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Multiplication Operators on Second Order Cesàro-Orlicz Sequence Spaces

Serkan Demiriz* and Emrah Evren Kara

Abstract
The main purpose of this paper is to characterize the compact, invertible, Fredholm and closed range multiplication operators on second Cesàro-Orlicz sequence spaces.

Keywords: *Compact operator; Fredholm multiplication operator; Invertible operator; Multiplication operator; Orlicz function; Second order Cesàro sequence space.*

AMS Subject Classification (2020): *47B38, 46A06*

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1. Preliminaries, background and notation

Over years, the interest on properties of multipliers between functional Banach spaces have increased. Let X and Y be Banach spaces consisting of sequences with real or complex terms. A numeric sequence $u = (u_n)$ such that $uf = (u_n f_n) \in Y$ for all $f \in X$ is called a multiplier for X and Y. Each multiplier $u = (u_n)$ induces a linear operator $M_u : X \to Y$ by $M_u(f) = uf$. If M_u is continuous, it is called the *multiplication operator* with symbol u.

Several studies on multiplication operators have been carried out. Mostly, multipliers of spaces of measurable functions have been thoroughly examined. In Halmos's monograph [1], one can find important knowledge about multiplication operators on the Hilbert space of square integrable measurable functions with respect to a given measure. In [2, 3], Singh and Kumar present good works on properties of multiplication operators on spaces of measurable functions and they study compactness and closedness of the range of multiplication operators on certain Hilbert spaces. Mursaleen et al. [4], İlkhan et al. [5] have studied multiplication operators on Cesàro function spaces. Further, Castillo et al. [6–8], obtained significant results and modified the techniques used by the others to study multiplication operators on Orlicz-Lorentz spaces, weak L_p spaces and variable Lebesgue spaces.

The Cesàro sequence space Ces_p was firstly introduced by Shiue [9] as the set of all real sequences $x = (x_n)$ satisfying

$$\|x\|_{Ces_p} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty,$$

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where $1 \leq p < \infty$. Some topological and geometrical properties of Cesàro spaces were studied by Shiue [9] , Leibowitz [10], Jagers [11], Cui and Pluciennik [12], Cui and Hudzik [13], Altay and Kama [14], Kama [15].

A continuous, non-decreasing and convex function $\varphi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it satisfies the following conditions:

- $\varphi(0) = 0$,
- $\varphi(x) > 0$ for $x > 0$,
- $\varphi(x) \to \infty$ as $x \to \infty$.

Additionally, if there exists $K > 0$ such that $\varphi(Lx) \leq KL\varphi(x)$ for all $x \geq 0$ and for $L > 1$, then we say that Orlicz function satisfies the δ_2-condition. We write $e = (e_k)$ and $e^n = (e^n_k)$ for the sequences with $e_k = 1$ for all k, and $e^n_n = 1$ and $e^n_k = 0$ for $k \neq n$.

Lindenstrauss and Tzafriri [16] define the Orlicz sequence space

$$\ell_\varphi = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \varphi\left(\frac{|x_k|}{\lambda}\right) < \infty, \text{ for some } \lambda > 0 \right\}$$

using the idea of Orlicz function. Here and what follows, the space of all complex sequences is denoted by ω. The Orlicz space ℓ_φ with the norm

$$\|x\| = \inf\left\{ \lambda > 0 : \sum_{k=1}^{\infty} \varphi\left(\frac{|x_k|}{\lambda}\right) \leq 1 \right\}$$

is a Banach space.

The space

$$Ces_\varphi(\mathbb{N}) = \left\{ x = (x_k) \in \omega : \sum_{m=1}^{\infty} \varphi\left(\frac{1}{m}\sum_{k=1}^{m} |\lambda x_k|\right) < \infty \right\}$$

is called the Cesàro-Orlicz sequence space which is a Banach space with the norm

$$\|x\|_{Ces_\varphi} = \inf\left\{ \lambda > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{m}\sum_{k=1}^{m}|x_k|}{\lambda}\right) \leq 1 \right\}$$

(see [17]). If $\varphi(x) = |x|^p \quad (p > 1)$, then the Cesàro-Orlicz sequence space $Ces_\varphi(\mathbb{N})$ reduces to the Cesàro sequence space Ces_p.

After Lim and Lee [18] found the dual spaces of Cesàro-Orlicz sequence spaces $Ces_\varphi(\mathbb{N})$, Cui et al. [19] and Damian [20] investigated some properties of these spaces. Later, the authors in [21] studied the multiplication operators on Cesàro-Orlicz sequence spaces.

In 2016, N. Braha [22] defined the second-order Cesàro sequence space as

$$Ces^2(p) = \left\{ x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (n+1-k)|x_k| \right)^p < \infty \right\}$$

for $1 \leq p < \infty$ and he examined some topological and geometrical properties of the space $Ces^2(p)$.

Now, we define the second-order Cesàro-Orlicz sequence space by

$$Ces^2_\varphi(\mathbb{N}) = \left\{ x = (x_k) \in \omega : \sum_{m=1}^{\infty} \varphi\left(\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|\lambda x_k| \right) < \infty \right\}.$$

It is clear that the sequence space $Ces^2_\varphi(\mathbb{N})$ is a Banach space with the norm

$$\|x\|_{Ces^2_\varphi} = \inf\left\{ \lambda > 0 : \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda} \right) \leq 1 \right\}.$$

In this paper, we give the characterization of the boundedness, compactness, closed range and Fredholmness for the multiplication operators $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ defined by $M_u f = uf$ for any $u \in \omega$.

2. Boundedness of Multiplication Operators

In this section, we will prove the theorems related to isometry and boundedness of multiplication operators.

Theorem 2.1. *Given any sequence $u \in \omega$, the multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is bounded if and only if the sequence u is bounded.*

Proof. Let M_u be a bounded operator. On the contrary, assume that u is not a bounded sequence. Then, given any $n \in \mathbb{N}$, there exists some $k_n \in \mathbb{N}$ such that $|u_{k_n}| > n$. It is clear that $\|e^{k_n}\|_{Ces^2_\varphi} = \sum_{m=k_n}^{\infty} \frac{m+1-k}{(m+1)(m+2)\lambda\varphi^{-1}(1)}$. Set $\widehat{e}^{k_n} = \frac{e^{k_n}}{\|e^{k_n}\|_{Ces^2_\varphi}}$. Then, we have $\|\widehat{e}^{k_n}\|_{Ces^2_\varphi} = 1$. It follows that

$$\begin{aligned}\|M_u\widehat{e}^{k_n}\|_{Ces^2_\varphi} &= \frac{\|M_u e^{k_n}\|_{Ces^2_\varphi}}{\|e^{k_n}\|_{Ces^2_\varphi}} \\ &= \frac{\sum_{m=k_n}^{\infty} \frac{(m+1-k)|u_{k_n}|}{(m+1)(m+2)\lambda\varphi^{-1}(1)}}{\|e^{k_n}\|_{Ces^2_\varphi}} \\ &= |u_{k_n}| > n.\end{aligned}$$

This contradicts the fact that M_u is a bounded operator. Hence, we conclude that u is bounded.

Conversely, let u be a bounded sequence. Then, there exists $K > 0$ such that $|u_n| \le K$ for all $n \in \mathbb{N}$. Given any $x \in Ces^2_\varphi(\mathbb{N})$, we obtain that

$$\begin{aligned}\|M_u x\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|(ux)_k|}{\lambda}\right) \\ &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_k||x_k|}{\lambda}\right) \\ &\le K\sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda}\right) \\ &= K\|x\|_{Ces^2_\varphi}\end{aligned}$$

which implies that M_u is a bounded operator. □

Theorem 2.2. *The multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is an isometry if and only if $|u_n| = 1$ for all $n \in \mathbb{N}$.*

Proof. On the contrary, assume that $|u_{n_0}| \neq 1$ for some $n_0 \in \mathbb{N}$. Clearly, we have $\|e^{n_0}\|_{Ces^2_\varphi} = \sum_{m=n_0}^{\infty} \frac{m+1-k}{(m+1)(m+2)\lambda\varphi^{-1}(1)}$. Let $|u_{n_0}| > 1$. Then,

$$\begin{aligned}\|M_u e^{n_0}\|_{Ces^2_\varphi} &= \left(\sum_{m=n_0}^{\infty} \frac{(m+1-k)|u_{n_0}|}{(m+1)(m+2)\lambda\varphi^{-1}(1)}\right) \\ &> \sum_{m=n_0}^{\infty} \frac{m+1-k}{(m+1)(m+2)\lambda\varphi^{-1}(1)} \\ &= \|e^{n_0}\|_{Ces^2_\varphi}\end{aligned}$$

holds. Similarly, if $|u_{n_0}| < 1$, $\|M_u e^{n_0}\|_{Ces^2_\varphi} < \|e^{n_0}\|_{Ces^2_\varphi}$ holds. Thus, we obtain a contradiction. Hence, we conclude that $|u_n| = 1$ for all $n \in \mathbb{N}$.

Now, suppose that $|u_n| = 1$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|M_u x\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|u_k x_k|}{\lambda}\right) \\ &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)}\sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda}\right) \\ &= \|x\|_{Ces^2_\varphi}.\end{aligned}$$

Therefore, $\|M_u x\|_{Ces^2_\varphi} = \|x\|_{Ces^2_\varphi}$ for all $x \in Ces^2_\varphi(\mathbb{N})$ and hence M_u is an isometry. □

3. Compactness of Multiplication Operators

Before we prove our main result in this section, remember the definition of a compact operator.

Let X be a Banach space and B_1 be the closed unit ball in X. If the closure of the set $T(B_1)$ is compact, then the bounded linear operator $T : X \to X$ is said to be *compact*.

By $B(Ces^2_\varphi(\mathbb{N}))$ we denote the set of all bounded linear operators from $Ces^2_\varphi(\mathbb{N})$ into itself. Now, we give our main results about the compactness of the multiplication operator.

Theorem 3.1. *A bounded linear multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is compact if and only if $u_n \to 0$ as $n \to \infty$.*

Proof. Firstly, let M_u be a compact operator. On the contrary, assume that $u_n \nrightarrow 0$ as $n \to \infty$. Then, there exists $\varepsilon_0 > 0$ such that the set $N_{\varepsilon_0} = \{k \in \mathbb{N} : |u_k| \geq \varepsilon_0\}$ is an infinite set and we can write $N_{\varepsilon_0} = \{p_1, p_2, ..., p_n, ...\}$. Then, the set $\{e^{p_n} : p_n \in N_{\varepsilon_0}\}$ is bounded in $Ces^2_\varphi(\mathbb{N})$. It follows that

$$\begin{aligned}
& \|M_u e^{p_n} - M_u e^{p_s}\|_{Ces^2_\varphi} \\
= & \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|u(k)e^{p_n}(k) - u(k)e^{p_s}(k)|}{\lambda}\right) \\
= & \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|u(k)||e^{p_n}(k) - e^{p_s}(k)|}{\lambda}\right) \\
\geq & \ \varepsilon_0 \|e^{p_n} - e^{p_s}\|_{Ces^2_\varphi}
\end{aligned}$$

for all $p_n, p_s \in N_{\varepsilon_0}$. This shows that $\{M_u e^{p_n} : p_n \in N_{\varepsilon_0}\}$ cannot have a convergent subsequence. This contradicts the fact that M_u is a compact operator. Thus, $u_n \to 0$ as $n \to \infty$ holds.

Conversely, let $u_n \to 0$ as $n \to \infty$. Then, for every $\varepsilon > 0$, the set $N_\varepsilon = \{n \in \mathbb{N} : |u_n| \geq \varepsilon\}$ is a finite set. Hence, the space $Ces^2_\varphi(N_\varepsilon)$ is finite dimensional and so $M_u|Ces^2_\varphi(N_\varepsilon)$ is a compact operator. Let $u_n \in \omega$ be defined by

$$u_n(m) = \begin{cases} u(m) & , \quad \forall m \in N_{\frac{1}{n}} \\ 0 & , \quad \forall m \notin N_{\frac{1}{n}} \end{cases}$$

for each $n \in \mathbb{N}$. M_{u_n} is a compact operator since the space $Ces^2_\varphi(N_{\frac{1}{n}})$ is finite dimensional for each $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
& \|(M_{u_n} - M_u)x\|_{Ces^2_\varphi} \\
= & \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|u_n(k)x_k - u(k)x_k|}{\lambda}\right) \\
= & \sum_{m \in N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|u_n(k)x_k - u(k)x_k|}{\lambda}\right) \\
+ & \sum_{m \notin N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|u_n(k)x_k - u(k)x_k|}{\lambda}\right) \\
= & \sum_{m \notin N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|u(k)x_k|}{\lambda}\right) \\
< & \ \frac{1}{n} \sum_{m \notin N_{\frac{1}{n}}}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m}(m+1-k)|x_k|}{\lambda}\right) \\
\leq & \ \frac{1}{n}\|x\|_{Ces^2_\varphi}.
\end{aligned}$$

Hence, we have $\|(M_{u_n} - M_u)\|_{Ces^2_\varphi} \leq \frac{1}{n}$ and so M_u is a compact operator. □

Theorem 3.2. *A bounded linear multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ has closed range if and only if u is bounded away from zero on $S = \{k \in \mathbb{N} : u_k \neq 0\}$.*

Proof. If the range of M_u is closed, then M_u is bounded away from zero on $(kerM_u)^\perp = Ces^2_\varphi(S)$. This means that there exists $\varepsilon > 0$ such that

$$\|M_u x\|_{Ces^2_\varphi} \geq \varepsilon \|x\|_{Ces^2_\varphi} \tag{3.1}$$

for all $x \in Ces^2_\varphi(S)$. Set $H = \{k \in S : |u_k| < \frac{\varepsilon}{2}\}$. If $H \neq \emptyset$, then for $n_0 \in H$, we have

$$\begin{aligned}
\|M_u e^{n_0}\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|u(k)e^{n_0}(k)|}{\lambda} \right) \\
&= \sum_{m=n_0}^{\infty} \frac{(m+1-k)|u(n_0)|}{(m+1)(m+2)\lambda\varphi^{-1}(1)} \\
&< \varepsilon \sum_{m=n_0}^{\infty} \frac{(m+1-k)}{(m+1)(m+2)\lambda\varphi^{-1}(1)} \\
&= \varepsilon \|e^{n_0}\|_{Ces^2_\varphi}.
\end{aligned}$$

That is, $\|M_u e^{n_0}\|_{Ces^2_\varphi} < \|e^{n_0}\|_{Ces^2_\varphi}$ which contradicts (3.1). Hence, $H = \emptyset$ so that $|u_k| \geq \varepsilon$ for all $k \in S$.

For the converse, let u be bounded away from zero on S. Then, there exists $\varepsilon > 0$ such that $|u_n| \geq \varepsilon$ for all $n \in S$. Choose a limit point z in range of M_u. Then there exists a sequence $(M_u x^n)$ which converges to z. Clearly, the sequence $\{M_u x^n\}$ is a Cauchy sequence. We obtain that

$$\begin{aligned}
\|M_u x^n - M_u x^m\|_{Ces^2_\varphi} &= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|u_k x^n_k - u_k x^m_k|}{\lambda} \right) \\
&= \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{\substack{k=0 \\ k\in S}}^{m} (m+1-k)|u_k||x^n_k - x^m_k|}{\lambda} \right) \\
&\geq \varepsilon \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{\substack{k=0 \\ k\in S}}^{m} (m+1-k)|x^n_k - x^m_k|}{\lambda} \right) \\
&= \varepsilon \sum_{m=1}^{\infty} \varphi\left(\frac{\frac{1}{(m+1)(m+2)} \sum_{k=0}^{m} (m+1-k)|\widetilde{x^n_k} - \widetilde{x^m_k}|}{\lambda} \right) \\
&= \varepsilon \|\widetilde{x^n} - \widetilde{x^m}\|_{Ces^2_\varphi},
\end{aligned}$$

where

$$\widetilde{x^n_k} = \begin{cases} x^n_k & , \quad k \in S \\ 0 & , \quad k \notin S. \end{cases}$$

Hence, $\{\widetilde{x_n}\}$ is a Cauchy sequence in $Ces^2_\varphi(\mathbb{N})$. Since $Ces^2_\varphi(\mathbb{N})$ is a complete space, the sequence $\{\widetilde{x_n}\}$ converges to a point $x \in Ces^2_\varphi(\mathbb{N})$. By continuity of M_u, $M_u\widetilde{x_n} \to M_u x$. Also, we have $M_u x^n = M_u \widetilde{x_n} \to z$ and so $M_u x = z$. Hence, $z \in ranM_u$ which means that the range of M_u is closed. □

4. Invertible and Fredholm Multiplication Operators

Before we prove our main results in this section, remember the definition of the Fredholm operator.

If T has closed range, $\dim(kerT)$ and co-$\dim(ranT)$ are finite, then the bounded linear operator $T : X \to X$ is said to be a Fredholm operator.

Theorem 4.1. *Given any sequence $u \in \omega$, the multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is invertible if and only if there exist $K_1 > 0$ and $K_2 > 0$ such that $K_1 < u_n < K_2$ for all $n \in \mathbb{N}$.*

Proof. Let M_u be an invertible operator. Then, the range of M_u is $Ces^2_\varphi(\mathbb{N})$ and so it is closed. From Theorem 3.2, there exists $\varepsilon > 0$ such that $|u_n| \geq \varepsilon$ for all $n \in S$. If $u_k = 0$, for some $k \in \mathbb{N}$, we have $e^k \in kerM_u$ which is a contradiction, since $kerM_u$ is trivial. Hence, we have $|u_n| \geq \varepsilon$ for all $n \in \mathbb{N}$. By boundedness of M_u and Theorem 2.1, there exists $K > 0$ such that $|u_n| \leq K$ for all $n \in \mathbb{N}$. Thus, we conclude that $\varepsilon \leq |u_n| \leq K$ for all $n \in \mathbb{N}$.

For the converse, define a sequence $\gamma \in \omega$ as $\gamma_n = \frac{1}{u_n}$. Theorem 2.1 implies that M_u and M_γ are bounded linear operators. Also $M_u \cdot M_\gamma = M_\gamma \cdot M_u = I$ which means M_u is invertible and M_γ is its inverse. □

Theorem 4.2. *A bounded multiplication operator $M_u : Ces^2_\varphi(\mathbb{N}) \to Ces^2_\varphi(\mathbb{N})$ is a Fredholm operator if and only if*

(i) the set $\{k \in \mathbb{N} : u_k = 0\}$ is finite,

(ii) $|u_n| \geq \varepsilon$, for all $n \in S$.

Proof. Let M_u be a Fredholm operator. If the set $\{k \in \mathbb{N} : u_k = 0\}$ is infinite, then $M_u e^n = (0, 0, ..., 0, ...)$ for all $n \in \mathbb{N}$ with $u_n = 0$. Since e^n's are linearly independent, the space $\{x \in Ces^2_\varphi(\mathbb{N}) : M_u x = (0, 0, ..., 0, ...)\}$ is infinite dimensional. This is a contradiction. Thus, we conclude that (i) holds. Also, from Theorem 3.2, (ii) holds.

Conversely, let the conditions (i) and (ii) hold. By Theorem 3.2 and the condition (ii), we obtain that the range of M_u is closed. The condition (i) implies that $kerM_u$ and $kerM_u^*$ are finite dimensional. Hence, we conclude that M_u is Fredholm. □

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Numerical Simulation of Two Dimensional Coupled Burgers Equations by Rubin-Graves Type Linearization

Nuri Murat Yağmurlu*, Abdulnasır Gagir

Abstract
In the present article, the numerical solution of the two-dimensional coupled Burgers equation has been sought by finite difference method based on Rubin-Graves type linearization. Three models with appropriate initial and boundary conditions are applied to the problem. In order to show the accuracy of the method, the error norms L_2, L_∞ are computed. The error norms L_2, L_∞ of the obtained numerical solutions are compared with the error norms of some of the numerical solutions in the literature.

Keywords: *Two-dimensional Burgers equation; Rubin-Graves type linearization; Finite difference method.*

AMS Subject Classification (2020): *Primary: 35Q51 ; Secondary: 74J35; 33F10.*

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1. Introduction

In nature, some of the pysical phenomena such as gas dynamics, traffic flow, Brusselator chemical reaction-diffusion and shock waves are modelled by nonlinear partial differential equation systems among others such as the two-dimensional coupled Burgers equation (2D-CBE). There are many theoretical and numerical studies about the 2D-CBE equation in the literature. Fletcher [1] has found its analytical solution by applying the two-dimensional Hopf-Cole transform to the two-dimensional coupled Burgers equation. 2D-CBE has been solved numerically by several scholars by means of various methods and techniques. Among others, Fletcher [2] have conducted a work for comparing finite difference and finite element methods. Goyon [3] applied multi level alternating direction implicit methods. Ali et al. [4] have used the collation method via the radial base functions. Jain and Holla [5] have implemented two algorithms using the cubic spline function technique. Bahadır [6] has dealt with the problem by a fully implicit finite difference method. Khater et al. [7] have found out the numerical solution of some Burgers type nonlinear partial differential equations by Chebyshev spectral collocation method.Mittal and Jiwari [8] have applied the differential quadrature method using the Chebyshev-Gauss-Lobatto nodal points. Liao [9] obtained the numerical solution of the two-dimensional coupled Burgers equation by solving the two-dimensional linear heat equation obtained by applying the two-dimensional Hopf-Cole transformation to the

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two-dimensional coupled Burgers equation using the fourth-dimensional finite difference method. Zhu et al. [10] applied the discrete Adomian decomposition method. Srivastava et al. have applied [11] Crank-Nicolson finite difference method, Tamsir and Srivastava [12] have used semi-implicit finite difference method, Srivastava and Tamsir [13] have utilized Crank-Nicolson semi-implicit finite difference method, Thakar and Wani [14] have used linear finite difference method, Srivastava et al. [15] have applied implicit logarithmic finite difference method, Srivastava et al. [16] have used implicit exponential finite difference method, Srivastava and Singh [17] have used explicit-implicit finite difference method, Zhang et al. [18] have used full finite difference and non-standard finite difference methods, Mittal and Tripathi [19] have applied modified bi-cubic B-spline collocation method, Tamsir et al. [20] have used exponential modified cubic-B-spline differential quadrature method, Zhanlav et al. [21] have applied high order explicit finite difference method and Ngondiep [22] has utilized three-level explicit time-split MacCormack algorithm. Saqib et al. [23] have dealt with numerical solutions of 2-dimensional time dependent coupled non-linear systems. Wubs and Goede [24], in their article, considered the fully explicit method resulting from the truncation in the solution process and chosen one of the test problems as the 2-dimensional coupled Burgers' equation. Chai and Ouyang [25] have used proper stabilized Galerkin methods.

The rest of this article is organized as follows: In the first section , the method based on Rubin-Graves type linearization together with finite difference method and used for the numerical solution of two dimensional coupled Burgers equation is presented . Then to see the performance accuracy of the method, the numerical solution of three test model problems has been made and presented in tables by calculating the pointwise values and the error norms L_2 and L_∞ of the model problems of which the analytical solution are known. In addition, comparisons have been made with the error norms of the numerical solutions obtained by various methods available in the literature. In the last section, a brief conclusion is given.

2. Application of the Method

In this article, we consider the the two-dimensional coupled Burgers equation of the general form given as

$$u_t + uu_x + vu_y = \varepsilon(u_{xx} + u_{yy}), \quad (x, y) \epsilon\, \Omega, t > 0 \tag{2.1}$$

$$v_t + uv_x + vv_y = \varepsilon(v_{xx} + v_{yy}), \quad (x, y) \epsilon\, \Omega, t > 0 \tag{2.2}$$

together with the initial

$$\begin{aligned} u(x, y, 0) &= \psi_1(x, y); \quad (x, y) \epsilon\, \Omega \\ v(x, y, 0) &= \psi_2(x, y); \quad (x, y) \epsilon\, \Omega \end{aligned}$$

and the boundary conditions

$$\begin{aligned} u(x, y, t) &= \xi(x, y, t); \quad (x, y) \epsilon\, \partial\Omega \\ v(x, y, t) &= \zeta(x, y, t); \quad (x, y) \epsilon\, \partial\Omega \end{aligned}$$

where $u\,(x, y, t)$ and $v\,(x, y, t)$ denote velocity components. Over the solution domain $\Omega = \{(x, y) : a \leqslant x \leqslant b,\, c \leqslant y \leqslant d\}$ together with its boundary $\partial\Omega$. ψ_1, ψ_2, ξ and ζ are known smooth functions. Re denotes the Reynold number. As it is widely known, at the large values of the Reynold number, a shock wave having a cusp results in and numerical stability near this shock wave is nearly always difficult to obtain.

For the solution process, the domain of the problem in $x-$direction $[a, b]$ is divided into N_x parts having equal length h_x, and in $y-$direction $[c, d]$ is divided into N_y parts having equal length h_y, $x_i = a + ih_x$, $i = 0(1)N_x$; $y_j = c + jh_y$, $j = 0(1)N_y$; a smooth grid is created in the solution domain of the problem with the help of nodal points (x_i, y_j). The step length Δt is taken in the direction of the time variable for $t_n = n\Delta t$, $n = 0(1)N$,. Then, all the numerical calculations to be made in each time step t_n are obtained at the nodes of this smooth grid. The numerical solution of $u\,(x, y, t)$ and $v\,(x, y, t)$ at any node (x_i, y_j, t_n) is shown by $U\,^n_{i,j}$ and $V\,^n_{i,j}$, respectively.

When the finite difference method based on Rubin-Graves type linearization technique is applied, a linear algebraic equation system results in since the related finite difference approaches are written in place of the derivatives in the equation. In the proposed method, the nonlinear partial differential equation is written in the appropriate form and after applying the finite difference method, an iterative relationship between the $(n+1)^{th}$ and

$(n)^{th}$ time level steps of the dependent variables is obtained. This newly obtained iterative relationship resulted in a linear algebraic equation system, which can be easily solved by a symbolic programming language such as MatLab.

Now, for 2D-CBE

$$\begin{aligned} u_t + uu_x + vu_y &= \varepsilon(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= \varepsilon(v_{xx} + v_{yy}) \end{aligned}$$

in place of non-linear terms uu_x, vu_y, uv_x and vv_y Rubin-Graves type [26] linearization technique are used.

In place of u_t an approximation as $u_t \cong (U_{i,j}^{n+1} - U_{i,j}^n)/k$ and in place of v_t an approximation as $v_t \cong (V_{i,j}^{n+1} - V_{i,j}^n)/k$ and in places of the terms uu_x, vu_y, uv_x and vv_y the following Rubin-Graves type approximations

$$uu_x \cong U_{i,j}^{n+1}\left[\frac{U_{i+1,j}^n - U_{i-1,j}^n}{2h_x}\right] + U_{i,j}^n\left[\frac{U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}}{2h_x}\right] - U_{i,j}^n\left[\frac{U_{i+1,j}^n - U_{i-1,j}^n}{2h_x}\right]$$

$$vu_y \cong V_{i,j}^{n+1}\left[\frac{U_{i+1,j}^n - U_{i-1,j}^n}{2h_y}\right] + V_{i,j}^n\left[\frac{U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}}{2h_y}\right] - V_{i,j}^n\left[\frac{U_{i+1,j}^n - U_{i-1,j}^n}{2h_y}\right]$$

$$uv_x \cong U_{i,j}^{n+1}\left[\frac{V_{i+1,j}^n - V_{i-1,j}^n}{2h_x}\right] + U_{i,j}^n\left[\frac{V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1}}{2h_x}\right] - U_{i,j}^n\left[\frac{V_{i+1,j}^n - V_{i-1,j}^n}{2h_x}\right]$$

$$vv_y \cong V_{i,j}^{n+1}\left[\frac{V_{i+1,j}^n - V_{i-1,j}^n}{2h_y}\right] + V_{i,j}^n\left[\frac{V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1}}{2h_y}\right] - V_{i,j}^n\left[\frac{V_{i+1,j}^n - V_{i-1,j}^n}{2h_y}\right]$$

are written. Then in place of the derivatives u_{xx}, u_{yy}, v_{xx} and v_{yy} their central finite difference approximations

$$u_{xx} \cong \frac{U_{i-1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i+1,j}^{n+1}}{h_x^2}$$

$$u_{yy} \cong \frac{U_{i,j-1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j+1}^{n+1}}{h_y^2}$$

$$v_{xx} \cong \frac{V_{i-1,j}^{n+1} - 2V_{i,j}^{n+1} + V_{i+1,j}^{n+1}}{h_x^2}$$

$$v_{yy} \cong \frac{V_{i,j-1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j+1}^{n+1}}{h_y^2}$$

are written. Finally, the terms on the $(n+1)^{th}$ time level are taken on the left hand side and $(n)^{th}$ time level terms are taken on the right hand side. After some simpliciation process, the following

$$\begin{aligned} & U_{i-1,j}^{n+1}\left(-\frac{k}{2h_x}U_{i,j}^n - \frac{\varepsilon k}{h_x^2}\right) + U_{i,j}^{n+1}\left(1 + k\left(\frac{U_{i+1,j}^n - U_{i-1,j}^n}{2h_x}\right) + 4\frac{\varepsilon k}{h_x^2}\right) \\ & + U_{i+1,j}^{n+1}\left(\frac{k}{2h_x}U_{i,j}^n - \frac{\varepsilon k}{h_x^2}\right) + U_{i,j-1}^{n+1}\left(-\frac{k}{2h_y}V_{i,j}^n - \frac{\varepsilon k}{h_y^2}\right) \\ & + U_{i,j+1}^{n+1}\left(\frac{k}{2h_y}V_{i,j}^n - \frac{\varepsilon k}{h_y^2}\right) + V_{i,j}^{n+1}\left(\frac{k(U_{i,j+1}^n - U_{i,j-1}^n)}{2h_y}\right) \\ & = U_{i,j}^n\left[1 + k\left(\frac{U_{i+1,j}^n - U_{i-1,j}^n}{2h_x}\right)\right] + V_{i,j}^n\left[k\left(\frac{U_{i,j+1}^n - U_{i,j-1}^n}{2h_y}\right)\right] \end{aligned}$$

and

$$V_{i-1,j}^{n+1}(-\frac{k}{2h_x}U_{i,j}^n-\frac{\varepsilon k}{h_x^2})+V_{i,j}^{n+1}(1+k(\frac{V_{i,j+1}^n-V_{i,j-1}^n}{2h_y})+4\frac{\varepsilon k}{h_x^2})$$
$$+V_{i+1,j}^{n+1}(\frac{k}{2h_x}U_{i,j}^n-\frac{\varepsilon k}{h_x^2})-V_{i,j-1}^{n+1}(\frac{k}{2h_y}V_{i,j}^n+\frac{\varepsilon k}{h_y^2})$$
$$+V_{i,j+1}^{n+1}(\frac{k}{2h_y}V_{i,j}^n-\frac{\varepsilon k}{h_y^2})+U_{i,j}^{n+1}(\frac{k(V_{i+1,j}^n-U_{i-1,j}^n)}{2h_x})$$
$$=V_{i,j}^n\left[1+k(\frac{V_{i+1,j}^n-V_{i-1,j}^n}{2h_x})\right]+U_{i,j}^n\left[k(\frac{V_{i,j+1}^n-V_{i,j-1}^n}{2h_y})\right]$$

linearized schemes are obtained, where $i,j=1(1)M-1$. In these schemes $h_x=h_y$, $\varepsilon k/h_x^2=\varepsilon k/h_y^2=a$, $k/2h_x=k/2h_y=b$ and $\varepsilon=1/\text{Re}$ are taken as some simplifications are carried out. Finally, the following

$$-U_{i-1,j}^{n+1}\left[bU_{i,j}^n+a\right]+U_{i,j}^{n+1}\left[1+4a+b(U_{i+1,j}^n-U_{i-1,j}^n)\right]+U_{i+1,j}^{n+1}\left[bU_{i,j}^n-a\right]$$
$$-U_{i,j-1}^{n+1}\left[bV_{i,j}^n+a)\right]+U_{i,j+1}^{n+1}\left[bV_{i,j}^n-a\right]+V_{i,j}^{n+1}\left[b(U_{i,j+1}^n-U_{i,j-1}^n)\right]$$
$$=U_{i,j}^n\left[1+b(U_{i+1,j}^n-U_{i-1,j}^n)\right]+V_{i,j}^n\left[b(U_{i,j+1}^n-U_{i,j-1}^n)\right]$$

and

$$-V_{i-1,j}^{n+1}\left[bU_{i,j}^n+a\right]+V_{i,j}^{n+1}\left[1+4a+b(V_{i,j+1}^n-V_{i,j-1}^n)\right]+V_{i+1,j}^{n+1}\left[bU_{i,j}^n-a\right]$$
$$-V_{i,j-1}^{n+1}\left[bV_{i,j}^n+a)\right]+V_{i,j+1}^{n+1}\left[bV_{i,j}^n-a\right]+U_{i,j}^{n+1}\left[b(V_{i+1,j}^n-U_{i-1,j}^n)\right]$$
$$=V_{i,j}^n\left[1+b(V_{i,j+1}^n-V_{i,j-1}^n)\right]+U_{i,j}^n\left[b(V_{i+1,j}^n-V_{i-1,j}^n)\right]$$

schemes are obtained. Using the known U^n and V^n values in the finite difference diagrams obtained as a result of this linearization, the unknown values of U^{n+1} and V^{n+1} at the desired time t are obtained for all three model problems.

3. Numerical Results

In this section, the numerical solution of the two-dimensional coupled Burgers equation given by the equations (2.1)-(2.2), for three problems with appropriate initial and boundary conditions using the finite difference method based on Rubin-Graves type linearization has been obtained. In order to show the accuracy of the obtained numerical solutions, the following error norms L_2 and L_∞ are calculated

$$L_2=\sqrt{\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}|U_{ij}-(u_{exact})_{ij}|^2}$$

and

$$L_\infty=\max_{i,j}|U_{i,j}-(u_{exact})_{i,j}|$$

where u_{ij}^n are analytical solutions and U_{ij}^n are approximate solutions at the nodal points (x_i,y_j,t_n) [27].

Problem I: As the first problem, finite difference method has been applied to 2D-CBE having the following exact solution over the region $\Omega=[0,1]\times[0,1]$ [6]

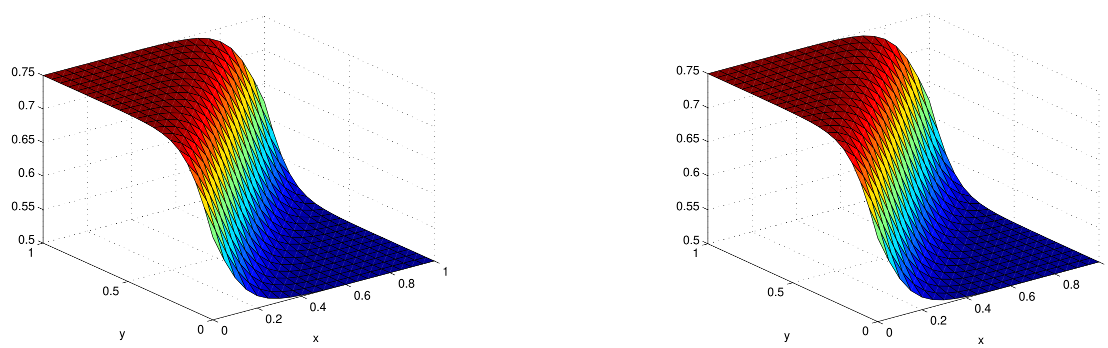
$$u(x,y,t)=\frac{3}{4}-\frac{1}{4\left[1+\exp((-4x+4y-t)\text{Re}/32\right]}\tag{3.1}$$

$$v(x,y,t)=\frac{3}{4}+\frac{1}{4\left[1+\exp((-4x+4y-t)\text{Re}/32\right]}.\tag{3.2}$$

The initial and boundary conditions required for the application of the method are obtained from the analytical solution given by the equations (3.1)-(3.2). Table (1) presents the numerical solutions of Problem I for u for values of $h_x=h_y=0.05$, Re= 10, $\Delta t=10^{-4}$ at times $t=0.01$, 0.5 and 1.0. From the table it is clearly seen that both the

Table 1. Numerical solutions of Problem I for u for values of $h_x = h_y = 0.05$, Re= 10, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.624805	0.624805	0.615254	0.615254	0.605626	0.605626
$(0.5, 0.1)$	0.594202	0.594202	0.585396	0.585396	0.576840	0.576840
$(0.9, 0.1)$	0.567082	0.567082	0.559837	0.559837	0.553017	0.553017
$(0.3, 0.3)$	0.624805	0.624805	0.615255	0.615254	0.605627	0.605626
$(0.7, 0.3)$	0.594202	0.594202	0.585396	0.585396	0.576840	0.576840
$(0.1, 0.5)$	0.655431	0.655431	0.646276	0.646275	0.636685	0.636685
$(0.5, 0.5)$	0.624805	0.624805	0.615256	0.615254	0.605628	0.605626
$(0.9, 0.5)$	0.594202	0.594202	0.585396	0.585396	0.576840	0.576840
$(0.3, 0.7)$	0.655431	0.655431	0.646277	0.646275	0.636687	0.636685
$(0.7, 0.7)$	0.624805	0.624805	0.615256	0.615254	0.605629	0.605626
$(0.1, 0.9)$	0.682611	0.682611	0.674814	0.674814	0.666353	0.666353
$(0.5, 0.9)$	0.655431	0.655431	0.646277	0.646275	0.636687	0.636685
$(0.9, 0.9)$	0.624805	0.624805	0.615255	0.615254	0.605627	0.605626
L_2	8.419211×10^{-8}		2.169158×10^{-6}		2.354379×10^{-6}	
L_∞	6.693449×10^{-8}		2.451640×10^{-6}		2.804863×10^{-6}	



(a) **(b)**

Figure 1. (a) Exact and (b) numerical solutions for u of Problem 1 for values of $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.5$.

numerical and analytical solutions at selected points for each time level are very close to each other. Moreover, it is also seen that the computed error norms L_2 and L_∞ are small enough to be acceptable. Table (2) presents the numerical solutions of Problem I for v for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 10$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0. Again from the table it can be observed that the numerical results are very close to their exact counterparts and computed error norms are small enough. Tables (3-4) show also pointwise values and the error norms L_2 and L_∞ of u and v but now for a larger value of Reynold number $\mathrm{Re} = 100$, respectively. As it is seen from the tables, both of the error norms increase as the Reynold number increases. Figures (1-2) show first exact and then numerical solutions for u and v of Problem 1 for values of $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.0$, respectively.

Problem II: Rubin-Graves type linearization finite difference method has been applied to 2D-CBE on the solution domain $\Omega = [0, 0.5] \times [0, 0.5]$ with the following initial

$$u(x, y, 0) = \sin \pi x + \cos \pi y,\ v(x, y, 0) = x + y \tag{3.3}$$

and boundary conditions

$$\left.\begin{array}{ll} u(0, y, t) = \cos(\pi y), & u(0.5, y, t) = 1 + \cos(\pi y) \\ v(0, y, t) = y, & v(0.5, y, t) = 0.5 + y \end{array}\right\} 0 \le y \le 0.5, t \ge 0 \tag{3.4}$$

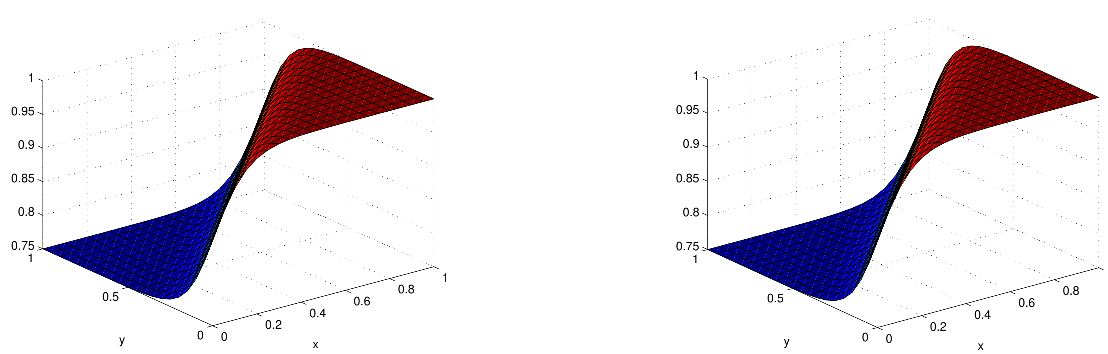
$$\left.\begin{array}{ll} u(x, 0, t) = 1 + \sin(\pi x) & u(x, 0.5, t) = \sin(\pi x) \\ v(x, 0, t) = x & v(x, 0.5, t) = x + 0.5 \end{array}\right\} 0 \le x \le 0.5, t \ge 0 \tag{3.5}$$

Table 2. Numerical solutions of Problem I for v for values of $h_x = h_y = 0.05$, Re$= 10$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.875195	0.875195	0.884746	0.884746	0.894374	0.894374
$(0.5, 0.1)$	0.905798	0.905798	0.914604	0.914604	0.923160	0.923160
$(0.9, 0.1)$	0.932918	0.932918	0.940163	0.940163	0.946983	0.946983
$(0.3, 0.3)$	0.875195	0.875195	0.884745	0.884746	0.894373	0.894374
$(0.7, 0.3)$	0.905798	0.905798	0.914604	0.914604	0.923160	0.923160
$(0.1, 0.5)$	0.844569	0.844569	0.853724	0.853725	0.863315	0.863315
$(0.5, 0.5)$	0.875195	0.875195	0.884744	0.884746	0.894372	0.894374
$(0.9, 0.5)$	0.905798	0.905798	0.914604	0.914604	0.923160	0.923160
$(0.3, 0.7)$	0.844569	0.844569	0.853723	0.853725	0.863313	0.863315
$(0.7, 0.7)$	0.875195	0.875195	0.884744	0.884746	0.894371	0.894374
$(0.1, 0.9)$	0.817389	0.817389	0.825186	0.825186	0.833647	0.833647
$(0.5, 0.9)$	0.844569	0.844569	0.853723	0.853725	0.863313	0.863315
$(0.9, 0.9)$	0.875195	0.875195	0.884145	0.884146	0.894373	0.894374
L_2	6.013832×10^{-8}		1.511454×10^{-6}		1.599711×10^{-6}	
L_∞	6.693447×10^{-8}		2.451640×10^{-6}		2.804862×10^{-6}	

Table 3. Numerical solutions of Problem I for u for values of $h_x = h_y = 0.05$, Re$= 100$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.623106	0.623047	0.543002	0.543322	0.500470	0.500482
$(0.5, 0.1)$	0.501617	0.501622	0.500341	0.500353	0.500003	0.500003
$(0.9, 0.1)$	0.500011	0.500011	0.500002	0.500002	0.500000	0.500000
$(0.3, 0.3)$	0.623106	0.623047	0.642692	0.543322	0.500441	0.500482
$(0.7, 0.3)$	0.501617	0.501622	0.500317	0.500353	0.500003	0.500003
$(0.1, 0.5)$	0.748272	0.748274	0.742150	0.742214	0.555153	0.555675
$(0.5, 0.5)$	0.623106	0.623047	0.542509	0.543322	0.500414	0.500482
$(0.9, 0.5)$	0.501617	0.501622	0.500304	0.500353	0.500003	0.500003
$(0.3, 0.7)$	0.748272	0.748274	0.742114	0.742214	0.554816	0.555675
$(0.7, 0.7)$	0.623106	0.623047	0.542463	0.543322	0.500384	0.500482
$(0.1, 0.9)$	0.749988	0.749988	0.749945	0.749946	0.744196	0.744256
$(0.5, 0.9)$	0.748272	0.748274	0.742103	0.742214	0.554504	0.555675
$(0.9, 0.9)$	0.623106	0.623047	0.542282	0.543322	0.500525	0.500482
L_2	3.811712×10^{-5}		1.070747×10^{-3}		1.097702×10^{-3}	
L_∞	6.071263×10^{-5}		2.031654×10^{-3}		2.240898×10^{-3}	



(a) **(b)**

Figure 2. (a) Exact and (b) numerical solutions for v of Problem 1 for values of $h_x = h_y = 0.05$, Re$= 100$, $\Delta t = 10^{-4}$ at $t = 0.5$.

Table 4. Numerical solutions of Problem I for v for values of $h_x = h_y = 0.05$, Re$= 100$, $\Delta t = 10^{-4}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 2.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.876894	0.876953	0.956998	0.956678	0.999530	0.999518
$(0.5, 0.1)$	0.998383	0.998378	0.999659	0.999647	0.999997	0.999997
$(0.9, 0.1)$	0.999989	0.999989	0.999998	0.999998	1.000000	1.000000
$(0.3, 0.3)$	0.876894	0.876953	0.957308	0.956678	0.999559	0.999518
$(0.7, 0.3)$	0.998383	0.998378	0.999683	0.999647	0.999997	0.999997
$(0.1, 0.5)$	0.751728	0.751726	0.757850	0.757786	0.944847	0.944325
$(0.5, 0.5)$	0.876894	0.876953	0.957491	0.956678	0.999586	0.999518
$(0.9, 0.5)$	0.998383	0.998378	0.999696	0.999647	0.999997	0.999997
$(0.3, 0.7)$	0.751728	0.751726	0.757886	0.757786	0.945184	0.944325
$(0.7, 0.7)$	0.876894	0.876953	0957537	0.956678	0.999616	0.999518
$(0.1, 0.9)$	0.750012	0.750012	0.750055	0.750054	0.755804	0.755744
$(0.5, 0.9)$	0.751728	0.751726	0.757897	0.757786	0.945496	0.944325
$(0.9, 0.9)$	0.876894	0.876953	0.957718	0.956678	0.999475	0.999518
L_2	2.736786×10^{-5}		7.126002×10^{-4}		6.043011×10^{-4}	
L_∞	6.071263×10^{-5}		2.031654×10^{-3}		2.240898×10^{-3}	

Table 5. A comparison of numerical solutions for u of Problem 2 for values of $h_x = h_y = 0.025$, Re$= 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$, $N = 40$ with those in Refs. [5, 6, 12].

(x, y)	u				
	Present	[5]	[5] N=40	[6]	[12]
$(0.15, 0.1)$	0.96870	0.95691	0.96066	0.96650	0.96870
$(0.3, 0.1)$	1.03204	0.95616	0.96852	1.02970	1.03200
$(0.1, 0.2)$	0.84618	0.84257	0.84104	0.84449	0.86178
$(0.2, 0.2)$	0.87813	0.86399	0.86866	0.87631	0.87813
$(0.1, 0.3)$	0.67920	0.67667	0.67792	0.67809	0.67920
$(0.3, 0.3)$	0.79944	0.76876	0.77254	0.79792	0.79945
$(0.15, 0.4)$	0.54675	0.54408	0.54543	0.54601	0.66039
$(0.2, 0.4)$	0.58958	0.58778	0.58564	0.58874	0.58958

[12]. There is no analytical solution to this problem. Since Problem II has not analytical solution in Table (5), a comparison of numerical solutions for u of Problem 2 for values of $h_x = h_y = 0.025$, $\text{Re} = 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$, $N = 40$ with those in Refs. [5, 6, 12] is presented. Again, due to the same reason, Table (6) presents a comparison of numerical solutions for v of Problem 2 for values of $h_x = h_y = 0.025$, $\text{Re} = 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [5, 6, 12]. Tables (7-8) show also pointwise values of u and v but now for a smaller value of Reynold number $\text{Re} = 50$, respectively. Figures (3) shows numerical solutions of u and v of Problem 2 for values of $h_x = h_y = 0.025$, Re$= 50$, $\Delta t = 10^{-4}$ at time $t = 0.625$, respectively.

Problem III: The solution domain of the third problem is $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and its analytical solution is of the form [12]

$$u(x, y, t) = -\frac{4\pi e^{-\frac{5\pi^2 t}{\text{Re}}} \cos(2\pi x) \sin(\pi y)}{\text{Re}(2 + e^{-\frac{5\pi^2 t}{\text{Re}}} \sin(2\pi x) \sin(\pi y)}$$
$$v(x, y, t) = -\frac{2\pi e^{-\frac{5\pi^2 t}{\text{Re}}} \sin(2\pi x) \cos(\pi y)}{\text{Re}(2 + e^{-\frac{5\pi^2 t}{\text{Re}}} \sin(2\pi x) \sin(\pi y)}$$

Table (9) presents numerical solutions of u of Problem 3 for values of $h_x = h_y = 0.05$, $\text{Re} = 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01, 0.5$ and 1.0. From the table one can easily see that the approximate and exact solutions are very close to each other and calculated error norms L_2 and L_∞ are small enough. In a similar manner, Table (10) presents numerical solutions of v of Problem 3 for values of $h_x = h_y = 0.05$, $\text{Re} = 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01, 0.5$ and 1.0. Again, one can see from this table that both of the approximate and exact pointwise values are in good agreement. Th error norms L_2 and L_∞ show the general consistency between the approximate and exact solutions

Table 6. A comparison of numerical solutions for v of Problem 2 for values of $h_x = h_y = 0.025$, Re= 500, $\Delta t = 10^{-4}$ at time $t = 0.625$, $N = 40$ with those in Refs. [5, 6, 12].

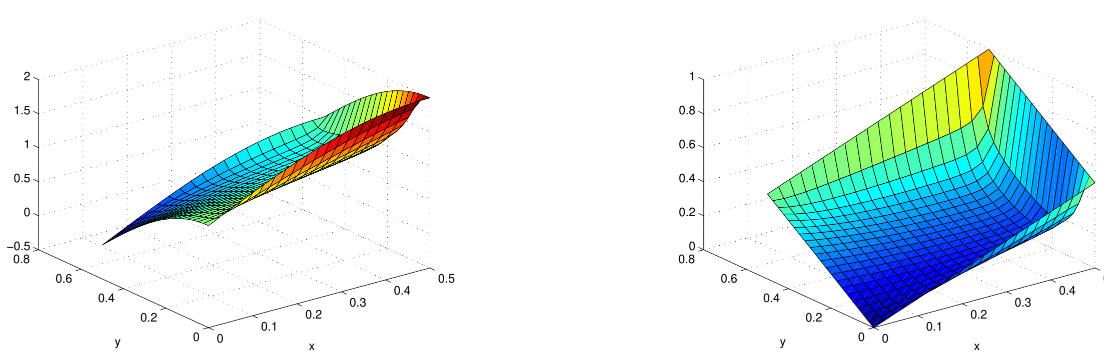
(x, y)	v				
	Present	[5]	[5] N=40	[6]	[12]
$(0.15, 0.1)$	0.09044	0.10177	0.08612	0.09020	0.09043
$(0.3, 0.1)$	0.10730	0.13287	0.07712	0.10690	0.10728
$(0.1, 0.2)$	0.18010	0.18503	0.17828	0.17972	0.17295
$(0.2, 0.2)$	0.16816	0.18169	0.16202	0.16777	0.16816
$(0.1, 0.3)$	0.26268	0.26560	0.26094	0.26222	0.26268
$(0.3, 0.3)$	0.23550	0.25142	0.21542	0.23497	0.23550
$(0.15, 0.4)$	0.31799	0.32084	0.31360	0.31753	0.29022
$(0.2, 0.4)$	0.30418	0.30927	0.29776	0.30371	0.30418

Table 7. A comparison of numerical solutions for u of Problem 2 for values of $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [5, 6, 12].

(x, y)	u			
	Present	[5]	[6]	[12]
$(0.1, 0.1)$	0.97146	0.97258	0.96688	0.97146
$(0.3, 0.1)$	1.15282	1.16214	1.14827	1.15280
$(0.2, 0.2)$	0.86308	0.86281	0.85911	0.86308
$(0.4, 0.2)$	0.97984	0.96483	0.97637	0.97984
$(0.1, 0.3)$	0.66316	0.66318	0.66019	0.66316
$(0.3, 0.3)$	0.77232	0.77030	0.76932	0.77232
$(0.2, 0.4)$	0.58181	0.58070	0.57966	0.58181
$(0.4, 0.4)$	0.75861	0.74435	0.75678	0.75860

Table 8. A comparison of numerical solutions for v of Problem 2 for values of $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [5, 6, 12].

(x, y)	v			
	Present	[5]	[6]	[12]
$(0.1, 0.1)$	0.09869	0.09773	0.09824	0.09869
$(0.3, 0.1)$	0.14158	0.14039	0.14112	0.14158
$(0.2, 0.2)$	0.16754	0.16660	0.16681	0.16754
$(0.4, 0.2)$	0.17110	0.17397	0.17065	0.17110
$(0.1, 0.3)$	0.26378	0.26294	0.26261	0.26378
$(0.3, 0.3)$	0.22654	0.22463	0.22576	0.22655
$(0.2, 0.4)$	0.32851	0.32402	0.32745	0.32851
$(0.4, 0.4)$	0.32500	0.31822	0.32441	0.32501



(a) **(b)**

Figure 3. Numerical solutions of (a) u and (b) v of Problem 2 for values of $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at time $t = 0.625$.

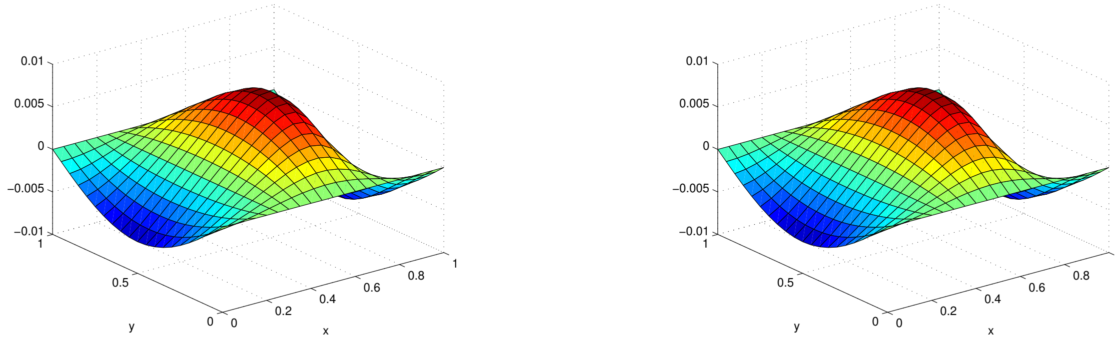
Table 9. Numerical solutions of u of Problem 3 for values of $h_x = h_y = 0.05$, Re= 1000, $\Delta t = 10^{-3}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	-0.001439	-0.001439	-0.001408	-0.001408	-0.001376	-0.001376
$(0.5, 0.1)$	0.001941	0.001941	0.001895	0.001894	0.001849	0.001848
$(0.9, 0.1)$	-0.001727	-0.001727	-0.001682	-0.001682	-0.001638	-0.001637
$(0.3, 0.3)$	0.001134	0.001134	0.001114	0.001114	0.001094	0.001094
$(0.7, 0.3)$	0.002551	0.002551	0.002458	0.002453	0.002368	0.002359
$(0.1, 0.5)$	-0.003927	-0.003927	-0.003854	-0.003854	-0.003780	-0.003781
$(0.5, 0.5)$	0.006280	0.006280	0.006130	0.006130	0.005981	0.005981
$(0.9, 0.5)$	-0.007194	-0.007194	-0.006960	-0.006953	-0.006731	-0.006718
$(0.3, 0.7)$	0.001134	0.001134	0.001114	0.001114	0.001094	0.001094
$(0.7, 0.7)$	0.002551	0.002551	0.002458	0.002453	0.002368	0.002359
$(0.1, 0.9)$	-0.001439	-0.001439	-0.001408	-0.001408	-0.001376	-0.001376
$(0.5, 0.9)$	0.001941	0.001941	0.001895	0.001894	0.001849	0.001848
$(0.9, 0.9)$	-0.001727	-0.001727	-0.001682	-0.001682	-0.001638	-0.001637
L_2	2.2105×10^{-5}		1.0312×10^{-3}		1.9287×10^{-3}	
L_∞	2.8241×10^{-7}		1.2663×10^{-5}		2.2938×10^{-5}	

throughout the solution domain. Figures (4-5) show first exact and then numerical solutions for u and v of Example 3 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at $t = 0.01$, respectively.

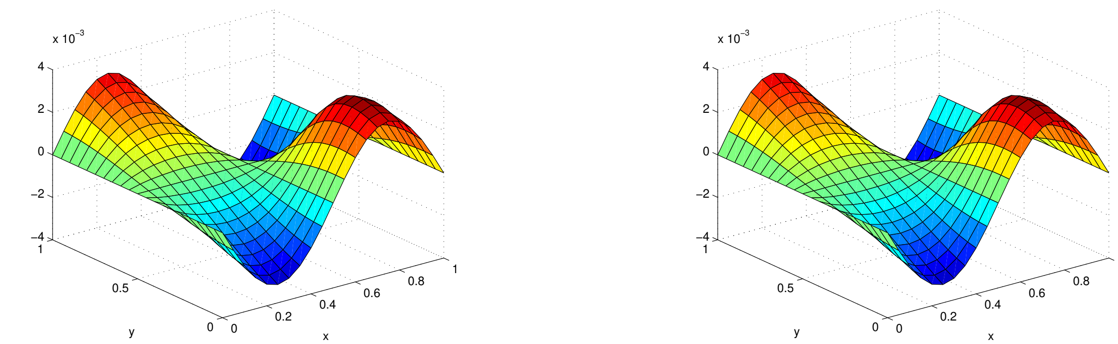
Table 10. Numerical solutions of v of Problem 3 for values of $h_x = h_y = 0.05$, Re= 1000, $\Delta t = 10^{-3}$ at times $t = 0.01$, 0.5 and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Ecaxt
$(0.1, 0.1)$	-0.001609	-0.001609	-0.001574	-0.001574	-0.001539	-0.001539
$(0.5, 0.1)$	-0.000000	-0.000000	-0.000000	-0.000000	-0.000001	-0.000000
$(0.9, 0.1)$	0.001931	0.001931	0.001880	0.001880	0.001830	0.001830
$(0.3, 0.3)$	-0.001268	-0.001268	-0.001246	-0.001246	-0.001223	-0.001224
$(0.7, 0.3)$	0.002852	0.002852	0.002746	0.002743	0.002643	0.002637
$(0.1, 0.5)$	-0.000000	-0.000000	-0.000000	-0.000000	-0.000000	-0.000000
$(0.5, 0.5)$	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000
$(0.9, 0.5)$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$(0.3, 0.7)$	0.001268	0.001268	0.001246	0.001246	0.001223	0.001224
$(0.7, 0.7)$	-0.002852	-0.002852	-0.002746	-0.002743	-0.002643	-0.002637
$(0.1, 0.9)$	0.001609	0.001609	0.001574	0.001574	0.001539	0.001539
$(0.5, 0.9)$	0.000000	0.000000	0.000000	0.000000	0.000001	0.000000
$(0.9, 0.9)$	-0.001931	-0.001931	-0.001880	-0.001880	-0.001830	-0.001830
L_2	1.2846×10^{-5}		6.0214×10^{-4}		1.1320×10^{-3}	
L_∞	9.3390×10^{-8}		4.1431×10^{-6}		7.3722×10^{-6}	



(a) **(b)**

Figure 4. (a) Exact and (b) numerical solutions of u of Problem 3 for values $h_x = h_y = 0.05$, Re$= 1000$, $\Delta t = 10^{-3}$ at time $t = 0.01$.



(a) **(b)**

Figure 5. (a) Exact and (b) numerical solutions of v of Problem 3 for values $h_x = h_y = 0.05$, Re$= 1000$, $\Delta t = 10^{-3}$ at time $t = 0.01$.

4. Conclusion

In this study, numerical solutions of two dimensional coupled Burgers equation has been obtained by using finite difference method based on a Rubin-Graves type linearization. To demonstrate the accuracy and efficiency of the method, this method has been applied to three test problems with known analytical solutions and to one test problem with unknown analytical solution. The error norms L_2 and L_∞ have been calculated. From these calculations, it is seen that the proposed method yield good enough results, and it is simple and easy to apply. In conclusion, numerical solution of two dimensional coupled nonlinear partial differential equations arises in physical sciences can be achieved easily and effectively by the proposed method. The algebraic systems found out by using the proposed schemes can be easily stored and solved by the software systems of nowadays. As a conclusion, the proposed method can be easily and successfully applied to this type of problems arising in applied mathematics, mathematical physics and engineering science.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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The Monoid Rank and Monoid Presentation of Order-Preserving and Order-Decreasing Full Contraction Mappings

Kemal Toker*

Abstract
Let $n \in \mathbb{Z}^+$ and $X_n = \{1, 2, \ldots, n\}$ be a finite set. Let $\mathcal{O}DCT_n$ be the order-preserving and order-decreasing full contraction mappings on X_n. It is well known that $\mathcal{O}DCT_n$ is a monoid. In this paper, we have found the monoid rank and monoid presentation of $\mathcal{O}DCT_n$. In particular, we have proved that monoid rank of $\mathcal{O}DCT_n$ is $n-1$ for $n \in \mathbb{Z}^+$ and $< a_1, a_2, \ldots, a_{n-1} \mid a_i a_{n-1} = a_i \ (1 \leq i \leq n-1), a_i a_j = a_{j+1} a_i \ (1 \leq i \leq j \leq n-2) >$ is a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$.

Keywords: *Contraction mappings; Generating set; Monoid presentation.*

AMS Subject Classification (2020): *20M20.*

***Corresponding author**

1. Introduction

Let X be a non-empty set and let $\mathcal{T}_X$ be the full transformation semigroup on X. Every semigroup is isomorphic to a subsemigroup of full transformation semigroup [7]. So, the full transformation semigroup is ubiquitous in the semigroup theory. Let $n \in \mathbb{Z}^+$ and $X_n = \{1, 2, \ldots, n\}$ be a finite set. We use $\mathcal{T}_n$ instead of $\mathcal{T}_{X_n}$ for convenience.

Let M be a monoid and A be any subset of M. Then the submonoid of M by generated A (which is the smallest submonoid of M containing A) is denoted by $< A >$. If $< A >= M$ while the cardinality of A is a finite number, then M is called finitely generated monoid. With a similar idea, by replacing M by a semigroup S, one may define finitely generated semigroup as well.

The monoid rank of finitely generated monoid M is defined by

$$\text{rank}_M(M) = \min\{|A| :< A >= M\}.$$

Let $\mathcal{C}T_n$ be the full contraction transformations on X_n, it is defined by

$$\mathcal{C}T_n = \{\alpha \in \mathcal{T}_n \mid (\forall x, y \in X_n) \ |x\alpha - y\alpha| \leq x - y\}$$

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and $\mathcal{CT}_n$ is a submonoid of $\mathcal{T}_n$. Let $\mathcal{O}_n$ be the order-preserving full transformations on X_n and it is defined by

$$\mathcal{O}_n = \{\alpha \in \mathcal{T}_n \mid (\forall x, y \in X_n)\ x \leq y \implies x\alpha \leq y\alpha\}.$$

Let $\mathcal{S}_n$ be the symmetric group on X_n. Gomes and Howie have found the semigroup rank of $\mathcal{O}_n \setminus \mathcal{S}_n = \mathcal{O}_n \setminus \{1_S\}$ where 1_S is the identity mapping of $\mathcal{S}_n$ [5]. Let $\mathcal{C}_n$ be the order-preserving and order-decreasing transformations on X_n, it is called Catalan monoid on X_n and it is defined by

$$\mathcal{C}_n = \{\alpha \in \mathcal{O}_n \mid (\forall x \in X_n)\ x\alpha \leq x\}.$$

There are some papers about $\mathcal{C}_n$, in the literature such as [2, 6]. Adeshola and Umar defined a semigroup which is $\mathcal{O}_n \cap \mathcal{CT}_n$ and they used $\mathcal{OCT}_n$ instead of $\mathcal{O}_n \cap \mathcal{CT}_n$. The cardinalities of some equivalences on $\mathcal{OCT}_n$ has been investigated by Adeshola and Umar [1]. Let

$$\mathcal{D}_n = \{\alpha \in \mathcal{T}_n \mid (\forall x \in X_n)\ x\alpha \leq x\}$$

be the subsemigroup of $\mathcal{T}_n$ consisting of all order-decreasing transformations of X_n. Moreover, Adeshola and Umar defined a semigroup which is $\mathcal{OCT}_n \cap \mathcal{D}_n$ and they used $\mathcal{ODCT}_n$ instead of $\mathcal{OCT}_n \cap \mathcal{D}_n$ [1]. $\mathcal{ODCT}_n$ is called order-preserving and order-decreasing full contraction mappings. Also, $\mathcal{ODCT}_n = \mathcal{CT}_n \cap \mathcal{C}_n$ thus $\mathcal{ODCT}_n$ is a submonoid of $\mathcal{OCT}_n$ and submonoid of $\mathcal{C}_n$.

Let A be a set, then we denote by A^* the free monoid on A. Let $R \subseteq A^* \times A^*$ is a set of pairs of words. An element (r, s) of R is called a relation, and is usually written $r = s$ instead of (r, s). Monoid presentation is an ordered pair $< A \mid R >$ which is the quotient monoid $A^*/R^\#$ where $R^\#$ is the smallest congruence on A^* containing R. Let M be the monoid defined by $< A \mid R >$. Let $w_1, w_2 \in A^*$, if w_1 and w_2 are identical words on A^* then we write $w_1 \equiv w_2$, and we write $w_1 = w_2$ if they represent the same element of the monoid M, that is $(w_1, w_2) \in R^\#$. If $u_1 \equiv xry$ and $u_2 \equiv xsy$ where $x, y \in A^*$ and $(r, s) \in R$ or $(s, r) \in R$ then, we say u_2 is obtained from u_1 by an application of one relation from R. We say that $w_1 = w_2$ is a consequence of R, if w_1 and w_2 are identical words or if there exists a sequence $w_1 \equiv u_1 \to u_2 \to \ldots \to u_k \equiv w_2$ where each u_{i+1} is obtained from u_i $(1 \leq i \leq k-1)$ by an application of one relation from R. Let T be any monoid, let B be a generating set for T, and let $\phi : A \to B$ be an onto mapping. ϕ can be extended in a unique way $\bar{\phi} : A^* \to T$. The monoid T is said to satisfy relations R if for each $(u, v) \in R$ we have $u\bar{\phi} = v\bar{\phi}$. We refer the readers to two theses about semigroup and monoid presentations [3, 8].

2. Preliminaries

Let $\alpha \in \mathcal{T}_n$, then the kernel and image of α are defined by

$$\ker(\alpha) = \{(x, y) \in X_n \times X_n \mid x\alpha = y\alpha\}$$

$$\text{im}(\alpha) = \{x\alpha \mid x \in X_n\}.$$

Moreover, it is well known that if $\alpha, \beta \in \mathcal{T}_n$ then $\text{im}(\alpha\beta) \subseteq \text{im}(\beta)$ and $\ker(\alpha\beta) \supseteq \ker(\alpha)$.

Definition 2.1. Let A be a non-empty subset of X_n. If $x, y \in A$ and $x \leq z \leq y \implies z \in A$ for all $x, y \in A$, then A is called a convex subset of X_n.

If $\alpha \in \mathcal{T}_n$ is a contraction mapping then $\text{im}(\alpha)$ is a convex subset of X_n [4]. Thus if $\alpha \in \mathcal{ODCT}_n$ then $\text{im}(\alpha)$ is a convex subset of X_n. Moreover, from the definition of $\mathcal{ODCT}_n$ it is easy to see that if $\alpha \in \mathcal{ODCT}_n$ then $\text{im}(\alpha) = \{1, 2, \ldots, r\}$ for $1 \leq r \leq n$ and each equivalence kernel classes of α are convex subsets of X_n. Thus if $\alpha \in \mathcal{ODCT}_n$ then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ 1 & 2 & \ldots & r \end{pmatrix}$$

for $1 \leq r \leq n$. Moreover, we have $x \geq i$ for $\forall x \in A_i$ and $\{A_1, A_2, \ldots A_r\}$ is a partition of X_n, if $a \in A_i$ and $b \in A_j$ for $1 \leq i < j \leq n$ then $a < b$.

3. The Monoid Rank of $ODCT_n$

In this section, we have found a minimal generating set of $\mathcal{O}DCT_n$ and we obtained the monoid rank of $\mathcal{O}DCT_n$. It is clear that $\mathcal{O}DCT_1 = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ which is a clearly generated by empty set as a monoid and $\mathcal{O}DCT_2 = \{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\}$ which is clearly generated by the element $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ as a monoid. Let $n \geq 3$ and $\mathcal{F}_r = \{\alpha \in \mathcal{O}DCT_n : |\text{im}(\alpha)| = r\}$ for $1 \leq r \leq n$. Notice that $\mathcal{F}_n = \{\epsilon = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}\}$ where ϵ is the identity element of $\mathcal{O}DCT_n$.

Lemma 3.1. *Let $n \geq 3$. If $\alpha \in \mathcal{F}_r$ then $\alpha \in < \mathcal{F}_{r+1} >$ for $1 \leq r \leq n-2$.*

Proof. Let $n \geq 3$ and $\alpha \in \mathcal{F}_r$ for $1 \leq r \leq n-2$. Then we have

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix}$$

where $1 \leq r \leq n-2$, so there exists i such that $|A_i| \geq 2$ for $1 \leq i \leq r$. Let x_i be the maximum element in A_i. Let β be a mapping such that

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \setminus \{x_i\} & \{x_i\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix}$$

for $i > 1$ and

$$\beta = \begin{pmatrix} A_1 \setminus \{x_1\} & \{x_1\} & A_2 & \dots & A_r \\ 1 & 2 & 3 & \dots & r+1 \end{pmatrix}$$

for $i = 1$. Then it is clear that $\beta \in \mathcal{F}_{r+1}$. Let γ be the mapping defined as

$$j\gamma = \begin{Bmatrix} j & \text{if } 1 \leq j \leq i \\ i & \text{if } j = i+1 \\ j-1 & \text{if } i+2 \leq j \leq r+1 \\ r+1 & \text{if } j > r+1 \end{Bmatrix}$$

then it is clear that $\gamma \in \mathcal{F}_{r+1}$ and $\alpha = \beta\gamma$, so $\alpha \in < \mathcal{F}_{r+1} >$. □

Corollary 3.1. *$\mathcal{F}_r \subseteq < \mathcal{F}_{r+1} >$ for each $1 \leq r \leq n-2$.*

Corollary 3.2. *Since $\mathcal{F}_n$ is the set that has only the identity mapping of $\mathcal{O}DCT_n$ then we have $< \mathcal{F}_{n-1} > = \mathcal{O}DCT_n$ for $n \geq 3$.*

Corollary 3.3 ([1]). *$|\mathcal{F}_r| = \binom{n-1}{r-1}$ for $1 \leq r \leq n$.*

Corollary 3.4. *$\text{rank}_M(\mathcal{O}DCT_n) \leq n-1$ for $n \in \mathbb{Z}^+$ since $|\mathcal{F}_{n-1}| = n-1$.*

Corollary 3.5 ([1]). *$|\mathcal{O}DCT_n| = 2^{n-1}$ for $n \geq 1$.*

Theorem 3.1. *$\text{rank}_M(\mathcal{O}DCT_n) = n-1$ for $n \in \mathbb{Z}^+$.*

Proof. If $n = 1$ or $n = 2$ then result is clear, let $n \geq 3$. We have $\text{rank}_M(\mathcal{O}DCT_n) \leq n-1$ from Corollary 3.4. Let

$$\mathcal{O}DCT_{(n,r)} = \{\alpha \in \mathcal{O}DCT_n : |\text{im}(\alpha)| \leq r\}$$

for $1 \leq r \leq n-1$. It is clear that $\mathcal{O}DCT_{(n,r)}$ is an ideal of $\mathcal{O}DCT_n$. In particular, $\mathcal{O}DCT_{(n,n-2)}$ is an ideal of $\mathcal{O}DCT_n$. Moreover, there are $n-1$ different kernel classes in $\mathcal{F}_{n-1}$ and we have $\mathcal{F}_n = \{\epsilon\}$, so $\text{rank}_M(\mathcal{O}DCT_n) \geq n-1$. Thus we have concluded that $\text{rank}_M(\mathcal{O}DCT_n) = n-1$ for $n \in \mathbb{Z}^+$. □

4. The Monoid Presentation of $ODCT_n$

In this section, we have found a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$.

Proposition 4.1 ([8]). *Let A be a set and let M be any monoid. Then any mapping $\phi : A \to M$ can be extended in a unique way to a homomorphism $\overline{\phi} : A^* \to M$.*

Definition 4.1. Let M be any monoid, let B be a generating set of M, and let $\phi : A \to B$ be an onto mapping. By Proposition 4.1 the mapping ϕ can be extended in a unique way to an epimorphism $\overline{\phi} : A^* \to M$. Let $R \subseteq A^* \times A^*$ be a set of relations. The monoid M is said to satisfy relations R if for each $(u, v) \in R$ we have $u\overline{\phi} = v\overline{\phi}$.

Let M be a finite monoid, $A \subseteq M$ and $< A >= M$. Let $R \subseteq A^* \times A^*$ be a set of relations, and let $W \subseteq A^*$. It is well known that if

(i) the generators A of M satisfy all the relations from R

(ii) for each word $w \in A^*$ there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R

(iii) $|W| \leq |M|$

then $< A \mid R >$ is a monoid presentation of M.

Let $n \geq 3$ and α_i be the mapping defined as

$$\alpha_i = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & i+2 & \dots & n \\ 1 & \dots & i-1 & i & i & i+1 & \dots & n-1 \end{pmatrix}$$

for $2 \leq i \leq n-1$ and

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 2 & \dots & n-1 \end{pmatrix},$$

then it is clear that $\mathcal{F}_{n-1} = \{\alpha_i \mid 1 \leq i \leq n-1\}$.

Lemma 4.1. *Let $n \geq 3$ and α_i be defined as above then $\alpha_i\alpha_{n-1} = \alpha_i$ for $1 \leq i \leq n-1$. In particular, $(\alpha_{n-1})^2 = \alpha_{n-1}$.*

Proof. Let $n \geq 3$ and α_i be defined as above, then

$$\alpha_{n-1} = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & n-1 \end{pmatrix}.$$

$1(\alpha_i\alpha_{n-1}) = 1$ and $n(\alpha_i\alpha_{n-1}) = n-1$, we have im$(\alpha_i\alpha_{n-1}) = \{1, 2, \dots, n-1\}$ from the definition of $\mathcal{O}DCT_n$. Moreover, $i(\alpha_i\alpha_{n-1}) = i$ and $(i+1)(\alpha_i\alpha_{n-1}) = i$, so $\alpha_i\alpha_{n-1} = \alpha_i$ for $1 \leq i \leq n-1$. □

Lemma 4.2. *Let $n \geq 3$ and α_i be defined as above then $\alpha_i\alpha_j = \alpha_{j+1}\alpha_i$ for $1 \leq i \leq j \leq n-2$.*

Proof. Let $n \geq 3$, α_i be defined as above and $1 \leq i \leq j \leq n-2$. It is clear that $1(\alpha_i\alpha_j) = 1$ and $n(\alpha_i\alpha_j) = n-1(\alpha_j) = n-2$ since $1 \leq i \leq j \leq n-2$. Thus im$(\alpha_i\alpha_j) = \{1, 2, \dots, n-2\}$ from the definition of $\mathcal{O}DCT_n$. Moreover we have

$$i(\alpha_i\alpha_j) = i\alpha_j = i$$

$$(i+1)(\alpha_i\alpha_j) = i\alpha_j = i$$

$$(j+1)(\alpha_i\alpha_j) = j\alpha_j = j$$

$$(j+2)(\alpha_i\alpha_j) = (j+1)\alpha_j = j.$$

Also, $1(\alpha_{j+1}\alpha_i) = 1$ and $n(\alpha_{j+1}\alpha_i) = (n-1)\alpha_i = n-2$ thus im$(\alpha_{j+1}\alpha_i) = \{1, 2, \dots, n-2\}$ from the definition of $\mathcal{O}DCT_n$. Moreover we have

$$i(\alpha_{j+1}\alpha_i) = i\alpha_i = i$$

$$(i+1)(\alpha_{j+1}\alpha_i) = (i+1)\alpha_i = i$$

$$(j+1)(\alpha_{j+1}\alpha_i) = (j+1)\alpha_i = j$$

$$(j+2)(\alpha_{j+1}\alpha_i) = (j+1)\alpha_i = j.$$

Therefore, $x(\alpha_i\alpha_j) = x(\alpha_{j+1}\alpha_i)$ for $\forall x \in X_n$. It follows that $\alpha_i\alpha_j = \alpha_{j+1}\alpha_i$ for $1 \leq i \leq j \leq n-2$. □

Definition 4.2. Let A be a finite set and $w = a_1a_2 \dots a_k$ for $a_i \in A$ and $1 \leq i \leq k$. Length of w is defined as k and we write $l(w) = k$ and if w is empty word then the length of w is defined as 0 (zero) and we write $l(w) = 0$.

Theorem 4.1. *Let $n \geq 3$. Let $A = \{a_1, a_2, \dots, a_{n-1}\}$ and $R = \{a_ia_{n-1} = a_i \ (1 \leq i \leq n-1), a_ia_j = a_{j+1}a_i \ (1 \leq i \leq j \leq n-2)\}$. Then $< A \mid R >$ is a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$.*

Proof. Let $n \geq 3$. Let $A = \{a_1, a_2, \dots, a_{n-1}\}$ and $R = \{a_ia_{n-1} = a_i \ (1 \leq i \leq n-1), a_ia_j = a_{j+1}a_i \ (1 \leq i \leq j \leq n-2)\}$. Let $f : A \to \mathcal{F}_{n-1}$ be the mapping such that $a_if = \alpha_i$. There exists a unique epimorpish $\overline{f} : A^* \to \mathcal{O}DCT_n$ extending the f. Thus $\mathcal{O}DCT_n$ satisfies all the relations from R since Lemma 4.1 and Lemma 4.2. Let ε is the empty word and

$$W = \{a_{j_k}a_{j_{k-1}} \dots a_{j_1} \mid n-1 \geq j_k > j_{k-1} > \dots > j_1 \geq 1\} \cup \{\varepsilon\}.$$

Thus it is clear that $W \subseteq A^*$ and $|W| = 2^{n-1}$. Let $w \in A^*$ and $l(w) = m$. We will show that there exists $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R. We use induction on m. If $m = 0$ or $m = 1$, then the result is clear. Let $m \geq 2$, then $w \equiv w_1w_2$ where $l(w_1) = m-1$ and $l(w_2) = 1$. Thus $w_2 \in A$, moreover we have $w_1 = \overline{w_1}$ such that $\overline{w_1} \in W$ from the induction hypothesis. So $w = \overline{w_1}w_2$. If $\overline{w_1} \equiv \varepsilon$ then result is clear. Let $\overline{w_1} \not\equiv \varepsilon$. Then,

$$\overline{w_1} \equiv a_{t_p}a_{t_{p-1}} \dots a_{t_1}$$

where $n-1 \geq t_p > t_{p-1} > \dots > t_1 \geq 1$ and $w = a_{t_p}a_{t_{p-1}} \dots a_{t_1}w_2$. If $w_2 \equiv a_{n-1}$ then

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_1}a_{n-1}$$

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_1}$$

so in this case $w = \overline{w_1}$ and $\overline{w_1} \in W$. Let $w_2 \equiv a_i$ where $1 \leq i \leq n-2$. Then

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_1}a_i,$$

if $t_1 > i$ then we have $\overline{w} \equiv a_{t_p}a_{t_{p-1}} \dots a_{t_1}a_i$ and $w = \overline{w}$, $\overline{w} \in W$. If $t_1 \leq i$ then

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{t_1}a_i$$

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{i+1}a_{t_1}.$$

If $p = 1$ then result is clear, let $p \geq 2$. If $t_2 > i+1$ then we have $\overline{w} \equiv a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{i+1}a_{t_1}$ and $w = \overline{w}$, $\overline{w} \in W$. If $i+1 = n-1$ then

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{i+1}a_{t_1}$$

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{n-1}a_{t_1}$$

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{t_1},$$

so in this case $\overline{w} \equiv a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{t_1}$ and $w = \overline{w}$, $\overline{w} \in W$. If $t_2 \leq i+1 < n-1$ then we have

$$w = a_{t_p}a_{t_{p-1}} \dots a_{t_2}a_{i+1}a_{t_1}$$

$$w = a_{t_p}a_{t_{p-1}} \dots a_{i+2}a_{t_2}a_{t_1}.$$

If we use the same algorithm, it is clear that finally we conclude that there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R. Moreover, $|W| = |\mathcal{O}DCT_n| = 2^{n-1}$, it follows that $< A \mid R >$ is a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$. □

5. Conclusion

In this paper we have found monoid rank of $\mathcal{O}DCT_n$ for $n \in \mathbb{Z}^+$. Moreover since $\mathcal{O}DCT_1$ is a trivial monoid and $\mathcal{O}DCT_2$ is a monogenic monoid, we give a monoid presentation of $\mathcal{O}DCT_n$ for $n \geq 3$. Recently, the rank of $\mathcal{O}CT_n$ and the rank of $\mathcal{O}RCT_n$ have been found, finding presentation problem can be considered on those semigroups as a future work.

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On Some Classes of Series Representations for $1/\pi$ and π^2

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Abstract
We propose some classes of series representations for $1/\pi$ and π^2 by using a new WZ-pair. As examples, among many others. we prove that

$$\frac{3}{2}\sum_{n=1}^{\infty}\frac{n}{16^n(n+1)(2n-1)}\binom{2n}{n}^2=\frac{1}{\pi},$$

$$1-\frac{1}{4}\sum_{n=0}^{\infty}\frac{3n+2}{(n+1)^2}\binom{2n}{n}^2\frac{1}{16^n}=\frac{1}{\pi}$$

and

$$4\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)}\frac{4^n}{\binom{2n}{n}}=\pi^2.$$

Furthermore, our results lead to new combinatorial identities and binomial sums involving harmonic numbers.

Keywords: *Ramanujan-type series; binomial sums; gamma function; digamma function; combinatorial identities; Legendre's duplication formula.*

AMS Subject Classification (2020): *Primary: 33C05; 33C20; 33C90; Secondary: 05A19.*

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1. Introduction

In 1914 in his famous paper [25] Indian genius mathematician Srinivasa Ramanujan proposed 17 extraordinary series for $1/\pi$ without giving a complete proof. The most well known two of them were as follows:

$$\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}\frac{1103+26390n}{99^{4n}}=\frac{1}{\pi}$$

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and

$$\frac{1}{16}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{(1)_n^3}\frac{42n+5}{64^n}=\frac{1}{\pi}.$$

Here $(a)_n$ stands for the Pochhammer symbol defined by

$$(a)_0=1 \quad \text{and} \quad (a)_n=a(a+1)(a+2)...(a+n-1) \quad ,n\geq 1.$$

Ramanujan's series for $1/\pi$ have not received much interest from mathematical community until 1985. In 1985 Gosper used one of Ramanujan's series to calculate 17,526,100 digits of π, which is at that time was a world record [2]. In 1987 Peter and Jon Borwein [5] provided rigorous proofs of all 17 of Ramanujan's series for $1/\pi$ for the first time and also offered many new series representations for this constant; see [3, 4, 6]. J. Guillera provided the proofs of 11 of Ramanujan series by using the WZ-method [19, Tables I,II]. At about the same time as the Borweins were devising their proofs, David and Gregory Chudnovsky [9] derived new series representations for $1/\pi$ and used the following their Ramanujan-type series

$$\frac{1}{\pi}=12\sum_{n=0}^{\infty}(-1)^n\frac{(6n)!}{(n!)^3(3n)!}\frac{13591409+545140134n}{640320^3n+3/2}$$

to calculate 2,260,331,336 digits of π, which was a world record even in 1989. It should be remarked that before Ramanujan in 1859 G. Bauer [1], and in 1905 W. L. Glaisher [10] had given series representations for $1/\pi$. The studies on Ramanujan-like series for $1/\pi$ are continuing intensively today, too and recently, many new series of this type have been published, see for example [7,8,11-23]. The aim of this paper is to derive new classes of series representations for $1/\pi$ and π^2 by using the WZ-method. Our results enable us to establish infinity many of new Ramanujan type series for the constants $1/\pi$ and π^2. Our results also lead to some new combinatorial identities involving harmonic numbers. The remainder of this paper organized as follows. In the next section, we explain how the WZ-method works briefly. In Section 3, we present our main theorems. In the final section choosing particular values for a free parameter, we offer many series representations for the constants $1/\pi$ and π^2. In this paper we shall frequently use the generalized binomial coefficient

$$\binom{s}{t}=\frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)},$$

where t and s are real numbers which are not negative integers, and the Legendre's duplication formula for the classical gamma function $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi},\ n\in\mathbb{N}\cup\{0\}. \tag{1.1}$$

2. The WZ-method (Wilf-Zeilberger Method)

In this section we want to explain the WZ-method briefly. A discrete function $A(n,k)$ is hypergeometric if both

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are rational functions in both n and k. A pair (F,G) of hypergeometric functions is said to be a $WZ-$ pair (Wilf-Zeilberger pair) if for all $k\in\mathbb{Z}$ and $n=0,1,2,...$ they satisfy

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{2.1}$$

In this case Wilf and Zeilberger [24, Chapter 7] and [27] proved that there exists a rational function $C(n,k)$ such that $G(n,k)=C(n,k)F(n,k)$. Wilf and Zeilberger called $C(n,k)$ as certificate of the pair (F,G). Summing on $n\geq 0$ both sides of (2.1), one gets

$$\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\sum_{n=0}^{\infty}\{F(n+1,k)-F(n,k)\}=\lim_{n\to\infty}F(n,k)-F(0,k). \tag{2.2}$$

In most applications it is usually very easy to evaluate $F(0,k)$ and $\lim_{n\to\infty} F(n,k)$. So, taking particular values for k in (2.2), we can obtain many identities. We can also sum both sides of (2.1) over k's and in this case we get

$$\sum_{k=0}^{\infty}\{F(n+1,k)-F(,k)\}=\sum_{k=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\lim_{k\to\infty}G(n,k)-G(n,0).$$

If $G(n,0)=0$ and $\lim_{k\to\infty} G(n,k)=0$, we get

$$\sum_{k=0}^{\infty}\{F(n+1,k)-F(n,k)\}=0\,(n=0,1,2,3,...),$$

which implies that $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Let us say $\sum_{k=0}^{\infty} F(n,k)=C$. Usually, it is very easy to evaluate this constant by choosing a particular value for k (usually k=0), in other cases we evaluate it by taking the limit as $k\to\infty$. Please refer to [24] and [27] for more information about the WZ-method.

3. Main results

In this section we collect our main results.

Theorem 3.1. *Let a be any real number, which is not zero and a negative integer. Then we have*

$$\sum_{n=0}^{\infty}\frac{(3n+2a+1)\Gamma(n+1/2)\Gamma(n+a+1)}{(n+a)\Gamma(n+2)\Gamma(n+a+3/2)}=\frac{4\sqrt{\pi}\Gamma(a)}{\Gamma(a+1/2)}-\frac{2}{a}. \tag{3.1}$$

Proof. Consider the following discrete function.

$$F(n,k)=\frac{1}{2\pi}\frac{(n+2a)\Gamma(k+1/2)\Gamma(n-k+1/2)\Gamma(n+a+1)\Gamma(a+1/2)}{(k+a)(n-k+a)\Gamma(a)\Gamma(k+1)\Gamma(n-k+1)\Gamma(n+a+1/2)}. \tag{3.2}$$

The package EKHAD [24] allows us to obtain the companion

$$G(n,k)=\frac{-1}{2\pi}\frac{(3n+2a-2k+3)\Gamma(n-k+3/2)\Gamma(n+a+1)\Gamma(k+1/2)\Gamma(a+1/2)}{(n-k+a+1)(n+1)\Gamma(k)\Gamma(a)\Gamma(n+a+3/2)\Gamma(n-k+2)}, \tag{3.3}$$

where $k\in\mathbb{Z}$ and $n\in\mathbb{N}\cup\{0\}$. That is, (F,G) is a WZ-pair, so that, we have

$$F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k). \tag{3.4}$$

Summing over n both sides of (3.4), we get

$$\sum_{n=0}^{\infty}\{F(n+1,k)-F(n,k)\}=\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}$$

or

$$\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\lim_{n\to\infty}F(n,k)-F(0,k). \tag{3.5}$$

By Stirling's formula $n!\sim n^n e^{-n}\sqrt{2\pi n}$, we can easily find that

$$\lim_{n\to\infty}\frac{(n+2a)\Gamma(n-k+1/2)\Gamma(n+a+1)}{(n-k+a)\Gamma(n-k+1)\Gamma(n+a+1/2)}=1, \tag{3.6}$$

which yields

$$\lim_{n\to\infty}F(n,k)=\frac{1}{2\pi}\frac{\Gamma(k+1/2)\Gamma(a+1/2)}{(k+a)\Gamma(a)\Gamma(k+1)}.$$

We therefore have

$$\sum_{n=0}^{\infty}\{G(n,k+1)-G(n,k)\}=\frac{1}{2\pi}\frac{\Gamma(k+1/2)\Gamma(a+1/2)}{(k+a)\Gamma(a)\Gamma(k+1)}-F(0,k), \tag{3.7}$$

For $k=0$ this immediately gives

$$\sum_{n=0}^{\infty}\{G(n,1)-G(n,0)\}=\frac{1}{2\pi}\frac{\Gamma(1/2)\Gamma(a+1/2)}{\Gamma(a+1)}-1.$$

But since $G(n,0)=0$ and $\Gamma(1/2)=\sqrt{\pi}$, we get

$$\sum_{n=0}^{\infty}G(n,1)=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(a+1/2)}{\Gamma(a+1)}-1.$$

or by (3.3)

$$\sum_{n=0}^{\infty}\frac{(3n+2a+1)\Gamma(n+1/2)\Gamma(n+a+1)}{(n+a)\Gamma(n+2)\Gamma(n+a+3/2)}=\frac{4\sqrt{\pi}\Gamma(a)}{\Gamma(a+1/2)}-\frac{2}{a}.$$

□

If we substitute $a=m-1/2$ ($m\in\mathbb{Z}$) in (3.1), we get

Corollary 3.2. *Let m be any integer. Then, we have*

$$\sum_{n=0}^{\infty}\frac{3n+2m}{(n+1)(2n+2m-1)}\binom{2n}{n}\binom{2n+2m}{n+m}\frac{1}{16^n}=\frac{4^{m+1}\Gamma(m+1/2)}{\sqrt{\pi}(2m-1)\Gamma(m)}-\frac{4^{m+1}}{2(2m-1)}\frac{1}{\pi}. \tag{3.8}$$

In particular, if m is zero or a negative integer, we have

$$\frac{2-4m}{4^{m+1}}\sum_{n=0}^{\infty}\frac{3n+2m}{(n+1)(2n+2m-1)}\binom{2n}{n}\binom{2n+2m}{n+m}\frac{1}{16^n}=\frac{1}{\pi}. \tag{3.9}$$

Theorem 3.3. *Let a be any real number, which is not a negative integer, then we have*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{k+a}=\frac{2^{2n}\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.10}$$

Proof. let F and G be as in (3.3) and (3.4). Summing both sides (3.4) on $k=0,1,2,\ldots$, we get

$$\sum_{k=0}^{\infty}\{F(n+1,k)-F(n,k)\}=\lim_{k\to\infty}G(n,k)-G(n,0).$$

By using Stirling formula it is very easy to see that $\lim_{k\to\infty}G(n,k)=0$. Clearly, we also have $G(n,0)=0$. Then for all $n=0,1,2,\ldots$, we get

$$\sum_{k=0}^{\infty}F(n,k)=\sum_{k=0}^{\infty}F(n+1,k)=\sum_{k=0}^{\infty}F(n+2,k)=\cdots,$$

which implies that $\sum_{k=0}^{\infty}F(n,k)$ is a constant. Let $\sum_{k=0}^{\infty}F(n,k)=A$. We can evaluate the constant A by setting $n=0$, so that we obtain

$$A=\sum_{k=0}^{\infty}F(0,k)=\frac{1}{2\pi}\frac{2a\Gamma(a+1)\Gamma(a+1/2)}{\Gamma(a)\Gamma(a+1/2)}\sum_{k=0}^{\infty}\frac{\Gamma(k+1/2)\Gamma(-k+1/2)}{(k+a)(a-k)\Gamma(k+1)\Gamma(1-k)}.$$

Notice that this sum is zero for $k=1,2,\ldots$ except $k=0$. Hence we get

$$A=\frac{1}{\pi}\frac{\Gamma(a+1)a}{\Gamma(a)}\frac{\Gamma(1/2)^2}{a^2}=\frac{1}{\pi}\frac{\Gamma(a+1)a}{\Gamma(a+1)}\frac{\pi}{a}=1$$

Hence, we conclude that for all $n=0,1,2,...$

$$\sum_{k=0}^{\infty}F(n,k)=1.$$

From this identity, by the help of (1.1), we obtain

$$\sum_{k=0}^{n}\frac{\Gamma(k+1/2)\Gamma(n-k+1/2)}{(k+a)(n-k+a)\Gamma(k+1)\Gamma(n-k+1)}=\frac{2\pi\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}.$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(k+a)(n-k+a)}=\frac{2^{2n+1}\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.11}$$

Since

$$\frac{1}{(k+a)(n-k+a)}=\frac{1}{n+2a}\Big(\frac{1}{k+a}+\frac{1}{n-k+a}\Big),$$

we get from (3.11)

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{k+a}=\frac{4^n\Gamma(a)\Gamma(n+a+1/2)}{\Gamma(n+a+1)\Gamma(a+1/2)},$$

which is the desired result. □

Corollary 3.4. *Let a be any real number, which is not zero and a negatve integer. Then we have*

$$\sum_{n=0}^{n}\frac{\binom{2n}{n}}{4^n(n+a)}=\frac{\sqrt{\pi}\Gamma(a)}{\Gamma(a+1/2)} \tag{3.12}$$

Proof. Multiplying by $\sqrt{n}4^{-n}$ both sides of (3.10) and taking infinity the upper bound of the summation, we get

$$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{\sqrt{n}}{4^n}\binom{2n-2k}{n-k}\frac{1}{k+a}=\frac{\Gamma(a)}{\Gamma(a+1/2)}\frac{\sqrt{n}\Gamma(n+a+1/2)}{\Gamma(n+a+1)}. \tag{3.13}$$

Since, by Stirling's formula,

$$\lim_{n\to\infty}\frac{\sqrt{n}}{4^n}\binom{2n-2k}{n-k}=\frac{4^{-k}}{\sqrt{\pi}}\quad\text{and}\quad\lim_{n\to\infty}\frac{\sqrt{n}\Gamma(n+a+1/2)}{\Gamma(n+a+1)}=1,$$

the proof follows from (3.13) by letting taking the limit of both ides as $n\to\infty$. □

Corollary 3.5. *Let a be a non-zero real number such that $2a$ is not a negative integer. Then we have*

$$\sum_{n=0}^{\infty}\binom{2n+2a}{n+a}\frac{4^{-n}}{n+2a}=\frac{\sqrt{\pi}2^{2a-1}\Gamma(a)}{\Gamma(a+1/2)} \tag{3.14}$$

Proof. From (3.11), we have

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}}{4^k(k+a)}.\frac{\binom{2n-2k}{n-k}}{4^{n-k}(n-k+a)}=\frac{2\Gamma(a)\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)\Gamma(a+1/2)}. \tag{3.15}$$

Summing both sides (3.15) over n, it follows that

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\frac{\binom{2k}{k}}{4^k(k+a)}\cdot\frac{\binom{2n-2k}{n-k}}{4^{n-k}(n-k+a)}\Big)=\frac{2\Gamma(a)}{\Gamma(a+1/2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)}.$$

Since the left side is a Cauchy product of two series, we conclude

$$\Big(\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(n+a)}\Big)^2=\frac{2\Gamma(a)}{\Gamma(a+1/2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a+1/2)}{(n+2a)\Gamma(n+a+1)}.$$

Now the proof follows from (3.12) by the help of (1.1). the result by using (3.4). □

Remark 3.6. The identity (3.12) can also be obtained from the Gauss hypergeometric series but we want to give a proof because of the method we used can be employed in other places.

4. Applications

4.1 Series for $1/\pi$

Taking particular values for m in (3.7) and (3.8) we can obtain many series for $1/\pi$ by the help of the duplication formula (1.1).

Example 1. If we set $m = 0$ in (3.9), we get

$$\frac{3}{2}\sum_{n=1}^{\infty}\frac{n}{16^n(n+1)(2n-1)}\binom{2n}{n}^2 = \frac{1}{\pi}$$

Example 2. If we set $m = -1$ in (3.9), we get

$$\frac{3}{4}\sum_{n=0}^{\infty}\frac{n(3n-2)}{16^n(2n-3)(2n-1)(n+1)}\binom{2n}{n}^2 = \frac{1}{\pi}.$$

Example 3. If we set $m = -2$ in (3.9), we get

$$\frac{5}{8}\sum_{n=0}^{\infty}\frac{(2n+3)(3n+2)(2n+1)}{(n+1)(n+2)(n+3)(2n-1)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

Example 4. If we set $m = -3$ in (3.9), we get

$$\frac{21}{6}\sum_{n=0}^{\infty}\frac{(2n+5)(2n+3)(2n+1)}{(2n-1)(n+4)(n+3)(n+2)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}$$

Example 5. If we set $m = -4$ in (3.9), we get

$$\frac{9}{32}\sum_{n=0}^{\infty}\frac{(3n+4)(2n+7)(2n+5)(2n+3)(2n+1)}{(2n-1)(n+5)(n+4)(n+3)(n+2)(n+1)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi},$$

Example 6. If we set $m = 1$ in (3.8), we get

$$1 - \frac{1}{4}\sum_{n=0}^{\infty}\frac{3n+2}{(n+1)^2}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}$$

Example 7. If we set $m = 2$ in (3.8), we get

$$\frac{3}{2} - \frac{3}{8}\sum_{n=0}^{\infty}\frac{(2n+1)(3n+4)}{(n+1)^2(n+2)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

Example 8. If we set $m = 3$ in (3.8), we get

$$\frac{15}{8} - \frac{15}{16}\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)}{(n+1)^2(n+3)}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

Example 9. If we set $m = 4$ in (3.8), we get

$$\frac{35}{16} - \frac{7}{32}\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)(2n+5)(3n+8)}{(n+2)(n+3)(n+4)(n+1)^2}\binom{2n}{n}^2\frac{1}{16^n} = \frac{1}{\pi}.$$

4.2 Series for π^2

Taking particular values for a in (3.14) we can obtain many series for π^2.

Example 1. Setting $a = 1/2$ in (3.14) we get

$$4\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)}\frac{4^n}{\binom{2n}{n}} = \pi^2.$$

Example 2. If we set $a = 3/2$ in (3.14), we get

$$16\sum_{n=0}^{\infty}\frac{n+1}{(n+3)(2n+1)(2n+3)}\frac{4^n}{\binom{2n}{n}} = \pi^2.$$

Example 3. If we set $a = 5/2$ in (3.14), we get

$$\frac{128}{3}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{(n+5)(2n+1)(2n+3)(2n+5)}\frac{4^n}{\binom{2n}{n}} = \pi^2.$$

Example 4. If we set $a = 7/2$ in (3.14), we get

$$\frac{512}{5}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)}{(n+7)(2n+1)(2n+3)(2n+5)(2n+7)}\frac{4^n}{\binom{2n}{n}} = \pi^2.$$

4.3 Combinatorial identities involving harmonic numbers

Differentiating w.r.t a both sides of (3.10), we get

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(k+a)^2} = \frac{\binom{2n+2m}{n+m}}{m\binom{2m}{m}}\{\psi(a)+\psi(n+a+1/2)-\psi(n+a+1)-\psi(a+1/2)\}, \tag{4.1}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Substituting particular values for a in (4.1) and using the following duplication formula for the digamma function

$$\psi\left(n+\frac{1}{2}\right) = 2\psi(2n) - 2\log 2 - \psi(n) = -\gamma + 2H_{2n} - H_n - 2\log 2,$$

where $\gamma = 0.57721...$ is the Euler constant, we obtain the following combinatorial identities involving harmonic numbers.

Example 1. If we substitute $a = 1/2$ in (4.1), we get

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(2k+1)^2} = \frac{16^n\{H_{2n+1}-H_n\}}{(2n+1)\binom{2n}{n}}. \tag{4.2}$$

Example 2. If we substitute $a = m \in \mathbb{N}$ in (4.1), we get

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(k+m)^2} = \frac{2\binom{2m+2n}{m+n}}{m\binom{2m}{m}}\left(H_{m+n}+H_{2m}-H_m-H_{2m+2n}+\frac{1}{2m}\right).$$

Example 3.

$$\sum_{n=0}^{\infty}\frac{2^n\{H_{2n+1}-H_n\}}{(2n+1)\binom{2n}{n}} = \frac{\pi}{4}\log 2 + G,$$

where G is the Catalan constant defined by $G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$. We want to give a proof of this identity. Summing both sides of (4.2), after dividing by 8^n, we get

$$\sum_{n=0}^{\infty}\frac{2^n\{H_{2n+1}-H_n\}}{(2n+1)\binom{2n}{n}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{8^k(2k+1)^2}\frac{\binom{2n-2k}{n-k}}{8^{n-k}} = \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k(2k+1)^2}\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8k}. \tag{4.3}$$

By [3, pg. 386], we have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8k} = \sqrt{2} \tag{4.4}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{2k+1}x^{2k} = \frac{\arcsin(2x)}{2x}. \tag{4.5}$$

Integrating both sides of (4.5) over $(0, \sqrt{2}/4)$, and making the change of variable $\arcsin(2x) = u$, we get

$$\frac{1}{2\sqrt{2}} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n (2n+1)^2} = \frac{1}{2} \int_0^{\pi/4} u \cot u du. \tag{4.6}$$

From [26, pg. 44,45] we have

$$\int_0^{\pi/4} u \cot u du = \frac{\pi}{8} \log 2 + \frac{G}{2}. \tag{4.7}$$

Combining the identities (4.3)-(4.7), the result is obtained.

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Some Remarks on the Equalities of Predictors in Linear Mixed Models

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Abstract
Consider a transformed linear mixed model (TLMM) obtained pre-multiplying a linear mixed model (LMM) $\mathcal{M} : \mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{R}\boldsymbol{\gamma} + \mathbf{e}$ by a given matrix. This work concerns the problem of the equalities of linear predictors under the considered two LMMs under general assumptions. We characterize the equalities between the best linear unbiased predictors (BLUPs) under the LMM and its TLMM by using various rank formulas of block matrices and elementary matrix operations.

Keywords: *BLUP; equalities; linear mixed model; random vectors; transformed model.*

AMS Subject Classification (2020): *Primary: 62J05 ; Secondary: 62H12; 15A03.*

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1. Introduction

Throughout this study, the symbol $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. $\mathbf{A}'$, $\mathbf{A}^+$, $r(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$ stand for the transpose, the Moore–Penrose generalized inverse, the rank, and the column space of $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ refers the $m \times m$ identity matrix. Furthermore, $\mathbf{E_A} = \mathbf{A}^\perp = \mathbf{I}_m - \mathbf{A}\mathbf{A}^+$ represents the orthogonal projector of $\mathbf{A} \in \mathbb{R}^{m\times n}$.

A linear mixed model (LMM) containing both fixed and random effects is formulated by

$$\mathcal{M} : \mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{R}\boldsymbol{\gamma} + \mathbf{e}, \tag{1.1}$$

where $\boldsymbol{\alpha}$ is a fixed effect and $\boldsymbol{\gamma}$ is a random effect. In statistical inferences of analysis requirements, LMMs may need to be transformed. Several transformation methods can be used such as linear transformation. By doing this, the transformed linear mixed model (TLMM) of $\mathcal{M}$ is expressed as

$$\mathcal{T} : \mathbf{T}\mathbf{y} = \mathbf{T}\mathbf{Z}\boldsymbol{\alpha} + \mathbf{T}\mathbf{R}\boldsymbol{\gamma} + \mathbf{T}\mathbf{e}, \tag{1.2}$$

which is obtained pre-multiplying $\mathcal{M}$ by a matrix $\mathbf{T}$. In two LMMs $\mathcal{M}$ and $\mathcal{T}$, $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{Z} \in \mathbb{R}^{n\times k}$, $\mathbf{R} \in \mathbb{R}^{n\times p}$, and $\mathbf{T} \in \mathbb{R}^{m\times n}$ are known matrices of arbitrary rank, $\boldsymbol{\alpha} \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters, $\boldsymbol{\gamma} \in \mathbb{R}^{p\times 1}$ is a vector of unobservable random effects, and $\mathbf{e} \in \mathbb{R}^{n\times 1}$ is

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an unobservable vector of random errors. We will make the following general assumptions on expectations and dispersion matrices of random vectors in considered models

$$\mathrm{E}\begin{bmatrix}\gamma\\ \mathbf{e}\end{bmatrix}=\mathbf{0} \text{ and } \mathrm{D}\begin{bmatrix}\gamma\\ \mathbf{e}\end{bmatrix}=\mathrm{cov}\left\{\begin{bmatrix}\gamma\\ \mathbf{e}\end{bmatrix},\begin{bmatrix}\gamma\\ \mathbf{e}\end{bmatrix}\right\}=\begin{bmatrix}\boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_2\\ \boldsymbol{\Sigma}_3 & \boldsymbol{\Sigma}_4\end{bmatrix}:=\boldsymbol{\Sigma}, \tag{1.3}$$

where $\boldsymbol{\Sigma}_i$ are known and $\boldsymbol{\Sigma}\in\mathbb{R}^{(n+p)\times(n+p)}$ is a positive semi-definite matrix of arbitrary rank, $i=1,\ldots,4$. Let $\mathbf{A}=\begin{bmatrix}\mathbf{R}, & \mathbf{I}_n\end{bmatrix}$ and then

$$\mathrm{E}(\mathbf{y})=\mathbf{Z}\boldsymbol{\alpha},\ \ \mathrm{D}(\mathbf{y})=\begin{bmatrix}\mathbf{R}, & \mathbf{I}_n\end{bmatrix}\boldsymbol{\Sigma}\begin{bmatrix}\mathbf{R}, & \mathbf{I}_n\end{bmatrix}'=\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'. \tag{1.4}$$

Further, assume that $\mathcal{M}$ is consistent, i.e.,

$$\mathbf{y}\in\mathcal{C}\begin{bmatrix}\mathbf{Z}, & \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\end{bmatrix} \text{ holds with probability 1 (wp 1)}, \tag{1.5}$$

see, e.g., [16]. The consistency assumption of the transformed model $\mathcal{T}$ is provided with $\mathbf{Ty}\in\mathcal{C}\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}'\end{bmatrix}$ wp 1. It is easy to see that TLMM is consistent under the assumption of consistency of LMM.

In this study, we investigate the relations between the models $\mathcal{M}$ and $\mathcal{T}$. In order to characterize predictors simultaneously under two LMMs $\mathcal{M}$ and $\mathcal{T}$, the following vector can be considered

$$\mathbf{u}=\mathbf{J}\boldsymbol{\alpha}+\mathbf{G}\boldsymbol{\gamma}+\mathbf{He}=\mathbf{J}\boldsymbol{\alpha}+\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\begin{bmatrix}\gamma\\ \mathbf{e}\end{bmatrix} \tag{1.6}$$

for given $\mathbf{J}\in\mathbb{R}^{s\times k}$, $\mathbf{G}\in\mathbb{R}^{s\times p}$, and $\mathbf{H}\in\mathbb{R}^{s\times n}$. Let $\mathbf{B}=\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}$, from (1.3) and (1.4), we obtain

$$\mathrm{E}(\mathbf{u})=\mathbf{J}\boldsymbol{\alpha},\ \ \mathrm{D}(\mathbf{u})=\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\boldsymbol{\Sigma}\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}'=\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}', \tag{1.7}$$

$$\mathrm{cov}(\mathbf{u},\mathbf{y})=\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\boldsymbol{\Sigma}\begin{bmatrix}\mathbf{R}, & \mathbf{I}_n\end{bmatrix}'=\mathbf{B}\boldsymbol{\Sigma}\mathbf{A}'. \tag{1.8}$$

The predictability requirement of vector $\mathbf{u}$ under $\mathcal{M}$ is described as holding the inclusion

$$\mathcal{C}(\mathbf{J}')\subseteq\mathcal{C}(\mathbf{Z}'). \tag{1.9}$$

Let $\mathbf{u}$ be predictable under $\mathcal{M}$. If there exists $\mathbf{Fy}$ such that

$$\mathrm{D}(\mathbf{Fy}-\mathbf{u})=\min \text{ subject to } \mathrm{E}(\mathbf{Fy}-\mathbf{u})=\mathbf{0} \tag{1.10}$$

holds in the Löwner partial ordering, then the best linear unbiased predictor (BLUP) of $\mathbf{u}$ is defined as $\mathbf{Fy}$ and is denoted by $\mathbf{Fy}=\mathrm{BLUP}_{\mathcal{M}}(\mathbf{u})=\mathrm{BLUP}_{\mathcal{M}}(\mathbf{J}\boldsymbol{\alpha}+\mathbf{G}\boldsymbol{\gamma}+\mathbf{He})$, originated from [6]. If $\mathbf{G}=\mathbf{0}$ and $\mathbf{H}=\mathbf{0}$, $\mathbf{Fy}$ corresponds the best linear unbiased estimator (BLUE) of $\mathbf{J}\boldsymbol{\alpha}$, denoted by $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{J}\boldsymbol{\alpha})$, under $\mathcal{M}$.

Although predictors under LMMs and their TLMMs have different properties, observable random vectors in TLMMs may contain enough information to predict unknown vectors under LMMs. Within this context, establishing the results on the relations between these models can be considered as one of the important issues among others in linear regression analysis; see, e.g., [4, 7, 22, 24]. We may also refer to the following works on relations between predictors under different LMMs; [2, 8–10, 12, 25]. The problems of relations between original LMMs and their TLMMs are also closely connected to the concept of linear sufficiency, which was first introduced by [3, 5], see, also [11].

In this study, considering comparison problem of predictors under LMMs and their TLMMs, we derive the results on the equality characterizations between the BLUPs under $\mathcal{M}$ and $\mathcal{T}$. In order to characterize relations between BLUPs, we establish the results for the equality of coefficient matrices in the expressions of BLUPs under these models. For that purpose, we use the following expression on equality of random vectors.

$$\mathbf{F}_1\mathbf{b}=\mathbf{F}_2\mathbf{b} \text{ holds definitely if } \mathbf{F}_1=\mathbf{F}_2 \text{ for a random vector } \mathbf{b}. \tag{1.11}$$

(1.11) means directly to solve the matrix equation $\mathbf{F}_1=\mathbf{F}_2$. We note that there are several types of equalities between two linear predictions $\mathbf{F}_1\mathbf{b}$ and $\mathbf{F}_2\mathbf{b}$ of a random vector $\mathbf{b}$, for details see, e.g., [4]. These equalities are defined according to different criteria for random vectors from the statistical point of view and (1.11) is one of these equality criteria. If coefficient matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ in (1.11) are not unique, then $\mathbf{F}_1=\mathbf{F}_2$ can be divided into following four possible situations

$$\{\mathbf{F}_1\}\cap\{\mathbf{F}_2\}\neq\emptyset,\quad \{\mathbf{F}_1\}\subset\{\mathbf{F}_2\},\quad \{\mathbf{F}_1\}\supset\{\mathbf{F}_2\},\quad \{\mathbf{F}_1\}=\{\mathbf{F}_2\}, \tag{1.12}$$

where $\{\mathbf{F}_1\}$ and $\{\mathbf{F}_2\}$ stand for the collections of all solutions of the equations. In accordance with (1.12), the equality between $\mathbf{F}_1\mathbf{b}$ and $\mathbf{F}_2\mathbf{b}$ can be divided into similar situations to (1.12). Considering the situations in (1.12), we give a comprehensive investigation in theoretical point of view to comparison of the BLUPs under the model $\mathcal{M}$ and its transformed model $\mathcal{T}$ by using various rank formulas of block matrices and elementary matrix operations. Various rank formulas for partitioned matrices provide us effective tools for simplifying complicated matrix expressions composed by matrices and their Moore-Penrose generalized inverses. The rank of matrices are one of the basic concepts in linear algebra and matrix theory, and also plays an essential role in problems on establishing equalities and inequalities occurred in statistical analysis; see, e.g., [4, 7, 17, 26].

2. Preliminary Results

This section briefly reviews the well-known results on linear matrix equations, some rank formulas of matrices, and the fundamental results on BLUP equations of $\mathbf{u}$ and related properties under models $\mathcal{M}$ and $\mathcal{T}$ that we will need for main results. The following lemma is given by [14].

Lemma 2.1. *The linear matrix equation* $\mathbf{MZ} = \mathbf{N}$ *is consistent* $\Leftrightarrow \boldsymbol{r}\begin{bmatrix}\mathbf{M}, & \mathbf{N}\end{bmatrix} = \boldsymbol{r}(\mathbf{M})$*, or equivalently,* $\mathbf{MM}^+\mathbf{N} = \mathbf{N}$*. In this case, the general solution of* $\mathbf{MZ} = \mathbf{N}$ *can be written as*

$$\mathbf{Z} = \mathbf{M}^+\mathbf{N} + (\mathbf{I} - \mathbf{M}^+\mathbf{M})\mathbf{U},$$

where $\mathbf{U}$ *is an arbitrary matrix.*

Let $\mathbf{u}$ in (1.6) be predictable under $\mathcal{M}$, i.e., (1.9) holds. Note that (1.10) is in fact a quadratic matrix optimization problem. The constrained covariance matrix minimization problem in (1.10) corresponds to a well-known fundamental BLUP equation, i.e.,

$$\mathrm{E}(\mathbf{Fy} - \mathbf{u}) = \mathbf{0} \ \text{and} \ \mathrm{D}(\mathbf{Fy} - \mathbf{u}) = \min \ \Leftrightarrow \mathbf{F}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'Z^\perp}\end{bmatrix} = \begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^\perp}\end{bmatrix}. \tag{2.1}$$

According to Lemma 2.1, the general solution of (2.1) is written as

$$\mathbf{F} = \begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^\perp}\end{bmatrix}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'Z^\perp}\end{bmatrix}^+ + \mathbf{U}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'Z^\perp}\end{bmatrix}^\perp, \tag{2.2}$$

where $\mathbf{U} \in \mathbb{R}^{s\times n}$ is an arbitrary matrix, and the BLUP of $\mathbf{u}$ under $\mathcal{M}$ is written as $\mathrm{BLUP}_{\mathcal{M}}(\mathbf{u}) = \mathbf{Fy}$ from (1.10). Further, we can add the following obvious results related to (2.1) and (2.2).

(a) The equation in (2.1) is always consistent.

(b) $\mathbf{F}$ in (2.2) is unique $\Leftrightarrow \boldsymbol{r}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'Z^\perp}\end{bmatrix} = n$.

(c) $\mathrm{BLUP}_{\mathcal{M}}(\mathbf{u})$ is unique wp 1 $\Leftrightarrow \mathcal{M}$ is consistent, i.e., (1.5) holds.

(d) $\boldsymbol{r}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'Z^\perp}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma}\end{bmatrix}$.

see, e.g., [15, 20, 21].

Let us consider TLMM $\mathcal{T}$. The predictability requirement of $\mathbf{u}$ under $\mathcal{T}$ is expressed as

$$\mathcal{C}(\mathbf{J}') \subseteq \mathcal{C}(\mathbf{Z}'\mathbf{T}'). \tag{2.3}$$

It is evident that the predictability of $\mathbf{u}$ under a TLMM shows predictability of $\mathbf{u}$ under an original LMM. Let $\mathbf{u}$ in (1.6) be predictable under $\mathcal{T}$. The expression in (1.10) and the equation in (2.1) can be adapted for model $\mathcal{T}$ and thereby the fundamental BLUP equation under $\mathcal{T}$ is written as:

$$\mathrm{E}(\mathbf{F}_t\mathbf{Ty} - \mathbf{u}) = \mathbf{0} \ \text{and} \ \mathrm{D}(\mathbf{F}_t\mathbf{Ty} - \mathbf{u}) = \min \ \Leftrightarrow \mathbf{F}_t\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA\Sigma A'T'(TZ)^\perp}\end{bmatrix} = \begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^\perp}\end{bmatrix}. \tag{2.4}$$

The matrix equation in (2.4) is always consistent. According to Lemma 2.1, the general solution of (2.4) is written as

$$\mathbf{F}_t = \begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^\perp}\end{bmatrix}\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA\Sigma A'T'(TZ)^\perp}\end{bmatrix}^+ + \mathbf{U}_t\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA\Sigma A'T'(TZ)^\perp}\end{bmatrix}^\perp \tag{2.5}$$

where $\mathbf{U}_t \in \mathbb{R}^{s\times m}$ is an arbitrary matrix, and $\mathrm{BLUP}_{\mathcal{T}}(\mathbf{u}) = \mathbf{F}_t\mathbf{Ty}$. Further, the expressions in (b)-(d) above for model $\mathcal{M}$ can be similarly expressed for model $\mathcal{T}$.

The requirement in (1.9) corresponds to the estimability of vector $\mathbf{J}\boldsymbol{\alpha}$ under $\mathcal{M}$; see, e.g., [1], and, similarly, the requirement in (2.3) corresponds to the estimability of vector $\mathbf{J}\boldsymbol{\alpha}$ under $\mathcal{T}$. We also note that the estimability of vector $\mathbf{Z}\boldsymbol{\alpha}$ under both the models $\mathcal{M}$ and $\mathcal{T}$ is

$$r(\mathbf{Z}) = r(\mathbf{TZ}). \tag{2.6}$$

Let $\mathbf{J}\boldsymbol{\alpha}$ be estimable under $\mathcal{T}$ (also estimable under $\mathcal{M}$). The BLUEs of $\mathbf{J}\boldsymbol{\alpha}$ under models $\mathcal{M}$ and $\mathcal{T}$ are expressed as $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{J}\boldsymbol{\alpha}) = \mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}}\mathbf{y}$ and $\mathrm{BLUE}_{\mathcal{T}}(\mathbf{J}\boldsymbol{\alpha}) = \mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}\mathbf{Ty}$, respectively, where

$$\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}} = \begin{bmatrix}\mathbf{J}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{Z}, & \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{Z}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{Z}, & \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{Z}^{\perp}\end{bmatrix}^{\perp} \tag{2.7}$$

and

$$\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}} = \begin{bmatrix}\mathbf{J}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}'(\mathbf{TZ})^{\perp}\end{bmatrix}^{+} + \mathbf{U}_t\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}'(\mathbf{TZ})^{\perp}\end{bmatrix}^{\perp}. \tag{2.8}$$

Let $\mathbf{Z}\boldsymbol{\alpha}$ be estimable under $\mathcal{T}$ (also estimable under $\mathcal{M}$). The BLUEs of $\mathbf{Z}\boldsymbol{\alpha}$ under models $\mathcal{M}$ and $\mathcal{T}$ are expressed as $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{Z}\boldsymbol{\alpha}) = \mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}}\mathbf{y}$ and $\mathrm{BLUE}_{\mathcal{T}}(\mathbf{Z}\boldsymbol{\alpha}) = \mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}\mathbf{Ty}$, respectively, where

$$\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}} = \begin{bmatrix}\mathbf{Z}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{Z}, & \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{Z}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{Z}, & \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{Z}^{\perp}\end{bmatrix}^{\perp} \tag{2.9}$$

and

$$\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}} = \begin{bmatrix}\mathbf{Z}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}'(\mathbf{TZ})^{\perp}\end{bmatrix}^{+} + \mathbf{U}_t\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}'(\mathbf{TZ})^{\perp}\end{bmatrix}^{\perp}. \tag{2.10}$$

The following lemma is related to the characterizations in (1.12) based on (1.11); see, [19].

Lemma 2.2. *Let* $\mathbf{M} \in \mathbb{R}^{m\times n_1}$, $\mathbf{N} \in \mathbb{R}^{p\times n_1}$, $\mathbf{P} \in \mathbb{R}^{m\times n_2}$, *and* $\mathbf{Q} \in \mathbb{R}^{p\times n_2}$ *be given. Then,*

(a) Matrix equations $\mathbf{ZM} = \mathbf{N}$ *and* $\mathbf{ZP} = \mathbf{Q}$ *have a common solution* $\Leftrightarrow \mathcal{C}\begin{bmatrix}\mathbf{N}'\\ \mathbf{Q}'\end{bmatrix} \subseteq \mathcal{C}\begin{bmatrix}\mathbf{M}'\\ \mathbf{P}'\end{bmatrix} \Leftrightarrow r\begin{bmatrix}\mathbf{M} & \mathbf{P}\\ \mathbf{N} & \mathbf{Q}\end{bmatrix} = r\begin{bmatrix}\mathbf{M}, & \mathbf{P}\end{bmatrix}$.

(b) Any solution of the matrix equation $\mathbf{ZP} = \mathbf{Q}$ *is a solution of* $\mathbf{ZM} = \mathbf{N} \Leftrightarrow r\begin{bmatrix}\mathbf{M} & \mathbf{P}\\ \mathbf{N} & \mathbf{Q}\end{bmatrix} = r(\mathbf{P})$.

In matrix algebra, some formulas of ranks of matrices are very helpful for facilitating complicated matrix equations. Within this framework, we use the following rank equalities for partitioned matrices; see [13].

Lemma 2.3. *Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$, $\mathbf{N} \in \mathbb{R}^{m\times k}$, *and* $\mathbf{P} \in \mathbb{R}^{l\times n}$. *Then,*

$$r\begin{bmatrix}\mathbf{M}, & \mathbf{N}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{E_M N}) = r(\mathbf{N}) + r(\mathbf{E_N M}), \tag{2.11}$$

$$r\begin{bmatrix}\mathbf{M}\\ \mathbf{P}\end{bmatrix} = r(\mathbf{M}) + r(\mathbf{PE_{M'}}) = r(\mathbf{P}) + r(\mathbf{ME_{P'}}). \tag{2.12}$$

3. Equality Relations of BLUPs under LMM and its TLMM

In this section, the main results on the equalities between BLUPs, related to the characterizations in (1.12), under models $\mathcal{M}$ and $\mathcal{T}$ are given .

Theorem 3.1. *Let us consider* $\mathcal{M}$ *in* (1.1) *and* $\mathcal{T}$ *in* (1.2). *Assume that* $\mathbf{u}$ *in* (1.6) *is predictable under these models. Let the coefficients* $\mathbf{F}$ *and* $\mathbf{F}_t$ *be as given in* (2.2) *and* (2.5), *respectively. Then*

$$\{\mathbf{F}\} \cap \{\mathbf{F}_t\mathbf{T}\} \neq \emptyset \Leftrightarrow r\begin{bmatrix}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{0} & \mathbf{Z} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}' & \mathbf{0} & \mathbf{TZ} & \mathbf{T}\\ \mathbf{Z}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ -\mathbf{B}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}' & -\mathbf{J} & \mathbf{J} & \mathbf{0}\end{bmatrix} = r\begin{bmatrix}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' & \mathbf{0} & \mathbf{Z} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{T}' & \mathbf{0} & \mathbf{TZ} & \mathbf{T}\\ \mathbf{Z}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{3.1}$$

In this case, $\{\mathrm{BLUP}_{\mathcal{M}}(\mathbf{u})\} \cap \{\mathrm{BLUP}_{\mathcal{T}}(\mathbf{u})\} \neq \emptyset$ *holds definitely.*

Proof. From (2.2) and (2.5), $\mathbf{F}-\mathbf{F}_t\mathbf{T}$ is written as

$$\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^{\perp}}\end{bmatrix}\mathbf{W}^{+}-\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^{\perp}}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{T}+\mathbf{UW}^{\perp}-\mathbf{U}_t\mathbf{W}_t^{\perp}\mathbf{T}, \tag{3.2}$$

where $\mathbf{W}=\begin{bmatrix}\mathbf{Z}, & \mathbf{A\Sigma A'Z^{\perp}}\end{bmatrix}$ and $\mathbf{W}_t=\begin{bmatrix}\mathbf{TZ}, & \mathbf{TA\Sigma A'T'(TZ)^{\perp}}\end{bmatrix}$. Then applying the formula $\min_{\mathbf{U}} \boldsymbol{r}(\mathbf{C}+\mathbf{UD}) = \boldsymbol{r}\begin{bmatrix}\mathbf{C}\\ \mathbf{D}\end{bmatrix}-\boldsymbol{r}(\mathbf{D})$, given in [18] and [23], to (3.2) and simplifying the block matrices by Lemma 2.3, we obtain

$$\begin{aligned}
&=\min_{\mathbf{U},\mathbf{U}_t}\boldsymbol{r}\left(\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^{\perp}}\end{bmatrix}\mathbf{W}^{+}-\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^{\perp}}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{T}+\begin{bmatrix}\mathbf{U}, & -\mathbf{U}_t\end{bmatrix}\begin{bmatrix}\mathbf{W}^{\perp}\\ \mathbf{W}_t^{\perp}\mathbf{T}\end{bmatrix}\right)\\
&=\boldsymbol{r}\begin{bmatrix}\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^{\perp}}\end{bmatrix}\mathbf{W}^{+}-\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^{\perp}}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{T}\\ \mathbf{W}^{\perp}\\ \mathbf{W}_t^{\perp}\mathbf{T}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{W}^{\perp}\\ \mathbf{W}_t^{\perp}\mathbf{T}\end{bmatrix}\\
&=\boldsymbol{r}\begin{bmatrix}\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^{\perp}}\end{bmatrix}\mathbf{W}^{+}-\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^{\perp}}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{T} & \mathbf{0} & \mathbf{0}\\ \mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T} & \mathbf{0} & \mathbf{W}_t\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T} & \mathbf{0} & \mathbf{W}_t\end{bmatrix}\\
&=\boldsymbol{r}\begin{bmatrix}\mathbf{0} & -\begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'Z^{\perp}}\end{bmatrix} & \begin{bmatrix}\mathbf{J}, & \mathbf{B\Sigma A'T'(TZ)^{\perp}}\end{bmatrix}\\ \mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T} & \mathbf{0} & \mathbf{W}_t\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T} & \mathbf{0} & \mathbf{W}_t\end{bmatrix}\\
&=\boldsymbol{r}\begin{bmatrix}\mathbf{0} & -\mathbf{J} & -\mathbf{B\Sigma A'} & \mathbf{J} & \mathbf{B\Sigma A'T'}\\ \mathbf{I}_n & \mathbf{Z} & \mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{0}\\ \mathbf{T} & \mathbf{0} & \mathbf{0} & \mathbf{TZ} & \mathbf{TA\Sigma A'T'}\\ \mathbf{0} & \mathbf{0} & \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z'T'}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{I}_n & \mathbf{Z} & \mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{0}\\ \mathbf{T} & \mathbf{0} & \mathbf{0} & \mathbf{TZ} & \mathbf{TA\Sigma A'T'}\\ \mathbf{0} & \mathbf{0} & \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z'T'}\end{bmatrix}\\
&=\boldsymbol{r}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{Z} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA\Sigma A'T'} & \mathbf{0} & \mathbf{TZ} & \mathbf{T}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ -\mathbf{B\Sigma A'} & \mathbf{B\Sigma A'T'} & -\mathbf{J} & \mathbf{J} & \mathbf{0}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{Z} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA\Sigma A'T'} & \mathbf{0} & \mathbf{TZ} & \mathbf{T}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}.
\end{aligned} \tag{3.3}$$

The required result is seen from (3.3). □

Corollary 3.1. *Let us consider $\mathcal{M}$ in (1.1) and $\mathcal{T}$ in (1.2).*

(a) Let $\mathbf{J}\boldsymbol{\alpha}$ be estimable under $\mathcal{T}$ (also estimable under $\mathcal{M}$). Let the coefficients $\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}}$ and $\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}$ be as given in (2.7) and (2.8), respectively. Then the following holds.

$$\begin{aligned}
&\{\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}}\}\cap\{\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}\mathbf{T}\}\neq\emptyset\\
&\Leftrightarrow \boldsymbol{r}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{Z} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA\Sigma A'T'} & \mathbf{0} & \mathbf{TZ} & \mathbf{T}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{J} & \mathbf{J} & \mathbf{0}\end{bmatrix}=\boldsymbol{r}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{Z} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA\Sigma A'T'} & \mathbf{0} & \mathbf{TZ} & \mathbf{T}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}.
\end{aligned} \tag{3.4}$$

In this case, $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{J}\boldsymbol{\alpha})\}\cap\{\mathrm{BLUP}_{\mathcal{T}}(\mathbf{J}\boldsymbol{\alpha})\}\neq\emptyset$ holds definitely.

(b) If $\mathbf{Z}\boldsymbol{\alpha}$ is estimable under the models $\mathcal{M}$ and $\mathcal{T}$ then (2.6) holds. Let the coefficients $\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}}$ and $\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}$ be as given in (2.9) and (2.10), respectively. Then the following holds.

$$\{\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}}\}\cap\{\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}\mathbf{T}\}\neq\emptyset\Leftrightarrow\boldsymbol{r}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{TA\Sigma A'T'} & \mathbf{T}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0}\end{bmatrix}=\boldsymbol{r}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{Z} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{0} & \mathbf{T}\end{bmatrix}+\boldsymbol{r}(\mathbf{Z}). \tag{3.5}$$

In this case, $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{Z}\boldsymbol{\alpha})\}\cap\{\mathrm{BLUE}_{\mathcal{T}}(\mathbf{Z}\boldsymbol{\alpha})\}\neq\emptyset$.

Theorem 3.2. *Let us consider $\mathcal{M}$ in (1.1) and $\mathcal{T}$ in (1.2). Assume that $\mathbf{u}$ in (1.6) is predictable under these models. Let the coefficients $\mathbf{F}$ and $\mathbf{F}_t$ be as given in (2.2) and (2.5), respectively. Then*

$$\{\mathbf{F}\} \subset \{\mathbf{F}_t\mathbf{T}\} \Leftrightarrow r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} \\ \mathbf{B\Sigma A'} & \mathbf{B\Sigma A'T'} & \mathbf{J} \end{bmatrix} = r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} \end{bmatrix} + r(\mathbf{Z}). \tag{3.6}$$

In this case, $\{\mathrm{BLUP}_{\mathcal{M}}(\mathbf{u})\} \subset \{\mathrm{BLUP}_{\mathcal{T}}(\mathbf{u})\}$ *holds.*

Proof. From Lemma 2.2 (b), all solutions of the equation in (2.1) are the solutions of the equation in (2.4) $\Leftrightarrow$

$$r\begin{bmatrix} \mathbf{Z} & \mathbf{A\Sigma A'Z^{\perp}} & \mathbf{Z} & \mathbf{A\Sigma A'T'(TZ)^{\perp}} \\ \mathbf{J} & \mathbf{B\Sigma A'Z^{\perp}} & \mathbf{J} & \mathbf{B\Sigma A'T'(TZ)^{\perp}} \end{bmatrix} = r\begin{bmatrix} \mathbf{Z}, & \mathbf{A\Sigma A'Z^{\perp}} \end{bmatrix}. \tag{3.7}$$

(3.7) equivalently written as

$$r\begin{bmatrix} \mathbf{Z} & \mathbf{A\Sigma A'} & \mathbf{Z} & \mathbf{A\Sigma A'T'} \\ \mathbf{J} & \mathbf{B\Sigma A'} & \mathbf{J} & \mathbf{B\Sigma A'T'} \\ \mathbf{0} & \mathbf{Z'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z'T'} \end{bmatrix} - r(\mathbf{Z}) - r(\mathbf{TZ}) = r\begin{bmatrix} \mathbf{Z} & \mathbf{A\Sigma A'} \\ \mathbf{0} & \mathbf{Z'} \end{bmatrix} - r(\mathbf{Z}), \tag{3.8}$$

which is equivalent to (3.6). □

Corollary 3.2. *Let us consider $\mathcal{M}$ in (1.1) and $\mathcal{T}$ in (1.2).*

(a) Let $\mathbf{J}\boldsymbol{\alpha}$ be estimable under $\mathcal{T}$ (also estimable under $\mathcal{M}$). Let the coefficients $\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}}$ and $\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}$ be as given in (2.7) and (2.8), respectively. Then the following holds.

$$\{\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}}\} \subset \{\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}\mathbf{T}\} \Leftrightarrow r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} = r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} \end{bmatrix} + r(\mathbf{Z}). \tag{3.9}$$

In this case, $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{J}\boldsymbol{\alpha})\} \subset \{\mathrm{BLUE}_{\mathcal{T}}(\mathbf{J}\boldsymbol{\alpha})\}$ *holds.*

(b) If $\mathbf{Z}\boldsymbol{\alpha}$ is estimable under the models $\mathcal{M}$ and $\mathcal{T}$ then (2.6) holds. Let the coefficients $\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}}$ and $\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}$ be as given in (2.9) and (2.10), respectively. Then the following holds.

$$\{\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}}\} \subset \{\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}\mathbf{T}\} \Leftrightarrow r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} \\ \mathbf{Z'} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z'T'} \end{bmatrix} = r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} \end{bmatrix}. \tag{3.10}$$

In this case, $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{Z}\boldsymbol{\alpha})\} \subset \{\mathrm{BLUE}_{\mathcal{T}}(\mathbf{Z}\boldsymbol{\alpha})\}$ *holds.*

Theorem 3.3. *Let us consider $\mathcal{M}$ in (1.1) and $\mathcal{T}$ in (1.2). Assume that $\mathbf{u}$ in (1.6) is predictable under these models. Let the coefficients $\mathbf{F}$ and $\mathbf{F}_t$ be as given in (2.2) and (2.5), respectively. Then*

$$\mathbf{F} \in \{\mathbf{F}_t\mathbf{T}\} \Leftrightarrow r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} \\ \mathbf{B\Sigma A'} & \mathbf{B\Sigma A'T'} & \mathbf{J} \end{bmatrix} = r\begin{bmatrix} \mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} & \mathbf{Z} \\ \mathbf{Z'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0} \end{bmatrix}. \tag{3.11}$$

Then, $\mathrm{BLUP}_{\mathcal{M}}(\mathbf{u}) \in \{\mathrm{BLUP}_{\mathcal{T}}(\mathbf{u})\}$ *holds.*

Proof. From Lemma 2.2 (a), the equations in (2.1) and (2.4) have a common solution $\Leftrightarrow$

$$r\begin{bmatrix} \mathbf{Z} & \mathbf{A\Sigma A'Z^{\perp}} & \mathbf{Z} & \mathbf{A\Sigma A'T'(TZ)^{\perp}} \\ \mathbf{J} & \mathbf{B\Sigma A'Z^{\perp}} & \mathbf{J} & \mathbf{B\Sigma A'T'(TZ)^{\perp}} \end{bmatrix} = r\begin{bmatrix} \mathbf{Z} & \mathbf{A\Sigma A'Z^{\perp}} & \mathbf{Z} & \mathbf{A\Sigma A'T'(TZ)^{\perp}} \end{bmatrix}. \tag{3.12}$$

(3.12) equivalently written as

$$r\begin{bmatrix}\mathbf{Z} & \mathbf{A\Sigma A'} & \mathbf{Z} & \mathbf{A\Sigma A'T'}\\ \mathbf{J} & \mathbf{B\Sigma A'} & \mathbf{J} & \mathbf{B\Sigma A'T'}\\ \mathbf{0} & \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z'T'}\end{bmatrix} - r(\mathbf{Z}) - r(\mathbf{TZ}) = r\begin{bmatrix}\mathbf{Z} & \mathbf{A\Sigma A'} & \mathbf{Z} & \mathbf{A\Sigma A'T'}\\ \mathbf{0} & \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z'T'}\end{bmatrix} - r(\mathbf{Z}) - r(\mathbf{TZ}), \tag{3.13}$$

which is equivalent to (3.11). □

Corollary 3.3. *Let us consider $\mathcal{M}$ in (1.1) and $\mathcal{T}$ in (1.2).*

(a) Let $\mathbf{J}\boldsymbol{\alpha}$ be estimable under $\mathcal{T}$ (also estimable under $\mathcal{M}$). Let the coefficients $\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}}$ and $\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}$ be as given in (2.7) and (2.8), respectively. Then the following holds.

$$\mathbf{F}_{\mathbf{J}\boldsymbol{\alpha}} \in \{\mathbf{F}_{t\mathbf{J}\boldsymbol{\alpha}}\mathbf{T}\} \Leftrightarrow r\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} & \mathbf{Z}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{J}\end{bmatrix} = r\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'} & \mathbf{Z}\\ \mathbf{Z'} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'} & \mathbf{0}\end{bmatrix}. \tag{3.14}$$

In this case, $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{J}\boldsymbol{\alpha}) \in \{\mathrm{BLUE}_{\mathcal{T}}(\mathbf{J}\boldsymbol{\alpha})\}$ holds.

(b) If $\mathbf{Z}\boldsymbol{\alpha}$ is estimable under the models $\mathcal{M}$ and $\mathcal{T}$ then (2.6) holds. Let the coefficients $\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}}$ and $\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}$ be as given in (2.9) and (2.10), respectively. Then the following holds.

$$\mathbf{F}_{\mathbf{Z}\boldsymbol{\alpha}} \in \{\mathbf{F}_{t\mathbf{Z}\boldsymbol{\alpha}}\mathbf{T}\} \Leftrightarrow \mathcal{C}\begin{bmatrix}\mathbf{A\Sigma A'} & \mathbf{A\Sigma A'T'}\\ \mathbf{Z'} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z'T'}\end{bmatrix} \cap \mathcal{C}(\mathbf{Z}) = \{\mathbf{0}\}. \tag{3.15}$$

In this case, $\mathrm{BLUP}_{\mathcal{M}}(\mathbf{Z}\boldsymbol{\alpha}) \in \{\mathrm{BLUP}_{\mathcal{T}}(\mathbf{Z}\boldsymbol{\alpha})\}$ holds.

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