MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

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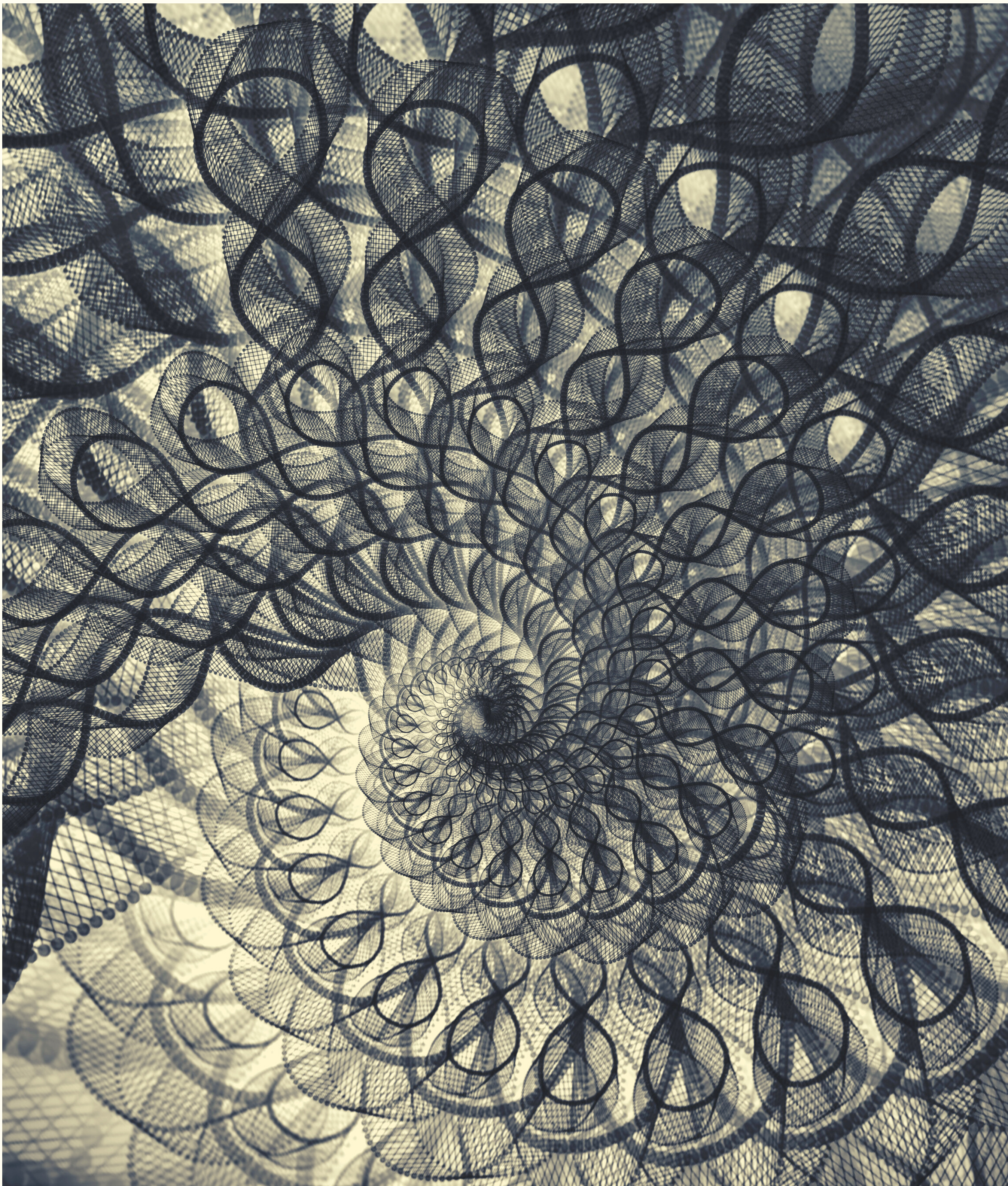

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Solvability of a Three-Dimensional System of Nonlinear Difference Equations

Merve Kara

Abstract
In this paper, we solve the following three-dimensional system of difference equations

$$
\begin{aligned}
x_n &= \frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a_n + b_n z_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)},\\
y_n &= \frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha_n + \beta_n x_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)},\\
z_n &= \frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A_n + B_n y_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)},\ n \in \mathbb{N}_0,
\end{aligned}
$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-j}, y_{-j}, $j = \overline{1,5}$, are real numbers. In addition, the constant coefficients of the mentioned system is solved in closed form. Finally, we also describe the forbidden set of solutions of the system of difference equations.

Keywords: *System of difference equations; Closed-form; Forbidden set.*

AMS Subject Classification (2020): *Primary: 39A10 ; Secondary: 39A20; 39A23.*

1. Introduction

Difference equations emerge from generation functions, numerical solutions of differential equations or mathematical models of physical events. Therefore, difference equations or systems of difference equations are important for many researchers. Because they use them in economics, physics, biology, engineering. Especially, mathematicians are interested in system of difference equations or difference equations [1–8, 10–18, 20–22, 25–39]. For example, the difference equation

$$
x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)},\ n \in \mathbb{N}_0, \tag{1.1}
$$

was studied in [9]. Elsayed have shown that this difference equation can be solved in closed form by using the method of induction.

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In addition, Stević found the general solution of following extension of difference equations (1.1)

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n\left(a+bx_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)},\ n\in\mathbb{N}_0, \tag{1.2}$$

where the parameters a, b and the initial values x_{-j}, $j=\overline{0,4}$, are complex numbers in [24].

The authors of [19] found formulas for exact solutions of the following equations

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n\left(a_n+b_nx_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)},\ n\in\mathbb{N}_0, \tag{1.3}$$

where a_n and b_n are real sequences.

Moreover, in [40], the following system of difference equations

$$x_n=\frac{x_{n-4}y_{n-5}}{y_{n-1}\left(a_n+b_nx_{n-2}y_{n-3}x_{n-4}y_{n-5}\right)},\ y_n=\frac{y_{n-4}x_{n-5}}{x_{n-1}\left(\alpha_n+\beta_ny_{n-2}x_{n-3}y_{n-4}x_{n-5}\right)},\ n\in\mathbb{N}_0, \tag{1.4}$$

was solved by Yazlik and Kara where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, y_{-i}, $i=\overline{1,5}$, are real numbers. Further, we investigated asymptotic behavior and periodicity of solutions of system (1.4) when all sequences are constant.

In this paper, we study the following system of difference equations

$$\begin{aligned}
x_n&=\frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)},\\
y_n&=\frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)},\\
z_n&=\frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)},\ n\in\mathbb{N}_0,
\end{aligned} \tag{1.5}$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-j}, y_{-j}, $j=\overline{1,5}$, are real numbers. System (1.5) is a generalization of equation (1.1), equation (1.2), equation (1.3) and system (1.4). Our aim in this paper is to show that system (1.5) is solvable in closed form by using the method of transformation. In addition, the forbidden set of initial values for solutions of system (1.5) is described. Then, for the case when all the coefficients are constant, solutions of system (1.5) are obtained.

Lemma 1.1. *[23] Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and the sequences y_{km+i}, $i=\overline{0,k-1}$, be solutions of the equations*

$$y_{km+i}=a_{km+i}y_{k(m-1)+i}+b_{km+i},\ m\in\mathbb{N}_0. \tag{1.6}$$

Then, for each fixed $i=\overline{0,k-1}$ and $m\geq -1$, equation (1.6) has the general solution

$$y_{km+i}=y_{i-k}\prod_{j=0}^{m}a_{kj+i}+\sum_{s=0}^{m}b_{ks+i}\prod_{j=s+1}^{m}a_{kj+i}.$$

Further, if $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant and $i=\overline{0,k-1}$, $m\geq -1$, then

$$y_{km+i}=\begin{cases}a^{m+1}y_{i-k}+b\frac{1-a^{m+1}}{1-a}, & \text{if } a\neq 1,\\ y_{i-k}+b\left(m+1\right), & \text{if } a=1.\end{cases}$$

2. Closed-Form Solutions of System (1.5)

In this section, we show that the system (1.5) is solvable in closed form. We will deal only with well-defined solutions to system (1.5). Hence, we assume that

$$x_n\neq 0,\ y_n\neq 0,\ z_n\neq 0,\ n\geq -5,$$

and

$$a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}\neq 0,\ \ \alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}\neq 0,\ \ A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}\neq 0,\ n\in\mathbb{N}_0.$$

Let $\{(x_n, y_n, z_n)\}_{n\geq -5}$ be solutions of system (1.5). If at least one of the initial values x_{-k}, y_{-k}, z_{-k}, $k=\overline{1,5}$ is equal to zero, then the solutions of system (1.5) is not defined. For instance, if $x_{-5}=0$, then $y_0=0$ and so x_1 is not defined. Similarly, if $y_{-5}=0$ (or $z_{-5}=0$) then $z_0=0$ (or $x_0=0$) and so y_1 (or z_1) is not defined. For $k=\overline{1,4}$, the other cases are similar.

On the other hand, if $x_{n_1}=0$ ($n_1\in\mathbb{N}_0$), $x_n\neq 0$, for every $n<n_1$. Then according to the first equation in (1.5) we get that $y_{n_1-4}=0$ or $z_{n_1-5}=0$. If $y_{n_1-4}=0$, then according to the second equation in (1.5) we get that $z_{n_1-8}=0$. If $z_{n_1-5}=0$, then according to the third equation in (1.5) we get that $y_{n_1-10}=0$. Repeating this procedure, we have a $i_1\in\{1,2,3,4,5\}$ such that $y_{-i_1}=0$ or $z_{-i_1}=0$. Similarly, if $y_{n_2}=0$ ($n_2\in\mathbb{N}_0$), $y_n\neq 0$, for every $n<n_2$. Then according to the second equation in (1.5) we get that $z_{n_2-4}=0$ or $x_{n_2-5}=0$. If $z_{n_2-4}=0$, then according to the third equation in (1.5) we get that $x_{n_2-8}=0$. If $x_{n_2-5}=0$, then according to the first equation in (1.5) we get that $z_{n_2-10}=0$. Repeating this procedure, we have a $i_2\in\{1,2,3,4,5\}$ such that $z_{-i_2}=0$ or $x_{-i_2}=0$. If $z_{n_3}=0$ ($n_3\in\mathbb{N}_0$), $z_n\neq 0$, for every $n<n_3$. Then according to the third equation in (1.5) we get that $x_{n_3-4}=0$ or $y_{n_3-5}=0$. If $x_{n_3-4}=0$, then according to the first equation in (1.5) we get that $y_{n_3-8}=0$. If $y_{n_3-5}=0$, then according to the second equation in (1.5) we get that $x_{n_3-10}=0$. Repeating this procedure, we have a $i_3\in\{1,2,3,4,5\}$ such that $x_{-i_3}=0$ or $y_{-i_3}=0$. Repeating this procedure we find a $i\in\{1,2,3,4,5\}$ such that $x_{-i}=0$ or $y_{-i}=0$ or $z_{-i}=0$. As we have proved above, such solutions are not defined. Hence, of some interest is the case when

$$x_n\neq 0,\ y_n\neq 0,\ z_n\neq 0,\ n\geq -5.$$

Note that the system (1.5) can be written in the form

$$\begin{aligned}
x_n y_{n-1} z_{n-2} x_{n-3} &= \frac{z_{n-2}x_{n-3}y_{n-4}z_{n-5}}{(a_n+b_n z_{n-2}x_{n-3}y_{n-4}z_{n-5})},\\
y_n z_{n-1} x_{n-2} y_{n-3} &= \frac{x_{n-2}y_{n-3}z_{n-4}x_{n-5}}{(\alpha_n+\beta_n x_{n-2}y_{n-3}z_{n-4}x_{n-5})},\\
z_n x_{n-1} y_{n-2} z_{n-3} &= \frac{y_{n-2}z_{n-3}x_{n-4}y_{n-5}}{(A_n+B_n y_{n-2}z_{n-3}x_{n-4}y_{n-5})},\ n\in\mathbb{N}_0.
\end{aligned}\tag{2.1}$$

Employing the change of variables

$$u_n=\frac{1}{x_n y_{n-1} z_{n-2} x_{n-3}},\ v_n=\frac{1}{y_n z_{n-1} x_{n-2} y_{n-3}},\ w_n=\frac{1}{z_n x_{n-1} y_{n-2} z_{n-3}},\ n\geq -2,\tag{2.2}$$

system (1.5) is transformed into the following system of linear difference equations

$$u_n=a_n w_{n-2}+b_n,\ v_n=\alpha_n u_{n-2}+\beta_n,\ w_n=A_n v_{n-2}+B_n,\ n\in\mathbb{N}_0,\tag{2.3}$$

from system (2.3), we get

$$u_{n+6}=a_{n+6}A_{n+4}\alpha_{n+2}u_n+a_{n+6}A_{n+4}\beta_{n+2}+a_{n+6}B_{n+4}+b_{n+6},\ n\geq -2,\tag{2.4}$$
$$v_{n+6}=\alpha_{n+6}a_{n+4}A_{n+2}v_n+\alpha_{n+6}a_{n+4}B_{n+2}+\alpha_{n+6}b_{n+4}+\beta_{n+6},\ n\geq -2,\tag{2.5}$$
$$w_{n+6}=A_{n+6}\alpha_{n+4}a_{n+2}w_n+A_{n+6}\alpha_{n+4}b_{n+2}+A_{n+6}\beta_{n+4}+B_{n+6},\ n\geq -2,\tag{2.6}$$

which are nonhomogeneous linear sixth-order difference equations with variable coefficient. If we apply the decomposition of indexes $n\to 6n+j$, for some $n\in\mathbb{N}_0$ and $j=\overline{-2,3}$ to (2.4) and (2.6), then they become

$$u_{6(n+1)+j}=a_{6n+j+6}A_{6n+j+4}\alpha_{6n+j+2}u_{6n+j}+a_{6n+j+6}A_{6n+j+4}\beta_{6n+j+2}+a_{6n+j+6}B_{6n+j+4}+b_{6n+j+6},\tag{2.7}$$
$$v_{6(n+1)+j}=\alpha_{6n+j+6}a_{6n+j+4}A_{6n+j+2}v_{6n+j}+\alpha_{6n+j+6}a_{6n+j+4}B_{6n+j+2}+\alpha_{6n+j+6}b_{6n+j+4}+\beta_{6n+j+6},\tag{2.8}$$
$$w_{6(n+1)+j}=A_{6n+j+6}\alpha_{6n+j+4}a_{6n+j+2}w_{6n+j}+A_{6n+j+6}\alpha_{6n+j+4}b_{6n+j+2}+A_{6n+j+6}\beta_{6n+j+4}+B_{6n+j+6},\tag{2.9}$$

for $n\in\mathbb{N}_0$, which are first-order 6-equations. Let $u_n^{(j)}=u_{6n+j}$, $v_n^{(j)}=v_{6n+j}$, $w_n^{(j)}=w_{6n+j}$ for $n\in\mathbb{N}_0$ and $j=\overline{-2,3}$ and

$$\begin{aligned}
\gamma_n^{(j)} &= a_{6n+j+6}A_{6n+j+4}\alpha_{6n+j+2},\\
\delta_n^{(j)} &= a_{6n+j+6}A_{6n+j+4}\beta_{6n+j+2}+a_{6n+j+6}B_{6n+j+4}+b_{6n+j+6},
\end{aligned}\tag{2.10}$$

$$\begin{aligned}\widehat{\gamma}_n^{(j)} &= \alpha_{6n+j+6}a_{6n+j+4}A_{6n+j+2},\\ \widehat{\delta}_n^{(j)} &= \alpha_{6n+j+6}a_{6n+j+4}B_{6n+j+2}+\alpha_{6n+j+6}b_{6n+j+4}+\beta_{6n+j+6},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\widetilde{\gamma}_n^{(j)} &= A_{6n+j+6}\alpha_{6n+j+4}a_{6n+j+2},\\ \widetilde{\delta}_n^{(j)} &= A_{6n+j+6}\alpha_{6n+j+4}b_{6n+j+2}+A_{6n+j+6}\beta_{6n+j+4}+B_{6n+j+6}.\end{aligned} \tag{2.12}$$

Then equations in (2.7)-(2.9) can be written in the form

$$u_{n+1}^{(j)} = \gamma_n^{(j)}u_n^{(j)}+\delta_n^{(j)},\ n\in\mathbb{N}_0, \tag{2.13}$$

$$v_{n+1}^{(j)} = \widehat{\gamma}_n^{(j)}v_n^{(j)}+\widehat{\delta}_n^{(j)},\ n\in\mathbb{N}_0, \tag{2.14}$$

$$w_{n+1}^{(j)} = \widetilde{\gamma}_n^{(j)}w_n^{(j)}+\widetilde{\delta}_n^{(j)},\ n\in\mathbb{N}_0, \tag{2.15}$$

for $j=\overline{-2,3}$.
From (2.13)-(2.15) and Lemma 1.1, we have

$$u_n^{(j)} = \left(\prod_{k=0}^{n-1}\gamma_k^{(j)}\right)u_0^{(j)}+\sum_{i=0}^{n-1}\left(\prod_{k=i+1}^{n-1}\gamma_k^{(j)}\right)\delta_i^{(j)}, \tag{2.16}$$

$$v_n^{(j)} = \left(\prod_{k=0}^{n-1}\widehat{\gamma}_k^{(j)}\right)v_0^{(j)}+\sum_{i=0}^{n-1}\left(\prod_{k=i+1}^{n-1}\widehat{\gamma}_k^{(j)}\right)\widehat{\delta}_i^{(j)}, \tag{2.17}$$

$$w_n^{(j)} = \left(\prod_{k=0}^{n-1}\widetilde{\gamma}_k^{(j)}\right)w_0^{(j)}+\sum_{i=0}^{n-1}\left(\prod_{k=i+1}^{n-1}\widetilde{\gamma}_k^{(j)}\right)\widetilde{\delta}_i^{(j)}, \tag{2.18}$$

for $n\in\mathbb{N}_0$, $j=\overline{-2,3}$. Using (2.10)-(2.12) in equations (2.16)-(2.18), we obtain

$$\begin{aligned}u_{6n+j} &= \left(\prod_{k=0}^{n-1}\left(a_{6k+j+6}A_{6k+j+4}\alpha_{6k+j+2}\right)\right)u_j\\ &+\sum_{i=0}^{n-1}\left(\prod_{k=i+1}^{n-1}\left(a_{6k+j+6}A_{6k+j+4}\alpha_{6k+j+2}\right)\right)\left(a_{6i+j+6}A_{6i+j+4}\beta_{6i+j+2}+a_{6i+j+6}B_{6i+j+4}+b_{6i+j+6}\right),\end{aligned} \tag{2.19}$$

$$\begin{aligned}v_{6n+j} &= \left(\prod_{k=0}^{n-1}\left(\alpha_{6k+j+6}a_{6k+j+4}A_{6k+j+2}\right)\right)v_j\\ &+\sum_{i=0}^{n-1}\left(\prod_{k=i+1}^{n-1}\left(\alpha_{6k+j+6}a_{6k+j+4}A_{6k+j+2}\right)\right)\left(\alpha_{6i+j+6}a_{6i+j+4}B_{6i+j+2}+\alpha_{6i+j+6}b_{6i+j+4}+\beta_{6i+j+6}\right),\end{aligned} \tag{2.20}$$

$$\begin{aligned}w_{6n+j} &= \left(\prod_{k=0}^{n-1}\left(A_{6k+j+6}\alpha_{6k+j+4}a_{6k+j+2}\right)\right)w_j\\ &+\sum_{i=0}^{n-1}\left(\prod_{k=i+1}^{n-1}\left(A_{6k+j+6}\alpha_{6k+j+4}a_{6k+j+2}\right)\right)\left(A_{6i+j+6}\alpha_{6i+j+4}b_{6i+j+2}+A_{6i+j+6}\beta_{6i+j+4}+B_{6i+j+6}\right),\end{aligned} \tag{2.21}$$

for $n\in\mathbb{N}_0$, $j=\overline{-2,3}$.
When the coefficients are constants i.e., $a_n=a$, $b_n=b$, $\alpha_n=\alpha$, $\beta_n=\beta$, $A_n=A$ and $B_n=B$, formulas (2.19)-(2.21) becomes

$$u_{6n+j} = \begin{cases}(a\alpha A)^n u_j+\frac{1-(a\alpha A)^n}{1-a\alpha A}\left(aA\beta+aB+b\right), & a\alpha A\neq 1,\\ u_j+\left(aA\beta+aB+b\right)n, & a\alpha A=1,\end{cases}\quad n\in\mathbb{N}_0, \tag{2.22}$$

$$v_{6n+j}=\begin{cases}(a\alpha A)^n v_j+\frac{1-(a\alpha A)^n}{1-a\alpha A}(\alpha aB+\alpha b+\beta), & a\alpha A\neq 1,\\ v_j+(\alpha aB+\alpha b+\beta)n, & a\alpha A=1,\end{cases}\quad n\in\mathbb{N}_0, \tag{2.23}$$

$$w_{6n+j}=\begin{cases}(a\alpha A)^n w_j+\frac{1-(a\alpha A)^n}{1-a\alpha A}(A\alpha b+A\beta+B), & a\alpha A\neq 1,\\ w_j+(A\alpha b+A\beta+B)n, & a\alpha A=1,\end{cases}\quad n\in\mathbb{N}_0, \tag{2.24}$$

for $j=\overline{-2,3}$. From equalities in (2.2), we get

$$x_n=\frac{1}{u_ny_{n-1}z_{n-2}x_{n-3}}=\frac{v_{n-1}}{u_n}y_{n-4}=\frac{v_{n-1}w_{n-5}}{u_nv_{n-4}}z_{n-8}=\frac{v_{n-1}w_{n-5}u_{n-9}}{u_nv_{n-4}w_{n-8}}x_{n-12}, \tag{2.25}$$

$$y_n=\frac{1}{v_nz_{n-1}x_{n-2}y_{n-3}}=\frac{w_{n-1}}{v_n}z_{n-4}=\frac{w_{n-1}u_{n-5}}{v_nw_{n-4}}x_{n-8}=\frac{w_{n-1}u_{n-5}v_{n-9}}{v_nw_{n-4}u_{n-8}}y_{n-12}, \tag{2.26}$$

$$z_n=\frac{1}{w_nx_{n-1}y_{n-2}z_{n-3}}=\frac{u_{n-1}}{w_n}x_{n-4}=\frac{u_{n-1}v_{n-5}}{w_nu_{n-4}}y_{n-8}=\frac{u_{n-1}v_{n-5}w_{n-9}}{w_nu_{n-4}v_{n-8}}z_{n-12}, \tag{2.27}$$

for $n\geq 7$, from which it follows that

$$\begin{aligned}x_{12m+6l+r}=x_{6l+r-12}\prod_{s=0}^{m}&\frac{v_{6(2s+l+1+\lfloor\frac{r-5}{6}\rfloor)+r-7-6\lfloor\frac{r-5}{6}\rfloor}}{u_{6(2s+l+1+\lfloor\frac{r-4}{6}\rfloor)+r-6-6\lfloor\frac{r-4}{6}\rfloor}}\frac{w_{6(2s+l+\lfloor\frac{r-3}{6}\rfloor)+r-5-6\lfloor\frac{r-3}{6}\rfloor}}{v_{6(2s+l+\lfloor\frac{r-2}{6}\rfloor)+r-4-6\lfloor\frac{r-2}{6}\rfloor}}\\ &\times\frac{u_{6(2s+l+\lfloor\frac{r-7}{6}\rfloor)+r-9-6\lfloor\frac{r-7}{6}\rfloor}}{w_{6(2s+l+\lfloor\frac{r-6}{6}\rfloor)+r-8-6\lfloor\frac{r-6}{6}\rfloor}},\end{aligned} \tag{2.28}$$

$$\begin{aligned}y_{12m+6l+r}=y_{6l+r-12}\prod_{s=0}^{m}&\frac{w_{6(2s+l+1+\lfloor\frac{r-5}{6}\rfloor)+r-7-6\lfloor\frac{r-5}{6}\rfloor}}{v_{6(2s+l+1+\lfloor\frac{r-4}{6}\rfloor)+r-6-6\lfloor\frac{r-4}{6}\rfloor}}\frac{u_{6(2s+l+\lfloor\frac{r-3}{6}\rfloor)+r-5-6\lfloor\frac{r-3}{6}\rfloor}}{w_{6(2s+l+\lfloor\frac{r-2}{6}\rfloor)+r-4-6\lfloor\frac{r-2}{6}\rfloor}}\\ &\times\frac{v_{6(2s+l+\lfloor\frac{r-7}{6}\rfloor)+r-9-6\lfloor\frac{r-7}{6}\rfloor}}{u_{6(2s+l+\lfloor\frac{r-6}{6}\rfloor)+r-8-6\lfloor\frac{r-6}{6}\rfloor}},\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}z_{12m+6l+r}=z_{6l+r-12}\prod_{s=0}^{m}&\frac{u_{6(2s+l+1+\lfloor\frac{r-5}{6}\rfloor)+r-7-6\lfloor\frac{r-5}{6}\rfloor}}{w_{6(2s+l+1+\lfloor\frac{r-4}{6}\rfloor)+r-6-6\lfloor\frac{r-4}{6}\rfloor}}\frac{v_{6(2s+l+\lfloor\frac{r-3}{6}\rfloor)+r-5-6\lfloor\frac{r-3}{6}\rfloor}}{u_{6(2s+l+\lfloor\frac{r-2}{6}\rfloor)+r-4-6\lfloor\frac{r-2}{6}\rfloor}}\\ &\times\frac{w_{6(2s+l+\lfloor\frac{r-7}{6}\rfloor)+r-9-6\lfloor\frac{r-7}{6}\rfloor}}{v_{6(2s+l+\lfloor\frac{r-6}{6}\rfloor)+r-8-6\lfloor\frac{r-6}{6}\rfloor}},\end{aligned} \tag{2.30}$$

for every $m\in\mathbb{N}_0$, $l\in\{1,2\}$ and $r=\overline{1,6}$. Employing (2.22)-(2.24) in (2.28)-(2.30), we have

$$x_{12m+6l+r}=x_{6l+r-12}\prod_{s=0}^{m}\frac{D_{s,l,r}}{C_{s,l,r}}\frac{E_{s,l,r}}{G_{s,l,r}}\frac{F_{s,l,r}}{H_{s,l,r}}, \tag{2.31}$$

$$y_{12m+6l+r}=y_{6l+r-12}\prod_{s=0}^{m}\frac{\widehat{D}_{s,l,r}}{\widehat{C}_{s,l,r}}\frac{\widehat{E}_{s,l,r}}{\widehat{G}_{s,l,r}}\frac{\widehat{F}_{s,l,r}}{\widehat{H}_{s,l,r}}, \tag{2.32}$$

$$z_{12m+6l+r}=z_{6l+r-12}\prod_{s=0}^{m}\frac{\widetilde{D}_{s,l,r}}{\widetilde{C}_{s,l,r}}\frac{\widetilde{E}_{s,l,r}}{\widetilde{G}_{s,l,r}}\frac{\widetilde{F}_{s,l,r}}{\widetilde{H}_{s,l,r}}, \tag{2.33}$$

for every $m \in \mathbb{N}_0$, $l \in \{1,2\}$ and $r = \overline{1,6}$, where

$$C_{s,l,r} = \left(\prod_{k=0}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(a_{6k+r-6\lfloor \frac{r-4}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-4}{6} \rfloor} \alpha_{6k+r-4-6\lfloor \frac{r-4}{6} \rfloor} \right) \right) u_{r-6-6\lfloor \frac{r-4}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l+\lfloor \frac{r-4}{6} \rfloor} \left(a_{6k+r-6\lfloor \frac{r-4}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-4}{6} \rfloor} \alpha_{6k+r-4-6\lfloor \frac{r-4}{6} \rfloor} \right) \right)$$
$$\times \left(a_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} A_{6i+r-2-6\lfloor \frac{r-4}{6} \rfloor} \beta_{6i+r-4-6\lfloor \frac{r-4}{6} \rfloor} + a_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} B_{6i+r-2-6\lfloor \frac{r-4}{6} \rfloor} + b_{6i+r-6\lfloor \frac{r-4}{6} \rfloor} \right),$$

$$D_{s,l,r} = \left(\prod_{k=0}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\alpha_{6k+r-1-6\lfloor \frac{r-5}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-5}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-5}{6} \rfloor} \right) \right) v_{r-7-6\lfloor \frac{r-5}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l+\lfloor \frac{r-5}{6} \rfloor} \left(\alpha_{6k+r-1-6\lfloor \frac{r-5}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-5}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-5}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} a_{6i+r-3-6\lfloor \frac{r-5}{6} \rfloor} B_{6i+r-5-6\lfloor \frac{r-5}{6} \rfloor} + \alpha_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} b_{6i+r-3-6\lfloor \frac{r-5}{6} \rfloor} + \beta_{6i+r-1-6\lfloor \frac{r-5}{6} \rfloor} \right),$$

$$E_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(A_{6k+r+1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6k+r-1-6\lfloor \frac{r-3}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-3}{6} \rfloor} \right) \right) w_{r-5-6\lfloor \frac{r-3}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-3}{6} \rfloor} \left(A_{6k+r+1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6k+r-1-6\lfloor \frac{r-3}{6} \rfloor} a_{6k+r-3-6\lfloor \frac{r-3}{6} \rfloor} \right) \right)$$
$$\times \left(A_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \alpha_{6i+r-1-6\lfloor \frac{r-3}{6} \rfloor} b_{6i+r-3-6\lfloor \frac{r-3}{6} \rfloor} + A_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \beta_{6i+r-1-6\lfloor \frac{r-3}{6} \rfloor} + B_{6i+r+1-6\lfloor \frac{r-3}{6} \rfloor} \right),$$

$$F_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(a_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right) u_{r-9-6\lfloor \frac{r-7}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(a_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} A_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right)$$
$$\times \left(a_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} A_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} \beta_{6i+r-7-6\lfloor \frac{r-7}{6} \rfloor} + a_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} B_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} + b_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} \right),$$

$$G_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\alpha_{6k+r+2-6\lfloor \frac{r-2}{6} \rfloor} a_{6k+r-6\lfloor \frac{r-2}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-2}{6} \rfloor} \right) \right) v_{r-4-6\lfloor \frac{r-2}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\alpha_{6k+r+2-6\lfloor \frac{r-2}{6} \rfloor} a_{6k+r-6\lfloor \frac{r-2}{6} \rfloor} A_{6k+r-2-6\lfloor \frac{r-2}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} a_{6i+r-6\lfloor \frac{r-2}{6} \rfloor} B_{6i+r-2-6\lfloor \frac{r-2}{6} \rfloor} + \alpha_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} b_{6i+r-6\lfloor \frac{r-2}{6} \rfloor} + \beta_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} \right),$$

$$H_{s,l,r}=\left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(A_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}a_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)w_{r-8-6\lfloor\frac{r-6}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(A_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}a_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)$$
$$\times\left(A_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}b_{6i+r-6-6\lfloor\frac{r-6}{6}\rfloor}+A_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}\beta_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}+B_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}\right),$$

$$\widehat{C}_{s,l,r}=\left(\prod_{k=0}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(\alpha_{6k+r-6\lfloor\frac{r-4}{6}\rfloor}a_{6k+r-2-6\lfloor\frac{r-4}{6}\rfloor}A_{6k+r-4-6\lfloor\frac{r-4}{6}\rfloor}\right)\right)v_{r-6-6\lfloor\frac{r-4}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(\alpha_{6k+r-6\lfloor\frac{r-4}{6}\rfloor}a_{6k+r-2-6\lfloor\frac{r-4}{6}\rfloor}A_{6k+r-4-6\lfloor\frac{r-4}{6}\rfloor}\right)\right)$$
$$\times\left(\alpha_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}a_{6i+r-2-6\lfloor\frac{r-4}{6}\rfloor}B_{6i+r-4-6\lfloor\frac{r-4}{6}\rfloor}+\alpha_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}b_{6i+r-2-6\lfloor\frac{r-4}{6}\rfloor}+\beta_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\right),$$

$$\widehat{D}_{s,l,r}=\left(\prod_{k=0}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(A_{6k+r-1-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6k+r-3-6\lfloor\frac{r-5}{6}\rfloor}a_{6k+r-5-6\lfloor\frac{r-5}{6}\rfloor}\right)\right)w_{r-7-6\lfloor\frac{r-5}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(A_{6k+r-1-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6k+r-3-6\lfloor\frac{r-5}{6}\rfloor}a_{6k+r-5-6\lfloor\frac{r-5}{6}\rfloor}\right)\right)$$
$$\times\left(A_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6i+r-3-6\lfloor\frac{r-5}{6}\rfloor}b_{6i+r-5-6\lfloor\frac{r-5}{6}\rfloor}+A_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}\beta_{6i+r-3-6\lfloor\frac{r-5}{6}\rfloor}+B_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}\right),$$

$$\widehat{E}_{s,l,r}=\left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(a_{6k+r+1-6\lfloor\frac{r-3}{6}\rfloor}A_{6k+r-1-6\lfloor\frac{r-3}{6}\rfloor}\alpha_{6k+r-3-6\lfloor\frac{r-3}{6}\rfloor}\right)\right)u_{r-5-6\lfloor\frac{r-3}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(a_{6k+r+1-6\lfloor\frac{r-3}{6}\rfloor}A_{6k+r-1-6\lfloor\frac{r-3}{6}\rfloor}\alpha_{6k+r-3-6\lfloor\frac{r-3}{6}\rfloor}\right)\right)$$
$$\times\left(a_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}A_{6i+r-1-6\lfloor\frac{r-3}{6}\rfloor}\beta_{6i+r-3-6\lfloor\frac{r-3}{6}\rfloor}+a_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}B_{6i+r-1-6\lfloor\frac{r-3}{6}\rfloor}+b_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}\right),$$

$$\widehat{F}_{s,l,r}=\left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-7}{6}\rfloor}\left(\alpha_{6k+r-3-6\lfloor\frac{r-7}{6}\rfloor}a_{6k+r-5-6\lfloor\frac{r-7}{6}\rfloor}A_{6k+r-7-6\lfloor\frac{r-7}{6}\rfloor}\right)\right)v_{r-9-6\lfloor\frac{r-7}{6}\rfloor}$$
$$+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-7}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-7}{6}\rfloor}\left(\alpha_{6k+r-3-6\lfloor\frac{r-7}{6}\rfloor}a_{6k+r-5-6\lfloor\frac{r-7}{6}\rfloor}A_{6k+r-7-6\lfloor\frac{r-7}{6}\rfloor}\right)\right)$$
$$\times\left(\alpha_{6i+r-3-6\lfloor\frac{r-7}{6}\rfloor}a_{6i+r-5-6\lfloor\frac{r-7}{6}\rfloor}B_{6i+r-7-6\lfloor\frac{r-7}{6}\rfloor}+\alpha_{6i+r-3-6\lfloor\frac{r-7}{6}\rfloor}b_{6i+r-5-6\lfloor\frac{r-7}{6}\rfloor}+\beta_{6i+r-3-6\lfloor\frac{r-7}{6}\rfloor}\right),$$

$$\begin{aligned}\widehat{G}_{s,l,r} &= \left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(A_{6k+r+2-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6k+r-6\lfloor\frac{r-2}{6}\rfloor}a_{6k+r-2-6\lfloor\frac{r-2}{6}\rfloor}\right)\right)w_{r-4-6\lfloor\frac{r-2}{6}\rfloor}\\ &+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-2}{6}\rfloor}\left(A_{6k+r+2-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6k+r-6\lfloor\frac{r-2}{6}\rfloor}a_{6k+r-2-6\lfloor\frac{r-2}{6}\rfloor}\right)\right)\\ &\times\left(A_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\alpha_{6i+r-6\lfloor\frac{r-2}{6}\rfloor}b_{6i+r-2-6\lfloor\frac{r-2}{6}\rfloor}+A_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\beta_{6i+r-6\lfloor\frac{r-2}{6}\rfloor}+B_{6i+r+2-6\lfloor\frac{r-2}{6}\rfloor}\right),\end{aligned}$$

$$\begin{aligned}\widehat{H}_{s,l,r} &= \left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(a_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}A_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)u_{r-8-6\lfloor\frac{r-6}{6}\rfloor}\\ &+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-6}{6}\rfloor}\left(a_{6k+r-2-6\lfloor\frac{r-6}{6}\rfloor}A_{6k+r-4-6\lfloor\frac{r-6}{6}\rfloor}\alpha_{6k+r-6-6\lfloor\frac{r-6}{6}\rfloor}\right)\right)\\ &\times\left(a_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}A_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}\beta_{6i+r-6-6\lfloor\frac{r-6}{6}\rfloor}+a_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}B_{6i+r-4-6\lfloor\frac{r-6}{6}\rfloor}+b_{6i+r-2-6\lfloor\frac{r-6}{6}\rfloor}\right),\end{aligned}$$

$$\begin{aligned}\widetilde{C}_{s,l,r} &= \left(\prod_{k=0}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(A_{6k+r-6\lfloor\frac{r-4}{6}\rfloor}\alpha_{6k+r-2-6\lfloor\frac{r-4}{6}\rfloor}a_{6k+r-4-6\lfloor\frac{r-4}{6}\rfloor}\right)\right)w_{r-6-6\lfloor\frac{r-4}{6}\rfloor}\\ &+\sum_{i=0}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l+\lfloor\frac{r-4}{6}\rfloor}\left(A_{6k+r-6\lfloor\frac{r-4}{6}\rfloor}\alpha_{6k+r-2-6\lfloor\frac{r-4}{6}\rfloor}a_{6k+r-4-6\lfloor\frac{r-4}{6}\rfloor}\right)\right)\\ &\times\left(A_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\alpha_{6i+r-2-6\lfloor\frac{r-4}{6}\rfloor}b_{6i+r-4-6\lfloor\frac{r-4}{6}\rfloor}+A_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\beta_{6i+r-2-6\lfloor\frac{r-4}{6}\rfloor}+B_{6i+r-6\lfloor\frac{r-4}{6}\rfloor}\right),\end{aligned}$$

$$\begin{aligned}\widetilde{D}_{s,l,r} &= \left(\prod_{k=0}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(a_{6k+r-1-6\lfloor\frac{r-5}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6k+r-5-6\lfloor\frac{r-5}{6}\rfloor}\right)\right)u_{r-7-6\lfloor\frac{r-5}{6}\rfloor}\\ &+\sum_{i=0}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l+\lfloor\frac{r-5}{6}\rfloor}\left(a_{6k+r-1-6\lfloor\frac{r-5}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-5}{6}\rfloor}\alpha_{6k+r-5-6\lfloor\frac{r-5}{6}\rfloor}\right)\right)\\ &\times\left(a_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}A_{6i+r-3-6\lfloor\frac{r-5}{6}\rfloor}\beta_{6i+r-5-6\lfloor\frac{r-5}{6}\rfloor}+a_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}B_{6i+r-3-6\lfloor\frac{r-5}{6}\rfloor}+b_{6i+r-1-6\lfloor\frac{r-5}{6}\rfloor}\right),\end{aligned}$$

$$\begin{aligned}\widetilde{E}_{s,l,r} &= \left(\prod_{k=0}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\alpha_{6k+r+1-6\lfloor\frac{r-3}{6}\rfloor}a_{6k+r-1-6\lfloor\frac{r-3}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-3}{6}\rfloor}\right)\right)v_{r-5-6\lfloor\frac{r-3}{6}\rfloor}\\ &+\sum_{i=0}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\prod_{k=i+1}^{2s+l-1+\lfloor\frac{r-3}{6}\rfloor}\left(\alpha_{6k+r+1-6\lfloor\frac{r-3}{6}\rfloor}a_{6k+r-1-6\lfloor\frac{r-3}{6}\rfloor}A_{6k+r-3-6\lfloor\frac{r-3}{6}\rfloor}\right)\right)\\ &\times\left(\alpha_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}a_{6i+r-1-6\lfloor\frac{r-3}{6}\rfloor}B_{6i+r-3-6\lfloor\frac{r-3}{6}\rfloor}+\alpha_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}b_{6i+r-1-6\lfloor\frac{r-3}{6}\rfloor}+\beta_{6i+r+1-6\lfloor\frac{r-3}{6}\rfloor}\right),\end{aligned}$$

$$\widetilde{F}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(A_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} a_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right) w_{r-9-6\lfloor \frac{r-7}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-7}{6} \rfloor} \left(A_{6k+r-3-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6k+r-5-6\lfloor \frac{r-7}{6} \rfloor} a_{6k+r-7-6\lfloor \frac{r-7}{6} \rfloor} \right) \right)$$
$$\times \left(A_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} \alpha_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} b_{6i+r-7-6\lfloor \frac{r-7}{6} \rfloor} + A_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} \beta_{6i+r-5-6\lfloor \frac{r-7}{6} \rfloor} + B_{6i+r-3-6\lfloor \frac{r-7}{6} \rfloor} \right),$$

$$\widetilde{G}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(a_{6k+r+2-6\lfloor \frac{r-2}{6} \rfloor} A_{6k+r-6\lfloor \frac{r-2}{6} \rfloor} \alpha_{6k+r-2-6\lfloor \frac{r-2}{6} \rfloor} \right) \right) u_{r-4-6\lfloor \frac{r-2}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-2}{6} \rfloor} \left(a_{6k+r+2-6\lfloor \frac{r-2}{6} \rfloor} A_{6k+r-6\lfloor \frac{r-2}{6} \rfloor} \alpha_{6k+r-2-6\lfloor \frac{r-2}{6} \rfloor} \right) \right)$$
$$\times \left(a_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} A_{6i+r-6\lfloor \frac{r-2}{6} \rfloor} \beta_{6i+r-2-6\lfloor \frac{r-2}{6} \rfloor} + a_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} B_{6i+r-6\lfloor \frac{r-2}{6} \rfloor} + b_{6i+r+2-6\lfloor \frac{r-2}{6} \rfloor} \right),$$

$$\widetilde{H}_{s,l,r} = \left(\prod_{k=0}^{2s+l-1+\lfloor \frac{r-6}{6} \rfloor} \left(\alpha_{6k+r-2-6\lfloor \frac{r-6}{6} \rfloor} a_{6k+r-4-6\lfloor \frac{r-6}{6} \rfloor} A_{6k+r-6-6\lfloor \frac{r-6}{6} \rfloor} \right) \right) v_{r-8-6\lfloor \frac{r-6}{6} \rfloor}$$
$$+ \sum_{i=0}^{2s+l-1+\lfloor \frac{r-6}{6} \rfloor} \left(\prod_{k=i+1}^{2s+l-1+\lfloor \frac{r-6}{6} \rfloor} \left(\alpha_{6k+r-2-6\lfloor \frac{r-6}{6} \rfloor} a_{6k+r-4-6\lfloor \frac{r-6}{6} \rfloor} A_{6k+r-6-6\lfloor \frac{r-6}{6} \rfloor} \right) \right)$$
$$\times \left(\alpha_{6i+r-2-6\lfloor \frac{r-6}{6} \rfloor} a_{6i+r-4-6\lfloor \frac{r-6}{6} \rfloor} B_{6i+r-6-6\lfloor \frac{r-6}{6} \rfloor} + \alpha_{6i+r-2-6\lfloor \frac{r-6}{6} \rfloor} b_{6i+r-4-6\lfloor \frac{r-6}{6} \rfloor} + \beta_{6i+r-2-6\lfloor \frac{r-6}{6} \rfloor} \right).$$

The previous computations prove the next theorem.

Theorem 2.1. *Suppose that $\{(x_n, y_n, z_n)\}_{n \geq -5}$ is a well-defined solution of system (1.5). Then, the general solutions of system (1.5) are given by equations in (2.31)-(2.33).*

By the following theorem, we characterize the forbidden set of the initial values for system (1.5).

Theorem 2.2. *Assume that $a_n \neq 0$, $b_n \neq 0$, $\alpha_n \neq 0$, $\beta_n \neq 0$, $A_n \neq 0$, $B_n \neq 0$, for every $n \in \mathbb{N}_0$. Then the forbidden set of the initial values for system (1.5) is given by the set*

$$\mathcal{F} = \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{1} \Big\{ (x_{-5}, x_{-4}, \ldots, x_{-1}, y_{-5}, y_{-4}, \ldots, y_{-1}, z_{-5}, z_{-4}, \ldots, z_{-1}) \in \mathbb{R}^{15} :$$
$$z_{i-2}x_{i-3}y_{i-4}z_{i-5} = \frac{1}{c_m}, x_{i-2}y_{i-3}z_{i-4}x_{i-5} = \frac{1}{d_m},\ y_{i-2}z_{i-3}x_{i-4}y_{i-5} = \frac{1}{e_m} \text{where}$$
$$c_m := -\sum_{j=0}^{m} \left(\frac{B_{6j+i+2} + A_{6j+i+2}b_{6j+i} + A_{6j+i+2}\alpha_{6j+i}b_{6j+i-2}}{A_{6j+i+2}\alpha_{6j+i}a_{6j+i-2}} \right) \prod_{l=0}^{j-1} \frac{1}{A_{6l+i+2}\alpha_{6l+i}a_{6l+i-2}} \neq 0,$$
$$d_m := -\sum_{j=0}^{m} \left(\frac{b_{6j+i+2} + a_{6j+i+2}\beta_{6j+i} + a_{6j+i+2}A_{6j+i}\beta_{6j+i-2}}{a_{6j+i+2}A_{6j+i}\alpha_{6j+i-2}} \right) \prod_{l=0}^{j-1} \frac{1}{a_{6l+i+2}A_{6l+i}\alpha_{6l+i-2}} \neq 0,,$$
$$e_m := -\sum_{j=0}^{m} \left(\frac{\beta_{6j+i+2} + \alpha_{6j+i+2}B_{6j+i} + \alpha_{6j+i+2}a_{6j+i}B_{6j+i-2}}{\alpha_{6j+i+2}a_{6j+i}A_{6j+i-2}} \right) \prod_{l=0}^{j-1} \frac{1}{\alpha_{6l+i+2}a_{6l+i}A_{6l+i-2}} \neq 0$$
$$\bigcup \bigcup_{j=1}^{5} \Big\{ (x_{-5}, x_{-4}, \ldots, x_{-1}, y_{-5}, y_{-4}, \ldots, y_{-1}, z_{-5}, z_{-4}, \ldots, z_{-1}) \in \mathbb{R}^{15} : x_{-j} = 0,\ y_{-j} = 0,\ z_{-j} = 0 \Big\}. \quad (2.34)$$

Proof. At the beginning of Section 2, we have obtained that the set

$$\bigcup_{j=1}^{5}\Big\{(x_{-5},x_{-4},\dots,x_{-1},y_{-5},y_{-4},\dots,y_{-1},z_{-5},z_{-4},\dots,z_{-1})\in\mathbb{R}^{15}: x_{-j}=0,\ y_{-j}=0,\ z_{-j}=0\Big\}$$

belongs to the forbidden set of the initial values for system (1.5). Now, we suppose that $x_n\neq 0$, $y_n\neq 0$ and $z_n\neq 0$. Note that the system (1.5) is not defined, when the conditions $a_n+b_nz_{n-2}x_{n-3}y_{n-4}z_{n-5}=0$, $\alpha_n+\beta_nx_{n-2}y_{n-3}z_{n-4}x_{n-5}=0$ or $A_n+B_ny_{n-2}z_{n-3}x_{n-4}y_{n-5}=0$, that is, $z_{n-2}x_{n-3}y_{n-4}z_{n-5}=-\frac{a_n}{b_n}$, $x_{n-2}y_{n-3}z_{n-4}x_{n-5}=-\frac{\alpha_n}{\beta_n}$ or $y_{n-2}z_{n-3}x_{n-4}y_{n-5}=-\frac{A_n}{B_n}$, for some $n\in\mathbb{N}_0$, are satisfied(Here we consider that $b_n\neq 0$, $\beta_n\neq 0$ and $B_n\neq 0$ for every $n\in\mathbb{N}_0$). From this and equations in (2.2), we get

$$u_{6m+i}=-\frac{\beta_{6m+i+2}}{\alpha_{6m+i+2}},\ v_{6m+i}=-\frac{B_{6m+i+2}}{A_{3m+i+2}},\ w_{6m+i}=-\frac{b_{6m+i+2}}{a_{6m+i+2}}, \tag{2.35}$$

for some $m\in\mathbb{N}_0$ and $i=\overline{-2,3}$. Hence, we can determine the forbidden set of the initial values for system (1.5) by using the substitution $u_n=\frac{1}{x_ny_{n-1}z_{n-2}x_{n-3}}$, $v_n=\frac{1}{y_nz_{n-1}x_{n-2}y_{n-3}}$, $w_n=\frac{1}{z_nx_{n-1}y_{n-2}z_{n-3}}$. Now, we consider the functions

$$\begin{aligned} f_{6m+i+2}(t) &:= a_{6m+i+2}t+b_{6m+i+2},\\ g_{6m+i+2}(t) &:= \alpha_{6m+i+2}t+\beta_{6m+i+2},\\ h_{6m+i+2}(t) &:= A_{6m+i+2}t+B_{6m+i+2}, \end{aligned} \tag{2.36}$$

for $m\in\mathbb{N}_0$, $i=\overline{-2,3}$, which correspond to the system (2.3). From (2.35) and (2.36), we can write

$$u_{6m+i}=f_{6m+i}\circ h_{6m+i-2}\circ g_{6m+i-4}\cdots\circ f_i\circ h_{i-2}\circ g_{i-4}(u_{i-6}), \tag{2.37}$$

$$v_{6m+i}=g_{6m+i}\circ f_{6m+i-2}\circ h_{6m+i-4}\cdots\circ g_i\circ f_{i-2}\circ h_{i-4}(v_{i-6}), \tag{2.38}$$

$$w_{6m+i}=h_{6m+i}\circ g_{6m+i-2}\circ f_{6m+i-4}\cdots\circ h_i\circ g_{i-2}\circ f_{i-4}(w_{i-6}), \tag{2.39}$$

where $m\in\mathbb{N}_0$, and $i=\overline{4,9}$. By using (2.35) and implicit forms (2.37)-(2.39) and considering $f^{-1}_{6m+i+2}(0)=-\frac{b_{6m+i+2}}{a_{6m+i+2}}$, $g^{-1}_{6m+i+2}(0)=-\frac{\beta_{6m+i+2}}{\alpha_{6m+i+2}}$, $h^{-1}_{6m+i+2}(0)=-\frac{B_{6m+i+2}}{A_{6m+i+2}}$, for $m\geq -1$ and $i=\overline{4,9}$, we get

$$u_{i-6}=g^{-1}_{i-4}\circ h^{-1}_{i-2}\circ f^{-1}_i\circ\cdots\circ g^{-1}_{6m+i-4}\circ h^{-1}_{6m+i-2}\circ f^{-1}_{6m+i}(0), \tag{2.40}$$

$$v_{i-6}=h^{-1}_{i-4}\circ f^{-1}_{i-2}\circ g^{-1}_i\circ\cdots\circ h^{-1}_{6m+i-4}\circ f^{-1}_{6m+i-2}\circ g^{-1}_{6m+i}(0), \tag{2.41}$$

$$w_{i-6}=f^{-1}_{i-4}\circ g^{-1}_{i-2}\circ h^{-1}_i\circ\cdots\circ f^{-1}_{6m+i-4}\circ g^{-1}_{6m+i-2}\circ h^{-1}_{6m+i}(0), \tag{2.42}$$

where $f^{-1}_{6m+i+2}(t)=\frac{t-b_{6m+i+2}}{a_{6m+i+2}}$, $g^{-1}_{6m+i+2}(t)=\frac{t-\beta_{6m+i+2}}{\alpha_{6m+i+2}}$, $h^{-1}_{6m+i+2}(t)=\frac{t-B_{6m+i+2}}{A_{6m+i+2}}$, $m\geq -1$, $i=\overline{4,9}$. From (2.40)-(2.42), we get

$$u_{i-6} = -\sum_{j=0}^{m}\left(\frac{b_{6j+i}+a_{6j+i}B_{6j+i-2}+a_{6j+i}A_{6j+i-2}\beta_{6j+i-4}}{a_{6j+i}A_{6j+i-2}\alpha_{6j+i-4}}\right)\prod_{l=0}^{j-1}\frac{1}{a_{6l+i}A_{6l+i-2}\alpha_{6l+i-4}},$$

$$v_{i-6} = -\sum_{j=0}^{m}\left(\frac{\beta_{6j+i}+\alpha_{6j+i}b_{6j+i-2}+\alpha_{6j+i}a_{6j+i-2}B_{6j+i-4}}{\alpha_{6j+i}a_{6j+i-2}A_{6j+i-4}}\right)\prod_{l=0}^{j-1}\frac{1}{\alpha_{6l+i}a_{6l+i-2}A_{6l+i-4}},$$

$$w_{i-6} = -\sum_{j=0}^{m}\left(\frac{B_{6j+i}+A_{6j+i}\beta_{6j+i-2}+A_{6j+i}\alpha_{6j+i-2}b_{6j+i-4}}{A_{6j+i}\alpha_{6j+i-2}a_{6j+i-4}}\right)\prod_{l=0}^{j-1}\frac{1}{A_{6l+i}\alpha_{6l+i-2}a_{6l+i-4}},$$

for some $m\in\mathbb{N}_0$ and $i=\overline{4,9}$. This means that if one of the conditions in (2.40)-(2.42) holds, then m-th iteration or $(m+1)$-th iteration in system (1.5) can not be calculated. $\square$

3. Solutions of System (1.5) with Constant Coefficients

In this section, we give the forms of solutions of system (1.5) when all the coefficients are constant. We assume that $a_n = a$, $b_n = b$, $\alpha_n = \alpha$, $\beta_n = \beta$, $A_n = A$ and $B_n = B$ for every $n \in \mathbb{N}_0$. In this case, system (1.5) is written as in the form

$$\begin{aligned}
x_n &= \frac{y_{n-4}z_{n-5}}{y_{n-1}\left(a + bz_{n-2}x_{n-3}y_{n-4}z_{n-5}\right)},\\
y_n &= \frac{z_{n-4}x_{n-5}}{z_{n-1}\left(\alpha + \beta x_{n-2}y_{n-3}z_{n-4}x_{n-5}\right)},\\
z_n &= \frac{x_{n-4}y_{n-5}}{x_{n-1}\left(A + By_{n-2}z_{n-3}x_{n-4}y_{n-5}\right)},\ n \in \mathbb{N}_0.
\end{aligned} \tag{3.1}$$

In (2.28)-(2.30), if we replace the formulas given in (2.22)-(2.24), then the solution of system (3.1) is given by

$$\begin{aligned}
x_{12m+6l+r} = x_{6l+r-12}\prod_{s=0}^{m} &\frac{(a\alpha A)^{2s+l+1+\lfloor\frac{r-5}{6}\rfloor} v_{r-7-6\lfloor\frac{r-5}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor\frac{r-5}{6}\rfloor}\right)(\alpha a B + \alpha b + \beta)}{(a\alpha A)^{2s+l+1+\lfloor\frac{r-4}{6}\rfloor} u_{r-6-6\lfloor\frac{r-4}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor\frac{r-4}{6}\rfloor}\right)(aA\beta + aB + b)}\\
&\times\frac{(a\alpha A)^{2s+l+\lfloor\frac{r-3}{6}\rfloor} w_{r-5-6\lfloor\frac{r-3}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-3}{6}\rfloor}\right)(A\alpha b + A\beta + B)}{(a\alpha A)^{2s+l+\lfloor\frac{r-2}{6}\rfloor} v_{r-4-6\lfloor\frac{r-2}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-2}{6}\rfloor}\right)(\alpha a B + \alpha b + \beta)}\\
&\times\frac{(a\alpha A)^{2s+l+\lfloor\frac{r-7}{6}\rfloor} u_{r-9-6\lfloor\frac{r-7}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-7}{6}\rfloor}\right)(aA\beta + aB + b)}{(a\alpha A)^{2s+l+\lfloor\frac{r-6}{6}\rfloor} w_{r-8-6\lfloor\frac{r-6}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-6}{6}\rfloor}\right)(A\alpha b + A\beta + B)},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
y_{12m+6l+r} = y_{6l+r-12}\prod_{s=0}^{m} &\frac{(a\alpha A)^{2s+l+1+\lfloor\frac{r-5}{6}\rfloor} w_{r-7-6\lfloor\frac{r-5}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor\frac{r-5}{6}\rfloor}\right)(A\alpha b + A\beta + B)}{(a\alpha A)^{2s+l+1+\lfloor\frac{r-4}{6}\rfloor} v_{r-6-6\lfloor\frac{r-4}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor\frac{r-4}{6}\rfloor}\right)(\alpha a B + \alpha b + \beta)}\\
&\times\frac{(a\alpha A)^{2s+l+\lfloor\frac{r-3}{6}\rfloor} u_{r-5-6\lfloor\frac{r-3}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-3}{6}\rfloor}\right)(aA\beta + aB + b)}{(a\alpha A)^{2s+l+\lfloor\frac{r-2}{6}\rfloor} w_{r-4-6\lfloor\frac{r-2}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-2}{6}\rfloor}\right)(A\alpha b + A\beta + B)}\\
&\times\frac{(a\alpha A)^{2s+l+\lfloor\frac{r-7}{6}\rfloor} v_{r-9-6\lfloor\frac{r-7}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-7}{6}\rfloor}\right)(\alpha a B + \alpha b + \beta)}{(a\alpha A)^{2s+l+\lfloor\frac{r-6}{6}\rfloor} u_{r-8-6\lfloor\frac{r-6}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-6}{6}\rfloor}\right)(aA\beta + aB + b)},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
z_{12m+6l+r} = z_{6l+r-12}\prod_{s=0}^{m} &\frac{(a\alpha A)^{2s+l+1+\lfloor\frac{r-5}{6}\rfloor} u_{r-7-6\lfloor\frac{r-5}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor\frac{r-5}{6}\rfloor}\right)(aA\beta + aB + b)}{(a\alpha A)^{2s+l+1+\lfloor\frac{r-4}{6}\rfloor} w_{r-6-6\lfloor\frac{r-4}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+1+\lfloor\frac{r-4}{6}\rfloor}\right)(A\alpha b + A\beta + B)}\\
&\times\frac{(a\alpha A)^{2s+l+\lfloor\frac{r-3}{6}\rfloor} v_{r-5-6\lfloor\frac{r-3}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-3}{6}\rfloor}\right)(\alpha a B + \alpha b + \beta)}{(a\alpha A)^{2s+l+\lfloor\frac{r-2}{6}\rfloor} u_{r-4-6\lfloor\frac{r-2}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-2}{6}\rfloor}\right)(aA\beta + aB + b)}\\
&\times\frac{(a\alpha A)^{2s+l+\lfloor\frac{r-7}{6}\rfloor} w_{r-9-6\lfloor\frac{r-7}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-7}{6}\rfloor}\right)(A\alpha b + A\beta + B)}{(a\alpha A)^{2s+l+\lfloor\frac{r-6}{6}\rfloor} v_{r-8-6\lfloor\frac{r-6}{6}\rfloor}(1-a\alpha A) + \left(1-(a\alpha A)^{2s+l+\lfloor\frac{r-6}{6}\rfloor}\right)(\alpha a B + \alpha b + \beta)},
\end{aligned} \tag{3.4}$$

if $a\alpha A \neq 1$,

$$
\begin{aligned}
x_{12m+6l+r} = x_{6l+r-12} \prod_{s=0}^{m} & \frac{v_{r-7-6\lfloor\frac{r-5}{6}\rfloor} + (\alpha aB + \alpha b + \beta)\left(2s + l + 1 + \lfloor\frac{r-5}{6}\rfloor\right)}{u_{r-6-6\lfloor\frac{r-4}{6}\rfloor} + (aA\beta + aB + b)\left(2s + l + 1 + \lfloor\frac{r-4}{6}\rfloor\right)} \\
& \times \frac{w_{r-5-6\lfloor\frac{r-3}{6}\rfloor} + (A\alpha b + A\beta + B)\left(2s + l + \lfloor\frac{r-3}{6}\rfloor\right)}{v_{r-4-6\lfloor\frac{r-2}{6}\rfloor} + (\alpha aB + \alpha b + \beta)\left(2s + l + \lfloor\frac{r-2}{6}\rfloor\right)} \\
& \times \frac{u_{r-9-6\lfloor\frac{r-7}{6}\rfloor} + (aA\beta + aB + b)\left(2s + l + \lfloor\frac{r-7}{6}\rfloor\right)}{w_{r-8-6\lfloor\frac{r-6}{6}\rfloor} + (A\alpha b + A\beta + B)\left(2s + l + \lfloor\frac{r-6}{6}\rfloor\right)},
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
y_{12m+6l+r} = y_{6l+r-12} \prod_{s=0}^{m} & \frac{w_{r-7-6\lfloor\frac{r-5}{6}\rfloor} + (A\alpha b + A\beta + B)\left(2s + l + 1 + \lfloor\frac{r-5}{6}\rfloor\right)}{v_{r-6-6\lfloor\frac{r-4}{6}\rfloor} + (\alpha aB + \alpha b + \beta)\left(2s + l + 1 + \lfloor\frac{r-4}{6}\rfloor\right)} \\
& \times \frac{u_{r-5-6\lfloor\frac{r-3}{6}\rfloor} + (aA\beta + aB + b)\left(2s + l + \lfloor\frac{r-3}{6}\rfloor\right)}{w_{r-4-6\lfloor\frac{r-2}{6}\rfloor} + (A\alpha b + A\beta + B)\left(2s + l + \lfloor\frac{r-2}{6}\rfloor\right)} \\
& \times \frac{v_{r-9-6\lfloor\frac{r-7}{6}\rfloor} + (\alpha aB + \alpha b + \beta)\left(2s + l + \lfloor\frac{r-7}{6}\rfloor\right)}{u_{r-8-6\lfloor\frac{r-6}{6}\rfloor} + (aA\beta + aB + b)\left(2s + l + \lfloor\frac{r-6}{6}\rfloor\right)},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
z_{12m+6l+r} = z_{6l+r-12} \prod_{s=0}^{m} & \frac{u_{r-7-6\lfloor\frac{r-5}{6}\rfloor} + (aA\beta + aB + b)\left(2s + l + 1 + \lfloor\frac{r-5}{6}\rfloor\right)}{w_{r-6-6\lfloor\frac{r-4}{6}\rfloor} + (A\alpha b + A\beta + B)\left(2s + l + 1 + \lfloor\frac{r-4}{6}\rfloor\right)} \\
& \times \frac{v_{r-5-6\lfloor\frac{r-3}{6}\rfloor} + (\alpha aB + \alpha b + \beta)\left(2s + l + \lfloor\frac{r-3}{6}\rfloor\right)}{u_{r-4-6\lfloor\frac{r-2}{6}\rfloor} + (aA\beta + aB + b)\left(2s + l + \lfloor\frac{r-2}{6}\rfloor\right)} \\
& \times \frac{w_{r-9-6\lfloor\frac{r-7}{6}\rfloor} + (A\alpha b + A\beta + B)\left(2s + l + \lfloor\frac{r-7}{6}\rfloor\right)}{v_{r-8-6\lfloor\frac{r-6}{6}\rfloor} + (\alpha aB + \alpha b + \beta)\left(2s + l + \lfloor\frac{r-6}{6}\rfloor\right)},
\end{aligned} \tag{3.7}
$$

if $a\alpha A = 1$, for every $m \in \mathbb{N}_0$, $l \in \{1, 2\}$ and $r = \overline{1, 6}$.

4. Conclusion

In this paper, we have studied the following system of difference equations

$$
\begin{aligned}
x_n &= \frac{y_{n-4} z_{n-5}}{y_{n-1}\left(a_n + b_n z_{n-2} x_{n-3} y_{n-4} z_{n-5}\right)}, \\
y_n &= \frac{z_{n-4} x_{n-5}}{z_{n-1}\left(\alpha_n + \beta_n x_{n-2} y_{n-3} z_{n-4} x_{n-5}\right)}, \\
z_n &= \frac{x_{n-4} y_{n-5}}{x_{n-1}\left(A_n + B_n y_{n-2} z_{n-3} x_{n-4} y_{n-5}\right)}, \ n \in \mathbb{N}_0,
\end{aligned}
$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-j}, y_{-j}, $j = \overline{1, 5}$, are real numbers.

Firstly, we have solved above system in closed form by using suitable transformation. In addition, we also characterize the forbidden set of solutions of the system of difference equations. Finally, we have obtained solutions of aforementioned system with constant coefficients.

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On the Existence of the Solutions of A Nonlinear Fredholm Integral Equation in Hölder Spaces

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Abstract
In this article, we prove the theorem concerning the existence of the solutions for some nonlinear integral equations. As an application, we investigate the problem of existence of solutions of Fredholm integral equations using the technique of relative compactness in conjunction with fixed point theorem. Our solutions are placed in the space of functions satisfying the Hölder condition. Our work is more general than the previous works in [1–3]. In the last section, we show the efficiency of this approach on one numerical example.

Keywords: *Hölder condition; Fredholm integral equation; Schauder fixed point theorem.*

AMS Subject Classification (2020): *Primary: 45B05 ; Secondary: 45G10; 47H10.*

1. Introduction and Preliminaries

Integral equations appear in most applied areas and are as important as differential equations. Nonlinear integral equations are frequently studied in research articles [1–32].

The symbol $\mathbb{R}$ will stand for the set of real numbers and put $\mathbb{R}^+ = [0, \infty)$. Let's give some inequalities that we use in some sections of the article.

Lemma 1.1. *Let u, v be arbitrary real numbers such that $1 \leq v < u$. Moreover, let a be an arbitrarily fixed nonnegative number. Then, the following inequality is satisfied*

$$\left|(x^v + a)^{\frac{1}{u}} - (y^v + a)^{\frac{1}{u}}\right| \leq |x - y|^{\frac{v}{u}} \tag{1.1}$$

for all $x, y \in \mathbb{R}$,[4].

Lemma 1.2. *[4] Observe that using the notation of the generalized root of an arbitrary degree $u(u > 0)$, i.e. putting $\sqrt[u]{x} = x^{\frac{1}{u}}$ for $x \in \mathbb{R}^+$, we can represent inequality (1.1) in a more transparent form*

$$\left|\sqrt[u]{(x^v + a)} - \sqrt[u]{(y^v + a)}\right| \leq \sqrt[u]{|x - y|^v}.$$

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Observe that in the case when v is a natural even number inequality (1.1) can be extended to the whole real axis $\mathbb{R}$ i.e., if $v = 2n$, where $n \in \mathbb{N}$, then for an arbitrary number $u > 2n$ the following inequality is satisfied

$$\left|(x^{2n}+a)^{\frac{1}{u}} - (y^{2n}+a)^{\frac{1}{u}}\right| \leq |x-y|^{\frac{2n}{u}}$$

that is

$$\left|\sqrt[u]{(x^{2n}+a)} - \sqrt[u]{(y^{2n}+a)}\right| \leq \sqrt[u]{(x-y)^{2n}}$$

for all $x, y \in \mathbb{R}$ and $a \geq 0$.

In the case when $a = 0$ we have that $f(x) = x^{\frac{v}{u}}$. Applying the standard methods of mathematical analysis (second derivative, the concavity and the subadditivity of the function f) we can easily show that

$$\left|x^{\frac{v}{u}} - y^{\frac{v}{u}}\right| \leq |x-y|^{\frac{v}{u}}$$

for all $x, y \in \mathbb{R}^+$. The following known definitions are available in [1, 2, 31, 32].

Let $[\lambda, \mu]$ be a closed interval in $\mathbb{R}$, by $C[\lambda, \mu]$ we indicate the space of continuous functions defined on $[\lambda, \mu]$ equipped with the supremum norm, i.e.,

$$\|x\|_\infty = \sup\{|x(u)| : u \in [\lambda, \mu]\}$$

for $x \in C[\lambda, \mu]$. For a fixed α with $0 < \alpha \leq 1$, by $H_\alpha[\lambda, \mu]$ we will state the spaces of the real functions x defined on $[\lambda, \mu]$ and satisfying the Hölder condition, that is, those functions x for which there exists a constant H_x^α such that

$$|x(u) - x(v)| \leq H_x^\alpha |u-v|^\alpha \tag{1.2}$$

for all $u, v \in [\lambda, \mu]$. It is well proved that $H^\alpha[\lambda, \mu]$ is a linear subspaces of $C[\lambda, \mu]$. Also, for $x \in H^\alpha[\lambda, \mu]$, by H_x^α we will state the least possible stable for which inequality (1.2) is satisfied. Rather, we put

$$H_x^\alpha = \sup\left\{\frac{|x(u) - x(v)|}{|u-v|^\alpha} : u, v \in [\lambda, \mu] \text{ and } u \neq v\right\}. \tag{1.3}$$

The space $H_\alpha[\lambda, \mu]$ with $0 < \alpha \leq 1$ may be equipped with the norm

$$\|x\|_\alpha = |x(\lambda)| + H_x^\alpha$$

for $x \in H_\alpha[\lambda, \mu]$. Here, H_x^α is defined by (1.3). In [1], the authors show that $(H_\alpha[\lambda, \mu], \|\cdot\|_\alpha)$ with $0 < \alpha \leq 1$ is a Banach space.

Theorem 1.1 (Schauder's Fixed Point Theorem). *Let E be a nonempty, compact subset of a Banach space $(X, \|\cdot\|)$, convex and let $T : E \to E$ be a continuity mapping. Then T has at least one fixed point in E,[9].*

Lemma 1.3. *For $0 < \alpha < \beta \leq 1$, we have*

$$H_\beta[\lambda, \mu] \subset H_\alpha[\lambda, \mu] \subset C[\lambda, \mu].$$

Furthermore, for $x \in H_\beta[\lambda, \mu]$, we have:

$$\|x\|_\alpha \leq \max\left(1, (\mu - \lambda)^{\beta - \alpha}\right) \|x\|_\beta.$$

Particularly, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\beta$ is satisfied for $\lambda = 0$ and $\mu = 1$, [1].

Lemma 1.4. *Let's assume that $0 < \alpha < \beta \leq 1$ and E is a bounded subset in $H_\beta[\lambda, \mu]$, then E is a relatively compact subset in $H_\alpha[\lambda, \mu]$, [2].*

Lemma 1.5. *Assume that $0 < \alpha < \beta \leq 1$ and by B_r^β we state the ball centered at θ and radius r in the space $H_\beta[\lambda, \mu]$, i.e., $B_r^\beta = \{x \in H_\beta[\lambda, \mu] : \|x\|_\beta \leq r\}$. B_r^β is a closed subset of $H_\alpha[\lambda, \mu]$, [2].*

Corollary 1.1. *Assume that $0 < \alpha < \beta \leq 1$ and B_r^β is a relatively compact subset in $H_\alpha[\lambda, \mu]$ and is a closed subset of $H_\alpha[\lambda, \mu]$, then B_r^β is a compact subset in the space $H_\alpha[\lambda, \mu]$, [2].*

2. Main Result

J. Banaś and R. Nalepa et al. [1] study the following equation;

$$x(u) = p(u) + x(u) \int_{\lambda}^{\mu} k(u,\tau)x(\tau)d\tau. \tag{2.1}$$

Also, J. Caballero, M. Darwish and K. Sadarangani et al. [2] study the following equation;

$$x(u) = p(u) + x(u) \int_{0}^{1} k(u,\tau)x(r(\tau))d\tau. \tag{2.2}$$

Further, S. Peng, J. Wang and J. Chen et al. [3] study the following equation;

$$x(u) = f(u, x(u)) + x(u) \int_{\lambda}^{\mu} k(u,\tau)x(\tau)d\tau. \tag{2.3}$$

The purpose of this paper is to examine the existence of solutions of the following integral equation of Fredholm type with a changed argument,

$$x(u) = (Gx)(u) + x(u) \int_{0}^{1} k(u,\tau)x(q(\tau))d\tau, \quad u \in I = [0,1]. \tag{2.4}$$

The equation (2.4) is more general than many equations considered up to now and includes (2.1), (2.2) and (2.3) as special cases. Notice that the equation (2.1) in [1] for $\lambda = 0$ and $\mu = 1$ is a particular case of (2.4) with $q(\tau) = \tau$ and $(Gx)(u) = p(u)$. Also, it should be noted that the equation (2.4) is the more general than the equation (2.2) considered in [2]. If we take $(Gx)(u) = p(u)$, then the equation

$$x(u) = p(u) + x(u) \int_{0}^{1} k(u,\tau)x(r(\tau))d\tau$$

is obtained from the equation (2.4). Further, notice that equation (2.3) in study [3] for $\lambda = 0$ and $\mu = 1$ is a particular case of (2.4), for $(Gx)(u) = f(u, x(u)), q(\tau) = \tau$.

Theorem 2.1. *Assume that the following conditions $(i) - (iv)$ are satisfied:*

(i) *The operator $G : H_\beta[0,1] \to H_\beta[0,1]$ is continuous on the space $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$ and there exists function $w : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-decreasing such that it holds the inequality*

$$\|Gx\|_\beta \leq w\left(\|x\|_\beta\right),$$

for any $x \in H_\beta[0,1]$.

(ii) *$k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function such that there exists a constant k_β such that*

$$|k(u,\tau) - k(v,\tau)| \leq k_\beta |u - v|^\beta,$$

for any $u, v, \tau \in [0,1]$.

(iii) *$q : [0,1] \to [0,1]$ is a measurable function.*

(iv) *There exists a positive solution r_0 of the inequality*

$$w(r) + (2K + k_\beta)r^2 \leq r,$$

where the constant K is defined by

$$\sup\left\{\int_0^1 |k(u,\tau)|d\tau : u \in [0,1]\right\} \leq K.$$

Then the equation (2.4) has at least one solution $x = x(u)$ belonging to space $H_\alpha[0,1]$.

Proof. Note that we suppose unless stated otherwise that α and β are arbitrarily fixed numbers such that $0 < \alpha < \beta \leq 1$. Now, let us consider $x \in H_\beta[0,1]$ and the operator N defined on the space $H_\beta[0,1]$ by the formula:

$$(Nx)(u) = (Gx)(u) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau,$$

for $u \in [0,1]$. Then for arbitrarily fixed $u, v \in [0,1], (u \neq v)$, in view of our assumptions we get

$$\begin{aligned}
(Nx)(u) - (Nx)(v) &= (Gx)(u) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau - (Gx)(v) - x(v)\int_0^1 k(v,\tau)x(q(\tau))d\tau \\
&= (Gx)(u) - (Gx)(v) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau - x(v)\int_0^1 k(v,\tau)x(q(\tau))d\tau \\
&\quad + x(v)\int_0^1 k(u,\tau)x(q(\tau))d\tau - x(v)\int_0^1 k(u,\tau)x(q(\tau))d\tau \\
&= (Gx)(u) - (Gx)(v) + (x(u) - x(v))\int_0^1 k(u,\tau)x(q(\tau))d\tau \\
&\quad + x(v)\int_0^1 (k(u,\tau) - k(v,\tau))x(q(\tau))d\tau
\end{aligned}$$

and

$$\begin{aligned}
\frac{|(Nx)(u) - (Nx)(v)|}{|u-v|^\beta} &\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + \frac{|x(u) - x(v)|}{|u-v|^\beta}\int_0^1 |k(u,\tau)||x(q(\tau))|d\tau \\
&\quad + \frac{|x(v)|}{|u-v|^\beta}\int_0^1 |k(u,\tau) - k(v,\tau)||x(q(\tau))|d\tau \\
&\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + \|x\|_\infty \|x\|_\beta \int_0^1 |k(u,\tau)|d\tau \\
&\quad + \|x\|_\infty \|x\|_\infty \int_0^1 \frac{|k(u,\tau) - k(v,\tau)|}{|u-v|^\beta}d\tau \\
&\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + \|x\|_\beta^2 K + \|x\|_\beta^2 \int_0^1 k_\beta \frac{|u-v|^\beta}{|u-v|^\beta}d\tau \\
&\leq \frac{|(Gx)(u) - (Gx)(v)|}{|u-v|^\beta} + (K + k_\beta)\|x\|_\beta^2. \qquad (2.5)
\end{aligned}$$

Considering the (i) hypothesis, this demonstrates that the operator N maps $H_\beta[0,1]$ into itself. Besides, for any $x \in H_\beta[0,1]$, we get

$$\begin{aligned}
|(Nx)(0)| &\leq |(Gx)(0)| + |x(0)|\int_0^1 |k(0,\tau)||x(q(\tau))|d\tau \\
&\leq |(Gx)(0)| + \|x\|_\infty \|x\|_\infty \int_0^1 |k(0,\tau)|d\tau \\
&\leq |(Gx)(0)| + \|x\|_\beta^2 K. \qquad (2.6)
\end{aligned}$$

By the inequalities by (2.5) and (2.6), we derive that

$$\|Nx\|_\beta \leq \|Gx\|_\beta + (2K + k_\beta)\|x\|_\beta^2. \qquad (2.7)$$

Since positive number r_0 is the solution of the inequality given in hypothesis (iv), from (2.7) and function $w : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-decreasing, we conclude that the inequality

$$\|Nx\|_\beta \leq w(r_0) + (2K + k_\beta)r_0^2 \leq r_0. \qquad (2.8)$$

As a results, it follows that N transform the ball

$$B_{r_0}^\beta = \{x \in H_\beta[0,1] : \|x\|_\beta \leq r_0\}$$

into itself. That is, $N : B^{\beta}_{r_0} \to B^{\beta}_{r_0}$. Thus, we have that the set $B^{\beta}_{r_0}$ is relatively compact in $H_\alpha[0,1]$ for any $0 < \alpha < \beta \le 1$. Furthermore, $B^{\beta}_{r_0}$ is a compact subset in $H_\alpha[0,1]$.

In the sequel, we will demonstrate that the operator N is continuous on $B^{\beta}_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \le 1$.

Let $y \in B^{\beta}_{r_0}$ be an arbitrary point in $B^{\beta}_{r_0}$. Then, we get

$$\begin{aligned}(Nx)(u) - (Ny)(u) - ((Nx)(v) - (Ny)(v)) &= (Gx)(u) + x(u)\int_0^1 k(u,\tau)x(q(\tau))d\tau \\ &\quad -(Gy)(u) - y(u)\int_0^1 k(u,\tau)y(q(\tau))d\tau \\ &\quad -(Gx)(v) - x(v)\int_0^1 k(v,\tau)x(q(\tau))d\tau \\ &\quad +(Gy)(v) + y(v)\int_0^1 k(v,\tau)y(q(\tau))d\tau \end{aligned} \tag{2.9}$$

for any $x \in B^{\beta}_{r_0}$ and $u, v \in [0,1]$. The equality (2.9) can be rewritten as

$$\begin{aligned}&(Nx)(u) - (Ny)(u) - ((Nx)(v) - (Ny)(v)) \\ &= (Gx)(u) - (Gy)(u) - ((Gx)(v) - (Gy)(v)) + (x(u) - y(u))\int_0^1 k(u,\tau)x(q(\tau))d\tau \\ &+y(u)\left[\int_0^1 k(u,\tau)x(q(\tau))d\tau - \int_0^1 k(u,\tau)y(q(\tau))d\tau\right] \\ &-(x(v) - y(v))\int_0^1 k(v,\tau)x(q(\tau))d\tau \\ &-y(v)\left[\int_0^1 k(v,\tau)x(q(\tau))d\tau - \int_0^1 k(v,\tau)y(q(\tau))d\tau\right]. \end{aligned} \tag{2.10}$$

By (2.10), we have

$$\begin{aligned}(Nx)(u) - (Ny)(u) - ((Nx)(v) - (Ny)(v)) &= (Gx)(u) - (Gy)(u) - ((Gx)(v) - (Gy)(v)) \\ &\quad +[x(u) - y(u) - (x(v) - y(v))]\int_0^1 k(u,\tau)x(q(\tau))d\tau \\ &\quad +(x(v) - y(v))\left[\int_0^1 k(u,\tau)x(q(\tau))d\tau - \int_0^1 k(v,\tau)x(q(\tau))d\tau\right] \\ &\quad +y(u)\int_0^1 k(u,\tau)(x(q(\tau)) - y(q(\tau))d\tau \\ &\quad -y(v)\int_0^1 k(v,\tau)(x(q(\tau)) - y(q(\tau))d\tau. \end{aligned} \tag{2.11}$$

(2.11) yields the following inequality:

$$\begin{aligned}|(Nx)(u) - (Ny)(u)) - ((Nx)(v) - (Ny)(v)| &\le |(Gx)(u) - (Gy)(u) - ((Gx)(v) - (Gy)(v))| \\ &\quad +|x(u) - y(u) - (x(v) - y(v))|\int_0^1 |k(u,\tau)||x(q(\tau))|d\tau \\ &\quad +|x(v) - y(v)|\int_0^1 |k(u,\tau) - k(v,\tau)||x(q(\tau))|d\tau \\ &\quad +|y(u) - y(v)|\int_0^1 |k(u,\tau)||(x(q(\tau)) - y(q(\tau))|d\tau \\ &\quad +|y(v)|\int_0^1 |k(u,\tau) - k(v,\tau)||(x(q(\tau)) - y(q(\tau))|d\tau. \end{aligned} \tag{2.12}$$

Hence, taking into account (2.12), we can write:

$$\begin{aligned}\frac{|(Nx)(u)-(Ny)(u)-((Nx)(v)-(Ny)(v))|}{|u-v|^{\alpha}} &\leq \frac{|(Gx)(u)-(Gy)(u)-((Gx)(v)-(Gy)(v))|}{|u-v|^{\alpha}}\\ &+\frac{|(x(u)-y(u))-(x(v)-y(v))|}{|u-v|^{\alpha}}\|x\|_{\infty}K\\ &+\|u-v\|_{\infty}\|x\|_{\infty}\int_0^1\frac{|k(u,\tau)-k(v,\tau)|}{|u-v|^{\alpha}}d\tau+\frac{|y(u)-y(v)|}{|u-v|^{\alpha}}\|x-y\|_{\infty}K\\ &+\|y\|_{\infty}\|x-y\|_{\infty}\int_0^1\frac{|k(u,\tau)-k(v,\tau)|}{|u-v|^{\alpha}}d\tau \qquad (2.13)\end{aligned}$$

for all $u, v \in [0,1]$ with $u \neq v$. Therefore the equality

$$\begin{aligned}(Nx)(0)-(Ny)(0) &= (Gx)(0)-(Gy)(0)+x(0)\int_0^1 k(0,\tau)x(q(\tau))d\tau-y(0)\int_0^1 k(0,\tau)y(q(\tau))d\tau\\ &= (Gx)(0)-(Gy)(0)+(x(0)-y(0))\int_0^1 k(0,\tau)x(q(\tau))d\tau\\ &\quad +y(0)\int_0^1 k(0,\tau)[x(q(\tau))-y(q(\tau))]d\tau\end{aligned}$$

holds. So, we get the inequality

$$|(Nx)(0)-(Ny)(0)| \leq |(Gx)(0)-(Gy)(0)|+|x(0)-y(0)|K\|x\|_{\infty}+|y(0)|\|x-y\|_{\infty}K. \qquad (2.14)$$

Moreover, since $\|x\|_{\infty} \leq \|x\|_{\alpha} \leq r_0$, $\|y\|_{\infty} \leq \|y\|_{\alpha} \leq r_0$ and $\|x-y\|_{\infty} \leq \|x-y\|_{\alpha}$, from (2.13) and (2.14), we have that

$$\begin{aligned}\|Nx-Ny\|_{\alpha} &= |(Nx-Ny)(0)|+H^{\alpha}_{Nx-Ny}\\ &= |(Nx)(0)-(Ny)(0)|\\ &\quad +\sup\left\{\frac{|(Nx)(u)-(Ny)(u)-((Nx)(v)-(Ny)(v))|}{|u-v|^{\alpha}} : u,v\in[0,1] \text{ and } u\neq v\right\}\\ &\leq \|Gx-Gy\|_{\alpha}+\|x-y\|_{\alpha}K\|x\|_{\infty}+\|y\|_{\alpha}\|x-y\|_{\infty}K\\ &\quad +\|x-y\|_{\infty}(\|x\|_{\infty}+\|y\|_{\infty})\\ &\leq \|Gx-Gy\|_{\alpha}+\|x-y\|_{\alpha}(\|x\|_{\alpha}+\|y\|_{\alpha})(K+1)\\ &\leq \|Gx-Gy\|_{\alpha}+\|x-y\|_{\alpha}2r_0(K+1). \qquad (2.15)\end{aligned}$$

Since the operator $G : H_{\beta}[0,1] \to H_{\beta}[0,1]$ is continuous on $H_{\beta}[0,1]$ with respect to the norm $\|\cdot\|_{\alpha}$, it is also continuous at the point $y \in B^{\beta}_{r_0}$. Let us take an arbitrary $\varepsilon > 0$. There exists $\delta > 0$ such that the inequality:

$$\|Gx-Gy\|_{\alpha} < \frac{\varepsilon}{2}$$

where $\|x-y\|_{\alpha} < \delta$ and

$$0 < \delta < \frac{\varepsilon}{4r_0(K+1)}.$$

Then, taking into account (2.15), we derive the following inequality:

$$\|Nx-Ny\|_{\alpha} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

As a results, we infer that the operator N is continuous at the point $y \in B^{\beta}_{r_0}$. Because y was chosen arbitrarily, we deduce that N is continuous on $B^{\beta}_{r_0}$ with respect to the norm $\|\cdot\|_{\alpha}$. As $B^{\beta}_{r_0}$ is compact in $H_{\alpha}[0,1]$, from the classical Schauder fixed point theorem, we get the desired result. □

3. An Example

Example 3.1. Let us consider the following nonlinear quadratic integral equation:

$$x(u) = \frac{1}{3}\left(x^2(u) + x(u) + \frac{3}{2^{10}}\right) + x(u)\int_0^1 \sqrt[7]{mu^4 + \tau x}\left(\sqrt{\frac{1}{\tau+1}}\right) d\tau, \tag{3.1}$$

where $u \in I = [0,1]$ and m is the real number.

Set $(Gx)(u) = \frac{1}{3}\left(x^2(u) + x(u) + \frac{3}{2^{10}}\right)$, $k(u,\tau) = \sqrt[7]{mu^4+\tau}$ and $q(\tau) = \sqrt{\frac{1}{\tau+1}}$. We will show that the operator G continuous according to be norm with $\|.\|_\alpha$. To do this, fix arbitrarily $\varepsilon > 0$ and $y \in H_\beta[0,1]$. Assume that $x \in H_\beta[0,1]$ is an arbitrary function and $\|x-y\|_\alpha < \delta$, where δ is a positive number such that

$$0 < \delta \le \frac{1}{12}\left(-(6\|y\|_\alpha + 2) + \sqrt{(6\|y\|_\alpha + 2)^2 + 36\varepsilon}\right).$$

Then, for arbitrary $u, v \in [0,1]$ we obtain

$$\begin{aligned}
3(Gx-Gy)(u) - 3(Gx-Gy)(v) &= x^2(u) + x(u) + \frac{3}{2^{10}} - y^2(u) - y(u) - \frac{3}{2^{10}} \\
&\quad - \left(x^2(v) + x(v) + \frac{3}{2^{10}} - y^2(v) - y(v) - \frac{3}{2^{10}}\right) \\
&= x^2(u) - y^2(u) - (x^2(v) - y^2(v)) + x(u) - y(u) - (x(v) - y(v)) \\
&= (x(u)-y(u))(x(u)+y(u)) - (x(u)-y(v))(x(v)+y(v)) \\
&\quad + (x(u) - y(u) - (x(v) - y(v))) \\
&= [x(u) - y(v) - (x(v) - y(v))](x(u)+y(u)) \\
&\quad + (x(v)-y(v))(x(u)+y(u)-x(v)-y(v)) \\
&\quad + x(u) - y(u) - (x(v) - y(v)) \\
&= [x(u) - y(u) - (x(v) - y(v))](x(u)+y(u)+1) \\
&\quad + (x(v)-y(v))(x(u)+y(u)-x(v)-y(v)).
\end{aligned} \tag{3.2}$$

By (3.2), we get

$$\begin{aligned}
3|(Gx-Gy)(u) - (Gx-Gy)(v)| &\le (\|x+y\|_\infty + 1)|x(u) - y(u) - (x(v) - y(v))| \\
&\quad + \|x-y\|_\infty |x(u)+y(u)-x(v)-y(v)| \\
&\le (\|x+y\|_\alpha + 1)|x(u) - y(u) - (x(v) - y(v))| \\
&\quad + \|x-y\|_\alpha |x(u)+y(u)-x(v)-y(v)|.
\end{aligned} \tag{3.3}$$

By (3.3), we have:

$$3\sup\left\{\frac{|(Gx-Gy)(u) - (Gx-Gy)(v)|}{|u-v|^\alpha} : u,v \in [0,1] \text{ and } u \ne v\right\}$$

$$\begin{aligned}
&\le (\|x+y\|_\alpha + 1)\sup\left\{\frac{|x(u) - y(u) - (x(v) - y(v))|}{|u-v|^\alpha} : u,v \in [0,1] \text{ and } u \ne v\right\} \\
&\quad + \|x-y\|_\alpha \sup\left\{\frac{|x(u) + y(u) - (x(v) + y(v))|}{|u-v|^\alpha} : u,v \in [0,1] \text{ and } u \ne v\right\} \\
&\le (\|x+y\|_\alpha + 1)\|x-y\|_\alpha + \|x-y\|_\alpha\|x+y\|_\alpha \\
&\le \|x-y\|_\alpha(2\|x+y\|_\alpha + 1).
\end{aligned} \tag{3.4}$$

From (3.4), we obtain the following inequality:

$$
\begin{aligned}
3\|Gx-Gy\|_\alpha &= 3|(Gx-Gy)(0)|+3\sup\left\{\frac{|(Gx-Gy)(u)-(Gx-Gy)(v)|}{|u-v|^\alpha}:u,v\in[0,1]\text{ and }u\neq v\right\}\\
&\le |x^2(0)-y^2(0)|+|x(0)-y(0)|+\|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\le |x(0)-y(0)||x(0)+y(0)|+|x(0)-y(0)|+\|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\le \|x-y\|_\infty(\|x+y\|_\infty+1)+\|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\le \|x-y\|_\alpha(\|x+y\|_\alpha+1)+\|x-y\|_\alpha(2\|x+y\|_\alpha+1)\\
&\le \|x-y\|_\alpha(3\|x+y\|_\alpha+2)\\
&\le \|x-y\|_\alpha(3\|x-y\|_\alpha+6\|y\|_\alpha+2)\\
&< 3\varepsilon
\end{aligned}
$$

which yields that the operator G is continuous on $H_\beta[0,1]$ with respect to the norm $\|.\|_\alpha$. Also,

$$
\begin{aligned}
3|(Gx)(0)| &= |x^2(0)+x(0)|+\frac{3}{2^{10}}\\
&\le |x^2(0)|+|x(0)|+\frac{3}{2^{10}}\\
&\le \|x\|_\infty^2+\|x\|_\infty+\frac{3}{2^{10}}\\
&\le \|x\|_\beta^2+\|x\|_\beta+\frac{3}{2^{10}}
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
3\sup\left\{\frac{|(Gx)(u)-(Gx)(v)|}{|u-v|^\beta}\right\} &= \frac{\left|x^2(u)+x(u)+\frac{3}{2^{10}}-x^2(v)-x(v)-\frac{3}{2^{10}}\right|}{|u-v|^\beta}\\
&= \frac{|x^2(u)-x^2(v)+x(u)-x(v)|}{|u-v|^\beta}\\
&= \frac{|(x(u)-x(v))(x(u)+x(v))+x(u)-x(v)|}{|u-v|^\beta}\\
&= \frac{|(x(u)-x(v))||(x(u)+x(v)+1)|}{|u-v|^\beta}\\
&\le \sup\left\{\frac{|(x(u)-x(v))|}{|u-v|^\beta}\right\}(2\|x\|_\infty+1)\\
&\le \|x\|_\beta(2\|x\|_\beta+1)\\
&\le 2\|x\|_\beta^2+\|x\|_\beta.
\end{aligned}
\tag{3.6}
$$

From (3.5) and (3.6), we get

$$
\begin{aligned}
3\|Gx\|_\beta &\le \|x\|_\beta^2+\|x\|_\beta+\frac{3}{2^{10}}+2\|x\|_\beta^2+\|x\|_\beta\\
&= 3\|x\|_\beta^2+2\|x\|_\beta+\frac{3}{2^{10}}
\end{aligned}
$$

which implies

$$
\|Gx\|_\beta\le\|x\|_\beta^2+\frac{2}{3}\|x\|_\beta+\frac{1}{2^{10}}.
$$

Therefore, there exists the function $w:\mathbb{R}_+\to\mathbb{R}_+,\ w(x)=x^2+\frac{2}{3}x+\frac{1}{2^{10}}$ which is non-decreasing such that it holds the inequality

$$
\|Gx\|_\beta\le w\left(\|x\|_\beta\right)
$$

for any $x \in H_\beta[0,1]$. So, the assumption (i) of Theorem 2.1 holds.

Further, we have

$$\begin{aligned} |k(u,\tau)-k(v,\tau)| &= \left|\sqrt[7]{mu^4+\tau}-\sqrt[7]{mv^4+\tau}\right| \\ &\leq \left|\sqrt[7]{m(u^4-v^4)}\right| \\ &\leq \sqrt[7]{m}\sqrt[7]{|(u^4-v^4)|} \\ &\leq \sqrt[7]{4m}|u-v|^{\frac{1}{7}} \end{aligned}$$

for all $u, v, \tau \in [0,1]$. The assumption (ii) of Theorem 2.1 holds with the constant $k_\beta = k_{\frac{1}{7}} = \sqrt[7]{4m}$.

The function $q : [0,1] \to [0,1], q(\tau) = \sqrt{\frac{1}{\tau+1}}$ decreasing function is measurable and this satisfies assumption (iii).

Further, we can calculate that

$$\begin{aligned} \sup\left\{\int_0^1 |k(u,\tau)|d\tau : u \in [0,1]\right\} &= \sup\left\{\int_0^1 \left|\sqrt[7]{mu^4+\tau}\right| d\tau : u \in [0,1]\right\} \\ &= \sup\left\{\frac{7}{8}\left(\sqrt[7]{(mu^4+1)^8}-\sqrt[7]{(mu^4)^8}\right) : u \in [0,1]\right\} \\ &\leq \sup\left\{\frac{7}{8}\sqrt[7]{(mu^4+1)^8} : u \in [0,1]\right\} \\ &= \frac{7}{8}\sqrt[7]{(m+1)^8} \\ &\leq \sqrt[7]{(m+1)^8} \\ &= K. \end{aligned}$$

In this case, the inequality appearing in assumption (vi) of Theorem 2.1 takes the following form:

$$w(r) + (2K + k_\beta)r^2 \leq r$$

which is equivalent to

$$r^2 + \frac{2}{3}r + \frac{1}{2^{10}} + \left(2\sqrt[7]{(m+1)^8} + \sqrt[7]{4m}\right)r^2 \leq r. \tag{3.7}$$

There exists a positive number r_0 satisfying (3.7) provided that the constant m is chosen as suitable. For example, if one chose $m = \frac{1}{10^{49}}$, then the inequality

$$r^2 + \frac{2}{3}r + \frac{1}{2^{10}} + \left(2\sqrt[7]{\left(\frac{1}{10^{49}}+1\right)^8} + \sqrt[7]{\frac{4}{10^{49}}}\right)r^2 \leq r$$

holds for $r = r_0 = 0.10 \in [0.0030113,\ 0.1081]$. Therefore, using Theorem 2.1, we infer that there is at least one solution x of the equation (3.1) in the space $H_\alpha[0,1]$ with $0 < \alpha < \frac{1}{7}$.

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Absolute Lucas Spaces with Matrix and Compact Operators

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Abstract
The main purpose of this study is to introduce the absolute Lucas series spaces and to investigate their some algebraic and topological structure such as some inclusion relations, $BK-$ to this space, duals and Schauder basis. Also, the characterizations of matrix operators related to these space with their norms are given. Finally, by using Hausdorff measure of noncompactness, the necessary and sufficient conditions for a matrix operator on them to be compact are obtained.

Keywords: *Absolute summability; Lucas numbers; matrix transformations; sequence spaces; bounded operators; operator norm; Hausdorff measures of noncompactness.*

AMS Subject Classification (2020): *Primary: 40C05; 11B39; 40D25; 40F05; 46A45.*

1. Introduction

Let ω be the set of all sequences of complex numbers. A vector subspace of ω is called *a sequence space*. The spaces l_∞, c, c_0, Ψ, bs, cs, l and l_p $(p > 1)$ stand for the classes of all bounded, convergent, null and finite sequences and the classes of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively.

Let X and Y be two sequence spaces and $A = (a_{nv})$ be an arbitrary infinite matrix with complex components for all $n, v \in \mathbb{N} = \{0, 1, 2, ...\}$. If the series

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

converges for all $n \in \mathbb{N}$, then, by $A(x) = (A_n(x))$, we indicate the A-transform of the sequence $x = (x_v)$. Also, if $Ax = (A_n(x)) \in Y$ for every $x \in X$, then, A is called a matrix transformation from the sequence space X into the sequence space Y, and the class of all infinite matrices from X into Y is denoted by (X, Y).

A summability method is denoted by the matrix A if the transform sequence $A(x)$ converges to a real number.

The multiplier space of X and Y is identified by

$$S(X, Y) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in Y \text{ for all } x \in X\}.$$

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According to this notation, duals of the space X are described as

$$X^{\alpha} = S(X, l), \;\; X^{\beta} = S(X, cs), \;\; X^{\gamma} = S(X, bs).$$

If $a_{nn} \neq 0$ for all n and $a_{nv} = 0$ for $n < v$, then it is said that A is a triangle.

The concept of the domain of an infinite matrix A in the sequence space X is described as

$$X_A = \{x = (x_n) \in \omega : A(x) \in X\}$$

which is a new sequence space. In this connection, by means of the concept of the matrix domain, different new sequence spaces have been presented and their topological, algebraic structure and matrix transformations have been studied in literature. For example, one can see some of these spaces in references ([1, 2], [10–12], [23]).

A sequence space X is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ for all $n \in \mathbb{N}$. Further, an FK-space X whose metric is given by a norm is said to be a BK-space. The theory of FK- and BK-spaces has an important role in summability theory. For example, the operators between BK-spaces are continuous and the matrix domain of a triangle A in the BK-space X is also a BK-space and its norm is given by

$$\|x\|_{X_A} = \|A(x)\|_X \,,$$

[4]. Let X be a normed sequence space and (b_k) be a sequence in X. If there exists a unique sequence of coefficients (x_k) such that, for each $x \in X$,

$$\left\| x - \sum_{k=0}^{n} x_k b_k \right\| \to 0, \;\; n \to \infty$$

then, the sequence (b_k) is called the Schauder basis (or briefly basis) for X, and in this case it is written that $x = \sum\limits_{k=0}^{\infty} x_k b_k$. It is said that an FK-space X, consisting all finite sequences, has AK property if every sequence $x = (x_k) \in X$ has a unique representation $x = \sum\limits_{j=0}^{\infty} x_j e^{(j)}$, where $e^{(j)}$ is the sequence whose only non-zero term is 1 in the jth place for each $j \in \mathbb{N}$. This means that the sequence $(e^{(j)})$ is a Schauder basis for any $FK-$ space with AK. For example, $(e^{(j)})$ is the Schauder basis of the space l_p, but the space l_∞ doesn't have the Schauder basis [20].

For arbitrary two Banach spaces X and Y, $\mathcal{B}(X, Y)$ denotes the set of all continuous linear operators from the space X into the space Y, and the operator norm of $A \in \mathcal{B}(X, Y)$ is stated by

$$\|A\| = \sup_{x \neq 0} \frac{\|A(x)\|_Y}{\|x\|_X}.$$

In the special case $Y = \mathbb{C}$, it is written that $X^* = \mathcal{B}(X, \mathbb{C})$, the set of all continuous linear functional on X.

If $a \in \omega$ and $X \supset \Psi$ is a BK-space, then

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|$$

provided the right hand side of the equation exists, where S_X is the unit sphere in X, and it is finite for $a \in X^{\beta}$.

Throughout the whole paper, we suppose that $\phi = (\phi_n)$ is a sequence of positive numbers and p^* is conjugate of p, that is, $1/p + 1/p^* = 1$, $p > 1$, and $1/p^* = 0$ for $p = 1$.

Let take $\sum x_v$ as an infinite series with nth partial sum s_n. Then, the series $\sum x_v$ is said to be summable $|A, \phi_n|_p$, if (see[29])

$$\sum_{n=0}^{\infty} \phi_n^{p-1} |A_n(s) - A_{n-1}(s))|^p < \infty, \;\; A_{-1}(s) = 0.$$

This method includes some well known methods. For instance, if A is the matrix of weighted mean $\left(\bar{N}, p_n\right)$ (resp. $\phi_n = P_n/p_n$), then it is reduced to the summability $\left|\bar{N}, p_n, \phi_n\right|_p$ [31] (the summability $\left|\bar{N}, p_n\right|_p$ [3]). Also if we take A as the matrix of Cesàro mean of order $\alpha > -1$ and $\phi_n = n$, then we get the summability $|C, \alpha|_p$ in Flett's notation [5]. The choice of the Fibonacci matrix instead of A leads to the absolute Fibonacci summability method [7]. In addition to the aforementioned spaces, several absolute series spaces have also taken place in the literature (see [6, 8, 19, 25, 27–29]).

The Lucas sequence (L_n) is one of the most interesting number sequences in mathematics and is named after the mathematician François Edouard Anatole Lucas (1842-1891). It is given by the Fibonacci recurrence relation with different initial condition such that

$$L_0 = 2,\ L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2,$$

also, the terms of the Lucas sequence have the following important properties

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3,\ \sum_{k=1}^{n} L_{2k-1} = L_{2n} - 2,$$
$$\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1,\ \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2,$$
$$L_{n-1}^2 + L_n L_{n-1} - L_n^2 = 5(-1)^{n+1}, n \geq 1,$$
$$L_{n-1} L_{n+1} - L_n^2 = 5(-1)^{n+1}, n \geq 1.$$

We refer reader to [13] for other properties. Additionally, just like the Fibonacci numbers, the rates of successive Lucas numbers converges to the golden ratio which is one of the most interesting irrationals playing an important role in number theory, algorithms, network theory, etc. Using Lucas numbers, the Lucas matrix $\hat{E}(r,s) = (\hat{e}_{nk}(r,s))$ has recently been defined [12] as

$$\hat{e}_{nk}(r,s) = \begin{cases} s\frac{L_n}{L_{n-1}}, & k = n-1 \\ r\frac{L_{n-1}}{L_n}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

where L_n be the nth Lucas number for every $n \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$.

The aim of this paper is to define the absolute sumability space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and investigate its some inclusion relations, $\alpha-, \beta-, \gamma-$ duals and basis. Also, some matrix and compact operators on this space are characterized and their operator norms and Hausdorff measures of noncompactness are determined.

It is required the following lemmas in proving theorems.

Lemma 1.1. *[18] Let T be a triangle, X and Y be two arbitrary subsets of ω. Then, we have*

(a) $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.

(b) Further, if X and Y are BK-spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

Lemma 1.2. *[30] Let $1 < p < \infty$. Then,*

1. *$A \in (l, c) \Leftrightarrow$ (i) $\lim_n a_{nv}$ exists for $v \geq 0$, (ii) $\sup_{n,v} |a_{nv}| < \infty$,*

2. *$A \in (l, l_\infty) \Leftrightarrow$ (ii) holds,*

3. *$A \in (l, c_0) \Leftrightarrow$ (iii) $\lim_n a_{nv} = 0$ for all $v \geq 0$ and (ii) hold,*

4. *$A \in (l_p, c) \Leftrightarrow (i)$ holds, $(iv) \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{p^*} < \infty$,*

5. *$A \in (l_p, l_\infty) \Leftrightarrow (iv)$ holds,*

6. $A \in (l_p, c_0) \Leftrightarrow$ *(iii)* and *(iv)* hold.

Lemma 1.3. *[14] Let $1 \leq p < \infty$. Then, $A \in (l, l_p)$ if and only if*

$$\|A\|_{(l,l_p)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^p \right\}^{\frac{1}{p}}.$$

Lemma 1.4. *[30] Let $1 < p < \infty$. Then, $A \in (l_p, l)$ if and only if*

$$\|A\|_{(l_p,l)} = \sup_{N \in \mathfrak{F}} \left\{ \sum_{v=0}^{\infty} \left| \sum_{n=0}^{\infty} a_{nv} \right|^{p^*} \right\}^{1/p^*}$$

where $\mathfrak{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

Lemma 1.5. *[27] Let $1 < p < \infty$. Then, $A \in (l_p, l)$ if and only if*

$$\|A\|'_{(l_p,l)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{p^*} \right\}^{1/p^*} < \infty.$$

Moreover since $\|A\|_{(l_p,l)} \leq \|A\|'_{(l_p,l)} \leq 4 \|A\|_{(l_p,l)}$, there exists $1 \leq \xi \leq 4$ such that $\|A\|'_{(l_p,l)} = \xi \|A\|_{(l_p,l)}$.

Lemma 1.6. *[18] Let $1 < p < \infty$ and p^* denote the conjugate of p. Then, $l_p^\beta = l_{p^*}$ and $l_\infty^\beta = c^\beta = c_0^\beta = l$, $l^\beta = l_\infty$. Also, let X denote any of the spaces l_∞, c, c_0, l and l_p. Then, we have*

$$\|a\|_X^* = \|a\|_{X^\beta}$$

for all $a \in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the X^β.

Lemma 1.7. *[15] Let X and Y be BK-spaces. Then, we have*

(a) $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = A(x)$ for all $x \in X$.

(b) If X has AK, then $\mathcal{B}(X, Y) \subset (X, Y)$, that is, for every operator $L \in \mathcal{B}(X, Y)$ there exists a matrix $A \in (X, Y)$ such that by $L(x) = A(x)$ for all $x \in X$.

Lemma 1.8. *[4] Let $X \supset \Psi$ be a BK-space and Y be any of the spaces ℓ_∞, c, c_0. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

2. Hausdorff Measure of Noncompactness

If S and R are subsets of a metric space (X, d) and, for every $r \in R$, there exists an $s \in S$ such that $d(r, s) < \varepsilon$ then, S is called an ε-net of R; if S is finite, then the ε-net S of R is called a finite ε-net of R. Let X, Y be two Banach spaces. It is said that a linear operator $L : X \to Y$ is compact if its domain is all of X and the sequence $(L(x_n))$ has a convergent subsequence in Y, for every bounded sequence (x_n) in X. The class of such operators is denoted by $\mathcal{C}(X, Y)$. If Q is any bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is given by

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon-\text{net in } X\},$$

and χ is named the Hausdorff measure of noncompactness. Using the Hausdorff measure of noncompactness, some compact operators on various sequence spaces are characterized by many authors. For example, Mursaleen and Noman in [21, 22], Malkowsky and Rakocevic in [17] have used the Hausdorff measure of noncompactness method to characterize the class of compact operators on some known spaces, (see also [7, 8, 15, 26]).

The following lemma is very important to calculate the Hausdorff measure of noncompactness of any bounded subset of the space l_p.

Lemma 2.1. *([24]) Let Q be a bounded subset of the normed space X where $X = l_p$ for $1 \leq p < \infty$ or $X = c_0$. If $P_r : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ...x_r, 0, 0, ...)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - P_r)(x)\| \right).$$

Let X and Y be two Banach spaces, χ_1 and χ_2 be Hausdorff measures on X and Y, the linear operator $L : X \to Y$ is said to be (χ_1, χ_2)- bounded if $L(Q)$ is a bounded subset of Y and there exists a constant $M > 0$ such that $\chi_2(L(Q)) \leq M\chi_1(Q)$ for every bounded subset Q of X. If an operator L is (χ_1, χ_2)- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = inf\left\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded set } Q \subset X\right\}$$

is called the (χ_1, χ_2)-measure noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$ then it is written that $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

There is a significant relation between compact operators and Hausdorff measure of noncompactness. The following lemma gives this relation.

Lemma 2.2. *[18] Let X and Y be two Banach spaces and $L \in \mathcal{B}(X, Y)$. Also, let the set $S_x = \{x \in X : \|x\| \leq 1\}$ be the unit sphere in X. Then,*

$$\|L\|_\chi = \chi(L(S_x))$$

and

$$L \in \mathcal{C}(X, Y) \Leftrightarrow \|L\|_\chi = 0.$$

Lemma 2.3. *[16] Let X be a normed sequence space, $T = (t_{nv})$ be an infinite triangle matrix, χ_T and χ define the Hausdorff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.*

Lemma 2.4. *[22] Let $X \supset \Psi$ be a BK-space with AK or $X = l_\infty$. If $A \in (X, c)$, then, we have*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ exists for all } k,$$

$$\alpha = (\alpha_k) \in X^\beta,$$

$$\sup_n \|A_n - \alpha\|_X^* < \infty,$$

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for every } x = (x_k) \in X.$$

Lemma 2.5. *[22] Let $X \supset \Psi$ be a BK-space. Then,*

(a) If $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \lim_{r\to\infty} \left(\sup_{n>r} \|A_n\|^* \right).$$

(b) If X has AK or $X = l_\infty$ and $A \in (X, c)$, then

$$\frac{1}{2} \lim_{r\to\infty} \left(\sup_{n\geq r} \|A_n - a\|^* \right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty} \left(\sup_{n\geq r} \|A_n - a\|^* \right)$$

where $a = (a_k)$ defined by $a_k = \lim_{n\to\infty} a_{nk}$, for all $n \in \mathbb{N}$.
(c) If $A \in (X, l_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \lim_{r\to\infty} \left(\sup_{n>r} \|A_n\|^* \right).$$

3. Absolute Lucas summability spaces

In this section, firstly, the summability space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ as the set of all series summable by absolute Lucas method is introduced, and it is proved that this space is a BK-space which is linearly isomorphic to l_p for $1 \le p < \infty$. Also, giving some inclusion relations, $\alpha-$, $\beta-$ and $\gamma-$ duals and Schauder basis of this space are investigated.

If the Lucas matrix is taken instead of A, then $|A, \phi_n|_p$ summability is reduced to the absolute Lucas summability. Then, since (s_n) is a sequence of partial sum of the series $\sum x_k$, it follows that

$$\begin{aligned}
\hat{E}_n(r,s)(s) = \sum_{k=1}^{n} \hat{e}_{nk}(r,s)s_k &= \sum_{k=1}^{n} x_k \sum_{v=k}^{n} \hat{e}_{nv}(r,s) \\
&= x_n \hat{e}_{nn}(r,s) + \sum_{k=1}^{n-1} (\hat{e}_{nn}(r,s) + \hat{e}_{n,n-1}(r,s))x_k \\
&= x_n r\tfrac{L_{n-1}}{L_n} + \sum_{k=1}^{n-1} \left(s\tfrac{L_n}{L_{n-1}} + r\tfrac{L_{n-1}}{L_n} \right) x_k \\
&= \sum_{k=1}^{n} l_{nk} x_k
\end{aligned}$$

where the matrix $\mathcal{L}(r,s) = (l_{nk})$ is given by

$$l_{nk} = \begin{cases} r\frac{L_{n-1}}{L_n}, & k = n \\ s\frac{L_n}{L_{n-1}} + r\frac{L_{n-1}}{L_n}, & 1 \le k \le n-1 \\ 0, & k > n. \end{cases} \tag{3.1}$$

So, we get

$$\begin{aligned}
\hat{E}_n(s) - \hat{E}_{n-1}(s) &= x_n r\tfrac{L_{n-1}}{L_n} + x_{n-1}\left(s\tfrac{L_n}{L_{n-1}} + r\tfrac{5(-1)^{n+1}}{L_n L_{n-1}} \right) + \sum_{k=1}^{n-2} \tfrac{5(-1)^n}{L_{n-1}} \left(\tfrac{s}{L_{n-2}} - \tfrac{r}{L_n} \right) x_k \\
&= \sum_{k=1}^{n} \xi_{nk} x_k
\end{aligned}$$

where

$$\xi_{nk} = \begin{cases} r\frac{L_{n-1}}{L_n}, & k = n \\ s\frac{L_n}{L_{n-1}} + r\frac{5(-1)^{n+1}}{L_n L_{n-1}}, & k = n-1 \\ \frac{5(-1)^n}{L_{n-1}} \left(\frac{s}{L_{n-2}} - \frac{r}{L_n} \right), & 1 \le k \le n-2 \\ 0, & k > n. \end{cases} \tag{3.2}$$

Hence, the space $|\mathcal{L}(r,s)|_p$ can be stated by

$$|\mathcal{L}(r,s)|_p = \left\{ x \in \omega : \left(\phi_n^{1/p^*} \sum_{k=1}^{n} \xi_{nk} x_k \right) \in l_p \right\}.$$

On the other hand, according to the matrix domain, this space is redefined by

$$\left|\mathcal{L}^{\phi}(r,s)\right|_p = (l_p)_{E^{(p)} \circ \mathcal{L}(r,s)}$$

where

$$e_{nk}^{(p)} = \begin{cases} \phi_n^{1/p^*}, & k = n \\ -\phi_n^{1/p^*}, & k = n-1 \\ 0, & k \ne n, n-1. \end{cases} \tag{3.3}$$

Also, we note

$$(E^{(p)} \circ \mathcal{L}(r,s))_n(x) = \phi_n^{1/p^*} (\mathcal{L}(r,s)_n(x) - \mathcal{L}_{n-1}(r,s)(x)).$$

Moreover, since every triangle matrix has a unique triangle inverse [32], the matrices $\mathcal{L}(r,s)$ and $E^{(p)}$ have unique inverses $\tilde{\mathcal{L}}(r,s)=(\tilde{l}_{nk})$ and $\tilde{E}^{(p)}=(\tilde{e}_{nk})$ whose terms are given by

$$\tilde{l}_{nk}=\begin{cases}\frac{1}{r}\frac{L_n}{L_{n-1}}, & k=n\\ \frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_kL_{k-1}}\left(\frac{s}{r}L_n^2+L_{n-1}^2\right), & 1\leq k\leq n-1\\ 0, & k>n\end{cases}\tag{3.4}$$

$$\tilde{e}^{(p)}_{nv}=\begin{cases}\phi_v^{-1/p^*}, & 1\leq v\leq n\\ 0, & v>n\end{cases}\tag{3.5}$$

respectively.

First, to understand the space better, we exibit some relations between the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and l_p.

Theorem 3.1. *Let $\phi\in l_\infty$ and $1\leq p<\infty$. Then, $l_p\subset\left|\mathcal{L}^{\phi}(r,s)\right|_p$.*

Proof. For $p=1$, it is clear, it is omitted. Let $p>1$. By the properties of Lucas numbers, the series $\sum\limits_n\frac{1}{L_n}$ is convergent and also $(\frac{1}{L_n})$ is a decreasing sequence. So, it follows from Abel's Theorem that $\frac{n}{L_n}\to 0$ as $n\to\infty$. This gives $\sum\limits_{k=0}^{n}|\xi_{nk}|=O(1)$ and $\sum\limits_{n=k}^{\infty}|\xi_{nk}|=O(1)$. Hence, by Hölder's inequality, it is obtained that

$$\begin{aligned}\|x\|_{|\mathcal{L}^{\phi}(r,s)|_p} &= \left\{\sum_{n=1}^{\infty}\left|\phi_n^{1/p^*}\sum_{k=1}^{n}\xi_{nk}x_k\right|^p\right\}^{1/p}\\ &\leq \left\{\sum_{n=1}^{\infty}\phi_n^{p-1}\sum_{k=1}^{n}|\xi_{nk}|\,|x_k|^p\left(\sum_{k=1}^{n}|\xi_{nk}|\right)^{p/p^*}\right\}^{1/p}\\ &= O(1)\left\{\sum_{k=1}^{\infty}|x_k|^p\sum_{n=k}^{\infty}|\xi_{nk}|\right\}^{1/p}\\ &= O(1)\left\{\sum_{k=1}^{\infty}|x_k|^p\right\}^{1/p}=O(1)\,\|x\|_{l_p},\end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $1\leq p\leq q<\infty$. If there is a constant $M>0$ such that $\phi_n\leq M$ for all $n\in\mathbb{N}$, then $\left|\mathcal{L}^{\phi}(r,s)\right|_p\subset\left|\mathcal{L}^{\phi}(r,s)\right|_q$.*

Proof. To prove the inclusion, take $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$. Since $l_p\subset l_q$ for $1\leq p\leq q<\infty$, it is clear that $\left(\phi_n^{1/p^*}\sum\limits_{j=0}^{n}\xi_{nj}x_j\right)\in l_q$. Also, by considering $\phi_n\leq M$ for all $n\in\mathbb{N}$, it can be written that

$$M^{\frac{q}{p^*}-\frac{q}{q^*}}\left|\phi_n^{\frac{1}{q^*}}\sum_{j=1}^{n}\xi_{nj}x_j\right|^q\leq\left|\phi_n^{1/p^*}\sum_{j=1}^{n}\xi_{nj}x_j\right|^q$$

which implies that $x\in|\mathcal{L}(r,s)|_q$. □

The following result is useful to determine a Schauder basis for the matrix domain of a special triangular matrix in a linear metric space.

Lemma 3.1. *([9]). Let T be a triangular matrix and S be its inverse. If (b_k) is a Schauder basis of the metric space (X,d), then $(S(b_k))$ is a basis of X_T with respect to the metric d_T given by $d_T(z_1,z_2)=d(Tz_1,Tz_2)$ for all $z_1,z_2\in X_T$.*

Theorem 3.3. *Let $1\leq p<\infty$. Then, the set $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ is a linear space with coordinate-wise addition and scalar multiplication. Also, it is a BK-space with respect to the norm*

$$\|x\|_{|\mathcal{L}^{\phi}(r,s)|_p}=\left\|E^{(p)}\circ\mathcal{L}(r,s)(x)\right\|_{l_p}.$$

Moreover, the sequence $b^{(j)}=(b_n^{(j)})$ *defined by*

$$b_n^{(j)}=\begin{cases}\phi_j^{-1/p^*}\left(\frac{1}{r}\frac{L_n}{L_{n-1}}+\sum\limits_{k=j}^{n-1}\frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_kL_{k-1}}\left(\frac{s}{r}L_n^2+L_{n-1}^2\right)\right), & 1\leq j\leq n-1\\ \phi_n^{-1/p^*}\frac{1}{r}\frac{L_n}{L_{n-1}}, & j=n\\ 0, & j>n,\end{cases}$$

is a Schauder basis for the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$.

Proof. Since the space l_p is a BK-space for $1\leq p<\infty$ and $E^{(p)}\circ\mathcal{L}^{\phi}(r,s)$ is a triangle matrix, it follows from Theorem 4.3.2 of [32], $\left|\mathcal{L}^{\phi}(r,s)\right|_p=(l_p)_{E^{(p)}\circ\mathcal{L}(r,s)}$ is a BK-space. On the other hand, it is known that the sequence $(e^{(j)})$ is the Schauder basis of the space l_p. So, it can be obtained by Lemma 3.1 that $b^{(j)}=((\tilde{L}(r,s)\circ\tilde{E}^{(p)})_n(e^{(j)}))$ is a Schauder basis of the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$. □

Theorem 3.4. *Let* $1\leq p<\infty$. *Then, there exists a linear isomorphism between the spaces* $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ *and* l_p *i.e.,* $\left|\mathcal{L}^{\phi}(r,s)\right|_p\cong l_p$.

Proof. To prove this, it should be shown that the existence of a linear bijection between the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and l_p where $1\leq p<\infty$. Let consider the maps $\mathcal{L}(r,s):\left|\mathcal{L}^{\phi}(r,s)\right|_p\to(l_p)_{E^{(p)}}$, $E^{(p)}:(l_p)_{E^{(p)}}\to l_p$ given by (3.1) and (3.3) . Since the matrices corresponding to these maps are triangles, these are linear bijections. So, the composite function $E^{(p)}\circ\mathcal{L}(r,s)$ has the same property. Further, one can see that the norm is preserved. This completes the proof. □

We use the following notations in the rest of the paper.

$$\eta_{nj}=\frac{1}{r}\frac{L_n}{L_{n-1}}+\sum_{k=j}^{n-1}\frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_kL_{k-1}}\left(\frac{s}{r}L_n^2+L_{n-1}^2\right),$$

$$D_1=\left\{\epsilon\in\omega:\sum_{n=j+1}^{\infty}\eta_{nj}\epsilon_n\text{ exists for all } j\right\},$$

$$D_2=\left\{\epsilon\in\omega:\sup_m\left\{\phi_m^{-1}\left|\frac{1}{r}\frac{L_m}{L_{m-1}}\epsilon_m\right|^{p^*}+\sum_{j=1}^{m-1}\phi_j^{-1}\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j+\sum_{n=j+1}^{m}\eta_{nj}\epsilon_n\right|^{p^*}\right\}<\infty\right\},$$

$$D_3=\left\{\epsilon\in\omega:\sup_{m,j}\left\{\left|\frac{1}{r}\frac{L_m}{L_{m-1}}\epsilon_m\right|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j+\sum_{n=j+1}^{m}\eta_{nj}\epsilon_n\right|\right\}<\infty\right\},$$

$$D_4=\left\{\epsilon\in\omega:\sum_{j=1}^{\infty}\frac{1}{\phi_j}\left\{\sum_{n=j+1}^{\infty}|\eta_{nj}\epsilon_n|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j\right|\right\}^{p^*}<\infty\right\},$$

$$D_5=\left\{\epsilon\in\omega:\sup_j\left\{\sum_{n=j+1}^{\infty}|\eta_{nj}\epsilon_n|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j\right|\right\}<\infty\right\}.$$

Theorem 3.5. *Let* $1<p<\infty$ *and* $\phi=(\phi_n)$ *be a sequence of positive numbers. Then,*
(i) $\{\left|\mathcal{L}^{\phi}(r,s)\right|\}^{\alpha}=D_5$, $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\alpha}=D_4$,
(ii) $\{\left|\mathcal{L}^{\phi}(r,s)\right|\}^{\beta}=D_1\cap D_3$, $\left\{\left|\mathcal{L}(r,s)\right|_p\right\}^{\beta}=D_1\cap D_2$,
(iii) $\{\left|\mathcal{L}^{\phi}(r,s)\right|\}^{\gamma}=D_3$, $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\gamma}=D_2$.

Proof. Since the proofs of the other parts are similar, we just calculate the β-dual of the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$. Recall that $\epsilon \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ iff $\epsilon x = (\epsilon_n x_n) \in cs$ for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$. Take $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$, $\mathcal{L}(r,s)(x) = y$ and $z = E^{(p)}(y)$. Then, $z \in l_p$. It follows from (3.4) and (3.5) that

$$
\begin{aligned}
\sum_{n=1}^{m} \epsilon_n x_n &= \epsilon_1 x_1 + \sum_{n=2}^{m} \epsilon_n \left(\frac{1}{r}\frac{L_n}{L_{n-1}} y_n + \sum_{k=1}^{n-1} \frac{(-1)^{n-k}}{r} \left(\frac{s}{r}\right)^{n-1-k} \frac{1}{L_k L_{k-1}} \left(\frac{s}{r} L_n^2 + L_{n-1}^2\right) y_k \right) \\
&= \sum_{j=1}^{m} \phi_j^{-1/p^*} \sum_{n=j}^{m} \epsilon_n \frac{1}{r}\frac{L_n}{L_{n-1}} z_n \\
&+ \sum_{j=1}^{m-1} \phi_j^{-1/p^*} \left(\sum_{n=j+1}^{m} \sum_{k=j}^{n-1} \epsilon_n \frac{(-1)^{n-k}}{r} \left(\frac{s}{r}\right)^{n-1-k} \frac{1}{L_k L_{k-1}} \left(\frac{s}{r} L_n^2 + L_{n-1}^2\right) \right) z_j \\
&= \phi_m^{-1/p^*} \epsilon_m \frac{1}{r}\frac{L_m}{L_{m-1}} z_m + \sum_{j=1}^{m-1} \phi_j^{-1/p^*} \left(\epsilon_j \frac{1}{r}\frac{L_j}{L_{j-1}} + \sum_{n=j+1}^{m} \epsilon_n \eta_{nj} \right) z_j \\
&= \sum_{j=1}^{m} h_{mj} z_j
\end{aligned}
$$

where the matrix $H = (h_{mj})$ is defined by

$$
h_{mj} = \begin{cases} \phi_j^{-1/p^*} \left(\epsilon_j \frac{1}{r}\frac{L_j}{L_{j-1}} + \sum_{n=j+1}^{m} \epsilon_n \eta_{nj} \right), & 1 \le j \le m-1 \\ \phi_m^{-1/p^*} \epsilon_m \frac{1}{r}\frac{L_m}{L_{m-1}}, & j = m \\ 0, & j > m. \end{cases}
$$

This means that $\epsilon \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ iff $H \in (l_p, c)$. Thus, by applying Lemma 1.2 to the matrix H, we obtain $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta} = D_1 \cap D_2$. This completes the proof. □

4. Matrix transformations on space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$

In this section, we characterize some classes of matrix operators on that space and compute their norms.

Lemma 4.1. *Let $1 < p < \infty$. If $a = (a_k) \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$, then, for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$, $\tilde{a}^{(p)} = (\tilde{a}_k^{(p)}) \in l_{p^*}$, $\tilde{a}^{(1)} \in l_\infty$ and*

$$\sum_k a_k x_k = \sum_k \tilde{a}_k^{(p)} z_k$$

holds, where $z = E^{(p)}(\mathcal{L}(r,s)(x)) \in l_p$ and

$$\tilde{a}_k^{(p)} = \phi_k^{-1/p^*} \left(a_k \frac{1}{r}\frac{L_k}{L_{k-1}} + \sum_{n=k+1}^{\infty} a_n \eta_{nk} \right).$$

Lemma 4.2. *Assume that $1 < p < \infty$. Then, we have $\|a\|^*_{|\mathcal{L}^{\phi}(r,s)|_p} = \left\|\tilde{a}^{(p)}\right\|_{l_{p^*}}$ for all $a \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ and $\|a\|^*_{|\mathcal{L}^{\phi}(r,s)|} = \left\|\tilde{a}^{(1)}\right\|_{l_\infty}$ for all $a \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|\right\}^{\beta}$ where $\tilde{a}^{(p)}$ as in Lemma 4.1.*

Proof. Let $a \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$. It can be immediately seen from Lemma 4.1, $\tilde{a}^{(p)} \in l_{p^*}$ and $\tilde{a}^{(1)} \in l_\infty$. So, using Lemma 1.6 and Lemma 4.1 , we get

$$\|a\|^*_{|\mathcal{L}^{\phi}(r,s)|_p} = \sup_{x \in S_{|\mathcal{L}^{\phi}(r,s)|_p}} \left| \sum_{v=0}^{\infty} a_v x_v \right| = \sup_{z \in S_{l_p}} \left| \sum_{v=0}^{\infty} \tilde{a}_v^{(p)} z_v \right| = \left\|\tilde{a}^{(p)}\right\|^*_{l_p} = \left\|\tilde{a}^{(p)}\right\|_{l_{p^*}}.$$

The proof for the case $k = 1$ is quite easy, so it is omitted. □

Theorem 4.1. *Let $1<p<\infty$, $A=(a_{nk})$ be an infinite matrix of complex numbers for each $n,k\in\mathbb{N}$ and define the matrix $B^{(n)}=\left(b_{mk}^{(n)}\right)$, $\bar{B}=(\bar{b}_{nk})$ and $\hat{B}=(\hat{b}_{nk})$ as follows:*

$$b_{mk}^{(n)}=\begin{cases}\phi_k^{-1/p^*}\left(a_{nk}\frac{1}{r}\frac{L_k}{L_{k-1}}+\sum\limits_{j=k+1}^{m}a_{nj}\eta_{jk}\right), & 0\le k\le m-1\\ \phi_m^{-1/p^*}a_{nm}\frac{1}{r}\frac{L_m}{L_{m-1}}, & k=m\\ 0, & j>m,\end{cases}$$

$$\bar{b}_{nk}=\lim_{m\to\infty}b_{mk}^{(n)},$$

$$\hat{B}=E^{(1)}\circ\mathcal{L}(r,s)\circ\bar{B}.$$

Then, $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$ if and only if

$$\sum_{j=k+1}^{\infty}\eta_{jk}a_{nj}\ \textit{ exists for all } k, \tag{4.1}$$

$$\sup_m\left\{\frac{1}{\phi_m}\left|\frac{1}{r}\frac{L_m}{L_{m-1}}a_{nm}\right|^{p^*}+\sum_{k=1}^{m-1}\frac{1}{\phi_k}\left|\frac{1}{r}\frac{L_k}{L_{k-1}}a_{nk}+\sum_{j=k+1}^{m}\eta_{jk}a_{nj}\right|^{p^*}\right\}, \tag{4.2}$$

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\left|\hat{b}_{nk}\right|\right)^{p^*}<\infty. \tag{4.3}$$

If $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$, then A defines a bounded linear operator L_A such that $L_A(x)=A(x)$ and

$$\|A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)}=\left\|\hat{B}\right\|_{(l_p,l)}.$$

Proof. $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$ if and only if $A(x)$ is well defined and belongs to the space $\left|\mathcal{L}^{\phi}(r,s)\right|$ for all $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$. By Theorem 3.5, $A(x)$ is well defined, or, $(a_{nk})_{k=0}^{\infty}\in\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ if and only if (4.1) and (4.2) hold.

Beside, for any matrix $R=(r_{nv})\in(l_p,c)$, the remaining term of the series tends to zero uniformly in n, that is

$$\left|\sum_{v=m}^{\infty}r_{nv}x_v\right|\le\left(\sum_{v=m}^{\infty}|r_{nv}|^{p^*}\right)^{\frac{1}{p^*}}\left(\sum_{v=m}^{\infty}|x_v|^{k}\right)^{\frac{1}{p}}\to 0,(m\to\infty)$$

which gives the series $R_n(x)=\sum\limits_{v=0}^{\infty}r_{nv}x_v$ converges uniformly in n. So we have

$$\lim_n R_n(x)=\sum_{v=0}^{\infty}\lim_n r_{nv}x_v. \tag{4.4}$$

It follows from (3.4), (3.5) and (4.4)

$$A_n(x)=\lim_m\sum_{k=0}^{m}a_{nk}x_k=\lim_m\sum_{r=0}^{m}b_{mr}^{(n)}z_r=\sum_{r=0}^{\infty}\bar{b}_{nr}z_r.$$

Taking into $\left|\mathcal{L}^{\phi}(r,s)\right|_p\cong l_p$ for $1\le p<\infty$, it follows that $A(x)\in\left|\mathcal{L}^{\phi}(r,s)\right|$ for all $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$ iff $\bar{B}\in\left(l_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$. In other words, since $\left|\mathcal{L}^{\phi}(r,s)\right|_p=(l_p)_{E^{(p)}\circ\mathcal{L}(r,s)}$, $A(x)\in\left|\mathcal{L}^{\phi}(r,s)\right|$ for all $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$ iff $\hat{B}\in(l_p,l)$. Also, a few calculations show that the matrix $\hat{B}$ is expressed as

$$b_{nk}^{*}=\sum_{v=0}^{n}l_{nv}(r,s)\bar{b}_{vk}=r\frac{L_{n-1}}{L_n}\bar{b}_{nk}+\sum_{v=0}^{n-1}\left(s\frac{L_n}{L_{n-1}}+r\frac{L_{n-1}}{L_n}\right)\bar{b}_{vk},$$

$$\hat{b}_{nr}=\phi_k^{1/p^*}\left(b^*_{nk}-b^*_{n-1,k}\right),\ \ n\geq 1 \text{ and } \hat{b}_{0k}=b^*_{0k}.$$

Now, if we apply Lemma 1.3 to the matrix $\hat{B}$, we get the condition (4.3). So, the first part of the proof is completed.

On the other hand, since the spaces $\left|\mathcal{L}^\phi(r,s)\right|_p$ and $\left|\mathcal{L}^\phi(r,s)\right|$ are BK-spaces, if $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|_p,\left|\mathcal{L}^\phi(r,s)\right|\right)$, then, by Theorem 4.2.8 of [32], L_A defines a bounded operator such that $L_A(x)=A(x)$. To calculate the operator norm of A, we consider the isomorphisms $\mathcal{L}(r,s):\left|\mathcal{L}^\phi(r,s)\right|_p\to(l_p)_{E^{(p)}}$, $E^{(p)}:(l_p)_{E^{(p)}}\to l_p$. Now, it is clear to see that $A=\tilde{\mathcal{L}}(r,s)\circ\tilde{E}^{(1)}\circ\hat{B}\circ E^{(p)}\circ\mathcal{L}(r,s)$ and so

$$\begin{aligned}\|A\|_{\left(\left|\mathcal{L}^\phi(r,s)\right|_p,\left|\mathcal{L}^\phi(r,s)\right|\right)} &= \sup_{x\neq 0}\frac{\|A(x)\|_{\left|\mathcal{L}^\phi(r,s)\right|}}{\|x\|_{\left|\mathcal{L}^\phi(r,s)\right|_p}}\\ &= \sup_{x\neq 0}\frac{\left\|\tilde{\mathcal{L}}(r,s)\circ\tilde{E}^{(1)}\circ\hat{B}\circ E^{(p)}\circ\mathcal{L}(r,s)(x)\right\|_{\left|\mathcal{L}^\phi(r,s)\right|}}{\|x\|_{\left|\mathcal{L}^\phi(r,s)\right|_p}}\\ &= \sup_{z\neq 0}\frac{\left\|\hat{B}(z)\right\|_l}{\|z\|_{l_p}}=\left\|\hat{B}\right\|_{(l_p,l)}\ (z=E^{(p)}\circ\mathcal{L}(r,s)(x))\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let $1\leq p<\infty$, $A=(a_{nk})$ be an infinite matrix with complex components for all $n,k\in\mathbb{N}$, $B^{(n)}=\left(b^{(n)}_{mk}\right)$ and $\bar{B}=(b_{nk})$ be as in Theorem 4.1 with $1/p^*=0$. Besides, define $\tilde{B}=E^{(p)}\circ\mathcal{L}\circ\bar{B}$. Then, $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|,\left|\mathcal{L}^\phi(r,s)\right|_p\right)$ if and only if*

$$\sum_{v=j+1}^{\infty}\eta_{vj}a_{nv}\ \text{ exists for all }\ j \tag{4.5}$$

$$\sup_{m,j}\left\{\left|\frac{1}{r}\frac{L_m}{L_{m-1}}a_{nm}\right|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}a_{nj}+\sum_{k=j+1}^{m}\eta_{kj}a_{nk}\right|\right\}<\infty, \tag{4.6}$$

$$\sup_j\sum_{n=1}^{\infty}\left|\tilde{b}_{nj}\right|^p<\infty. \tag{4.7}$$

Moreover, if $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|,\left|\mathcal{L}^\phi(r,s)\right|_p\right)$, then A denotes a bounded linear operator L_A such that $L_A(x)=A(x)$ and

$$\|L_A\|_{\left(\left|\mathcal{L}^\phi(r,s)\right|,\left|\mathcal{L}^\phi(r,s)\right|_p\right)}=\left\|\tilde{B}\right\|_{(l,l_p)}.$$

Proof. $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|,\left|\mathcal{L}^\phi(r,s)\right|_p\right)$ if and only if $A_n=(a_{nv})_{v=0}^\infty\in\left\{\left|\mathcal{L}^\phi(r,s)\right|\right\}^\beta$ and $A(x)\in\left|\mathcal{L}^\phi(r,s)\right|_p$ where $x\in\left|\mathcal{L}^\phi(r,s)\right|$. By Theorem 3.5, it is clear that $A_n\in\left\{\left|\mathcal{L}^\phi(r,s)\right|\right\}^\beta$ iff (4.5) and (4.6) hold. Also, if any matrix $R=(r_{nv})\in(l,c)$, then the series $R_n(x)=\sum\limits_{v=0}^{\infty}r_{nv}x_v$ converges uniformly in n. Because, the remaining term of the series tends to zero uniformly in n, since

$$\left|\sum_{v=m}^{\infty}r_{nv}x_v\right|\leq\sup_v|r_{nv}|\sum_{v=m}^{\infty}|x_v|\to 0\ \ (m\to\infty)$$

and so

$$\lim_n R_n(x)=\sum_{v=0}^{\infty}\lim_n r_{nv}x_v. \tag{4.8}$$

Considering the equation (4.8), it can be written

$$A_n(x)=\lim_m\sum_{k=0}^{m}a_{nk}x_k=\lim_m\sum_{r=0}^{m}b^{(n)}_{mr}z_r=\sum_{r=0}^{\infty}\bar{b}_{nr}z_r.$$

Since $\left|\mathcal{L}^{\phi}(r,s)\right| \cong l$, then, it is obtained $A(x) \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ whenever $x \in \left|\mathcal{L}^{\phi}(r,s)\right|$ iff $\bar{B}(z) \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ i.e., $\tilde{B}(z) = E^{(p)} \circ \mathcal{L}(r,s) \circ \bar{B}(z) \in l_p$ for all $z \in l$, where $z = E^{(p)} \circ \mathcal{L}(r,s)(x)$, or, equivalently, $\tilde{B} \in (l, l_p)$. So, if we apply Lemma 1.5 to the matrix $\tilde{B}$, the last condition is immediately obtained, which completes the first part of the proof.

Since the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$, $1 \le p < \infty$, are BK-space, by Theorem 4.2.8 of [32], L_A defines a bounded operator such that $L_A(x) = A(x)$.

Moreover, from Theorem 3.4, it can be seen that $A = \tilde{\mathcal{L}}(r,s) \circ \tilde{E}^{(p)} \circ \tilde{B} \circ E^{(1)} \circ \mathcal{L}(r,s)$ and so,

$$\begin{aligned} \|L_A\|_{\left(|\mathcal{L}^{\phi}(r,s)|, |\mathcal{L}^{\phi}(r,s)|_p\right)} &= \sup_{x \neq 0} \frac{\|A(x)\|_{|\mathcal{L}^{\phi}(r,s)|_p}}{\|x\|_{|\mathcal{L}^{\phi}(r,s)|}} = \sup_{x \neq 0} \frac{\left\|\tilde{B} \circ E^{(1)} \circ \mathcal{L}(r,s)(x)\right\|_{l_p}}{\left\|E^{(p)} \circ \mathcal{L}(r,s)(x)\right\|_l} \\ &= \sup_{z \neq 0} \frac{\left\|\tilde{B}(z)\right\|_{l_p}}{\|z\|_l} = \left\|\tilde{B}\right\|_{(l,l_p)}, (z = E^{(1)} \circ \mathcal{L}(r,s)(x)). \end{aligned}$$

□

Theorem 4.3. *Let $1 \le p < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$ and $B = (b_{nv})$ be a matrix satisfying the following relation*

$$b_{nk} = \phi_n^{1/p^*} \sum_{v=0}^{n} \xi_{nv} a_{vk}. \tag{4.9}$$

Then, for any sequence spaces λ, $A \in \left(\lambda, \left|\mathcal{L}^{\phi}(r,s)\right|_p\right)$ if and only if $B \in (\lambda, l_p)$.

Proof. Take $x \in \lambda$. It follows from (4.9) that

$$\sum_{k=0}^{\infty} b_{nk} x_k = \phi_n^{1/p^*} \sum_{v=0}^{n} \xi_{nv} \sum_{k=0}^{\infty} a_{vk} x_k.$$

By (3.2), it is seen immediately that $B_n(x) = \left(E^{(p)} \circ \mathcal{L}(r,s)\right)_n (A(x))$ for all $x \in \lambda$. So, it is obtained that $A_n(x) \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ whenever $x \in \lambda$ if and only if $B(x) \in l_p$ whenever $x \in \lambda$, which completes the proof of the theorem. □

Theorem 4.4. *Let $1 \le p < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$. Then, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X\right)$ if and only if*

$$V^{(n)} \in (l_p, c) \text{ for all } n \in \mathbb{N},$$
$$\tilde{A}^{(p)} \in (l_p, X),$$

where the matrices $V^{(n)}$ and $\tilde{A}$ are defined as

$$\tilde{a}_{nk}^{(p)} = \phi_k^{-1/p^*} \left(\frac{1}{r} \frac{L_k}{L_{k-1}} a_{nk} + \sum_{j=k+1}^{\infty} a_{nj} \eta_{jk} \right)$$

and

$$v_{mk}^{(n)} = \begin{cases} \phi_k^{-1/p^*} \left(a_{nk} \frac{1}{r} \frac{L_k}{L_{k-1}} + \sum_{j=k+1}^{m} a_{nj} \eta_{jk} \right), & 0 \le k \le m-1 \\ \phi_m^{-1/p^*} a_{nm} \frac{1}{r} \frac{L_m}{L_{m-1}}, & k = m \\ 0, & k > m. \end{cases}$$

Proof. Let $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X\right)$ and $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$. Note that $\left|\mathcal{L}^{\phi}(r,s)\right|_p = (l_p)_{E^{(p)} \circ \mathcal{L}(r,s)}$. Considering (3.4) and (3.5), we get

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} v_{mk}^{(n)} z_k \tag{4.10}$$

for all $n, m \in \mathbb{N}$. It can be seen immediately that Ax is well defined for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ iff $V^{(n)} \in (l_p, c)$. Further, letting $m \to \infty$, it is seen from (4.10) that $Ax = \tilde{A}^{(p)} z$. Since $Ax \in X$, then $\tilde{A}^{(p)} z \in X$, that is $\tilde{A} \in (l_p, X)$.

Conversely, let $V^{(n)} \in (l_p, c)$ and $\tilde{A}^{(p)} \in (l_p, X)$. Since $V^{(n)} \in (l_p, c)$ with (4.10), we get $A_n \in \left\{ \left|\mathcal{L}^{\phi}(r,s)\right|_p \right\}^{\beta}$, for all n, which gives that Ax exists. Besides, we deduced from $\tilde{A}^{(p)} \in (l_p, X)$ and (4.10) as $m \to \infty$, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X \right)$.

$\square$

Now, we list the following notations:

1. $\lim\limits_{n\to\infty} \tilde{a}_{nv}^{(p)}$ exists for all $v \in \mathbb{N}$
2. $\lim\limits_{n\to\infty} \tilde{a}_{nv}^{(p)} = 0$ for all $v \in \mathbb{N}$
3. $\sup\limits_{n} \sum\limits_{v=0}^{\infty} \left|\tilde{a}_{nv}^{(p)}\right|^{p^*} < \infty$
4. $\sup\limits_{n,v} \left|\tilde{a}_{nv}^{(p)}\right| < \infty$
5. $\sup\limits_{N} \sum\limits_{v} \left| \sum\limits_{n\in N} \tilde{a}_{nv}^{(p)} \right|^{p^*} < \infty$
6. $\sup\limits_{v} \sum\limits_{n} \left|\tilde{a}_{nv}^{(p)}\right| < \infty$
7. $\sup\limits_{m} \sum\limits_{v=0}^{\infty} \left|v_{mv}^{(n)}\right|^{p^*} < \infty$
8. $\sup\limits_{m,v} \left|v_{mv}^{(n)}\right| < \infty$
9. $\lim\limits_{m\to\infty} v_{mv}^{(n)}$ exists for all $v, n \in \mathbb{N}$

By Theorem 4.4, we obtain following results giving the necessary and sufficient conditions for $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), X \right)$ with $X \in \{l_\infty, c_0, c, l, cs, bs\}$.

Theorem 4.5. *Let $1 < p < \infty$. The following statements hold:*

(i) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, c \right) \Leftrightarrow (1), (3), (7)$ and (9) hold.
(ii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, c_0 \right) \Leftrightarrow (2), (3), (7)$ and (9) hold.
(iii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l_\infty \right) \Leftrightarrow (3), (7)$ and (9) hold.
(iv) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l \right) \Leftrightarrow (5), (7)$ and (9) hold.
(v) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, c \right) \Leftrightarrow (1), (4), (8)$ and (9) hold.
(vi) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, c_0 \right) \Leftrightarrow (2), (4), (8)$ and (9) hold.
(vii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, l_\infty \right) \Leftrightarrow (4), (8)$ and (9) hold.
$(viii)$ $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, l \right) \Leftrightarrow (6), (8)$ and (9) hold.

Corollary 4.1. *Put $a(n,k) = \sum\limits_{j=0}^{n} a_{jk}$ instead of a_{nk} for all n, k in $V^{(n)} = (v_{mv}^{(n)})$ and $\tilde{A}^{(p)} = (\tilde{a}_{nv}^{(p)})$. Then,*

(i) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, cs \right) \Leftrightarrow (1), (3), (7)$ and (9) hold.
(ii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, bs \right) \Leftrightarrow (3), (7)$ and (9) hold.
(iii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, cs \right) \Leftrightarrow (1), (4), (8)$ and (9) hold.
(iv) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, bs \right) \Leftrightarrow (4), (8)$ and (9) hold.

Theorem 4.6. *(i) Let $1 < p < \infty$ and X be one of the sequence spaces c_0, c, l_∞.*

$$A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, X\right) \Rightarrow \|L_A\| = \|A\|_{\left(|\mathcal{L}^\phi(r,s)|_p, l_\infty\right)} = \sup_n \left\|\tilde{A}_n^{(p)}\right\|_{l_{p^*}},$$

$$A \in \left(\left|\mathcal{L}^\phi(r,s)\right|, X\right) \Rightarrow \|L_A\| = \|A\|_{\left(|\mathcal{L}^\phi(r,s)|, l_\infty\right)} = \sup_n \left\|\tilde{A}_n^{(1)}\right\|_{l_\infty}.$$

(ii) Let $1 < p < \infty$. There exists $1 \leq \xi \leq 4$ such that

$$A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, l\right) \Rightarrow \|L_A\| = \|A\|_{\left(|\mathcal{L}^\phi(r,s)|_p, l\right)} = \left\|\tilde{A}^{(p)}\right\|_{(l_p,l)} = \frac{1}{\xi}\left\|\tilde{A}^{(p)}\right\|'_{(l_p,l)},$$

$$A \in \left(\left|\mathcal{L}^\phi(r,s)\right|, l_p\right) \Rightarrow \|L_A\| = \|A\|_{\left(|\mathcal{L}^\phi(r,s)|, l_p\right)} = \left\|\tilde{A}_n^{(1)}\right\|_{(l,l_p)}.$$

Proof. The proof of the theorem is obtained together with Lemma 1.8, Lemma 1.3 and Lemma 1.5. □

5. Compact Operators on absolute Lucas series spaces

The aim of this section is to establish some identities or estimates for the Hausdorff measures of noncompactness of the matrix operators on the $\left|\mathcal{L}^\phi(r,s)\right|_p$ and also to characterize certain classes of compact operators by using the Hausdorff measure of noncompactness.

Theorem 5.1. *Under the hypothesis of Theorem 4.1, if $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, \left|\mathcal{L}^\phi(r,s)\right|\right)$, then*

$$\|L_A\|_\chi = \frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\hat{b}_{nr}\right|\right)^{p^*}\right\}^{\frac{1}{p^*}}$$

and

$$L_A \text{ is compact iff } \lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\hat{b}_{nr}\right|\right)^{p^*}\right\}^{\frac{1}{p^*}} = 0.$$

Proof. To determine the Hausdorff measure of noncompactness of L_A, take $S_{|\mathcal{L}^\phi(r,s)|_p}$ as a unique ball in the space $\left|\mathcal{L}^\phi(r,s)\right|_p$. By using Lemma 2.1, Lemma 2.3 and Lemma 1.3, it is obtained that

$$\begin{aligned}
\|A\|_\chi &= \chi\left(A\left(S_{|\mathcal{L}^\phi(r,s)|_p}\right)\right)\\
&= \chi\left(E^{(1)}\circ\mathcal{L}(r,s)\circ A\left(S_{|\mathcal{L}^\phi(r,s)|_p}\right)\right)\\
&= \chi\left(\hat{B}\circ E^{(p)}\circ\mathcal{L}(r,s)\left(S_{|\mathcal{L}^\phi(r,s)|_p}\right)\right)\\
&= \lim_{v\to\infty}\left(\sup_{z\in E^{(p)}(\mathcal{L}(r,s)(S_{|\mathcal{L}^\phi(r,s)|_p}))}\left\|(I-P_v)\left(\hat{B}(z)\right)\right\|\right)\\
&= \tfrac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\hat{b}_{nr}\right|\right)^{p^*}\right\}^{\frac{1}{p^*}}.
\end{aligned}$$

Finally, by using Lemma 2.2, the compact operators in this class can be immediately characterized. □

Theorem 5.2. *Under the hypothesis of Theorem 4.2, if $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|, \left|\mathcal{L}^\phi(r,s)\right|_p\right)$, then*

$$\|A\|_\chi = \lim_{v\to\infty}\left\{\sup_j\sum_{n=v+1}^{\infty}\left|\tilde{b}_{nj}\right|^p\right\}^{\frac{1}{p}}$$

and

$$L_A \text{ is compact iff } \lim_{v\to\infty}\left\{\sup_j \sum_{n=v+1}^{\infty}\left|\tilde{b}_{nj}\right|^p\right\}^{\frac{1}{p}} = 0.$$

Proof. Let $S_{|\mathcal{L}^\phi(r,s)|}$ be a unit sphere in $\left|\mathcal{L}^\phi(r,s)\right|$. Since $E^{(p)}\circ\mathcal{L}(r,s)\circ AS_{|\mathcal{L}^\phi(r,s)|} = \tilde{B}\circ E^{(1)}\circ\mathcal{L}(r,s)S_{|\mathcal{L}^\phi(r,s)|}$, it follows by Lemma 2.1, Lemma 2.3 and Lemma 1.5 that

$$\begin{aligned}
\|A\|_\chi &= \chi\left(AS_{|\mathcal{L}^\phi(r,s)|}\right)\\
&= \chi\left(E^{(p)}\circ\mathcal{L}(r,s)\circ AS_{|\mathcal{L}^\phi(r,s)|}\right)\\
&= \chi\left(\tilde{B}\circ E^{(1)}\circ\mathcal{L}(r,s)S_{|\mathcal{L}^\phi(r,s)|}\right)\\
&= \lim_{v\to\infty}\left(\sup_{z\in E^{(1)}\circ\mathcal{L}(r,s)\left(S_{|\mathcal{L}^\phi(r,s)|}\right)}\left\|(I-P_v)(\tilde{B}(z))\right\|_{l_p}\right)\\
&= \lim_{v\to\infty}\left\{\sup_j \sum_{n=v+1}^{\infty}\left|\tilde{b}_{nj}\right|^p\right\}^{\frac{1}{p}}.
\end{aligned}$$

Using Lemma 2.2, the last part of the proof is completed. □

Theorem 5.3. *Let $1<p<\infty$. Then,*

(a) If $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|_p, c_0\right)$, then

$$\|L_A\|_\chi = \lim_{j\to\infty}\sup_{n>j}\left\|\tilde{A}_n^{(p)}\right\|_{l_{p^*}} = \lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*},$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\sup \sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*} = 0.$$

(b) If $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|_p, c\right)$, then

$$\frac{1}{2}\lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}-\tilde{a}_k\right|^{p^*} \le \|L_A\|_\chi \le \lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}-\tilde{a}_k\right|^{p^*}$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\sup \sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}-\tilde{a}_k\right|^{p^*} = 0$$

where $\tilde{a}=(\tilde{a}_k)$ is defined by $\tilde{a}_k=\lim_{n\to\infty}\tilde{a}_{nk}$, for all $n\in\mathbb{N}$.

(c) If $A\in\left(\left|\mathcal{L}^\phi(r,s)\right|_p, l_\infty\right)$, then

$$0\le\|L_A\|_\chi \le \lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*},$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty}\sup \sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*} = 0.$$

(d) *If* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, l_p\right)$, $1 \le p < \infty$, *then*

$$\|L_A\|_{\chi} = \lim_{j\to\infty}\left(\sup_v\left(\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(1)}\right|^p\right)^{1/p}\right),$$

and

$$L_A \text{ is compact iff } \lim_{j\to\infty}\sup_v \sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(1)}\right|^p = 0.$$

(e) *If* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l\right)$, $1 < p < \infty$, *then there exists* $1 \le \xi \le 4$ *such that*

$$\|L_A\|_{\chi} = \frac{1}{\xi}\lim_{j\to\infty}\left(\sum_{v=1}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(p)}\right|\right)^{p^*}\right)^{1/p},$$

and

$$L_A \text{ is compact iff } \lim_{j\to\infty}\sum_{v=1}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(p)}\right|\right)^{p^*} = 0.$$

Proof. The proof of the theorem can be obtained by combining Lemma 4.2 and Lemma 2.5, so it has been left to reader. □

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An Analytical Approach to an Elastic Circular Rod Equation

Zehra Pınar

Abstract
The size-dependent longitudinal and torsional dynamic problems for small-scaled rods have importance in two-phase media. The special case of the elastic rod equation such as magneto-electro circular equation are seen in the literature commonly, but in this work, the generalized form of the nonlinear elastic circular equation, which was not studied in the literature, is considered. The exact solutions are obtained via Mathieu approximation method with a novel proposed ansatz. Obtained solutions are discussed and illustrated in details. We believe that the proposed results will be key part of further analytical and numerical studies for waves in the dispersive medium with reaction.

Keywords: *Mathieu approximation method; the elastic rod equation; travelling wave solutions.*

AMS Subject Classification (2020): *Primary: 35CXX ; Secondary: 35BXX; 35QXX.*

1. Introduction

Modeling a wide range of problems related to different research areas such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and chemistry, is done by using partial differential equations (PDEs). Therefore, PDEs are of paramount importance among researchers. In particular, nonlinear wave propagation, diffusion, reaction and convection are very important. Thus, the longitudinal wave equation (LWE) arising in a magneto-electro-elastic circular rod is a major problem that rods have been used for transmission and owing to their superior electrical, mechanical, optical and other physical and chemical properties, areas of use vary [18]. Firstly, the models of LWE can be examined. The most known model for long finite-amplitude waves is the Korteweg-de Vries (KdV) equation but the dispersive term is ignored, so the non-linearity is dominant. Then, there were many attempts to improve the model via using extra nonlinear terms involving second-order and a third-order derivative. Hence, the aim is focused on finite-length and finite-amplitude waves. Respect to material of the rod, the equation is reduced to Benjamin-Bona-Mahony (BBM) equation which is known as an alternative to the KdV equation for modelling long finite-amplitude waves. Although, no matter how thin the rod is, it is always assumed three-dimensional, when the rod diameter is much smaller than the axial length scale, it is reasonable to expect that approximate one-dimensional equations (rod equations) can give a good description of the motion of the rod. Many modifications of rod equations are seen in the literature and these equations ignore many properties

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of rod and constraining terms [8, 12, 13, 17, 18, 19]. In this work, the considered rod equation was not studied before in the literature and it contains physical and chemical properties of the rod. The generalized form of the elastic rod equation [2, 3, 7] is determined

$$u_{tt} - c_0^2(1 + na_n(u_x)^2)_{xx} - \frac{\nu^2 J_\rho}{s} u_{ttxx} = F(t), \tag{1.1}$$

where $F(t)$ is the forcing term, s is the cross-section area of the rod, J_ρ is the polar moment of inertia, $c_0^2(= \frac{E}{\rho})$ is the square of the linear elastic longitudinal wave velocity, E is the modulus of elasticity (Young's modulus), ρ is the density of the rod, is the Poisson ratio, n is an integer, a_n is material constants of the rod.

In case of the soft-nonlinear materials a_n is less than zero, whereas for hard-nonlinear materials such as rubbers polymers and some metals a_n is greater than zero. As it is seen that, Eq.(1.1) is also material depended equation (model). As can be seen, when $F(t) = 0$, Eq.(1.1) is reduced to the well-known rod equation and also when $a_n = 0$ under adequate assumptions can be converted to classical wave equation for an elastic thin rod [7]. In the literature, solitary wave solutions of the Eq.(1.1) are obtained for $n = 2$ and $n = 3$ by simplicity [17, 19, 20]. Generally, the ansatz is determined after reducing and integrating the Eq.(1.1) [1, 6, 7]. But the degree of the ansatz is fractional that is not valid. In this work, the Eq.(1.1) is considered in the original form and the degree of the ansatz is determined in a novel way. Till now, the degree of ansatz is determined by balancing principle which is generally given for the power-law non-linearity. The reason of the transformations and integration is to reduce the Eq.(1.1) to the equation with power non-linearity to use the known balancing principle. The balancing principle, determined in our previous work, works not only positive integer power but also negative and/or fractional powers. Moreover, the power is least numeral satisfying the finite expansion in most cases [16]. Hence in the similar manner, the novel balancing principle for non-power-law non-linearity is proposed

$$N + n = qN + psN, (modn), \tag{1.2}$$

where the highest order term is $\frac{\partial^n u}{\partial x^n}$ and the highest order nonlinear term is $\frac{\partial^q u}{\partial x^q}(\frac{\partial^s u}{\partial x^s})^p$ .By the novel proposed balancing principle (Eq. (1.2)), the power is obtained as the least numeral and positive integer, so the computational cost decreases. Additionally, the novel balancing principle works for all types of non-linearity and general cases. We exemplified some examples either does not work with already existing balancing principles in the literature. In this work, we consider the auxiliary equation method based on the Mathieu equation. In order to explain the idea of the auxiliary equation method, using the wave transformation,

$$u(x,t) = u(\xi), \xi = x - \mu t. \tag{1.3}$$

A nonlinear partial differential equation (PDE), $M(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0$, is reduced to a nonlinear ODE, $N(u, u_\xi, u_{\xi\xi}, ...) = 0$. Assuming that the exact solution of equation nonlinear ODE has the simple finite expansion as

$$u(\xi) = \sum_{i=0}^{N} g_i z(\xi)^i, \tag{1.4}$$

where g_i are unknown constants to be determined later. Also, $z(\xi)$ defines the exact solution of the new proposed auxiliary equation. The unknown coefficients are determined in main three steps: (1) substituting the proposed auxiliary equation into the reduced nonlinear ODE. (2) Equating each coefficient of power of $z(\xi)$ to zero. (3) Solving the corresponding algebraic system, the coefficients are obtained. Also, the main step is determination of the integer N , which indicates the number of terms will be used in Eq.(1.4), basically by balancing the term with the highest order derivative and the term with the highest power non-linearity in reduced nonlinear ODE. But, in this work, the novel proposed balancing principle (Eq. (2)) is used to determine the integer so the considered generalized rod equation has non-power-law non-linearity. Generally, the function $z(\xi)$ is used as the exact solution of proposed auxiliary equation in the literature. Since nonlinear PDEs cannot be recovered by only one auxiliary ordinary differential equation, there have been many studies utilizing different exactly solvable auxiliary equations [14, 15].

As it is known that Eq. (1.1) has solitary and periodic type travelling wave equation, Mathieu equation [14],

$$\frac{d^2 z(\xi)}{d\xi^2} + (a - 2qcos(\xi))z(\xi) = 0, \tag{1.5}$$

which has a solution as $z(\xi) = C_1 MathieuC(a, q, \xi) + C_2 MathieuS(a, q, \xi)$, where $MathieuC$ and $MathieuS$ are periodic functions for countably many values of (a function of q). In this work, the rod equation is solved using the novel ansatz and the proposed method which has the same idea with different auxiliary equation.

2. Solutions

In this section, using proposed method and the novel balancing principle, the analytical solutions of the following generalized elastic rod equation is obtained

$$u_{tt} - c_0^2(1 + na_n(u_x)^2)_{xx} - \frac{\nu^2 J_\rho}{s} u_{ttxx} = F(t). \tag{2.1}$$

Till now, the solutions of the elastic rod equation (Eq. (1.1)) is obtained after reducing to ODE, transformation $u' = v$ and integrations are used. Hence instead of the Eq. (1), the reduced equation is solved i.e. the original equation is not solved [8, 12, 13, 17, 18, 19]. Using the novel balancing principle for non-power-law non-linearity, Eq. (1.1) is solved directly. To our knowledge, this study is the first attempt to investigate the analytical solutions of the generalized nonlinear elastic rod equation (Eq. (1.1)). With the wave transformation, $u(x,t) = u(\xi), \xi = x - \mu t$, Eq. (2.1) is reduced as below,

$$\mu^2 u'' - c_0^2 u'' - nc_0^2 (u')^{n-1} u'' - \frac{\nu^2 J_\rho}{s} \mu^2 u^{(4)} = 0, \tag{2.2}$$

respect to the novel balancing principle (Eq. (1.2)) $N = \frac{4}{n}(mod4)$.

Case 1.In the case $n = 2$ as generally considered in the literature, using the novel balancing principle $N = \frac{4}{2} = 2(mod4)$ is obtained, so the ansatz is $u(\xi) = g_0 + g_1 z(\xi) + g_2 z(\xi)^2$. Applying the given procedure, the parameters are obtained as the solution of the nonlinear algebraic system.

$$g_1 = 0, C_1 = \frac{-C_2 MathieuSPrime(a, q, \xi)}{MathieuCPrime(a, q, \xi)}, \mu = -c_0. \tag{2.3}$$

The plot of the obtained solution is given by Figure 1. Figure 1. The 3D ((a)-(b)) and contour (c) plots of the solution of Eq. (2.2) via Mathiue approximation method for Case 1 $g_0 = 2, g_2 = 1, s = 2, c_0 = \sqrt{3}, \nu = 2, a = 1, q = 1.2, a_n = 1, J_\rho = 1.$

Case 2.. As a novel case $n = 4$ is considered, using the novel balancing principle $N = \frac{4}{4} = 1(mod4)$ is obtained, so the ansatz is $u(\xi) = g_0 + g_1 z(\xi)$. Applying the given procedure, the plot of the solution is given by Figure 2.

Figure 2. The 3D ((a)-(b)) and contour (c) plots of the solution of Eq. (2.2) via Mathiue approximation method for Case 2 $g_0 = 2, g_1 = 1, s = 2, c_0 = 3, \nu = 1, a = 1.8, a_n = 1, J_\rho = 1, \mu = 0.1.$

3. Conclusion

The main idea of this study is based on obtaining the exact solutions of generalized elastic rod equation containing non-power non-linearity by using the exact solutions of different type equations as an ansatz. By means of Mathieu equation with the wave transformation, the exact solutions of the generalized elastic rod equation are obtained. The generalized elastic rod equation is not an usual equation that contains parameters refer to physical and chemical properties of rod material. In this work, the novel balancing principle for non-power non-linearity to determine degree of ansatz is proposed for the first time in the literature. We believe that the obtained solutions of the generalized elastic rod equation will play key role in further analytical and numerical studies. Future studies will be based on investigating the analytical solutions of the nonlinear partial differential equations containing the non-power non-linearity via the proposed novel balancing principle in this study.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Local Asymptotic Stability and Sensitivity Analysis of a New Mathematical Epidemic Model Without Immunity

Sümeyye Çakan

Abstract
With this study it is aimed to introduce and analyze a new SIS epidemic model including vaccination effect. Vaccination considered in the model provides a temporary protection effect and is administered to both susceptible and new members of the population. The study provides a different aspect to the SIS models used to express, mathematically, some infectious diseases which are not eradicated by the immune system. The model given this study is designed by considering varying processes from person to person in the disease transmission, the recovery from disease (recovery without immunity) and in the loss of protective effect provided by the vaccine. The processes that change according to individuals are explained by distributed delays used in the relevant differential equations that provide the transition between compartments. The differences in the model are especially evident in these parts. In analyzing the model, firstly, the disease-free and endemic equilibrium points related to the model are determined. Then, the basic reproduction number $\mathcal{R}_0$ is calculated with the next generation matrix method. Next, the dynamics about locally asymptotically stable of the model at the disease-free and endemic equilibriums are examined according to the basic reproduction number $\mathcal{R}_0$. Attempts intended to reduce the spread of the disease are, of course, in the direction supporting the lowering the value $\mathcal{R}_0$. In this context, the reducing and enhancing effects of the parameters used in the model on the value $\mathcal{R}_0$ have been interpreted mathematically and suggestions were made to implement control measures in this direction. Also, in order to evaluate the support provided by the vaccine during the spread of the disease, the model has been examined as vaccinated and unvaccinated, and by some mathematical process, it has been seen that the vaccination has a crucial effect on disease control by decreasing the basic reproduction number. In other respects, by explored that the effect of parameters related to vaccination on the change of $\mathcal{R}_0$, a result about the minimum vaccination ratio of new members required for the elimination of the disease in the population within the scope of the target of $\mathcal{R}_0 < 1$ has been obtained.

Keywords: *Local Asymptotic Stability; Sensitivity Analysis; SIS model; Vaccine Effect; Disease-free equilibrium point; Endemic equilibrium point; Basic Reproduction Number.*

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1. Introduction

Mathematical modeling has been used to describe and analyze behaviors of many phenomena in the practical application areas such as theoretical ecology, mathematical epidemiology, economics, medicine, physics, chemical, biology, engineering and so on, [1–7]. Specially, the technique of compartmental modeling has become substantial tools in mathematical epidemiology for analyzing of the spread and control of infectious diseases. The modeling idea related to epidemic disease transmitted in a closed population consisting of susceptibles (S), infectives (I), and recovereds (R) were firstly considered by Kermack and McKendrick in 1927, [8]. Then, a lot of authors have dealt with various details to carry further forward this modeling technique. Along with, the historical adventure of compartmental modeling in mathematical epidemiology has proceeded from basic models to more detailed models. It is usually difficult or almost impossible the analytical examination of detailed models and so their usefulness for theoretical objectives is restricted, even though they may include substantial strategic values. On the other hand simple models may be inadequate for public health authorities who are faced with the need to make recommendations on strategies to deal with a particular situation. Therefore, the researches on the dynamics of basic but slightly more detailed models have folded day by day. Especially, it has been concentrated on seeing whether the variations in the models which are studied can lead to significant differences in behaviors related to qualitative and stability, with respect to models in classical type. Hereby, by using the general principles of modeling of epidemics, various models to describe the course of some epidemic diseases have been formulated, [9–18].

With the details studied in the epidemic models, specific principles including factors such as vaccination, quarantine, treatment; differences in systems reflecting transmission dynamics (such as being difference, differential, integral or integro-differential equations) or using of the delay element in the projected system ... etc. are meant. Vaccination appears as one of the significant factors between control measures for the dynamics in diseases transmission. Li and Ma studied on SIS epidemic model with vaccination in [19]. Cai and Li [20] examined the global stability of their $SEIV$ epidemic model with a nonlinear incidence rate.

In this paper, we formulate a new SIS model with distributed delays by adding the vaccination effect, too. To do this we use three distribute functions. Vaccination strategy in the model presented in this study base on administering to both susceptible and new members of the population. Also, in the model, we assume that the vaccinated individuals have temporary immunity and the losing of efficacy of vaccination varies from individual to individual depending on the fact that efficiency of any vaccine does not usually continue lifetime of the individual. On the other hand, it is thought that the infectiousness period in the transition from S to I and the recovery without immunity in transition from I to S vary from individual to individual. The fact that the system consists of integro-differential equations is originated from these effects varying according to individuals.

In the literature, there are studies that take into account the relative infectivity, [21–23], as well as the studies that assume that the immunity formed after vaccination is not permanent, [24].

On the other hand, by using nonlinear classical differential equations, models in which the delay period is the same and constant for all individuals can be made. However, nonlinear integro-differential equations are needed to express the delay process with distributed manner, provided that the delay process remains within a certain interval and varies according to individuals.

In the model introduced in this study, it is assumed that both the infectivity differs according to the individuals over time and the protection provided by the vaccination that does not create permanent immunity changes over time. In addition, the assumption that vaccinated individuals become relatively susceptible again with the loss of immunity is also reflected in the model. The study aims to contribute to the mathematical epidemiology literature with these novel aspects.

We continue this study to which we begin with introducing the model, with the qualitative and stability analysis of the model. In what follows, we evaluate the impact of vaccination on the model dynamics and discuss sensitivity analysis utilizing the normalized forward sensitivity index.

2. The Main Results Related to Research

The model which have been constructed by using the distribution function in three directions of transmission and adding vaccination effect is governed by a system of nonlinear integro-differential equations below.

$$\frac{dS}{dt} = (1-p)\,b - \beta S(t)\int_0^{h_1} f(\tau)\,I(t-\tau)\,d\tau - \sigma S(t) - \mu S(t) + \xi\int_0^{h_2} g(\theta)\,V(t-\theta)\,d\theta + \eta\int_0^{h_3} k(\gamma)\,I(t-\gamma)\,d\gamma,$$

$$
\begin{aligned}
\frac{dI}{dt} &= \beta S(t)\int_0^{h_1} f(\tau)\, I(t-\tau)\, d\tau - \eta \int_0^{h_3} k(\gamma)\, I(t-\gamma)\, d\gamma - \delta I(t) - \mu I(t), \qquad (2.1)\\
\frac{dV}{dt} &= pb + \sigma S(t) - \xi \int_0^{h_2} g(\theta)\, V(t-\theta)\, d\theta - \mu V(t).
\end{aligned}
$$

According to the model, the population was divided into three categories: Susceptible (S), Vaccinated (V), and Infectious (I) individuals.

The susceptible class consists the individuals who are susceptible to the disease and have not any immunity. By infectious individuals, it is meant the individuals who are infected by the disease and are able to spread the disease to susceptible individuals.

Here $S(t)$, $I(t)$ and $V(t)$ represent the numbers of susceptible, infectious and vaccinated individuals at time t, respectively. The total population size at time t is $N(t)$ and for all $t \geq 0$, $N(t) = S(t) + I(t) + V(t)$. Also it is assumed that all functions and parameters used in the model are nonnegative. The inclusion of all newborn individuals into the population is provided by giving input to the susceptible and vaccinated classes with the constant rate b in total. The rates of natural death and the disease induced death are represented by μ and δ, respectively. β denotes the effective contact rate between infectious and susceptible individuals.

h_1 is maximum infectiousness period and τ indicates the period of time for each individual becomes infectious such that $0 \leq \tau \leq h_1$. By using f which is first distribution function used in the model, the density of individuals whose infectious period τ is indicated with $f(\tau)$. Classically, it is supposed that f is non-negative and continuous on $[0, h_1]$. Also f satisfies $\int_0^{h_1} f(\tau)\, d\tau = 1$. The term $f(\tau)\, I(t-\tau)$ corresponds to number of surviving individuals at time t who infected at time $t-\tau$ and have infectiousness period τ. The integral $\beta S(t) \int_0^{h_1} f(\tau)\, I(t-\tau)\, d\tau$ is expression that reflects transition of individuals to the compartment I as a result of effective contact between the susceptible and infectious individuals within their infectiousness period.

The model envisions a vaccination strategy in which the vaccine is effective on all individuals and vaccinated individuals are not become infected during their protection period. But the effectiveness of the vaccination loses over time. p shows the vaccination rate of newborns while $(1-p)\, b$ represents the inclusion rate of newborns without vaccination to the susceptibles. Also σ is the vaccination rate of individuals in susceptible group and ξ is the losing rate of effectiveness of the vaccine.

Besides these, g is the second distribution function such that $g(\theta)$ shows the ratio of individuals whose protection period provided by the vaccine is θ. h_2 is the maximum protection period provided by vaccination. So $\theta = 0$ means that the vaccine is completely ineffective. Also, $0 < \theta \leq h_2$ means that the vaccinated individuals gain only a finite protection period (partial protection). Classically it is supposed that g is non-negative and continuous on $[0, h_2]$ in addition that g satisfies $\int_0^{h_2} g(\theta)\, d\theta = 1$. The term $g(\theta)\, V(t-\theta)$ corresponds to number of surviving individuals at time t who have been vaccinated at time $t-\theta$ and whose protection period is θ.

According to this model, the vaccination does not provide a protective effect that will last forever. So, when the protection period is finished, the vaccinated individuals who no longer has any protection turns again to the susceptible compartment. To reflect this transition, we have used the expression $\xi \int_0^{h_2} g(\theta)\, V(t-\theta)\, d\theta$ in the model.

On the other hand, with entering the individuals to the recovery process, the amount of pathogens in the host individual's body become sufficiently low in the rate that the individual is no longer capable of transmitting the disease. Individuals who complete the recovery process return to class S because they have not acquire any immunity to the disease. η indicates the recovery rate of infectious individuals (recovery without immunity) and h_3 is maximum recovery period. γ indicates the time of recovery period of each individual with $0 \leq \gamma \leq h_3$. k is third distribution function used in the model such that $k(\gamma)$ denotes the density of individuals whose their recovery period is γ. Again, classically, it is supposed that k denotes non-negative and k is continuous on $[0, h_3]$, such that k satisfies $\int_0^{h_3} k(\gamma) d\gamma = 1$. The term $k(\gamma) I(t-\gamma)$ represents the number of surviving individuals at time t who have been infectious at time $t-\gamma$ and whose recovery period is γ. According to our model the recovery period is also a process that varies according to the individuals, just like the infectiousness period. Thus we use the mathematical expression $\eta \int_0^{h_3} k(\gamma) I(t-\gamma)\, d\gamma$ in the transition from I to S to reflect the changing of the recovery process according to individuals.

2.1 The Qualitative Analysis Results

Before moving on to the analysis of the model, we have to be sure that solutions of the system (2.1) remain in a biologically feasible region for all parameters t belong to time. After preparation to this particular, we determine the equilibrium points and basic reproduction number of the model.

2.1.1 Feasible Positive Invariant Region for the Model

Theorem 2.1. *The bounded set*

$$\Theta = \left\{ (S,I,V) : S \in C(\mathbb{R}_+, \mathbb{R}_+),\ I \in C([-\max\{\tau,\gamma\},\infty), \mathbb{R}_+),\ V \in C([-\theta,\infty), \mathbb{R}_+) : N(t) \le \frac{b}{\mu} \right\} \tag{2.2}$$

is positively invariant for the model, where $\mathbb{R}_+ = [0,\infty)$.

Proof. By the sum of the differential equations that make up the system (2.1), the differential inequality

$$\begin{aligned} N'(t) &= \frac{dS}{dt} + \frac{dI}{dt} + \frac{dV}{dt} \\ &= b - \mu(S(t) + I(t) + V(t)) - \delta I(t) \\ &\le b - \mu(N(t)) \end{aligned} \tag{2.3}$$

is obtained. The solution of this differential inequality is achieved from solving the differential equation

$$N'(t) + \mu N(t) = b.$$

Then, we get the solution

$$N(t) = N(0)e^{-\mu t} + \frac{b}{\mu}\left(1 - e^{-\mu t}\right) \tag{2.4}$$

for the initial condition $t = 0$. Standard Comparison Theorem [25] says that the right side of the equality (2.4) is the maximal solution of inequality (2.3). Thus we write

$$N(t) \le N(0)e^{-\mu t} + \frac{b}{\mu}\left(1 - e^{-\mu t}\right)$$

for all $t \ge 0$.

It is obvious that $N(t) \le b/\mu$ for all $t > 0$ when $N(0) \le b/\mu$. Hence, Θ is positively invariant for the system (2.1).

On the other hand, it can be derived that $N(t)$ is bounded above with b/μ.

Consequently Θ is an asymptotic global attractor for all solutions of (2.1). Thus examining of the dynamics of (2.1) in the region Θ would be appropriate epidemiologically. □

2.1.2 Disease-Free Equilibrium Point

Since an equilibrium point of the system (2.1) is a constant solution of the system, it holds the equations constituting the system and so it is written as:

$$\begin{aligned} 0 &= (1-p)b - \beta S_0 I_0 - \sigma S_0 - \mu S_0 + \xi V_0 + \eta I_0, \\ 0 &= pb + \sigma S_0 - \xi V_0 - \mu V_0. \end{aligned}$$

From first and second equations, it is obtained respectively that

$$S_0 = \frac{(1-p)b + \xi V_0}{\sigma + \mu} \tag{2.5}$$

and

$$V_0 = \frac{pb + \sigma S_0}{\xi + \mu}, \tag{2.6}$$

for $I_0 \neq 0$. Substituting the equality (2.6) into (2.5), we get

$$S_0 \left[\sigma\xi + \sigma\mu + \mu\xi + \mu^2 - \sigma\xi\right] = b\xi + b\mu\,(1-p)$$

and so

$$S_0 = \frac{b\,(\xi + \mu\,(1-p))}{\mu\,(\xi + \mu + \sigma)}.$$

If this value is rewritten in (2.6), it is obtained that

$$V_0 = \frac{b\,(p\mu + \sigma)}{\mu\,(\xi + \mu + \sigma)}.$$

Hence, the disease-free equilibrium point of the model is

$$DFE = (S_0, I_0, V_0) = \left(\frac{b\,(\xi + \mu\,(1-p))}{\mu\,(\xi + \mu + \sigma)}, 0, \frac{b\,(p\mu + \sigma)}{\mu\,(\xi + \mu + \sigma)}\right). \tag{2.7}$$

2.1.3 Basic Reproduction Number

The basic reproduction number denoted by $\mathcal{R}_0$ is described as the average number of new cases (secondary infections) created from one infectious individual in the wholly susceptible population through the entire length of him/her infectiousness period.

In this part, we determine the basic reproduction number of the model by using the next generation matrix approach, [26].

The dynamic system given by (2.1) can be written in matrix form as

$$\frac{dW}{dt} = \begin{bmatrix} \dot{I} \\ \dot{S} \\ \dot{V} \end{bmatrix},$$

where $W = (I, S, V)^T$.

For the system written in the form

$$\frac{dW}{dt} = \mathcal{Y}(W) - \mathcal{Z}(W),$$

$\mathcal{Y}(W)$ and $\mathcal{Z}(W)$ are the following matrices, respectively:

$$\mathcal{Y}(W) = \begin{bmatrix} \beta S\,(t) \int\limits_0^{h_1} f\,(\tau)\,I\,(t-\tau)\,d\tau \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathcal{Z}(W) = \begin{bmatrix} \mathcal{Z}(W)_{11} \\ \mathcal{Z}(W)_{21} \\ \mathcal{Z}(W)_{31} \end{bmatrix},$$

where

$$\begin{aligned}
\mathcal{Z}(W)_{11} &= \eta \int_0^{h_3} k\,(\gamma)\,I\,(t-\gamma)\,d\gamma + (\delta + \mu)\,I\,(t)\,, \\
\mathcal{Z}(W)_{21} &= \beta S\,(t) \int_0^{h_1} f\,(\tau)\,I\,(t-\tau)\,d\tau + \sigma S\,(t) - \xi \int_0^{h_2} g\,(\theta)\,V\,(t-\theta)\,d\theta - \eta \int_0^{h_3} k\,(\gamma)\,I\,(t-\gamma)\,d\gamma + \mu S\,(t) - (1-p)\,b, \\
\mathcal{Z}(W)_{31} &= \xi \int_0^{h_2} g\,(\theta)\,V\,(t-\theta)\,d\theta + \mu V\,(t) - \sigma S\,(t) - pb.
\end{aligned}$$

In this splitting, $\mathcal{Y}(W)$ is the matrix formed by writing of the partitionings in which new infections appear in compartments I, S and V, respectively; and $\mathcal{Z}(W)$ is the matrix formed by writing of the partitionings in which other transitions between compartments I, S, V, and other compartments, respectively.

Now we find the correspondences at the DFE of the derivative matrices of $\mathcal{Y}(W)$ and $\mathcal{Z}(W)$ with respect to I, S and V, respectively.

$$d\mathcal{Y}(DFE) = \begin{bmatrix} \beta S_0 & \beta I_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$d\mathcal{Z}(DFE) = \begin{bmatrix} \eta+\delta+\mu & 0 & 0 \\ \beta S_0 - \eta & \beta I_0 + \sigma + \mu & -\xi \\ 0 & -\sigma & \xi+\mu \end{bmatrix}.$$

Now, considering that the infection can only exist in compartment I, let us constitute the block matrices Y and Z as

$$Y = \mathcal{Y}_{11} = [\beta S_0]$$

and

$$Z = \mathcal{Z}_{11} = [\eta+\delta+\mu].$$

Hence

$$YZ^{-1} = \left[\frac{\beta S_0}{\eta+\mu+\delta}\right].$$

From the biological meanings of Y and Z, it follows that Y is entrywise non-negative and Z is a non-singular M-matrix, so Z^{-1} is entrywise non-negative. Let $\phi(0)$ shows the number of initially infected individuals. Then $YZ^{-1}\phi(0)$ is an entrywise non-negative vector giving the expected number of new infections. The matrix YZ^{-1} has $(1;1)$ entry equal to the expected number of secondary infections in compartments I produced by an infected individual introduced in compartments I. Thus YZ^{-1} is the next generation matrix and $\mathcal{R}_0 = \rho\left(YZ^{-1}\right)$; where ρ denotes the spectral radius.

Considering the component

$$S_0 = \frac{b(\xi+\mu(1-p))}{\mu(\xi+\mu+\sigma)}$$

of the DFE, the basic reproduction number of the system (2.1) is obtained as

$$\begin{aligned} \mathcal{R}_0 &= \rho\left(YZ^{-1}\right) \\ &= \frac{\beta S_0}{\eta+\mu+\delta} \\ &= \frac{b\beta(\xi+\mu(1-p))}{\mu(\xi+\mu+\sigma)(\eta+\mu+\delta)}. \end{aligned}$$

2.1.4 Existence and Uniqueness of Endemic Equilibrium Point

Now, we handle the problem of existence and uniqueness of endemic equilibrium point of the presented model. The endemic equilibrium $EE(S^*, I^*, V^*)$ which is a constant solution of differential equations constituting the system (2.1) satisfies the algebraic equations

$$\begin{aligned} 0 &= (1-p)b - \beta S^* I^* - \sigma S^* - \mu S^* + \xi V^* + \eta I^*, \\ 0 &= \beta S^* I^* - \eta I^* - (\mu+\delta) I^*, \\ 0 &= pb + \sigma S^* - \xi V^* - \mu V^*, \end{aligned} \tag{2.8}$$

such that $I^* \neq 0$. From second equation of this algebraic system, we write

$$I^*(\beta S^* - \eta - (\mu+\delta)) = 0.$$

So, it must be

$$\beta S^* - \eta - (\mu+\delta) = 0.$$

Then

$$S^* = \frac{\eta + \mu + \delta}{\beta}. \tag{2.9}$$

If S^* obtained in (2.9) is written in third equation of (2.8), V^* is found as

$$V^* = \frac{pb\beta + \sigma(\eta + \mu + \delta)}{\beta(\xi + \mu)}.$$

On the other hand, by considering $\mathcal{R}_0$, S^* and V^* are written as

$$S^* = \frac{b(\xi + \mu(1-p))}{\mu(\xi + \mu + \sigma)\mathcal{R}_0}$$

and

$$V^* = \frac{pb\mu(\xi + \mu + \sigma)\mathcal{R}_0 + \sigma b(\xi + \mu(1-p))}{\mu(\xi + \mu)(\xi + \mu + \sigma)\mathcal{R}_0}.$$

Now, by using these equalities we have obtained, we will focus on the first equation of the system (2.8).

$$\left[\beta\frac{b(\xi + \mu(1-p))}{\mu(\xi + \mu + \sigma)\mathcal{R}_0} - \eta\right] I^* = (1-p)b - (\sigma + \mu)\frac{b(\xi + \mu(1-p))}{\mu(\xi + \mu + \sigma)\mathcal{R}_0} + \xi\frac{pb\mu(\xi + \mu + \sigma)\mathcal{R}_0 + \sigma b(\xi + \mu(1-p))}{\mu(\xi + \mu)(\xi + \mu + \sigma)\mathcal{R}_0}.$$

After regulations, we write

$$I^* = \frac{\begin{array}{c}(1-p)b\mu(\xi + \mu + \sigma)(\xi + \mu)\mathcal{R}_0 - b(\sigma + \mu)(\xi + \mu(1-p))(\xi + \mu) \\ +\xi\left[pb\mu(\xi + \mu + \sigma)\mathcal{R}_0 + \sigma b(\xi + \mu(1-p))\right]\end{array}}{(\xi + \mu)\left[b\beta(\xi + \mu(1-p)) - \eta\mu(\xi + \mu + \sigma)\mathcal{R}_0\right]}.$$

Precisely in this part, it has great importance to regulate the numerator of this fraction with careful operations. The numerator part of I^* can be written as

$$(1-p)b\mu(\xi + \mu + \sigma)(\xi + \mu)\mathcal{R}_0 - b(\sigma + \mu)\xi(\xi + \mu) - b\mu(1-p)(\sigma + \mu)(\xi + \mu) + \xi pb\mu(\xi + \mu + \sigma)\mathcal{R}_0 + \xi\sigma b(\xi + \mu(1-p)).$$

If the first and fourth terms of the numerator consisting of five sums are taken into the common factor ($(\xi + \mu + \sigma)\mathcal{R}_0$) parenthesis, it is obtained the term $(\xi + \mu + \sigma)\mathcal{R}_0 b\mu(\xi + \mu(1-p))$. From second and third terms, it comes $-b(\xi + \mu)(\sigma + \mu)(\xi + \mu(1-p))$. If this last term and the fifth term of the sum are considered together, it is obtained that $-b\mu(\xi + \sigma + \mu)(\xi + \mu(1-p))$.

So with the last rearrangement of the numerator part, we obtain

$$\begin{aligned} I^* &= \frac{b\mu(\xi + \mu(1-p))(\xi + \mu + \sigma)\left[\mathcal{R}_0 - 1\right]}{(\xi + \mu)\left[\underbrace{b\beta(\xi + \mu(1-p))}_{\mu(\xi+\mu+\sigma)(\eta+\mu+\delta)\mathcal{R}_0} - \eta\mu(\xi + \mu + \sigma)\mathcal{R}_0\right]} \\ &= \frac{b(\xi + \mu(1-p))(\mathcal{R}_0 - 1)}{(\mu + \delta)(\xi + \mu)\mathcal{R}_0}. \end{aligned}$$

Hence I^* is meaningful for only $\mathcal{R}_0 > 1$. Thus, we say that the system (2.1) has a unique endemic equilibrium point formulated by equality

$$\begin{aligned} EE &= (S^*, I^*, V^*) \\ &= \left(\frac{b(\xi + \mu(1-p))}{\mu(\xi + \mu + \sigma)\mathcal{R}_0}, \frac{b(\xi + \mu(1-p))(\mathcal{R}_0 - 1)}{(\mu + \delta)(\xi + \mu)\mathcal{R}_0}, \frac{pb\mu(\xi + \mu + \sigma)\mathcal{R}_0 + \sigma b(\xi + \mu(1-p))}{\mu(\xi + \mu)(\xi + \mu + \sigma)\mathcal{R}_0}\right), \end{aligned}$$

when $\mathcal{R}_0 > 1$.

2.2 The Stability Analysis Results

In this section, we explore the asymptotic behaviors of the equilibrium points for the model (2.1).

Theorem 2.2. *The disease-free equilibrium point DFE is locally asymptotically stable in Θ for $\mathcal{R}_0 < 1$.*

Proof. For the system (2.1), the Jacobian matrix at $DFE = (S_0, I_0, V_0)$ is

$$J\left(DFE\right) = \begin{bmatrix} -\beta I_0 - \sigma - \mu & -\beta S_0 + \eta & \xi \\ \beta I_0 & \beta S_0 - \eta - \mu - \delta & 0 \\ \sigma & 0 & -\xi - \mu \end{bmatrix}.$$

Since $I_0 = 0$, the characteristic equation which is correspond to this Jacobian matrix is

$$\begin{aligned} \det\left(J\left(DFE\right) - \lambda I_3\right) &= \begin{vmatrix} -\left(\sigma+\mu\right) - \lambda & -\beta S_0 + \eta & \xi \\ 0 & \beta S_0 - \eta - \mu - \delta - \lambda & 0 \\ \sigma & 0 & -\xi - \mu - \lambda \end{vmatrix} \\ &= \left(\beta S_0 - \eta - \mu - \delta - \lambda\right)\left[\left(\sigma + \mu + \lambda\right)\left(\xi + \mu + \lambda\right) - \sigma\xi\right] \\ &= 0. \end{aligned} \tag{2.10}$$

From hence, for the roots of characteristic equation given by (2.10), we write

$$\begin{aligned} \lambda_1 &= \beta S_0 - \left(\eta + \mu + \delta\right) \\ &= \left(\eta + \mu + \delta\right)\left(\mathcal{R}_0 - 1\right). \end{aligned}$$

The remaining roots are obtained from the equation

$$\lambda^2 + \left(\xi + \sigma + 2\mu\right)\lambda + \mu\xi + \sigma\mu + \mu^2 = 0.$$

For this quadratic equation,

$$\lambda_2 + \lambda_3 = -\left(\xi + \sigma + 2\mu\right) < 0$$

and

$$\lambda_2\lambda_3 = \mu\left(\xi + \sigma + \mu\right) > 0.$$

While $\mathcal{R}_0 < 1$, all roots of the characteristic equation always have the negative sign. Therefore DFE is locally asymptotically stable for $\mathcal{R}_0 < 1$. □

To prove that the EE is locally asymptotically stable when $\mathcal{R}_0 > 1$, we will use the criteria which is well known in the literature and given by Routh and Hurwitz.

Theorem 2.3. *The endemic equilibrium point EE is locally asymptotically stable in Θ for $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix of system (2.1) at $EE = (S^*, I^*, V^*)$ is

$$J\left(EE\right) = \begin{bmatrix} -\beta I^* - \sigma - \mu & -\beta S^* + \eta & \xi \\ \beta I^* & \beta S^* - \eta - \mu - \delta & 0 \\ \sigma & 0 & -\xi - \mu \end{bmatrix}.$$

Thus, the characteristic equation which is correspond to $J\left(EE\right)$ is

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{2.11}$$

where

$$C_1 = \beta I^* + \xi + \sigma + 2\mu,$$

$$C_2 = \mu\beta I^* + \delta\beta I^* + \mu\xi + \sigma\xi + \beta\xi I^* + \mu^2 + \mu\sigma + \mu\beta I^*$$

and

$$C_3 = \mu\beta\xi I^* + \delta\beta\xi I^* + \mu^2\beta I^* + \mu\delta\beta I^*.$$

Since C_1, C_2, C_3 are positive, we can determine stability of the system (2.1) by using Routh-Hurwitz Criteria. According to this criteria,

$$H_1 = C_1 > 0$$

and

$$H_2 = \frac{C_1C_2 - C_3}{C_1}.$$

After required simplifications, the numerator part of the value H_2 is obtained as

$$\begin{aligned} C_1C_2 - C_3 &= \mu\left(\beta I^*\right)^2 + \delta\left(\beta I^*\right)^2 + \mu\xi\beta I^* + \sigma\xi\beta I^* + \xi\left(\beta I^*\right)^2 + \mu^2\beta I^* + \mu\sigma\beta I^* + \mu\left(\beta I^*\right)^2 + \sigma\mu\beta I^* + \sigma\delta\beta I^* \\ &+\sigma\mu\xi + \sigma^2\xi + \sigma\xi\beta I^* + \sigma\mu^2 + \mu\sigma^2 + \sigma\mu\beta I^* + \mu^2\beta I^* + \mu\delta\beta I^* + 2\mu^2\xi + 2\mu\sigma\xi + 2\mu\beta\xi I^* \\ &+2\mu^3 + 2\mu^2\sigma + 2\mu^2\beta I^* + \mu\beta\xi I^* + \mu\xi^2 + \sigma\xi^2 + \xi^2\beta I^* + \mu^2\xi + \sigma\mu\xi \\ &> 0. \end{aligned}$$

and so

$$H_2 > 0.$$

Finally,

$$H_3 = C_3 > 0.$$

Thus, according to Routh-Hurwitz stability criteria, all eigenvalues of the Jacobian matrix of system (2.1) at the endemic equilibrium point EE, that is, each of the roots of equation (2.11) have negative real parts. Consequently, if $\mathcal{R}_0 > 1$ then the endemic equilibrium $EE = (S^*, I^*, V^*)$, which is unique equilibria for the system (2.1), is locally asymptotically stable. □

2.3 The Effect of Vaccination on the Spread of Disease

When the model is considered without vaccine (in this case, $\sigma = p = 0$ and so $\xi = 0$)) it transforms to SIS epidemic model in the following form:

$$\begin{aligned} \frac{dS}{dt} &= b - \beta S(t)\int_0^{h_1} f(\tau) I(t-\tau)\,d\tau + \eta\int_0^{h_3} k(\gamma) I(t-\gamma)\,d\gamma - \mu S(t), \\ \frac{dI}{dt} &= \beta S(t)\int_0^{h_1} f(\tau) I(t-\tau)\,d\tau - \eta\int_0^{h_3} k(\gamma) I(t-\gamma)\,d\gamma - \delta I(t) - \mu I(t) \end{aligned}$$

and for this model, the basic reproduction number is

$$\breve{\mathcal{R}}_0 = \frac{b\beta}{\mu(\eta + \mu + \delta)}.$$

It can be easily seen that there exists the relationship

$$\mathcal{R}_0 = \left(1 - \frac{\mu p + \sigma}{\xi + \mu + \sigma}\right)\breve{\mathcal{R}}_0$$

between $\mathcal{R}_0$ and $\breve{\mathcal{R}}_0$. Here $\mathcal{R}_0 < \breve{\mathcal{R}}_0$ and this mathematical result indicates that, obviously, vaccination has a crucial effect on disease control by decreasing the basic reproduction number. Thus, with the appropriate vaccination strategy, the disease can be eradicated in the population by keeping the value $\mathcal{R}_0$ below 1.

Several mathematical operations give us:

$$\begin{aligned} \mathcal{R}_0 &< 1 \\ &\Leftrightarrow \left(\breve{\mathcal{R}}_0 - \frac{(\mu p + \sigma)\breve{\mathcal{R}}_0}{\xi + \mu + \sigma}\right) < 1 \\ &\Leftrightarrow (\xi + \mu)\left(\breve{\mathcal{R}}_0 - 1\right) < \sigma + \mu p\breve{\mathcal{R}}_0. \end{aligned}$$

Thus, within the scope of the target of $\mathcal{R}_0 < 1$, the value $p_{\min}$ that comes with the inequality

$$p_{\min} > \frac{(\xi+\mu)\left(\breve{\mathcal{R}}_0 - 1\right) - \sigma}{\mu\breve{\mathcal{R}}_0} \tag{2.12}$$

is the minimum vaccination ratio of new members required for the elimination of the disease in the population. We note obviously that the parameters which define $p_{\min}$ in (2.12) should be chosen such that $0 < p_{\min} < 1$. Also, since the other parameter determined the number of vaccinated individuals is σ, the choosing of parameters $p_{\min}$ and σ should be considered together in (2.12). The result obtained about $p_{\min}$ means that, with increasing of σ and with decreasing of ξ, $\mathcal{R}_0$ decreases and so the spread of the disease gradually decreases in the population. Therefore it is meaningful that the efforts to increasing σ or decreasing ξ. This result will be seen again from the mathematical explanations in a different perspective in the following part.

2.4 Sensitivity Analysis

One of the main objectives of the epidemic investigations is to suggest strategies such that it will ensure that the necessary control measures are taken to stop the epidemic and to prevent possible outbreaks in the future. Attempts intended to reduce the spread of the disease are, of course, in the direction supporting the lowering the value $\mathcal{R}_0$. Considering that there are many negative conditions brought about by the disease, together with the difficulty of completely eliminating the epidemic in a population in a short time, attempts to reduce the spread of the disease are very important. In this sense, with various control measures which will be implemented; lowering the value $\mathcal{R}_0$ is one of the most fundamental issues. Thus, it has a major significance to explore the effect of parameters on the change of $\mathcal{R}_0$ and to apply control measures in this direction. To this, in the followings, we will evaluate the influence aspects of the parameters that affect $\mathcal{R}_0$ by determining the normalized forward sensitivity index of it. The normalized forward sensitivity index of the variable $\mathcal{R}_0$ with respect to the parameter ϑ is defined as

$$Q_{\vartheta}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \vartheta} \times \frac{\vartheta}{\mathcal{R}_0},$$

by using partial derivative. Where ϑ represents the basic parameters constituting $\mathcal{R}_0$. In that case,

$$\mathcal{Q}_{\beta}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \beta} \times \frac{\beta}{\mathcal{R}_0} = 1 > 0$$

and

$$\begin{aligned} \mathcal{Q}_{\xi}^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \xi} \times \frac{\xi}{\mathcal{R}_0} \\ &= \frac{(\sigma+\mu p)\,\xi}{(\xi+\mu+\sigma)\,(\xi+\mu\,(1-p))} > 0. \end{aligned}$$

By increasing of these parameters that have additive effect on the spread of disease, $\mathcal{R}_0$ increases and so the disease gets out of control in the population. Therefore, the control measures which will be established should be aimed at reducing of the parameters β and ξ.

Now let us concentrate to the effect of parameters related to vaccine on $\mathcal{R}_0$. If we calculate, the normalized forward sensitivity index taking account of the derivatives of $\mathcal{R}_0$ with respect to p and σ, we get

$$\begin{aligned} \mathcal{Q}_{p}^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0} \\ &= -\frac{\mu p}{\xi+\mu\,(1-p)} < 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_{\sigma}^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \sigma} \times \frac{\sigma}{\mathcal{R}_0} \\ &= -\frac{\sigma}{\xi+\mu+\sigma} < 0. \end{aligned}$$

Thus the disease can be eliminated with some favorable and adequate vaccination strategies. For example, one of the necessary conditions for disease elimination is given in the result of mathematical calculation in (2.12). Improvements in these two parameters that depend on the efficacy of vaccines may lead to disease eradication.

On the other hand

$$\begin{aligned} \mathcal{Q}_{\eta}^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \eta} \times \frac{\eta}{\mathcal{R}_0} \\ &= -\frac{\eta}{\eta + \mu + \delta} < 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{Q}_{\delta}^{\mathcal{R}_0} &= \frac{\partial \mathcal{R}_0}{\partial \delta} \times \frac{\delta}{\mathcal{R}_0} \\ &= -\frac{\delta}{\eta + \mu + \delta} < 0. \end{aligned}$$

The parameters η and δ that its sensitivity indices are negative will bring about the decrease in $\mathcal{R}_0$. Therefore, strategies and actions developed on these two parameters will be useful in order that the spread of disease enters a downward course.

3. Concluding Remarks

While expressing dynamic systems mathematically, nonlinear and moreover delayed differential equations are needed to construct closer models to reality in the expression of complex phenomena. Because of the fact that nonlinearity and the existence of delay in a system may lead to being much more complex of analysis and control of the system, in particular, studying with nonlinear differential equations with delays is quite coercive mathematically.

All these difficulties aside, the dynamic analysis of nonlinear systems is often examined by looking at the local stability of the system. To reach conclusions related to local stabilities, it is needed to look at the linearized equivalent of any equilibrium point of the nonlinear system. Thus it can be reached a conclusion about the local dynamics of the system.

In this paper, a new mathematical epidemic model under the vaccine effect is constructed. Also asymptotic behaviors of solutions by evaluating the local stabilities of equilibrium points for mentioned model are examined.

Subsequently, in order to evaluate the support provided by the vaccine during the spread of the disease, the model has been considered as vaccinated and unvaccinated, and it has been seen that the vaccination has a crucial effect on disease control by decreasing the basic reproduction number with several mathematical operations. Thus, with the appropriate vaccination strategy, the disease can be eradicated in the population by keeping the value $\mathcal{R}_0$ below 1. Also, within the scope of the target of $\mathcal{R}_0 < 1$, a result about the minimum vaccination ratio of new members required for the elimination of the disease in the population has been obtained.

Also in this part, the effects on $\mathcal{R}_0$ of the parameters σ and p which represents the vaccination rate of susceptible individuals and of the parameter ξ which the losing rate of protective effect provided by the vaccine have been determined; and the control measures which will can be applied on these parameters have been interpreted.

One of the main objectives of the epidemic investigations is to suggest strategies such that it will ensure that the necessary control measures are taken to decrease and if it is possible to stop the epidemic and to prevent possible outbreaks in the future. Attempts intended to reduce the spread of the disease are, of course, in the direction supporting the lowering the value $\mathcal{R}_0$. In this context, the reducing and enhancing effects of the parameters used in the model on the value $\mathcal{R}_0$ have been interpreted mathematically and suggestions were made to implement control measures in this direction.

Nowadays, with the advancement of science, the desires and efforts of individuals have been increased in solving and analyzing more complex problems. In this sense, the various nonlinear dynamic systems have been formed to explain the more complex mechanisms in the struggle against epidemics and it have been examined the stability behaviors of these new models. As a matter of course, the several details such as adding some different compartments (exposed, asymptomatic infectious, etc.) or adding some parameters reflecting various control measures (isolation etc.) may be considered to carry forward this model.

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Author's contributions

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