MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

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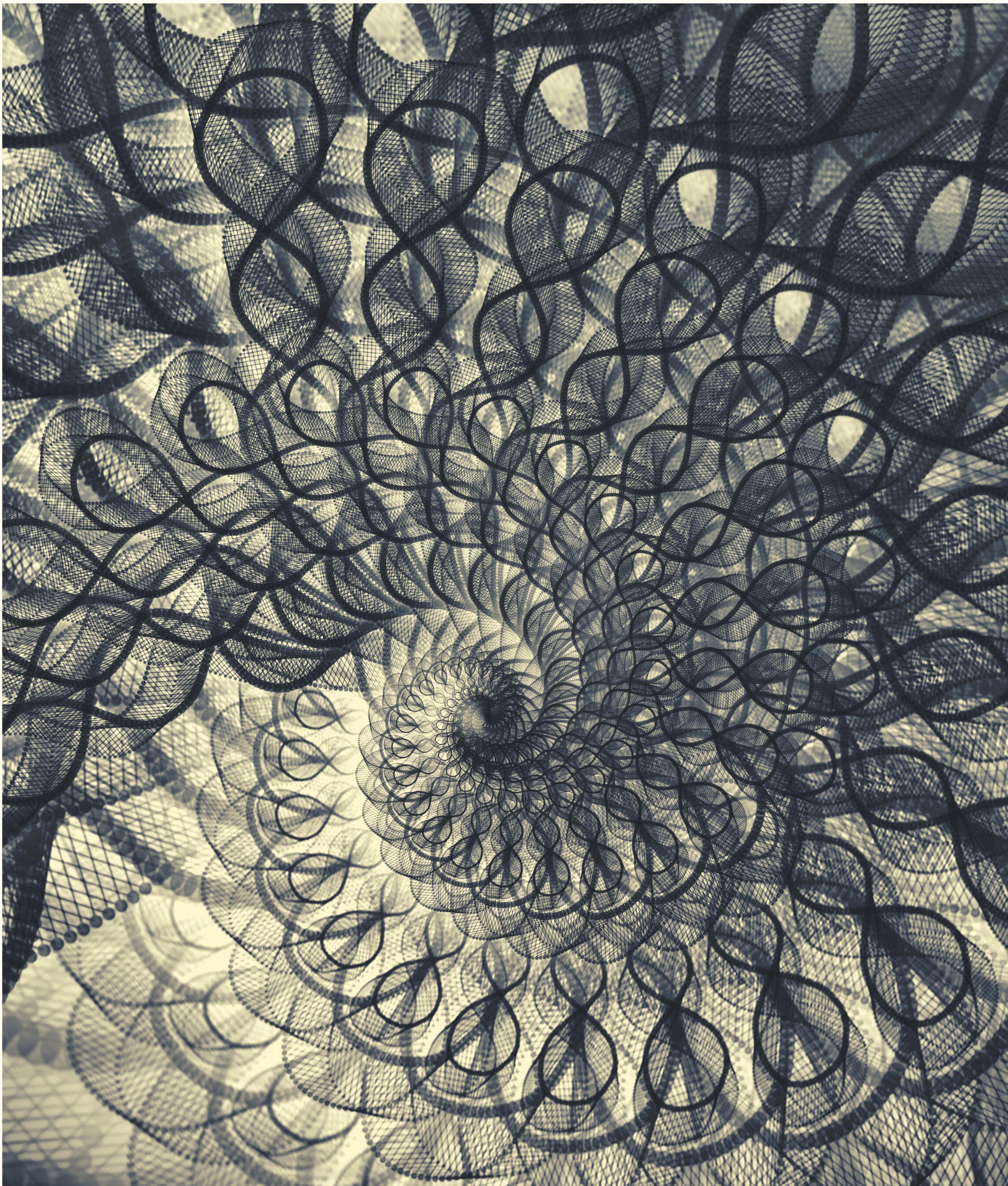

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(m_1, m_2)-Geometric Arithmetically Convex Functions and Related Inequalities

Mahir Kadakal

Abstract
In this manuscript, we introduce and study the concept of (m_1, m_2)-geometric arithmetically (GA) convex functions and their some algebric properties. In addition, we obtain Hermite-Hadamard type inequalities for the newly introduced this type of functions whose derivatives in absolute value are the class of (m_1, m_2)-GA-convex functions by using both well-known power mean and Hölder's integral inequalities.

Keywords: *Convex function; m-convex function; (m_1, m_2)-GA convex function; Hermite-Hadamard inequality.*

AMS Subject Classification (2020): *26A51; 26D10; 26D15.*

1. Preliminaries and fundamentals

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. Hermite-Hadamard integral inequality is very important in the convexity theory. Readers can find more informations in [1–6, 8, 9, 12, 13, 16] and references therein regarding both convexity theory and H-H integral inequalities.

Definition 1.1 ([10, 11])**.** $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is called GA-convex on I if

$$f\left(a^{\xi} b^{1-\xi}\right) \leq \xi f(a) + (1-\xi) f(b)$$

holds for all $a, b \in I$ and $\xi \in [0, 1]$.

Definition 1.2 ([14])**.** $f : [0, b] \to \mathbb{R}$ is called m-convex for $m \in (0, 1]$ if the following inequality

$$f(\xi x_1 + m(1-\xi)x_2) \leq \xi f(x_1) + m(1-\xi)f(x_2)$$

holds for all $x_1, x_2 \in [0, b]$ and $\xi \in [0, 1]$.

Definition 1.3 ([7])**.** $f : [0, b] \to \mathbb{R}$, $b > 0$, is caled (m_1, m_2)-convex function, if

$$f(m_1\xi\theta + m_2(1-\xi)\vartheta) \leq m_1\xi f(\theta) + m_2(1-\xi)f(\vartheta)$$

for all $\theta, \vartheta \in I$, $\xi \in [0, 1]$ and $(m_1, m_2) \in (0, 1]^2$.

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The purpose of this manuscript is to give the concept of (m_1, m_2)-geometric arithmetically (GA) convex functions and find some results connected with new inequalities similar to the well-known H-H inequality for these classes of functions.

2. Some properties of (m_1, m_2)-GA convex functions

Here, we will definite a new concept, which is called (m_1, m_2)-GA convex functions and we give by setting some algebraic properties for the (m_1, m_2)-GA convex functions.

Definition 2.1. Let the function $f : [0, b] \to \mathbb{R}$ and $(m_1, m_2) \in (0, 1]^2$. If

$$f\left(a^{m_1 t} b^{m_2(1-t)}\right) \leq m_1 t f(a) + m_2 (1-t) f(b). \tag{2.1}$$

for all $[a, b] \subset [0, b]$ and $t \in [0, 1]$, then the function f is called (m_1, m_2)-GA convex function, if this inequality reversed, then the function f is called (m_1, m_2)-GA concave function.

We discuss some connections between the class of the (m_1, m_2)-GA convex functions and other classes of generalized convex functions.

Remark 2.1. When $m_1 = m_2 = 1$, the (m_1, m_2)-GA convex (concave) function becomes a GA convex (concave) function in defined [10, 11].

Remark 2.2. When $m_1 = 1$, $m_2 = m$, the (m_1, m_2)-GA convex (concave) function becomes the (α, m)-GA convex (concave) function defined in [15].

Proposition 2.1. *$f : I \subset (0, \infty) \to \mathbb{R}$ is (m_1, m_2)-GA convex on $I \iff f \circ \exp : \ln I \to \mathbb{R}$ is (m_1, m_2)-convex on the interval $\ln I = \{\ln x | \, x \in I\}$.*

Proof. $(\Rightarrow)$ Suppose $f : I \subset (0, \infty) \to \mathbb{R}$ is (m_1, m_2)-GA convex function. Then, we get

$$\begin{aligned} (f \circ \exp)(m_1 t \ln a + m_2(1-t) \ln b) &\leq m_1 t (f \circ \exp)(\ln a) + m_2(1-t)(f \circ \exp)(\ln b) \\ f\left(a^{m_1 t} b^{m_2(1-t)}\right) &\leq m_1 t f(a) + m_2 (1-t) f(b). \end{aligned}$$

Therefore, the function $f \circ \exp$ is (m_1, m_2)-convex function on $\ln I$.

$(\Leftarrow)$ Let $f \circ \exp : \ln I \to \mathbb{R}$, (m_1, m_2)-convex function on $\ln I$. Then, we get

$$\begin{aligned} f\left(a^{m_1 t} b^{m_2(1-t)}\right) &= f\left(e^{m_1 t \ln a + m_2(1-t) \ln b}\right) \\ &= (f \circ \exp)(m_1 t \ln a + m_2(1-t) \ln b) \\ &\leq m_1 t f\left(e^{\ln a}\right) + m_2 (1-t) f\left(e^{\ln b}\right) \\ &= m_1 t f(a) + m_2 (1-t) f(b). \end{aligned}$$

□

Theorem 2.1. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$. If f and g are (m_1, m_2)-geometric arithmetically convex functions, then*
(i) $f + g$ is an (m_1, m_2)-geometric arithmetically convex function,
(ii) For $c \in \mathbb{R}$ ($c \geq 0$) cf is an (m_1, m_2)-geometric arithmetically convex function.

Proof. (i) Let f, g be (m_1, m_2)-geometric arithmetically convex functions, then

$$\begin{aligned} (f+g)\left(a^{m_1 t} b^{m_2(1-t)}\right) &= f\left(a^{m_1 t} b^{m_2(1-t)}\right) + g\left(a^{m_1 t} b^{m_2(1-t)}\right) \\ &\leq m_1 t f(a) + m_2(1-t) f(b) + m_1 t g(a) + m_2(1-t) g(b) \\ &= m_1 t (f+g)(a) + m_2(1-t)(f+g)(b) \end{aligned}$$

(ii) Let f be (m_1, m_2)-GA convex function and $c \in \mathbb{R}$ ($c \geq 0$), then

$$\begin{aligned} (cf)\left(a^{m_1 t} b^{m_2(1-t)}\right) &\leq c\left[m_1 t f(x) + m_2(1-t) f(y)\right] \\ &= m_1 t (cf)(x) + m_2(1-t)(cf)(y). \end{aligned}$$

□

Theorem 2.2. *Let $f, g : I \to \mathbb{R}$ are nonnegative and monotone increasing. If f and g are (m_1, m_2)-GA convex functions, then fg is (m_1, m_2)-GA convex function.*

Proof. If $\vartheta_1 \le \vartheta_2$ ($\vartheta_2 \le \vartheta_1$ is similar) then

$$f(\vartheta_1)g(\vartheta_2) + f(\vartheta_2)g(\vartheta_1) \le f(\vartheta_1)g(\vartheta_1) + f(\vartheta_2)g(\vartheta_2). \tag{2.2}$$

Therefore, for $a, b \in I$ and $t \in [0, 1]$,

$$\begin{aligned}(fg)\left(a^{m_1 t} b^{m_2(1-t)}\right) &= f\left(a^{m_1 t} b^{m_2(1-t)}\right) g\left(a^{m_1 t} b^{m_2(1-t)}\right) \\ &\le [m_1 t f(a) + m_2(1-t) f(a)]\,[m_1 t g(a) + m_2(1-t) g(b)] \\ &= m_1 m_1 t^2 f(a)g(a) + m_1 m_2 t(1-t) f(a)g(b) + m_2 m_1 t(1-t) f(b)g(a) \\ &\quad + m_2 m_2 (1-t)^2 f(b)g(b) \\ &= m_1^2 t^2 f(a)g(a) + m_1 m_2 t(1-t)\,[f(b)g(a) + f(a)g(b)] + m_2^2(1-t)^2 f(b)g(b).\end{aligned}$$

Using now the inequality (2.2), we obtain,

$$\begin{aligned}(fg)\,(m_1 t a + m_2(1-t)b) &\le m_1^2 t^2 f(a)g(a) + m_1 m_2 t(1-t)\,[f(a)g(a) + f(b)g(b)] \\ &\quad + m_2^2(1-t)^2 f(b)g(b) \\ &= m_1 t\,[m_1 t + m_2(1-t)]\, f(a)g(a) + m_2(1-t)\,[m_1 t + m_2(1-t)]\, f(b)g(b).\end{aligned}$$

Since $m_1 t + m_2(1-t) \le m \le 1$, where $m = \max\{m_1, m_2\}$. Therefore, we get

$$\begin{aligned}(fg)\,(m_1 t a + m_2(1-t)b) &\le m_1 t f(a)g(a) + m_2(1-t) f(b)g(b) \\ &= m_1 t\,(fg)\,(a) + m_2(1-t)\,(fg)\,(b).\end{aligned}$$

□

Theorem 2.3. *Let $b > 0$ and $f_\alpha : [a, b] \to \mathbb{R}$ be an arbitrary family of (m_1, m_2)-geometric arithmetically convex functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $J = \{u \in [a, b] : f(u) < \infty\}$ is nonempty, then J is an interval and f is an (m_1, m_2)-geometric arithmetically convex function on J.*

Proof. Let $t \in [0, 1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned}f\left(a^{m_1 t} b^{m_2(1-t)}\right) &= \sup_\alpha f_\alpha \left(a^{m_1 t} b^{m_2(1-t)}\right) \\ &\le \sup_\alpha\,[m_1 t f_\alpha(a) + m_2(1-t) f_\alpha(b)] \\ &\le m_1 t \sup_\alpha f_\alpha\,(a) + m_2(1-t) \sup_\alpha f_\alpha\,(b) \\ &= m_1 t f\,(a) + m_2(1-t) f\,(b) < \infty.\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that f is an (m_1, m_2)-GA convex on J. □

Theorem 2.4. *If the function $f : [a^{m_1}, b^{m_2}] \to \mathbb{R}$ is an (m_1, m_2)-GA, then f is bounded on the interval $[a^{m_1}, b^{m_2}]$.*

Proof. Let $M = \max\{m_1 f(a), m_2 f(b)\}$ and $x \in [a^{m_1}, b^{m_2}]$ is an arbitrary point. Then there exist a $t \in [0, 1]$ such that $x = a^{m_1 t} b^{m_2(1-t)}$. Thus, since $m_1 t \le 1$ and $m_2(1-t) \le 1$ we have

$$f(x) = f\left(a^{m_1 t} b^{m_2(1-t)}\right) \le m_1 t f(a) + m_2(1-t) f(b) \le M.$$

Also, for every $x \in [a^{m_1}, b^{m_2}]$ there exist a $\lambda \in \left[\sqrt{\frac{a^{m_1}}{b^{m_2}}}, 1\right]$ such that $x = \lambda\sqrt{a^{m_1} b^{m_2}}$ and $x = \frac{\sqrt{a^{m_1} b^{m_2}}}{\lambda}$. Without loss of generality we can suppose $x = \lambda\sqrt{a^{m_1} b^{m_2}}$. So, we get

$$f\left(\sqrt{a^{m_1} b^{m_2}}\right) = f\left(\sqrt{\left[\lambda\sqrt{a^{m_1} b^{m_2}}\right]\left[\frac{\sqrt{a^{m_1} b^{m_2}}}{\lambda}\right]}\right) \le \frac{1}{2}\left[f\,(x) + f\left(\frac{\sqrt{a^{m_1} b^{m_2}}}{\lambda}\right)\right].$$

Using M as the upper bound, we get

$$f(x) \geq 2f\left(\sqrt{a^{m_1}b^{m_2}}\right) - f\left(\frac{\sqrt{a^{m_1}b^{m_2}}}{\lambda}\right) \geq 2f\left(\sqrt{a^{m_1}b^{m_2}}\right) - M = m.$$

□

3. Hermite-Hadamard inequality for (m_1, m_2)-GA-convex function

In this section, we will obtain some inequalities of similar to the H-H type integral inequalities for (m_1, m_2)-GA-convex.

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be an (m_1, m_2)-GA-convex function. If $a < b$ and $f \in L[a, b]$, then the following H-H type integral inequalities hold:*

$$f\left(\sqrt{a^{m_1}b^{m_2}}\right) \leq \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du \leq \frac{m_1 f(a)}{2} + \frac{m_2 f(b)}{2}. \tag{3.1}$$

Proof. Firstly, from the property of the (m_1, m_2)-GA convex function of f, we get

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &= f\left(\sqrt{a^{m_1 t}b^{m_2(1-t)}a^{m_1(1-t)}b^{m_2 t}}\right) \\ &\leq \frac{f\left(a^{m_1 t}b^{m_2(1-t)}\right) + f\left(a^{m_1(1-t)}b^{m_2 t}\right)}{2}. \end{aligned}$$

Now, if we take integral in the above inequality with respect to $t \in [0, 1]$, we deduce that

$$\begin{aligned} f\left(\sqrt{a^{m_1}b^{m_2}}\right) &\leq \frac{1}{2}\int_0^1 f\left(a^{m_1 t}b^{m_2(1-t)}\right) dt + \frac{1}{2}\int_0^1 \left(a^{m_1(1-t)}b^{m_2 t}\right) dt \\ &= \frac{1}{2}\left[\frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du + \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du\right] \\ &= \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du. \end{aligned}$$

Secondly, from the property of the (m_1, m_2)-GA convex function of f, if the variable is changed as $u = a^{m_1 t}b^{m_2(1-t)}$, then

$$\begin{aligned} \frac{1}{\ln b^{m_2} - \ln a^{m_1}} \int_{a^{m_1}}^{b^{m_2}} \frac{f(u)}{u} du &= \int_0^1 f\left(a^{m_1 t}b^{m_2(1-t)}\right) dt \\ &\leq \int_0^1 [m_1 t f(a) + m_2(1-t) f(b)]\, dt \\ &= m_1 f(a) \int_0^1 t dt + m_2 f(b) \int_0^1 (1-t) dt \\ &= \frac{m_1 f(a)}{2} + \frac{m_2 f(b)}{2}. \end{aligned}$$

□

4. Some new inequalities for (m_1, m_2)-GA convex functions

The aim of this section is to establish new estimates that refine Hermite-Hadamard integral inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is (m_1, m_2)-GA convex function. Ji et al. [15] used the following lemma:

Lemma 4.1 ([15]). *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be differentiable function and $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then*

$$\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x) dx = \frac{\ln b - \ln a}{2} \int_0^1 a^{3(1-t)} b^{3t} f'\left(a^{1-t}b^t\right) dt.$$

Theorem 4.1. *Let the function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|$ is (m_1, m_2)-GA convex on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1, m_2] \in (0, 1]^2$, then the following integral inequalities hold*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{m_1}{6}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left[L\left(a^3, b^3\right) - a^3\right] + \frac{m_2}{6}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left[b^3 - L\left(a^3, b^3\right)\right], \quad (4.1)$$

where L is the logarithmic mean.

Proof. By using Lemma 4.1 and the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right| = \left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right| \leq m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right| + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|,$$

we get

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(a^{1-t}b^t\right)\right| dt \\
\leq\ & \frac{\ln(b/a)}{2}\int_0^1 a^{3(1-t)}b^{3t}\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right| + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\right] dt \\
=\ & m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\frac{\ln(b/a)}{2}\int_0^1 (1-t)a^{3(1-t)}b^{3t}dt + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\frac{\ln(b/a)}{2}\int_0^1 t a^{3(1-t)}b^{3t}dt \\
=\ & m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\frac{\ln(b/a)}{2}\left[\frac{b^3 - a^3 - a^3\left(\ln b^3 - \ln a^3\right)}{\left(\ln b^3 - \ln a^3\right)^2}\right] + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\frac{\ln(b/a)}{2}\left[\frac{b^3\left(\ln b^3 - \ln a^3\right) - \left(b^3 - a^3\right)}{\left(\ln b^3 - \ln a^3\right)^2}\right] \\
=\ & \frac{m_1}{6}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left[L\left(a^3, b^3\right) - a^3\right] + \frac{m_2}{6}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left[b^3 - L\left(a^3, b^3\right)\right].
\end{aligned}$$

□

Corollary 4.1. *By considering the conditions of Theorem 4.1, If we take $m_1 = m$ and $m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{m}{6}\left|f'\left(a^{\frac{1}{m}}\right)\right|\left[L\left(a^3, b^3\right) - a^3\right] + \frac{1}{6}\left|f'(b)\right|\left[b^3 - L\left(a^3, b^3\right)\right].$$

Corollary 4.2. *By considering the conditions of Theorem 4.1, If we take $m_1 = m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{|f'(a)|}{6}\left[L\left(a^3, b^3\right) - a^3\right] + \frac{|f'(b)|}{6}\left[b^3 - L\left(a^3, b^3\right)\right].$$

Theorem 4.2. *Let the function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^q$ is (m_1, m_2)-GA convex on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1, m_2] \in (0, 1]^2$ and $q \geq 1$ then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{\left(b^3 - a^3\right)^{1-\frac{1}{q}}}{6}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^3, b^3\right) - a^3\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^3 - L\left(a^3, b^3\right)\right)\right]^{\frac{1}{q}},$$

where L is the logarithmic mean.

Proof. By using Lemma 4.1, power mean inequality and the (m_1, m_2)-GA convexity of $|f'|^q$ on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, that is, the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right| = \left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q \leq m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q,$$

we get

$$\begin{aligned}
&\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}\left[\int_0^1 a^{3(1-t)}b^{3t}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}\left[\int_0^1 a^{3(1-t)}b^{3t}\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)}b^{3t}dt\right]^{1-\frac{1}{q}}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\int_0^1(1-t)a^{3(1-t)}b^{3t}dt+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\int_0^1 t a^{3(1-t)}b^{3t}dt\right]^{\frac{1}{q}} \\
=\ & \frac{\left(b^3-a^3\right)^{1-\frac{1}{q}}}{6}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^3,b^3\right)-a^3\right)+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^3-L\left(a^3,b^3\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 4.3. *By considering the conditions of Theorem 4.2, If we take $q=1$, then,*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \leq \left[\frac{m_1}{6}\left|f'\left(a^{\frac{1}{m_1}}\right)\right|\left(L\left(a^3,b^3\right)-a^3\right)+\frac{m_2}{6}\left|f'\left(b^{\frac{1}{m_2}}\right)\right|\left(b^3-L\left(a^3,b^3\right)\right)\right].$$

This inequality coincides with the inequality (4.1).

Corollary 4.4. *By considering the conditions of Theorem 4.2, If we take $m_1=m$ and $m_2=1$, then,*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \leq \frac{\left(b^3-a^3\right)^{1-\frac{1}{q}}}{6}\left[m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q\left(L\left(a^3,b^3\right)-a^3\right)+\left|f'(b)\right|^q\left(b^3-L\left(a^3,b^3\right)\right)\right]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [15].

Theorem 4.3. *Let the function $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable function and $f'\in L([a,b])$ for $0<a<b<\infty$. If $|f'|^q$ is (m_1,m_2)-GA convex on $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1,m_2]\in(0,1]^2$ and $q>1$, then,*

$$\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \leq \frac{\ln(b/a)}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)A^{\frac{1}{q}}\left(m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q, m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right)$$

where L is the logarithmic mean, A is the arithmetic mean and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 4.1, Hölder inequality and the (m_1,m_2)-GA-convexity of the function $|f'|^q$ on the interval $\left[0,\max\left\{a^{\frac{1}{m_1}},b^{\frac{1}{m_2}}\right\}\right]$, that is, the inequality

$$\left|f'\left(a^{1-t}b^t\right)\right|=\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q\leq m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q,$$

we get

$$\begin{aligned}
&\left|\frac{b^2 f(a)-a^2 f(b)}{2}-\int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1\left(a^{3(1-t)}b^{3t}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q+m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3p(1-t)}b^{3pt}dt\right]^{\frac{1}{p}}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\int_0^1(1-t)dt+m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\int_0^1 t\,dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}L^{\frac{1}{p}}\left(a^{3p},b^{3p}\right)A^{\frac{1}{q}}\left(m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q, m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right).
\end{aligned}$$

□

Corollary 4.5. *By considering the conditions of Theorem 4.3, If we take $m_1 = m$ and $m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{\ln(b/a)}{2} L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) A^{\frac{1}{q}}\left(m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q, |f'(b)|^q\right)$$

Corollary 4.6. *By considering the conditions of Theorem 4.3, If we take $m_1 = m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{\ln(b/a)}{2} L^{\frac{1}{p}}\left(a^{3p}, b^{3p}\right) A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right)$$

Theorem 4.4. *Let the function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function and $f' \in L([a, b])$ for $0 < a < b < \infty$. If $|f'|^q$ is (m_1, m_2)-GA convex on $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$ for $[m_1, m_2] \in (0, 1]^2$ and $q > 1$, then the following integral inequalities hold*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}$$
$$\times\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}},$$

where L is the logarithmic mean and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 4.1, Hölder inequality and the (m_1, m_2)-GA-convexity of the function $|f'|^q$ on the interval $\left[0, \max\left\{a^{\frac{1}{m_1}}, b^{\frac{1}{m_2}}\right\}\right]$, we get

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln(b/a)}{2}\left(\int_0^1 1dt\right)^{\frac{1}{p}}\left[\int_0^1 a^{3q(1-t)} b^{3qt}\left|f'\left(\left(a^{\frac{1}{m_1}}\right)^{m_1(1-t)}\left(b^{\frac{1}{m_2}}\right)^{m_2 t}\right)\right|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{\ln(b/a)}{2}\left[\int_0^1 a^{3(1-t)q} b^{3tq}\left[m_1(1-t)\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q + m_2 t\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\right]dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln(b/a)}{2}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\int_0^1(1-t)a^{3q(1-t)}b^{3qt}dt + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\int_0^1 t a^{3q(1-t)}b^{3qt}dt\right]^{\frac{1}{q}} \\
=\ & \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}\left[m_1\left|f'\left(a^{\frac{1}{m_1}}\right)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + m_2\left|f'\left(b^{\frac{1}{m_2}}\right)\right|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 4.7. *By considering the conditions of Theorem 4.4, If we take $m_1 = m$ and $m_2 = 1$, then,*

$$\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \leq \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}$$
$$\times\left[m\left|f'\left(a^{\frac{1}{m}}\right)\right|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + |f'(b)|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}}.$$

Corollary 4.8. *By considering the conditions of Theorem 4.3, If we take $m_1 = m_2 = 1$, then,*

$$\begin{aligned}
&\left|\frac{b^2 f(a) - a^2 f(b)}{2} - \int_a^b x f(x)dx\right| \\
\leq\ & \frac{\ln^{1-\frac{1}{q}}(b/a)}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}}\left[|f'(a)|^q\left(L\left(a^{3q}, b^{3q}\right) - a^{3q}\right) + |f'(b)|^q\left(b^{3q} - L\left(a^{3q}, b^{3q}\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

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Approximation for q-Chlodowsky Operators via Statistical Convergence with Respect to Power Series Method *

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Abstract
Many results which are obtained or unable to obtained by classical calculus have also been studied by q-calculus. It is effective to use q-calculus since it acts as a bridge between mathematics and physics. The q-analog of Chlodowsky operators has been introduced and the approximation properties of these operators have been studied in [12]. Then in [23], the q-analog of Stancu-Chlodowsky operators has been introduced and some approximation results of these operators have been studied via A-statistical convergence which is a more general setting. In this paper, we present the approximation properties of q-Chlodowsky operators via statistical convergence with respect to power series method. It is noteworthy to mention that statistical convergence and statistical convergence with respect to power series method are incompatible.

Keywords: *q-calculus; Chlodowsky operators; approximation theory; power series method; statistical convergence.*

AMS Subject Classification (2020): *Primary: 40G10; Secondary: 41A36.*

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1. Introduction and Preliminaries

In approximation theory, Bernstein operators have different applications. With the use of these operators, it is possible to give an understandable and easy proof of Weierstrass's theorem. This is the most important application of these operators. The classical Bernstein operators have been introduced and discussed in detailed in [1, 2, 5, 13, 15, 25]. Since q-calculus acts as a bridge between mathematics and physics, the q-analog of Bernstein operators have been introduced by Lupaş [16]. Different type of q-Bernstein operators has also been introduced by Phillips [21] and Ostrovska [19] have investigated the approximation properties of these operators. Karsli and Gupta [12] have introduced q-Chlodowsky operators which extend q-Bernstein operators to an unbounded interval.

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The important results in approximation theory have also been studied by using different concepts of convergences such as statistical convergence, ideal convergence, summation process [3, 4, 8, 11, 20, 24]. It is effective to use these concepts since they make a nonconvergent sequence to converge.

In the present paper, we study the approximation properties of q-Chlodowsky operators via statistical convergence with respect to power series method. In [26], such examples have been provided to show that statistical convergence and statistical convergence with respect to power series method do not imply each other.
This paper is organized as follows:
The first section is devoted to basic definitions, notations and also well known results. The second section is devoted to our main results and in the third section we will give an application.

Now, let us recall basic definitions, notations and also the well known results which we need throughout the paper.

The density of the subset $E \subseteq \mathbb{N}_0$ is given by

$$\delta(E) := \lim_{n\to\infty} \frac{1}{n+1} |\{j \leqslant n : j \in E\}|$$

whenever the limit exists where the vertical bars indicate the cardinality of enclosed set and $\mathbb{N}_0$ is the set of all nonnegative integers. A sequence $x = (x_j)$ is called statistically convergent to L [9, 10, 22] if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{j \leqslant n : |x_j - L| \geqslant \varepsilon\}| = 0$$

that is, $\delta(E_\varepsilon) = 0$ for any $\varepsilon > 0$ where $E_\varepsilon = \{j \in \mathbb{N}_0 : |x_j - L| \geqslant \varepsilon\}$.

By assuming that (p_j) is nonnegative real sequence such that $p_0 > 0$ and the corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leqslant \infty$. Now the definition of power series method is as follows :

Let

$$C_p := \left\{ f : (-R, R) \to \mathbb{R} | \lim_{0<t\to R^-} \frac{1}{p(t)} f(t) \quad \text{exists} \right\}$$

and

$$C_{P_p} := \left\{ x = (x_k) |\, p_x(t) := \sum_{j=0}^{\infty} p_j t^j x_j \quad \text{has radius of convergence} \geqslant R \quad \text{and} \quad p_x \in C_p \right\}.$$

The functional $P_p - \lim : C_{P_p} \to \mathbb{R}$ (for short P_p) defined by

$$P_p - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and x is said to be P_p-convergent [6], [14].

A power series method P_p is said to be regular if $P_p - \lim x = L$ provided that $\lim x = L$ [6].

By combining these concepts, Ünver and Orhan [26] have recently introduced P_p-statistical convergence and have proved a Korovkin type theorem for a sequence of positive linear operators defined on $C[0, 1]$, the space of all continuous functions on the interval $[0, 1]$.
Now let us recall the statistical convergence with respect to power series method, i.e., P_p-statistical convergence.
Let P_p be a regular power series method and $E \subseteq \mathbb{N}_0$. If the limit

$$\delta_{P_p}(E) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in E} p_j t^j$$

exists then $\delta_{P_p}(E)$ is called the P_p-density of E. Notice that by the definition of a power series method and P_p-density it is obvious that $0 \leqslant \delta_{P_p}(E) \leqslant 1$ whenever it exists [26].
Let $x = (x_j)$ be a real sequence and let P_p be a regular power series method. Then x is said to be P_p-statistically convergent to L if for any $\varepsilon > 0$

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j\in E_\varepsilon} p_j t^j = 0$$

that is, $\delta_{P_p}(E_\varepsilon) = 0$ for any $\varepsilon > 0$. In the case we write $st_{P_p} - \lim x = L$ [26].

Before recalling the q-Chlodowsky operators, it is useful to mention certain properties of q-calculus. For any fixed real number $q > 0$ and nonnegative integer r, the q-integer of the number r is defined by

$$[r]_q = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1. \end{cases}$$

The q-factorial is defined by

$$[r]_q! = \begin{cases} [r]_q[r-1]_q \cdots [1]_q, & r = 1, 2, \ldots, \\ 1, & r = 0 \end{cases}$$

and q-binomial coefficient can be defined as

$$\begin{bmatrix} k \\ r \end{bmatrix}_q = \frac{[k]_q!}{[r]_q![k-r]_q!}$$

for integers $k \geqslant r \geqslant 0$ and q-binomial coefficients satisfy the following properties:

$$\begin{bmatrix} k+1 \\ r \end{bmatrix}_q = q^{k-r+1} \begin{bmatrix} k \\ r-1 \end{bmatrix}_q + \begin{bmatrix} k \\ r \end{bmatrix}_q$$

and

$$\begin{bmatrix} k+1 \\ r \end{bmatrix}_q = \begin{bmatrix} k \\ r-1 \end{bmatrix}_q + q^r \begin{bmatrix} k \\ r \end{bmatrix}_q.$$

The q-analog of $(1-a)^n$ is the polynomial

$$(1-a)_q^n = \begin{cases} 1, & n = 0 \\ \prod_{s=0}^{n-1}(1-q^s a), & n \geqslant 1. \end{cases}$$

The Bernstein-Chlodowsky operators were defined by Chlodowsky on an unbounded set in 1937 [7] as follows:

$$C_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right) \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

where $0 \leq x \leq b_n$, (b_n) is a positive increasing sequence with the properties that, $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{n} = 0$. The q-Bernstein operators have also been defined by [21] and have been studied by many researchers (see e.g. [17, 18, 27] etc.)

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x).$$

With the same motivation in the classical procedure, the q-Chlodowsky operators have been defined as:

$$C_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}b_n\right) \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right)$$

where $0 \leq x \leq b_n$, (b_n) is a positive increasing sequence with the property that $\lim_{n\to\infty} b_n = \infty$.
In [12] the following theorem has been obtained for q-Chlodowsky operators.

Theorem 1. *$C_{n,q}$ operators satisfy the following equalities*

$$\begin{aligned} C_{n,q}(1;x) &= 1, \\ C_{n,q}(t;x) &= x, \\ C_{n,q}(t^2;x) &= x^2 + \frac{x(b_n - x)}{[n]_q}. \end{aligned}$$

From Theorem 1 and by direct computations, we have the following equalities:

$$C_{n,q}((t-x)^2;x) = \frac{x(b_n - x)}{[n]_q}, \quad C_{n,q}((t-x);x) = 0.$$

One can easily observe that $[n]_q \to \dfrac{1}{1-q}$ as $n \to \infty$ for $0 < q < 1$ and it implies that $C_{n,q}(t^2;x)$ and $C_{n,q}((t-x)^2;x)$ do not converge to x^2 and 0 respectively, as $n \to \infty$.

In order to overcome this difficulty, we replace q by (q_n) where (q_n) is a sequence of real numbers such that $0 < q_n < 1$,

$$st_{P_p} - \lim q_n = 1$$

and

$$st_{P_p} - \lim \frac{b_n}{[n]_{q_n}} = 0.$$

Now let us recall the modulus of continuity of f, $\omega(f, \delta)$ is defined by

$$\omega(f,\delta) = \sup_{\substack{|x-y|\leq \delta \\ x,y\in[0,B]}} |f(x) - f(y)|.$$

It is well known that for a function $f \in C[0, B]$,

$$\lim_{\delta\to 0^+} \omega(f,\delta) = 0,$$

and for any $\lambda > 0$

$$\omega(f, \lambda\delta) \leq (1+\lambda)\omega(f,\delta).$$

2. Main Results

In this section, we present our main results which fill the gaps in the existing literature. First of all, we recall the following theorem which states the necessary and sufficient condition for the convergence of a sequence of positive linear operators on $C[0, 1]$.

Theorem 2. *[26] Let P_p be a regular power series method and let (L_n) be a sequence of positive linear operators on $C[a, b]$ such that for $e_i(x) = x^i, i = 0, 1, 2$*

$$st_{P_p} - \lim \|L_n e_i - e_i\| = 0$$

then for any $f \in C[a, b]$, we have

$$st_{P_p} - \lim \|L_n f - f\| = 0.$$

Under the light of Theorem 2, we are ready to present and prove the following:

Theorem 3. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$, $st_{P_p} - \lim q_n = 1$. Then for any $f \in C[0, \infty)$, we have*

$$st_{P_p} - \lim \|C_{n,q_n}(f) - f\|_{C[0,B]} = 0$$

where B is positive real number.

Proof. Using Theorem 1 and Theorem 2 and since $st_{P_p} - \lim q_n = 1$ and $st_{P_p} - \lim \dfrac{b_n}{[n]_{q_n}} = 0$, we obtain the desired result. This completes the proof. □

Theorem 4. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $st_{P_p} - \lim q_n = 1$. If $f \in C[0, \infty)$, then we have*

$$|C_{n,q_n}(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{\frac{x(b_n - x)}{[n]_{q_n}}}\right).$$

Proof. By using simple calculations, we have

$$
\begin{aligned}
|C_{n,q_n}(f;x)-f(x)| &= \left|\sum_{k=0}^{n} f\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n\right)\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}-f(x)\right| \\
&= \left|\sum_{k=0}^{n}\left[f\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n\right)-f(x)\right]\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right| \\
&\le \sum_{k=0}^{n}\left|f\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n\right)-f(x)\right|\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\le \sum_{k=0}^{n}\omega\left(f,\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\right)\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&= \sum_{k=0}^{n}\omega\left(f,\frac{\delta\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|}{\delta}\right)\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\le \sum_{k=0}^{n}\left(1+\frac{\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|}{\delta}\right)\omega(f,\delta)\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&= \sum_{k=0}^{n}\omega(f,\delta)\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\quad +\sum_{k=0}^{n}\frac{\omega(f,\delta)}{\delta}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&= \omega(f,\delta)\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\quad +\frac{\omega(f,\delta)}{\delta}\sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}.
\end{aligned}
$$

It is also well known that

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}=\left[\frac{x}{b_n}+\left(1-\frac{x}{b_n}\right)\right]_{q_n}^{n}=1$$

and by using Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k} \\
&\le \sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right|\left\{\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2}\left\{\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2} \\
&\le \left\{\sum_{k=0}^{n}\left(\frac{[k]_{q_n}}{[n]_{q_n}}b_n-x\right)^2\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2}\left\{\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{q_n}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_{q_n}^{n-k}\right\}^{1/2} \\
&= \left\{\frac{x(b_n-x)}{[n]_{q_n}}\right\}^{1/2}.
\end{aligned}
$$

Then

$$|C_{n,q_n}(f;x)-f(x)|\le\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{\frac{x(b_n-x)}{[n]_{q_n}}\right\}^{1/2}$$

holds. By taking $\delta = \left\{ \frac{x(b_n - x)}{[n]_{q_n}} \right\}^{1/2}$, we have

$$|C_{n,q_n}(f;x) - f(x)| \leq 2\omega(f,\delta) = 2\omega\left(f, \sqrt{\frac{x(b_n - x)}{[n]_{q_n}}}\right).$$

This completes the proof. □

Theorem 5. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $st_{P_p} - \lim q_n = 1$. If $f \in C[0,b_n]$, then we have*

$$|C_{n,q_n}(f;x_0) - f(x_0)| \leq 2\omega\left(f, \sqrt{\frac{x_0 b_n}{[n]_{q_n}}}\right)$$

where $x_0 \in [0,b_n]$ and x_0 is a fixed point.

Proof. The validity of the following is obvious:

$$\frac{x(b_n - x)}{[n]_{q_n}} \leq \frac{x_0 b_n}{[n]_{q_n}}$$

for any fixed point x_0. One can obtain the remaining part in a similar way in [12]. Therefore we omit the details here. □

Theorem 6. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $st_{P_p} - \lim q_n = 1$. If $f \in C[0,\infty)$, then we have, for sufficiently large n*

$$\|C_{n,q_n}(f) - f\|_{C[0,b_n]} \leq 2\omega\left(f, \sqrt{\frac{B b_n}{[n]_{q_n}}}\right)$$

where $B > 0$ is a constant being appeared in Theorem 3.

In [23], the q-analog of the Stancu type Bernstein-Chlodowsky operators have been introduced as follows:

$$C_{n,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q + [\alpha]_q}{[n]_q + [\beta]_q} b_n\right) \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

where $0 \leq x \leq b_n$, (b_n) is a positive increasing sequence with the property that $\lim_{n\to\infty} b_n = \infty$ and α, β are positive integers such that $0 \leqslant \alpha \leqslant \beta$.
Observe that when we take $\alpha = \beta = 0$, q-analog of the Stancu type Bernstein-Chlodowsky operators coincide with q-Bernstein-Chlodowsky operators.

In [23] the following theorem has been obtained for the Stancu type Bernstein-Chlodowsky operators.

Theorem 7. *The followings are satisfied for $C_{n,q}^{\alpha,\beta}$;*

$$\begin{aligned} C_{n,q}^{\alpha,\beta}(1;x) =& 1, \\ C_{n,q}^{\alpha,\beta}(t;x) =& \frac{[n]_q}{[n]_q + [\beta]_q} x + \frac{[\alpha]_q}{[n]_q + [\beta]_q} b_n, \\ C_{n,q}^{\alpha,\beta}(t^2;x) =& \frac{{[n]_q}^2}{([n]_q + [\beta]_q)^2}\left(x^2 + \frac{x(b_n - x)}{[n]_q}\right) + \frac{2[\alpha]_q[\beta]_q}{([n]_q + [\beta]_q)^2} + \frac{{[n]_q}^2}{([n]_q + [\beta]_q)^2} {b_n}^2. \end{aligned}$$

Since

$$st_{P_p} - \lim \frac{b_n}{[n]_{q_n}} = 0,\ 0 < q_n < 1 \text{ and } st_{P_p} - \lim q_n = 1,$$

we can also obtain analogous results those given in [23] in a similar manner for the concept of P_p-statistical convergence.

Recall that for $f \in C[a,b]$ and $t > 0$, the Peetre-K Functional is defined by

$$K(f,\delta) := \inf_{g \in C^2[a,b]} \{\|f-g\|_{C[a,b]} + t\|g\|_{C^2[a,b]}\}$$

where $C^2[a,b] = \{f \in C[a,b] : f', f'' \in C[a,b]\}$, with the norm

$$\|g\|_{C^2[a,b]} := \|g\|_{C[a,b]} + \|g'\|_{C[a,b]} + \|g''\|_{C[a,b]}.$$

It is obtained in [12] that for $g \in C^2[0,b_n]$, then we have

$$|C_{n,q}(g;x) - g(x)| \leqslant \frac{x(b_n - x)}{2[n]_q}\|g\|_{C^2[0,b_n]}.$$

Theorem 8. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and*

$$st_{P_p} - \lim q_n = 1.$$

If $f \in C[0,\infty)$ and $B > 0$ is a constant, then we have

$$\|C_{n,q_n}(f) - f\|_{C[0,b_n]} \leqslant 2K\left(f, \frac{Bb_n}{2[n]_{q_n}}\right).$$

Proof. From [12], it is known that

$$|C_{n,q_n}(f;x) - f(x)| \leqslant \|f-g\|_{C[0,b_n]}|C_{n,q_n}(1;x)| + \|f-g\|_{C[0,b_n]} + |C_{n,q_n}(g;x) - g(x)|$$

and

$$|C_{n,q_n}(f;x) - f(x)| \leqslant 2\|f-g\|_{C[0,b_n]} + \frac{x(b_n - x)}{2[n]_q}\|g\|_{C^2[0,b_n]}$$

and hence

$$|C_{n,q_n}(f;x) - f(x)| \leqslant 2\|f-g\|_{C[0,b_n]} + \frac{Bb_n}{2[n]_{q_n}}\|g\|_{C^2[0,b_n]}.$$

Taking infimum over all $g \in C^2[0,b_n]$, we obtain the desired result which completes the proof. □

Also recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be uniform Lipschitz continuous of order $\gamma > 0$ if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leqslant M|x-y|^\gamma$$

for any x and y in $\mathbb{R}$. In this case, we write $f \in Lip(\gamma, \mathbb{R})$.

Theorem 9. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and*

$$st_{P_p} - \lim q_n = 1.$$

If $f \in Lip_M[0,b_n]$ and $x \in [0,B]$, $B > 0$ is a constant, then we have

$$\|C_{n,q_n}(f) - f\|_{C[0,b_n]} \leqslant M\left\{\frac{Bb_n}{[n]_{q_n}}\right\}^{\dfrac{\alpha}{2}}.$$

Proof. The proof follows in a similar manner used in [12]. Therefore we omit the details here. □

Theorem 10. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and*

$$st_{P_p} - \lim q_n = 1.$$

Also let $\omega(\delta)$ is the modulus of continuity of f on $[0,B]$ and $f(x)$ has continuous derivative as $f'(x)$, then we have

$$|C_{n,q_n}(f;x) - f(x)| \leqslant M\sqrt{\frac{b_n}{[n]_{q_n}}}\,\omega\left(f, \sqrt{\frac{b_n}{[n]_{q_n}}}\right).$$

3. Conclusions

In this section, we provide an example such that the sequence (q_n) satisfies neither the conditions of the results obtained in [12] nor the conditions of the results obtained in [23].

Example 1. *Let the sequences (p_n) and (q_n) defined as follows:*

$$p_n = \begin{cases} 1 & , \quad n = 2k \\ 0 & , \quad n = 2k+1 \end{cases}, \qquad q_n = \begin{cases} 0 & , \quad n = 2k+1 \\ 1 - \frac{1}{n} & , \quad n = 2k \end{cases}.$$

It is easy to see that the method P_p is regular and one can easily see that

$$\delta_{P_p}(E_\varepsilon) = 0$$

where $E_\varepsilon = \{n \in \mathbb{N}_0 : |q_n - 1| \geqslant \varepsilon\}$ holds for every $\varepsilon > 0$. That is $st_{P_p} - \lim q_n = 1$. Notice that (q_n) is not convergent in the ordinary sense or statistically convergent.
In [26], a sequence of positive linear operators has been presented which satisfies neither the conditions of Theorem 1 of [11] nor the conditions of the classical Korovkin theorem (Theorem 4 of [11]) but it satisfies the conditions of Theorem 5 of [11].

Here it is remarkable to mention that our results cannot be deduced from the results in [23] since (q_n) is not convergent or statistically convergent.

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Soft Quasilinear Operators

Hacer Bozkurt

Abstract
In this paper, we have introduced a new concept, called soft quasilinear operator over soft quasilinear spaces which extends the notion of quasilinear operator. Also, we studied some properties of soft quasilinear operators with illustrating examples. Further, we have defined inverse of a soft quasilinear operator and its some different properties from inverse of soft linear operators are obtained.

Keywords: *Quasilinear space; Soft quasilinear space; Normed quasilinear space; Soft normed quasilinear space; Quasilinear operator; Soft quasilinear operator.*

AMS Subject Classification (2020): *Primary: 06B99 ; Secondary: 03E72; 08A72; 06F99; 04A99; 04B40.*

1. Introduction

In 1986, Aseev [1] introduced the concept of quasilinear spaces, normed quasilinear spaces and quasilinear operators which are generalization of the linear spaces, normed linear spaces and linear operators, respectively. Additionally, in [2], [3], [4], [5], [6], [7], [8], the authors introduced some new concepts and results on quasilinear spaces. Recently, in [9], Yılmaz et all. introduced the notion of inner product quasilinear space and investigated some basic properties of inner product quasilinear spaces. Also, in [10], Levent and Yılmaz deal with bounded quasilinear interval-valued functions and analized the Hahn Banach extension theorem for interval valued functions.

Molodtsov [11] initiated a new theory of linear functional analysis by starting the theory of soft sets. Then, Maji et all. [12], [13] introduced several operations on soft sets. After that, many research works have been done in soft set theory such as [14], [15], [16] . Also, Das and Samanta introduced the idea of soft linear spaces in [17]. Next, Samanta et all. [18], [19] presented some new concepts about the soft set theory such as soft convex set, soft semi norm, soft Minkowski's functionals on a soft linear space and soft pseudo metric.

Based on our studies with related to quasilinear spaces and studies of Samanta and Das, in [21], Bozkurt defined soft quasilinear spaces and soft normed quasilinear spaces which are generalization of the soft linear spaces and soft normed linear spaces, respectively. In the same study, Bozkurt obtained new results about soft quasilinear spaces.

In this paper, we have introduced a concept of soft quasilinear operator over soft quasilinear spaces which extends the notion of quasilinear operator. Also, we studied some properties of soft quasilinear operators with illustrating examples. Further, we have defined inverse of a soft quasilinear operator and its some different properties from inverse of soft linear operators are obtained.

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2. Preliminaries

Firstly, we give the definition of quasilinear space, normed quaslinear space and some its basic properties given by Aseev [1]. After, we give the concepts of soft quasilinear space and soft normed quaslinear space given by [21]. Now, let's continue with the definition of Aseev:

Definition 2.1. [1] A quasilinear space over a field $\mathbb{R}$ is a set Q with a partial order relation "$\preceq$", with the operations of addition $Q \times Q \to Q$ and scalar multiplication $\mathbb{R} \times Q \to Q$ satisfying the following conditions:
$(Q1)$ $q \preceq q$,
$(Q2)$ $q \preceq z$ if $q \preceq w$ and $w \preceq z$,
$(Q3)$ $q = w$ if $q \preceq w$ and $w \preceq q$,
$(Q4)$ $q + w = w + q$,
$(Q5)$ $q + (w + z) = (q + w) + z$,
$(Q6)$ there exists an element $\theta \in Q$ such that $q + \theta = q$,
$(Q7)$ $\alpha \cdot (\beta \cdot q) = (\alpha \cdot \beta) \cdot q$,
$(Q8)$ $\alpha \cdot (q + w) = \alpha \cdot q + \alpha \cdot w$,
$(Q9)$ $1 \cdot q = q$,
$(Q10)$ $0 \cdot q = \theta$,
$(Q11)$ $(\alpha + \beta) \cdot q \preceq \alpha \cdot q + \beta \cdot q$,
$(Q12)$ $q + z \preceq w + v$ if $q \preceq w$ and $z \preceq v$,
$(Q13)$ $\alpha \cdot q \preceq \alpha \cdot w$ if $q \preceq w$,
for every $q, w, z, v \in Q$ and every $\alpha, \beta \in \mathbb{R}$.

If an element q has an inverse, then it is called *regular*. If an element q has no inverse, then it is called *singular*. Also, Q_r express for the set of all regular elements in Q and Q_s imply the sets of all singular elements in Q. Besides, Q_r, Q_d and $Q_s \cup \{0\}$ are subspaces of Q, where Q_r regular subspace of Q, Q_d symmetric subspace of Q and $Q_s \cup \{0\}$ singular subspace of Q [2].

Definition 2.2. [1] Let Q be a quasilinear space. A function $\|.\|_Q : Q \longrightarrow \mathbb{R}$ is named a norm if the following circumstances hold:
$(NQ1)$ $\|q\|_Q > 0$ if $q \neq 0$,
$(NQ2)$ $\|q + w\|_Q \leq \|q\|_Q + \|w\|_Q$,
$(NQ3)$ $\|\alpha \cdot q\|_Q = |\alpha| \cdot \|q\|_Q$,
$(NQ4)$ if $q \preceq w$, then $\|q\|_Q \leq \|w\|_Q$,
$(NQ5)$ if for any $\varepsilon > 0$ there exists an element $q_\varepsilon \in Q$ such that, $q \preceq w + q_\varepsilon$ and $\|q_\varepsilon\|_Q \leq \varepsilon$ then $q \preceq w$ for any elements $q, w \in Q$ and any real number $\alpha \in \mathbb{R}$.

A quasilinear space Q is called normed quasilinear space with a norm defined on it. Let Q be a normed quasilinear space. Then, Hausdorff or norm metric on Q is defined by

$$h_Q(q, w) = \inf \{r \geq 0 : q \preceq w + a_1^r, w \preceq q + a_2^r, \|a_i^r\| \leq r\}.$$

Definition 2.3. [1] Let Q and W be quasilinear spaces. Then a quasilinear operator $\lambda : Q \to W$ is a function satisfying
$(QO1)$ $\lambda(\alpha \cdot q) = \alpha \cdot \lambda(q)$,
$(QO2)$ $\lambda(q + w) \preceq \lambda(q) + \lambda(w)$,
$(QO3)$ $\lambda(q) \preceq \lambda(w)$ if $q \preceq w$ for any $q, w \in Q$ and $\alpha \in \mathbb{R}$.

Definition 2.4. [11] Let U be an universe and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is mapping given by $F : A \to P(U)$. A soft set (F, E) over U is said to be absolute soft set denoted by $\widetilde{U}$ if for all $\varepsilon \in E$, $F(\varepsilon) = U$.

Definition 2.5. [20] Let X be a non-empty set and E be a non-empty parameter set. Then a function $\varepsilon : E \to X$ is said to be a soft element of X. A soft element ε of X is said to belongs to a soft set A of X, which is denoted by $\varepsilon \widetilde{\in} A$, if $\varepsilon(e) \in A(e), \forall e \in E$.

Now, we will give the notion of soft quasilinear space, soft normed quasilinear space, soft quasi vector and some results related this notions.

Definition 2.6. [21] Let (G, P) be a non-null soft set over a quasilinear space Q. Then (G, P) is called a soft quasilinear space over Q if $G(p)$ is a subquasilinear space of Q for every $p \in Supp(G, P)$.

Definition 2.7. [21] Let (G, P) is a soft quasilinear space of Q. A soft element of Q is said to be a soft quasi vector of (G, P). A soft element of the soft set $(\mathbb{R}, P)$ is said to be a soft scalar.

Definition 2.8. [21] Let (G, P) is a soft quasilinear space of Q over $\mathbb{R}$ and $(F, P) \subseteq (G, P)$ is a soft set over (G, P). Then (F, P) is called a soft subquasilinear space of (G, P) whenever (F, P) is quasilinear space with identical partial ordering and identical operations on Q.

Proposition 2.1. *[21] Let (G, P) be a soft quasilinear space over Q. Then*
a) $0 \cdot q = \Theta$, for all $q \in (G, P)$,
b) $k \cdot \Theta = \Theta$, for all soft scalar k,
c) $(-1) \cdot q = -q$, for all $q \in (G, P)$.

Let Q be a quasilinear space, Q is also our initial uiverse set and P be the non-empty set of parameters. Let $\widetilde{Q}$ be the absolute soft quasilinear space i.e., $G(p) = Q$, $\forall p \in P$, where $(G, P) = \widetilde{Q}$. Let $SQV\left(\widetilde{Q}\right)$ be the collection all soft quasi vectors over $\widetilde{Q}$. We use the notation $\widetilde{q}, \widetilde{w}$ to denote soft quasi vectors of a soft quasilinear space and $\widetilde{\alpha}$ to denote soft real numbers whereas $\overline{\alpha}$ will denote a particular type of soft real numbers such that $\overline{\alpha}(\lambda) = \alpha$, for all $\lambda \in P$.

Theorem 2.1. *[21] The set $SQV\left(\widetilde{Q}\right)$ is a quasilinear space with the relation "$\widetilde{\preceq}$"*

$$\widetilde{q}\widetilde{\preceq}\widetilde{w} \Leftrightarrow \widetilde{q}(\lambda) \preceq \widetilde{w}(\lambda)$$

the sum operation

$$(\widetilde{q} + \widetilde{w})(\lambda) = \widetilde{q}(\lambda) + \widetilde{w}(\lambda)$$

and the soft real-scalar multiplication

$$(\widetilde{\alpha} \cdot \widetilde{q})(\lambda) = \widetilde{\alpha}(\lambda) \cdot q(\lambda)$$

for every $\widetilde{q}, \widetilde{w}$ soft vectors of $SQV\left(\widetilde{Q}\right)$, $\forall \lambda \in P$ and for every soft real numbers $\widetilde{\alpha}$.

Definition 2.9. [21] Let $SQV\left(\widetilde{Q}\right)$ be a soft quasilinear space and $\widetilde{N} \subset SQV\left(\widetilde{Q}\right)$ be a subset. If $\widetilde{N}$ is a soft quasilinear space, then $\widetilde{N}$ is said to be a soft quasilinear subspace of $SQV\left(\widetilde{Q}\right)$ and stated by $SQV\left(\widetilde{N}\right) \subset SQV\left(\widetilde{Q}\right)$.

Definition 2.10. [21] Let $SQV\left(\widetilde{Q}\right)$ be a soft quasilinear space. Then a mapping $\|.\| : SQV\left(\widetilde{Q}\right) \to \mathbb{R}^+(\mathbb{R})$ is said to be a soft norm on the soft quasilinear space $SQV\left(\widetilde{Q}\right)$, if $\|.\|$ satisfies the following conditions:

(SNQ1) $\|\widetilde{q}\| \widetilde{>} \widetilde{0}$ if $\widetilde{q} \neq \widetilde{\theta}$ for all $\widetilde{q} \in SQV\left(\widetilde{Q}\right)$,

(SNQ2) $\|\widetilde{q} + \widetilde{w}\| \widetilde{\leq} \|\widetilde{q}\| + \|\widetilde{w}\|$ for all $\widetilde{q}, \widetilde{w} \in SQV\left(\widetilde{Q}\right)$,

(SNQ3) $\|\widetilde{\alpha} \cdot \widetilde{q}\| = |\widetilde{\alpha}| \, \|\widetilde{q}\|$ for every $\widetilde{q} \in SQV\left(\widetilde{Q}\right)$ and for every soft scalar $\widetilde{\alpha}$,

(SNQ4) if $\widetilde{q}\widetilde{\preceq}\widetilde{w}$, then $\|\widetilde{q}\| \widetilde{\leq} \|\widetilde{w}\|$ for all $\widetilde{q}, \widetilde{w} \in SQV\left(\widetilde{Q}\right)$,

(SNQ5) if for any $\widetilde{\epsilon} \widetilde{>} \widetilde{0}$ there exists an element $\widetilde{z}$ $SQV\left(\widetilde{Q}\right)$ such that $\widetilde{q}\widetilde{\preceq}\widetilde{w} + \widetilde{z}$ and $\|\widetilde{z}\| \widetilde{\leq} \widetilde{\epsilon}$ then $\widetilde{q}\widetilde{\preceq}\widetilde{w}$.

Definition 2.11. [21] Let $\left(\widetilde{Q}, \|.\|\right)$ be a soft normed quasilinear space. Soft Hausdorff metric or soft norm metric on $\widetilde{Q}$ is defined by equality

$$h_{\widetilde{Q}}(\widetilde{q}, \widetilde{w}) = \inf\left\{\widetilde{r} \geq \widetilde{0} : \widetilde{q}\widetilde{\preceq}\widetilde{w} + \widetilde{a_1^r}, \widetilde{w}\widetilde{\preceq}\widetilde{q} + \widetilde{a_2^r}, \left\|\widetilde{a_i^r}\right\| \widetilde{\leq} \widetilde{r}\right\}.$$

Same as the definition of Hausdorff metric on normed quasilinear space, we obtain $\widetilde{q}\widetilde{\preceq}\widetilde{w} + (\widetilde{q} - \widetilde{w})$ and $\widetilde{w}\widetilde{\preceq}\widetilde{q} + (\widetilde{w} - \widetilde{q})$ for every $\widetilde{q}, \widetilde{w} \in SQV\left(\widetilde{Q}\right)$.

$$h_{\widetilde{Q}}(\widetilde{q}, \widetilde{w}) \widetilde{\leq} \|\widetilde{q} - \widetilde{w}\|.$$

Here, we should note that $h_{\widetilde{Q}}(\widetilde{q},\widetilde{w})$ may not equal to $\|\widetilde{q}-\widetilde{w}\|$ since $\widetilde{Q}$ is a soft quasilinear space.

Definition 2.12. [21] A sequence of soft elements $\{\widetilde{q}_n\}$ in a soft normed quasilinear space $\left(\widetilde{Q},\|.\|\right)$ is said to be converges to a soft element $\widetilde{q}_0$ if $h_{\widetilde{Q}}(\widetilde{q}_n,\widetilde{q}_0)\to\widetilde{0}$ as $n\to\infty$.

Definition 2.13. [21] A sequence of soft elements $\{\widetilde{q}_n\}$ in a soft normed quasilinear space $\left(\widetilde{Q},\|.\|\right)$ is said to be a Cauchy sequence if corresponding to every $\widetilde{\epsilon}\widetilde{>}\widetilde{0}$, $\exists m\in\mathbb{N}$ such that $h_{\widetilde{Q}}(\widetilde{q}_i,\widetilde{q}_j)\widetilde{\leq}\widetilde{\epsilon}$ for all $i,j>m$ i.e. $h_{\widetilde{Q}}(\widetilde{q}_i,\widetilde{q}_j)\to\widetilde{0}$ as $i,j\to\infty$.

3. Main Results

Let Q and W be two soft quasilinear spaces over field $\mathbb{R}$, P be a nonempty set of parameters, $\widetilde{Q}$ and $\widetilde{W}$ be the corresponding absolute soft quasilinear spaces i.e. $\widetilde{Q}(\lambda)=Q$ and $\widetilde{W}(\lambda)=W$ for every $\lambda\in P$. We use the notations $\widetilde{q},\widetilde{w}$ and $\widetilde{z}$ to denote soft quasi vectors of a soft quasilinear space.

Definition 3.1. Let $\chi: SQV(\widetilde{Q})\to SQV(\widetilde{W})$ be an operator. Then χ is said to be soft quasilinear if
(SQO1) $\chi(\widetilde{q}+\widetilde{w})\widetilde{\preceq}\chi(\widetilde{q})+\chi(\widetilde{w})$,
(SQO2) $\chi(\widetilde{c}\cdot\widetilde{q})=\widetilde{c}\cdot\chi(\widetilde{q})$ for every soft scalar $\widetilde{c}$,
(SQO3) $\widetilde{q}\widetilde{\preceq}\widetilde{w}\Rightarrow\chi(\widetilde{q})\widetilde{\preceq}\chi(\widetilde{w})$,
for every $\widetilde{q},\widetilde{w}\in SQV(\widetilde{Q})$.

Example 3.1. If $\widetilde{Q}$ be a soft normed quasilinear space. Then the identity operator $\chi: SQV(\widetilde{Q})\to SQV(\widetilde{Q})$ such that $\chi(\widetilde{q})=\widetilde{q}$, for every soft quasi element $\widetilde{q}\in\widetilde{Q}$, is a soft quasilinear operator.

Example 3.2. Let $\mathbb{R}(P)$ be the set of all soft real numbers defined over the parameter set P and consider the absolute soft quasi set generated by $\Omega_C(\mathbb{R})$ i.e. $\widetilde{\Omega_C(\mathbb{R})}(\lambda)=\Omega_C(\mathbb{R})$. Let an operator

$$\begin{aligned}\chi &: \mathbb{R}(P)\to SQV(\widetilde{\Omega_C(\mathbb{R})})\\ \widetilde{r} &\to \chi(\widetilde{r})=\widetilde{r}\cdot\widetilde{[1,2]}\end{aligned}$$

for a soft quasi vector $\widetilde{[1,2]}\in\widetilde{\Omega_C(\mathbb{R})}$. For every $\widetilde{r},\widetilde{m}\in\mathbb{R}(P)$, we have

$$\begin{aligned}\chi(\widetilde{r}+\widetilde{m}) &= (\widetilde{r}+\widetilde{m})\cdot\widetilde{[1,2]}\\ &\widetilde{\preceq}\widetilde{r}\cdot\widetilde{[1,2]}+\widetilde{m}\cdot\widetilde{[1,2]}\\ &= \chi(\widetilde{r})+\chi(\widetilde{m}).\end{aligned}$$

For every soft scalar $\widetilde{c}$, we get

$$\begin{aligned}\chi(\widetilde{c}\widetilde{r}) &= \widetilde{c}\widetilde{r}\cdot\widetilde{[1,2]}\\ &= \widetilde{c}\left(\widetilde{r}\cdot\widetilde{[1,2]}\right)\\ &= \widetilde{c}\cdot\chi(\widetilde{r}).\end{aligned}$$

For every $\widetilde{r},\widetilde{m}\in\mathbb{R}(P)$, if $\widetilde{r}=\widetilde{m}$ then $\widetilde{r}\cdot\widetilde{[1,2]}=\widetilde{m}\cdot\widetilde{[1,2]}$ since $\mathbb{R}(P)$ is a soft quasilinear space with relation "=". So, we obtain $\chi(\widetilde{r})\widetilde{\preceq}\chi(\widetilde{m})$.

Definition 3.2. The operator $\chi: SQV(\widetilde{Q})\to SQV(\widetilde{W})$ is said to be continuous at $\widetilde{q}\in\widetilde{Q}$ if for every sequence $\{\widetilde{q}_n\}$ of soft element of $\widetilde{Q}$ with $\widetilde{q}_n\to\widetilde{q}$ as $n\to\infty$, we have $\chi(\widetilde{q}_n)\to\chi(\widetilde{q})$ as $n\to\infty$ i.e., $h(\widetilde{q}_n,\widetilde{q})\to\overline{0}$ as $n\to\infty$ implies $h(\chi(\widetilde{q}_n),\chi(\widetilde{q}))\to\overline{0}$ as $n\to\infty$. If χ is continuous at every soft quasi element of $\widetilde{Q}$, then χ is said to be a continuous quasilinear operator.

Example 3.3. The identity operator given in Example 3.1 is continuous since $h(\chi(\widetilde{q}_n),\chi(\widetilde{q}))=h(\widetilde{q}_n,\widetilde{q})\to\overline{0}$ as $n\to\infty$.

Theorem 3.1. *Let $\widetilde{Q}$ and $\widetilde{W}$ be two soft normed quasilinear spaces. If $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be a soft quasilinear operator, then $\chi\left(\sum\limits_{k=1}^{n} \widetilde{c_k} \cdot \widetilde{q_k}\right) \widetilde{\leq} \sum\limits_{k=1}^{n} \widetilde{c_k}\chi\left(\widetilde{q_k}\right)$, $\widetilde{c_k}$ are soft scalars.*

Proof. For $n = 1$ the inequality is satisfied. We consider that the conclusion is true for $(n-1)$ i.e.,

$$\chi\left(\sum_{k=1}^{n-1} \widetilde{c_k} \cdot \widetilde{q_k}\right) \widetilde{\leq} \sum_{k=1}^{n-1} \widetilde{c_k}\chi\left(\widetilde{q_k}\right), \widetilde{c_k}.$$

From here,

$$\begin{aligned}
\chi\left(\sum_{k=1}^{n} \widetilde{c_k} \cdot \widetilde{q_k}\right) &= \chi\left(\sum_{k=1}^{n-1} \widetilde{c_k} \cdot \widetilde{q_k} + \widetilde{c_n} \cdot \widetilde{q_n}\right) \\
&\widetilde{\leq} \chi\left(\sum_{k=1}^{n-1} \widetilde{c_k} \cdot \widetilde{q_k}\right) + \chi\left(\widetilde{c_n} \cdot \widetilde{q_n}\right) \\
&= \sum_{k=1}^{n-1} \widetilde{c_k} \cdot \chi\left(\widetilde{q_k}\right) + \widetilde{c_n} \cdot \chi\left(\widetilde{q_n}\right) \\
&= \sum_{k=1}^{n} \widetilde{c_k} \cdot \chi\left(\widetilde{q_k}\right).
\end{aligned}$$

□

Definition 3.3. Let $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be a soft quasilinear operator, where $\widetilde{Q}$ and $\widetilde{W}$ are soft normed quasilinear spaces. The operator χ is called bounded if there exists some positive soft real number $\widetilde{N}$ such that for all $\widetilde{q} \in \widetilde{Q}$, $\|\chi\left(\widetilde{q}\right)\| \widetilde{\leq} \widetilde{N}\ \|\widetilde{q}\|$.

Theorem 3.2. *Let $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be a soft quasilinear operator, where $\widetilde{Q}$ and $\widetilde{W}$ are soft normed quasilinear spaces. If χ is bounded then χ is continuous.*

Proof. Assume that χ is bounded. Then there exists a positive soft real number $\widetilde{N}$ such that for all $\widetilde{q} \in \widetilde{Q}$, $\|\chi\left(\widetilde{q}\right)\| \widetilde{\leq} \widetilde{N}$ $\|\widetilde{q}\|$. Let $\widetilde{q_n} \to \widetilde{q}$ as $n \to \infty$ i.e., for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$\widetilde{q_n} \widetilde{\leq} \widetilde{q} + \widetilde{q_{1n}^{\epsilon}},\ \widetilde{q} \widetilde{\leq} \widetilde{q_n} + \widetilde{q_{2n}^{\epsilon}},\ \left\|\widetilde{q_{in}^{\epsilon}}\right\| \widetilde{\leq} \frac{\widetilde{\epsilon}}{\widetilde{N}}$$

for all $n \geq n_0$. Then

$$\chi\left(\widetilde{q_n}\right) \widetilde{\leq} \chi\left(\widetilde{q}\right) + \chi\left(\widetilde{q_{1n}^{\epsilon}}\right),\ \chi\left(\widetilde{q}\right) \widetilde{\leq} \chi\left(\widetilde{q_n}\right) + \chi\left(\widetilde{q_{2n}^{\epsilon}}\right)$$

and

$$\left\|\chi\left(\widetilde{q_{in}^{\epsilon}}\right)\right\| \widetilde{\leq} \widetilde{N} \left\|\widetilde{q_{in}^{\epsilon}}\right\| \leq \widetilde{\epsilon}.$$

Therefore, $\chi\left(\widetilde{q_n}\right) \to \chi\left(\widetilde{q}\right)$ as $n \to \infty$. So χ is continuous at $\widetilde{q} \in \widetilde{Q}$. Since $\widetilde{q} \in \widetilde{Q}$ is arbitrary, χ is continuous. □

Theorem 3.3. *Suppose a soft quasilinear operator $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$, where $\widetilde{Q}$ and $\widetilde{W}$ are soft normed quasilinear spaces, satisfies the condition: for $\mu \in Q$ and $\lambda \in P$,*

$$\left\{\chi\left(\widetilde{q}\right)(\lambda) : \widetilde{q} \widetilde{\in} \widetilde{Q} \text{ such that } \widetilde{q}(\lambda) = \mu\right\}$$

is a singleton set. Then for each $\lambda \in P$, $\chi_\lambda : Q \to W$ defined by $\chi_\lambda(\mu) = \chi\left(\widetilde{q}\right)(\lambda)$, for all $\mu \in Q$, $\widetilde{q} \widetilde{\in} \widetilde{Q}$ such that $\widetilde{q}(\lambda) = \mu$, is a quasilinear operator.

Proof. From the above condition, χ_λ is well defined for every $\lambda \in P$. Since χ is a soft quasilinear operator, χ_λ satisfies soft quasilinear operator conditions for $\forall \lambda \in P$:

For every $\mu,\nu\in Q$ and soft scalar $\widetilde{c}$, we get

$$\chi_\lambda(\mu+\nu)=\chi\left(\widetilde{q}+\widetilde{q^{\imath}}\right)(\lambda)\,\widetilde{\leq}\,\chi\left(\widetilde{q}\right)(\lambda)+\chi\left(\widetilde{q^{\imath}}\right)(\lambda)=\chi_\lambda(\mu)+\chi_\lambda(\nu),$$
$$\chi_\lambda(\widetilde{c}\cdot\mu)=\chi\left(\widetilde{c}\cdot\widetilde{q}\right)(\lambda)=\widetilde{c}\cdot\chi\left(\widetilde{q}\right)(\lambda)=\widetilde{c}\cdot\chi_\lambda(\mu),$$
$$\mu\leq\nu\Rightarrow\mu=\widetilde{q}\,(\lambda)\,\widetilde{\leq}\,\widetilde{q^{\imath}}(\lambda)=\nu\Rightarrow\chi\left(\widetilde{q}\right)(\lambda)\,\widetilde{\leq}\,\chi\left(\widetilde{q^{\imath}}\right)(\lambda)\Rightarrow\chi_\lambda(\mu)\leq\chi_\lambda(\nu).$$

Therefore, the soft quasilinear operator χ satisfying above condition gives a parametrized family of crisp quasilinear operators. □

Theorem 3.4. *Let $\{\chi_\lambda:Q\to W,\lambda\in P\}$ be a family of crisp quasilinear operators from quasilinear space Q to the quasilinear space W. Then there exists a soft quasilinear operator $\chi:SQV(\widetilde{Q})\to SQV(\widetilde{W})$, defined by $\chi\left(\widetilde{q}\right)(\lambda)=\chi_\lambda(\mu)$ if $\widetilde{q}\,(\lambda)=\mu$ and $\lambda\in P$; which satisfies Theorem 3.3 and $\chi\,(\lambda)=\chi_\lambda$ for every $\lambda\in P$.*

Proof. Let $\widetilde{q}\,\widetilde{\in}\,\widetilde{Q}$ be an arbitrary soft quasi element and $\chi:SQV(\widetilde{Q})\to SQV(\widetilde{W})$, by $\chi\left(\widetilde{q}\right)(\lambda)=\chi_\lambda(\mu)$ if $\widetilde{q}\,(\lambda)=\mu$ for every $\lambda\in P$. Also, $\widetilde{q^{\imath}}\,\widetilde{\in}\,\widetilde{Q}$ be any soft quasi element, $\lambda\in P$ and $\widetilde{q^{\imath}}\,(\lambda)=\nu$. Then, we get

$$\begin{aligned}\chi\left(\widetilde{q}+\widetilde{q^{\imath}}\right)(\lambda) &= \chi\left(\widetilde{q}\,(\lambda)+\widetilde{q^{\imath}}\,(\lambda)\right)\\ &= \chi_\lambda(\mu+\nu)\\ &\widetilde{\leq}\,\chi_\lambda(\mu)+\chi_\lambda(\nu)\\ &= \chi\left(\widetilde{q}\right)(\lambda)+\chi\left(\widetilde{q^{\imath}}\right)(\lambda).\end{aligned}$$

For every soft scalar $\widetilde{c}$, we obtain

$$\begin{aligned}\chi\left(\widetilde{c}\cdot\widetilde{q}\right)(\lambda) &= \chi\left(\widetilde{c}\,(\lambda)\cdot\widetilde{q}\,(\lambda)\right)\\ &= \chi_\lambda(\widetilde{c}\,(\lambda)\cdot\mu)\\ &= \widetilde{c}\,(\lambda)\cdot\chi_\lambda(\mu)\\ &= \widetilde{c}\,(\lambda)\cdot\chi\left(\widetilde{q}\right)(\lambda).\end{aligned}$$

Let us consider $\widetilde{q}\,\widetilde{\leq}\,\widetilde{q^{\imath}}$ such that $\widetilde{q}\,(\lambda)=\mu$ and $\widetilde{q^{\imath}}\,(\lambda)=\nu$ for arbitrary $\widetilde{q},\widetilde{q^{\imath}}\,\widetilde{\in}\,\widetilde{Q}$ and arbitrary $\lambda\in P$. Then, we have $\widetilde{q}\,(\lambda)\,\widetilde{\leq}\,\widetilde{q^{\imath}}\,(\lambda)$ for $\lambda\in P$. From here, we get $\chi_\lambda(\mu)\widetilde{\leq}\chi_\lambda(\nu)$ since χ_λ is a soft quasilinear operator. So, we have $\chi\left(\widetilde{q}\right)(\lambda)\,\widetilde{\leq}\,\chi\left(\widetilde{q^{\imath}}\right)(\lambda)$. Therefore, $\chi:SQV(\widetilde{Q})\to SQV(\widetilde{W})$ is a soft quasilinear operator. □

Lemma 3.1. *Let $\left(\widetilde{Q},\|.\|,P\right)$ be a soft normed quasilinear space and a soft quasi norm $\|.\|$ satisfies the condition:*

For $\mu\in Q$ and $\lambda\in P$, $\{\|\widetilde{q}\|\,(\lambda):\widetilde{q}\,(\lambda)=\mu\}$ is a singleton set.

Then for every $\lambda\in P$, $\|.\|_\lambda:Q\to\mathbb{R}^+$ defined by $\|\mu\|_\lambda=\|\widetilde{q}\|\,(\lambda)$, for every $\mu\in Q$ and $\widetilde{q}\,\widetilde{\in}\,\widetilde{Q}$ such that $\widetilde{q}\,(\lambda)=\mu$, is a quasi norm on Q.

Proof. Let $\|.\|_\lambda:Q\to\mathbb{R}^+$ defined by $\|\mu\|_\lambda=\|\widetilde{q}\|\,(\lambda)$, for every $\mu\in Q$, $\lambda\in P$ and $\widetilde{q}\,\widetilde{\in}\,\widetilde{Q}$ such that $\widetilde{q}\,(\lambda)=\mu$. For every $\mu\in Q$, $\|\mu\|_\lambda=\|\widetilde{q}\|\,(\lambda)\,\widetilde{\geq}\,\overline{0}$. If $\|\mu\|_\lambda=0$, then $\|\widetilde{q}\|\,(\lambda)=\|\widetilde{q}\,(\lambda)\|=\theta=\mu$. For every soft scalar $\widetilde{c}$, we obtain $\|\widetilde{c}\cdot\mu\|_\lambda=\|\widetilde{c}\cdot\widetilde{q}\|\,(\lambda)=\widetilde{c}\cdot\|\widetilde{q}\|\,(\lambda)=\widetilde{c}\cdot\|\mu\|_\lambda$. Also, $\widetilde{q^{\imath}}\,\widetilde{\in}\,\widetilde{Q}$ be any soft quasi element, $\lambda\in P$ and $\widetilde{q^{\imath}}\,(\lambda)=\nu$. Then, for every $\mu,\nu\in Q$, we get $\|\mu+\nu\|_\lambda=\left\|\widetilde{q}+\widetilde{q^{\imath}}\right\|(\lambda)=\left\|\widetilde{q}\,(\lambda)+\widetilde{q^{\imath}}\,(\lambda)\right\|\,\widetilde{\leq}\,\|\widetilde{q}\,(\lambda)\|+\left\|\widetilde{q^{\imath}}\,(\lambda)\right\|=\|\widetilde{q}\|\,(\lambda)+\left\|\widetilde{q^{\imath}}\right\|(\lambda)=\|\mu\|_\lambda+\|\nu\|_\lambda$. If $\mu\leq\nu$, then $\widetilde{q}\,(\lambda)\,\widetilde{\leq}\,\widetilde{q^{\imath}}\,(\lambda)$. Since $\|.\|$ is a soft quasi norm, we obtain $\|\widetilde{q}\|\,(\lambda)\,\widetilde{\leq}\,\left\|\widetilde{q^{\imath}}\right\|(\lambda)$. So, we have $\|\mu\|_\lambda\leq\|\nu\|_\lambda$. Lastly, for every $\epsilon\geq0$ there exist an element $\xi_\epsilon\in Q$ such that $\mu\leq\nu+\xi_\epsilon$ and $\|\xi_\epsilon\|_\lambda\leq\epsilon$. Here, there exist an element $\widetilde{q_\epsilon}\in\widetilde{Q}$ such that $\widetilde{q_\epsilon}\,(\lambda)=\xi_\epsilon$. Thus, we get $\widetilde{q}\,(\lambda)\,\widetilde{\leq}\,\widetilde{q^{\imath}}\,(\lambda)+\widetilde{q_\epsilon}\,(\lambda)$ for $\widetilde{q}\,(\lambda)=\mu$, $\widetilde{q^{\imath}}\,(\lambda)=\nu$ and $\widetilde{q_\epsilon}\,(\lambda)=\xi_\epsilon$. On the other hand, we obtain $\|\xi_\epsilon\|_\lambda=\|\widetilde{q_\epsilon}\|\,(\lambda)=\|\widetilde{q_\epsilon}\,(\lambda)\|\,\widetilde{\leq}\,\widetilde{\epsilon}$ since $\|\xi_\epsilon\|_\lambda\leq\epsilon$. From $\left(\widetilde{Q},\|.\|,P\right)$ is a soft normed quasilinear space, we have $\widetilde{q}\,(\lambda)\,\widetilde{\leq}\,\widetilde{q^{\imath}}\,(\lambda)$. This gives $\mu\leq\nu$. □

Theorem 3.5. *Let $\widetilde{Q}$ and $\widetilde{W}$ be soft normed quasilinear space which for $\mu\in Q$ and $\lambda\in P$, $\{\|\widetilde{q}\|\,(\lambda):\widetilde{q}\,(\lambda)=\mu\}$ is a singleton set. Let $\chi:SQV(\widetilde{Q})\to SQV(\widetilde{W})$ be a soft quasilinear operator satisfying for $\mu\in Q$ and $\lambda\in P$,*

$$\left\{\chi\left(\widetilde{q}\right)(\lambda):\widetilde{q}\,\widetilde{\in}\,\widetilde{Q}\text{ such that }\widetilde{q}\,(\lambda)=\mu\right\}$$

is a singleton set. If χ is continuous then χ is bounded.

Proof. The proof is similar to the soft linear counterpart. □

Let's examine the inverse of a soft quasilinear operator χ. Let $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be a soft quasilinear operator where $\widetilde{Q}$ and $\widetilde{W}$ are soft normed quasilinear spaces. $\left\{\chi(\widetilde{q}) : \widetilde{q}\widetilde{\in}\widetilde{Q}\right\}$ is the range set of χ.

Theorem 3.6. *Let $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be q soft quasilinear operator. If χ^{-1} exists then*

$$\chi(\widetilde{q}) = \theta \textit{ implies } \widetilde{q} = \theta.$$

Proof. Assume χ^{-1} exists i.e. $\chi(\widetilde{q}) = \chi(\widetilde{q'})$ implies $\widetilde{q} = \widetilde{q'}$. Let $\widetilde{q'} = \theta$, then

$$\chi(\widetilde{q}) = \chi(\theta) = \theta$$

implying thereby

$$\widetilde{q} = \theta.$$

But, the converse of above theorem is not true. That is, if $\chi(\widetilde{q}) = \theta$ implies $\widetilde{q} = \theta$, then χ^{-1} may not be exists. Clearly, we know that this requirement exists in linear soft quasilinear spaces, that is, in soft linear spaces. Let's give an example related to soft quasilinear operators and it's inverse. □

Example 3.4. $\mathbb{R}(P)$ be the set of all soft real numbers defined over the parameter set P. Let an operator

$$\begin{array}{ccl} \chi & : & \mathbb{R}(P) \to SQV(\widetilde{\Omega_C(\mathbb{R})}) \\ \widetilde{r} & \to & \chi(\widetilde{r}) = \begin{cases} [-\widetilde{r}, \widetilde{r}] : \widetilde{r} \geq \widetilde{0} \\ [\widetilde{r}, -\widetilde{r}] : \widetilde{r} < \widetilde{0}. \end{cases} \end{array}$$

Clearly, $\chi(\widetilde{r}) \in SQV(\widetilde{\Omega_C(\mathbb{R})})$ for every $\widetilde{r} \in \mathbb{R}(P)$. Now, for every $\widetilde{r}, \widetilde{m} \in \mathbb{R}(P)$:

1) If $\widetilde{r}, \widetilde{m} \geq \widetilde{0}$, then $\widetilde{r} + \widetilde{m} \geq \widetilde{0}$. So, we get

$$\begin{aligned} \chi(\widetilde{r} + \widetilde{m}) &= [-(\widetilde{r} + \widetilde{m}), (\widetilde{r} + \widetilde{m})] \\ &= [-\widetilde{r} - \widetilde{m}, \widetilde{r} + \widetilde{m}] \\ &= [-\widetilde{r}, \widetilde{r}] + [-\widetilde{m}, \widetilde{m}] \\ &= \chi(\widetilde{r}) + \chi(\widetilde{m}). \end{aligned}$$

2) If $\widetilde{r}, \widetilde{m} < \widetilde{0}$, then $\widetilde{r} + \widetilde{m} < \widetilde{0}$. So, we have

$$\begin{aligned} \chi(\widetilde{r} + \widetilde{m}) &= [(\widetilde{r} + \widetilde{m}), -(\widetilde{r} + \widetilde{m})] \\ &= [\widetilde{r} + \widetilde{m}, -\widetilde{r} - \widetilde{m}] \\ &= [\widetilde{r}, -\widetilde{r}] + [\widetilde{m}, -\widetilde{m}] \\ &= \chi(\widetilde{r}) + \chi(\widetilde{m}). \end{aligned}$$

3) Let $\widetilde{r} \geq \widetilde{0}$ and $\widetilde{m} < \widetilde{0}$. If $\widetilde{r} + \widetilde{m} < \widetilde{0}$, then, we get

$$\begin{aligned} \chi(\widetilde{r} + \widetilde{m}) &= [(\widetilde{r} + \widetilde{m}), -(\widetilde{r} + \widetilde{m})] \\ &= [\widetilde{r} + \widetilde{m}, -\widetilde{r} - \widetilde{m}] \\ &\widetilde{\preceq} [\widetilde{r}, -\widetilde{r}] + [\widetilde{m}, -\widetilde{m}] \\ &= \chi(\widetilde{r}) + \chi(\widetilde{m}). \end{aligned}$$

If $\widetilde{r} + \widetilde{m} \geq \widetilde{0}$, then, we get

$$\begin{aligned} \chi(\widetilde{r} + \widetilde{m}) &= [(\widetilde{r} + \widetilde{m}), -(\widetilde{r} + \widetilde{m})] \\ &= [\widetilde{r} + \widetilde{m}, -\widetilde{r} - \widetilde{m}] \\ &\widetilde{\preceq} [\widetilde{r}, -\widetilde{r}] + [\widetilde{m}, -\widetilde{m}] \\ &= \chi(\widetilde{r}) + \chi(\widetilde{m}). \end{aligned}$$

4) Let $\widetilde{r} < \widetilde{0}$ and $\widetilde{m} \geq \widetilde{0}$. If $\widetilde{r} + \widetilde{m} < \widetilde{0}$, then, we get

$$\begin{aligned}
\chi(\widetilde{r} + \widetilde{m}) &= [(\widetilde{r} + \widetilde{m}), -(\widetilde{r} + \widetilde{m})] \\
&= [\widetilde{r} + \widetilde{m}, -\widetilde{r} - \widetilde{m}] \\
&\widetilde{\preceq} [\widetilde{r}, -\widetilde{r}] + [\widetilde{m}, -\widetilde{m}] \\
&= \chi(\widetilde{r}) + \chi(\widetilde{m}).
\end{aligned}$$

If $\widetilde{r} + \widetilde{m} \geq \widetilde{0}$, then, we get

$$\begin{aligned}
\chi(\widetilde{r} + \widetilde{m}) &= [(\widetilde{r} + \widetilde{m}), -(\widetilde{r} + \widetilde{m})] \\
&= [\widetilde{r} + \widetilde{m}, -\widetilde{r} - \widetilde{m}] \\
&\widetilde{\preceq} [\widetilde{r}, -\widetilde{r}] + [\widetilde{m}, -\widetilde{m}] \\
&= \chi(\widetilde{r}) + \chi(\widetilde{m}).
\end{aligned}$$

If $\widetilde{r} \geq \widetilde{0}$, then $\widetilde{c}.\widetilde{r} \geq \widetilde{0}$ for every soft positive scalar $\widetilde{c}$. Thus, we get

$$\chi(\widetilde{c}.\widetilde{r}) = [-\widetilde{c}.\widetilde{r}, \widetilde{c}.\widetilde{r}] = \widetilde{c} \cdot [-\widetilde{r}, \widetilde{r}] = \widetilde{c} \cdot \chi(\widetilde{r}).$$

If $\widetilde{r} < \widetilde{0}$, then $\widetilde{c}.\widetilde{r} < \widetilde{0}$ for every soft positive scalar $\widetilde{c}$. Thus, we get

$$\chi(\widetilde{c}.\widetilde{r}) = [\widetilde{c}.\widetilde{r}, -\widetilde{c}.\widetilde{r}] = \widetilde{c} \cdot [\widetilde{r}, -\widetilde{r}] = \widetilde{c} \cdot \chi(\widetilde{r}).$$

If $\widetilde{r} \geq \widetilde{0}$, then $\widetilde{c}.\widetilde{r} < \widetilde{0}$ for every soft negative scalar $\widetilde{c}$. Thus, we get

$$\chi(\widetilde{c}.\widetilde{r}) = [\widetilde{c}.\widetilde{r}, -\widetilde{c}.\widetilde{r}] = \widetilde{c} \cdot [\widetilde{r}, -\widetilde{r}] = \widetilde{c} \cdot \chi(\widetilde{r}).$$

If $\widetilde{r} < \widetilde{0}$, then $\widetilde{c}.\widetilde{r} \geq \widetilde{0}$ for every soft negative scalar $\widetilde{c}$. Thus, we get

$$\chi(\widetilde{c}.\widetilde{r}) = [-\widetilde{c}.\widetilde{r}, \widetilde{c}.\widetilde{r}] = \widetilde{c} \cdot [-\widetilde{r}, \widetilde{r}] = \widetilde{c} \cdot \chi(\widetilde{r}).$$

If $\widetilde{c} = \widetilde{0}$, then

$$\chi(\widetilde{c}.\widetilde{r}) = \chi\left(\widetilde{0}.\widetilde{r}\right) = \chi\left(\widetilde{\theta}\right) = \left\{\widetilde{0}\right\}.$$

For every $\widetilde{r}, \widetilde{m} \in \mathbb{R}(P)$, if $\widetilde{r} = \widetilde{m}$ then

$$\chi(\widetilde{r}) = \begin{cases} [-\widetilde{r}, \widetilde{r}] : \widetilde{r} \geq \widetilde{0} \\ [\widetilde{r}, -\widetilde{r}] : \widetilde{r} < \widetilde{0}. \end{cases}$$

equal to

$$\chi(\widetilde{m}) = \begin{cases} [-\widetilde{m}, \widetilde{m}] : \widetilde{m} \geq \widetilde{0} \\ [\widetilde{m}, -\widetilde{m}] : \widetilde{m} < \widetilde{0}. \end{cases}$$

Therefore, we obtain $\chi(\widetilde{r})\widetilde{\preceq}\chi(\widetilde{m})$ for every $\widetilde{r}, \widetilde{m} \in \mathbb{R}(P)$. So, the operator χ is a soft quasilinear operator. Further, if $\chi(\widetilde{r}) = \widetilde{\theta}$ then $\widetilde{r} = \widetilde{0}$. But, χ is not an one to one mapping. Because, $-\widetilde{2} \neq \widetilde{2}$ for $-\widetilde{2}, \widetilde{2} \in \mathbb{R}(P)$, but $\chi\left(-\widetilde{2}\right) = \left[-\widetilde{2}, \widetilde{2}\right] = \chi\left(\widetilde{2}\right)$.

Remark 3.1. We know from the soft linear operators, if χ^{-1} exists for a soft linear operator χ, then χ^{-1} is a soft linear. But, this situation may not be true for a soft quasilinear operators. Now, let 's give an example to illlustrate this situation.

Example 3.5. Let $\mathbb{R}(P)$ be the set of all soft real numbers defined over the parameter set P and consider the absolute soft quasi set generated by $\Omega_C(\mathbb{R})$ i.e. $\widetilde{\Omega_C(\mathbb{R})}(\lambda) = \Omega_C(\mathbb{R})$. Let an operator

$$\begin{aligned}
\chi \quad &: \quad \mathbb{R}(P) \to SQV(\widetilde{\Omega_C(\mathbb{R})}) \\
\widetilde{r} \quad &\to \quad \chi(\widetilde{r}) = \widetilde{r} \cdot \widetilde{[-1, 0]}
\end{aligned}$$

for a soft quasi vector $\widetilde{[-1, 0]} \in \widetilde{\Omega_C(\mathbb{R})}$. χ is a soft quasilinear operator. So, χ^{-1} is exists since χ is an one to one soft quasilinear operator. Also, for $\widetilde{1} \cdot \widetilde{[-1, 0]} \in SQV(\widetilde{\Omega_C(\mathbb{R})})$ and $\widetilde{\frac{1}{2}} \cdot \widetilde{[-1, 0]} \in SQV(\widetilde{\Omega_C(\mathbb{R})})$, we have

$$\widetilde{\frac{1}{2}} \cdot \widetilde{[-1,0]} \widetilde{\subseteq} \widetilde{1} \cdot \widetilde{[-1,0]}$$

but

$$\widetilde{\frac{1}{2}} \neq \widetilde{1}.$$

Thus, χ^{-1} is not a soft quasilinear operator since χ^{-1} does not satisfy condition $\widetilde{q} \widetilde{\preceq} \widetilde{w} \Rightarrow \chi(\widetilde{q}) \widetilde{\preceq} \chi(\widetilde{w})$.

Theorem 3.7. *Let $\widetilde{Q}$ and $\widetilde{W}$ be a soft normed quasilinear spaces which satisfy the condition: $\{\|\widetilde{q}\|(\lambda) : \widetilde{q}(\lambda) = \mu\}$ is a singleton set for $\mu \in Q$ and $\lambda \in P$. Let $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be a soft quasilinear operator, χ^{-1} be a continuous soft quasilinear operator and χ^{-1} which satisfy the condition:*

$$\left\{\chi^{-1}(\widetilde{w})(\lambda) : \widetilde{w} \widetilde{\in} \widetilde{W} \text{ such that } \widetilde{w}(\lambda) = \kappa\right\}$$

is a singleton set for $\kappa \in W$ and $\lambda \in P$. Then there exists a soft real number $\widetilde{k} \widetilde{\geq} \widetilde{0}$ such that $\widetilde{k}\,\|\widetilde{q}\| \widetilde{\leq} \|\chi(\widetilde{q})\|$ for every $\widetilde{q} \widetilde{\in} \widetilde{Q}$.

Proof. Assume that χ^{-1} is exists and continuous. We obtain χ^{-1} bounded by Theorem 3.5 since $\{\|\widetilde{q}\|(\lambda) : \widetilde{q}(\lambda) = \mu\}$ is a singleton set for $\mu \in Q$ and $\lambda \in P$ and $\left\{\chi^{-1}(\widetilde{w})(\lambda) : \widetilde{w} \widetilde{\in} \widetilde{W} \text{ such that } \widetilde{w}(\lambda) = \kappa\right\}$ is a singleton set for $\kappa \in W$ and $\lambda \in P$. Thus, there exists a positive soft real number $\widetilde{N} \widetilde{\geq} \widetilde{0}$ such that $\left\|\chi^{-1}(\widetilde{w})\right\| \widetilde{\leq} \widetilde{N}\,\|\widetilde{w}\|$ for every $\widetilde{w} \in \widetilde{W}$. There exists $\widetilde{q} \widetilde{\in} \widetilde{Q}$ such that $\chi^{-1}(\widetilde{w}) = \widetilde{q}$. Thus, we obtain $\|\widetilde{q}\| \widetilde{\leq} \widetilde{N}\,\|\chi(\widetilde{q})\|$. If we take $\widetilde{k} = \frac{1}{\widetilde{N}}$, then we get $\widetilde{k}\,\|\widetilde{q}\| \widetilde{\leq} \|\chi(\widetilde{q})\|$. □

Theorem 3.8. *Let $\widetilde{Q}$ be a soft Banach quasilinear space and $\widetilde{W}$ be a soft normed quasilinear spaces which satisfy the condition: $\{\|\widetilde{q}\|(\lambda) : \widetilde{q}(\lambda) = \mu\}$ is a singleton set for $\mu \in Q$ and $\lambda \in P$. Also, $\chi : SQV(\widetilde{Q}) \to SQV(\widetilde{W})$ be a bijective continuous soft quasilinear operator satisfying the condition: $\left\{\chi(\widetilde{q})(\lambda) : \widetilde{q} \widetilde{\in} \widetilde{Q} \text{ such that } \widetilde{q}(\lambda) = \mu\right\}$ is a singleton set for $\mu \in Q$ and $\lambda \in P$. If χ^{-1} is continuous then $\widetilde{W}$ is a soft Banach quasilinear space.*

Proof. Let $\{\widetilde{w_n}\}$ be a Cauchy sequence in $\widetilde{W}$. Then there exists a $n_0 \in \mathbb{N}$ such that

$$\widetilde{w_n} \widetilde{\leq} \widetilde{w_m} + \widetilde{w_{1n}^{\epsilon}},\ \widetilde{w_m} \widetilde{\leq} \widetilde{w_n} + \widetilde{w_{2n}^{\epsilon}},\ \left\|\widetilde{w_{in}^{\epsilon}}\right\| \widetilde{\leq} \frac{\widetilde{\epsilon}}{\widetilde{N}}$$

for all $n, m \geq n_0$. Since χ^{-1} is continuous

$$\chi^{-1}(\widetilde{w_n}) \widetilde{\leq} \chi^{-1}(\widetilde{w_m}) + \chi^{-1}\left(\widetilde{w_{1n}^{\epsilon}}\right),\ \chi^{-1}(\widetilde{w_m}) \widetilde{\leq} \chi^{-1}(\widetilde{w_n}) + \chi^{-1}\left(\widetilde{w_{2n}^{\epsilon}}\right)$$

i.e.

$$\widetilde{q_n} \widetilde{\leq} \widetilde{q_m} + \widetilde{q_{1n}^{\epsilon}},\ \widetilde{q_m} \widetilde{\leq} \widetilde{q_n} + \widetilde{q_{2n}^{\epsilon}}$$

for every $\chi(\widetilde{q_n}) = \widetilde{w_n}$ for $\widetilde{q_n} \widetilde{\in} \widetilde{Q}$. There exists $\widetilde{q_{in}^{\epsilon}} \widetilde{\in} \widetilde{Q}$ such that $\chi^{-1}\left(\widetilde{w_{in}^{\epsilon}}\right) = \widetilde{q_{in}^{\epsilon}}$. Thus, from the above theorem, we obtain

$$\left\|\widetilde{q_{in}^{\epsilon}}\right\| \widetilde{\leq} \widetilde{N}\left\|\chi\left(\widetilde{q_{1n}^{\epsilon}}\right)\right\| = \widetilde{N}\left\|\widetilde{w_{in}^{\epsilon}}\right\| \widetilde{\leq} \widetilde{\epsilon}.$$

Thus, $\{\widetilde{q_n}\}$ is a Cauchy sequence in $\widetilde{Q}$. Since $\widetilde{Q}$ is complete, $\widetilde{q_n} \to \widetilde{q_0}$ for some $\widetilde{q_0} \in \widetilde{Q}$. So, there exists a $n_0 \in \mathbb{N}$ such that

$$\widetilde{q_n} \widetilde{\leq} \widetilde{q_0} + \widetilde{q_{1n}^{\epsilon}},\ \widetilde{q_0} \widetilde{\leq} \widetilde{q_n} + \widetilde{q_{2n}^{\epsilon}},\ \left\|\widetilde{q_{in}^{\epsilon}}\right\| \widetilde{\leq} \widetilde{\epsilon}$$

for all $n \geq n_0$. Thus, we get

$$\widetilde{w_n} \widetilde{\leq} \widetilde{w_0} + \widetilde{w_{1n}^{\epsilon}},\ \widetilde{w_0} \widetilde{\leq} \widetilde{w_n} + \widetilde{w_{2n}^{\epsilon}},\ \left\|\widetilde{w_{in}^{\epsilon}}\right\| \widetilde{\leq} \widetilde{\epsilon}$$

for $\chi(\widetilde{q_n}) = \widetilde{w_n}$ and $\chi(\widetilde{q_0}) = \widetilde{w_0}$ since χ is continuous. Therefore, $\widetilde{W}$ is a soft Banach quasilinear space. □

4. Conclusion

In this work, the notion of soft quasilinear operator is defined. Also, some consistent theorems and conclusions related with soft quasilinear operators are obtained. Lastly, the inverse of a soft quasilinear operator is described and its some basic properties are worked.

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Some Properties of Two Dimensional Interval Numbers

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Abstract
In this paper, we will introduce the notion of convergence of two dimensional interval sequences and show that the set of all two dimensional interval numbers is a metric space. Also, some ordinary vector norms will be extended to the set of two dimensional interval vectors. Furthermore, we will give definitions of statistical convergence, statistically Cauchy and Cesàro summability for the two dimensional interval numbers and we will get the relationships between them.

Keywords: *Convergent sequence; Metric spaces; Interval number sequence; Two dimensional interval number.*

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1. Introduction

It is known that many mathematical structures have been constructed with real or complex numbers. In recent years, these mathematical structures were replaced by interval numbers and these mathematical structures have been very popular for three decades.

In order fully and effectively to utilize pure mathematics for the analysis of natural phenomena, we must be aware that there are many phases concerning which mathematics and reality do not perfectly agree. For example neither one point on the real number axis is sufficient to represent a physical quantity, no any trace of a moving body described completely as a continuous function of time having no breadth. The concept of an interval is more fundamental than that of a real number. The concept of an interval is fundamental is not only in the case of numerical calculation. It is better to use an interval instead of real number. Interval arithmetic was first suggested by Dwyer [2]. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [11] and Moore and Yang [12]. Furthermore, Moore and others [2, 3, 6, 7, 10] have developed applications to differential equations. Chiao [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval numbers. Markov [8, 9] studied on interval arithmetic. Şengönül and Eryilmaz [15] introduced and studied convergent and bounded sequence spaces of interval numbers and showed that these spaces are complete metric space. The concept of a two dimensional interval is more fundamental than that of a one dimensional interval. In this paper, we introduce and study two dimensional interval sequences.

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The set of all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ satisfying the condition $\xi_{1\underline{\ell}} \leq \xi_1 \leq \xi_{1\underline{r}}$ and $\xi_{2\overline{\ell}} \leq \xi_2 \leq \xi_{2\overline{r}}$ is called the two dimensional interval (or two dimensional interval vector) and is denoted by $\big([\xi_{1\underline{\ell}}, \xi_{1\underline{r}}], [\xi_{2\overline{\ell}}, \xi_{2\overline{r}}]\big)$. Let's denote the set of all closed two dimensional intervals by $\mathfrak{R}^2$. Any elements of $\mathfrak{R}^2$ are called a closed rectangle or closed two dimensional interval and it denoted by $\overline{\xi}$. That is

$$\overline{\xi} = \big\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_{1\underline{\ell}} \leq \xi_1 \leq \xi_{1\underline{r}} \text{ and } \xi_{2\overline{\ell}} \leq \xi_2 \leq \xi_{2\overline{r}}\big\}.$$

For all $\overline{\xi}_1, \overline{\xi}_2 \in \mathfrak{R}^2$, we have $\overline{\xi}_1 = \overline{\xi}_2$ if and only if $\xi_{1\underline{\ell}} = \xi_{2\underline{\ell}}$, $\xi_{1\underline{r}} = \xi_{2\underline{r}}$, $\xi_{1\overline{\ell}} = \xi_{2\overline{\ell}}$ and $\xi_{1\overline{r}} = \xi_{2\overline{r}}$.

$$\overline{\xi}_1 + \overline{\xi}_2 = \big\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_{1\underline{\ell}} + \xi_{2\underline{\ell}} \leq \xi_1 \leq \xi_{1\overline{r}} + \xi_{2\overline{r}},\ \xi_{1\overline{\ell}} + \xi_{2\overline{\ell}} \leq \xi_2 \leq \xi_{1\overline{r}} + \xi_{2\overline{r}}\big\}.$$

If $\alpha > 0$ then

$$\alpha\overline{\xi} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \alpha\xi_{\underline{l}} \leq \xi_1 \leq \alpha\xi_{\underline{r}} \text{ and } \alpha\xi_{\overline{l}} \leq \xi_2 \leq \alpha\xi_{\overline{r}}\}$$

and $\alpha < 0$ then

$$\alpha\overline{\xi} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \alpha\xi_{\underline{r}} \leq \xi_1 \leq \alpha\xi_{\underline{l}} \text{ and } \alpha\xi_{\overline{r}} \leq \xi_2 \leq \alpha\xi_{\overline{l}}\}.$$

$$\begin{aligned}\overline{\xi}_1\overline{\xi}_2 \quad = \quad & \Big(\big[\min\{\xi_{1\underline{\ell}}\xi_{2\underline{\ell}}, \xi_{1\underline{\ell}}\xi_{2\underline{r}}, \xi_{1\underline{r}}\xi_{2\underline{r}}, \xi_{1\underline{r}}\xi_{2\underline{\ell}}\},\ \max\{\xi_{1\underline{\ell}}\xi_{2\underline{\ell}}, \xi_{1\underline{\ell}}\xi_{2\underline{r}}, \xi_{1\underline{r}}\xi_{2\underline{r}}, \xi_{1\underline{r}}\xi_{2\underline{\ell}}\}\big], \\ & \big[\min\{\xi_{1\overline{\ell}}\xi_{2\overline{\ell}}, \xi_{1\overline{\ell}}\xi_{2\overline{r}}, \xi_{1\overline{r}}\xi_{2\overline{r}}, \xi_{1\overline{r}}\xi_{2\overline{\ell}}\},\ \max\{\xi_{1\overline{\ell}}\xi_{2\overline{\ell}}, \xi_{1\overline{\ell}}\xi_{2\overline{r}}, \xi_{1\overline{r}}\xi_{2\overline{r}}, \xi_{1\overline{r}}\xi_{2\overline{\ell}}\}\big]\Big).\end{aligned}$$

The absolute value of a two dimensional interval is defined by

$$|\overline{\xi}| = \max\big\{|\xi_{\underline{\ell}}|, |\xi_{\underline{r}}|, |\xi_{\overline{\ell}}|, |\xi_{\overline{r}}|\big\}. \tag{1.1}$$

2. Main Results

Theorem 2.1. *The set of all two dimensional interval numbers $\mathfrak{R}^2$ is a metric space with the metric d defined by*

$$d(\overline{\xi}_1, \overline{\xi}_2) = \max\big\{|\xi_{1\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{1\underline{\ell}} - \xi_{2\underline{\ell}}|, |\xi_{1\overline{r}} - \xi_{2\overline{r}}|, |\xi_{1\underline{r}} - \xi_{2\underline{r}}|\big\}.$$

Proof. The positivity and symmetry of d are obvious. Let's show the triangle inequality. If $\overline{\xi}_1$, $\overline{\xi}_2$ and $\overline{\xi}_3 \in \mathfrak{R}^2$, then

$$d(\overline{\xi}_1, \overline{\xi}_2) = \max\big\{|\xi_{1\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{1\underline{\ell}} - \xi_{2\underline{\ell}}|, |\xi_{1\overline{r}} - \xi_{2\overline{r}}|, |\xi_{1\underline{r}} - \xi_{2\underline{r}}|\big\},$$

$$\begin{aligned} d(\overline{\xi}_3, \overline{\xi}_1) + d(\overline{\xi}_3, \overline{\xi}_2) \quad = \quad & \max\big\{|\xi_{3\overline{\ell}} - \xi_{1\overline{\ell}}|, |\xi_{3\underline{\ell}} - \xi_{1\underline{\ell}}|, |\xi_{3\overline{r}} - \xi_{1\overline{r}}|, |\xi_{3\underline{r}} - \xi_{1\underline{r}}|\big\} \\ & + \max\big\{|\xi_{3\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{3\underline{\ell}} - \xi_{2\underline{\ell}}|, |\xi_{3\overline{r}} - \xi_{2\overline{r}}|, |\xi_{3\underline{r}} - \xi_{2\underline{r}}|\big\} \\ = \quad & \max\big\{|\xi_{3\overline{\ell}} - \xi_{1\overline{\ell}}| + |\xi_{3\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{3\overline{\ell}} - \xi_{1\overline{\ell}}| + |\xi_{3\underline{\ell}} - \xi_{2\underline{\ell}}|, \\ & |\xi_{3\overline{\ell}} - \xi_{1\overline{\ell}}| + |\xi_{3\overline{r}} - \xi_{2\overline{r}}|, |\xi_{3\overline{\ell}} - \xi_{1\overline{\ell}}| + |\xi_{3\underline{r}} - \xi_{2\underline{r}}|, \\ & |\xi_{3\underline{\ell}} - \xi_{1\underline{\ell}}| + |\xi_{3\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{3\underline{\ell}} - \xi_{1\underline{\ell}}| + |\xi_{3\underline{\ell}} - \xi_{2\underline{\ell}}|, \\ & |\xi_{3\underline{\ell}} - \xi_{1\underline{\ell}}| + |\xi_{3\overline{r}} - \xi_{2\overline{r}}|, |\xi_{3\underline{\ell}} - \xi_{1\underline{\ell}}| + |\xi_{3\underline{r}} - \xi_{2\underline{r}}|, \\ & |\xi_{3\overline{r}} - \xi_{1\overline{r}}| + |\xi_{3\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{3\overline{r}} - \xi_{1\overline{r}}| + |\xi_{3\underline{\ell}} - \xi_{2\underline{\ell}}|, \\ & |\xi_{3\overline{r}} - \xi_{1\overline{r}}| + |\xi_{3\overline{r}} - \xi_{2\overline{r}}|, |\xi_{3\overline{r}} - \xi_{1\overline{r}}| + |\xi_{3\underline{r}} - \xi_{2\underline{r}}|, \\ & |\xi_{3\underline{r}} - \xi_{1\underline{r}}| + |\xi_{3\overline{\ell}} - \xi_{2\overline{\ell}}|, |\xi_{3\underline{r}} - \xi_{1\underline{r}}| + |\xi_{3\underline{\ell}} - \xi_{2\underline{\ell}}|, \\ & |\xi_{3\underline{r}} - \xi_{1\underline{r}}| + |\xi_{3\overline{r}} - \xi_{2\overline{r}}|, |\xi_{3\underline{r}} - \xi_{1\underline{r}}| + |\xi_{3\underline{r}} - \xi_{2\underline{r}}|\big\}. \end{aligned}$$

Now since

$$\begin{aligned} |\xi_{1\overline{\ell}} - \xi_{2\overline{\ell}}| &\leq |\xi_{3\overline{\ell}} - \xi_{1\overline{\ell}}| + |\xi_{3\overline{\ell}} - \xi_{2\overline{\ell}}|, \\ |\xi_{1\underline{\ell}} - \xi_{2\underline{\ell}}| &\leq |\xi_{3\underline{\ell}} - \xi_{1\underline{\ell}}| + |\xi_{3\underline{\ell}} - \xi_{2\underline{\ell}}|, \\ |\xi_{1\overline{r}} - \xi_{2\overline{r}}| &\leq |\xi_{3\overline{r}} - \xi_{1\overline{r}}| + |\xi_{3\overline{r}} - \xi_{2\overline{r}}| \end{aligned}$$

and

$$|\xi_{1\underline{r}} - \xi_{2\underline{r}}| \leq |\xi_{3\underline{r}} - \xi_{1\underline{r}}| + |\xi_{3\underline{r}} - \xi_{2\underline{r}}|,$$

we have

$$d(\overline{\xi}_1, \overline{\xi}_2) \leq d(\overline{\xi}_3, \overline{\xi}_1) + d(\overline{\xi}_3, \overline{\xi}_2).$$

□

In the special case

$$\overline{\xi}_1 = \big([\xi_{1\underline{\ell}}, \xi_{1\underline{r}}], [0,0]\big) = [\xi_{1\underline{\ell}}, \xi_{1\underline{r}}]$$

and

$$\overline{\xi}_2 = \big([\xi_{2\underline{\ell}}, \xi_{2\underline{r}}], [0,0]\big) = [\xi_{2\underline{\ell}}, \xi_{2\underline{r}}],$$

we obtain the metric

$$d(\overline{\xi}_1, \overline{\xi}_2) = \max\big\{|\xi_{1\underline{\ell}} - \xi_{2\underline{\ell}}|, |\xi_{1\underline{r}} - \xi_{2\underline{r}}|\big\}$$

defined on the set of all interval numbers.

Definition 2.1. A sequence of two dimensional interval numbers

$$\big\{\overline{\xi}_k = \big([\xi_{k\underline{\ell}}, \xi_{k\underline{r}}], [\xi_{k\overline{\ell}}, \xi_{k\overline{r}}]\big)\big\}_{k=1}^{\infty}$$

is said to be bounded if there exists a real number $M > 0$ such that $|\overline{\xi}_k| \le M$ for all $k \in \mathbb{N}$.

Definition 2.2. The infinite sequence of two dimensional interval numbers

$$\big\{\overline{\xi}_k = \big([\xi_{k\underline{\ell}}, \xi_{k\underline{r}}], [\xi_{k\overline{\ell}}, \xi_{k\overline{r}}]\big)\big\}_{k=1}^{\infty}$$

is said to be convergent to a bounded two dimensional interval number

$$\overline{\xi} = \big([\xi_{\underline{\ell}}, \xi_{\underline{r}}], [\xi_{\overline{\ell}}, \xi_{\overline{r}}]\big)$$

if for each $\epsilon > 0$ there exists a positive integer N such that $d(\overline{\xi}_k, \overline{\xi}) < \epsilon$ for all $k \ge N$. In this case, we write $\lim_{k\to\infty} \overline{\xi}_k = \overline{\xi}$.

Thus,

$$\lim_{k\to\infty} \overline{\xi}_k = \overline{\xi} \Leftrightarrow \lim_{k\to\infty} \xi_{k\underline{\ell}} = \xi_{\underline{\ell}},\ \ \lim_{k\to\infty} \xi_{k\underline{r}} = \xi_{\underline{r}},\ \ \lim_{k\to\infty} \xi_{k\overline{\ell}} = \xi_{\overline{\ell}} \text{ and } \lim_{k\to\infty} \xi_{k\overline{r}} = \xi_{\overline{r}}.$$

As an example, let

$$\left\{\overline{\xi}_k = \left(\left[\frac{1}{k+1}, \frac{k+1}{k}\right], \left[\frac{k}{k+1}, \frac{2k}{k+1}\right]\right)\right\},$$

then

$$\lim_{k\to\infty} \overline{\xi}_k = \big([0,1],[1,2]\big).$$

A two dimensional interval sequence $\{\overline{\xi}_k\}$ is nested if $\overline{\xi}_{k+1} \subseteq \overline{\xi}_k$ for all k.

Every nested two dimensional interval sequence $\{\overline{\xi}_k\}$ converges and has the limit $\bigcap_{k=1}^{\infty} \overline{\xi}_k$.

For a sequence of (x_k) of real numbers, it is easy to see that

$$\lim_{k\to\infty} x_k = x \Leftrightarrow \lim_{k\to\infty} |x_k - x| = 0.$$

This can be extended to the sequences of two dimensional intervals.

Theorem 2.2. *Let the sequence of two dimensional intervals*

$$\big\{\overline{\xi}_k = \big([\xi_{k\underline{\ell}}, \xi_{k\underline{r}}], [\xi_{k\overline{\ell}}, \xi_{k\overline{r}}]\big)\big\}_{k=1}^{\infty}$$

be convergent to a bounded two dimensional interval

$$\overline{\xi} = \big([\xi_{\underline{\ell}}, \xi_{\underline{r}}], [\xi_{\overline{\ell}}, \xi_{\overline{r}}]\big),$$

then

$$\lim_{k\to\infty} \overline{\xi}_k = \overline{\xi} \Leftrightarrow \lim_{k\to\infty} |\overline{\xi}_k - \overline{\xi}| = 0.$$

Proof.

$$
\begin{aligned}
\lim_{k\to\infty} \overline{\xi}_k = \overline{\xi} \quad &\Leftrightarrow \quad \lim_{k\to\infty} \xi_{k\underline{\ell}} = \xi_{\underline{\ell}},\ \lim_{k\to\infty} \xi_{k\underline{r}} = \xi_{\underline{r}},\ \lim_{k\to\infty} \xi_{k\overline{\ell}} = \xi_{\overline{\ell}} \text{ and } \lim_{k\to\infty} \xi_{k\overline{r}} = \xi_{\overline{r}} \\
&\Leftrightarrow \quad \lim_{k\to\infty} |\xi_{k\underline{\ell}} - \xi_{\underline{\ell}}| = 0,\ \lim_{k\to\infty} |\xi_{k\underline{r}} - \xi_{\underline{r}}| = 0,\ \lim_{k\to\infty} |\xi_{k\overline{\ell}} - \xi_{\overline{\ell}}| = 0 \text{ and } \lim_{k\to\infty} |\xi_{k\overline{r}} - \xi_{\overline{r}}| = 0 \\
&\Leftrightarrow \quad \max\left\{ \lim_{k\to\infty} |\xi_{k\underline{\ell}} - \xi_{\underline{\ell}}|,\ \lim_{k\to\infty} |\xi_{k\underline{r}} - \xi_{\underline{r}}|,\ \lim_{k\to\infty} |\xi_{k\overline{\ell}} - \xi_{\overline{\ell}}|,\ \lim_{k\to\infty} |\xi_{k\overline{r}} - \xi_{\overline{r}}| \right\} = 0 \\
&\Leftrightarrow \quad \lim_{k\to\infty} \max\left\{ |\xi_{k\underline{\ell}} - \xi_{\underline{\ell}}|,\ |\xi_{k\underline{r}} - \xi_{\underline{r}}|, |\xi_{k\overline{l}} - \xi_{\overline{l}}|,\ |\xi_{k\overline{r}} - \xi_{\overline{r}}| \right\} = 0 \\
&\Leftrightarrow \quad \lim_{k\to\infty} |\overline{\xi}_k - \overline{\xi}| = 0.
\end{aligned}
$$

□

Definition 2.3. The infinite sequence of two dimensional interval numbers

$$\left\{ \overline{\xi}_k = \left([\xi_{k\underline{\ell}}, \xi_{k\underline{r}}], [\xi_{k\overline{\ell}}, \xi_{k\overline{r}}] \right) \right\}_{k=1}^{\infty}$$

is said to be two dimensional interval Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that $d(\overline{\xi}_i, \overline{\xi}_j) < \epsilon$ for all $i, j \geq N$.

An n-dimensional two dimensional interval vector is an ordered n-tuples of two dimensional intervals.

$$\overline{\xi} = [\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_n]^T = \left[\left([\xi_{1\underline{\ell}}, \xi_{1\underline{r}}], [\xi_{1\overline{\ell}}, \xi_{1\overline{r}}] \right), \left([\xi_{2\underline{\ell}}, \xi_{2\underline{r}}], [\xi_{2\overline{\ell}}, \xi_{2\overline{r}}] \right), \ldots, \left([\xi_{n\underline{\ell}}, \xi_{n\underline{r}}], [\xi_{n\overline{\ell}}, \xi_{n\overline{r}}] \right) \right]^T.$$

If $|\overline{\xi}| = 0$, then $\overline{\xi}$ is said to be a zero two dimensional interval and $\overline{\xi}$ must be a degenerate interval $\overline{0} = \left([0,0], [0,0] \right)$.

Some well-known inequalities can be extended to the two dimensional interval vectors as in the following two lemmas with absolute value defined in (1.1).

Lemma 2.1. *Let $\overline{\xi}_1$ and $\overline{\xi}_2$ be two dimensional intervals, then*

$$|\overline{\xi}_1 + \overline{\xi}_2| \leq |\overline{\xi}_1| + |\overline{\xi}_2|.$$

Proof.

$$
\begin{aligned}
|\overline{\xi}_1 + \overline{\xi}_2| &= \max\left\{ |\xi_{1i\underline{\ell}} + \xi_{2i\underline{\ell}}|, |\xi_{1i\underline{r}} + \xi_{2i\underline{r}}|, |\xi_{1i\overline{\ell}} + \xi_{2i\overline{\ell}}|, |\xi_{1i\overline{r}} + \xi_{2i\overline{r}}| \right\} \\
&\leq \max\left\{ |\xi_{1i\underline{\ell}}| + |\xi_{2i\underline{\ell}}|, |\xi_{1i\underline{r}}| + |\xi_{2i\underline{r}}|, |\xi_{1i\overline{\ell}}| + |\xi_{2i\overline{\ell}}|, |\xi_{1i\overline{r}}| + |\xi_{2i\overline{r}}| \right\} \\
&= \max\left\{ |\xi_{1i\underline{\ell}}|, |\xi_{1i\underline{r}}|, |\xi_{1i\overline{\ell}}|, |\xi_{1i\overline{r}}| \right\} + \max\{ |\xi_{2i\underline{\ell}}|, |\xi_{2i\underline{r}}|, |\xi_{2i\overline{\ell}}|, |\xi_{2i\overline{r}}| \} \\
&= |\overline{\xi}_1| + |\overline{\xi}_2|
\end{aligned}
$$

□

The inner product of two dimensional interval vectors is defined trough the two dimensional interval multiplication as follows:

$$
\begin{aligned}
< \overline{\xi}_1, \overline{\xi}_2 > &= \sum_{i=1}^{n} \overline{\xi}_{1i} \overline{\xi}_{2i} \\
&= \sum_{i=1}^{n} \left([\overline{\xi}_{1i\underline{\ell}}, \overline{\xi}_{1i\underline{2}}], [\overline{\xi}_{1i\overline{\ell}}, \overline{\xi}_{1i\overline{r}}] \right) \left([\overline{\xi}_{2i\underline{\ell}}, \overline{\xi}_{2i\underline{2}}], [\overline{\xi}_{2i\overline{\ell}}, \overline{\xi}_{2i\overline{r}}] \right) \\
&= \sum_{i=1}^{n} \left([\min\{ \xi_{1i_{\underline{\ell}}} \xi_{2i_{\underline{\ell}}}, \xi_{1i_{\underline{\ell}}} \xi_{2i_{\underline{r}}}, \xi_{1i_{\underline{r}}} \xi_{2i_{\underline{r}}}, \xi_{1i_{\underline{r}}} \xi_{2i_{\underline{\ell}}} \},\ \max\{ \xi_{1i_{\underline{\ell}}} \xi_{2i_{\underline{\ell}}}, \xi_{1i_{\underline{\ell}}} \xi_{2i_{\underline{r}}}, \xi_{1i_{\underline{r}}} \xi_{2i_{\underline{r}}}, \xi_{1i_{\underline{r}}} \xi_{2i_{\underline{\ell}}} \}], \right. \\
&\qquad \left. [\min\{ \xi_{1i_{\overline{\ell}}} \xi_{2i_{\overline{\ell}}}, \xi_{1i_{\overline{\ell}}} \xi_{2i_{\overline{r}}}, \xi_{1i_{\overline{r}}} \xi_{2\overline{r}}, \xi_{1\overline{r}} \xi_{2\overline{\ell}} \},\ \max\{ \xi_{1\overline{\ell}} \xi_{2i_{\overline{\ell}}}, \xi_{1i_{\overline{\ell}}} \xi_{2i_{\overline{r}}}, \xi_{1i_{\overline{r}}} \xi_{2i_{\overline{r}}}, \xi_{1i_{\overline{r}}} \xi_{2i_{\overline{\ell}}} \}] \right).
\end{aligned}
$$

Lemma 2.2. *Let $\overline{\xi}_1$ and $\overline{\xi}_2$ be two dimensional interval vectors in $(\Re^2)^n$. Then*

$$| < \overline{\xi}_1, \overline{\xi}_2 > | \leq | < \overline{\xi}_1, \overline{\xi}_1 > |^{\frac{1}{2}}\, | < \overline{\xi}_2, \overline{\xi}_2 > |^{\frac{1}{2}}.$$

Proof. Let α be any real number and the following inequality is always true

$$|<\alpha\overline{\xi}_1+\overline{\xi}_2, \alpha\overline{\xi}_1+\overline{\xi}_2>|\geq 0.$$

Then

$$\begin{aligned}
0 &\leq |<\alpha\overline{\xi}_1+\overline{\xi}_2, \alpha\overline{\xi}_1+\overline{\xi}_2>| \\
&\leq |<\alpha^2\overline{\xi}_1, \overline{\xi}_2> +2\alpha<\overline{\xi}_1, \overline{\xi}_2> + <\overline{\xi}_1, \overline{\xi}_2>| \\
&\leq \alpha^2|<\overline{\xi}_1, \overline{\xi}_2> +2|\alpha||<\overline{\xi}_1, \overline{\xi}_2>|+|<\overline{\xi}_1, \overline{\xi}_2>|.
\end{aligned}$$

The right hand side of the last inequality is a quadratic form for $|\alpha|$ and it is always greater than or equal to 0. Therefore

$$|<\overline{\xi}_1, \overline{\xi}_2>|^2 - |<\overline{\xi}_1, \overline{\xi}_1>||<\overline{\xi}_2, \overline{\xi}_2>|\leq 0,$$

thus

$$|<\overline{\xi}_1, \overline{\xi}_2>|\leq |<\overline{\xi}_1, \overline{\xi}_1>|^{\frac{1}{2}}|<\overline{\xi}_2, \overline{\xi}_2>|^{\frac{1}{2}}.$$

□

Let $(\mathfrak{R}^2)^n$ be the set of two dimensional interval vectors. Some properties of the classical vector norms can be extended to the two dimensional interval vectors. The max-norm of a two dimensional interval vector on $(\mathfrak{R}^2)^n$ is a non-negative valued function

$$\|.\| : (\mathfrak{R}^2)^n \to \mathbb{R}^+ \cup \{0\}$$

that satisfies following properties:

(1) $\forall\overline{\xi} \in (\mathfrak{R}^2)^n - \{\overline{0}\},\ \ \|\overline{\xi}\| > 0,$

(2) $\forall\overline{\xi} \in (\mathfrak{R}^2)^n$ and $\alpha \in \mathbb{R},\ \ \|\alpha\overline{\xi}\| = |\alpha|\|\overline{\xi}\|,$

(3) $\forall\overline{\xi},\overline{\zeta} \in (\mathfrak{R}^2)^n,\ \ \|\overline{\xi}+\overline{\zeta}\| \leq \|\overline{\xi}\| + \|\overline{\zeta}\|.$

Theorem 2.3. $(\mathfrak{R}^2)^n$ *with the following norm is a normed space*

$$\|\overline{\xi}\| = \left(\sum_{i=1}^{n} |\overline{\xi}_i|^{\frac{1}{2}}\right)^2.$$

Proof. If $\overline{\xi}\neq\overline{0}$, then $\|\overline{\xi}\| > 0$.

$$\begin{aligned}
\|\alpha\overline{\xi}\| &= \left(\sum_{i=1}^{n}\left(\max\left\{|\alpha\xi_{i_{\underline{\ell}}}|, |\alpha\xi_{i_{\underline{r}}}|, |\alpha\xi_{i_{\overline{\ell}}}|, |\alpha\xi_{i_{\overline{r}}}|\right\}\right)^2\right)^{\frac{1}{2}} \\
&= |\alpha|\left(\sum_{i=1}^{n}\left(\max\left\{|\xi_{i_{\underline{\ell}}}|, |\xi_{i_{\underline{r}}}|, |\xi_{i_{\overline{\ell}}}|, |\xi_{i_{\overline{r}}}|\right\}\right)^2\right)^{\frac{1}{2}} \\
&= |\alpha|\|\overline{\xi}\|.
\end{aligned}$$

$$\begin{aligned}
\|\bar{\xi}+\bar{\zeta}\| &= \left(\sum_{i=1}^{n}|\bar{\xi}_i+\bar{\zeta}_i|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n}\left(\max\left\{|\xi_{i_{\underline{\ell}}}+\zeta_{i_{\underline{\ell}}}|, |\xi_{i_{\underline{r}}}+\zeta_{i_{\underline{r}}}|, |\xi_{i_{\overline{\ell}}}+\zeta_{i_{\overline{\ell}}}|, |\xi_{i_{\overline{r}}}+\zeta_{i_{\overline{r}}}|\right\}\right)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n}|\bar{\xi}_{i_{m_i}}+\bar{\zeta}_{i_{m_i}}|^2\right)^{\frac{1}{2}} \qquad (m_i=\underline{\ell},\ \overline{\ell},\ \underline{r}\ \text{ or }\ \overline{r}) \\
&= \|\xi_m+\zeta_m\|_2 \quad \text{(ordinary vector norm)} \\
&\leq \|\xi_m\|_2+\|\zeta_m\|_2 \\
&= \left(\sum_{i=1}^{n}|\bar{\xi}_{i_{m_i}}|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}|\bar{\zeta}_{i_{m_i}}|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n}\max\left\{|\xi_{i_{\underline{\ell}}}|^2, |\xi_{i_{\underline{r}}}|^2, |\xi_{i_{\overline{\ell}}}|^2, |\xi_{i_{\overline{r}}}|^2\right\}\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{i=1}^{n}\max\left\{|\zeta_{i_{\underline{\ell}}}|^2, |\zeta_{i_{\underline{r}}}|^2, |\zeta_{i_{\overline{\ell}}}|^2, |\zeta_{i_{\overline{r}}}|^2\right\}\right)^{\frac{1}{2}} \\
&= \|\bar{\xi}\|+\|\bar{\zeta}\|
\end{aligned}$$

where $\xi_m=[\xi_{1_{m_1}},\xi_{2_{m_2}},\ldots,\xi_{n_{m_n}}]^T$, $\zeta_m=[\zeta_{1_{m_1}},\zeta_{2_{m_2}},\ldots,\zeta_{n_{m_n}}]^T$. Thus, axioms of the norm function are hold and the proof is completed. □

Observe that

$$\|\bar{\xi}\|_1=\sum_{i=1}^{n}|\bar{\xi}_i|$$

and

$$\|\bar{\xi}\|_\infty=\max_i\left\{\max\left\{|\xi_{i\underline{\ell}}|, |\xi_{i\underline{r}}|, |\xi_{i\overline{\ell}}|, |\xi_{i\overline{r}}|\right\}\right\}$$

are the other norms on $(\Re^2)^n$.

3. Statistical Convergence

Statistical convergence of sequences of numbers was introduced by Fast [4]. In [13], Schoenberg established some basic properties of statistical convergence and also studied the concept as a summability method.

A sequence (x_k) is said to be statistically convergent to the number ℓ if for every $\epsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n: |x_k-\ell|\geq\epsilon\}|=0.$$

In this case, we write $st-\lim x_k=\ell$. $\lim x_k=\ell$ implies $st-\lim x_k=\ell$, so statistical convergence may be considered as a regular summability method. This was observed in [13] along with the fact that the statistical limit is a linear functional on some sequence spaces.

In [5], Fridy proved that if (x_k) is a statistically convergent sequence, then there is a convergent sequence (y_k) such that

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n: x_k\neq y_k\}|=0.$$

Now we will give definitions of statistical convergence, statistically Cauchy and Cesàro summability for the two dimensional interval numbers and we will get the relationships between them.

Definition 3.1. The infinite sequence of two dimensional interval numbers

$$\{\overline{\xi}_k = \big([\xi_{k\underline{\ell}}, \xi_{k\underline{r}}], [\xi_{k\overline{\ell}}, \xi_{k\overline{r}}]\big)\}_{k=1}^{\infty}$$

is said to be statistically convergent to a bounded two dimensional interval number

$$\overline{\xi} = \big([\xi_{\underline{\ell}}, \xi_{\underline{r}}], [\xi_{\overline{\ell}}, \xi_{\overline{r}}]\big)$$

if for each $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k \le n : d(\overline{\xi}_k, \overline{\xi}) \ge \epsilon\}\big| = 0,$$

where the vertical bars denote the number of elements in the enclosed set. In this case, we write $st - \lim_{k\to\infty} \overline{\xi}_k = \overline{\xi}$.

Thus,

$$st - \lim_{k\to\infty} \overline{\xi}_k = \overline{\xi} \Leftrightarrow st - \lim_{k\to\infty} \xi_{k\underline{\ell}} = \xi_{\underline{\ell}},\ \ st - \lim_{k\to\infty} \xi_{k\underline{r}} = \xi_{\underline{r}},\ \ st - \lim_{k\to\infty} \xi_{k\overline{\ell}} = \xi_{\overline{\ell}} \text{ and } st - \lim_{k\to\infty} \xi_{k\overline{r}} = \xi_{\overline{r}}.$$

Statistical convergence is a natural generalization of ordinary convergence. If $\lim_{k\to\infty} \overline{\xi}_k = \overline{\xi}$, then $st - \lim_{k\to\infty} \overline{\xi}_k = \overline{\xi}$. The converse does not hold in general. For example, let $(\overline{\xi}_k)$ be following sequence:

$$\overline{\xi}_k := \begin{cases} \big([k, k+1], [k+1, k+2]\big) & , \text{ if and } k \text{ is a square integer,} \\ \big([0,0],[0,0]\big) & , \text{ otherwise.} \end{cases}$$

This sequence of two dimensional interval numbers is not convergent. But since

$$\frac{1}{n}|\{k \le n : d\big(\overline{\xi}_k, \big([0,0],[0,0]\big)\big) \ge \epsilon| \le \frac{\sqrt{n}}{n},$$

this sequence is statistically convergent to the $\big([0,0],[0,0]\big)$.

Definition 3.2. The infinite sequence of two dimensional interval numbers

$$\{\overline{\xi}_k = \big([\xi_{k\underline{\ell}}, \xi_{k\underline{r}}], [\xi_{k\overline{\ell}}, \xi_{k\overline{r}}]\big)\}_{k=1}^{\infty}$$

is said to be statistically Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer N such that

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k \le n : d(\overline{\xi}_k, \overline{\xi}_N) \ge \epsilon\}\big| = 0.$$

Theorem 3.1. *The following statements are equivalent:*

i. $(\overline{\xi}_k)$ *is a statistically convergent sequence,*

ii. $(\overline{\xi}_k)$ *is a statistically Cauchy sequence,*

iii. $(\overline{\xi}_k)$ *is a sequence for which there is a convergent sequence* $(\overline{\zeta}_k)$ *such that* $\lim_{n\to\infty} \frac{1}{n}\big|\{k \le n : \overline{\xi}_k \ne \overline{\zeta}_k\}\big| = 0.$

The theorem can be proved using techniques similar to those in [5], so we omit it.

Definition 3.3. We say that $(\overline{\xi}_k)$ is Cesàro summable to $\overline{\xi}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(\overline{\xi}_k, \overline{\xi}) = 0.$$

Theorem 3.2. .

i. If $(\overline{\xi}_k)$ *is Cesàro summable to* $\overline{\xi}$*, then it is statistically convergent to* $\overline{\xi}$*,*

ii. If $(\overline{\xi}_k)$ *is bounded and statistically convergent to* $\overline{\xi}$*, then it is Cesàro summable to* $\overline{\xi}$*.*

The theorem can be proved using techniques similar to those in [13], so we omit it.

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Group Invariant Solutions and Local Conservation Laws of Heat Conduction Equation Arising Laser Heating Carbon Nanotubes Using Lie Group Analysis

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Abstract
In this study, based on the continuous transformations of Lie groups, the exact analytic solutions of the laser heating carbon nanotubes formulated by using the classical heat conduction equation with various physical properties were constructed. These solutions are the type of group invariant solutions. The constructed solutions have expanded and enriched the solution forms of this new model existing in the literature. With the help of the Maple package program, 3D, density, and contour graphs were drawn for the special values of the parameters in the solutions, and the physical structures of the solutions obtained in this way were also observed. The solutions obtained can be used in the explanation of physical phenomena occurring in cancer investigations.

Keywords: *Carbon nanotubes; Lie groups; Conservation laws.*

AMS Subject Classification (2020): *Primary: 37L20; 35C05 ; Secondary: 37K06; 34A05*

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1. Introduction

It is well known that the evolution differential equations (EDEs) mathematically model many physical phenomena that occur in nature. Many analytical and numerical solution techniques such as Hirota bilinear method, Backlund transformations, Darboux transformations, Painleve property, variational iteration method, $tanh$ method, invariant subspace method, Lie symmetry groups etc. have been developed over time to solve those equations [1–14]. Among the methods listed above, the Lie symmetry groups method is an effective approach in obtaining exact solutions (special group invariant solutions) of the considered differential equations (systems), regardless of the order, degree and linearity types [14, 15].

Carbon nanotubes (CNTs) have an important place in nanomaterials science due to their mechanical, electrical, optical and magnetic properties. CNTs have effective applications in the field of medicine, drug distribution, and contrast agents. One of the physical applications that these EDEs address is cancer disease. We know from experimental studies that CNTs are promising nanomaterials for warming agents in photothermal therapy (PTT)

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and contrast agents in photoacoustic (PA) imaging. In the experiments, the temperature of the agents used in both PTT and PA imaging during laser irradiation was examined in the tissue. It is also known from experimental studies that cancer cells can be destroyed by increasing the temperature in the tissue (with the help of the agents in PTT) to 41-47 ° C. Thus, cancer cells become hyperthermic and suffer significant damage [16].

When we look at the studies in the literature, in the study [17], the heat analysis of multi-walled CNTs during pulsed laser heating was investigated using the finite element method (FEM) for the classical heat conduction equation. The dynamics of pulsed nanosecond laser heating process was simulated by the solution of the heat conduction equation. In addition, the FEM is applied to compute the temperature profiles as a function of depth x and time t in the sample (multi-walled CNT) [18]. Also, in the literature optical soliton-like solutions for the system of ring-cavity fiber laser using carbon nanotubes for passive mode locking have also been studied [19]. In [20], the authors investigated more information on the system of carbon nanotubes conveying fluid by using Lie symmetry groups.

In this study, we examine the temperature profiles using the Lie symmetry groups method and obtain analytical solutions, based on the classical heat conduction equation that explains the laser-heated CNT model previously discussed in the study [16]. In the study [16], the authors considered the temperature function as dependent only on the radial variable and treated the model as a simple ordinary differential equation (ODE). Besides, we will try to obtain the conservation laws of the model.

The study is organized as follows: The laser heated governing model discussed in Section 2 is presented. The solution of the model will be explored in Section 3. Section 4 is devoted to the conservation laws of the model through the multiplier method. The results and remarks are presented in Section 5.

2. Governing equation

We will examine the CNT, which is exposed to laser heating and has a cylindrical structure. Here we will assume that there are cancerous tissues around the CNT. Besides, it will be assumed that the length of the CNT is greater than its radius, and the temperature T_b is 37 degrees at a unit distance b from the center. Considering the above-mentioned assumptions, CNTs exposed to laser heating can be formulated with the heat conduction equation given below:

$$k_1 k_2 u_t = k_3 \frac{1}{x}(x u_x)_x + f(x,t), \quad 0 < x < a, \tag{2.1}$$

where k_1 is the density of CNT, k_2 is the concentration of CNT, k_3 is the thermal conductivity of CNTs and $u(x,t)$ and $f(x,t)$ denote the temperature function and source term respectively, where x is the distance measured from the center of the cylinder and t is the time variable. In our work we will assume that $f(x,t)$ is a constant let say f (in fact, we learn from [16] that based on the physical meaning of $f(x,t)$, its mathematical formulation is in the form of $f(x,t) = (1-R)I_0\alpha$ where R denotes reflectivity, I_0 is the laser intensity, α is optical absorption coefficient of CNTs) (see, [16] for further details).

3. Lie point symmetries of Eq.(2.1)

Now consider the continuous Lie transformations with one small parameter given below:

$$\begin{aligned} \bar{x} &= x + \mu\xi\,(t,x,u) + O(\mu)^2, \\ \bar{t} &= t + \mu\tau\,(t,x,u) + O(\mu)^2, \\ \bar{u} &= u + \mu\eta\,(t,x,u) + O(\mu)^2. \end{aligned} \tag{3.1}$$

In this case, the Lie point symmetry generator of Eq. (2.1) associated with (3.1) is generated by the vector field of the form

$$X = \tau\,(t,x,u)\,\frac{\partial}{\partial t} + \xi\,(t,x,u)\,\frac{\partial}{\partial x} + \eta\,(t,x,u)\,\frac{\partial}{\partial u}, \tag{3.2}$$

where τ, ξ and η depend on only t, x and u. Applying the second prolongation $pr^{(2)}X$

$$pr^{(2)}X = X + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}}, \tag{3.3}$$

to Eq. (2.1), i.e,

$$pr^{(2)}X \left[k_1 k_2 u_t - k_3 \frac{1}{x}(x u_x)_x - f \right] |_{k_1 k_2 u_t = k_3 \frac{1}{x}(x u_x)_x + f} = 0, \tag{3.4}$$

we find that the coefficient functions $\xi(x,t,u), \tau(x,t,u)$ and $\eta(x,t,u)$ must satisfy the following linearized symmetry condition

$$k_1k_2\eta^t + k_3\frac{1}{x^2}u_x\xi - k_3\frac{1}{x}\eta^x - k_3\eta^{xx} = 0, \tag{3.5}$$

where η^t, η^x, and η^{xx} are the extended coefficients of $pr^{(2)}X$. In a simplified form these coefficients can be written in the following format:

$$\begin{aligned}
\eta^t &= D_t\eta - u_xD_t\xi - u_tD_t\tau, \\
\eta^x &= D_x\eta - u_xD_x\xi - u_tD_x\tau, \\
\eta^{xx} &= D_x^2\eta - u_xD_x^2\xi - u_tD_t^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau,
\end{aligned} \tag{3.6}$$

where $D_x,\ D_t$ are the total derivatives with respect to x and t, respectively and are given as follows:

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + ..., \\
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + ...
\end{aligned} \tag{3.7}$$

Here, we would like to point out that, for a differential equation (or system) of order n, the coefficient functions of $pr^{(n)}\ (X)$ prolongation is given as the follows:

$$\eta^{\alpha}_{i_1...i_s} = D_{i_s}(\eta^{\alpha}_{i_1...i_{s-1}}) - u_{ji_1...i_{s-1}}D_{i_s}(\xi^j),\ s > 1. \tag{3.8}$$

If we write the prolonged coefficients (3.6) in the linearized invariance condition (3.5) and equal the coefficients of the derivatives of u with respect to x to zero, we arrive at the following linear over-determined partial differential equation system:

$$\begin{aligned}
\eta_{uu} &= 0, \\
\eta_{xu} &= -\frac{\tau_{tt}k_1k_2x}{4k_3}, \\
\eta_{xx} &= \frac{xk_2k_1\eta_t - fx\eta_u + fx\tau_t - k_3\eta_x}{k_3x}, \\
\eta_{tu} &= -\frac{\tau_{tt}}{2}, \\
\tau_u &= 0, \\
\tau_x &= 0, \\
\tau_{ttt} &= 0, \\
\xi &= \frac{x\tau_t}{2}.
\end{aligned} \tag{3.9}$$

If the above over-determined system of equations is solved for $\xi(x,t)$, $\tau(t)$ and $\eta(x,t,u)$, the following useful and important Lie vector fields are obtained (in fact, 6 dimensional Lie vector algebras are obtained but some trivial vector fields are omitted). Hence, the point symmetry generators admitted by the heat equation (2.1) are given by

$$\begin{aligned}
X &= \frac{\partial}{\partial t} + \left(\frac{x^2}{4} + \frac{k_3t}{k_1k_2}\right)\frac{\partial}{\partial u}, \\
Y &= t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \left(\frac{x^2}{4} + \frac{k_3t}{k_1k_2}\right)\frac{\partial}{\partial u}, \\
Z &= \left(u + \frac{x^2}{4} + \frac{k_3t}{k_1k_2}\right)\frac{\partial}{\partial u}.
\end{aligned} \tag{3.10}$$

Consider the infinitesimal generator Y in (3.10). The corresponding one-parameter Lie group of point transformations is obtained by solving the initial value problem for the first order system of ODEs,

$$\frac{dx^*}{d\varepsilon} = \frac{x^*}{2}, \tag{3.11}$$

$$\frac{dt^*}{d\varepsilon} = t^*, \tag{3.12}$$

$$\frac{du^*}{d\varepsilon} = \frac{(x^*)^2}{4} + \frac{k_3 t^*}{k_1 k_2}, \tag{3.13}$$

with $u^* = u$, $x^* = x$, $t^* = t$ at $\varepsilon = 0$. This yields

$$x^* = X(x,t,u;\varepsilon) = C_3 \exp\left(\frac{\varepsilon}{2}\right), \tag{3.14}$$

$$t^* = T(x,t,u;\varepsilon) = C_2 \exp(\varepsilon), \tag{3.15}$$

$$u^* = U(x,t,u;\varepsilon) = \left(\frac{C_3^2}{4} + \frac{C_2 k_3}{k_1 k_2}\right)\exp(\varepsilon) + C_1. \tag{3.16}$$

Now we find the invariant solutions $u(x,t)$ of the model (2.1).

4. Exact solutions

The group invariant solutions of the laser heating CNTs formulated by using the classical heat conduction equation with various physical properties are constructed with the help of the invariant form method, direct substitution method, λ-symmetry reductions and first integrals.

4.1 Invariant form method

The required invariant surface condition [10, 15] $\eta - \xi u_x - \tau u_t = 0$ for X vector field becomes

$$u_t = (\frac{x^2}{4} + \frac{k_3 t}{k_1 k_2}). \tag{4.1}$$

The corresponding characteristic equations of Eq. (2.1) are given by

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{\frac{x^2}{4} + \frac{k_3 t}{k_1 k_2}}. \tag{4.2}$$

We yield two invariants of X by solving above the characteristic equations:

$$\zeta = x, \quad v = u - \frac{k_3 t^2}{2k_1 k_2} - \frac{x^2}{4} t. \tag{4.3}$$

Hence, the solution of the invariant surface condition (4.1) is represented by the invariant form

$$u - \frac{k_3 t^2}{2k_1 k_2} - \frac{x^2}{4} t = \phi(x), \tag{4.4}$$

or

$$u = \Theta(x,t) = \frac{k_3 t^2}{2k_1 k_2} + \frac{x^2}{4} t + \phi(\zeta), \tag{4.5}$$

in terms of the similarity variable (one of the invariants) $\zeta = x$. Plugging of (4.5) into the classical heat conduction equation (2.1) leads to $\phi(\zeta)$ satisfying the reduced ODE which converts Eq.(2.1) to the second-order variable coefficient ODE

$$k_3 \phi''(\zeta) + \frac{k_3}{\zeta}\phi'(\zeta) - \frac{1}{4} k_1 k_2 \zeta^2 + f = 0. \tag{4.6}$$

Thus, the invariant solution of PDE (2.1), resulting from its invariance under X, is presented by

$$u = \Theta(x,t) = \frac{k_3 t^2}{2k_1 k_2} + \frac{x^2}{4} t + \frac{1}{64}\frac{x^4 k_1 k_2}{k^3} - \frac{1}{4}\frac{x^2 f}{k_3} + C_1 \ln(x) + C_2, \tag{4.7}$$

where C_1 and C_2 are arbitrary constants.

At this point, we also note that if $\xi u_x + \tau u_t - \eta = 0$ invariant surface condition is used for the Z Lie vector field, then it is readily seen that $u = -\frac{x^2}{4} - \frac{k_3 t}{k_1 k_2}$ group invariant solution is obtained.

4.2 Direct substitution method

In this alternative way [10, 15], we first express the invariant surface condition in a solved form u_t (we consider the case of Y Lie vector field):

$$u_t = -\frac{x}{2t}u_x + \frac{x^2}{4t} + \frac{k_3}{k_1 k_2}. \tag{4.8}$$

After plugging (4.8) into heat equation (2.1), we obtain the following ODE with t playing the role of a parameter:

$$k_3 u_{xx} + (k_3\frac{1}{x} + k_1 k_2 \frac{x}{2t})u_x - (\frac{x^2}{4t}k_1 k_2 + k_3) + f = 0. \tag{4.9}$$

The general solution of the parametric ODE (4.9) is given by

$$u = \frac{1}{4}x^2 - \frac{2ftln(x)}{k_1 k_2} - \frac{1}{2}A(t)Ei(1, \frac{x^2 k_1 k_2}{4k_3 t}) + B(t), \tag{4.10}$$

where $A(t)$ and $B(t)$ are arbitrary functions. Substitution of (4.10) into the invariant surface condition (4.8) yields

$$-A'(t)Ei(1, \frac{x^2 k_1 k_2}{4k_3 t})k_1 k_2 + 2B'(t)k_1 k_2 - 4fln(x) - 2f - 2k_3 = 0, \tag{4.11}$$

where the exponential integrals, $Ei(a, z)$, are defined for $0 < R(z)$ by

$$Ei(a, z) = \int_1^{\infty} e^{-k_1 z} k_1^{-a} dk_1. \tag{4.12}$$

We now find the one-parameter (ε) family of solutions $u = \Theta(x, t; \varepsilon)$, resulting from the invariance of the model equation (2.1) under the point symmetry X, obtained from any solution $u(x, t)$ that is not of the form (4.7). Let

$$\hat{x} = X(x, t, u; \varepsilon) = C_3 \exp\left(\frac{\varepsilon}{2}\right), \tag{4.13}$$

$$\hat{t} = T(x, t, u; \varepsilon) = C_2 \exp(\varepsilon), \tag{4.14}$$

$$\hat{u} = \Theta(\hat{x}, \hat{t}). \tag{4.15}$$

Then

$$u = \Phi(x, t; \varepsilon) = U\left(\hat{x}, \hat{t}, \hat{u}; -\varepsilon\right) = \frac{\exp(\varepsilon)}{\frac{C_3^2}{4} + \frac{C_2 k_3}{k_1 k_2}} \Theta(C_3 \exp\left(\frac{\varepsilon}{2}\right), C_2 \exp(\varepsilon)). \tag{4.16}$$

4.3 λ-Symmetry Reductions and First Integrals Using Lie Symmetry

In this case, the Lie point symmetry generators of Eq.(4.6) are generated by the vector field of the form (see, [21, 22])

$$V = \xi(\zeta, \phi)\frac{\partial}{\partial \zeta} + \eta(\zeta, \phi)\frac{\partial}{\partial \phi}, \tag{4.17}$$

where ξ and η depend on ζ and ϕ. Applying the second prolongation $pr^{(2)}V$

$$pr^{(2)}V = X + \eta^{\zeta}\frac{\partial}{\partial \phi_{\zeta}} + \eta^{\zeta\zeta}\frac{\partial}{\partial \phi_{\zeta\zeta}}, \tag{4.18}$$

to Eq. (4.6), i.e,

$$pr^{(2)}V\left(k_3\phi''(\zeta) + \frac{k_3}{\zeta}\phi'(\zeta) - \frac{1}{4}k_1 k_2 \zeta^2 + f\right)\Big|_{\phi''(\zeta) = -\frac{\phi'(\zeta)}{\zeta} + \frac{k_1 k_2 \zeta^2}{4k_3} - \frac{f}{k_3}} = 0, \tag{4.19}$$

we get that the coefficient functions $\xi(\zeta, \phi)$ and $\eta(\zeta, \phi)$ should fulfill the following linearized symmetry equation

$$\xi\left(-\frac{k_3}{\zeta^2}\phi_{\zeta} - \frac{1}{2}k_1 k_2 \zeta\right) + \frac{k_3 \eta^{\zeta}}{\zeta} + k_3 \eta^{\zeta\zeta} = 0, \tag{4.20}$$

where η^{ζ} and $\eta^{\zeta\zeta}$ are the coefficients of $pr^{(2)}V$. In a simplified form these coefficients can be written in the following format:

$$\begin{aligned} \eta^{\zeta} &= D_{\zeta}\eta - \phi_{\zeta} D_{\zeta}\xi, \\ \eta^{\zeta\zeta} &= D_{\zeta}^{2}\eta - \phi_{\zeta} D_{\zeta}^{2}\xi - 2\phi_{\zeta\zeta} D_{\zeta}\xi, \end{aligned} \tag{4.21}$$

where D_{ζ} is the total derivatives with respect to ζ and is given as follows:

$$D_{\zeta} = \frac{\partial}{\partial\zeta} + \phi_{\zeta}\frac{\partial}{\partial\phi} + \phi_{\zeta\zeta}\frac{\partial}{\partial\phi_{\zeta}} + \ldots. \tag{4.22}$$

If we write the prolonged coefficients (4.21) in the linearized invariance condition (4.20) and equal the coefficients of the derivatives of ϕ with respect to ζ to zero, we arrive at the following linear over-determined PDE system:

$$\begin{aligned} \eta_{\phi,\phi,\phi} &= 0, \\ \eta_{\phi,\phi,\zeta} &= 0, \\ \xi_{\phi,\phi} &= 0, \\ \xi_{\phi,\zeta} &= \frac{\zeta\eta_{\phi,\phi} + 2\xi_{\phi}}{2\zeta}, \\ \xi_{\zeta,\zeta} &= \frac{-3\zeta^{4}k_1k_2\xi_{\phi} + 12f\zeta^{2}\xi_{\phi} + 8k_3\zeta^{2}\eta_{\phi,\zeta} + 4\zeta k_3\xi_{\zeta} - 4k_3\xi_{\zeta}}{4\zeta^{2}k_3}, \\ \eta_{\zeta,\zeta} &= \frac{-k_1k_2\zeta^{3}\eta_{\phi} + 2\zeta^{3}k_1k_2\xi_{\zeta} + 2\zeta^{2}k_1k_2\xi_{\zeta} + 4f\zeta\eta_{\phi} - 8f\zeta\xi_{\zeta} - 4k_3\eta_{\zeta}}{4\zeta k_3}. \end{aligned} \tag{4.23}$$

If the above over-determined system of equations is solved for $\xi(\zeta,\phi)$ and $\eta(\zeta,\phi)$, the following useful and important Lie vector fields are obtained (in fact, 6 dimensional Lie vector algebras are obtained). Hence, the point symmetry generators admitted by the equation (4.6) are given by

$$\begin{aligned} V_1 &= \frac{\partial}{\partial\phi}, \\ V_2 &= \ln(\zeta)\frac{\partial}{\partial\phi}, \\ V_3 &= 4k_3\zeta\frac{\partial}{\partial\zeta} - \frac{\zeta^{2}(-k_1k_2^{2}\zeta + 8f)}{4}\frac{\partial}{\partial\phi}, \\ V_4 &= (16\zeta\ln(\zeta) - 8\zeta)k_1k_2k_3\frac{\partial}{\partial\zeta} + (k_1^{2}k_2^{2}\zeta^{4}\ln(\zeta) \\ &\quad -\frac{k_1^{2}k_2^{2}\zeta^{4}}{2} - 8k_1k_2f\zeta^{2}\ln(\zeta) + 4fk_1k_2\zeta^{2} + 8f^{2}\ln(\zeta))\frac{\partial}{\partial\phi}. \end{aligned} \tag{4.24}$$

Now we use the relationship between Lie point symmetries and λ-symmetries to get λ-symmetries of Equation (4.6). Let us consider X_3. Then we have

$$\xi = 4k_3\zeta, \tag{4.25}$$

$$\eta = -\frac{\zeta^{2}(-k_1k_2^{2}\zeta + 8f)}{4}, \tag{4.26}$$

and the characteristic function of V_3

$$\begin{aligned} Q &= \eta - \xi\phi_\zeta \\ &= -\frac{\zeta^2(-k_1k_2^2\zeta+8f)}{4} - 4k_3\zeta\phi_\zeta, \end{aligned} \tag{4.27}$$

and the total derivative operator

$$D_\zeta = \frac{\partial}{\partial\zeta} + \phi_\zeta\frac{\partial}{\partial\phi} + \phi_{\zeta\zeta}\frac{\partial}{\partial\phi_\zeta} = \frac{\partial}{\partial\zeta} + \phi_\zeta\frac{\partial}{\partial\phi} + \left(-\frac{\phi_\zeta}{\zeta} + \frac{k_1k_2\zeta^2}{4k_3} - \frac{f}{k_3}\right)\frac{\partial}{\partial\phi_\zeta}. \tag{4.28}$$

The symmetry $\upsilon = \dfrac{\partial}{\partial\phi}$ is the λ-symmetry when

$$\lambda = \frac{D_\zeta\,(Q)}{Q} = -\frac{\zeta k_1k_2(-3k_2+4\zeta)}{\zeta^2k_1k_2^2 - 8\zeta f - 16k_3\phi_\zeta}. \tag{4.29}$$

In this stage, we wish to calculate a first integral from λ. Plugging Eq. (4.29) into

$$\omega_\phi + \lambda\omega_{\phi_\zeta} = 0, \tag{4.30}$$

we obtain

$$\omega_\phi - \frac{\zeta k_1k_2(-3k_2+4\zeta)}{\zeta^2k_1k_2^2 - 8\zeta f - 16k_3\phi_\zeta}\omega_{\phi_\zeta} = 0. \tag{4.31}$$

Integrating the characteristic equation of (4.31)

$$\frac{d\phi}{1} = -\frac{\left(\zeta^2k_1k_2^2 - 8\zeta f - 16k_3\phi_\zeta\right)d\phi_\zeta}{\zeta k_1k_2(-3k_2+4\zeta)}, \tag{4.32}$$

we deduce a special solution

$$\omega\left(\zeta,\phi,\phi_\zeta\right) = \frac{-\zeta^2\phi_\zeta k_1{k_2}^2 - 4\zeta^2\phi k_1k_2 + 3\zeta\phi k_1{k_2}^2 + 8\zeta\phi_\zeta f + 8\phi_\zeta^2k_3}{\zeta k_1k_2\left(-3k_2+4\zeta\right)}. \tag{4.33}$$

Secondly, calculating function $D[\omega]$, one can get

$$D[\omega] = \frac{\partial\omega}{\partial\zeta} + \phi_\zeta\frac{\partial\omega}{\partial\phi} + \left(-\frac{\phi_\zeta}{\zeta} + \frac{k_1k_2\zeta^2}{4k_3} - \frac{f}{k_3}\right)\frac{\partial\omega}{\partial\phi_\zeta} \tag{4.34}$$

$$\begin{aligned} &= -\frac{1}{4\zeta^2k_1k_2\left(-3k_2+4\zeta\right)^2k_3}(4\zeta^6{k_1}^2{k_2}^3 - 3\zeta^5{k_1}^2{k_2}^4 \\ &\quad -32\zeta^5fk_1k_2 + 8\zeta^4fk_1{k_2}^2 - 64\zeta^3\phi_\zeta k_1{k_2}^2k_3 + 12\zeta^3fk_1{k_2}^3 \\ &\quad +36\zeta^2\phi_\zeta k_1{k_2}^3k_3 + 128\zeta^3f^2 + 512\zeta^2\phi_\zeta fk_3 - 96\zeta^2f^2k_2 \\ &\quad +512\zeta\phi_\zeta^2{k_3}^2 - 288\zeta\phi_\zeta fk_2k_3 - 288\phi_\zeta^2k_2{k_3}^2) \end{aligned} \tag{4.35}$$

$$\begin{aligned} &= -\frac{1}{4\zeta k_3\left(-3k_2+4\zeta\right)k_1k_2}({k_1}^2{k_2}^3\zeta^4 - 8fk_1k_2\zeta^3 - 4\zeta^2fk_1{k_2}^2 \\ &\quad +64\zeta\phi k_1k_2k_3 + 64\zeta\omega k_1k_2k_3 - 36\phi k_1{k_2}^2k_3 - 36\omega k_1{k_2}^2k_3 + 32\zeta f^2) \end{aligned} \tag{4.36}$$

$$= F\left(\zeta,\phi,\omega\right). \tag{4.37}$$

Next, calculating the first-order partial differential equation

$$G_\zeta + \phi_\zeta G_\phi - \frac{\begin{pmatrix} {k_1}^2{k_2}^3\zeta^4 - 8fk_1k_2\zeta^3 - 4\zeta^2 fk_1{k_2}^2 + 64\zeta\phi k_1k_2k_3 \\ +64\zeta\omega k_1k_2k_3 - 36\phi k_1{k_2}^2k_3 - 36\omega k_1{k_2}^2k_3 + 32\zeta f^2 \end{pmatrix} G_\omega}{4\zeta k_3\left(-3k_2+4\zeta\right)k_1k_2} = 0, \tag{4.38}$$

and solving the corresponding characteristic equation, we get a special solution

$$\begin{aligned} G\left(\zeta,\phi,\phi_\zeta,\omega\right) &= \frac{\zeta^3}{420k_1k_2k_3}(15{k_1}^2{k_2}^3\zeta^4 + 1344\phi_\zeta\zeta^2k_1k_2k_3 - 945\phi_\zeta\zeta\,k_1{k_2}^2k_3 \\ &\quad -140\,fk_1k_2\zeta^3 - 84\,\zeta^2fk_1{k_2}^2 + 1680\,(-\zeta\,\phi_\zeta + \phi)\,\zeta\,k_1k_2k_3 \\ &\quad -1260\,(-\zeta\,\phi_\zeta + \phi)\,k_1{k_2}^2k_3 + 1680\,\zeta\,wk_1k_2k_3 - 1260\,wk_1{k_2}^2k_3 + 840\,\zeta\,f^2). \end{aligned} \tag{4.39}$$

Finally, substituting (4.33) into (4.39), we get the first integral

$$I = \frac{\zeta^2\begin{pmatrix} 15\zeta^5{k_1}^2{k_2}^3 - 140\zeta^4fk_1k_2 - 336\phi_\zeta\zeta^3k_1k_2k_3 - 84\zeta^3fk_1{k_2}^2 \\ -105\phi_\zeta\zeta^2k_1{k_2}^2k_3 + 840\zeta^2f^2 + 3360\phi_\zeta fk_3\zeta + 3360\phi_\zeta^2{k_3}^2 \end{pmatrix}}{420k_1k_2k_3}. \tag{4.40}$$

We get the invariant solution of Eq. (4.6) by integrating Eq. (4.40) as follows

$$\phi\left(\zeta\right) = \int \frac{P\left(\zeta\right)}{6720k_3}d\zeta + C_1, \tag{4.41}$$

where

$$\begin{aligned} P\left(\zeta\right) &= 336\zeta^3k_1k_2 + 105\zeta^2k_1{k_2}^2 - 3360f\zeta \\ &\quad \pm\zeta\sqrt{-21\zeta k_1k_2\left(-5376\zeta^3k_1k_2 + 6240\zeta^2k_1{k_2}^2 - 525\zeta k_1{k_2}^3 + 17920f\zeta - 20160fk_2\right)}. \end{aligned} \tag{4.42}$$

Similarly, we get the invariant solution of Eq. (2.1) by using Eq. (4.5) as follows

$$u(x,t) = \frac{k_3t^2}{2k_1k_2} + \frac{x^2}{4}t + \int \frac{P\left(x\right)}{6720k_3}dx + C_1, \tag{4.43}$$

where

$$\begin{aligned} P\left(x\right) &= 336x^3k_1k_2 + 105x^2k_1{k_2}^2 - 3360fx \\ &\quad \pm x\sqrt{-21xk_1k_2\left(-5376x^3k_1k_2 + 6240x^2k_1{k_2}^2 - 525xk_1{k_2}^3 + 17920fx - 20160fk_2\right)}. \end{aligned} \tag{4.44}$$

5. Conservation laws of Eq.(2.1)

In order to produce conserved vectors, we perform multiplier approach [14], [23] depend upon on the famous result that the Euler–Lagrange operator annihilates a total divergence. Firstly, if (T^t, T^x) is a conserved vector related with a conservation law, then

$$D_tT^t + D_xT^x = 0, \tag{5.1}$$

on the solutions of Eq. (2.1). Furthermore, if there exists a nontrivial differential function Λ, defined a multiplier or characteristic function such that $E_u\left(\Lambda G\right) = 0$, then ΛG is a total divergence, i.e., $\Lambda G = D_tT^t + D_xT^x$ for some (conserved) vector (T^t, T^x) and E_u is the respective Euler–Lagrange operator. Thus, knowledge of each multiplier $\Lambda(x, t, u, u_x)$ leads to conserved vectors computed by a homotopy operator [14, 15, 23]. For Eq. (2.1), we obtain the multipliers, Λ_1, Λ_2, that are given by

$$\Lambda_1(x, t, u, u_x, u_t) = \frac{xJ_0\left(\sqrt{-c_1}x\right)}{\mathrm{e}^{\frac{k_3c_1t}{k_1k_2}}}, \tag{5.2}$$

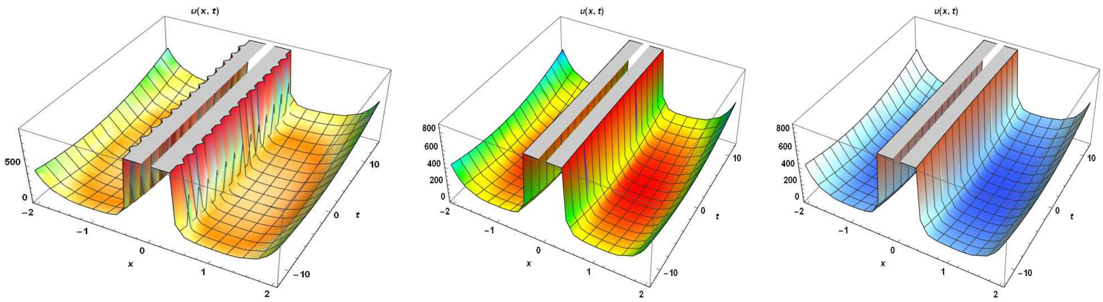

Figure 1. Three-dimensional plots of the solution (4.7) setting all arbitrary parameters to unity.

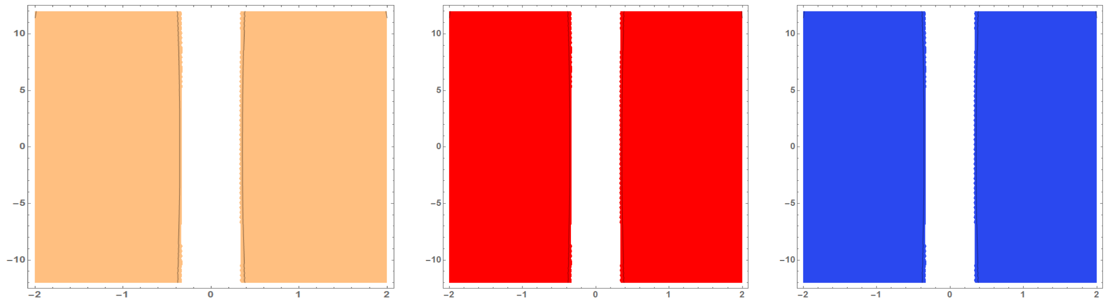

Figure 2. Contour plots of the solution (4.7) setting all arbitrary parameters to unity.

$$\Lambda_2(x,t,u,u_x,u_t) = \frac{xY_0\left(\sqrt{-c_1}x\right)}{\mathrm{e}^{\frac{k_3 c_1 t}{k_1 k_2}}}, \tag{5.3}$$

where J_0 and Y_0 are the first and second kind Bessel functions, respectively. Thus, corresponding to the above multipliers we have the following conservation laws of Eq. (2.1):

$$T^x = \frac{1}{f}\left(\begin{array}{c} -f\left(C_2u + xC_4 + C_3\,t + C_{13}\right)u_t - 2\,xu_x\left(C_{10} + \frac{C_6}{2}\right)k_3 \\ +\left(-uC_3 + C_{10}\,x^2 + \left(-2C_5\,t - C_{14}\right)x + C_{11}\,t^3 + C_9\,t^2 + C_8\,t + C_1\right)f \end{array}\right), \tag{5.4}$$

$$T^t = \frac{1}{f}\left(\begin{array}{c} f\left(C_2\,u + xC_4 + C_3\,t + C_{13}\right)u_x + \left(C_4\,f + 2\,k_1xk_2\left(C_{10} + \frac{C_6}{2}\right)\right)u \\ +\left(C_7\,x^2 + \left(C_6\,t + C_{15}\right)x + C_6\,t^2 + C_{14}\,t + C_{12}\right)f \end{array}\right). \tag{5.5}$$

6. Conclusion

In this study, laser heated CNT equation used in cancer research has been discussed and Lie group analysis has been applied in detail. In this sense, group-invariant solutions i.e, Eqs. (4.7), (4.10) and (4.43) were obtained. The accuracy of the solutions obtained has been tested and verified in Maple program. The obtained analytical solutions have been established for the first time in the literature and can be used in experimental research for cancer studies. It may also be useful in the production of suitable carbon nanotubes that can be used in later stages. Besides, local

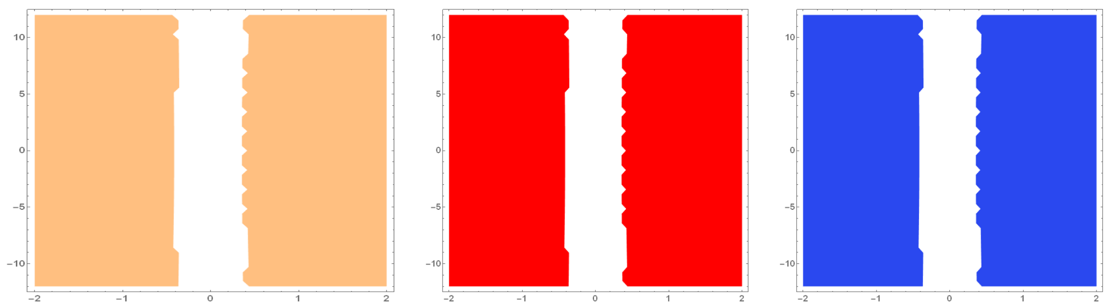

Figure 3. Density plots of the solution (4.7) setting all arbitrary parameters to unity.

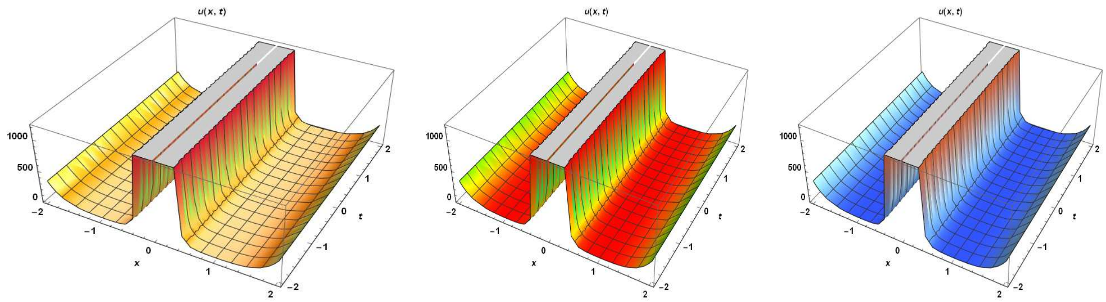

Figure 4. Three-dimensional plots of the solution (4.43) setting all arbitrary parameters to unity.

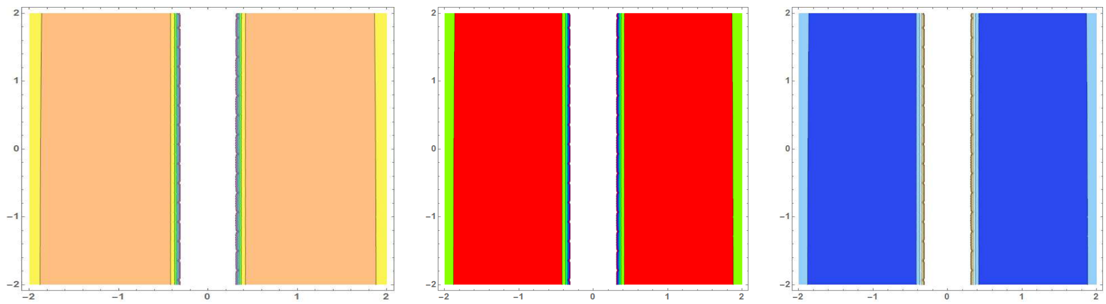

Figure 5. Contour plots of the solution (4.43) setting all arbitrary parameters to unity.

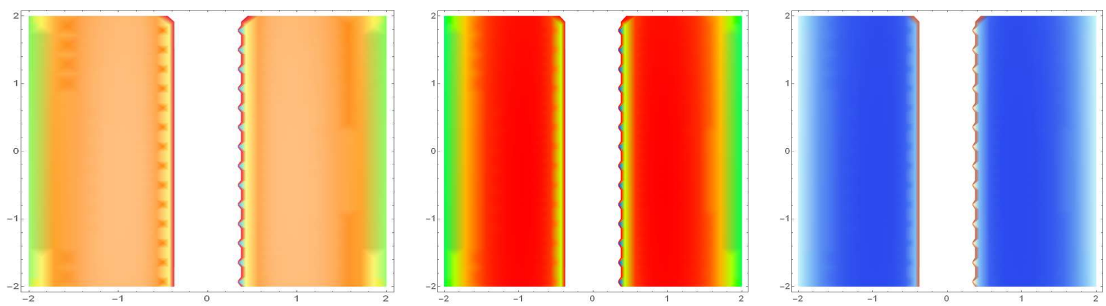

Figure 6. Density plots of the solution (4.43) setting all arbitrary parameters to unity.

conservation laws of the model were obtained. In addition, an open problem for the model is the μ-symmetry concept [24], which is a new type of symmetry, and these open questions will be addressed in future studies.

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Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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