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Solution of a Solvable System of Difference Equations

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Difference equations, Periodicity, System of difference equations, Asymptotic behavior

Abstract — In this study we give solutions for the following difference equation sytem

$$x_{n+1} = \frac{a x_n y_{n-3}}{y_{n-2} - \alpha} + \beta, \;\; y_{n+1} = \frac{b x_{n-3} y_n}{x_{n-2} - \beta} + \alpha, \;\; n \in \mathbb{N}_0$$

where the parameters a, b, α, β and initial values $x_{-i},\ y_{-i},\ i = 0, 1, 2, 3$ are non-zero real numbers. We show the asymptotic behavior of the system of equation.

Subject Classification (2020): 39A10.

1. Introduction

Although difference equations are look simple, it is very difficult fully understand the behaviors of their solutions. Continuous events in engineering, finance, physics, biology etc., is modelled by using differential equations. But, discontinuous events can be formed by a difference equations. Also, difference equations can be used to solve differential equations numerically. So, there is a great recent interest in difference equations(see [1-27]). In this study we investigate following system of the equation which is motivated by Haddad at al. [5].

$$x_{n+1} = \frac{a x_n y_{n-3}}{y_{n-2} - \alpha} + \beta, \;\; y_{n+1} = \frac{b x_{n-3} y_n}{x_{n-2} - \beta} + \alpha, \;\; n \in \mathbb{N}_0 \tag{1.1}$$

where the parameters a, b, α, β and initial values $x_{-i},\ y_{-i},\ i = 0, 1, 2, 3$ are non-zero real numbers.

Let's give following well known lemma which be used to prove our theorems.

Lemma 1.1. *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and consider the linear difference equation $a_n \neq 0$ for $\forall n \in \mathbb{N}_0$*

$$y_{n+1} = a_n y_n + b_n.$$

Then

$$y_n = (\prod_{i=0}^{n-1} a_i) y_0 + \sum_{r=0}^{n-1} (\prod_{i=r+1}^{n-1} a_i) b_r.$$

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Moreover, if $(a_n)_{n\in\mathbb{N}_0}$ *and* $(b_n)_{n\in\mathbb{N}_0}$ *are constant (i.e* $a_n = a$ *and* $b_n = b$ *for some real numbers* a *and* b *for all* $n \in \mathbb{N}_0$*), then*

$$y_n = \begin{cases} y_0 + bn, & a = 1, \\ a^n y_0 + (\frac{a^n-1}{a-1})b, & a \neq 1 \end{cases} \quad n \in \mathbb{N}_0.$$

2. Solutions of the system

In this section we give well-defined solutions of the system

Theorem 2.1. *Let* (x_n, y_n) *be a well-defined solution of the system* (1.1)*. Then for* $n \in \mathbb{N}_0$

$$x_{6n+i} = \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{6t+i} d\right) x_i + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{6t+i} d\right) F_i^r(u, \frac{a}{b})\beta,$$

$$y_{6n+i} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{6t+i} e\right) y_i + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{6t+i} e\right) F_i^r(v, \frac{b}{a})\alpha,$$

where $u_i = \frac{x_{i+1}-\beta}{x_i},\ v_i = \frac{y_{i+1}-\alpha}{y_i},\ i \in \{0,1,2,3,4,5\},\ d = u_5u_4u_3u_2u_1u_0,\ e = v_5v_4v_3v_2v_1v_0$ *and*

$$F_0^n(x, y) = y^{5n}x_5x_4x_3x_2x_1 + y^{4n}x_5x_4x_3x_2 + y^{3n}x_5x_4x_3 + y^{2n}x_5x_4 + y^n x_5 + 1$$

$$F_1^n(x, y) = y^{5n+1}x_5x_4x_3x_2x_0 + y^{4n+1}x_5x_4x_3x_0 + y^{3n+1}x_5x_4x_0 + y^{2n+1}x_5x_0 + y^{n+1}x_0 + 1$$

$$F_2^n(x, y) = y^{5n+2}x_5x_4x_3x_1x_0 + y^{4n+2}x_5x_4x_1x_0 + y^{3n+2}x_5x_1x_0 + y^{2n+2}x_1x_0 + y^{n+1}x_1 + 1$$

$$F_3^n(x, y) = y^{5n+3}x_5x_4x_2x_1x_0 + y^{4n+3}x_5x_2x_1x_0 + y^{3n+3}x_2x_1x_0 + y^{2n+2}x_2x_1 + y^{n+1}x_2 + 1$$

$$F_4^n(x, y) = y^{5n+4}x_5x_3x_2x_1x_0 + y^{4n+4}x_3x_2x_1x_0 + y^{3n+3}x_3x_2x_1 + y^{2n+2}x_3x_2 + y^{n+1}x_3 + 1$$

$$F_5^n(x, y) = y^{5n+5}x_4x_3x_2x_1x_0 + y^{4n+4}x_4x_3x_2x_1 + y^{3n+3}x_4x_3x_2 + y^{2n+2}x_4x_3 + y^{n+1}x_4 + 1.$$

Proof.
Firstly let's write system (1.1) as

$$\frac{x_{n+1}-\beta}{x_n} = \frac{ay_{n-3}}{y_{n-2}-\alpha},\ \frac{y_{n+1}-\alpha}{y_n} = \frac{bx_{n-3}}{x_{n-2}-\beta}.$$

Assume

$$u_n = \frac{x_{n+1}-\beta}{x_n},\ v_n = \frac{y_{n+1}-\alpha}{y_n},\quad n \in \mathbb{N}_0 \tag{2.1}$$

then

$$u_n = \frac{a}{v_{n-3}},\ v_n = \frac{b}{u_{n-3}} \Rightarrow u_{n+3} = \frac{a}{v_n},\ v_{n+3} = \frac{b}{u_n},\quad n \in \mathbb{N}_0 \tag{2.2}$$

and

$$u_{n+6} = \frac{a}{b}u_n,\ v_{n+6} = \frac{b}{a}v_n,\quad n \in \mathbb{N}_0. \tag{2.3}$$

Hence for $n \in \mathbb{N}_0$

$$u_{6n+i} = \left(\frac{a}{b}\right)^n u_i,\ v_{6n+i} = \left(\frac{b}{a}\right)^n v_i,\ i = 0,1,2,3,4,5. \tag{2.4}$$

Rearranging equation (2.1), we get

$$x_{n+1} = u_nx_n + \beta \tag{2.5}$$

$$y_{n+1} = v_n y_n + \alpha \tag{2.6}$$

Replacing n by $6n+i$ for $i \in \{0,1,2,3,4,5\}$

$$x_{6n+i+1} = u_{6n+i} x_{6n+i} + \beta = \left(\frac{a}{b}\right)^n u_i x_{6n+i} + \beta, \quad n \in \mathbb{N}_0,\ i \in \{0,1,2,3,4,5\},$$

$$y_{6n+i+1} = v_{6n+i} y_{6n+i} + \alpha = \left(\frac{b}{a}\right)^n v_i y_{6n+i} + \alpha, \quad n \in \mathbb{N}_0,\ i \in \{0,1,2,3,4,5\}.$$

$$\begin{aligned} x_{6n+6} = {} & \left(\frac{a}{b}\right)^{6n} u_5u_4u_3u_2u_1u_0x_{6n} + \beta\left(\left(\frac{a}{b}\right)^{5n} u_5u_4u_3u_2u_1 + \left(\frac{a}{b}\right)^{4n} u_5u_4u_3u_2\right. \\ & \left. + \left(\frac{a}{b}\right)^{3n} u_5u_4u_3 \left(\frac{a}{b}\right)^{2n} u_5u_4 + \left(\frac{a}{b}\right)^{n} u_5 + 1\right), \end{aligned} \tag{2.7}$$

$$\begin{aligned} x_{6n+7} = {} & \left(\frac{a}{b}\right)^{6n+1} u_5u_4u_3u_2u_1u_0x_{6n+1} + \beta\left(\left(\frac{a}{b}\right)^{5n+1} u_5u_4u_3u_2u_0 + \left(\frac{a}{b}\right)^{4n+1} u_5u_4u_3u_0\right. \\ & \left. + \left(\frac{a}{b}\right)^{3n+1} u_5u_4u_0 + \left(\frac{a}{b}\right)^{2n+1} u_5u_0 + \left(\frac{a}{b}\right)^{n+1} u_0 + 1\right), \quad n \in \mathbb{N}_0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} x_{6n+8} = {} & \left(\frac{a}{b}\right)^{6n+2} u_5u_4u_3u_2u_1u_0x_{6n+2} + \beta\left(\left(\frac{a}{b}\right)^{5n+2} u_5u_4u_3u_1u_0 + \left(\frac{a}{b}\right)^{4n+2} u_5u_4u_1u_0\right. \\ & \left. + \left(\frac{a}{b}\right)^{3n+2} u_5u_1u_0 + \left(\frac{a}{b}\right)^{2n+2} u_1u_0 + \left(\frac{a}{b}\right)^{n+1} u_1 + 1\right), \quad n \in \mathbb{N}_0 \end{aligned} \tag{2.9}$$

$$\begin{aligned} x_{6n+9} = {} & \left(\frac{a}{b}\right)^{6n+3} u_5u_4u_3u_2u_1u_0x_{6n+3} + \beta\left(\left(\frac{a}{b}\right)^{5n+3} u_5u_4u_2u_1u_0 + \left(\frac{a}{b}\right)^{4n+3} u_5u_2u_1u_0\right. \\ & \left. + \left(\frac{a}{b}\right)^{3n+3} u_2u_1u_0 + \left(\frac{a}{b}\right)^{2n+2} u_2u_1 + \left(\frac{a}{b}\right)^{n+1} u_2 + 1\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} x_{6n+10} = {} & \left(\frac{a}{b}\right)^{6n+4} u_5u_4u_3u_2u_1u_0x_{6n+4} + \beta\left(\left(\frac{a}{b}\right)^{5n+4} u_5u_3u_2u_1u_0 + \left(\frac{a}{b}\right)^{4n+4} u_3u_2u_1u_0\right. \\ & \left. + \left(\frac{a}{b}\right)^{3n+3} u_3u_2u_1 + \left(\frac{a}{b}\right)^{2n+2} u_3u_2 + \left(\frac{a}{b}\right)^{n+1} u_3 + 1\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} x_{6n+11} = {} & \left(\frac{a}{b}\right)^{6n+5} u_5u_4u_3u_2u_1u_0x_{6n+5} + \beta\left(\left(\frac{a}{b}\right)^{5n+5} u_4u_3u_2u_1u_0 + \left(\frac{a}{b}\right)^{4n+4} u_4u_3u_2u_1\right. \\ & \left. + \left(\frac{a}{b}\right)^{3n+3} u_4u_3u_2 + \left(\frac{a}{b}\right)^{2n+2} u_4u_3 + \left(\frac{a}{b}\right)^{n+1} u_4 + 1\right), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.12}$$

$$y_{6n+6}=\left(\frac{b}{a}\right)^{6n}v_5v_4v_3v_2v_1v_0y_{6n}+\alpha\left(\left(\frac{b}{a}\right)^{5n}v_5v_4v_3v_2v_1+\left(\frac{b}{a}\right)^{4n}v_5v_4v_3v_2+\left(\frac{b}{a}\right)^{3n}v_5v_4v_3+\left(\frac{b}{a}\right)^{2n}v_5v_4+\left(\frac{b}{a}\right)^{n}v_5+1\right),\quad n\in\mathbb{N}_0, \tag{2.13}$$

$$y_{6n+7}=\left(\frac{b}{a}\right)^{6n+1}v_5v_4v_3v_2v_1v_0y_{6n+1}+\alpha\left(\left(\frac{b}{a}\right)^{5n+1}v_5v_4v_3v_2v_0+\left(\frac{b}{a}\right)^{4n+1}v_5v_4v_3v_0+\left(\frac{b}{a}\right)^{3n+1}v_5v_4v_0+\left(\frac{b}{a}\right)^{2n+1}v_5v_0+\left(\frac{b}{a}\right)^{n+1}v_0+1\right),\quad n\in\mathbb{N}_0, \tag{2.14}$$

$$y_{6n+8}=\left(\frac{b}{a}\right)^{6n+2}v_5v_4v_3v_2v_1v_0y_{6n+2}+\alpha\left(\left(\frac{b}{a}\right)^{5n+2}v_5v_4v_3v_1v_0+\left(\frac{b}{a}\right)^{4n+2}v_5v_4v_1v_0+\left(\frac{b}{a}\right)^{3n+2}v_5v_1v_0+\left(\frac{b}{a}\right)^{2n+2}v_1v_0+\left(\frac{b}{a}\right)^{n+1}v_1+1\right),\quad n\in\mathbb{N}_0, \tag{2.15}$$

$$y_{6n+9}=\left(\frac{b}{a}\right)^{6n+3}v_5v_4v_3v_2v_1v_0y_{6n+3}+\alpha\left(\left(\frac{b}{a}\right)^{5n+3}v_5v_4v_2v_1v_0+\left(\frac{b}{a}\right)^{4n+3}v_5v_2v_1v_0+\left(\frac{b}{a}\right)^{3n+3}v_2v_1v_0+\left(\frac{b}{a}\right)^{2n+2}v_2v_1+\left(\frac{b}{a}\right)^{n+1}v_2+1\right),\quad n\in\mathbb{N}_0, \tag{2.16}$$

$$y_{6n+10}=\left(\frac{b}{a}\right)^{6n+4}v_5v_4v_3v_2v_1v_0y_{6n+4}+\alpha\left(\left(\frac{b}{a}\right)^{5n+4}v_5v_3v_2v_1v_0+\left(\frac{b}{a}\right)^{4n+4}u_3u_2u_1u_0+\left(\frac{b}{a}\right)^{3n+3}v_3v_2v_1+\left(\frac{b}{a}\right)^{2n+2}v_3v_2+\left(\frac{b}{a}\right)^{n+1}v_3+1\right),\quad n\in\mathbb{N}_0, \tag{2.17}$$

$$y_{6n+11}=\left(\frac{b}{a}\right)^{6n+5}v_5v_4v_3v_2v_1v_0y_{6n+5}+\alpha\left(\left(\frac{b}{a}\right)^{5n+5}v_4v_3v_2v_1v_0+\left(\frac{b}{a}\right)^{4n+4}u_4u_3u_2u_1+\left(\frac{b}{a}\right)^{3n+3}v_4v_3v_2+\left(\frac{b}{a}\right)^{2n+2}v_4v_3+\left(\frac{b}{a}\right)^{n+1}v_4+1\right),\quad n\in\mathbb{N}_0. \tag{2.18}$$

Let

$$x_{6n}=K_n^0,\ x_{6n+1}=K_n^1,\ x_{6n+2}=K_n^2,\ x_{6n+3}=K_n^3,\ x_{6n+4}=K_n^4,\ x_{6n+5}=K_n^5,$$

$$y_{6n}=L_n^0,\ y_{6n+1}=L_n^1,\ y_{6n+2}=L_n^2,\ y_{6n+3}=L_n^3,\ y_{6n+4}=L_n^4,\ y_{6n+5}=L_n^5,\ n\in\mathbb{N}_0.$$

$$F_0^n(x,y)=y^{5n}x_5x_4x_3x_2x_1+y^{4n}x_5x_4x_3x_2+y^{3n}x_5x_4x_3+y^{2n}x_5x_4+y^nx_5+1,$$

$$F_1^n(x,y)=y^{5n+1}x_5x_4x_3x_2x_0+y^{4n+1}x_5x_4x_3x_0+y^{3n+1}x_5x_4x_0+y^{2n+1}x_5x_0+y^{n+1}x_0+1,$$

$$F_2^n(x,y)=y^{5n+2}x_5x_4x_3x_1x_0+y^{4n+2}x_5x_4x_1x_0+y^{3n+2}x_5x_1x_0+y^{2n+2}x_1x_0+y^{n+1}x_1+1,$$

$$F_3^n(x,y)=y^{5n+3}x_5x_4x_2x_1x_0+y^{4n+3}x_5x_2x_1x_0+y^{3n+3}x_2x_1x_0+y^{2n+2}x_2x_1+y^{n+1}x_2+1,$$

$$F_4^n(x,y) = y^{5n+4}x_5x_3x_2x_1x_0 + y^{4n+4}x_3x_2x_1x_0 + y^{3n+3}x_3x_2x_1 + y^{2n+2}x_3x_2 + y^{n+1}x_3 + 1,$$

$$F_5^n(x,y) = y^{5n+5}x_4x_3x_2x_1x_0 + y^{4n+4}x_4x_3x_2x_1 + y^{3n+3}x_4x_3x_2 + y^{2n+2}x_4x_3 + y^{n+1}x_4 + 1.$$

Then, from (2.7)-(2.18) we get

$$K_{n+1}^i = \left(\frac{a}{b}\right)^{6n} u_5u_4u_3u_2u_1u_0K_n^i + \beta F_i^n(u,\frac{a}{b}), \quad n\in\mathbb{N}_0,\ i\in\{0,1,2,3,4,5\},$$

$$L_{n+1}^i = \left(\frac{b}{a}\right)^{6n} v_5v_4v_3v_2v_1v_0L_n^i + \alpha F_i^n(v,\frac{b}{a}), \quad n\in\mathbb{N}_0,\ i\in\{0,1,2,3,4,5\}.$$

From Lemma 1.1 we have

$$x_{6n+i} = \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{6t+i} u_5u_4u_3u_2u_1u_0\right)x_i + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{6t+i} u_5u_4u_3u_2u_1u_0\right)F_i^r(u,\frac{a}{b})\beta,$$

$$y_{6n+i} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{6t+i} v_5v_4v_3v_2v_1v_0\right)y_i + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{6t+i} v_5v_4v_3v_2v_1v_0\right)F_i^r(v,\frac{b}{a})\alpha,$$

this ends the proof.

3. Asymptotic behaviour of (1.1) for $a = b$

We study here asymptotic behavior and periodicity the case when $a = b$ of system (1.1). Let's give the following corollary, which is a direct result of Theorem 2.1.

Corollary 3.1. *Let (x_n, y_n) be a well-defined solution of the system (1.1) with $a = b$. Then, for $n\in\mathbb{N}_0$, $i\in\{0,1,2,3,4,5\}$*

$$x_{6n+i} = \begin{cases} x_i + F_i^r(u,1)\beta n, & d=1 \\ d^n x_i + \left(\frac{d^n-1}{d-1}\right)F_i^r(u,1)\beta, & otherwise \end{cases} \tag{3.1}$$

$$y_{6n+i} = \begin{cases} y_i + F_i^r(v,1)\beta n, & e=1 \\ e^n y_i + \left(\frac{e^n-1}{e-1}\right)F_i^r(v,1)\beta, & otherwise \end{cases} \tag{3.2}$$

Now we study the limits of solutions of system (1.1).

Theorem 3.2. *Let (x_n, y_n) be a well-defined solution of the system (1.1) with a=b. Then, the following statements are true.*

a) *Let's assume $d = 1$. When $F_i^r(u,1) \neq 0$ then $|x_{6n+i}| \to \infty$ as $n\to\infty$ for $i\in\{0,1,2,3,4,5\}$. When $F_i^r(u,1) = 0$ then $x_{6n+i} = x_i$ for all $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3,4,5\}$.*

b) *When $(d-1)x_i + F_i^r(u,1)\beta \neq 0$ then*

$$\lim_{n\to\infty}|x_{6n+i}| = \begin{cases} \left|\frac{F_i^r(u,1)\beta}{d-1}\right|, & |d|<1, \\ \infty, & |d|>1. \end{cases}$$

When $(d-1)x_i + F_i^r(u,1)\beta = 0$ then $x_{6n+i} = x_i$ for all $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3,4,5\}$.

c) *Let's assume* $e=1$. *If* $F_i^r(v,1)\neq 0$ *then* $\left|y_{6n+i}\right|\to\infty$ *as* $n\to\infty$ *for* $i\in\{0,1,2,3,4,5\}$. *Otherwise, if* $F_i^r(v,1)=0$ *then* $y_{6n+i}=y_i$ *for* $i\in\{0,1,2,3,4,5\}$.

d) *If* $(e-1)\,y_i+F_i^r(v,1)\alpha\neq 0$ *then*

$$\lim_{n\to\infty}\left|y_{6n+i}\right|=\begin{cases}\left|\frac{F_i^r(v,1)\alpha}{d-1}\right|, & |d|<1,\\ \infty, & |d|>1.\end{cases}$$

Else $(e-1)\,y_i+F_i^r(v,1)\alpha=0$ *then* $y_{6n+i}=y_i$ *for all* $n\in\mathbb{N}_0$ *and* $i\in\{0,1,2,3,4,5\}$.

Proof.
We are going to prove a) and b) rest can be done with same manner.

a) Assuming $d=1$ and $F_i^r(u,1)\neq 0$ we have from (3.1)

$$x_{6n+i}=x_i+F_i^r(u,1)\beta n\neq 0,$$

when $n\to\infty$ in this equation $|x_{6n+i}|\to\infty$. If $F_i^r(u,1)=0$ then

$$x_{6n+i}=x_i+0.\beta n=x_i$$

for all $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3,4,5\}$.

b) Suppose that $(d-1)\,x_i+F_i^r(u,1)\beta\neq 0$ for $i\in\{0,1,2,3,4,5\}$. Then, it shows $x_{6n+i}\neq 0$. From (3.1)

$$\begin{aligned}\lim_{n\to\infty}|x_{6n+i}| &= \lim_{n\to\infty}\left|\frac{(d-1)x_i+F_i^r(u,1)\beta}{d-1}d^n+\frac{F_i^r(u,1)\beta}{1-d}\right|\\ &= \left|\frac{(d-1)x_i+F_i^r(u,1)\beta}{d-1}\lim_{n\to\infty}d^n+\lim_{n\to\infty}\frac{F_i^r(u,1)\beta}{1-d}\right|\\ &= \begin{cases}\left|\frac{F_i^r(u,1)\beta}{d-1}\right|, & |d|<1,\\ \infty, & |d|>1.\end{cases}\end{aligned}$$

If $(d-1)\,x_i+F_i^r(u,1)\beta=0$ and $d\neq 1$. Then

$$\begin{aligned}x_{6n+i} &= d^n x_i+\left(\frac{d^n-1}{d-1}\right)F_i^r(u,1)\beta=d^n x_i+\left(\frac{d^n-1}{d-1}\right)(-(d-1)\,x_i)\\ &= d^n x_i-(d^n-1)\,x_i=x_i,\quad \forall n\in\mathbb{N}_0,\ i\in\{0,1,2,3,4,5\}.\end{aligned}$$

this ends the proof.

Corollary 3.3. *Let* (x_n,y_n) *be a well-defined solution of the system* (1.1) *with a=b. Then, the following statements are true.*

a) If $d=-1$ then for all $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3,4,5\}$,

$$\begin{cases}x_{12n+i}=x_i,\\ x_{12n+6+i}=-x_i+F_i^r(u,1)\beta.\end{cases}$$

b) If $e = -1$ then for all $n \in \mathbb{N}_0$ and $i \in \{0,1,2,3,4,5\}$,

$$\begin{cases} y_{12n+i} = y_i, \\ y_{12n+6+i} = -y_i + F_i^r(v,1)\alpha. \end{cases}$$

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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The Gershgorin type theorem on localization of the eigenvalues of infinite matrices and zeros of entire functions

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Keywords
Infinite matrix, Compact operator, Localization of spectrum, Zeros of entire functions

Abstract — Let $1<p<\infty$, $1/p+1/p'=1$ and $A=(a_{jk})_{j,k=1}^{\infty}$ be a p-Hille-Tamarkin infinite matrix, i.e.

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{jk}|^{p'}\right)^{p/p'}<\infty.$$

It is proved that the spectrum of A lies in the union of the discs

$$\left\{z\in\mathbb{C}:|a_{jj}-z|\le\left[\sum_{j=1}^{\infty}\Big(\sum_{k=1,k\neq j}^{\infty}|a_{jk}|^{p'}\Big)^{p/p'}\right]^{1/p}\right\}\ (j=1,2,\dots).$$

In addition, an application of that result to finite order entire functions is discussed. An illustrative example is also presented.

Subject Classification (2020): 47A10; 47A11; 30B10.

1. Introduction and statement of the main result

Let $1<p<\infty$, $1/p+1/p'=1$ and $A=(a_{jk})_{j,k=1}^{\infty}$ be a p-Hille-Tamarkin infinite matrix, i.e.

$$c_p(A):=\left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{jk}|^{p'}\right)^{p/p'}\right]^{1/p}<\infty.$$

The paper is devoted to the localization of the eigenvalues of such matrices.

The literature on the localization of the eigenvalues of finite and infinite matrices is very rich, cf. [1, 3, 5, 9, 14, 15, 18, 19] and the references which are given therein. At the same time, to the best of our knowledge, the location of the eigenvalues of Hille-Tamarkin matrices has not been considered in the available literature.

As is well-known, Hille-Tamarkin matrices represent numerous integral operators, arising in various applications, cf. [17]. About properties of Hille-Tamarkin matrices, see for instance, [17], [6],[7, Section 18]. In particular, in the well-known book [17], the convergence of the powers of the eigenvalues of these matrices is investigated. The works [6, 7] deal with infinite matrices, whose upper-triangular parts are Hille-Tamarkin matrices. Besides, the invertibility and positive invertibility conditions are explored, as well as

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upper bounds for the spectra have been derived.

Denote

$$\tau_p(A) := \left[\sum_{j=1}^{\infty} \left(\sum_{k=1,k\neq j}^{\infty} |a_{jk}|^{p'} \right)^{p/p'} \right]^{1/p}.$$

Throughout the paper $\lambda_k(A)$ $(k = 1,2,...)$ are the eigenvalues of A taken with their multiplicities and enumerated in the non-increasing way of the absolute values: $|\lambda_{k+1}(A)| \le |\lambda_k(A)|$ $(k = 1,2,...)$, and $\sigma(A)$ is the spectrum of A as the operator in l^p. Recall that l^p is the Banach space of the sequences $x = (x_k)_{k=1}^{\infty}$ of complex numbers with the finite norm

$$\|x\|_{l^p} = \sum_{k=1}^{\infty} (|x_k|^p)^{1/p}.$$

The following theorem is the main result of this paper.

Theorem 1.1. Let $c_p(A) < \infty$ for a finite $p > 1$. Then with the notation

$$U_{j,p}(A) := \{z \in \mathbb{C} : |a_{jj} - z| \le \tau_p(A)\} \ (j = 1,2,...),$$

one has

$$\sigma(A) \subset \cup_{j=1}^{\infty} U_{j,p}(A).$$

The proof of this theorem is presented in the next section.

For a positive integer n, let $\mathbb{C}^{n\times n}$ be the set of $n \times n$-matrices and $A_n = (a_{jk})_{j,k=1}^n \in \mathbb{C}^{n\times n}$. Recall that by the Gershgorin theorem [15],

$$\sigma(A_n) \subset \cup_{j=1}^{n} \hat{U}_j(A_n),$$

where

$$\hat{U}_j(A_n) := \{z \in \mathbb{C} : |a_{jj} - z| \le \sum_{k=1,k\neq j}^{n} |a_{jk}|\} \ (j = 1,...,n).$$

This result can be easily extended to the infinite dimensional case, provided

$$\sup_j \sum_{k=1}^{\infty} |a_{jk}| < \infty.$$

Thus, Theorem 1.1 can be considered as an extending of the Gershgorin theorem to a finite $p > 1$. Furthermore, the quantity $\hat{s}(A) := \sup_{j,k} |\lambda_j(A) - \lambda_k(A)|$ is called the spread of A. In the finite dimensional case the spread plays an essential role, cf. [15, Section III.4]. Since

$$|\lambda_j(A) - \lambda_k(A)| \le |\lambda_j(A) - a_{jj}| + |\lambda_k(A) - a_{kk}| + |a_{jj} - a_{kk}|,$$

for a p-Hille-Tamarkin matrix A Theorem 1.1 gives us the inequality

$$\hat{s}(A) \le \sup_{j,k} |a_{jj} - a_{kk}| + 2\tau_p(A). \tag{1.1}$$

Similarly, let $A = (a_{jk})$ and $B = (b_{jk})$ be two p-Hille-Tamarkin matrices. Then the quantity $s(A,B) := \sup_{j,k} |\lambda_j(A) -$

$\lambda_k(B)|$ will be called the mutual spread of A and B. Since

$$|\lambda_j(A) - \lambda_k(B)| \le |\lambda_j(A) - a_{jj}| + |\lambda_k(B) - b_{kk}| + |a_{jj} - b_{kk}|,$$

Theorem 1.1 gives us the inequality

$$s(A,B) \le \sup_{j,k} |a_{jj} - b_{kk}| + \tau_p(A) + \tau_p(B). \tag{1.2}$$

Let $r_s(A)$ denote the spectral radius of A. Then clearly, $|r_s(A) - r_s(B)| \le s(A,B)$. Now we can apply inequality (1.2).

2. Proof of Theorem 1.1

Let $A_n = (a_{jk})_{j,k=1}^n \in \mathbb{C}^{n\times n}$, $\mu \in \sigma(A_n)$ and $x = (x_j)$ be the eigenvector of A_n corresponding to μ:

$$\sum_{k=1}^n a_{jk}x_k = \mu x_j \ \ (j = 1, \dots n).$$

Then

$$(\mu - a_{jj})x_j = \sum_{k=1, k\ne j}^n a_{jk}x_k$$

and

$$|a_{jj} - \mu||x_j| \le \sum_{k=1, k\ne j}^n |a_{jk}||x_k| \ \ (j = 1, \dots, n).$$

So by the Hőlder inequality,

$$|a_{jj} - \mu|^p |x_j|^p \le (\sum_{k=1, k\ne j}^n |a_{jk}|^{p'})^{p/p'} \sum_{i=1}^n |x_i|^p$$

and

$$\sum_{j=1}^n |a_{jj} - \mu|^p |x_j|^p \le \tau_p^p(A_n) \sum_{i=1}^n |x_i|^p.$$

Here

$$\tau_p(A_n) := \left[\sum_{j=1}^n \left(\sum_{k=1, k\ne j}^n |a_{jk}|^{p'} \right)^{p/p'} \right]^{1/p}.$$

Consequently,

$$\min_j |a_{jj} - \mu| \le \tau_p(A_n).$$

In other words, for any eigenvalue μ of A_n, there is an integer $m \le n$, such that $|a_{mm} - \mu| \le \tau_p(A_n)$. We thus have proved the following result.

Lemma 2.1. Let $A_n \in \mathbb{C}^{n\times n}$. Then for any finite $p > 1$ we have

$$\sigma(A_n) \subset \cup_{j=1}^n U_{j,p}(A_n),$$

where

$$U_{j,p}(A_n) = \{z \in \mathbb{C} : |a_{jj} - z| \le \tau_p(A_n)\}.$$

Proof of Theorem 1.1: By the Hőlder inequality we have

$$\|A\|_{l^p} \le c_p(A), \tag{2.1}$$

where $\|A\|_{l^p}$ means the operator norm of A in l^p. Since $\tau_p(A_n) \to \tau_p(A)$ as $n \to \infty$, according to (2.1), $A_n \to A$ in the operator norm and therefore, by the upper semicontinuity of spectra [11, Theorem IV.3.1], for any finite k we have $\lambda_k(A_n) \to \lambda_k(A)$ as $n \to \infty$. Now Lemma 2.1 implies the required result. □

3. Applications to entire functions

Let us consider the entire function

$$h(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{(k!)^\alpha} \ (0 < \alpha \le 1,\ z \in \mathbb{C},\ a_0 = 1, a_k \in \mathbb{C}, k \ge 1). \tag{3.1}$$

Denote the zeros of h with the multiplicities in non-decreasing order of their absolute values by $z_k(h)$: $|z_k(h| \le |z_{k+1}(h)|$ $(k = 1, 2, ...)$ and assume that for a $p > 1/\alpha$,

$$\theta_p(h) := \sum_{k=2}^{\infty} |a_k|^{p'} < \infty \ (1/p + 1/p' = 1). \tag{3.2}$$

Introduce the quantity

$$b_p(h) := [\theta_p^{p/p'}(h) + \sum_{j=2}^{\infty} \frac{1}{j^{\alpha p}}]^{1/p} = [\theta_p^{p/p'}(h) + \zeta(\alpha p) - 1]^{1/p},$$

where

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z} \ (\mathrm{Re}\, z > 1)$$

is the Riemann zeta function. Our aim in this section is to prove the following theorem.

Theorem 3.1. Let h be defined by (3.1) and condition (3.2) hold. Then for any zero $z(h)$ of h we have either $|a_1 - \frac{1}{z(h)}| \le b_p(h)$ or $|z(h)| \ge \frac{1}{b_p(h)}$.

To prove this theorem introduce the polynomial

$$f_n(z) = \sum_{k=0}^{n} \frac{a_k z^{n-k}}{(k!)^\alpha}$$

and $n \times n$-matrix

$$F_n = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1/(2^\alpha) & 0 & \dots & 0 & 0 \\ 0 & 1/(3^\alpha) & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1/(n^\alpha) & 0 \end{pmatrix}.$$

Let $z_k(f_n)$ $(k = 1, ..., n)$ be the zeros of f_n with their multiplicities enumerated in non-increasing order of their absolute values, and $\lambda_k(F_n)$ be the eigenvalues of F_n taken with the multiplicities enumerated in the non-increasing order of their absolute values.

Lemma 3.2. One has $\lambda_k(F_n) = z_k(f_n)$ $(k = 1, ..., n)$.

Proof.

Clearly, f_n is the characteristic polynomial of the matrix

$$B = \begin{pmatrix} -a_1 & -\frac{a_2}{2^\alpha} & \dots & -\frac{a_{n-1}}{((n-1)!)^\alpha} & -\frac{a_n}{(n!)^\alpha} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Following [8, Lemma 5.2.1, p. 117], put

$$m_k = \frac{1}{k^\alpha} \text{ and } \psi_k = \frac{1}{(k!)^\alpha} = m_1 m_2 \cdots m_k \ \ (k = 1, ..., n).$$

Then

$$F_n = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ m_2 & 0 & \dots & 0 & 0 \\ 0 & m_3 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & m_n & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -a_1 & -a_2\psi_2 & \dots & -a_{n-1}\psi_{n-1} & -a_n\psi_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let μ be an eigenvalue of B, i.e. for the eigenvector $(x_k)_{k=1}^n \in \mathbb{C}^n$, we have

$$-a_1 x_1 - a_2\psi_2 x_2 - \dots - a_{n-1}\psi_{n-1} x_{n-1} - a_n \psi_n x_n = \mu x_1,$$

$$x_k = \mu x_{k+1} \ \ (k = 1, ..., n-1).$$

Since $\psi_1 = 1$, substituting $x_k = y_k/\psi_k$, we obtain

$$-a_1 y_1 - a_2 y_2 \dots - a_{n-1} y_{n-1} - a_n y_n = \mu y_1$$

and

$$\frac{y_k}{\psi_k} = \mu \frac{y_{k+1}}{\psi_{k+1}} \ \ (k = 1, ..., n-1).$$

Or

$$m_{k+1} y_k = \frac{y_k \psi_{k+1}}{\psi_k} = \mu y_{k+1} \ \ (k = 1, ..., n-1).$$

These equalities are equivalent to the equality $F_n y = \mu y$ with $y = (y_k)$. In other words $TBT^{-1} = F_n$, where

$T = \operatorname{diag}(1, \psi_2, ..., \psi_n)$ and therefore

$$T^{-1} = \operatorname{diag}(1, \frac{1}{\psi_2}, ..., \frac{1}{\psi_n}).$$

This proves the lemma. □

The simple calculations show that

$$\tau_p(F_n) = [(\sum_{k=2}^{n} |a_k|^{p'})^{p/p'} + \sum_{j=2}^{n} \frac{1}{j^{\alpha p}}]^{1/p}.$$

Due to Lemmas 2.1 and 3.2, for any zero $z(f_n)$ of f_n either

$$|a_1 - z(f_n)| \le \tau_p(F_n), \text{ or } |z(f_n)| \le \tau_p(F_n). \tag{3.3}$$

With

$$h_n(z) = z^n f_n(1/z) = \sum_{k=0}^{n} \frac{a_k z^k}{(k!)^\alpha},$$

we obtain

$$z_k(h_n) = \frac{1}{z_k(f_n)} = \frac{1}{\lambda_k(F_n)} \quad (k = 1, ..., n).$$

Here $z_k(h_n)$ are the zeros of h_n with their multiplicities enumerated in non-decreasing order of their absolute values. According to (3.3) for any zero $z(h_n)$ of h_n either

$$|a_1 - \frac{1}{z(h_n)}| \le \tau_p(F_n), \text{ or } |z(h_n)| \ge \tau_p(F_n). \tag{3.4}$$

Proof of Theorem 3.1: Clearly, $\tau_p(F_n) \to b_p(h)$ as $n \to \infty$. In each compact domain $\Omega \in \mathbb{C}$, we have $h_n(z) \to h(z)$ $(n \to \infty)$ uniformly in Ω. Due to the Hurwitz theorem [16, p. 5] $z_k(h_n) \to z_k(h)$ for $z_k(h) \in \Omega$. Now (3.4) prove the theorem. □

From Theorem 3.1 it follows

$$\inf |z_j(h)| \ge \frac{1}{|a_1| + b_p(h)}$$

So the disc $|z| < \frac{1}{|a_1|+b_p(h)}$ is a zero-free domain.

Note that the classical results on the zeros of entire functions are presented in [13]; the recent investigations on localization of the zeros of entire functions can be found, for instance, in the works [2, 4, 8, 10, 12] and the references which are given therein.

4. Example

The following example characterizes the sharpness of Theorem 1.1.

Let $A = \operatorname{diag}(B_j)_{j=1}^{\infty}$, where

$$B_j = \begin{pmatrix} \frac{1}{j} & \frac{\sqrt{3}}{2j} \\ \frac{\sqrt{3}}{2j} & \frac{1}{j} \end{pmatrix} \quad (j = 1, 2, ...).$$

Under consideration we have

$$\tau_2(A) = [2\sum_{j=1}^{\infty}\frac{3}{4j^2}]^{1/2} = \sqrt{3\zeta(2)/2} \approx \sqrt{3\cdot 1.645/2} \leq 1.570.$$

By Theorem 1.1

$$\sigma(A) \subseteq \cup_{j=1}^{\infty}\{z \in \mathbb{C} : |z - \frac{1}{j}| \leq 1.570\}.$$

Simple calculations show that $\lambda_1(B_j) = \frac{3}{2j}$, $\lambda_2(B_j) = \frac{1}{2j}$ $(j = 1,2,...)$.

Author Contributions

The author read and approved the final version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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On Quasi 2-Crossed Modules for Lie Algebras and Functorial Relations

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Quasi 2-Crossed Modules of Lie Algebras, 2-Crossed Modules of Lie Algebras

Abstract — In this paper, we have introduced the category of quasi 2-crossed modules for Lie algebras and we have constructed a pair of adjoint functors between this category and that of 2-crossed modules Lie algebras.

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1. Introduction

Crossed modules have been introduced for Lie algebras by Kassel and Loday in [6], as well as they initially originate in Whitehead's work for groups,[10]. It is known that the notion of crossed modules modelling homotopy 2-type has become an important tool in various contexts. Some of related works with crossed modules of Lie algebras are [2], [8], and [9]. The notion of 2-crossed modules of groups based on that of crossed modules has been introduced by Conduche [3] as an algebraic models of homotopy 3-types. In [5], Ellis has also presented the Lie algebra version of that for getting the equivalence between the category of 2-crossed modules and that of simplicial Lie algebras with Moore complex of length 2. Akça and Arvasi apply higher order Peiffer elements in simplicial Lie algebras to the Lie 2-crossed module in [1].

In this paper, we invented the concept of quasi 2-crossed modules of Lie algebras. In [4], Carrasco and Porter have mentioned this notion for group cases. We have also intend to use it to work on functorial relations, similar to how algebraic models of homotopy 2-types are used. We will see that the roles of quasi 2-crossed modules in Lie algebras and those of pre-crossed modules are similar except dimensionally.

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2. Preliminaries

2.1. Crossed modules of Lie algebras

If Y and Z are two Lie algebras, then a left Lie algebra action of Z on Y is a k-bilinear map

$$\begin{array}{rcl} Z \times Y & \longrightarrow & Y \\ (z,y) & \longmapsto & z * y, \end{array}$$

that satisfies the following two axioms:

L1) $z * [y, y'] = [z * y, y'] + [y, z * y']$,

L2) $[z, z'] * y = z * (z' * y) - z' * (z * y)$

for each $z, z' \in Z$ and each $y, y' \in Y$.

A pre-crossed module over Lie algebras (Y, Z, ∂) is given by a Lie homomorphism $\partial : Y \to Z$, together with a left Lie algebra action of Z on Y such that the condition

XMod$_L$1 $\partial(z * y) = [z, \partial(y)]$ is satisfied for each $z \in Z$ and each $y \in Y$.

A crossed module over Lie algebras (Y, Z, ∂) is a pre-crossed module satisfying, in addition "Peiffer identity" condition:

XMod$_L$2 $\partial(y) * y' = [y, y']$

for all $y, y' \in Y$.

Example 2.1. An inclusion map $i : I \longrightarrow Z$ is a crossed module where I is any ideal of a Lie algebra Z. Conversely given any crossed module $\partial : I \to Z$, one can easily verify that $\partial(Y) = I$ is an ideal in Z.

Example 2.2. Any Z-module Y can be considered as a Lie algebra with zero multiplication, and then $\mathbf{0} : Y \to Z$ is a crossed module by $\mathbf{0}(y) * y' = 0y' = [y, y']$ and $\mathbf{0}(z * y) = 0 = [z, \mathbf{0}(y)]$, for all $y, y' \in Y, z \in Z$.

Example 2.3. A Lie k-algebra morphism

$$\begin{array}{rcccccl} \mu : & S & \to & Der(S) & & & \\ & s & \mapsto & \mu(s) = \mu_s : & S & \to & S \\ & & & & s' & \mapsto & \mu_s(s') = [s, s'] \end{array}$$

with the action of Der(S) on S given as

$$\begin{array}{rcl} Der(S) \times S & \to & S \\ (d, s) & \mapsto & d * s = d(s) \end{array}$$

is a crossed module where $Der(S)$ is a set of derivations of S,i.e.

$$Der(S) = \{d \mid d : S \to S,\ d([s_1, s_2]) = [s_1, ds_2] + [ds_1, s_2]\ s_1, s_2 \in S\}.$$

(See for detail [7].)

A crossed module morphism $f:(Y,Z,\partial)\to(Y',Z',\partial')$ is a pair $(f_1:Y\to Y', f_0:Z\to Z')$ of Lie algebra morphisms, making the diagram below commutative:

$$\begin{tikzcd} Y \arrow[r, "\partial"] \arrow[d, "f_1"'] & Z \arrow[d, "f_0"] \\ Y' \arrow[r, "\partial'"'] & Z' \end{tikzcd}$$

also preserving action of Z on Y.

Although the following discussion may be found in various algebraic cases, we include it here since we will need to generalize it later.

If $\partial: M\to P$ is a pre-crossed module of Lie algebras then $\bar{\partial}: M/\bar{M}\to P$ given by $\bar{\partial}([m])=\partial(m)$ is a crossed module where $\bar{M}$ is the ideal generated by the elements $[m,m']-\partial(m)*m'$, for $m,m'\in M$. It is not difficult to see that following equations are satisfied

$$\bar{\partial}([m])*[m']=\partial(m)*[m']=[\partial(m)*m']=[[m,m']]=[[m],[m']]$$

$$\bar{\partial}(p*[m])=\bar{\partial}([p*m])=\partial(p*m)=[p,\partial(m)].$$

For any pre-crossed module morphism $(f_1,f_0):(M,P,\partial)\to(M',P',\partial')$, we get the crossed module morphism $(\bar{f}_1,f_0):(M/\bar{M},P,\partial)\to(M'/\bar{M}',P',\partial')$, where $\bar{f}_1([m])=[f_1(m)], m\in M$. Since

$$\begin{aligned} f_1([m,m']-\partial(m)*m') &= f_1([m,m'])-f_1(\partial(m)*m') \\ &= [f_1(m),f_1(m')]-f_0(\partial(m))*f_1(m') \\ &= [f_1(m),f_1(m')]-\partial'(f_1(m))*f_1(m')\in\bar{M}' \end{aligned}$$

$\bar{f}_1$ is well-defined morphism. Thus, it can be given a functor

$$F: PXMOD\to XMOD$$

defined as $F((M,P,\partial))=(M'/\bar{M}',P',\partial')$ on object and as $F((f_1,f_0))=(\bar{f}_1,f_0)$ on morphism.

Furthermore, it is clear that there is forgetful functor $G: XMOD\to PXMOD$ and the functor F is left adjoint to G.

2.2. 2-Crossed Modules of Lie algebras

In this section, we recall the definition of 2-crossed modules over Lie algebras given [5].

A pair of Lie homomorphisms $L\xrightarrow{\partial_2} M\xrightarrow{\partial_1} P$ with an action of P on M and L, and a bilinear function $\{,\}: M\times M\to L$ such that below axioms are satisfied for every $m,m',m''\in M$, $l,l'\in L$ and $p\in P$ are defined as a 2-crossed module of Lie algebras

1. $\partial_1\partial_2=0$
2. $\partial_2({}^p l)={}^p(\partial_2 l)$, $\partial_1({}^p m)=[p,\partial_1(m)]$

3. $\partial_2\{m,m'\} = {}^{(\partial_1 m)}m' - [m,m']$

4. $\{\partial_2 l, \partial_2 l'\} = [l,l']$

5. $\{\partial_2 l, m\} + \{m, \partial_2 l\} = {}^{\partial_1 m} l$

6. ${}^{p}\{m,m'\} = \{{}^{p}m, m'\} + \{m, {}^{p}m'\}$

7. $\{[m,m'],m''\} = {}^{\partial_1 m}\{m',m''\} + \{m,[m',m'']\} - {}^{\partial_1 m'}\{m,m'\} - \{m',[m,m'']\}$

8. $\{m,[m',m'']\} = {}^{\partial_1 m'}\{m,m''\} - {}^{\partial_1 m''}\{m,m'\} - \{m', {}^{\partial_1 m}m'' - [m,m'']\} + \{m'', {}^{\partial_1 m}m' - [m,m']\}$

It is denoted by $(L,M,P,\partial_2,\partial_1,\{,\})$. If the below diagram is commutative

$$\begin{tikzcd}
M\times M \arrow[r, "\{,\}"] \arrow[d, "f_1\times f_1"'] & L \arrow[r, "\partial_2"] \arrow[d, "f_2"] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P \arrow[d, "f_0"] \\
M'\times M' \arrow[r, "\{,\}'"'] & L' \arrow[r, "\partial_2'"'] & M' \arrow[r, "\partial_1'"'] & P'
\end{tikzcd}$$

that is, the equations

$$\partial_1{}' f_1 = f_0\partial_1$$

$$\partial_2{}' f_2 = f_1\partial_2,$$

$$f_2\{,\} = \{,\}'(f_1,f_1)$$

are satisfied and

$$f_1({}^{p}m) = {}^{f_0(p)}f_1(m)$$

$$f_2({}^{p}l) = {}^{f_0(p)}f_2(l)$$

then a triple (f_2,f_1,f_0) is called by the morphism of between 2-crossed modules $(L,M,P,\partial_2,\partial_1,\{,\})$ and $(L',M',P',\partial_2',\partial_1',\{,\}')$.

As a result, the category of 2-crossed modules is obtained, with 2-crossed modules as objects and morphisms between them as morphisms and it is denoted by L2XMOD.

When the morphisms f_1 and f_0 above are the identity, we will get a subcategory $L2XMOD/(M,P)$, the category of 2-crossed modules, over fixed pre-crossed module $\partial_1 : M \to P$.

2.3. Quasi 2-Crossed Modules of Lie Algebras

A quasi 2-crossed module of Lie algebras is a sequence $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ of Lie algebra homomorphisms together with a bilinear map $\{,\} : M\times M \to L$ satisfying the below axioms

LQ2X1) $\partial_1\partial_2 = 0$

LQ2X2) $\partial_2({}^{p}l) = {}^{p}(\partial_2 l)$, $\partial_1({}^{p}m) = [p,\partial_1(m)]$

LQ2X3) ${}^{p}\{m_0,m_1\} = \{{}^{p}m_0, m_1\} + \{m_0, {}^{p}m_1\}$

LQ2X4) $\partial_2\{m_0,m_1\} = {}^{\partial_1 m_0}m_1 - [m_0,m_1]$

LQ2X5) $\{m_0,[m_1,m_2]\} = {}^{\partial_1 m_1}\{m_0,m_2\} - {}^{\partial_1 m_2}\{m_0,m_1\} - \{m_1, {}^{\partial_1}m_2 - [m_0,m_2]\}$
$+\{m_2, {}^{\partial_1 m_0}m_1 - [m_0,m_1]\}$

LQ2X6) $\{[m_0,m_1],m_2\} = {}^{\partial_1 m_0}\{m_1,m_2\} + \{m_0,[m_1,m_2]\} - {}^{\partial_1 m_1}\{m_1,m_2\}$
$-\{m_1,[m_0,m_2]\}$

LQ2X7) $[\{m_0,m_1\}, {}^{\partial_1 m_0}(m_1 \lhd l)] = \{{}^{\partial_1 m_0}[m_1,\partial_2 l], \partial_2\{m_0,m_1\}\}$

for all $m, m_0, m_1, m_2 \in M$ and $l \in L$. Also, the action on L of M is given by

$$m \lhd l = {}^{\partial_1 m}l - \{m,\partial_2 l\} = \{\partial_2 l, m\}.$$

We get the category $LQ2XMOD$ quasi 2-crossed modules of Lie algebras by defining whose morphisms similar to that of L2XMOD and it is obtained a subcategory $L2XMOD/(M,P)$ with base ÃÂ $\partial_1 : M \to P$, [7].

Proposition 2.4. Every 2-crossed module is a quasi 2-crossed module.

Proof.
Let $(L, M, P, \partial_2, \partial_1, \{,\})$ be a 2-crossed module. To complete the proof, just axiom 7 has to be verified.

$$\begin{aligned}
[\{m_0,m_1\}, {}^{\partial m_0}(m_1 \lhd l)] &= \partial_2\{m_0,m_1\} \lhd {}^{\partial m_0}\{\partial_2 l, m\} \\
&= \partial_2\{m_0,m_1\} \lhd \{{}^{\partial m_0}\partial_2 l, m\} + \{\partial_2 l, {}^{\partial m_0}m_1\} \\
&= \partial_2\{m_0,m_1\} \lhd \{{}^{\partial m_0}\partial_2 l, m\} + \partial_2\{m_0,m_1\} \lhd \{\partial_2 l, {}^{\partial m_0}m_1\} \\
&= \left\{\partial_2\left(\{{}^{\partial m_0}\partial_2 l, m\}\right), \partial_2\{m_0,m_1\}\right\} + \\
&\quad \left\{\partial_2\left(\{\partial_2 l, {}^{\partial m_0}m_1\}\right), \partial_2\{m_0,m_1\}\right\} \\
&= \left\{{}^{(\partial_1({}^{\partial m_0}\partial_2 l))}m - [{}^{\partial m_0}\partial_2 l, m]\right\} + \\
&\quad \left\{{}^{(\partial_1(\partial_2 l))}({}^{\partial m_0}m_1) - [\partial_2 l, {}^{\partial m_0}m_1]\right\} \\
&= \left\{{}^{[\partial m_0, \partial_1(\partial_2 l)]}m - [{}^{\partial m_0}\partial_2 l, m], {}^{\partial m_0}m_1 - [m_0,m_1]\right\} + \\
&\quad \{0 - [\partial_2 l, {}^{\partial m_0}m_1], {}^{\partial m_0}m_1 - [m_0,m_1]\} \\
&= \{-[{}^{\partial m_0}\partial_2 l, m], {}^{\partial m_0}m_1 - [m_0,m_1]\} + \\
&\quad \{-[\partial_2 l, {}^{\partial m_0}m_1], {}^{\partial m_0}m_1 - [m_0,m_1]\} \\
&= \{[m_1, {}^{\partial m_0}\partial_2 l] + [{}^{\partial m_0}m_1, \partial_2 l], \partial_2\{m_0,m_1\}\} \\
&= \{{}^{\partial m_0}[m_1,\partial_2 l], \partial_2\{m_0,m_1\}\}
\end{aligned}$$

for all $m, m_0, m_1 \in M$ and $l \in L$.

Proposition 2.5. If $(L, M, P, \partial_2, \partial_1, \{,\})$ is a Lie quasi 2-crossed module, then ideal $\overline{L}$ generated by the elements of the type

$$m * l = {}^{\partial_1 m}l - \{m,\partial_2 l\} - \{\partial_2 l, m\}$$

$$l \circledast l' = [l, l'] - \{\partial_2 l, \partial_2 l'\}$$

is a P-invariant ideal in Lie algebra L, for all $l, l' \in L$, $m \in M$.

Proof.

$$\begin{aligned}
{}^{p}(m*l) &= {}^{p}({}^{\partial_1 m}l-\{m,\partial_2 l\}-\{\partial_2 l,m\}) \\
&= {}^{p}({}^{\partial_1 m}l)-{}^{p}\{m,\partial_2 l\}-{}^{p}\{\partial_2 l,m\} \\
&= \{p,\partial_1 m\}\cdot l+(\partial_1 m)\cdot(p\cdot l)-\{{}^{p}m,\partial_2 l\}-\{m,\partial_2({}^{p}l)\}-\{\partial_2({}^{p}l),m\}-\{\partial_2 l,{}^{p}m\} \\
&= m*{}^{p}l+{}^{\partial_1({}^{p}m)}l-\{p_m,\partial_2 l\}-\{\partial_2 l,{}^{p}m\} \\
&= m*{}^{p}l+({}^{p}m*l)\in\overline{L}
\end{aligned}$$

for $p\in P$, $m\in M$, $l\in L$, and we also get

$$\begin{aligned}
{}^{p}(l\circledast l') &= {}^{p}([l,l']-\{\partial_2 l,\partial_2 l'\}) \\
&= [l,{}^{p}l']-\{{}^{p}\partial_2 l,\partial_2 l'\}-\{\partial_2 l,{}^{p}\partial_2 l'\} \\
&= [{}^{p}l,l']+[l,{}^{p}l']-\{\partial_2({}^{p}l),\partial_2 l'\}-\{\partial_2 l,\partial_2({}^{p}l')\} \\
&= [{}^{p}l,l']-\{\partial_2({}^{p}l),\partial_2 l'\}+[l,{}^{p}l']-\{\partial_2 l,\partial_2({}^{p}l')\} \\
&= ({}^{p}l\circledast l')+(l\circledast {}^{p}l')\in\overline{L}
\end{aligned}$$

for $p\in P$, $l,l'\in L$

Theorem 2.6. Let $(L,M,P,\partial_2,\partial_1,\{,\})$ be a Lie quasi 2-crossed module and $\overline{L}$ be as in previous proposition. Then $(L/\overline{L},M,P,\overline{\partial},\partial_1,\overline{\{,\}})$ is a 2-crossed module where $\overline{\partial}:L/\overline{L}\longrightarrow M$, is given by $\overline{\partial}(l+\overline{L})=\partial_2 l$ and $\overline{\{,\}}:M\times M\longrightarrow L/\overline{L}$ is defined by $\overline{\{,\}}(m_1,m_2)=\{m_1,m_2\}+\overline{L}$ for $l\in L$ and $m_1,m_2\in M$,respectively.

Proof.

$$\begin{aligned}
\partial_2(m*l) &= \partial_2({}^{\partial_1 m}l-\{m,\partial_2 l\}-\{\partial_2 l,m\}) \\
&= \partial_2({}^{\partial_1 m}l)-\partial_2(\{m,\partial_2 l\})-\partial_2(\{\partial_2 l,m\}) \\
&= {}^{\partial_1 m}\partial_2 l-{}^{\partial_1 m}\partial_2 l+[m,\partial_2 l]-{}^{\partial_1(\partial_2 l)}m+[\partial_2 l,m] \\
&= 0+[m,\partial_2 l]-{}^{0}m-[m,\partial_2 l] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_2(l\circledast l') &= \partial_2([l,l']-\{\partial l,\partial l'\}) \\
&= \partial_2([l,l'])-\partial_2(\{\partial l,\partial l'\}) \\
&= [\partial_2 l,\partial_2 l']-{}^{\partial_1(\partial_2 l)}\partial_2 l'-[\partial_2 l,\partial_2 l'] \\
&= 0
\end{aligned}$$

for all $m\in M,l,l'\in L$, that is $\partial_2(\overline{L})=0$. Thus

$$\begin{aligned}
\overline{\partial}:\quad L/\overline{L} &\rightarrow M \\
l+\overline{L} &\mapsto \overline{\partial}(l+\overline{L})=\partial_2 l
\end{aligned}$$

is well-defined. It is seen that some of the axioms of the 2-crossed module are verified.

$$\begin{aligned}
\overline{\{\overline{\partial}(l+\overline{L}),\overline{\partial}(l'+\overline{L})\}} &= \overline{\{\partial_2 l,\partial_2 l'\}} \\
&= \{\partial_2 l,\partial_2 l'\}+\overline{L} \\
&= [l,l']+\overline{L} \qquad (\because \{\partial_2 l,\partial_2 l'\}-[l,l']\in\overline{L})
\end{aligned}$$

$$\begin{aligned}
\overline{\{\overline{\partial}(l+\overline{L}),m\}}+\overline{\{m,\overline{\partial}(l+\overline{L})\}} &= \overline{\{\partial_2 l,m\}}+\overline{\{m,\partial_2 l\}} \\
&= \{\partial_2 l,m\}+\overline{L}+\{m,\partial_2 l\}+\overline{L} \\
&= {}^{\partial_1 m}l+\overline{L} \\
&= {}^{\partial_1 m}(l+\overline{L})
\end{aligned}$$

$$\begin{aligned}
\overline{\partial}\,\overline{\{m_1,m_2\}} &= \overline{\partial}(\{m_1,m_2\}+\overline{L}) \\
&= \partial_2(\{m_1,m_2\}) \\
&= {}^{\partial_1 m_1}m_2-[m_1,m_2]
\end{aligned}$$

for all $m,m_1,m_2 \in M,\ l+\overline{L},l^{'}+\overline{L} \in L/\overline{L}$. The validity of other axioms can be seen similarly. Therefore we have following result:

Corollary 2.7. There is (F,G) adjoint functor pair,

$$LQ2XMOD \underset{G}{\overset{F}{\rightleftarrows}} L2XMOD.$$

Proof.

Let $\mathscr{L}=(L,M,P,\partial_2,\partial_1,\{,\})$ and $\mathscr{L}^{'}=(L^{'},M^{'},P^{'},\partial_2^{'},\partial_1^{'},\{,\}^{'})$ be two Lie quasi 2-crossed module and (f_2,f_1,f_0) be morphism between them. The functor

$$F: LQ2XMOD \to L2XMOD$$

is given by $F(\mathscr{L})=(L/\overline{L},M,P,\overline{\partial},\partial_1,\overline{\{,\}}),F(\mathscr{L}^{'})=(L^{'}/\overline{L^{'}},M^{'},P^{'},\overline{\partial}^{'},\partial_1^{'},\overline{\{,\}^{'}})$ and $F(f_2,f_1,f_0)=(f_2^{*},f_1,f_0)$ where $f_2^{*}(l+\overline{L})=f_2(l)+\overline{L^{'}}$. We have

$$\begin{aligned}
f_2(m*l) &= f_2({}^{\partial_1 m}l-\{m,\partial_2 l\}-\{\partial_2 l,m\}) \\
&= f_2({}^{\partial_1 m}l)-f_2(\{m,\partial_2 l\})-f_2(\{\partial_2 l,m\}) \\
&= {}^{f_0(\partial_1 m)}f_2(l)-\{f_1(m),f_1(\partial_2 l)\}^{'}-\{f_1(\partial_2 l),f_1(m)\}^{'} \\
&= {}^{\partial_1^{'}f_1(m)}f_2(l)-\{f_1(m),\partial_2^{'}(f_2(l))\}^{'}-\{\partial_2^{'}(f_2(l)),f_1(m)\}^{'} \\
&= f_1(m)*f_2(l) \in \overline{L^{'}}
\end{aligned}$$

and

$$\begin{aligned}
f_2(l_1 \circledast l_2) &= f_2([l_1,l_2^{'}]-\{\partial_2 l,\partial_2 l^{'}\}) \\
&= f_2[l_1,l_2]-f_2(\{\partial_2 l_1,\partial_2 l_2\}) \\
&= [f_2 l_1,f_2 l_2]-(f_2(\{\})(\partial_2 l_1,\partial_2 l_2)) \\
&= [f_2 l_1,f_2 l_2]-\{\}^{'}(f_1,f_2)(\partial_2 l_1,\partial_2 l_2) \\
&= [f_2 l_1,f_2 l_2]-\{f_1\partial_2 l_1,f_1\partial_2 l_2\}^{'} \\
&= [f_2 l_1,f_2 l_2]-\{\partial_2^{'}f_2 l_1,\partial_2^{'}f_2 l_2\}^{'} \\
&= f_2(l_1)*f_2(l_2) \in \overline{L^{'}}
\end{aligned}$$

for $m*l$ and $l_1 \circledast l_2 \in \overline{L}$, and so $f_2(\overline{L}) \subseteq \overline{L^{'}}$.

The morphism $f_2^{*}: L/\overline{L} \to L^{'}/\overline{L^{'}}$ given by $f_2^{*}(l+\overline{L})=f_2(l)+\overline{L^{'}}$ is well-defined, since $f_2(l_1-l_2) \in f_2(\overline{L}) \subseteq \overline{L^{'}}$ for $l_1-l_2 \in \overline{L}$.

We have

$$\begin{aligned}\overline{\partial_2'}(f_2^*(l+\overline{L})) &= \overline{\partial_2'}((f_2 l)+\overline{L'})\\ &= \partial_2'(f_2 l)\\ &= f_1(\partial_2(l))\\ &= f_1(\overline{\partial}(l+\overline{L}))\end{aligned}$$

and also

$$\partial_1' f_1 = f_0\partial_1$$

since (f_2, f_1, f_0) is a morphism of quasi 2-crossed modules of Lie algebras. Therefore we get following commutative diagram:

$$\begin{tikzcd} L/\overline{L} \arrow[r, "\overline{\partial}"] \arrow[d, "f_2^*"] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P \arrow[d, "f_0"] \\ L'/\overline{L'} \arrow[r, "\overline{\partial}_2'"'] & M' \arrow[r, "\partial_1'"'] & P' \end{tikzcd}$$

Furthermore we have below equations:

$$\begin{aligned}f_2^*\overline{\{,\}}(m_1,m_2) &= f_2^*(\{m_1,m_2\}+\overline{L})\\ &= f_2(\{m_1,m_2\})+\overline{L}\\ &= \{f_1(m_1),f_1(m_2)\}'\\ &= \{,\}(f_1,f_1)(m_1,m_2)\end{aligned}$$

$$\begin{tikzcd} M\times M \arrow[r, "\{,\}"] \arrow[d, "f_1\times f_1"'] & L/\overline{L} \arrow[r, "\overline{\partial}"] \arrow[d, "f_2^*"] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P \arrow[d, "f_0"] \\ M'\times M' \arrow[r, "\{,\}'"'] & L'/\overline{L'} \arrow[r, "\overline{\partial}_2'"'] & M' \arrow[r, "\partial_1'"'] & P' \end{tikzcd}$$

Thus (f_2^*, f_1, f_0) is a morphism of 2-crossed modules, as seen above.

For $\mathscr{K} = (K, N, Q, \partial_2', \partial_1', \{,\}')$ and $(f, f_1, f_0) : F(\mathscr{L}) \to \mathscr{K} \in Mor(L2XMOD)$, the morphism $(f q_{_L}, f_1, f_0) : \mathscr{L} \to \mathscr{K}$ is in $Mor(LQ2XMOD)$, where $q_{_L} : L \to L/\overline{L}$. Conversely, for $(f_2, f_1, f_0) : \mathscr{L} \to G(\mathscr{K}) \in Mor(LQ2XMOD)$,

$$(f_2^*, f_1, f_0) : (L/\overline{L}, M, P, \overline{\partial}, \partial, \overline{\{,\}}) \to (K, N, Q, \partial_2', \partial_1', \{,\}')$$

is a morphism in $Mor(L2XMOD)$. Thus, we get the bijection

$$L2XMOD(F(\mathscr{L}), \mathscr{K}) \cong LQ2XMOD(\mathscr{L}, G(\mathscr{K}))$$

such that this family of bijections is natural in $\mathscr{L}$ and $\mathscr{K}$. Clearly; for $h : (h_2, h_1, h_0) = \mathscr{L}' \to \mathscr{L} \in Mor(LQ2XMOD)$, we have following commutative diagram

$$\begin{tikzcd} L2XMOD(F(\mathscr{L}),\mathscr{K}) \arrow[r, "\eta_{\mathscr{L},\mathscr{K}}"] \arrow[d, "F(h)^*=-\circ F(h)"'] & LQ2XMOD(\mathscr{L},G(\mathscr{K})) \arrow[d, "-\circ h=h^*"] \\ L2XMOD(F(\mathscr{L}'),\mathscr{K}) \arrow[r, "\eta_{\mathscr{L}',\mathscr{K}}"'] & LQ2XMOD(\mathscr{L}',G(\mathscr{K})) \end{tikzcd}$$

since

$$\begin{aligned} f_2 h_2^* q_L(l') &= f_2 h_2^*(l' + \overline{L'}) \\ &= f_2(h_2(l') + \overline{L}) \\ &= f_2(q_L(h_2(l'))), \end{aligned}$$

and for $k : (k_2, k_1, k_0) = \mathscr{K} \to \mathscr{K}' \in Mor(LX_2MOD)$, we get commutative diagram

$$\begin{tikzcd} L2XMOD(F(\mathscr{L}),\mathscr{K}) \arrow[r, "\eta_{\mathscr{L},\mathscr{K}}"] \arrow[d, "k^* = k\circ -"'] & LQ2XMOD(\mathscr{L}, G(\mathscr{K})) \arrow[d, "G(k)\circ - = G(k)^*"] \\ L2XMOD(F(\mathscr{L}),\mathscr{K}') \arrow[r, "\eta_{\mathscr{L},\mathscr{K}'}"'] & LQ2XMOD(\mathscr{L}, G(\mathscr{K}')) \end{tikzcd}$$

because of

$$(k_2 f_2) q_L = k_2(f_2 q_L).$$

Hence, it is concluded that there is an adjunction between LQ2XMOD and L2XMOD.

3. Conclusion

In this paper, the category of quasi 2-crossed modules for Lie algebras has been introduced, and an adjunction between this category and that of 2-crossed modules for Lie algebras is constructed. It is concluded that this category has a similar role to that of pre-crossed modules in corresponding adjunction to their 1-dimensional analogous.

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Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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