MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

VOLUME X - ISSUE III

2022-2

VOLUME X ISSUE III
ISSN 2147-6268

September 2022
www.dergipark.org.tr/en/pub/mathenot

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

Mohammed Mursaleen
Aligarh Muslim University,
India

Ioan Raşa
Technical University of Cluj-Napoca,
Romania

Reza Saadati
Iran University of Science and Technology,
Iran

Huriye Kadakal
Bayburt University,
Turkey

Contents

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
https://doi.org/10.36753/msaen.931071
10 (3) 114-124 (2022) - Research Article
ISSN: 2147-6268

Fibonacci Lacunary Ideal Convergence of Double Sequences in Intuitionistic Fuzzy Normed Linear Spaces

Ömer Kişi

Abstract
The purpose of this article is to research the concept of Fibonacci lacunary ideal convergence of double sequences in intuitionistic fuzzy normed linear spaces (IFNS). Additionally, a new concept, called Fibonacci lacunary convergence, is examined. Fibonacci lacunary $\mathcal{I}_2$-limit points and Fibonacci lacunary $\mathcal{I}_2$-cluster points for double sequences in IFNS have been defined and the significant results have been given. Additionally, Fibonacci lacunary Cauchy and Fibonacci lacunary $\mathcal{I}_2$-Cauchy double sequences in IFNS are worked.

Keywords: *Fibonacci sequence; intuitionistic fuzzy normed linear space; limit point; cluster point.*

AMS Subject Classification (2020): *Primary: 40A30; Secondary: 40G15.*

1. Introduction and Background

Statistical convergence of single and double real sequences was firstly studied by Fast [5] and Mursaaleen and Edely [17], respectively.

$\mathcal{I}$-convergence idea was firstly considered by Kostyrko et al. [16]. Tripathy et al. [29] gave the concept of ideal convergence of double sequences in a metric space and examined fundamental features.

Using lacunary sequence, Fridy and Orhan [6] examined lacunary statistical convergence. Lacunary statistical convergence of double sequences was worked at initial stage by Savaş and Patterson [24]. Lacunary ideal convergence of real sequences was introduced by Tripathy et al. [28]. This kind of convergence extended from single to double sequences with the study of Hazarika [8]. For different studies on these topic we refer to [3, 4, 21].

After the original study of Zadeh [30], a huge number of research works have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy sets (FS), have been extensively applied in different disciplines and technologies. The theory of intuitionistic fuzzy sets (IFS) was presented by Atanassov [1]. The FS and IFS have been extensively to analyse many complex problems associated with different fields, particularly in decision-making. In IFS, membership degrees are described with a pair of a membership degree and

Received : *01-05-2021,* ***Accepted :*** *03-03-2022*

a nonmembership degree. Intuitionistic fuzzy metric space was investigated by Park [22]. In [23], motivated by Park's definition of an IF-metric, Lael and Nourouzi first defined an IF-normed space. Statistical convergence of single and double sequences in IFNS was defined by Karakuş et al [11] and Mursaleen et al [18], respectively. Some researches of convergence of sequences in some normed linear spaces in a fuzzy settings can be found in [2, 19, 20]. Also, similar works worked by some authors, see [26, 27].

Fibonacci gave Fibonacci sequences which was published in the book 'Liber Abaci'. This sequences were earlier stated as Virahanka numbers by Indian mathematics [7]. The sequence

$$(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...)$$

is known as Fibonacci sequence [15].

Kara and Başarır [9] prensented the first applications of Fibonacci sequence in the sequence spaces. Then, Kara [10] acquired the Fibonacci difference matrix $\widehat{F}$ via Fibonacci sequence (f_n) and described some new sequence spaces in this connection. Recently, Kirişci [12] thought the Fibonacci statistical convergence on IFNS. Kişi and Tuzcuoğlu [13] examined Fibonacci lacunary statistical convergence on IFNS. Additionally, Fibonacci ideal convergence of double sequences in IFNS was worked by Kişi and Güler [14].

Let us start with fundamental definitions from the literature.

Let $\emptyset \neq S$ be a set, and then $\emptyset \neq \mathcal{I} \subseteq P(S)$ is said to be an ideal on S iff (i) $\emptyset \in \mathcal{I}$, (ii) $\mathcal{I}$ is additive under union, (iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we get $B \in \mathcal{I}$. A non-empty family of sets $\mathcal{F}$ is called filter on S iff (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we get $A \cap B \in \mathcal{F}$, (iii) for every $A \in \mathcal{F}$ and each $B \supseteq A$, we obtain $B \in \mathcal{F}$. Correlation between ideal and filter is specified as follows:

$$\mathcal{F}(\mathcal{I}) = \{K \subset S : K^c \in \mathcal{I}\},$$

where $K^c = S - K$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is named as strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

Throughout the paper, we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Let (X, ρ) be a metric space. A double sequence $x = (x_{mn})$ is named as $\mathcal{I}_2$-convergent to ξ, if for any $\varepsilon > 0$ we get $P(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, \xi) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = \xi.$$

A double sequence $\overline{\theta} = \theta_{us} = \{(k_u, l_s)\}$ is named as double lacunary sequence if there are two increasing sequences of integers (k_u) and (l_s) such that

$$k_0 = 0,\, h_u = k_u - k_{u-1} \to \infty \text{ and } l_0 = 0,\, \overline{h}_s = l_s - l_{s-1} \to \infty, \quad u, s \to \infty.$$

We utilize the subsequent notations

$$k_{us} := k_u l_s,\, h_{us} := h_u \overline{h}_s$$

and θ_{us} is determined by

$$J_{us} := \{(k, l) : k_{u-1} < k \leq k_u \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_u := \frac{k_u}{k_{u-1}},\, \overline{q}_s := \frac{l_s}{l_{s-1}} \text{ and } q_{us} := q_u \overline{q}_s.$$

Throughout the paper, by $\theta_2 = \theta_{us} = \{(k_u, l_s)\}$ we will indicate a double lacunary sequence.

Schweizer and Sklar [25] defined continuous t-norm and t-conorm. Using the continuous t-norm and t-conorm, Lael and Nourouzi [23] defined the concept of IFNS as follows:

The five-tuple $(X, \phi, \omega, *, \diamond)$ is named as IFNS if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and ϕ, ω are fuzzy sets on $X \times (0, \infty)$ fulfilling the subsequent conditions: For every $a, b \in X$ and $p, q > 0$:

(i) $\phi(a,q) + \omega(a,q) \leq 1$,

(ii) $\phi(a,q) > 0$,

(iii) $\phi(a, q) = 1$ if and only if $a = 0$,

(iv) $\phi(ca, q) = \phi\left(a, \frac{q}{|c|}\right)$ if $c \neq 0$,

(v) $\phi(a, q) * \phi(b, p) \leq \phi(a + b, q + p)$,

(vi) $\phi(a,.) : (0, \infty) \to [0, 1]$ is continuous in q;

(vii) $\lim_{q\to\infty} \phi(a, q) = 1$ and $\lim_{q\to 0} \phi(a, q) = 0$,

$(viii)\ \omega(a,q)<1,$
$(ix)\ \omega(a,q)=0$ if and only if $a=0,$
$(x)\ \omega(ca,q)=\omega\left(a,\frac{q}{|c|}\right)$ if $c\neq 0,$
$(xi)\ \omega(a,q)\diamond\omega(b,p)\geq\omega(a+b,q+p),$
$(xii)\ \omega(a,.):(0,\infty)\to[0,1]$ is continuous in q;
$(xiii)\ \lim_{q\to\infty}\omega(a,q)=0$ and $\lim_{q\to 0}\omega(a,q)=1.$

2. Main Results

Definition 2.1. Let $(X,\phi,\omega,*,\diamond)$ be an IFNS. A double sequence $x=(x_{kl})$ in X is named as Fibonacci lacunary convergent to ξ with regards to the IFN (ϕ,ω) if, for every $t>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon$$

for all $u,s\geq r_0$. In this case, we write $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$.

Theorem 2.1. *If $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$, then $(\phi,\omega)^{\theta_{us}}-\lim Fx$ is unique.*

Proof. Presume that $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi_1$ and $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi_2$. Given $\varepsilon>0$, select $\gamma\in(0,1)$ such that $(1-\gamma)*(1-\gamma)>1-\varepsilon$ and $\gamma\diamond\gamma<\varepsilon$. Now, for all $t>0$, there is $r_1\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi_1,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi_1,t\right)<\varepsilon$$

for all $u,s\geq r_1$. Also, there is $r_2\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi_2,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi_2,t\right)<\varepsilon$$

for all $u,s\geq r_2$. Consider $r_0=\max\{r_1,r_2\}$. Then, for $u,s\geq r_0$, we take a $(m,p)\in\mathbb{N}\times\mathbb{N}$ such that

$$\phi\left(\widehat{F}x_{mp}-\xi_1,\frac{t}{2}\right)>\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi_1,\frac{t}{2}\right)>1-\gamma$$

and

$$\phi\left(\widehat{F}x_{mp}-\xi_2,\frac{t}{2}\right)>\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi_2,\frac{t}{2}\right)>1-\gamma.$$

Then, we obtain

$$\phi(\xi_1-\xi_2,t)\geq\phi\left(\widehat{F}x_{mp}-\xi_1,\tfrac{t}{2}\right)*\phi\left(\widehat{F}x_{mp}-\xi_2,\tfrac{t}{2}\right)$$

$$>(1-\gamma)*(1-\gamma)>1-\varepsilon.$$

Since $\varepsilon>0$ is abitrary, we have $\phi(\xi_1-\xi_2,t)=1$ for every $t>0$, which gives that $\xi_1=\xi_2$. □

Definition 2.2. A double sequence $x=(x_{kl})$ in IFNS is named as Fibonacci lacunary $\mathcal{I}_2$-convergent to ξ with regards to the IFN (ϕ,ω) if, for each $\varepsilon>0$ and $t>0$, the set

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\in\mathcal{I}_2.$$

ξ is named the Fibonacci $\mathcal{I}_\theta$-limit of the sequence of (x_{kl}), and we note $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)}-\lim x=\xi$.

Lemma 2.1. *For every $\varepsilon > 0$ and $t > 0$, the following demonstrations are equivalent.*

(a) $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi,$

(b) $\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(x_{kl} - \xi, t\right) \leq 1 - \varepsilon\right\} \in \mathcal{I}_2$ *and*

$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(x_{kl} - \xi, t\right) \geq \varepsilon\right\} \in \mathcal{I}_2,$

(c) $\left\{\begin{array}{c} (u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1 - \varepsilon \\ \text{and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon \end{array}\right\} \in \mathcal{F}(\mathcal{I}_2),$

(d) $\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1 - \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2)$ *and*

$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2)$ *and*

(e) $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim \phi\left(\widehat{F}x_{kl} - \xi, t\right) = 1$ *and* $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim \omega\left(\widehat{F}x_{kl} - \xi, t\right) = 0.$

Theorem 2.2. *If a sequence $x = (x_{kl})$ in IFNS is Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ,ω), then $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x$ is unique.*

Proof. Assume that $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_1$ and $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_2$. Given $\varepsilon \in (0,1)$, select $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \diamond \gamma < \varepsilon$. Then, for any $t > 0$, take the following sets:

$$K_{\phi,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \tfrac{t}{2}\right) \leq 1 - \gamma\right\},$$

$$K_{\phi,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \tfrac{t}{2}\right) \leq 1 - \gamma\right\},$$

$$K_{\omega,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi_1, \tfrac{t}{2}\right) \geq \gamma\right\},$$

$$K_{\omega,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi_2, \tfrac{t}{2}\right) \geq \gamma\right\}.$$

Since $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_1$, applying Lemma 2.1, we get $K_{\phi,1}(\gamma,t) \in \mathcal{I}_2$ and $K_{\omega,1}(\gamma,t) \in \mathcal{I}_2$ for every $t > 0$. Using $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi_2$, we have $K_{\phi,2}(\gamma,t) \in \mathcal{I}_2$ and $K_{\omega,2}(\gamma,t) \in \mathcal{I}_2$ for all $t > 0$.

Now, take $K_{\phi,\omega}(\gamma,t) = \left(K_{\phi,1}(\gamma,t) \cup K_{\phi,2}(\gamma,t)\right) \cap \left(K_{\omega,1}(\gamma,t) \cup K_{\omega,2}(\gamma,t)\right)$. Then, $K_{\phi,\omega}(\gamma,t) \in \mathcal{I}_2$. This gives that $K_{\phi,\omega}^c(\gamma,t) \neq \emptyset$ in $\mathcal{F}(\mathcal{I}_2)$. If $(u,s) \in K_{\phi,\omega}^c(\gamma,t)$, first, contemplate the case $(u,s) \in \left(K_{\phi,1}^c(\gamma,t) \cap K_{\phi,2}^c(\gamma,t)\right)$. Then, we get

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \tfrac{t}{2}\right) > 1 - \gamma \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \tfrac{t}{2}\right) > 1 - \gamma.$$

Now, obviously, we will get a $(p,q) \in \mathbb{N}\times\mathbb{N}$ such that

$$\phi\left(\widehat{F}x_{pq} - \xi_1, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_1, \frac{t}{2}\right) > 1 - \gamma$$

and

$$\phi\left(\widehat{F}x_{pq} - \xi_2, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi_2, \frac{t}{2}\right) > 1 - \gamma$$

(That is, consider $\max\left\{\phi\left(\widehat{F}x_{kl}-\xi_1,\frac{t}{2}\right),\phi\left(\widehat{F}x_{kl}-\xi_2,\frac{t}{2}\right):(k,l)\in J_{us}\right\}$ and select that (k,l) as (p,q) for which the maximum occurs).

Then, we obtain

$$\begin{aligned}\phi(\xi_1-\xi_2,t) &\geq \phi\left(\widehat{F}x_{pq}-\xi_1,\tfrac{t}{2}\right)*\phi\left(\widehat{F}x_{pq}-\xi_2,\tfrac{t}{2}\right)\\ &> (1-\gamma)*(1-\gamma)>1-\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get $\phi(\xi_1-\xi_2,t)=1$ for each $t>0$, which gives that $\xi_1=\xi_2$. At the same time, if $(u,s)\in K^c_{\phi,1}(\gamma,t)\cap K^c_{\phi,2}(\gamma,t)$, then by using the similar method, it can be demonstrated that $\omega(\xi_1-\xi_2,t)<\varepsilon$, for arbitrary $\varepsilon>0$ and for every $t>0$, and so $\xi_1=\xi_2$. Hence, in all cases, we deduce that $F\mathcal{I}^{(\phi,\omega)}_{\theta_{us}}-\lim x$ is unique. □

Theorem 2.3. *If $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$, then $F\mathcal{I}^{(\phi,\omega)}_{\theta_{us}}-\lim x=\xi$.*

Proof. Let $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$. Then, for every $t>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon$$

for all $u,s\geq r_0$. Therefore, we obtain

$$\begin{aligned}A&=\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\\ &\subseteq\{(1,1),(2,2),...,(k_0-1,k_0-1)\}.\end{aligned}$$

But, with $\mathcal{I}_2$ being admissible ideal, we get $A\in\mathcal{I}_2$. Hence, $F\mathcal{I}^{(\phi,\omega)}_{\theta_{us}}-\lim x=\xi$. □

Theorem 2.4. *If $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$, then there exists a subsequence $\left(x_{k'(u)l'(s)}\right)$ of x such that $(\phi,\omega)^{\theta_{us}}-\lim Fx_{k'(u)l'(s)}=\xi$.*

Proof. Let $(\phi,\omega)^{\theta_{us}}-\lim Fx=\xi$. Then, for every $t>0$ and $\varepsilon\in(0,1)$, there exists $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon$$

for all $u,s\geq r_0$. Obviously, for each $u,s\geq r_0$, we can select $(k'(u),l'(s))\in J_{us}$ such that

$$\phi\left(\widehat{F}x_{k'(u)l'(s)}-\xi,t\right)>\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)>1-\varepsilon$$

$$\text{and } \omega\left(\widehat{F}x_{k'(u)l'(s)}-\xi,t\right)<\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)<\varepsilon.$$

It follows that $(\phi,\omega)^{\theta_{us}}-\lim Fx_{k'(u)l'(s)}=\xi$. □

Definition 2.3. A double sequence $x=(x_{jk})$ in IFNS is named as Fibonacci lacunary Cauchy with regards to the IFN (ϕ,ω) if, for each $\varepsilon>0$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that, for all $j,p\geq N$, $k,q\geq M$,

$$\frac{1}{h_{us}}\sum_{(j,k),(p,q)\in J_{us}}\phi\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)>1-\varepsilon \text{ and } \frac{1}{h_{us}}\sum_{(j,k),(p,q)\in J_{us}}\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)<\varepsilon.$$

Definition 2.4. A double sequence $x=(x_{jk})$ in IFNS is named as Fibonacci lacunary $\mathcal{I}_2$-Cauchy with regards to the IFN (ϕ,ω) if, for every $\varepsilon\in(0,1)$ and $t>0$, there exists $(p,q)\in\mathbb{N}\times\mathbb{N}$ fulfilling

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\phi\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)>1-\varepsilon\\ \text{and } \frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{pq},t\right)<\varepsilon\end{array}\right\}\in\mathcal{F}(\mathcal{I}_2).$$

Definition 2.5. A double sequence $x = (x_{jk})$ in IFNS is named as Fibonacci lacunary $\mathcal{I}_2^*$-Cauchy with regards to the IFN (ϕ, ω) if there is a subset $M = \{(j_m, k_m) : j_1 < j_2 < ...; k_1 < k_2 < ...\}$ of $\mathbb{N} \times \mathbb{N}$ such that the set $M' = \{(u, s) \in \mathbb{N} \times \mathbb{N} : (j_u, k_s) \in J_{us}\} \in \mathcal{F}(\mathcal{I}_2)$ and the subsequence $(x_{j_u k_s})$ is a Fibonacci lacunary Cauchy sequence with regards to the IFN (ϕ, ω).

Theorem 2.5. *A double sequence $x = (x_{jk})$ is Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ, ω) iff it is Fibonacci lacunary $\mathcal{I}_2$-Cauchy with regards to (ϕ, ω).*

Proof. Let $x = (x_{jk})$ be Fibonacci lacunary $\mathcal{I}_2$-convergent to ξ with regards to the IFN (ϕ, ω). Then

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \le 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \ge \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

Specifically, for $j = M$, $k = N$

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(M,N) \in J_{us}} \phi\left(\widehat{F}x_{MN} - \xi, \frac{t}{2}\right) \le 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(M,N) \in J_{us}} \omega\left(\widehat{F}x_{MN} - \xi, \frac{t}{2}\right) \ge \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

Since

$$\begin{aligned} \phi\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) &= \phi\left(\widehat{F}x_{jk} - \xi - \widehat{F}x_{MN} + \xi, \frac{t}{2} + \frac{t}{2}\right) \\ &\ge \phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) * \phi\left(\widehat{F}x_{MN} - \xi, \frac{t}{2}\right) \end{aligned}$$

and

$$\omega\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \le \omega\left(x_{jk} - \xi, \frac{t}{2}\right) \diamond \omega\left(x_{MN} - \xi, \frac{t}{2}\right),$$

we obtain

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \mu\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \le 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \ge \varepsilon \end{array} \right\} \in \mathcal{I}_2.$$

That is, x is Fibonacci $\mathcal{I}_2$-lacunary Cauchy with regards to (ϕ, ω).

In contrast, let $x = (x_{jk})$ be Fibonacci $\mathcal{I}_2$-lacunary Cauchy but not Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ, ω). Then, there are N and M such that the set $A(\varepsilon, t) \in \mathcal{I}_2$, where

$$A(\varepsilon, t) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \phi\left(x_{jk} - x_{MN}, t\right) \le 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(x_{kl} - x_{MN}, t\right) \ge \varepsilon \end{array} \right\}$$

and also $B(\varepsilon, t) \in \mathcal{I}_2$, where

$$B(\varepsilon, t) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N} : \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \le 1 - \varepsilon \\ \text{or } \dfrac{1}{h_{us}} \sum\limits_{(j,k) \in J_{us}} \omega\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) \ge \varepsilon \end{array} \right\}.$$

Since

$$\phi\left(\widehat{F}x_{jk} - \widehat{F}x_{MN}, t\right) \ge 2\phi\left(\widehat{F}x_{jk} - \xi, \frac{t}{2}\right) > 1 - \varepsilon,$$

and

$$\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{MN},t\right)\leq 2\omega\left(\widehat{F}x_{jk}-\xi,\frac{t}{2}\right)<\varepsilon,$$

if $\phi\left(\widehat{F}x_{jk}-\xi,\frac{t}{2}\right)>\frac{(1-\varepsilon)}{2}$ and $\omega\left(\widehat{F}x_{jk}-\xi,\frac{t}{2}\right)<\frac{\varepsilon}{2}$. Therefore,

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\dfrac{1}{h_{us}}\sum\limits_{(j,k)\in J_{us}}\phi\left(\widehat{F}x_{jk}-\widehat{F}x_{MN},t\right)>1-\varepsilon\\ \text{or }\dfrac{1}{h_{us}}\sum\limits_{(j,k)\in J_{us}}\omega\left(\widehat{F}x_{jk}-\widehat{F}x_{MN},t\right)<\varepsilon\end{array}\right\}\in\mathcal{I}_2,$$

that is, $A^c(\varepsilon,t)\in\mathcal{I}_2$ and hence, $A(\varepsilon,t)\in\mathcal{F}(\mathcal{I}_2)$, which leads to a contradiction. Hence x must be Fibonacci lacunary $\mathcal{I}_2$-convergent with regards to the IFN (ϕ,ω). □

Theorem 2.6. *If (ρ_{us}) is a double lacunary refinement of θ_{us} and $F\mathcal{I}_{\rho_{us}}^{(\phi,\omega)}-\lim x=\xi$, then $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)}-\lim x=\xi$.*

Proof. Presume that each $\mathcal{I}_{us}$ of θ_{us} involves the points $\left(\overline{k}_{u,i},\overline{l}_{s,j}\right)_{i,j=1}^{v(u),w(s)}$ of (ρ_{us}) so that

$$k_{u-1}<\overline{k}_{u,1}<\overline{k}_{u,2}<...<\overline{k}_{u,v(u)}=k_u,\text{ where }\overline{I}_{u,i}=\left(\overline{k}_{u,i-1},\overline{k}_{u,i}\right],$$

$$l_{s-1}<\overline{l}_{s,1}<\overline{l}_{s,2}<...<\overline{l}_{s,w(s)}=l_s,\text{ where }\overline{J}_{s,j}=\left(\overline{l}_{s,j-1},\overline{l}_{s,j}\right],$$

and

$$\overline{J}_{u,s,i,j}=\left\{(k,l):\overline{k}_{u,i-1}<k\leq\overline{k}_u;\overline{l}_{s,j-1}<l\leq\overline{l}_s\right\}$$

for all u,s and $v(u)\geq 1,w(s)\geq 1$ this gives that $(k_u,l_s)\subseteq\left(\overline{k}_u,\overline{l}_s\right)$. Let $\left(\overline{J}_{ij}\right)_{i,j=1,1}^{\infty,\infty}$ be the sequence of abutting blocks of $\left(\overline{J}_{u,s,i,j}\right)$ ordered by increasing a lower right index points. Since $F\mathcal{I}_{\rho_{us}}^{(\phi,\omega)}-\lim x=\xi$, we obtain the following for each $t>0$ and $\varepsilon\in(0,1)$

$$\left\{\begin{array}{c}(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\in\mathcal{I}_2. \tag{2.1}$$

As before, we take $h_{us}=h_u\overline{h}_s;\overline{h}_{ui}=\overline{k}_{ui}-\overline{k}_{u,i-1};\overline{h}_{sj}=\overline{l}_{s,j}-\overline{l}_{s,j-1}$.

For each $t>0$ and $\varepsilon\in(0,1)$ we get

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum\limits_{(k,l)\in J_{us}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{h_{us}}\sum\limits_{(k,l)\in J_{us}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}$$

$$\subseteq\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum\limits_{(k,l)\in J_{us}}\{(i,j)\in\mathbb{N}\times\mathbb{N}:\\ \left.\begin{array}{c}\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in\overline{J}_{ij}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{\overline{h}_{ij}}\sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in\overline{J}_{ij}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right\}\end{array}\right\}.$$

By (2.1), for each $t>0$ and $\varepsilon\in(0,1)$ if we define

$$t_{ij}=\left(\begin{array}{c}\frac{1}{\overline{h}_{ij}}\sum\limits_{(k,l)\in\overline{J}_{ij}}\phi\left(\widehat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \text{or }\frac{1}{\overline{h}_{i,j}}\sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in\overline{J}_{ij}}\omega\left(\widehat{F}x_{kl}-\xi,t\right)\geq\varepsilon\end{array}\right)_{i,j=1}^{\infty,\infty},$$

then $(t_{i,j})$ is a Pringsheim null sequence. The transformation

$$(At)_{us} = \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \begin{pmatrix} \overline{h}_{ij} \frac{1}{\overline{h}_{ij}} \sum\limits_{(k,l)\in \overline{J}_{ij}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) \leq 1-\varepsilon \\ \text{or } \overline{h}_{ij} \frac{1}{\overline{h}_{ij}} \sum\limits_{\overline{J}_{ij}\subset J_{us};(k,l)\in \overline{J}_{ij}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) \geq \varepsilon \end{pmatrix}$$

fulfills all situations for a matrix transformation to map a Pringsheim null sequence into a Pringsheim null sequence. Hence, $F\mathcal{I}_{\theta_{us}}^{(\phi,\omega)} - \lim x = \xi$. □

Definition 2.6. Let $(X,\phi,\omega,*,\diamond)$ be an IFNS.

(a) An element $\xi \in X$ is named as Fibonacci lacunary $\mathcal{I}_2$-limit point of $x = (x_{kl})$ if there is set $M = \{(k_1,l_1) < (k_2,l_2) < ... < (k_u,l_s) < ...\} \subset \mathbb{N}\times\mathbb{N}$ such that the set

$$M' = \{(u,s) \in \mathbb{N}\times\mathbb{N} : (k_u,l_s) \in J_{us}\} \notin \mathcal{I}_2$$

and $(\phi,\omega)^{\theta_{us}} - \lim Fx_{k_u l_s} = \xi$.

(b) $\xi \in X$ is named as Fibonacci lacunary $\mathcal{I}_2$-cluster point of $x = (x_{kl})$ if, for every $t > 0$ and $\varepsilon \in (0,1)$, we get

$$\left\{ \begin{array}{c} (u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1-\varepsilon \\ \text{and } \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon \end{array} \right\} \notin \mathcal{I}_2.$$

$\Lambda^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$ and $\Gamma^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$ indicate the set of all Fibonacci lacunary $\mathcal{I}_2$-limit points and the set of all Fibonacci lacunary $\mathcal{I}_2$-cluster points in IFNS, respectively.

Theorem 2.7. *For every sequence $x = (x_{kl})$ in IFNS, we have $\Lambda^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x) \subseteq \Gamma^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$.*

Proof. Let $\xi \in \Lambda^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$. Then, there is a set $M \subset \mathbb{N}\times\mathbb{N}$ such that the set $M' \notin \mathcal{I}_2$, where M and M' are as in Definition 2.6, fulfills $(\phi,\omega)^{\theta_{us}} - \lim Fx_{k_u l_s} = \xi$. Hence, for every $t > 0$ and $\varepsilon \in (0,1)$, there are $u_0, s_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{k_u l_s} - \xi, t\right) > 1-\varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{k_u l_s} - \xi, t\right) < \varepsilon$$

for all $u \geq u_0$, $s \geq s_0$. Therefore,

$$B = \left\{ \begin{array}{c} (u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1-\varepsilon \\ \text{and } \frac{1}{h_{us}} \sum\limits_{(k,l)\in J_{us}} \omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon \end{array} \right\}$$

$$\supseteq M' \backslash \{(k_1,l_1) < (k_2,l_2) < ... < (k_{u_0}, l_{s_0})\}.$$

Now, with $\mathcal{I}_2$ being admissible, we must have $M' \backslash \{(k_1,l_1) < (k_2,l_2) < ... < (k_{u_0}, l_{s_0})\} \notin \mathcal{I}_2$ and as such $B \notin \mathcal{I}_2$. Hence, $\xi \in \Gamma^{F\mathcal{I}_2}_{(\phi,\omega)^{\theta_{us}}}(x)$. □

Theorem 2.8. *The following observations are equivalent.*

(a) ξ is Fibonacci lacunary $\mathcal{I}_2$-limit point of x.

(b) There are two sequences $y = (y_{kl})$ and $z = (z_{kl})$ in IFNS such that $x = y + z$ and $(\phi,\omega)^{\theta_{us}} - \lim Fy = \xi$ and

$$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : (k,l) \in J_{us}, z_{kl} \neq \overline{0}\right\} \in \mathcal{I}_2.$$

Proof. Presume that (a) holds. Then there are M and M' are as in Definition 2.6 such that $M' \notin \mathcal{I}_2$ and $(\phi, \omega)^{\theta_{us}} - \lim Fx = \xi$. Take the sequences y and z as follows:

$$y_{kl} = \begin{cases} x_{kl}, & \text{if } (k,l) \in J_{us},\ (u,s) \in M' \\ \xi, & \text{otherwise.} \end{cases}$$

and

$$z_{kl} = \begin{cases} \overline{0}, & \text{if } (k,l) \in J_{us},\ (u,s) \in M' \\ x_{kl} - \xi, & \text{otherwise.} \end{cases}$$

It is adequate to think the case $(k,l) \in J_{us}$ such that $(u,s) \in \mathbb{N} \times \mathbb{N} \setminus M'$. Then, for each $t > 0$ and $\varepsilon \in (0,1)$. Then, we have $\phi\left(\widehat{F}y_{kl} - \xi, t\right) = 1 > 1 - \varepsilon$ and $\omega\left(\widehat{F}y_{kl} - \xi, t\right) = 0 < \varepsilon$. Thus, we write

$$\frac{1}{h_{us}} \sum_{(k,l) \in J_{us}} \phi\left(\widehat{F}y_{kl} - \xi, t\right) = 1 > 1 - \varepsilon \text{ and } \frac{1}{h_{us}} \sum_{(k,l) \in J_{us}} \omega\left(\widehat{F}y_{kl} - \xi, t\right) = 0 < \varepsilon.$$

Hence, $(\phi, \omega)^{\theta_{us}} - \lim y = \xi$. Now, we have

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : (k,l) \in J_{us},\ z_{kl} \neq \overline{0}\right\} \subset \mathbb{N} \times \mathbb{N} \diagdown M'.$$

But $\mathbb{N} \times \mathbb{N} \diagdown M' \in \mathcal{I}_2$, and so

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : (k,l) \in J_{us},\ z_{kl} \neq \overline{0}\right\} \in \mathcal{I}_2.$$

Now, presume that (b) holds. Let $M' = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : (k,l) \in J_{us},\ z_{kl} = \overline{0}\right\}$. Then, obviously $M' \in \mathcal{F}(\mathcal{I}_2)$ and so it is an infinite set. Construct the set

$$M = \{(k_1, l_1) < (k_2, l_2) < \ldots < (k_u, l_s) < \ldots\} \subset \mathbb{N} \times \mathbb{N}$$

such that $(k_u, l_s) \in J_{us}$ and $z_{k_u l_s} = \overline{0}$. Since $x_{k_u l_s} = y_{k_u l_s}$ and $(\phi, \omega)^{\theta_{us}} - \lim Fy = \xi$ we obtain $(\phi, \omega)^{\theta_{us}} - \lim Fx_{k_u l_s} = \xi$. □

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Atanassov, K.T.: *Intuitionistic fuzzy sets*. Fuzzy Sets and Syst. **20**, 87-96 (1986).

[2] Debnath, P.: *Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces*. Comput. Math. Appl. **63**, 708-715 (2012).

[3] Dündar, E., Ulusu, U., Pancaroğlu, N.: *Strongly $\mathcal{I}_2$-lacunary convergence and $\mathcal{I}_2$-lacunary Cauchy double sequences of sets*. Aligarh Bull. Math. **35** (1-2), 1-15 (2016).

[4] Dündar, E., Ulusu, U., Aydın, B.: *$\mathcal{I}_2$-lacunary statistical convergence of double sequences of sets*. Konuralp J. Math. **5** (1), 1-10 (2017).

[5] Fast, H.: (1951). *Sur la convergence statistique*. Colloq. Math. **2**, 241-244 (1951).

[6] Fridy, J.A., Orhan, C.: *Lacunary statistical convergence.* Pacific J. Math. **160** (1), 43-51 (1993).

[7] Goonatilake S.: Toward a Global Science. Indiana University Press. (1998).

[8] Hazarika, B.: *Lacunary ideal convergence of multiple sequences*. J. Egyptian Math. Soc. 24, 54-59 (2016).

[9] Kara, E.E., Başarır, M.: *An application of Fibonacci numbers into infinite Toeplitz matrices*. Casp. J. Math. Sci. **1** (1), 43-47 (2012).

[10] Kara, E.E.: *Some topological and geometrical properties of new Banach sequence spaces*. J. Inequal. Appl. **38** (2013), (2013).

[11] Karakuş, S., Demirci, K., Duman, O.: *Statistical convergence on intuitionistic fuzzy normed spaces*. Chaos Solitons Fractals. **35**, 763-769 (2008).

[12] Kirişçi, M., Karaisa, A.: *Fibonacci statistical convergence and Korovkin type approximation theorems*. J. Inequal. Appl. **2017** (229), 1-15 (2017).

[13] Kişi, Ö., Tuzcuoğlu, İ.: *Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces*. J. Progress Res. Soc. Sci. **16** (3), 3001-3007 (2020).

[14] Kişi, Ö., Güler, E.: *On Fibonacci ideal convergence of double sequences in intuitionistic fuzzy normed linear spaces*. Turk. J. Math. Comput. Sci. **11**, 46-55 (2019).

[15] Koshy, T.: Fibonacci and Lucas Numbers with Applications. Wiley, New York. (2001).

[16] Kostyrko, P., Salat, T., Wilczynsski, W.: *$\mathcal{I}$-convergence*. Real Anal. Exchange. **6** (2), 669-686 (2001).

[17] Mursaleen, M., Edely, O.H.: *Statistical convergence of double sequences*. J. Math. Anal. Appl. **288**, 223-231 (2003).

[18] Mursaleen, M., Mohiuddine, S.A.: *Statistical convergence of double sequences in intuitionistic fuzzy normed space*. Chaos Solitons Fractals. **41**, 2414-2421 (2009).

[19] Mursaleen, M., Mohiuddine, S.A., Edely, O.H.H.: *On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces.* Comput. Math. Appl. **59**, 603-611 (2010).

[20] Mursaleen, M., Mohiuddine, S.A.: *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*. J. Comput. Appl. Math. **233** (2), 142-149 (2009).

[21] Nuray, F., Ulusu, U., Dündar, E.: *Lacunary statistical convergence of double sequences of sets*. Soft Comput. **20**, 2883-2888 (2016).

[22] Park, J.H.: *Intuitionistic fuzzy metric spaces*. Chaos Solitons Fractals. **22**, 1039-1046 (2004).

[23] Lael, F., Nourouzi, K.: *Some results on the IF-normed spaces*. Chaos Solitons Fractals. **37**, 931-939 (2008).

[24] Savaş, E., Patterson, R.F.: *Lacunary statistical convergence of double sequences*. Math. Commun. **10**, 55-61 (2005).

[25] Schweizer, B., Sklar, A.: *Statistical metric spaces*. Pacific J. Math. **10**, 314-334. (1960).

[26] Temizer Ersoy, M., Furkan, H.: *Distinguished supspaces in topological sequence spaces theory*. AIMS Math. **5** (4), 2858-2868 (2020).

[27] Temizer Ersoy, M.: Some abelian, tauberian and core theorems related to the (V, λ) summability. Univers. J. Math. Appl. **4** (2), 70-75.

[28] Tripathy, B.C., Hazarika, B., Choudhary, B.B.: *Lacunary $\mathcal{I}$-convergent sequences*. Kyungpook Math. J. **2** (4), 473-482 (2012).

[29] Tripathy, B.K., Tripathy, B.C.: *On $\mathcal{I}$-convergent double sequences*. Soochow J. Math. **31** (4), 549-560 (2005).

[30] Zadeh, L.A.: *Fuzzy sets*. Inf.Control. **8**, 338-353 (1965). (2021).

Affiliations

ÖMER KIŞI
ADDRESS: Bartın University, Dept. of Mathematics, 74100, Bartın-Turkey.
E-MAIL: okisi@bartin.edu.tr
ORCID ID:0000-0001-6844-3092

Mathematical Sciences and Applications E-Notes
https://doi.org/10.36753/msaen.893469
10 (3) 125-134 (2022) - Research Article
ISSN: 2147-6268

Gould-Hopper Based Degenerate Truncated Bernoulli Polynomials

Uğur Duran

Abstract
In this study, we consider Gould-Hopper based truncated degenerate Bernoulli polynomials and examine diverse properties and formulas covering addition formulas, correlations and derivation property. Then, we derive some interesting implicit summation formulas and symmetric identities. Moreover, we define Gould-Hopper based truncated degenerate Bernoulli polynomials of order r and give some of their properties and relations.

Keywords: *Degenerate exponential function; truncated exponential function; Bernoulli polynomials; Gould-Hopper polynomials; exponential generating function.*

AMS Subject Classification (2020): *Primary: 11B73; Secondary: 11B68; 33B10.*

1. Introduction

Along this paper, the usual notations $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$, are referred to the set of all natural numbers, the set of all non-negative integers, the set of all real numbers and the set of all complex numbers, respectively.

The truncated form of the exponential polynomials $e_n(z)$ are the first $(n+1)$ terms of the Taylor series for e^z (*cf.* [3]) at $z=0$, namely

$$e_n(z)=\sum_{k=0}^{n}\frac{z^k}{k!}. \tag{1.1}$$

One can see [3] to get the detailed information about $e_n(z)$.

For $\lambda\in\mathbb{C}$, the λ-falling factorial $(z)_{n,\lambda}$ is defined by $(z)_{n,\lambda}=z(z-\lambda)(z-2\lambda)\cdots(z-(n-1)\lambda)$ for $n\in\mathbb{N}$ with $(z)_{0,\lambda}=1$, *cf.* [1,4,8,12]. In the case $\lambda=1$, the λ-falling factorial becomes to the usual falling factorial given by $(z)_{n,1}:=(z)_n=z(z-1)\cdots(z-n+1)$ with $(z)_{0,1}=1$.

Let $\lambda\in\mathbb{R}/\{0\}$. The degenerate form of the exponential function $e_\lambda^z(z)$ is defined by (*cf.* [1,4,5,8,10-14])

$$e_\lambda^\omega(z)=(1+\lambda z)^{\frac{\omega}{\lambda}} \text{ and } e_\lambda^1(z):=e_\lambda(z). \tag{1.2}$$

Received : *09-03-2021,* ***Accepted :*** *06-07-2021*

We note that $\lim_{\lambda\to 0} e_\lambda^\omega(z) = e^{\omega z}$. From (1.2), we attain

$$e_\lambda^\omega(z) = \sum_{n=0}^{\infty} (\omega)_{n,\lambda} \frac{z^n}{n!}. \tag{1.3}$$

The degenerate truncated form of the exponential polynomials (also called the $Detr$-exponential polynomials) are considereed as the first $(n+1)$ terms of the Mac Laurin series expansion of $e_\lambda(z)$ in (1.3) (*cf.* [8]):

$$e_{n,\lambda}(z) = \sum_{k=0}^{n} (1)_{k,\lambda} \frac{z^k}{k!}. \tag{1.4}$$

Also, when $\lambda \to 0$, the polynomials $e_{n,\lambda}(z)$ (1.4) become the polynomials $e_n(z)$ in (1.1). To get more detailed information about the $Detr$-exponential polynomials and their properties, see [8].

The Stirling numbers $S_2(n,k)$ and polynomials $S_2(n,k:\omega)$ of the second kind are provided as follows (*cf.* [8,12,17]):

$$\sum_{n=0}^{\infty} S_2(n,k) \frac{z^n}{n!} = \frac{(e^z-1)^k}{k!} \text{ and } \sum_{n=0}^{\infty} S_2(n,k:\omega) \frac{z^n}{n!} = \frac{(e^z-1)^k}{k!} e^{z\omega}. \tag{1.5}$$

The degenerate form of the Stirling polynomials of the second kind are given below (*cf.* [6-8,11-13]):

$$\sum_{n=0}^{\infty} S_{2,\lambda}(n,k:\omega) \frac{z^n}{n!} = \frac{(e_\lambda(z)-1)^k}{k!} e_\lambda^\omega(z). \tag{1.6}$$

The degenerate truncated form of the Stirling polynomials of the second kind are considered as follows (*cf.* [8]):

$$\sum_{n=0}^{\infty} S_{2,m;\lambda}(n,k:\omega) \frac{z^n}{n!} = \frac{(e_\lambda(z)-1-e_{m-1,\lambda}(z))^k}{k!} e_\lambda^\omega(z). \tag{1.7}$$

The Gould-Hopper polynomials $H_n^{(j)}(\omega,\theta)$ are defined by (see [4,6-8,15]):

$$\sum_{n=0}^{\infty} H_n^{(j)}(\omega,\theta) \frac{z^n}{n!} = e^{\omega z+\theta z^j},$$

where $j \in \mathbb{N}$ with $j \geq 2$. Choosing $j = 1$ in (2.14), the polynomials $H_n^{(j)}(\omega,\theta)$ reduce to the Newton binomial formula. Moreover, taking $j = 2$ in (2.14), the polynomials $H_n^{(j)}(\omega,\theta)$ become the Hermite polynomials $H_n(\omega,\theta)$ (*cf.* [15]). The two polynomials $H_n^{(j)}(\omega,\theta)$ and $H_n(\omega,\theta)$ have been utilized to generalize multifarious special polynomials including Bell, Bernoulli, Genocchi and Euler polynomials (see [4,6-8,15]).

Let $j \in \mathbb{N}$ and $\lambda \in \mathbb{R}\setminus\{0\}$. The degenerate Gould-Hopper polynomials $H_{n,\lambda}^{(j)}(\omega,\theta)$ are defined below (*cf.* [4,6,7]):

$$\sum_{n=0}^{\infty} H_{n,\lambda}^{(j)}(\omega,\theta) \frac{z^n}{n!} = e_\lambda^x(z)\, e_\lambda^y(z^j). \tag{1.8}$$

Diverse applications and properties of the polynomials $H_{n,\lambda}^{(j)}(\omega,\theta)$ are investigated in [4,6,7].

2. The Gould-Hopper Based Degenerate Truncated Bernoulli Polynomials

In this chapter, we consider the Gould-Hopper based degenerate truncated Bernoulli polynomials and examine diverse formulas and correlations such as implicit summation formulas, derivation rule and symmetric identities.

The Bernoulli polynomials are defined below (*cf.* [1,2,6-10,14,16,18]):

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z}{e^z-1} e^{xz}. \qquad (|z| < 2\pi)$$

The degenerate form of the Bernoulli polynomials are given below (*cf.* [1,8,10,14]):

$$\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{z^n}{n!} = \frac{2z}{e_\lambda(z)+1}e_\lambda^x(z).$$

The truncated form of the Bernoulli polynomials $B_{m,n}(x)$ are provided below (*cf.* [3,9]):

$$\sum_{n=0}^{\infty} B_{m,n}(x)\frac{z^n}{n!} = \frac{\frac{z^{m+1}}{(m+1)!}}{e^z-1-e_{m-1}(z)}e^{xz}. \tag{2.1}$$

Thanks to many mathematicians, recently, multifarious truncated and degenerate extensions of the Bernoulli polynomials have been considered and invesitgated in [1,4,6-10,14,16].

The degenerate truncated Bernoulli polynomials are defined below (*cf.* [8]):

$$\sum_{n=0}^{\infty} B_{m,n,\lambda}(x)\frac{z^n}{n!} = \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^x(z). \tag{2.2}$$

When $x=0$, we have $B_{m,n,\lambda}(0) := B_{m,n,\lambda}$ called the degenerate truncated Bernoulli numbers provided by

$$\sum_{n=0}^{\infty} B_{m,n,\lambda}\frac{z^n}{n!} = \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}. \tag{2.3}$$

The polynomials $B_{m,n,\lambda}(x)$ in conjuction with the several identities and formulas are analyzed in [8] with details.

We now introduce the Gould-Hopper based degenerate truncated Bernoulli polynomials as follows.

Definition 2.1. Let x and y be two independent variables and $j \in \mathbb{N}_0$. The Gould-Hopper based degenerate truncated Bernoulli polynomials are defined below:

$$\sum_{n=0}^{\infty} {}_HB_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} = \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^x(z)\,e_\lambda^y\left(z^j\right). \tag{2.4}$$

We choose to call the Gould-Hopper based $Detr$-Bernoulli polynomials instead of the Gould-Hopper based degenerate truncated Bernoulli polynomials.

Remark 2.1. When $x=0$ in Definition 2.1, the Gould-Hopper based $Detr$-Bernoulli polynomials $B_{m,n,\lambda}(x)$ reduce to the following polynomials which is also new extension of the $Detr$-Bernoulli polynomials:

$$\sum_{n=0}^{\infty} B_{m,n,\lambda}^{(j)}(y)\frac{z^n}{n!} = \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^y\left(z^j\right). \tag{2.5}$$

Remark 2.2. Taking $x=y=0$ in Definition 2.1, the polynomials ${}_HB_{m,n,\lambda}^{(j)}(x,y)$ reduce to the degenerate truncated Bernoulli numbers in (2.3).

Theorem 2.1. *The following summation formulae holds for $n \in \mathbb{N}_0$:*

$${}_HB_{m,n,\lambda}^{(j)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}\ {}_HB_{m,n-k,\lambda}^{(j)}(y) \tag{2.6}$$

and

$${}_HB_{m,n,\lambda}^{(j)}(x,y) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k,\lambda}^{(j)}(x,y)\,B_{m,k,\lambda}.$$

Proof. By Definition 2.1 and utilizing the (2.5) and (2.3), we attain

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{H}B_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} &= \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^x(z)\,e_\lambda^y\left(z^j\right) \\
&= \sum_{n=0}^{\infty} {}_{H}B_{m,n,\lambda}^{(j)}(y)\frac{z^n}{n!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}\ {}_{H}B_{m,n-k,\lambda}^{(j)}(y)\right)\frac{z^n}{n!}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{H}B_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} &= \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^x(z)\,e_\lambda^y\left(z^j\right) \\
&= \sum_{n=0}^{\infty} H_{n,\lambda}^{(j)}(x,y)\frac{z^n}{n!}\sum_{n=0}^{\infty}B_{m,n,\lambda}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}H_{n-k,\lambda}^{(j)}(x,y)\,B_{m,k,\lambda}\right)\frac{z^n}{n!},
\end{aligned}$$

which complete the proof. □

We give the following lemma.

Lemma 2.1. *(cf. [6]) The following series manipulation is valid:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/j\rfloor}A(k,n-jk), \tag{2.7}$$

where $\lfloor\cdot\rfloor$ is the Gauss symbol, and shows the maximum integer that does not exceed the number in the square brackets.

We give the following theorem.

Theorem 2.2. *We have*

$${}_{H}B_{m,n,\lambda}^{(j)}(x,y)=n!\sum_{k=0}^{\lfloor n/j\rfloor}\frac{(y)_{n-jk,\lambda}}{k!\,(n-jk)!}B_{m,k,\lambda}(x). \tag{2.8}$$

Proof. By applying (2.7) and using the following equality

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_{H}B_{m,n,\lambda}^{(j)}(x,y)\frac{z^n}{n!} &= \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^x(z)\,e_\lambda^y\left(z^j\right) \\
&= \left(\sum_{n=0}^{\infty}B_{m,n,\lambda}(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y)_{n,\lambda}\frac{z^{jn}}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(n!\sum_{k=0}^{\lfloor n/j\rfloor}\frac{(y)_{n-jk,\lambda}}{k!\,(n-jk)!}B_{m,k,\lambda}(x)\right)\frac{z^n}{n!},
\end{aligned}$$

which is the claimed result (2.8). □

Theorem 2.3. *We have*

$${}_{H}B_{m,n,\lambda}^{(j)}(x_1+x_2,y_1+y_2)=\sum_{k=0}^{n}\binom{n}{k}\ {}_{H}B_{m,k,\lambda}^{(j)}(x_1,y_1)\,H_{n-k,\lambda}^{(j)}(x_2,y_2). \tag{2.9}$$

Proof. Using the following equality

$$\frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^{x_1+x_2}(z)\,e_\lambda^{y_1+y_2}\left(z^j\right)=\frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}e_\lambda^{x_1}(z)\,e_\lambda^{y_1}\left(z^j\right)e_\lambda^{x_2}(z)\,e_\lambda^{y_2}\left(z^j\right),$$

the proof is similar to Theorem 2.1. We, therefore, choose to omit details involved. □

Theorem 2.4. *We have*

$$\frac{\partial}{\partial x}\,{}_HB_{m,n,\lambda}^{(j)}(x,y)=n!\sum_{s=1}^{\infty}{}_HB_{m,n-s,\lambda}^{(j)}(x,y)\,\frac{(-1)^{s+1}}{(n-s)!s}\lambda^{s-1}. \tag{2.10}$$

Proof. By appliying the operator $\frac{\partial}{\partial x}$ to both sides of (2.4), we then derive

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\,{}_HB_{m,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!}&=\frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}e_\lambda^y\left(z^j\right)}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}\frac{\partial}{\partial x}(1+\lambda z)^{\frac{x}{\lambda}}\\&=\frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}e_\lambda^y\left(z^j\right)}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}(1+\lambda z)^{\frac{x}{\lambda}}\ln(1+\lambda z)^{\frac{1}{\lambda}}\\&=\sum_{n=0}^{\infty}{}_HB_{m,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!}\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}\lambda^{s-1}z^s\\&=\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}{}_HB_{m,n,\lambda}^{(j)}(x,y)\,\frac{(-1)^{s+1}}{s}\lambda^{s-1}\frac{z^{n+s}}{n!},\end{aligned}$$

which means the assertion in (2.10). □

Theorem 2.5. *For $n,m\in\mathbb{N}_0$, we have*

$$\begin{aligned}{}_HB_{m+1,n,\lambda}^{(j)}(x,y)&=n\frac{1-(m+1)\lambda}{m+2}\,{}_HB_{m,n-1,\lambda}^{(j)}(x,y)\\&\quad+\frac{(m+1)!}{(1-m\lambda)}\sum_{k=0}^{n}\binom{n+1}{k}B_{m,k;\lambda}\,{}_HB_{m+1,n+1-k,\lambda}^{(j)}(x,y).\end{aligned} \tag{2.11}$$

Proof. Utilizing the following equality

$$\begin{aligned}(1)_{m+2,\lambda}\frac{z^{m+2}}{(m+2)!}e_\lambda^x(z)\,e_\lambda^y\left(z^j\right)&=\left(e_\lambda(z)-1-e_{m,\lambda}(z)\right)\sum_{n=0}^{\infty}{}_HB_{m+1,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!}\\&=\left(e_\lambda(z)-1-e_{m-1,\lambda}(z)\right)\sum_{n=0}^{\infty}{}_HB_{m+1,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!}\\&\quad-(1)_{m,\lambda}\frac{z^m}{m!}\sum_{n=0}^{\infty}{}_HB_{m+1,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!},\end{aligned}$$

the proof is similar to Theorem 2.1. We, therefore, choose to omit details involved. □

Theorem 2.6. *For $n,m\in\mathbb{N}_0$, we have*

$$\frac{(1)_{m+1,\lambda}}{(m+1)!}H_{n,\lambda}^{(j)}(x,y)=\sum_{k=0}^{n+1}n!\,(1)_{k+m,\lambda}\frac{{}_HB_{m,n+1-k,\lambda}^{(j)}(x,y)}{(k+m)!\,(n+1-k)!}-n!\frac{{}_HB_{m,n+m+1,\lambda}^{(j)}(x,y)}{(n+m+1)!}. \tag{2.12}$$

Proof. By Definition 2.1, we have

$$\begin{aligned}(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}e_\lambda^x(z)\,e_\lambda^y\left(z^j\right)&=\left(e_\lambda(z)-1-e_{m-1,\lambda}(z)\right)\sum_{n=0}^{\infty}{}_HB_{m,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!}\\&=\sum_{n=m}^{\infty}(1)_{n,\lambda}\frac{z^n}{n!}\sum_{n=0}^{\infty}{}_HB_{m,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!}-\sum_{n=0}^{\infty}{}_HB_{m,n,\lambda}^{(j)}(x,y)\,\frac{z^n}{n!},\end{aligned}$$

which yields the asserted result (2.12). □

In recent years, many mathematicians have been studied special polynomials to acquire some of their symmetric identities and implicit summation formulas, *cf.* [5,15] and see also each of the references cited therein. We now derive some the mentioned formulas and identities for the polynomials ${}_{H}B^{(j)}_{m,n,\lambda}(x,y)$.

Theorem 2.7. *For $n,m\in\mathbb{N}_0$, we have*

$$ {}_{H}B^{(j)}_{m,n,\lambda}(x,y)=\sum_{l=0}^{n}\sum_{k=0}^{n}\binom{n}{k}B^{(j)}_{m,n-k,\lambda}(y)\,S_{2;\lambda}(k,l:-l)\,(x)^{(l)}, \tag{2.13}$$

where $(x)^{(l)}=x(x+1)(x+2)\cdots(x+(l-1))$ for $l\in\mathbb{N}$ with $(x)^{(l)}:=1$ (cf. [8]).

Proof. From Definition 2.1 and utilizing (1.6) and (2.5), we acquire

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{H}B^{(j)}_{m,n,\lambda}(x,y)\frac{z^n}{n!} &= \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}e^{y}_{\lambda}(z^j)}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}\left(e^{-1}_{\lambda}(z)-1+1\right)^{x}\\ &= \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}e^{y}_{\lambda}(z^j)}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}\sum_{l=0}^{\infty}\binom{x+l-1}{l}\left(1-e^{-1}_{\lambda}(z)\right)^{l}\\ &= \frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}e^{y}_{\lambda}(z^j)}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}\sum_{l=0}^{\infty}\binom{x+l-1}{l}\frac{(e_\lambda(z)-1)^{l}}{l!}e^{-l}_{\lambda}(z)\,l!\\ &= \sum_{l=0}^{\infty}(x)^{(l)}\sum_{n=0}^{\infty}B^{(j)}_{m,n,\lambda}(y)\frac{z^n}{n!}\sum_{n=0}^{\infty}S_{2;\lambda}(n,l:-l)\frac{z^n}{n!}\\ &= \sum_{l=0}^{\infty}(x)^{(l)}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B^{(j)}_{m,n-k,\lambda}(y)\,S_{2;\lambda}(k,l:-l)\right)\frac{z^n}{n!},\end{aligned}$$

which means the assertion (2.13). □

Note that (*cf.* [5,15])

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{n,s=0}^{\infty}f(n+s)\frac{x^n}{n!}\frac{y^s}{s!}. \tag{2.14}$$

We give the following theorem.

Theorem 2.8. *We have*

$$ {}_{H}B^{(j)}_{m,k+l,\lambda}(x,y)=\sum_{n,s=0}^{k,l}\binom{k}{n}\binom{l}{s}(\mu-x)_{n+s,\lambda}\;{}_{H}B^{(j)}_{m,k+l-n-s,\lambda}(\mu,y). \tag{2.15}$$

Proof. Taking z by $z+\omega$ in (2.4), it yields

$$\frac{(1)_{m+1,\lambda}\frac{(z+\omega)^{m+1}}{(m+1)!}}{e_\lambda(z+\omega)-1-e_{m-1,\lambda}(z+\omega)}e^{y}_{\lambda}\left((z+\omega)^{j}\right)=e^{\mu}_{\lambda}(z+\omega)\sum_{k,l=0}^{\infty}{}_{H}B^{(j)}_{m,k+l,\lambda}(\mu,y)\frac{z^k}{k!}\frac{\omega^l}{l!}$$

and similarly we acquire

$$\frac{(1)_{m+1,\lambda}\frac{(z+\omega)^{m+1}}{(m+1)!}}{e_\lambda(z+\omega)-1-e_{m-1,\lambda}(z+\omega)}e^{y}_{\lambda}\left((z+\omega)^{j}\right)=e^{x}_{\lambda}(z+\omega)\sum_{k,l=0}^{\infty}{}_{H}B^{(j)}_{m,k+l,\lambda}(x,y)\frac{z^k}{k!}\frac{\omega^l}{l!}.$$

By the last two equalities, we write

$$\begin{aligned}\sum_{k,l=0}^{\infty}{}_{H}B^{(j)}_{m,k+l,\lambda}(x,y)\frac{z^k}{k!}\frac{\omega^l}{l!} &= e^{\mu-x}_{\lambda}(z+\omega)\sum_{k,l=0}^{\infty}{}_{H}B^{(j)}_{m,k+l,\lambda}(\mu,y)\frac{z^k}{k!}\frac{\omega^l}{l!}\\ &= \sum_{n,s=0}^{\infty}(\mu-x)_{n+s,\lambda}\frac{z^n}{n!}\frac{\omega^m}{s!}\sum_{k,l=0}^{\infty}{}_{H}B^{(j)}_{m,k+l,\lambda}(\mu,y)\frac{z^k}{k!}\frac{\omega^l}{l!}.\end{aligned}$$

By using (2.14), we acquire

$$\sum_{k,l=0}^{\infty} {}_HB^{(j)}_{m,k+l,\lambda}(x,y)\frac{z^k}{k!}\frac{\omega^l}{l!} = \sum_{k,l=0}^{\infty}\sum_{n,s=0}^{k,l}\frac{(\mu-x)_{n+s,\lambda}\ {}_HB^{(j)}_{m,k+l-n-s,\lambda}(\mu,y)}{n!s!(k-l)!(l-s)!}z^k\omega^l,$$

whichmeans the assertion (2.15). □

Theorem 2.9. *The following symmetric identity holds for $n \in \mathbb{N}_0$ and $a, b \in \mathbb{R}$:*

$$\sum_{k=0}^{n}\binom{n}{k} {}_HB^{(j)}_{m,n-k,\lambda}(bx,y)\ {}_HB^{(j)}_{m,k,\lambda}(ax,y)\,a^{n-k}b^k = \sum_{k=0}^{n}\binom{n}{k} {}_HB^{(j)}_{m,n-k,\lambda}(ax,y)\ {}_HB^{(j)}_{m,k,\lambda}(bx,y)\,b^{n-k}a^k. \tag{2.16}$$

Proof. Let

$$\Upsilon = \frac{(az)^{m+1}(bz)^{m+1}\left(\frac{(1)_{m+1,\lambda}}{(m+1)!}\right)^2 e^{bx}_{\lambda}(az)\,e^{ax}_{\lambda}(bz)\,e^{y}_{\lambda}\left(a^jz^j\right)e^{y}_{\lambda}\left(b^jz^j\right)}{\left(e_\lambda(az)-1-e_{m-1,\lambda}(az)\right)\left(e_\lambda(bz)-1-e_{m-1,\lambda}(bz)\right)}$$

Then, thanks to Υ being symmetric in a and b, we have two expansions of Υ as follows:

$$\begin{aligned}\Upsilon &= \sum_{n=0}^{\infty} {}_HB^{(j)}_{m,n,\lambda}(bx,y)\frac{(az)^n}{n!}\sum_{n=0}^{\infty} {}_HB^{(j)}_{m,n,\lambda}(ax,y)\frac{(bz)^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_HB^{(j)}_{m,n-k,\lambda}(bx,y)\ {}_HB^{(j)}_{m,k,\lambda}(ax,y)\,a^{n-k}b^k\frac{z^n}{n!}\end{aligned}$$

and similarly

$$\Upsilon = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_HB^{(j)}_{m,n-k,\lambda}(ax,y)\ {}_HB^{(j)}_{m,k,\lambda}(bx,y)\,b^{n-k}a^k\frac{z^n}{n!},$$

which means the assertion (2.16).

□

For $k, m \in \mathbb{N}_0$, we define the numbers $S_{m,k,\lambda}(n)$ as follows:

$$\frac{e_\lambda((n+1)z)-1-e_{m-1,\lambda}((n+1)z)}{e_\lambda(z)-1-e_{m-1,\lambda}(z)} = \sum_{k=0}^{\infty}S_{m,k,\lambda}(n)\frac{z^k}{k!}, \tag{2.17}$$

which, for $\lambda \to m = 0$, reduces to the power sum $\lim_{\lambda\to 0} S_{0,k,\lambda}(n) := S_k(n)$ given by (*cf.* [5])

$$\sum_{k=0}^{\infty}S_k(n)\frac{z^k}{k!} = \frac{e^{(n+1)z}-1}{e^z-1}.$$

A symmetric identity for ${}_HB^{(j)}_{m,n,\lambda}(x,y)$ is stated below.

Theorem 2.10. *For a, b being two integers and $n, m \in \mathbb{N}_0$, we have*

$$\begin{aligned}&\sum_{u=0}^{n}\sum_{j=0}^{n-u}\sum_{i=0}^{u}\binom{n}{u}\binom{n-u}{j}\binom{n}{i} {}_HB^{(j)}_{m,n-u-j,\lambda}(bx,by)\,S_{m,j,\lambda}(b-1)\\ &\times {}_HB^{(j)}_{m,u-i,\lambda}(ax,ay)\,S_{m,i,\lambda}(a-1)a^{n-u}b^u\\ =\ &\sum_{u=0}^{n}\sum_{j=0}^{n-u}\sum_{i=0}^{u}\binom{n}{u}\binom{n-u}{j}\binom{n}{i} {}_HB^{(j)}_{m,n-u-j,\lambda}(ax,ay)\,S_{m,j,\lambda}(a-1)\\ &{}_HB^{(j)}_{m,u-i,\lambda}(bx,by)\,S_{m,i,\lambda}(b-1)b^{n-u}a^u.\end{aligned}$$

Proof. Let

$$\begin{aligned}\Psi &= \frac{(az)^{m+1}(bz)^{m+1}\left(\frac{(1)_{m+1,\lambda}}{(m+1)!}\right)^2 (e_\lambda(abz)-1-e_{m-1,\lambda}(abz))^2}{(e_\lambda(az)-1-e_{m-1,\lambda}(az))^2(e_\lambda(bz)-1-e_{m-1,\lambda}(bz))^2} \\ &\quad \times e_\lambda^{bx}(az)\,e_\lambda^{ax}(bz)\,e_\lambda^{by}\left(a^j z^j\right)e_\lambda^{ay}\left(b^j z^j\right) \\ &= \frac{(az)^{m+1}\frac{(1)_{m+1,\lambda}}{(m+1)!}e_\lambda^{bx}(az)\,e_\lambda^{by}\left(a^j z^j\right)}{e_\lambda(az)-1-e_{m-1,\lambda}(az)}\frac{e_\lambda(abz)-1-e_{m-1,\lambda}(abz)}{e_\lambda(az)-1-e_{m-1,\lambda}(az)} \\ &\quad \times \frac{(bz)^{m+1}\frac{(1)_{m+1,\lambda}}{(m+1)!}e_\lambda^{ax}(bz)\,e_\lambda^{ay}\left(b^j z^j\right)}{e_\lambda(bz)-1-e_{m-1,\lambda}(bz)}\frac{e_\lambda(abz)-1-e_{m-1,\lambda}(abz)}{e_\lambda(bz)-1-e_{m-1,\lambda}(bz)}\end{aligned}$$

Thus we have

$$\begin{aligned}\Psi &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\,{}_HB^{(j)}_{m,n-l,\lambda}(bx,by)\,S_{m,l,\lambda}(b-1)\frac{(az)^n}{n!} \\ &\quad \times \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\,{}_HB^{(j)}_{m,n-l,\lambda}(ax,ay)\,S_{m,l,\lambda}(a-1)\frac{(bz)^n}{n!},\end{aligned}$$

which completes the proof. □

3. Further Remarks

Now, we introduce the Gould-Hopper based $Detr$-Bernoulli polynomials $B^{(r)}_{m,n,\lambda}(x)$ of order r as follows:

$$\sum_{n=0}^{\infty}{}_HB^{(j,r)}_{m,n,\lambda}(x,y)\frac{z^n}{n!} = \left(\frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}\right)^r e_\lambda^x(z)\,e_\lambda^y\left(z^j\right). \tag{3.1}$$

We note that ${}_HB^{(j,1)}_{m,n,\lambda}(x,y) := {}_HB^{(j)}_{m,n,\lambda}(x,y)$. Also, upon letting $x=y=0$, the polynomials in (3.1) reduce to the Gould-Hopper based $Detr$-Bernoulli numbers of order r below:

$$\sum_{n=0}^{\infty}B^{(r)}_{m,n,\lambda}\frac{z^n}{n!} = \left(\frac{(1)_{m+1,\lambda}\frac{z^{m+1}}{(m+1)!}}{e_\lambda(z)-1-e_{m-1,\lambda}(z)}\right)^r.$$

We first give the following summation formula.

Theorem 3.1. *We have*

$${}_HB^{(j,r)}_{m,n,\lambda}(x,y) = \sum_{l=0}^{n}\binom{n}{l}B^{(r)}_{m,n-l,\lambda}H^{(j)}_{l,\lambda}(x,y)$$

Proof. By using (1.8) and (3.1), the proof is similar to Theorem 2.1. We, therefore, choose to omit details involved. □

Addition property of the Gould-Hopper based $Detr$-Bernoulli polynomials of order r is given below.

Theorem 3.2. *We have*

$${}_HB^{(j,r_1+r_2)}_{m,n,\lambda}(x_1+x_2,y_1+y_2) = \sum_{u=0}^{n}\binom{n}{u}\,{}_HB^{(j,r_1)}_{m,u,\lambda}(x_1,y_1)\;{}_HB^{(j,r_2)}_{m,n-u,\lambda}(x_2,y_2).$$

Proof. By utlizing (1.8) and (3.1), the proof is similar to Theorem 2.1. We, therefore, choose to omit details involved. □

Theorem 3.3. *We have*

$$H_{n,\lambda}^{(j)}\left(x+\mu,y\right)=\frac{n!r!\left(\left(1\right)_{m+1,\lambda}\right)^{r}}{\left(n+\left(m+1\right)\right)!\left(\left(m+1\right)!\right)^{r}} \tag{3.2}$$
$$\times\sum_{l=0}^{n+(m+1)r}\binom{n+(m+1)}{l}S_{2,m;\lambda}\left(n+\left(m+1\right)-l,r:\mu\right)\ {}_{H}B_{m,l,\lambda}^{(j,r)}\left(x,y\right).$$

Proof. By (1.7), (1.8) and (3.1), we investigate

$$\begin{aligned}\sum_{n=0}^{\infty}S_{2,m;\lambda}\left(n,r:\mu\right)\frac{z^{n}}{n!}\sum_{n=0}^{\infty}{}_{H}B_{m,n,\lambda}^{(j,r)}\left(x,y\right)\frac{z^{n}}{n!} &= \frac{\left(e_{\lambda}\left(z\right)-1-e_{m-1,\lambda}\left(z\right)\right)^{r}}{r!}e_{\lambda}^{\mu}\left(z\right)\\ &\quad\times\frac{\left(\left(1\right)_{m+1,\lambda}\right)^{r}\frac{z^{(m+1)r}}{\left(\left(m+1\right)!\right)^{r}}}{\left(e_{\lambda}\left(z\right)-1-e_{m-1,\lambda}\left(z\right)\right)^{r}}e_{\lambda}^{x}\left(z\right)e_{\lambda}^{y}\left(z^{j}\right)\\ &= e_{\lambda}^{x+\mu}\left(z\right)e_{\lambda}^{y}\left(z^{j}\right)\frac{\left(\left(1\right)_{m+1,\lambda}\right)^{r}}{r!}\frac{z^{(m+1)r}}{\left(\left(m+1\right)!\right)^{r}}\\ &= \sum_{n=0}^{\infty}H_{n,\lambda}^{(j)}\left(x+\mu,y\right)\frac{z^{n+(m+1)r}}{n!}\frac{\left(\left(1\right)_{m+1,\lambda}\right)^{r}}{r!\left(\left(m+1\right)!\right)^{r}},\end{aligned}$$

which implies the assertion (3.2). □

4. Conclusion

In this study, we have introduced the Gould-Hopper based truncated degenerate Bernoulli polynomials and have examined diverse properties and formulas covering addition formulas, derivation rule and relationships with the Gould-Hopper polynomials and the degenerate Stirling numbers of the second. Then, we have derived some interesting symmetric relations and implicit summation identities. Moreover, we have defined Gould-Hopper based truncated degenerate Bernoulli polynomials of order r and have given some of their properties and relations.

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author’s contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Carlitz, L.: *Degenerate Stirling, Bernoulli and Eulerian numbers.* Utilitas Mathematica. **15**, 51-88 (1975).

[2] Cheon,G.-S.: *A note on the Bernoulli and Euler polynomials.* Applied Mathematics Letters. **16**, 365-368 (2003).

[3] Dattoli, G., Ceserano, C., Sacchetti, D.: *A note on truncated polynomials.* Applied Mathematics and Computation. **134**, 595-605 (2003).

[4] Duran, U, Acikgoz, M.: *Generalized Gould-Hopper based fully degenerate central Bell polynomials.* Turkish Journal of Analysis and Number Theory. **7** (5), 124-134 (2019).

[5] Duran, U., Araci, S., Acikgoz, M.: *Bell-Based Bernoulli polynomials with applications.* Axioms. **10**, no. 29 (2021).

[6] Duran, U., Sadjang, P.N.: *On Gould-Hopper-based fully degenerate poly-Bernoulli polynomials with a q-parameter.* Mathematics. **7**, no. 121 (2019).

[7] Duran, U., Acikgoz, M.: *On generalized degenerate Gould-Hopper based fully degenerate Bell polynomials.* Journal of Mathematics and Computer Science. **21** (3), 243-257 (2020).

[8] Duran, U., Acikgoz, M.: *On Degenerate Truncated Special Polynomials.* Mathematics. **8** (1), no. 144 (2020).

[9] Hassen, A., Nguyen, H.D.: *Hypergeometric Bernoulli polynomials and Appell sequences.* International Journal of Number Theory. **4**, 767-774 (2008).

[10] Howard, F.T.: *Explicit formulas for degenerate Bernoulli numbers.* Discrete Mathematics. **162**, 175-185 (1996).

[11] Kim, T., Kim, D.S., Jang, L.-C., Kwon, H.I.: *Extended degenerate stirling numbers of the second kind and extended degenerate Bell polynomials.* Utilitas Mathematica. **106**, 11-21 (2018).

[12] Kim, T., Yao, Y., Kim, D.S., Jang, G.-W.: *Degenerate r-Stirling numbers and r-Bell polynomials.* Russian Journal of Mathematical Physics. **25**, 44-58 (2018).

[13] Kim, T.: *A note on degenerate Stirling polynomials of the second kind.* Proceedings of the Jangjeon Mathematical Society. **20**, 319-331 (2017).

[14] Kurt, B: *Explicit relations for the modified degenerate Apostol-type polynomials.* Journal of Balikesir University Institute of Science and Technology. **20**, 401-412 (2018).

[15] Pathan, M.A., Khan, W.A.: *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials.* Mediterranean Journal of Mathematics. **12**, 679-695 (2015).

[16] Srivastava, H.M., Araci, S., Khan, W.A., Acikgoz, M.: *A note on the truncated-exponential based Apostol-type polynomials.* Symmetry. **11**, no. 538 (2019).

[17] Srivastava, H.M., Choi, J.: Zeta and q-Zeta functions and associated series and integrals. Elsevier Science Publishers. Amsterdam (2012).

[18] Srivastava, H.M., Pinter, A.: *Remarks on some relationships between the Bernoulli and Euler polynomials.* Applied Mathematics Letters. **17**, 375-380 (2004).

Affiliation

UĞUR DURAN

ADDRESS: Iskenderun Technical University, Faculty of Engineering and Natural Sciences, Department of the Basic Concepts of Engineering, TR-31200 Hatay,Turkey.

E-MAIL: mtdrnugur@gmail.com & ugur.duran@iste.edu.tr

ORCID ID: 0000-0002-5717-1199

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
https://doi.org/10.36753/msaen.1025072
10 (3) 135-145 (2022) - Research Article
ISSN: 2147-6268

A Trigonometric Approach to Time Fractional FitzHugh-Nagumo Model on Nerve Pulse Propagation

Berat Karaağac

Abstract
The aim of this paper is to put on display the numerical solutions and dynamics of time fractional Fitzhugh-Nagumo model, which is an important nonlinear reaction-diffusion equation. For this purpose, finite element method based on trigonometric cubic B-splines are used to obtain numerical solutions of the model. In this model, the derivative which is fractional order is taken in terms of Caputo. Thus, time dicretization is made using $L1$ algorithm for Caputo derivative and space discretization is made using trigonometric cubic B- spline basis. Also, the non-linear term in the model is linearized by the Rubin Graves type linearization. The error norms are calculated for measuring the accuracy of the finite element method. The comparison of numerical and exact solutions are exhibited via tables and graphics.

Keywords: *Time fractional Fitzhugh-Nagumo model; finite element method; collocation; trigonometric B-splines*

AMS Subject Classification (2020): *Primary: 65L60 ; Secondary: 65A05; 41A15.*

1. Introduction

The roots of Fractional calculus (FC) dates back to 1965. This date is not far from the emerging date of traditional calculus. Nonetheless; though its fascinating nature and contribution of mathematicians, physicists and engineers, it has not been a huge attraction. Recent decades have seen a dramatically accelerating pace in the development of fractional calculus due to frequently appearing in various applications in fields of biomechanics, viscoelasticity, control theory, aerodynamics, physics and engineering and so on [1, 2]. The popularity of FC has attracted many researchers from all over the world, thus, this attraction produced research papers and books covering all areas of science and motivated and development of numerical methods [3, 4]. Especially, since fractional derivatives are more convenient and economical, solving fractional order differential equations has a vital role in modeling various phenomena. One can get a brief glimpse to numerical methods for solving fractional differential equations in Refs[5–14].

In this paper, Time fractional Fitzhugh–Nagumo model is going to be considered via initial and boundary conditions. In order to obtain numerical solutions a framework of combination collocation method and finite element method (FEM) based on cubic trigonometric B-spline basis will be used. The error norms will be calculated

Received : *17-11-2021,* ***Accepted :*** *28-02-2022*

to know how well accurate the exact solutions and numerical ones close to each other. Comparison of exact and numerical results and dynamics of the solutions is going to be presented via tables and graphics.

2. Time fractional Fitzhugh–Nagumo model and Application of the method

The FitzHugh–Nagumo model [18]

$$\varepsilon \frac{\partial u(x,t)}{\partial t} = \Psi\left(u\left(x,t\right)\right) - v\left(x,t\right) + I,$$

$$\frac{\partial v(x,t)}{\partial t} = u\left(x,t\right) - \zeta v\left(x,t\right)$$

where ε and ζ are positive constants and $\Psi\left(u\left(x,t\right)\right) = u\left(x,t\right)\left(1-u\left(x,t\right)\right)\left(u\left(x,t\right)-r\right)$ for $r \in \left(0,\frac{1}{2}\right)$. The model takes its name from R. FitzHugh [15] and J. Nagumo [16] who proposed a model for emulating the current signal observed in a living organism's excitable cells in 1961. A lipid bilayer membrane distinct nerve cells from the extracellular region. When the cells do not run the signals, there are a a potential difference. This potential difference is known as resting potential of cells. Positively charged sodium and the potassium ions and negatively charged protein ions keep in existence of resting potential. When there is a external disturbation to the cells, depolarization and repolarization process begins. Depolarization rises up spikes toward a positive value, depolarization passes to resting potential. In a simple description, the FitzHugh-Nagumo model is a simplified model of activation and deactivation dynamics in a spiking neuron [17]. In literature, I is a constant external stimulus. $u\left(x,t\right)$ and $v\left(x,t\right)$ are unknowns and they measure the potential difference across the cell membrane transmembrane currents which influence the tendency of the cell to regain before being able to fire again, respectively [18]. It will be useful to explain that many variations of FitzHugh-Nagumo model have been derived from the original one.

There are various approaches to FitzHugh-Nagumo model. One can read and get information in [19–22] and there in. For the reason that the main issue of this paper, it will be important to take a quick glance at fractional FitzHugh-Nagumo model; Liu et al. [23] have considered the model in two dimensional and obtained solutions via implicit numerical method. Kumar et al. [24] have focused on solutions of the model using q-homotopy analysis approach and Laplace transform approach. Injrou et al. [25] proposed a finite difference scheme to obtain numerical solutions of the model.

In this paper, a reduced problem of FitzHugh-Nagumo model in fractional form will be taken into consideration. As the starting point, time fractional nonlinear reaction–diffusion equations of the form is considered as following [25]

$$D_t^{\gamma} u_t\left(x,t\right) = \kappa u_{xx}\left(x,t\right) + \Psi\left(u\left(x,t\right)\right) \tag{2.1}$$

where x and t denote space and time derivative, respectively. κ is a physical constant which represents diffusion coefficient , Ψ accounts for all local reactions and γ is the fractional order and $0<\gamma<1$.In order to calculate wave solutions, the function $\Psi\left(u\left(x,t\right)\right)$ has to be specified; if $\kappa=1$ and $\Psi\left(u\left(x,t\right)\right) = u\left(x,t\right)\left(1-u\left(x,t\right)\right)\left(u\left(x,t\right)-r\right)$, the equation (2.1) is called as time fractional Fitzhugh–Nagumo model. Thus, time fractional initial-boundary Fitzhugh–Nagumo model [24, 26, 27] will be considered and it is formulated as

$$\begin{array}{lll} D_t^{\gamma} u\left(x,t\right) = \kappa u_{xx}\left(x,t\right) + \mu u\left(x,t\right)\left(1-u\left(x,t\right)\right)\left(u\left(x,t\right)-r\right), & & \\ & & \\ u\left(x_L,t\right) = f_1\left(t\right), & u\left(x_R,t\right) = f_2\left(t\right), & t\in[0,T] \\ u_x\left(x_L,t\right) = g_1\left(t\right), & u_x\left(x_R,t\right) = g_2\left(t\right), & x\in[x_L,x_R] \\ u\left(x,0\right) = h\left(x\right). & & \end{array} \tag{2.2}$$

As highlighted above, there are several variants of the FitzHugh-Nagumo equations. In this subsection finite element collocation method will be considered for the model given in (2.2) with its initial and boundary conditions. In order to compute the numerical solutions of the FitzHugh-Nagumo model via FEM on fixed points on a specific interval $[x_L, x_R]$, the interval should be divided N subelements with $h = \frac{x_R - x_L}{N}$ such that

$$x_L = x_0 < x_1 < x_2 < \cdots < x_{N-2} < x_{N-1} < x_N = x_R$$

where $\{x_m\}|_{m=0}^{N}$ are distinct grid points and number of grid points is $(N+1)$. The procedure for obtaining a numerical scheme is to compute numerical solutions define an approximate solution for the mentioned model. Let

$w(x,t)$ is an approximate solution to exact solution $u(x,t)$ is determined as follows

$$u(x,t) \approx w(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)\, TB_j^3(x)$$

where $\delta_j(t)$ are time dependent parameters which will be determined for obtaining numerical solutions at the point (x_j, t_n) and for this problem $TB_j^3(x)$ are cubic trigonometric B-spline basis given in [28] as follows

$$TB_m^3(x) = \frac{1}{\phi}\begin{cases} p^2(x_{m-2}) - p^2(x_{m-2})\,p(x_m), & x_{m-2} < x < x_{m-1} \\ -p(x_{m-2})\,p(x_{m+1})\,p(x_{m-1}) - p(x_{m+2})\,p^2(x_{m-1}), & x_{m-1} < x < x_m \\ p(x_{m-2})\,p^2(x_{m+1}) + p(x_{m+2})\,p(x_{m-1})\,p(x_{m+1}), & x_m < x < x_{m+1} \\ +p^2(x_{m+2})\,p(x_m), & \\ -p^3(x_{m+2}) & x_{m+1} < x < x_{m+2} \\ 0 & \text{otherwise} \end{cases}$$

where $p(xm) = \sin\left(\frac{x-x_m}{2}\right)$ and $\phi = \sin(\frac{h}{2})\sin(h)\sin\left(\frac{3h}{2}\right)$ for $m = 0, 1, 2, \ldots, N$. $TB_j(x)$ basis are zero out of interval $[x_{m-2}, x_{m+2}]$. On the ground of local support property of basis $TB_m(x)$, the approximate solution can be defined over a sub interval $[x_m, x_{m+1}]$ as

$$w(x,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\, TB_j^3(x)$$

Now, when the value of $w(x,t)$ at point x_m is expressed as w_m^n, it and its required derivatives can be obtained with some calculations as follows

$$\begin{aligned} w_m^n &= \alpha_1\delta_{m-1}(t) + \alpha_2\delta_m(t) + \alpha_1\delta_{m+1}(t), \\ (w_m^n)' &= \beta_1\delta_{m-1}(t) + \beta_2\delta_{m+1}(t), \\ (w_m^n)'' &= \eta_1\delta_{m-1}(t) + \eta_2\delta_m(t) + \eta_1\delta_{m+1}(t) \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \alpha_1 &= \sin^2\left(\tfrac{h}{2}\right)\csc(h)\csc\left(\tfrac{3h}{2}\right), & \alpha_2 &= \tfrac{2}{1+\cos(h)}, \\ \beta_1 &= -\tfrac{3}{4}\csc\left(\tfrac{3h}{2}\right), & \beta_2 &= \tfrac{3}{4}\csc\left(\tfrac{3h}{2}\right), \\ \eta_1 &= \tfrac{3\left((1+3\cos(h))\csc^2\left(\frac{h}{2}\right)\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}, & \eta_2 &= \tfrac{3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}. \end{aligned}$$

The solution of the problem given in (2.2) will be obtained by substituting approximate solution and its derivatives into discretization of the problem. Thus, for a first task, Fitzhugh–Nagumo model should be discretized.

3. Discretization of the Fitzhugh–Nagumo model and Application of the method

In order to compute numerical solutions of Fitzhugh–Nagumo model, it is crucial to discritize the model and obtain recursive equation system i.e numerical scheme. For this purpose, the derivatives according to space will be discretize using Crank-Nicolson method, which is a average in time, by reason of obtaining the second order accurate and the fractional order derivative according to time will be discretized using $L1$ algorithm [29] as follows

$$D_t^\gamma f(t_m) = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} b_k^\gamma \left[f(t_{n-k}) - f(t_{n-1-k})\right]$$

where $b_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}$, $t_n = n\Delta t \ (n = 0, 1.2, ..., M)$, and final time $T = M\Delta t$

When Crank-Nicolson method and $L1$ algorithm applied to Fitzhugh–Nagumo model given in (2.2) , it yields;

$$\begin{aligned} \tfrac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \textstyle\sum_{k=0}^{n-1} b_k^\gamma \left[u^{n-k} - u^{n-1-k}\right] &= \tfrac{\kappa}{2}\left((u_{xx})^{n+1} + (u_{xx})^n\right) + \tfrac{\mu}{2}\left((u^2)^{n+1} + (u^2)^n\right) \\ &+ \tfrac{\mu r}{2}\left(u^{n+1} + u^n\right) + \tfrac{\mu}{2}\left((u^3)^{n+1} + (u^3)^n\right) - \tfrac{\mu r}{2}\left((u^2)^{n+1} + (u^2)^n\right) \end{aligned} \tag{3.1}$$

From this point of view, nonlinear terms seen in (3.1) at $(n+1)$ time level will be linearized via Rubin-Graves linearization method as following way;

$$\left(u^2\right)^{n+1} = u^{n+1}u^n + u^n u^{n+1} - u^n u^n,$$

$$\left(u^3\right)^{n+1} = u^{n+1}u^n u^n + u^n u^{n+1}u^n + u^n u^n u^{n+1} - 2u^n u^n u^n.$$

Putting approximate solution and its derivatives (2.3) into (3.1) , and a bit of simple arrangements gives a recursive equation system as follows

$$\begin{aligned}
&\delta_{m-1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\mu\alpha_1\rho_1 - \kappa\eta_1\right)\right] + \delta_m^{n+1}\left[\alpha_2 + \tfrac{S}{2}\left(\mu\alpha_2\rho_1 - \kappa\eta_2\right)\right] + \delta_{m+1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\mu\alpha_1\rho_1 - \kappa\eta_1\right)\right] \\
&\quad = \delta_{m-1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\alpha_1\rho_2 + \kappa\eta_1\right)\right] + \delta_m^{n+1}\left[\alpha_2 + \tfrac{S}{2}\left(\alpha_2\rho_2 + \kappa\eta_2\right)\right] + \delta_{m+1}^{n+1}\left[\alpha_1 + \tfrac{S}{2}\left(\alpha_1\rho_2 + \kappa\eta_1\right)\right] \\
&\quad - \sum_{k=1}^{n-1} b_k^{\gamma}\left[\left(\alpha_1\delta_{m-1}^{n-k} + \alpha_2\delta_m^{n-k} + \alpha_1\delta_{m+1}^{n-k}\right) - \left(\alpha_1\delta_{m-1}^{n-1-k} + \alpha_2\delta_m^{n-1-k} + \alpha_1\delta_{m+1}^{n-1-k}\right)\right].
\end{aligned} \tag{3.2}$$

The sub indexes seen in (3.2) are $m = 0,1,2,...,N$ and $t = 0,1,2,...,M$. Here, T is final time and Δt is time step such as $T = M.\Delta t$ coefficients of unknowns seen in recursive equation system are $S = (\Delta t)^{\gamma}\,\Gamma(2-\gamma)$, $\rho_1 = 3\left(u^n\right)^2 - 2ru^n - 2u^n + r$ and $\rho_2 = \mu u^n\left(u^n - r\right)$.

3.1 Boundary condition at $x = x_L$

As in with the finite difference method, boundary conditions at boundary of the interval are so essential in applying Finite element method, the numerical scheme obtained via FEM have to satisfy predefined conditions applied on their boundary.

When one glimpse at the system in (3.2), it can be realized that systems consist of $(N+1)$ equations with $(N+3)$ unknowns. For a solvable system, there are two ways, first one is adding two equation into the system and the second one is eliminating two unknown from the system. For two choice, boundary conditions will be used to achieve to aim. In this section, eliminating two unknown choice will apply. The boundary condition $u(x_L,t) = f_1(t)$ can be written using approximate solution at grid (x_m,t_n) as

$$u(x_m,t_n) = \alpha_1\delta_{m-1}(t) + \alpha_2\delta_m(t) + \alpha_1\delta_{m+1}(t), \qquad m = 0,1,2,...,N \tag{3.3}$$

Equation (3.3) is rewritten by $m = 0$ for right boundary

$$f_1(t) = \alpha_1\delta_{-1}(t) + \alpha_2\delta_0(t) + \alpha_1\delta_1(t).$$

Thus this gives;

$$\delta_{-1}(t) = \frac{f_1(t)}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\delta_0(t) - \delta_1(t). \tag{3.4}$$

3.2 Boundary condition at $x = x_R$

When the boundary condition $u(x_R,t) = f_2(t)$ is put into equation (3.3) with $m = N$, it yields;

$$f_2(t) = \alpha_1\delta_{N-1}(t) + \alpha_2\delta_N(t) + \alpha_1\delta_{N+1}(t).$$

Thus,

$$\delta_{N+1}(t) = \frac{f_2(t)}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\delta_N(t) - \delta_{N-1}(t). \tag{3.5}$$

When substituting (3.4) and (3.5) into recursive system (3.2), there will be a system consisting of $(N+1)$ equations with $(N+1)$ unknowns. When recursive system (3.2) is used with boundary conditions, it gives following numerical scheme which will derive numerical solutions as follows

$$\begin{cases} \delta_0^{n+1}(t)\left[\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right]=\delta_0^{n}(t)\left[-\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right] & \\ -\sum_{k=1}^{n-1}b_k^{\gamma}\left[u_0^{n-k}-u_0^{n-1-k}\right]+\Upsilon_1, & m=0 \end{cases}$$

$$\begin{cases} \delta_{m-1}^{n+1}\left[\alpha_1+\frac{S}{2}(\mu\alpha_1\rho_1-\kappa\eta_1)\right]+\delta_m^{n+1}\left[\alpha_2+\frac{S}{2}(\mu\alpha_2\rho_1-\kappa\eta_2)\right] & \\ +\delta_{m+1}^{n+1}\left[\alpha_1+\frac{S}{2}(\mu\alpha_1\rho_1-\kappa\eta_1)\right] & \\ =\delta_{m-1}^{n}\left[\alpha_1+\frac{S}{2}(\alpha_1\rho_2+\kappa\eta_1)\right]+\delta_m^{n}\left[\alpha_2+\frac{S}{2}(\alpha_2\rho_2+\kappa\eta_2)\right]+ & 1<m<N-1 \\ \delta_{m+1}^{n}\left[\alpha_1+\frac{S}{2}(\alpha_1\rho_2+\kappa\eta_1)\right] & \\ -\sum_{k=1}^{n-1}b_k^{\gamma}\left[\left(\alpha_1\delta_{m-1}^{n-k}+\alpha_2\delta_m^{n-k}+\alpha_1\delta_{m+1}^{n-k}\right)-\left(\alpha_1\delta_{m-1}^{n-1-k}+\alpha_2\delta_m^{n-1-k}+\alpha_1\delta_{m+1}^{n-1-k}\right)\right] & \end{cases}$$

$$\begin{cases} \delta_N^{n+1}(t)\left[\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right]=\delta_0^{n}(t)\left[-\kappa\frac{S}{2\alpha_1}(\alpha_2\eta_1-\alpha_1\eta_2)\right] & \\ -\sum_{k=1}^{n-1}b_k^{\gamma}\left[u_N^{n-k}-u_N^{n-1-k}\right]+\Upsilon_2. & m=N \end{cases} \tag{3.6}$$

Now, the system (3.6) is solvable. Of course, (3.6) can be given in the form of a matrix in briefly such as

$$A\Lambda^{n+1}=B\Lambda^{n}+\Upsilon \tag{3.7}$$

where vector $\Lambda=\{\delta_0,\delta_1,...,\delta_{N-1},\delta_N\}$ and vector Υ consist of boundary terms and values which produced via sum term. Using (3.7) , the unknown vector Λ^{n+1} will be determined by values of vector Λ^n.

4. Initial State

All recursive formulae must always state on initial terms. In this section, an initial vector must be derived. Thus, subsequent terms of the sequence can be found via initial vector. The initial vector $\Lambda^{(0)}=\left\{\delta_0^{(0)},\delta_1^{(0)},...,\delta_{N-1}^{(0)},\delta_N^{(0)}\right\}$ will be computed using the initial condition $u(x,0)=h(x)$. For this process, approximate solution and its derivatives are used at point x_m as follows;

$$u(x_m,0)=w(x_m,0)=h(x_m),\qquad m=0,1,2,...,N \tag{4.1}$$

When (4.1) is written in an explicit way, one can see that the system is composed of $(N+1)$ equations with $(N+3)$ unknowns. In order to eliminate two unknowns $\delta_{-1}^{(0)}$ and $\delta_{N+1}^{(0)}$ from the system, first derivative according to space variable will be used such that

$$\begin{aligned} \left(w_m^{(0)}\right)'&=\beta_1\delta_{-1}^{(0)}+\beta_2\delta_1^{(0)}=h(x_0), && \text{for } i=0 \\ \left(w_m^{(0)}\right)'&=\beta_1\delta_{N-1}^{(0)}+\beta_2\delta_{N+1}^{(0)}=h(x_N), && \text{for } i=N \end{aligned} \tag{4.2}$$

Thus,(4.2) yields

$$\delta_{-1}^{(0)}=\frac{1}{\beta_1}\left(h(x_0)-\beta_2\delta_1^{(0)}\right),$$

$$\delta_{N+1}^{(0)}=\frac{1}{\beta_2}\left(h(x_N)-\beta_1\delta_{N-1}^{(0)}\right).$$

At the end of process, all calculations result in a system involving $(N+1)$ equations with $(N+1)$ unknowns. Solving the system results in deriving initial vector $\Lambda^{(0)}$.

5. Numerical Approximation

In this section, the cubic trigonometric B-splines will be used to calculate the numerical solutions of FitzHugh–Nagumo model. Hence, accuracy and efficiency of the method will be tested of proposed numerical scheme. So as to calculate error norms, following formulas will be used;

$$L_2 = \sqrt{h \sum_{m=0}^{N} \left|u_m - w_m\right|^2}, \qquad L_\infty = \max_m \left|u_m - w_m\right|.$$

5.1 Case 1:

In this section, we apply the proposed technique on two cases. Consider the following time fractional FitzHugh-Nagumo model given in (2.2) under the following conditions $\kappa = \mu = 1, r = 4/10$, Thus, the problem is translated into following form;

$$D_t^\gamma u(x,t) = u_{xx}(x,t) + u(x,t)\left(1 - u(x,t)\right)\left(u(x,t) - 4/10\right),$$
$$\begin{cases} u(x_L,t) = f_1(t), \quad u(x_R,t) = f_2(t), \\ u_x(x_L,t) = g_1(t), \quad u_x(x_R,t) = g_2(t), \quad (x,t) = [0,1]\times[0,1] \\ u(x,0) = h(x) \end{cases} \tag{5.1}$$

where $h(x) = 1/e^{-\frac{x}{\sqrt{2}}}$ and exact solution of the model is

$$u(x,t) = \frac{1}{e^{-\frac{x}{\sqrt{2}}}} - \frac{t^\gamma (2r-1) e^{\frac{x}{\sqrt{2}}}}{2\Gamma(\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2} - \frac{t^{2\gamma}(2r-1)^2\left(e^{\frac{x}{\sqrt{2}}}-1\right)}{4\Gamma(2\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}$$
$$-\frac{t^{3\gamma}(2r-1)^3 e^{\frac{x}{\sqrt{2}}}\left(e^{\sqrt{2}x}+4e^{\frac{x}{\sqrt{2}}}+1\right)}{16\Gamma(2\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4} + \frac{t^{4\gamma}(2r-1)^4 e^{\frac{x}{\sqrt{2}}}\left(-11e^{\sqrt{2}x}+e^{\frac{3x}{\sqrt{2}}}+11e^{\frac{x}{\sqrt{2}}}-1\right)}{96\Gamma(2\gamma+1)\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}.$$

Boundary conditions can be seen taking $x = x_L$ and $x = x_R$ at exact solution.

As the first example, the interval which problem discussed on is chosen as $\Omega = [0,1]\times[0,1]$. The numerical solutions of (5.1) for different values of space step partition N and time step Δt and presented in Table 5.1-2 for $\gamma = 0.25$. Then different values for fractional order derivative are presented in Table 3 for $N = 800$ and $\Delta t = 0.001$. In last table i.e Table 4, comparisons are presented between [25] and present method for $\gamma = 0.9$ and different values of time step Δt.

It can be seen clearly from the table 5.1-2 that, the error norms L_2 and L_∞ are getting smaller with increasing number of space and time discretization. So that, for $N = 50$ and $\Delta t = 1/10$, L_2 =2.06214883$\times 10^{-5}$, $L_\infty =$ 3.53959227$\times 10^{-5}$and for $N = 800$ and $\Delta t = 1/1000$, $L_2 = 1.26949713 \times 10^{-5}$, $L_\infty = 1.70531113{\times}10^{-5}$. We can conclude that the finite element method using trigonometric cubic splines derives quite accurate solutions to exact ones. Additionally, the results given in Table 4 are calculated using "mean absolute error" given in [25]. It can be seen from the table 4, the error norms which are obtained using trigonometric cubic splines are in agreement with the error norms given in [25] and obtained more accurate solutions for many values of Δt.

Table 1. Case 1: Error norms L_2 and L_∞ of Time fractional FitzHugh-Nagumo model for different values of Δt and N.

$\gamma = 0.25$	$\Delta t = 0.01$		$\Delta t = 0.01$	
N	$L_2 \times 10^5$	$L_\infty \times 10^5$	$L_2 \times 10^5$	$L_\infty \times 10^5$
50	2.06214883	3.53959227	1.34963953	1.89004865
100	2.01958305	3.44947107	1.28629959	1.80653553
200	2.00910969	3.42689595	1.27040051	1.78548711
400	2.00650198	3.42151241	1.26641599	1.78020364
800	2.00585031	3.42009868	1.26541792	1.77888708

Table 2. Case 1: Error norms L_2 and L_∞ of Time fractional FitzHugh-Nagumo model for different values of Δt and N.

$\gamma = 0.25$	$\Delta t = 0.05$		$\Delta t = 0.001$	
N	$L_2 \times 10^5$	$L_\infty \times 10^5$	$L_2 \times 10^{-5}$	$L_\infty \times 10^5$
50	1.31240768	1.84955706	1.28811292	1.81625015
100	1.24911235	1.76581920	1.22486871	1.73269800
200	1.23322360	1.74499717	1.20899298	1.71185235
400	1.22924146	1.73972543	1.20501403	1.70662887
800	1.22824392	1.73841335	1.20401729	1.70531113

Table 3. Case 1: Error norms L_2 and L_∞ of Time fractional FitzHugh-Nagumo model for different values of Δt and $N = 800$.

$N = 800$	$\Delta t = 0.01$		$\Delta t = 0.001$	
γ	$L_2 \times 10^5$	$L_\infty \times 10^5$	$L_2 \times 10^5$	$L_\infty \times 10^5$
0.1	1.10005315	1.58375582	1.04760205	1.49860510
0.3	1.30542358	1.83538958	1.24669134	1.75903888
0.5	1.41379340	1.96531926	1.36063388	1.89242299
0.8	1.40460709	1.92744306	1.33323670	1.82888194
0.9	1.33077714	1.82130939	1.26949713	1.73682731

Table 4. Case 1: A comparison between error norms L_2 and L_∞ for Time fractional FitzHugh-Nagumo model

$\gamma = 0.9$	Present method	[25]
Δt	MAE	MAE
1/10	$1.86895787 \times 10^{-5}$	5.35×10^{-5}
1/20	$1.51876756 \times 10^{-5}$	2.81×10^{-5}
1/40	$1.33900185 \times 10^{-5}$	1.48×10^{-5}
1/80	$1.25240772 \times 10^{-5}$	7.74×10^{-6}
1/120	$1.22481462 \times 10^{-5}$	4.04×10^{-6}

5.2 Case 2:

For this case, the solution interval is chosen as $x = [-10, 10]$ and $T = 1$. Also, the parameters seen in the model are chosen as $\kappa = \mu = 1, r = 0.8$ [30]. Similar to case 1, Table 5 is prepared to check effect of time partition on numerical solution. Thus, Table is prepared for various values of Δt . In table 6, numerical solutions of FitzHugh-Nagumo model for several values of fractional derivative are demonstrated.

Figures show numerical behaviour of Time fractional FitzHugh-Nagumo model for $r = 0.8, \gamma = 0.9, \Delta t = 0.01, N = 400$ and different final times in . Figure clears behaviour of model at final time $T = 5$, and is the 3-dimensional representation of the model.

Table 5. Case 2: Error norms L_2 and L_∞ for different values of Δt and γ for $N = 400$.

$N = 400$	$\gamma = 0.5$		$\gamma = 0.9$	
Δt	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.1	11.9763931	7.29847571	5.10996928	3.09912528
0.01	8.30005134	5.34968587	3.10841816	2.02770504
0.005	8.10092924	5.24171639	2.96691736	1.94862517
0.0025	8.00168118	5.18768771	2.89126522	1.90592140
0.001	7.94222728	5.15525135	2.84271156	1.87833803

Table 6. Case 2: A comparison between values of solutions of Time fractional FitzHugh-Nagumo model and absolute errors for $\gamma = 0.8$ and $x = 0.01$

			[30]			
t	u_{RPSM}	u_{HAM}	u_{FVIM}	u_{NIM}	u_{FEMT}	$\lvert u(x,t) - w(x,t) \rvert_{FEM}$
0.01	0.499745	0.499765	0.499774	0.497779	0.500038	0.0002929366
0.05	0.494437	0.494699	0.494541	0.487317	0.495499	0.0010628858
0.1	0.489004	0.489798	0.489186	0.476613	0.490853	0.0018553202
0.15	0.484113	0.485631	0.484366	0.46698	0.486670	0.0025749224
0.2	0.479543	0.481948	0.479864	0.457985	0.482762	0.0032545373

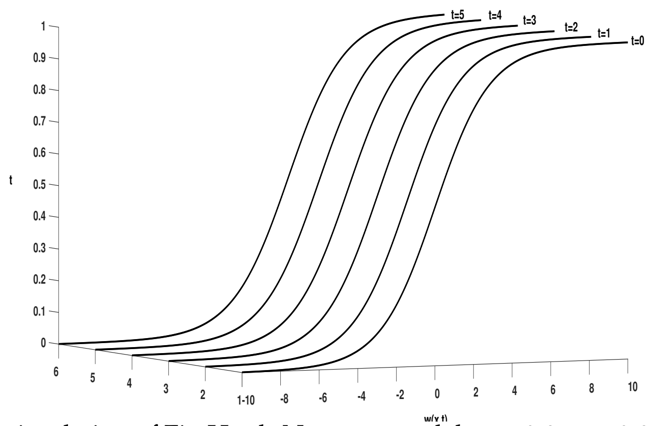


Figure 1. Numerical simulation of FitzHugh-Nagumo model: $r = 0.8, \gamma = 0.9, \Delta t = 0.01, N = 400$

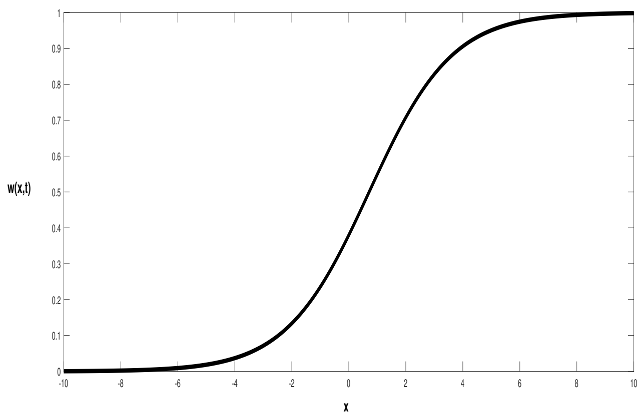


Figure 2. Numerical simulation of FitzHugh-Nagumo model: $r = 0.8, \gamma = 0.25, \Delta t = 0.01, N = 400$

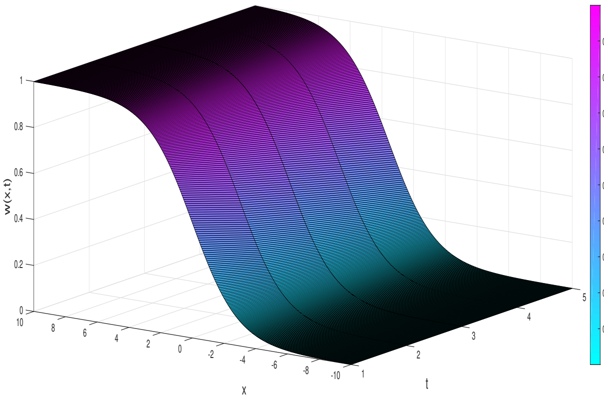


Figure 3. Numerical simulation of FitzHugh-Nagumo model: $r = 0.8$, $\gamma = 0.9$, $\Delta t = 0.01$,$N = 400$

6. Conclusion

In the present paper, finite element method based on trigonometric cubic B-spline is applied to time fractional FitzHugh-Nagumo model. Because the fractional derivative is taken in terms of Caputo sense, time discretization is made by $L1$ algorithm. Numerical solutions with comparison tables and error norms are presented. As it is seen from the numerical results, the results demonstrate that finite element method with trigonometric cubic basis is accurate and effective among existing methods.

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Freed, A. D., Diethelm K.: *Fractional calculus in biomechanics: a 3D viscoelastic model using regularized fractional derivative kernels with application to the human calcaneal fat pad.* Biomechanics and modeling in mechanobiology. **5**(4), 203-215 (2006).

[2] Ullah, N., Khan W., Wang S.: *High performance direct torque control of electrical aerodynamics load simulator using fractional calculus.* Acta Polytechnica Hungarica **11** (10), 59-78 (2014).

[3] Baleanu, D., Atangana, A., Oukouomi Noutchie,S. C., Kurulay, M., Bildik, N.,Kilicman, A.: Fractional calculus: models and numerical methods. World Scientific, **3**, 2012.

[4] Changpin, L., Zeng, F.: Numerical methods for fractional calculus. Chapman and Hall/CRC, 2019.

[5] Oruç, Ö., Esen, A., Bulut F. : *Highly accurate numerical scheme based on polynomial scaling functions for equal width equation*. Wave Motion: **102760**,(2021).

[6] Günerhan, H., Çelik, E.: *Analytical and approximate solutions of fractional partial differential-algebraic equations*. Applied Mathematics and Nonlinear Sciences **5** (1): 109-120 (2020).

[7] Mahdy, Amr M. S. :*Numerical solutions for solving model time-fractional Fokker–Planck equation*. Numerical Methods for Partial Differential Equations **37** (2), 1120-1135 (2021).

[8] Dokuyucu, M. A., Baleanu,D., Çelik E. : *Analysis of Keller-Segel model with Atangana-Baleanu fractional derivative*, Filomat **32** (16): 5633-5643 (2018).

[9] Owolabi, K. M., Hammouch, Z.: *Spatiotemporal patterns in the Belousov–Zhabotinskii reaction systems with Atangana–Baleanu fractional order derivative*. Physica A: Statistical Mechanics and its Applications **523**, 1072-1090(2019).

[10] Onal, M., Esen, A. : *A Crank-Nicolson approximation for the time fractional Burgers equation*, Applied Mathematics and Nonlinear Sciences **5** (2), 177-184(2020).

[11] Oruç,Ö., Esen, A., Bulut, F.: *A unified finite difference Chebyshev wavelet method for numerically solving time fractional Burgers' equation*. Discrete & Continuous Dynamical Systems-S **12** (3), 533(2019).

[12] Owolabi, Kolade M., Atangana A. : *Analysis and application of new fractional Adams–Bashforth scheme with Caputo–Fabrizio derivative*. Chaos, Solitons & Fractals **105**,111-119(2017).

[13] Oruç,Ö., Esen, A., Bulut, F.: *A Haar wavelet approximation for two-dimensional time fractional reaction–subdiffusion equation*, Engineering with Computers **35** (1), 75-86(2019).

[14] Owolabi, K. M.: *Robust and adaptive techniques for numerical simulation of nonlinear partial differential equations of fractional order*. Communications in Nonlinear Science and Numerical Simulation **44**, 304-317(2017).

[15] FitzHugh R.: *Mathematical models of threshold phenomena in the nerve membrane*. Bull. Math. Biophysics, **17**,257-278(1955)

[16] Nagumo J., Arimoto S., Yoshizawa S: *An active pulse transmission line simulating nerve axon*. Proc. IRE. **50**,2061–2070(1962).

[17] Xu, B., Binczak, S., Jacquir, S., Pont, O., Yahia, H.: *Parameters analysis of FitzHugh-Nagumo model for a reliable simulation*. Annu Int Conf IEEE Eng Med Biol Soc. 2014;2014:4334-7. doi: 10.1109/EMBC.2014.6944583. PMID: 25570951.

[18] Alphen, F. I. M: Model order reduction on FitzHugh-Nagumo model. BS thesis. University of Twente, 2020.

[19] Singh, S. : *Mixed-Type Discontinuous Galerkin Approach for Solving the Generalized FitzHugh–Nagumo Reaction–Diffusion Model.* International Journal of Applied and Computational Mathematics **7** (5), 1-16 (2021).

[20] Yongheng, W., Cai, L., Feng, X., Luo,X., Gao, H.: *A ghost structure finite difference method for a fractional FitzHugh-Nagumo monodomain model on moving irregular domain*. Journal of Computational Physics **428**, 110081 (2021).

[21] Hemami, M., Parand,K., Rad J. A.: *An efficient meshfree machine learning approach to simulate the generalized Fitzhugh-Nagumo equation inspired by neuroscience*. In: The 51th Annual Iranian Mathematics Conference, 16–19 February, University of Kashan .1-4,(2021)

[22] Jiwari, R., Gupta, R. K., Kumar V.: *Polynomial differential quadrature method for numerical solutions of the generalized Fitzhugh–Nagumo equation with time-dependent coefficients*. Ain Shams Engineering Journal **5** (4), 1343-1350(2014).

[23] Liu, F., Turner, I., Yang, Q., Burrage, K.:*A numerical method for the fractional Fitzhugh–Nagumo monodomain model*. Anziam Journal **54**, C608-C629(2012).

[24] Kumar, D., Singh,J., Baleanu, D.:*A new numerical algorithm for fractional Fitzhugh–Nagumo equation arising in transmission of nerve impulses*. Nonlinear Dynamics **91** (1), 307-317(2018).

[25] Injrou, S., Mahmoud, S., Kassem, A.:*A Stable Numerical Scheme for Fitzhugh-Nagumo Time-Fractional Partial Differential Equation*. Tishreen University Journal-Basic Sciences Series **41** (4), (2019).

[26] Chapwanya, M., Jejeniwa, O. A., Appadu, A. R., Lubuma, J. M.-S.: *An explicit nonstandard finite difference scheme for the FitzHugh–Nagumo equations*. International Journal of Computer Mathematics **96** (10), 1993-2009(2019).

[27] Jiwari, R., Gupta, R. K., Kumar, V.:*Polynomial differential quadrature method for numerical solutions of the generalized Fitzhugh–Nagumo equation with time-dependent coefficients*. Ain Shams Engineering Journal **5** (4), 1343-1350(2014).

[28] Ersoy Hepson, O., Dağ, İ, Saka B., Ay B.:*The Cubic B-spline Least Squares Finite Element Method for the Numerical Solutions of Regularized Long Wave Equation*. International Journal of Computer Mathematics, 1-12 (2021). https://doi.org/10.1080/00207160.2021.1940979

[29] Yuste, Santos B., Acedo, L. *An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations*. SIAM Journal on Numerical Analysis **42** (5), 1862-1874(2005).

[30] Tchier, F., Inc M., Korpinar Z.S.:*Solutions of the time fractional reaction–diffusion equations with residual power series method*. Advances in Mechanical Engineering **8** (10) : 1687814016670867(2016).

Affiliations

BERAT KARAAGAC
ADDRESS: Adıyaman University, Dept. of Mathematics Education, Adıyaman-Turkey.
E-MAIL: bkaraagac@adiyaman.edu.tr
ORCID ID:0000-0002-6020-3243

MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753 /msaen.981131
10 (3) 146-158 (2022) - Research Article
ISSN: 2147-6268

A Finite Difference Approximation for Numerical Simulation of 2D Viscous Coupled Burgers Equations

Nuri Murat Yağmurlu* and Abdulnasır Gagir

Abstract
Many of the physical phenomena in nature are usually expressed in terms of algebraic, differential or integral equations.Several nonlinear phenomena playing a very important role in engineering sciences, physics and computational mathematics are usually modeled by those non-linear partial differential equations (PDEs). It is usually difficult and problematic to examine and find out nalytical solutions of initial-boundary value problems consisting of PDEs. In fact, there is no a certain method or technique working well for all these type equations. For this reason, their approximate solutions are usually preferred rather than analytical ones of such type equations. Thus, many researchers are concentrated on approximate methods and techniques to obtain numerical solutions of non-linear PDEs. In the present article, the numerical simulation of the two-dimensional coupled Burgers equation (2D-cBE) has been sought by finite difference method based on Crank-Nicolson type approximation. Widely used three test examples given with appropriate initial and boundary conditions are used for the simulation process. During the simulation process,the error norms L_2, L_∞ are calculated if the exact solutions are already known, otherwise the pointwise values and graphics are provided for comparison. The newly obtained error norms L_2, L_∞ by the presented schemes are compared with those of some of the numerical solutions in the literature. A good consistency and accuracy are observed both by numerical values and visual illustrations.

Keywords: *2D viscous Burgers equation; Crank-Nicolson;Finite difference method.*

AMS Subject Classification (2020): *Primary: 35Q51 ; Secondary: 74J35, 33F10.*

**Corresponding Author*

1. Introduction

Most of the natural principles and laws in the real world are modelled by differential equations and only a few of them could be solved analytically. Thus obtaining numerical solutions for those equations has become more important. Throughout history of mankind, scientists and engineers have utilized mathematics for describing the physical characteristics of the universe by searching appropriate modelling methods and techniques. In this context, it is seen that numerous fundamental phenomena in ecology, physics, finance, data science, mechanical engineering

Received : *10-08-2021,* ***Accepted :*** *03-03-2022*

etc. could be modelled by means of various differential equations. Thus, finding analytical and numerical solutions of those differential equations has become a significant part of scientific studies. Becuase of this fact, over the years, a wide range of efficient and effective methods have been proposed and developed for solving those equations. Among others, one of those differential equations is the two-dimensional coupled Burgers equation (2D-cBE). There are many theoretical and numerical studies about the 2D-cBE equation in the literature. Fletcher [1] has presented its exact solution via applying the two-dimensional Hopf-Cole transform to the 2D coupled Burgers equation. 2D-cBE has been solved approximately by many researchers using several methods. For instance, Yagmurlu and Gagir [2] have sought the numerical solution of the two-dimensional coupled Burgers equation by finite difference method based on Rubin-Graves type linearization. Fletcher [3] has made a study to compare finite difference and finite element methods. Goyon [4] applied multi level alternating direction implicit methods. Ali et al. [5] have utilized collation method based on the radial base functions. Jain and Holla [6] have proposed two different schemes based on the cubic spline basis function. Bahadır [7] has solved the problem by means of fully implicit finite difference method. Khater et al. [8] have put forward the approximate solution of some Burgers type nonlinear partial differential equations using Chebyshev spectral collocation method. Mittal and Jiwari [9] have used the differential quadrature method using the Chebyshev-Gauss-Lobatto nodal points. Liao [10] obtained the numerical solution of the two-dimensional coupled Burgers equation by solving the two-dimensional linear heat equation obtained by applying the two-dimensional Hopf-Cole transformation to the two-dimensional coupled Burgers equation using the fourth-dimensional finite difference method. Zhu et al. [11] applied the discrete Adomian decomposition method. Srivastava et al. have applied [12] Crank-Nicolson finite difference method, Tamsir and Srivastava [13] have used semi-implicit finite difference method, Srivastava and Tamsir [14] have utilized Crank-Nicolson semi-implicit finite difference method, Thakar and Wani [15] have used linear finite difference method, Srivastava et al. [16] have applied implicit logarithmic finite difference method, Srivastava et al. [17] have used implicit exponential finite difference method, Srivastava and Singh [18] have used explicit-implicit finite difference method, Zhang et al. [19] have used full finite difference and non-standard finite difference methods, Mittal and Tripathi [20] have applied modified bi-cubic B-spline collocation method, Tamsir et al. [21] have used exponential modified cubic-B-spline differential quadrature method, Zhanlav et al. [22] have applied high order explicit finite difference method, Chai and Ouyang [34] have used proper stabilized Galerkin methods, and Ngondiep [23] has utilized three-level explicit time-split MacCormack algorithm. Saqib et al. [24] have dealt with numerical solutions of 2-dimensional time dependent coupled non-linear systems. Wubs and Goede [25], in their article, considered the fully explicit method resulting from the truncation in the solution process and chosen one of the test examples as the 2-dimensional coupled Burgers' equation. Kutluay and Yagmurlu [26] have proposed and succesfully applied the modified bi-quintic B-spline base functions for the two dimensional unsteady Burgers' equation using the Galerkin method. Başhan [27] has solved coupled viscous Burgers' equations numerically in the presence of very large Reynolds numbers. Başhan et al. [28] have applied quintic B-spline differential quadrature method to find the numerical solution of the modified Burgers' equation. Uçar et al. [29] have aimed to obtain the numerical approximate solutions of the nonlinear modified Burgers' equation via the modified cubic B-spline differential quadrature methods. Karakoç et al. [30] have obtained a numerical solution of the modified Burgers' equation by using quartic B-spline subdomain finite element method over which the nonlinear term is locally linearized and using quartic B-spline differential quadrature method. Başhan. [31] has modified quintic B-spline base function to use for numerical solution of the Burgers' equation. Karakoç and Bhowmik [32] have studied solitary-wave solutions of the nonlinear Benjamin-Bona-Mahony-Burgers equation based on a lumped Galerkin technique using cubic B-spline finite elements for the spatial approximation. Bhowmik and Karakoç [33] have developed and analyzed a powerful numerical scheme for the nonlinear GRLW equation by Petrov-Galerkin method in which the element shape functions are cubic and weight functions are quadratic B-splines.

The present paper is outlined as follows: The first section presented the method based on Crank-Nicolson type approximation together with finite difference method and used for the numerical solution of two dimensional coupled Burgers equation. To test the efficiency and effectiveness of the method, the approximate solution of three model problems has been found out and given in tabular form by computing the nodal values and also the error norms L_2 and L_∞ of the model examples of which the analytical solutions are known In addition, a comparison is made with the error norms of the numerical solutions obtained by several methods available in the literature. In the end of the article, a brief summary and plans about possible future studies are presented.

2. A concise description of the Method

In the present article, 2D coupled Burgers equation of the following form [13]

$$u_t + \mathrm{u}\, u_x + v u_y = \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}), \quad (x, y) \,\epsilon\, \Omega, t > 0, \tag{2.1}$$

$$v_t + \mathrm{u}\, v_x + v v_y = \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}), \quad (x, y) \,\epsilon\, \Omega, t > 0, \tag{2.2}$$

with the appropriate initial

$$\begin{aligned} u(x, y, 0) &= \psi_1(x, y); \quad (x, y) \,\epsilon\, \Omega, \\ v(x, y, 0) &= \psi_2(x, y); \quad (x, y) \,\epsilon\, \Omega, \end{aligned}$$

and the boundary conditions

$$\begin{aligned} u(x, y, t) &= \xi(x, y, t); \quad (x, y) \,\epsilon\, \partial\Omega, \\ v(x, y, t) &= \zeta(x, y, t); \quad (x, y) \,\epsilon\, \partial\Omega, \end{aligned}$$

will be considered, where $u\,(x, y, t)$ and $v\,(x, y, t)$ are velocity components given over the solution domain of the problem $\Omega = \{(x, y) : a \leqslant x \leqslant b,\, c \leqslant y \leqslant d\}$ together with its boundary $\partial\Omega$. ψ_1, ψ_2, ξ and ζ are known smooth functions.Re is the Reynold number and $\varepsilon = \frac{1}{\mathrm{Re}}$. For the large values of the Reynold number, a shock wave having a cusp results in and numerical stability near this shock wave is nearly always difficult to obtain. This is obvious in many studies in the literature and also observed in the present study.

The solution domain of the problem in $x-$direction is divided into N_x equal parts with length h_x, and in $y-$direction is divided into N_y equal parts with length h_y such that $x_i = a + ih_x$, $i = 0(1)N_x$ and $y_i = c + jh_y$, $j = 0(1)N_y$. As a result of these division processes, a rectangular grid is constructed over the solution domain of the problem by means of grid points (x_i, y_j). For the time dimension, again a fixed step length Δt is taken such that $t_n = n\Delta t$, $n = 0(1)N$. Throughout the paper, the numerical computations are going to be carried out at each t_n time step and the results are going to be obtained at the grid points of this rectangular grid. From now on, the numerical solutions of $u\,(x, y, t)$ and $v\,(x, y, t)$ at the grid point (x_i, y_j, t_n) are going to be denoted by uppercase $U_{\,i,j}^{\,n}$ and $V_{\,i,j}^{\,n}$, respectively.

Now, 2D-cBE given as follows

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y &= \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}), \end{aligned}$$

is first discretized by explicit finite difference method (EFDM)

$$\begin{aligned} U_{i,j}^{n+1} &= U_{i-1,j}^{n}(b_1 U_{i,j}^{n} + a_1) + U_{i,j}^{n}(1 - 4a_1) - U_{i+1,j}^{n}(b_1 U_{i,j}^{n} - a_1) + U_{i,j-1}^{n}(b_1 V_{i,j}^{n} + a_1) \\ &\quad - U_{i,j+1}^{n}(bV_{i,j}^{n} - a_1), \\ V_{i,j}^{n+1} &= V_{i-1,j}^{n}(b_1 U_{i,j}^{n} + a_1) + V_{i,j}^{n}(1 - 4a_1) - V_{i+1,j}^{n}(b_1 U_{i,j}^{n} - a_1) + V_{i,j-1}^{n}(b_1 V_{i,j}^{n} + a_1) \\ &\quad - V_{i,j+1}^{n}(b_1 V_{i,j}^{n} - a_1), \end{aligned}$$

and then by implicit finite difference method (IFDM)

$$\begin{aligned} &- U_{i-1,j}^{n+1}(b_1 U_{i,j}^{n} + a_1) + U_{i,j}^{n+1}(1 + 4a_1) + U_{i+1,j}^{n+1}(b_1 U_{i,j}^{n} - a_1) - U_{i,j-1}^{n+1}(b_1 V_{i,j}^{n} + a_1) \\ &+ U_{i,j+1}^{n+1}(b_1 V_{i,j}^{n} - a_1) = U_{i,j}^{n}, \end{aligned}$$

$$\begin{aligned} &- V_{i-1,j}^{n+1}(b_1 U_{i,j}^{n} + a_1) + V_{i,j}^{n+1}(1 + 4a_1) + V_{i+1,j}^{n+1}(b_1 U_{i,j}^{n} - a_1) - V_{i,j-1}^{n+1}(b_1 V_{i,j}^{n} + a_1) \\ &+ V_{i,j+1}^{n+1}(b_1 V_{i,j}^{n} - a_1) = V_{i,j}^{n}, \end{aligned}$$

where $h_x = h_y$, $a_1 = \varepsilon k/h_x^2 = \varepsilon k/h_y^2$, $b_1 = k/2h_x = k/2h_y$ and $\varepsilon = 1/\operatorname{Re}$ for $i, j = 1(1)M-1$ .When the average of EFDM and IFDM is taken, the following Crank-Nicolson finite difference schemes are obtained

$$\begin{aligned}
&- U_{i-1,j}^{n+1}(d_1U_{i,j}^n + c_1) + U_{i,j}^{n+1}(1+4c_1) + U_{i+1,j}^{n+1}(d_1U_{i,j}^n - c_1)\\
&- U_{i,j-1}^{n+1}(d_1V_{i,j}^n + c_1) + U_{i,j+1}^{n+1}(d_1V_{i,j}^n - c_1)\\
&= U_{i-1,j}^n(d_1U_{i,j}^n + c_1) + U_{i,j}^n(1-4c_1) - U_{i+1,j}^n(d_1U_{i,j}^n - c_1)\\
&+ U_{i,j-1}^n(d_1V_{i,j}^n + c_1) - U_{i,j+1}^n(d_1V_{i,j}^n - c_1),
\end{aligned}$$

and

$$\begin{aligned}
&- V_{i-1,j}^{n+1}(d_1U_{i,j}^n + c_1) + V_{i,j}^{n+1}(1+4c_1) + V_{i+1,j}^{n+1}(d_1U_{i,j}^n - c_1)\\
&- V_{i,j-1}^{n+1}(d_1V_{i,j}^n + c_1) + V_{i,j+1}^{n+1}(d_1V_{i,j}^n - c_1)\\
&= V_{i-1,j}^n(d_1U_{i,j}^n + c_1) + V_{i,j}^n(1-4c_1) - V_{i+1,j}^n(d_1U_{i,j}^n - c_1)\\
&+ V_{i,j-1}^n(d_1V_{i,j}^n + c_1) - V_{i,j+1}^n(d_1V_{i,j}^n - c_1),
\end{aligned}$$

where $h_x = h_y$, $c_1 = \varepsilon k/2h_x^2 = \varepsilon k/2h_y^2$, $d_1 = k/4h_x = k/4h_y$ and $\varepsilon = 1/\operatorname{Re}$ for $i, j = 1(1)M-1$.Using the known U^n and V^n values in the finite difference schemes obtained as a result of this approximation, the unknown values of U^{n+1} and V^{n+1} at the desired time t were obtained for all three model examples.

3. Numerical Examples and Results

In this section, the numerical solution of 2D coupled Burgers equation given by the equations (2.1)-(2.2), for three examples with appropriate initial and boundary conditions using the finite difference method based on Crank-Nicolson type approximation has been obtained. For all computations, the MATLAB software is used. In order to show the accuracy of the obtained numerical solutions, the following error norms L_2 and L_∞ are calculated

To show how good the numerical results obtained by the present method, the error norms L_2 and L_∞ given as

$$L_2 = \sqrt{\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} |U_{ij} - (u_{exact})_{ij}|^2},$$

and

$$L_\infty = \max_{i,j} |U_{i,j} - (u_{exact})_{i,j}|,$$

where u_{ij}^n are exact solutions and U_{ij}^n are numerical solutions at the nodal points (x_i, y_j, t_n) [35].The proposed numerical scheme is applied to the test problems and the computed approximate results for different values of the time step size k and partition number N at some values of T on the solution domains are displayed in tables.

Example I: Firstly, finite difference method has been applied to 2D-cBE having the following exact solution over the region $\Omega = [0,1] \times [0,1]$ [7]

$$u(x,y,t) = \frac{3}{4} - \frac{1}{4\left[1 + \exp((-4x + 4y - t)\operatorname{Re}/32\right]}, \tag{3.1}$$

$$v(x,y,t) = \frac{3}{4} + \frac{1}{4\left[1 + \exp((-4x + 4y - t)\operatorname{Re}/32\right]}. \tag{3.2}$$

Since the Example I has an exact solution, its initial and boundary conditions required for the application of the method are obtained from the analytical solution. Table (1) shows the numerical solutions of Example I for u for $h_x = h_y = 0.05$, $\operatorname{Re} = 10$, $\Delta t = 10^{-4}$ at $t = 0.01$ and 1.0. One can see from this table that both the numerical and analytical solutions at selected points at given times are very close to each other. Besides, it is obvious that the computed error norms L_2 and L_∞ are small enough. In a similar way, Table (2) gives the numerical solutions of Example I for v for $h_x = h_y = 0.05$, $\operatorname{Re} = 10$, $\Delta t = 10^{-4}$ at $t = 0.01$ and 1.0. One can observe that the numerical results are very close to their exact counterparts and computed error norms are small enough. It is also observed that the numerical solutions found out by the presented scheme are getting closer and closer to analytical values as the mesh sizes are refined.Tables (3-4) present nodal values and the error norms L_2 and L_∞ of u and v but now for

Table 1. Some nodal values u of Example 1 with the error norms L_2 and L_∞ for $h_x = h_y = 0.05$, Re= 10, $\Delta t = 10^{-4}$ at $t = 0.01$ and 1.0.

(x, y)	$t = 0.01$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.624805	0.624805	0.605626	0.605626
$(0.5, 0.1)$	0.594202	0.594202	0.576840	0.576840
$(0.9, 0.1)$	0.567082	0.567082	0.553017	0.553017
$(0.3, 0.3)$	0.624805	0.624805	0.605627	0.605626
$(0.7, 0.3)$	0.594202	0.594202	0.576840	0.576840
$(0.1, 0.5)$	0.655431	0.655431	0.636685	0.636685
$(0.5, 0.5)$	0.624805	0.624805	0.605628	0.605626
$(0.9, 0.5)$	0.594202	0.594202	0.576840	0.576840
$(0.3, 0.7)$	0.655431	0.655431	0.636687	0.636685
$(0.7, 0.7)$	0.624805	0.624805	0.605629	0.605626
$(0.1, 0.9)$	0.682611	0.682611	0.666353	0.666353
$(0.5, 0.9)$	0.655431	0.655431	0.636687	0.636685
$(0.9, 0.9)$	0.624805	0.624805	0.605627	0.605626
L_2	8.649162×10^{-8}		2.409775×10^{-6}	
L_∞	6.878261×10^{-8}		2.872069×10^{-6}	

Table 2. Some nodal values v of Example 1 with the error norms L_2 and L_∞ for $h_x = h_y = 0.05$, Re= 10, $\Delta t = 10^{-4}$ at $t = 0.01$ and 1.0.

(x, y)	$t = 0.01$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.875195	0.875195	0.894374	0.894374
$(0.5, 0.1)$	0.905798	0.905798	0.923160	0.923160
$(0.9, 0.1)$	0.932918	0.932918	0.946983	0.946983
$(0.3, 0.3)$	0.875195	0.875195	0.894373	0.894374
$(0.7, 0.3)$	0.905798	0.905798	0.923160	0.923160
$(0.1, 0.5)$	0.844569	0.844569	0.863315	0.863315
$(0.5, 0.5)$	0.875195	0.875195	0.894372	0.894374
$(0.9, 0.5)$	0.905798	0.905798	0.923160	0.923160
$(0.3, 0.7)$	0.844569	0.844569	0.863313	0.863315
$(0.7, 0.7)$	0.875195	0.875195	0.894371	0.894374
$(0.1, 0.9)$	0.817389	0.817389	0.833647	0.833647
$(0.5, 0.9)$	0.844569	0.844569	0.863313	0.863315
$(0.9, 0.9)$	0.875195	0.875195	0.894373	0.894374
L_2	6.178088×10^{-8}		1.637351×10^{-6}	
L_∞	6.878261×10^{-8}		2.872070×10^{-6}	

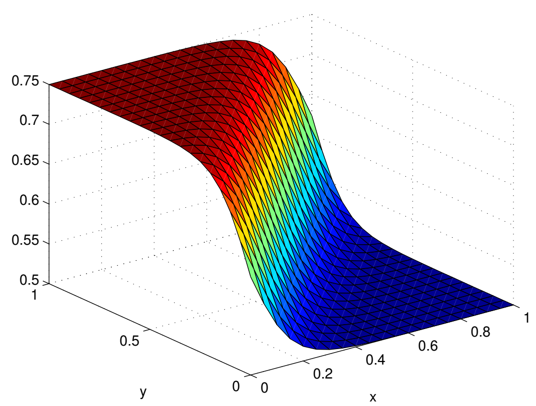
Table 3. Comparison of the approximate and exact solutions u of Example 1 for $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.01$ and 1.0.

(x, y)	$t = 0.01$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.623106	0.623047	0.510307	0.510522
$(0.5, 0.1)$	0.501617	0.501622	0.500072	0.500074
$(0.9, 0.1)$	0.500011	0.500011	0.500000	0.500000
$(0.3, 0.3)$	0.623106	0.623047	0.509823	0.510522
$(0.7, 0.3)$	0.501617	0.501622	0.500067	0.500074
$(0.1, 0.5)$	0.748272	0.748274	0.716948	0.716759
$(0.5, 0.5)$	0.623106	0.623047	0.509497	0.510522
$(0.9, 0.5)$	0.501617	0.501622	0.500063	0.500074
$(0.3, 0.7)$	0.748272	0.748274	0.717267	0.716759
$(0.7, 0.7)$	0.623106	0.623047	0.509311	0.510522
$(0.1, 0.9)$	0.749988	0.749988	0.749738	0.749742
$(0.5, 0.9)$	0.748272	0.748274	0.717532	0.716759
$(0.9, 0.9)$	0.623106	0.623047	0.509170	0.510522
L_2	3.822706×10^{-5}		1.341393×10^{-3}	
L_∞	6.086191×10^{-5}		2.903955×10^{-3}	

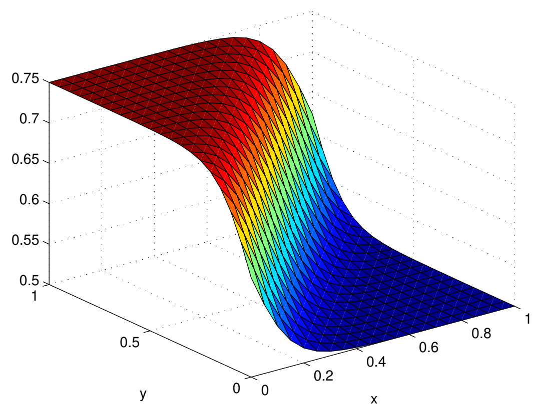
Table 4. Comparison of the approximate and exact solutions v of Example 1 for $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.01$ and 1.0.

(x, y)	$t = 0.01$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	0.876894	0.876953	0.989693	0.989478
$(0.5, 0.1)$	0.998383	0.998378	0.999928	0.999926
$(0.9, 0.1)$	0.999989	0.999989	1.000000	1.000000
$(0.3, 0.3)$	0.876894	0.876953	0.990177	0.989478
$(0.7, 0.3)$	0.998383	0.998378	0.999933	0.999926
$(0.1, 0.5)$	0.751728	0.751726	0.783052	0.783241
$(0.5, 0.5)$	0.876894	0.876953	0.990503	0.989478
$(0.9, 0.5)$	0.998383	0.998378	0.999937	0.999926
$(0.3, 0.7)$	0.751728	0.751726	0.782733	0.783241
$(0.7, 0.7)$	0.876894	0.876953	0.990689	0.989478
$(0.1, 0.9)$	0.750012	0.750012	0.750262	0.750258
$(0.5, 0.9)$	0.751728	0.751726	0.782468	0.783241
$(0.9, 0.9)$	0.876894	0.876953	0.990830	0.989478
L_2	2.744679×10^{-5}		8.302825×10^{-4}	
L_∞	6.086191×10^{-5}		2.903955×10^{-3}	

a larger value of Reynold number $\mathrm{Re} = 100$, respectively. From those tables one can see that both of the error norms increase as the Reynold number increases. In the Figures (1-2), one can see first exact and then numerical solutions for u and v of Example 1 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 100$, $\Delta t = 10^{-4}$ at $t = 0.5$,respectively.



(a)



(b)

Figure 1. The profiles of Example 1 of (a) exact and (b) numerical solutions for u for $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.5$.

Table 5. A comparison of numerical solutions for u of Example 2 with those given in [6, 7, 13] for $h_x = h_y = 0.025$, Re= 500, $\Delta t = 10^{-4}$ at $t = 0.625$.

(x, y)	u				
	Present	[6]	[6] N=40	[7]	[13]
$(0.15, 0.1)$	0.96870	0.95691	0.96066	0.96650	0.96870
$(0.3, 0.1)$	1.03202	0.95616	0.96852	1.02970	1.03200
$(0.1, 0.2)$	0.84619	0.84257	0.84104	0.84449	0.86178
$(0.2, 0.2)$	0.87814	0.86399	0.86866	0.87631	0.87813
$(0.1, 0.3)$	0.67920	0.67667	0.67792	0.67809	0.67920
$(0.3, 0.3)$	0.79947	0.76876	0.77254	0.79792	0.79945
$(0.15, 0.4)$	0.54674	0.54408	0.54543	0.54601	0.66039
$(0.2, 0.4)$	0.58959	0.58778	0.58564	0.58874	0.58958

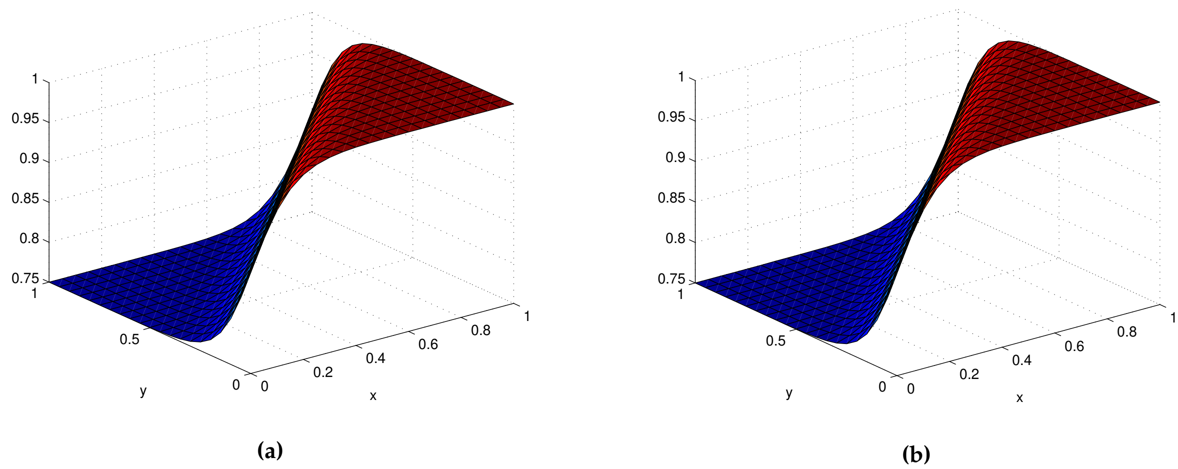


Figure 2. The profiles of Example 1 of (a) exact and (b) numerical solutions for v for values of $h_x = h_y = 0.05$, Re= 100, $\Delta t = 10^{-4}$ at $t = 0.5$.

Example II: Secondly, Crank-Nicolson finite difference method has been applied to 2D-cBE on the solution domain $\Omega = [0, 0.5] \times [0, 0.5]$ with the following initial

$$u\left(x, y, 0\right) = \sin \pi x + \cos \pi y,\ v\left(x, y, 0\right) = x + y \tag{3.3}$$

and boundary conditions

$$\left.\begin{array}{ll} u(0, y, t) = \cos(\pi y), & u(0.5, y, t) = 1 + \cos(\pi y) \\ v(0, y, t) = y, & v(0.5, y, t) = 0.5 + y \end{array}\right\} 0 \leq y \leq 0.5, t \geq 0, \tag{3.4}$$

$$\left.\begin{array}{ll} u(x, 0, t) = 1 + \sin(\pi x) & u(x, 0.5, t) = \sin(\pi x) \\ v(x, 0, t) = x & v(x, 0.5, t) = x + 0.5 \end{array}\right\} 0 \leq x \leq 0.5, t \geq 0. \tag{3.5}$$

[13]. Unlike the Example I, Example II has no analytical solution. Due to this fact, Table (5) only gives a comparison of numerical solutions for u of Example II for $h_x = h_y = 0.025$, Re= 500, $\Delta t = 10^{-4}$ at $t = 0.625$ with those in Refs. [6, 7, 13]. Again, due to the same reason, Table (6) presents a comparison of numerical solutions for v of Example II for values of $h_x = h_y = 0.025$, $\mathrm{Re} = 500$, $\Delta t = 10^{-4}$ at time $t = 0.625$ with those in Refs. [6, 7, 13]. Tables (7-8) show also pointwise values of u and v but now for a smaller value of $\mathrm{Re} = 50$, respectively. Figures (3) shows numerical solutions of u and v of Example II for $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at $t = 0.625$, respectively.

Table 6. A comparison of numerical solutions for v of Example 2 with those given in [6, 7, 13] for $h_x = h_y = 0.025$, Re$= 500$, $\Delta t = 10^{-4}$ at $t = 0.625$.

(x,y)	v				
	Present	[6]	[6] N=40	[7]	[13]
$(0.15, 0.1)$	0.09043	0.10177	0.08612	0.09020	0.09043
$(0.3, 0.1)$	0.10728	0.13287	0.07712	0.10690	0.10728
$(0.1, 0.2)$	0.18010	0.18503	0.17828	0.17972	0.17295
$(0.2, 0.2)$	0.16816	0.18169	0.16202	0.16777	0.16816
$(0.1, 0.3)$	0.26268	0.26560	0.26094	0.26222	0.26268
$(0.3, 0.3)$	0.23550	0.25142	0.21542	0.23497	0.23550
$(0.15, 0.4)$	0.31799	0.32084	0.31360	0.31753	0.29022
$(0.2, 0.4)$	0.30419	0.30927	0.29776	0.30371	0.30418

Table 7. A comparison of numerical solutions for u of Example 2 with those given in [6, 7, 13] for $h_x = h_y = 0.025$, Re$= 50$, $\Delta t = 10^{-4}$ at $t = 0.625$.

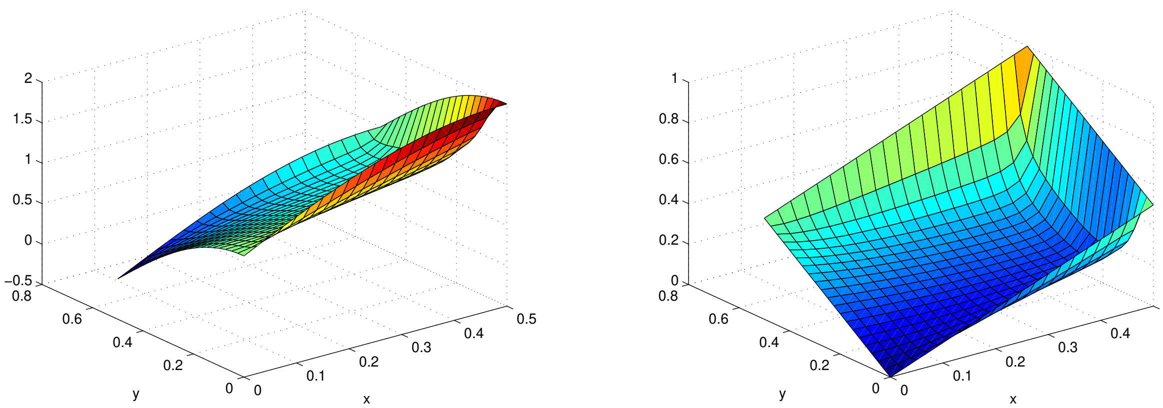
(x,y)	u			
	Present	[6]	[7]	[13]
$(0.1, 0.1)$	0.97146	0.97258	0.96688	0.97146
$(0.3, 0.1)$	1.15282	1.16214	1.14827	1.15280
$(0.2, 0.2)$	0.86307	0.86281	0.85911	0.86308
$(0.4, 0.2)$	0.97981	0.96483	0.97637	0.97984
$(0.1, 0.3)$	0.66316	0.66318	0.66019	0.66316
$(0.3, 0.3)$	0.77230	0.77030	0.76932	0.77232
$(0.2, 0.4)$	0.58180	0.58070	0.57966	0.58181
$(0.4, 0.4)$	0.75855	0.74435	0.75678	0.75860

Table 8. A comparison of numerical solutions for v of Example 2 with those given in [6, 7, 13] for $h_x = h_y = 0.025$, Re$= 50$, $\Delta t = 10^{-4}$ at $t = 0.625$.

(x,y)	v			
	Present	[6]	[7]	[13]
$(0.1, 0.1)$	0.09869	0.09773	0.09824	0.09869
$(0.3, 0.1)$	0.14158	0.14039	0.14112	0.14158
$(0.2, 0.2)$	0.16754	0.16660	0.16681	0.16754
$(0.4, 0.2)$	0.17109	0.17397	0.17065	0.17110
$(0.1, 0.3)$	0.26378	0.26294	0.26261	0.26378
$(0.3, 0.3)$	0.22654	0.22463	0.22576	0.22655
$(0.2, 0.4)$	0.32851	0.32402	0.32745	0.32851
$(0.4, 0.4)$	0.32499	0.31822	0.32441	0.32501

Table 9. Some nodal values u of Example 3 with the error norms L_2 and L_∞ for $h_x = h_y = 0.05$, Re= 1000, $\Delta t = 10^{-3}$ at $t = 0.01, 0.5$ and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
$(0.1, 0.1)$	-0.001439	-0.001439	-0.001408	-0.001408	-0.001376	-0.001376
$(0.5, 0.1)$	0.001941	0.001941	0.001895	0.001894	0.001849	0.001848
$(0.9, 0.1)$	-0.001727	-0.001727	-0.001682	-0.001682	-0.001638	-0.001637
$(0.3, 0.3)$	0.001134	0.001134	0.001114	0.001114	0.001094	0.001094
$(0.7, 0.3)$	0.002551	0.002551	0.002458	0.002453	0.002368	0.002359
$(0.1, 0.5)$	-0.003927	-0.003927	-0.003854	-0.003854	-0.003780	-0.003781
$(0.5, 0.5)$	0.006280	0.006280	0.006130	0.006130	0.005981	0.005981
$(0.9, 0.5)$	-0.007194	-0.007194	-0.006960	-0.006953	-0.006731	-0.006718
$(0.3, 0.7)$	0.001134	0.001134	0.001114	0.001114	0.001094	0.001094
$(0.7, 0.7)$	0.002551	0.002551	0.002458	0.002453	0.002368	0.002359
$(0.1, 0.9)$	-0.001439	-0.001439	-0.001408	-0.001408	-0.001376	-0.001376
$(0.5, 0.9)$	0.001941	0.001941	0.001895	0.001894	0.001849	0.001848
$(0.9, 0.9)$	-0.001727	-0.001727	-0.001682	-0.001682	-0.001638	-0.001637
L_2	2.2082×10^{-5}		1.0301×10^{-3}		1.9265×10^{-3}	
L_∞	2.8221×10^{-7}		1.2650×10^{-5}		2.2915×10^{-5}	



(a) **(b)**

Figure 3. The numerical profiles of (a) u and (b) v of Example 2 for $h_x = h_y = 0.025$, Re= 50, $\Delta t = 10^{-4}$ at $t = 0.625$.

Example III: Thirdly, the solution domain is taken as $\Omega = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ and its exact solution is [13]

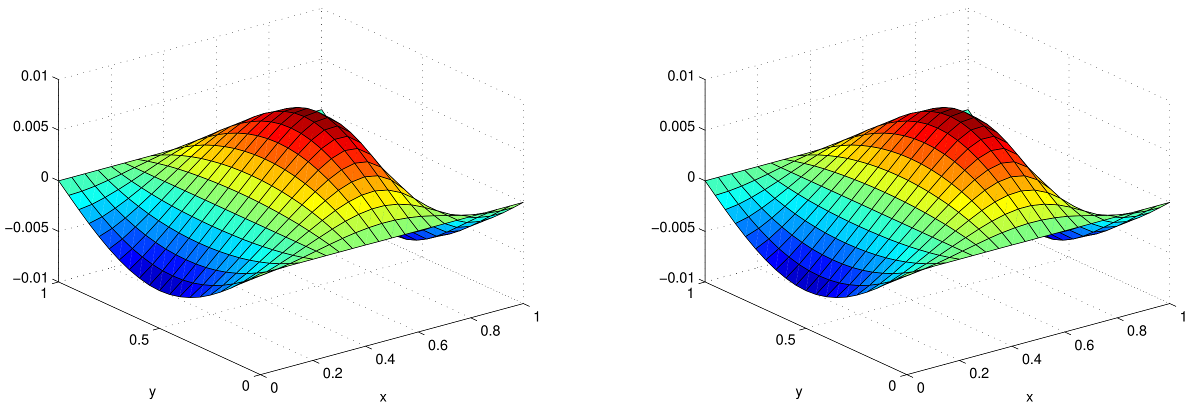
$$u(x, y, t) = -\frac{4\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \cos(2\pi x)\sin(\pi y)}{\mathrm{Re}(2 + e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \sin(2\pi x)\sin(\pi y)},$$

$$v(x, y, t) = -\frac{2\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \sin(2\pi x)\cos(\pi y)}{\mathrm{Re}(2 + e^{-\frac{5\pi^2 t}{\mathrm{Re}}} \sin(2\pi x)\sin(\pi y)},$$

Table (9) gives approximate solutions of u of Example III for $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at $t = 0.01$, 0.5 ve 1.0. From the table one can easily see that the approximate and exact solutions are very close to each other and calculated error norms L_2 and L_∞ are small enough. In a similar manner, Table (10) presents numerical solutions of v of Example 3 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at times $t = 0.01, 0.5$ ve 1.0. Again, one can see from this table that both of the approximate and exact pointwise values are in good agreement. Th error norms L_2 and L_∞ show the general consistency between the approximate and exact solutions throughout the solution domain. Figures (4-5) show (a) exact and (b) approximate solutions for u and v of Example 3 for values of $h_x = h_y = 0.05$, $\mathrm{Re} = 1000$, $\Delta t = 10^{-3}$ at $t = 0.01$,respectively.

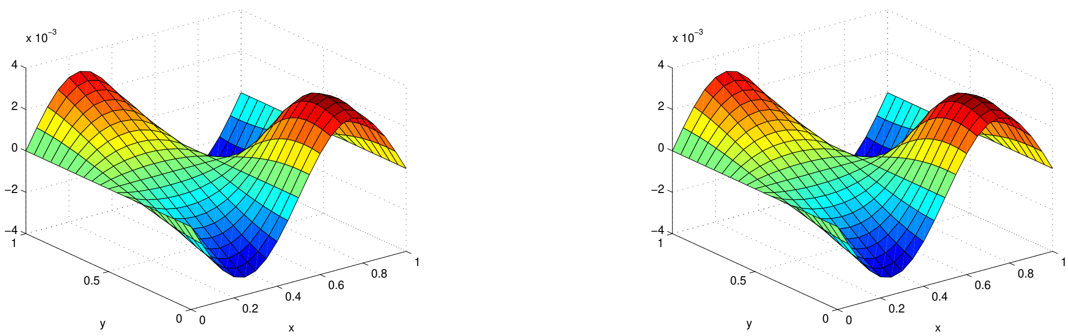
Table 10. Some nodal values v of Example 3 with the error norms L_2 and L_∞ for $h_x = h_y = 0.05$, Re= 1000, $\Delta t = 10^{-3}$ at $t = 0.01, 0.5$ and 1.0.

(x, y)	$t = 0.01$		$t = 0.5$		$t = 1.0$	
	Approx.	Exact	Approx.	Exact	Approx.	Ecaxt
$(0.1, 0.1)$	-0.001609	-0.001609	-0.001574	-0.001574	-0.001539	-0.001539
$(0.5, 0.1)$	-0.000000	-0.000000	-0.000000	-0.000000	-0.000001	-0.000000
$(0.9, 0.1)$	0.001931	0.001931	0.001880	0.001880	0.001830	0.001830
$(0.3, 0.3)$	-0.001268	-0.001268	-0.001246	-0.001246	-0.001223	-0.001224
$(0.7, 0.3)$	0.002852	0.002852	0.002743	0.002743	0.002643	0.002637
$(0.1, 0.5)$	-0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000
$(0.5, 0.5)$	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000
$(0.9, 0.5)$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$(0.3, 0.7)$	0.001268	0.001268	0.001246	0.001246	0.001223	0.001224
$(0.7, 0.7)$	-0.002852	-0.002852	-0.002746	-0.002743	-0.002643	-0.002637
$(0.1, 0.9)$	0.001609	0.001609	0.001574	0.001574	0.001539	0.001539
$(0.5, 0.9)$	0.000000	0.000000	0.000000	0.000000	0.000001	0.000000
$(0.9, 0.9)$	-0.001931	-0.001931	-0.001880	-0.001880	-0.001830	-0.001830
L_2	1.2840×10^{-5}		6.0180×10^{-4}		1.1312×10^{-3}	
L_∞	9.3384×10^{-8}		4.1425×10^{-6}		7.3706×10^{-6}	



(a) (b)

Figure 4. The profiles of Example 3 of (a) exact and (b) numerical solutions of u for $h_x = h_y = 0.05$, Re= 1000, $\Delta t = 10^{-3}$ at $t = 0.01$.



(a) (b)

Figure 5. The profiles of Example 3 of (a) exact and (b) numerical solutions of v for $h_x = h_y = 0.05$, Re= 1000, $\Delta t = 10^{-3}$ at $t = 0.01$.

4. Conclusion

In this study, the proposed scheme resulting an implicit linear algebraic system has been successfully applied to obtain the approximate solutions of two dimensional Burgers equation. The error norms L_2 and L_∞ of the presented scheme are calculated. The three numerical experiments showed that the approximate solutions are in very good agreement with the analytical ones, and also the error norms are adequately small. The obtained results support that the numerical accuracy of the scheme is in consistency with its theoretical value and that the scheme is also unconditionally stable. In conclusion, the present numerical scheme, which can be easily implemented, produces accurate and reliable results. As a future work, the method can be successfully used to find approximate solutions of such combined partial differential equations that play an important role in describing nonlinear wave propagation encountered in physics and applied mathematics.

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Fletcher, C. A. J.: *Generating exact solutions of the two-dimensional Burgers' equations*. International Journal for Numerical Methods in Fluids. **3**, 213-216 (1983). https://doi.org/10.1002/fld.1650030302

[2] Yagmurlu, N. M., Gagir A.: *Numerical Simulation of Two Dimensional Coupled Burgers Equations by Rubin-Graves Type Linearization*. Mathematical Sciences and Applications E-notes. **9** (4), 158-169 (2021). https://doi.org/10.36753/mathenot.947552

[3] Fletcher, C. A. J.: *A comparison of finite element and finite difference solutions of the one- and two-dimensional Burgers' equations*. Journal of Computational Physics. **51** (1), 159-188 (1983). https://doi.org/10.1016/0021-9991(83)90085-2

[4] Goyon, O.: *Multilevel Schemes for Solving Unsteady Equations*. International Journal for Numerical Methods in Fluids. **22** (10), 937-959 (1996).

[5] Arshed A., Siraj-ul-Islam, Sirajul H.: *A Computational Meshfree Technique for the Numerical Solution of the Two-Dimensional Coupled Burgers' Equations*. International Journal for Computational Methods in Engineering Science and Mechanics. **10** (5), 406-422 (2009). https://doi.org/10.1080/15502280903108016

[6] Jain, P. C., Holla, D. N.: *Numerical solutions of coupled Burgers' equation*. International Journal of Non-Linear. Meechanics. **13 (**4), 213-222 (1978), https://doi.org/10.1016/0020-7462(78)90024-0.

[7] Bahadır, A. R.: *A fully implicit finite-difference scheme for two-dimensional Burgers' equations. Applied Mathematics and Computation*. **137** (1)**,** 131–137 (2003). https://doi.org/10.1016/S0096-3003(02)00091-7

[8] Khater, A. H., Temsah, R. S, Hassan, M.M.: *Chebyshev spectral collocation method for solving Burgers'-type equations*. Journal of Computational and Applied Mathematics. **222** (2) 333–350 (2008). https://doi.org/10.1016/j.cam.2007.11.007https://doi.org/10.1016/j.cam.200711.007

[9] Mittal, R. C., Jiwari, R.: *Differential Quadrature Method for Two-Dimensional Burgers' Equations*. International Journal for Computational Methods in Engineering Science and Mechanics. **10** (6), 450–459 (2009). https://doi:10.1080/15502280903111424

[10] Liao, W.: *A fourth-order finite-difference method for solving the system of two-dimensional Burgers' equations*. International Jlournal Numerical Methods in Fluids. **64** (5) 565–590 (2010). https://doi.org/10.1002/fld.2163

[11] Zhu, H., Shu, H., Ding, M.: *Numerical solutions of two-dimensional Burgers' equations by discrete Adomian decomposition method*. Computers and Mathematics with Applications. **60** (3) 840-848 (2010). https://doi.org/10.1016/j.camwa.2010.05.031

[12] Srivastava, V. K., Tamsir, M., Bhardwaj, U., Sanyasiraju, Y.: *Crank-Nicolson Scheme for Numerical Solutions of Two-dimensional Coupled Burgers' Equations*. International Journal of Scientific & Engineering Research. **2** (5), 1-6 (2011).

[13] Tamsir, M., Srivastava, V. K.: *A semi-implicit finite-difference approach for two-dimensional coupled Burgers equations*. International Journal of Scientific & Engineering Research. **2** (6), 46-51 (2011). ISSN 2229-5518

[14] Srivastava, V. K., Tamsir, M.: *Crank-Nicolson Semi-Implicit Approach For Numerical Solutions of Two- Dimensional Coupled Nonlinear Burgers Equations*. International Journal of Applied Mechanics and Engineering. **17** (2), 571-581 (2012).

[15] Thakar, S., Wani, S.: *Linear Method For Two Dimensional Burgers Equation*. Ultra Scientist. **25** (1)A, 156-168 (2013).

[16] Srivastava, V. K., Awasthi, M.K., Singh, S.: *An implicit logarithmic finite-difference technique for two dimensional coupled viscous Burgers' equation*. AIP Advances. **3** (12), 122105 (2013). https://doi:10.1063/1.4842595

[17] Srivastava, V. K., Singh, S., Awasthi, M. K.: *Numerical solutions of coupled Burgers equations by an implicit finite difference scheme*. AIP Advances. **3** (8), 082131 (2013). https://doi: 10.1063/1.4820355

[18] Srivastava, V. K., Singh, B. K.: *A robust finite difference scheme for the numerical solutions of two dimensional time dependent coupled nonlinear Burgers equations*. International Journal of Applied Mathematics and Mechanics. **10** (7), 28-39 (2014).

[19] Zhang, L., Wang, L., Ding, X.:*Exact finite-difference scheme and nonstandard finite-difference scheme for coupled Burgers equation*. Advances in Difference Equations. **122** (2014) (2014). https://doi:10.1186/1687-1847-2014-122

[20] Mittal, R. C., Tripathi, A.: *Numerical solutions of two-dimensional Burgers' equations using modified Bi-cubic B-spline finite elements*. Engineering Computations. **32** (5), 1275 - 1306.(2015). https://doi.org/10.1108/EC-04-2014-0067

[21] Tamsir, M., Srivastava, V. K., Jiwari, R.: *An algorithm based on exponential modified cubic B-spline differential quadrature method for nonlinear Burgers' equation*. Applied Mathematics and Computation. **290**, 111–124 (2016). https://doi.org/10.1016/j.amc.2016.05.048

[22] Zhanlav, T., Chuluunbaatar, O., Ulziibayar, V.: *Higher-Order Numerical Solution of Two-Dimensional Coupled Burgers Equations*. American Journal of Computational Mathematics. **6** (2), 120-129 (2016). https://doi:10.4236/ajcm.2016.62013

[23] Ngondiep, E.: *An efficient three-level explicit time-split scheme for solving two-dimensional unsteady nonlinear coupled Burgers' equations*. Int J Numer Meth Fluids. **92** (4), 266–284 (2020). https://doi.org/10.1002/fld.4783

[24] Saqib, M., Hasnain, S., Mashat, D. S.: *Highly Efficient Computational Methods for Two Dimensional Coupled Nonlinear Unsteady Convection-Diffusion Problems. IEEE Access. **5**, 7139-7148 (2017)*. https://doi:10.1109/ACCESS.2017.2699320

[25] Wubs, F. W., de Goede, E. D.: *An explicit-implicit method for a class of time-dependent partial differential equations*. Appl. Numer. Math**.** **9** (2)**,** 157-181 (1992). https://doi.org/10.1016/0168-9274(92)90012-3

[26] Kutluay, S., Yağmurlu, N. M.: *The Modified Bi-quintic B-Splines for Solving the Two-Dimensional Unsteady Burgers' Equation*. European International Journal of Science and Technology. **1**(2), 23-39 (2012).

[27] Başhan, A.: *A numerical treatment of the coupled viscous Burgers' equation in the presence of very large Reynolds number*. Physica A: Statistical Mechanics and its Applications. **545**, (2020). https://doi.org/10.1016/j.physa.2019.123755

[28] Başhan, A., Karakoç, S. B. G., Geyikli, T.: *B-spline Differential Quadrature Method for the Modified Burgers' Equation*. Çankaya University Journal of Science and Engineering. **12** (1), 001–013 (2015) .

[29] Uçar, Y., Yağmurlu, N. M., Başhan, A.: *Numerical Solutions and Stability Analysis of Modified Burgers Equation via Modified Cubic B-Spline Differential Quadrature Methods*. Sigma J Eng & Nat Sci. **37** (1), 129-142 (2019).

[30] Karakoç, S. B. G., Başhan, A., Geyikli, T.: *Two Different Methods for Numerical Solution of the Modified Burgers' Equation*. The Scientific World Journal. **2014** (5), 1-13 (2014)**.** https://doi.org/10.1155/2014/780269.

[31] Başhan, A.: *Nonlinear dynamics of the Burgers' equation and numerical experiments*. Math Sci (2021). https://doi.org/10.1007/s40096-021-00410-8.

[32] Karakoç, B., Bhowmik, S. K.: *Galerkin Finite Element Solution for Benjamin-Bona-Mahony-Burgers Equation with Cubic B-Splines*. Computers and Mathematics with Applications. **77** (7), 1917–1932 (2019).

[33] Bhowmik, S. K., Karakoç, S. B. G.: *Numerical approximation of the generalized regularized long wave equation using Petrov Galerkin finite element method*. Numerical Methods for Partial Differential Equations. **35** (6), 2236–2257 (2019).

[34] Chai, Y., Ouyang, J.: *Appropriate stabilized Galerkin approaches for solving two-dimensional coupled Burgers' equations at high Reynolds numbers*. Computers and Mathematics with Applications. **79** (5), 1287–1301 (2020). https://doi.org/10.1016/j.camwa.2019.08.036

[35] Shukla, H. S., Tamsir, M., Srivastava, V. K., Kumar, J.: *Numerical Solution of two dimensional coupled viscous Burgers' Equation using the Modified Cubic B-Spline Differential Quadrature Method*. AIP Advances. **4** (11), 1-10 (2014).https://doi.org/10.1063/1.4902507

Affiliations

NURI MURAT YAĞMURLU
ADDRESS: Inonu University, Dept. of Mathematics, 44200, Malatya-TURKEY.
E-MAIL: murat.yagmurlu@inonu.edu.tr
ORCID ID: 0000-0003-1593-0254

ABDULNASIR GAGIR
ADDRESS: Inonu University, Dept. of Mathematics, 44200, Malatya-TURKEY.
E-MAIL: ansrggr@gmail.com
ORCID ID: 0000-0003-1029-4447

Mathematical Sciences and Applications E-Notes
https://doi.org/10.36753 /msaen.988711
10 (3) 159-169 (2022) - Research Article
ISSN: 2147-6268

An Application of Path Analysis in Gaugeing Stimulated Travel Demand

Enver Cenan İnce*, İbrahim Demir and Hüseyin Murat Çelik

Abstract
This article aims to designate a methodological baseline for gaugeing stimulated travel demand in urban areas in the case of İstanbul in addition to the intention of detecting the primary factors affecting motorized trips in urban spaces so as to re-formulate current travel demand management policies. In this sense, this article exhibits an investigation on gaugeing stimulated travel demand in the case of İstanbul by interrogating the inter-relationship between two variables called amount of trips and travel time. Such an empirical investigation is expected to construct a baseline for determining the optimum transportation project with regard to the concern of minimizing the amount of travel time spent by communities. At this juncture, an amount of applied econometric models for measuring stimulated travel demand were exhibited in detail. According to the results of path analysis as an advanced version of simultaneous equations system, approximately 27% decrease in travel time causes 71% additional increase in amount of trips in a day. That is to say, a ten minutes decrese in travel time will trigger 108,869 more motorized trips per day in İstanbul. This result once more reveals that unending supply of transportation infrastructure in urban areas causes a kind of vicious circle in the long run, which necessitates further travel demand management strategies in all over the world.

Keywords: *Applied econometric models; travel demand models; simultaneous equations model system, path analysis.*

AMS Subject Classification (2020): *Primary: 00A00 ; Secondary: 00B00; 00C00; 00D00; 00E00; 00F00.*

**Corresponding author*

1. Introduction

This article aims to open a door for gaugeing stimulated travel demand in a new way by investigating the mathemetical relationship between daily amount of trips and travel time per individual. From this point forth, the marginal effect of time allocated for daily trips on amount of trips per individual has been calculated with the help of a new methodological way called path analysis as an advanced version of simultaneous equations system, which is an extended version of classical simultaneous equations model, was designated. In this way, the estimations derived from this model can be benefitted in the selection the optimum large scale transportation project among of a bundle of alternatives in an urban scale. Also, it was taken into consideration that the time allocated for trips is a

Received : *30-08-2021,* ***Accepted :*** *03-03-2022*

kind of friction parameter referring to a proxy measure of travel cost.

There are three main aims of this paper. Firstly, it is aimed to constitute a methodological baseline for gaugeing stimulated travel demand in urban areas in the case of İstanbul. Secondly, it is aimed to constitute a baseline for detection of primary parameters affecting motorized flow in urban spaces so as to re-formulate current travel demand management policies. Thirdly, the findings of this article can make classical travel demand forecasting models more dynamic and realistic.

In the light of this motivation, this article has been structured by five sections. In pursuit of introduction part, the basic theoretical background of gaugeing stimulated travel demand was reviewed. Then the data, materials and method were discussed in detail. The fourth section exhibited the preliminary results of the designated model, while the last section concludes all these efforts with some recommendations for future studies.

2. Theoretical Background

It is asserted that generated traffic decreases the benefits of new transportation investments, which also re-increases the cost of travel in the long run within the considerations of stimulated travel demand [1]. Herein, the cost can be represented by the travel time as a proxy variable in urban systems. Therefore, measuring stimulated travel demand in urban spaces seems a vital requirement so as to perceive the long run effects of any new large scale transportation investments in urban areas.

In this sense, within the considerations of gaugeing stimulated travel demand in literature, the variables included in the analyses were generally vehicle miles travelled as the dependent variable with the explanatory ones called travel time, travel speed, additional lane miles, time lagged geographical variables with follows to the socioconomical parameters [2, 3]. Besides, the model structures in their analyses are called matched pair analysis, growth comparison analysis [2, 4], two/three stages least squares regression models, auto-regressive models, and travel demand models [3, 5]. Also, it is seen that in most of these models, the dependent variable, which is generally vehicle miles travelled, is modelled in the log-linear functional forms [3, 5].

On the other hand, there are some inefficiencies in these approaches. The first is about the study areas in that some of these studies focused on the corridor based analyses with grasping the marginal elasicity of vehicle miles travelled [2, 4], which refers to a kind of short run partial urban equilibrium disregarding the dynamics of whole urban areas. That is to say, urban area itself ought to be defined as the case area as a whole so as to grasp the long run urban equilibriums in such these considerations [3, 5]. The second is about the aggregation of the data structures in that some researches aggregated the travel survey data by summing all observations on the urban communities scale [2, 6], which makes estimations errors increase dramatically [5, 6]. The third is about disregarding the potential erogeneity of explanatory variables such as lane miles additions, travel time, etc. in their models [2, 5**?** –15], which is ought to be taken into consideration in such these related analyses [16–18].

From this point forth, in the studies gaugeing stimulated travel demand can be categorized into three approaches [2, 3]. First is about the data structure in that whether the data is collected in an aggregated scale or in a behavioral units scale. In other words, whether the travel survey structure is designated by urban communities scale or individuals/household scale matters. Second is about whether the study area is defined as a whole urban area or just as a corridor/ highway based line in related analyses. Third is about the model structures in related analyses in that whether the erogeneity of any explanatory variable affecting stimulated travel demand is taken into consideration or not [2, 3].

In the light of these views, our paper investigates the marginal interaction between amount of trips (instead of vehicle miles travelled) and travel time, since amount of trips is much more direct measure of stimulated demand rather than the vehicle miles travelled in urban spaces with taking the followings into consideration:

i. Taking the endogeneity of travel time into account [16–18].

ii. Instead of the aggregated data structures, the travel surveys are needed to be produced by the behavioral units scale with regard to individuals [5, 6] instead of aggregation towards urban communities.

iii. Study area ought to be defined as the whole urban area [3, 5] instead of the corridor based ones [2, 4].

3. Data, Materials and Method

According to the research design of this article, the term traveller should firstly be defined. In this study, traveller is defined as the one, who is above 16 years old. Secondly, the term trip is defined as the motorized mobility, which does not return to its origin within 15 minutes. From this viewpoints, the sample was selected by the procedure of stratified simple random sampling in İstanbul metropolitan region and the survey was coordinated by Transportation Department of the Metropolitan Municipality. The sample referred to 90,000 households with the sampling ratio 3 % and two-stages random cluster sampling procedure was implemented. In the first stage, 4,000 primary household units were randomly selected, while 30 household units per 90 were systematically selected in the second stage. Thus, 450 traffic zones were constructed as the spatial analyzing units with refers to 356,000 trips with the records of origins and destinations.

From this point forth, the variables in the analyses were defined as in the followings:

- $number\_trip$: amount of trips exhibited per individual per day.
- $travel\_time$: amount of minutes spent on travel per day.
- $travel\_distance$: length of km travelled per day.
- sex: dummy variable; 1: male, 0: female.
- $household\_head$: dummy variable; 1: household head, 0: others in the family.
- $household\_income$: monthly household income in Turkish Lira per month.
- $vehicle\_number$: number of automobiles owned.
- $household\_size$: number of individuals in the family.
- age: age of the person.
- $home\_base\_work$: dummy variable; 1: if the recorded trip is from home to work, 0: otherwise.
- $home\_base\_school$: dummy variable; 1: if the recorded trip is from home to school, 0: otherwise.
- $education\_year$: amount of years spent in education by the individual.
- $free\_flow\_time$: the hypothetical travel time in minutes experienced in the free flow speed as the only vehicle in the highway system.
- $motorized\_flow$: dummy variable; 1: if at least one motorized trip is exhibited by the individual, 0: otherwise.

In the light of these variables, to begin with, number of trips behaves as a discrete count variable and it involves excess amount of zero observations in that almost 63 % of them is zero (see Figure 1). Secondly, unlike amount of trips, travel time is a kind of gaussian distributed variable.

Figure 1. Daily trip frequencies produced by STATA 15.

Thirdly, amount of minutes spent in travel is a kind of endogenous variable in explaining amount of trips, which necessitates simultaneous system. But, classical simultaneous equations system assumes that both dependent and the endogenous explanatory variable(s) are gaussian distributed continuous variables, which is not the case in this study since amount of trips is a count variable.

In the light of these views, non linear regression models come into agenda in modelling the count variable called amount of daily trips. These models are poisson regression and negative binomial regression [19, 20].

In the structure of poisson regression, the dependent variable for i^{th} observation called y_i is assumed to be poisson distributed with mean λ_i, then Equation 3.1 is:

$$f(Y_i = y_i) = \frac{e^{-\lambda_i} \times \lambda_i^{y_i}}{y_i!}; y_i = 0, 1, 2, \ldots \tag{3.1}$$

f : *probability distrubution function*

Then, the mathematical equation between λ_i and x_i is as in Equation 3.2:

$$\ln(\lambda_i) = \beta^{'} \times x_i \tag{3.2}$$

In addition, poisson regression structure necessitates the condition that conditional mean of the dependent variable (given the explanatory variables x_i's) is equal to conditional variance of this variable as represented in equation 3.3:

$$E[y_i|x_i] = Var(y_i|x_i) = \lambda_i = e^{\beta^{'} \times x_i} \tag{3.3}$$

Then, the marginal elasticity of amount of trips (y_i) with regard to the marginal change in travel time is explained by Equation 3.4:

$$\frac{\partial E[y_i|x_i]}{\partial x_i} = Var(y_i|x_i) = \lambda_i \times \beta = \lambda_i \times e^{\beta^{'} \times x_i} \tag{3.4}$$

So, the maximum likelihood function in poisson regression system becomes as revealed in Eequation 3.5:

$$\ln(L) = \sum_{i=1}^{n} \left(-\lambda_i + y_i \times \beta^{'} \times x_i - \ln(y_i!) \right) \tag{3.5}$$

On the other hand, the assumption of poisson regression system, which asserts that conditional mean of the dependent variable is equal to conditional variance of this variable, is not usually realistic in most cases. In this sense, it is indicated that the condition of overdispersion is observed in most cases, which makes negative binomial regression system more attractive [21]. This system can be revealed as in the following Equation 3.6 with the help of θ as a scale variable:

$$f(y_i|x_i) = \frac{r(\theta + y_i)}{r(\theta) \times r(y_i + 1)} \times r_i^{y} \times (1 - r_i)^{\theta} \tag{3.6}$$

where $r_i = \frac{\lambda_i}{\lambda_i + \theta_i}$. So the related mean still stays as λ_i as in the poisson system, while the variance takes its new form as $\lambda_i \times (1 + \frac{1}{\theta}) \times \lambda_i$ due to the case of overdispersion. Then, the related marginal elasticity can still be derived as revealed in Equation 3.4 [19].

In addition, the most commonly used test in the selection between the poisson and negative binomial regression systems is likelihood ratio (LR) test [19, 22], which is represented as in Equation 3.7:

$$LR = 2 \times [\ln(L_{NBM}) - \ln(L_{PM})] \tag{3.7}$$

Herein, if the calculated value of LR is larger than the critical value in the asserted confidence interval, then this signs that there is the case of overdispersion that favours negative binomial system.

As it is explained in the following section of the article, although the negative binomial system is preferred as the single equation model in modelling number of trips, it can not deal with the excess zero observations in this variable. This situation makes zero truncated models come into account, since it can cope with excessive amount of zero counts in the dependent variable. [20].The mathematical function behind this model structure is as revealed in Equation 3.8:

$$F(y|\theta, y \geq 1) = \frac{f(y|\theta)}{1 - F(0|\theta)}, y = 1, 2, \cdots \tag{3.8}$$

Here $f(y|\theta)$, probability distribution function, $F(y|\theta) = Prob[Y \leq y]$, cumulative probability distribution function of y, and θ is a vector of parameters.

On the other hand, all these asserted regression systems refer to single equations system, which disregards the erogeneity of an independent variable in the system [19, 20, 22–24], which is minutes spent in daily travel in our case. Therefore, simultaneous equations system, which can cope with the problem of such this erogeneity, comes into considerations. Unlike single equation models, endogenous explanatory variable can be taken as the other dependent variable that is also affected by the priority defined dependent variable in such this equations system. This can be succeeded by asserting convenient instrumental variable(s) in the system [19, 20]. On the other hand, the failure of this system in our case is due to the assumption of classical simultaneous equations system asserting that both dependent and endogenous variables are continuous ones, since trip counts is not Gaussian distributed. Therefore, an extended version of classical simultaneous equations system, which is called path analysis as an advanced version of simultaneous equations system, comes into considerations for our case.

In the designation of this model structure, firstly, trip counts per person is estimated by negative binomial regression system. Then, these estimated values (e_number_trip) are used as further explanatory observations in modelling the endogenous variable called minutes spent in travel (see Figure 2).

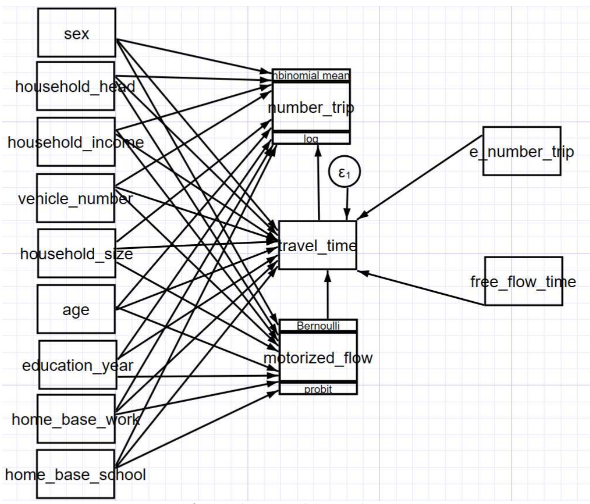


Figure 2. Designation of path analysis by STATA-15 SEM Builder.

Furthermore, in pursuit of finalization of the structure of the path analysis, the mathematical mechanism for grasping marginal elasticities between related variables involving especially the marginal effect of minutes spent in travel on number of trips come into considerations.

In this sense, to begin with, the marginal elasticity calculation with regard to the non-linear models can basically be represented by Equation 3.9:

$$ME = \frac{\partial E[Y|X]}{\partial X_j}. \tag{3.9}$$

Here, ME represents marginal elasticity and $E(.)$ stands as the expected value function. From this point forth, this equation gets its new form as in Equation 3.10 by indicating a function $g(.)$ that represents the function of non-linear mean values:

$$E[Y|X] = g(X' \times \beta), \frac{\partial E[Y|X]}{\partial X_j} = g'(X' \times \beta) \times \beta_j. \tag{3.10}$$

Then, the model structures involving multi explanatory variables requires the calculation of relative effects of changes in regressors, which can be represented by Equation 3.11:

$$\frac{\partial E[Y|X]/\partial X_j}{\partial E[Y|X]/\partial X_k} = \frac{\beta_j}{\beta_k} \times \frac{g'(X' \times \beta)}{g'(X' \times \beta)} = \frac{\beta_j}{\beta_k}. \tag{3.11}$$

On the other hand, within the concept of finite difference method, Equation 3.11 is modified by Equation 3.12:

$$\frac{\Delta E[Y|X]}{\Delta X_j} = g(X + e_j, \beta) - g(X, \beta). \tag{3.12}$$

Here, e_j represents the j^{th} element of the vector, in which the other elements are zero. Lastly, in the exponential distribution functions the equation above is transferred to Equation 3.13 below [20]:

$$E[Y|X] = e^{X' \times \beta}, \frac{\partial E[Y|X]}{\partial X_j} = E[Y|X] \times \beta_j. \tag{3.13}$$

In the light of these mathematical derivations, the marginal elasticity estimation between $number\_trip$ and $travel\_time$ per each traveller i in our case can be represented by Equation 3.14:

$$\frac{1}{N} \times \sum_{i=1}^{N} \frac{\partial E[number\_trip_i|X_i, travel\_time_i]}{\partial(travel\_time_i)} \tag{3.14}$$

Also, the marginal elasticity of each explanatory variable X_i on the number of trips per each individual i can be indicated by Equation 3.15:

$$\frac{\partial E[number\_trip_i|X_i, travel\_time_i]}{\partial(X_i)} \tag{3.15}$$

To sum, the path analysis as an advanced and extended version of classical simultaneous equations system is the most convenient model concept in our case since it can cope with non-linearity of daily trip counts, excess zero amounts in trip counts, and erogeneity of travel time (see Table 1).

Table 1. Results of Poisson and Negative Binomial Regression Systems.

Poisson Regression	Coefficient	Standard Error	z	Negative Binomial	Coefficient	Standard Error	z
sex	0.0817799	0.0083001	9.85	sex	0.029871	0.0071096	4.2
$household\_head$	0.1405562	0.009524	14.76	$household\_head$	0.1075885	0.0076744	14.02
$household\_income$	6.79e-06	1.58e-06	4.3	$household\_income$	7.80e-06	1.93e-06	4.05
$vehicle\_number$	0.1617351	0.0053751	30.09	$vehicle\_number$	0.1146227	0.0049238	23.28
$household\_size$	-0.0137819	0.0020895	-6.6	$household\_size$	-0.0138107	0.0017256	-8
age	-0.0020655	0.0003179	-6.5	age	0.0020207	0.0002602	-7.77
$education\_year$	0.0311067	0.001005	30.95	$education\_year$	0.0240298	0.0007979	30.12
$travel\_time$	0.0054185	0.0000725	73.77	$travel\_time$	0.0085438	0.0000896	95.37
$home\_base\_work$	0.9484354	0.0098287	96.5	$home\_base\_work$	0.7390091	0.0087332	84.62
$home\_base\_school$	0.8404261	0.0147617	56.93	$home\_base\_school$	0.6927652	0.0117506	58.96
$free\_flow\_time$	0.1482236	0.0206504	7.18	$free\_flow\_time$	1.516752	0.0416975	36.38
$\_constant$	-1473.417	0.0201662	-73.06	$\_constant$	-1.712916	0.0163486	-104.77
				$/lnalpha$	-1.368784	0.0397883	
				$alpha$	0.2544163	0.0101228	
	$L_{PRM} = 182318.45$				$L_{NBRM} = 173215.72$		

4. Results

To begin with, as seen from the results of poisson and negative binomial regression systems (see Table 1), there exists the case of overdispersion, which makes negative binomial system more preferable (remember Equation 3.7). That is why in the path analysis design, the equation for number of trips is exhibited by negative binomial system.

$LRTest(Equation3.7)Result:$
$2 \times [\ln(173215.72) - \ln(182318.45)] = 0.1024343$ and this value is larger than the critiqual chi square value for 2 degrees of freedom in 95 % confidence level ($0.1024 > 0.10$).

In addition, according to the results of path analysis (see Table 2), it is seen that all coefficient of related variables are statistically significant in at least 99 % confidence level.

For the considerations of signs of the coefficients, the sign of the estimated coefficient of household size is negative in the model of $motorized\_flow$ that reveals the probability to produce at least one motorized flow decreases as the household size increases (see Table 2). This will be the effect of budget constraints in the Turkish families. Also, the sign of the age is positive in the same model of motorized flow, which shows that the probability of exhibiting motorized flow in a day explicitly increases as age increases. On the other hand, the sign of age is negative in the model of $number\_trips$. This will be the case since the retired ageing population tend to exhibit more recreation trips as a replacement of the home based work ones in the case of Istanbul.

Furthermore, estimated coefficient of travel time is negative in the model of $number\_trip$, which is already expected theoretically in that as the travel time decreases, amount of trips increases with regard to the considerations of stimulated travel demand.

In addition, according to the marginal elasticity estimations in pursuit of results of path analysis, a unit decrease in the minutes spent on travel causes an additional increase in the amount of trips by 0.000953 (see Table **??**). That is to say, an approximately 27% decrease in travel time stimulates 70.4% additional motorized flow in the case of Istanbul.

Furthermore, the leading parameters enhancing the level of motorized flow are sex, household head, and $vehicle\_number$. According to the estimations, per male intends to produce 0.1597 more trips than per female and the household head is 0.256 more inclined to produce a motorized flow than the other members in the family in a day. Also, additional vehicle purchased by the family causes an increase of 0.1884 more motorized flow in the family in Istanbul (see Table **??**).

In addition, the marginal effect of household size and age on motorized flow are 0.028 and 0.0017 with negative signs (see Table **??**). This reveals that as one more traveller, who is above 16 years old, is involved in the family, the amount of motorized flow decreases by 0,028 and if the individuals gets one year older, then the amount of daily trips decreases by 0.0017.

Besides, income per month of each household seems as the least prominent parameter stimulating daily motorized flow with the marginal elasticity estimations 0.000025, which means that an amount of 10,000 TL increase in income of the family per month brings about only 0.025 additional trip per day per person in the family. This result is interesting, since it is unlike for most of the cities of developed countries in that income seems as the leading parameter affecting daily mobility, which is not the case in Istanbul.

Table 2. Results of Path Analysis.

Variable	Coefficient	Standard Error	z
	$motorized\_flow$		
sex	0.0876364	0.0088781	9.98
$household\_head$	0.1824174	0.0098624	18.50
$household\_income$	0.0000381	5.01e-06	7.60
$vehicle\_number$	0.2149051	0.0060338	35.62
$household\_size$	-0.0340989	0.0020449	-16.68
age	0.000905	0.0002754	3.29
$education\_year$	0.0537412	0.0010146	52.97
$home\_base\_work$	1.448814	0.0078503	184.55
$home\_base\_school$	1.230701	0.0139649	88.13
$\_constant$	-1.498341	0.0181383	-82.61
	$number\_trip$		
sex	0.094116	0.0078328	12.02
$household\_head$	0.1502352	0.0086166	17.44
$household\_income$	0.0000143	2.67e-06	5.35
$vehicle\_number$	0.1087255	0.0053448	20.34
$household\_size$	-0.0162595	0.0019109	-8.51
age	-0.0010601	0.0002899	-3.66
$education\_year$	0.0430605	0.0008928	48.23
$home\_base\_work$	1.052222	0.0101024	104.16
$home\_base\_school$	1.042079	0.0122483	85.08
$free\_flow\_time$	2.269169	0.0490115	46.30
$\_constant$	-1.65532	0.0177808	-93.10
$travel\_time$	-0.0023983	0.0000523	-45.89
	$free\_flow\_time$		
$travel\_distance$	0.0102411	0.000213	48.07
$motorized\_flow$	0.2824551	0.0030349	93.07
$\_constant$	-7.54e-18	1.28e-19	-58.70
	$travel\_time$		
g_number_trip	7.415484	0.4303755	17.23
$\_constant$	30.81517	0.3910488	17.23
	$/number\_trip$		
$lnalpha$	-0.8130799	0.030545	
$var(e\_free\_flow\_time)$	0.0407052	0.0113116	
$var(e\_travel\_time)$	3392.502	41.8371	
	$Logpseudolikelihood = -1314459.5$		

Lastly, the justification of the model specification called path analysis is summarized in the table below (Table 4).

Table 4. Model Comparions.

Model	Competing with the non-linearity of daily amount of trips per head	Competing with excess amount of zero observations in the trip counts	Taking endogeneity into account
Poisson Regression Model (PRM)	Tick	X	X
Negative Binomial Regression Model (NBRM)	Tick	X	X
Zero Truncated Models	Tick	X	X
Classical Simultaneous Equations Model (SEM)	x	X	Tick
Path Analysis	Tick	Tick	Tick

5. Concluding Remarks

The findings of this article has three multiplicative effects. First is about the way of gaugeing the stimulated travel demand in a new methodological proposal. In this sense, the variables and the designation of the path analysis can be extended and generalized in the cases of other cities of different countries in all over the World. Second multiplicative effect is about the detection of the prominent parameters affecting motorized mobility in an urban area. Herein, sex, household head and number of vehicles in the family come into prominence stimulating daily motorized flow in the case of Istanbul. This can be extended and verified in other cities of Turkey, and then the common leading parameters will be taken into considerations as the control parameters in the travel demand management formulations. To illustrate, the cities can be categorized by their scores called per cent of males, per cent of household head among population, and automobile ownership ratio per household, and then the provinces with the highest scores can be prioritized in the applications of travel demand management policies. Same procedure also Works for any province(s) of any country in the World.

The third multiplicative effect of this article is that the marginal elasticity estimation between amount of trips and travel time can be integrated into the classical travel demand models in that the first stage of motorized trip production shapes the last stage called network assignment with its related optimum total system travel time, and then the new finding of travel time re-affects the first stage of trip production. In this sense, the findings of this article can make classical four stages travel demand forecasting models much more dynamic and realistic.

Acknowledgment.

We thank the reviewer for their insightful comments and suggestions that helped us improve the paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Litman, T. (2021). Generated Traffic and Induced Travel Implications for Transport Planning. Victoria Transport Policy Institute.

[2] Pells, S. (1989). User response to new road capacity: A review of published evidence. University of Leeds, Institute of Transport Studies, Workin Paper 283.

[3] Vos, J., & Witlox, F. (2013). Transportation policy as spatial planning tool; reducing urban sprawl by increasing travel costs and clustering infrastructure and public transportation. Journal of Transport Geography, 33, 117-125.

[4] Mokhtarian, P., Samaniego, F., Shumway, R., Willits, N., & Azari, R. (2000). Analyzing induced traffic from capacity enhancements used matched pairs: A California Study. Davis: University of California, Institute of Transport Studies .

[5] Hansen, M., & Huang, Y. (1997). Road supply and traffic in California urban areas. Transportation Research Part A , 31, 205-218.

[6] Barr, L. (2000). Testing for the significance of induced highway travel demand in metropolitan areas. Transportation Research Record: Transportation Planning, Public Participation, and Telecommuting(1706), 1-8.

[7] Hansen, M., Gillen, D., Dobbins, A., Huang, Y., & Puvathingal, M. (1993). The air quality impacts of urban highway capacity expansion: Traffic Generation and land use change. Berkley: University of California, Institute of Transportation Studies.

[8] Kroes, E., Daily, A., Gunn, H., & Van der Hoorn, T. (1996). The opening of the Amsterdam ring road: A case study of short term impacts on removing bottleneck. Transportation, 23, 71-82.

[9] Luk, J., & Chung, E. (1997). Induced demand and road investment: An initial appraisal. ARR 299. Vermont South, Australia: ARRB Transportation Research

[10] Fulton, L., Meszler, D., Noland, R., & Thomas, J. (2000). A statistical analysis of induced travel effects in the U.S. mid- Atlantic region. Journal of Transportation and Statistics, 3(1), 1-14.

[11] Silva, A., & Costa, G. (2007, September). Urban sprawl and energy use for transportation in the largest Brazilian cities. Energy for Sustainable Development , 3(6), 44-50.

[12] Ozuysal, M., & Tanyel, S. (2008). Induced travel demand in developing countries : Study on state highways in Turkey. Journal of Urban Planning and Development -ASCE, 134(2), 78-87.

[13] Holcombe, R., & Williams, D. (2010). Urban Sprawl and Transportation Externalities. The Review of Regional Studies, 3(40), 257-273.

[14] Hymel, K., Small, K., & Dender, K. (2010, February 5). Induced Demand and Rebound Effects in Road Transport .

[15] Melo, P., Graham, D., & Canavan, S. (2012). The Effects of Road Investments on Economic Output and Induced Travel Demand: Evidence for Urbanized Areas in the US. Transportation Research Board: Journal of the Transportation Research Board.

[16] Noland, R., & Cowart, W. (2000). Analysis of metropolitan highway capacity and the growth in vehicle miles of travel. Transportation, 27(4), 363-390.

[17] Cervero, R., & Hansen, M. (2002). Induced travel demand an induced road investment: A simultaneous-equation analysis. Journal of Transport Economics and Policy, 36, 469-490.

[18] Cervero, R. (2003). Road expansion, urban growth, and induced travel: A path analysis. Journal of the American Planning Association, 69(2), 145-163.

[19] Green, W. (2007). Limdep version 9.0. Econometric modeling guide , v.2. New York: Plainview.

[20] Cameron, C., & Trivedi, P. (2005). Microeconometrics:Methods and Applications. New York: Cambridge University Press.

[21] Kennedy, P. (1998). A guide to econometrics. Cambridge, Massachusetts: The MIT Press.

[22] Green, W. (2003). Econometric Analysis (5th ed.). New Jersey: Pearson.

[23] Long, J. (1997). Regression models for categorical and limited dependent variables. London, New Delhi: Thousand Oaks: Syaş Publications.

[24] Winkelmann, R. (2008). Econometric analysis of count data (Cilt 5). Verlag, Berlin, Heidelberg: S pringer.

Affiliations

ENVER CENAN İNCE
ADDRESS: Sivas Cumhuriyet University, Dept. of City and Regional Planning, Sivas, Turkey
E-MAIL: :eince@cumhuriyet.edu.tr
ORCID ID:0000-0002-8264-6707

IBRAHIM DEMIR
ADDRESS: Yildiz Technical University, Dept. of Statistics, Istanbul, Turkey.
E-MAIL: idemir@gmail.com
ORCID ID:0000-0002-2734-4116

HÜSEYIN MURAT ÇELIK
ADDRESS: Istanbul Technical University, Dept. of City and Regional Planning, Istanbul, Turkey
E-MAIL: celikhus@itu.edu.tr
ORCID ID:0000-0001-9647-868X