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Double Edge-Vertex Domination on Middle and Splitting Graphs of Path and Cycle

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Abstract

An edge $e = uv$ of graph $G = (V,E)$ is said to be edge-vertex dominate vertices u and v, as well as all vertices adjacent to u and v. A set $S \subseteq E$ is a double edge-vertex dominating set if every vertex of V is edge-vertex dominated by at least two edges of S. The minimum cardinality of a double edge-vertex dominating set of G is the double edge-vertex domination number and is denoted by $\gamma_{dev}(G)$. In this paper, we present results for middle graphs of path and cycle and some splitting graphs of path and cycle on double edge-vertex domination numbers.

1. Introduction

The "domination" was first used by Oystein Ore [1]. After that, studies on domination increased rapidly. The first textbook on dominance is published in 1998 [2]. There are lots of studies on domination but small number of them on edge-vertex domination.

Double edge-vertex domination is a quite new concept of domination. It was introduced in 2020. We studied on double edge-vertex domination of some basic graph classes, double edge-vertex domination of corona pruduct and double edge-vertex domination of cartesian product our previous studies [3, 4]. In this paper, we study on double edge-vertex domination of middle and splitting graphs of paths and cycle.

Let $G = (V,E)$ be a simple graph. The set $N(v) = \{v \in V | uv \in E\}$ is open neighborhood and $N[v] = N(v) \cup N(u)$ is closed neigboorhood of $v \in V$. For any edge $e \in E$, the open edge neighborhood $N(e)$ of e is the set of edges adjacent to e [5].

For $S \subseteq V$ of vertices in a graph $G = (V,E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or adjacent to an element of S. The minimum cardinality of a dominating set of G is called the domination number and is denoted by $\gamma(G)$ [2].

A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G [6].

A vertex v of $G = (V,E)$ is said to be vertex-edge dominate every edge incident to v, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a vertex-edge dominating set, if every edge of E is vertex-edge dominated by at least one vertex of S. The minimum cardinality of a vertex-edge dominating set of G is the vertex-edge domination number and is denoted by $\gamma_{ve}(G)$ [7].

An edge $e = uv$ of graph $G = (V,E)$ is said to be edge-vertex dominate vertices u and v, as well as all vertices adjacent to u and v. A set $S \subseteq E$ is an edge-vertex dominating set, if every vertex of V is edge-vertex dominated by at least an edge of S. The minimum cardinality of an edge-vertex dominating set of G is the edge-vertex domination number and is denoted by $\gamma_{ev}(G)$ [8].

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A subset $D \subseteq V$ is a double vertex-edge dominating set of G if every edge of E is vertex-edge dominated by at least two vertices of D. The double vertex-edge domination number of G, is the minumum cardinality of a double vertex-edge dominating set of G and is denoted by $\gamma_{dve}(G)$ [9].
An edge $e = uv$ of graph $G = (V,E)$ is said to be edge-vertex dominate vertices u and v, as well as all vertices adjacent to u and v. A set $S \subseteq E$ is a double edge-vertex dominating set, if every vertex of V is edge-vertex dominated by at least two edges of S. The minimum cardinality of a double edge-vertex dominating set of G is the double edge-vertex domination number and is denoted by $\gamma_{dev}(G)$ [3].
For each vertex v of a graph G, take a new vertex v' and join v' to all vertices of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph of G [10].
The middle graph of a graph G is a graph whose vertex set is $V(G) \cup E(G)$, and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it, and it is denoted by $M(G)$ [11].

2. Double edge-vertex domination on middle graph of P_n and C_n

In this section, we study on double edge- domination on middle graphs of P_n and C_n. We found results on double edge-vertex domination number of this graph class and we prove that.

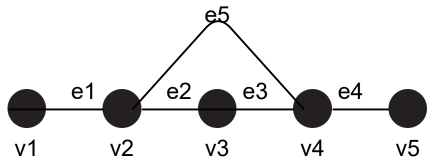

Figure 2.1: $M(P_3)$

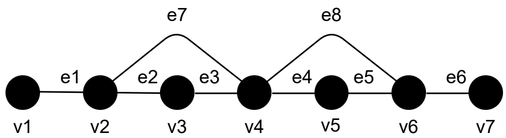

Figure 2.2: $M(P_4)$

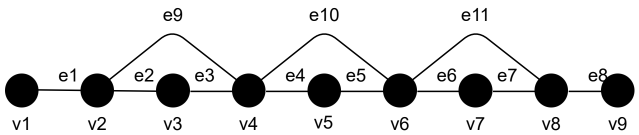

Figure 2.3: $M(P_5)$

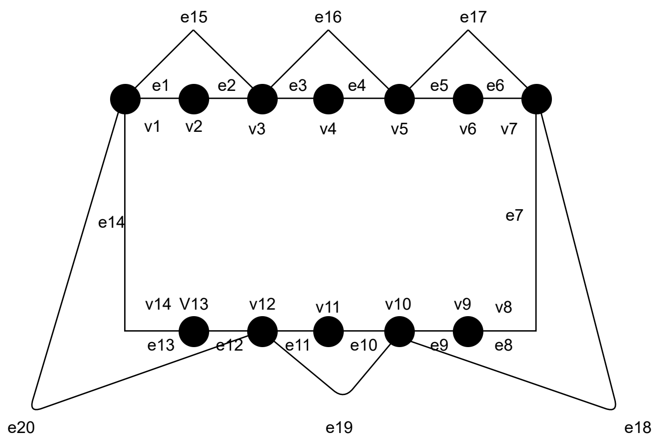

Figure 2.4: $M(C_6)$

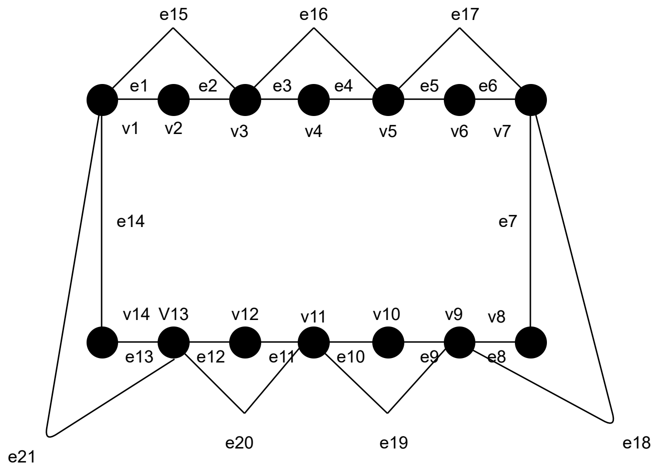

Figure 2.5: $M(C_7)$

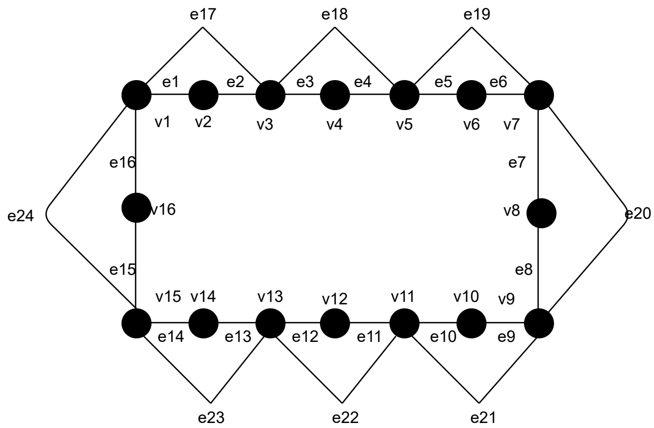

Figure 2.6: $M(C_8)$

Theorem 2.1. *For a middle graph of path with order $n \geq 3$*

$$\gamma_{dev}(M(P_n)) = \begin{cases} \frac{2n}{3}+1, & if\ n \equiv 0(mod3) \\ \lceil \frac{2n+2}{3} \rceil, & otherwise \end{cases}$$

Proof. Let's prove this theorem by mathematical induction.

Case 1: For $n = 3k$, $n \equiv 0(mod3)$, result is true for $n = 3$.

$$\gamma_{dev}(M(P_3)) = \frac{2.3}{3}+1 = 3.$$

We choose e_2, e_3, e_5 edges in Figure 2.1. Our assumption asserts that,

$$\gamma_{dev}(M(P_{3k})) = \frac{2.3k}{3}+1.$$

Our goal is show that for $n = 3(k+1) = 3k+3$, double edge-vertex domination number of middle graph of path is,

$$\gamma_{dev}(M(P_{3k+3})) = \frac{2.(3k+3)}{3}+1.$$

We added 6 vertices to $3k$, from Figure 2.1 we must choose at least two edges. Hence,

$$\gamma_{dev}(M(P_{3k+3})) \geq \gamma_{dev}(M(P_{3k})+2$$

$$=\frac{2.3k}{3}+1+2=\frac{2.3k}{3}+2+1=\frac{2.3k}{3}+\frac{2}{1}+1=\frac{2.3k}{3}+\frac{6}{3}+1=\frac{2.3k+6}{3}+1=\frac{2.3k+2.3}{3}+1=\frac{2.(3k+3)}{3}+1.$$

Case 2: For $n=3k+1$, $n\equiv 1(mod3)$, result is true for $n=4$.

$$\gamma_{dev}(M(P_4))=\left\lceil\frac{2.4+2}{3}\right\rceil=4.$$

We choose e_2, e_5, e_7, e_8 edges in Figure 2.2. Our assumption asserts that,

$$\gamma_{dev}(M(P_{3k+1}))=\left\lceil\frac{2.(3k+1)+2}{3}\right\rceil.$$

We want to prove that for $n=3(k+1)+1=3k+4$, double edge-vertex domination number of middle graph of path is,

$$\gamma_{dev}(M(P_{3k+4}))=\left\lceil\frac{2.(3k+4)+2}{3}\right\rceil.$$

We added 6 vertices to $3k+1$, from Figure 2.2 we must choose at least two edges. Hence,

$$\gamma_{dev}(M(P_{3k+4}))\geq\gamma_{dev}(M(P_{3k+1})+2$$

$$=\left\lceil\frac{2.(3k+1)+2}{3}\right\rceil+2=\left\lceil\frac{2.(3k+1)+2}{3}\right\rceil+\frac{6}{3}\geq\left\lceil\frac{2.(3k+1)+2}{3}+\frac{6}{3}\right\rceil$$

$$=\left\lceil\frac{2.(3k+1)+2+6}{3}\right\rceil=\left\lceil\frac{2.(3k+1)+6+2}{3}\right\rceil=\left\lceil\frac{2.(3k+1)+2.3+2}{3}\right\rceil=\left\lceil\frac{2.(3k+1+3)+2}{3}\right\rceil=\left\lceil\frac{2.(3k+4)+2}{3}\right\rceil.$$

Case 3: For $n=3k+2$, $n\equiv 1(mod3)$, result is true for $n=5$.

$$\gamma_{dev}(M(P_5))=\left\lceil\frac{2.5+2}{3}\right\rceil=4.$$

We choose e_2, e_7, e_9, e_{11} edges in Figure 2.3. Our assumption asserts that,

$$\gamma_{dev}(M(P_{3k+2}))=\left\lceil\frac{2.(3k+2)+2}{3}\right\rceil.$$

We want to prove that for $n=3(k+2)+1=3k+5$, double edge-vertex domination number of middle graph of path is,

$$\gamma_{dev}(M(P_{3k+5}))=\left\lceil\frac{2.(3k+5)+2}{3}\right\rceil.$$

We added 6 vertices to $3k+2$, from Figure 2.3 we must choose at least two edges. Hence,

$$\gamma_{dev}(M(P_{3k+5}))\geq\gamma_{dev}(M(P_{3k+2})+2$$

$$=\left\lceil\frac{2.(3k+2)+2}{3}\right\rceil+2=\left\lceil\frac{2.(3k+2)+2}{3}\right\rceil+\frac{6}{3}\geq\left\lceil\frac{2.(3k+2)+2}{3}+\frac{6}{3}\right\rceil$$

$$=\left\lceil\frac{2.(3k+2)+2+6}{3}\right\rceil=\left\lceil\frac{2.(3k+2)+6+2}{3}\right\rceil=\left\lceil\frac{2.(3k+2)+2.3+2}{3}\right\rceil=\left\lceil\frac{2.(3k+2+3)+2}{3}\right\rceil=\left\lceil\frac{2.(3k+5)+2}{3}\right\rceil.$$

□

Theorem 2.2. *For a middle graph of cycle with order $n \geq 6$*

$$\gamma_{dev}(M(C_n)) = \begin{cases} \frac{2n}{3}, & if\ n \equiv 0(mod3) \\ \lceil \frac{2n+2}{3} \rceil - 1, & otherwise \end{cases}$$

Proof. Let's prove this theorem by mathematical induction.

Case 1: For $n = 3k$, $n \equiv 0(mod3)$, result is true for $n = 6$.

$$\gamma_{dev}(M(C_6)) = \frac{2.6}{3} = 4.$$

We choose $e_{15}, e_{16}, e_{18}, e_{19}$ edges in Figure 2.4. Our assumption asserts that,

$$\gamma_{dev}(M(C_{3k})) = \frac{2.3k}{3}.$$

We want to prove that for $n = 3(k+1) = 3k+3$, double edge-vertex domination number of middle graph of cycle is,

$$\gamma_{dev}(M(C_{3k+3})) = \frac{2.(3k+3)}{3}.$$

We added 6 vertices to $3k$, from Figure 2.4 we must choose at least two edges. Hence,

$$\gamma_{dev}(M(C3k+3)) \geq \gamma_{dev}(M(C_{3k}) = \frac{2.(3k)}{3} + 2 = \frac{2.(3k)}{3} + \frac{6}{3}$$

$$= \frac{2.(3k)+6}{3} = \frac{2.(3k)+2.3}{3} = \frac{2.(3k+3)}{3}.$$

Case 2: For $n = 3k+1$, $n \equiv 1(mod3)$, result is true for $n = 7$.

$$\gamma_{dev}(M(C_7) = \left\lceil \frac{2.7+2}{3} \right\rceil - 1 = \left\lceil \frac{14+2}{3} \right\rceil - 1 = \left\lceil \frac{16}{3} \right\rceil - 1 = 6 - 1 = 5.$$

We choose $e_{15}, e_{16}, e_{18}, e_{19}, e_{21}$ edges in Figure 2.5. Our assumption asserts that,

$$\gamma_{dev}(M(C_{3k+1})) = \left\lceil \frac{2.(3k+1)+2}{3} \right\rceil - 1.$$

We want to prove that for $n = 3(k+1)+1 = 3k+4$, double edge-vertex domination number of middle graph of cycle is,

$$\gamma_{dev}(M(C_{3k+4})) = \left\lceil \frac{2.(3k+4)+2}{3} \right\rceil - 1.$$

We added 6 vertices to $3k+1$, from Figure 2.5 we must choose at least two edges. Hence,

$$\gamma_{dev}(M(C_{3k+4})) \geq \gamma_{dev}(M(C_{3k12}) + 2$$

$$= \left\lceil \frac{2.(3k+1)+2}{3} \right\rceil - 1 + 2 = \left\lceil \frac{2.(3k+1)+2}{3} \right\rceil + 2 - 1 = \left\lceil \frac{2.(3k+1)+2}{3} \right\rceil + \frac{6}{3} \geq \left\lceil \frac{2.(3k+1)+2}{3} + \frac{6}{3} \right\rceil - 1$$

$$= \left\lceil \frac{2.(3k+1)+2+6}{3} \right\rceil - 1 = \left\lceil \frac{2.(3k+1)+6+2}{3} \right\rceil - 1 = \left\lceil \frac{2.(3k+1)+2.3+2}{3} \right\rceil - 1 = \left\lceil \frac{2.(3k+1+3)+2}{3} \right\rceil - 1$$

$$= \left\lceil \frac{2.(3k+4)+2}{3} \right\rceil - 1.$$

Case 3: For $n = 3k+2$, $n \equiv 2(mod3)$, result is true for $n = 8$.

$$\gamma_{dev}(M(C_8) = \left\lceil \frac{2.8+2}{3} \right\rceil - 1 = \left\lceil \frac{16+2}{3} \right\rceil - 1 = \left\lceil \frac{18}{3} \right\rceil - 1 = 6 - 1 = 5.$$

We choose e_{17}, e_{18}, e_{20}, e_{22}, e_{23} edges in Figure 2.6. Our assumption asserts that,

$$\gamma_{dev}(M(C_{3k+2})) = \left\lceil \frac{2.(3k+2)+2}{3} \right\rceil - 1.$$

We want to prove that for $n = 3(k+2)+1 = 3k+5$, double edge-vertex domination number of middle graph of cycle is,

$$\gamma_{dev}(M(C_{3k+5})) = \left\lceil \frac{2.(3k+5)+2}{3} \right\rceil - 1.$$

We added 6 vertices to $3k+2$, from Figure 2.6 we must choose at least two edges. Hence,

$$\gamma_{dev}(M(C_{3k+5})) \geq \gamma_{dev}(M(C_{3k+2})+2 = \left\lceil \frac{2.(3k+2)+2}{3} \right\rceil - 1 + 2$$

$$= \left\lceil \frac{2.(3k+2)+2}{3} \right\rceil + 2 - 1 = \left\lceil \frac{2.(3k+2)+2}{3} \right\rceil + \frac{6}{3} \geq \left\lceil \frac{2.(3k+2)+2}{3} + \frac{6}{3} \right\rceil - 1$$

$$= \left\lceil \frac{2.(3k+2)+2+6}{3} \right\rceil - 1 = \left\lceil \frac{2.(3k+2)+6+2}{3} \right\rceil - 1 = \left\lceil \frac{2.(3k+2)+2.3+2}{3} \right\rceil - 1 = \left\lceil \frac{2.(3k+2+3)+2}{3} \right\rceil - 1$$

$$= \left\lceil \frac{2.(3k+5)+2}{3} \right\rceil - 1.$$

□

3. Double edge-vertex domination on splitting graph of P_n and C_n

In this section, we study on splitting graph of P_n and C_n. We found results on double edge-vertex domination number of this graph class and we prove that.

Theorem 3.1. *For a splitting graph of path with order $n \geq 4$*

$$\gamma_{dev}(S(P_n)) = \begin{cases} \lceil \frac{2n}{4} \rceil, & if\ n \equiv 3(mod4) \\ \lceil \frac{2n}{4} \rceil + 1, & otherwise \end{cases}$$

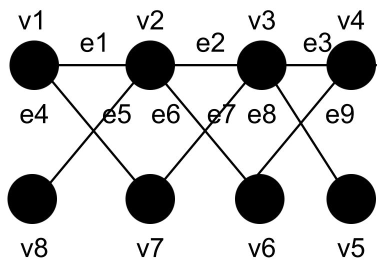

Figure 3.1: $S(P_4)$

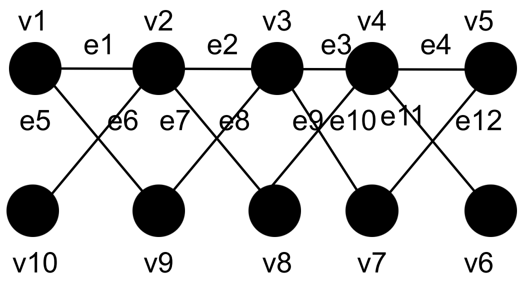

Figure 3.2: $S(P_5)$

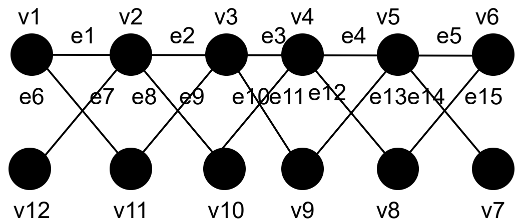

Figure 3.3: $S(P_6)$

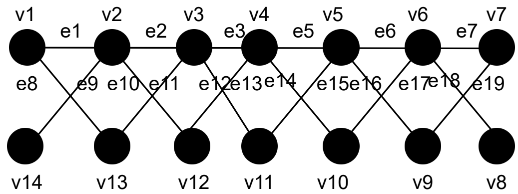

Figure 3.4: $S(P_7)$

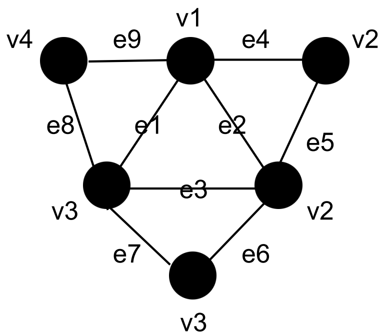

Figure 3.5: $S(C_3)$

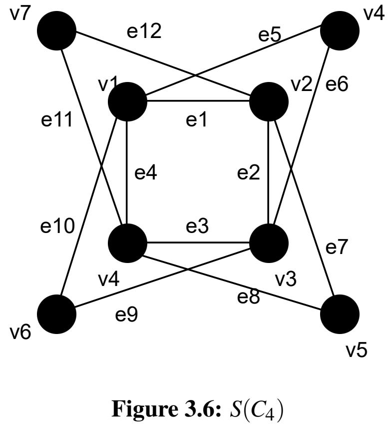

Figure 3.6: $S(C_4)$

Proof. We begin the selection edges of P_n, because degree of the edge of P_n are the most.

Case 1: For $n = 4k$, $n \equiv 0(mod4)$, result is true for $n = 4$.

$$\gamma_{dev}(S(P_4)) = \left\lceil \frac{2.4}{4} \right\rceil + 1 = 3.$$

We choose e_1, e_2, e_3 edges in Figure 3.1. Our assumption asserts that,

$$\gamma_{dev}(S(P_{4k})) = \left\lceil \frac{2.4k}{4} \right\rceil + 1.$$

We want tho prove that for $n = 4(k+1) = 4k+4$, double edge-vertex domination number of splitting graph of path is,

$$\gamma_{dev}(S(P_{4k+4})) = \left\lceil \frac{2.(4k+4)}{4} \right\rceil + 1.$$

We added 8 vertices to $4k$, from Figure 3.1 we must choose at least two edges. Hence,

$$\gamma_{dev}(S(P_{4k+4})) \geq \gamma_{dev}(S(P_{4k}) + 2$$

$$= \left\lceil \frac{2.4k}{4} \right\rceil + 1 + 2 = \left\lceil \frac{2.4k}{4} \right\rceil + 2 + 1 = \left\lceil \frac{2.4k}{4} \right\rceil + \frac{8}{4} + 1 \geq \left\lceil \frac{2.4k}{4} + \frac{8}{4} \right\rceil + 1$$

$$= \left\lceil \frac{2.4k + 2.4}{4} \right\rceil + 1 = \left\lceil \frac{2.4k+4}{4} \right\rceil + 1.$$

Case 2: For $n = 4k+1$, $n \equiv 1(mod4)$, result is true for $n = 5$.

$$\gamma_{dev}(S(P_5)) = \left\lceil \frac{2.5}{4} \right\rceil + 1 = 3 + 1 = 4.$$

We choose e_1, e_2, e_3, e_4 edges in Figure 3.2. Our assumption asserts that,

$$\gamma_{dev}(S(P_{4k+1})) = \left\lceil \frac{2.(4k+1)}{4} \right\rceil + 1.$$

We want the prove that for $n = 4(k+1)+1 = 4k+5$, double edge-vertex domination number of splitting graph of path is,

$$\gamma_{dev}(S(P_{4k+5})) = \left\lceil \frac{2.(4k+5)}{4} \right\rceil + 1$$

We added 8 vertices to $4k+1$, from Figure 3.2 we must choose at least two edges. Hence,

$$\gamma_{dev}(S(P_{4k+5})) \geq \gamma_{dev}(S(P_{4k+1}) + 2$$

$$= \left\lceil \frac{2.(4k+1)}{4} \right\rceil + 1 + 2 = \left\lceil \frac{2.(4k+1)}{4} \right\rceil + 2 + 1 = \left\lceil \frac{2.(4k+1)}{4} \right\rceil + \frac{8}{4} + 1 \geq \left\lceil \frac{2.(4k+1)}{4} + \frac{8}{4} \right\rceil + 1$$

$$= \left\lceil \frac{2.(4k+1)+8}{4} \right\rceil + 1 = \left\lceil \frac{2.(4k+1)+2.4}{4} \right\rceil + 1 = \left\lceil \frac{2.(4k+1+4)}{4} \right\rceil + 1 = \left\lceil \frac{2.(4k+5)}{4} \right\rceil + 1.$$

Case 3: For $n = 4k+2$, $n \equiv 2(mod4)$, result is true for $n = 6$.

$$\gamma_{dev}(S(P_6)) = \left\lceil \frac{2.6}{4} \right\rceil + 1 = 3 + 1 = 4.$$

We choose e_1, e_2, e_3, e_4, e_5 edges in Figure 3.3. Our assumption asserts that,

$$\gamma_{dev}(S(P_{4k+2})) = \left\lceil \frac{2.(4k+2)}{4} \right\rceil + 1.$$

We want the prove that for $n = 4(k+2)+1 = 4k+6$, double edge-vertex domination number of splitting graph of path is,

$$\gamma_{dev}(S(P_{4k+6})) = \left\lceil \frac{2.(4k+6)}{4} \right\rceil + 1.$$

We added 8 vertices to $4k+2$, from Figure 3.3 we must choose at least two edges. Hence,

$$\gamma_{dev}(S(P_{4k+6})) \geq \gamma_{dev}(S(P_{4k+2}) + 2$$

$$= \left\lceil \frac{2.(4k+2)}{4} \right\rceil + 1 + 2 = \left\lceil \frac{2.(4k+2)}{4} \right\rceil + 2 + 1 = \left\lceil \frac{2.(4k+2)}{4} \right\rceil + \frac{8}{4} + 1 \geq \left\lceil \frac{2.(4k+2)}{4} + \frac{8}{4} \right\rceil + 1$$

$$= \left\lceil \frac{2.(4k+2)+8}{4} \right\rceil + 1 = \left\lceil \frac{2.(4k+2)+2.4}{4} \right\rceil + 1 = \left\lceil \frac{2.(4k+2+4)}{4} \right\rceil + 1 = \left\lceil \frac{2.(4k+6)}{4} \right\rceil + 1.$$

Case 4: For $n = 4k+3$, $n \equiv 3(mod4)$, result is true for $n = 7$.

$$\gamma_{dev}(S(P_7)) = \left\lceil \frac{2.7}{4} \right\rceil = 4. \tag{3.1}$$

We choose $e_1, e_2, e_3, e_5, e_6, e_7$ edges in Figure 3.4. Our assumption asserts that,

$$\gamma_{dev}(S(P_{4k+3})) = \left\lceil \frac{2.(4k+3)}{4} \right\rceil.$$

We want the prove that for $n = 4(k+3)+1 = 4k+7$, double edge-vertex domination number of splitting graph of path is,

$$\gamma_{dev}(S(P_{4k+7})) = \left\lceil \frac{2.(4k+7)}{4} \right\rceil.$$

We added 8 vertices to $4k+3$, from (3.1) we must choose at least two edges. Hence,

$$\gamma_{dev}(S(P_{4k+7})) \geq \gamma_{dev}(S(P_{4k+3}) + 2$$

$$= \left\lceil \frac{2.(4k+3)}{4} \right\rceil + 2 = \left\lceil \frac{2.(4k+3)}{4} \right\rceil + 2 = \left\lceil \frac{2.(4k+3)}{4} \right\rceil + \frac{8}{4} \geq \left\lceil \frac{2.(4k+3)}{4} + \frac{8}{4} \right\rceil$$

$$= \left\lceil \frac{2.(4k+3)+8}{4} \right\rceil = \left\lceil \frac{2.(4k+3)+2.4}{4} \right\rceil = \left\lceil \frac{2.(4k+3+4)}{4} \right\rceil = \left\lceil \frac{2.(4k+7)}{4} \right\rceil.$$

□

Theorem 3.2. *For a splitting graph of cycle with order $n \geq 3$*

$$\gamma_{dev}(S(C_n)) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. We begin the selection edges of C_n, because degree of the edge of C_n are the most.

Case 1: For $n = 2k+1$, result is true for $n = 3$.

$$\gamma_{dev}(S(C_3)) = \left\lceil \frac{3}{2} \right\rceil = 2.$$

We choose e_1, e_2 edges in Figure 3.5. Our assumption asserts that,

$$\gamma_{dev}(S(C_{2k+1})) = \left\lceil \frac{2k+1}{2} \right\rceil.$$

We want the prove that for $n = 2(k+1)+1 = 2k+3$, double edge-vertex domination number of splitting graph of cycle is,

$$\gamma_{dev}(M(C_{2k+3})) = \left\lceil \frac{2k+3}{2} \right\rceil.$$

We added 2 vertices to $2k+1$, therefore we must choose at least one edge. Hence,

$$\gamma_{dev}(M(C_{2k+3})) \geq \gamma_{dev}(M(C_{2k+1})+1$$

$$= \left\lceil \frac{2k+1}{2} \right\rceil + 1 = \left\lceil \frac{2k+1}{2} \right\rceil + \frac{2}{2} \geq \left\lceil \frac{2k+1}{2} + \frac{2}{2} \right\rceil = \left\lceil \frac{2k+1+2}{2} \right\rceil$$

$$= \left\lceil \frac{2k+3}{2} \right\rceil .$$

Case 2: For $n = 2k$, result is true for $n = 4$.

$$\gamma_{dev}(M(C_4)) = \left\lceil \frac{4}{2} \right\rceil = 2.$$

We choose e_1, e_3 edges in Figure 3.6. Our assumption asserts that,

$$\gamma_{dev}(M(C_{2k})) = \left\lceil \frac{2k}{2} \right\rceil .$$

We want the prove that for $n = 2(k+1) = 2k+2$, double edge-vertex domination number of splitting graph of cycle is,

$$\gamma_{dev}(M(C_{2k+2})) = \left\lceil \frac{2k+2}{2} \right\rceil .$$

We added 2 vertices to $2k$, therefore we must choose at least one edge. Hence,

$$\gamma_{dev}(M(C_{2k+2})) \geq \gamma_{dev}(M(C_{2k})+1$$

$$= \left\lceil \frac{2k}{2} \right\rceil + 1 = \left\lceil \frac{2k}{2} \right\rceil + \frac{2}{2} = \left\lceil \frac{2k}{2} + \frac{2}{2} \right\rceil = \left\lceil \frac{2k+2}{2} \right\rceil .$$

□

4. Conclusion

Graphs, can be used to model communication networks. We can represent any network with graphs. Each center in the structure of the network is represented as a vertex, and each connection as an edge in graph. For axample, a telecommunication system, GPS/Google maps, connections of social media and many more can be represented as a graph. It is important to ensure persistence of communication in networks. In graph theory, various metrics are used to strengthen the stability of communication networks. Domination is one of the most studied topics of them. Domination has wide range of applications. For instance, to compute a wireless network with minimal power, to decide the locations where the radio stations in the region will be located so that the radio messages can be broadcast to all places in the region, to define the route of shuttle buses. There are lots of studies on domination concept. We have also worked on topic in our previous studies of double edge-vertex domination. In this study we find double edge-vertex domination number of middle and splitting graph of P_n and C_n. In future study we plan to generalize the results on middle and splitting graphs of any given graphs.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Modelling Sport Events with Supervised Machine Learning

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Abstract

It has been very important to understand the change of multivariable systems to make predictions accordingly. The goal of supervised machine learning is to build a model of changing classes of observations depending on various variables and to make predictions about the coming situations. Due to the fact that sports are followed by the whole world modelling sports events and studies about predicting the results of future matches have gained importance. In this study, match statistics of the teams in the Turkey Super League were used, and it was examined how successfully the outcome of the match was predicted using a decision tree, random forest, k-nearest neighbor, naive Bayes, support vector machine. According to the tests done in Turkey Super League, the support vector machine performs the best.

1. Introduction

Nowadays, machine learning has been one of the fastest growing fields of computer science with the data growth and large scaled applications. In general, machine learning is unfoldment of the hidden structure in the data. Machine learning is the field of research devoted to the formal study of learning systems. This is a highly interdisciplinary field which borrows and builds upon ideas from statistics, computer science, engineering, cognitive science, optimization theory and many other disciplines of science and mathematics [1].

There are several applications for Machine Learning (ML), the most significant of which is data mining [2]. Machine learning is examined under three main headings: Supervised machine learning, unsupervised machine learning, reinforcement machine learning. Machine learning algorithms differ from each other according to the way to get the required results. Supervised machine learning has a algorithm that process by matching the outputs and inputs. While the system was being trained, the outputs and inputs of each sample in data set are given. The problem with supervised machine learning is tackled as classification problem[3].

The training set given for unsupervised leaning is the unlabeled dataset. Unsupervised learning aims at clustering, probability density estimation, finding association among features, and dimensionality reduction. In general, an unsupervised algorithm may simultaneously learn more than one properties listed above, and the results from unsupervised learning could be further used for supervised learning [3].

Score prediction with machine learning has frequently been the subject of academic studies because of collecting data of past sport events is easier than other fields with sport being followed by the whole world and the opportunities provided by technology. Although the most of the studies is about football, basketball, baseball and American football, there are also studies for different sports. Analyses such as athlete injury risks, ticket sales prediction, success evaluation, score prediction

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are performed. In addition to these, machine learning are utilized for specifying game strategies from sports data and making selection athlete, trainer, equipment [4].

In modelling sport events, generally supervised machine learning algorithms such as regression models based Poisson, decision tree, neural network and models completely explained by data are used.

2. Literature

There are many statistical and machine learning studies about modelling sports events. Linear models based on probability distribution ise used in some of these. Another its part is tackled as a classification problem in form of predicting the match outcome with machine learning algorithms. Harville [5] has developed one of the first linear models based on past matches data. He used the model to predict match results. Knorr-Held [6] and Koning [7] developed models based on different statistical eventuations with match results in the form of win-lose-draw and predicted future match results.

The number of goals scored by teams in a football match are Poisson distributed and Poisson variables of these teams depend offensive power of one of these teams and defensive power of other team. Maher [8] developed the first Poisson-based regression model depending the number of goals scored and conceded by the teams in the match and he used for score prediction in football. Crowder, Dixon et.al. [9] added the time factor to the Poisson-based regression model and used to predict football match results. After Karlis and Ntzoufras [10, 11] have done a study on score prediction in football, they made evaluations for outcome estimation per applying the Poisson-based model to water polo.

In Poisson-based models and time series models, outcome estimations based on team strength determined by data based on goals scored and conceded by the teams in the past matches, and in linear regression model, outcome estimations based directly on the match results are made. Especially, Poisson-based models and time series models are used often for outcome prediction in football matches. However factors such as foul, shot, corner, offside point, percentage of passes, possession of the ball, goal attempt, players morale and the position of the team in the league affect the match results, accordingly these models are inadequate in outcome estimation. Bayesian network models aim to predict match results by determining the offensive and defensive strength of teams by considering different factors. Bayesian network models give successful results because of they also pay regard to the relationships between variables. Rue and Salvesen [12], have used Bayesian linear model for English Premier League. Baio and Blangiargo [13] brought a Bayesian approach to the Poisson based model and they have established the score prediction model in football.

In modelling studies in field of sport, when the variable selection based on expert knowledge was made, the ideas that the models give more successful results have became prominent. Josephs, Fenton et.al. [14]'s study is a good example for this. Using 2006 FIFA World Cup data, the matches that ended together were extracted, and the winnings of the home team or the visiting team were estimated using data based machine learning algorithms and Bayesian network based on expert knowledge. While achieving 60% success with machine learning algorithms, 76.9% success was achieved with Bayesian networks based on expert knowledge. Thats why Bayesian network has been stated that it is more successful in modelling. Huang [15] modeled the Tottenhnam Hotspur's which is Englang Premier League team game between 1995-1997 with Bayesian network. In her study, while Bayesian networks created with the experts of the subject made prediction with 59% success, the other networks remained in the range of $40-50\%$. Similarly, Constantinou et.al [16] developed Bayesian networks model based on expert knowledge with English Premier League seasonal data. The model has been successful with high accuracy rate. Points that the teams will collect during the season has been estimated via expert knowledge based Bayesian network model developed in the study made by Contantinou ve Fenton [17]. It has been established that the model can be used to predict both the season ranking and the outcome of individual matches. Karabiyik and Yet [18] developed Bayesian network model based on expert knowledge which called FutBa for Turkey Super League. The model predicts the outcome of past and future matches with an accuracy of $60-70\%$.

The use of artificial neural networks in modelling sports events is also quite common. In Purucker [19]'s study, one of the first examples, the matches belonging to the first eight weeks of the NFL 1994 season were modeled with artificial neural networks and it was intended to predict the winner of the NFL games. Back-propagation network model predicts with 60.7% accuracy. Kahn [20] developed artificial neural networks model with data from 2003 NFL season in first 14 week and that model has achieved 75% success in predicting seasonal averages at 14th week. McCabe and Trevathan [21] modeled rugby matches with artificial neural networks. Tests were applied out for four different rugby leagues in their study. The model succeeded between 54% and 65%.

Various machine learning algorithms have been used in modelling sport events. Hamadoni [22] used logistic regression and support vector machine algorithms in NFL for predict which team winning game from 2003 to 2005 seasons. In comparison made by Hamadoni, support vector machine algorithm has shown a test success rate of 67.08% for 2003, 61.37% for 2004, 65.83% for 2005. Similarly Sierra, Forco et.al. [23] have developed support vector machine model and logistic regression model to predict the NFL game results. Linear support vector machine algorithm was more successful than logistic regression model and other support vector machine models. Smith, Lipscomb et.al. [24] used naive Bayes algorithms to predict which team wins in MLB (Major League Baseball) and they tested with passed matches data between 1967 and 2006. The model predicted winners 80% correctly. Hucaljuk ve Rakipovic [25] modeled 96 matches in UEFA Champions League using 30 features with different machine learning algorithms. Models have shown success rate between 60% and 50%. Cao [26] modeled with NBA data from 2006 to 2010 seasons. Logistic regression made predictions with 67.82% accuracy, support vector machine made predictions with ½67.22%½ accuracy, multilayer perceptron neural network made predictions with 66.67% accuracy and naive Bayes made predictions with 65.82% accuracy. Yezus [27] modeled English Premier League ames using 9 features with k-nearest neighbors and random forest algorithms. Models have shown success rate 55.8% and 63.4% respectively. Ulmer and Hernandez [28] also modeled English Premier Leauge. Linear classification model they developed has achieved 48% success. Support vector machine model predicted with 50% accuracy and random forest model made predictions with 50% accuracy. Karaoğlu [29] modeled 16 different league with data from 2013-2014 season and 2015-2016 season using machine learning algorithms. The best model performance was 52%. Vaidya, Sanghavi et.al. [30] also modeled English Premier League using data from 2006 to 2010 seasons with logistic regression, random forest and naive Bayes algorithms. The models have shown 49.37%, 47.11% and 47.11% success respectively. Soto Valero [31] aimed to assess the predictive capabilities of four machine learning methods for predicting outcomes in MLB regular season games and he used k-nearest neighbors, artificial neural network, support vector machine algorithms. As a result of him study, it was revealed that the classification algorithms make more successful predictions than regression models. The support vector machine model from four machine learning algorithms was the most successful model by making predictions with approximately 60% accuracy.

In this study decision tree, random forest, k-nearest neighbors, naive Bayes, support vector machine will be used from supervised machine learning algorithms in the modelling of Turkish Super League. This paper has a unique structure due to the variety of algorithms used in this study and the lack of modelling studies in Turkey Football League.

3. Method

Each observation used by machine learning algorithms is represented with the same variable set. Variables in the variable set can have different structures such as categorical, continuous or bivalent. When one of the variables is considered as output and the others are also considered as inputs, if the label of output variable of the observations is known it is supervised machine learning method. Supervised machine learning methods are called as classification algorithms because of supervised machine learning methods works with labeled dataset. The main purpose of supervised machine learning is that build model that can predict the class values of the test set with training set.

In supervised machine learning, the training data with labeled output variable is processed with machine learning algorithms and thus a prediction model is created. The model is tested using unlabeled test data and the class labels of observations in test data is predicted. This process has been shown in Figure 3.1.

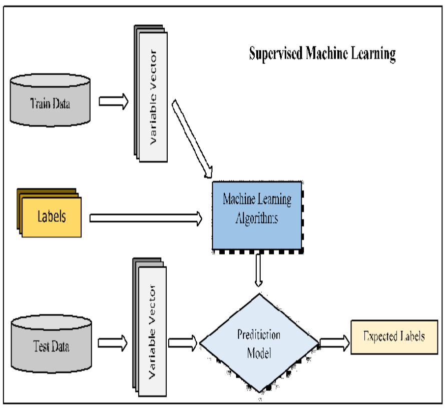

Figure 3.1: Supervised machine learning process

In this study, decision tree, random forest, k-nearest neighbor, naive Bayes, support vector machine widely used have been concentrated.

3.1. Decision tree

Decision tree is one of the commonly used supervised machine learning methods in classification. In this case, its properties such as simplicity of implementation, understandability, no using parameters, availability for mixed data types, being faster than other methods plays a role. Despite these positive properties, decision tree algorithm contains some problems. Not possible to obtain outputs containing more than one feature, causing changeable results, being sensitive to small alterations, giving complex output for numerical dataset are some of these problems.

In decision trees, it is aimed to create a tree according to the features of the data in the training set. Decision trees decide which class new data belongs by determining separation rules based on historical data. It acts on questions and answers and it creates rules by combining asked questions with answers. It can also be said that the resulting tree is set of rules consisting of many if-then. When it is decided which variable in the data to start asking question, the related variable creates the root node of the tree. Starting from the root node, new node are created according to the answers received by asking questions whose answer in the dataset. Then, each node is split up two node or more. If new question can not be asked after the node are created, branching is over [32].
As a result of this process, a classification tree as in Table 1 is obtained.

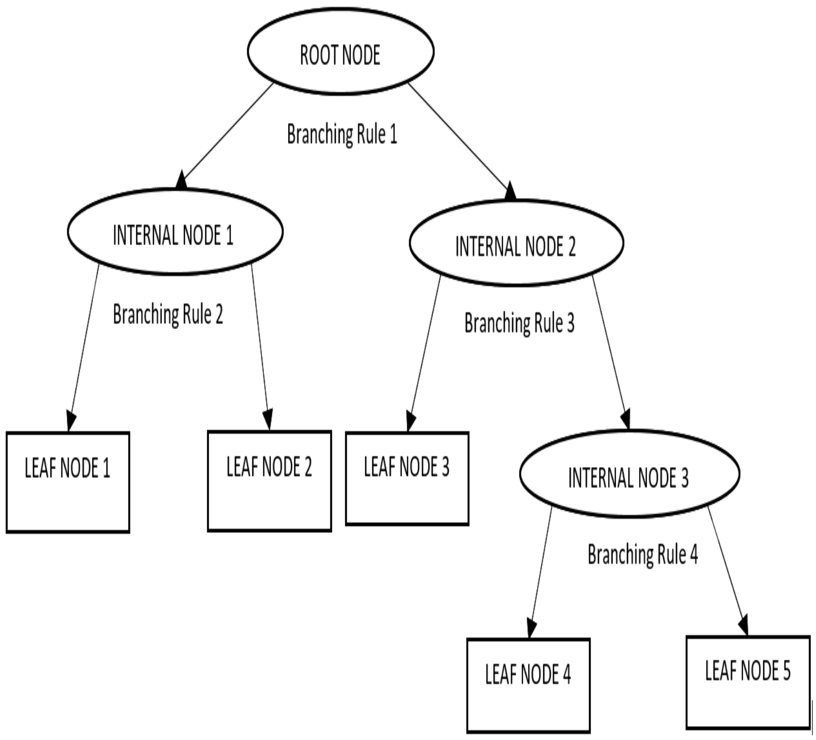

Figure 3.2: Decision tree structure

There are various decision tree algorithms that differ according to the criteria they use. First, Automatic Interaction Detector (AID) was used in decision tree applications. Then many algorithms were developed. The main ones are CART (Classification and Regression Trees), CHAID (Chi-Squared Automatic Interaction Detector), Exhaustive CHAID (Chi-Squared Automatic Interaction Detection), ID3 (Iterative Dichotomiser 3), C4.5, MARS (Multivariate Adaptive Regression Splines), QUEST (Quick, Unbiased, Efficient Statistical Tree), C5.0, SLIQ (Supervised Learning in Quest), SPRINT (Scalable Parallelizable Induction of Decision Trees).

3.2. Random forest

In multivariate different data groups, the success of classifier varies. In this situation, it can be efficient to use single classifier in terms of results. Ensemble algorithms uses sets of classifier together instead of a classifier to resolve the problem. The most common of these are bagging, boosting and random forest.
Random forest comprises of the union of many decision trees. Trees are formed by drawing independently from each other with boostrap method. After the trees are voted by one, the winning classifier is chosen. Random forest determines variables to use in branching by selecting the randomly choosed m pieces from all the data in the data set. m is usually taken as the square root of the variable number. The advantage of random forest over the bagging algorithm is that it puts randomness into training stage [33].

Random forest algorithm can be used for purposes such as finding error rate of the algorithm, determining the importance levels of variable, specifying the outliers and detecting the missing value.

In random forest, if the original data set does not have a test set of its own, from the original dataset n samples are selected by the bootstrap method. 2/3 of each sample is used to create tree and the remaining 1/3 is used to calculate the error rate. If the original data set has its own test set, error rate of this test can also be calculated with the set. The variable that gives the best information among the randomly selected variables from the training set is used as the branching variable. These are performed simultaneously and iterated until the most successful tree is obtained. The flow chart of this process is given in Figure 3.3 below.

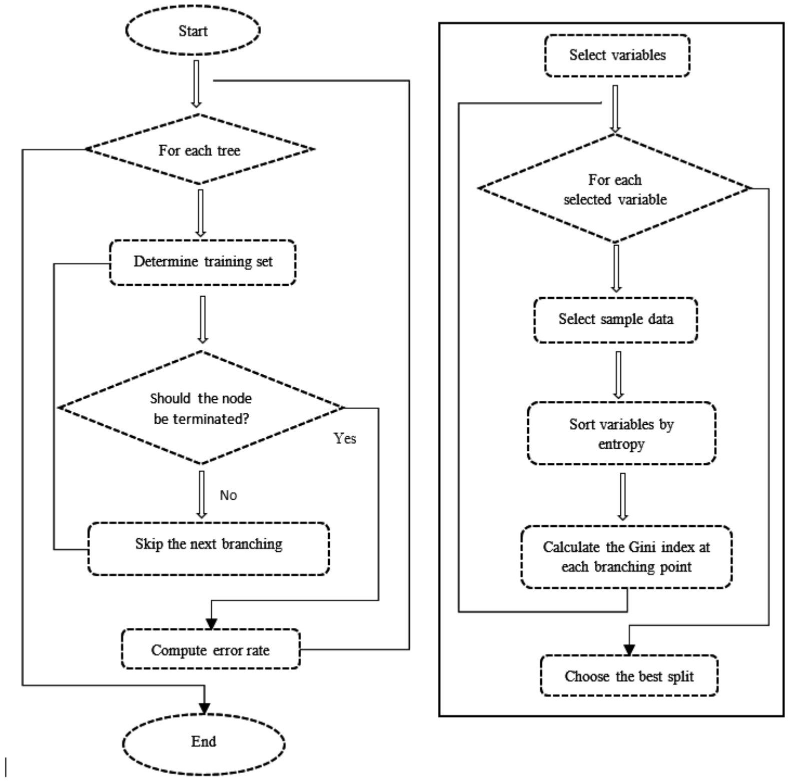

Figure 3.3: The random forest flow chart, [34]

3.3. k – nearest neighbor

k-nearest neighbor is a non-parametric, sample based classification algorithm. k-nearest neighbor is predicated on the principle that samples which are close to each other will be similar. When an sample without a class label is given, a sample space based on the classifier k created with k-nearest neighbor algorithm is created and if which class is repeated mostly in this sample space, this observation is assigned that class label. It would be appropriate to visualize it as in Figure 3.3.

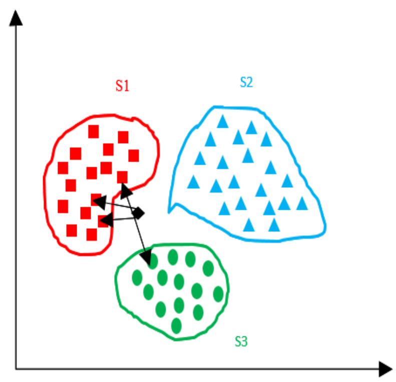

Figure 3.4: k-nearest neighbor classification visualization

The chose of k is very important. However there is no optimal k value, its value varies according to the nature of the problem. The chose of k effects the performance of k-nearest neighbor algorithm. If k is small value, it causes classification error. This problem can be solved by choosing a greater k. If k is great value, the proportion of classes at a specified distance will decrease and the instances of other classes will gain the majority. This is also causes a classification error again. It can be solved by choosing a smaller k [35].

3.4. Naive Bayes algorithms

Naive Bayes networks are the simplest Bayesian networks. Naive Bayes classifier is composed without a cyclical relationship with only one parent and several children with a strong assumption of independence among child nodes in the context of their parent.

Accordingly, for example, the naive bayes network for the data set consisting of one output variable and four input variables should be handled as in the Table 2. This input variables affect the output variable independently of each other.

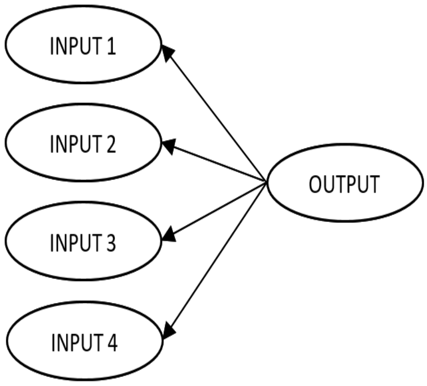

Figure 3.5: Naive Bayes algorithm network

The naive Bayes is based on Bayes Theorem. As part of the theorem, naive Bayes algorithm makes predictions by calculating the probabilities of classes with the probabilities obtained from a labeled training set. Bayes Theorem is shown below.

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

where, Y: Some hypothesis, such that data tuple X belongs to specified class C, X: Some evidence, describe by measure on set of attributes, $P(Y|X)$: The posterior probability that the hypothesis H holds given the evidence X, $P(Y)$: Prior probability of H, independent on X, $P(X|Y)$: The posterior probability that of X conditioned on H. The algorithm calculates the following two probabilities and compares them.

$$R = \frac{P(m|X)}{P(n|X)} = \frac{P(m)P(X|m)}{P(n)P(X|n)} = \frac{P(m)\prod P(X|m)}{P(n)\prod P(X|n)}.$$

As a result of comparing these probabilities, predicted class label is the class of higher probability. If $R > 1$, prediction is m, otherwise prediction is n. The major advantage of the naive Bayes classifier is its short computational time for training. In addition, since the model has the form of a product, it can be converted into a sum through the use of logarithms with significant consequent computational advantages. It does not need any complicated iterative parameter estimation schemes, so can be applied to large data set. Easy interpretation of knowledge representation. It is also easy to present and understand due to the interpretation of knowledge representation is easy. It may not be best classifier in any particular application, but it does well and robust. Otherwise, naive Bayes has some disadvantages. Theoretically, naive Bayes classifier have minimum error rate comparing to other classifier, but practically it is not always true, because of assumption of class conditional independence and the lack of available probability data. It has less accurate compare to other classifier [36]-[38].

3.5. Support vector machine

Support vector machines are one of the most powerful supervised machine learning algorithms. In addition, it has many application fields such as classification, regression, variable selection, detection of outliers. Support vector machine is algorithm that revolve around the notion of margin and create optimal separating hyperplane according to this. Margin is the distance between the hyperplane and the closest observation point on any side of the plane. The classifier chooses hyperplane in accordance the data used to detect optimum margin. It is examined as cases where data is separated completely linear and data can not be separated linearly [39]. SVM searches for the optimal separating hyperplane that correctly classifies the data as shown in Figure 3.6.

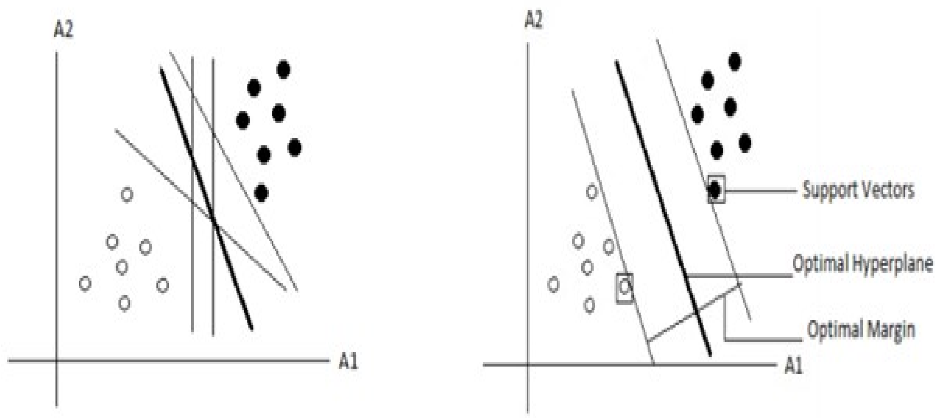

Figure 3.6: The optimal seperating hyperplane for linearly seperable data [38]

In real world problem, linearly separated data are not encountered. In this case support vector machine can not use linear learning and it is extended to learn non-linear decision onto a high dimensional sample space using Kernel functions. The common Kernel functions are used in support vector machine are linear, polynomial, radial, sigmoid. In case of the data can not be separated linearly, the support vector machine illustration designed using Kernel functions is shown in Figure 3.7.

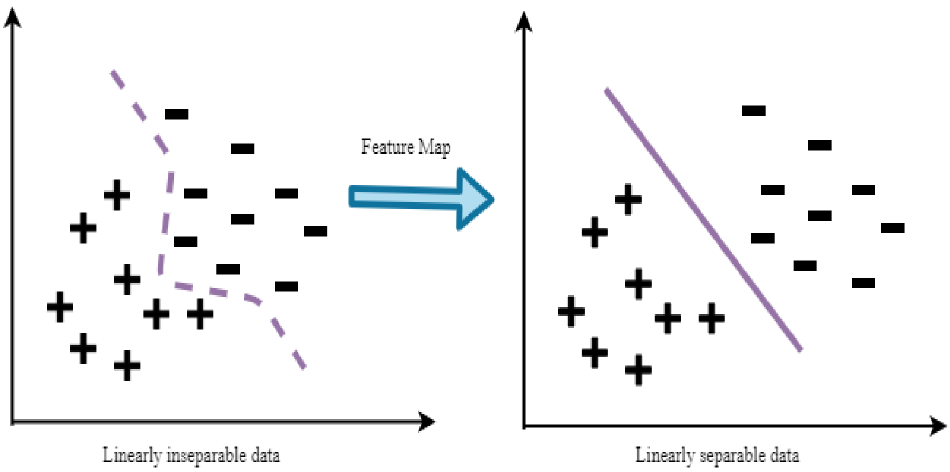

Figure 3.7: Illustration of nonlinear support vector mavhine consept [41]

Support vector machine has a sturdy theoretical basis. It requires only examples and is insensitive to the size. Support vector machine is the best algorithm to classify the members of two classes in the training set. It is less prone to overfilling than other methods. However, it is computationally expensive and tedious. It learns extremely slow. Its results may be poorly interpreted. [36, 40]

4. Data management

Modelling Turkey football matches with supervised machine learning algorithms using Turkey Super Lig data and comparison of prediction success of the models are aimed. The study is carried out in R Studio.

4.1. Data collection

The data is taken from WhoScored.com. Turkey Super Lig 2018-2019 season statistics have been used. There are 612 observations belong to 306 matches played during Turkey Super Lig 2018-2019 season. Each observation consists of 14 variables. These variables are team name and the match results in form of win-lose-draw, home ownership, number of goal, number of offside, percentage of pass, crossing, ball hawking, number of shot, goal conversion rate, possession of the ball, number of red-card, number of yellow-card, number of foul.

4.2. Data arrangement

Choosing the variables to be used in organizing data is very important. It can be explained with two subject. The first one is the requirement for the model to be simple. Second, using significant variables in the model can show a more meaningful model performance.

The first thing to look at is the correlation between variables. In consequence of determination of correlation between number of goal and goal conversion rate with percentage of pass and possession of the ball, goal conversion rate and possession of the ball variables have left out of analysis. After controlling correlation, choosing the variables should be made. There is various variable selection method can be used. The methods can be organized into three categories, depending on how they combine the feature selection search with the construction of the classification model: filter methods, wrapper methods and embedded methods [42].

Forward selection method has been used in the study. Forward selection algorithm works in the way deciding the final model, starting from empty model, adding each variable. The model involved all variables has been decided. The variables in the processed data set are given below.

```
     result            team                goal        homeownership
 Min.   :0.0000   Length:612         Min.   :0.000   Min.   :0.0
 1st Qu.:0.0000   Class :character   1st Qu.:0.000   1st Qu.:0.0
 Median :1.0000   Mode  :character   Median :1.000   Median :0.5
 Mean   :0.9363                      Mean   :1.338   Mean   :0.5
 3rd Qu.:2.0000                      3rd Qu.:2.000   3rd Qu.:1.0
 Max.   :2.0000                      Max.   :7.000   Max.   :1.0

     ofsaid           pass           crossing         hawking
 Min.   : 0.000   Min.   :0.5400   Min.   : 1.000   Min.   : 4.00
 1st Qu.: 1.000   1st Qu.:0.7400   1st Qu.: 6.000   1st Qu.:13.00
 Median : 2.000   Median :0.7900   Median : 8.000   Median :16.00
 Mean   : 1.894   Mean   :0.7765   Mean   : 8.572   Mean   :16.43
 3rd Qu.: 3.000   3rd Qu.:0.8200   3rd Qu.:11.000   3rd Qu.:19.00
 Max.   :11.000   Max.   :0.9200   Max.   :28.000   Max.   :35.00

      shot             red             yellow            foul
 Min.   : 2.00   Min.   :0.0000   Min.   : 0.000   Min.   :  4.0
 1st Qu.: 9.00   1st Qu.:0.0000   1st Qu.: 1.000   1st Qu.: 11.0
 Median :12.00   Median :0.0000   Median : 2.000   Median : 14.0
 Mean   :12.74   Mean   :0.1307   Mean   : 2.248   Mean   : 14.1
 3rd Qu.:16.00   3rd Qu.:0.0000   3rd Qu.: 3.000   3rd Qu.: 16.0
 Max.   :34.00   Max.   :2.0000   Max.   :20.000   Max.   :141.0
```

Figure 4.1: Descriptive statistics of the variables

Dependent variable:
result (0: Lose, 1: Win, 2: Draw)

Independent variables:
team (Teams in Turkey Super Lig 2018-2019 season)
home ownership (0: No, 1: Yes)
goal (The number of goals a team scored in the match)
offside (The number of offside a team has committed in the match)
pass (The pass rate of a team in the match)
crossing (The number of opponents passed by a team player without losing the ball)
hawking (The number of times a team takes the ball from opponent)
shot (The number of shot for a team in a match)
red (The number of red card a team committed in the match)
yellow (The number of yellow card a team committed in the match)
fault (The number of fouls a team has committed in the match)

5. Model building

It is aspired to classify the match results belonged to the teams in form of win/lose/draw through the instrument of other match statistics. Hence classification algorithms from supervised machine learning algorithms that suitable for the purpose of study are used. At this stage prediction model will be developed using decision tree, random forest, k-nearest neighbors, naive Bayes, support vector machine algorithms. Data set consisting of 612 observations is divided into 75% training set and 25% test set to use in modelling.

5.1. Decision trees algorithms

The variables given in the data management phase have been used in analysis. The CART algorithm from decision tree have been applied to the data. Because of the dependent variable is categorical, the Gini criterion was chosen as the classification scale. The tree structure that minimizes the error rates and maximizes the classification success was specified applying 10-fold cross validation test. As a consequence of the analysis, it is determined that the most important variable affecting the class variable which is dependent variable is goal. Team and pass variables supervene on goal. When classification success was controlled, the accuracy rate was found to be 60.1%. Respectively, sensitivity values of the classes are 52.2%, 79.3%, 41.3% and selectivity values of the classes are 80.7%, 86.6%, 73.8%. These are given in Table 1.

	Lose	Win	Draw
Sensitivity	0.5227	0.7937	0.4130
Selectivity	0.8073	0.8667	0.7383

Table 1: Decision tree sensitivity and selectivity values

5.2. Random forest algorithms

Although the random forest algorithm is onerous, because of data management is done in previous stages, the algorithm has been simply applied to the data. After the necessary libraries are loaded in R Studio, the codes are written using the parameters in random forest package to train model. Then the fitting codes is written. When classification success was controlled, the accuracy rate was found to be 60.1%. Respectively, sensitivity values of the classes are 61.3%, 79%, 34% and selectivity values of the classes are 75.2%, 87.9%, 78.3%. These are given below in table. The results obtained are given in Table 2.

	Lose	Win	Draw
Sensitivity	0.6136	0.7903	0.3404
Selectivity	0.7523	0.8791	0.7830

Table 2: Random forest sensitivity and selectivity values

5.3. k-nearest neighbors algorithms

The variables given in the data management phase have been used in analysis. As a result of cross validation control, the optimum value for k is decided to be 7. Right after classification, it is fixed that the accuracy rate of the model is 42.4%. Respectively, sensitivity values of the classes are 30.6%, 58%, 41.4% ve and selectivity values of the classes are 72.5%, 67.3%, 73.2% These are given below in table. The results obtained are given in Table 3.

	Lose	Win	Draw
Sensitivity	0.3065	0.5800	0.4146
Selectivity	0.7253	0.6736	0.7321

Table 3: k-nearest neighbors sensitivity and selectivity values

5.4. Naive Bayes algorithms

After data management phase, football data is separated as training set and test set. In the system the data has been classified with R Studio packages required for naive Bayes algorithm based upon the labeled class data. Then the classes of the unlabeled observation in test set has been predicted via the naive Bayes model. The classification success is determined as 55.6%. Respectively, sensitivity values of the classes are 33.3%, 90.7%, 37.9% and selectivity values of the classes are 72.5%, 86.9%, 74.7%. These are given below in Table 4.

	Lose	Win	Draw
Sensitivity	0.3333	0.9074	0.3788
Selectivity	0.7250	0.8687	0.7471

Table 4: Naive bayes sensitivity and selectivity values

5.5. Support vector machine algorithms

The system is modeled with the train data set using support vector machines algorithm. After the modelling of the system is completed, the basis function of support vector machines is improved in accordance with the model, radial kernel function has been used. Train and test errors vary related to gamma and cost values used in the function. With the calculations, the best gamma and cost values were found. The ideal situation would be that there be no training errors and minimal test errors. The aim is to make both minimums and thus the best classification model will be developed. Support vector machine training was repeated using best gamma and cost values. The model was accepted and controls were made with the test set. Accuracy rate was found to be 61.4% when the classification success was monitored(control). Sensitivity rates for lose, win, draw are 49.2%, 86%, 42.4%, and the selectivity rates are 85.6%, 86.5%, 72.5%, respectively. The classification results obtained are shown in Table 5.

	Lose	Win	Draw
Sensitivity	0.3333	0.9074	0.3788
Selectivity	0.7250	0.8687	0.7471

Table 5: Support vector machine sensitivity and selectivity values

5.6. Model comparison

In this study, we aimed to apply supervised machine learning algorithms to the field of sports and to evaluate the performance of the models through it. While evaluating the success of the models, accuracy, sensitivity and selectivity measures were used. The output which we get from these metrics is the complexity matrix. In the complexity matrix, there is a case of comparing the predicted groups made by a classification algorithm with the real-case groups. A complexity matrix is shown in the table. TP defines true-positive, TN indicates the true-negative which represents the correct number of classified samples. FP means false-positive. It shows that the positively predicted samples are in the negative class. FN means false negative and gives the number of samples that were predicted negatively when in fact in the positive class. Generally, the diagonal of the complexity matrix gives the number of correctly estimated samples, while the others give the number of incorrectly estimated samples.

		Prediction	
		Positive	Negative
Fact	Positive	TP	FN
	Negative	FP	TN

Table 6: Confusion matrix

The accuracy rate used in measuring model success is the ratio of the number of correctly classified samples to the total number of samples.

$$Accuracy = \frac{TP+TN}{TP+TN+FP+FN}.$$

Sensitivity is indicated the ratio of the number of correctly classified positive samples to the total number of positive samples.

$$Sensitivity = \frac{TP}{TP+FN}.$$

The selectivity criterion used to measure the success of the model is explained as the ratio of the number of correctly classified negative samples to the total number of negative samples.

$$Selectivity = \frac{TN}{TN+FP}.$$

6. Performance measure

The confusion matrices of the models were given in detail during the model building phase. At this stage, the accuracy rates are shown in the table below in order to compare the models. As a result of the tests, the best performing model was support vector machine, while the worst performing was k-nearest neighbor.

	Accuracy Rate
Decision tree model	0.6013072
Random forest model	0.6013072
k-Nearest neighbor model	0.4248366
Naive Bayes model	0.5555556
Support Vector Machine Model	0.6143791

Table 7: Performance of classification algorithms

In order to evaluate the models correctly, it is necessary to pay attention to the working area. Football is a sport that affected by a wide variety of factors. At the same time, according to experts, Turkey Super League teams are not very stable as team success. Teams that are the leaders of the league may suddenly show low performance and risk falling to lower ranks. In addition, there are major differences between the top three and the last three of the league. All of these pose a big problem in predicting the match results. Therefore, 61.4% success in terms of this study is a satisfactory result. Such as the English Premier League teams which are more stable and more homogeneous structure in themselves can show better model success within the same scope studies.

7. Results

Machine learning algorithms have diversified in progress of time and have been applied to different fields. Correspondingly, studies on which of the algorithms are more successful have a great place in literature. Although there are so many studies on the success of the algorithms, no consensus could be reached on this issue. There may be various reasons of that, but the main reason is even if a study is not an unique, each study has its idiosyncratic structure. Factors such as the data source used in the study, the preprocessing method, parameters used in the algorithms form the structure of the study and these affect the performance of the process. Therefore, it is normal for studies to give different results, assessments should be made by considering these.

In this study, modelling sport events via classification methods and comparison of the prediction success of the models are aimed. In line with this purpose, the five of supervised machine learning algorithms are commonly used in literature were applied. These are decision tree, random forest, k-nearest neighbor, naive Bayes and support vector machine algorithms. As a result of the test, it has been determined that support vector machines have the best performance with 64% success rate.

Considering this study and experimental studies in the literature, it is not possible to talk about the superiority of any algorithm over another. However, comparison of the models in solving a problem contributes both of the study and academic literature in the terms of consequences.

8. Discussion

The use of machine learning is getting common of sports area. Therefore, a lots of algorithm been developed and improved has a wide place in literature. Linear models, Poisson-based models, time series models used frequently in the beginning. As a result of the inadequacy of these models in predicting outcomes, Bayesian networks have gained importance and then Bayesian networks based on expert knowledge have showed off themselves. Joseph, Fenton et.al. predicted which of the home team and guest team will win with 2006 FIFA World Cup data's by removing the draw status from the data's. In their study while other supervised machine learning algorithms have shown 60% success, Bayesian networks were 76.9% successful. In this regard one of studies in Turkey is FutBa Model based on expert knowledge is developed by Karabiyik and Yet in 2019. The model has estimated with success rates in the range $60-70\%$. If the studies concentrated on supervised machine learning is viewed, Hucalijuk and Rakipovic's study draw attention. They modeled UEFA Champions League games and they made estimation in the range of $50-60\%$. Secondly, Yezus used k- nearest neighbor and random forest algorithms in 2014 to predict English Premier League results. The models have shown 55.8% and 68.4% estimation success. The same year Ulmer and Fernandez modeled English Premier League in their study. Linear classification model 48%, support vector machines 50%, random forest 50% were successful. Vaidya, Sanghavi et.al. modeled English Premier League for 2006-2010 seasons with logistic regression, random forest and naive Bayes algorithms. The accuracy of these models was 49.37%, 47.11%, 47.11% respectively. These are some examples about soccer from literature, nevertheless there is no consensus could be reached on the success of the models. In this study, sport events were modeled with five supervised machine learning algorithms which are decision tree, k-nearest neighbor, naive Bayes, support vector machine. After the test, it has been observed that support vector machine with 61.4% have the best performance. Consequently, it is not possible to talk about the superiority of any algorithm over another.

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Curve Couples of Bézier Curves in Euclidean $2-$Space

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Abstract

The goal of this paper is to characterize the evolute, involute and parallel curves of a Bezier curve which is applicable to computer graphics and related subjects. Especially, these curve couples are investigated at the endpoints. Moreover, the curvatures of these curve couples are given.

1. Introduction

Geometry of curves is very essential because it has many important applications in many different areas. Therefore, various curves and surfaces have been studied by many authors for many years. Recently, due to its different structure, Bézier curves have attracted the attention of many researchers. Bézier curves are introduced firstly by Pierre Bézier in 1968. Bézier curves are the most important mathematical representations of curves which are applied to computer graphics and related areas.
C. Huygens, who is also known for his studies in optics, investigated the concepts of evolutes and involutes [1]. In classical differential geometry, the evolute of a regular curve in the Euclidean plane is given by not only the locus of all its centres of the curvature, but also the envelope of normal lines of the regular curve, namely, the locus of singular loci of parallel curves. On the other side, the involute of a regular curve is to replace the taut string by a line segment that is tangent to the curve on one end, while the other end traces out the involute. Two curves are said to be parallel of one another if any curve normal to one is normal to the other. Kılıçoğlu and Şenyurt studied the involute of the cubic Bézier curve in Euclidean $3-$space [2]. In [3], the evolute-involute curve couples of Bézier curves in Euclidean $3-$space are investigated. In this study, curve couples of Bézier curves are examined in the Euclidean $2-$ space in which the Bézier curve couples need not to be unit speed and suitable for giving examples.
The rest part of the paper is given as follows: Section 2 gives some basic notations and definitions for needed throughout the study. Section 3 gives the Serret-Frenet frame of a planar Bézier curve. Section 4 characterizes evolute curve of a planar Bézier curve and investigate at end points. Moreover, the Frenet apparatus of this curve couple is given. Section 5 characterizes involute curve of a planar Bézier curve and investigate at end points. In addition, the Frenet apparatus of this curve couple is handled. Section 6 constructs the parallel curve of a planar Bézier curve. Especially, the Frenet apparatus of this curve couple is given. In the final section, we conclude our work and talk about our future works.

2. Preliminaries

A classical Bézier curve of degree m with control points p_j is defined as

$$B(t)=\sum_{j=0}^{m} p_j B_j^m(t), t\in[0,1] \tag{2.1}$$

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where

$$B_{i,n}(t)=\begin{cases}\frac{n!}{(n-i)!i!}(1-t)^{n-i}t^{i}, & \text{if } 0\leq i\leq n\\ 0, & \text{otherwise}\end{cases}$$

are called the Bernstein basis functions of degree m. The polygon formed by joining the control points $p_0,p_1,...,p_m$ in the specified order is called the Bézier control polygon.
If a curve is differentiable at its each point in an open interval, in this case a set of orthogonal unit vectors can be obtained. And these unit vectors are called Frenet frame. The rates of these frame vectors along the curve define curvatures of the curves. The set of these vectors and curvatures of a curve, is called Frenet apparatus of the curve.

Definition 2.1. *The first derivative $B'(t)$ of a degree-n Bézier curve $B(t)$ is clearly a degree $m-1$ curve. Such a curve can be written in Bézier form as*

$$B'(t)=m\sum_{i=0}^{m-1}\triangle p_iB_i^{m-1}(t)$$

where $\triangle p_i=p_{i+1}-p_i$, $i=0,1,...,m-1$ are the control points of $B^{'}(t)$[4].

Definition 2.2. *$J:E^2\to E^2$ is a linear transformation which is defined by the following equation*

$$J(P_1,P_2)=(-P_2,P_1)[5].$$

Definition 2.3. *Let $\beta:I\to E^2$ be a non-unit speed planar curve. The Serret-Frenet frame $\{T(t),N(t)\}$ and curvature $\kappa(t)$ of $\beta(t)$ for $\forall t\in I$ are defined by the following equations [5]:*

$$T(t)=\frac{\beta'(t)}{\|\beta'(t)\|},N(t)=\frac{J\beta'(t)}{\|\beta'(t)\|},\kappa(t)=\frac{<\beta''(t),J\beta'(t)>}{\|\beta'(t)\|^3}. \tag{2.2}$$

Definition 2.4. *For a plane regular curve $\beta(t)$ with $\kappa\neq 0$, the central curve*

$$\beta^*(t)=\beta(t)+\frac{1}{\kappa(t)}N(t) \tag{2.3}$$

where N is the normal of the curve β is called the evolute of β [6].

Definition 2.5. *For a plane regular curve $\beta(t)$ with $\kappa(t)\neq 0$, $t\in[t_1,t_2]$ and $a\in(t_1,t_2)$*

$$\beta^*(t)=\beta(t)-\frac{\beta'(t)}{\|\beta'(t)\|}\int_a^t\|\beta'(w)\|dw \tag{2.4}$$

is called the involute of β [7].

Definition 2.6. *The parallel at an oriented distance c to the left of a regular curve $\beta(t)$ is defined by the following equation*

$$\beta^*(t)=\beta(t)+cN(t) \tag{2.5}$$

[7].

For further information on curve couples see [6]-[8].

From now on, we will say Bézier curve instead of a non-unit speed planar Bézier curve of degree m throughout the paper.

3. The Serret-Frenet frame of a planar Bézier curve

In this section, the Serret-Frenet frame and curvature of a Bézier curve is given.

Theorem 3.1. *A Bézier curve with control points $p_0,p_1,...,p_m$ has the following Serret-Frenet frame $\{T(t),N(t)\}$ and curvature $\kappa(t)$ of Bézier curve with control points $p_0,p_1,...,p_m$ defined by (2.1) for $\forall t\in\mathbb{R}$ are*

$$T(t)=\frac{\sum\limits_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j}{(\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{1}{2}}}, \tag{3.1}$$

$$N(t)=\frac{\sum\limits_{j=0}^{m-1}B_j^{m-1}(t)J\triangle p_j}{(\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{1}{2}}} \tag{3.2}$$

and

$$\kappa(t)=\frac{m-1}{m}\frac{\sum\limits_{j=0}^{m-2}B_j^{m-2}(t)\sum\limits_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>}{(\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{3}{2}}} \tag{3.3}$$

where $\triangle p_j = p_{j+1}-p_j$ *and* $\triangle^2 p_j = p_{j+2}-2p_{j+1}+p_j$ *[9].*

4. Evolute of a planar Bézier curve

In this section, we characterize evolute curve of a planar Bézier curve and give its curvature. Moreover, we investigate this curve at $t=0$ and $t=1$.

Theorem 4.1. *The evolute* $B^*(t)$ *of a Bézier curve with control points* $p_0,p_1,...,p_m$ *defined by (2.1) for* $\forall t\in R$ *is*

$$B^*(t)=\sum_{j=0}^{m}p_jB_j^m(t)+\frac{m}{m-1}\frac{\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>\sum\limits_{j=0}^{m-1}B_j^{m-1}(t)J\triangle p_j}{\sum\limits_{j=0}^{m-2}B_j^{m-2}(t)\sum\limits_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>}. \tag{4.1}$$

Proof. Taking into consideration the equations (2.1), (3.2) and (3.3) in (2.3), it can be proved. □

Remark 4.2. *The evolute* $B^*(t)$ *of a Bézier curve which is defined by (2.1) with control points* $p_0,p_1,...,p_m$ *is*

$$B^*(0)=p_0+\frac{m}{m-1}\frac{\|\triangle p_0\|^2 J\triangle p_0}{<\triangle p_1,J\triangle p_0>}$$

at $t=0$.

Remark 4.3. *The evolute* $B^*(t)$ *of a Bézier curve which is defined by (2.1) with control points* $p_0,p_1,...,p_m$ *is*

$$B^*(1)=p_m+\frac{m}{m-1}\frac{\|\triangle p_{m-1}\|^2 J\triangle p_{m-1}}{<\triangle p_{m-1},J\triangle p_{m-2}>}$$

at $t=1$.

Theorem 4.4. *The curvature of evolute* $B^*(t)$ *of a Bézier curve with control points* $p_0,p_1,...,p_m$ *defined by (2.1) for* $\forall t\in R$ *is*

$$\kappa_*(t)=\varepsilon_\kappa\left|\frac{(m^2.(m-1)\sum\limits_{j=0}^{m-2}B_j^{m-2}(t)\sum\limits_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>)^3}{\begin{array}{c}m^3(\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{3}{2}}.(m^4(m-1)(m-2)\sum\limits_{j=0}^{m-3}B_j^{m-3}(t)\sum\limits_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^3 p_j,J\triangle p_i>)\\ (\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)-3m^7(m-1)^2(\sum\limits_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{3}{2}}\\ .[(\sum\limits_{j=0}^{m-1}B_j^{m-1}(t)\triangle_x p_j)(\sum\limits_{i=0}^{m-2}B_j^{m-2}(t)\triangle_x^2 p_i)+(\sum\limits_{j=0}^{m-1}B_j^{m-1}(t)\triangle_y p_j)(\sum\limits_{i=0}^{m-2}B_j^{m-2}(t)\triangle_y^2 p_i)]\\ .(\sum\limits_{j=0}^{m-2}B_j^{m-2}(t)\sum\limits_{i=0}^{m-1}B_i^{m-1}(t)<\triangle^2 p_j,J\triangle p_i>)\end{array}}\right|$$

where ε_κ *is the sign of the curvature of the Bézier curve,* $\triangle_x p_j=(p_{j+1})_x-(p_j)_x$, $\triangle_y p_j=(p_{j+1})_y-(p_j)_y$, $\triangle_x^2 p_j=(p_{j+2})_x-2(p_{j+1})_x+(p_j)_x$ *and* $\triangle_y^2 p_j=(p_{j+2})_y-2(p_{j+1})_y+(p_j)_y$.

Proof. Taking into consideration the equations (2.2) and (4.1) together, it can be proved. □

Example 4.5. *For given control points* $p_0=(0,0),p_1=(\frac{1}{2},0),p_2=(\frac{1}{2},\frac{1}{2})$, *we have the following quadratic planar Bézier curve* $B(t)$

$$B(t)=\sum_{j=0}^{2}p_jB_j^2(t) \tag{4.2}$$

and the evolute of $B(t)$ *is given by the following equation*

$$B^*(t)=(\frac{3}{2}t^2-2t^3,1-3t+\frac{9}{2}t^2-2t^3).$$

The tanget of $B(t)$ *at* $t=0$ *is* $T=(1,0)$ *and the tanget of* $B^*(t)$ *at* $t=0$ *is* $T^*=(0,1)$. *The tanget of* $B(t)$ *at* $t=1$ *is* $T=(0,1)$ *and the tanget of* $B^*(t)$ *at* $t=1$ *is* $T^*=(1,0)$. *Therefore, the tangets at the end points are perpendicular.*

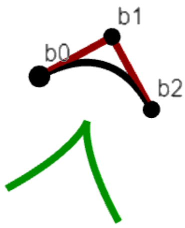

Figure 4.1: Bézier curve and the evolute couple are given by black and green color, respectively.

5. Involute of a planar Bézier curve

In this section, we characterize involute curve of a planar Bézier curve and give its curvature. In addition, we give an example.

Theorem 5.1. *The involute $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$B^*(t) = \sum_{i=0}^{m} p_i B_i^m(t) - \frac{\sum\limits_{i=0}^{m-1} \triangle p_i B_i^{m-1}(t)}{(\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}}} \int_a^t \|m \sum_{i=0}^{m-1} \triangle p_i B_i^{m-1}(w)\| dw. \tag{5.1}$$

Proof. Taking into consideration the equations (2.1) and (3.1) in (2.4), it can be proved. □

Theorem 5.2. *The curvature of involute $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$\kappa_*(t) = \frac{\varepsilon_\kappa}{m \int_a^t (\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(w) B_i^{m-1}(w) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}} dw}$$

where ε_κ is the sign of the curvature of the Bézier curve.

Proof. Taking into consideration the equations (2.2) and (5.1) together, it can be proved. □

Example 5.3. *The equation of involute couple of $B(t)$ which is given by (4.2)*

$$B^*(t) = (t - \frac{t^2}{2} - \frac{(1-t)[\sqrt{2} arcsinh(2t-1) + (4t-2)\sqrt{2t^2-2t+1}]}{4\sqrt{1-2t+2t^2}}, \frac{t^2}{2} - \frac{t[\sqrt{2} arcsinh(2t-1) + (4t-2)\sqrt{2t^2-2t+1}]}{4\sqrt{1-2t+2t^2}})$$

where $a = \frac{1}{2}$.

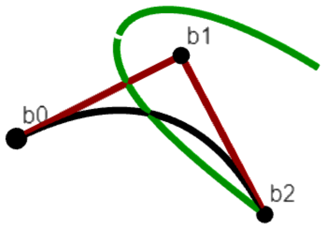

Figure 5.1: Bézier curve and the involute couple are given by black and green color, respectively.

6. Parallel of a planar Bézier curve

In this section, we characterize parallel curve of a planar Bézier curve and give its curvature. Moreover, we investigate this curve at $t = 0$ and $t = 1$.

Theorem 6.1. *The parallel $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$B^*(t) = \sum_{j=0}^{m} p_j B_j^m(t) + \frac{c \sum\limits_{j=0}^{m-1} B_j^{m-1}(t) J \triangle p_j}{(\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{1}{2}}} \tag{6.1}$$

where c is a constant.

Proof. Taking into consideration the equations (2.1) and (3.2) in (2.5), it can be proved. □

Remark 6.2. *The parallel $B^*(t)$ of a Bézier curve which is defined by (2.1) with control points $p_0, p_1, ..., p_m$ is*

$$B^*(0) = p_0 + \frac{cJ\triangle p_0}{\|\triangle p_0\|}$$

at $t = 0$.

Remark 6.3. *The parallel $B^*(t)$ of a Bézier curve which is defined by (2.1) with control points $p_0, p_1, ..., p_m$ is*

$$B^*(1) = p_m + \frac{cJ\triangle p_{m-1}}{\|\triangle p_{m-1}\|}$$

at $t = 1$.

Theorem 6.4. *The curvature κ_* of parallel curve $B^*(t)$ of a Bézier curve with control points $p_0, p_1, ..., p_m$ defined by (2.1) for $\forall t \in R$ is*

$$\kappa_*(t) = \frac{(m-1)\sum\limits_{j=0}^{m-2} B_j^{m-2}(t) \sum\limits_{i=0}^{m-1} B_i^{m-1}(t) < \triangle^2 p_j, J\triangle p_i >}{m(\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(t) B_i^{m-1}(t) < \triangle p_j, \triangle p_i >)^{\frac{3}{2}} - c(m-1)\sum\limits_{j=0}^{m-2} B_j^{m-2}(t) \sum\limits_{i=0}^{m-1} B_i^{m-1}(t) < \triangle^2 p_j, J\triangle p_i >}.$$

Proof. Taking into consideration the equations (2.2) and (6.1) together, it can be proved. □

Example 6.5. *The equation of parallel couple of $B(t)$ which is given by (4.2)*

$$B^*(t) = (t - \frac{t^2}{2} + \frac{t}{\sqrt{1-2t+2t^2}}, \frac{t^2}{2} - \frac{t}{\sqrt{1-2t+2t^2}})$$

where $c = -1$.

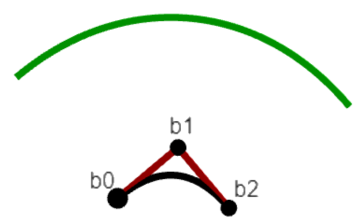

Figure 6.1: Bézier curve and the parallel couple are given by black and green color, respectively.

7. Conclusion

In this paper evolute, involute and parallel curves of a Bézier curve are characterized and investigated at the beginning and the ending points. In addition, these curve couples curvatures are obtained. In our future work, we will study the other curve couples of Bézier curve.

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The Super-Connectivity of the Double Vertex Graph of Complete Bipartite Graphs

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Abstract

Let $G=(V,E)$ be a graph. The double vertex graph $F_2(G)$ of G is the graph whose vertex set consists of all 2-subsets of $V(G)$ such that two vertices are adjacent in $F_2(G)$ if their symmetric difference is a pair of adjacent vertices in G. The super–connectivity of a connected graph is the minimum number of vertices whose removal results in a disconnected graph without an isolated vertex. In this paper, we determine the super–connectivity of the double vertex graph of the complete bipartite graph $K_{m,n}$ for $m \geq 4$ where $n \geq m+2$.

1. Introduction

Throughout this paper, let G be a simple finite graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges, respectively. A set $S \subset V(G)$ is a vertex–cut of G, if $G-S$ is disconnected or has only one vertex. The neighbourhood of a vertex v is the set $N_G(v)=\{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by $\deg_G(v)$, is the cardinality of $N_G(v)$. Let $\delta(G)$ denote the minimum vertex degree in G. Two paths are internally disjoint if they have no common vertex except the end vertices. A set of paths is called internally disjoint if these paths are pairwise internally disjoint.

The *double vertex graph* $F_2(G)$ of G is the graph whose vertex set consists of all the 2-subsets of $V(G)$ and two vertices are adjacent in $F_2(G)$ if their symmetric difference is a pair of adjacent vertices in G. That is, the vertices $\{u,v\}$ and $\{x,y\}$ of $F_2(G)$ are adjacent if and only if $|\{u,v\} \cap \{x,y\}|=1$ with $u=x$ and $vy \in E(G)$ (See Fig 1.1 for an example).

The notion of double vertex graph was introduced and studied by Alavi *et al.* [1]-[3]. The same concept was used by Rudolph to study the graph isomorphism problem under the name of symmetric power of a graph [4]. Later, Rudolph *et al.* [5] defined *symmetric k^{th} power* of a graph G as a generalization of symmetric power. In 2012, Fabila-Monroy *et al.* [6] introduced the notion of k–token graphs, which was a redefinition of symmetric k^{th} powers of graphs. The *k–token graph* $F_k(G)$ of G (or, symmetric k^{th} power of a graph G) is the graph whose vertices are all k–subsets of $V(G)$, where two vertices are adjacent if their symmetric difference is an edge in $E(G)$. Obviously, double vertex graphs correspond to 2–token graphs.

Note that if G is a connected graph, then its double vertex graph is bipartite if and only if G is bipartite. Also note that the degree of a vertex $\omega=\{x,y\}$ in $F_2(G)$ is given by

$$deg_{F_2(G)}\omega = \begin{cases} deg_G(x)+deg_G(y), & \text{if } xy \notin E(G), \\ deg_G(x)+deg_G(y)-2, & \text{if } xy \in E(G). \end{cases}$$

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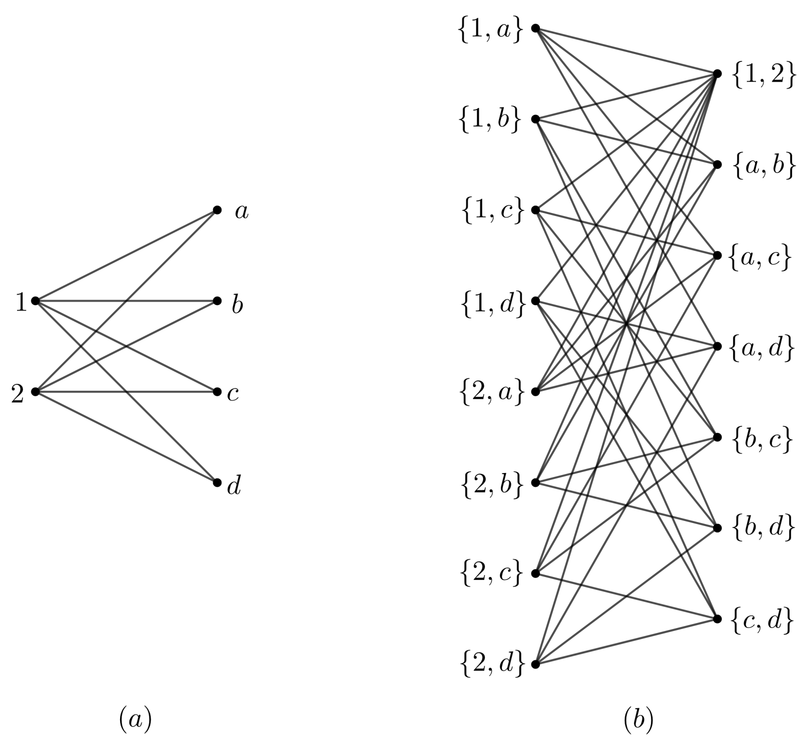

Figure 1.1: (a) Complete bipartite graph $K_{2,4}$ (b) Double vertex graph of $K_{2,4}$

Token graphs have been extensively studied especially in terms of the combinatorial parameters such as connectivity, diameter, cliques, chromatic number, Hamiltonian paths and Cartesian product (see [7]-[14] and the references therein).

The connectivity, $\kappa(G)$, of a graph G is the minimum number of vertices whose removal from G results in a disconnected graph or an isolated vertex. It is an important factor to determine the fault–tolerance of a network. In 1983, Harary introduced conditional connectivity as a generalization of the classical connectivity concept by imposing some conditions on the remaining graph. Let G be a connected graph, and let P be a given graph-theoretical property. The conditional connectivity of a graph G is the size of a minimum vertex–cut S of G (if it exists), where $G-S$ is disconnected and every component of $G-S$ has the property P [15]. Motivated by this definition, various types of conditional connectivity have been extensively studied in literature. The case when the condition is that the remaining graph does not have an isolated vertex corresponds to the super-connectivity notion.

The super–connectivity, $\kappa'(G)$, of a graph G is the size of a minimum vertex–cut S such that the resulting graph $G-S$ has no isolated vertices. If such a vertex–cut exists, it is referred to as a super vertex–cut; otherwise we write $\kappa'(G) = +\infty$. The super–connectivity has been studied for various families of graphs, including circulant graphs [16], hypercubes [17, 18], product graphs [19]-[21].

Considering the connectivity aspect of token graphs, it is known that if G is a k–connected graph, then $F_2(G)$ is $(2k-2)$–connected, where $k \geq 3$ [3]. In 2012, Fabila-Monroy *et al.* [6] presented several families of graphs of order n which are t–connected and have k–token graphs with connectivity exactly $k(t-k+1)$ whenever $k \leq t$. They also conjectured that $F_k(G)$ is at least $k(t-k+1)$–connected for all $k \leq t$. In 2018, Leaños and Trujillo-Negrete [22] proved that their conjecture is true. In [23], Leaños and Ndjatchi proved an analogous result for edge connectivity; they showed that if G is t–edge connected for $t \geq k$, then $F_k(G)$ is at least $k(t-k+1)$–edge connected. Later Fabila-Monroy *et al.* [24] proved that if G is a tree, then the connectivity of $F_k(G)$ is equal to the minimum degree of $F_k(G)$. Although the connectivity of k–token graphs has been studied in several papers, super–connectivity of this class has not yet been investigated. Recently, we fully determined the super–connectivity of Johnson graphs, which corresponds to a special case of k–token graphs [25]. More precisely, if G is the complete graph on n vertices, then k–token graph corresponds to the Johnson graph $J(n,k)$. In this paper, we continue to investigate token graphs by determining the super–connectivity of 2–token graph of complete bipartite graphs.

In the rest of the paper, a vertex ω of $F_2(G)$ corresponding to the 2-subset $\{x,y\} \in V(F_2(G))$ will be denoted by $\omega = xy$. While constructing the paths, it is assumed that the subscripts of the vertices are taken modulo n or m, depending on the size of the set we consider.

2. Main results

Let $K_{m,n}$ be the complete bipartite graph with partition $V = A \cup B$ such that $A = \{x_1, \ldots, x_m\}$ and $B = \{y_1, \ldots, y_n\}$, where $m \le n$. Letting $\mathscr{G} = F_2(K_{m,n})$, we have a bipartite graph $\mathscr{G}$ with partition $V(\mathscr{G}) = \mathscr{A} \cup \mathscr{B}$ such that

$$\mathscr{A} = \{x_i y_j \in V(\mathscr{G}) : x_i \in A \text{ and } y_j \in B\} \qquad \text{and} \qquad \mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2,$$

where

$$\mathscr{B}_1 = \{x_i x_j \in V(\mathscr{G}) : i \neq j\} \qquad \text{and} \qquad \mathscr{B}_2 = \{y_i y_j \in V(\mathscr{G}) : i \neq j\}.$$

It is easy to see that $\delta(\mathscr{G}) = \min\{2m, 2n, m+n-2\}$. Since $\kappa(K_{m,n}) = m$ when $m \le n$, we know that the graph $\mathscr{G}$ is $(2m-2)$–connected for $n \ge m \ge 2$. We know that the connectivity of a graph is at most the minimum degree of it. Thus, we have $2m-2 \le \kappa(\mathscr{G}) \le 2m$ when $n \ge m+2$. Moreover, if $m = n$, then $\kappa(\mathscr{G}) = 2m-2$ and if $m = n-1$, then $2m-2 \le \kappa(\mathscr{G}) \le 2m-1$. It is quite natural to ask whether every minimum vertex–cut of a graph G corresponds to the neighbourhood of a vertex. If the answer is yes, then every vertex–cut isolates a vertex in G and thus the super–connectivity of G is strictly greater than the connectivity.

Both the Remark 2.1 and the explanation before it are given in [25]. Although it is easy to observe, it plays an important role in the proof of our main result.

Let S be a minimum super vertex–cut S of a connected graph G. Note that S contains a vertex v having at least one neighbour in the resulting graph $G-S$ for otherwise G would be disconnected. Let C be a component of $G-S$ and suppose that v does not have a neighbour in C. Now consider the set $T = S - \{v\}$. Since C is a component of $G-T$, it is obvious that T is a vertex–cut of G which does not isolate a vertex. Thus, T is a super vertex–cut of G and this contradicts the minimality of S. Hence, the remark below follows.

Remark 2.1. *[25] Let G be a connected graph. A minimum super vertex–cut S of G contains a vertex having at least a neighbour in every component of $G-S$. Moreover, if a vertex v in a minimum super vertex–cut S of G has a neighbour in one component of $G-S$, then it has at least one neighbour in every component of $G-S$.*

We now prove our main result on the super–connectivity of the double vertex graph of complete bipartite graphs.

Theorem 2.2. *Let $\mathscr{G}$ be the double vertex graph of the complete bipartite graph $K_{m,n}$, where $n \ge m+2$ and $m \ge 4$. Then $\kappa'(\mathscr{G}) = 3m+n-4$.*

Proof. Let S be a super vertex–cut of $\mathscr{G} = F_2(K_{m,n})$ where $n \ge m+2$ and $m \ge 4$. By Remark 2.1, we know that there exists a vertex, say ω, in S having at least a neighbour in every component of $\mathscr{G} - S$. Let C_1 and C_2 be two components of $\mathscr{G} - S$. Consider a neighbour of ω from each of the components C_1 and C_2, say $u_1 \in C_1$ and $u_2 \in C_2$. Since S is a super vertex–cut, each component of the resulting graph $\mathscr{G} - S$ has at least two vertices. Thus, each of u_1 and u_2 has at least a neighbour in C_1 and C_2, respectively. Let $v_1 \in C_1$ and $v_2 \in C_2$ such that $v_1 \in N_{\mathscr{G}}(u_1)$ and $v_2 \in N_{\mathscr{G}}(u_2)$. Note that the intersection $v_1 \cap u_2 = \emptyset$, otherwise there will be an edge between the components C_1 and C_2. Similarly, $u_1 \cap v_2 = \emptyset$. Since $\mathscr{G}$ is a bipartite graph, ω is either in $\mathscr{A}$ or in $\mathscr{B}$.

First, we suppose that ω is in $\mathscr{A}$. Without loss of generality, let $\omega = x_1 y_1$. For the vertices u_1 and u_2, there are three cases to consider:

(1) Both of u_1 and u_2 are in $\mathscr{B}_1$,

(2) One of them is in $\mathscr{B}_1$ and the other one is in $\mathscr{B}_2$,

(3) Both of u_1 and u_2 are in $\mathscr{B}_2$.

Next, we suppose that ω is in $\mathscr{B}$. Then, either $\omega \in \mathscr{B}_1$ or $\omega \in \mathscr{B}_2$. In both of these two cases, we get the same subcases for the vertices $u_1, v_1 \in C_1$ and $u_2, v_2 \in C_2$. Thus, it is enough to consider only one of them, say $\omega \in \mathscr{B}_1$. Without loss of generality, we assume that $\omega = x_1 x_2$. Consider the neighbours of ω in the resulting graph $\mathscr{G} - S$, in particular $u_1 \in C_1$ and $u_2 \in C_2$. Due to the shared index of u_1 and u_2, there are three cases to consider:

(4) $u_1 \cap u_2 \subset A$,

(5) $u_1 \cap u_2 \subset B$,

(6) $u_1 \cap u_2 = \emptyset$.

Let us assume that $\omega = x_1 y_1$ and consider the first three cases (1-3) given below.

Case 1. Without loss of generality, assume that $u_1 = x_1 x_2$ and $u_2 = x_1 x_3$. Since we have $v_1 \cap u_2 = \emptyset$ and $u_1 \cap v_2 = \emptyset$, we let $v_1 = x_2 y_k$ and $v_2 = x_3 y_\ell$. Without loss of generality, we assume that $k = 1$. Thus, we have either $\ell = k$ or $\ell \neq k$. In the latter case we let, without loss of generality, $\ell = 2$.

First we investigate the common paths that can be constructed when either $\ell = k$ or $\ell \neq k$.

- $u_1 \sim x_1y_j \sim u_2$ for all $j \in \{1,\dots,n\}$
- $v_1 \sim x_2x_3 \sim v_2$
- $v_1 \sim x_2x_j \sim y_2x_j \sim x_3x_j \sim v_2$ for all $j \in \{4,\dots,m\}$

Note that if $\ell = k$, then the vertices v_1 and v_2 have common neighbours, and the additional paths that can be constructed particularly in this case are given in (a). Similarly, the additional paths constructed only when $\ell = 2$ are given in (b).

(a) If $\ell = k$, then consider the extra paths given below:
- $v_1 \sim y_1y_j \sim v_2$ for all $j \in \{2,\dots,n\}$
- $u_1 \sim x_2y_j \sim y_jy_{j+1} \sim x_3y_j \sim u_2$ for all $j \in \{2,\dots,n\}$
  When $j = n$, use the vertex y_jy_{j+2} instead of y_jy_{j+1} since y_1y_n is used already.

(b) If $\ell \neq k$ (note that ℓ is assumed to be 2 above), then consider the extra paths given below:
- $u_1 \sim x_2y_2 \sim y_2y_3 \sim v_2$ and $u_1 \sim x_2y_3 \sim y_3y_4 \sim x_3y_3 \sim u_2$
- $v_1 \sim y_1y_2 \sim v_2$ and $v_1 \sim y_1y_3 \sim x_3y_1 \sim u_2$
- $v_1 \sim y_1y_j \sim x_3y_j \sim u_2$ for all $j \in \{4,\dots,n\}$
- $u_1 \sim x_2y_j \sim y_2y_j \sim v_2$ for all $j \in \{4,\dots,n\}$

Thus, in both cases, we have constructed $3n+m-4$ internally disjoint paths.

Case 2. Without loss of generality, we let $u_1 = x_1x_2$ and $u_2 = y_1y_2$. Since we have $v_1 \cap u_2 = \emptyset$ and $u_1 \cap v_2 = \emptyset$, we assume that $v_1 = x_1y_3$ and $v_2 = x_3y_1$. Consider the following paths:
- $u_1 \sim x_1y_i \sim y_2y_i \sim x_iy_2 \sim u_2$ for all $i \in \{4,\dots,m\}$
- $u_1 \sim x_2y_j \sim y_1y_j \sim v_2$ for all $j \in \{4,\dots,n\}$
- $v_1 \sim x_1x_i \sim x_iy_1 \sim u_2$ for all $i \in \{4,\dots,m\}$
- $v_1 \sim y_3y_i \sim x_iy_3 \sim x_3x_i \sim v_2$ for all $i \in \{4,\dots,m\}$
- $u_1 \sim x_2y_3 \sim x_2x_3 \sim v_2$ and $v_1 \sim y_2y_3 \sim x_3y_2 \sim u_2$
- $u_1 \sim a \sim u_2$ for each $a \in \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$
- $v_1 \sim x_1x_3 \sim v_2$ and $v_1 \sim y_1y_3 \sim v_2$

Thus, we have constructed $3m+n-4$ internally disjoint paths.

Case 3. Without loss of generality, we let $u_1 = y_1y_2$ and $u_2 = y_1y_3$. Since we have $v_1 \cap u_2 = \emptyset$ and $u_1 \cap v_2 = \emptyset$, we let $v_1 = x_ky_2$ and $v_2 = x_\ell y_3$. Without loss of generality, we assume that $k = 1$. Thus, we have either have $\ell = k$ or $\ell \neq k$. In the latter case we let, without loss of generality, $\ell = 2$.

First we investigate the common paths that can be constructed when either $\ell = k$ or $\ell \neq k$

- $u_1 \sim x_iy_1 \sim u_2$ for all $i \in \{1,\dots,m\}$
- $v_1 \sim y_2y_3 \sim v_2$
- $v_1 \sim y_2y_i \sim x_1y_i \sim y_3y_i \sim v_2$ for all $i \in \{4,\dots,n\}$

Note that if $\ell = k$, then the vertices v_1 and v_2 have common neighbours, and the additional paths that can be constructed particularly in this case are given in (a). Similarly, the additional paths constructed only when $\ell = 2$ are given in (b).

(a) If $\ell = k$, then consider the extra paths given below:

- $v_1 \sim x_1x_i \sim v_2$ for all $i \in \{2,\dots,m\}$
- $u_1 \sim x_iy_2 \sim x_ix_{i+1} \sim x_iy_3 \sim u_2$ for all $i \in \{2,\dots,m\}$.
  When $i = m$, use the vertex x_ix_{i+2} instead of x_ix_{i+1} since x_1x_m is used already.

(b) If $\ell \neq k$ (ℓ is assumed to be 2 above), then consider the extra paths given below:

- $u_1 \sim x_2y_2 \sim x_2x_3 \sim v_2$ and $v_1 \sim x_1x_3 \sim x_1y_3 \sim u_2$
- $u_1 \sim x_3y_2 \sim x_3x_4 \sim x_3y_3 \sim u_2$
- $u_1 \sim x_4y_2 \sim x_2x_4 \sim v_2$ and $v_1 \sim x_1x_4 \sim x_4y_3 \sim u_2$
- $v_1 \sim x_1x_2 \sim v_2$
- $u_1 \sim x_iy_2 \sim x_{i-1}x_i \sim x_iy_3 \sim u_2$ for all $i \in \{5,\dots,m\}$

- $v_1 \sim x_1x_i \sim x_iy_4 \sim x_2x_i \sim v_2$ for all $i \in \{5,\dots,m\}$

Thus, in both cases, we have constructed $3m+n-4$ internally disjoint paths.

Now we assume that $\omega = x_1x_2$ in order to consider the latter three cases (4-6) given below.

Case 4. Let $u_1 \cap u_2 \subset A$, say $u_1 \cap u_2 = \{x_1\}$. Since both of $u_1,u_2 \in \mathscr{A}$, without of loss of generality, we assume that $u_1 = x_1y_1$ and $u_2 = x_1y_2$. Since we have $v_1 \cap u_2 = \emptyset$ and $v_2 \cap u_1 = \emptyset$, we have either $|v_1 \cap v_2| = 1$ or $|v_1 \cap v_2| = 0$. Let $v_1 = y_1y_k$ and $v_2 = y_2y_\ell$. Note that $k,\ell \notin \{1,2\}$. Thus, without loss of generality, we let $k = 3$.

(*a*) If $\ell = k$, then the paths here can be constructed similarly as in Case 3(a), such that the vertices $\{x_1,y_1,y_2,y_3\}$ of this case correspond to the vertices $\{x_1,y_2,y_3,y_1\}$ of Case 3(a), respectively.

(*b*) If $\ell \neq k$, then we have $\ell \notin \{1,2,3\}$. Thus, without loss of generality, we let $\ell = 4$.

Consider the following paths:

- $u_1 \sim x_1x_i \sim u_2$ for all $i \in \{2,\dots,m\}$
- $u_1 \sim y_1y_2 \sim u_2$
- $u_1 \sim y_1y_4 \sim x_1y_4 \sim v_2$
- $v_1 \sim x_1y_3 \sim y_2y_3 \sim u_2$
- $u_1 \sim y_1y_i \sim x_1y_i \sim y_2y_i \sim u_2$ for all $i \in \{5,\dots,n\}$
- $v_1 \sim x_iy_1 \sim x_ix_{i+1} \sim x_iy_2 \sim v_2$ for all $i \in \{2,\dots,m\}$
  When $i = m$, use the vertex x_ix_{i+2} instead of x_ix_{i+1} since x_1x_m is used already.
- $v_1 \sim x_2y_3 \sim y_3y_4 \sim x_2y_4 \sim v_2$
- $v_1 \sim x_iy_3 \sim y_3y_{i+2} \sim x_2y_{i+2} \sim y_4y_{i+2} \sim x_iy_4 \sim v_2$ for all $i \in \{3,\dots,m\}$

Thus, in both of the cases, we have constructed $3m+n-4$ internally disjoint paths.

Case 5. Let $u_1 \cap u_2 \subset B$, say $u_1 \cap u_2 = \{y_1\}$. Since both of $u_1,u_2 \in \mathscr{A}$, without of loss of generality, we assume that $u_1 = x_1y_1$ and $u_2 = x_2y_1$. Since $v_1 \cap u_2 = \emptyset$ and $v_2 \cap u_1 = \emptyset$, we have either $|v_1 \cap v_2| = 1$ or $|v_1 \cap v_2| = 0$. Let $v_1 = x_1x_k$ and $v_2 = x_2x_\ell$. Note that $k,\ell \notin \{1,2\}$. Thus, without loss of generality, we let $k = 3$.

(*a*) If $\ell = k$ then the paths here can be constructed similarly as in Case 1(a), such that the vertices $\{y_1,x_1,x_2,x_3\}$ of this case correspond to the vertices $\{y_1,x_2,x_3,x_1\}$ of Case 1(a), respectively.

(*b*) If $\ell \neq k$, then we have $\ell \notin \{1,2,3\}$. Thus, without loss of generality, we let $\ell = 4$.

Consider the following paths:

- $u_1 \sim x_1x_2 \sim u_2$
- $u_1 \sim y_1y_i \sim u_2$ for all $i \in \{2,\dots,n\}$
- $u_1 \sim x_1x_4 \sim x_4y_m \sim v_2$ and $v_1 \sim x_3y_m \sim x_2x_3 \sim u_2$
- $u_1 \sim x_1x_i \sim x_iy_1 \sim x_2x_i \sim u_2$ for all $i \in \{5,\dots,m\}$
- $v_1 \sim x_3y_1 \sim x_3x_4 \sim x_4y_1 \sim v_2$
- $v_1 \sim x_1y_i \sim y_iy_{i+1} \sim x_2y_i \sim v_2$ for all $i \in \{2,\dots,n\}$
  When $i = n$, use the vertex y_iy_{i+2} instead of y_iy_{i+1} since y_1y_n is used already.
- $v_1 \sim x_3y_i \sim y_iy_{i+2} \sim x_4y_i \sim v_2$ for all $i \in \{2,\dots,n-1\}$
  When $i = n-1$, use the vertex y_iy_{i+3} instead of y_iy_{i+2} since y_1y_{n-1} is used already.

Thus, in both of the cases, we have constructed $3n+m-4$ internally disjoint paths.

Case 6. Let $u_1 \cap u_2 = \emptyset$. Since both of $u_1,u_2 \in \mathscr{A}$, the vertices v_1 and v_2 are in $\mathscr{B}$. There are three subcases to consider: (a) Both of v_1,v_2 are in $\mathscr{B}_1$, (b) One of v_1,v_2 is in $\mathscr{B}_1$ and the other one is in $\mathscr{B}_2$, (c) Both of v_1,v_2 are in $\mathscr{B}_2$.

First, without loss of generality, we let $u_1 = x_1y_1$ and $u_2 = x_2y_2$.

(*a*) Assume that $v_1,v_2 \in \mathscr{B}_1$. Since $v_1 \cap u_2 = \emptyset$ and $v_2 \cap u_1 = \emptyset$, we have either $|v_1 \cap v_2| = 1$ or $|v_1 \cap v_2| = 0$. Let $v_1 = x_1x_k$ and $v_2 = x_2x_\ell$. Note that we have $k,\ell \notin \{1,2\}$. Thus, without loss of generality, we let $k = 3$.

(*i*) If $\ell = k$, then the paths here can be constructed similarly as in Case 1(b), such that the vertices $\{x_1,x_2,x_3,y_1,y_2\}$ of this case correspond to the vertices $\{x_2,x_3,x_1,y_1,y_2\}$ of Case 1(b), respectively.

(*ii*) If $\ell \neq k$, then we have $\ell \notin \{1,2,3\}$. Thus, without loss of generality, let $\ell = 4$.
Consider the following paths:

- $u_1 \sim x_1x_2 \sim u_2$ and $u_1 \sim y_1y_2 \sim u_2$
- $u_1 \sim x_1x_4 \sim x_4y_1 \sim v_2$ and $v_1 \sim x_3y_1 \sim x_2x_3 \sim u_2$
- $u_1 \sim x_1x_i \sim x_iy_1 \sim x_2x_i \sim u_2$ for all $i \in \{5,\ldots,m\}$
- $u_1 \sim y_1y_3 \sim x_2y_1 \sim v_2$ and $v_1 \sim x_1y_2 \sim y_2y_3 \sim u_2$
- $u_1 \sim y_1y_i \sim x_4y_i \sim v_2$ for all $i \in \{4,\ldots,n\}$
- $v_1 \sim x_3y_i \sim y_2y_i \sim u_2$ for all $i \in \{4,\ldots,n\}$
- $v_1 \sim x_1y_i \sim y_iy_{i+1} \sim x_2y_i \sim v_2$ for all $i \in \{3,\ldots,n\}$
  When $i = n$, use the vertex y_iy_{i+3} instead of y_iy_{i+1} since y_1y_n is used already.
- $v_1 \sim x_3y_2 \sim x_3x_4 \sim x_4y_2 \sim v_2$
- $v_1 \sim x_3y_3 \sim y_3y_5 \sim x_4y_3 \sim v_2$

Thus, in both of the cases, we have constructed $3n+m-4$ internally disjoint paths.

(b) Assume that $v_1 \in \mathscr{B}_1$ and $v_2 \in \mathscr{B}_2$. Since $v_1 \cap u_2 = \emptyset$ and $v_2 \cap u_1 = \emptyset$, we let $v_1 = x_1x_3$ and $v_2 = y_2y_3$. The paths here can be constructed similarly as in Case 2, such that the vertices $\{x_1,x_2,x_3,y_1,y_2,y_3\}$ of this case correspond to the vertices $\{x_1,x_3,x_2,y_3,y_1,y_2\}$ of Case 2, respectively. Thus, we can construct $3m+n-4$ internally disjoint paths.

(c) Assume that $v_1,v_2 \in \mathscr{B}_2$. Since $v_1 \cap u_2 = \emptyset$ and $v_2 \cap u_1 = \emptyset$, we have either $|v_1 \cap v_2| = 1$ or $|v_1 \cap v_2| = 0$. Let $v_1 = y_1y_k$. Note that $k \notin \{1,2\}$. Thus, without loss of generality, we let $k = 3$.

Since $v_2 \in \mathscr{B}_2$ by the assumption, we have $v_2 = y_2y_\ell$ such that $\ell \notin \{1,2\}$. Then we have either $\ell = k$ or $\ell \neq k$.

(i) If $\ell = k$, then the paths here can be constructed similarly as in Case $3(b)$, such that the vertices $\{x_1,x_2,y_1,y_2,y_3\}$ of this case correspond to the vertices $\{x_1,x_2,y_2,y_3,y_1\}$ of Case $3(b)$, respectively.

(ii) If $\ell \neq k$, then we have $\ell \notin \{1,2,3\}$. Thus, without loss of generality, we let $v_2 = y_2y_4$.
Consider the following paths:

- $u_1 \sim x_1x_2 \sim u_2$ and $u_1 \sim y_1y_2 \sim u_2$
- $u_1 \sim x_1x_3 \sim x_1y_2 \sim v_2$ and $v_1 \sim x_2y_1 \sim x_2x_3 \sim u_2$
- $u_1 \sim x_1x_i \sim x_iy_2 \sim v_2$ for all $i \in \{4,\ldots,m\}$
- $u_1 \sim y_1y_4 \sim x_1y_4 \sim v_2$ and $v_1 \sim x_1y_3 \sim y_2y_3 \sim u_2$
- $v_1 \sim x_2y_3 \sim y_3y_4 \sim x_2y_4 \sim v_2$ and $v_1 \sim x_3y_1 \sim x_3x_4 \sim x_3y_2 \sim v_2$
- $u_1 \sim y_1y_i \sim x_1y_i \sim y_2y_i \sim u_2$ for all $i \in \{5,\ldots,n\}$
- $v_1 \sim x_iy_3 \sim y_3y_{i+2} \sim x_2y_{i+2} \sim y_4y_{i+2} \sim x_iy_4 \sim v_2$ for all $i \in \{3,\ldots,m\}$
- $v_1 \sim x_iy_1 \sim x_2x_i \sim u_2$ for all $i \in \{4,\ldots,m\}$

Thus, in both cases, we have constructed $3m+n-4$ internally disjoint paths.

In each of the six cases above, we presented either $3m+n-4$ or $3n+m-4$ internally disjoint paths between C_1 and C_2. Since $m \leq n-2$ by the assumption, this implies that there exist at least $3m+n-4$ internally disjoint paths between C_1 and C_2. Thus, $\kappa'(\mathscr{G}) \geq 3m+n-4$.

On the other hand, consider two adjacent vertices α and β of $\mathscr{G}$ such that $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}_1$. Let $S = (N_{\mathscr{G}}(\alpha) \cup N_{\mathscr{G}}(\beta)) - \{\alpha,\beta\}$. It is easy to see that the set S disconnects the graph without isolating a vertex, that is, S is a super–vertex cut of $\mathscr{G}$. Hence, we get $\kappa'(\mathscr{G}) \leq |S| = 3m+n-4$ and this finishes the proof. □

3. Conclusion

In our main result, it is proved that the super–connectivity of the double vertex graph of complete bipartite graph $K_{m,n}$ is equal to $3m+n-4$ where $m \geq 4$ and $n \geq m+2$. This result also implies that the double vertex graph of complete bipartite graph $F_2(K_{m,n})$ is super–connected, i.e., each minimum vertex–cut of $F_2(K_{m,n})$ isolates a vertex. It would be interesting to determine the super–connectivity of k–token graphs for larger graph classes. Note that the well studied Johnson graph $J(n,k)$ is a special case of k–token graphs, where G is the complete graph K_n. In [25], we fully determined the super–connectivity of $J(n,k)$. Thus, the results given in [25] might be generalized by a possible study on k–token graphs of larger graph classes.

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On the Framed Normal Curves in Euclidean 4-space

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Abstract

In this paper, we introduce the adapted frame of framed curves and we give the relations between the adapted frame and Frenet type frame of the framed curve in four-dimensional Euclidean space. Moreover, we define the framed normal curves in four-dimensional Euclidean space. We obtain some characterizations of framed normal curves in terms of their framed curvature functions. Furthermore, we give the necessary and sufficient condition for a framed curve to be a framed normal curve.

1. Introduction

The most basic building blocks of classical differential geometry are curves. There are many studies on Frenet curves and they are useful for investigating the geometric properties of regular curves. Especially, the subject of curves with singular points was discussed in the 20th century. However, recently, the subject of curves with singular points, for which a Frenet frame cannot be formed at a particular point, has been discussed from a different perspective. Honda and Takahashi introduced the concept of framed curves to examine curves with singular points in terms of differential geometry [1]. These curves, called Framed curves expressed by Honda and Takahashi, are a natural generalization of Frenet curves. Moreover, Wang et. al obtained a moving adapted frame to investigate the properties of rectifying curve with singular points in $\mathbb{R}^3$, and this frame was used to analyze some special curves with singular points [2]. For more details on the notion of framed curves, see [3]-[9].

In Euclidean space $\mathbb{R}^3$, curves whose position vector is always in the normal plane are normal curves, and also these curves are spherical curves [10]. Analogously, timelike normal curves in three-dimensional Minkowski space is defined as the curves whose normal planes always contain a fixed point. Therefore, the position vector of such curves with respect to some chosen origin always lies in its normal plane [11]. In particular, timelike normal curves lie in the pseudosphere in $\mathbb{R}^3_1$. In addition, the characterizations of timelike and spacelike normal curves in Minkowski space $\mathbb{R}^3_1$ have been examined recently, [12], [13]. Bahar et. al has been studied framed normal curve in $\mathbb{R}^3$ [4].

In this paper, inspired by [2] and using Euler angles [14], we obtained moving adapted frame for framed curves in $\mathbb{R}^4$. We define generalized Frenet vectors and framed curvatures to investigate the geometric properties of framed curves in $\mathbb{R}^4$. After that, we introduce framed normal curves in $\mathbb{R}^4$. We give some characterizations for framed normal curves. we obtained the necessary and sufficient conditions for such framed curves to be framed normal curves.

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2. Preliminaries

Let $\mathbb{R}^4$ be the 4−dimensional Euclidean space equipped with the inner product

$$<x,y>=\sum_{i=1}^{4}x_iy_i,$$

where $x=(x_1,x_2,x_3,x_4),y=(y_1,y_2,y_3,y_4)\in\mathbb{R}^4$ and norm of $x\in\mathbb{R}^4$ is given by $\|x\|=\sqrt{<x,x>}$. Vector product in $\mathbb{R}^4$ is given by

$$x\times y\times z=\begin{vmatrix} e_1 & e_2 & e_3 & e_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix}$$

where $x=(x_1,x_2,x_3,x_4),y=(y_1,y_2,y_3,y_4),z=(z_1,z_2,z_3,z_4)\in\mathbb{R}^4$ and e_1,e_2,e_3,e_4 are the canonical basis vectors of $\mathbb{R}^4$. Let Δ_3 be a 6−dimensional smooth manifold as follows

$$\Delta_3=\left\{\mu=(\mu_1,\mu_2,\mu_3)\in\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4\middle|<\mu_i,\mu_j>=\delta_{ij},i,j=1,2,3\right\}.$$

We can define a unit vector $\nu=\mu_1\times\mu_2\times\mu_3$ such that $\det(\nu,\mu_1,\mu_2,\mu_3)=1$.

Definition 2.1. *$(\gamma,\mu):I\to\mathbb{R}^4\times\Delta_3$ is said to be framed curve if $<\gamma'(s),\mu_i(s)>=0$ for all $s\in I$ and $i=1,2,3$. $\gamma:I\to\mathbb{R}^4$ is called as a framed base curve if there exists $\mu:I\to\Delta_3$ such that (γ,μ) is a framed curve, [1].*

By following similar way as the curvatures of regular curve, we can define smooth function for a framed curve. Let $\{\nu(s),\mu(s)\}$ be a moving frame along the framed base curve $\gamma(s)$. Then, we have the Frenet-Serret type formula, which is given by

$$\begin{bmatrix}\mu_1'(s)\\ \mu_2'(s)\\ \mu_3'(s)\\ \nu'(s)\end{bmatrix}=\begin{bmatrix}0 & f(s) & g(s) & h(s)\\ -f(s) & 0 & j(s) & k(s)\\ -g(s) & -j(s) & 0 & l(s)\\ -h(s) & -k(s) & -l(s) & 0\end{bmatrix}\begin{bmatrix}\mu_1(s)\\ \mu_2(s)\\ \mu_3(s)\\ \nu(s)\end{bmatrix}$$

where $f(s),g(s),h(s),j(s),k(s)$ and $l(s)$ are smooth curvature functions. Moreover, there exists a smooth mapping $\alpha:I\to\mathbb{R}$ such that $\gamma'(s)=\alpha(s)\nu(s)$. $(f(s),g(s),h(s),j(s),k(s),l(s),\alpha(s))$ are called curvatures of γ at $\gamma(s)$. Clearly, s_0 is singular points of γ iff $\alpha(s_0)=0$. $(f(s),g(s),h(s),j(s),k(s),l(s),\alpha(s))$ are useful to investigate the framed curve and its singularities.

Theorem 2.2. *Let $(f,g,h,j,k,l,\alpha):I\to\mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma,\mu):I\to\mathbb{R}^4\times\Delta_3$ whose associated curvature of the framed curve is (f,g,h,j,k,l,α) [1].*

Theorem 2.3. *Let (γ,μ) and $(\widetilde{\gamma},\eta):I\to\mathbb{R}^4\times\Delta_3$ be framed curves whose curvatures of the framed curves (f,g,h,j,k,l,α) and $\left(\widetilde{f},\widetilde{g},\widetilde{h},\widetilde{j},\widetilde{k},\widetilde{l},\widetilde{\alpha}\right)$ coincide. Then, (γ,μ) and $(\widetilde{\gamma},\eta)$ are congruent as framed curves [1].*

Let $(\gamma,\mu):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with (f,g,h,j,k,l,α). By using Euler angels $\eta=(\eta_1,\eta_2,\eta_3)\in\Delta_3$ is defined by

$$\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{bmatrix}=\begin{bmatrix}\cos\theta\cos\psi & -\cos\varphi\sin\psi+\sin\varphi\cos\psi\sin\theta & \sin\varphi\sin\psi+\cos\varphi\cos\psi\sin\theta\\ \cos\theta\sin\psi & \cos\varphi\cos\psi+\sin\varphi\sin\psi\sin\theta & -\sin\varphi\cos\psi+\cos\varphi\sin\psi\sin\theta\\ -\sin\theta & \sin\varphi\cos\theta & \cos\varphi\cos\theta\end{bmatrix}\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix}$$

where θ,φ and ψ are smooth functions. By straightforward calculations,

$$\widetilde{\nu}=\eta_1\times\eta_2\times\eta_3=\mu_1\times\mu_2\times\mu_3=\nu.$$

So, $(\gamma,\eta):I\to\mathbb{R}^4\times\Delta_3$ is also a framed curve. Assume that

$$\frac{\tan\psi}{\cos\theta}=l\sin\varphi-k\cos\varphi$$

and

$$h=\cot\theta\,(l\cos\varphi+k\sin\varphi)$$

are satisfied for given smooth function θ,φ and ψ (Euler angle), the adapted frame along $\gamma(s)$ is given by

$$\begin{bmatrix} \nu'(s) \\ \eta_1'(s) \\ \eta_2'(s) \\ \eta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & p(s) & 0 & 0 \\ -p(s) & 0 & q(s) & 0 \\ 0 & -q(s) & 0 & r(s) \\ 0 & 0 & -r(s) & 0 \end{bmatrix} \begin{bmatrix} \nu(s) \\ \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{bmatrix} \tag{2.1}$$

where $(p(s),q(s),r(s),\alpha(s))$ are framed curvature of $\gamma(s)$ and their expression are

$$\begin{aligned} p &= -h\sec\theta\sec\psi, \\ q &= -(j-\varphi')\sin\theta - \psi', \\ r &= \tfrac{\cos\theta}{\cos\psi}(j-\varphi') \end{aligned}$$

and the following equalities

$$\begin{aligned} f &= -\sin\varphi(\theta' - r\sin\psi), \\ g &= -\cos\varphi(\theta' - r\sin\psi), \\ j &= r\tfrac{\cos\psi}{\cos\theta} + \theta' \end{aligned}$$

hold. We can call the vectors ν,η_1,η_2,η_3 the generalized tangent, the generalized principal normal, the generalized first binormal, and the generalized second binormal vectors of the framed curve, respectively.
In order to give a definition of the framed spherical curve, let us recall that a $4-$dimensional hypersphere S^3 is

$$S^3 = \left\{ x \in \mathbb{R}^4 \middle| < x-m, x-m >= c^2 \right\}$$

where $c \in \mathbb{R}^+$ is the radius and m is the center of hypersphere.

Definition 2.4. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve. We call γ a framed spherical curve if the framed base curve γ is a curve on S^3 [2].*

3. Framed normal curves in Euclidean 4-space

In this section, we characterize the framed normal curve with non-zero framed curvatures in $\mathbb{R}^4$.
Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then its position vector satisfies

$$\gamma(s) = \lambda(s)\eta_1(s) + \mu(s)\eta_2(s) + \rho(s)\eta_3(s). \tag{3.1}$$

By differentiating of (3.1), we obtain

$$\alpha\nu = (-p\lambda)\nu + (\lambda' - q\mu)\eta_1 + (q\lambda + \mu' - r\rho)\eta_2 + (r\mu + \rho')\eta_3$$

and therefore

$$-p\lambda = \alpha\,,\ \ \lambda' - q\mu = 0\,,\ \ q\lambda + \mu' - r\rho = 0\,,\ \ r\mu + \rho' = 0. \tag{3.2}$$

From the first three equations, we obtain

$$\begin{aligned} \lambda(s) &= -\tfrac{\alpha(s)}{p(s)}\,,\ \ \mu(s) = -\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)', \\ \rho(s) &= -\tfrac{1}{r(s)}\left(\tfrac{\alpha(s)q(s)}{p(s)} + \left(\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\right)'\right). \end{aligned}$$

By using the above relations, we obtain that

$$\gamma(s) = -\frac{\alpha(s)}{p(s)}\eta_1(s) - \frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\eta_2(s) - \frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\eta_3(s). \tag{3.3}$$

Then we can give the following theorem.

Theorem 3.1. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then, (γ,η) is congruent to a framed normal curve iff*

$$-\frac{r(s)}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)' = \left(\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\right)'. \tag{3.4}$$

Proof. First assume that (γ,η) is congruent to a framed normal curve. Then substituting (3.3) into the fourth equation of (3.2), we obtain imply that (3.4) holds.
Conversely, assume that relation (3.4) holds. Let the vector m be given by

$$m(s)=\gamma(s)+\frac{\alpha(s)}{p(s)}\eta_1(s)+\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\eta_2(s)+\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)}+\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\eta_3(s). \tag{3.5}$$

By differentiating (3.5) with respect to s and by applying (2.1), we obtain

$$m'(s)=\frac{r(s)}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\eta_3(s)+\left(\frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)}+\left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right)\right)'\eta_3(s).$$

From relation (3.4) it is easily seen that m is a constant vector. So, (γ,η) is congruent to a framed normal curve. □

Theorem 3.2. *Let $(\gamma,\eta): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. If (γ,η) is a framed normal curve, then the following statements hold:*
(i) the generalized principal normal and the generalized first binormal component of the position vector γ are respectively given by

$$\begin{aligned}&<\gamma,\eta_1>=-\tfrac{\alpha(s)}{p(s)},\\ &<\gamma,\eta_2>=-\tfrac{1}{q(s)}\left(\tfrac{\alpha}{p(s)}\right)'\end{aligned} \tag{3.6}$$

(ii) the generalized first binormal and the generalized second binormal component of the position vector γ are respectively given by

$$\begin{aligned}&<\gamma,\eta_2>=-\tfrac{1}{q(s)}\left(\tfrac{\alpha}{p(s)}\right)',\\ &<\gamma,\eta_3>=-\tfrac{1}{r(s)}\left(\tfrac{\alpha(s)q(s)}{p(s)}+\left(\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\right)'\right).\end{aligned}$$

Conversely, if one of the statements (i) or (ii) holds, then (γ,η) is a framed normal curve.

Proof. Let (γ,η) be a framed normal curve with non-zero framed curvatures. Statements (i) and (ii) are easily obtained from (3.3).
Conversely, if statement (i) holds, differentiating the first equation of (3.6) and by using (2.1), we obtain $<\gamma,\upsilon>=0$ which means that (γ,η) is a framed normal curve. If statement (ii) holds, in a similar way it is seen that (γ,η) is a framed normal curve. □

Theorem 3.3. *Let $(\gamma,\eta): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures and γ has at least one non-singular point. Then (γ,η) is a framed normal curve if and only if γ lies on S^3.*

Proof. Let $(\gamma,\eta): I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures. By using (3.4), we obtain

$$2\frac{\alpha}{p}\left(\frac{\alpha}{p}\right)'+2\frac{1}{q}\left(\frac{\alpha}{p}\right)'\left(\frac{1}{q}\left(\frac{\alpha}{p}\right)'\right)'+2\frac{1}{\mathrm{r}}\left(\frac{\alpha\mathrm{q}}{\mathrm{p}}+\left(\frac{1}{\mathrm{q}}\left(\frac{\alpha}{\mathrm{p}}\right)'\right)'\right)\left(\frac{1}{\mathrm{r}}\left(\frac{\alpha\mathrm{q}}{\mathrm{p}}+\left(\frac{1}{\mathrm{q}}\left(\frac{\alpha}{\mathrm{p}}\right)'\right)'\right)\right)'=0.$$

The above equation is differential of the equation

$$\left(\frac{\alpha}{p}\right)^2+\left(\frac{1}{q}\left(\frac{\alpha}{p}\right)'\right)^2+\left(\frac{1}{r}\left(\frac{\alpha q}{p}+\left(\frac{1}{q}\left(\frac{\alpha}{p}\right)'\right)'\right)\right)^2=c^2 \quad, c\in\mathbb{R}^+.$$

By using (3.5), it is easily seen that $<\gamma-m,\gamma-m>=c^2$. So, this implies that γ lies on S^3 in $\mathbb{R}^4$.
Conversely, let γ lies on S^3 in $\mathbb{R}^4$, then

$$<\gamma-m,\gamma-m>=c^2 \quad ,c\in\mathbb{R}^+, \tag{3.7}$$

where $m\in\mathbb{R}^4$ is the constant vector. By taking the derivative of (3.7) and γ has at least one non-singular point, we get $<\gamma-m,\nu>=0$, which implies that (γ,η) is a framed normal curve. □

Lemma 3.4. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be a framed curve with non-zero framed curvatures. Then, (γ,η) is congruent to a framed normal curve iff there exists a differentiable function $\xi(s)$ such that*

$$\begin{aligned}\xi(s)r(s) &= \tfrac{\alpha(s)q(s)}{p(s)} + \left(\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\right)', \\ \xi'(s) &= -\tfrac{r(s)}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'.\end{aligned} \tag{3.8}$$

In the following theorem, we obtain the necessary and sufficient conditions for such framed curves to be framed normal curves.

Theorem 3.5. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero framed curvatures. (γ,η) is congruent to framed normal curve iff there exist constants a_0, $b_0 \in \mathbb{R}$ such that*

$$-\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)' = \left(a_0 + \int \frac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds\right)\cos\omega(s) + \left(b_0 + \int \frac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds\right)\sin\omega(s), \tag{3.9}$$

where $\omega(s) = \int_0^s r(s)\,ds$.

Proof. If (γ,η) is congruent to a framed normal curve, according to Lemma 3.4 there exists a differentiable function $\xi(s)$ such that relation (3.8) holds. Let us define differentiable functions $\omega(s)$, $a(s)$ and $b(s)$ by

$$\begin{aligned}&\omega(s) = \int_0^s r(s)\,ds, \\ &a(s) = -\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\cos\omega(s) + \xi(s)\sin\omega(s) - \int \tfrac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds, \\ &b(s) = -\tfrac{1}{q(s)}\left(\tfrac{\alpha(s)}{p(s)}\right)'\sin\omega(s) - \xi(s)\cos\omega(s) - \int \tfrac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds.\end{aligned} \tag{3.10}$$

By using (3.8), we easily find $\omega'(s) = r(s)$, $a'(s) = 0$, $b'(s) = 0$ and thus

$$a(s) = a_0\,,\ \ b(s) = b_0\,,\ \ a_0, b_0 \in \mathbb{R}. \tag{3.11}$$

By multiplying the last two equations in (3.10), respectively with $\cos\omega(s)$ and $\sin\omega(s)$, adding the obtained equations and using (3.11), we conclude that relation (3.9) holds.
Conversely, assume that there exist constants a_0, $b_0 \in \mathbb{R}$ such that relation (3.9) holds. By taking the derivative of (3.9), we find

$$-\frac{\alpha(s)q(s)}{p(s)} - \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)' = r(s)\begin{bmatrix} -\left(a_0 + \int \frac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds\right)\sin\omega(s) \\ +\left(b_0 + \int \frac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds\right)\cos\omega(s)\end{bmatrix}. \tag{3.12}$$

Let us define the differentiable function $\xi(s)$ by

$$\xi(s) = \frac{1}{r(s)}\left(\frac{\alpha(s)q(s)}{p(s)} + \left(\frac{1}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'\right)'\right). \tag{3.13}$$

Next, relations (3.12) and (3.13) imply

$$\xi(s) = \left(a_0 + \int \frac{\alpha(s)q(s)}{p(s)}\cos\omega(s)ds\right)\sin\omega(s) - \left(b_0 + \int \frac{\alpha(s)q(s)}{p(s)}\sin\omega(s)ds\right)\cos\omega(s).$$

By using the above equation and (3.9), we obtain $\xi'(s) = -\frac{r(s)}{q(s)}\left(\frac{\alpha(s)}{p(s)}\right)'$. Finally, Lemma 3.4 implies that (γ,η) is congruent to a framed normal curve. □

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Some Results on $\mathscr{D}$-Homothetic Deformation On Almost Paracontact Metric Manifolds

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Abstract

In this paper, we investigate the effect of $\mathscr{D}$-homothetic deformation on almost para-contact metric manifolds. The main results of the paper are about some classes of almost paracontact metric manifolds in which the characteristic vector field is parallel. It is shown that certain classes are invariant under the $\mathscr{D}$-homothetic deformation.

1. Introduction

Almost paracontact structures were first studied by [1] (Kaneyuki, 1985) and after the work of Zamkovoy in [2] (Zamkovoy, 2009), many authors have made contributions to the subject. In the literature, there are many studies on almost paracontact manifolds from different perspectives in various dimensions. For recent studies, see [3]-[8]. In [9], Zamkovoy and Nakova classified almost paracontact metric structures into the 2^{12} classes by considering the covariant derivative of the fundamental 2-form Φ of the structure with respect to the Levi-Civita connection. The main goal of this work is to study $\mathscr{D}$-homothetic deformations on these structures. We examine the almost paracontact metric structure after the deformation and investigate some certain classes after the deformation. Mostly, we focused on the classes having parallel characteristic vector fields.

2. Almost paracontact metric structures

Definition 2.1. *A differentiable manifold M of dimension $(2n+1)$ is said to be have an almost paracontact structure (ϕ,ξ,η), if it has an endomorphism ϕ, a 1-form η and a vector field ξ such that*

$$\phi^2 = I - \eta\otimes\xi, \quad \eta(\xi)=1, \quad \phi(\xi)=0, \quad \eta\circ\phi=0, \tag{2.1}$$

and there exists a distribution $\mathbb{D}: p\in M \longrightarrow \mathbb{D}_p = Ker\eta$ such that an almost paracomplex structure is induced by the tensor field ϕ. The vector field ξ is said to be the Reeb (or characteristic) vector field of (ϕ,ξ,η).

For each $p\in M$, the tangent space T_pM can be stated as the direct sum

$$T_pM = \mathbb{D}_p \oplus Span_{\mathbb{R}}\{\xi(p)\}$$

and a vector $U\in T_pM$ can be uniquely decomposed as

$$u = hU + vU,$$

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where $hU=\phi^2U\in\mathbb{D}_p$ and $vU=\eta(U)\xi(p)\in Span_{\mathbb{R}}\{\xi(p)\}$ [9]. Let g be a semi-Riemannian metric of signature $(n,n+1)$ on an almost paracontact manifold M with

$$g(\phi U,\phi V)=-g(U,V)+\eta(U)\eta(V). \tag{2.2}$$

Then the metric g is said to be a compatible metric and the quadruple (ϕ,ξ,η,g) is called an almost paracontact metric structure on M. The 2-form Φ given with

$$\Phi(U,V):=g(\phi U,V)$$

is called the fundamental 2-form of the structure.
The basis (namely, ϕ-basis) $\{e_1,\phi e_1,\cdots,e_n,\phi e_n,\xi\}$ with

$$g(e_i,e_j)=-g(\phi e_i,\phi e_j)=\delta_{ij},\quad g(e_i,\phi e_j)=0,\quad i,j=1,\cdots,n,$$

is an orthonormal basis on (ϕ,ξ,η,g) see [2]. For the almost contact case, see [10]. It can be seen that the $(0,3)$- tensor F (the fundamental tensor) given with

$$F(U,V,W)=(\nabla_U\Phi)(V,W)=g((\nabla_U\phi)V,W),$$

satisfies the followings

$$\begin{aligned}F(U,V,W)&=-F(U,W,V),\\ F(U,\phi V,\phi W)&=F(U,V,W)+\eta(V)F(U,W,\xi)-\eta(W)F(U,V,\xi),\end{aligned} \tag{2.3}$$

for any $U,V,W\in TM$. In [9], Zamkovoy and Nakova classified almost paracontact metric manifolds by considering the space $\mathscr{F}$ of tensors F which satisfy (2.3). Initially, they decomposed this space into four subspaces $\mathscr{W}_i$ $(i=1,2,3,4)$, i.e.

$$\mathscr{F}=\mathscr{W}_1\oplus\mathscr{W}_2\oplus\mathscr{W}_3\oplus\mathscr{W}_4,$$

where $\mathscr{W}_i$'s are defined by

$$\mathscr{W}_1=\{F\in\mathscr{F}|F(U,V,W)=F(hU,hV,hW)\},$$

$$\mathscr{W}_2=\{F\in\mathscr{F}|F(U,V,W)=-\eta(V)F(hU,hW,\xi)+\eta(W)F(hU,hV,\xi)\},$$

$$\mathscr{W}_3=\mathscr{G}_{11}=\{F\in\mathscr{F}|F(U,V,W)=\eta(U)F(\xi,hV,hW)\},$$

$$\mathscr{W}_4=\mathscr{G}_{12}=\{F\in\mathscr{F}|F(U,V,W)=\eta(U)[(\eta(V)F(\xi,\xi,hW)-\eta(W)F(\xi,\xi,hV)]\}.$$

Then $\mathscr{W}_1$ and $\mathscr{W}_2$ are written as sums of $U(n)\times 1$ irreducible components $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3,\mathscr{G}_4$ and $\mathscr{G}_5,\cdots,\mathscr{G}_{10}$ respectively, where $U(n)$ is the paraunitary group, with the following relations [9]:

$\mathscr{G}_1$	$F(U,V,W)=\frac{1}{2n-1}[g(U,\phi V)\theta_F(\phi W)-g(U,\phi W)\theta_F(\phi V)-g(\phi U,\phi V)\theta_F(hW)+g(\phi U,\phi W)\theta_F(hU)]$
$\mathscr{G}_2$	$F(\phi U,\phi V,W)=-F(U,V,W),\quad \theta_F=0$
$\mathscr{G}_3$	$F(\xi,V,W)=F(U,\xi,W)=0,\quad F(U,V,W)=-F(V,U,W)$
$\mathscr{G}_4$	$F(\xi,V,W)=F(U,\xi,W)=0,\quad \mathfrak{S}_{(U,V,W)}F(U,V,W)=0$
$\mathscr{G}_5$	$F(U,V,W)=\frac{\theta_F(\xi)}{2n}[\eta(V)g(\phi U,\phi W)-\eta(W)g(\phi U,\phi V)]$
$\mathscr{G}_6$	$F(U,V,W)=-\frac{\theta_F^*(\xi)}{2n}[\eta(V)g(U,\phi W)-\eta(W)g(U,\phi V)]$
$\mathscr{G}_7$	$F(U,V,W)=-\eta(V)F(U,W,\xi)+\eta(W)F(U,V,\xi),;F(U,V,\xi)=-F(V,U,\xi)=-F(\phi U,\phi V,\xi),\quad \theta_F^*(\xi)=0$
$\mathscr{G}_8$	$F(U,V,W)=-\eta(V)F(U,W,\xi)+\eta(W)F(U,V,\xi),;F(U,V,\xi)=F(V,U,\xi)=-F(\phi U,\phi V,\xi),\quad \theta_F(\xi)=0$
$\mathscr{G}_9$	$F(U,V,W)=-\eta(V)F(U,W,\xi)+\eta(W)F(U,V,\xi),;F(U,V,\xi)=-F(V,U,\xi)=F(\phi U,\phi V,\xi)$
$\mathscr{G}_{10}$	$F(U,V,W)=-\eta(V)F(U,W,\xi)+\eta(W)F(U,V,\xi);F(U,V,\xi)=F(V,U,\xi)=F(\phi U,\phi V,\xi)$
$\mathscr{G}_{11}$	$F(U,V,W)=\eta(U)F(\xi,\phi V,\phi W)$
$\mathscr{G}_{12}$	$F(U,V,W)=\eta(U)[\eta(V)F(\xi,\xi,W)-\eta(W)F(\xi,\xi,V)]$

where $\theta_F(U)=g^{ij}F(e_i,e_j,U)$, $\theta_F^*(U)=g^{ij}F(e_i,\phi e_j,U)$, (called Lee forms of the structure).

3. The projection maps of the structure tensor F

In this section, we recall the projection maps of the tensor F. The vector space $\mathscr{F}$ is decomposed as the direct sums of the subspaces $\mathscr{W}_i\ (i=1,2,3,4)$ and $\mathscr{G}_j\ (j=1,...,12)$ mean that any $F\in\mathscr{F}$ can be uniquely represented in the form

$$F(U,V,W)=\sum_{i=1}^{4}F^{\mathscr{W}_i}(U,V,W),$$

and

$$F(U,V,W)=\sum_{j=1}^{12}F^{i}(U,V,W)$$

respectively, where $F^{\mathscr{W}_i}\in\mathscr{W}_i$ and $F^j\in\mathscr{G}_j$. Thereby, $M\in\mathscr{G}_i\oplus\mathscr{G}_j\oplus...$ if and only if the structure tensor F of M satisfies $F=F^i+F^j+....$ The projections ($F^i(i=1,...,12)$) are defined as follows [9]

$$\begin{aligned}
F^1(U,V,W) &= \frac{1}{2n-1}[g(U,\phi V)\theta_{F^1}(\phi W)-g(U,\phi W)\theta_{F^1}(\phi V)\\
&\quad -g(\phi U,\phi V)\theta_{F^1}(\phi^2 W)+g(\phi U,\phi W)\theta_F(\phi^2 V)],
\end{aligned}$$

$$\begin{aligned}
F^2(U,V,W) &= \frac{1}{2}[F(\phi^2U,\phi^2V,\phi^2W)-F(\phi U,\phi^2V,\phi W)]\\
&\quad -\frac{1}{2n-1}[g(U,\phi V)\theta_{F^1}(\phi W)-g(U,\phi W)\theta_{F^1}(\phi V)\\
&\quad -g(\phi U,\phi V)\theta_{F^1}(\phi^2 W)+g(\phi U,\phi W)\theta_F(\phi^2 V)],
\end{aligned}$$

$$\begin{aligned}
F^3(U,V,W) &= \frac{1}{6}[F(\phi^2U,\phi^2V,\phi^2W)+F(\phi U,\phi^2V,\phi W)\\
&\quad +F(\phi^2V,\phi^2W,\phi^2U)+F(\phi V,\phi^2W,\phi U)\\
&\quad F(\phi^2W,\phi^2U,\phi^2V)+F(\phi W,\phi^2U,\phi V)],
\end{aligned}$$

$$\begin{aligned}
F^4(U,V,W) &= \frac{1}{2}[F(\phi^2U,\phi^2V,\phi^2W)+F(\phi U,\phi^2V,\phi W)]\\
&\quad -\frac{1}{6}[F(\phi^2U,\phi^2V,\phi^2W)+F(\phi U,\phi^2V,\phi W)\\
&\quad +F(\phi^2V,\phi^2W,\phi^2U)+F(\phi V,\phi^2W,\phi U)\\
&\quad F(\phi^2W,\phi^2U,\phi^2V)+F(\phi W,\phi^2U,\phi V)],
\end{aligned}$$

$$F^5(U,V,W) = \frac{\theta_{F^5}(\xi)}{2n}[\eta(V)g(\phi U,\phi W)-\eta(W)g(\phi U,\phi V)],$$

$$F^6(U,V,W) = -\frac{\theta^*_{F^6}(\xi)}{2n}[\eta(V)g(U,\phi W)-\eta(W)g(U,\phi V)],$$

$$\begin{aligned}
F^7(U,V,W) &= -\frac{1}{4}\eta(Y)[F(\phi^2U,\phi^2W,\xi)-F(\phi U,\phi W,\xi)\\
&\quad -F(\phi^2W,\phi^2U,\xi)+F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi)\\
&\quad -F(\phi U,\phi V,\xi)-F(\phi^2V,\phi^2U,\xi)+F(\phi V,\phi U,\xi)]\\
&\quad +\frac{\theta^*_{F^6}(\xi)}{2n}[\eta(V)g(U,\phi W)-\eta(W)g(U,\phi V)],
\end{aligned}$$

$$\begin{aligned}
F^8(U,V,W) &= -\frac{1}{4}\eta(V)[F(\phi^2U,\phi^2W,\xi)-F(\phi U,\phi W,\xi)\\
&\quad +F(\phi^2W,\phi^2U,\xi)-F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi)\\
&\quad -F(\phi U,\phi V,\xi)+F(\phi^2V,\phi^2U,\xi)-F(\phi V,\phi U,\xi)]\\
&\quad -\frac{\theta_{F^5}(\xi)}{2n}[\eta(V)g(\phi U,\phi W)-\eta(W)g(\phi U,\phi V)],
\end{aligned}$$

$$
\begin{aligned}
F^{9}(U,V,W) &= -\frac{1}{4}\eta(V)[F(\phi^2U,\phi^2W,\xi)+F(\phi U,\phi W,\xi)\\
&\quad -F(\phi^2W,\phi^2U,\xi)-F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi)\\
&\quad +F(\phi U,\phi V,\xi)-F(\phi^2V,\phi^2U,\xi)-F(\phi V,\phi U,\xi)],
\end{aligned}
$$

$$
\begin{aligned}
F^{10}(U,V,W) &= -\frac{1}{4}\eta(V)[F(\phi^2U,\phi^2W,\xi)+F(\phi U,\phi W,\xi)\\
&\quad +F(\phi^2W,\phi^2U,\xi)+F(\phi W,\phi U,\xi)]+\frac{1}{4}\eta(W)[F(\phi^2U,\phi^2V,\xi)\\
&\quad +F(\phi U,\phi V,\xi)+F(\phi^2V,\phi^2U,\xi)+F(\phi V,\phi U,\xi)],
\end{aligned}
$$

$$
F^{11}(U,V,W) = \eta(U)F(\xi,\phi^2V,\phi^2W),
$$

$$
F^{12}(U,V,W) = \eta(U)[\eta(V)F(\xi,\xi,\phi^2W)-\eta(W)F(\xi,\xi,\phi^2V)].
$$

4. Almost paracontact metric structures with parallel Reeb vector field

This section is dedicated to investigating the almost paracontact metric structures equipped with parallel Reeb vector field ξ. In [9], it is stated that the vector field ξ is Killing only in the classes $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3,\mathscr{G}_4,\mathscr{G}_5,\mathscr{G}_8,\mathscr{G}_9,\mathscr{G}_{11}$ and in their direct sums. As it is known, the vector field ξ is said to be parallel if $\nabla_U\xi=0$, and Killing if $g(\nabla_U\xi,V)+g(\nabla_V\xi,U)=0$, for any vector field U,V. So, as a natural result of these definitions, we can say that if a vector field is not Killing, then it is not parallel. Thus, the characteristic vector field ξ of the classes $\mathscr{G}_6,\mathscr{G}_7,\mathscr{G}_{10},\mathscr{G}_{12}$ and of their direct sums can not be parallel. So, let us consider the remaining classes.

For the classes $\mathscr{G}_i$ $(i=1,2,3,4,11)$, set $V=\xi$ and substitute W with ϕW. Then, we get

$$
F^{i}(U,\xi,\phi W)=g((\nabla_U\phi)(\xi),\phi W)=0.
$$

Since $\eta(\nabla_U\xi)=0$ for any U, and from the equation (2.2), we get $g(\nabla_U\xi,W)=0$, which means $\nabla\xi=0$, since g is non-degenerate.

For the class $\mathscr{G}_5$, set $V=\xi$ in the defining relation of $\mathscr{G}_5$. Then we get

$$
g(\phi(\nabla_U\xi),W)=\frac{\theta_F(\xi)}{2n}g(\phi^2U,W).
$$

From the equation (2.1), we get

$$
\nabla_U\xi=\frac{\theta_F(\xi)}{2n}\phi U,
$$

which is non-zero since the class $\mathscr{G}_5$ is non-trivial. Thus, the vector field ξ is not parallel in $\mathscr{G}_5$.

For the classes $\mathscr{G}_i$, $(i=8,9)$, assume that the vector field ξ is parallel. Under this assumption, one can easily see that $F^i=0$. However, since these classes are non-trivial, we come up with the result that ξ is not parallel in these classes.

In addition, it is known from [9] that, if an almost paracontact metric structure is of the classes $\mathscr{G}_i\oplus\mathscr{G}_j\oplus\ldots$, then the structure tensor F is of the form $F=F^i+F^j+\ldots$. So, it is clear that a class, which is a direct sum of some classes having a parallel characteristic vector field, is also equipped with a parallel characteristic vector field.

After all, we can give the following theorem:

Theorem 4.1. *The characteristic vector field ξ is parallel only in the classes $\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3,\mathscr{G}_4,\mathscr{G}_{11}$ and in their direct sums.*

5. $\mathscr{D}$-homothetic deformation on an almost paracontact metric structure

The idea of a $\mathscr{D}$-homothetic deformation on a contact metric manifold (especially on Sasakian and K-contact structures) was introduced by Tanno ([11], [12]).

Let (ϕ,ξ,η,g) be an almost paracontact metric structure on a $(2n+1)-$dimensional manifold M and $\lambda\neq 0$ be a positive constant. Set,

$$
\bar{\phi}=\phi,\quad \bar{\xi}=\frac{1}{\lambda}\xi,\quad \bar{\eta}=\lambda\eta,\quad \bar{g}=-\lambda g+\lambda(\lambda+1)\eta\otimes\eta.
$$

Then, it can be seen that

$$
Ker\bar{\eta}=Ker\eta,\quad \bar{\phi}^2=I-\bar{\eta}\otimes\bar{\xi},\quad \bar{\eta}(\bar{\xi})=1
$$

and for any $U,V\in\mathfrak{X}(M)$,

$$\bar{g}(\bar{\phi}U,\bar{\phi}V)=-\bar{g}(U,V)+\bar{\eta}(U)\bar{\eta}(V).$$

Hence, $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also an almost paracontact metric structure on M [2].
This is called a $\mathscr{D}-$homothetic deformation of (ϕ,ξ,η,g). In this paper, we consider this deformation. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of the metrics g and $\bar{g}$, respectively. Then by the Koszul formula and the definition of $\bar{g}$,

$$\begin{aligned} 2\bar{g}(\bar{\nabla}_UV,W) &= -2\lambda g(\bar{\nabla}_UV,W)+2\lambda(\lambda+1)\eta(\bar{\nabla}_UV)\eta(W) \\ &= -2\lambda g(\nabla_UV,W)+\lambda(\lambda+1)[2g(\nabla_UV,\xi)\eta(W) \\ &\quad +\eta(U)(g(\nabla_V\xi,W)-g(\nabla_W\xi,V)) \\ &\quad +\eta(V)(g(\nabla_U\xi,W)-g(\nabla_W\xi,U)) \\ &\quad +\eta(W)(g(\nabla_U\xi,V)+g(\nabla_V\xi,U))]. \end{aligned} \tag{5.1}$$

To obtain the relation between $\eta(\bar{\nabla}_UV)$ and $\eta(\nabla_UV)$, take $W=\xi$ in the equation (5.1). So we get,

$$\begin{aligned} \eta(\bar{\nabla}_UV) &= \eta(\nabla_UV)+\frac{\lambda+1}{2\lambda}[-\eta(U)g(\nabla_\xi\xi,V) \\ &\quad -\eta(V)g(\nabla_\xi\xi,U)+g(\nabla_U\xi,V)+g(\nabla_V\xi,U)]. \end{aligned} \tag{5.2}$$

If we apply the equation (5.2) into the equation (5.1), we get the following

$$\begin{aligned} g(\bar{\nabla}_UV,W) &= g(\nabla_UV,W)+\frac{(\lambda+1)^2}{2\lambda}\eta(W)[-\eta(U)g(\nabla_\xi\xi,V) \\ &\quad -\eta(V)g(\nabla_\xi\xi,U)+g(\nabla_U\xi,V)+g(\nabla_V\xi,U)] \\ &\quad -\frac{\lambda+1}{2}[\eta(U)(g(\nabla_V\xi,W)-g(\nabla_W\xi,V)) \\ &\quad +\eta(V)(g(\nabla_U\xi,W)-g(\nabla_W\xi,U))) \\ &\quad +\eta(W)(g(\nabla_U\xi,V)+g(\nabla_V\xi,U))]. \end{aligned} \tag{5.3}$$

By means of the equation (5.3), we may obtain the relations between $\bar{\nabla}$ and ∇ under some certain assumptions. The next section is devoted to studying the $\mathscr{D}-$ homothetic deformations of the structure with a parallel characteristic vector field.

6. $\mathscr{D}$-homothetic deformations of the structures with parallel Reeb vector field

In this section, we examine the $\mathscr{D}-$homothetic deformations of the almost paracontact structures with a parallel characteristic vector field.

Let (M,ϕ,ξ,η,g) be an almost paracontact metric manifold with parallel characteristic vector field ξ (i.e. $\nabla_U\xi=0$, for any $U\in\mathfrak{X}(M)$ and $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ be the $\mathscr{D}$-homothetic deformed structure as defined above. Then we state the followings:

Proposition 6.1. *Let (M,ϕ,ξ,η,g) be an almost paracontact metric manifold with parallel characteristic vector field ξ. Then the followings hold*

i $\bar{\nabla}_UV=\nabla_UV$,
ii $\bar{F}(U,V,W)=-\lambda F(U,V,W)$,
iii $\bar{\theta}_{\bar{F}}(U)=\theta_F(U)$,

for any $U,V,W\in\mathfrak{X}(M)$, where $\bar{F}$ and $\bar{\theta}$ are the fundamental tensor and the Lee form of the deformed structure, respectively.

Proof. By assuming $\nabla\xi=0$ in the equation (5.3), we directly get the equation (i).
For the equation (ii), we have the following

$$\begin{aligned} \bar{F}(U,V,W) &= \bar{g}((\bar{\nabla}_U\bar{\phi})(V),W)=-\lambda g((\nabla_U\phi)(V),W)+\lambda(\lambda+1)\eta((\nabla_U\phi)(V))\eta(W) \\ &= -\lambda F(U,V,W)+\lambda(\lambda+1)\eta((\nabla_U\phi)(V))\eta(W). \end{aligned}$$

On the other hand, since

$$0=U[g(\phi V,\xi)]=g(\nabla_U\phi V,\xi)+g(\phi V,\nabla_U\xi)\Rightarrow g(\nabla_U\phi V,\xi)=0,$$

we have

$$\eta((\nabla_U\phi)(V))=g((\nabla_U\phi)(V),\xi)=g(\nabla_U\phi V,\xi)-g(\phi(\nabla_UV),\xi)=0.$$

Thus, the equation (ii) is proved.

For the proof of (iii), consider the $\phi-$ basis $\{e_i, \phi e_i, \xi\}(i=1,...,n)$ for the structure (ϕ,ξ,η,g). Then $\{\bar{e}_i, \bar{\phi e_i}, \bar{\xi}\}$ is the $\phi-$ basis for the structure $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$, where

$$\bar{e}_i = \frac{1}{\sqrt{\lambda}}e_i, \bar{\phi e_i} = \frac{1}{\sqrt{\lambda}}\phi e_i, \bar{\xi} = \frac{1}{\lambda}\xi \qquad (\lambda > 0)$$

and

$$\bar{g}(\bar{e}_i,\bar{e}_i) = -\bar{g}(\bar{\phi e_i},\bar{\phi e_i}) = -\bar{g}(\bar{\xi},\bar{\xi}) = -1.$$

Since ξ is parallel, by (i) $\bar{\nabla} = \nabla$. So, by direct calculation we get

$$\bar{F}(\bar{e}_i,\bar{e}_i,U) = -F(e_i,e_i,U),$$

$$\bar{F}(\bar{\phi e_i},\bar{\phi e_i},U) = -F(\phi e_i,\phi e_i,U),$$

$$\bar{F}(\bar{\xi},\bar{\xi},U) = F(\xi,\xi,U) = 0.$$

So, by the definition of the from θ, we have

$$\begin{aligned}
\bar{\theta}_{\bar{F}}(U) &= -\sum_{i=1}^{n}\bar{F}(\bar{e}_i,\bar{e}_i,U) + \sum_{i=1}^{n}\bar{F}(\bar{\phi e_i},\bar{\phi e_i},U) \\
&= -\sum_{i=1}^{n}(-F(e_i,e_i,U)) + \sum_{i=1}^{n}(-F(\phi e_i,\phi e_i,U)) \\
&= \theta_F(U).
\end{aligned}$$

□

Theorem 6.2. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_1$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_1$.*

Proof. Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_1$. Then the fundamental tensor F satisfied the defining relation of the class $\mathscr{G}_1$, that is

$$F(U,V,W) = \frac{1}{2n-1}[g(U,\phi V)\theta_F(\phi W) - g(U,\phi W)\theta_F(\phi V) - g(\phi U,\phi V)\theta_F(hW) + g(\phi U,\phi W)\theta_F(hV)]. \tag{6.1}$$

On the other hand, by the proposition (4.1), the vector field ξ is parallel and so, the equations in the proposition (6.1) hold. By routine calculation, it can be seen that $\bar{F}$ also satisfies the equation (6.1). Thus, $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_1$. □

Theorem 6.3. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_2$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_2$.*

Proof. Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_2$. Then the fundamental tensor F satisfies the defining relation of the class $\mathscr{G}_2$, that is

$$F(\phi U,\phi V,W) = -F(U,V,W), \quad \theta_F = 0. \tag{6.2}$$

Since ξ is parallel, $\bar{F}(U,V,W) = -\lambda F(U,V,W)$ and $\bar{\theta}_{\bar{F}}(U) = \theta_F(U)$. Thus, $\bar{F}$ also satisfies the equation (6.2). □

Theorem 6.4. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_3$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_3$.*

Proof. Let F satisfies the defining relation of $\mathscr{G}_3$, that is,

$$F(\xi,V,W) = F(U,\xi,W) = 0, \quad F(U,V,W) = -F(V,U,W). \tag{6.3}$$

Since ξ is parallel in the class $\mathscr{G}_3$, $\bar{F}(U,V,W) = -\lambda F(U,V,W)$ and so $\bar{F}$ also satisfies (6.3). □

Theorem 6.5. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_4$. Then $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also in $\mathscr{G}_4$.*

Proof. It can be seen by direct calculation since ξ is parallel in $\mathscr{G}_4$. □

Theorem 6.6. *Let (ϕ,ξ,η,g) belongs to the class $\mathscr{G}_{11}$. Then so is the structure $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$.*

Proof. It can see seen from the definition class and the proposition (6.1). □

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A Study on Strongly Almost Convergent and Strongly Almost Null Binomial Double Sequence Spaces

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Abstract

The 4 dimensional (4d) binomial matrix and its domains on the classical double sequence spaces $\mathscr{L}_p$, $\mathscr{M}_u$, $\mathscr{C}_P$, $\mathscr{C}_{bP}$, $\mathscr{C}_r$, $\mathscr{C}_f$ and $\mathscr{C}_{f_0}$ have been described and examined by Demiriz and Erdem in the papers [1]-[3]. In this article, we describe two double sequence spaces with the aid of the aforementioned matrix and study some properties of these. After giving inclusion relations, we compute $\alpha-$, $\beta(bp)-$ and $\gamma-$duals and give some new matrix classes related them.

1. Introduction

The function F defined by $F:\mathbb{N}\times\mathbb{N}\to\zeta$, $(i,j)\mapsto F(i,j)=u_{ij}$ is called as *double sequence* where ζ denotes any nonempty set and $\mathbb{N}=\{0,1,2,\ldots\}$. Ω represents the vector space of all complex valued double sequences. $\mathscr{M}_u$, $\mathscr{C}_P$, $\mathscr{C}_r$ and $\mathscr{L}_p$ $(0<p<\infty)$ are the spaces of all bounded, convergent in the Pringsheim's sense (or shortly P-convergent), regularly convergent and p-absolutely summable double sequences, respectively. If any $u=(u_{ij})\in\Omega$ is P-convergent to a limit point L, it is stated by $P-\lim_{i,j\to\infty}u_{ij}=L$. It is worth mentioning that P-convergence does not require boundedness in double sequences. The bounded sequences which are also P-convergent are indicated by $\mathscr{C}_{bP}$. It is also significant to remember that the space $\mathscr{L}_u$ which was described by Zeltser [4] is the special case of the space $\mathscr{L}_p$ for $p=1$.

Throughout this article, it is used the summation $\sum_{i,j}$ instead of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$ and $\vartheta\in\{p,bP,r\}$. With the notation of Zeltser [4], we describe the double sequences $e^{kl}=(e^{kl}_{ij})$ and e by $e^{kl}_{ij}=1$ if $(k,l)=(i,j)$ and $e^{kl}_{ij}=0$ for other cases, and $e=\sum_{k,l}e^{kl}$ for every $i,j,k,l\in\mathbb{N}$. If $d_{klij}=0$ for $i>k$ or $j>l$ or both for every $k,l,i,j\in\mathbb{N}$, it is said that $D=(d_{klij})$ is a *triangular matrix* and also if $d_{klkl}\neq 0$ for every $k,l\in\mathbb{N}$, then the 4d matrix D is called *triangle*.

Now, we shall deal with the matrix mapping. Let us consider double sequence spaces Ψ and Λ and the 4d complex infinite matrix $D=(d_{klij})$. If for every $u=(u_{ij})\in\Psi$, $(Du)_{kl}=\vartheta-\sum_{i,j}d_{klij}u_{ij}$ is exists and is in Λ, then it is said that D is a matrix mapping from Ψ into Λ and is written as $D:\Psi\to\Lambda$.

Let $(\Psi,\Lambda)=\{D=(d_{klij})|D:\Psi\to\Lambda\}$. Here, $D\in(\Psi,\Lambda)$ if and only if $D_{kl}\in\Psi^{\beta(\vartheta)}$ and $Du\in\Lambda$ for all $u\in\Psi$, where $D_{kl}=(d_{klij})_{i,j\in\mathbb{N}}$ for every $k,l\in\mathbb{N}$.

The domain $\Psi_D^{(\vartheta)}$ of D in a double sequence space Ψ consists of whose D-transforms are in Ψ is defined by the following way:

$$\Psi_D^{(\vartheta)}:=\left\{u=(u_{ij})\in\Omega: Du:=\left(\vartheta-\sum_{ij}d_{klij}u_{ij}\right)_{k,l\in\mathbb{N}}\text{ exists and is in }\Psi\right\}.$$

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In the past, many authors were interested in double sequence spaces. Now, let us give some information about these studies. In her doctoral dissertation, Zeltser [5] has fundamentally examined the topological structure of double sequences. Recently, Altay and Başar [6] have been described the spaces $\mathscr{BS}$, $\mathscr{BS}(t)$, $\mathscr{CS}_P$, $\mathscr{CS}_{bP}$, $\mathscr{CS}_r$ and $\mathscr{BV}$ of double series. After that, Talebi [7] defined and examined the space $\mathscr{E}_p^{r,s}$ for $1 \leq p < \infty$ and also Yeşilkayagil and Başar [8] examined for $0 < p < 1$ where $\mathscr{E}_p^{r,s} = \{u = (u_{ij}) : E(r,s)u \in \mathscr{L}_p\}$. Here, $E(r,s)$ indicates the Euler mean. More recently, Tuğ [9]-[11] have defined and examined some domains of the 4d matrix $B(r,s,t,u)$.
On the other hand, Bişgin [12, 13] have introduced the sequence spaces $b_0^{r,s}, b_c^{r,s}, b_p^{r,s}$ and $b_\infty^{r,s}$ of single sequences whose 2d binomial matrix $B^{r,s}$-transforms are convergent to zero, convergent, absolutely p-summable and bounded, respectively. After that in [14], Bişgin have been examined the domains of $B^{r,s}$ on f and f_0. Here, f and f_0 symbolize the spaces of every almost convergent and almost null single sequences, respectively.
A generalization for convergence of a double sequence is almost convergence was firstly introduced by Mòricz and Rhoades [15]. It is said that $u \in \Omega$ is almost convergent if

$$P - \lim_{\rho,\rho'} \sup_{k,l>0} \left| \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho} \sum_{j=l}^{l+\rho'} u_{ij} - L \right| = 0$$

and stated by $f_2 - \lim u = L$. Every almost convergent $u \in \Omega$ are included by $\mathscr{C}_f$ which is defined by the following way:

$$\mathscr{C}_f = \left\{ u = (u_{ij}) \in \Omega : \exists L \in \mathbb{C} \ni \quad P - \lim_{\rho,\rho'} \sup_{k,l>0} \left| \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho} \sum_{j=l}^{l+\rho'} u_{ij} - L \right| = 0, \text{ uniformly in } k,l \right\}.$$

Moreover, the space of almost null double sequences $\mathscr{C}_{f_0}$ is obtained from $\mathscr{C}_f$ by taking $L = 0$.
It is significant to say that the convergence of a double sequence does not require its almost convergence. However, the inclusion $\mathscr{C}_{bP} \subset \mathscr{C}_f \subset \mathscr{M}_u$ is valid.
With the notion Başarır [16], it is said that $u = (u_{kl}) \in \Omega$ is strongly almost convergent to a limit point L_1 if

$$P - \lim_{\rho,\rho'} \sup_{k,l>0} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho} \sum_{j=l}^{l+\rho'} |u_{ij} - L_1| = 0, \quad \text{uniformly in } k,l \in \mathbb{N}.$$

In that case, this stuation is shown by $[f_2] - \lim u = L_1$.
Every strongly almost convergent $u \in \Omega$ are included by $[\mathscr{C}_f]$ which is defined by the following way:

$$[\mathscr{C}_f] = \left\{ u = (u_{ij}) \in \Omega : \exists L_1 \in \mathbb{C} \ni \quad P - \lim_{\rho,\rho'} \sup_{k,l>0} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho} \sum_{j=l}^{l+\rho'} |u_{ij} - L_1| = 0, \text{ uniformly in } k,l \right\}.$$

Furthermore, the space of strongly almost null double sequences $[\mathscr{C}_{f_0}]$ is obtained from $[\mathscr{C}_f]$ by taking $L_1 = 0$.
Between the mentioned spaces, the inclusion relations $\mathscr{C}_{bP} \subset [\mathscr{C}_{f_0}] \subset [\mathscr{C}_f] \subset \mathscr{M}_u$ and $\mathscr{C}_{bP} \subset \mathscr{C}_{f_0} \subset \mathscr{C}_f \subset \mathscr{M}_u$ strictly hold.
Moreover, the spaces $[\mathscr{C}_f]$ and $[\mathscr{C}_{f_0}]$ are Banach spaces with norm

$$\|u\|_{[\mathscr{C}_f]} = \sup_{\rho,\rho',k,l \in \mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho} \sum_{j=l}^{l+\rho'} |u_{ij}|.$$

For further information about single and double sequence spaces and related topics, the reader may refer to some of the papers [17]-[39] and references therein.
Our main purpose in this article is to investigate the domains of 4d binomial matrix on the spaces $[\mathscr{C}_f]$ and $[\mathscr{C}_{f_0}]$.

2. Strongly almost convergent binomial double sequence spaces

Let r, s and $r+s$ are nonzero real numbers. We have been defined the 4d binomial matrix $B^{(r,s)} = (b_{klij}^{r,s})$ of orders r, s in [1] as follows:

$$b_{klij}^{r,s} := \begin{cases} \frac{1}{(r+s)^{k+l}} \binom{k}{i} \binom{l}{j} s^{k+j-i} r^{l+i-j} & , \quad 0 \leq i \leq k \,,\ 0 \leq j \leq l, \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{2.1}$$

for every $k,l,i,j \in \mathbb{N}$. As can be understood from its definition, $B^{(r,s)}$ is a triangle. In that case, we write the $B^{(r,s)}$-transform of $u \in \Omega$ as

$$v_{kl} := (B^{(r,s)}u)_{kl} = \sum_{i,j}^{k,l} \frac{1}{(r+s)^{k+l}} \binom{k}{i} \binom{l}{j} s^{k+j-i} r^{l+i-j} u_{ij}, \tag{2.2}$$

for every $k,l \in \mathbb{N}$. We will assume unless stated otherwise that the double sequences $u=(u_{ij})$ and $v=(v_{ij})$ are connected with the relation (2.2) and r, s and $r+s$ are nonzero real numbers. We would like touch on a point, when it is chosen $r+s=1$, $B^{(r,s)}$ is reduced to the 4d Euler matrix $E(r,s)$. So, our matrix $B^{(r,s)}$ generalizes the $E(r,s)$. Consider that the 4d unit matrix $I=(\delta_{klij})$ defined by

$$\delta_{klij}=\begin{cases} 1 \quad , & (k,l)=(i,j), \\ 0 \quad , & \text{otherwise.}\end{cases}$$

From the equality

$$\delta_{klij}=\sum_{m,n} b^{r,s}_{klmn} c^{r,s}_{mnij},$$

one can see that the inverse $\{B^{(r,s)}\}^{-1}=C^{(r,s)}=(c^{r,s}_{klij})$ as

$$c^{r,s}_{klij}:=\begin{cases} (-1)^{k+l-(i+j)}\binom{k}{i}\binom{l}{j}s^{k-l-i}r^{l-k-j}(r+s)^{i+j} & , \quad 0\leq i\leq k, 0\leq j\leq l, \\ 0 & , \quad \text{otherwise,}\end{cases}$$

for every $k,l,i,j \in \mathbb{N}$.

A 4d matrix D is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit [22, 32]. In [1], it was proven that 4d binomial matrix described by (2.1) is RH-regular for $r.s>0$. In the rest of the study, it will be assumed that $r.s>0$.

Now, we introduce the sets $\mathscr{B}^{r,s}_{[f]}$ and $\mathscr{B}^{r,s}_{[f_0]}$ by

$$\mathscr{B}^{r,s}_{[f]} = \left\{u=(u_{ij})\in\Omega : \exists L\in\mathbb{C}\ni P-\lim_{\rho,\rho'}\sup_{k,l>0}\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left|\left(B^{(r,s)}u\right)_{ij}-L\right|=0,\ \text{uniformly in } k,l\right\},$$

$$\mathscr{B}^{r,s}_{[f_0]} = \left\{u=(u_{ij})\in\Omega : P-\lim_{\rho,\rho'}\sup_{k,l>0}\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left|\left(B^{(r,s)}u\right)_{ij}\right|=0,\ \text{uniformly in } k,l\right\}.$$

Theorem 2.1. *The sets $\mathscr{B}^{r,s}_{[f]}$ and $\mathscr{B}^{r,s}_{[f_0]}$ are linear spaces.*

Proof. Since it is easy to see, we omit it. □

Theorem 2.2. *The sequence spaces $\mathscr{B}^{r,s}_{[f]}$ and $\mathscr{B}^{r,s}_{[f_0]}$ are Banach spaces with the norm*

$$\|u\|_{\mathscr{B}^{r,s}_{[f]}}=\sup_{\rho,\rho',k,l\in\mathbb{N}}\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left|\left(B^{(r,s)}u\right)_{ij}\right|. \tag{2.3}$$

Proof. Since it can be similarly proven for the space $\mathscr{B}^{r,s}_{[f_0]}$, it will be proven for $\mathscr{B}^{r,s}_{[f]}$.

Consider any cauchy sequence $u^{(m)}=\left\{u^{(m)}_{ij}\right\}_{i,j\in\mathbb{N}}\in\mathscr{B}^{r,s}_{[f]}$. In that case, for $\varepsilon>0$ there exists a $N\in\mathbb{N}^+$ such that

$$\left\|u^{(m)}-u^{(n)}\right\|_{\mathscr{B}^{r,s}_{[f]}}=\sup_{\rho,\rho',k,l\in\mathbb{N}}\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left|\left(B^{(r,s)}u^{(m)}\right)_{ij}-\left(B^{(r,s)}u^{(n)}\right)_{ij}\right|<\varepsilon \tag{2.4}$$

for all $m,n>N$. Thus, it is concluded from (2.4), $\left\{\left(B^{(r,s)}u^{(m)}\right)_{ij}\right\}$ is also Cauchy in $[\mathscr{C}_f]$. Since, $[\mathscr{C}_f]$ is a Banach space, we can write

$$\left\{\left(B^{(r,s)}u^{(m)}\right)_{ij}\right\}\longrightarrow\left\{\left(B^{(r,s)}u\right)_{ij}\right\}$$

as $m\to\infty$ and using these infinitely many limit points, we can define double sequence $\left\{\left(B^{(r,s)}u\right)_{ij}\right\}$.

Now, by taking the limit as $n\to\infty$ on (2.4), we have

$$\sup_{\rho,\rho',k,l\in\mathbb{N}}\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left|\left(B^{(r,s)}u^{(m)}\right)_{ij}-\left(B^{(r,s)}u\right)_{ij}\right|<\varepsilon$$

for all $\varepsilon>0$, $m>N$ and $i,j\in\mathbb{N}$.

Furthermore, since $u^{(m)} \in \mathscr{B}^{r,s}_{[f]}$, it is clear that $B^{(r,s)}u^{(m)} \in [\mathscr{C}_f]$ and

$$\sup_{\rho,\rho',k,l\in\mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \left(B^{(r,s)}u^{(m)}\right)_{ij} \right| \leq M$$

for a positive real number M. Now, we can say by taking supremum over $\rho,\rho',k,l \in \mathbb{N}$ on the inequality

$$\begin{aligned} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \left(B^{(r,s)}u\right)_{ij} \right| &\leq \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \left(B^{(r,s)}u^{(m)}\right)_{ij} - \left(B^{(r,s)}u\right)_{ij} \right| \\ &+ \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \left(B^{(r,s)}u^{(m)}\right)_{ij} \right| < \varepsilon + M \end{aligned}$$

that $B^{(r,s)}u \in [\mathscr{C}_f]$, that is $u \in \mathscr{B}^{r,s}_{[f]}$. Thus, it is concluded that $\mathscr{B}^{r,s}_{[f]}$ is a Banach space with the norm $\|u\|_{\mathscr{B}^{r,s}_{[f]}}$ defined by (2.3). □

Theorem 2.3. *The double sequence spaces $\mathscr{B}^{r,s}_{[f]}$ and $\mathscr{B}^{r,s}_{[f0]}$ are linearly norm isomorphic to the spaces $[\mathscr{C}_f]$ and $[\mathscr{C}_{f_0}]$, respectively.*

Proof. Because it can be similarly shown for the space $\mathscr{B}^{r,s}_{[f]}$, we give the proof only for $\mathscr{B}^{r,s}_{[f0]}$. For the claim of theorem, we must see that there is a linear bijection which preserves the norm from one to the other for the spaces $\mathscr{B}^{r,s}_{[f0]}$ and $[\mathscr{C}_{f_0}]$.
For this purpose, let us take the map $T : \mathscr{B}^{r,s}_{[f0]} \to [\mathscr{C}_{f_0}]$, $u \mapsto v = Tu = B^{(r,s)}u$. The linearity of T is clear. Consider the equality $Tu = \theta$ which yields us that $u_{ij} = 0$ for every $i, j \in \mathbb{N}$. So, $u = \theta$ and therefore, T is injective. Let us consider $v \in [\mathscr{C}_{f_0}]$. It is clear by defining

$$u_{kl} = \sum_{i,j=0}^{k,l} (-1)^{k+l-(i+j)} \binom{k}{i}\binom{l}{j} s^{k-l-i} r^{l-k-j} (r+s)^{i+j} v_{ij} \tag{2.5}$$

that $Tu = v$ and $u \in \mathscr{B}^{r,s}_{[f0]}$ for every $k,l \in \mathbb{N}$. So, the map T is surjective. Furthermore, by bearing in mind the following equality

$$\begin{aligned} \|u\|_{\mathscr{B}^{r,s}_{[f0]}} &= \sup_{\rho,\rho',k,l\in\mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \left(B^{(r,s)}u\right)_{ij} \right| \\ &= \sup_{\rho,\rho',k,l\in\mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} |v_{ij}| = \|v\|_{[\mathscr{C}_{f_0}]} \end{aligned}$$

that, T preserves the norm. As a result, the assertion of the theorem has been proved. □

Theorem 2.4. *The inclusion $\mathscr{B}^{r,s}_{[f0]} \subset \mathscr{B}^{r,s}_{[f]}$ holds.*

Proof. Consider any sequence $u = (u_{ij}) \in \mathscr{B}^{r,s}_{[f0]}$. In that case, from the relation (2.2), there exists a double sequence $v \in [\mathscr{C}_{f_0}]$ such that $v = (v_{kl}) = \left(B^{(r,s)}u\right)_{kl}$. Since, $[\mathscr{C}_{f_0}] \subset [\mathscr{C}_f]$, then $v \in [\mathscr{C}_f]$ and this says us that $u \in \mathscr{B}^{r,s}_{[f]}$ which is the desired result. □

Theorem 2.5. *The inclusion $\mathscr{M}_u \subset \mathscr{B}^{r,s}_{[f0]}$ strictly holds.*

Proof. From the inequality

$$\begin{aligned} \|u\|_{\mathscr{B}^{r,s}_{[f0]}} &= \sup_{\rho,\rho',k,l\in\mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \left(B^{(r,s)}u\right)_{ij} \right| \\ &\leq \sup_{\rho,\rho',k,l\in\mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \sum_{m=0}^{i}\sum_{n=0}^{j} b^{r,s}_{ijmn} \right| |u_{mn}| \\ &\leq \sup_{m,n\in\mathbb{N}} |u_{mn}| \sup_{\rho,\rho',k,l\in\mathbb{N}} \frac{1}{(\rho+1)(\rho'+1)} \sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'} \left| \sum_{m=0}^{i}\sum_{n=0}^{j} b^{r,s}_{ijmn} \right| \\ &= \|u\|_{\infty}, \end{aligned}$$

it is seen that any double sequence u taken in $\mathcal{M}_u$ is in $\mathscr{B}^{r,s}_{[f0]}$.

Now, let us select the sequence $u=(u_{kl})=\frac{(-s-r)^{k+l}}{r^k s^l}$ to show the strictness. In that case, we see that $u\notin\mathcal{M}_u$ but its $B^{(r,s)}$-transform $B^{(r,s)}u=\frac{(-1)^{k+l}r^k s^l}{(r+s)^{k+l}}$ is in $\mathcal{M}_u\cap\mathcal{C}_P=\mathcal{C}_{bP}\subset\left[\mathcal{C}_{f_0}\right]$ which means that $u\in\mathscr{B}^{r,s}_{[f0]}$. In the light of all this said, it is seen that $u\in\mathscr{B}^{r,s}_{[f0]}-\mathcal{M}_u$ and the inclusion is strict, as claimed. □

Combining Theorem 2.4 and Theorem 2.5, we may give the following corollary:

Corollary 2.6. *The inclusion $\mathcal{M}_u\subset\mathscr{B}^{r,s}_{[f]}$ strictly holds.*

3. Dual spaces

In the current section, we deal with the computation of the α, $\beta(bP)$ and γ-duals of the space $\mathscr{B}^{r,s}_{[f]}$. Before these, let us give some information related duals.

The $\alpha-$, $\beta(bP)-$ and $\gamma-$duals of a $\Psi\subset\Omega$ are described as

$$\begin{aligned}
\Psi^{\alpha} &:= \left\{t=(t_{ij})\in\Omega:\sum_{i,j}|t_{ij}u_{ij}|<\infty \quad \text{for all} \quad (u_{ij})\in\Psi\right\},\\
\Psi^{\beta(bP)} &:= \left\{t=(t_{ij})\in\Omega: bP-\sum_{i,j}t_{ij}u_{ij} \quad \text{exists for all} \quad (u_{ij})\in\Psi\right\},\\
\Psi^{\gamma} &:= \left\{t=(t_{ij})\in\Omega:\sup_{k,l\in\mathbb{N}}\left|\sum_{i,j=0}^{k,l}t_{ij}u_{ij}\right|<\infty \quad \text{for all} \quad (u_{ij})\in\Psi\right\},
\end{aligned}$$

respectively. It is well known that $\Psi^{\alpha}\subset\Psi^{\gamma}$ and if $\Psi\subset\Lambda$, then $\Lambda^{\alpha}\subset\Psi^{\alpha}$ for the double sequence spaces Ψ and Λ.

Theorem 3.1. $\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}=\mathcal{L}_u$.

Proof. To show the inclusion $\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}\subset\mathcal{L}_u$, assume the sequence $t=(t_{kl})\in\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}-\mathcal{L}_u$. So, $\sum_{k,l}|t_{kl}u_{kl}|<\infty$ for all $u=(u_{kl})\in\mathscr{B}^{r,s}_{[f]}$. If we consider $e=\sum_{k,l}e^{kl}$, we see that $e\in\mathscr{B}^{r,s}_{[f]}$. Since $te=t\notin\mathcal{L}_u$, we obtain from the equality $\sum_{k,l}|t_{kl}e|=\sum_{k,l}|t_{kl}|=\infty$ that $t\notin\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}$ which is a contradiction. Thus, it must be $t\in\mathcal{L}_u$ and the inclusion $\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}\subset\mathcal{L}_u$ is valid.

For the sufficiency part, let us take the sequences $t=(t_{kl})\in\mathcal{L}_u$ and $u=(u_{kl})\in\mathscr{B}^{r,s}_{[f]}$. Then, there exist a double sequence $v=(v_{kl})\in\mathcal{C}_f$ with the relation $v_{kl}=(B^{(r,s)}u)_{kl}$. Since $\mathcal{C}_f\subset\mathcal{M}_u$, then $\sup_{k,l}|v_{kl}|<M_1$, where $M_1\in\mathbb{R}^+$. Therefore,

$$\begin{aligned}
\sum_{k,l}|t_{kl}u_{kl}| &= \sum_{k,l}|t_{kl}|\left|\sum_{i,j=0}^{k,l}(-1)^{k+l-(i+j)}\binom{k}{i}\binom{l}{j}s^{k-l-i}r^{l-k-j}(r+s)^{i+j}v_{ij}\right|\\
&\le \sum_{k,l}|t_{kl}|\left|\frac{1}{r^k s^l}\sum_{i,j=0}^{k,l}\binom{k}{i}\binom{l}{j}(-s)^{k-i}(r+s)^{i}(-r)^{l-j}(r+s)^{j}\right||v_{ij}|\\
&\le M_1\sum_{k,l}|t_{kl}|\left|\frac{1}{r^k s^l}\sum_{i=0}^{k}\binom{k}{i}(-s)^{k-i}(r+s)^{i}\sum_{j=0}^{l}\binom{l}{j}(-r)^{l-j}(r+s)^{j}\right|\\
&= M_1\sum_{k,l}|t_{kl}|
\end{aligned}$$

and this says us that $t\in\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}$. Thus, it is seen that $\mathcal{L}_u\subset\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\alpha}$. □

Definition 3.2. *[16] A subset $E\subset\mathbb{N}^+\times\mathbb{N}^+$ is said to be uniformly of zero density if and only if the number of elements of E which lie in the rectangle R is $o(\lambda\mu)$ as $\lambda,\mu\to\infty$, uniformly in $k,l\ge 0$, where $R=\{(i,j):k\le i\le k+\lambda-1,\quad l\le j\le l+\mu-1\}$ and $\mathbb{N}^+=\{1,2,3,\ldots\}$.*

Now, let us describe the sets $w_1 - w_7$ that will be used in calculating $\beta(bP)-$ and $\gamma-$duals.

$$\begin{aligned}
w_1 &= \left\{ t=(t_{ij}) \in \Omega : P-\lim_{k,l\to\infty} \chi(k,l,i,j,m,n) = 0 \right\},\\
w_2 &= \left\{ t=(t_{ij}) \in \Omega : P-\lim_{k,l\to\infty} \sum_{i,j} \chi(k,l,i,j,m,n) = 1 \right\},\\
w_3 &= \left\{ t=(t_{ij}) \in \Omega : P-\lim_{k,l\to\infty} \sum_{i} |\chi(k,l,i,j,m,n)| = 0, \quad \forall j \in \mathbb{N} \right\},\\
w_4 &= \left\{ t=(t_{ij}) \in \Omega : P-\lim_{k,l\to\infty} \sum_{j} |\chi(k,l,i,j,m,n)| = 0, \quad \forall i \in \mathbb{N} \right\},\\
w_5 &= \left\{ t=(t_{ij}) \in \Omega : \exists M_2, M_3 \in \mathbb{N} \ni \sum_{i,j>M_2} |\chi(k,l,i,j,m,n)| < M_3 \right\},\\
w_6 &= \left\{ t=(t_{ij}) \in \Omega : bP-\lim_{k,l\to\infty} \sum_{i\in E}\sum_{j\in E} |\triangle_{10}\chi(k,l,i,j,m,n)| = 0 \right\},\\
w_7 &= \left\{ t=(t_{ij}) \in \Omega : bP-\lim_{k,l\to\infty} \sum_{i\in E}\sum_{j\in E} |\triangle_{01}\chi(k,l,i,j,m,n)| = 0 \right\},
\end{aligned}$$

where

$$\begin{aligned}
\chi(k,l,i,j,m,n) &= \sum_{m=i}^{k}\sum_{n=j}^{l} (-1)^{m+n-(i+j)} \binom{m}{i}\binom{n}{j} s^{m-n-i} r^{n-m-j} (r+s)^{i+j} t_{mn},\\
\triangle_{10}\chi(k,l,i,j,m,n) &= \chi(k,l,i,j,m,n) - \chi(k,l,i+1,j,m,n),\\
\triangle_{01}\chi(k,l,i,j,m,n) &= \chi(k,l,i,j,m,n) - \chi(k,l,i,j+1,m,n)
\end{aligned}$$

and E is uniformly of zero density.

Theorem 3.3. $\left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\beta(bP)} = \bigcap_{k=1}^{7} w_k$

Proof. Suppose that $t=(t_{kl}) \in \Omega$ and $u=(u_{kl}) \in \mathscr{B}^{r,s}_{[f]}$. Thus, $v=(v_{kl}) \in \left[\mathscr{C}_f\right]$ with $B^{(r,s)}u = v$. We obtain by the relation (2.5) that

$$\begin{aligned}
z_{kl} &= \sum_{i,j=0}^{k,l} t_{ij}u_{ij}\\
&= \sum_{i,j=0}^{k,l} t_{ij} \left\{ \sum_{m,n=0}^{i,j} (-1)^{i+j-(m+n)} \binom{i}{m}\binom{j}{n} s^{i-j-m} r^{j-i-n} (r+s)^{m+n} v_{mn} \right\}\\
&= \sum_{i,j=0}^{k,l} \left\{ \sum_{m=i}^{k}\sum_{n=j}^{l} (-1)^{m+n-(i+j)} \binom{m}{i}\binom{n}{j} s^{m-n-i} r^{n-m-j} (r+s)^{i+j} t_{mn} \right\} v_{ij}\\
&= (O^{r,s}v)_{kl}
\end{aligned} \tag{3.1}$$

for all $k,l \in \mathbb{N}$, where $O^{r,s} = \left(o^{r,s}_{klij}\right)$ defined by

$$o^{r,s}_{klij} = \begin{cases} \chi(k,l,i,j,m,n) &, \quad 0 \le i \le k,\ 0 \le j \le l,\\ 0 &, \quad \text{otherwise,} \end{cases}$$

for every $k,l,i,j \in \mathbb{N}$. In that case, by bearing in mind (3.1), it is infered that $tu = (t_{kl}u_{kl}) \in \mathscr{CS}_{bP}$ whenever $u=(u_{kl}) \in \mathscr{B}^{r,s}_{[f]}$ if and only if $z=(z_{kl}) \in \mathscr{C}_{bP}$ whenever $v=(v_{kl}) \in \left[\mathscr{C}_f\right]$. This implies that $t=(t_{kl}) \in \left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\beta(bP)}$ if and only if $O^{r,s} \in \left(\left[\mathscr{C}_f\right], \mathscr{C}_{bP}\right)$ and the proof is completed in view of Theorem 1 in [16].

□

Lemma 3.4. *[11] A 4d matrix $D=(d_{klij}) \in \left(\left[\mathscr{C}_f\right], \mathscr{M}_u\right)$ if and only if $D_{kl} \in \left\{\left[\mathscr{C}_f\right]\right\}^{\beta(\vartheta)}$ for all $k,l \in \mathbb{N}$ and*

$$\sup_{k,l\in\mathbb{N}} \sum_{i,j} \left|d_{klij}\right| < \infty. \tag{3.2}$$

Theorem 3.5. *$\left\{\mathscr{B}_{[f]}^{r,s}\right\}^{\gamma}=w_8\cap\mathscr{C}\mathscr{S}_{\vartheta}$, where*

$$w_8=\left\{t=(t_{ij})\in\Omega:\sup_{k,l\in\mathbb{N}}\sum_{i,j}|\chi(k,l,i,j,m,n)|<\infty\right\}.$$

Proof. We easily reach the proof by the aid of (ii) of Theorem 4.4 in [3]. So, we omit it. □

4. Matrix transformations

In this part, it will be given the classes $\left(\mathscr{B}_{[f]}^{r,s},\mathscr{C}_f\right)_{reg}$ and $\left(\mathscr{B}_{[f]}^{r,s},\mathscr{M}_u\right)$. Before these, it is needed to give the following lemma which will be used in Theorem 4.2.

Lemma 4.1. *[11] A 4d matrix $D=(d_{klij})\in\left(\left[\mathscr{C}_f\right],\mathscr{C}_f\right)_{reg}$ if and only if $D\in\left(\mathscr{C}_{bP},\mathscr{C}_f\right)_{reg}$ and $\sum_{i,j\in E}\left|\triangle_{11}d_{klij}\right|\to 0$ as $k,l\to\infty$ for each set E which is uniformly zero density where*

$$\triangle_{11}d_{klij}=d_{klij}-d_{kl,i+1,j}-d_{kli,j+1}+d_{kl,i+1,j+1}.$$

Theorem 4.2. *Consider the 4d infinite matrices $D=(d_{klij})$ and $H=(h_{klij})$ whose elements are connected with the equality*

$$h_{klij}=\sum_{a=i}^{\infty}\sum_{b=j}^{\infty}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}.$$

In that case, a 4d matrix $D=(d_{klij})\in\left(\mathscr{B}_{[f]}^{r,s},\mathscr{C}_f\right)_{reg}$ if and only if

$$D_{kl}\in\{\mathscr{B}_{[f]}^{r,s}\}^{\beta(\vartheta)}, \tag{4.1}$$

$$\sup_{k,l\in\mathbb{N}}\sum_{i,j}\left|h_{klij}\right|<\infty, \tag{4.2}$$

$$bP-\lim_{\rho,\rho'\to\infty}\sigma(i,j,\rho,\rho',m,n)=0,\quad \text{uniformly in } m,n\in\mathbb{N}, \tag{4.3}$$

$$bP-\lim_{\rho,\rho'\to\infty}\sum_{i,j}\sigma(i,j,\rho,\rho',m,n)=1,\quad \text{uniformly in } m,n\in\mathbb{N}, \tag{4.4}$$

$$bP-\lim_{\rho,\rho'\to\infty}\sum_{i}\left|\sigma(i,j,\rho,\rho',m,n)\right|=0,\quad \text{uniformly in } m,n\in\mathbb{N}, \tag{4.5}$$

$$bP-\lim_{\rho,\rho'\to\infty}\sum_{j}\left|\sigma(i,j,\rho,\rho',m,n)\right|=0,\quad \text{uniformly in } m,n\in\mathbb{N}, \tag{4.6}$$

$$\sum_{i,j\in E}\left|\triangle_{11}h_{klij}\right|\to 0,\quad k,l\to\infty \tag{4.7}$$

for each set E which is uniformly of zero density where $\sigma(i,j,\rho,\rho',m,n)=\sum_{k=m}^{m+\rho}\sum_{l=n}^{n+\rho'}\frac{h_{klij}}{(\rho+1)(\rho'+1)}$.

Proof. Suppose that the matrix $D=(d_{klij})\in\left(\mathscr{B}_{[f]}^{r,s},\mathscr{C}_f\right)_{reg}$. Then, Du exists and is in $\mathscr{C}_f$ for all $u=(u_{kl})\in\mathscr{B}_{[f]}^{r,s}$, which implies that $v=B^{(r,s)}u\in\left[\mathscr{C}_f\right]$ and $D_{kl}\in\left\{\mathscr{B}_{[f]}^{r,s}\right\}^{\beta(\vartheta)}$. Thus, condition (4.1) holds. We have the following equality derived from the $(\varsigma,\xi)th-$partial sums of the series $\sum_{i,j}d_{klij}u_{ij}$ by taking into account the relation between the terms of the sequences u and v,

$$\begin{aligned}\sum_{i,j}^{\varsigma,\xi}d_{klij}u_{ij} &= \sum_{i,j}^{\varsigma,\xi}d_{klij}\left[\sum_{a,b=0}^{i,j}(-1)^{i+j-(a+b)}\binom{i}{a}\binom{j}{b}s^{i-j-a}r^{j-i-b}(r+s)^{a+b}v_{ab}\right]\\ &= \sum_{i,j}^{\varsigma,\xi}\left[\sum_{a=i}^{\varsigma}\sum_{b=j}^{\xi}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}\right]v_{ij}\end{aligned} \tag{4.8}$$

for all $k,l,m,n\in\mathbb{N}$. Let us define the 4d matrix

$$h_{klij}:=\begin{cases}\sum_{a=i}^{\infty}\sum_{b=j}^{\infty}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab} &, \quad 0\leq i\leq k, 0\leq j\leq l,\\ 0 &, \quad \text{otherwise}\end{cases} \tag{4.9}$$

for all $k,l,i,j \in \mathbb{N}$. In that case, by taking f_2-limit on (4.8) as $\varsigma,\xi \to \infty$, it is seen that $Du = Hv$. Thus, if we take into account the fact that $D \in \left(\mathscr{B}^{r,s}_{[f]}, \mathscr{C}_f\right)_{reg}$ if and only if $H \in \left(\left[\mathscr{C}_f\right], \mathscr{C}_f\right)_{reg}$ with Lemma 4.1 and Theorem 3.1 in [39], we can reach the conditions (4.2)-(4.7).
Conversely, from the condition (4.1), Du exists for all $u=(u_{kl}) \in \mathscr{B}^{r,s}_{[f]}$ such that $v = B^{(r,s)}u \in \left[\mathscr{C}_f\right]$ and from (4.8) and (4.9), we see that $Du = Hv$. Furthermore, we reach that $H \in \left(\mathscr{C}_{bP}, \mathscr{C}_f\right)_{reg}$ by the aid of the conditions (4.2)-(4.6) and $H \in \left(\left[\mathscr{C}_f\right], \mathscr{C}_f\right)_{reg}$ from (4.7). Thus, $D \in \left(\mathscr{B}^{r,s}_{[f]}, \mathscr{C}_f\right)_{reg}$. □

Theorem 4.3. *A 4d matrix $D=(d_{klij}) \in \left(\mathscr{B}^{r,s}_{[f]}, \mathscr{M}_u\right)$ if and only if $D_{kl} \in \left\{\mathscr{B}^{r,s}_{[f]}\right\}^{\beta(\vartheta)}$ for all $k,l \in \mathbb{N}$ and the condition (3.2) holds.*

Proof. If we take into account the Lemma 3.4 with the 4d matrix H defined in Theorem 4.2 in place of the 4d matrix D, we can easily reach the proof. □

5. Conclusion

The concept of matrix domain was examined by several researchers on some single sequence spaces by using some special matrices. As we have mentioned some of them in the current paper, double sequence spaces which are obtained by using the domains of triangular 4d matrices have been studied by some authors recently. In the light of these and similar studies, as a natural continuation of the papers [1]-[3], we described two double sequence spaces by using the domain of 4d binomial matrix on the spaces of strongly almost convergent and strongly almost null double sequences. Moreover, we investigated their some properties and inclusion relations related them, computed duals and characterized some matrix classes. We conclude that the results obtained from the 4d binomial matrix $B^{(r,s)}$ is more general and extensive than the existent results obtained from the 4d Euler matrix $E(r,s)$. We expect that our results might be a reference for further studies in this field.

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Exploring a Simple Stochastic Mathematical Model Including Fear with a Linear Functional Response

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Abstract

This paper concentrates on a simple population model incorporating fear. Firstly, positivity and steady state analysis are performed, where the theoretical investigations show that change in the level of fear in prey population does not effect the local stability of the system around each equilibria (either stable or unstable). For the deterministic model, the numerical simulations are plotted for the density of prey species as a function of various system parameters. The stability analysis of the coexisting state shows that only transcritical bifurcation, where the steady states intersect, is observed. Secondly, the model is analysed with Gaussian noise term incorporated in the prey's death rate. The model comprising noise term turns the system into stochastic differential equations and irregular noise related oscillations are observed in the densities of both species.

1. Introduction

Due to the complexity of the ecosystem, it is often challenging to predict the effects of the various parameters in the dynamics of species using only statistical information. Therefore it is very crucial to incorporate mathematical formulations into research to better comprehend the reasons and outcomes of the dynamics of species in the ecosystem. In particular, theoretical formulations may help to foresee the required conditions for controlling ecological balance. In this context, the latest theories regarding the evolution of predator-prey type interactions shed lights on the prediction of population dynamics for multiple species, including their response to environmental conditions.

In mathematical terms, the impact of prey species response to predator species has been analysed using a wide variety functional responses. Here, a *functional response* refers to the consumption rate of a predator on prey [1]. The most common functional responses that have been used to model predator and prey interactions are Holling type I (i), Holling type II (ii) and Holling type III (iii) functional responses, classified by Crawford Stanley Holling [2]-[4]. In this paper, Holling type I, leading to a linear functional response will be taken into consideration.

The presence of prey species directly or indirectly depends on the presence of predators. Most of the available models analyse the direct influence of predator species on the prey species such as predation via direct killing. However, the change in the prey density may have also been affected due to indirect factors including psychological conditions such as fear. In fact, scared prey species may change their habitat and live in various conditions, where the quality of the new habitat may lead an increase or decrease in the density of prey population. There are many experimental studies to explore the role of fear in a population [5]-[7]. From the mathematical point of view, the fear effect has been thoroughly explored in recent years. For example, the role of fear in a prey-predator system with BeddingtonDeAngelis functional response has been studied by Pal *et al.* [8]. The instability of a population dynamics due to large fear parameters has been analysed by Wang and Zou

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[9]. The impact of group defence along with fear has been discussed by Sasmal and Takeuchi [10]. See [11]-[13] for other comprehensive works taking the fear effect into account.

Although previously studied deterministic models do not take stochastic effects into account, the random fluctuations may occur in nature due to climate change or some short term diseases which may affect the evolution of species. In fact these fluctuations may appear in any biological process. Therefore, it is worth to incorporate environmental noise in the model to better capture the dynamics of species in ecology. Many scientist have studied models with Gaussian noise from various point of view, including stochastic models with stage structure [14], foraging [15], anti-predator defence [13], intra-specific competition [16], leading to stochastic differential equations.

This paper is organised as follows: The deterministic model given in [17] is revisited in Section 2, where positivity of the model, its steady states and local stability analysis are derived respectively in Sections 2.1, 2.2 and 2.3. In Section 3, incorporation of the white Gaussian noise in the prey's death rate is discussed. Numerical simulations resulting from local stability analysis, as well as the comparison between deterministic model and stochastic model are demonstrated in Section 5. Furthermore the conditions under which prey and predator species go to extinction are provided in Section 4.

2. Deterministic dynamics of the model

A deterministic model based on the paper [17] is written as in the following:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r_0 x f_1(k,y) - dx - ax^2 - f_2(x)y := \mathscr{A}_1(x,y,t), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(-m + cf_2(x)) := \mathscr{B}_1(x,y,t), \tag{2.1}$$

where parameters r_0, k, d, a and m respectively stand for the birth rate for prey, the level of fear due to predator, the natural death rate of prey, the death rate of prey as a result of intra-species competition and the natural death rate of predator. Here c denotes the conversion rate from prey to predator biomass. All parameters stated in the model are positive for their biological meaning. The function f_1 denotes the fear factor and function f_2 represents linear functional response.

The function $f_1(k,y)$ satisfies the following conditions:

$$\begin{aligned} &f_1(0,y) = 1, \qquad f_1(k,0) = 1, \\ &\lim_{k\to\infty} f_1(k,y) = 0, \qquad \lim_{y\to\infty} f_1(k,y) = 0, \\ &\frac{\partial f_1(k,y)}{\partial k} < 0, \qquad \frac{\partial f_1(k,y)}{\partial y} < 0. \end{aligned} \tag{2.2}$$

and $f_2(x) = px$ is taken as a linear function. To make the system mathematically attainable, fear factor is adopted in a form of $f_1(k,y) = 1/1+ky$, that satisfies the conditions given in equation (2.2) [17].

2.1. Positivity

It is straightforward to show that all solutions $(x(t), y(t))$ of the model given by (2.1) are non-negative with positive initial conditions, i.e. $(x_0, y_0) \in \mathbb{R}^2_+$ for $\forall t \in \mathbb{R}_+$. Using the first equation in model (2.1), one can write that

$$\frac{\mathrm{d}x}{x} = \mathscr{A}_2(x,y,t)\mathrm{d}t, \tag{2.3}$$

where $\mathscr{A}_2(x,y,t) = r_0/(1+ky) - d - ax - py$. Integrating both sides of equation (2.3) it follows that

$$\ln x - \ln x_0 = \int_0^t \mathscr{A}_2(x,y,s')\mathrm{d}s'. \tag{2.4}$$

Subtracting x from equation (2.4) leads to

$$x(t) = x_0 \exp\left\{ \int_0^t \mathscr{A}_2(x,y,s')\mathrm{d}s' \right\}, \quad \forall t > 0.$$

Similarly, positivity of the predator variable y can be shown as

$$\frac{\mathrm{d}y}{y} = \mathscr{B}_2(x,t)\mathrm{d}t, \quad \Rightarrow \quad y = y_0 \exp\left\{ \int_0^t \mathscr{B}_2(x,s')\mathrm{d}s' \right\}, \quad \forall t > 0.$$

where $\mathscr{B}_2(x,s) = -m + cpx$. Since $(x_0, y_0) \in \mathbb{R}^2_+$ for $\forall t > 0$, it is obtained that $x(t) > 0$ and $y(t) > 0$. Thus the interior of $\mathbb{R}^2_+$ is an invariant set of model (2.1). This is also biologically meaningful as the densities of prey and predator species are expected to be non-negative.

2.2. Steady state analysis

The model given by (2.1) has three steady states which can be found using $\mathscr{A}_1(x,y,t)=0$ and $\mathscr{B}_1(x,y,t)=0$. The first steady state is the trivial equilibrium $E_0=(0,0)$ where both prey and predator populations go to extinction. The second steady state is semi trivial state $S_1=((r_0-d)/a,0)$ bywhich prey population still exists in the system in the absence of predator. The last steady state is coexisting state, say $S_2=(x*,y*)$, where both prey and predator populations exist in the system. Here $x*=m/cp$ can be easily found using $\mathscr{B}_1(x,y,t)=0$ for $y\neq 0$. The critical value for predator $y*$ are obtained solving

$$\frac{r_0}{1+ky}-d-ax-py=0, \tag{2.5}$$

which can be rewritten for coexisting state as

$$(py*+h)(ky+1)=r_0, \qquad h=d+am/cp.$$

Here equation (2.5) has two roots one of which may be positive (thus biologically meaningful) under the condition $r_0>h$, for which coexisting state $S_2=(x*,y*)$ exists.

In addition the nullclines of the model can be similarly found:

$$\begin{aligned} x=m/cp, &\quad \text{(straight line for predator nullcline)},\\ pky^2+(hk+p)y+h-r_0=0, &\quad \text{(parabolic curve for prey nullcline)}. \end{aligned}$$

2.3. Local stability analysis around coexisting state

Stability of the system (2.1) can be determined using linearisation argument, where the prey and predator densities are perturbed from their steady state. Considering

$$x=x*+\tilde{x}, \quad \text{and} \quad y=y*+\tilde{y},$$

where accents $\tilde{}$ represent the perturbed variables. Substituding these in the original model the following linearised system of equations are obtained:

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= \left(\frac{r_0}{1+ky*}-d-2ax*-py*\right)\tilde{x}+\left(-\frac{kr_0x*}{(1+ky*)^2}-px*\right)\tilde{y},\\ \frac{d\tilde{y}}{dt} &= cpy*\tilde{x}+(cpx*-m)\tilde{y}. \end{aligned}$$

The coefficient matrix for the above linear system of equations can be written as

$$\mathscr{M}_J=\begin{pmatrix} \frac{r_0}{1+ky*}-d-2ax*-py* & -\frac{kr_0x*}{(1+ky*)^2}-px* \\ cpy* & cpx*-m \end{pmatrix}$$

for which corresponding characteristic polynomial are obtained using $\mathrm{Det}\left(\mathscr{M}_J-\mu\mathscr{I}\right)=0$ where $\mathscr{I}$ is a 2×2 unit matrix and μ is the eigenvalues of the system for local stability. The characteristic equation can be explicitly written as

$$\mu^2-\mathscr{E}_1(\mathscr{M}_J)\mu+\mathscr{E}_2(\mathscr{M}_J)=0, \tag{2.6}$$

where

$$\begin{aligned} \mathscr{E}_1 &= \frac{r_0}{1+ky*}-d-2ax*-py*+cpx*-m,\\ \mathscr{E}_2 &= \left(\frac{r_0}{1+ky*}-d-2ax*-py*\right)(cpx*-m)+\left(\frac{kr_0x*}{(1+ky*)^2}+px*\right)cpy* \end{aligned}$$

and the roots for the equation (2.6) is given as

$$\mu_{1,2}=\frac{1}{2}\left[\mathscr{E}_1(\mathscr{M}_J)\pm\sqrt{\mathscr{E}_1(\mathscr{M}_J)-4\mathscr{E}_2(\mathscr{M}_J)}\right].$$

It is worth noting that $\mathscr{E}_1$ and $\mathscr{E}_2$ respectively stand for trace and determinant of the matrix $\mathscr{M}_J$.

In Figure 2.1, stability of the prey density is shown with regard to parameters p,k,a,r_0 that respectively stand for the strength of linear functional response, rate of fear due to predator, the death rate of prey due to intraspecies competition and the birth rate of prey. As seen there is a switch in the stability when eigenvalues crosses the transcritical bifurcation, that is a typical case in population models referring to invasion. In fact transcritical bifurcation demonstrates the onset of coexisting state in the system [18, 19]. Here the dashed line indicates unstable state where the number of eigenvalues with positive real part is 1, whereas the straight line indicates the stable state with both eigenvalues are found with negative real part.

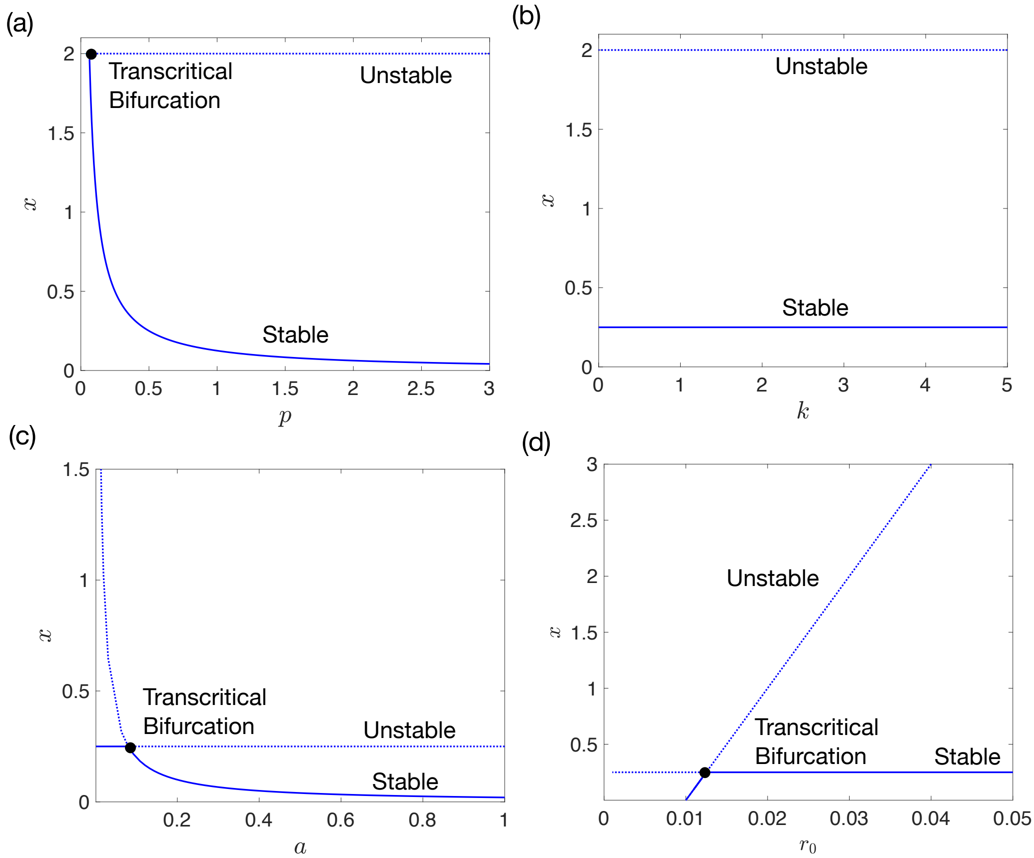

Figure 2.1: Transcritical bifurcation occurs in the model (2.1) with respect to various parameters, i.e. p, k, a and r_0. See text around the system (2.1) for biological interpretations of these parameters.

3. Analysis of the model with white Gaussian noise

Deterministic model provided in Section 2 does not comprise the role of environmental fluctuations which may be any unpredictable factor including the quality of food, climate change, diseases as well as temperature [14, 20]. Hence incorporating random noise in model parameters may significantly alter dynamics of both prey and predator species. Although any parameter of the model may be affected by environmental noise, the uncertain growth and death rates may be particularly influenced, see for example [16, 21, 22].

$$\mathrm{d}x = \left[r_0 x f_1(k,y) - dx - ax^2 - pxy\right]\mathrm{d}t - \varepsilon x \mathrm{d}\beta, \qquad \mathrm{d}y = \left[cpxy - my\right]\mathrm{d}t. \tag{3.1}$$

which can be also written as

$$\mathrm{d}x = \mathscr{A}_1(x,y,t)\mathrm{d}t - \varepsilon x \mathrm{d}\beta, \qquad \mathrm{d}y = \mathscr{B}_1(x,y,t)\mathrm{d}t. \tag{3.2}$$

where $\beta = \{\beta(t); t > 0\}$ denotes standart Wiener process. Here ε represents noise parameters. Here we assume that the death rate of prey could be noisy, thus $d = d + \varepsilon\dot{\beta}(t)$. The presence of noise terms turns the model (2.1) into a system of stochastic differential equations. The numerical solutions of the system (3.1) can be found using Euler Maruyama method [23].

4. Extinction probability

The notion of extinction is one of the key subjects in population dynamics. In biological terms the extinction occurs in a population if there is no individual that can reproduce or create a new generation in a long term [12]. In mathematical terms a population goes to extinction with probability one if $\lim_{t\to\infty} \mathscr{X}(t) = 0$, where $\mathscr{X}$ is the density of a population. Thus the conditions for which the prey and predator species go to extinction can be found in a fluctuating environment as given in system (3.1).

In this context from the first bit of stochastic equation (3.1) it can be written that

$$\mathrm{d}x = \left[r_0 x f_1(k,y) - dx - ax^2 - pxy\right]\mathrm{d}t - \varepsilon x \mathrm{d}\beta. \tag{4.1}$$

Considering $U(t)=\ln x(t)$ and $V(t)=\ln y(t)$ and applying Itô formula, the equation (4.1) can be written as

$$\begin{aligned}
\mathrm{d}\ln x(t) &= \left[\frac{\partial U}{\partial t}+\mathscr{A}_1(x,y,t)\frac{\partial U}{\partial x}+\frac{1}{2}(-\varepsilon x)^2\frac{\partial^2 u}{\partial x^2}\right]\mathrm{d}t+(-\varepsilon x)\frac{\partial U}{\partial x}\mathrm{d}\beta, &(4.2)\\
&= \left(\mathscr{A}_2(x,y,t)-\frac{\varepsilon^2}{2}\right)\mathrm{d}t-\varepsilon\mathrm{d}\beta, &(4.3)
\end{aligned}$$

with initial values $U(0)=\ln U_0$ and $V(0)=\ln V_0$. Equation (4.3) can be rewritten as

$$\begin{aligned}
\mathrm{d}U(t) &= \left(\mathscr{A}_2(\mathrm{e}^U,\mathrm{e}^V,t)-\frac{\varepsilon^2}{2}\right)-\varepsilon\mathrm{d}\beta,\\
&= \left(\frac{r_0}{1+k\mathrm{e}^V}-d-a\mathrm{e}^U-p\mathrm{e}^V-\frac{\varepsilon^2}{2}\right)\mathrm{d}t-\varepsilon\mathrm{d}\beta,\\
&\leq \left(r_0-d-a\mathrm{e}^U-\frac{\varepsilon^2}{2}\right)\mathrm{d}t-\varepsilon\mathrm{d}\beta,\\
\mathrm{d}\ln x &\leq \left(r_0-d-ax-\frac{\varepsilon^2}{2}\right)\mathrm{d}t-\varepsilon\mathrm{d}\beta. &(4.4)
\end{aligned}$$

Taking $\tilde{f}(x)=r_0-d-ax-\varepsilon^2/2$ and finding the supremum of $\tilde{f}$ one can derive that $\tilde{f}'(x)=-a<0$. This implies that $\tilde{f}$ is an decreasing function and has maximum at $x=0$. Thus it can be written that $\tilde{f}(0)=r_0-d-\varepsilon^2/2$. From equation (4.4) it follows that

$$\ln x\leq \ln x_0+\left(r_0-d-\frac{\varepsilon^2}{2}\right)t-\varepsilon\beta:=\mathscr{H}(t),$$

from this

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t}\leq \limsup_{t\to\infty}\mathscr{H}(t)=r_0-d-\frac{\varepsilon^2}{2}.$$

thus this concludes that one can have $\limsup\limits_{t\to\infty}\frac{\ln x}{t}\leq 0$ with the condition $r_0\leq d+\frac{\varepsilon^2}{2}$ and this leads to $\lim\limits_{t\to\infty}x(t)=0$.

5. Numerical simulations

In Figure 5.1, the evolution of prey density with respect to time is shown for different noise strengths where $\varepsilon=0$ (a), $\varepsilon=0.002$ (b), $\varepsilon=0.01$ (c) and $\varepsilon=0.02$ (d). The red and magenta lines represent the density of prey species deterministic and stochastic models respectively. The system is stable with damping oscillations, leading to a stable spiral with complex eigenvalues having negative real part, in the absence of noise as demonstrated in Fig. 5.1(a), corresponding to deterministic model. For a small perturbation, e.g. $\varepsilon=0.002$, the stochastic model demonstrates similar behaviour to deterministic model. Increasing noise term a bit further, irregular and large amplitude oscillatory dynamics for a stochastic model is observed, see 5.1(c,d).

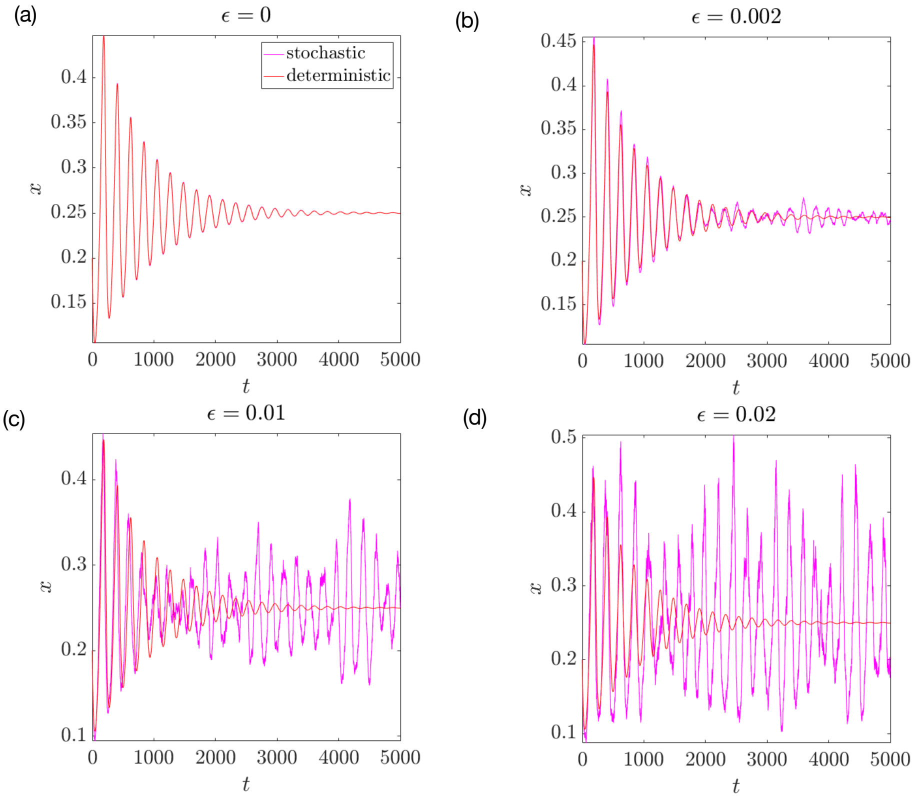

Figure 5.1: Comparison for time evolution of prey (x) species between the deterministic model (2.1) and stochastic model (3.1) based on different values of noise parameter $\varepsilon = 0$(a), $\varepsilon = 0.002$(b), $\varepsilon = 0.01$(c) and $\varepsilon = 0.02$(d) with initial conditions $(x_0, y_0) = (0.2, 0.1)$. Red and magenta colors respectively correspond to deterministic and stochastic dynamics of prey population.

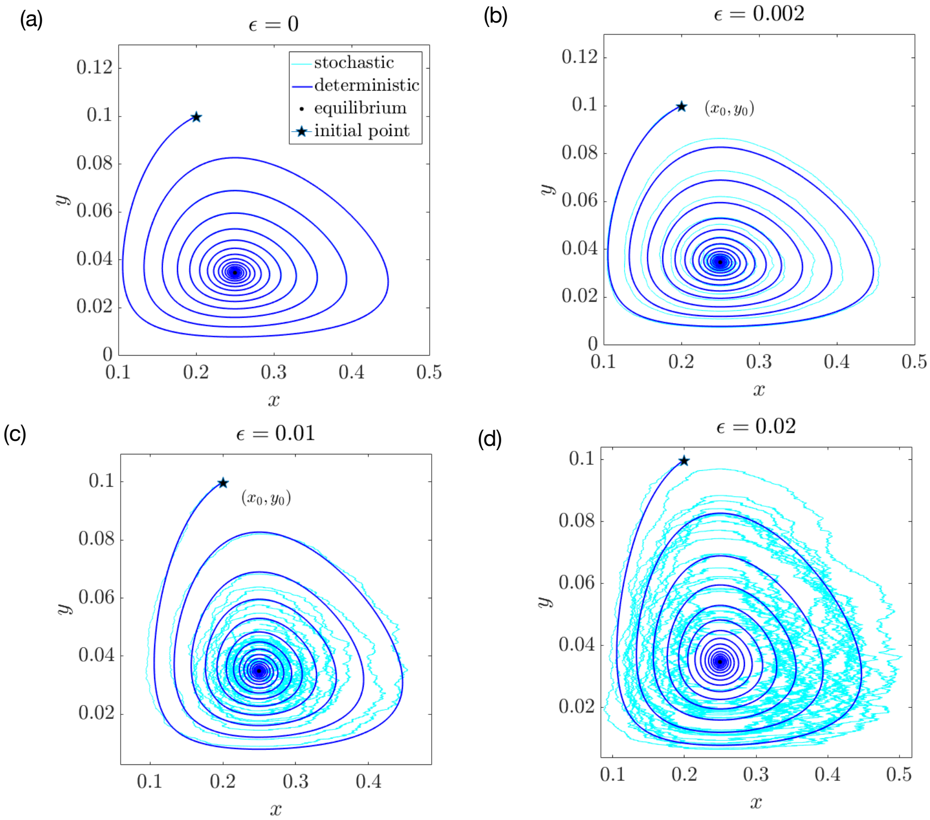

Figure 5.2: The comparison between phase portraits of two-component deterministic and stochastic prey-predator model, respectively presented in (2.1) and (3.1) for different values of noise terms $\varepsilon = 0$(a), $\varepsilon = 0.002$(b), $\varepsilon = 0.01$(c) and $\varepsilon = 0.02$(d) with initial conditions $(x_0, y_0) = (0.2, 0.1)$. (black star). Steady state of the system is given by black point at $(x_s, y_s) = (0.25, 0.0443)$.

In Section 4 the analytical conditions for extinction of prey and predator species for the stochastic model are developed. Here it has been discussed that the extinction of species depends on the strength of noise term. It is found that high values of perturbation may lead extinction of both species. In Figure 5.3, it is demonstrated that prey and predator are wiped out from the system with the condition $r_0 \leq d+\varepsilon^2/2$ where $\varepsilon = 0.2$, $r_0 = 0.03$ and $d = 0.01$.

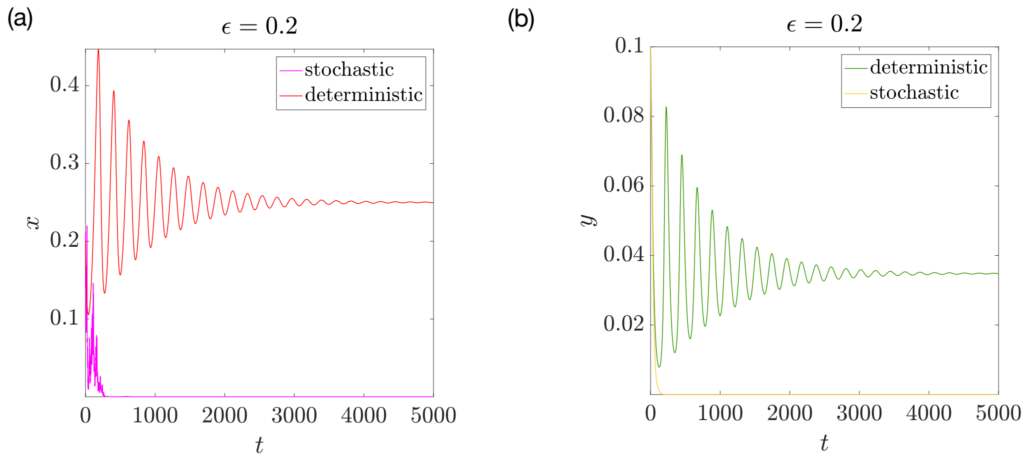

Figure 5.3: Extinction of prey and predator species with the condition $r_0 \leq d+\varepsilon^2/2$, where $\varepsilon = 0.2$, $r_0 = 0.03$ and $d = 0.01$.

In Figure 5.4, the corresponding phase portrait of the extinction case given in Fig. 5.3 is plotted for $\varepsilon = 0.2$. The blue and cyan color respectively stand for deterministic and stochastic phase diagram. Here the initial condition is $(x_0, y_0) = (0.2, 0.1)$. (black star) and the steady state of the system is given by black point at $(x_s, y_s) = (0.25, 0.0443)$. Increasing the noise term, irregular nonperiodic random peaks are observed more frequently which can be also observed from the phase planes.

All simulations are performed using parameters: $r_0 = 0.03$, $k = 0.1$, $d = 0.01$, $a = 0.01$, $p = 0.5$, $m = 0.05$, $c = 0.4$ with increasing values of noise strengths (ε).

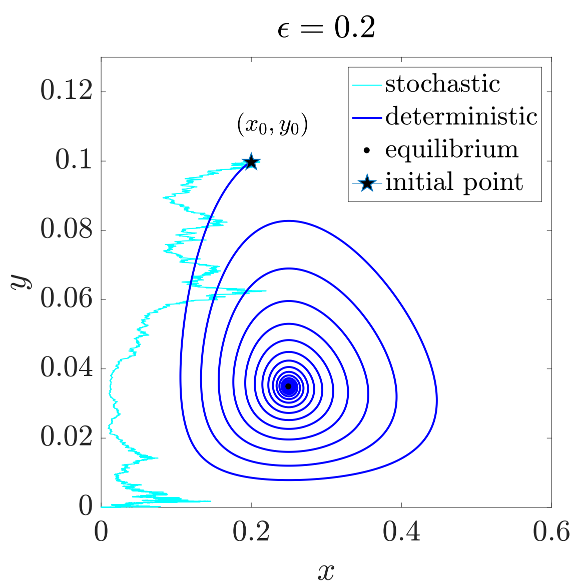

Figure 5.4: Phase diagram of the extinction state for prey and predator populations corresponding to Figure (5.3).

6. Conclusion

Since natural fluctuations appear in many biological system, analyses performed on prey-predator type interactions with stochastic effects are much realistic compared to deterministic models. In this paper, the fear effect in a population model with a linear functional response is considered with random perturbation in prey's death rate. The model in the absence of white Gaussian noise is based on the paper written by Wang *et al.* [17], though without taking stochastic effects into account. Here a further analysis of the deterministic model including its positivity is presented. Compared to its deterministic version, it is also demonstrated that the high and low values of noise strength, denoted with ε, in the stochastic system give rise to rich spectrum of interesting results. Furthermore, excessive noise strength may induces both species to undergo extinction with $r_0 \leq d + \varepsilon^2/2$. This result is also biologically understandable as high environmental fluctuations may have a drastic impact on the populations and may lead extinction.

Numerical bifurcation is performed for different parameters and a transcritical point where stability changes is observed. In fact, linear functional response is the simplest choice and one could expect more interesting dynamics with other functional responses such as Holling type II and Holling type III. Although it is more challenging to perform analytical results for stochastic system, the theoretical conditions where prey and predator species undergo extinction are determined. Here it is found that high level of noise increase the probability of both species to be wiped out from the system.

One straightforward extension of this work would be to incorporate local and non-local delay terms in the model. In fact the interactions between prey and predator are not straightforward and require some time lag, e.g. gestation period. Then the stochastic model would be extended to comprise delay terms. It is well known that delay term in the model supports periodic oscillations,where Hopf bifurcation may occur through a limit cycle around the coexisting state. Therefore, more complex behaviour in the dynamics of both species are expected [24].

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Competing interests

No potential conflict of interest was reported by the author.

Author's contributions

The author developed the mathematical formulas for stochastic model and performed corresponding numerical simulations, the author has also written the original manuscript and gave final approval of the current version and any revised version to be submitted to the journal.

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