 4$. To show $\chi'_{as}(DSF_n) \leq 4$.

We consider two cases first define $\sigma : E(DSF_n) \to \{1,2,3,4\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\ldots,n\}$

$$\sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(e'_i) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(g_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(h_i) = \begin{cases} 3 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,3,\ldots,n\}$, $S_\sigma(x_i) = \{1,2,3,4\}$

$$S_\sigma(y_i) = \begin{cases} \{1,2,3\} & \text{if } i \text{ is odd} \\ \{2,3,4\} & \text{if } i \text{ is even} \end{cases}$$

$$S_\sigma(x_i') = \begin{cases} \{2,3,4\} & \text{if } i \text{ is odd} \\ \{1,2,3\} & \text{if } i \text{ is even} \end{cases}$$

Therefore σ is an AVD proper edge-coloring of DSF_n. Hence, $\chi'_{as}(DSF_n) = 4$.

Case 2. If n is odd

$\sigma(e_1) = 1, \sigma(e_1') = 3$

For $i \in \{2,3,\ldots,n\}$, $\sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_i') = 4$,

For $i \in \{1,2,\ldots,n\}$, $\sigma(f_i) = \begin{cases} 3 & \text{if } i \neq 2 \\ 2 & \text{if } i = 2 \end{cases}$

$\sigma(g_1) = 4$,

For $i \in \{2,3,\ldots,n\}$, $\sigma(g_i) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(h_1) = 2, \sigma(h_2) = 3$,

For $i \in \{3,4,\ldots,n\}$, $\sigma(h_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,\ldots,n\}$, $S_\sigma(x_i) = \{1,2,3,4\}$

$S_\sigma(y_1) = \{1,2,3\}, S_\sigma(y_2) = \{1,3,4\}$

For $i \in \{3,4,\ldots,n\}$, $S_\sigma(y_i) = \{1,2,4\}$

$S_\sigma(x_1') = \{2,3,4\}$

For $i \in \{2,3,\ldots,n\}$, $S_\sigma(x_i') = \{1,2,3\}$.

Therefore σ is an AVD proper edge-coloring of DSF_n. Hence, $\chi'_{as}(DSF_n) = 4$.

2.4. AVD Proper Edge-chromatic Index of Triangular Winged Prism

By a triangular winged prism of order n denoted by TWP_n, is a graph obtained from the prism graph D_n, by adding some outsider middle vertices z_i on edge $y_i y_{i+1}$ and adding z_i to both vertices y_i and y_{i+1}.

Theorem 2.4. $\chi'_{as}(TWP_n) = 6,\ for\ n \geq 4$.

Proof. Let $C_n = x_1 x_2 \ldots x_n x_1$ For $n \geq 4$, $x_1', x_2', \ldots, x_n'$ and $y_1, y_2, \ldots, y_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \ldots, x_n$ to form TWP_n. In TWP_n, for $i \in \{1,2,\ldots,n\}$, let $e_i = x_i x_{i+1}$, $e_i' = x_i' x_{i+1}'$, $f_i = x_i x_i'$, $g_i = x_i' y_i$ and $h_i = x_{i+1}' y_i$, where $x_{n+1} = x_1$, $x_{n+1}' = x_1'$.

For $n \geq 4$, since $\Delta(TWP_n) = 5$, by observation 1.1. $\chi'_{as}(TWP_n) \geq 6$. To show $\chi'_{as}(TWP_n) \leq 6$. we consider two cases first define $\sigma : E(TWP_n) \to \{1,2,3,4,5,6\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\dots,n\}$

$$\sigma(e_i) = \sigma(e_i') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(g_i) = 5$$

$$\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,3,\dots,n\}$, $S_\sigma(x_i) = \begin{cases} \{1,3,4\} & \text{if } i \text{ is odd} \\ \{1,2,3\} & \text{if } i \text{ is even} \end{cases}$

$$S_\sigma(x_i') = \begin{cases} \{1,3,4,5,6\} & \text{if } i \text{ is odd} \\ \{1,2,3,4,5\} & \text{if } i \text{ is even} \end{cases}$$

For $i \in \{1,2,\dots,n\}$, $S_\sigma(y_i) = \begin{cases} \{4,5\} & \text{if } i \text{ is odd} \\ \{5,6\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of TWP_n. Hence, $\chi'_{as}(TWP_n) = 6$.

Case 2. If n is odd

For $i \in \{1,2,3,\dots,n-1\}$

$$\sigma(e_i) = \sigma(e_i') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(e_n) = \sigma(e_n') = 2,$$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

$\sigma(f_n) = 4,$

For $i \in \{1,2,\dots,n\}$, $\sigma(g_i) = 5,$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$

$\sigma(h_n) = 6.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,4\}$

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3\} & \text{if } i \text{ is even} \\ \{1,3,4\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n) = \{2,3,4\}$

$S_\sigma(x_1') = \{1,2,4,5,6\}$

For $i \in \{2,3,\dots,n-1\}, S_\sigma(x_i') = \begin{cases} \{1,2,3,4,5\} & \text{if } i \text{ is even} \\ \{1,3,4,5,6\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n') = \{2,3,4,5,6\}$

For $i \in \{1,2,\dots,n-1\}$, $S_\sigma(y_i) = \begin{cases} \{4,5\} & \text{if } i \text{ is odd} \\ \{5,6\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(y_n) = \{5,6\}$.

Therefore σ is an AVD proper edge-coloring of TWP_n. Hence, $\chi'_{as}(TWP_n) = 6$.

2.5. AVD Proper Edge-chromatic Index of Rectangular Winged Prism Graph

By a rectangular winged prism graph of order n denoted by RWP_n, is a graph obtained from the prism graph D_n, by adding an edge a_ib_i corresponding to the edge y_iy_{i+1} and adding an edge a_i to y_i and b_i to y_{i+1}.

Theorem 2.5. $\chi'_{as}(RWP_n) = 6,\ for\ n \geq 4$.

Proof. Let $C_n = x_1x_2 \dots x_nx_1$, For $n \geq 4$ and $x_1', x_2', \dots, x_n'$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$. Let $y_1, y_2, \dots, y_n$ and $z_1, z_2, \dots, z_n$ be newly added vertices corresponding to the vertices $x_1', x_2', \dots, x_n'$ to form RWP_n. In RWP_n, for $i \in \{1,2,\dots,n\}$, let $e_i = x_ix_{i+1}$, $e_i' = x_i'x_{i+1}'$, $e_i'' = y_iz_i$, $f_i = x_ix_i'$, $g_i = x_i'y_i$ and $h_i = x_{i+1}'z_i$, where $x_{n+1} = x_1$, $x_{n+1}' = x_1'$.

For $n \geq 4$, since $\Delta(RWP_n) = 5$, by observation 1.1. $\chi'_{as}(RWP_n) \geq 6$. To show $\chi'_{as}(RWP_n) \leq 6$. we consider two cases first define $\sigma : E(RWP_n) \to \{1,2,3,4,5,6\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\dots,n\}$

$$\sigma(e_i) = \sigma(e_i') = \sigma(e_i'') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$\sigma(g_i) = 5$

$\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,3,\dots,n\}, S_\sigma(x_i) = \begin{cases} \{1,3,4\} & \text{if } i \text{ is odd} \\ \{1,2,3\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(x_i') = \begin{cases} \{1,3,4,5,6\} & \text{if } i \text{ is odd} \\ \{1,2,3,4,5\} & \text{if } i \text{ is even} \end{cases}$

For $i \in \{1,2,\dots,n\}$, $S_\sigma(y_i) = \begin{cases} \{1,5\} & \text{if } i \text{ is odd} \\ \{3,5\} & \text{if } i \text{ is even} \end{cases}$

For $i \in \{1,2,\dots,n\}$, $S_\sigma(z_i) = \begin{cases} \{1,4\} & \text{if } i \text{ is odd} \\ \{3,6\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of RWP_n. Hence, $\chi'_{as}(RWP_n) = 6$.

Case 2. If n is odd

For $i \in \{1,2,\dots,n-1\}$, $\sigma(e_i) = \sigma(e_i') = \sigma(e_i'') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$

$\sigma(e_n) = \sigma(e_n') = \sigma(e_n'') = 2$,

For $i \in \{1,2,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

$\sigma(f_n) = 4$,

For $i \in \{1,2,\dots,n\}$, $\sigma(g_i) = 5$,

For $i \in \{1,2,\dots,n-1\}$, $\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$

$\sigma(h_n) = 6$.

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,4\}$

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3\} & \text{if } i \text{ is even} \\ \{1,3,4\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n) = \{2,3,4\}$

$S_\sigma(x_1') = \{1,2,4,5,6\}$

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(x_i') = \begin{cases} \{1,2,3,4,5\} & \text{if } i \text{ is even} \\ \{1,3,4,5,6\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n') = \{2,3,4,5,6\}$

For $i \in \{1,2,\dots,n-1\}$, $S_\sigma(y_i) = \begin{cases} \{1,5\} & \text{if } i \text{ is odd} \\ \{3,5\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(y_n) = \{2,5\}$

For $i \in \{1,2,\dots,n-1\}$, $S_\sigma(z_i) = \begin{cases} \{1,4\} & \text{if } i \text{ is odd} \\ \{3,6\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(z_n) = \{2,6\}$

Therefore σ is an AVD proper edge-coloring of RWP_n. Hence, $\chi'_{as}(RWP_n) = 6$.

3. AVD Proper Edge-chromatic Index of Polygonal Snake Graph

In this section, we investigate AVD proper edge-coloring of Polygonal snake graph only. A graph is obtained from a path P_m with vertex set $x_1, x_2, \dots, x_m$ by joining all consecutive vertices by path P_n with vertex set $y_1, y_2, \dots, y_n$ in such a way that merging y_1 with x_i and y_n with x_{i+1}, $i \in \{1,2,\dots,n-1\}$ and so on. Then $P_m(S_n)$, $\forall\, m,n$ is called as polygonal snake graph. [8]

Theorem 3.1. $\chi'_{as}(P_m(S_n)) = 5,\ for\ m \geq 3, n \geq 5$.

Proof. Let $P_m: x_1x_2 \dots x_m$, For $n \geq 5$, $P_n: y_1y_2 \dots y_n$ be attached to an edge x_ix_{i+1}, $i \in \{1,3,\dots,m-1\}$, m is even, where $x_i = y_1,\ x_{i+1} = y_n$ and $P'_n: y'_1y'_2 \dots y'_n$ be attached to an edge x_ix_{i+1}, $i \in \{2,4,\dots,m-1\}$, m is odd, where $x_i = y'_1, x_{i+1} = y'_n$ to form $P_m(S_n)$. In $P_m(S_n)$, for $i \in \{1,2,\dots,m-1\}$, let $e_i = x_ix_{i+1}$. For $i \in \{1,2,\dots,n-1\}$, $f_i = y_iy_{i+1}$, $f'_i = y'_iy'_{i+1}$.

For $m \geq 3, n \geq 5$, since $\Delta(P_m(S_n)) = 4$, by observation 1.1. $\chi'_{as}(P_m(S_n)) \geq 5$. To show $\chi'_{as}(P_m(S_n)) \leq 5$. we consider five cases and in each case, we first define $\sigma : E(P_m(S_n)) \to \{1,2,3,4,5\}$ as follows:

Case 1: For $n \equiv 5 \pmod 6$

For $i \in \{1,2,\dots,m-1\}$, $\sigma(e_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$$\sigma(f'_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(y_i) = \begin{cases} \{1,2\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(y'_i) = \begin{cases} \{2,3\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_1) = \{1,3\}$,

For $i \in \{2,3,\dots,m-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2,3,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_m) = \begin{cases} \{2,4\} & \text{if } m \equiv 3 \pmod 6 \\ \{1,5\} & \text{if } m \equiv 4 \pmod 6 \\ \{2,3\} & \text{if } m \equiv 5 \pmod 6 \\ \{1,4\} & \text{if } m \equiv 0 \pmod 6 \\ \{2,5\} & \text{if } m \equiv 1 \pmod 6 \\ \{1,3\} & \text{if } m \equiv 2 \pmod 6 \end{cases}$

Therefore σ is an AVD proper edge-coloring of $P_m(S_n)$. Hence, $\chi'_{as}\big(P_m(S_n)\big) = 5$.

Case 2: For $n \equiv 0 \pmod 6$

For $i \in \{1,2,\ldots,m-1\}$, $\sigma(e_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

For $i \in \{1,2,\ldots,n-1\}$, $\sigma(f_i) = \sigma(f_i') = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{2,3,\ldots,n-1\}$, $S_\sigma(y_i) = S_\sigma(y_i') = \begin{cases} \{1,2\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_1) = \{1,3\}$,

For $i \in \{2,3,\ldots,m-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2,3,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_m) = \begin{cases} \{2,4\} & \text{if } m \equiv 0 \pmod 3 \\ \{2,5\} & \text{if } m \equiv 1 \pmod 3 \\ \{2,3\} & \text{if } m \equiv 2 \pmod 3 \end{cases}$

Therefore σ is an AVD proper edge-coloring of $P_m(S_n)$. Hence, $\chi'_{as}\big(P_m(S_n)\big) = 5$.

Case 3: For $n \equiv 1 \pmod 6$

For $i \in \{1,2,\ldots,m-1\}$, $\sigma(e_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

For $i \in \{1,2,\ldots,n-4\}$, $\sigma(f_i) = \sigma(f_i') = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$\sigma(f_{n-3}) = \sigma(f_{n-3}') = 4, \sigma(f_{n-2}) = \sigma(f_{n-2}') = 1, \sigma(f_{n-1}) = \sigma(f_{n-1}') = 2.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{2,3,\ldots,n-4\}$, $S_\sigma(y_i) = S_\sigma(y_i') = \begin{cases} \{1,2\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(y_{n-3}) = S_\sigma(y_{n-3}') = \{3,4\}, S_\sigma(y_{n-2}) = S_\sigma(y_{n-2}') = \{1,4\}, S_\sigma(y_{n-1}) = S_\sigma(y_{n-1}') = \{1,2\}$

$S_\sigma(x_1) = \{1,3\}$,

For $i \in \{2,3,\ldots,m-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2,3,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$$S_\sigma(x_m) = \begin{cases} \{2,4\} & \text{if } m \equiv 0 \pmod 3 \\ \{2,5\} & \text{if } m \equiv 1 \pmod 3 \\ \{2,3\} & \text{if } m \equiv 2 \pmod 3 \end{cases}$$

Therefore σ is an AVD proper edge-coloring of $P_m(S_n)$. Hence, $\chi'_{as}\big(P_m(S_n)\big) = 5$.

Case 4: For $n \equiv 2 \pmod 6$

Proof is similar to case 1. $n \equiv 5 \pmod 6$

Case 5: For $n \equiv 3 \pmod 6$

Proof is similar to case 2. $n \equiv 0 \pmod 6$

Case 6: For $n \equiv 4 \pmod 6$

Proof is similar to case 3. $n \equiv 1 \pmod 6$

4. Conclusion

In this paper, I investigate the AVD proper edge-chromatic index of Anti-prism, sunflower graph, double sunflower graph, triangular winged prism and rectangular winged prism. And I also investigate AVD Proper edge-chromatic index of Polygonal snake graph. The investigation of analogous results for different graphs and different operation of above families of graphs are still open.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Embedding Orthomorphisms d-Algebra in Biorthomorphisms as Ordered Ideal

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Keywords:
d-Algebra,
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Orthomorphism.

Abstract — In the historical development of Riesz spaces, we can trace the history of ordered vector spaces to the International Mathematical Congress in Bologna in 1928. Studies related with f −Algebras for the Dedekind complete ordered vector space defined in Riesz spaces were initiated by Nakano and their current definition was made by Amemiya, Birkhoff and Pierce.The revival of f −algebras, which had a tendency to slow down for a period of time, emerged as a result of Pagter's doctoral thesis [9] and the examination of Alkansas lecture notes by Luxemburg. The concepts of homomorphism, isomorphism, orthomorphism and biorthomorphism in Riesz spaces are defined by Zaanen, Huijsmans, Boulabiar, Buskes and Triki. Algebraic structure of biorthomorphisms defined on Riesz space examined by [8]. f-Algebra on $Orth(X,X)$ were studied by [8] and [6]. [6] demonstrated that biorthomorphisms space have an f −algebraic structure with the help of the product defined as $(T_1 *_e T_2)(x,y) = T_1(x, T_2(e,y))$ for $e \in X^+$, $\forall x,y \in X$ and $T_1, T_2 \in Orth(X,X)$. [8] showed that if X are semiprime Dedekind complete f −algebras, $Orth(X)$ is an ordered ideal in biorthomorphisms. [6] developed an alternative proof for this situation. If X Archimedean Riesz space, $Orth(X)$ is an f −algebra according to compound operation with unit element. [11] showed that if X is a semiprime f −algebra, it is a d −algebra. In this study, we investigated embedding orthomorphism in biorthomorphisms when X is uniformly complete d −algebra.

Subject Classification (2020): 06B05, 06B10.

1. Introduction

In this section, we gave some definitions about Riesz space and Riesz algebra.

Definition 1. Let X is a set different from empty. If relation $\leq$ defined on X satisfy the following properties, relation $\leq$ is called the (partial) order relation, the pair $(X, \leq)$ is called the (partial) ordered set. For $\forall x, y, z \in X$

i. $x \leq x$ for $\forall x \in X$.

ii. If $x \leq y$ and $y \leq x$, $x = y$.

iii. If $x \leq y$ and $y \leq z$, $x \leq z$ [1].

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Definition 2. Let $(X, \leq)$ is an ordered vector space. If every finite subset different from empty has a supremum, X is called a Riesz space (or a vector lattice). Supremum of $\{x, y\}$ is demonstrated with $x \vee y$ in classic Riesz space as notation [1].

Definition 3. Let X is an Riesz space. If $nx \leq y \Longrightarrow x = 0$ is satisfy for $\forall n \in \mathbb{N}$ where $x, y \in X^+$, X is called Archimedean Riesz space [1].

Definition 4. A Riesz space is called Dedekind complete if every non-empty upper bound (bottom bounded) subset has a supremum (infimum) [1].

Definition 5. Let X is an Riesz space. If X is an associative algebra and $xy \in X^+$ for $\forall x, y \in X^+$, X is called Riesz algebra (or ordered lattice algebra) [1].

Definition 6. Let X is an Riesz algebra. If $x \cdot x = x^2 \in X^+$for $\forall x \in X$, X is called positive square or X have the positive square property.

Definition 7. Let X is an Riesz algebra. If $cx \wedge y = xc \wedge y = 0$ is satisfy for $\forall x, y \in X, x \wedge y = 0$ and $\forall c \in X^+$, X is called f $-$algebra [4].

Definition 8. Let X is an Riesz algebra. If $xy = 0$ is satisfy for $\forall x, y \in X, x \wedge y = 0$, X is called almost f $-$algebra [5].

Definition 9. Let X is an Riesz algebra. If $cx \wedge cy = xc \wedge yc = 0$ is satisfy for $\forall x, y \in X$, $x \wedge \mathrm{y} = 0$ and $\forall \mathrm{c} \in X^+$, X is called d $-$algebra [12].

For more information about d $-$algebra, [3] can be given as reference.

Definition 10. Let X is an Riesz algebra. If $x = 0$ when $x^k = 0$ for $x \in X$ and $\exists k \in \mathbb{N}$, X is called semi prime [1].

Theorem 1. If X is a semiprime f $-$algebra, the following statements are are equivalent to each other.

i. X is a f $-$algebra.

ii. X is a d $-$algebra.

iii. X is an almost f $-$algebra [11].

Definition 11. If the operator T transforms every ordered bounded subset of X to an ordered bounded subset of Y, where X and Y are ordered bounded vector spaces, transformation T is called an ordered bounded operator. The set of all ordered bounded operators is denoted by $L_b(X, Y)$ [1].

[13] can be examined about biliner operators.

Definition 12. Let X is an Riesz space. If $g \in S$ for $|g| \leq |f|$ where $S \subseteq X$, $g \in X$ and $f \in S$, S is called solid [1].

Definition 13. Let X is an Riesz space and $Y \subseteq X$. If Y is a solid linear subspace, Y is called ideal in X.

If A and B are ideals in Y, $A \cap B$ and $A + B$ are ideals in Y. $A \cap B$ is an ideal in Y because of $A \cap B \subseteq Y$ for $A \subseteq Y$ and $B \subseteq Y$. $A + B = \{A \cup B: A \cap B = \emptyset, A, B \in Y\}$ is an ideal in Y because of $A + B \subseteq A \cup B \subseteq Y$[1].

Definition 14. Let X is an Riesz space and Y is an ideal of X.If any subset of Y has a supremum in X and this supremum is an element of Y, in other words, if $f \in Y$ is satisfied when $Z \subseteq Y$ and $f = supZ$, then the ideal Y is called band in X [14].

Definition 15. Let X is an Riesz space and $T: X \to X$ is an linear operator. If $T(B) \subseteq B$ for $\forall B \subseteq X$, in other words if operator T leaves all bands of X unchanged, T is called band preserving operator [1].

Definition 16. Let X an Y are two Riesz spaces and $T: X \to Y$ is an linear operator. For $\forall x, y \in X$, if operator T is satisfy $T(x \vee y) = Tx \vee Ty$, operator T is called Riesz homomorphism [1].

Definition 17. A band preserving ordered bounded operator is called an orthomorphism. So, let X is a Riesz space and $T: X \to X$ is an bounded operator. In X, $Tx \perp y$ is provided when $x \perp y$. Also, if the orthomorphism T is positive at the same time, T is called a positive orthomorphism. In other words, T is a positive orthomorphism if and only if $x \wedge y = 0$ then $Tx \wedge y = 0$ on X. The set of all orthomorphisms on X is denoted by $Orth(X)$ [1].

Definition 18. Let X is an Archimedean Riesz space. If bilinear transformation $T: X \times X \to X$ is an orthomorphism in each component of X, T is called biorthomorphism. In other words, if $T(x, .), T(., x) \in Orth(X)$ for $\forall x \in X$, bilinear transformation $T: X \times X \to X$ is called biorthomorphism on X. The set of all biorthomorphism on X is denoted by $Orth(X, X)$.

Note. $\rho: Orth(X) \to Orth(X, X)$ defined with $\rho(T)(x, y) = T(xy) = T(x)y$ is one-to-one Riesz homomorphism for $\forall T \in Orth(X)$ and $(x, y) \in X \times X$ [8].

$$\rho: (Orth(X), X \times X) \to Orth(X, X)$$

$$\big(T, (x, y)\big) \to \rho(T)(x, y) = T(xy) = T(x)y.$$

Notation. $K(T) = \{x \in X: T(x, x) = 0\}$ for $\forall T \in Orth(X, X)$ [6].

Lemma 1. Let X is an Archimedean Riesz space and $T \in Orth(X, X)$.

$$K(T) = \{x \in X: T(x, y) = 0, \forall y \in X\}.$$

Specially, $K(T)$ is an ordered ideal in X [6].

Lemma 2. Let X is an Archimedean Riesz space and $T \in Orth(X, X)$. If $x \in X$, $T(x, x) = 0 \Leftrightarrow T(x, x) \in K(T)$ [6].

Proposition 1.

i. If X is a semiprime f −algebra, $Orth(X)$ is Riesz space in $Orth(X, X)$.

ii. If X is a semiprime Dedekind complete f −algebra, $Orth(X)$ is an ordered ideal in $Orth(X, X)$ [8].

Theorem 2. Let X is a semiprime Dedekind complete f −algebra. $\rho(Orth(X))$ is an ordered ideal in $Orth(X,X)$ [2].

2. Embedding Orthomorphisms in Biorthomorphisms

Let X is a semiprime f −algebra. Specially, $xT(y) = T(xy) = yT(x)$ is provided for $\forall x, y \in X$ and $T \in Orth(X)$ from $\rho: Orth(X) \to Orth(X)$, $\rho(\pi)(x,y) = \pi(xy) = \pi(x)y$ for $\forall \pi \in Orth(X)$ and $(x,y) \in X \times X$. If transformation $\hat{T}: X \times X \to X$ is satisfy $\hat{T}(x,y) = T(xy) = T(x)y$, $\hat{T}$ is called biorthomorphism. Therefore, if $\rho(T)(x,y) = \hat{T}(x,y)$,$\rho(T) = \hat{T}$. Transformation $\rho: Orth(X) \to Orth(X,X)$ is an one-to-one Riesz homomorphism [8]. So, $Orth(X)$ as Riesz subspace is embedded in $Orth(X,X)$ under transformation ρ. Then, $T \in Orth(X)$ determined with $\hat{T} \in Orth(X,X)$.

Theorem 3. If X Riesz space is uniformly complete, $Orth(X,X)$ is uniformly complete [6].

2.1. Embedding Orthomorphisms in Biorthomorphisms when X is Uniformly Complete Semiprime f −Algebra

Let X is an uniformly complete semiprime f −algebra and $X^{\odot} = \{xy : x, y \in X\}$. Then the set $X^{\odot}$ is a Riesz subspace of X with positive cone $\{xx = x^2 : x \in X\}$ [7].

Theorem 4. Let X is an uniformly complete semiprime f −algebra and transformation $T: X \times X \to X$. T is a (positive) biorthomorphism on X $\Leftrightarrow$ There is only one positive biorthomorphism $T^{\odot}: X^{\odot} \to X$ satisfying property $T(x,y) = T^{\odot}(xy)$ for $\forall x, y \in X$ [6].

Proof. $\Longrightarrow$ Let T is a biorthomorphism on X. Let positive biorthomorphism $T^{\odot}: X^{\odot} \to X$ is satisfy property $T(x,y) = T^{\odot}(xy)$ for $\forall x, y \in X$. Let us now show that there is only one biorthomorphism. Let ${T_1}^{\odot}$ ve ${T_2}^{\odot}$ are two biorthomorphisms satisfying $T(x,y) = T^{\odot}(xy)$ on X for $\forall x, y \in X$. ${T_1}^{\odot}(xy) = T_1(x,y) = T_1(xy)$ is provided for transformation ${T_1}^{\odot}: X^{\odot} \to X$. Similarly, ${T_2}^{\odot}(xy) = T_2(x,y) = T_2(xy)$ is provided for ${T_2}^{\odot}: X^{\odot} \to X$. If $T_1(xy) = T_2(xy)$, $T_1 = T_2$. Then ${T_1}^{\odot} = {T_2}^{\odot}$.

$\Longleftarrow$ If T is a positive biorthomorphism on X, T is called orthosymmetric Riesz bimorphism. Transformation $\odot: X \times X \to X$ defined with $\odot\,(x,y) = xy$ is orthosymmetric Riesz bimorphism, $(X^{\odot},\odot)$ is a Riesz space and square of X [7]. Therefore, there is an only Riesz homomorphism $T^{\odot}: X^{\odot} \to X$ defined with $T(x,y) = T^{\odot}(xy)$ for $\forall x, y \in X$. Moreover $T^{\odot}$ is an orthomorphism. Indeed, let $|x| \wedge |v| = 0$ for $x \in X$ and $v \in X^{\odot}$. $T(x,v) = 0$ from T orthosymmetric. On the other hand, there are $y, z \in X$ satisfying property $v = yz$ from definition $X^{\odot}$ for $v \in X^{\odot}$.From $\rho(\pi)(x,y) = \pi(xy) = \pi(x)y$ for $\forall \pi \in Orth(X)$ and $(x,y) \in X \times X$, then $xT(y,z) = T(xy,z) = yT(x,z) = T(x,yz) = T(x,v) = 0$ for $T \in Orth(X,X)$. Since X is semiprime,

$$|x| \wedge |T^{\odot}(v)| = |x| \wedge |T^{\odot}(yz)| = |x| \wedge |T(y,z)| = 0.$$

This demonstrated that operator $T^{\odot}$ is a positive orthomorphism.

Conclusion 1. Let X is Dedekind complete semiprime f −algebra. $Orth(X)$ is an ordered ideal of $Orth(X,X)$ [6].

Proof. Since $Orth(X)$ is a Riesz subspace of $Orth(X,X)$, it is sufficient to prove the theorem to show that $Orth(X)$ is a solid in $Orth(X,X)$. For this let $T \in Orth(X,X)$. By definition of solid, let $0 \leq T \leq f$ for $f \in Orth(X)$. Here we have to show that $T \in Orth(X)$. Since X is uniformly complement, there is an only positive orthomorphism $T^{\odot}: X^{\odot} \to X$ satisfying $T(x,y) = T^{\odot}(xy)$ for $\forall x,y \in X$. From here,

$$0 \leq T^{\odot}(x^2) = T^{\odot}(xx) = T(x,x) \leq f(x^2)$$

is provided. In other words, $0 \leq T^{\odot}(v) \leq f(v)$ is provided for $\forall\ 0 \leq v = x^2 \in X^{\odot}$. The operator $T^{\odot}$ has an extension to a positive operator that satisfies the property $0 \leq T^{\odot} \leq f$, which we can denote again with $T^{\odot}$. From here, $T^{\odot} \in Orth(X)$ and $\widehat{T^{\odot}} = T$ is obtained. Consequently, $T \in Orth(X)$ is found. This proves that $Orth(X)$ is a solid in $Orth(X,X)$ and therefore an ordered ideal.

2.2. Embedding Orthomorphism in Biorthomorphisms when X is Uniformly Complete Semiprime f −Algebra with Weak Ordered Unit

Theorem 5. If X is uniformly complete semiprime f −algebra with weak ordered unit, $Orth(X)$ is an ordered ideal in $Orth(X,X)$ [6].

Proof. $Orth(X,X)$ is an uniformly complement semiprime f −algebra. Let $e \in X$ is an positive weak ordered unit. $(Orth(X,X), *_e)$ is an semiprime f −algebra [6]. Let us show that $Orth(X)$ is a ring ideal in $(Orth(X,X), *_e)$. Let $f \in Orth(X)$ and $T \in Orth(X,X)$. From here, $(\hat{f} *_e T)(x,y) = \hat{f}\big(x, T(e,y)\big) = f\big(xT(e,y)\big) = xf\big(T(e,y)\big) = x(f\text{o}T(e,.))(y) = f\text{o}\widehat{T(e,.)}(x,y)$ is provided from $xf(y) = f(xy) = yf(x)$ for $f\text{o}T(e,.) \in Orth(X)$ and $\forall x,y \in X$. Then, $\hat{f} *_e T = fo\widehat{T(e,.)} \in Orth(X)$. This shows that $Orth(X)$ is a ring ideal in $Orth(X,X)$. On the other hand, since $Orth(X)$ is an uniformly complete f −algebra with unit element, $Orth(X)$ is square root closed. At the same time, ring ideal $Orth(X)$ is an ordered ideal. From this, it is concluded that $Orth(X)$ is an ordered ideal in $Orth(X,X)$.

2.3. Embedding Orthomorphism in Biorthomorphisms when X is Uniformly Complete d −Algebra

In this section, we defined the subspace $X^d = \{cx \wedge cy : x \wedge y = 0, c \in X^+, x, y \in X\}$ of X, which is an d −algebra. Following conclusion is obtained from Theorem 1.

Conclusion 2. Let X is an uniformly complete d −algebra. For $c \in X^+, x, y \in X$;

i. If $y = x$, $X^d = X^{\odot}$.
ii. If $y = x = c$ then set X^d is a Riesz subspace of space X with positive cone $\{cc = c^2 : c \in X\}$.

Proof.
i. $X^d = \{cx \wedge cy : x \wedge y = 0, c \in X^+, x, y \in X\}$

$= \{cx \wedge cx : x \wedge x = 0, c \in X^+, x \in X\}\ (y = x)$

$= \{cx : c \in X^+, x \in X\}$

From here, $X^d = X^{\odot}$ is provided.

ii. $X^d = \{cx \wedge cy : x \wedge y = 0, c \in X^+, x, y \in X\}$

$= \{cc \wedge cc : c \wedge c = 0, c \in X^+\}\ (y = x = c)$

$= \{c^2 : \ c \in X^+\}$
From here, X^d is a Riesz subspace of X with positive cone $\{cc = c^2 : c \in X\}$.

Conclusion 3. Let X is uniformly complete $d-$algebra, $T: X \times X \to X$ is an transformation and T is an biorthomorphism on X. For $\forall x, y \in X$ and $c \in X^+$;

i. If $y = x$, there is an only positive biorthomorphism $T^d: X^d \to X$ satisfying property $T(c, x) = T^d(cx \wedge cy)$.

ii. If there is an only biorthomorphism $T^d: X^d \to X$ satisfying property $T(c, x) = T^d(cx \wedge cy)$ for $y = x$, T is a positive biorthomorphism on X.

iii. If $y = x$ for $\forall x, y \in X$ and $c \in X^+$, $T^{\odot} = T^d$.

Proof.

i. Let $y = x$, T_1^d and T_2^d are two biorthomorphisms satisfying transformation $T^d: X^d \to X$.

$$T_1^d(cx \wedge cy) = T_1^d(cx \wedge cx) = T_1^d(cx) = T_1(c, x)$$

$$T_2^d(cx \wedge cy) = T_2^d(cx \wedge cx) = T_2^d(cx) = T_2(c, x)$$

$$T_1^d = T_2^d \text{ for } T_1^d(cx) = T_2^d(cx)$$

ii. Since X is semiprime $f-$algebra, X is $d-$algebra. In that case, proof is similar from Theorem 4. If T is a biorthomorphism on X, then T is an orthosymmetric Riesz bimorphism. In other words, if $x \wedge y = 0$ on X for $\forall c \in X^+$, $T(c, x) \wedge T(c, y) = 0$. If $y = x$, $T: X \times X \to X$ defined with $T(c, x) = cx$ is an orthosymmetric Riesz bimorphism. In addition, X^d for $y = x = c$ is Riesz space and X^d is square of X.

Therefore, there is an only Riesz homomorphism $T^d: X^d \to X$ defined with $T(c, x) = T^d(cx \wedge cy)$ for $\forall x, y \in X$ and $c \in X^+$. Moreover T^d is an orthomorphism. Indeed, let $|n| \wedge |v| = 0$ for $n \in X$ and $v \in X^d$. Since T is orthosymmetric, $T(n, v) =0$. On the other hand, there are $x, y \in X$ and $c \in X^+$ satisfying property $v = cx \wedge cy$ from definition X^d for $v \in X^d$. From [7] and $\forall c \in X^+$ for $y = x$, $nT(c, x) = T(nc, x) = cT(n, x) = T(n, cx) = T(n, cx \wedge cy) = 0$ is provided for $T \in Orth(X, X)$. Since X is semiprime for $y = x$,

$$|n| \wedge \left|T^d(v)\right| = |n| \wedge \left|T^d(cx \wedge cy)\right| = |n| \wedge \left|T^d(cx)\right| = |n| \wedge |T(c, x)| = 0$$

is obtained. This showed that operator T^d is a positive orthomorphism.

iii. For $y = x$,

$$T^d(cx \wedge cy) = T^d(cx \wedge cx) = T^d(cx) = T(c, x)$$
$$T^{\odot}(cx) = T(c, x)$$

$T^d = T^{\odot}$ from $T^d(cx) = T^{\odot}(cx)$.

Conclusion 4. Let X is a Dedekind complete $d-$algebra. $Orth(X)$ is an ordered ideal of $Orth(X, X)$.

Proof. It suffices to show that $Orth(X)$ is a solid in $Orth(X, X)$ since $Orth(X)$ is a Riesz subspace of $Orth(X, X)$, as in the case of X being a Dedekind complete semiprime $f-$algebra [8]. Let $T \in Orth(X, X)$. By definition of solid, $0 \leq T \leq f$ for $f \in Orth(X)$. We have to show that $T \in Orth(X)$. Since X is uniformly complete $d-$algebra, there is an only positive orthomorphism $T^d: X^d \to X$ satisfying property $T(c, x) = T^d(cx \wedge cy)$ for $\forall x, y \in X$ and $\forall c \in X^+$. From here for $y = x = c$,

$$0 \leq T^d(cx \wedge cy) = T^d(cc \wedge cc) = T^d(c^2) = T(c, c) \leq T(c^2) \leq f(c^2)$$

is provided. In other words, $0 \leq T^d(v) \leq f(v)$ for $\forall\ 0 \leq v = c^2 \in X^d$. The operator T^d has the extension to a positive operator, again denoted by T^d, which satisfies the property $0 \leq T^d \leq f$. From here, $T^d \in Orth(X)$ and $\widehat{T^d} = T$ is obtained. Consequently, $T \in Orth(X)$. This proves that $Orth(X)$ is a solid in $Orth(X, X)$ and therefore an ordered ideal.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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