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A Note on Horadam Hybrinomials

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Abstract

This paper ensures an extensive survey of the generalization of the various hybrid numbers and hybrid polynomials especially as part of its enhancing importance in the disciplines of mathematics and physics. In this paper, by using the Horadam polynomials, we define the Horadam hybrid polynomials called Horadam hybrinomials. We obtain some special cases and algebraic properties of the Horadam hybrinomials such as recurrence relation, generating function, exponential generating function, Binet formula, summation formulas, Catalan's identity, Cassini's identity and d'Ocagne's identity, respectively. Moreover, we give some applications related to the Horadam hybrinomials in matrices.

1. Introduction

Horadam defined the sequence $w_n = w_n(a,b;p,q)$ by the recurrence relation

$$w_n = pw_{n-1} + qw_{n-2}, \quad n \geq 2$$

with the initial values $w_0 = a$ and $w_1 = b$. For different values $p,q,a,b \in \mathbb{Z}$, Horadam sequence turns into several well-known sequences such as Fibonacci, Lucas, Pell and so on. These sequences are studied in many areas such as physics, number theory, algebra, geometry, and combinatorics. For more details, we refer to [1]-[6].
In [7], the Horadam polynomials $h_n(x) = h_n(x;a,b;p,q)$ are defined by the recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad n \geq 3 \tag{1.1}$$

with the initial values $h_1(x) = a$ and $h_2(x) = bx$. Let $\alpha = \frac{px+\sqrt{p^2x^2+4q}}{2}$ and $\beta = \frac{px-\sqrt{p^2x^2+4q}}{2}$ be the real roots of the characteristic equation $t^2 - pxt - q = 0$. The Binet formula for the polynomial $h_n(x)$ is given by

$$h_n(x) = A\alpha^{n-1} + B\beta^{n-1}, \tag{1.2}$$

where $A = \frac{bx-a\beta}{\sqrt{p^2x^2+4q}}$ and $B = \frac{a\alpha-bx}{\sqrt{p^2x^2+4q}}$.
The generating function of the Horadam polynomials is

$$\sum_{n=0}^{\infty} h_n(x)t^n = \frac{a+xt(b-ap)}{1-pxt-qt^2}. \tag{1.3}$$

Hybrid numbers were studied by Ozdemir in [8], extensively. A hybrid number is defined as

$$\mathbb{K} = \left\{a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a,b,c,d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = \mathbf{hi} = \varepsilon + \mathbf{i}\right\}.$$

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Addition and subtraction of hybrid numbers are done by adding and subtracting corresponding terms. Two hybrid numbers are equal if all their components are equal, one by one.
Using the equalities $\mathbf{i}^2=-1$, $\varepsilon^2=0$, $\mathbf{h}^2=0$, $\mathbf{ih}=-\mathbf{hi}=\varepsilon+\mathbf{i}$, the multiplication table of the basis of hybrid numbers is as follows:

Table 1: Multiplication table for $\mathbb{K}$

.	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	$\mathbf{h}+\mathbf{1}$	0	$-\varepsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\varepsilon-\mathbf{i}$	ε	$\mathbf{1}$

Recently, many researchers have studied related to hybrid numbers. For example, in [9] Szynal-Liana and Wloch considered the Fibonacci hybrid numbers and obtained some properties of this numbers. In [10, 11] the authors also defined and examined the Jacosthal and Jacosthal–Lucas hybrid numbers and the Pell and Pell–Lucas hybrid numbers respectively. In [12] Szynal-Liana generalized their results and defined the Horadam hybrid numbers. In [13] Kızılateş introduced the another generalization of hybrid numbers and gave miscellaneous properties of these numbers. For more details, we refer to [8]-[23].
We now turn to a recent investigation by Szynal-Liana and Wloch [24], who defined and studied a family of the special polynomials and the special numbers which are related to the Fibonacci hybrinomials and Lucas hybrinomials. The Fibonacci hybrinomials and Lucas hybrinomials are defined as follows:

$$FH_n(x)=F_n(x)+F_{n+1}(x)\mathbf{i}+F_{n+2}(x)\varepsilon+F_{n+3}(x)\mathbf{h},$$

and

$$LH_n(x)=L_n(x)+L_{n+1}(x)\mathbf{i}+L_{n+2}(x)\varepsilon+L_{n+3}(x)\mathbf{h}.$$

For $n\geq 2$, the recurrence relations of the Fibonacci hybrinomials and the Lucas hybrinomials are

$$FH_n(x)=xFH_{n-1}(x)+FH_{n-2}(x),$$

and

$$LH_n(x)=xLH_{n-1}(x)+LH_{n-2}(x),$$

with the initial values $FH_0(x)=\mathbf{i}+x\varepsilon+(x^2+1)\mathbf{h}$, $FH_1(x)=1+x\mathbf{i}+(x^2+1)\varepsilon+(x^3+2x)\mathbf{h}$, $LH_0(x)=2+x\mathbf{i}+(x^2+2)\varepsilon+(x^3+3x)\mathbf{h}$ and $LH_1(x)=x+(x^2+2)\mathbf{i}+(x^3+3x)\varepsilon+(x^4+4x^2+2)\mathbf{h}$, respectively. The Fibonacci hybrinomials and the Lucas hybrinomials, namely polynomials, are a generalization of the Fibonacci hybrid and Lucas hybrid numbers.
Motivated by some of the above-mentioned recent papers, we introduce here new polynomials which are called Horadam hybrinomials. This definition brings about a more general hybrid polynomial sequence by taking components from Horadam polynomials. Thanks to this generalization, we obtain the Fibonacci hybrinomials $FH_n(x)$, the Lucas hybrinomials $LH_{n-1}(x)$, the Pell hybrinomials $PH_n(x)$, the Pell-Lucas hybrinomials $QH_{n-1}(x)$, the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$, the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$ and the Balancing hybrinomials $BH_n(x)$. We also obtain various results for the Horadam hybrinomials. Moreover, we give some applications of Horadam hybrinomials in matrices.

2. Horadam hybrinomials

In this section, we define the Horadam hybrinomials. Then we give some special cases of Horadam hybrinomials such as the Fibonacci hybrinomials, the Fibonacci hybrid numbers, the Lucas hybrinomials, the Lucas hybrid numbers, the Pell hybrinomials, the Pell hybrid numbers, the Pell-Lucas hybrinomials, the Pell-Lucas hybrid numbers, the Chebyshev hybrinomials of the first kind, the Chebyshev hybrid numbers of the first kind, the Chebyshev hybrinomials of the second kind, the Chebyshev hybrid numbers of the second kind, the Balancing hybrinomials and the Balancing hybrid numbers. Finally we obtain some algebraic properties of Horadam hybrinomials.

Definition 2.1. *For $n\geq 1$, the n^{th} Horadam hybrinomials are defined by*

$$\mathbb{H}_n(x)=h_n(x)+h_{n+1}(x)\mathbf{i}+h_{n+2}(x)\varepsilon+h_{n+3}(x)\mathbf{h}. \tag{2.1}$$

Some special cases of Horadam hybrinomials are as follows:

1. For $a=b=p=q=1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Fibonacci hybrinomials $FH_n(x)$,
2. For $a=2$ and $b=p=q=1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Lucas hybrinomials $LH_{n-1}(x)$,

3. For $a=q=1$ and $b=p=2$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Pell hybrinomials $PH_n(x)$,
4. For $a=b=p=2$ and $q=1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Pell-Lucas hybrinomials $QH_{n-1}(x)$,
5. For $a=b=1$, $p=2$, and $q=-1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$,
6. For $a=1$, $b=p=2$, and $q=-1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$,
7. For $a=1$, $b=p=6$, and $q=-1$, the Horadam hybrinomials $\mathbb{H}_n(x)$ become the Balancing hybrinomials $BH_n(x)$,
8. For $x=1$, the Fibonacci hybrinomials $FH_n(x)$, reduce to the Fibonacci hybrid numbers FH_n,
9. For $x=1$, the Lucas hybrinomials $LH_{n-1}(x)$, reduce to the Lucas hybrid numbers LH_{n-1},
10. For $x=1$, the Pell hybrinomials $PH_n(x)$, reduce to the Pell hybrid numbers PH_n,
11. For $x=1$, the Pell-Lucas hybrinomials $QH_{n-1}(x)$, reduce to the Pell-Lucas hybrid numbers QH_{n-1},
12. For $x=1$, the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$, reduce to the Chebyshev hybrid numbers of the first kind TH_{n-1},
13. For $x=1$, the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$, reduce to the Chebyshev hybrid numbers of the second kind UH_{n-1},
14. For $x=1$, the Balancing hybrinomials $BH_n(x)$, reduce to the Balancing hybrid numbers BH_n.

Using (2.1) and (1.1), we obtain that for $n>2$,

$$\begin{aligned}\mathbb{H}_n(x) &= pxh_{n-1}(x)+qh_{n-2}(x)+(pxh_n(x)+qh_{n-1}(x))\,\mathbf{i}\\ &\quad+(pxh_{n+1}(x)+qh_n(x))\,\varepsilon+(pxh_{n+2}(x)+qh_{n+1}(x))\,\mathbf{h}\\ &= px\mathbb{H}_{n-1}(x)+q\mathbb{H}_{n-2}(x)\end{aligned}$$

and so

$$\mathbb{H}_n(x)=px\mathbb{H}_{n-1}(x)+q\mathbb{H}_{n-2}(x),$$

with the initial values $\mathbb{H}_1(x)=a+bx\mathbf{i}+(bpx^2+aq)\varepsilon+(bp^2x^3+(apq+bq)x)\mathbf{h}$ and $\mathbb{H}_2(x)=bx+(bpx^2+aq)\mathbf{i}+(bp^2x^3+(apq+bq)x)\varepsilon+(bp^3x^4+(ap^2q+2bpq)x^2+aq^2)\mathbf{h}$.

Theorem 2.2. *The Binet formula for the Horadam hybrinomial* $\mathbb{H}_n(x)$ *is*

$$\mathbb{H}_n(x)=A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}, \tag{2.2}$$

where $\tilde{\alpha}=1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h}$ and $\tilde{\beta}=1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h}$.

Proof. Due to (1.2) and (2.1), we find that

$$\begin{aligned}\mathbb{H}_n(x) &= (A\alpha^{n-1}+B\beta^{n-1})+(A\alpha^n+B\beta^n)\mathbf{i}+(A\alpha^{n+1}+B\beta^{n+1})\varepsilon+(A\alpha^{n+2}+B\beta^{n+2})\mathbf{h}\\ &= A\alpha^{n-1}(1+\alpha\mathbf{i}+\alpha^2\varepsilon+\alpha^3\mathbf{h})+B\beta^{n-1}(1+\beta\mathbf{i}+\beta^2\varepsilon+\beta^3\mathbf{h})\\ &= A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}.\end{aligned}$$

□

We now give the generating function and exponential generating function for the Horadam hybrinomials.

Theorem 2.3. *The generating function for the Horadam hybrinomial* $\mathbb{H}_n(x)$ *is*

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\frac{\mathbb{H}_0(x)+(\mathbb{H}_1(x)-px\mathbb{H}_0(x))\,t}{1-pxt-qt^2}. \tag{2.3}$$

Proof. Suppose that the generating function for the Horadam hybrinomials $\left\{\mathbb{H}_n(x)_{n=0}^{\infty}\right\}$, has the following formal power series

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\mathbb{H}_0(x)+\mathbb{H}_1(x)t+\cdots+\mathbb{H}_k(x)t^k+\cdots. \tag{2.4}$$

Hence

$$pxt\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=px\mathbb{H}_0(x)t+px\mathbb{H}_1(x)t^2+\cdots+px\mathbb{H}_k(x)t^{k+1}+\cdots, \tag{2.5}$$

$$qt^2\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=q\mathbb{H}_0(x)t^2+q\mathbb{H}_1(x)t^3+\cdots+q\mathbb{H}_k(x)t^{k+2}+\cdots. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we find that

$$(1-pxt-qt^2)\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\mathbb{H}_0(x)+(\mathbb{H}_1(x)-px\mathbb{H}_0(x))\,t.$$

So

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)t^n=\frac{\mathbb{H}_0(x)+(\mathbb{H}_1(x)-px\mathbb{H}_0(x))\,t}{1-pxt-qt^2}.$$

□

Corollary 2.4. *([24, Theorem 2.10]) The generating function for the Fibonacci hybrinomial $FH_n(x)$ is*

$$\sum_{n=0}^{\infty}FH_n(x)t^n=\frac{\mathbf{i}+x\varepsilon+(x^2+1)\mathbf{h}+(1+\varepsilon+x\mathbf{h})t}{1-xt-t^2}.$$

Proof. If we take $a=b=p=q=1$ in Equation (2.3), the proof is completed. □

Corollary 2.5. *([24, Theorem 2.11]) The generating function for the Lucas hybrinomial $LH_n(x)$ is*

$$\sum_{n=0}^{\infty}LH_n(x)t^n=\frac{LH_0(x)+(LH_1(x)-xLH_0(x))\,t}{1-xt-t^2}.$$

Proof. If we take $a=2$ and $b=p=q=1$ in Equation (2.3), the proof is completed. □

Theorem 2.6. *The exponential generating function for the Horadam hybrinomial $\mathbb{H}_n(x)$ is*

$$\sum_{n=0}^{\infty}\mathbb{H}_n(x)\frac{t^n}{n!}=A\alpha^{-1}\tilde{\alpha}e^{\alpha t}+B\beta^{-1}\tilde{\beta}e^{\beta t}.$$

Proof. Using the Equation (2.2), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{H}_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}(A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta})\frac{t^n}{n!}\\
&= \frac{A\tilde{\alpha}}{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}+\frac{B\tilde{\beta}}{\beta}\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!}\\
&= \frac{A\tilde{\alpha}}{\alpha}e^{\alpha t}+\frac{B\tilde{\beta}}{\beta}e^{\beta t}\\
&= A\alpha^{-1}\tilde{\alpha}e^{\alpha t}+B\beta^{-1}\tilde{\beta}e^{\beta t}.
\end{aligned}$$

So the proof is completed. □

We now give the following interesting identities.

Theorem 2.7. *(Catalan's Identity). For positive integers n and r, with $n\geq r$, the following identity is true:*

$$\mathbb{H}_{n+r}(x)\mathbb{H}_{n-r}(x)-\mathbb{H}_n^2(x)=(-q)^{n-1}AB\left(\tilde{\alpha}\tilde{\beta}\left(\left(\frac{\beta}{\alpha}\right)^r-1\right)+\tilde{\beta}\tilde{\alpha}\left(\left(\frac{\alpha}{\beta}\right)^r-1\right)\right). \tag{2.7}$$

Proof. Using the Equation (2.2), we obtain the LHS of the equality (2.7),

$$\begin{aligned}
\mathbb{H}_{n+r}(x)\mathbb{H}_{n-r}(x)-\mathbb{H}_n^2(x) &= \left(A\alpha^{n-r-1}\tilde{\alpha}+B\beta^{n-r-1}\tilde{\beta}\right)\left(A\alpha^{n+r-1}\tilde{\alpha}+B\beta^{n+r-1}\tilde{\beta}\right)\\
&\quad-\left(A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}\right)^2\\
&= AB\left(\alpha\beta\right)^{n-1}\alpha^{-r}\beta^{r}\tilde{\alpha}\tilde{\beta}+BA\left(\beta\alpha\right)^{n-1}\beta^{-r}\alpha^{r}\tilde{\beta}\tilde{\alpha}\\
&\quad-AB\left(\alpha\beta\right)^{n-1}\tilde{\alpha}\tilde{\beta}-BA\left(\beta\alpha\right)^{n-1}\tilde{\beta}\tilde{\alpha}.
\end{aligned}$$

Then, we have

$$\mathbb{H}_{n+r}(x)\mathbb{H}_{n-r}(x)-\mathbb{H}_n^2(x)=(-q)^{n-1}AB\left(\tilde{\alpha}\tilde{\beta}\left(\left(\frac{\beta}{\alpha}\right)^r-1\right)+\tilde{\beta}\tilde{\alpha}\left(\left(\frac{\alpha}{\beta}\right)^r-1\right)\right).$$

□

Theorem 2.8. *(Cassini's Identity). For $n \geq 1$, the following equality holds:*

$$\mathbb{H}_{n+1}(x)\mathbb{H}_{n-1}(x) - \mathbb{H}_n^2(x) = (-q)^{n-1}AB\left(\tilde{\alpha}\tilde{\beta}\left(\frac{\beta}{\alpha}-1\right)+\tilde{\beta}\tilde{\alpha}\left(\frac{\alpha}{\beta}-1\right)\right). \tag{2.8}$$

Proof. If we take $r=1$, in (2.7), we obtain the assertion of the theorem. □

Theorem 2.9. *(d'Ocagne's Identity) Let $m \geq 0$ and $n \geq 0$ be integers such that $m > n+1$. Then we have*

$$\mathbb{H}_m(x)\mathbb{H}_{n+1}(x) - \mathbb{H}_{m+1}(x)\mathbb{H}_n(x) = \sqrt{\Delta}AB(-q)^{n-1}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha} - \alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right), \tag{2.9}$$

where $\Delta = p^2x^2+4q$.

Proof. By virtue of Equation (2.2), we get

$$\begin{aligned}
\mathbb{H}_m(x)\mathbb{H}_{n+1}(x) - \mathbb{H}_{m+1}(x)\mathbb{H}_n(x) &= \left(A\alpha^{m-1}\tilde{\alpha}+B\beta^{m-1}\tilde{\beta}\right)\left(A\alpha^{n}\tilde{\alpha}+B\beta^{n}\tilde{\beta}\right)\\
&\quad -\left(A\alpha^{m}\tilde{\alpha}+B\beta^{m}\tilde{\beta}\right)\left(A\alpha^{n-1}\tilde{\alpha}+B\beta^{n-1}\tilde{\beta}\right)\\
&= AB\alpha^{m-1}\beta^{n}\tilde{\alpha}\tilde{\beta} - AB\alpha^{m}\beta^{n-1}\tilde{\alpha}\tilde{\beta}\\
&\quad +BA\alpha^{n}\beta^{m-1}\tilde{\beta}\tilde{\alpha} - BA\alpha^{n-1}\beta^{m}\tilde{\beta}\tilde{\alpha}.
\end{aligned}$$

After some calculations, we can easily see that

$$\mathbb{H}_m(x)\mathbb{H}_{n+1}(x) - \mathbb{H}_{m+1}(x)\mathbb{H}_n(x) = \sqrt{\Delta}AB(-q)^{n-1}\left(\beta^{m-n}\tilde{\beta}\tilde{\alpha} - \alpha^{m-n}\tilde{\alpha}\tilde{\beta}\right).$$

□

If we take $a=b=p=q=1$ in (2.7), (2.8) and (2.9), we obtain the Catalan, the Cassini and the d'Ocagne identities for the Fibonacci hybrinomials [24, Theorem 2.4], [24, Corollary 2.6] and [24, Theorem 2.7], respectively. Similarly, if we take $a=2$ and $b=p=q=1$ in (2.7), (2.8) and (2.9), we obtain the Catalan, the Cassini and the d'Ocagne identities for the Lucas hybrinomials [24, Theorem 2.5], [24, Corollary 2.6] and [24, Theorem 2.9], respectively.

Theorem 2.10. *Let $n \geq 2$ be an integer. Then we obtain*

$$\sum_{k=1}^{n-1}\mathbb{H}_k(x) = \frac{\mathbb{H}_1(x)-\mathbb{H}_n(x)+q\left(\mathbb{H}_0(x)-\mathbb{H}_{n-1}(x)\right)}{1-px-q}. \tag{2.10}$$

Proof. By virtue of Equation (2.2), we find that

$$\begin{aligned}
\sum_{k=1}^{n-1}\mathbb{H}_k(x) &= \sum_{k=1}^{n-1}\left(A\alpha^{k-1}\tilde{\alpha}+B\beta^{k-1}\tilde{\beta}\right)\\
&= A\tilde{\alpha}\sum_{k=1}^{n-1}\alpha^{k-1}+B\tilde{\beta}\sum_{k=1}^{n-1}\beta^{k-1}\\
&= A\tilde{\alpha}\left(\frac{1-\alpha^{n-1}}{1-\alpha}\right)+B\tilde{\beta}\left(\frac{1-\beta^{n-1}}{1-\beta}\right)\\
&= \frac{A\tilde{\alpha}(1-\beta)\left(1-\alpha^{n-1}\right)+B\tilde{\beta}(1-\alpha)\left(1-\beta^{n-1}\right)}{1-px-q}.
\end{aligned}$$

Utilizing the last equation, we have

$$\sum_{k=1}^{n-1}\mathbb{H}_k(x) = \frac{\mathbb{H}_1(x)-\mathbb{H}_n(x)+q\left(\mathbb{H}_0(x)-\mathbb{H}_{n-1}(x)\right)}{1-px-q}.$$

□

Corollary 2.11. *([24, Theorem 2.13]) Let $n \geq 2$ be an integer. Then we have*

$$\sum_{k=1}^{n-1}FH_k(x) = \frac{FH_n(x)+FH_{n-1}(x)-FH_0(x)-FH_1(x)}{x}.$$

Proof. If we take $a=b=p=q=1$ in Equation (2.10), the proof is completed. □

Corollary 2.12. *([24, Theorem 2.15]) Let $n \geq 2$ be an integer. Then we have*

$$\sum_{k=1}^{n-1} LH_k(x) = \frac{LH_n(x) + LH_{n-1}(x) - LH_0(x) - LH_1(x)}{x}.$$

Proof. If we take $a = 2$ and $b = p = q = 1$ in Equation (2.10), the proof is completed. □

Theorem 2.13. *For $n \geq 0$, we have*

$$q^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{px}{q}\right)^{n-i} \mathbb{H}_{n-i}(x) = \mathbb{H}_{2n}(x). \tag{2.11}$$

Proof. Because of the Binet formula of the Horadam hybrinomials, we have the LHS of the equality (2.11),

$$\begin{aligned}
& q^n \sum_{i=0}^{n} \binom{n}{i} (px)^{n-i} q^i \left(A\alpha^{n-i-1}\tilde{\alpha} + B\beta^{n-i-1}\tilde{\beta}\right) \\
= \; & A\tilde{\alpha}\alpha^{-1} \sum_{i=0}^{n} \binom{n}{i} (px\alpha)^{n-i} q^i + B\tilde{\beta}\beta^{-1} \sum_{i=0}^{n} \binom{n}{i} (px\beta)^{n-i} q^i \\
= \; & A\tilde{\alpha}\alpha^{-1} (px\alpha + q)^n + B\tilde{\beta}\beta^{-1} (px\beta + q)^n \\
= \; & A\tilde{\alpha}\alpha^{2n-1} + B\tilde{\beta}\beta^{2n-1} \\
= \; & \mathbb{H}_{2n}(x).
\end{aligned}$$

Thus the proof is completed. □

Corollary 2.14. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \binom{n}{i} x^{n-i} FH_{n-i}(x) = FH_{2n}(x).$$

Proof. If we take $a = b = p = q = 1$ in Equation (2.11), the proof is completed. □

Corollary 2.15. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \binom{n}{i} (2x)^{n-i} PH_{n-i}(x) = PH_{2n}(x).$$

Proof. If we take $a = q = 1$ and $b = p = 2$ in Equation (2.11), the proof is completed. □

Corollary 2.16. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} (-1)^n \binom{n}{i} (-6x)^{n-i} BH_{n-i}(x) = BH_{2n}(x).$$

Proof. If we take $a = 1$, $b = p = 6$, and $q = -1$ in Equation (2.11), the proof is completed. □

3. An application of Horadam hybrinomials in matrices

In this section, we derive the matrix representation of the Horadam hybrinomials. Then we obtain closed formula for the Horadam hybrinomials $\mathbb{H}_n(x)$, in terms of tridiagonal determinant (see [26]-[28]).

Theorem 3.1. *Let $n \geq 1$ be an integer. The following equality holds:*

$$\begin{bmatrix} \mathbb{H}_{n+3}(x) & \mathbb{H}_{n+2}(x) \\ \mathbb{H}_{n+2}(x) & \mathbb{H}_{n+1}(x) \end{bmatrix} = \begin{bmatrix} \mathbb{H}_3(x) & \mathbb{H}_2(x) \\ \mathbb{H}_2(x) & \mathbb{H}_1(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix}^n. \tag{3.1}$$

Proof. For the proof, we use induction method on n. The equality holds for $n = 1$. Now suppose that the equality is true for $n > 1$. Then we can verify it for $n+1$ as follows:

$$\begin{aligned}
\begin{bmatrix} \mathbb{H}_3(x) & \mathbb{H}_2(x) \\ \mathbb{H}_2(x) & \mathbb{H}_1(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix}^{n+1} &= \begin{bmatrix} \mathbb{H}_3(x) & \mathbb{H}_2(x) \\ \mathbb{H}_2(x) & \mathbb{H}_1(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix}^{n} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{H}_{n+3}(x) & \mathbb{H}_{n+2}(x) \\ \mathbb{H}_{n+2}(x) & \mathbb{H}_{n+1}(x) \end{bmatrix} \begin{bmatrix} px & 1 \\ q & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{H}_{n+4}(x) & \mathbb{H}_{n+3}(x) \\ \mathbb{H}_{n+3}(x) & \mathbb{H}_{n+2}(x) \end{bmatrix}.
\end{aligned}$$

So the proof is completed. □

Corollary 3.2. *([24, Theorem 2.16]) Let $n \geq 1$ be an integer. The following equality holds:*

$$\begin{bmatrix} FH_{n+3}(x) & FH_{n+2}(x) \\ FH_{n+2}(x) & FH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} FH_3(x) & FH_2(x) \\ FH_2(x) & FH_1(x) \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Proof. If we take $a = b = p = q = 1$ in Equation (3.1), the proof is completed. □

Corollary 3.3. *([24, Theorem 2.17]) Let $n \geq 1$ be an integer. The following equality holds:*

$$\begin{bmatrix} LH_{n+3}(x) & LH_{n+2}(x) \\ LH_{n+2}(x) & LH_{n+1}(x) \end{bmatrix} = \begin{bmatrix} LH_3(x) & LH_2(x) \\ LH_2(x) & LH_1(x) \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Proof. If we take $a = 2$ and $b = p = q = 1$ in Equation (3.1), the proof is completed. □

The n^{th} term of Horadam hybrinomial can be obtained via the computation of the determinant of the tridiagonal matrix $M_{n-1}(x)$.

Proposition 3.4. *The $n \times n$ tridiagonal matrices*

$$M\mathbb{H}_n(x) = \begin{pmatrix} \mathbb{H}_2(x) & \mathbb{H}_1(x) & & & & & \\ -q & px & 1 & & & & \\ & -q & px & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -q & px & 1 \\ & & & & & -q & px \end{pmatrix}, \tag{3.2}$$

satisfy

$$|M\mathbb{H}_n(x)| = \mathbb{H}_{n+1}(x).$$

Corollary 3.5. *The $n \times n$ tridiagonal matrices*

$$MF_n(x) = \begin{pmatrix} FH_2(x) & FH_1(x) & & & & & \\ -1 & x & 1 & & & & \\ & -1 & x & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -1 & x & 1 \\ & & & & & -1 & x \end{pmatrix},$$

satisfy

$$|MF_n(x)| = FH_{n+1}(x).$$

Proof. If we take $a = b = p = q = 1$ in Equation (3.2), the proof is completed. □

Corollary 3.6. *The $n \times n$ tridiagonal matrices*

$$MP_n(x) = \begin{pmatrix} PH_2(x) & PH_1(x) & & & & & \\ -1 & 2x & 1 & & & & \\ & -1 & 2x & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 2x & 1 \\ & & & & & -1 & 2x \end{pmatrix},$$

satisfy

$$|MP_n(x)| = PH_{n+1}(x).$$

Proof. If we take $a = q = 1$ and $b = p = 2$ in Equation (3.2), the proof is completed. □

Corollary 3.7. *The $n \times n$ tridiagonal matrices*

$$MB_n(x)=\begin{pmatrix} BH_2(x) & BH_1(x) & & & & & \\ 1 & 6x & 1 & & & & \\ & 1 & 6x & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 6x & 1 \\ & & & & & 1 & 6x \end{pmatrix},$$

satisfy

$$|MB_n(x)| = BH_{n+1}(x).$$

Proof. If we take $a=1$, $b=p=6$ and $q=-1$ in Equation (3.2), the proof is completed. □

Note that, Horadam hybrinomial can be obtained using the another tridiagonal matrix.

Proposition 3.8. *For $n \geq 1$, we have*

$$\mathbb{H}_n(x)=\begin{vmatrix} \mathbb{H}_1(x) & \mathbb{H}_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & q & 0 & \cdots & 0 & 0 \\ 0 & -1 & px & q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & px & q \\ 0 & 0 & 0 & 0 & \cdots & -1 & px \end{vmatrix}_{n\times n}. \tag{3.3}$$

Corollary 3.9. *For $n \geq 1$, we have*

$$FH_n(x)=\begin{vmatrix} FH_1(x) & FH_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix}_{n\times n}.$$

Proof. This follows from setting $a=b=p=q=1$ in the Equation (3.3). □

Corollary 3.10. *For $n \geq 1$, we have*

$$PH_n(x)=\begin{vmatrix} PH_1(x) & PH_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2x \end{vmatrix}_{n\times n}.$$

Proof. This follows from taking $a=q=1$ and $b=p=2$ in the Equation (3.3). □

Corollary 3.11. *For $n \geq 1$, we have*

$$BH_n(x)=\begin{vmatrix} BH_1(x) & BH_2(x) & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 6x & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 6x & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 6x \end{vmatrix}_{n\times n}.$$

Proof. This follows from setting $a=1$, $b=p=6$, and $q=-1$ in the Equation (3.3). □

Remark 3.12. *This paper is a slightly corrected and revised version of the electronic preprint [29].*

4. Conclusion

In our present research, we have studied Horadam hybrinomials which are defined by dint of the Horadam polynomials. We have obtained some properties of Horadam hybrinomials. Finally in Section 3, with the help of the two different tridiagonal matrix, we have obtained the n^{th} term of Horadam hybrinomials. According to the special cases of a, b, p and q, all the results given in Section 2 and Section 3 are applicable to all hybrinomials and hybrid numbers mentioned in this paper. The Horadam hybrinomials that we have defined include previously introduced the Fibonacci hybrinomials $FH_n(x)$, the Fibonacci hybrid numbers FH_n, the Lucas hybrinomials $LH_{n-1}(x)$, the Lucas hybrid numbers LH_{n-1}, the Pell hybrinomials $PH_n(x)$, the Pell hybrid numbers PH_n, the Pell-Lucas hybrinomials $QH_{n-1}(x)$, the Pell-Lucas hybrid numbers QH_{n-1} (see, [24, 25]). From the definition of the Horadam hybrinomials, we also have obtained the Chebyshev hybrinomials of the first kind $TH_{n-1}(x)$, the Chebyshev hybrid numbers of the first kind TH_{n-1}, the Chebyshev hybrinomials of the second kind $UH_{n-1}(x)$, the Chebyshev hybrid numbers of the second kind UH_{n-1}, the Balancing hybrinomials $BH_n(x)$ and the Balancing hybrid numbers BH_n.

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Tweaking Ramanujan's Approximation of $n!$

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Abstract

About 1730 James Stirling, building on the work of Abraham de Moivre, published what is known as Stirling's approximation of $n!$. He gave a good formula which is asymptotic to $n!$. Since then hundreds of papers have given alternative proofs of his result and improved upon it, including notably by Burside, Gosper, and Mortici. However, Srinivasa Ramanujan gave a remarkably better asymptotic formula. Hirschhorn and Villarino gave nice proof of Ramanujan's result and an error estimate for the approximation. In recent years there have been several improvements of Stirling's formula including by Nemes, Windschitl, and Chen. Here it is shown (i) how all these asymptotic results can be easily verified; (ii) how Hirschhorn and Villarino's argument allows tweaking of Ramanujan's result to give a better approximation; and (iii) that new asymptotic formulae can be obtained by further tweaking of Ramanujan's result. Tables are calculated displaying how good each of these approximations is for n up to one million.

1. Introduction

About 1730 the Scottish mathematician James Stirling (1692–1770), building on the work of the French mathematician Abraham de Moivre (1667–1754), published what is known as Stirling's approximation of $n!$. In fact, Stirling [1] proved that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$; that is, $n!$ is asymptotic to $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. De Moivre had been considering a gambling problem and needed to approximate $\binom{2n}{n}$ for large n. The Stirling approximation gave a very satisfactory solution to this problem.

The problem of extending the factorial from the positive integers to a wider class of numbers was first investigated by the German mathematicians Daniell Bernoulli (1700–1782) and Christian Goldbach (1690–1764) in the 1720s. In 1729 the Swiss polymath Leonhard Euler (1707–1783) succeeded and in 1730 he proved that for z any complex number with positive real part, $\Gamma(z) = \int_0^\infty t^{z-1}e^t\,dt$, where $\Gamma(n) = (n-1)!$, for any positive integer n. The name gamma function is due to the French mathematician Adrien-Marie Legendre (1752–1833).

In 1774 the French polymath Pierre-Simon Laplace (1749–1827) noticed that Stirling's formula for $n!$ has a generalization to the gamma function, namely that for x a positive real number, $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$. One of the most elementary proofs of Stirling's formula for the gamma function was published in 2008 by Reinhard Michel [2].

Most of the proofs in the literature of Stirling's formula and its extensions prove that they are asymptotic by establishing an error estimate such as

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+O\left(x^{-1}\right)\right).$$

In fact most of the effort goes into proving such error estimates.

In this paper our main result says that once one knows that Stirling's formula is asymptotic to $\Gamma(x+1)$, all of the other known asymptotic formulae can be verified trivially without the need to establish any error estimates.

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In 1917 William Burnside [3] published a modest improvement on Stirling's formula, namely $\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}$. How modest an improvement it is can be ascertained from our Table 1. In 1978 Ralph William (Bill) Gosper Jr, [4], published a significant improvement on Stirling and Burnside's formulae. It was that $\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt{2x+\frac{1}{3}}$. In a web post in 2002, Robert H. Windschitl, [5], gave an elegant and good asymptotic approximation of $n!$, namely that $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$. In 2010 Gergő Nemes gave an asymptotic approximation which is almost as good as Windschitl's but better than all the others at that time. It was that $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x$. An asymptotic formula of a different style, which is much better than Gosper's, was published in 2011 by Cristinel Mortici [6]. It was $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}+\frac{1}{12ex}\right)^x$.

Pierre-Simon Laplace discovered what is now known as the Stirling series for the gamma function.

$$\Gamma(x+1) \sim e^{-x}x^{x+\frac{1}{2}}\sqrt{2\pi}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\sum_{n=5}^{\infty}\frac{a_n}{b_n x^n}\right),$$

where the real numbers a_n and b_n are explicitly calculated in [7]. As stated in [8], "the performance deteriorates as the number of terms is increased beyond a certain value." We show how using up to the term x^{-4} in this divergent series compares with the other approximations.

A major advance in producing an asymptotic formula for $n!$ was made by the extraordinary Indian mathematician Srinivasa Ramanujan (1887–1920) in the last year of his life. Ramanujan's claim, recorded in [9, p. 339], was that

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{\theta_x}{30}\right)^{\frac{1}{6}},$$

where $\theta_x \to 1$ as $x \to \infty$ and $\frac{3}{10} < \theta_x < 1$ and he gave numerical evidence for his claim.

Ramanujan's approximation is substantially better than all those which were published in the subsequent 80 years. For example, when $n = 1$ million, the percentage error of Ramanujan's approximation is one million million times better than Gosper's.

In 2013 Michael Hirschhorn and Mark B. Villarino [10] proved the correctness of Ramanujan's claim above for positive integers. They showed that Ramanujan's θ_n satisfies for each positive integer n:

$$1-\frac{11}{8n}+\frac{79}{112n^2} < \theta_n < 1-\frac{11}{8n}+\frac{79}{112n^2}+\frac{20}{33n^3}.$$

Although they did not explicitly say it, it is clear from their work that $\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1-\frac{11}{8x}+\frac{79}{112x^2}}{30}\right)^{\frac{1}{6}}$, at least for positive integers. This approximation, as can be seen in Table 3, is better than all that preceded it. Indeed for $n = 1$ million, it has a percentage error at least one million times better than each one.

In 2016 Chao-Ping Chen [11] produced an asymptotic approximation which for $n = 1$ million has a percentage error one million times better than that of Hirschhorn and Villarino. His asymptotic approximation is

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}.$$

A more detailed analysis of Hirschhorn and Villarino's improvement on that of Ramanujan, suggests a tweaking of their approximation. That tweaking produces an approximation which is stated in Corollary 2.3 and is comparable to Chen's for $n = 1$ to $n = 10,000$ and much better than Chen's for $n = 1$ million, as is evidenced in Table 3.

Chen points out in [11] that Burnside's approximation involves an error of order $O(n^{-1})$, Ramunajan's approximation involves an error of $O(n^{-4})$, Nemes and Windschitl's approximations involves an error of $O(n^{-5})$, and his own approximation involves an error of order $O(n^{-7})$. However caution is needed. Consider the following extreme example:

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}} \left(1+\frac{10^{100}}{n^8}\right) \sim n!$$

and has an error of the order of $O(n^{-7})$ but is an absurdly bad approximation even for $n = 1$ million. The order estimate can be used to compare approximations for "very large" n, but does not tell us how large is "very large".

2. The approximations of $\Gamma(x+1)$

As suggested in §1, once we know Stirling's asymptotic formula for $\Gamma(x+1)$, all of the others follow trivially. This fact is captured in Theorem 2.1 .

Theorem 2.1. *Let f be a function from a subset (a,∞) to $\mathbb{R}$, where $a \in \mathbb{R}, a > 0$. If $\lim_{x\to\infty} f(x) = 1$, then $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x . f(x)$.*

Proof. This follows immediately from the Stirling asymptotic approximation, namely that $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$. □

As an immediate corollary of Theorem 2.1 we obtain that all of the other mentioned approximations are asymptotic to $\Gamma(x+1)$. Some of these were proved by the authors only for x a positive integer.

Corollary 2.2. *For x a positive real number:*

(i) Burnside [3]: $\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}$;

(ii) Gosper [4]: $\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt{2x+\frac{1}{3}}$;

(iii) Mortici [6]: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}+\frac{1}{12ex}\right)^x$;

(iv) Ramanujan [9]: $\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{\frac{1}{6}}$;

(v) Laplace (n): *Fix $n \in \mathbb{N}$. For $a_i, b_i \in \mathbb{N}$,*

$$\Gamma(x+1) \sim e^{-x}x^{x+\frac{1}{2}}\sqrt{2\pi}\left(1+\frac{1}{12x}+\frac{1}{288x^2}+\sum_{i=3}^{n}\frac{a_i}{b_i x^i}\right);$$

(vi) Nemes: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x$.

(vii) Windschitl [5]: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$.

(viii) Hirschhorn & Villarino [10] *:*

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1-\frac{11}{8x}+\frac{79}{112x^2}}{30}\right)^{\frac{1}{6}}.$$

(ix) Chen [11]: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}$.

Proof. In each case it is sufficient to determine the function f in Theorem 2.1 and observe that $\lim_{x\to\infty} f(x) = 1$.

(i) Use $f(x) = \left(1+\frac{1}{2x}\right)^x\left(\frac{1+\frac{1}{2x}}{e}\right)^{\frac{1}{2}}$.

(ii) Use $f(x) = \sqrt{1+\frac{1}{6x}}$.

(iii) Use $f(x) = \left(1+\frac{1}{12x^2}\right)^x$.

(iv) Use $f(x) = \left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}\right)^{\frac{1}{6}}$.

(v) Use $f(x) = \left(1+\frac{1}{12x}+\frac{1}{288x^2}+\sum_{i=3}^{n}\frac{a_i}{b_i x^i}\right)$.

(vi) Use $f(x) = \left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x$.

(vii) Use $f(x) = \left(x\sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$.

(viii) Use $f(x) = \left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1-\frac{11}{8x}+\frac{79}{112x^2}}{240x^3}\right)^{\frac{1}{6}}$.

(ix) Use $f(x)=\left(1+\dfrac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}$.

$\square$

With this result in hand we see than one can easily tweak any of the known asymptotic approximation to get others and this tweaking can be done to optimize the approximation for any value of n decided in advance. In the spirit of Ramanujan I will not include the details as they are not particularly important.
We can tweak Ramanujan's approximation again to get an even better approximation for $n=1$ million which we refer to in Table 2 as the SAM approximation. The proof of the corollary uses an obvious modification of the proof of (ix) above.

Corollary 2.3. *For x a positive real number,*

$$\Gamma(x+1)\sim\sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1-\dfrac{11}{8x}+\dfrac{79}{112x^2}+\dfrac{A}{x^3}}{30}\right)^{\frac{1}{6}},$$

where $A=\dfrac{380279456577}{722091376690}$. $\square$

3. Numerical analysis of the approximations

The tables in this section were calculated using the WolframAlpha software package. (See https://www.wolframalpha.com/.) They demonstrate the performance of the asymptotic approximations.

Each of the approximations gets further and further from $n!$ as n tends to infinity. So the quality of the approximations is best judged by considering the percentage error, that is $100\times\dfrac{\text{approximation}-n!}{n!}$.

In the tables S = Stirling, B = Burnside, G = Gosper, L4 = (Laplace) Stirling series up to x^{-4}, M = Mortici, N=Nemes, W=Windschitl, R = Ramanujan, HV = Hirschhorn and Villarino, C = Chen, and SAM = the author of this paper.

From the tables it is abundantly clear that Gosper's approximation is a much better approximation than Stirling's, and Mortici's elegant approximation is closer in accuracy to Ramanujan's. Ramanujan's approximation is amazingly good. The tweaking of Ramanujan's approximation using the Hirschhorn-Villarino results significantly improves the approximation. Chen's approximation is better than all that precede it. The SAM approximation obtained by extra tweaking of Ramanujan's approximation produces an approximation similar to Chen's up to $n=10{,}000$ and much better for $n=1{,}000{,}000$.

Table 1

n	$n!$	S %error	B %error	G %error
2	2	4.0	1.7	1.3×10^{-1}
5	1.2×10^{2}	1.7	7.6×10^{-1}	2.5×10^{-2}
10	3.6×10^{6}	8.3×10^{-1}	4.0×10^{-1}	6.6×10^{-3}
20	2.4×10^{18}	4.2×10^{-1}	2.0×10^{-1}	1.7×10^{-3}
50	3.0×10^{64}	1.7×10^{-1}	8.3×10^{-2}	2.7×10^{-4}
100	9.3×10^{157}	8.3×10^{-1}	4.1×10^{-2}	6.9×10^{-5}
10^3	4.0×10^{2567}	8.3×10^{-3}	4.2×10^{-3}	6.9×10^{-7}
10^4	2.8×10^{35659}	8.3×10^{-4}	4.2×10^{-4}	6.9×10^{-9}
10^6	8.3×10^{5565708}	8.3×10^{-6}	4.2×10^{-6}	6.9×10^{-13}

Table 2

n	$n!$	M %error	R % error	L4 %error	N %error
2	2	1.0×10^{-2}	3.3×10^{-3}	1.4×10^{-2}	1.7×10^{-3}
5	1.2×10^{2}	5.7×10^{-4}	1.2×10^{-4}	3.5×10^{-4}	2.0×10^{-5}
10	3.6×10^{6}	7.0×10^{-5}	8.6×10^{-6}	7.8×10^{-7}	6.5×10^{-7}
20	2.4×10^{18}	8.7×10^{-6}	5.7×10^{-7}	2.4×10^{-8}	2.0×10^{-8}
50	3.0×10^{64}	5.6×10^{-7}	1.5×10^{-8}	2.5×10^{-10}	2.1×10^{-10}
100	9.3×10^{157}	6.9×10^{-8}	9.5×10^{-10}	7.8×10^{-12}	6.5×10^{12}
10^3	4.0×10^{2567}	6.9×10^{-11}	9.5×10^{-14}	7.8×10^{-17}	6.5×10^{-17}
10^4	2.8×10^{35659}	6.9×10^{-14}	9.5×10^{-18}	7.8×10^{-22}	6.5×10^{-22}
10^6	8.3×10^{5565708}	6.9×10^{-20}	9.5×10^{-26}	7.8×10^{-32}	6.5×10^{-32}

Table 3

n	$n!$	W % error	HV %error	C % error	SAM %error
2	2	1.6×10^{-3}	1.6×10^{-4}	2.2×10^{-4}	2.9×10^{-4}
5	1.2×10^{2}	1.9×10^{-5}	1.5×10^{-6}	5.0×10^{-7}	6.0×10^{-7}
10	3.6×10^{6}	6.1×10^{-7}	3.0×10^{-8}	4.1×10^{-9}	4.9×10^{-9}
20	2.4×10^{18}	1.9×10^{-8}	5.2×10^{-10}	3.2×10^{-11}	3.8×10^{-11}
50	3.0×10^{64}	2.1×10^{-10}	2.3×10^{-12}	5.3×10^{-14}	6.3×10^{-14}
100	9.3×10^{157}	6.2×10^{-12}	3.6×10^{-14}	4.2×10^{-16}	4.9×10^{-16}
10^3	4.0×10^{2567}	6.2×10^{-17}	3.7×10^{-20}	4.17×10^{-23}	4.9×10^{-23}
10^4	2.8×10^{35659}	6.2×10^{-22}	3.7×10^{-26}	4.2×10^{-30}	4.9×10^{-30}
10^6	8.3×10^{5565708}	6.2×10^{-32}	3.7×10^{-38}	4.2×10^{-44}	1.3×10^{-50}

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On Hyperbolic Jacobsthal-Lucas Sequence

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Abstract

In this study, we define the hyperbolic Jacobsthal-Lucas numbers and we obtain recurrence relations, Binet's formula, generating function and the summation formulas for these numbers.

1. Introduction and preliminaries

In this study, we introduce hyperbolic Jacobsthal-Lucas numbers and give some properties of them. Firstly, we present some background information about hyperbolic numbers and Jacobsthal-Lucas numbers. One can see [1]-[8] for details. Jacobsthal-Lucas sequence J_n is defined by the second-order recurence relation

$$J_{n+2} = J_{n+1} + 2J_n$$

with initial values $J_0 = 2, J_1 = 1$. The first few terms of this sequence are given as follows:

$$2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, \ldots$$

Binet's formula and generating function of Jacobsthal-Lucas sequence are given by

$$J_n = 2^n + (-1)^n$$

and

$$\sum_{n=0}^{\infty} J_n x^n = \frac{2 - x^2}{1 - x - 2x^2}$$

respectively.
The set of hyperbolic numbers H can be described as

$$H = \{z = x + hy : h \notin R, h^2 = 1, x, y \in R\}.$$

Addition, substruction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2),$$
$$z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h(x_1y_2 + y_1x_2),$$

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and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2}=\frac{x_1+hy_1}{x_2+hy_2}=\frac{(x_1+hy_1)(x_2-hy_2)}{(x_2+hy_2)(x_2-hy_2)}=\frac{x_1x_2+y_1y_2}{x_2^2-y_2^2}+h\frac{x_1y_2+y_1x_2}{x_2^2-y_2^2}.$$

The hyperbolic conjugation of $z=x+hy$ is defined by

$$\bar{z}=x-hy.$$

For more information and properties related hyperbolic numbers, see [9]-[18].

2. Hyperbolic Jacobsthal-Lucas sequence

In [14], author presented hyperbolic Fibonacci sequence and examined its properties. In this study, we define hyperbolic Jacobsthal-Lucas sequence and examined some of its properties.
The hyperbolic Jacobsthal-Lucas numbers are defined by

$$HJ_n=J_n+hJ_{n+1}$$

with initial conditions $HJ_0=2+h$, $HJ_1=1+5h$ where $h^2=1$. Then the first few terms of hyperbolic Jacobsthal-Lucas numbers are

$$2+h,1+5h,5+7h,7+17h,17+31h,31+65h,65+127h,...,$$

It can be easily shown that

$$HJ_n=HJ_{n-1}+2HJ_{n-2}.$$

In fact, by using the definition of the hyperbolic Jacobsthal-Lucas numbers, we have

$$\begin{aligned}HJ_n=J_n+hJ_{n+1}&=J_{n-1}+2J_{n-2}+h(J_n+2J_{n-1})\\&=2J_{n-2}+h2J_{n-1}+J_{n-1}+hJ_n\\&=HJ_{n-1}+2HJ_{n-2}.\end{aligned}$$

Theorem 2.1. *Let HJ_n be $n-th$ hyperbolic Jacobsthal-Lucas number, then we obtain*

$$\lim_{x\to\infty}\frac{HJ_{n+1}}{HJ_n}=2.$$

Proof. We have

$$\lim_{x\to\infty}\frac{J_{n+1}}{J_n}=2.$$

for the Jacobsthal-Lucas sequence J_n. Then using this value for the hyperbolic Jacobsthal-Lucas HJ_n, we get

$$\begin{aligned}\lim_{x\to\infty}\frac{HJ_{n+1}}{HJ_n}&=\lim_{x\to\infty}\frac{J_{n+1}+hJ_{n+2}}{J_n+hJ_{n+1}}\\&=\lim_{x\to\infty}\frac{J_{n+1}+h(J_{n+1}+2HJ_n)}{J_n+hJ_{n+1}}\\&=\lim_{x\to\infty}\frac{(\frac{J_{n+1}}{J_n})+h((\frac{J_{n+1}}{J_n})+2)}{1+(h\frac{J_{n+1}}{J_n})}\\&=\frac{2+4h}{1+2h}=2.\end{aligned}$$

□

Theorem 2.2. *The Binet formula for the hyperbolic Jacobsthal-Lucas numbers is given by*

$$HJ_n=(1+2h)2^n+(1-h)(-1)^n. \tag{2.1}$$

Proof. By using the Binet formula of the Jacobsthal-Lucas numbers

$$J_n=2^n+(-1)^n,$$

we get

$$\begin{aligned}HJ_n&=J_n+hJ_{n+1}\\&=2^n+(-1)^n+h(2^{n+1}+(-1)^{n+1})\\&=(1+2h)2^n+(1-h)(-1)^n.\end{aligned}$$

□

Theorem 2.3. *The generating function for the hyperbolic Jacobsthal-Lucas sequence is given by*

$$\sum_{n=0}^{\infty} HJ_n x^n = \frac{2+h+(1-4h)x}{1-x-2x^2}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} HJ_n x^n$$

be generating function of hyperbolic Jacobsthal-Lucas numbers. Then we have the following equations:

$$\begin{aligned} g(x) &= HJ_0 + HJ_1 x + HJ_2 x^2 + HJ_3 x^3 + HJ_4 x^4 + \dots \\ -xg(x) &= -HJ_0 x - HJ_1 x^2 - HJ_2 x^3 - HJ_3 x^4 - HJ_4 x^5 - \dots \\ -2x^2 g(x) &= -2HJ_0 x^2 - 2HJ_1 x^3 - 2HJ_2 x^4 - 2HJ_3 x^5 - 2HJ_4 x^6 - \dots \\ (1-x-2x^2)g(x) &= HJ_0 + (HJ_1 - HJ_0)x. \end{aligned}$$

By rewriting the last equation, we get

$$g(x) = \frac{2+4h+(1-4h)x}{1-x-2x^2}$$

with $HJ_0 = 2+h$, $HJ_1 = 1+5h$. □

Theorem 2.4. *(Catalan's identity) The following identitiy holds for all natural numbers n and m:*

$$HJ_{n+m}HJ_{n-m} - HJ_n^2 = (-1+h)[(-2)^{n+m} + (-2)^{n-m} + (-2)^{n+1}].$$

Proof. By using the formula (2.1), we obtain

$$\begin{aligned} HJ_{n+m}HJ_{n-m} - HJ_n^2 &= \left((1+2h)\,2^{n+m} + (1-h)\,(-1)^{n+m}\right)\left((1+2h)\,2^{n-m} + (1-h)\,(-1)^{n-m}\right) \\ &\quad -((1+2h)2^n + (1-h)(-1)^n)^2 \\ &= ((5+4h)2^{2n} + (2-2h)(-1)^{2n} + (-1+h)2^n(-1)^n[2^m(-1)^{-m} + 2^{-m}(-1)^m]) \\ &\quad -((5+4h)2^{2n} + (2-2h)(-1)^{2n} + 2(-1+h)2^n(-1)^n) \\ &= (-1+h)[(-2)^{n+m} + (-2)^{n-m} + (-2)^{n+1}]. \end{aligned}$$

□

Theorem 2.5. *(d'Ocagne's identity) The following identitiy holds for any integers n and m:*

$$HJ_{m+1}HJ_n - HJ_m HJ_{n+1} = 3(-1+h)[(-2)^m(-1)^n - (-2)^n(-1)^m].$$

Proof. By the Binet formula (2.1), we get

$$\begin{aligned} HJ_{m+1}HJ_n - HJ_m HJ_{n+1} &= ((1+2h)2^{m+1} + (1-h)(-1)^{m+1})((1+2h)2^n + (1-h)(-1)^n) \\ &\quad -((1+2h)2^m + (1-h)(-1)^m)((1+2h)2^{n+1} + (1-h)(-1)^{n+1}) \\ &= 3(-1+h)[(-2)^m(-1)^n - (-2)^n(-1)^m]. \end{aligned}$$

□

Theorem 2.6. *(Gelin-Cesaro's identity) The following identitiy holds for any integers n and m:*

$$HJ_{n+2}HJ_{n+1}HJ_{n-1}HJ_{n-2} - HJ_n^4 = \frac{9}{8}(-1+h)(-2)^n[(2)^{2n+1} - 13(1-h)(-2)^n + 4(1-h)].$$

Proof. Using

$$\begin{aligned} HJ_n &= (1+2h)2^n + (1-h)(-1)^n, \\ HJ_n &= (1+2h)[2^n + (-1+h)(-1)^n], \end{aligned}$$

and by setting $a = 2^n, b = (-1+h)(-1)^n$ we obtain following values:

1.$HJ_{n+2} = (1+2h)[4a+b]$

2.$HJ_{n+1} = (1+2h)[2a-b]$

3.$HJ_{n-1} = (1+2h)[\frac{a}{2} - b]$

$4.HJ_{n-2} = (1+2h)[\frac{a}{4}+b]$

from the above values, we can easily calculate

$$\begin{aligned} HJ_{n+2}HJ_{n+1}HJ_{n--1}HJ_{n-2}-HJ_n^4 &= (1+2h)^4[(8a^2-2ab-b^2)(\tfrac{a^2}{8}+\tfrac{ab}{4}-b^2)-(a^4+b^4+4a^3b+6a^2b^2+4ab^3)] \\ &= \tfrac{9}{8}(-1+h)(-2)^n[(2)^{2n+1}-13(1-h)(-2)^n+4(1-h)]. \end{aligned}$$

□

Theorem 2.7. *(Melham's identity) The following identity holds for any integers n and m:*

$$HJ_{n+1}HJ_{n+2}HJ_{n+6}-HJ_{n+3}^3 = 9(1-h)(-2)^n[2^{n+3}+10(1-h)(-1)^n].$$

Proof. Using

$$\begin{aligned} HJ_n &= (1+2h)2^n+(1-h)(-1)^n, \\ HJ_n &= (1+2h)[2^n+(-1+h)(-1)^n], \end{aligned}$$

and by setting $a = 2^n, b = (-1+h)(-1)^n$ we obtain following values:

$1.HJ_{n+1} = (1+2h)[2a-b]$,

$2.HJ_{n+2} = (1+2h)[4a+b]$,

$3.HJ_{n+6} = (1+2h)[64a+b]$,

$4.HJ_{n+3} = (1+2h)[8a-b]$.

From the above values, we can easily calculate

$$\begin{aligned} HJ_{n+1}HJ_{n+2}HJ_{n+6}-HJ_{n+3}^3 &= (1+2h)^3[(8a^2-2ab-b^2)(64a+b)-(8a-b)^3] \\ &= (1+2h)^3 9ab[8a-10b] \\ &= 9(1-h)(-2)^n[2^{n+3}+10(1-h)(-1)^n]. \end{aligned}$$

□

Theorem 2.8. *For $n \geq 0$, we obtain*

$$\sum_{k=0}^{n} HJ_k = \frac{1}{2}(HJ_{n+2}-(1+5h)).$$

Proof. We use the mathematical induction on n. For $n = 0$, we have

$$HJ_0 = \frac{1}{2}[HJ_2-(1+5h)] = \frac{1}{2}[5+7h-1-5h] = 2+h.$$

Now assume that it is true for $n = k$, namelyand by setting

$$\sum_{k=0}^{k} HJ_k = \frac{1}{2}(HJ_{k+2}-(1+5h)).$$

From the induction hypothesis, we obtain

$$\begin{aligned} \sum_{k=0}^{k+1} HJ_k &= \frac{1}{2}(HJ_{k+2}-(1+5h))+HJ_{k+1} \\ &= \frac{1}{2}(HJ_{k+2}-(1+5h)+2HJ_{k+1}) \\ &= \frac{1}{2}(HJ_{k+3}-(1+5h)). \end{aligned}$$

□

3. Conclusion

The hyperbolic Jacobsthal-Lucas numbers with initial conditions $HJ_0 = 2+h, HJ_1 = +5h$ are defined by

$$HJ_n = J_n + hJ_{n+1}$$

where $h^2 = 1$.
In this paper, we give the hyperbolic Jacobsthal Lucas numbers and present some recurrence relations, Binet's formula, generating function and some special idetities for these numbers.

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Construction of Networks by Associating with Submanifolds of Almost Hermitian Manifolds

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Abstract

The idea that data lies in a non-linear space has brought up the concept of manifold learning as a part of machine learning and such notion is one of the most important research fields of today. The main idea here is to design the data as a submanifold model embedded in a high-dimensional manifold. On the other hand, graph theory is one of the most important research areas of applied mathematics and computer science. As a result, many researchers investigate new methods for machine learning on graphs. From the above information, it is seen that the theory of submanifolds and graph theory have become two important concepts in machine learning and nowadays, the geometric deep learning research area using these two concepts has emerged. By combining these two fields, this article aims to present the relationships between submanifolds of complex manifolds with the help of graphs. In this paper, we build some directed networks by identifying with submanifolds of almost Hermitian manifolds. Moreover, we give some results and relations among holomorphic submanifolds, totally real submanifolds, CR-submanifolds, slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds, and bi-slant submanifolds in almost Hermitian manifolds in terms of graph theory.

1. Introduction

Graph theory can be used to model computer networks, social networks, communications networks, information networks, software design, transportation networks, biological networks, etc. So this theory is applicable in many real-world mathematical modelling. Therefore, this theory is the most active area of mathematical research.

On the other hand, one of the most active research areas of differential geometry is the submanifold theory of complex manifolds. A submanifold of an almost Hermitian manifold is characterized by the behavior of tangent space of the submanifold of almost Hermitian manifold under the complex structure of the ambient manifold. In this way, we have various submanifolds titled as holomorphic, totally real, CR, slant, semi-slant, hemi-slant, bi-slant for almost Hermitian manifolds. In fact, the theory of submanifolds of almost Hermitian manifolds is still the main active area of complex differential geometry, see: [1]-[8] for recent results.

Manifold learning method is one of the most exciting developments in machine learning recently. Manifold learning has been applied in utilizing semi-supervised learning [9]. Moreover, manifolds also play an important role in public health. Fiorini has defined the Riemannian manifold, which is isomorphic to traditional information geometry Riemannian manifold, for noise reduction in theoretical computerized tomography providing many competitive computational advantages over the traditional Euclidean approach [10]. Besides, Monti et al. have introduced a general framework, geometric deep learning, enabling the design of convolutional deep architectures on manifolds and graphs [11]. Moreover, Shahzad et al. have simplified the complex chemical reaction by reducing it from a high dimension to the low by applying three well-established techniques based on

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manifolds [12], and they have investigated the different completion routes of reaction and overall reaction for dehydrogenation of butane to further extend towards the surfaces using the slow invariant manifold comparison [13].
Also, Carriazo and Fernandez [14] have constructed a relation between slant surface and graph theory. Later, they have related graph theory with vector spaces of even dimension [15, 16]. Their work was restricted to slant submanifolds. We believe that further use of graph theory is possible in the theory of submanifolds.
By considering vast literature of graph theory and submanifold theory, one expects more relations between these research areas. In this direction, the aim of this paper is to examine the relation among various submanifolds of almost Hermitian manifolds by using graph theory. We note that our approach is different from the approach considered in [14] and [16]. They only considered adapted frames of slant surface and they used them to characterize CR-submanifolds by means of trees. Later they have extended this approach for weakly associated graphs. In this paper, we give relations between submanifolds of Hermitian manifolds in terms of graph theory notions.

2. Preliminaries

In this section, we are going to recall certain notions used in graph theory to be used in this paper from [17]-[25]. For those who are not familiar with the theory of graphs (especially for readers working with the submanifolds theory), we specifically recall the basic definitions from graph theory.
A graph $G=(V,E)$ consists of a nonempty set V of vertices and a set E of edges. Each edge has either one or two vertices connected with it, called its endpoints. An edge connects its endpoints. Two distinct vertices u,v in a graph G are called adjacent (or neighbors) in G if there is an edge e between u and v, where the edge e is called incident with the vertices u and v and e connects u and v. The set of all neighbors of a vertex v of $G=(V,E)$ is denoted by $N(v)$. If $A\subset V$, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. The degree of a vertex in a graph is the number of edges incident with it. The degree of the vertex v is denoted by $d(v)$ and $d(v)=|N(v)|$. The graph theory can be divided into two branches as undirected and directed graphs [24].
A directed graph (digraph) D is a finite nonempty set of objects called vertices together with a set of ordered pairs of distinct vertices of D called directed edges or arcs. For a digraph $D=(V,A)$, the vertex set of D is denoted by $V(D)$ or simply V and the arc set of D is denoted by $A(D)$ or A. Each arc is an ordered pair of vertices. The arc (u,v) is said to start at u and end at v. The in-degree of a vertex v, $d^-(v)$, is the number of edges which end at v. The out-degree of v, $d^+(v)$, is the number of edges with v as their initial vertex. Also, for a vertex $v\in V(D)$, $N_D^-(v)$ and $N_D^+(v)$ are respectively called out-neighbors and in-neighbors where $N_D^-(v)=\{u|(u,v)\in A(D),u\in V(D)\}$ and $N_D^+(v)=\{u|(v,u)\in A(D),u\in V(D)\}$ [19, 22, 24, 26].
In a digraph $D=(V,A)$, given a pair of vertices u and v, whether or not there is a path from u to v in the digraph is useful to know. The transitive closure of D is to construct a new digraph, $D^*=(V,A^*)$, such that there is an arc (u,v) in D^* if and only if there is a path from u to v in D [23].
A walk $W=x_1a_1x_2a_2x_3...x_{k-1}a_{k-1}x_k$ is a sequence of vertices x_i and arcs a_j in D such that the tail and head of a_i is x_i and x_{i+1} for $\forall i<k$, respectively. The set of vertices and arcs of the walk W are denoted $V(W)$ and $A(W)$, respectively. W is denoted without arcs as $x_1x_2...x_k$ and shortly (x_1,x_k)-walk. If $x_1=x_k$ then W is a closed walk, and otherwise w is an open walk. If W is an open walk, the vertices x_1 and x_k are end-vertices and named as the initial and the terminal vertex of W, respectively. The length of a walk is the number of its arcs and the walk W above has length $k-1$ [19].
A trail is a walk in which all arcs are distinct. W is called a path, if the vertices of a trail $V(W)\subset V(D)$ are distinct. If the vertices $x_1,x_2,...,x_{k-1}$ are distinct, $k\geq 3$ and $x_1=x_k$, then W is a cycle. The longest path in D is a path of maximum length in D [19].

Proposition 2.1. *[19] Let D be a digraph and let x,y be a pair of distinct vertices in D. If D has an (x,y)-walk W, then D contains an (x,y)-path P such that $A(P)\subseteq A(W)$. If D has a closed $(x,x)-$walk W, then D contains a cycle C through x such that $A(C)\subseteq A(W)$.*

An oriented graph is a digraph with no cycle of length two [19]. For a digraph D, the underlying graph of D is the undirected graph engendered utilizing all vertices in $V(D)$, and superseding all of the arcs in $A(D)$ with undirected edges [21].
If a digraph D has an (x,y)-walk, then the vertex y is reachable from the vertex x. Every vertex is reachable from itself specifically. By Proposition 2.1, y is reachable from x if and only if D contains an (x,y)-path. If every pair of vertices in digraph D is mutually reachable then D is strongly connected (or shortly strong). A strong component of digraph D is a maximal induced strong subdigraph in D. If $D_1,...,D_t$ are the strong components of D, then precisely $V(D_1)\cup...\cup V(D_t)=V(D)$. If a digraph D is not strongly connected and if the underlying graph of D is connected, then D is said to be weakly connected [19, 26].
Pseudograph is a graph having parallel edges and loops, and multigraph is a pseudograph with no loops. If every pair of distinct vertices are adjacent in a multigraph then the multigraph is complete.
A multigraph H is called as $p-$partite if there is a partition into p sets $V(H)=V_1\cup V_2\cup...\cup V_p$ where $V_i\cap V_j=\emptyset$ for every $i\neq j$. In particular, when $p=2$ the graph is called a bipartite graph. A bipartite graph B is denoted by $B=(V_1,V_2;E)$. If the edge (x,y) is in $p-$partite multigraph H where all $x\in V_i$, $y\in V_j$ for $i\neq j$ then H is complete $p-$partite [19].
A digraph $D=(V,A)$ is symmetric if arc $(x,y)\in A$ implies arc $(y,x)\in A$. A matching M is an arc set having no common end-vertices and loops in D. Also, the arcs of M are independent if M is a matching. If a matching M implicates the highest number of arcs in D, then M is maximum. Besides, a maximum matching is perfect if it has $\frac{|A(D)|}{2}$ arcs. A set Q of vertices in

a directed pseudograph H is independent if there are no arcs between vertices in Q. The independence number of H is the size of the independent set having maximum cardinality in H. A coloring of a digraph H is a partition of $V(H)$ into disjoint independent sets. The minimum number of independent sets in the coloring of H is the chromatic number of H. A simple directed graph is a digraph that has no multiple arcs or loops. If a digraph contains no cycle, then it is acyclic and called acyclic digraph [19].
The eccentricity $e(v)$ of a vertex v is the distance from v to the farthest vertex from itself. The radius (rad) of D is the minimum eccentricity, and the diameter ($diam$) is the maximum eccentricity. Besides, a vertex v is central if $e(v) = rad(D)$, and v is peripheral if $e(v) = diam(D)$ [27].
Let $D = (V,A)$ be a digraph, $V(D) = n$ and $S \subset V(D)$. S is a dominating set of D if each vertex $v \in V(D) - S$ is dominated by at least a vertex in S. A dominating set of D having the smallest cardinality is called the minimum dominating set of D. Also, the cardinality of the minimum dominating set is called the domination number of D [28, 29].
Let r be a root vertex in D. A directed spanning tree T starting from r is a subdigraph of D such that the undirected form of T is a tree and there is a directed unique (r,v)-path in T for each $v \in V(T) - r$ [19].
The vertex-integrity of a digraph D is defined by $I(D) = min\{|F| + m(D-F) : F \subseteq V(D)\}$, where $m(D-F)$ indicates the maximum order of a strong component of $D-F$. If $I(D) = |F| + m(D-F)$ then F is called as an I-set of D. In addition, the arc-integrity of a digraph D, shortly $I^{'}(D)$, is described as the minimum value of $\{|F| + m(D-F) : F \subseteq A(D)\}$. The set F is called as an $I^{'}$-set of D if $I^{'}(D) = |F| + m(D-F)$ [30].

Proposition 2.2. *[30] If S is a subdigraph of D then $I(S) \leq I(D)$ and $I^{'}(S) \leq I^{'}(D)$.*

3. Networks built among submanifolds of almost Hermitian manifolds

Let (M,g) be a Riemannian manifold. (M,g) is called an almost Hermitian manifold if there is a (1,1) tensor field on M such that $J^2 = -I$, where I is the identity map on the tangent bundle of M, and $g(JX,JY) = g(X,Y)$ for vector fields X,Y on M. Moreover, if J is parallel with respect to any vector field X, then (M,J,g) is called a Kaehler manifold [31]. There are various submanifolds of an almost Hermitian manifold based on the behavior of the tangent space of the submanifold at a point under the almost complex structure J. Let N be a submanifold of an almost Hermitian manifold and T_pN the tangent space at a point p belongs to N. Then, if T_pN is invariant with respect to J_p for any point p, then N is called holomorphic (or complex) submanifold [31]. We denote the normal space at p by $T_pN^{\perp}$. A submanifold of an almost Hermitian manifold is called an anti-invariant submanifold if $JT_pN \subseteq T_pN^{\perp}$ [31]. As a generalization of holomorphic submanifold and anti-invariant submanifolds, a submanifold M of a Kaehler manifold N is called CR-submanifold [32] if there are two orthogonal complementary distributions $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $\mathscr{D}_1$ is invariant with respect to J and $\mathscr{D}_2$ is anti-invariant with respect to J for every point $p \in M$. It is clear that if $\mathscr{D}_1 = \{0\}$, then a CR-submanifold becomes an anti-invariant submanifold. If $\mathscr{D}_2 = \{0\}$, then M becomes a holomorphic submanifold. Another generalization of holomorphic submanifolds and anti-anti-invariant submanifolds is slant submanifolds. Let N be a submanifold of an almost Hermitian manifold M. The submanifold N is called slant [33] if for each non-zero vector X tangent to N the angle $\theta(X)$ between JX and T_pN is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pN$. θ is called the slant angle. It is clear that if $\theta(X) = 0$ then N becomes a holomorphic submanifold. If $\theta(X) = \pi/2$, N becomes an anti-invariant submanifold. We will use the v_1, v_2, v_3, and v_4 to represent the submanifolds holomorphic, CR, anti-invariant and slant, respectively.
Digraph $D_1 = (V,A)$ has four vertices, $V(D_1) = \{v_1, v_2, v_3, v_4\}$, and four arcs, $A(D_1) = \{(v_2,v_1),(v_2,v_3),(v_4,v_1),(v_4,v_3)\}$ in Fig. 3.1. D_1 has the maximum length of one as the longest path. D_1 has 2 vertices (v_2 and v_4) which are not reachable. Topological sort of D_1 is $v_4 - v_2 - v_3 - v_1$. $rad(D_1) = 1$, the radius of D_1 is $v_2 \rightarrow v_1$. $diam(D_1) = 1$, the diameter of D_1 is the same as the radius. Also, in D_1, there is no center vertex, but two peripheral vertices such as v_2 and v_4.

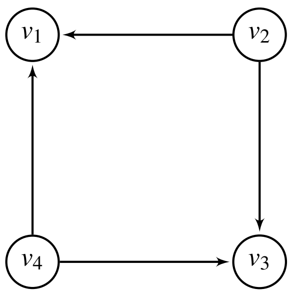

Figure 3.1: Digraph D_1 built by submanifolds holomorphic, CR, anti-invariant and slant

Theorem 3.1. *Let D_1 be a digraph constructed by the four submanifolds holomorphic, CR, anti-invariant and slant considering as the vertices v_1, v_2, v_3, and v_4, respectively. Then D_1 holds the following properties:*

1. *D_1 is a bipartite digraph as well as a complete bipartite digraph.*
2. *D_1 has a perfect matching.*

3. *The independence number of D_1 is 2.*
4. *The chromatic number of D_1 is 2.*
5. *D_1 has no directed spanning tree.*
6. *The domination number of D_1 is 2.*

Proof. 1. There exists a partition V_1 and V_2 of $V(D_1)$ into two partite sets for the submanifolds in D_1: $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$. Owing to $V(D_1) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, then D_1 is a bipartite digraph.
Besides, for every submanifold, $x \in V_1, y \in V_2$, a connection from x to y (i.e. an arc (x,y)) is in D_1. Therefore, D_1 is a complete bipartite digraph.
2. There is a matching $M = \{(v_2, v_1), (v_4, v_3)\} \subset A(D_1)$ in D_1. Each element (arc or connection between two submanifolds) in M is independent, i.e. no common vertices, and M is maximum. Also, M is perfect so that $|M| = \frac{|A(D_1)|}{2}$. It is obvious that D_1 has a perfect matching.
3. The subset $\widetilde{V} = \{v_2, v_4\} \subset V(D_1)$ is one of the independent sets having maximum cardinality and the size of maximum independent submanifolds set is 2. This also means that there is no relation between submanifolds v_2 and v_4. Then, the independence number of D_1 is 2.
4. $V_1 = \{v_2, v_4\}$ and $V_2 = \{v_1, v_3\}$ are two subsets of $V(D_1)$. $V_i (i = 1, 2)$ are all independent sets providing the minimum number of cardinality at the same time. Hence, the minimum number of independent sets of D_1 is 2. Then, the chromatic number of D_1 is 2.
5. There is no root vertex where a subdigraph T of D_1 contains a directed path from the root to any other vertex in $V(D_1)$. Then, D_1 has no directed spanning tree.
6. There is a subset $\widetilde{V} = \{v_2, v_4\} \subset V(D_1)$ that including minimum cardinality of vertices in D_1. Considering this subset, for each vertex $v \in \widetilde{V}$ and $u \in V(D_1) - \widetilde{V}$, (v,u) is an arc in D_1. The domination number is 2, because of no smaller cardinality of dominating sets in D_1.

□

Corollary 3.2. *In the submanifold network represented by D_1 in Fig. 3.1, the submanifolds, CR (v_2) and slant (v_4), cannot be derived by the other submanifolds, because the in-degree of these vertices (submanifolds) are zero in D_1, $d^-(v_2) = d^-(v_4) = 0$. In addition, whereas CR and slant subamnifolds cannot be mutually derived as between holomorphic (v_1) and anti-invariant (v_3), holomorphic and anti-invariant submanifolds can be derived separately from CR and slant from $N^-_{D_1}(v_1) = N^-_{D_1}(v_3) = \{v_2, v_4\}$.*

We now recall the notion of hemi-slant submanifolds of an almost Hermitian manifold. Let M be an almost Hermitian manifold and N a real submanifold of M. Then we say that N is a hemi-slant submanifold [34]-[37] if there exist two orthogonal distributions $\mathscr{D}^\perp$ and $\mathscr{D}^\theta$ on N such that

1. TN admits the orthogonal direct decomposition $TN = \mathscr{D}^\perp \oplus \mathscr{D}^\theta$.
2. The distribution $\mathscr{D}^\perp$ is an anti-invariant distribution, i.e., $J\mathscr{D}^\perp \subset TM^\perp$.
3. The distribution $\mathscr{D}^\theta$ is slant with slant angle θ.

It is easy to see that if $\mathscr{D}^\perp = \{0\}$, N becomes a slant submanifold with a slant angle θ. If $\mathscr{D}^\theta = \{0\}$, then N becomes an anti-invariant submanifold. Moreover if $\theta = 0$, then N becomes a CR-submanifold. Furthermore, if $\mathscr{D}^\perp = \{0\}$ and $\theta = 0$, then N becomes a holomorphic submanifold. We denote hemi-slant submanifolds by v_6.
Digraph $D_2 = (V, A)$ is an extension of D_1, and has five vertices, $V(D_2) = \{v_1, v_2, v_3, v_4, v_6\}$, and seven arcs, $A(D_2) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_6, v_1), (v_6, v_2), (v_6, v_3)\}$ in Fig. 3.2. D_2 has the maximum length of two as the longest path. It has 2 vertices (v_4 and v_6) which are not reachable. Topological sort of D_2 is $v_6 - v_4 - v_2 - v_3 - v_1$. $rad(D_2) = 1$, the radius of D_2 is $v_2 \to v_1$. $diam(D_2) = 1$, the diameter of D_2 is the same as the radius. Also, there is no center vertex but three peripheral vertices such as v_2, v_4 and v_6.

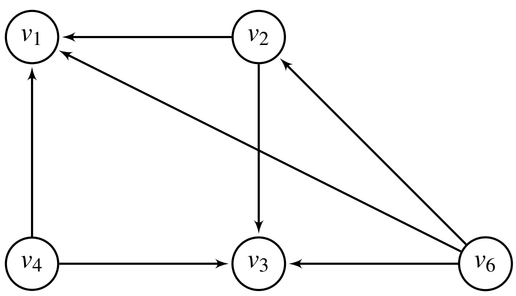

Figure 3.2: Digraph D_2 built by submanifolds in D_1 and the hemi-slant submanifold

Theorem 3.3. *Let D_2 be a digraph built by adding the hemi-slant submanifolds as vertex v_6 to the D_1. Then, D_2 satisfies the following properties:*

1. *D_2 is a three-partite digraph.*
2. *The maximum matching is 2.*
3. *The independence number is 2.*
4. *The chromatic number is 3.*
5. *D_2 has no directed spanning tree.*
6. *The domination number is 2.*

Proof. 1. There exists a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2\}$ and $V_3 = \{v_4, v_6\}$ of $V(D_2)$. These three subsets are three partite sets because of following attributes: $V(D_2) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). Then, D_2 is a three-partite digraph.
2. There is an arc subset $M = \{(v_6, v_1), (v_4, v_3)\}$ in D_2, and $|M| = 2$. In M, there is no common vertices and loops, that is M is a matching. Also, there is no arc subset having greater cardinality than M. Therefore, M is maximum matching in D_2.
3. The maximum independent set and independence number of D_2 is the same as D_1. See Theorem 3.1-iii.
4. The minimum number of disjoint independent sets of D_2 is three: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2\}$ and $V_3 = \{v_4, v_6\}$. Then, chromatic number of D_2 is 3.
5. No root vertex that contains a directed path from the root to any other vertex in $V(D_2)$. Then, D_2 has no directed spanning tree.
6. There is a subset $\widetilde{V} = \{v_4, v_6\} \subset V(D_2)$. Considering this subset, that including the minimum cardinality of vertices in D_2 as a dominating set, for each vertex $v \in \widetilde{V}$ and $u \in V(D_2) - \widetilde{V}$, (v, u) is an arc in D_2. Clearly, the domination number is 2.

□

Corollary 3.4. *In the submanifold network represented by D_2 in Fig. 3.2, the submanifolds, slant (v_4) and hemi-slant (v_6), cannot be derived by the other submanifolds, because $d^-(v_4) = d^-(v_6) = 0$ in D_2. Also, holomorphic (v_1) and anti-invariant (v_3) submanifolds can be derived separately by CR (v_2), slant and hemi-slant since $N^-_{D_2}(v_1) = N^-_{D_2}(v_3) = \{v_2, v_4, v_6\}$.*

To remind the notion of semi-slant submanifolds of an almost Hermitian manifold, let M be an almost Hermitian manifold and N a real submanifold of M. Then we say that N is a semi-slant submanifold [38] if there exist two orthogonal distributions $\mathscr{D}$ and $\mathscr{D}^\theta$ on N such that

1. TN admits the orthogonal direct decomposition $TN = \mathscr{D} \oplus \mathscr{D}^\theta$.
2. The distribution $\mathscr{D}$ is an invariant distribution, i.e., $J(\mathscr{D}) = \mathscr{D}$.
3. The distribution $\mathscr{D}^\theta$ is slant with slant angle θ.

It is easy to see that if $\mathscr{D} = \{0\}$, N becomes a slant submanifold with a slant angle θ. If $\mathscr{D}^\theta = \{0\}$, then N becomes a holomorphic submanifold. Moreover if $\theta = \frac{\pi}{2}$, then N becomes a CR-submanifold. Furthermore, if $\mathscr{D} = \{0\}$ and $\theta = \frac{\pi}{2}$, then N becomes an anti-invariant submanifold. We denote semi-slant submanifolds by v_5.
Digraph $D_3 = (V, A)$ is another extension of D_1, and has five vertices, $V(D_3) = \{v_1, v_2, v_3, v_4, v_5\}$, and seven arcs, $A(D_3) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_2), (v_5, v_3), (v_5, v_4)\}$ in Fig. 3.3. D_3 has the maximum length of two as the longest path. It has a vertex (v_5) which is not reachable. Using transitive closure, D_3 has only one new direct connection such as $v_5 \to v_1$. Topological sort of D_3 is $v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_3) = 1$, the radius of D_3 is $v_2 \to v_1$. $diam(D_3) = 2$, the diameter of D_3 is $v_5 \to v_2 \to v_1$. Also, in D_3, there are two center vertices as v_2 and v_4, and one peripheral vertex as v_5.

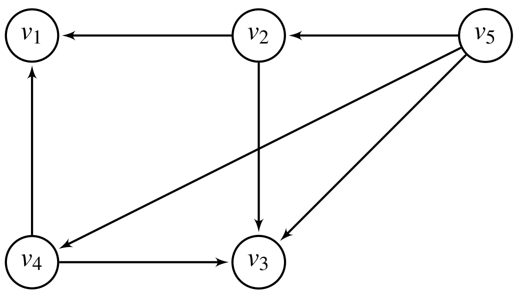

Figure 3.3: Digraph D_3 built by submanifolds in D_1 and the semi-slant submanifold

Theorem 3.5. *Let D_3 be a digraph created by adding the semi-slant submanifolds as vertex v_5 to the D_1. Then, D_3 holds the followings:*

1. *D_3 is a three-partite digraph.*
2. *The maximum matching is 2.*

3. *The independence number is 2.*
4. *The chromatic number is 3.*
5. *D_3 has a directed spanning tree.*
6. *The domination number is 2.*

Proof. 1. There exists a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5\}$ of $V(D_3)$ as three partite sets in D_3, and the subsets provide following properties: $V(D_3) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). In that case, D_3 is a three-partite digraph.
2. There is an arc subset $M = \{(v_2, v_1), (v_4, v_3)\}$ in D_3, and $|M| = 2$. Because of no common vertices and no loops in M, M is a matching. Furthermore, M has the maximum cardinality so that M is the maximum matching in D_3.
3. The maximum independent set and independence number of D_3 is the same as D_1. See Theorem 3.1-iii.
4. The minimum number of disjoint independent sets of D_3 is 3: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5\}$. It follows that the chromatic number of D_3 is 3.
5. D_3 has a unique directed spanning tree of length 4 and rooted at v_5 such as in Fig. 3.4. It also means that there is a transformation from submanifolds v_5 to all other submanifolds in D_3.

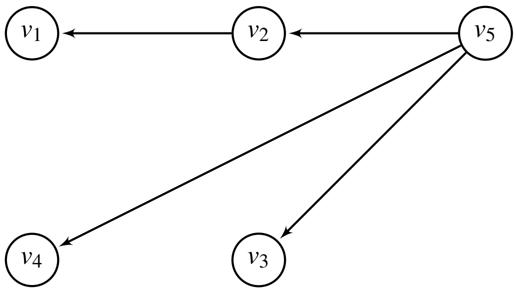

Figure 3.4: Directed spanning tree in D_3

6. There is a subset $\widetilde{V} = \{v_4, v_5\} \subset V(D_3)$. According to this subset, that having the minimum cardinality, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_3) - \widetilde{V}$, (v, u) is an arc in D_3, the domination number is 2.

□

Corollary 3.6. *In the submanifold network represented by D_3 in Fig. 3.3, while no submanifolds can be transformed to semi-slant (v_5) submanifold since $N^-_{D_3}(v_5) = \emptyset$, all other submanifolds (holomorphic (v_1), CR (v_2), anti-invariant (v_3) and slant (v_4)) can be obtained from semi-slant submanifold because of existence of a directed spanning tree with a root vertex v_5 (Fig. 3.4).*

Digraph $D_4 = (V, A)$ has six vertices, $V(D_4) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and 10 arcs, $A(D_4) = \{(v_2, v_1), (v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_2), (v_5, v_3), (v_5, v_4), (v_6, v_1), (v_6, v_2), (v_6, v_3)\}$ in Fig. 3.5. D_4 has the maximum length of two as the longest path. It has 2 vertices (v_5 and v_6) which are not reachable. Using transitive closure, D_4 has only one new direct connection such as $v_5 \to v_1$. The topological sort of D_4 is $v_6 - v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_4) = 1$, the radius of D_4 is $v_2 \to v_1$. $diam(D_4) = 2$, the diameter of D_4 is $v_5 \to v_2 \to v_1$. Also, in D_4, there are three center vertices as v_2, v_4 and v_6, and one peripheral vertex as v_5.

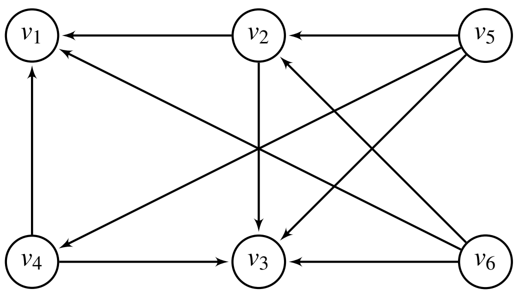

Figure 3.5: Digraph D_4 built by submanifolds in D_3 and the hemi-slant submanifold

Theorem 3.7. *Let D_4 be a digraph obtained by adding the hemi-slant submanifolds as vertex v_6 to the D_3. Then, D_4 provides the following properties:*

1. *D_4 is a three-partite digraph.*
2. *D_4 has a perfect matching.*
3. *The independence number is 2.*

4. *The chromatic number is 3.*
5. *D_4 has no directed spanning tree.*
6. *The domination number is 2.*

Proof. 1. There exists a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5, v_6\}$ of $V(D_4)$ as three subsets, and these subsets provide that $V(D_4) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ ($i, j = 1,2,3$ and $i \neq j$). Under these conditions, D_4 is a three-partite digraph.
2. There is an arc subset $M = \{(v_2,v_1),(v_5,v_4),(v_6,v_3)\}$ in D_4, and $|M| = 3$. On conditions that no common vertices and no loops in M and $|M| = \frac{|A(D_4)|}{2}$, M is perfect matching that's why D_4 has a matching also perfect.
3. The maximum independent set and the independence number of D_4 is the same as D_1. See Theorem 3.1-iii.
4. The minimum number of disjoint independent sets of D_4 is 3: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$ and $V_3 = \{v_5, v_6\}$. Then, the chromatic number of D_4 is 3.
5. No root vertex that contains a directed path from the root to any other vertex in $V(D_4)$. Then, D_4 has no directed spanning tree.
6. There is a subset $\widetilde{V} = \{v_5, v_6\} \subset V(D_4)$. According to this subset, that having the minimum cardinality, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_4) - \widetilde{V}$, (v,u) is an arc in D_4 so that the domination number is 2.

□

Corollary 3.8. *In the submanifold network represented by D_4 in Fig. 3.5, semi-slant (v_5) and hemi-slant (v_6) submanifolds cannot be obtained by any other submanifolds because $d^-(v_5) = d^-(v_6) = 0$. Besides, no submanifolds can be derived from holomorphic (v_1) and anti-invariant (v_3) submanifolds since $N^-_{D_4}(v_1) = N^-_{D_4}(v_3) = \emptyset$.*

Let M be an almost Hermitian manifold and N a real submanifold of M. Then we say that N is a bi-slant submanifold [34] if there exist two orthogonal distributions $\mathscr{D}^{\theta_1}$ and $\mathscr{D}^{\theta_2}$ on N such that

1. TN admits the orthogonal direct decomposition $TN = \mathscr{D}^{\theta_1} \oplus \mathscr{D}^{\theta_2}$.
2. The distributions $\mathscr{D}^{\theta_1}$ and $\mathscr{D}^{\theta_2}$ are slant distributions with slant angles θ_1 and θ_2.

It is easy to see that if $\mathscr{D}^{\theta_1} = \{0\}$ (or $\mathscr{D}^{\theta_2} = \{0\}$), N becomes a slant submanifold with a slant angle θ_1. If $\theta = \theta_1 = \theta_2 = \{0\}$, then N becomes a holomorphic submanifold. If $\theta = \theta_1 = \theta_2 = \frac{\pi}{2}$, then N becomes an anti-invariant submanifold. Moreover, if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, then N becomes a CR-submanifold. Furthermore, $\theta_1 = \frac{\pi}{2}$ and $\theta_1 = 0$, then N becomes a hemi-slant submanifold and semi-slant submanifold, respectively. We denote bi-slant submanifolds by v_7.
Digraph $D_5 = (V, A)$ has seven vertices, $V(D_5) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, and 12 arcs, $A(D_4) = \{(v_2,v_1),(v_2,v_3),(v_4,v_1),(v_4,v_3),(v_5,v_2),(v_5,v_3),(v_5,v_4),(v_6,v_1),(v_6,v_2),(v_6,v_3),(v_7,v_5),(v_7,v_6)\}$ in Fig. 3.6. D_5 has the maximum length of three as the longest path. It has a vertex (v_7) which is not reachable. Using transitive closure, D_5 has five new direct connections such as $v_5 \to v_1$, $v_7 \to v_1$, $v_7 \to v_2$, $v_7 \to v_3$ and $v_7 \to v_4$. Topological sort of D_5 is $v_7 - v_6 - v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_5) = 1$, the radius of D_5 is $v_2 \to v_1$. $diam(D_5) = 2$, the diameter of D_5 is $v_5 \to v_2 \to v_1$. Also, in D_5, there are three center vertices as v_2, v_4 and v_6, and two peripheral vertices as v_5 and v_7.

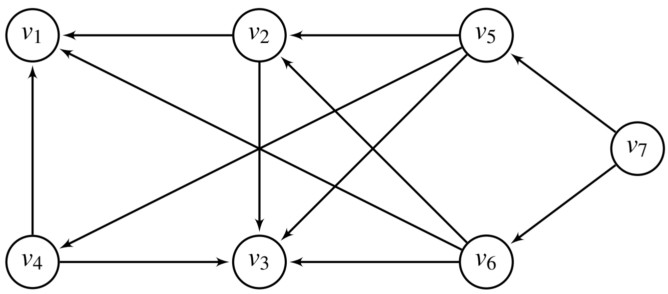

Figure 3.6: Digraph D_5 built by submanifolds in D_4 and the bi-slant submanifold

Theorem 3.9. *Let D_5 be a digraph constructed by adding the bi-slant submanifolds as vertex v_7 to the D_4. Then, D_5 holds the followings:*

1. *D_5 is a three-partite digraph.*
2. *The maximum matching is 3.*
3. *The independence number is 3.*
4. *The chromatic number is 3.*
5. *D_5 has a directed spanning tree.*
6. *The domination number is 3.*

Proof. 1. There is a partition $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4, v_7\}$ and $V_3 = \{v_5, v_6\}$ of $V(D_5)$ as three subsets, and these subsets support that $V(D_4) = \bigcup_{i=1}^{3} V_i$ and $V_i \cap V_j = \emptyset$ $(i, j = 1, 2, 3$ and $i \neq j)$. Then, D_5, containing the subsets, is actually a three-partite digraph.
2. $M = \{(v_2, v_1), (v_5, v_4), (v_6, v_3)\}$ is an arc subset in D_5, and $|M| = 3$. According to this, M, that includes no common vertices and no loops, is a matching. Since no other subset greater cardinality than M, D_5 has a maximum matching called M.
3. The subset $\widetilde{V} = \{v_2, v_4, v_7\}$ is an independent set having maximum cardinality. It also means that there is no direct relationship between any two elements, i.e. submanifolds, in $\widetilde{V}$. Then, the independence number of D_5 is 3, because $|\widetilde{V}| = 3$.
4. The minimum number of disjoint independent sets of D_5 is 3: $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4, v_7\}$ and $V_3 = \{v_5, v_6\}$. According to that, three different colors are needed to coloring D_5 and that's why the chromatic number of D_5 is 3.
5. D_5 has a directed spanning tree of length 6 and root at v_7 such as in Fig. 3.7. It also means that there is a transformation from submanifolds v_7 to all other submanifolds in D_5 at most two-step.

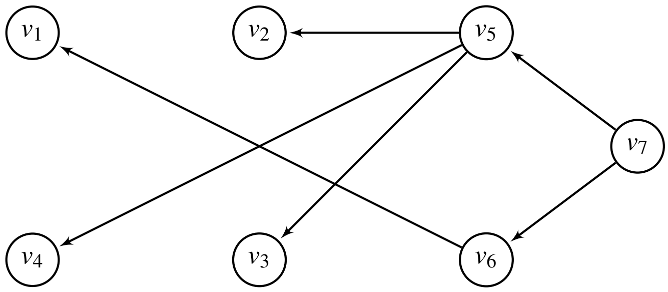

Figure 3.7: A directed spanning tree in D_5

6. There is a subset $\widetilde{V} = \{v_5, v_6, v_7\} \subset V(D_5)$. According to this subset, that having the minimum cardinality, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_5) - \widetilde{V}$, (v, u) is an arc in D_5. The domination number is 3.

□

Corollary 3.10. *In the submanifold network represented by D_5 in Fig. 3.6, all other submanifolds can be derivated from bi-slant (v_7) submanifold since v_7 is the root vertex of the directed spanning tree of D_5 and $N^+_{D_5}(v_7) = \{v_5, v_6\}$ in Fig. 3.7. Also, no submanifolds can be transformed to bi-slant because $N^-_{D_5}(v_7) = \emptyset$.*

Digraph $D_6 = (V, A)$ has also seven vertices as well as D_5, $V(D_6) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, and 12 arcs, $A(D_6) = \{(v_2, v_1),$ $(v_2, v_3), (v_4, v_1), (v_4, v_3), (v_5, v_1),$ $(v_5, v_2), (v_5, v_3), (v_5, v_4), (v_6, v_1), (v_6, v_2),$ $(v_6, v_3), (v_6, v_4), (v_7, v_5), (v_7, v_6)\}$ in Fig. 3.8. D_6 has the maximum length of three as the longest path. It has a vertex (v_7) which is not reachable. Using transitive closure, D_6 has four new direct connections such as $v_7 \to v_1$, $v_7 \to v_2$, $v_7 \to v_3$ and $v_7 \to v_4$. Topological sort of D_6 is $v_7 - v_6 - v_5 - v_4 - v_2 - v_3 - v_1$. $rad(D_6) = 1$, the radius of D_6 is $v_2 \to v_1$. $diam(D_6) = 2$, the diameter of D_6 is $v_7 \to v_5 \to v_1$. Also, in D_6, there are four center vertices as v_2, v_4, v_5 and v_6, and one peripheral vertex as v_7.

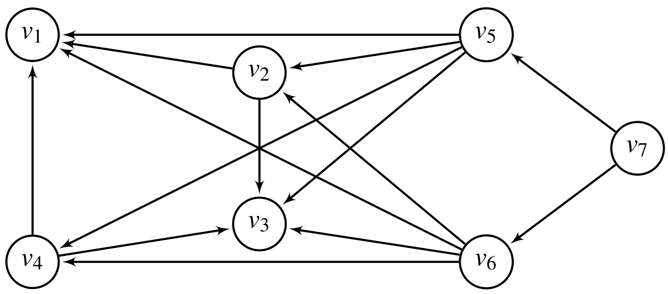

Figure 3.8: Digraph D_6 built by D_5 with arcs (v_5, v_1) and (v_6, v_4)

Theorem 3.11. *Let D_6 be a digraph created by adding two more relations from semi-slant to holomorphic and from hemi-slant to slant as arcs to the D_5. Then, D_6 satisfies the following properties:*

1. *D_6 is a three-partite digraph.*
2. *The maximum matching is 3.*
3. *The independence number is 3.*
4. *The chromatic number is 3.*
5. *D_6 contains a directed spanning tree.*

6. *The domination number is 2.*

Proof. The properties i, ii, iii and iv are clear from Theorem 3.9.

v. D_6 has a directed spanning tree having the same structure as in Fig. 3.7 (see Theorem 3.9-v).
vi. There is a subset $\widetilde{V} = \{v_5, v_7\} \subset V(D_6)$. According to that, the subset has the minimum cardinality while dominating all other vertices, and for each vertex $v \in \widetilde{V}$ and $u \in V(D_6) - \widetilde{V}$, (v, u) is an arc in D_6. The domination number is 2.

□

Corollary 3.12. *In the most comprehensive submanifold network represented by D_6 in Fig. 3.7, just two submanifolds, holomorphic (v_1) and anti-invariant (v_3), are not generative since $d^+(v_1) = d^+(v_3) = 0$. Besides, bi-slant (v_7) is the most productive submanifold owing to transforming to all other submanifolds.*

Using the seven submanifolds, named as holomorphic, CR, anti-invariant, slant hemi-slant, semi-slant and bi-slant, it is constructed six digraphs, called D_1, D_2, D_3, D_4, D_5 and D_6, whose vertices are submanifolds and arcs are connections among submanifolds from one to another.

Theorem 3.13. *Let $D \in \{D_1, D_2, D_3, D_4, D_5, D_6\}$ be a digraph. D provides the following properties:*

1. *Simple directed graph.*
2. *Directed acyclic graph.*
3. *Weakly connected.*

Proof. 1. In digraph D, there is no more than one relationship between any two submanifolds and no transformations from a submanifold to itself. According to that, D is a simple directed graph.
2. Given a transition list among submanifolds such as $v_1 v_2 ... v_k$, meaning that v_1 is the source submanifold and v_k is the sink submanifold. Because D doesn't have any transition list including the same submanifold is both source and also sink, D is acyclic. That's why D is a directed acyclic digraph.
3. D has one pair of submanifolds as a relation at least that they can not mutually be transformed from one to another submanifold. Hence, D is not strongly connected. However, when D, that considered as without direction of transformations, is connected, named connectedness of underlying graph because there are no isolated submanifolds. For this reason, D is weakly connected.

□

Corollary 3.14. *Among all digraphs D_1, D_2, D_3, D_4, D_5, and D_6, the digraph D_6 has*

- *the maximum vertex-integrity, and*
- *the maximum edge-integrity*

as well as the maximum size by Proposition 2.2.

Example 3.15. *Let H be a directed graph having vertex set $V(H) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and arc set $A(H) = \{(u_1, u_2), (u_1, u_4), (u_1, u_5), (u_2, u_4), (u_2, u_7), (u_3, u_2), (u_3, u_4), (u_3, u_7), (u_5, u_4), (u_5, u_7), (u_6, u_3), (u_6, u_7)\}$ as modeled in Fig. 3.9. Suppose that H' indicates an induced subgraph of H when we consider as $V(H') = U$ where $U = \{u_1, u_2, u_3, u_4, u_5, u_7\}$ is a vertex subset of $V(H)$. Accordingly, we attain that H' is isomorphic to the network created by submanifolds called as holomorphic, CR, anti-invariant, slant, hemi-slant and semi-slant. Thus, H' provides the same properties as D_4. This means that H contains bounds for some parameters such as independence, domination and chromatic numbers.*

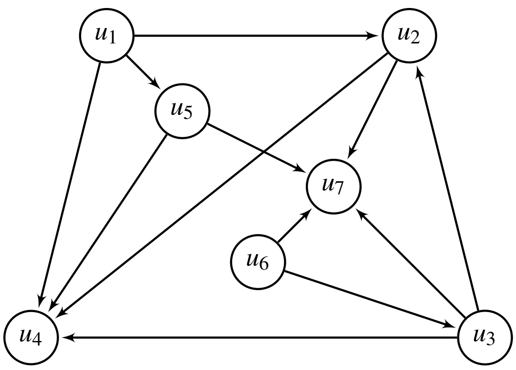

Figure 3.9: A sample network

4. Conclusion

Manifold learning plays an important role in analyzing data lying on a non-linear space as a part of machine learning. Moreover, the geometric deep learning yields using the concepts of manifolds and graphs together in building convolutional deep structures. In this paper, using holomorphic submanifolds, anti-invariant submanifolds, CR-submanifolds, slant submanifolds, semi-slant submanifolds, hemi-slant submanifolds and bi-slant submanifolds in almost Hermitian manifolds, it is given relations among them, six different digraphs are created as a network of these submanifolds, and main properties of them are first examined in terms of digraphs. Accordingly, some directed networks by identifying with submanifolds of almost Hermitian manifolds are established. We note that there is a much wider class that includes slant submanifolds. This class was first defined in [6] by Etayo as quasi-slant submanifolds. Later, these submanifolds were called pointwise slant submanifolds in [7] by Chen and Garay. Although we have excluded such submanifolds in this article, our next research will be to examine the connections between these submanifolds and graph theory.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Some Spectral Properties of Multiplicative Hermite Equation

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Abstract

We reconstruct the Multiplicative Hermite Equation from multiplicative Sturm-Liouville equation. A new representation of eigenfunctions for the constructed problem are obtained by the power series solution technique. While making these solutions, multiplicative Hermite polynomials were used strongly. We get a generator for multiplicative Hermite polynomials and construct integration representations for these polynomials. Finally, some spectral properties of the multiplicative Hermite problem are examined in detail.

1. Introduction

Multiplicative calculus was introduced by *Grossman and Katz* [1, 2] in 1967 as an alternative to usual calculus. This type of calculus is also known as non-Newtonian because of its difference from classical calculus of Newton and Leibniz. Multiplicative calculus is a useful supplement to usual calculus in that it is tailored to situations involving exponential functions in the same sense that the usual calculus is tailored to situations involving linear functions. Multiplicative calculus moves the roles of subtraction and addition to division and multiplication. There are actually many reasons to study multiplicative analysis. It improves the work of additive calculations indirectly. Problems that are difficult to solve in classical case can be solved with incredible ease in here. Every property in Newtonian case can be defined in multiplicative analysis within certain rules.
Many events in nature change exponentially. For example: populations of countries, magnitude of an earthquake [3] are events that behave in this manner. For this reason, using multiplicative analysis instead of usual analysis allows a better physical evaluation of these type events. This calculus also gives better results than usual case in many fields such as finance, economics, biology and demography. A very limited number of studies have been conducted on this analysis until the beginning of the 2000s. Recently, various studies have been carried out on it and quality and effective results have been obtained (see [4]- [13]).
A Sturm-Liouville equation is a second-order linear differential equation that allows us to find solutions that form an orthogonal system. While many mathematicians have studied various properties of this equation, some special cases of it produce some special equations [14]- [26]. One of these special equations is the Hermite equations. Hermite equations have many real world practical applications [27]- [33]. We will only focus on the methods of solution and use in a mathematical sense. In solving these equations explicit solutions cannot be found. That is solutions in terms of elementary functions cannot be found. In many cases, it is easier to find a numerical or series solution. Determining the generating function and integral representation of the multiplicative Hermite polynomials, which is one of the main aims of the present study, is very important in physical problems. For example, in the classical case, one computes the bond energy of the atomic nucleus with the use of Eden-Goldstone integral equation which contains Hermite polynomials on the integrals. In addition, the density of the probability distribution of the coordinate in the ensemble of quantum harmonic oscillators is represented by generating function of Hermite polynomials [34]. Therefore, Hermite equation, which has a very important place in classical sense, and its solutions will be dealt with in a multiplicative sense and will be examined in detail.

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In this study, apart from known analysis, Hermite equation will be established and its properties will be examined with methods similar to classical case. The equation we solve in multiplicative case is actually a much more complex equation in classical case, and solutions of the two equations coincide. Before moving on to basic results, let's firstly express the concepts and important theorems of multiplicative analysis that we will use in our study.

Definition 1.1. *[4] Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be differentiable in usual case and $f(x) > 0$ for all x. If the below limit exists and positive*

$$f^*(x) = \lim_{h\to 0}\left[\frac{f(x+h)}{f(x)}\right]^{\frac{1}{h}},$$

$f^*(x)$ is called multiplicative (or *) derivative of f at x.

Lemma 1.2. *[4] Let $f : A \to \mathbb{R}$ be positive and usual differentiable at x. Then, there is following relation between classical and *derivatives.*

$$f^*(x) = e^{(\ln o f)'(x)}.$$

Theorem 1.3. *[4] Let f,h be *differentiable and p be usual differentiable at x. The following expressions are provided for *derivative.*
i. $(cf)^*(x) = f^*(x),\ c \in \mathbb{R}^+$,
ii. $(fg)^*(x) = f^*(x)g^*(x)$,
iii. $(f/g)^*(x) = f^*(x)/g^*(x)$,
iv. $\left(f^h\right)^*(x) = f^*(x)^{h(x)} f(x)^{h'(x)}$,
v. $(f \circ h)^*(x) = f^*(h(x))^{h'(x)}$,
vi. $(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}} g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$.

Since multiplicative integration will emerge while obtaining Hermite polynomials for multiplicative Hermite equation, let's express fundamental properties of multiplicative integration.

Definition 1.4. *[4] Let f be a positive, bounded function on $[a,b]$ where $-\infty < a < b < \infty$. Then, the symbol $\int_a^b f(x)^{dx}$ is called multiplicative integral or *integral of f on $[a,b]$. By this definition, if f is positive and Riemann integrable on $[a,b]$, then it is *integrable on $[a,b]$ and*

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln o f)(x)dx}.$$

Conversely, one can show that if f is Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)dx = \ln \int_a^b \left(e^{f(x)}\right)^{dx}.$$

Theorem 1.5. *[4] Let $f,g > 0$ be bounded, *integrable and $h > 0$ be usual differentiable on $[a,b]$. Then, below expressions hold:*
i. $\int_a^b \left[f(x)^k\right]^{dx} = \left[\int_a^b f(x)^{dx}\right]^k$,
ii. $\int_a^b [f(x)g(x)]^{dx} = \int_a^b f(x)^{dx} \int_a^b g(x)^{dx}$,
iii. $\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}$,
iv. $\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \int_c^b f(x)^{dx}$,
v. $\int_a^b \left[f^*(x)^{g(x)}\right]^{dx} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \left\{\int_a^b \left[f(x)^{g'(x)}\right]^{dx}\right\}^{-1}$,
*where $k \in \mathbb{R}$ is a constant and $c \in [a,b]$. The expression **v** is known as *integration by parts formula.*

In order to avoid any difficulties in expressing main parts of study, inner product function will be defined and inner product space used throughout study will be given in multiplicative case.

Definition 1.6. *Let X be a non empty set and $<,>_*: X\times X\to\mathbb{R}^+$ be a function such that below axioms hold for $\forall x,y,z\in X$*
i. $<x,x>_*\geq 1$,
ii. $<x,x>_*=1$ *if and only if* $x=1$,
iii. $<x\oplus y,z>_*=<x,z>_*\oplus<y,z>_*$,
iv. $<e^{\alpha}\odot x,y>_*=e^{\alpha}\odot<x,y>_*,\alpha\in\mathbb{R}$,
v. $<x,y>_*=<y,x>_*$.
Then, $(X,<,>_*)$ is called *inner product space and $<,>_*$ is *inner product on X.

Lemma 1.7. $L_2^*[a,b]=\left\{f:\int\limits_a^b[f(x)\odot f(x)]^{dx}<\infty\right\}$ *is an *inner product space with*

$$<,>_*: L_2^*[a,b]\times L_2^*[a,b]\to\mathbb{R}^+,\qquad <f,h>_*=\int\limits_a^b[f(x)\odot h(x)]^{dx},$$

where $f,h\in L_2^*[a,b]$ are positive functions.

Proof. Proof can be easily demonstrated using properties of multiplicative inner product and definition of given space. □

$n-$th order linear homogeneous multiplicative differential equation is denoted by

$$\left(y^{*(n)}\right)\left(y^{*(n-1)}\right)^{a_{n-1}(x)}\cdots\left(y^{**}\right)^{a_2(x)}\left(y^*\right)^{a_1(x)}y^{a_0(x)}=1, \tag{1.1}$$

where $y^{*(n)}(x)=e^{(\ln\circ f)^{(n)}(x)}$ and $a_{n-1}(x),\cdots,a_2(x),a_1(x),a_0(x)$ are functions of x [35].

Definition 1.8. *[35] Let $x_0\in[a,b]$, $N(x_0)$ be a neighbourhood of x_0 and $f(x)$ be a real function defined on $[a,b]$. In this case $f(x)$ is said to multiplicative-analytic at x_0 if $f(x)$ can be expressed as a series of natural powers of $(x-x_0)$ for all $x\in N(x_0)$. In other words, $f(x)$ can be expressed as following:*

$$f(x)=\prod_{n=0}^{\infty}(c_n)^{(x-x_0)^n},(c_n\in\mathbb{R}^+).$$

There exists $\delta>0$ such that this series is convergent for all x satisfying $|x-x_0|<\delta$ and divergent for $|x-x_0|>\delta$. δ is the radius of convergence of the series.

Definition 1.9. *[35] Let $x_0\in[a,b]$ and functions $a_k(x)$ be multiplicative-analytic at $x_0\in[a,b]$ for $k=0,1,2,...,(n-1)$. In this case, the point $x_0\in[a,b]$ is said to be a multiplicative-ordinary point of (1.1). If a point $x_0\in[a,b]$ is not a multiplicative-ordinary point, then it is said to be multiplicative singular.*

Theorem 1.10. *[35] Let $p(x),q(x)$ be analytic functions such as*

$$p(x)=\sum_{k=0}^{\infty}p_k(x-x_0)^k,\qquad (t\in[x_0,x_0+\delta_1];\delta_1>0),$$

$$q(x)=\sum_{k=0}^{\infty}q_k(x-x_0)^k,\qquad (t\in[x_0,x_0+\delta_2];\delta_2>0)$$

and let x_0 be a multiplicative-ordinary point of the equation

$$y^{**}(y^*)^{p(x)}y^{q(x)}=1 \tag{1.2}$$

Then, there exists a solution to (1.2) as

$$y=\prod_{k=0}^{\infty}c_k^{(x-x_0)^k}$$

for $t\in(x_0,x_0+\rho)$ with $\rho=\min\{\delta_1,\delta_2\}$ and initial conditions $y(x_0)=c_0$, $y^*(x_0)=c_1$.

The rest of this study is organized as follows: In the second section, we reconstruct multiplicative Hermite equation by multiplicative Sturm-Liouville equation. A new asymptotic formula for eigenfunction of multiplicative Hermite equation is established by series technique. Moreover, for multiplicative Hermite equation, a generator function is obtained and an integral representation is constructed. In the last section, some spectral properties of multiplicative Hermite equation are examined.

2. Multiplicative Hermite equation

Here, multiplicative Hermite equation will be established from multiplicative Sturm-Liouville equation by some algebraic structures and Hermite polynomials of constructed problem will be obtained. That way, let's express multiplicative algebraic structures that we will encounter while establishing and solving multiplicative Hermite equation. Arithmetic operations created with exponential functions are called multiplicative algebraic operations. Let's show some properties of these operations with a multiplicative arithmetic table for $x,y \in \mathbb{R}^+$:

$$x \ominus y = \tfrac{x}{y}, \quad x \oplus y = xy, \quad x \odot y = x^{\ln y} = y^{\ln x}.$$

These operations create some algebraic structures. If $\oplus : A \times A \to A$ is an operation where $A \neq \phi$ and $A \subset \mathbb{R}^+$, algebraic structure $(A, \oplus)$ is called multiplicative group. Similarly, $(A, \oplus, \odot)$ is a multiplicative ring. This situation gives us the opportunity to use these processes easily and define different structures [36].
Let

$$\left\{ \frac{d^*}{dx} \left(e^{p(x)} \odot \frac{d^* y}{dx} \right) \right\} \oplus \left(e^{q(x)} \odot y \right) \oplus \left(e^{\lambda w(x)} \odot y \right) = 1, \tag{2.1}$$

be multiplicative Sturm-Liouville equation where p,q,w are real valued, continuous functions [37]. Here, if p,q,w are chosen in a special way as $p(x) = w(x) = e^{-x^2}$, $q(x) = 0$, the equation (2.1) transforms into

$$L[y] = \left[(y^*(x))^{e^{-x^2}} \right]^* y^{\lambda e^{-x^2}} = 1, \ x \in \mathbb{R}$$

or

$$L[y] = y^{**} (y^*)^{-2x} y^{\lambda} = 1, \ x \in \mathbb{R}, \tag{2.2}$$

where $a,b \in \mathbb{R}$ and, λ is a spectral parameter. If it is set $\lambda = 2n$, (2.2) is called multiplicative Hermite equation [35]. Here, $y(x,\lambda)$ is solution of above equation which is called multiplicative Hermite polynomial. Although (2.2) has no multiplicative singular points, it is multiplicative singular Sturm-Liouville equation because of the range for which x is defined. $x = 0$ is a multiplicative ordinary point. In [35], the solutions on the neighborhood of this point are found as

$$y(x) = c_0 \prod_{k=1}^{\infty} \left(c_0^{(-1)^k \frac{2^k n(n-2)\dots(n-2k+2)}{(2k)!}} \right)^{x^{2k}} c_1^x \prod_{k=1}^{\infty} \left(c_1^{(-1)^k \frac{2^k (n-1)(n-3)\dots(n-2k+1)}{(2k+1)!}} \right)^{x^{2k+1}},$$

where c_0 and c_1 are arbitrary constants. The equation (2.2) will be considered together with the following condition:

$$\lim_{x \to \pm\infty} e^{e^{-x^2}} y(x,\lambda) = 1. \tag{2.3}$$

The equation (2.2) corresponds to following nonlinear differential equation in classical case. The spectral properties of multiplicative Hermite equation coincide with properties of this nonlinear equation.

$$\left[y'' - 2xy' + \lambda y \right] y - \left[\left(y' \right)^2 + \lambda y^2 (1 - \ln y) \right] = 0.$$

According to the cases where n is even and odd, let's define $\widetilde{H}_n(x)$ as follows

$$\widetilde{H}_n(x) = \prod_{k=0}^{\left[\frac{n}{2}\right]} e^{\frac{(-1)^k (n)!}{k!(n-2k)!} (2x)^{n-2k}}. \tag{2.4}$$

where

$$\left[\frac{n}{2} \right] = \begin{cases} \frac{n}{2}, & if \quad n \text{ is even} \\ \frac{n-1}{2}, & if \quad n \text{ is odd} \end{cases}.$$

Eigenfunctions $\widetilde{H}_n(x)$ are $n-$th degree multiplicative Hermite polynomials. These numbers and polynomials play an important role in various areas of mathematics and physics, including numerical theory, combinations, special functions, and differential equations. Many interesting properties about them have been explored. For example, in mathematics and physics, the Hermite polynomials are a classical orthogonal polynomial sequence. In probability, they appears as the Edgeworth series; in combinatorics, they arise in the umbral calculus as an example of an Appell sequence; in numerical analysis, they play a role in Gaussian quadrature; and in physics, they give rise to the eigenstates of the quantum harmonic oscillator. The polynomials arise in signal processing as Hermitian wavelets for wavelet transform analysis.
Now, let's obtain a generator function for multiplicative Hermite equation and find integral representations of multiplicative Hermite polynomial:

Lemma 2.1. *The generator functions of multiplicative Hermite polynomials have the following representation:*

$$H(x,t) = e^{e^{2xt-t^2}} = \prod_{n=0}^{\infty} \left\{ \widetilde{H}_n(x) \right\}^{\frac{t^n}{n!}}, \tag{2.5}$$

where $|x| < \infty$, $|t| < \infty$ *and* $\widetilde{H}_n(x)$ *is defined as (2.4).*

Proof. Consider the following function

$$H(x,t) = e^{e^{2xt-t^2}} = e^{\left[\sum\limits_{m=0}^{\infty} \frac{(2xt)^m}{m!}\right]\left[\sum\limits_{k=0}^{\infty} \frac{(-t^2)^k}{k!}\right]} = \prod_{n=0}^{\infty} \left\{ \widetilde{H}_n(x) \right\}^{\frac{t^n}{n!}}.$$

If it is proven that $\widetilde{H}_n(x)$ is the multiplicative Hermite polynomial (that is, it satisfies the equation (2.2)), the proof completes. Firstly, *differentiating (2.5) with respect to x, we get

$$e^{2te^{2xt-t^2}} = \prod_{n=1}^{\infty} \left\{ \widetilde{H}_n^*(x) \right\}^{\frac{t^n}{n!}}$$

or

$$\prod_{n=0}^{\infty} \left\{ \widetilde{H}_n(x) \right\}^{\frac{2t^{n+1}}{n!}} = \prod_{n=1}^{\infty} \left\{ \widetilde{H}_n^*(x) \right\}^{\frac{t^n}{n!}}.$$

From the last equality, if the exponents of t are compared, we obtain the recurrence relation

$$\widetilde{H}_{n+1}^*(x) = \left\{ \widetilde{H}_n(x) \right\}^{2(n+1)}. \tag{2.6}$$

Secondly, *differentiating (2.5) with respect to t, and then comparing the exponents of t, we have the other recurrence relation

$$\widetilde{H}_{n+1}(x) \left\{ \widetilde{H}_n(x) \right\}^{-2x} \left\{ \widetilde{H}_{n-1}(x) \right\}^{2n} = 1. \tag{2.7}$$

Thirdly, *differentiating (2.7) with respect to x, we have

$$\widetilde{H}_{n+1}^*(x) \left\{ \widetilde{H}_n^*(x) \right\}^{-2x} \left\{ \widetilde{H}_n(x) \right\}^{-2} \left\{ \widetilde{H}_{n-1}^*(x) \right\}^{2n} = 1.$$

Using (2.6) in the last equation, we take

$$\left\{ \widetilde{H}_n(x) \right\}^{2(n+1)} \left\{ \widetilde{H}_n^*(x) \right\}^{-2x} \left\{ \widetilde{H}_n(x) \right\}^{-2} \widetilde{H}_n^{**}(x) = 1$$

or

$$\widetilde{H}_n^{**}(x) \left\{ \widetilde{H}_n^*(x) \right\}^{-2x} \left\{ \widetilde{H}_n(x) \right\}^{2n} = 1.$$

This last equality shows that $\widetilde{H}_n(x)$ satisfies the equation (2.2). □

The following lemma has been proved by [35] using a different technique. Here, the proof is made using generator functions.

Lemma 2.2. *The multiplicative Rodrigues formula for* $\widetilde{H}_n(x)$ *has the formula*

$$\widetilde{H}_n(x) = \left[\frac{d^{*(n)}}{dx^n} e^{e^{-x^2}} \right]^{(-1)^n e^{x^2}}, \qquad n = 0, 1, 2, \cdots. \tag{2.8}$$

Proof. From the formula (2.5), we obtain

$$e^{e^{\left(x^2-(t-x)^2\right)}} = \prod_{n=0}^{\infty}\left\{\widetilde{H}_n(x)\right\}^{\frac{t^n}{n!}}$$

or

$$\left\{e^{e^{-(t-x)^2}}\right\}^{e^{x^2}} = \left\{\widetilde{H}_0(x)\right\}^{\frac{1}{0!}}\left\{\widetilde{H}_1(x)\right\}^{\frac{t}{1!}}\left\{\widetilde{H}_2(x)\right\}^{\frac{t^2}{2!}}\ldots\left\{\widetilde{H}_n(x)\right\}^{\frac{t^n}{n!}}\left\{\widetilde{H}_{n+1}(x)\right\}^{\frac{t^{n+1}}{(n+1)!}}\ldots. \tag{2.9}$$

By taking $n-$times *derivative of both side of (2.9) with respect to t, we get

$$\left\{\frac{\partial^{*(n)}}{\partial t^n}e^{e^{-(t-x)^2}}\right\}^{e^{x^2}} = 1.1\ldots\left\{\widetilde{H}_n(x)\right\}^{\frac{n!}{n!}}\left\{\widetilde{H}_{n+1}(x)\right\}^{\frac{(n+1)!}{(n+1)!}t}\ldots.$$

Then,

$$\widetilde{H}_n(x) = \left.\left\{\frac{\partial^{*(n)}}{\partial t^n}e^{e^{-(t-x)^2}}\right\}^{e^{x^2}}\right|_{t=0}$$

for $t=0$. After some basic calculations, taking the relation

$$\left.\left\{\frac{\partial^{*(n)}}{\partial t^n}e^{e^{-(t-x)^2}}\right\}\right|_{t=0} = \left\{\frac{d^{*(n)}}{dx^n}e^{e^{-x^2}}\right\}^{(-1)^n},$$

we prove (2.8). $\square$

Lemma 2.3. *The integral representation of multiplicative Hermite polynomials is as follows:*

$$\widetilde{H}_n(x) = \left\{\int_{-\infty}^{\infty}\left\{e^{(x+is)^n e^{-s^2}}\right\}^{ds}\right\}^{\frac{2^n}{\sqrt{\pi}}}.$$

Proof. Consider the integral representation of $e^{e^{-x^2}}$ as:

$$e^{e^{-x^2}} = \left\{\int_{-\infty}^{\infty}\left(e^{e^{-s(s+2ix)^2}}\right)^{ds}\right\}^{\frac{1}{\sqrt{\pi}}}. \tag{2.10}$$

Taking the n-th *derivative of (2.10) with respect to x, we have

$$\begin{aligned}\frac{d^{*(n)}}{dx^n}\left(e^{e^{-x^2}}\right) &= \left\{\int_{-\infty}^{\infty}\left(e^{(-i2s)^n e^{-s(s+2ix)}}\right)^{ds}\right\}^{\frac{1}{\sqrt{\pi}}}\\ &= \left\{\int_{-\infty}^{\infty}\left(e^{(is)^n e^{-(s+ix)^2}}\right)^{ds}\right\}^{(-1)^n\frac{2^n}{\sqrt{\pi}}e^{-x^2}}\\ &= \left\{\int_{-\infty}^{\infty}\left(e^{(x+is)^n e^{-s^2}}\right)^{ds}\right\}^{(-1)^n\frac{2^n}{\sqrt{\pi}}e^{-x^2}}.\end{aligned} \tag{2.11}$$

By considering the $(-1)^n e^{x^2}$ power of both sides for (2.11), we get

$$\left\{\frac{d^{*(n)}}{dx^n}\left(e^{e^{-x^2}}\right)\right\}^{(-1)^n e^{x^2}} = \left\{\int_{-\infty}^{\infty}\left(e^{(x+is)^n e^{-s^2}}\right)^{ds}\right\}^{\frac{2^n}{\sqrt{\pi}}}.$$

From (2.8), proof is completed. $\square$

Now, let's express some special cases of multiplicative Hermite polynomials, which have an important place in applications.

Remark 2.4. *[35] By (2.4), some multiplicative Hermite polynomials are as follows:*

$$\begin{aligned}
\widetilde{H}_0(x) &= e, \\
\widetilde{H}_1(x) &= e^{2x}, \\
\widetilde{H}_2(x) &= e^{4x^2-2}, \\
\widetilde{H}_3(x) &= e^{8x^3-12x}, \\
\widetilde{H}_4(x) &= e^{16x^4-48x^2+12}, \\
\widetilde{H}_5(x) &= e^{32x^5-160x^3+120x} \\
&\vdots
\end{aligned}$$

Remark 2.5. *Multiplicative Hermite polynomials provide the following properties:*
i. $\widetilde{H}_n(x) = e^{H_n(x)}$,
ii. $\widetilde{H}_n(-x) = [\widetilde{H}_n(x)]^{(-1)^n}, \qquad \widetilde{H}_n^*(-x) = \left[\widetilde{H}_n^*(x)\right]^{(-1)^{n+1}}$,
iii. $\widetilde{H}_{2n}(0) = e^{(-1)^n \frac{(2n)!}{n!}}, \qquad \widetilde{H}_{2n+1}(0) = 1$,
iv. $\widetilde{H}_{2n}^*(0) = 1, \qquad \widetilde{H}_{2n+1}^*(0) = e^{(-1)^n \frac{(2n+2)!}{(n+1)!}}$,
for $n = 0, 1, 2, \ldots$.

Proof. From multiplicative Rodrigues formula, the proofs of these features can be easily made similar to the classical situation. □

3. Some spectral properties of multiplicative Hermite problem

In this section, we consider the problem (2.2)-(2.3). As will be remembered from the previous section, general solution of the equation (2.2) is represented (2.4).
Now, let's express some spectral properties of multiplicative Hermite polynomial. The following lemma has been proved by [35] with different perspective. Here, the proof is made using multiplicative Sturm-Liouville equation.

Lemma 3.1. *The multiplicative Hermite polynomials $\widetilde{H}_n(x)$ and $\widetilde{H}_m(x)$ are orthogonal according to the weight function $e^{e^{-x^2}}$ on $(-\infty, \infty)$ for $m \neq n$. Furthermore,*

$$\int_{-\infty}^{\infty} \left[e^{e^{-x^2}} \odot \widetilde{H}_n(x) \odot \widetilde{H}_m(x) \right]^{dx} = \begin{cases} 1, & if\ m \neq n \\ e^{2^n n! \sqrt{\pi}}, & if\ m = n \end{cases}.$$

Proof. Let's do the proof separately for two cases.
i. Let $m \neq n$. Since multiplicative Hermite polynomials $\widetilde{H}_n(x)$ and $\widetilde{H}_m(x)$ are solutions of the equation (2.2), we can write

$$\left[\left(\widetilde{H}_n^*(x) \right)^{e^{-x^2}} \right]^* \left\{ \widetilde{H}_n(x) \right\}^{2ne^{-x^2}} = 1, \tag{3.1}$$

$$\left[\left(\widetilde{H}_m^*(x) \right)^{e^{-x^2}} \right]^* \left\{ \widetilde{H}_m(x) \right\}^{2me^{-x^2}} = 1. \tag{3.2}$$

Let us take $\ln \widetilde{H}_m(x)$ and $\ln \widetilde{H}_n(x)-$th powers of (3.1) and (3.2) respectively. Then, if we use multiplicative integration to both sides on $(-\infty, \infty)$ after the obtained relations are divided by side, we get

$$\left[\int_{-\infty}^{\infty} \left\{ \left(\widetilde{H}_n^{\ln \widetilde{H}_m} \right)^{e^{-x^2}} \right\}^{dx} \right]^{2(m-n)} = \int_{-\infty}^{\infty} \left[\frac{d^*}{dx} \left(W \left(\widetilde{H}_m, \widetilde{H}_n \right) \right)^{e^{-x^2}} \right]^{dx},$$

where $W\left(\widetilde{H}_m, \widetilde{H}_n\right) = \left(\widetilde{H}_m \odot \widetilde{H}_n^*\right) \ominus \left(\widetilde{H}_m^* \odot \widetilde{H}_n\right)$. Since $m \neq n$ and the conditions (2.3), it gives

$$\int_{-\infty}^{\infty} \left\{ \left(\widetilde{H}_n(x) \odot \widetilde{H}_m(x) \right)^{e^{-x^2}} \right\}^{dx} = 1. \tag{3.3}$$

So, the proof for $m \neq n$ is completed.

ii. Let $m = n$. After taking $n \to n-1$ in the equation (2.7), circle multiplying the equation (2.7) by $e^{e^{-x^2}} \odot \widetilde{H}_n(x)$, we have

$$\left\{e^{e^{-x^2}} \odot \widetilde{H}_n(x) \odot \widetilde{H}_n(x)\right\} \left\{\widetilde{H}_n(x) \odot \widetilde{H}_{n-1}(x)\right\}^{-2xe^{-x^2}} \left\{\widetilde{H}_n(x) \odot \widetilde{H}_{n-2}(x)\right\}^{2(n-1)e^{-x^2}} = 1.$$

Then, if we use (3.3) after *integrating to both sides of the last equation on $(-\infty, \infty)$, we get

$$\int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_n(x) \odot \widetilde{H}_n(x)\right)^{e^{-x^2}}\right\}^{dx} \int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_n(x) \odot \widetilde{H}_{n-1}(x)\right)^{-2xe^{-x^2}}\right\}^{dx} = 1. \tag{3.4}$$

On the other hand, let's circle multiply the equation (2.7) by $e^{e^{-x^2}} \odot \widetilde{H}_{n-1}(x)$, then, if we use (3.3) after *integrating both sides of this equation on $(-\infty, \infty)$, we have

$$\int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_{n-1}(x) \odot \widetilde{H}_{n-1}(x)\right)^{2ne^{-x^2}}\right\}^{dx} \int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_n(x) \odot \widetilde{H}_{n-1}(x)\right)^{-2xe^{-x^2}}\right\}^{dx} = 1. \tag{3.5}$$

Considering (3.4) and (3.5) together, the following recurrence relation is obtained:

$$\int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_n(x) \odot \widetilde{H}_n(x)\right)^{e^{-x^2}}\right\}^{dx} = \int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_{n-1}(x) \odot \widetilde{H}_{n-1}(x)\right)^{2ne^{-x^2}}\right\}^{dx}$$

After taking $n = 1, 2, 3, \ldots$ and using $\int_{-\infty}^{\infty} \left\{e^{e^{-x^2}}\right\}^{dx} = e^{\sqrt{\pi}}$, gives

$$\int_{-\infty}^{\infty} \left\{\left(\widetilde{H}_n(x) \odot \widetilde{H}_n(x)\right)^{e^{-x^2}}\right\}^{dx} = e^{2^n n! \sqrt{\pi}}.$$

It completes the proof. □

Lemma 3.2. **eigenvalues of the problem (2.2)-(2.3) are all real.*

Proof. Let λ be a complex eigenvalue for $y(x, \lambda)$. Then, $\overline{\lambda}$ is eigenvalue corresponding to $\overline{y(x, \lambda)}$. By (3.3),

$$\left\{\int_{-\infty}^{\infty} \left\{\left(y^{\ln \bar{y}}\right)^{e^{-x^2}}\right\}^{dx}\right\}^{\lambda - \overline{\lambda}} = 1.$$

By the notion of multiplicative integration,

$$\left\{\int_{-\infty}^{\infty} e^{-x^2} |\ln y|^2 dx\right\}^{\lambda - \overline{\lambda}} = 0.$$

Since y must be a non-trivial solution, i.e. $y \neq 1$, and $\int_{-\infty}^{\infty} e^{-x^2} |\ln y|^2 dx > 0$, we get $\lambda = \overline{\lambda}$. This completes the proof. □

Self-adjoint operators are used in functional analysis and quantum mechanics. In quantum mechanics their importance lies in which physical observables such as position, momentum, angular momentum and spin are represented by self-adjoint operators. Because of this importance, we examine the self-adjointness of the multiplicative Hermite operator.

Lemma 3.3. *Multiplicative Hermite operator L on (2.2) is self-adjoint in $L_2^*(\mathbb{R})$.*

Proof. Assume that u and v are positive multiplicative Hermite polynomials on $\mathbb{R}$ and $u, v \in C^{*(2)}$ where $C^{*(2)}$ is the set of all functions whose second order multiplicative derivatives are continuous. By the definition of multiplicative Hermite operator and multiplicative derivative, we get

$$\frac{(Lv)^{\ln u}}{(Lu)^{\ln v}} = \frac{d^*}{dx}\left[(W(u, v))^{e^{-x^2}}\right]. \tag{3.6}$$

By *integrating both sides of (3.6) on $\mathbb{R}$,

$$\int_{-\infty}^{\infty}\left[\frac{(Lv)^{\ln u}}{(Lu)^{\ln v}}\right]^{dx}=\frac{\lim\limits_{x\to-\infty}(W(u,v))^{e^{-x^2}}}{\lim\limits_{x\to\infty}(W(u,v))^{e^{-x^2}}}$$

and using the properties of limit, we acquire

$$<Lu,v>_*=<u,Lv>_*$$

This indicates that the given multiplicative Hermite operator is self-adjoint on $L_2^*(\mathbb{R})$. □

4. Conclusion

Hermite's differential equation is frequently encountered in physics and engineering. It arises in numerous problems, particularly in boundary value problems for spheres. Because Hermite polynomials with these equations and their solutions have such an important place, we carried these concepts to multiplicative analysis. First, we set up Hermite equation in multiplicative analysis. Then, we obtained multiplicative Hermite polynomials for different situations using multiplicative series methods. Finally, we examined some spectral properties of these multiplicative polynomials. In fact, these investigations coincide with spectral properties of a much more complex nonlinear equation in classical case.

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Characterizations of Adjoint Curves According to Alternative Moving Frame

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Abstract

In this paper, the adjoint curve is defined by using the alternative moving frame of a unit speed space curve in 3-dimensional Euclidean space. The relationships between Frenet vectors and alternative moving frame vectors of the curve are used to offer various characterizations. Besides, ruled surfaces are constructed with the curve and its adjoint curve, and their properties are examined. In the last section, there are examples of the curves and surfaces defined in the previous sections.

1. Introduction

In differential geometry, the theory of curves in the 3-dimensional Euclidean space E^3 is one of the leading fields of study. In terms of curves, the most interesting curves in recent years are helices and slant helices[1, 2]. However, curves associated with a given curve are also widely studied. Among these curves, the most studied are Bertrand curve pairs, Mannheim curve pairs and involute-evolute curve pairs [3, 4]. In addition to the aforementioned pairs of curves, there are associated curves that have gained a lot of popularity. We can list some of them as the principal normal-direction curve, binormal-direction curve, principal-donor curve and binormal-donor curve, which were defined by Choi and Kim in 2012 with the help of integral curves [5]. With the adjoint curves discussed in 2019, a new definition of binormal-direction curves has been introduced. Also in this study, characterizations of adjoint curves and ruled and tube surfaces associated with adjoint curves were studied [6]. The W-direction curves in Macit and Düldül's paper are another reference for integral curves. Here, the relationships between a curve and the integral curve of the vector W of this curve are given. The relationships between the curvatures of the associated curves are explained and the characterizations of the curves are studied [7].
The curves are generally characterized by a moving Frenet frame. However, it may not be possible to obtain characterizations by using this frame or it is difficult to characterize them in some cases. For this reason, it will be useful to examine the curves with the help of another moving frame. In 2016, Yaylı et al. defined an alternative moving frame on the curve in their study [8]. The ruled surfaces introduced by Monge is one of the most frequently studied topics in differential geometry. The ruled surfaces have application areas especially in kinematics, computer-aided geometric design, architecture and many other fields. Any ruled surface occurs as a result of the continuous movement of a line along a curve. The ruled surfaces have been studied in differential geometry in different spaces, different dimensions and different frames [9]-[14].
In this study, a new definition is given to the W-direction curve of a curve in an alternative moving frame. This curve, which is direction curve considered with an alternative moving frame, is defined and characterized as $W-$adjoint curve in E^3. The significant relationships are founded between alternative moving frame apparatus and Frenet frame apparatus of curve pairs occuring. Then, the ruled surfaces associated with these direction curves are studied. The ruled surfaces obtained by different

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variations of the associated curves of the base curve and the direction curve are given. Moreover, Maple software is applied to model the data in this paper.

2. Preliminaries

In this section, let's remember the basic concepts in differential geometry:
A curve α is defined by coordinate neighborhood (I,α) in E^n, where $I \subseteq R$ is an open interval and $\alpha : I \to E^n$ $(t \to \alpha(t))$ is differentiable function. A curve whose velocity vector at each point is nonzero is called a regular curve. That is, $\alpha'(s) \neq 0$ for $\forall s \in I$.
Let's the curve α be given with neighborhood (I,α). If $\|\alpha'(s)\| = 1$, for $\forall s \in I$. α is called a unit speed curve according to (I,α). In this case, the parameter $s \in I$ of the curve is called the arc length parameter.
The orthonormal basis vectors T,N,B, also known as the Frenet frame or TNB frame, correspond to each point of a unit speed curve in three-dimensional Euclidean space. Here, $T = \alpha^{'}$ is the unit tangent vector field, $N = \frac{\alpha''}{\|\alpha''\|}$ is the principal normal vector field, and $B = T \times N$ is the binormal vector field. Furthermore, the Frenet formulas $T^{'}(s) = \kappa(s)N(s)$, $N^{'}(s) = -\kappa(s)T(s) + \tau(s)B(s)$, and $B^{'}(s) = -\tau(s)N(s)$, where $\kappa(s) > 0$ and $\tau(s)$ are curvature and torsion at the point $\alpha(s)$, respectively. In terms of the Frenet-Serret apparatus, the Darboux vector w can be expressed as $w = \tau T + \kappa B$. Here, we can write

$$\kappa = \|w\| \cos\phi, \tau = \|w\| \sin\phi,$$

where ϕ is the angle between B and w. If the unit vector in the direction w is W, $W = \frac{\tau}{\|w\|}T + \frac{\kappa}{\|w\|}B$, where $\|w\| = \sqrt{\kappa^2 + \tau^2} \geq 0$ [15].
In Euclidean 3-space, the alternative moving frame along the curve α is given by $\{N,C,N \times C = W\}$. Here, the unit principal normal vector, the derivative of the principal normal vector, and the Darboux vector, respectively, are N, $C = \frac{N^{'}}{\|N'\|}$ and $W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}$. The following equations produce the derivative vectors of these vectors:

$$\begin{aligned} N^{'}(s) &= f(s)C(s), \\ C^{'}(s) &= -f(s)N(s) + g(s)W(s), \\ W^{'}(s) &= -g(s)C(s), \end{aligned}$$

where

$$f = \kappa\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}$$

and

$$g = \frac{\kappa^2}{\kappa^2 + \tau^2}\left(\frac{\tau}{\kappa}\right)'$$

are the first and second curvature of the curve $\alpha(s)$ with respect to alternative moving frame, respectively [8].
A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio of curvature to torsion is constant [15].

Definition 2.1. *Let α be a unit speed curve in E^3 with non-zero torsion and the Frenet frame of α be $\{T_\alpha, N_\alpha, B_\alpha\}$. The adjoint curve of α is defined as [6]*

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s)\, ds.$$

Theorem 2.2. *If α is a curve with arc length parameter s, then the arc length parameter of adjoint curve of α is also s [6].*

Theorem 2.3. *Let α be a curve with arc length s and β be the adjoint curve of α. If the Frenet vectors of α and β are $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$, the curvature and torsion are κ_α, τ_α and κ_β, τ_β respectively, then the following relations hold [6]:*

$$\begin{aligned} T_\beta &= B_\alpha \\ N_\beta &= -N_\alpha \\ B_\beta &= T_\alpha \end{aligned}$$

and

$$\begin{aligned} \kappa_\beta &= \tau_\alpha \\ \tau_\beta &= \kappa_\alpha. \end{aligned}$$

Corollary 2.4. *If a is a general helix parametrized by arc length parameter s, then the adjoint curve β of α is a general helix.*

Definition 2.5. *Let α be a Frenet curve in E^3 and W be the unit Darboux vector field of α. We call an integral curve of $W(s)$ the W-direction curve of α. Namely, if $\beta(s)$ is the W-direction curve of α, then*

$$W(s)=\beta'(s),$$

where $W=\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}}$ [7].

Theorem 2.6. *Let α be a Frenet curve in E^3 with the curvature κ and the torsion τ, and β be W-direction curve of α. If α is not a general helix, then the curvature κ_β and the torsion τ_β of β are given by [7]*

$$\kappa_\beta=\frac{|\tau\kappa'-\tau'\kappa|}{\kappa^2+\tau^2}$$

$$\tau_\beta=\sqrt{\kappa^2+\tau^2}.$$

Theorem 2.7. *Let β be the W-direction curve of a nonplanar curve α. Then α is a general helix if and only if β is a straight line [7].*

Definition 2.8. *A ruled surface in E^3 may therefore be represented in the form*

$$\varphi(\alpha,d):I\times E\to E^3,(s,v)\to\varphi(\alpha,d)(s,v)=\alpha(s)+vd(s)$$

such that $\alpha:I\to E^3,d:I\to E^3\setminus\{0\}$ are differentiable transformations. Here, α is called base curve and d is called the director curve [16].

The distribution parameter of a ruled surface parameterized by

$$\varphi(s,v)=\alpha(s)+vX(s),$$

where α is the base curve and X is the director curve, is the function D_X defined by

$$D_X=\frac{\det(\alpha',X,X')}{\|X'\|^2}.$$

A developable ruled surface is characterized by $D_X=0$ [15].

Definition 2.9. *Let $\alpha(s)$ be a curve with arc length in E^3 and $\{N,C,W\}$ be the alternative moving frame of α. The $C-$ruled surface can be given by the following parameterization as [17]*

$$\varphi(s,v)=\alpha(s)+vC(s),$$

and the $W-$ruled surface can be given by the following parameterization as [18]

$$\varphi(s,v)=\alpha(s)+vW(s).$$

3. $W-$adjoint curve

Let α be a unit speed curve in E^3 with non-zero torsion and the alternative moving frame of α be $\{N_\alpha,C_\alpha,W_\alpha\}$. The $W-$adjoint curve of α can be write as

$$\beta(s)=\int_{s_0}^{s}W_\alpha(s)\,ds,\tag{3.1}$$

where $C_\alpha(s)=\frac{N'_\alpha(s)}{\left\|N'_\alpha(s)\right\|}$, $W_\alpha(s)=\frac{\tau_\alpha(s)T_\alpha(s)+\kappa_\alpha(s)B_\alpha(s)}{\sqrt{(\kappa_\alpha(s))^2+(\tau_\alpha(s))^2}}$. We know that $\{T_\alpha,N_\alpha,B_\alpha\}$ is Frenet frame of α and κ_α, τ_α are curvature and torsion of α, respectively. The derivative vectors of $\{N_\alpha,C_\alpha,W_\alpha\}$ can also be given as:

$$N'_\alpha(s)=f_\alpha(s)C_\alpha(s)$$

$$C'_\alpha(s)=-f_\alpha(s)N_\alpha(s)+g_\alpha(s)W_\alpha(s),$$

$$W'_\alpha(s)=-g_\alpha(s)C_\alpha(s),\tag{3.2}$$

where

$$f_\alpha(s)=\sqrt{\kappa_\alpha(s)^2+\tau_\alpha(s)^2},$$

$$g_\alpha(s)=\frac{\kappa_\alpha(s)^2}{\sqrt{(\kappa_\alpha(s))^2+(\tau_\alpha(s))^2}}\left(\frac{\tau_\alpha(s)}{\kappa_\alpha(s)}\right)'.\tag{3.3}$$

Theorem 3.1. *If α is a curve with arc length parameter s, then the arc length parameter of $W-$adjoint curve of α is also s.*

Proof. By differentiating both sides of (3.1), we have

$$\frac{d}{ds}\beta(s) = W_\alpha(s).$$

Here, if we take the norm of both sides and we use $\|W_\alpha(s)\| = 1$, we obtain $\|\beta'(s)\| = 1$. This means that β is a unit speed curve and

$$T_\beta(s) = W_\alpha(s), \tag{3.4}$$

where $T_\beta(s)$ is unit tangent vector of β. □

Theorem 3.2. *Let α be a curve with arc length s and β be the $W-$adjoint curve of α. If the alternative moving frame vectors of α and β are $\{N_\alpha, C_\alpha, W_\alpha\}$ and $\{N_\beta, C_\beta, W_\beta\}$, curvatures according to the alternative moving frame of α and β are $\{f_\alpha, g_\alpha\}$ and $\{f_\beta, g_\beta\}$ respectively, then the following relations hold:*

$$N_\beta = -C_\alpha, \tag{3.5}$$

$$W_\beta = \frac{f_\alpha W_\alpha + g_\alpha N_\alpha}{\sqrt{{f_\alpha}^2 + {g_\alpha}^2}}, \tag{3.6}$$

and

$$C_\beta = \frac{f_\alpha N_\alpha - g_\alpha W_\alpha}{\sqrt{{f_\alpha}^2 + {g_\alpha}^2}}. \tag{3.7}$$

Proof. If we take derivative both sides of (3.4) and divide by their norms, we get

$$N_\beta(s) = \frac{W'_\alpha(s)}{\|W'_\alpha(s)\|}.$$

Considering (3.2), we have (3.5). We know that we can write

$$W_\beta = \frac{\tau_\beta T_\beta + \kappa_\beta B_\beta}{\sqrt{{\kappa_\beta}^2 + {\tau_\beta}^2}}. \tag{3.8}$$

Now let's write W_β in terms of alternative moving frame apparatus of α. In that case from (3.4), (3.5) and the equation $B_\beta = T_\beta \times N_\beta$, we obtain $B_\beta = W_\alpha \times (-C_\alpha)$ and

$$B_\beta = N_\alpha. \tag{3.9}$$

If we take derivative both sides of (3.4) and we use (3.2), we obtain $T'_\beta = -g_\alpha C_\alpha$. In last equation, if we take the norm of both sides, we get $\left\|T'_\beta\right\| = g_\alpha$. This means that

$$\kappa_\beta = g_\alpha. \tag{3.10}$$

If we take derivative both sides of (3.9) and we use Frenet and alternative moving frame formulae, we have

$$-\tau_\beta N_\beta = f_\alpha C_\alpha.$$

From (3.5), we get

$$\tau_\beta = f_\alpha. \tag{3.11}$$

Hence, if we use (3.4), (3.9), (3.10) and (3.11) in (3.8), we obtain (3.6).
On the other hand, it is known that

$$C_\beta = \frac{-\kappa_\beta T_\beta + \tau_\beta B_\beta}{\sqrt{{\kappa_\beta}^2 + {\tau_\beta}^2}}. \tag{3.12}$$

If we use (3.4), (3.9), (3.10) and (3.11) in (3.12), we get (3.7). □

Theorem 3.3. *The relationships between alternative moving frame curvatures $\{f_\alpha, g_\alpha\}$ and $\{f_\beta, g_\beta\}$ with respect to α and β are*

$$f_\beta = \sqrt{f_\alpha{}^2 + g_\alpha{}^2} \tag{3.13}$$

and

$$g_\beta = \frac{g_\alpha^2}{\sqrt{(g_\alpha)^2 + (f_\alpha)^2}} \left(\frac{f_\alpha}{g_\alpha}\right)'. \tag{3.14}$$

Proof. The relationships between the curvatures with respect to Frenet frame and alternative moving frame of the unit speed curve β are

$$f_\beta = \sqrt{\kappa_\beta^2 + \tau_\beta^2}, \tag{3.15}$$

$$g_\beta = \frac{\kappa_\beta^2}{\sqrt{(\kappa_\beta)^2 + (\tau_\beta)^2}} \left(\frac{\tau_\beta}{\kappa_\beta}\right)'. \tag{3.16}$$

If we use (3.10), (3.11) in the equations (3.15) and (3.16), we easily get (3.13) and (3.14). □

Theorem 3.4. *Let α be a nonplanar curve with arc length s in E^3. α is a helix if and only if $g_\alpha = 0$, where g_α is second curvature with respect to alternative moving frame of α.*

Proof. It is known that if α is helix, $\frac{\tau_\alpha}{\kappa_\alpha} = c$ $(c = constant)$. Here, if we take derivative of both sides we have

$$\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = 0.$$

Considering (3.3), we write $g_\alpha = 0$.
Conversely, if $g_\alpha = 0$, from (3.3) we get $\frac{\kappa_\alpha^2}{\sqrt{(\kappa_\alpha)^2+(\tau_\alpha)^2}} \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = 0$. Since α is a nonplanar curve, $\kappa \neq 0$ and $\tau \neq 0$. Then we can say $\frac{\tau_\alpha}{\kappa_\alpha} = c$ $(c = constant)$. Thus, α is a helix. □

Theorem 3.5. *Let α be a curve with arc length s in E^3 and β be W-adjoint curve of α. β is helix if and only if the ratio $\frac{f_\alpha}{g_\alpha}$ is constant.*

Proof. If β is helix, $\frac{\tau_\beta}{\kappa_\beta} = c$. From (3.10) ve (3.11), we obtain

$$\frac{f_\alpha}{g_\alpha} = c. \tag{3.17}$$

Conversely, given by (3.17). From (3.10) and (3.11), we have $\frac{\tau_\beta}{\kappa_\beta} = c$. Hence, β is a helix. This completes the proof. □

4. Ruled surfaces associated with $W-$adjoint curve

4.1. Ruled surface with base curve α and director curve β

We examine ruled surface created by a curve and $W-$adjoint curve of this curve under this heading. Let α be a curve with arc length s and β be $W-$adjoint curve of α. The ruled surface with the base curve α and the director curve β can be defined by

$$\phi(s,v) = \alpha(s) + v\beta(s). \tag{4.1}$$

Theorem 4.1. *Let α be a curve with arc length s and β be $W-$adjoint curve of α. Given by $\{T_\beta, N_\beta, B_\beta\}$ is the Frenet frame of β and $\{N_\alpha, C_\alpha, W_\alpha\}$ is alternative moving frame of α. Distribution parameter of the ruled surface given by (4.1) is*

$$D_\phi = \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \langle \beta, B_\beta \rangle, \tag{4.2}$$

where β is position vector of the curve β.

Proof. If we calculate distribution parameter of the ruled surface given by (4.1), we have

$$D_\phi = \frac{\det\left(\frac{d\alpha}{ds}, \beta, \frac{d\beta}{ds}\right)}{\left\|\frac{d\beta}{ds}\right\|^2}.$$

From (3.4), we get

$$D_\phi = \frac{\det\left(T_\alpha, \beta, W_\alpha\right)}{\left\|T_\beta\right\|^2}.$$

$$D_\phi = \det\left(T_\alpha, \beta, \frac{\tau_\alpha T_\alpha + \kappa_\alpha B_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}\right)$$

$$D_\phi = \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \det\left(T_\alpha, \beta, T_\alpha\right) + \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \det\left(T_\alpha, \beta, B_\alpha\right)$$

Since $\det(T_\alpha, \beta, T_\alpha) = 0$, we can write

$$D_\phi = \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \det\left(T_\alpha, \beta, B_\alpha\right).$$

Using determinant and mixed product properties, we have

$$D_\phi = \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \left\langle \beta, N_\alpha \right\rangle. \tag{4.3}$$

If we substitute the equation (3.9) in (4.3), we have (4.2). □

Corollary 4.2. *The ruled surface given by (4.1) is developable if and only if the position vector β and the binormal vector of β are orthogonal. In this case $D_\phi = 0$.*

Proof. The ruled surface is developable if and only if $D_\phi = 0$. Then, let us consider the equation (4.2). Since $\kappa_\alpha \neq 0$, $\left\langle \beta, B_\beta \right\rangle = 0$. Then, surface given by (4.1) is developable. □

4.2. Ruled surface with base curve β and director curve α

Let α be a curve with arc length s and β be $W-$adjoint curve of α. The ruled surface with the base curve β and the director curve α can be defined by

$$\psi(s, v) = \beta(s) + v\alpha(s). \tag{4.4}$$

Theorem 4.3. *Let α be a curve with arc length s and β be $W-$adjoint curve of α. Given by $\left\{T_\beta, N_\beta, B_\beta\right\}$ is the Frenet frame of β and $\{N_\alpha, C_\alpha, W_\alpha\}$ is alternative moving frame of α. Distribution parameter of the ruled surface given by (4.4) is*

$$D_\psi = -\frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} \left\langle \alpha, N_\alpha \right\rangle,$$

where α is position vector of the curve α.

Corollary 4.4. *The ruled surface given by (4.4) is developable if and only if the position vector α and the normal vector of α are orthogonal.*

5. Applications

Example 5.1. *Let us consider the curve $\alpha(s)$ with arc length s in E^3 given by*

$$\alpha(s) = \left(-\frac{1}{12}\cos(4s) - \frac{1}{3}\cos(2s), \frac{1}{12}\sin(4s) + \frac{1}{3}\sin(2s), -\frac{2\sqrt{2}}{3}\cos(s)\right)$$

(see Figure 5.1 and Figure 5.3). $W-$adjoint curve of α is

$$\beta(s) = \left(-\frac{1}{9}\sin(3s), \frac{1}{9}\cos(3s), -\frac{2\sqrt{2}}{3}s\right)$$

(see Figure 5.2 and Figure 5.3).

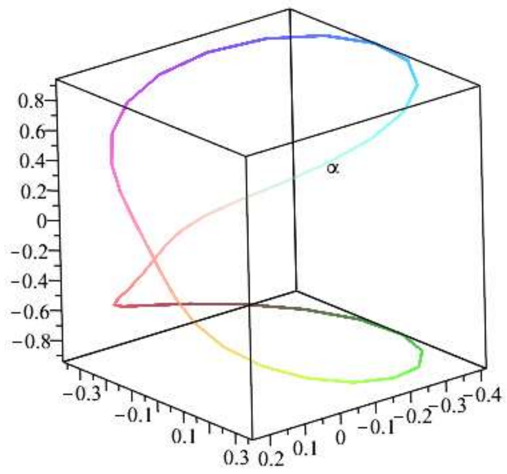

Figure 5.1: The curve $\alpha(s)$

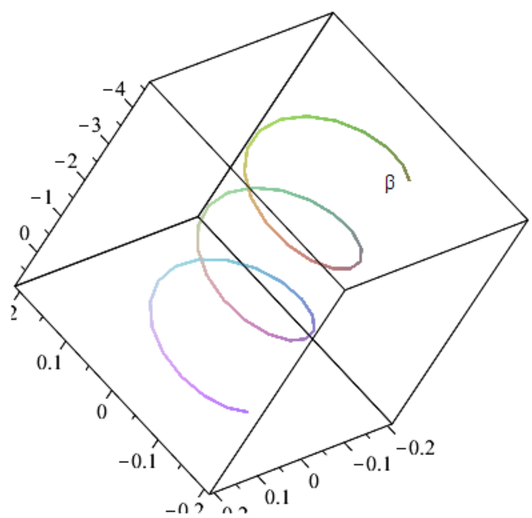

Figure 5.2: $W-$Adjoint curve of $\alpha(s)$

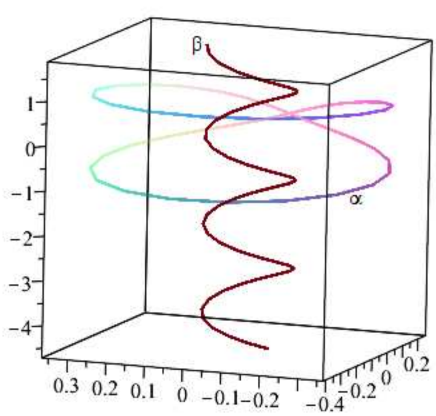

Figure 5.3: The curves $\alpha(s)$ and $\beta(s)$

Example 5.2. *Let's exemplify the ruled surfaces associated with the α and $W-$adjoint curve of α that we took in Example 5.1. First, let's write the ruled surface with base curve is α and director curve is $W-$adjoint curve of α*

$$\phi_\alpha(s,v)=(-\frac{1}{12}\cos(4s)-\frac{1}{3}\cos(2s),\frac{1}{12}\sin(4s)+\frac{1}{3}\sin(2s),-\frac{2\sqrt{2}}{3}\cos(s))+v(-\frac{1}{9}\sin(3s),\frac{1}{9}\cos(3s),-\frac{2\sqrt{2}}{3}s)$$

(see Figure 5.4). On the other hand, the ruled surface with the base curve W and the director curve α is

$$\phi(s,v)=(-\frac{1}{9}\sin(3s),\frac{1}{9}\cos(3s),-\frac{2\sqrt{2}}{3}s)+v(-\frac{1}{12}\cos(4s)-\frac{1}{3}\cos(2s),\frac{1}{12}\sin(4s)+\frac{1}{3}\sin(2s),-\frac{2\sqrt{2}}{3}\cos(s))$$

(see Figure 5.5).

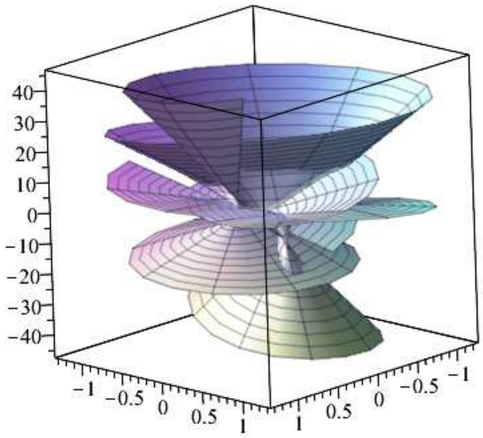

Figure 5.4: $\phi(s,v)=\alpha(s)+v\beta(s)$

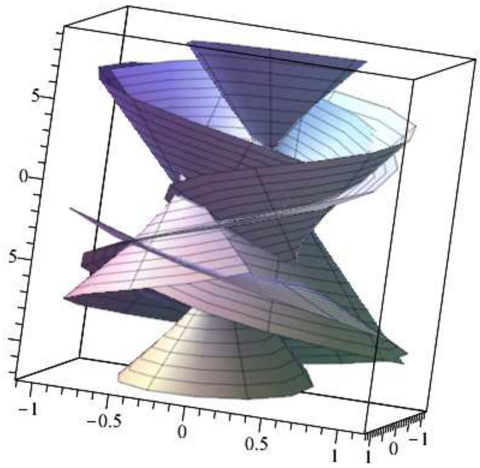

Figure 5.5: $\phi(s,v)=\beta(s)+v\alpha(s)$

6. Conclusion

In this study, the curve β is defined as the W-adjoint curve of the curve α with respect to alternative moving frame. The relationships are established between the alternative moving frame vectors of the curves α and β. In addition, connections between the curvatures defined in the alternative moving frame are constructed. The results relating to the helix curve are collected at this point. The ruled surfaces created with the curves α and β are obtained.It is found under which conditions the acquired ruled surfaces may be developable. In the last section, it is reinforced with examples.

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A New Aspect for Some Sequence Spaces Derived Using the Domain of the Matrix $\widehat{\widehat{B}}$

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Abstract

This study serves for analysing algebraic and topological characteristics of the sequence spaces $X(\widehat{\widehat{B}}(r,s))$ constituted by using non-zero real number r and s, where X denotes arbitrary of the classical sequence spaces ℓ_∞, c, c_0 and ℓ_p $(1 < p < \infty)$ of bounded, convergent, null and absolutely p-summable sequences, respectively and $X(\widehat{\widehat{B}})$ also is the domain of the matrix $\widehat{\widehat{B}}(r,s)$ in the sequence space X. Briefly, the β- and γ-duals of the space $X(\widehat{\widehat{B}})$ are computed, and Schauder bases for the spaces $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are determined, and some algebraic and topological properties of the spaces $c_0(\widehat{\widehat{B}})$, $\ell_1(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are studied. Additionally, it is observed that all these spaces have some remarkable features, including the classes $(X_1(\widehat{\widehat{B}}): X_2)$ and $(X_1(\widehat{\widehat{B}}) : X_2(\widehat{\widehat{B}}))$ of infinite matrices which are characterized, in which $X_1 \in \{\ell_\infty, c, c_0, \ell_p, \ell_1\}$ and $X_2 \in \{\ell_\infty, c, c_0, \ell_1\}$.

1. Background and historical developments

One of the non-classical approaches when building new sequence space used recently in summability is that of working with any infinite matrix. Although this technique is not easy, it provides a quick technique in obtaining certain results if the inverse of an infinite matrix is present. In addition to the different aspects of this technique used in the listed references at the end of the article, much more detailed information can be found in the five books of Başar [1], Başar and Dutta [2], Mursaleen and Başar [3], Mursaleen [4] and Malafosse et al. [5] published recently. We now remind some basic definitions and conclusions, which we will mainly use in the following sections. Any x sequence in X is a transformation $x : \mathbb{N} \to X$, where X is a non-empty set. The collection of all real or complex number sequences forms a vector space which we denote by w, under the operations of coordinate-wise addition and well-known scalar multiplication. The subspaces of w are important in such applications because each of them is called a sequence space. We denote ℓ_∞, c, c_0 and ℓ_p for the classical sequence spaces of all bounded, convergent, null and absolutely $p-$summable sequences, respectively. bv is the space consisting of all sequences (x_k) such that $(x_k - x_{k+1})$ in ℓ_1 and bv_0 is the intersection of the spaces bv and c_0 where $k \in \mathbb{N}$. Unless otherwise stated, all other chapters shall also be applicable to $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and utilize the fact that each term having negative subscript equals to zero.

Let us remember the definition of another concept we need. Given an infinite matrix $A = (a_{nk})$ of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, for any sequence x, we write

$$(Ax)_n := \sum a_{nk}x_k; \quad (n \in \mathbb{N}, x \in D_{00}(A)), \tag{1.1}$$

where $D_{00}(A)$ denotes the subspace of ω consisting of $x \in \omega$ for which the sum exists as a finite sum. For simplicity in illustration, here and wherever after that, the summation without limits runs from 0 to ∞.

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Now, let us continue by giving the definition of a matrix transformation between arbitrary sequence spaces X, Y that will be required in the following sections. Having supposed that Y is a normed sequence space, it is natural to consider the question of whether or not the sum in (1.1) is converges in the norm Y for $x \in X$, for this situation we write $D_Y(A)$. In this meaning, $(X : Y) := \{A : X \subseteq D_Y(A)\}$ is written for the space of all matrices that satisfy the condition mentioned in the previous row, which send the all of X into Y.

We need a definition and some of its results, which provide some advantage in our paper. A matrix $T = (t_{nk})$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n \in \mathbb{N}$. As the immediate consequences of this concept, we have the following useful results. Let U and V two triangle matrix and x any sequence then $U(Vx) = (UV)x$ is valid. Moreover, the inverse of such a matrix is always uniquely exist and at the same time has a triangle matrix as well. In a practical way, we can obtain, if the inverse of matrix U is V, then $x = U(Vx) = V(Ux)$ is always valid for all $x \in \omega$.

Now, we shall be concerned with certain properties of difference sequences. First of all, we define and discuss briefly the meaning of the concept of difference sequence spaces. In 1981 Kızmaz [6] defined new sequence spaces using the sequence $(x_k - x_{k+1})$ instead of working directly by a sequence $x = (x_k)$. Let X denote an arbitrary well-known classical sequence spaces ℓ_∞, c or c_0. Kızmaz [6] defined the sequence spaces $X(\Delta) = \{x = (x_k) \in w : \Delta x \in X\}$ where $\Delta x = (x_k - x_{k+1})$ and also showed that these are the Banach spaces with the norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty; \quad x = (x_k) \in X$. These spaces are called *difference sequence spaces*. His new method is an expansion of the classical sequence spaces, which are probably more familiar to most readers. In other words, the inclusion relation $X \subset X(\Delta)$ is strictly valid. Shows that many facts about difference sequence spaces. Kızmaz [6] obtained almost basic algebraic and topological properties in his work, including the $\alpha-$, $\beta-$ and $\gamma-$duals of the difference sequence spaces and $(X(\Delta) : Y)$ and $(Y : X(\Delta))$ of infinite matrices, where $X, Y \in \{\ell_\infty, c\}$. Following Kızmaz's Technique, Et [7] defined the sequence spaces $X(\Delta^2) = \{x = (x_k) \in w : \Delta^2 x \in X\}$ where $\Delta^2 x = (\Delta^2 x_k) = (\Delta x_k - \Delta x_{k+1})$ and $X \in \{\ell_\infty, c, c_0\}$. In 1987 Sarıgöl [8] introduced a new difference sequence spaces $X(\Delta_t) := \{x = (x_k) \in w : (\Delta_t x) = [k^t(x_k - x_{k+1})] \in X \ \ for \ \ t < 1\}$ which more complicated than the spaces of Kızmaz [6] and he observed its some algebraic and topological properties, where $X \in \{\ell_\infty, c, c_0\}$. It is the fact that it is easy to get both of the following inclusion relationships are valid: $X(\Delta_t) \subset X(\Delta)$, if $0 < t < 1$ and $X(\Delta) \subset X(\Delta_t)$, if $t < 0$. Simultaneously, the sequence spaces $X(p,\Delta)$ which are expanded from the previous ones defined by Ahmad and Mursaleen [9] and they studied various problems. Almost two years later, Malkowsky [10] introduced the sequence spaces $\ell_\infty(p,\Delta)$, $c_0(p,\Delta)$ and specified the Köthe-Toeplitz duals of them and proved characterization of the matrix transformations discussed in [9]. Later on, Choudhary and Mishra [11] examined certain characteristic of the sequence space $c_0(\Delta_t)$, for $t \geq 1$. In the same year, a characterization of BK-spaces involving a subspace which is isomorphic to $sc_0(\Delta)$ with respect to matrix maps obtained by Mishra [12] and a sufficient situation of a map from $s\ell_\infty(\Delta)$ into a BK-space for being compact operator. He proved that arbitrary matrix from $s\ell_\infty(\Delta)$ is compact, where $sX(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X, x_1 = 0 \ \ for \ \ X = \ell_\infty \ \ or \ \ c_0\}$. In the year 1996, Mursaleen et al. [13] interested in introducing and examining the sequence space $\ell_\infty(p,\Delta_r) = \{x = (x_k) \in w : \Delta_r x \in \ell_\infty(p)\}, \qquad (r > 0)$. Gnanaseelan and Srivastava [14] introduced and investigated the spaces $X(z,\Delta)$ for a non-complex numbers $z = (z_k)$ satisfying the following three conditions

(i) $\frac{|z_k|}{|z_{k+1}|} = 1 + O(1/k)$ for each $k \in \mathbb{N}_1 = \{1,2,3,\ldots\}$.
(ii) $k^{-1}|z_k|\sum_{i=0}^{k}|z_i|^{-1} = O(1)$.
(iii) $(k|z_k^{-1}|)$ is a sequence of positive numbers increasing monotonically to infinity.

Malkowsky [15] described the spaces $X(z,\Delta)$ for any fixed sequence $z = (z_k)$ not having restriction upon z in the same year. The author has also gave the proof of the fact that the sequence spaces $X(u,\Delta)$ are BK- spaces having the norm given by $\|x\| = \sup_{k\in\mathbb{N}} |u_{k-1}(x_{k-1} - x_k)| \quad$ with $\ u_0 = x_0 = 1$. Subsequently, Gaur and Mursaleen [16] defined a more general space $S_r(p,\Delta)$ using the space $S_r(\Delta)$, where $S_r(\Delta) = \{x = (x_k) \in w : (k^r|\Delta x_k|) \in c_0(p)\}, \quad (r \geq 1)$ and they characterized both the matrix classes $(S_r(p,\Delta) : \ell_\infty)$ and $(S_r(p,\Delta) : \ell_1)$. Almost simultaneously and independently of each other; Malkowsky et al. [17], and Asma and Çolak [18] defined the sequence spaces $X(p,u,\Delta)$ which is a generalization of the sequence spaces $X(u,\Delta)$ and examined some of their properties for $X \in \{\ell_\infty, c, c_0\}$. In 2001, the matrix classes $(\Delta X : Y)$ and $(\Delta X : \Delta Y)$ are characterized by Malkowsky and Mursaleen [19], where $X \in \{c_0(p), c(p), \ell_\infty(p)\}$ and $Y \in \{c_0(p), c(p), \ell_\infty(p)\}$.

What is very important for us and the framework of this study is the matrix domain. For this reason, in this paragraph is presented the definition of it. The connection between any sequence space X and any limitation matrix A as below lets us the concept known as a matrix domain X_A to describe;

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}, \tag{1.2}$$

which gives a sequence space. If X is a sequence space, then the continuous dual X_A^* of the space X_A is defined by $X_A^* := \{f : f = g \circ A, g \in X^*\}$

Now, the matrix domain concept is briefly analyzed. There may be a relationship between the new sequence space X_A made up of using the limitable matrix A and the original sequence spaces X. This relationship can come across us in different ways, depending on the choice of X and A. Let us explain what we have said with examples. In fact, we find the relation if $X \in \{\ell_\infty, c, c_0\}$ then we obtain that the inclusion we seek $X_S \subset X$ is strictly valid where $S = (s_{nk})$ is the summation matrix described by $s_{nk} = \begin{cases} 1 & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n). \end{cases}$ But, if X is an element of the set $X \in \{\ell_\infty, c, c_0, \ell_p\}$ then one can easily see that the inclusion $X \subset X_{\Delta^{(1)}}$ is strictly valid, where $\Delta^{(1)}_{nk} = \begin{cases} (-1)^{n-k} & , \quad (n-1 \le k \le n), \\ 0 & , \quad 0 \le k < n-1 \text{ or } (k > n). \end{cases}$ However, when we describe $X := c_0 \oplus span t$ together with $t = ((-1)^k)$ namely; $x \in X$ only when $x := z + \lambda x$ for some $z \in c_0$ and some $\lambda \in \mathbb{C}$, and take into consideration the given matrix A together with the rows A_n given by $A_n := (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, then we are going to obtain $Ae = t \in X$ but $At = e \notin X$, resulting in the conclusion that $t \in X \backslash X_A$ and $e \in X_A \backslash X$ in which $e = (1,1,1,\ldots)$ and $e^{(n)}$ is a given sequence of which its unique term different from zero is a 1 found in the nth position for every $n \in \mathbb{N}$. In view of this explanation, both of the sequence spaces X_A and X overlap each other; however, neither of them contains the other. The tendency to built a new sequence spaces from the old ones is a widely used method. One of the most popular of these methods is obtain new sequence spaces using the matrix domain of a certain limitation method.

Today, the tendency for building a new sequence spaces via matrix domain and its extensions in summability theory is really reaching a wide range. It is recommended to look at the references to see what a profusion of problems it solves and what a wide range of fields now it uses in the different mathematical models that bring understanding about it.

In analogy with the difference sequence spaces that you have seen in the paper of Kızmaz [6], the sequence spaces defined by Kirişçi and Başar [20] are a generalization of the previously defined spaces. With the r and s being non-zero real numbers, Kirişçi and Başar [20] constructed the respective spaces using the matrix $\widehat{B}$ defined as below

$$\widehat{b}_{nk}(r,s) = \begin{cases} r & , \quad (k=n), \\ s & , \quad (k=n-1), \\ 0 & , \quad (0 \le k < n-1 \ \text{ or } \ k > n), \end{cases}$$

for all $k,n \in \mathbb{N}$. We would like to multiply the matrix $\widehat{B}$ by itself and we want to use it. This old approach is nothing new. Here is the formal statement. Let r,s be non-zero reel numbers and define the band matrix $\widehat{\widehat{B}}(r,s) = \{\widehat{\widehat{b}}_{nk}(r,s)\}$ by

$$\widehat{\widehat{b}}_{nk}(r,s) = \begin{cases} r^2 & , \quad (n=k), \\ 2rs & , \quad (k=n-1), \\ s^2 & , \quad (k=n-2), \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for all $k,n \in \mathbb{N}$. To simplify the notation let us write $\widehat{\widehat{B}}(r,s) = \widehat{\widehat{B}}$ and so forth. We must state the fact that here, the newly defined matrix $\widehat{\widehat{B}}$ can be derived from the triple band matrix $B(r,s,t)$ used by Sönmez [21] and the main results are obtained independently from the paper of Sönmez [21].

Again in analogy with the sequence spaces that one can see in the paper of Kirişçi and Başar [20], the sequence spaces introduced by Candan [22] are a generalization of the previously defined spaces. With $\widetilde{r} = (r_n)_{n=0}^{\infty}$ and $\widetilde{s} = (s_n)_{n=0}^{\infty}$ being convergent sequences of positive real numbers. Candan [22] introduced the respective spaces using the matrix $\widetilde{B} = \widetilde{B}(\widetilde{r},\widetilde{s}) = \{b_{nk}(\widetilde{r},\widetilde{s})\}$ defined as below

$$b_{nk}(\widetilde{r},\widetilde{s}) = \begin{cases} r_n & , \quad (k=n), \\ s_n & , \quad (k=n-1), \\ 0 & , \quad (0 \le k < n-1 \ \text{ or } \ k > n), \end{cases}$$

for all $k,n \in \mathbb{N}$. The $\widetilde{B}(\widetilde{r},\widetilde{s})-$ transforms of a sequence $x = (x_k)$ is $\widetilde{B}(\widetilde{r},\widetilde{s})(x) = r_k x_k + s_{k-1} x_{k-1}$ for all $k \in \mathbb{N}$. In the last decade, Candan [22]-[24] has worked on many different studies using the $\widetilde{B}$ matrix.

The main emphasis in this paper is going to be on defining the sequence space $X(\widehat{\widehat{B}})$ and continuing with explanations of the properties accounting for their importance in scientific work, and determining the $\beta-$ and $\gamma-$duals of the spaces, in which $\widehat{\widehat{B}}$ denotes the any of the classical spaces ℓ_∞, c, c_0 or ℓ_p and $\widehat{\widehat{B}}$ is the band matrix $\widehat{\widehat{B}}(r,s)$. Moreover, the Schauder basis for the space $c(\widehat{\widehat{B}}), c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are obtained, and some topological properties of the spaces $c(\widehat{\widehat{B}}), c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are studied. Finally, some classes of matrix mappings on the space $X(\widehat{\widehat{B}})$ are calculated.

The present paper is roughly composed as follows: In Section 1, we explain the kinds of sequence spaces that arise in scientific study, including basic concepts, historical developments of some subjects and matrix domain etc. In Section 2, the domain $X(\widehat{\widehat{B}})$ within the sequence space X with $X \in \{\ell_\infty, c, c_0, \ell_p\}$ is going to be introduced, and the $\beta-$ and $\gamma-$ duals of $X(\widehat{\widehat{B}})$ will be determined. The Schauder basis of the spaces $c(\widehat{\widehat{B}}), c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are given after a proof is given about under which conditions the equality $X = X(\widehat{\widehat{B}})$ and inclusion $X \subset X(\widehat{\widehat{B}})$ are valid. In final section, some topological properties of those spaces $c_0(\widehat{\widehat{B}})$, $\ell_1(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ having $p > 1$ are investigated. In Section 3, a general theorem which characterizes the matrix transformations from the domain of a triangle matrix into any sequence spaces is stated and also proven. To present the application of this fundamental theorem, a table is given showing the necessary and sufficient conditions for a matrix transformations from $X(\widehat{\widehat{B}})$ to Y in which $X \in \{\ell_\infty, c, c_0, \ell_p\}$ and $Y \in \{\ell_\infty, c, c_0, \ell_p\}$.

2. Some new sequence spaces derived by the domain of the matrix $\widehat{\widehat{B}}(r,s)$

Section 2 is devoted to a quick review of the newly defined sequence spaces derived by using a band matrix $\widehat{\widehat{B}}$ defined above and its various properties. Briefly, the subject of this section is the definition the sequence spaces $\ell_\infty(\widehat{\widehat{B}})$, $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$, and it is seen that they are norm isomorphic to the spaces well-known classical sequence spaces. Moreover, one of the typical applications includes calculating the $\beta-$ and $\gamma-$duals of the spaces. Armed with elementary facts given earlier in the article, we can now describe the spaces as follows;

$$\ell_\infty(\widehat{\widehat{B}}) := \left\{ x=(x_k)\in w : \sup_{k\in\mathbb{N}} |s^2x_{k-2}+2rsx_{k-1}+r^2x_k| < \infty \right\},$$

$$c(\widehat{\widehat{B}}) := \left\{ x=(x_k)\in w : \exists l\in\mathbb{C} \ni \lim_{k\to\infty} |s^2x_{k-2}+2rsx_{k-1}+r^2x_k - l| = 0 \right\},$$

$$c_0(\widehat{\widehat{B}}) := \left\{ x=(x_k)\in w : \lim_{k\to\infty} |s^2x_{k-2}+2rsx_{k-1}+r^2x_k| = 0 \right\},$$

$$\ell_p(\widehat{\widehat{B}}) := \left\{ x=(x_k)\in w : \sum_k |s^2x_{k-2}+2rsx_{k-1}+r^2x_k|^p < \infty \right\}.$$

In other words; the sets defined above; of all sequences whose $\widehat{\widehat{B}}$-transforms are in the spaces ℓ_∞, c, c_0 and ℓ_p, respectively. Considering the matrix domain with the notation of (1.2), the remarkable feature of these sets is as follows $\ell_\infty(\widehat{\widehat{B}})$, $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ by $\ell_\infty(\widehat{\widehat{B}}) := \{\ell_\infty\}_{\widehat{\widehat{B}}}$, $c(\widehat{\widehat{B}}) := c_{\widehat{\widehat{B}}}$, $c_0(\widehat{\widehat{B}}) := \{c_0\}_{\widehat{\widehat{B}}}$ and $\ell_p(\widehat{\widehat{B}}) := \{\ell_p\}_{\widehat{\widehat{B}}}$. When $x=(x_k)$ is a sequence and the transformation $\widehat{\widehat{B}}$ of $x=(x_k)$ which is defined by matrix multiplication is the sequence $y=(y_k)$, we shall write

$$y_k := s^2x_{k-2}+2rsx_{k-1}+r^2x_k, \quad (k\in\mathbb{N}).$$

Before beginning the general theory, at first we should state the following fundamental theorem, showing that sets just described have an important role in their algebraic structures.

Theorem 2.1. *$\ell_\infty(\widehat{\widehat{B}})$, $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are sets which are linear spaces given by coordinatewise addition and also scalar multiplication, that is, those $\ell_\infty(\widehat{\widehat{B}})$, $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ sets are in fact the sequence spaces.*

Proof. Since the calculations involving coordinatewise addition and scalar multiplication are considerably simply, details of the proof will not be given here. □

Now it is time to give the definition of isomorphism between two linear spaces. Let U and V be linear spaces. We say that U is isomorphic to V if there exists a linear transformation $T: U\to V$ that is invertible. Such a linear transformation is called an isomorphism from U onto V and it is written $U\approx V$.
We now derive one of the most important properties of $X\in\{\ell_\infty, c, c_0, \ell_p\}$ and $X_{\widehat{\widehat{B}}}$ which are used extensively.

Theorem 2.2. *The newly defined sequence spaces $\ell_\infty(\widehat{\widehat{B}})$, $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are norm isomorphic to the classical sequence spaces ℓ_∞, c, c_0 and ℓ_p, respectively; that is, $\ell_\infty(\widehat{\widehat{B}})\approx\ell_\infty$, $c(\widehat{\widehat{B}})\approx c$, $c_0(\widehat{\widehat{B}})\approx c_0$ and $\ell_p(\widehat{\widehat{B}})\approx\ell_p$.*

Proof. When focusing on the proof, it is almost the same to show that the newly defined sequence spaces related to the classical sequence spaces are linear isomorphs, so we are going to prove only one here. What is needed to verify this allegation is to guarantee the existence by the technique used in solving previously published papers could also have been used in from the $c_0(\widehat{\widehat{B}})$ and c_0. When the transformation T described above is taken into consideration again, taking $y=T(x)=\widehat{\widehat{B}}x$ in the definition of T, we can see that T is a linear transformation between $c_0(\widehat{\widehat{B}})$ to c_0 when (1.2) is used. We shall not prove it in detail. Over and above, it is obtained $x=\theta$ from some basic calculations whenever $T(x)=\theta$. It follows from this fact that T is injective. Thus, we choose an $y=(y_k)\in c_0$ and inverse transformation provides us the possibility to identify the sequence $x=(x_k)$ by

$$x_k := \frac{1}{r^2}\sum_{j=0}^{k}(k-j+1)\left(-\frac{s}{r}\right)^{k-j} y_j \quad (k\in\mathbb{N}) \tag{2.1}$$

then from (2.1) we successively calculate

$$s^2x_{k-2}+2rsx_{k-1}+r^2x_k = s^2\left[\frac{1}{r^2}\sum_{j=0}^{k-2}(k-j-1)\left(-\frac{s}{r}\right)^{k-j-2}y_j\right]$$
$$+2rs\left[\frac{1}{r^2}\sum_{j=0}^{k-1}(k-j)\left(-\frac{s}{r}\right)^{k-j-1}y_j\right]+r^2\left[\frac{1}{r^2}\sum_{j=0}^{k}(k-j+1)\left(-\frac{s}{r}\right)^{k-j}y_j\right]=y_k$$

for every $k\in\mathbb{N}$. □

From the last calculations, we conclude that the $x=(x_k)$ defined by above lies $c_0(\widehat{\widehat{B}})$ since the $y=(y_k)$ lies c_0. This means that T is surjective. All that we have done so far is to show that the newly defined space $c_0(\widehat{\widehat{B}})$ and well-known space c_0 are linearly isomorphic. We accomplished the proof by carrying out the necessary steps.

Theorem 2.3. *Let $X\in\{\ell_\infty,c,c_0,\ell_p\}$ and the matrix $\widehat{\widehat{B}}$ defined above. Then,*

(i) $X=X(\widehat{\widehat{B}})$ *if* $\left|\frac{s}{r}\right|<1$.

(ii) $X\subset X(\widehat{\widehat{B}})$ *is strict if* $\left|\frac{s}{r}\right|\geq 1$.

Proof. Let $X\in\{\ell_\infty,c,c_0,\ell_p\}$. The usage of appropriate properties of maths leads to obtaining equations for the matrix $\widehat{\widehat{B}}$ that are briefly written below but actually require long calculations.

$$\sup_{n\in\mathbb{N}}\sum_k|\widehat{\widehat{b}}_{nk}|=(|r|+|s|)^2,\quad \lim_{n\to\infty}\widehat{\widehat{b}}_{nk}=0,$$

$$\lim_{n\to\infty}\sum_k\widehat{\widehat{b}}_{nk}=(r+s)^2\ \text{ and }\ \sup_{k\in\mathbb{N}}\sum_n|\widehat{\widehat{b}}_{nk}|=(|r|+|s|)^2,$$

$\widehat{\widehat{B}}\in(X:X)$. Because of the above explanation, we get $x\in X(\widehat{\widehat{B}})$ for any sequence $x\in X$. when these facts are used, we see that the inclusion $X\subset X(\widehat{\widehat{B}})$ is hold.

(i) We consider first the case where $\left|\frac{s}{r}\right|<1$. In proving that the following conditions are met for inverse matrix $\widehat{\widehat{B}}^{-1}:=(\widehat{\widehat{b}}_{nk}^{-1})$ of the matrix $\widehat{\widehat{B}}$, we follow a similar way to the above.

$$\sup_{n\in\mathbb{N}}\sum_k|\widehat{\widehat{b}}_{nk}^{-1}|=\frac{1}{r^2}\sum_k(k+1)\left|\frac{s}{r}\right|^k<\infty,$$

$$\begin{aligned}\lim_{n\to\infty}\widehat{\widehat{b}}_{nk}^{-1} &= \lim_{n\to\infty}(n-k+1)\left(-\frac{s}{r}\right)^{n-k}\\ &= \left(-\frac{s}{r}\right)^{-k}\lim_{n\to\infty}(n-k)\left(-\frac{s}{r}\right)^{n}\\ &= 0,\end{aligned}$$

$$\lim_{n\to\infty}\sum_k\widehat{\widehat{b}}_{nk}^{-1}=\frac{1}{r^2}\lim_{n\to\infty}\sum_{k=1}^{n}k\left(-\frac{s}{r}\right)^{k-1}\quad exists,$$

$$\sup_{k\in\mathbb{N}}\sum_n|\widehat{\widehat{b}}_{nk}^{-1}|=\frac{1}{r^2}\sum_n(n+1)\left|\frac{s}{r}\right|^n<\infty,$$

$\widehat{\widehat{B}}^{-1}\in(X:X)$, where

$$\widehat{\widehat{b}}_{nk}^{-1}:=\begin{cases}(n-k+1)\frac{1}{r^2}\left(-\frac{s}{r}\right)^{n-k}, & (0\leq k\leq n),\\ 0, & (k>n)\end{cases}$$

for every $k,n\in\mathbb{N}$. We know that $x\in X(\widehat{\widehat{B}})$ and when we apply the definition, we get $y=\widehat{\widehat{B}}x\in X$ and the next step makes the fact that $x=\widehat{\widehat{B}}^{-1}y\in X$. This means that the inclusion $X(\widehat{\widehat{B}})\subset X$ is fulfilled. This shows that the proof of Part (i) is over.

(ii) When getting results we need to use the following.

$$t^1 := \left\{\frac{n+1}{r^2}\left(-\frac{s}{r}\right)^n\right\},$$
$$t^2 := \left\{\frac{n+1}{r^2}\right\},\ t^3 := \{(-1)^n(n+2)\}\ \text{ and }\ t^4 := \left\{\frac{1+(-1)^n}{2}\right\}.$$

With a simple approximation, if $\left|\frac{s}{r}\right| > 1$ holds, then $\widehat{\widehat{B}}t^1 = e^{(0)} = (1,0,0,\dots,0,\dots) \in X$. Thus, we have $t^1 \in X(\widehat{\widehat{B}})$. But, the sequence t^1 is unbounded and then $t^1 \in X(\widehat{\widehat{B}}) \setminus X$.
Suppose that $\left|\frac{s}{r}\right| = 1$.

(a) Let $X = c_0, \ell_p$. Then, $t^1 \in X(\widehat{\widehat{B}})\setminus X$.
(b) Let $X = \ell_\infty, c$. Then,

i) When $s = -r$ is taken, the transformation of t^2 is $\widehat{\widehat{B}}t^2 = e^{(0)} = \big(1,0,0,\dots,0,\dots\big) \in X$. Hence $t^2 \in X(\widehat{\widehat{B}})\setminus X$.
ii) If $s = r$, then $\widehat{\widehat{B}}t^3 = \big(2r^2, r^2, 0,\dots,0,\dots\big) \in \ell_\infty$ and
$\widehat{\widehat{B}}t^4 = \big(r^2, 2r^2, 2r^2,\dots,2r^2,\dots\big) \in c$. Hence $t^3 \in \ell_\infty(\widehat{\widehat{B}})\setminus\ell_\infty$ and $t^4 \in c(\widehat{\widehat{B}})\setminus c$.

Clearly, from of these, we have precisely shown that the inclusion $X \subset X_{\widehat{\widehat{B}}}$ is strict. □

We have the following terminology used by almost everyone studying in this field. Let X and Y arbitrary two sequence spaces. Known as the multiplier space $S(X,Y)$ defined by

$$S(X,Y) := \{t = (t_k) \in w : xt = (x_k t_k) \in Y \ \text{ for all }\ x = (x_k) \in X\} \tag{2.2}$$

is set. We now look at some additional properties of multiplier space. Let X,Y and Z denote any three sequence spaces and $X \supset Z \supset Y$. In view of the preceding elementary knowledge, we see that the inclusions $S(X,Y) \subset S(Z,Y)$ and also $S(X,Y) \subset S(X,Z)$. The notation of (2.2) provides an easy of forming the duals as follows X^α, X^β and X^γ are defined by $X^\alpha = S(X,\ell_1)$, $X^\beta = S(X,cs)$ and $X^\gamma = S(X,bs)$.

Lemma 2.4. *Let X, Y be the sequence spaces and $\xi \in \{\alpha,\beta,\gamma\}$. If $X \subset Y$ then $Y^\xi \subset X^\xi$.*

The following list helps in dealing with some difficult situations:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}|^q < \infty \tag{2.3}$$

$$\sup_{k,n\in\mathbb{N}} |a_{nk}| < \infty \tag{2.4}$$

$$\lim_{n\to\infty} a_{nk} = a_k \ \text{ for each }\ k \in \mathbb{N} \tag{2.5}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k |a_k| \tag{2.6}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = a \tag{2.7}$$

The following lemma presented by Stieglitz and Tietz [25] is particularly useful in obtaining that certain properties.

Lemma 2.5. *The necessary and sufficient conditions for $A \in (X:Y)$ when $X \in \{\ell_\infty, c, c_0, \ell_p, \ell_1\}$ and $Y \in \{\ell_\infty, c\}$ can be read from Table 1, where*

1. *(2.3) with $q = 1$.*
2. *(2.3).*
3. *(2.4).*
4. *(2.5) and (2.6).*
5. *(2.3) with $q = 1$, (2.5) and (2.7).*
6. *(2.3) with $q = 1$ and (2.5).*
7. *(2.3) and (2.5).*
8. *(2.4) and (2.5).*

Table 1: The characterization of the class (X,Y) with $X \in \{\ell_\infty,\ c,\ c_0,\ \ell_p, \ell_1\}$ and $Y \in \{\ell_\infty, c\}$

From To	ℓ_∞	c	c_0	ℓ_p	ℓ_1
ℓ_∞	**1.**	**1.**	**1.**	**2.**	**3.**
c	**4.**	**5.**	**6.**	**7.**	**8.**

Lemma 2.6. *([8, Theorem 3.1]) Let $C=(c_{nk})$ be defined via a sequence $b=(b_k)\in w$ and inverse matrix $D=(d_{nk})$ of triangle matrix $U=(u_{nk})$ by*

$$c_{nk}:=\begin{cases}\sum_{j=k}^{n} b_j d_{jk} & , \quad (0\le k\le n),\\ 0 & , \quad (k>n),\end{cases}$$

for all $k,n\in\mathbb{N}$. Then,

$$\{X_U\}^{\gamma}:=\{b=(b_k)\in w : C\in(X:\ell_\infty)\},$$

$$\{X_U\}^{\beta}:=\{b=(b_k)\in w : C\in(X:c)\}.$$

When both Lemma 2.5 and Lemma 2.6 are considered together, it is seen that the following corollary will be obtained.

Corollary 2.7. *Define the sets $\widehat{\widehat{d}}_1$, $\widehat{\widehat{d}}_2$, $\widehat{\widehat{d}}_3$, $\widehat{\widehat{d}}_4$ and $\widehat{\widehat{d}}_5$ by*

$$\widehat{\widehat{d}}_1=\left\{a=(a_k)\in w \ : \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\frac{1}{r^2}\sum_{j=k}^{n}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_j\right|^{q}<\infty\right\},$$

$$\widehat{\widehat{d}}_2=\left\{a=(a_k)\in w \ : \lim_{n\to\infty}\frac{1}{r^2}\sum_{j=k}^{n}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_j \ \text{exist}\right\},$$

$$\widehat{\widehat{d}}_3=\left\{a=(a_k)\in w \ : \lim_{n\to\infty}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}b_{jk}^{-1}a_j\right|=\sum_{k=0}^{\infty}\left|\lim_{n\to\infty}\frac{1}{r^2}\sum_{j=k}^{n}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_j\right|\right\},$$

$$\widehat{\widehat{d}}_4=\left\{a=(a_k)\in w \ : \lim_{n\to\infty}\frac{1}{r^2}\sum_{k=0}^{n}\sum_{j=k}^{n}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_j \ \text{exist}\right\},$$

$$\widehat{\widehat{d}}_5=\left\{a=(a_k)\in w \ : \sup_{n,k\in\mathbb{N}}\frac{1}{r^2}\left|\sum_{j=k}^{n}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_j\right|<\infty\right\},$$

Then,

(i) $\left\{\ell_\infty(\widehat{\widehat{B}})\right\}^{\gamma}=c(\widehat{\widehat{B}})^{\gamma}=\left\{c_0(\widehat{\widehat{B}})\right\}^{\gamma}:=\widehat{\widehat{d}}_1$ *with $q=1$.*

(ii) $\left\{\ell_p(\widehat{\widehat{B}})\right\}^{\gamma}:=\widehat{\widehat{d}}_1$.

(iii) $\left\{\ell_1(\widehat{\widehat{B}})\right\}^{\gamma}:=\widehat{\widehat{d}}_5$.

(iv) $\left\{c_0(\widehat{\widehat{B}})\right\}^{\beta}:=\widehat{\widehat{d}}_1\cap\widehat{\widehat{d}}_2$ *with $q=1$.*

(v) $\left\{c(\widehat{\widehat{B}})\right\}^{\beta}:=\widehat{\widehat{d}}_1\cap\widehat{\widehat{d}}_2\cap\widehat{\widehat{d}}_4$ *with $q=1$.*

(vi) $\left\{\ell_p(\widehat{\widehat{B}})\right\}^{\beta}:=\widehat{\widehat{d}}_1\cap\widehat{\widehat{d}}_2$.

(vii) $\left\{\ell_1(\widehat{\widehat{B}})\right\}^{\beta}:=\widehat{\widehat{d}}_2\cap\widehat{\widehat{d}}_5$.

(viii) $\left\{\ell_\infty(\widehat{\widehat{B}})\right\}^{\beta}:=\widehat{\widehat{d}}_2\cap\widehat{\widehat{d}}_3$.

Sequence space X having a linear topology is known as $K-$space when every map $p_i : X\to\mathbb{C}$ described by $p_i(x)=x_i$ is continuous for every $i\in\mathbb{N}$. Again $AK-$space X is known as $FK-space$ when X satisfies the condition of being a complete linear metric space. When $FK-$space of which topology is known as $BK-$space. When a normed sequence space X including a sequence (b_n) having the characteristics of having a unique sequence of scalars (a_n) for each $x\in X$ such that

$$\lim_{n\to\infty}\left\|x-\left(\alpha_0 b_0+\alpha_1 b_1+\cdots+\alpha_n b_n\right)\right\|=0,$$

under this condition (b_n) is known as Schauder basis for X(or in short form only basis). Then the series $\sum a_k b_k$ having the summation x is known as the expansion of x in terms of (b_n), and denoted by $x=\sum a_k b_k$. Because of the fact that, the matrix domain X_A of a normed sequence space denoted by X has got a basis iff X has got a basis when $A=(a_{nk})$ is a triangle, one can obtain:

Corollary 2.8. *Let $\alpha_k(r) = \{\widehat{\widehat{B}}x\}_k$ for all $k \in \mathbb{N}$. Define the sequence $z = (z_k)$ and $\widehat{\widehat{b}}^{(k)} = \{\widehat{\widehat{b}}^{(k)}\}_{n\in\mathbb{N}}$ for every fixed $k \in \mathbb{N}$ by $z_n = \sum_{k=0}^{n} \widehat{\widehat{b}}_{nk}^{-1}$ and $\widehat{\widehat{b}}_n^{(k)} = \begin{cases} 0 & , \quad (n < k), \\ \widehat{\widehat{b}}_{nk}^{-1} & , \quad (n \geq k). \end{cases}$ Then,*

(a) The sequence $\{\widehat{\widehat{b}}^{(k)}\}_{n\in\mathbb{N}}$ is a basis for the spaces $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$, and any x in $c_0(\widehat{\widehat{B}})$ or in $\ell_p(\widehat{\widehat{B}})$ has a unique representation of the form

$$x := \sum_k \alpha_k(r)\widehat{\widehat{b}}^{(k)}.$$

(b) The set $\{z, \widehat{\widehat{b}}^{(k)}\}_{n\in\mathbb{N}}$ is a basis for the spaces $c(\widehat{\widehat{B}})$ and any x in $c(\widehat{\widehat{B}})$ has a unique representation of the form

$$x := lz + \sum_k [\alpha_k(r) - l]\widehat{\widehat{b}}^{(k)},$$

where $l = \lim_{k\to\infty}\{\widehat{\widehat{B}}x\}_k$.

It is known that the XY set means

$$XY := \{z = (z_k) \in w : z_k = x_k y_k, \ \ \forall k \in \mathbb{N}, \ x = (x_k) \in X, y = (y_k) \in Y\}$$

for the sequence spaces X and Y. When a $BK-$space $X \supset \phi$ is given, the nth section $x^{[n]}$ of the sequence $x = (x_k) \in X$ is described by $x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)}$ and it is said that x has got the characteristics:
AK if $\lim_{n\to\infty} \|x - x^{[n]}\|_X = 0$ (abschnittskonvergenz),
AB if $\sup_{n\in\mathbb{N}} \|x^{[n]}\|_X < \infty$ (abschnittsbeschränktheit),
AD if $x \in \phi$(clousure of $\phi \subset X$) (abschnittsdichte),
KB if the set $\{x_k e^{(k)}\}$ is bounded in X (koordinatenweise beschränkt).
It is said that the space X has a property if this property is held for each $x \in X$, (cf. [26]). One can obviously see that AK implies AD and AK if and only if $AB + AD$. To give an example for this fact, even though c_0 and ℓ_p are AK-spaces, c and ℓ_∞ are not AD-spaces.

Lemma 2.9. *([27, Theorem 2.1 and Lemma 4.1]). Let X, Y be the $BK-$spaces and $F_Y^U = (f_{nk})$ be defined via the sequence $\alpha = (\alpha_k) \in Y$ and the triangle matrix $U = (u_{nk})$ by*

$$f_{nk} = \sum_{j=k}^{n} \alpha_j u_{nj} d_{jk}$$

for all $k, n \in \mathbb{N}$. Then, the domain of the matrix U in the sequence space X has the following properties

(i) KB iff $F_{\ell_1}^U \in (X : X)$.
(ii) AB iff $F_{bv_0}^U \in (X : X)$.

From Lemma 2.9, we have:

Corollary 2.10. *Let $\left|\frac{s}{r}\right|$ be equal to 1. Under this condition, the space $\ell_1(\widehat{\widehat{B}})$ has both the $KB-$ and $AB-$ properties.*

Lemma 2.11. *[27, Theorem 2.2] Let X be a BK-space which has AK-property, U be a triangle matrix and $X_U \supset \phi$. Then the sequence space X_U has the AD-property if and only if the fact $tU = \theta$ for $t \in X^\beta$ implies the fact $t = \theta$.*

We know that both c_0 and ℓ_p have the AK-property, when $U = \widehat{\widehat{B}}$, we get the following corollary by applying Lemma 2.11.

Corollary 2.12. *$c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ $(p > 1)$ have the AD-property if and only if $\left|\frac{s}{r}\right| \leq 1$.*

3. Some matrix transformations related to the sequence spaces $\ell_\infty(\widehat{\widehat{B}}), c(\widehat{\widehat{B}}), c_0(\widehat{\widehat{B}})$ and $\ell_1(\widehat{\widehat{B}})$

The present section is devoted to the characterization of some classes of infinite matrices related with newly defined sequence spaces. The following theorem about matrix transformations is analogous to the corresponding theorem obtained by the previous ones. It tells us how to combine those results as necessary and sufficient condition.

Theorem 3.1. *Let us assume that X be an FK-space, U be a triangle matrix, D denotes its inverse matrix and Y be any subset of ω. Under these assumptions, one conclude that $A=(a_{nk})\in(X_U:Y)$ iff*

$$C^{(n)}:=(c_{mk}^{(n)})\in(X:c)\ \text{ for all }\ n\in\mathbb{N} \tag{3.1}$$

and

$$C=(c_{nk})\in(X:Y), \tag{3.2}$$

in which

$$c_{mk}^{(n)}:=\begin{cases}\sum_{j=k}^{m}a_{nj}d_{jk} & , \quad (0\le k\le m),\\ 0 & , \quad (k>m),\end{cases} \quad \text{and}\ \ c_{nk}:=\sum_{j=k}^{\infty}a_{nj}d_{jk}$$

for every $k,m,n\in\mathbb{N}$.

Proof. Suppose that $A=(a_{nk})\in(X_U:Y)$ and let us take $x\in X_U$. Under these assumptions, we leave it to the reader to verify that following equations are indeed satisfied

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k=0}^{m}a_{nk}\left(\sum_{j=0}^{k}d_{kj}y_j\right)=\sum_{k=0}^{m}\left(\sum_{j=k}^{m}a_{nj}d_{jk}\right)y_k=\sum_{k=0}^{m}c_{mk}^{(n)}y_k \tag{3.3}$$

for all $m,n\in\mathbb{N}$. Due to the fact that Ax exists, we deduce that $C^{(n)}$ must belong to the class $(X:c)$. By passing to limit $m\to\infty$ in the equality (3.3) we can easily deduce $Ax=Cy$. It is obtained that $Cy\in Y$, using $Ax\in Y$. This means that $C\in(X:Y)$.
In a converse way, let us assume that (3.1), (3.2) are met and let us consider any $x\in X_U$. In these conditions, we take $(c_{nk})_{k\in\mathbb{N}}\in X^{\beta}$ it is obtained that $(a_{nk})_{k\in\mathbb{N}}\in X_U^{\beta}$ for all $n\in\mathbb{N}$, using (3.1). This tells us the existence of the $A-$transform of x, namely Ax exists. Moreover, we derive from the equality (3.3) as $m\to\infty$ that $Ax=Cy$ and this indicates that $A\in(X_U:Y)$. □

Now, we list the following conditions:

$$\sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|\frac{1}{r^2}\sum_{j=k}^{m}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_{nj}\right|^{q}<\infty \tag{3.4}$$

$$\lim_{m\to\infty}\frac{1}{r^2}\sum_{j=k}^{m}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_{nj}=c_{nk} \tag{3.5}$$

$$\lim_{m\to\infty}\sum_{k=0}^{m}\left|\frac{1}{r^2}\sum_{j=k}^{m}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_{nj}\right|=\sum_{k}|c_{nk}|\ \text{ for each }\ n\in\mathbb{N} \tag{3.6}$$

$$\lim_{m\to\infty}\sum_{k=0}^{m}\frac{1}{r^2}\sum_{j=0}^{k}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_{nk}=\alpha_n\ \text{ for all }\ n\in\mathbb{N} \tag{3.7}$$

$$\sup_{m,k\in\mathbb{N}}\left|\frac{1}{r^2}\sum_{j=k}^{m}(j-k+1)\left(-\frac{s}{r}\right)^{j-k}a_{nj}\right|<\infty \tag{3.8}$$

$$\sup_{n\in\mathbb{N}}\sum_{k}|c_{nk}|^{q}<\infty \tag{3.9}$$

$$\lim_{n\to\infty}c_{nk}=\beta_k \tag{3.10}$$

$$\lim_{n\to\infty}\sum_{k}|c_{nk}|=\sum_{k}|\beta_k| \tag{3.11}$$

$$\lim_{n\to\infty}\sum_{k}c_{nk}=\beta \tag{3.12}$$

$$\sup_{n,k\in\mathbb{N}} |c_{nk}| < \infty \tag{3.13}$$

$$\sup_{k\in\mathbb{N}} \sum_{n} |c_{nk}| < \infty \tag{3.14}$$

$$\lim_{n\to\infty} \sum_{k} c_{nk} = 0 \tag{3.15}$$

$$\sup_{N,K\in\mathscr{F}} \left| \sum_{n\in N} \sum_{k\in K} c_{nk} \right| < \infty \tag{3.16}$$

$$\sup_{N\in\mathscr{F}} \sum_{k} \left| \sum_{n\in N} c_{nk} \right|^{q} < \infty, \tag{3.17}$$

in which the symbol $\mathscr{F}$ illustrates the collection of all finite subsets of $\mathbb{N}$. We note here that, Theorem 3.1 tells us that we will have the following table.

Table 2. The characterization of the class $A \in (X : Y)$ with $X \in \{\ell_\infty(\widehat{\widehat{B}}), c(\widehat{\widehat{B}}), c_0(\widehat{\widehat{B}}), \ell_p(\widehat{\widehat{B}})\}$ and $Y \in \{\ell_\infty, c, c_0, \ell_1\}$.

From / To	$\ell_\infty(\widehat{\widehat{B}})$	$c(\widehat{\widehat{B}})$	$c_0(\widehat{\widehat{B}})$	$\ell_p(\widehat{\widehat{B}})$	$\ell_1(\widehat{\widehat{B}})$
ℓ_∞	**1.**	**2.**	**3.**	**4.**	**5.**
c	**6.**	**7.**	**8.**	**9.**	**10.**
c_0	**11.**	**12.**	**13.**	**14.**	**15.**
ℓ_1	**16.**	**17.**	**18.**	**19.**	**20.**

Corollary 3.2. *The necessary and sufficient conditions for all $A \in (X : Y)$ when $X \in \left\{\ell_\infty(\widehat{\widehat{B}}), c(\widehat{\widehat{B}}), c_0(\widehat{\widehat{B}}), \ell_p(\widehat{\widehat{B}})\right\}$ and $Y \in \{\ell_\infty, c, c_0, \ell_1\}$ can be read from the Table 2 : where,*

1. *(3.5), (3.6) and (3.9) with $q = 1$.*
2. *(3.5), (3.7) and (3.4),(3.9) with $q = 1$.*
3. *(3.5) and (3.4), (3.9) with $q = 1$.*
4. *(3.4), (3.5) and (3.9).*
5. *(3.5), (3.8) and (3.13).*
6. *(3.5), (3.6), (3.10) and (3.11).*
7. *(3.5), (3.7), (3.10), (3.12) and (3.4), (3.9) with $q = 1$.*
8. *(3.5), (3.10) and (3.4), (3.9) with $q = 1$.*
9. *(3.4), (3.5), (3.9) and (3.10).*
10. *(3.5), (3.8), (3.10) and (3.13).*
11. *(3.5), (3.6) and (3.15).*
12. *(3.5), (3.7), (3.10) with $\beta_k = 0$ and(3.12) with $\beta = 0$ and (3.4), (3.9) with $q = 1$.*
13. *(3.5), (3.10) with $\beta_k = 0$ and (3.4), (3.9) with $q = 1$.*
14. *(3.4), (3.5), (3.9) and (3.10) with $\beta_k = 0$.*
15. *(3.5), (3.8), (3.10) with $\beta_k = 0$ and (3.13).*
16. *(3.5), (3.6) and (3.16).*
17. *(3.4) with $q = 1$, (3.5),(3.7) and (3.16).*
18. *(3.4) with $q = 1$, (3.5) and (3.16).*
19. *(3.4), (3.5) and (3.17).*
20. *(3.5), (3.8) and (3.14).*

Now, we are going to present the following lemma leading more quickly to the computation of the characterization of some new matrix classes, using the Corollary 3.2.

Lemma 3.3. *[28, Lemma 5.3] Let X, Y be arbitrary two sequence spaces, A be an infinite matrix and U a triangle matrix. Then, $A \in (X : Y_U)$ iff $UA \in (X : Y)$.*

Here we are able to give an ultimate note. When based on writing $r^2 a_{nk} + 2rs a_{n-1,k} + s^2 a_{n-2,k}$ instead of a_{nk} for all $k, n \in \mathbb{N}$ in Corollary 3.2, since $U = \widehat{\widehat{B}}$ is triangle matrix, we can actually find out the characterization of the class $(X(\widehat{\widehat{B}}) : Y(\widehat{\widehat{B}}))$ using Lemma 3.3.

4. Conclusion

In brief, the present manuscript has investigated algebraic and topological characteristics of the sequence space $X(\widehat{\widehat{B}}(r,s))$. The β- and γ-duals for these spaces have been calculated at the same time Schauder bases for those spaces $c(\widehat{\widehat{B}})$, $c_0(\widehat{\widehat{B}})$ and $\ell_p(\widehat{\widehat{B}})$ are found out. It has been noted that all of these spaces have got special characteristics. Some matrix transformations have been characterized.

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Exact Sequences of BCK-Modules

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Abstract

BCK-modules were introduced as an action of a BCK-algebra over an Abelian group. Homomorphisms of BCK-modules form an exact sequence which is called BCK-sequence. In this paper, we study homomorphisms of BCK-modules. We show that this homomorphisms have a module structure. Moreover, we show that sequences of Hom functors are BCK-sequences.

1. Introduction

BCK/BCI-algebras were introduced by Imai and Iseki [1, 2]. BCK/BCI-algebras have been studied by many authors, extensively. In 1994, the BCK-module structure of BCK-algebras was introduced as an action on an Abelian group [3]. In [4], exact sequences of BCK-modules were studied. Further, in [5],the authors studied the homomorphisms between BCK-modules and they showed that the set of homomorphisms of BCK-modules form a BCK-module. Later, in [6], homology theory of BCK-modules was investigated. In [7], the authors studied BCK-sequences and finitely presented BCK-modules.
The paper organized as follows; in section 2, we give general theory of BCK-algebras and BCK-modules. In section 3, we study the exactness of modules of homomorphisms between BCK-modules.

2. Preliminaries

In this section we introduce the background informations about BCK-algebras, BCK-modules and X-homomorphisms.

Definition 2.1. *[8] A BCK-algebra is an algebra $(X;*,0)$ of type $(2,0)$ which satisfies the following axioms: for all $p,q,r \in X$,*

1. $((p*q)*(p*r))*(r*q) = 0$,
2. $(p*(p*q))*q = 0$,
3. $(p*p) = 0$,
4. $p*q = 0 = q*p$ *implies* $p = q$.
5. $0*p = 0$.

Moreover, the relation $\leq$ can be defined as $p \leq q$ if and only if $p*q = 0$, for any $p,q \in X$, is a partial-order on X which is called *BCK-ordering* of X.

Definition 2.2. *[6] Let $(X;*,0)$ be a BCK-algebra and M be an Abelian group under addition $+$, then M is said to be an (left) X-module, if there is a mapping $(x,m) \mapsto xm$ from $X \times M \to M$ such that it satisfies the following conditions for all $x,x_1,x_2 \in X$ and $m,m_1,m_2 \in M$:*

1. $(x_1 \wedge x_2)m = x_1(x_2m)$,

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2. $x(m_1+m_2)=xm_1+xm_2$,
3. $0m=0$

where, $x_1\wedge x_2=x_2*(x_2*x_1)$. *If* X *is bounded with maximal element* 1, *then*

4. $1m=m$.

The right X-module can be defined similarly. This X-module M is an BCK-module. If a subgroup N of the X-module M is also an X-module, then N is called a *submodule*.
Let M and N be X-modules. A mapping $\phi: M\to N$ is said to be an X-homomorphism, if for any $x\in X$ and $m_1,m_2\in M$ the followings hold:

1. $\phi(m_1+m_2)=\phi(m_1)+\phi(m_2)$,
2. $\phi(xm_1)=x\phi(m_1)$.

If ϕ is both injective and surjective, then ϕ is an X-isomorphism. We say M is isomorphic to N if ϕ is an X-isomorphism and denote it by $M\cong N$.
The bounded implicative BCK-algebras form a BCK-module over itself (Abujabal et al., 1994). This section devoted to the examples of BCK-modules.

Example 2.3. *Let* $(X;*,0)$ *be a bounded implicative BCK-algebra with* $X=\{0,x,y,1\}$. *Let* $M=\{0,x\}$ *be a subset of* X. *If we define addition operation* $+$ *as* $x+y=(x*y)\vee(y*x)$ *and* $xm=x\wedge m$ *for all* $x\in X$, $m\in M$, *then* M *is an* X*-module. Cayley table of these operations are as follows:*

*	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

+	0	x
0	0	x
x	x	0

∧	0	x
0	0	0
x	0	x
y	0	0
1	0	x

3. Exact BCK-sequences

Definition 3.1. *[7] The sequence of* X*-module homomorphisms* $M_1\xrightarrow{f}M_2\xrightarrow{g}M_3$ *is said to be exact at* M_2, *if* $\mathrm{Im}(f)=\mathrm{Ker}(g)$. *A sequence of* X*-module homomorphisms,* $M_1\xrightarrow{f_1}M_2\xrightarrow{f_2}\dots\xrightarrow{f_{n-1}}M_n$ *is called exact sequence of* X*-modules, if* $\mathrm{Im}(f_i)=\mathrm{Ker}(f_{i+1})$ *for all* $i\in\{1,2,...,n\}$.

Theorem 3.2. *Let* X *be a BCK-algebra and* K,L *and* M *be* X*-modules. If* A *is an* X*-module and* $0\to K\xrightarrow{\psi}L\xrightarrow{\phi}M$ *is exact, then*

$$0\to\mathrm{Hom}(A,K)\xrightarrow{\psi_*}\mathrm{Hom}(A,L)\xrightarrow{\phi_*}\mathrm{Hom}(A,M)$$

is an exact sequence of X*-modules.*

Proof. First we show that ψ_* is a monomorphism. Let $\theta: A\to K$ be a X-homomorphism with $\psi_*\theta=0$. Since ψ is a monomorphism, then for any $a\in A$, the identity $\psi_*\theta(a)=0$ implies that $\theta(a)=0$. Thus $\theta=0$. Hence ψ_* is a monomorphism. Let $b\in\mathrm{Im}(\psi_*)\subseteq\mathrm{Hom}(A,L)$. Then there exists $a\in\mathrm{Hom}(A,K)$ such that $\psi_*(a)=b=\psi a$. Since $\phi_*(b)=\phi_*(\psi a)=\phi\psi a=0a=0$, we have $b\in\mathrm{Ker}(\phi_*)$. Hence $\mathrm{Im}(\psi_*)\subseteq\mathrm{Ker}(\phi_*)$. Let $u\in\mathrm{Ker}(\phi_*)\subseteq\mathrm{Hom}(A,L)$. Then $\phi_*(u)=0$ and $\phi u(a)=0$ for any $a\in A$. The exactness of the sequence gives that $\mathrm{Ker}(\phi)=\psi(K)$. Thus there exists an $x\in K$ which satisfies $\psi(x)=u(a)$. Then $v(a)=x$ defines a homomorphism $v: A\to K$ with $\psi_*(v)=u$. Thus $\mathrm{Ker}(\phi_*)\subseteq\mathrm{Im}(\psi_*)$. Therefore $\mathrm{Ker}(\phi_*)=\mathrm{Im}(\psi_*)$. □

Theorem 3.3. *Let* X *be a BCK-algebra and* K,L *and* M *be* X*-modules. If* A *is an* X*-module and* $K\xrightarrow{\psi}L\xrightarrow{\phi}M\to 0$ *is exact, then*

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{\phi_*} \mathrm{Hom}(L,A) \xrightarrow{\psi_*} \mathrm{Hom}(K,A)$$

is an exact sequence of X-modules.

Proof. First we show that ϕ_* is a monomorphism. Let $\theta : M \to A$ be an X-homomorphism and $\theta \in \mathrm{Ker}(\phi_*)$. Since $0 = \phi_*\theta = \theta\phi$, this implies that $\theta(\phi(l)) = 0$ for all $l \in L$. Thus $\theta(m) = 0$ for all $m \in \mathrm{Im}(\phi)$. The fact that ϕ is epimorphism implies that $\mathrm{Im}(\phi) = M$ and $\theta = 0$. Hence ϕ_* is a monomorphism.
Let $b \in \mathrm{Im}(\phi_*) \subseteq \mathrm{Hom}(L,A)$. Then there exists $a \in \mathrm{Hom}(M,A)$ such that $\phi_*(a) = b = a\phi$. Since $\psi_*(b) = \psi_*(a\phi)$ and $\psi_*(a\phi) = a\phi\psi = a0 = 0$, this implies that $b \in \mathrm{Ker}(\psi_*)$. Hence $\mathrm{Im}(\phi_*) \subseteq \mathrm{Ker}(\psi_*)$. Let $u \in \mathrm{Ker}(\psi_*) \subseteq \mathrm{Hom}(L,A)$. Then $\psi_*(u) = 0 = u\psi$. Following the diagram,

$$\begin{array}{ccccc} K \xrightarrow{\psi} & L & \xrightarrow{\phi} & M & \to 0 \\ & u\downarrow & \swarrow p & & \\ & A & & & \end{array}$$

There exists $p \in \mathrm{Hom}(M,A)$ such that $u = p\phi = \phi_*(p)$. This implies that $u \in \mathrm{Im}(\phi_*)$. Thus $\mathrm{Ker}(\psi_*) \subseteq \mathrm{Im}(\phi_*)$. Therefore $\mathrm{Ker}(\psi_*) = \mathrm{Im}(\phi_*)$.

□

Definition 3.4. *Let X be a BCK-algebra and M,N and K be X-modules. If the following sequence of X-modules is exact. Then*

$$0 \to M \to N \to K \to 0$$

is called short exact sequence.

Theorem 3.5. *Let X be a BCK-algebra and M,N and K be X-modules. If the short sequence of X-homomorphisms is exact;*

$$0 \to M \underset{\eta}{\overset{\psi}{\rightleftarrows}} N \underset{\theta}{\overset{\phi}{\rightleftarrows}} K \to 0$$

then followings are equivalent;

1. *There exists an X-homomorphism $\eta : N \to M$ such that $\eta\psi = 1_M$.*
2. *Submodule* $\mathrm{Im}(\psi)$ *is a direct summand of N.*
3. *There exists an X-homomorphism $\theta : K \to N$ suct that $\phi\theta = 1_K$.*

Moreover, we have $N \cong M \oplus K$.

Proof. $1 \Rightarrow 2$ Let $x \in N$ be any element. Since $\eta(x - \psi\eta(x)) = \eta(x) - ((\eta\psi)\eta(x)) = \eta(x) - \eta(x) = 0$, then we have $x - \psi\eta(x) \in \mathrm{Ker}(\eta)$. This implies that $x = \psi(\eta(x)) + (x - \psi\eta(x)) \in \mathrm{Im}(\psi) + \mathrm{Ker}(\eta)$.
Let $\psi(m) \in \mathrm{Im}(\psi) \cap \mathrm{Ker}(\eta)$. Since $m = \eta\psi(m) = \eta(\psi(m)) = 0$, one can conclude that $\mathrm{Im}(\psi) \cap \mathrm{Ker}(\eta) = 0$. Hence $N = \mathrm{Im}(\psi) \oplus \mathrm{Ker}(\eta)$.
$2 \Rightarrow 3$ Let N' be a submodule of N and $N = \mathrm{Im}(\psi) \oplus N'$. Now since $N' \cap \mathrm{Ker}(\phi) = N' \cap \mathrm{Im}(\psi) = 0$, the $\phi|_{N'}$ is a monomorphism. The fact that ϕ is a epimorphism implies that there exists x in N for every $y \in K$ such that $\phi(x) = y$. If we set $x = \psi(a) + b$ for $a \in M, b \in N'$. Then $y = \phi(x) = \phi(\psi(a) + b) = \phi\psi(a) + \phi(b) = \phi(b)$. This implies that $\phi|_{N'}$ is an epimorphism. Thus $\phi|_{N'}$ is an isomorphism. Since $\phi|_{N'}$is an isomorphism, we can conclude that $\phi|_{N'}$ has an inverse $(\phi|_{N'})^{-1} : K \to N$ for $\theta := (\phi|_{N'})^{-1} : K \to N$ then we have $\phi\theta = 1_K$.
$3 \Rightarrow 1$ Since $\phi(n - \theta\phi(n)) = \phi(n) - \phi(\theta\phi(n)) = 0$, we have $n - \theta\phi(n) \in \mathrm{Ker}(\phi) = \mathrm{Im}(\psi)$. Then there exists $m \in M$ such that $\psi(m) = n - \theta\phi(n)$. This m is unique, since ψ is a monomorphism. Set $\eta : N \to M$ and $\eta(n) = m$ with η is a homomorphism. The equality,

$$\psi(m) - \theta\phi(\psi(m)) = \psi(m) - \theta(\phi\psi(m)) = \psi(m) - \theta(0) = \psi(m), \text{ for every } m \text{ in } M.$$

holds, since $\phi\psi(n) = 0$. It follows that $\psi(m) = \psi(m) - \theta\phi(\psi(m))$, and combining this equality with $\psi(m) = n - \theta\phi(n)$, we can deduce that $\psi(m) = n$. Thus $\eta(\psi(m)) = m$, so we have $\eta\psi = 1_M$. Since ψ is a monomorphism, then $\mathrm{Im}(\psi) \cong M$. Therefore, $N \cong M \oplus K$.

□

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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