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On the Solutions of the Higher Order Fractional Differential Equations of Riesz Space Derivative with Anti-Periodic Boundary Conditions

Şuayip Toprakseven[1]*

Abstract
We present existence and uniqueness results for a class of higher order anti-periodic fractional boundary value problems with Riesz space derivative which is two-sided fractional operator. The obtained results are established by applying some fixed point theorems. Various numerical examples are given to illustrate the obtained results.

Keywords: Anti periodic boundary conditions, existence, fixed point Theorem, fractional boundary value problem, Riesz Caputo derivative
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[1] *Faculty of Engineering, Department of Computer Science, Artvin Çoruh University, Artvin, 08100, Turkey, ORCID:0000-0003-3901-9641*
***Corresponding author**: topraksp@artvin.edu.tr
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1. Introduction

Recently, many researchers have investigated a large range of problems including fractional differential equations. A variety of scientific areas such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, biology, control theory, etc involve fractional differential equations. Some applications and detailed explanation of fractional differential equations can be found in the books [1, 2, 3] and references [7, 29, 16]. Geometric and physical interpretation of fractional differentiation and integration can be found in the paper [27]. Existence results for fractional differential equations have studied and developed by many authors; see the books [26, 4, 2] and references [11, 12, 24, 15, 9, 26, 4, 2, 17, 18, 19, 30, 31, 39, 41, 42, 43, 44, 45] and references therein.

Much of recent works on fractional boundary/initial value problems involve Riemann-Lioville and Caputo derivatives in the literature. Unfortunately, these fractional operators are one-sided operators which hold either past or future memory effects. Unlike these fractional operators, the Riesz space fractional operator is two-sided operator which holds both the history and future non-local memory effects. This is important in the mathematical modelling for physical processes on a finite domain because the present states depend both on the past and future memory effects. As an example, Riesz fractional derivative has been used for the memory effects in both past and future concentrations in the anomalous diffusion problem [13, 5].

Numerical solutions of the fractional calculus, specifically in the anomalous diffusion that involves the Riesz derivative have been presented in [13, 8, 5, 38]. Analytical and numerical solutions for fractional differential equations using different definitions for fractional derivatives and integrals have been proposed and studied in the literature [28, 32, 33, 34, 35, 21, 36, 37]. Recently, there are papers on existence and positive solutions for the fractional boundary value problems with the Riesz-Caputo derivative [14, 25, 20].

The mathematical modelling of many physical phenomena can be expressed in terms of anti-periodic boundary value problems [10]. Recently, a large amount of papers are devoted to anti-periodic boundary value problems, for example, see [22, 23] and references therein.

In this paper, we study the existence and uniqueness of solutions for the following anti-periodic boundary value problem of the Riesz-Caputo fractional differential equations

$$\begin{aligned} &{}^{RC}_{0}D^{\nu}_{T}u(\eta) = F(\eta, u(\eta)) \quad \nu \in (2,3], \quad 0 \le \eta \le T, \\ &u(0)+u(T)=0, \quad u'(0)+u'(T)=0, \quad u''(0)+u''(T)=0, \end{aligned} \tag{1.1}$$

where ${}^{RC}_{0}D^{\nu}_{T}$ is the Riesz-Caputo derivative defined below and $F:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The remainder of paper is organized as follows. Section 2 introduces some preliminaries, definitions and lemmas which are useful in proving main results. Section 3 provides some sufficient conditions for the existence and the uniqueness of solutions of the problem (1.1) with anti-periodic boundary conditions. Finally, some numerical examples are given to illustrate the applications of the main results in the last section.

2. Preliminaries

This section is devoted to some important definitions and lemmas that will be needed in the sequel.

Definition 2.1. *[26] Let $\nu > 0$. The left and right Riemann-Liouville fractional integral of a function $f \in C[0,T]$ of order ν defined as, respectively*

$$I^{\nu}_{0} f(x) = \frac{1}{\Gamma(\nu)} \int_{0}^{x} (x-s)^{\nu-1} f(s)\, ds, \quad x \in [0,T].$$

$${}_{T}I^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{T} (s-x)^{\nu-1} f(s)\, ds, \quad x \in [0,T].$$

Definition 2.2. *(Riesz Fractional Integral) Let $\nu > 0$. The Riesz fractional integral of a function $f \in C[0,T]$ of order ν defined as*

$${}_{0}I^{\nu}_{T} f(x) = \frac{1}{2\Gamma(\nu)} \int_{0}^{T} |x-s|^{\nu-1} f(s)\, ds, \quad x \in [0,T].$$

Note that the Riesz fractional integral operator can be written as

$${}_{0}I^{\nu}_{T} f(x) = \frac{1}{2}\Big(I^{\nu}_{0} f(x) + {}_{T}I^{\nu} f(x)\Big) \tag{2.1}$$

Definition 2.3. *[26] Let $\nu \in (n, n+1], n \in \mathbb{N}$. The left and right Caputo fractional derivative of a function $f \in C^{n+1}[0,T]$ of order ν defined as, respectively*

$${}^{C}_{0}D^{\nu}_{x} f(x) = \frac{1}{\Gamma(n+1-\nu)} \int_{0}^{x} (x-s)^{n-\nu} f^{(n+1)}\, ds = (I^{n+1-\nu}_{0} D^{n+1}) u(x).$$

$${}^{C}_{x}D^{\nu}_{T} f(x) = \frac{(-1)^{n+1}}{\Gamma(n+1-\nu)} \int_{x}^{T} (s-x)^{n-\nu} f^{(n+1)}\, ds = (-1)^{n+1} ({}_{T}I^{n+1-\nu} D^{n+1}) u(x).$$

where D is the ordinary differential operator.

Definition 2.4. *Let $\nu \in (n, n+1], n \in \mathbb{N}$. The Riesz-Caputo fractional derivative ${}^{RC}_{0}D^{\nu} f$ of order ν of a function $f \in C^{n+1}[0,T]$ defined by*

$$\begin{aligned} {}^{RC}_{0}D^{\nu}_{T} f(x) &= \frac{1}{\Gamma(n+1-\nu)} \int_{0}^{T} |x-s|^{n-\nu} f^{(n+1)}(s)\, ds \\ &= \frac{1}{2}\Big({}^{C}_{0}D^{\nu}_{x} f(x) + (-1)^{n+1}\, {}^{C}_{x}D^{\nu}_{T} f(x)\Big) \\ &= \frac{1}{2}\Big((I^{n+1-\nu}_{0} D^{n+1}) u(x) + (-1)^{n+1} ({}_{T}I^{n+1-\nu} D^{n+1}) u(x)\Big). \end{aligned}$$

Lemma 2.5. *[26] Let $f \in C^n[0,T]$ and $\nu \in (n,n+1]$. Then we have the following relations*

$$I_0^{\nu}{}_0^C D_x^{\nu} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

$$_T I^{\nu}{}_x^C D_T^{\nu} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k.$$

In the case when $\nu \in (2,3]$ and $f(x) \in C^3(0,T)$ we have

$$\begin{aligned} {}_0I_T^{\nu}\, {}_0^{RC}D_T^{\nu} f(x) =& \frac{1}{2}\left(I_0^{\nu}{}_0^C D_x^{\nu} f(x) - {}_T I^{\nu}{}_x^C D_T^{\nu} f(x)\right) \\ =& f(x) - \frac{1}{2}(f(0)+f(T)) - \frac{1}{2}(f'(0)+f'(T))x + \frac{T}{2}f'(T) \\ & - \frac{1}{4}(f''(0)+f''(T))x^2 - \frac{T^2-2Tx}{4}f''(T). \end{aligned} \tag{2.2}$$

The following fixed point theorems will be needed to establish the existence results.

Theorem 2.6. *[6] Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be the operators such that*

(i) $Ax+By \in M$ whenever $x,y \in M$;

(ii) A is compact and continuous;

(iii) B is a contraction mapping.

Then there exists $u \in M$ such that $u = Au + Bu$.

Theorem 2.7. *[6] Let X be a Banach space. Assume that O is an open bounded subset of X with $\theta \in O$ and let $T : \overline{O} \to X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial O.$$

Then T has a fixed point in $\overline{O}$.

Lemma 2.8. *Assume that $g \in C([0,T],\mathbb{R})$. A unique solution $u \in C^3([0,T])$ of the following fractional boundary problem*

$$\begin{aligned} &{}_0^{RC}D_T^{\nu} u(\eta) = g(\eta) \quad \nu \in (2,3], \quad 0 \leq \eta \leq T, \\ &u(0)+u(T) = 0, \quad u'(0)+u'(T) = 0, \quad u''(0)+u''(T) = 0, \end{aligned} \tag{2.3}$$

is given as

$$\begin{aligned} u(\eta) =& \frac{2T^2-2T\eta}{4\Gamma(\nu-2)} \int_0^T (T-s)^{\nu-3} g(s)\,ds - \frac{T}{2\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} g(s)\,ds \\ & + \frac{1}{\Gamma(\nu)} \int_0^T |\eta-s|^{\nu-1} g(s)\,ds. \end{aligned} \tag{2.4}$$

Proof. We infer from (2.2) and (2.3) that

$$\begin{aligned} u(\eta) =& \frac{1}{2}(u(0)+u(T)) + \frac{1}{2}(u'(0)+u'(T))\eta - \frac{T}{2}u'(T) \\ & \frac{1}{4}(u''(0)-u''(T))\eta^2 + \frac{T^2-2T\eta}{4}u''(T) + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1} g(s)\,ds. \end{aligned} \tag{2.5}$$

The anti-periodic boundary conditions $u(0)+u(T)=0, \quad u'(0)+u'(T)=0, \quad u''(0)+u''(T)=0$ imply that

$$u(\eta) = -\frac{T}{2}u'(T) + \frac{T^2-2T\eta}{4}u''(T) + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1} g(s)\,ds. \tag{2.6}$$

Then,

$$\begin{aligned}
u'(\eta) &= -\frac{T}{2}u''(T) + \frac{1}{\Gamma(\nu-1)}\int_0^\eta (\eta-s)^{\nu-2}g(s)\,ds - \frac{1}{\Gamma(\nu-1)}\int_\eta^T (s-\eta)^{\nu-2}g(s)\,ds,\\
u''(\eta) &= \frac{1}{\Gamma(\nu-2)}\int_0^\eta (\eta-s)^{\nu-3}g(s)\,ds + \frac{1}{\Gamma(\nu-2)}\int_\eta^T (s-\eta)^{\nu-3}g(s)\,ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
u'(T) &= -\frac{T}{2}\Big(\frac{1}{\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}g(s)\,ds\Big) + \frac{1}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}g(s)\,ds,\\
u''(T) &= \frac{1}{\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}g(s)\,ds.
\end{aligned} \tag{2.7}$$

Plugging the equations in (2.7) into (2.6) gives

$$u(\eta) = \frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}g(s)\,ds - \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}g(s)\,ds \qquad + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}g(s)\,ds,$$

which completes the proof. □

3. Existence of Solutions

We prove the main results of the paper in this section. Let $C[0,T]$ be the space of continuous functions u defined on $[0,T]$ with the norm $\|u\| = \sup_{\eta\in[0,T]}|u(\eta)|$. We assume the following conditions on F are satisfied.

(H1) F satisfies a Lipschitz condition in the second variable, that is,

$$|F(\eta,u)-F(\eta,v)| \le L|u-v|, \forall \eta\in[0,T], u,v\in\mathbb{R}.$$

(H2) F is dominated by a L^1 function, that is,

$$|F(\eta,u)| \le \ell(\eta), \forall(\eta,u)\in[0,T]\times\mathbb{R}, \text{ and } \ell\in L^1([0,T],\mathbb{R}^+).$$

Theorem 3.1. *Let $F\in C([0,T]\times\mathbb{R},\mathbb{R})$ satisfy the assumption (H1) with*

$$L \le \frac{2\Gamma(\nu+1)}{T^\nu(8+\nu(\nu+1))}.$$

Then the problem (1.1) has a unique solution.

Proof. We convert the problem (1.1) into a fixed point solution of operator $\mathscr{T}: C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ defined by

$$\begin{aligned}
(\mathscr{T}u)(\eta) = &\frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}F(s,u(s))\,ds - \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}F(s,u(s))\,ds\\
&+\frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}F(s,u(s))\,ds, \quad \eta\in[0,T].
\end{aligned}$$

We shall prove that the operator $\mathscr{T}$ has a fixed point by showing that $\mathscr{T}$ is a contraction. To this end, we first demonstrate that $\mathscr{T}S_r \subset S_r$ where $S_r = \{u\in C([0,T],\mathbb{R}) : \|u\|\le r\}$ with $r \ge \dfrac{KT^\nu(8+\nu(\nu+1))}{2\Gamma(\nu+1)}$ and $K := \sup_{\eta\in[0,T]}|F(\eta,0)|$. For $u\in S_r$, we have

$$
\begin{aligned}
|(\mathscr{T}u)(\eta)| \leq & \frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}|F(s,u(s))|\,ds + \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds \\
& + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}|F(s,u(s))|\,ds \\
\leq & \frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}(|F(s,u(s))-F(s,0)|+|F(s,0)|)\,ds + \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}(|F(s,u(s)) \\
& -F(s,0)|+|F(s,0)|)\,ds + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}(|F(s,u(s))-F(s,0)|+|F(s,0)|)\,ds \\
\leq & (Lr+K)\Big(\frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}\,ds + \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}\,ds + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}\,ds\Big) \\
\leq & (Lr+K)\Big(\frac{T^\nu}{\Gamma(\nu+1)}(2+\frac{\nu(\nu+1)}{4})\Big) \leq r.
\end{aligned}
$$

Next, for $u,v \in C([0,T),\mathbb{R})$ and for any $\eta \in [0,T]$, we get

$$
\begin{aligned}
& \big|(\mathscr{T}u)(\eta)-(\mathscr{T}v)(\eta)\big| \\
\leq & \frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}|F(s,u(s))-F(s,v(s))|\,ds \\
& + \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))-F(s,v(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}|F(s,u(s))-F(s,v(s))|\,ds \\
\leq & L\|u-v\|\Big(\frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}\,ds + \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}\,ds + \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}\,ds\Big) \\
& \leq \Big(\frac{LT^\nu}{\Gamma(\nu+1)}(2+\frac{\nu(\nu+1)}{4})\Big)\|u-v\| < \|u-v\|.
\end{aligned}
$$

This shows that $\mathscr{T}$ is a contraction. Therefore, the Banach fixed point theorem tells us $\mathscr{T}$ has a fixed point which is a solution to the problem (1.1). □

Theorem 3.2. *Let $F \in C([0,T]\times\mathbb{R},\mathbb{R})$ be a completely continuous function. Assume that the conditions (H1) and (H2) hold with $\dfrac{LT^\nu(\nu+1)}{4\Gamma(\nu)} < 1$. Then the fractional boundary problem with anti-periodic boundary conditions (1.1) has a solution on $[0,T]$.*

Proof. Let $S_r = \{u \in C([0,T],\mathbb{R}) : \|u\| \leq r$ be the ball of radius r with $r \geq \dfrac{\|\ell\|_{L^1}T^\nu}{\Gamma(\nu+1)}(2+\dfrac{\nu(\nu+1)}{4})$, where $\|\ell\|_{L^1} = \int_0^T |\ell(s)|\,ds$. We define two operator $\mathscr{F}$ and $\mathscr{S}$ on S_r given by

$$
\begin{aligned}
(\mathscr{F}u)(\eta) :=& \frac{1}{\Gamma(\nu)}\int_0^T |\eta-s|^{\nu-1}F(s,u(s))\,ds, \\
(\mathscr{S}u)(\eta) :=& \frac{2T^2-2T\eta}{4\Gamma(\nu-2)}\int_0^T (T-s)^{\nu-3}F(s,u(s))\,ds - \frac{T}{2\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}F(s,u(s))\,ds.
\end{aligned}
$$

For any $u,v \in S_r$, as above, we have

$$
\|\mathscr{F}u+\mathscr{S}v\| \leq \frac{\|\ell\|_{L^1}T^\nu}{\Gamma(\nu+1)}(2+\frac{\nu(\nu+1)}{4}) \leq r.
$$

Hence, it follows that $\mathscr{F}u+\mathscr{S}v \in S_r$ whenever $u,v \in S_r$. It can easily be shown that $\mathscr{S}$ is a contraction using the assumption

$\dfrac{LT^{\nu}(\nu+1)}{4\Gamma(\nu)} < 1$. The continuity of $\mathscr{F}$ follows from the continuity of F. Moreover, $\mathscr{F}$ is uniformly bounded on S_r as follows.

$$\begin{aligned}|(\mathscr{F}u)(\eta)| \leq &\frac{1}{\Gamma(\nu)}\int_0^{\eta}(\eta-s)^{\nu-1}|F(s,u(s))|\,ds+\frac{1}{\Gamma(\nu)}\int_{\eta}^{T}(s-\eta)^{\nu-1}|F(s,u(s))|\,ds\\ \leq &\frac{\|\ell\|_{L_1}}{\Gamma(\nu)}\Big(\int_0^{\eta}(\eta-s)^{\nu-1}\,ds+\int_{\eta}^{T}(s-\eta)^{\nu-1}\,ds\Big)\\ \leq &\frac{\|\ell\|_{L_1}}{\Gamma(\nu+1)}(\eta^{\nu}+(T-\eta)^{\nu}) \leq \frac{2\|\ell\|_{L_1}T^{\nu}}{\Gamma(\nu+1)}.\end{aligned}$$

We now show that the operator $\mathscr{F}$ is compact on S_r. For $u \in S_r$, we first estimate the derivative $(\mathscr{F}u)'(\eta)$:

$$\begin{aligned}\big|(\mathscr{F}u)'(\eta)\big| \leq &\frac{1}{\Gamma(\nu-1)}\int_0^{\eta}(\eta-s)^{(\nu-2)}\big|F\big(s,u(s)\big)\big|\,ds+\frac{1}{\Gamma(\nu-1)}\int_{\eta}^{T}(s-\eta)^{(\nu-2)}\big|F\big(s,u(s)\big)\big|\,ds\\ \leq &\Big(\frac{\eta^{\nu-1}}{\Gamma(\nu)}+\frac{(T-\eta)^{\nu-1}}{\Gamma(\nu)}\Big)L \leq \frac{2T^{\nu-1}L}{\Gamma(\nu)} := \beta_{T,L,\nu},\end{aligned} \tag{3.1}$$

where $\beta_{T,L,\nu}$ is independent of the function u. Therefore, for any $\eta_1,\eta_2 \in [0,T]$ with $\eta_1 < \eta_2$, we have

$$|(\mathscr{F}u)(\eta_1)-(\mathscr{F}u)(\eta_2)| = \int_{\eta_1}^{\eta_2}|(\mathscr{F}u)'(s)|\,ds \leq \beta_{T,L,\nu}(\eta_2-\eta_1).$$

Hence, $\mathscr{F}$ is relatively compact on S_r. It follows form Arzela Ascoli Theorem that $\mathscr{F}$ is compact on S_r. As a consequence of Theorem 2.6, we infer that $\mathscr{F}+\mathscr{S}$ has a fixed point which is a solution of the problem (1.1) on $[0,T]$. Thus the proof is completed. □

Theorem 3.3. *Assume that* $\lim_{u\to 0}\dfrac{F(\eta,u)}{u} = 0$. *Then the problem (1.1) has one solution.*

Proof. $\lim_{u\to 0}\dfrac{F(\eta,u)}{u} = 0$ implies that there is a $\delta > 0$ such that $|F(\eta,u)| \leq \varepsilon|u|$ for $0 < |u| < \delta$, where ε is chosen such that

$$\Big(\frac{T^{\nu}}{\Gamma(\nu+1)}(2+\frac{\nu(\nu+1)}{4})\Big)\varepsilon \leq 1. \tag{3.2}$$

Set $S_r = \{u \in C([0,T],\mathbb{R}) : \|u\| < r$ and let $u \in \partial S_r$, that is $\|u\| = r$. As before, the continuity of the operator $\mathscr{T}$ follows from the continuity of F, and, as before, it can be shown that $\mathscr{T} = \mathscr{F}+\mathscr{S}$ is bounded on S_r. Note that $|(\mathscr{T}u)' = (\mathscr{F}u)'+(\mathscr{S}u)'|$ where $(\mathscr{F}u)'$ is given by (3.1) and $(\mathscr{S}u)'$ is given as

$$|(\mathscr{S}u)'(\eta)| = \frac{T}{2\Gamma(\nu-2)}\int_0^{T}(T-s)^{(\nu-3)}\big|F\big(s,u(s)\big)\big|\,ds \leq \frac{T^{\nu-1}L}{\Gamma(\nu-1)}.$$

Hence,

$$|(\mathscr{T}u)'(\eta)| \leq \frac{(\nu+1)T^{\nu-1}L}{\Gamma(\nu-1)} := L_1.$$

Therefore, for $\eta_1,\eta_2 \in [0,T]$ with $\eta_1 < \eta_2$, we have

$$|(\mathscr{T}u)'(\eta_1)-(\mathscr{T}u)'(\eta_2)| \leq \int_{\eta_1}^{\eta_2}|(\mathscr{T}u)'(s)|\,ds \leq L_1(\eta_2-\eta_1).$$

We deduce that $\mathscr{T}$ is equicontinuous on $[0,T]$. Hence, in view of the Arzela–Ascoli theorem, the operator $\mathscr{T}$ is completely continuous. Morevover, we have

$$|(\mathscr{T}u)(\eta)| \leq \Big(\frac{T^{\nu}}{\Gamma(\nu+1)}(2+\frac{\nu(\nu+1)}{4})\Big)\varepsilon\|u\|,$$

which implies $\|\mathscr{T}u\| \leq \|u\|$ for $u \in \partial S_r$ in light of (3.2). As a consequence of Theorem 2.7, the operator $\mathscr{T}$ has a fixed point which is solution of the problem (1.1). □

Remark 3.4. *The results in this paper can be applied to obtain the existence results for nonlinear third-order ordinary differential equations with anti-periodic boundary conditions [40] by taking* $\nu = 3$

$$\begin{aligned} u'''(\eta) &= F(\eta, u(\eta)) \quad 0 \le \eta \le T, \\ u(0) + u(T) = 0, \quad & u'(0) + u'(T) = 0, \quad u''(0) + u''(T) = 0. \end{aligned}$$

4. Numerical Examples

In this section, numerical examples are given to show the applications of the result of this paper.

Example 4.1. *Consider the following fractional boundary problem with anti-periodic boundary conditions*

$$\begin{aligned} &{}^{RC}_{0}D^{\frac{5}{2}}_{1} u(\eta) = \frac{1}{(2+\eta)^2}\frac{u(\eta)}{2+u(\eta)} \quad 0 \le \eta \le 1, \\ &u(0) + u(1) = 0, \quad u'(0) + u'(1) = 0, \quad u''(0) + u''(1) = 0. \end{aligned} \tag{4.1}$$

Here, $F(s,u(s)) = \dfrac{1}{(2+\eta)^2}\dfrac{u(\eta)}{2+u(\eta)}$, $\quad T = 1$ and $\nu = \frac{5}{2}$. We have $|F(s,u) - F(s,v)| \le \frac{1}{4}\|u - v\|$,hence the condition (H1) is fulfilled with $L = \frac{1}{4}$. Also, we calculate $\frac{LT^\nu}{\Gamma(\nu+1)}(2 + \dfrac{\nu(\nu+1)}{4}) \approx 0.3150 < 1$. Therefore, the fractional boundary value problem (4.1) has a solution by Theorem 3.1.

Example 4.2. *Consider the following fractional boundary problem with anti-periodic boundary conditions*

$$\begin{aligned} &{}^{RC}_{0}D^{\nu}_{1} u(\eta) = u^{3/2}(\eta) + 3(\eta+2)(u(\eta) - \tan u(\eta)), \quad \nu \in (2,3], \quad 0 \le \eta \le 1, \\ &u(0) + u(1) = 0, \quad u'(0) + u'(1) = 0, \quad u''(0) + u''(1) = 0, \end{aligned} \tag{4.2}$$

where $F(s,u(s)) = u^{3/2}(\eta)) + 3(\eta+2)(u(\eta) - \tan u(\eta))$, $\quad T = 1$ *and* $\nu \in (2,3]$ *is any real number. We have* $\lim_{u\to 0} \dfrac{F(\eta,u)}{u} = 0$*, hence the condition of Theorem 3.3 holds. As a result of Theorem 3.3, the fractional boundary value problem (4.2) has at least one solution.*

5. Conclusion

This paper concerns with the existence and uniqueness for fractional differential equations with the Riesz space with anti-periodic boundary conditions in Banach spaces. With the help of Banach's contraction principle and some fixed point theorems, existence results have been presented. As a special value of the fractional order, the results are extended to nonlinear third order ordinary differential equation with anti-periodic boundary conditions. Some examples are given to illustrate the theoretical results.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] R. L. Magin, *Fractional calculus in bioengineering*, Begell House Publisher, Inc., Connecticut, 2006.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

[3] G. M. Zaslavsky, *Hamiltonian chaos and fractional dynamics*, Oxford University Press, Oxford 2005.

[4] K. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley, NY, 1993.

[5] S. Shen, F. Liu, V. Anh, *Numerical approximations and solution techniques for the Caputo-time Riesz–Caputo fractional advection–diffusion equation*, Numer. Algorithms, **56** (2011), 383-403.

[6] D. R. Smart, *Fixed point Theorems*, Cambridge University Press, Cambridge 1980.

[7] C. Pinto, A. R. M. Carvalho, *New findings on the dynamics of HIV and TB coinfection models*, Appl. Math. Comput., **242**(2014), 36-46.

[8] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, *Discrete random walk models for space–time fractional diffusion*, Chem. Phys., **284** (2012), 521-541.

[9] L. Guo, L. Liu, W. Ye, *Uniqueness of iterative positive solutions for the singular fractional differential equations with integral boundary conditions*, Comput. Math. Appl., **59**(8) (2010), 2601–2609.

[10] J. W. Negele, E. Vogt (Eds.), Volume 23 of advances in the physics of particles and nuclei, Advances in nuclear physics, Springer Science and Business Media, 1996.

[11] R. Agarwal, D, O'Regan, S. Stanek, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl., **371** (2010), 57-68.

[12] A. Babakhani, V. Gejji, *Existence of positive solutions of nonlinear fractional differential equations*, J. Math. Anal.Appl., **278** (2003), 434-442.

[13] C. Celik, M. Duman, *Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative*, J. Comput. Phys., **231** (2012), 1743–1750.

[14] F. Chen, A. Chen, X. Wu, *Anti-periodic boundary value problems with Riesz-Caputo derivative*, Adv. Dif. Eq., **2019** (2019), 119.

[15] M.Darwish, S. Ntouyas, *On initial and boundary value problems for fractional order mixed type functional differential inclusion*, Comput. Math. Appl., **59** (2010), 1253–1265.

[16] H. Sun, S. Hu, Y. Chen, W. Chen, Z. Yu, *A dynamic–order fractional dynamic system Chinese Phys. Lett.*, **30** (2013), Article 046601 pp.4.

[17] S.Toprakseven, *Existence and uniqueness of solutions to anti-periodic Riezs-Caputo impulsive fractional boundary value problems*, Tbil. Math. J. **14**(1) (2021), 71-82.

[18] S. Toprakseven, *Existence and uniqueness of solutions to Riesz-Caputo impulsive fractional boundary value problems*, Journal of Interdisciplinary Mathematics, (2021), DOI: 10.1080/09720502.2020.1826629.

[19] S. Toprakseven, *Positive solutions for two-point conformable fractional differential equations by monotone iterative scheme*, TWMS J. App. Eng. Math., **11**(1) (2021), 289-301.

[20] S. Toprakseven, *Solvability of fractional boundary value problems for a combined caputo derivative*, Konuralp J. Math., **9**(1) (2021), 119-126.

[21] F. Usta, M. Z. Sarıkaya, *The analytical solution of Van der Pol and Lienard differential equations within conformable fractional operator by retarded integral inequalities*, Demo. Math., **52**(1) (2019), 204–212.

[22] B. Ahmad, *Existence of solutions for fractional differential equations of order $q \in (2,3]$ with anti-periodic boundary conditions*, J. Appl. Math. Comput., **34** (2010), 385-391.

[23] Y. Chen, J.J. Nieto, D. O'Regan, *Anti-periodic solutions for evolution equations associated with maximal monotone mappings*, Appl. Math. Lett., **24** (3) (2011), 302-307.

[24] Y. Cui, *Uniqueness of solution for boundary value problems for fractional differential equations*, Appl. Math. Lett., **51** (2016), 48–54.

[25] C. Gu, G. Wu, *Positive solutions of fractional differential equations with the Riesz space derivative*, Appl. Math. Lett., **95** (2019), 59–64.

[26] A. Kilbas, H. H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204, North–Holland mathematics studies, Elsevier, Amsterdam, 2006.

[27] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calculus Appl. Anal., **5** (2002), 367–386.

[28] M. Z. Sarıkaya, F. Usta, *On comparison theorems for conformable fractional differential equations*, Int. J. Anal. App., **12**(2) (2016), 207-214.

[29] Sierociuk, T. Skovranek, M. Macias, I. Podlubny, I. Petras, A. Dzielinski, P. Ziubinski, *Diffusion process modeling by using fractional–order models*, Appl. Math. Comput., **15** (257) (2015), 2-11.

[30] S. Toprakseven, *The existence and uniqueness of initial-boundary value problems of the fractional Caputo-Fabrizio differential equations*, Uni. J. Math. App., **2** (2) (2019), 100-106.

[31] S. Toprakseven, *The existence of positive solutions and a Lyapunov-type inequality for boundary value problems of the fractioanl Caputo-Fabrizio differential equations*, Sigma J. Eng. Nat. Sci., **37** (4) (2019), 1125-1133.

[32] F. Usta, *Numerical analysis of fractional Volterra integral equations via Bernstein approximation method*, J. Comput. Appl. Math., **384**(2021), 113198, DOI: 10.1016/j.cam.2020.113198.

[33] F. Usta, *Fractional type Poisson equations by radial basis functions Kansa approach*, J. Ineq. Special Func., **7**(4) (2016), 143-149.

[34] F. Usta, *Numerical solution of fractional elliptic PDE's by the collocation method*, Applications and Applied Mathematics: An International Journal, **12**(1) (2017), 470- 478.

[35] F. Usta, H. Budak, M. Z. Sarıkaya, *Yang-Laplace transform method Volterra and Abel's integro-differential equations of fractional order*, Int. J. Nonlinear Anal. App., **9**(2) (2018), 203-214, DOI: 10.22075/ijnaa.2018.13630.1709.

[36] F. Usta, *A mesh free technique of numerical solution of newly defined conformable differential equations*, Konuralp J. Math., **4**(2) (2016), 149-157.

[37] M. Yavuz, T. A. Sulaiman, F. Usta, H. Bulut, *[*Analysis and numerical computations of the fractional regularized long wave equation with damping term, Math. Meth. Appl. Sci., In Press, DOI: 10.1002/mma.6343.

[38] G. Wu, D. Baleanu et al., *Lattice fractional diffusion equation in terms of a Riesz–Caputo difference*, Physics A., **438** (2015), 335-339.

[39] X. Zhang, L. Liu, Y. Wu, *The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium*, Appl. Math. Lett., **37** (2014), 26–33.

[40] A. R. Aftabizadeh, Y. K. Huang, N. H. Pavel, *Nonlinear third-order differential equations with anti-periodic boundary conditions and some optimal control problems*, J. Math. Anal. Appl., **192** (1995), 266-293.

[41] M. Yavuz, N. Özdemir, H.M. Baskonus, *Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel*, Eur. Phys. J. Plus, **133**(6) (2018), 1-11.

[42] M. Yavuz, *Characterizations of two different fractional operators without singular kernel*, Math. Model. Nat. Phenom, **14**(3) (2019), 302.

[43] M. Yavuz, N. Özdemir *Comparing the new fractional derivative operators involving exponential and Mittag-Leffler kernel*, Discrete Contin. Dyn. Syst. Ser. S, **13**(3) (2020), 995-1006.

[44] A. Yokuş, *Construction of different types of traveling wave solutions of the relativistic wave equation associated with the Schrödinger equation*, Math. Model. Numer. Simul. Appl., **1**(1) (2021), 24-31.

[45] P. Kumar, V.S. Erturk, *Dynamics of cholera disease by using two recent fractional numerical methods*, Math. Model. Numer. Simul. Appl., **1**(2) (2021), 102-111.

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Results on the Behavior of the Solutions for Linear Impulsive Neutral Delay Differential Equations with Constant Coefficients

Ali Fuat Yeniçerioğlu[1]*

Abstract
We have given some results regarding the behavior of solutions for first order linear impulsive neutral delay differential equations with constant coefficients. These results were obtained using two different real roots of the corresponding characteristic equation. Finally, two examples are given for solutions of impulsive neutral delay differential equations.

Keywords: Behavior of solutions, characteristic equation, neutral delay differential equation
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[1] *Faculty of Education, Kocaeli University, Kocaeli, 41380, Turkey, ORCID: 0000-0002-1063-0538*
***Corresponding author**: fuatyenicerioglu@kocaeli.edu.tr
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1. Introduction and Preliminaries

The author [1] has recently obtained some results regarding asymptotic behavior and stability for solutions of first order linear impulsive neutral delay differential equations with constant coefficient and constant delay. These results are obtained using a real root of the corresponding characteristic equations. Our aim in this article is to obtain different results from the article in [1] by using two different real roots of the corresponding characteristic equation.

Consider the linear impulsive neutral delay differential equation

$$[x(t)+cx(t-\sigma)]' = ax(t)+bx(t-\tau)\ , \quad t \neq t_k\ , \quad t \geq 0, \tag{1.1}$$

$$\triangle x(t_k) = \ell_k\ , \quad k \in \mathbb{Z}^+ = \{1,2,\cdots\}, \tag{1.2}$$

where σ and τ are positive constants, a,b,c and ℓ_k are real constants, $x(t) \in \mathbb{R}$ and $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$. The impulse points t_k satisfy $0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$ and $\lim_{k\to\infty} t_k = \infty$ and also $t_k - \sigma$ be not impulsive points for all $k \in \mathbb{Z}^+$.

Let's introduce the positive constant h defined by $h = \max\{\sigma, \tau\}$. Together with (1.1), an initial condition is indicated, i.e.

$$x(t) = \phi(t)\ , \quad -h \leq t \leq 0, \tag{1.3}$$

where *the initial function* ϕ is any given continuous real-valued function on the interval $[-h,0]$.

With the equation (1.1) we associate its *characteristic equation*

$$\lambda\left(1+ce^{-\lambda\sigma}\right) = a+be^{-\lambda\tau}. \tag{1.4}$$

Equation (1.4) is obtained from (1.1) by looking for solutions of the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$.

The authors in [2]-[6] obtained interesting results for the solutions of linear impulsive neutral delay differential equations in the form of (1.1). The authors in [2] examined some classes of integro-functional inequalities of the Gronwall type for piecewise continuous functions, and through the results obtained from them, they made estimates for the solutions of impulsive functional differential equations. As an application, they have proven the existence of solutions of certain nonlinear equations with arbitrarily long lifespan for sufficiently small initial functions. Later on, in the article [3] made by the same authors, the problem of stability under persistent disturbances of an impulsive systems of differential-difference equations of neutral type is investigated. An as application the existence of a global solutions of a systems with quadratic nonlinearities is proved for sufficiently small initial data. In [4], by means of Lyapunov's direct method sufficient conditions for uniform asymptotic stability of the zero solution of impulsive systems of differential-difference equations of neutral type are found. In [5], the authors examined the asymptotic behavior of positive solutions of first-order neutral impulsive differential equations with constant coefficients and constant delays, and established the necessary and sufficient conditions for the existence of such solutions. Finally, the authors in [6] established some criteria for the asymptotic stability of a neutral delay control system by applying the Lyapunov functions and Razumikhin technique, which combine with impulsive feedback control. They also showed that the stability behavior of the system can be controlled by appropriate impulsive perturbations.

In this paper, we construct estimates for (1.1)-(1.3) solutions using two different real roots of the corresponding characteristic equation. We obtained the results using the methods in [1, 7, 8]. Sufficient information about the delay or neutral impulsive differential equations and initial value problem (1.1)-(1.3) is given in [1]. For more results regarding delay or neutral impulsive differential equations, we refer the reader to [9]-[16] and references therein.

2. The Main Result

In this section, before going to the main result, we will give an lemma about two different real roots of the characteristic equation (1.4) by Philos and Purnaras [8]. In the following lemma, only the first part of the lemma in [8] is considered.

Lemma 2.1. *([8], Lemma 3.1) Suppose that* $c \leq 0$ *and* $b < 0$*. Let* λ_0 *be a nonpositive real root of the characteristic equation* (1.4) *and let* $\beta(\lambda_0) = b\tau e^{-\lambda_0 \tau} + ce^{-\lambda_0 \sigma}(1 - \lambda_0 \sigma)$*. Then*

$$1 + \beta(\lambda_0) > 0$$

if (1.4) *has another real root less than* λ_0*, and*

$$1 + \beta(\lambda_0) < 0$$

if (1.4) *has another nonpositive real root greater than* λ_0*.*

Now, our main conclusion in this article is that we can give the following theorem.

Theorem 2.2. *Suppose that*

$$c \leq 0 \quad \text{and} \quad b < 0.$$

Let λ_0 *be a nonpositive real root of the characteristic equation* (1.4) *with* $1 + \beta(\lambda_0) \neq 0$ *where* $\beta(\lambda_0)$ *is defined as in Lemma 2.1, and let*

$$L(\lambda_0;\phi) = \phi(0) + c\phi(-\sigma) + be^{-\lambda_0 \tau}\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)ds - c\lambda_0 e^{-\lambda_0 \sigma}\int_{-\sigma}^{0} e^{-\lambda_0 s}\phi(s)ds.$$

Let also λ_1 *be a nonpositive real root of* (1.4) *with* $\lambda_0 \neq \lambda_1$*.*

(I) Assume that $\lambda_0 > \lambda_1$ *and* $\ell_i > 0$ *for* $i \in \mathbb{Z}^+$*. Also let there be a number* $d_1 > 0$ *such that it is provided*

$$1 + \beta(\lambda_0) \geq \frac{1}{d_1}\sum_{i=1}^{\infty} \ell_i e^{-\lambda_0 t_i}, \tag{2.1}$$

then, for any $\phi \in C\big([-h,0],\mathbb{R}\big)$ *such that*

$$\phi(t) \leq e^{\lambda_0 t}\left[d_1 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right] \quad for \quad t \in [-h,0], \tag{2.2}$$

the solution x of (1.1)-(1.3) *satisfies*

$$D_1(\lambda_0,\lambda_1;\phi) \le e^{-\lambda_1 t}\left[x(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le d_1 e^{(\lambda_0-\lambda_1)t} \quad for\ all\ t\ge 0, \tag{2.3}$$

where

$$D_1(\lambda_0,\lambda_1;\phi) = \min_{-h\le t\le 0}\left\{e^{-\lambda_1 t}\left[\phi(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right]\right\}.$$

Note: Since $\lambda_0>\lambda_1$, according to the Lemma 2.1 is $1+\beta(\lambda_0)>0$.

(II) Assume that $\lambda_0<\lambda_1$ and $\ell_i<0$ for $i\in\mathbb{Z}^+$. Also let there be a number $d_2>0$ such that it is provided

$$1+\beta(\lambda_0) \le \frac{1}{d_2}\sum_{i=1}^{\infty}\ell_i e^{-\lambda_0 t_i}, \tag{2.4}$$

then, for any $\phi\in C\big([-h,0],\mathbb{R}\big)$ such that

$$e^{\lambda_0 t}\left[d_2+\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right] \le \phi(t) \quad for \quad t\in[-h,0], \tag{2.5}$$

the solution x of (1.1)-(1.3) *satisfies*

$$d_2 e^{(\lambda_0-\lambda_1)t} \le e^{-\lambda_1 t}\left[x(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le D_2(\lambda_0,\lambda_1;\phi) \quad for\ all\ t\ge 0, \tag{2.6}$$

where

$$D_2(\lambda_0,\lambda_1;\phi) = \max_{-h\le t\le 0}\left\{e^{-\lambda_1 t}\left[\phi(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right]\right\}.$$

Note: Since $\lambda_0<\lambda_1$, according to the Lemma 2.1 is $1+\beta(\lambda_0)<0$.

Proof. (Proof of Part (I) of the Theorem 2.2): We will show that the double inequality (2.3) is first

$$D_1(\lambda_0,\lambda_1;\phi) \le e^{-\lambda_1 t}\left[x(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \quad \text{for all } t\ge 0,$$

and

$$e^{-\lambda_1 t}\left[x(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t\ge 0,$$

respectively. Let $\phi\in C\big([-h,0],\mathbb{R}\big)$ such that satisfies (2.2) and x be the solution of (1.1)-(1.3). Furthermore, let $y(t)=e^{-\lambda_0 t}x(t)$ for $t\ge -h$. As it has been shown ([1], Lemma 1.1), the fact that x satisfies (1.1)-(1.3) for $t\ge 0$ is equivalent to the fact that y satisfies

$$\begin{aligned} y(t)+&ce^{-\lambda_0\sigma}y(t-\sigma) = L(\lambda_0;\phi)+\sum_{i=1}^{n(t)}\ell_i e^{-\lambda_0 t_i} \\ &+(a-\lambda_0)\int_0^t y(s)ds+be^{-\lambda_0\tau}\int_0^{t-\tau}y(s)ds-c\lambda_0 e^{-\lambda_0\sigma}\int_0^{t-\sigma}y(s)ds \end{aligned} \tag{2.7}$$

where

$$n(t)=max\{k\in\mathbb{Z}^+: \quad t_k\le t\} \quad \text{and} \quad n(t)=0 \quad \text{if } t<t_1.$$

In addition, the initial condition (1.3) can be made equivalent to

$$y(t)=e^{-\lambda_0 t}\phi(t) \quad \text{for } t\in[-h,0].$$

Later on, by using the fact that λ_0 is root of (1.4) and by using $z(t)=y(t)-\frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}$ for $t\ge -h$, then (2.7) becomes

$$z(t)+ce^{-\lambda_0\sigma}z(t-\sigma)=\sum_{i=1}^{n(t)}\ell_i e^{-\lambda_0 t_i}-be^{-\lambda_0\tau}\int_{t-\tau}^{t}z(s)ds+c\lambda_0 e^{-\lambda_0\sigma}\int_{t-\sigma}^{t}z(s)ds \quad \text{for } t\ge 0 \tag{2.8}$$

and we immediately see that the initial condition (1.3) becomes

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for } t \in [-h,0]. \tag{2.9}$$

Next, let us define

$$w(t) = e^{(\lambda_0-\lambda_1)t}z(t) \quad \text{for } t \geq -h.$$

By the use of the function w, (2.8) becomes

$$\begin{aligned} w(t) + ce^{-\lambda_1\sigma}w(t-\sigma) &= e^{(\lambda_0-\lambda_1)t}\sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &\quad - be^{-\lambda_0\tau}\int_{t-\tau}^{t} e^{(\lambda_0-\lambda_1)(t-s)}w(s)ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t-\sigma}^{t} e^{(\lambda_0-\lambda_1)(t-s)}w(s)ds \quad \text{for } t \geq 0. \end{aligned} \tag{2.10}$$

Also, (2.9) takes the following equivalent form

$$w(t) = e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \quad \text{for } t \in [-h,0].$$

By way of the definitions of y, z and w, we have

$$w(t) = e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \quad \text{for } t \geq -h. \tag{2.11}$$

Thus, from the definition of the constant $D_1(\lambda_0,\lambda_1;\phi)$, it follows that the double inequality (2.3) in the conclusion of our theorem can equivalently be written as follows

$$\min_{-h\leq s\leq 0} w(s) \leq w(t) \leq d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t \geq 0. \tag{2.12}$$

The proof of the theorem will be accomplished by proving the double inequality (2.12). First, let's prove the following inequality of the double inequality (2.12)

$$\min_{-h\leq s\leq 0} w(s) \leq w(t) \quad \text{for all } t \geq 0. \tag{2.13}$$

To prove (2.13), we consider an arbitrary real number A such that $A < \min_{-h\leq s\leq 0} w(s)$. Clearly,

$$A < w(t) \quad \text{for } -h \leq t \leq 0. \tag{2.14}$$

We will show that

$$A < w(t) \quad \text{for all } t \geq 0. \tag{2.15}$$

To this end, let us assume that (2.15) fails to hold. Then, because of (2.14), there exists a point $t^* > 0$ so that

$$A < w(t) \quad \text{for } -h \leq t < t^*, \quad \text{and} \quad w(t^*) = A.$$

Thus, by using the hypothesis that $c \leq 0$, $b < 0$, $\ell_i > 0 \;$ for $i \in \mathbb{Z}^+$ and taking into account the fact that $\lambda_0 \leq 0$, from (2.10) we obtain

$$
\begin{aligned}
A = w(t^*) &= -ce^{-\lambda_1\sigma}w(t^*-\sigma) + e^{(\lambda_0-\lambda_1)t^*}\sum_{i=1}^{n(t^*)}\ell_i e^{-\lambda_0 t_i} \\
&\quad - be^{-\lambda_0\tau}\int_{t^*-\tau}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}w(s)ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t^*-\sigma}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}w(s)ds \\
&> A\left\{-ce^{-\lambda_1\sigma} - be^{-\lambda_0\tau}\int_{t^*-\tau}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t^*-\sigma}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds\right\} + e^{(\lambda_0-\lambda_1)t^*}\sum_{i=1}^{n(t^*)}\ell_i e^{-\lambda_0 t_i} \\
&> A\left\{-ce^{-\lambda_1\sigma} - be^{-\lambda_0\tau}\int_{t^*-\tau}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds + c\lambda_0 e^{-\lambda_0\sigma}\int_{t^*-\sigma}^{t^*} e^{(\lambda_0-\lambda_1)(t^*-s)}ds\right\} \\
&= A\left\{-ce^{-\lambda_1\sigma} - be^{-\lambda_0\tau}\left(\frac{1}{\lambda_1-\lambda_0}\right)\left[1-e^{(\lambda_0-\lambda_1)\tau}\right] + c\lambda_0 e^{-\lambda_0\sigma}\left(\frac{1}{\lambda_1-\lambda_0}\right)\left[1-e^{(\lambda_0-\lambda_1)\sigma}\right]\right\} \\
&= \frac{A}{\lambda_1-\lambda_0}\left\{-c\lambda_1 e^{-\lambda_1\sigma} + be^{-\lambda_1\tau} + c\lambda_0 e^{-\lambda_0\sigma} - be^{-\lambda_0\tau}\right\} \\
&= \frac{A}{\lambda_1-\lambda_0}\left\{(\lambda_1-a)+(a-\lambda_0)\right\} = A.
\end{aligned}
$$

This is a contradiction and hence (2.15) is always satisfied. We have thus proved that (2.15) holds true for all real numbers A with $A < \min_{-h\le s\le 0} w(s)$. This guarantees that (2.13) is fulfilled and so, the first part of the double inequality (2.12) (or, (2.3)) is proved.

Now, let's prove the second part of the double inequality (2.3). Property (2.2) implies $\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t} \le d_1 e^{\lambda_0 t}$. So, if both sides of this inequality are multiplied by $e^{-\lambda_1 t}$, using the definition (2.11), it follows that

$$
e^{-\lambda_1 t}\left[\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}e^{\lambda_0 t}\right] \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for } t\in[-h,0]
$$

or by way of the definition of w, we have

$$
w(t) \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for } t\in[-h,0]. \tag{2.16}
$$

We will show that $d_1 e^{(\lambda_0-\lambda_1)t}$ is a bound of w on the whole interval $[-h,\infty]$, namely that

$$
w(t) \le d_1 e^{(\lambda_0-\lambda_1)t} \quad \text{for all } t \ge -h. \tag{2.17}
$$

For the sake of contradiction suppose that there exists a $\bar{t} > 0$ such that $w(\bar{t}) > d_1 e^{(\lambda_0-\lambda_1)\bar{t}}$. Let

$$
t_* = \inf\left\{\bar{t}: \quad w(\bar{t}) > d_1 e^{(\lambda_0-\lambda_1)\bar{t}}\right\}.
$$

Now, by right continuity, either $w(t_*) = d_1 e^{(\lambda_0-\lambda_1)t_*}$ if there is no impulsive point at t_*, or $w(t_*) \ge d_1 e^{(\lambda_0-\lambda_1)t_*}$ as a consequence of a t_*. Whatever the case, using right continuity, we thus have $w(t) \le d_1 e^{(\lambda_0-\lambda_1)t}$ for $t\in[-h,t_*)$, where $w(t_*) = d_1 e^{(\lambda_0-\lambda_1)t_*}$ if this occors at a non-impulsive point. Then, by using the hypothesis that $c \le 0$, $b<0$, $\ell_i > 0$ for $i\in\mathbb{Z}^+$ and taking into account the fact that $\lambda_0 \le 0$, and also using (2.1), from (2.10) we have that

$$
\begin{aligned}
d_1 e^{(\lambda_0-\lambda_1)t_*} = w(t_*) &= -ce^{-\lambda_1\sigma} w(t_*-\sigma) + e^{(\lambda_0-\lambda_1)t_*} \sum_{i=1}^{n(t_*)} \ell_i e^{-\lambda_0 t_i} \\
&\quad - be^{-\lambda_0\tau} \int_{t_*-\tau}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} w(s)ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t_*-\sigma}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} w(s)ds \\
&\leq -cd_1 e^{-\lambda_1\sigma} e^{(\lambda_0-\lambda_1)(t_*-\sigma)} + e^{(\lambda_0-\lambda_1)t_*} \sum_{i=1}^{n(t_*)} \ell_i e^{-\lambda_0 t_i} \\
&\quad - bd_1 e^{-\lambda_0\tau} \int_{t_*-\tau}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} e^{(\lambda_0-\lambda_1)s} ds + cd_1\lambda_0 e^{-\lambda_0\sigma} \int_{t_*-\sigma}^{t_*} e^{(\lambda_0-\lambda_1)(t_*-s)} e^{(\lambda_0-\lambda_1)s} ds \\
&< d_1 e^{(\lambda_0-\lambda_1)t_*} \left\{ -ce^{-\lambda_0\sigma} - be^{-\lambda_0\tau}\tau + c\lambda_0 e^{-\lambda_0\sigma}\sigma + \frac{1}{d_1} \sum_{i=1}^{\infty} \ell_i e^{-\lambda_0 t_i} \right\} \\
&= d_1 e^{(\lambda_0-\lambda_1)t_*} \left\{ -\beta(\lambda_0) + \frac{1}{d_1} \sum_{i=1}^{\infty} \ell_i e^{-\lambda_0 t_i} \right\} \leq d_1 e^{(\lambda_0-\lambda_1)t_*}.
\end{aligned}
$$

This gives us the desired contradiction, since we proved $w(t_*) < d_1 e^{(\lambda_0-\lambda_1)t_*}$, and we assumed $w(t_*) = d_1 e^{(\lambda_0-\lambda_1)t_*}$ if t_* is a continuity point, or $w(t_*) \geq d_1 e^{(\lambda_0-\lambda_1)t_*}$ if t_* is a discontinuity point. So (2.17) is true and the second part of the double inequality (2.12) (or, (2.3)) is proved. As a result, the Part (I) of Theorem 2.2 has been proven.

(Proof of Part (II) of the Theorem 2.2): As in Part (I), the double inequality (2.6) can be shown to be

$$
e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t}\right] \leq D_2(\lambda_0,\lambda_1;\phi) \quad \text{for all } t \geq 0,
$$

and

$$
d_2 e^{(\lambda_0-\lambda_1)t} \leq e^{-\lambda_1 t}\left[x(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t}\right] \quad \text{for all } t \geq 0
$$

respectively. So, from the definition (2.11), it follows that the double inequality (2.6) in the conclusion of our theorem can equivalently be written as follows

$$
w(t) \leq \max_{-h\leq t\leq 0} w(t) \quad \text{for all } t \geq 0,
$$

and

$$
d_2 e^{(\lambda_0-\lambda_1)t} \leq w(t) \quad \text{for all } t \geq 0,
$$

respectively. Thus, using the hypothesis in the Part (II), it can be proved similarly as in the Part (I). As a result, the proof of the Part (II) of Theorem 2.2 here is omitted.

□

It is immediately clear that the following corollary of double inequalities ((2.3) and (2.6)) in Theorem 2.2 can be written as equivalent.

Corollary 2.3. *Assume that the conditions in Theorem 2.2 are provided. Then the solution of* (1.1)-(1.3) *satisfies*
(I) for $\lambda_1 < \lambda_0$

$$
D_1(\lambda_0,\lambda_1;\phi)\, e^{\lambda_1 t} + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t} \leq x(t) \leq e^{\lambda_0 t}\left(d_1 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right) \quad \textit{for all } t \geq 0,
$$

(II) for $\lambda_0 < \lambda_1$

$$
e^{\lambda_0 t}\left(d_2 + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}\right) \leq x(t) \leq D_2(\lambda_0,\lambda_1;\phi)\, e^{\lambda_1 t} + \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} e^{\lambda_0 t} \quad \textit{for all } t \geq 0.
$$

Also, if $\lambda_0, \lambda_1 < 0$*, then from (I) and (II) the solution of* (1.1)-(1.3) *satisfies*

$$
\lim_{t\to\infty} x(t) = 0.
$$

Example 2.4. Consider

$$[x(t)-\frac{1}{3}x(t-\frac{1}{4})]' = \frac{1}{2}x(t)-\frac{1}{2}x(t-\frac{1}{2}), \quad t \neq t_k, \quad t \geq 0, \tag{2.18}$$

$$\triangle x(t_k) = \left(\frac{1}{4}\right)^k, \quad k \in \mathbb{Z}^+, \tag{2.19}$$

$$x(t) = \phi(t), \quad -\frac{1}{2} \leq t \leq 0,$$

where $\phi \in C\left([-\frac{1}{2},0],\mathbb{R}\right)$ and t_k are arbitrary impulsive points, such that $t_k - \frac{1}{4}$ are not impulsive points for all $k \in \mathbb{Z}^+$.

The characteristic equation of (2.18) is

$$2\lambda\left(3-e^{-\frac{\lambda}{4}}\right) = 3\left(1-e^{-\frac{\lambda}{2}}\right). \tag{2.20}$$

We see that $\lambda = 0$ and $\lambda \approx -2.08$ are real roots of (2.20). Let $\lambda_0 = 0$ and $\lambda_1 = -2.08$. Let's choose the number $d_1 = 1$. We have $\lambda_0 > \lambda_1$, $\ell_i = \left(\frac{1}{4}\right)^i > 0 \quad i \in \mathbb{Z}^+$ and from (2.1)

$$1+\beta(0) = 1-\frac{1}{4}-\frac{1}{3} = \frac{5}{12} > \sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i = \frac{1}{3}.$$

Thus, by applying Theorem 2.2-(I) and Corollary 2.3-(I), we obtain the following results:

According to (2.2), for any $\phi \in C\left([-\frac{1}{2},0],\mathbb{R}\right)$ such that

$$\phi(t) \leq \left[1+\frac{L(0;\phi)}{5/12}\right], \quad \text{for } t \in \left[-\frac{1}{2},0\right], \tag{2.21}$$

the solution x of (2.18)-(2.19) satisfies

$$D_1(0,-2.08;\phi) \leq e^{2.08t}\left[x(t)-\frac{L(0;\phi)}{5/12}\right] \leq e^{2.08t} \quad \text{for all } t \geq 0,$$

or equivalent

$$D_1(0,-2.08;\phi)\,e^{-2.08t}+\frac{L(0;\phi)}{5/12} \leq x(t) \leq 1+\frac{L(0;\phi)}{5/12} \quad \text{for all } t \geq 0,$$

where

$$L(0;\phi) = \phi(0)-\frac{1}{3}\phi\left(-\frac{1}{4}\right)-\frac{1}{2}\int_{-\frac{1}{2}}^{0}\phi(s)ds$$

and

$$D_1(0,-2.08;\phi) = \min_{-\frac{1}{2}\leq t\leq 0}\left\{e^{2.08t}\left[\phi(t)-\frac{L(0;\phi)}{5/12}\right]\right\}.$$

Now let's take the special case of $\phi(t) = 1$. Then

$$L(0;1) = 1-\frac{1}{3}-\frac{1}{2}\int_{-\frac{1}{2}}^{0}ds = \frac{5}{12} \quad \text{and} \quad D_1(0,-2.08;1) = \min_{-\frac{1}{2}\leq t\leq 0}\left\{e^{2.08t}\left[1-\frac{L(0;1)}{5/12}\right]\right\} = 0.$$

Thus, for $\phi(t) = 1$ the inequality (2.21) is provided and the solution x of (2.18)-(2.19) satisfies

$$0 \leq e^{2.08t}[x(t)-1] \leq e^{2.08t} \quad \text{for all } t \geq 0,$$

or equivalent

$$1 \leq x(t) \leq 2 \quad \text{for all } t \geq 0.$$

Example 2.5. Consider

$$[x(t)-e^{-\frac{1}{4}}x(t-\frac{1}{2})]' = -x(t)+e^{-\frac{1}{4}}x(t-\frac{1}{2})\,, \quad t \neq t_k = k\,, \quad t \geq 0, \tag{2.22}$$

$$\triangle x(t_k) = -\left(\frac{1}{e^2}\right)^k\,, \quad k \in \mathbb{Z}^+, \tag{2.23}$$

$$x(t) = \phi(t)\,, \quad -\frac{1}{2} \leq t \leq 0,$$

where $\phi \in C\left([-\frac{1}{2},0],\mathbb{R}\right)$.

The characteristic equation of (2.22) is

$$\lambda\left(1-e^{-\frac{1}{4}}e^{-\frac{\lambda}{2}}\right) = -1+e^{-\frac{1}{4}}e^{-\frac{\lambda}{2}}$$

or

$$\lambda-(\lambda+1)\,e^{-\frac{1}{4}(2\lambda+1)}+1=0. \tag{2.24}$$

We see that $\lambda=-1$ and $\lambda=-\frac{1}{2}$ are real roots of (2.24). Let $\lambda_0=-1$ and $\lambda_1=-\frac{1}{2}$. Let's choose the number $d_2=\frac{5}{4}$. We have $\lambda_0<\lambda_1$, $\ell_i=-\frac{1}{2}\left(\frac{1}{e^2}\right)^i<0 \quad i\in\mathbb{Z}^+$ and from (2.4)

$$\begin{aligned} 1+\beta(-1) &= 1-e^{\frac{1}{4}} \approx -0.284 \\ &< -\frac{4}{5}\sum_{i=1}^{\infty}\frac{1}{2}\left(\frac{1}{e^2}\right)^i e^i = -\frac{2}{5}\sum_{i=1}^{\infty}\frac{1}{e^i} \approx -0.232. \end{aligned}$$

Thus, by applying Theorem 2.2-(II) and Corollary 2.3-(II), we obtain the following results:

According to (2.5), for any $\phi\in C\big([-\frac{1}{2},0],\mathbb{R}\big)$ such that

$$e^{-t}\left[\frac{5}{4}+\frac{L(-1;\phi)}{1-e^{\frac{1}{4}}}\right] \leq \phi(s) \quad \text{for } t\in\left[-\frac{1}{2},0\right], \tag{2.25}$$

the solution x of (2.22)-(2.23) satisfies

$$\frac{5}{4}e^{-\frac{t}{2}} \leq e^{\frac{t}{2}}\left[x(t)+\frac{L(-1;\phi)}{1-e^{\frac{1}{4}}}e^{-t}\right] \leq D_2\left(-1,-\frac{1}{2};\phi\right)$$

for all $t\geq 0$, where

$$\begin{aligned} L(-1;\phi) &= \phi(0)-e^{-\frac{1}{4}}\phi\left(-\frac{1}{2}\right)+e^{\frac{1}{4}}\int_{-\frac{1}{2}}^{0}e^s\phi(s)ds-e^{\frac{1}{4}}\int_{-\frac{1}{2}}^{0}e^s\phi(s)ds \\ &= \phi(0)-e^{-\frac{1}{4}}\phi\left(-\frac{1}{2}\right) \end{aligned}$$

and

$$D_2\left(-1,-\frac{1}{2};\phi\right) = \max_{-\frac{1}{2}\leq t\leq 0}\left\{e^{\frac{t}{2}}\left[\phi(t)+\frac{L(-1;\phi)}{1-e^{\frac{1}{4}}}e^{-t}\right]\right\}.$$

Now let's take the special case of $\phi(t)=1$. Then $L(-1;1)=1-e^{-\frac{1}{4}}$ and

$$D_2\left(-1,-\frac{1}{2};\phi\right) = \max_{-\frac{1}{2}\leq t\leq 0}\left\{e^{\frac{t}{2}}\left[1+\frac{1-e^{-\frac{1}{4}}}{1-e^{\frac{1}{4}}}e^{-t}\right]\right\} = \max_{-\frac{1}{2}\leq t\leq 0}\left\{e^{\frac{t}{2}}\left[1+e^{-\left(t+\frac{1}{4}\right)}\right]\right\} = 1+e^{-\frac{1}{4}}.$$

Thus, for $\phi(t)=1$ the inequality (2.25) is provided, i.e.

$$e^{-t}\left[\frac{5}{4}-e^{-\frac{1}{4}}\right] \leq 1 \quad \text{for} \quad t\in\left[-\frac{1}{2},0\right],$$

and for $\phi(t) = 1$ the solution x of (2.22)-(2.23) satisfies

$$\frac{5}{4}e^{-\frac{t}{2}} \leq e^{\frac{t}{2}}\left[x(t)+e^{-\left(t+\frac{1}{4}\right)}\right] \leq 1+e^{-\frac{1}{4}} \quad \text{for all } \; t \geq 0$$

or from Corollary 2.3-(II) it follows that

$$e^{-t}\left(d_2+\frac{L(-1;\phi)}{1+\beta(-1)}\right) \leq x(t) \leq D_2\left(-1,-\frac{1}{2};\phi\right)e^{-\frac{t}{2}}+\frac{L(-1;\phi)}{1+\beta(-1)}e^{-t},$$

$$e^{-t}\left(\frac{5}{4}+\frac{1-e^{-\frac{1}{4}}}{1-e^{\frac{1}{4}}}\right) \leq x(t) \leq \left(1+e^{-\frac{1}{4}}\right)e^{-\frac{t}{2}}+\frac{1-e^{-\frac{1}{4}}}{1-e^{\frac{1}{4}}}e^{-t},$$

$$e^{-t}\left(\frac{5}{4}-e^{-\frac{1}{4}}\right) \leq x(t) \leq \left(1+e^{-\frac{1}{4}}\right)e^{-\frac{t}{2}}-e^{-\left(t+\frac{1}{4}\right)} \quad \text{for all } \; t \geq 0.$$

Also, from the last double inequality we get

$$\lim_{t\to\infty} x(t) = 0.$$

3. Conclusion

In this paper, an important result is obtained for the behavior of the solutions by making use of two appropriate real roots of the characteristic equation and two examples were given. The real roots used in this paper play an important role in determining the results.

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References

[1] A.F. Yeniçerioğlu, *Stability of linear impulsive neutral delay differential equations with constant coefficients*, J. Math. Anal. Appl., **479** (2019), 2196–2213.

[2] D. D. Bainov, V. Covachev, I.M. Stamova, *Estimates of the solutions of impulsive quasilinear functional differential equations*, Annales de la Facultb des Sciences de Toulouse, **2** (1991), 149-161.

[3] D. D. Bainov, V. Covachev, I. M. Stamova, *Stability under persistent disturbances of impulsive differential- difference equations of neutral type*, J. Math. Anal. Appl., **187** (1994), 790-808.

[4] D. D. Bainov, I. M. Stamova, *Uniform asymptotic stability of impulsive differential-difference equations of neutral type by Lyapunov's direct method*, J. Comput. Appl. Math., **62** (1995), 359-369.

[5] M. B. Dimitrova, V. I. Donev, *Existence and asymptotic behavior of positive solutions of neutral impulsive differential equations*, Nonlinear Oscillations, **8**(3) (2005), 302-317.

[6] I. M. Stamova, T. G. Stamov, *Asymptotic stability of impulsive control neutral-type systems*, Inter. J. Control, **87**(1) (2014), 25-31.

[7] I. G. E. Kordonis, N. T. Niyianni, Ch. G. Philos, *On the behavior of the solutions of scalar first order linear autonomous neutral delay differential equations*, Arch. Math., (Basel) **71** (1998), 454-464.

[8] Ch. G. Philos, I. K. Purnaras, *On the behavior of the solutions for certain first order linear autonomous functional differential equations*, Rocky Mountain J. Math., **36** (2006), 1999-2019.

[9] R. P. Agarwal, F. Karakoç, *A survey on oscillation of impulsive delay differential equations*, Comput. Math. Appl., **60** (2010), 1648-1685.

[10] G.H. Ballinger, *Qualitative Theory of impulsive delay differential Equations*, National Library of Canada, 1999.

[11] X. Liu, J. Shen, *Asymptotic behavior of solutions of impulsive neutral differential equations*, Appl. Math. Letters, **12** (1999), 51-58.

[12] X. Liu, Y. M. Zeng, *Linear multistep methods for impulsive delay differential equations*, Appl. Math. Comput., **321** (2018), 555-563.

[13] F. Jiang, J. Shen,*Asymptotic behavior of solutions for a nonlinear differential equation with constant impulsive jumps*, Acta Math. Hungar., **138**(1–2) (2013), 1–14.

[14] I. M. Stamova, *Stability analysis of impulsive functional differential equations*, Walter de Gruyter, Berlin, 2009.

[15] Z. You, J. Wang, *Stability of impulsive delay differential equations*, J. Appl. Math. Comput., **56** (2018), 253-268.

[16] G.L. Zhang, M.H. Song, M. Z. Liu, *Exponential stability of the exact solutions and the numerical solutions for a class of linear impulsive delay differential equations*, J. Comput. Appl. Math., **285** (2015), 32-44.

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Padovan and Perrin Hybrid Number Identities

Renata Passos Machado Vieira[1]*, Milena Carolina dos Santos Mangueira[2], Francisco Regis Vieira Alves[3], Paula Maria Machado Cruz Catarino[4]

Abstract
This work investigates the numbers of Padovan and Perrin hybrids. At first, the hybrid numbers, the sequences in the hybrid form and their matrix forms are ordered as studied sequences. Thus, it was possible to display the negative index hybrids, define some identities belonging to these hybrid sequences, develop novel theorems and present them as binomial sums of the Padovan and Perrin hybrids.

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[1] *Department of Mathematics, Federal Institute of Education, Science and Techonology of State of Ceara - IFCE, ORCID:0000-0002-1966-7097*
[2] *Department of Mathematics, Federal Institute of Education, Science and Techonology of State of Ceara - IFCE, ORCID:0000-0002-4446-155X*
[3] *Department of Mathematics, Federal Institute of Education, Science and Techonology of State of Ceara - IFCE, ORCID:0000-0003-3710-1561*
[4] *University of Tras-os-Montes and Alto Douro - UTAD, ORCID:0000-0001-6917-5093*
***Corresponding author**: re.passosm@gmail.com
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1. Introduction

A recursive linear sequence has an infinite number of terms and is generated by a linear recurrence, called a recurrence formula, which allows you to calculate the terms of the sequence from its predecessors. Thus, in order to be able to calculate the terms of a sequence, it is necessary to know its initial terms. For mathematics, sequences are found in the area of number theory and have applicability in several areas.

In the mathematical scope, the Fibonacci sequence is the most explored sequence, however, the Padovan sequence is considered a prime Fibonacci sequence which is a linear and recurrent type sequence of third order, of integers. The Padovan Sequence, named after the Italian architect Richard Padovan (1935 - ?) [13], his work and contributions have important repercussions for research in Mathematics.

On the other hand, we have the Perrin sequence, which is a linear and recurrent sequence of integers and presents the same recurrence relation as the Padovan sequence, differing only in the terms of the sequence. This sequence was defined in 1899 by the French mathematician Olivier Raoul Perrin (1841-1910) [16]. Due to the similarity between the Padovan and Perrin sequences, in the works of [2, 6, 12] properties and identities between these numbers are presented.

So we have the recurrences of the Padovan and Perrin sequences defined below.

Definition 1.1. *The recurrence of the sequence of Padovan and Perrin, respectively, is given by:*

$$P_n = P_{n-2} + P_{n-3}, n \geqslant 3,$$

$$Pe_n = Pe_{n-2} + Pe_{n-3}, n \geq 3$$

being $P_0 = P_1 = P_2 = 1$, $Pe_0 = 3$, $Pe_1 = 0$ and $Pe_2 = 2$ the initial conditions.

Since these sequences have the same recurrence relation, we can perform algebraic operations in order to obtain the characteristic polynomial of these sequences [1].

Definition 1.2. *The characteristic polynomial of the Padovan and Perrin Sequence is defined as:*

$$x^3 - x - 1 = 0,$$

having three roots, two complex and one real.

And yet, in the literature of pure mathematics, there is the set of hybrid numbers, defined by [10], which presents the complex, hyperbolic and dual numbers together, combined with each other.

Definition 1.3. *A hybrid number is defined as:*

$$\mathbb{K} = \{z = a + bi + c\varepsilon + dh : a,b,c,d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}$$

From the definition of hybrid numbers, it is possible to perform some operations with these numbers, namely: addition, subtraction, multiplication by scalar. As for the multiplication between two hybrid numbers, this product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and using the equalities $i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i$.

From the multiplication of the imaginary units, we can present the table of the multiplication of a hybrid number, as shown in the Table 1.

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon + i$
ε	ε	$1+h$	0	$-\varepsilon$
h	h	$-\varepsilon - i$	ε	1

Table 1. Multiplication table for $\mathbb{K}$.

Furthermore, from the hybrid numbers it is possible to present their conjugate, denoted by $\bar{z}$ and is defined as

$$\bar{z} = a - bi - c\varepsilon - dh$$

and the real number

$$C(z) = z\bar{z} = \bar{z}z = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is called the hybrid number character, where the root of the absolute value of that real number will be the hybrid number norm. z, so we have to: $\|z\| = \sqrt{|C(z)|}$.

From the linear recursive sequences and the hybrid numbers, the hybridization process of the sequences is then carried out, as seen in [4, 7, 8, 9, 14, 15]. For the Padovan and Perrin sequence we can define these numbers as:

Definition 1.4. *The hybrid numbers of Padovan and Perrin, denoted by PH_n and PeH_n, are defined as:*

$$PH_n = P_n + P_{n+1}i + P_{n+2}\varepsilon + P_{n+3}h,$$

$$PeH_n = Pe_n + Pe_{n+1}i + Pe_{n+2}\varepsilon + Pe_{n+3}h,$$

where $PH_0 = 1 + i + \varepsilon + 2h$ and $PH_1 = 1 + i + 2\varepsilon + 2h$ and $PH_2 = 1 + 2i + 2\varepsilon + 3h$ the initial conditions for hybrids of Padovan and $PeH_0 = 3 + 2\varepsilon + 3h$ and $PeH_1 = 2i + 3\varepsilon + 2h$ and $PeH_2 = 2 + 3i + 2\varepsilon + 5h$ the initial conditions for Perrin hybrids.

Based on the work of [11], we can present the hybrid Padovan matrix form (Q_n) given by:

$$Q_n = \begin{bmatrix} PH_{n+2} & PH_{n+1} & PH_n \\ PH_{n+3} & PH_{n+2} & PH_{n+1} \\ PH_{n+1} & PH_n & PH_{n-1} \end{bmatrix}, \text{ for } n \geqslant 1.$$

That satisfies equality $Q_n = U^n Q$, where:

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix},$$

So, for $n = 1$, you have that:

$$\begin{aligned} Q_1 = U^1 Q &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 2+3i+4\varepsilon+5h & 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h \\ 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \end{bmatrix} \\ &= \begin{bmatrix} PH_3 & PH_2 & PH_1 \\ PH_4 & PH_3 & PH_2 \\ PH_2 & PH_1 & PH_0 \end{bmatrix} \end{aligned}$$

Assuming it is valid for $n = k, (k \in \mathbb{Z})$:

$$\begin{aligned} Q_k = U^k Q &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} P_{n-2} & P_{n-3} & P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{bmatrix} \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} PH_{k+2} & PH_{k+1} & PH_k \\ PH_{k+3} & PH_{k+2} & PH_{k+1} \\ PH_{k+1} & PH_k & PH_{k-1} \end{bmatrix} \end{aligned}$$

In this way, it is shown that it is valid for $n = k+1, (k \in \mathbb{Z})$:

$$\begin{aligned} Q_{k+1} = U^{k+1} Q &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} P_{n-2} & P_{n-3} & P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} P_{n-1} & P_{n-2} & P_{n-3} \\ P_n & P_{n-1} & P_{n-2} \\ P_{n-2} & P_{n-3} & P_{n-4} \end{bmatrix} \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix} \\ &= \begin{bmatrix} PH_{k+3} & PH_{k+2} & PH_{k+1} \\ PH_{k+4} & PH_{k+3} & PH_{k+2} \\ PH_{k+2} & PH_{k+1} & PH_k \end{bmatrix} \end{aligned}$$

Furthermore, the Perrin hybrid matrix (W_n) is given by:

$$W_n = \begin{bmatrix} PeH_{n+2} & PeH_{n+1} & PeH_n \\ PeH_{n+3} & PeH_{n+2} & PeH_{n+1} \\ PeH_{n+1} & PeH_n & PeH_{n-1} \end{bmatrix}, \text{ for } n \geqslant 1.$$

That satisfies equality $W_n = A^n W$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, W = \begin{bmatrix} 2+3i+2\varepsilon+5h & 2i+3\varepsilon+2h & 3+2\varepsilon+3h \\ 3+2i+5\varepsilon+5h & 2+3i+2\varepsilon+5h & 2i+3\varepsilon+2h \\ 2i+3\varepsilon+2h & 3+2\varepsilon+3h & -1+3i+2h \end{bmatrix}.$$

With this, based on what was previously presented and based on the work of [17], in this work, we will present several identities on Padovan and Perrin hybrid numbers, their extension to hybrids with negative indices and identities around them.

2. Results

Based on the definitions seen in the introduction referring to the hybrid numbers of Padovan and Perrin, some theorems are then developed with the aim of carrying out a new investigative study on these numbers.

Definition 2.1. *Hybrid numbers of Padovan and Perrin with negative indices are defined by:*

$$PH_{-n} = P_{-n} + P_{-n+1}i + P_{-n+2}\varepsilon + P_{-n+3}h$$
$$PeH_{-n} = Pe_{-n} + Pe_{-n+1}i + Pe_{-n+2}\varepsilon + Pe_{-n+3}h$$

Based on this, we can obtain the binomial sums of the Padovan hybrid numbers in the following theorem.

Theorem 2.2. *The following identities are valid:*
$(i) \sum_{k=0}^{n} \binom{n}{k} PH_k = PH_{3n}$,
$(ii) \sum_{k=0}^{n} \binom{n}{k} PH_{k+1} = PH_{3n+1}$,

Proof. (i) According to Binet's formula of Padovan hybrid numbers $PH_n = Ax_1^n + Bx_2^n + Cx_3^n$, one has that:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} PH_k &= \sum_{k=0}^{n} \binom{n}{k} (Ax_1^k + Bx_2^k + Cx_3^k) \\ & A\sum_{k=0}^{n} \binom{n}{k} x_1^k + B\sum_{k=0}^{n} \binom{n}{k} x_2^k + C\sum_{k=0}^{n} \binom{n}{k} x_3^k \\ & A(1+x_1)^n + B(1+x_2)^n + C(1+x_3)^n. \end{aligned}$$

Considering the infinite interactions of the cubic roots given by the expression $\psi = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+...}}}$, one can establish the relation $\psi^3 = 1+\psi$, where ψ represents the real Padovan root [5]. Hence, $1+x_1 = x_1^3, 1+x_2 = x_2^3$ and $1+x_3 = x_3^3$. With that, $\sum_{k=0}^{n} \binom{n}{k} PH_k = PH_{3n}$.
(ii) Similarly to demonstration (i), this Identity can be validated.

□

Thus, we have the binomial sums of the Perrin hybrid numbers in the following proposition. Since the proof of these sums is similar to the Padovan hybrid numbers discussed in the previous Theorem, we omit the proof.

Theorem 2.3. *The following identities are valid:*
$(i) \sum_{k=0}^{n} \binom{n}{k} PeH_k = PeH_{3n}$,
$(ii) \sum_{k=0}^{n} \binom{n}{k} PeH_{k+1} = PeH_{3n+1}$.

Theorem 2.4. *For $m \geqslant 3$, $n \geqslant 3$, one can:*
$PH_{n+2}PH_m + PH_{n+1}PH_{m+1} + PH_nPH_{m-1} = PH_{n+m} + PH_{n+m+1}i + PH_{n+m+2}\varepsilon + PH_{n+m+3}h$.

Proof. According to the matrix form of the Padovan hybrid numbers:

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix},$$

$$U^n = \begin{bmatrix} P_{n-2} & P_{n-3} & P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{bmatrix}.$$

Resulting in $Q_n = U^n Q$:

$$\begin{bmatrix} PH_{n+2} & PH_{n+1} & PH_n \\ PH_{n+3} & PH_{n+2} & PH_{n+1} \\ PH_{n+1} & PH_n & PH_{n-1} \end{bmatrix}$$

Thus, performing $(Q_n)(Q_m)$, one can equal the term a_{13}, obtaining:

$$\begin{bmatrix} PH_{n+2} & PH_{n+1} & PH_n \\ PH_{n+3} & PH_{n+2} & PH_{n+1} \\ PH_{n+1} & PH_n & PH_{n-1} \end{bmatrix} \begin{bmatrix} PH_{m+2} & PH_{m+1} & PH_m \\ PH_{m+3} & PH_{m+2} & PH_{m+1} \\ PH_{m+1} & PH_m & PH_{m-1} \end{bmatrix} = PH_{n+2}PH_m + PH_{n+1}PH_{m+1} + PH_nPH_{m-1}.$$

By definition, one has to $PH_n = PH_{n+m} + PH_{n+m+1}i + PH_{n+m+2}\varepsilon + PH_{n+m+3}h$.
Soon:

$$PH_{n+2}PH_m + PH_{n+1}PH_{m+1} + PH_nPH_{m-1} = PH_{n+m} + PH_{n+m+1}i + PH_{n+m+2}\varepsilon + PH_{n+m+3}h.$$

□

Theorem 2.5. *For $m \geqslant 0$, $n \geqslant 0$, one has to:*
$PeH_{n+2}PeH_m + PeH_{n+1}PeH_{m+1} + PeH_nPeH_{m-1} = PeH_{n+m} + PeH_{n+m+1}i + PeH_{n+m+2}\varepsilon + PeH_{n+m+3}h.$

Proof. According to the matrix form of Perrin's hybrid numbers:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, W = \begin{bmatrix} 2+3i+2\varepsilon+5h & 2i+3\varepsilon+2h & 3+2\varepsilon+3h \\ 3+2i+5\varepsilon+5h & 2+3i+2\varepsilon+5h & 2i+3\varepsilon+2h \\ 2i+3\varepsilon+2h & 3+2\varepsilon+3h & -1+3i+2h \end{bmatrix},$$

$$A^nW = \begin{bmatrix} PeH_{n+2} & PeH_{n+1} & PeH_n \\ PeH_{n+3} & PeH_{n+2} & PeH_{n+1} \\ PeH_{n+1} & PeH_n & PeH_{n-1} \end{bmatrix}.$$

Thus, performing $(W_n)(W_m)$, one can equal the term a_{13}, obtaining:

$$\begin{bmatrix} PeH_{n+2} & PeH_{n+1} & PeH_n \\ PeH_{n+3} & PeH_{n+2} & PeH_{n+1} \\ PeH_{n+1} & PeH_n & PeH_{n-1} \end{bmatrix} \begin{bmatrix} PeH_{m+2} & PeH_{m+1} & PeH_m \\ PeH_{m+3} & PeH_{m+2} & PeH_{m+1} \\ PeH_{m+1} & PeH_m & PeH_{m-1} \end{bmatrix} = PeH_{n+2}PeH_m + PeH_{n+1}PeH_{m+1} + PeH_nPeH_{m-1}$$

By definition, one has to $PeH_n = PeH_{n+m} + PeH_{n+m+1}i + PeH_{n+m+2}\varepsilon + PeH_{n+m+3}h$. Soon:

$$PeH_{n+2}PeH_m + PeH_{n+1}PeH_{m+1} + PeH_nPeH_{m-1} = PeH_{n+m} + PeH_{n+m+1}i + PeH_{n+m+2}\varepsilon + PeH_{n+m+3}h.$$

□

Theorem 2.6. *For $m \geqslant 3$, $n \geqslant 3$, one has to:*
$(i) PH_{m+n} - PH_{m+n+1}i - PH_{m+n+2}\varepsilon - PH_{m+n+3}h = \overline{PH}_{m+2}PH_n + \overline{PH}_{m+1}PH_{n+1} + \overline{PH}_mPH_{n-1},$
$(ii) \overline{PH}_{n+m} + \overline{PH}_{n+m+1}i + \overline{PH}_{n+m+2}\varepsilon + \overline{PH}_{n+m+3}h = PH_{m+2}\overline{PH}_n + PH_{m+1}\overline{PH}_{n+1} + PH_m\overline{PH}_{n-1}.$

Proof. Using the conjugate of the matrix Q, called $\overline{Q}$, where:

$$Q = \begin{bmatrix} 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h \\ 2+2i+3\varepsilon+4h & 1+2i+2\varepsilon+3h & 1+i+2\varepsilon+2h \\ 1+i+2\varepsilon+2h & 1+i+\varepsilon+2h & i+\varepsilon+h \end{bmatrix}.$$

Thus, performing:

$$U^n\overline{Q} = \begin{bmatrix} P_{n-2} & P_{n-3} & P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{bmatrix} \begin{bmatrix} 1-2i-2\varepsilon-3h & 1-i-2\varepsilon-2h & 1-i-\varepsilon-2h \\ 2-2i-3\varepsilon-4h & 1-2i-2\varepsilon-3h & 1-i-2\varepsilon-2h \\ 1-i-2\varepsilon-2h & 1-i-\varepsilon-2h & i-\varepsilon-h \end{bmatrix}$$

$$= \begin{bmatrix} \overline{PH}_{n+2} & \overline{PH}_{n+1} & \overline{PH}_n \\ \overline{PH}_{n+3} & \overline{PH}_{n+2} & \overline{PH}_{n+1} \\ \overline{PH}_{n+1} & \overline{PH}_n & \overline{PH}_{n-1} \end{bmatrix}.$$

Thus, considering the element a_{13}, it can be said that:
(i)$\overline{Q}(U^{m+n}Q)=(\overline{Q}U^m)(U^nQ)$,

$$\begin{aligned}
\overline{Q}(U^{m+n}Q)&=\overline{Q}\begin{bmatrix}P_{n+m-2}&P_{n+m-3}&P_{n+m-4}\\P_{n+m-1}&P_{n+m-2}&P_{n+m-3}\\P_{n+m-3}&P_{n+m-4}&P_{n+m-5}\end{bmatrix}\begin{bmatrix}1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h&1+i+\varepsilon+2h\\2+2i+3\varepsilon+4h&1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h\\1+i+2\varepsilon+2h&1+i+\varepsilon+2h&i+\varepsilon+h\end{bmatrix}\\
&=\begin{bmatrix}1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h&1-i-\varepsilon-2h\\2-2i-3\varepsilon-4h&1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h\\1-i-2\varepsilon-2h&1-i-\varepsilon-2h&i-\varepsilon-h\end{bmatrix}\begin{bmatrix}PH_{n+m+2}&PH_{n+m+1}&PH_{n+m}\\PH_{n+m+3}&PH_{n+m+2}&PH_{n+m+1}\\PH_{n+m+1}&PH_{n+m}&PH_{n+m-1}\end{bmatrix}\\
&=\begin{bmatrix}\overline{PH}_{n+m+2}&\overline{PH}_{n+m+1}&\overline{PH}_{n+m}\\\overline{PH}_{n+m+3}&\overline{PH}_{n+m+2}&\overline{PH}_{n+m+1}\\\overline{PH}_{n+m+1}&\overline{PH}_{n+m}&\overline{PH}_{n+m-1}\end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
(\overline{Q}U^m)(U^nQ)&=\begin{bmatrix}1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h&1-i-\varepsilon-2h\\2-2i-3\varepsilon-4h&1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h\\1-i-2\varepsilon-2h&1-i-\varepsilon-2h&i-\varepsilon-h\end{bmatrix}\begin{bmatrix}P_{m-2}&P_{m-3}&P_{m-4}\\P_{m-1}&P_{m-2}&P_{m-3}\\P_{m-3}&P_{m-4}&P_{m-5}\end{bmatrix}(U^nQ)\\
&=\begin{bmatrix}\overline{PH}_{m+2}&\overline{PH}_{m+1}&\overline{PH}_{m}\\\overline{PH}_{m+3}&\overline{PH}_{m+2}&\overline{PH}_{m+1}\\\overline{PH}_{m+1}&\overline{PH}_{m}&\overline{PH}_{m-1}\end{bmatrix}\begin{bmatrix}P_{n-2}&P_{n-3}&P_{n-4}\\P_{n-1}&P_{n-2}&P_{n-3}\\P_{n-3}&P_{n-4}&P_{n-5}\end{bmatrix}\begin{bmatrix}1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h&1+i+\varepsilon+2h\\2+2i+3\varepsilon+4h&1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h\\1+i+2\varepsilon+2h&1+i+\varepsilon+2h&i+\varepsilon+h\end{bmatrix}\\
&=\begin{bmatrix}\overline{PH}_{m+2}&\overline{PH}_{m+1}&\overline{PH}_{m}\\\overline{PH}_{m+3}&\overline{PH}_{m+2}&\overline{PH}_{m+1}\\\overline{PH}_{m+1}&\overline{PH}_{m}&\overline{PH}_{m-1}\end{bmatrix}\begin{bmatrix}PH_{n+2}&PH_{n+1}&PH_{n}\\PH_{n+3}&PH_{n+2}&PH_{n+1}\\PH_{n+1}&PH_{n}&PH_{n-1}\end{bmatrix}
\end{aligned}$$

(ii)$(QU^{m+n})\overline{Q}=(QU^m)(U^n\overline{Q})$,

$$\begin{aligned}
(QU^{m+n})\overline{Q}&=\begin{bmatrix}1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h&1+i+\varepsilon+2h\\2+2i+3\varepsilon+4h&1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h\\1+i+2\varepsilon+2h&1+i+\varepsilon+2h&i+\varepsilon+h\end{bmatrix}\begin{bmatrix}P_{m+n-2}&P_{m+n-3}&P_{m+n-4}\\P_{m+n-1}&P_{m+n-2}&P_{m+n-3}\\P_{m+n-3}&P_{m+n-4}&P_{m+n-5}\end{bmatrix}\overline{Q}\\
&=\begin{bmatrix}PH_{m+n+2}&PH_{m+n+1}&PH_{m+n}\\PH_{m+n+3}&PH_{m+n+2}&PH_{m+n+1}\\PH_{m+n+1}&PH_{m+n}&PH_{m+n-1}\end{bmatrix}\begin{bmatrix}1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h&1-i-\varepsilon-2h\\2-2i-3\varepsilon-4h&1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h\\1-i-2\varepsilon-2h&1-i-\varepsilon-2h&i-\varepsilon-h\end{bmatrix}\\
&=\begin{bmatrix}\overline{PH}_{m+n+2}&\overline{PH}_{m+n+1}&\overline{PH}_{m+n}\\\overline{PH}_{m+n+3}&\overline{PH}_{m+n+2}&\overline{PH}_{m+n+1}\\\overline{PH}_{m+n+1}&\overline{PH}_{m+n}&\overline{PH}_{m+n-1}\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
(QU^m)(U^n\overline{Q})&=\begin{bmatrix}1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h&1+i+\varepsilon+2h\\2+2i+3\varepsilon+4h&1+2i+2\varepsilon+3h&1+i+2\varepsilon+2h\\1+i+2\varepsilon+2h&1+i+\varepsilon+2h&i+\varepsilon+h\end{bmatrix}\begin{bmatrix}P_{m-2}&P_{m-3}&P_{m-4}\\P_{m-1}&P_{m-2}&P_{m-3}\\P_{m-3}&P_{m-4}&P_{m-5}\end{bmatrix}(U^n\overline{Q})\\
&=\begin{bmatrix}PH_{m+2}&PH_{m+1}&PH_{m}\\PH_{m+3}&PH_{m+2}&PH_{m+1}\\PH_{m+1}&PH_{m}&PH_{m-1}\end{bmatrix}\begin{bmatrix}P_{n-2}&P_{n-3}&P_{n-4}\\P_{n-1}&P_{n-2}&P_{n-3}\\P_{n-3}&P_{n-4}&P_{n-5}\end{bmatrix}\begin{bmatrix}1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h&1-i-\varepsilon-2h\\2-2i-3\varepsilon-4h&1-2i-2\varepsilon-3h&1-i-2\varepsilon-2h\\1-i-2\varepsilon-2h&1-i-\varepsilon-2h&i-\varepsilon-h\end{bmatrix}\\
&=\begin{bmatrix}PH_{m+2}&PH_{m+1}&PH_{m}\\PH_{m+3}&PH_{m+2}&PH_{m+1}\\PH_{m+1}&PH_{m}&PH_{m-1}\end{bmatrix}\begin{bmatrix}\overline{PH}_{n+2}&\overline{PH}_{n+1}&\overline{PH}_{n}\\\overline{PH}_{n+3}&\overline{PH}_{n+2}&\overline{PH}_{n+1}\\\overline{PH}_{n+1}&\overline{PH}_{n}&\overline{PH}_{n-1}\end{bmatrix}
\end{aligned}$$

□

Theorem 2.7. *For $m \geqslant 0$, $n \geqslant 0$, one has to:*
(i)$PeH_{m+n}-PeH_{m+n+1}i-PeH_{m+n+2}\varepsilon-PeH_{m+n+3}h=\overline{PeH}_{m+2}PeH_n+\overline{PeH}_{m+1}PeH_{n+1}+\overline{PeH}_mPeH_{n-1}$,
(ii)$\overline{PeH}_{n+m}+\overline{PeH}_{n+m+1}i+\overline{PeH}_{n+m+2}\varepsilon+\overline{PeH}_{n+m+3}h=PeH_{m+2}\overline{PeH}_n+PeH_{m+1}\overline{PeH}_{n+1}+PeH_m\overline{PeH}_{n-1}$.

Proof. Using the conjugate of the matrix W, called $\overline{W}$, where:
$W = \begin{bmatrix} 2+3i+2\varepsilon+5h & 2i+3\varepsilon+2h & 3+2\varepsilon+3h \\ 3+2i+5\varepsilon+5h & 2+3i+2\varepsilon+5h & 2i+3\varepsilon+2h \\ 2i+3\varepsilon+2h & 3+2\varepsilon+3h & -1+3i+2h \end{bmatrix}$.

Thus, performing:

$$A^n\overline{W} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} 2-3i-2\varepsilon-5h & 2i-3\varepsilon-2h & 3-2\varepsilon-3h \\ 3-2i-5\varepsilon-5h & 2-3i-2\varepsilon-5h & 2i-3\varepsilon-2h \\ 2i-3\varepsilon-2h & 3-2\varepsilon-3h & -1-3i-2h \end{bmatrix} = \begin{bmatrix} \overline{PeH}_{n+2} & \overline{PeH}_{n+1} & \overline{PeH}_{n} \\ \overline{PeH}_{n+3} & \overline{PeH}_{n+2} & \overline{PeH}_{n+1} \\ \overline{PeH}_{n+1} & \overline{PeH}_{n} & \overline{PeH}_{n-1} \end{bmatrix}.$$

Thus, considering the element a_{13}, it can be said that:
(i)$\overline{W}(A^{m+n}W) = (\overline{W}A^m)(A^nW)$,
(ii)$W(A^{m+n}\overline{W}) = (WA^m)(A^n\overline{W})$. □

3. Conclusion

This work carried out an investigative study around the hybrid numbers of Padovan and Perrin, developing new and proven theorems based on fundamental algebraic operations and their respective matrix forms. Thus, the results presented here have the bias of motivating further studies on the numbers of Padovan and Perrin hybrids, improving the investigation for other numerical sequences.

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Author's contributions

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References

[1] J. Aarts, R. Fokkink, G. Kruijtzer, *Morphic numbers*, Nieuw Archief voor Wiskunde, **5**(2) (2001), 56-58.

[2] K. Adegoke, *Summation identities involving Padovan and Perrin numbers*, arXiv preprint arXiv:1812.03241, (2018).

[3] P. Catarino, *On k-pell hybrid numbers*, J. Disc. Math. Sci. Cryp., Taylor & Francis, (2019), 1-7.

[4] G. Cerda-Morales, *Investigation of generalized hybrid Fibonacci numbers and their properties*, arXiv preprint arXiv:1806.02231, (2018).

[5] R. Ferreira, *Números mórficos*, Dissertação de Mestrado Profissional em Matemática, Universidade Federal da Paraíba, João Pessoa, 2015.

[6] K. Khompungson, B. Rodjanadid, S. Sompong, *Some matrices in term of Perrin and Padovan sequences*, Thai J. Math., **17**(3) (2019), 767-774.

[7] M. C. dos S. Mangueira, et al., *A generalização da forma matricial da sequência de Perrin*, Revista Sergipana de Matemática e Educação Matemática, **5**(1) (2020), 384-392.

[8] M. C. dos S. Mangueira, R. P. M. Vieira, F. R. V. Alves, P. M. M. C. Catarino, *The hybrid numbers of Padovan and some identities*, Annales Mathematicae Silesianaei 1(ahead-of-print), Sciendo, (2020).

[9] M. C. dos S. Mangueira, F. R. V. Alves, P. M. M. C. Catarino. *Números híbridos de Mersenne*, C.Q.D.-Revista Eletrônica Paulista de Matemática, **18** (2020), 1-11.

[10] M. Özdemir. *Introducion to hybrid numbers*, Adv. App. Cliff. Alg., **28** (2018).

[11] T. D. Senturk, et al. *A Study on Horadam hybrid numbers*, Turk. J. Math., **44**(4) (2020), 1212-1221.

[12] K. Sokhuma, *Matrices formula for Padovan and Perrin sequences*, App. Math. Sci., **7**(142) (2013), 7093-7096.

[13] I. Stewart, *Tales of a neglected number*, Scientific American, **274**(6) (1996), 102-103, 1996.

[14] A. Szynal-Liana, *The Horadam hybrid numbers*, Discussiones Mathematicae-General Algebra and App., Sciendo, **38**(1) (2018), 91-98.

[15] A. Szynal-Liana, I. Wloch. *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, In: Annales Mathematicae Silesianae, Sciendo, (2019), 276-283.

[16] R. Vieira, M. Mangueira, F. Alves, P. Catarino. *Perrin n-dimensional relations*, Fund. J. Math. App. **4**(2) (2021), 100-109.

[17] N. Yilmaz. *More identities on Fibonacci and Lucas hybrid numbers*i Notes on Number Theory and Discrete Math. **27**(2) (2021), 159-167.

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On Recursive Hyperbolic Fibonacci Quaternions

Ahmet Daşdemir[1]*

Abstract
Many quaternions with the coefficients selected from special integer sequences such as Fibonacci and Lucas sequences have been investigated by a great number of researchers. This article presents new classes of quaternions whose components are composed of symmetrical hyperbolic Fibonacci functions. In addition, the Binet's formulas, certain generating matrices, generating functions, Cassini's and d'Ocagne's identities for these quaternions are given.

Keywords: Binet's formula, Cassini's identity, generating Matrix, hyperbolic Fibonacci functions, quaternion
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[1] *Department of Mathematics, Faculty of Science and Arts, Kastamonu University, Kastamonu, Turkey, ORCID: 0000-0001-8352-2020*
***Corresponding author**: ahmetdasdemir37@gmail.com
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1. Introduction

Quaternions are defined as a 4-tuple of real numbers and represented by a linear combination of the elements of the standard orthonormal basis such as

$$q = q_0 + iq_1 + jq_2 + kq_3$$

with the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \tag{1.1}$$

where q_0, q_1, q_2 and q_3 are any real numbers, q_0 is called the scalar part of q and $iq_1 + jq_2 + kq_3$ is the vector part. Note that its scalar and vector parts are abbreviated as $Sc(q)$ and $Vec(q)$, respectively. The conjugate of q is

$$q^* = q_0 - iq_1 - jq_2 - kq_3$$

and its norm is

$$N(q) = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{1.2}$$

Eq. (1.1) indicates that the real quaternions form a non-commutative division algebra, even the skew field in the set of quaternions. Due to the loss of commutativity, it is very difficult to study them.

Quaternions are useful tools in many science areas such as mathematics, physics and computer sciences. The monographs in [1] and [2] present well-known systematic investigations on the subject. In recent decades, several researchers investigate different types of quaternions. In [3], Horadam gave the Fibonacci quaternions in the form

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

where F_n is the usual Fibonacci number defined as

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geqslant 0$$

by the initial terms $F_0 = 0$ and $F_1 = 1$. In addition, the Lucas numbers are defined by the same recurrence relation but with the initial conditions $L_0 = 2$ and $L_1 = 1$. Certain important results on the Fibonacci quaternions are presented in the references [3]-[8] and many other related references which are not given here.

In the current literature, many interesting generalizations of the Fibonacci and Lucas numbers and their various types can be found. However, one of the most interesting approaches to the topics is given by Stakhov and Tkachenko in [9]. By Binet's formulas of the Fibonacci and Lucas numbers, they introduced a new concept called hyperbolic Fibonacci and hyperbolic Lucas functions. The monograph [10] offers a detailed review. Furthermore, Stakhov and Rozin improve this approach to symmetrical hyperbolic Fibonacci and symmetrical hyperbolic Lucas functions. According to the authors in [11], the symmetrical hyperbolic functions are defined as follows:

$$\text{Symmetrical Fibonacci sine functions: } sFs\,(x) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}, \tag{1.3}$$

$$\text{Symmetrical Fibonacci cosine functions: } cFs\,(x) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}, \tag{1.4}$$

$$\text{Symmetrical Lucas sine functions: } sLs\,(x) = \alpha^x - \alpha^{-x} \tag{1.5}$$

and

$$\text{Symmetrical Lucas cosine functions: } cLs\,(x) = \alpha^x + \alpha^{-x}, \tag{1.6}$$

where x is any real number and α is the golden ratio. It should be noted that $sFs\,(2k) = F_{2k}$, $cLs\,(2k) = L_{2k}$, $cFs\,(2k+1) = F_{2k+1}$, and $sLs\,(2k+1) = L_{2k+1}$ can be written for $k \in Z$, respectively. Since the hyperbolic functions play a great role in modern sciences such as mathematics and physics, these special functions are very important. Note that throughout the paper, we will omit the letter "s" in right-hand sides of the representations "$sFs\,(x)$" and "$cFs\,(x)$" for combinatorial simplicity, e.g. $sF\,(x)$ and $cF\,(x)$, respectively.

Based on the above developments, Daşdemir introduced the symmetrical hyperbolic Lucas sine and cosine quaternions as

$$sPs\,(x) = sLs\,(x) + isLs\,(x+1) + jsLs\,(x+2) + ksLs\,(x+3)$$

and

$$cPs\,(x) = cLs\,(x) + icLs\,(x+1) + jcLs\,(x+2) + kcLs\,(x+3)\,,$$

respectively [12]. Here, x is any real number, and $sLs\,(x)$ and $cLs\,(x)$ were respectively defined in (1.5) and (1.6).

In this paper, we define new classes of quaternions. The coefficients of these quaternions are chosen from the symmetrical hyperbolic Fibonacci functions. Note that our definitions give the quaternions regarded as a combinations of real valued-functions, not integer valued-functions. Further, we give hyperbolic properties and some identities including the Binet's formulas, the generating functions, the Cassini's and d'Ocagne's identities for these quaternions.

2. Main Results

Consider the symmetrical hyperbolic Fibonacci functions given in (1.3) and (1.4). Hence, we give the following definition.

Definition 2.1. *The symmetrical hyperbolic Fibonacci sine and cosine quaternion functions are defined by the relations*

$$S\,(x) = sF\,(x) + isF\,(x+1) + jsF\,(x+2) + ksF\,(x+3) \tag{2.1}$$

and

$$C\,(x) = cF\,(x) + icF\,(x+1) + jcF\,(x+2) + kcF\,(x+3)\,, \tag{2.2}$$

respectively.

For simplicity, we shall call these quaternions in (2.1) and (2.2) s-Fibonacci quaternions and c-Fibonacci quaternions, respectively. It should be noted that x is regarded as any real number throughout the paper.

We can present the following fundamental properties of the quaternions defined above.

Theorem 2.2. *Let x be any real number. Hence, we have the following relations:*

$$S(x+2) = C(x+1) + S(x) \tag{2.3}$$

and

$$C(x+2) = S(x+1) + C(x), \tag{2.4}$$

respectively.

Proof. The proof is completed by employing the Binet's formulas given in (1.3) and (1.4). □

Remark 2.3. *According to Theorem 2.2, it is possible to exchange symmetrically $S(x)$ with $C(x)$ and $C(x)$ with $S(x)$ for all linear relations of the s-Fibonacci and c-Fibonacci quaternions.*

Note that the recurrence relations given in (2.3) and (2.4) are inhomogeneous. If we apply the corresponding relation to the first term of its right-hand-side, we obtain new structure of each equation, in the homogeneous form, as

$$S(x+2) = 3S(x) - S(x-2) \tag{2.5}$$

and

$$C(x+2) = 3C(x) - C(x-2). \tag{2.6}$$

We can represent the recurrence relations in (2.3)-(2.6) by generating matrix technique. To do this, we introduce the matrices

$$\mathbf{Mc}(x) = \begin{bmatrix} C(x+1) & S(x) \\ S(x) & C(x-1) \end{bmatrix},$$

$$\mathbf{Ms}(x) = \begin{bmatrix} S(x+1) & C(x) \\ C(x) & S(x-1) \end{bmatrix},$$

$$\mathbf{Ns}(x) = \begin{bmatrix} S(x+1) & S(x) \\ S(x-1) & S(x-2) \end{bmatrix}$$

and

$$\mathbf{Nc}(x) = \begin{bmatrix} C(x+1) & C(x) \\ C(x-1) & C(x-2) \end{bmatrix}.$$

Hence, we can write

$$\mathbf{Mc}(x) = \mathbf{P}.\mathbf{Ms}(x-1),\ \mathbf{Ms}(x) = \mathbf{P}.\mathbf{Mc}(x-1), \tag{2.7}$$

$$\mathbf{Ns}(x) = \mathbf{R}.\mathbf{Ns}(x-2) \text{ and } \mathbf{Nc}(x) = \mathbf{R}.\mathbf{Nc}(x-2), \tag{2.8}$$

where $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$. Hence, we can give the following theorem.

Theorem 2.4. *Let x and n be a real positive integer, respectively. Then,*

$$\mathbf{Mc}(x) = \mathbf{P}^m.\begin{cases} \mathbf{Ms}(\mu), & \text{if } m \text{ is odd} \\ \mathbf{Mc}(\mu), & \text{if } m \text{ is even} \end{cases},$$

$$\mathbf{Ms}(x) = \mathbf{P}^m.\begin{cases} \mathbf{Mc}(\mu), & \text{if } m \text{ is odd} \\ \mathbf{Ms}(\mu), & \text{if } m \text{ is even} \end{cases},$$

$$\mathbf{Ns}(x) = \mathbf{R}^m.\mathbf{Ns}(\mu),$$

and

$$\mathbf{Nc}(x) = \mathbf{R}^m.\mathbf{Nc}(\mu).$$

Proof. Extending the right side of the matrix equations in (2.7) and (2.8) to the right, the desired results are obtained by a property of matrix multiplication. □

Remark 2.5. *It follows from Theorem 2.4 that all the determinantal identities obtained for s-Fibonacci quaternions are equal to negative sign of those for c-Fibonacci quaternions. Hence, it is sufficient that we only give the next results for the s-Fibonacci quaternions. But keep in mind that negative signs of right-hand side for these equations are valid for c-Fibonacci quaternions.*

The Binet's formulas for the s-Fibonacci and c-Fibonacci quaternions are given in the following theorem.

Theorem 2.6 (Binet's formula)**.** *The Binet's formulas of s-Fibonacci and c-Fibonacci quaternions are given as follows:*

$$S(x) = \frac{\boldsymbol{A}\alpha^x - \boldsymbol{B}\alpha^{-x}}{\sqrt{5}} \text{ and } C(x) = \frac{\boldsymbol{A}\alpha^x + \boldsymbol{B}\alpha^{-x}}{\sqrt{5}}, \tag{2.9}$$

where $\boldsymbol{A} = 1 + i\alpha + j\alpha^2 + k\alpha^3$*,* $\boldsymbol{B} = 1 - i\beta + j\beta^2 - k\beta^3$ *and* $\beta = -\alpha^{-1}$*.*

Proof. From the definition of s-Fibonacci quaternions and the Binet's formula of $sF(x)$, we can write

$$\begin{aligned} S(x) &= sF(x) + isF(x+1) + jsF(x+2) + ksF(x+3) \\ &= \tfrac{\alpha^x - \alpha^{-x}}{\sqrt{5}} + i\tfrac{\alpha^{x+1} - \alpha^{-(x+1)}}{\sqrt{5}} + j\tfrac{\alpha^{x+2} - \alpha^{-(x+2)}}{\sqrt{5}} + k\tfrac{\alpha^{x+3} - \alpha^{-(x+3)}}{\sqrt{5}} \\ &= \tfrac{\alpha^x\left(1 + i\alpha + j\alpha^2 + k\alpha^3\right) - \alpha^{-x}\left(1 + i\alpha^{-1} + j\alpha^{-2} + k\alpha^{-3}\right)}{\sqrt{5}}. \end{aligned}$$

Substituting $\beta = -\alpha^{-1}$ into the last equation, the first equation is attained. By employing the same procedure, the other is obtained. So, the proof is completed. □

From the Binet's formulas in (2.9), we conclude that the c-Fibonacci quaternions are an even function, but nothing can be said for the s-Fibonacci quaternions. In addition, considering the Binet's formulas given in (2.9), we can write

$$C(x) = S(x) + \frac{2\mathbf{B}}{\sqrt{5}}\beta^x. \tag{2.10}$$

This result indicates that a study of the one involves familiarity with the other one. Note that all the results obtained in this paper are transformed to the other by employing Eq. (2.10).

From the Binet's formulas given in (2.9), we conclude that the s-Fibonacci and the c-Fibonacci quaternions have the same form except for the sign of α^{-x}. Hence, we can enter an auxiliary function that possesses 1 and -1 for consecutive integer values of x into the Binet's formulas to guarantee continuous condition. To do this, the function $\cos(\pi x)$ may be the best choice. Consequently, the following definition arises naturally.

Definition 2.7. *The quasi-sine Fibonacci quaternion is defined as*

$$\mathscr{Q}(x) = \frac{\boldsymbol{A}\alpha^x - \cos(\pi x)\boldsymbol{B}\alpha^{-x}}{\sqrt{5}} \tag{2.11}$$

Here, we can say that the definition in (2.11) satisfies the same recurrence relation in (2.3). For even or odd integer values of x, the quasi-sine Fibonacci quaternion reduces to the s-Fibonacci and the c-Fibonacci quaternions, respectively.

Theorem 2.8. *For any real number x, we have*

$$N(S(x)) = \sqrt{3cF(2x+3) - \frac{8}{5}} \tag{2.12}$$

and

$$N(C(x)) = \sqrt{3cF(2x+3) + \frac{8}{5}}. \tag{2.13}$$

Proof. Considering definition in (1.2), we can write

$$\begin{aligned} [N(S(x))]^2 &= [sF(x)]^2 + [sF(x+1)]^2 + [sF(x+2)]^2 + [sF(x+3)]^2 \\ &= \left[\frac{\alpha^x - \alpha^{-x}}{\sqrt{5}}\right]^2 + \left[\frac{\alpha^{(x+1)} - \alpha^{-(x+1)}}{\sqrt{5}}\right]^2 + \left[\frac{\alpha^{(x+2)} - \alpha^{-(x+2)}}{\sqrt{5}}\right]^2 + \left[\frac{\alpha^{(x+3)} - \alpha^{-(x+3)}}{\sqrt{5}}\right]^2 \end{aligned}$$

Here, expanding the square terms and using property $\alpha^2+1=\sqrt{5}\alpha$ yields to

$$
\begin{aligned}
[N(S(x))]^2 &= \frac{\alpha^{2x}\left(1+\alpha^2+\alpha^4+\alpha^6\right)+\alpha^{-2x}\left(1+\alpha^{-2}+\alpha^{-4}+\alpha^{-6}\right)-8}{5} \\
&= \frac{3\sqrt{5}\left(\alpha^{(2x+3)}+\alpha^{-(2x+3)}\right)-8}{5}.
\end{aligned}
$$

The last equation gives Eq. (2.12). Repeating the same procedure, (2.13) can be demonstrated. □

Next theorem only presents some linear elementary properties for the s-Fibonacci quaternions to reduce the size of the paper. Note that according to Remark 2.3, a property known for the one can also be written for the other due to the exchange rule for all the linear properties.

Theorem 2.9. *Let x be any real numbers. Then, we have*

$$sPs(x) = S(x+1)+S(x-1), \tag{2.14}$$

$$2S(x+1) = C(x)+sPs(x),$$

$$5S(x) = sPs(x+1)+sPs(x-1)$$

and

$$cPs(x)+5S(x) = 2sPs(x+1).$$

Proof. We only prove Eq. (2.14) since the others can be showed similarly. From the Binet's formula in (2.9), we can write

$$\frac{\mathbf{A}\alpha^{x+1}-\mathbf{B}\alpha^{-(x+1)}}{\sqrt{5}}+\frac{\mathbf{A}\alpha^{x-1}-\mathbf{B}\alpha^{-(x-1)}}{\sqrt{5}}=\frac{\left(\alpha^2+1\right)\left(\mathbf{A}\alpha^{x-1}-\mathbf{B}\alpha^{-x-1}\right)}{\sqrt{5}}$$

and using $\alpha^2+1=\sqrt{5}\alpha$, the proof is completed. □

Theorem 2.10 (Hyperbolic version of the Pythagorean Theorem)**.** *The following property holds for any real number x:*

$$C(x)^2-S(x)^2 = -\frac{4}{\sqrt{5}}[C(0)]^*.$$

Proof. To show the validity of theorem, we use an interesting property of quaternions: Let p be any quaternion of the components p_0, p_1, p_2, and p_3. Then, $p^2=2p_0p-[N(p)]^2$. Hence, we can write

$$
\begin{aligned}
C(x)^2-S(x)^2 &= 2cF(x)C(x)-[N(C(x))]^2-2sF(x)S(x)-[N(S(x))]^2 \\
&= 2cF(x)C(x)-2sF(x)S(x)-\frac{16}{5}.
\end{aligned}
$$

It can be showed that $cF(x)cF(x+y)-sF(x)sF(x+y)=\frac{2}{\sqrt{5}}cF(y)$ for all real numbers x any y by using the Binet's formulas in (1.3) and (1.4). Considering this identity, we obtain

$$C(x)^2-S(x)^2 = 2cF(x)C(x)-2sF(x)S(x)-\frac{16}{5}=\frac{4}{\sqrt{5}}C(0)-\frac{16}{5}$$

Since $\frac{2}{\sqrt{5}}cF(0)=\frac{4}{5}$, the proof is completed. □

Theorem 2.11 (Moivre-type formula)**.** *Let x be any real number and n be any $(positive\,or\,negative)$ integers. Then,*

$$[C(x)+S(x)]^n = \left(\frac{2}{\sqrt{5}}\boldsymbol{A}\right)^{n-1}(C(nx)+S(nx)) \tag{2.15}$$

and

$$[C(x)-S(x)]^n = \left(\frac{2}{\sqrt{5}}\boldsymbol{B}\right)^{n-1}(C(nx)-S(nx)). \tag{2.16}$$

Proof. Using the Binet's formulas in (2.9), we can write

$$\left[\frac{\mathbf{A}\alpha^{x}+\mathbf{B}\alpha^{-(x)}}{\sqrt{5}}+\frac{\mathbf{A}\alpha^{x}-\mathbf{B}\alpha^{-x}}{\sqrt{5}}\right]^{n}=\left[\frac{2\mathbf{A}}{\sqrt{5}}\right]^{n}\alpha^{nx}$$
$$=\left[\frac{2\mathbf{A}}{\sqrt{5}}\right]^{n-1}\frac{2\mathbf{A}\alpha^{nx}+\mathbf{B}\alpha^{-nx}-\mathbf{B}\alpha^{-nx}}{\sqrt{5}}$$
$$=\left[\frac{2\mathbf{A}}{\sqrt{5}}\right]^{n-1}\frac{\mathbf{A}\alpha^{nx}+\mathbf{B}\alpha^{-nx}+\mathbf{A}\alpha^{nx}-\mathbf{B}\alpha^{-nx}}{\sqrt{5}},$$

which is Eq. (2.15). Similarly, Eq. (2.16) can be proved. □

Now we give two important theorems that will be reduced to some special cases.

Theorem 2.12 (Vajda identity)**.** *Let x, y, z, and t be any real numbers. Then, we have*

$$S(x+z)S(y-t)-S(x)S(y+z-t)=\frac{2}{\sqrt{5}}[C(u)-C(v)]^{*}+\frac{1}{\sqrt{5}}(sF(u)-sF(v))(2i+5k), \tag{2.17}$$

where $u=x-y+z+t$ *and* $v=x-y-z+t$.

Proof. By employing the Binet's formula in (1.3) for each part after applying the multiplication rule in (1.1) to the left-hand side of Eq. (2.17), the desired result is obtained directly. □

From Vajda identity, we also have the following special identities:

- For $y-t=y$ and $z=1$, we recover the d'Ocagne's identity:

$$S(x+1)S(y)-S(x)S(y+1)=\frac{1}{\sqrt{5}}\left\{2[S(x-y)]^{*}+cF(x-y)(2i+5k)\right\}.$$

- For $x=y$ and $r=s$, we find the Catalan's identity:

$$S(x+z)S(x-z)-S(x)^{2}=\frac{1}{\sqrt{5}}\left\{2[C(2z)]^{*}+sF(2z)(2i+5k)+2\gamma\right\},$$

  where $\gamma=-\frac{2}{\sqrt{5}}+i+\frac{3}{\sqrt{5}}j+2k$.

- For $x=y$ and $r=s=1$, we find the Cassini's identity:

$$S(x+1)S(x-1)-S(x)^{2}=\frac{1}{5}\left(2-8j-\sqrt{5}k\right).$$

Theorem 2.13 (Mixed-Vajda identity)**.** *Let x, y, z, and t be any real numbers. Then, the mixed-Vajda identity of first kind is*

$$S(x+z)S(y-t)-C(x)C(y+z-t)=\frac{2}{\sqrt{5}}[C(u)+C(v)]^{*}+\frac{1}{\sqrt{5}}(sF(u)+sF(v))(2i+5k)$$

and the mixed-Vajda identity of second kind is

$$C(x+z)S(y-t)-C(x)S(y+z-t)=\frac{2}{\sqrt{5}}[S(u)-S(v)]^{*}+\frac{1}{\sqrt{5}}(cF(u)-cF(v))(2i+5k),$$

where $u=x-y+z+t$ *and* $v=x-y-z+t$.

Proof. The proof can be completed by repeating the same procedure in Theorem 2.12. □

Particular cases of the mixed-Vajda identity of first kind are as follows:

- For $y-t=y$ and $z=1$, we obtain the d'Ocagne's identity of first kind:

$$S(x+1)S(y)-C(x)C(y+1)=\frac{1}{\sqrt{5}}\left\{\begin{array}{l}2[C(x-y+1)+C(x-y-1)]^{*}\\+(sF(x-y+1)+sF(x-y-1))(2i+5k)\end{array}\right\}.$$

- For $x=y$ and $r=s$, we find the Catalan's identity of first kind:

$$S(x+z)S(x-z)-C(x)^2=\frac{1}{\sqrt{5}}\left\{2[C(2z)]^*+sF(2z)(2i+5k)-2\gamma\right\}.$$

where $\phi=i-\sqrt{5}j+k$.

- For $x=y$ and $r=s=1$, we find the Cassini's identity of first kind:

$$S(x+1)S(x-1)-C(x)^2=\frac{1}{\sqrt{5}}\left(2\sqrt{5}-4i-4\sqrt{5}j-9k\right).$$

Similarly, we have the following particular cases of the mixed-Vajda identity of second kind are as follows:

- For $y-t=y$ and $z=1$, we obtain the d'Ocagne's identity of second kind:

$$C(x+1)S(y)-C(x)S(y+1)=\frac{1}{\sqrt{5}}\left\{2[C(x-y)]^*+sF(x-y)(2i+5k)\right\}.$$

- For $x=y$ and $r=s$, we find the Catalan's identity of second kind:

$$C(x+z)S(x-z)-C(x)S(x)=\frac{1}{\sqrt{5}}\left\{2[S(2z)]^*+cF(2z)(2i+5k)-\frac{2}{\sqrt{5}}\phi\right\}.$$

where $\phi=i-\sqrt{5}j+k$.

- For $x=y$ and $r=s=1$, we find the Cassini's identity of second kind:

$$C(x+1)S(x-1)-C(x)S(x)=\frac{1}{5}\left(2\sqrt{5}-4i-4\sqrt{5}j-9k\right).$$

Theorem 2.14 (General bilinear formula)**.** *Let x, y, z, t, and w be any integers satisfying $x+y=z+t$. Then, we have*

$$sF(x)C(y)-cF(z)S(t)=sF(x-w)C(y-w)-cF(z-w)S(t-w)$$

and

$$cF(x)S(y)-sF(z)C(t)=cF(x-w)S(y-w)-sF(z-w)C(t-w).$$

Proof. The proof is seen easily by applying the similar technique used in Theorem 2.12. □

We define the following functions:

$$G_s(x,t)=\sum_{n=0}^{\infty}S(x+n)t^n,\ G_c(x,t)=\sum_{n=0}^{\infty}C(x+n)t^n \tag{2.18}$$

and

$$g_s(x,t)=\sum_{n=0}^{\infty}S(x+n)\frac{t^n}{n!}\text{ and }g_c(x,t)=\sum_{n=0}^{\infty}C(x+n)\frac{t^n}{n!}.$$

Note that generating functions are powerful tools for solving linear homogeneous recurrence relations with constant coefficients. Let us introduce generating functions of the s-Fibonacci and c-Fibonacci quaternions. Hence we can write the following theorem.

Theorem 2.15. *The generating functions for the s-Fibonacci and c-Fibonacci quaternions are as follows:*

$$G_s(x,t)=\frac{S(x)+S(x+1)t-S(x-2)t^2-S(x-1)t^3}{(1+x-x^2)(1-x-x^2)} \tag{2.19}$$

and

$$G_c(x,t)=\frac{C(x)+C(x+1)t-C(x-2)t^2-C(x-1)t^3}{(1+x-x^2)(1-x-x^2)}.$$

Proof. We only show the validity of Eq. (2.19) since other can be proven in a similar way. First, we take $t^4 G_s(x,t)$ and $-3t^2 G_s(x,t)$ into account. Hence, from Eqs. (2.5), (2.18) and the last equations, we have

$$\left(1-3x^2+x^4\right) G_s(x,t) = S(x)+S(x+1)t-S(x-2)t^2-S(x-1)t^3,$$

which is desired result. □

We give the exponential generating function for $S(x)$ and $C(x)$ in the following theorem.

Theorem 2.16. *The exponential generating functions for the s-Fibonacci and c-Fibonacci quaternions are as follows:*

$$g_s(x,t) = \frac{\boldsymbol{A}\alpha^x e^{\alpha t} - \boldsymbol{B}\alpha^{-x} e^{-\beta t}}{\sqrt{5}}$$

and

$$g_c(x,t) = \frac{\boldsymbol{A}\alpha^x e^{\alpha t} + \boldsymbol{B}\alpha^{-x} e^{-\beta t}}{\sqrt{5}},$$

where t is any real number and e is the famous Euler's constant.

Proof. Considering MacLaurin expansion for the exponential function, we can write

$$\begin{aligned}
g_s(t) = \sum_{n=0}^{\infty} S(x+n)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\mathbf{A}\alpha^{x+n} - \mathbf{B}\alpha^{-(x+n)}}{\sqrt{5}} \frac{t^n}{n!} &= \frac{1}{\sqrt{5}}\left(\mathbf{A}\alpha^x \sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} - \mathbf{B}\alpha^{-x}\sum_{n=0}^{\infty}\frac{\left(\alpha^{-1}t\right)^n}{n!}\right) \\
&= \frac{1}{\sqrt{5}}(\mathbf{A}\alpha^x e^{\alpha t} - \mathbf{B}\alpha^{-x} e^{\alpha^{-1}t}),
\end{aligned}$$

which is the first equation. Using the same procedure, the second equation can be obtained. □

We give the Honsberger formula for s-Fibonacci and c-Fibonacci quaternions in the following theorem. Note that there are many applications in physic and statistics.

Theorem 2.17 (Honsberger formula)**.** *Let x and y be any real numbers. Then,*

$$S(x+y) = sF(x+1)C(y) + cF(x)S(y-1)$$

and

$$C(x+y) = sF(x+1)S(y) + cF(x)C(y-1).$$

Proof. Using the Binet's formulas in (1.3), (1.4) and (2.9), we can write

$$\begin{aligned}
sFs(x+1)C(y) + cFs(x)S(y-1) &= \frac{\alpha^{x+1}-\alpha^{-(x+1)}}{\sqrt{5}}\frac{\mathbf{A}\alpha^y+\mathbf{B}\alpha^{-y}}{\sqrt{5}} + \frac{\alpha^x+\alpha^{-x}}{\sqrt{5}}\frac{\mathbf{A}\alpha^{y-1}-\mathbf{B}\alpha^{-(y-1)}}{\sqrt{5}} \\
&= \frac{1}{5}\left\{\mathbf{A}\alpha^{x+y+1} - \mathbf{B}\alpha^{-(x+y+1)} + \mathbf{A}\alpha^{x+y-1} - \mathbf{B}\alpha^{-(x+y-1)}\right\} \\
&= \frac{1}{5}\left\{\mathbf{A}\alpha^{x+y-1}\left(\alpha^2+1\right) - \mathbf{B}\alpha^{-(x+y-1)}\left(\alpha^{-2}+1\right)\right\}.
\end{aligned}$$

It can be proved easily that $\alpha^2+1=\sqrt{5}\alpha$ and $\alpha^{-2}+1=\sqrt{5}\alpha^{-1}$ are satisfied. Substituting these properties, the result follows. The second equation can be proved similarly. □

Let z be any real number. Substituting $(x,y)=(x-z,y+z)$ into the Honsberger formulas we can give a more general version of Theorem 2.17 in the following.

Corollary 2.18. *For any real numbers x, y and z, we have*

$$S(x+y) = sF(x-z+1)C(y+z) + cF(x-z)S(y+z-1)$$

and

$$C(x+y) = sF(x-z+1)S(y+z) + cF(x-z)C(y+z-1).$$

We now give summation formula in the following theorem.

Theorem 2.19. *Let x be any real number and n be any positive integer. Then, we have*

$$\sum_{k=0}^{n-1} S(x+k) = C(x+n+1) + C(x+n) - C(x) - C(x-1)$$

and

$$\sum_{k=0}^{n-1} C(x+k) = S(x+n+1) + S(x+n) - S(x) - S(x-1).$$

Proof. Summing all the equations after writing Eq. (2.5) for $x, x+1, \ldots, x+n$, with some mathematical arrangements, we obtain

$$\sum_{k=0}^{n} S(x+k) = -S(x) - S(x+1) + S(x+n+1) + S(x+n+2) + S(x-2) + S(x-1) - S(x+n-1) - S(x+n).$$

Applying the recurrence relation in (2.3) to the last equation, the result follows. □

3. Conclusion

In this paper, we defined the hyperbolic Fibonacci and the quasi-sine Fibonacci quaternion and try to develop some matrix equations to these definitions. Also, we investigated some identities including Binet's formulas, the generating functions, the Pythagorean-type and Moivre-type formulas. In particular, we presented Vajda-type identities that can be reduced to some important well-known forms, including Catalan's or Cassini's identities.

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References

[1] W. R. Hamilton, *Lectures on quaternions*, Hodges and Smith, Dublin, 1853.

[2] J. H. Conway, *Quaternions and octonions*, A K Peters/CRC Press, Canada, 2003.

[3] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly, **70** (1963), 289-291.

[4] M. R. Iyer, *Some results on Fibonacci quaternions*, Fibonacci Quart., **2** (1969), 201-210.

[5] M. R. Iyer, *A note on Fibonacci quaternions*, Fibonacci Quart., **3** (1969), 225–229.

[6] C. Flaut, V. Shpakivskyi, *On generalized Fibonacci quaternions and Fibonacci–Narayana quaternions*, Adv. Appl. Clifford Alg., **23** (2013), 673-688.

[7] P. Catarino, *A note on h(x)-Fibonacci quaternion polynomials*, Chaos Solitons Fractals, **77** (2015), 1-5.

[8] J. L. Ramirez, *Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions*, An. St. Univ. Ovidius Constanta, **23** (2015), 201-212.

[9] A. P. Stakhov, I. S. Tkachenko, *Hyperbolic Fibonacci trigonometry*, Rep. Ukr. Acad. Sci., **208** (1993), 9-14.

[10] A. P. Stakhov, *Hyperbolic Fibonacci and Lucas functions: A new mathematics for the living nature*, ITI, Vinnitsa, 2003.

[11] A. P. Stakhov, B. Rozin, *On a new class of hyperbolic functions*, Chaos Solitons Fractals, **23** (2005), 379-389.

[12] A. Daşdemir, *On hyperbolic Lucas quaternions*, Ars Combin., **150** (2020), 77-84.

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A Nonlinear $r(x)$-Kirchhoff Type Hyperbolic Equation: Stability Result and Blow up of Solutions with Positive Initial Energy

Mohammad Shahrouzi[1]*, Jorge Ferreira[2]

Abstract
In this paper we consider $r(x)-$Kirchhoff type equation with variable-exponent nonlinearity of the form

$$u_{tt}-\Delta u-\big(a+b\int_{\Omega}\frac{1}{r(x)}|\nabla u|^{r(x)}dx\big)\Delta_{r(x)}u+\beta u_t=|u|^{p(x)-2}u,$$

associated with initial and Dirichlet boundary conditions. Under appropriate conditions on $r(.)$ and $p(.)$, stability result along the solution energy is proved. It is also shown that regarding arbitrary positive initial energy and suitable range of variable exponents, solutions blow-up in a finite time.

Keywords: blow-up, Kirchhoff equation, stability result, variable exponents
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[1] *Department of Mathematics, Jahrom University, Jahrom Iran, P.O.Box: 74137-66171*
[2] *Department of Mathematics, Federal University Fluminense, Volta Redonda, Brazil*
***Corresponding author**: mshahrouzi@jahromu.ac.ir
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1. Introduction

Let Ω be a bounded domain of $R^n (n\geq 1)$ with a smooth boundary $\partial\Omega$. Consider the following $r(x)-$Kirchhoff type hyperbolic boundary value problem

$$u_{tt}-\Delta u-\big(a+b\int_{\Omega}\frac{1}{r(x)}|\nabla u|^{r(x)}dx\big)\Delta_{r(x)}u+\beta u_t=|u|^{p(x)-2}u,\ \ (x,t)\in\Omega\times(0,\infty), \tag{1.1}$$

$$u(x,t)=0\ \ (x,t)\in\partial\Omega\times(0,\infty), \tag{1.2}$$

$$u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\ x\in\Omega, \tag{1.3}$$

where a,b,β are positive constants and $\Delta_{r(x)}$ is called $r(x)-$Laplace operator defined as

$$\Delta_{r(x)}u=div(|\nabla u|^{r(x)-2}\nabla u).$$

Here, we have the following condition on the variable exponents:
$(A1)$ the exponents $r(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ such that:

$$2 < r_1 \leq r(x) \leq r_2 < \infty,$$

$$2 < p_2 \leq p(x) \leq p_2 < \infty,$$

with

$$r_1 := essinf_{x\in\overline{\Omega}} r(x),\ r_2 := esssup_{x\in\overline{\Omega}} r(x),$$

$$p_1 := essinf_{x\in\overline{\Omega}} p(x),\ \ p_2 := esssup_{x\in\overline{\Omega}} p(x).$$

Before going any further, it is worth pointing out some results about the Kirchhoff-type equations. Kirchhoff equation

$$u_{tt} - M\Big(\int_\Omega |\nabla_x u|^2 dx\Big)\Delta_x u = f(x,t), \tag{1.4}$$

where $M(s) = as + b,\ a, b > 0$, was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. In the last decade many papers in the literature have investigated the existence of solutions and blow-up results to the Kirchhoff-type problem. For example, Matsuyama and Ikehata [2] considered the following initial-boundary value problem

$$u_{tt} - M\Big(\|\nabla u(t)\|_2^2\Big)\Delta u + \delta|u_t|^{p-1}u_t = \mu|u|^{q-1}u, \quad t \geq 0,\ x \in \Omega,$$

$$u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x),\ \ x \in \Omega,$$

$$u(t,x)|_{\partial\Omega} = 0,\ \ t \geq 0.$$

They proved a global solvability in the class $H^2 \times H_0^1$ and energy decay of the problem without the smallness of the initial data in a certain sense. Ono [3] investigated the global existence, decay properties, and blow-up of solutions to the nonlinear Kirchhoff strings with nonlinear dissipation. Pişkin [4] considered the initial-boundary value problem for the following extensible beam equation with nonlinear damping and source terms

$$u_{tt} + \Delta^2 u - M\big(\|\nabla u\|^2\big)\Delta u + |u_t|^{p-1}u_t = |u|^{q-1}u\ (x,t) \in \Omega \times (0.T),$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad x \in \Omega,$$

$$u(x,t) = \frac{\partial}{\partial \nu}u(x,t) = 0 \quad x \in \partial\Omega,$$

author established the existence of the solution by Banach contraction mapping principle and the decay estimates of the solution by using Nakao's inequality. Moreover, under suitable conditions on the initial datum, the blow up of solutions in finite time has been proved.
In another study, the following initial boundary value problem for a Kirchhoff type plate equation has been considered by Zhou [5]:

$$u_{tt} + \alpha\Delta^2 u - a\Delta u - b\Big(\int_\Omega |\nabla u|^2 dx\Big)^\gamma \Delta u + \lambda u_t = \mu|u|^{p-2}u \ \ in\ \Omega_T,$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad in\ \Omega,$$

$$u(x,t) = \partial_\nu u(x,t) = 0 \quad on\ \Gamma.$$

He proved the blow-up of solutions and the lifespan estimates for three different ranges of initial energy. Global existence of solutions has been proved by the potential well theory, and decay estimates of the energy function have been established by using Nakao's inequality. For more results about the Kirchhoff type equations we refer the readers to [6, 7, 8, 9, 10].
On the other hand, it is known that modeling of some physical phenomena such as flows of electro-rheological fluids, nonlinear viscoelasticity and image processing give rise to equation with nonstandard growth conditions, that is, equations with variable exponents of nonlinearities. In [11], Shahrouzi and Kargarfard proved the blow-up result for the following Kirchhoff type problem:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta_{m(x)}u + h(x,t,u,\nabla u) + \beta u_t = \phi_{p(x)}(u), \ \ in\ \ \Omega \times (0,+\infty)$$

$$\begin{cases} u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,+\infty) \\ M(\|\nabla u\|^2)\frac{\partial u}{\partial n}(x,t) = \alpha u - |\nabla u|^{m(x)}\frac{\partial u}{\partial n}, & (x,t) \in \Gamma_1 \times (0,+\infty) \end{cases}$$

$$u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\quad in\ \ \Omega,$$

and proved the blow up of solutions with positive initial energy and suitable conditions on datas. Recently, Antontsev et. al [12], investigated the following nonlinear Timoshenko equation with variable exponents

$$u_{tt}+\Delta^2 u+M\Big(\|\nabla u\|^2_{L^2(\Omega)}\Big)\Delta u+|u_t|^{m(x)-2}u_t=|u|^{q(x)-2}u,$$

and by using the Faedo–Galerkin method, they proved the local existence of the solution under suitable conditions. Also, the nonexistence of solutions with negative initial energy has been investigated. (see also [13, 14, 15])
Dai and Hao [16] studied the following problem

$$-M\big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\big)div(|\nabla u|^{p(x)-2}\nabla u)=f(x,u),\ \ in\ \Omega$$

$$u=0,\ \ on\ \partial\Omega.$$

By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, they established conditions ensuring the existence and multiplicity of solutions for the problem. Recently, Hamdani et. al. [17] investigated the following nonlocal $p(x)-$Kirchhoff problem:

$$-(a-b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)div(|\nabla u|^{p(x)-2}\nabla u)=\lambda|u|^{p(x)-2}u+g(x,u),\ \ in\ \Omega$$

$$u=0,\ \ on\ \partial\Omega.$$

They obtained a nontrivial weak solution by using the Mountain Pass theorem. For more results in Kirchhoff type equations with variable-exponents nonlinearities we refer the reader to [18, 19, 20, 21, 22, 23] and references therein.

Motivated by the aforementioned works, in the present paper, we study a $r(x)-$ Kirchhoff type equation with variable-exponent nonlinearities. Under appropriate conditions on the initial data and variable exponents, we prove asymptotic stability and blow up of solutions with positive initial energy.
The rest of paper is organized as follows. In Section 2, we recall some definitions and Lemmas about the variable-exponent Lebesgue space, $L^{p(.)}(\Omega)$, the Sobolev space, $W^{1,p(.)}(\Omega)$ and additional conditions that be use for main results. In Section 3, we prove the asymptotic stability of solutions for appropriate initial data and variable exponents. Finally, the blow-up result has been proved with positive initial energy and suitable conditions on data and variable exponents, in fourth Section.

2. Preliminaries

Throughout this work, all the functions considered are real-valued. We denote by $\|.\|_q$ the L^q-norm over Ω . In particular, the L^2-norm is denoted $\|.\|$ in Ω. In order to study problem (1.1)-(1.3), we need some theories about Lebesgue and Sobolev spaces with variable-exponents (for detailed, see [24, 25, 26, 27, 28]). Let $p(x)\geq 1$ and measurable, we assume that

$$C_+(\overline{\Omega})=\{h|h\in C(\overline{\Omega}),\ h(x)>1\ \forall x\in\overline{\Omega}\},$$

$$h^+=\max_{\overline{\Omega}}h(x),\ \ h^-=\min_{\overline{\Omega}}h(x)\quad for\ any\ h\in C(\overline{\Omega}),$$

$$L^{p(x)}(\Omega)=\Big\{u|\ u\ is\ a\ measurable\ real-valued\ function,\ \int_\Omega|u(x)|^{p(x)}dx<\infty\Big\}.$$

We equip the Lebesgue space with a variable exponent, $L^{p(x)}(\Omega)$, with the following Luxembourg-type norm

$$\|u\|_{p(x)}:=\inf\Big\{\lambda>0\Big|\int_\Omega\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}dx\leq 1\Big\}$$

Lemma 2.1. *[24, 28] Let Ω be a bounded domain in R^n*
(i) the space $(L^{p(x)}(\Omega),\|.\|_{p(x)})$ is a Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{p(x)}=1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have

$$\Big|\int_\Omega uvdx\Big|\leq\big(\frac{1}{p^-}+\frac{1}{q^-}\big)\|u\|_{p(x)}\|v\|_{q(x)}.$$

(ii) If $p,q\in C_+(\overline{\Omega})$, $q(x)\leq p(x)$ for any $x\in\overline{\Omega}$, then $L^{p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, and the imbedding is continuous.

The variable-exponent Lebesgue Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Furthermore, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p'(x)}(\Omega)$, by the same way as the usual Sobolev spaces, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
If we define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p^+ < N \\ \infty, & p^+ \geq N, \end{cases}$$

then we have

Lemma 2.2. *[24, 28] Let Ω be a bounded domain in R^n then for any measurable bounded exponent $p(x)$ we have*
(i) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable Banach spaces;
(ii) if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous;
(iii) if $p(x)$ is uniformly continuous in Ω then there exists a constant $C > 0$, such that

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

By (iii) of Lemma 2.2, we know that the space $W_0^{1,p(x)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)}$.
We recall the trace Sobolev embedding in Lebesgue space with a constant exponent

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1) \qquad for \qquad 2 \leq q < \frac{2(n-1)}{n-2}$$

where

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

and the embedding inequality

$$\|u\|_{q,\Gamma_1} \leq B_q \|\nabla u\|_2, \tag{2.1}$$

where B_q is the optimal constant.
We sometimes use the Young's inequality

$$ab \leq \beta a^q + C(\theta, q) b^{q'}, \;\; a, b \geq 0, \;\; \theta > 0, \;\; \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.2}$$

where $C(\theta,q) = \frac{1}{q'}(\theta q)^{-\frac{q'}{q}}$ are constants.

3. Asymptotic stability

In this section we prove a stability result for the solution energy. For this goal we make the following assumptions:
(A2) There exist $\varepsilon > 0$ sufficiently small and β sufficiently large such that

$$p_2 \leq \frac{1}{\varepsilon(\beta - \varepsilon)} \leq r_1 \leq r(x) \leq r_2 \leq 2\varepsilon(\beta - \varepsilon) r_1^2.$$

The energy associated with problem (1.1)-(1.3) is given by

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|\nabla u\|^2\right) + \left(a + \frac{b}{2}\int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)}dx\right)\int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)}dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx. \tag{3.1}$$

Our main result in this section reads in the following theorem:

Theorem 3.1. *Let the conditions (A1) and (A2) are satisfied. Then the energy $E(t)$ of problem (1.1)-(1.3) tends to zero as time goes to infinity.*

To prove the above theorem, we need following Lemmas. First, we define

$$F(t) = E(t) + \varepsilon \int_{\Omega} u u_t dx,$$

for some $\varepsilon > 0$.

Lemma 3.2. *Let u be the solution of (1.1)-(1.3). Then the energy functional satisfies*

$$E'(t) = -\beta \int_{\Omega} |u_t|^2 dx \leq 0. \tag{3.2}$$

Proof. By multiplying equation (1.1) by u_t and integrating over Ω, using integration by parts, we obtain (3.2) for any regular solution. This equality remains valid for weak solutions by a simple density argument. □

The following Lemma estimates an appropriate upper bound for $F'(t)$:

Lemma 3.3. *Under the assumptions of Theorem 3.1, $F(t)$ satisfies, along the solution, the estimate*

$$F'(t) \leq -(\beta-\varepsilon)\|u_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\beta \int_{\Omega} u u_t dx - \varepsilon \int_{\Omega} |u|^{p(x)} dx - \varepsilon(a + b\int_{\Omega} \frac{1}{r(x)} |\nabla u|^{r(x)} dx) \int_{\Omega} |\nabla u|^{r(x)} dx. \tag{3.3}$$

Proof. To prove this Lemma, at first we differentiate $F(t)$ to obtain

$$F'(t) = E'(t) + \varepsilon\|u_t\|^2 + \varepsilon \int_{\Omega} u u_{tt} dx,$$

thanks to Lemma 3.2, we get

$$F'(t) \leq -(\beta-\varepsilon)\|u_t\|^2 + \varepsilon \int_{\Omega} u u_{tt} dx. \tag{3.4}$$

By multiplying (1.1) in u, it is easy to see that

$$\int_{\Omega} u u_{tt} dx = -\|\nabla u\|^2 - (a + b\int_{\Omega} \frac{1}{r(x)} |\nabla u|^{r(x)} dx) \int_{\Omega} |\nabla u|^{r(x)} dx - \beta \int_{\Omega} u u_t dx + \int_{\Omega} |u|^{p(x)} dx. \tag{3.5}$$

Combining (3.5) with (3.4), proof is completed. □

Now, from definition of $F(t)$ and Lemma 3.3, we have

$$F'(t) + \frac{1}{(\beta-\varepsilon)} F(t) \leq -\Big(\beta - \frac{1}{2(\beta-\varepsilon)} - \varepsilon\Big)\|u_t\|^2 - \Big(\varepsilon - \frac{1}{2(\beta-\varepsilon)}\Big)\|\nabla u\|^2 - a\Big(\varepsilon - \frac{1}{r_1(\beta-\varepsilon)}\Big) \int_{\Omega} |\nabla u|^{r(x)} dx$$

$$-b\Big(\frac{\varepsilon}{r_2} - \frac{1}{2r_1^2(\beta-\varepsilon)}\Big)\Big(\int_{\Omega} |\nabla u|^{r(x)} dx\Big)^2 - \Big(\frac{1}{p_2(\beta-\varepsilon)} - \varepsilon\Big) \int_{\Omega} |u|^{p(x)} dx - \int_{\Omega} u u_t dx. \tag{3.6}$$

Using the Young and Poincaré inequalities, we get

$$|\int_{\Omega} u u_t dx| \leq \frac{\varepsilon}{2}\|\nabla u\|^2 + \frac{B_2^2}{2\varepsilon}\|u_t\|^2, \tag{3.7}$$

where B_2 is the best constant in Poincaré inequality.
Utilizing (3.7) into (3.6) to get

$$F'(t) + \frac{1}{(\beta-\varepsilon)} F(t) \leq -\Big(\beta - \frac{1}{2(\beta-\varepsilon)} - \varepsilon - \frac{B_2^2}{2\varepsilon}\Big)\|u_t\|^2 - \Big(\frac{\varepsilon}{2} - \frac{1}{2(\beta-\varepsilon)}\Big)\|\nabla u\|^2 - a\Big(\varepsilon - \frac{1}{r_1(\beta-\varepsilon)}\Big) \int_{\Omega} |\nabla u|^{r(x)} dx$$

$$-b\Big(\frac{\varepsilon}{r_2} - \frac{1}{2r_1^2(\beta-\varepsilon)}\Big)\Big(\int_{\Omega} |\nabla u|^{r(x)} dx\Big)^2 - \Big(\frac{1}{p_2(\beta-\varepsilon)} - \varepsilon\Big) \int_{\Omega} |u|^{p(x)} dx. \tag{3.8}$$

Thanks to the (A2), we deduce

$$F'(t) + \frac{1}{(\beta-\varepsilon)} F(t) \leq 0. \tag{3.9}$$

Integrating over (0,t), we get from (3.9)

$$F(t) \leq F(0) e^{\frac{-t}{\beta-\varepsilon}},$$

according to $(A2)$, this inequality show that $F(t) \to 0$ as $t \to \infty$. Since $E(t) \leq C_0 F(t)$, thus the proof of Theorem 3.1 has been completed.

4. Blow-up

In this section we are going to prove that for appropriate initial data some of the solutions blow up in a finite time. To prove the blow-up result for certain solutions with positive initial energy, we assumed that:
(A3)

$$p_1 \geq r_2+2, \quad r_2^2 \leq 2r_1^2 \leq 2r_2^2.$$

At this point, we shall add a new variable $v(x,t)$ to the system (1.1)-(1.3). Let us define for any $\lambda > 0$

$$v(x,t) = e^{-\lambda t} u(x,t). \tag{4.1}$$

A direct computation by substituting (4.1) into the problem (1.1)-(1.3), yields

$$v_{tt}+(2\lambda+\beta)v_t+\lambda(\lambda+\beta)v-\Delta v-(a+b\int_\Omega \frac{1}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx)div(|e^{\lambda t}\nabla v|^{r(x)-2}\nabla v) = |e^{\lambda t}v|^{p(x)-2}v, \ \ (x,t)\in\Omega\times(0,\infty) \tag{4.2}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega\times(0,\infty), \tag{4.3}$$

$$v(x,0) = u_0(x), \ \ v_t(x,0) = u_1(x)-\lambda u_0(x), \quad x\in\Omega. \tag{4.4}$$

The energy function related with problem (4.2)-(4.4) is given by

$$E_\lambda(t) = e^{-2\lambda t}\int_\Omega \frac{|e^{\lambda t}v|^{p(x)}}{p(x)}dx - \frac{1}{2}I(t), \tag{4.5}$$

where

$$I(t) = \|v_t\|^2+\lambda(\lambda+\beta)\|v\|^2+\|\nabla v\|^2+2ae^{-2\lambda t}\int_\Omega \frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx+be^{-2\lambda t}\Big(\int_\Omega \frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx\Big)^2$$

Now we are in a position to state our blow-up result as follows:

Theorem 4.1. *Let the conditions (A1) and (A3) are satisfied. Assume that $E_\lambda(0) > 0$. Then there exists a finite time t^* such that the solution of the problem (1.1)-(1.3) blows up in a finite time, that is*

$$\|u(t)\| \to +\infty \ \ as \ \ t\to t^*.$$

To prove the blow-up result, we need the following Lemma.

Lemma 4.2. *Under the conditions of Theorem 4.1, the energy functional $E_\lambda(t)$, defined by (4.5), satisfies*

$$E_\lambda(t) \geq e^{r_2\lambda t}E_\lambda(0) \qquad \forall t\in R^+. \tag{4.6}$$

Proof. A multiplication of equation (4.2) by v_t and integrating over Ω gives

$$E'_\lambda(t) = (2\lambda+\beta)\|v_t\|^2 - b\lambda e^{-2\lambda t}\Big(\int_\Omega \frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx\Big)^2 - ae^{-2\lambda t}\int_\Omega \frac{\lambda(r(x)-2)}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx$$

$$+e^{-2\lambda t}\int_\Omega \frac{\lambda(p(x)-2)}{p(x)}|e^{\lambda t}v|^{p(x)}dx - be^{-2\lambda t}\Big(\int_\Omega \frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx\Big)\Big(\int_\Omega \frac{\lambda(r(x)-2)}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx\Big),$$

next, for any $\varepsilon > 0$, we have

$$E'_\lambda(t)-\varepsilon E_\lambda(t) = (2\lambda+\beta+\frac{\varepsilon}{2})\|v_t\|^2+b(\frac{\varepsilon}{2}-\lambda)e^{-2\lambda t}\Big(\int_\Omega \frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx\Big)^2+\frac{\varepsilon}{2}\lambda(\lambda+\beta)\|v\|^2+\frac{\varepsilon}{2}\|\nabla v\|^2$$

$$+a\varepsilon e^{-2\lambda t}\int_\Omega \frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx - ae^{-2\lambda t}\int_\Omega \frac{\lambda(r(x)-2)}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx + e^{-2\lambda t}\int_\Omega \frac{\lambda(p(x)-2)}{p(x)}|e^{\lambda t}v|^{p(x)}dx$$

$$-\varepsilon e^{-2\lambda t}\int_{\Omega}\frac{|e^{\lambda t}v|^{p(x)}}{p(x)}dx-be^{-2\lambda t}\big(\int_{\Omega}\frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx\big)\big(\int_{\Omega}\frac{\lambda(r(x)-2)}{r(x)}|e^{\lambda t}\nabla v|^{r(x)}dx\big). \tag{4.7}$$

Utilizing additional condition $(A1)$, we get

$$E_{\lambda}'(t)-\varepsilon E_{\lambda}(t)\geq b(\frac{\varepsilon}{r_2^2}-\frac{\lambda(r_2-2)}{r_2^2}-\frac{\lambda}{r_1^2})e^{-2\lambda t}\big(\int_{\Omega}|e^{\lambda t}\nabla v|^{r(x)}dx\big)^2+a(\frac{\varepsilon}{r_2}-\frac{\lambda(r_2-2)}{r_2})e^{-2\lambda t}\int_{\Omega}|e^{\lambda t}\nabla v|^{r(x)}dx$$
$$+\frac{1}{p_1}(\lambda(p_1-2)-\varepsilon)e^{-2\lambda t}\int_{\Omega}|e^{\lambda t}v|^{p(x)}dx.$$

Finally, if we set $\varepsilon:=r_2\lambda$ then by using $(A3)$ we arrive at

$$E_{\lambda}'(t)-\varepsilon E_{\lambda}(t)\geq 0,$$

and by integration over $(0,t)$ we obtain the desired result. □

Proof of Theorem 4.1. For obtaining the blow-up result, the choice of the following functional is standard

$$\psi(t)=\|v(t)\|^2, \tag{4.8}$$

then

$$\psi'(t)=2\int_{\Omega}vv_t dx, \tag{4.9}$$

$$\psi''(t)=2\int_{\Omega}vv_{tt}dx+2\|v_t\|^2. \tag{4.10}$$

A multiplication of equation (4.2) by v and integrating over Ω gives

$$\int_{\Omega}v_{tt}v dx=-(2\lambda+\beta)\int_{\Omega}vv_t dx-\lambda(\lambda+\beta)\|v\|^2-\|\nabla v\|^2-e^{-2\lambda t}(a+b\int_{\Omega}\frac{|e^{\lambda t}\nabla v|^{r(x)}}{r(x)}dx)\int_{\Omega}|e^{\lambda t}\nabla v|^{r(x)}dx.$$

$$+e^{-2\lambda t}\int_{\Omega}|e^{\lambda t}v|^{p(x)}dx \tag{4.11}$$

By using definition of $E_{\lambda}(t)$ in (4.11), we have

$$\int_{\Omega}v_{tt}v dx\geq r_2E_{\lambda}(t)+\frac{r_2}{2}\|v_t\|^2+\lambda(\frac{r_2}{2}-1)(\lambda+\beta)\|v\|^2+(\frac{r_2}{2}-1)\|\nabla v\|^2+\frac{b(r_1-2)}{2r_1}e^{-2\lambda t}\big(\int_{\Omega}|e^{\lambda t}\nabla v|^{r(x)}dx\big)^2$$
$$+(1-\frac{r_2}{p_1})e^{-2\lambda t}\int_{\Omega}|e^{\lambda t}v|^{p(x)}dx-(2\lambda+\beta)\int_{\Omega}vv_t dx,$$

and taking into account $(A3)$ to obtain

$$\int_{\Omega}v_{tt}v dx\geq r_2E_{\lambda}(t)+\frac{r_2}{2}\|v_t\|^2+\lambda(\frac{r_2}{2}-1)(\lambda+\beta)\|v\|^2-(2\lambda+\beta)\int_{\Omega}vv_t dx. \tag{4.12}$$

By substituting (4.8)-(4.10) in (4.12) we get

$$\begin{aligned}\psi''(t) &\geq 2r_2E_{\lambda}(t)+(r_2+2)\|v_t\|^2+\lambda(\lambda+\beta)(r_2-2)\psi(t)-(2\lambda+\beta)\psi'(t)\\ &\geq (r_2+2)\|v_t\|^2+\lambda(\lambda+\beta)(r_2-2)\psi(t)-(2\lambda+\beta)\psi'(t),\end{aligned} \tag{4.13}$$

where Lemma 4.2 and hypotheses of Theorem 4.1 about initial energy have been used.
Multiplying (4.13) in $\psi(t)$, we get

$$\psi(t)\psi''(t)\geq(r_2+2)\|v\|^2\|v_t\|^2+\lambda(\lambda+\beta)(r_2-2)\psi^2(t)-(2\lambda+\beta)\psi(t)\psi'(t), \tag{4.14}$$

and finally we obtain

$$\psi(t)\psi''(t) \geq \frac{(r_2+2)}{4}(\psi'(t))^2 + \lambda(\lambda+\beta)(r_2-2)\psi^2(t) - (2\lambda+\beta)\psi(t)\psi'(t),$$

where the inequality $(\psi'(t))^2 \leq 4\|v\|^2\|v_t\|^2$ has been used.
Thus by the modified concavity method we deduce that there exists a finite time t^* such that

$$\lim_{t\to t^*}\psi(t) = \infty.$$

Consequently, the proof of Theorem 4.1 has been completed.

References

[1] G. Kirchhoff, *Vorlesungen uber Mechanik*, Teubner, Leipzig , (1883).

[2] T. Matsuyama, R. Ikehata, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms*, J. Math. Anal. Appl. **204**, 729–753 (1996).

[3] K. Ono, *On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation*, J. Math. Anal. Appl. **216**, 321–342 (1997).

[4] E. Pişkin, *Existence, decay and blow up of solutions for the extensible beam equation with nonlinear damping and source terms*, Open Math. **13**, 408–420 (2015).

[5] J. Zhou, *Global existence and blow-up of solutions for a Kirchhoff type plate equation with damping*, Appl. Math. Comput. **265**, 807–818 (2015).

[6] R. Ikehata, *A note on the global solvability of solutions to some nonlinear wave equations with dissipative terms*, Differ. Integral Equ. **8**, 607–616 (1995).

[7] S. A. Messaoudi, *Blow-up of solutions for the Kirchhoff equation of $q-$Laplacian type with nonlinear dissipation*, Colloq. Math. **94**, 103–109 (2002).

[8] K. Ono, *On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation*, Math. Meth. Appl. Sci. **20**, 151–177 (1997).

[9] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma, *Positive Solutions for a Quasilinear Elliptic Equation of Kirchhoff Type*, Comput. Math. Appl. **49**, 85–93 (2005).

[10] A. Yang, Z. Gong, *Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with arbitrary positive initial energy*, Electron. J. Differ. Equ. **332**, 1–8 (2016).

[11] M. Shahrouzi, F. Kargarfard, *Blow-up of solutions for a Kirchhoff type equation with variable-exponent nonlinearities*, J. App. Anal. **27(1)**, 97–105 (2021).

[12] S. N. Antontsev, J. Ferreira, E. Pişkin, S. M. S. Cordeiro, *Existence and non-existence of solutions for Timoshenko-type equations with variable exponents*, Nonlinear Anal. Real World Appl. **61**, 103341, (2021).

[13] E. Pişkin, *Finite time blow up of solutions of the Kirchhoff-type equation with variable exponents*, Int. J. Nonlinear Anal. Appl. **11(1)**, 37–45 (2020).

[14] M. Shahrouzi, *Blow up of solutions for a $r(x)-$Laplacian Lamé equation with variable-exponent nonlinearities and arbitrary initial energy level*, Int. J. Nonlinear Anal. Appl. **13(1)**, 441–450 (2022).

[15] M. Shahrouzi, *General decay and blow up of solutions for a class of inverse problem with elasticity term and variable-exponent nonlinearities*, Math. Meth. Appl. Sci. 2021. DOI: https://doi.org/10.1002/mma.7891.

[16] G. Dai, R. Hao, *Existence of solutionsfor a $p(x)-$Kirchhoff-type equation*, J. Math. Anal. Appl. **359**, 275–284 (2009).

[17] M. K. Hamdani, A. Harrabi, F. Mtiri, D. D. Repově, *Existence and multiplicity results for a new $p(x)-$Kirchhoff problem*, Nonlinear Analysis **190**, 111598 (2020).

[18] G. Dai, R. Ma, *Solutions for a $p(x)-$Kirchhoff type equation with Neumann boundary data*, Nonlinear Analysis: Real World Applications **12** , 2666–2680 (2011).

[19] A. Ghanmi, Nontrivial solutions for Kirchhoff-type problems involving the $p(x)-$Laplace operator, Rocky Mount. J. Math. **48(4)**, 1145–1158 (2018).

[20] E. J. Hurtado, O. H. Miyagaki, R. D. S. Rodrigues, *Existence and Asymptotic Behaviour for a Kirchhoff Type Equation With Variable Critical Growth Exponent*, Milan J. Math. **85**, 71–102 (2017).

[21] S. Heidarkhani, A. L. A. De Araujo, G. A. Afrouzi, S. Moradi, *Multiple solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions*, Math. Nachr. 291(2), 326–342 (2018).

[22] M. Shahrouzi, *On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities*, Comp. Math. Appl. **75**, 3946–3956 (2018).

[23] M. Shahrouzi, *Global nonexistence of solutions for a class of viscoelastic Lamé system*, Indian J. Pure Appl. Math. **51(4)**, 1383–1397 (2020).

[24] L. Diening, P. Hasto, et al. *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag, (2011).

[25] D. Edmunds, J. Rakosnik, *Sobolev embeddings with variable exponent*, Stud. Math. **143**, 267–293 (2000).

[26] D. Edmunds, J. Rakosnik, *Sobolev embeddings with variable exponent II*, Math. Nachr. **246**, 53–67 (2002).

[27] X. Fan, D. Zhao, *On the spaces $L^{p(x)}$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263**, 424–446 (2001).

[28] X. Fan, Q. H. Zhang, *Existence of solutions for $p(x)-$Laplacian Dirichlet problem*, Nonlinear Anal. **52**, 1843–1852 (2003).