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Blow-up for a Generalized Dullin-Gottwald-Holm Equation

Nurhan Dündar

Abstract
In this paper, the blow up of solutions for a generalized version of the Dullin-Gottwald-Holm equation which is a nonlinear shallow water wave equation is studied. The precise blow-up scenario and a result of blow-up solutions are described. The blow-up occurs as wave breaking. This means the solution (representing the wave) remains bounded but its slope becomes infinite in finite time. We use an approach devised in [1].

Keywords: *Generalized Dullin-Gottwald-Holm equation; shallow water wave; blow-up.*

AMS Subject Classification (2020): *Primary: 35B44 ; Secondary: 35Q35; 35G20.*

1. Introduction

Dullin et al. in [3] presented the following the nonlinear dispersive evolution equation, then called the Dullin-Gottwald-Holm (DGH) equation:

$$u_t - \beta^2 u_{xxt} + k_0 u_x + 3uu_x + \Gamma u_{xxx} = \beta^2 \left(2u_x u_{xx} + uu_{xxx}\right), \qquad t > 0,\ x \in \Re. \tag{1.1}$$

The DGH equation is an equation modeling the unidirectional propagation of shallow water waves on a flat bottom. $u = u\left(t, x\right)$ is fluid velocity, where t and x are variables related to time and space respectively. β, Γ and k_0 are some physical positive parameters.

In equation (1.1), if $\beta = 0$ and $\Gamma \neq 0$, the Korteweg-de Vries (KdV) equation is obtained, and if $\beta = 1$ and $\Gamma = 0$, the Camassa-Holm (CH) equation is obtained. As can be seen, equation (1.1) contains two different integrable soliton equations for shallow water waves. The DGH equation (1.1) combines the linear dispersion of the KdV equation with the nonlinear/nonlocal dispersion of the CH equation. Equation (1.1) has important properties. Some of these important features are: It has the bi-Hamiltonian structure and soliton solutions and it is completely integrable [3]. For this equation, blow up occurs in the form of wave breaking: This means: while the solution u representing the wave remains bounded, u_x, which is its first derivative with respect to x becomes infinite in finite time [1, 12, 15].

Since the equation (1.1) was discovered, a great deal of space has been devoted to it in the literature and this equation has been the subject of intense research. Its mathematical behaviors such as local well-posedness, global

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strong solutions, global weak solutions, blow up solutions in finite time and stability of peakons have been studied in many works [8, 11, 12, 15–18].

In present paper, we study the following initial value problem for the generalized DGH equation:

$$\begin{cases} u_t - \beta^2 u_{xxt} + (P(u))_x + \Gamma u_{xxx} = \beta^2 \left(\frac{Q'(u)}{2} u_x^2 + Q(u) u_{xx} \right)_x, & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re, \end{cases} \tag{1.2}$$

where $P(u),\ Q(u)\ : \Re \to \Re$ are given C^3-functions. For $P(u) = 2\omega u + \frac{3}{2}u^2$ (where $2\omega = k_0$) and $Q(u) = u$, it is seen that the (1.2) turns into equation equation (1.1). Some mathematical behaviors of equation (1.2) have been studied by many authors before. In [13, 14], the authors established the well-posedness a finite time for (1.2) by using Kato's theory. Furthermore, the stability of peakons of (1.2) was discussed with $P(u) = 2\omega u + \frac{a+2}{2}u^{a+1}$ and $Q(u) = u^a$ in [13]. In [4], Dündar and Polat investigated the blow up of the solutions of (1.2) with $Q(u) = u$. Also in the same article, they proved stability of solitary waves by using the method in [7] for $P(u) = 2\omega u + \frac{a+2}{2}u^{a+1}$ and $Q(u) = u^a$.

In (1.2), if the weak dispersive term Γu_{xxx} is changed into the strong dispersive term $\Gamma\left(u - \beta^2 u_{xx}\right)_{xxx}$, we obtain

$$\begin{cases} u_t - \beta^2 u_{xxt} + (P(u))_x + \Gamma\left(u - \beta^2 u_{xx}\right)_{xxx} = \beta^2 \left(\frac{Q'(u)}{2} u_x^2 + Q(u) u_{xx} \right)_x, & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re. \end{cases} \tag{1.3}$$

Dündar and Polat studied the well-posedness for (1.3) a finite time in [6]. Also, they showed the existence of solitary waves and proved the stability of solitary wave solutions of (1.3) in [5].

The main aim of this paper is to investigate the blow up of the solutions of (1.2) in finite time. In [4], authors obtained the blow up of the strong solutions of (1.2) with $Q(u) = u$. In this paper, we remove this restriction and obtain more general results.

The content of this article is as follows: In Section 2, we will give the notations and some basic informations, and recall some necessary conclusions. In Section 3, we will examine the blow up of solutions of (1.2).

2. Preliminaries

We introduce by summarizing some notations. The convolution is denoted by $*$. $\|.\|_{\mathfrak{B}}$ denotes the norm of Banach space $\mathfrak{B}$. Since all space of functions are over $\Re$, for convenience, we will not use $\Re$ in our notations of function spaces if there is no equivocalness. We denote the norm in the Sobolev space H^s by

$$\|v\|_s = \|v\|_{H^s} = \left(\int_{\Re} \left(1 + |\xi|^2\right)^s |\widehat{v}(\xi)|^2 \, d\xi \right)^{1/2}$$

for $s \in \Re$. Here $\widehat{v}(\xi)$ is the Fourier transform of v. We use the $\|.\|_{L^p}$ for the norm of the space L^p, $1 \le p \le \infty$. We define the operator Λ^s by the formula $\Lambda^s = \left(1 - \partial_x^2\right)^{\frac{s}{2}}$, $s \in \Re$.

From now on, throughout this article, we assume $\beta = 1$ for convenience. Note that if $f(x) = \frac{1}{2}e^{-|x|}$, $x \in \Re$, then $\left(1 - \partial_x^2\right)^{-1} v = f * v$ for all $v \in L^2$. Then (1.2) can be rewritten as follows:

$$\begin{cases} u_t + (Q(u) - \Gamma) u_x = f * [Q(u) u_x] - \partial_x f * \left[\frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u \right], & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re. \end{cases} \tag{2.1}$$

Or in the equivalent form:

$$\begin{cases} u_t + (Q(u) - \Gamma) u_x = \left(1 - \partial_x^2\right)^{-1} [Q(u) u_x] - \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u \right], & t > 0,\ x \in \Re, \\ u(0,x) = u_0(x), & x \in \Re. \end{cases} \tag{2.2}$$

It can be seen that (1.2) is equivalent to (2.1) (or (2.2)) for $\beta = 1$. So, we will investigate the blow up of solutions of (2.1) (or (2.2)).

2.1 Local well-posedness for the Cauchy problem of (2.1)

Theorem 2.1. *[14]. Let $n \geq 2$ be a natural number, $s \in \left(\frac{3}{2}, n\right)$, and $P, Q \in C^{n+3}$, with $P(0) = 0$. If $u_0 \in H^s$, there exists a maximal $T = T(u_0) > 0$, and a unique solution u to (2.1) (or (2.2)) such that*

$$u = u(., u_0) \in C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(., u_0) : H^s \to C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right)$$

is continuous.

In [13], Liu and Yin obtained the local well-posedness theorem of the Cauchy problem (2.1) with the constraint $Q(0) = 0$ by applying Kato's theory [10]. Later, in [14] (Theorem 1.2 and Corollary 1.1), the authors removed the limiting condition $Q(0) = 0$, which makes an improvement in the results in [13].

Theorem 2.2. *Let $n \geq 2$ be a natural number, $s \in \left(\frac{3}{2}, n\right)$, and $P, Q \in C^{n+3}$, with $P(0) = 0$. Then T in Theorem 2.1 may be chosen independent of s in the following sense. If*

$$u = u(., u_0) \in C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right)$$

to 2.1 (or 2.2), and if $u_0 \in H^{s'}$ for some $s' \neq s$, $\frac{3}{2} < s' < n$, then

$$u \in C\left([0, T); H^{s'}\right) \cap C^1\left([0, T); H^{s'-1}\right)$$

and with the same T. In particular, if $P,\ Q \in C^\infty$ and let $u_0 \in H^\infty = \cap_{s \geq 0} H^s$, $u \in C([0, T); H^\infty)$.

Proof. For $\beta = 1$, since (1.2) can be rewritten as

$$\frac{dw}{dt} + K(t)\, w + L(t)\, w = R(t), \qquad w(0) = \Lambda^2 u(0),$$

where

$$K(t)\, w = \partial_x\left((Q(u) - \Gamma)\, w\right), \qquad L(t)\, w = Q'(u)\, u_x w,$$

and

$$R(t) = u_x\left(\frac{1}{2} Q''(u)\, u_x^2 - P'(u) + 2Q'(u)\, u + Q(u) - \Gamma\right),$$

thus the proof of Theorem 2.2 is alike to the proof of Theorem 1.2 of [6]. The proof is completed with reference the proof of Theorem 1.2 in [6].

2.2 Some lemmas

We will now give some lemmas that we will use in this paper. We list below without proof.

Lemma 2.1. *[9]. Let $s > 0$. Then we have*

$$\left\|\left[\Lambda^s, y\right] z\right\|_{L^2} \leq K\left(\|\partial_x y\|_{L^\infty} \left\|\Lambda^{s-1} z\right\|_{L^2} + \|\Lambda^s y\|_{L^2} \|z\|_{L^\infty}\right).$$

Here K is constant depending only on s.

Lemma 2.2. *[9]. Let $s > 0$. Then $H^s \cap L^\infty$ is an algebra. Moreover*

$$\|yz\|_s \leq K\left(\|y\|_{L^\infty} \|z\|_s + \|y\|_s \|z\|_{L^\infty}\right)$$

where K is constant depending only on s.

Lemma 2.3. *[2]. Assume that $G \in C^{n+2}$ with $G(0) = 0$. Then for every $\frac{1}{2} < s \leq n$, we have that*

$$\|G(u)\|_s \leq \tilde{G}\left(\|u\|_{L^\infty}\right) \|u\|_s, \qquad u \in H^s,$$

where $\tilde{G}$ is a monotone increasing function depending only on G and s.

Lemma 2.4. *[1]. Let $T > 0$ and $u \in C^1\left([0,T);H^2\right)$. Then for every $t \in [0,T)$, there exist at least one pair points $\theta(t)$, $\Theta(t) \in \Re$, such that*

$$j(t) = \inf_{x\in\Re} u_x(t,x) = u_x(t,\theta(t)), \qquad J(t) = \sup_{x\in\Re} u_x(t,x) = u_x(t,\Theta(t)),$$

and $j(t)$, $J(t)$ are absolutely continuous on $(0,T)$. Furthermore,

$$\frac{dj(t)}{dt} = u_{tx}(t,\theta(t)), \qquad \frac{dJ(t)}{dt} = u_{tx}(t,\Theta(t)), \qquad a.e. on\,(0,T).$$

Lemma 2.5. *[13]. Let $u(t,x)$ be a solution of (1.2). Then the functionals*

$$\mathcal{E}(u) = \int_\Re \left(u^2 + \beta^2 u_x^2\right) dx,$$

$$\mathcal{F}(u) = \int_\Re \left(2\mathfrak{P}(u) + \beta^2 Q(u)\, u_x^2 - \Gamma u_x^2\right) dx$$

are constant with respect to t, where $\mathfrak{P}'(s) = P(s)$.

3. Blow-up analysis

In this section, we examine the blow-up phenomena of the (2.1) (or (2.2)).

Remark 3.1. Given in Lemma (2.5), $\mathcal{E}(u) = \int_\Re \left(u^2 + u_x^2\right) dx$ $(\beta = 1)$ is an invariant for equation (2.1). So, we have that

$$\|u\|_{L^\infty}^2 \le \int_\Re \left(u^2 + u_x^2\right) = \mathcal{E}(u) = \mathcal{E}(u_0) = \|u_0\|_1^2.$$

Remark 3.2. Since $Q \in C^{n+3}$ with $n \ge 2$, by using $\|u\|_{L^\infty} \le \|u\|_1 = \|u_0\|_1$ which can be seen in Remark 3.1, a positive constant $a_1 > 0$ can be found such that

$$|Q'(u)| \le \sup_{|z|\le\|u_0\|_1} |Q'(z)| \le a_1. \tag{3.1}$$

We will first give the following theorem.

Theorem 3.1. *Let $P, Q \in C^{n+3}$, $n \ge 2$, $P(0) = 0$ and $u_0 \in H^s$, $\frac{3}{2} < s \le n$. Then the solution $u(t,x)$ of (2.2) blows up in finite time $T < \infty$ if and only if*

$$\lim_{t\to T} \sup_{0\le\tau\le t} \|u_x(\tau,x)\|_{L^\infty} = +\infty. \tag{3.2}$$

Moreover, if $T < \infty$, then

$$\int_0^T \left(\|u_x(t,x)\|_{L^\infty} + 1\right)^2 dt = +\infty.$$

Proof. Let $\Gamma = Q(0)$. We can rewrite (2.2) as

$$u_t + (Q(u) - Q(0))\, u_x = \left(1-\partial_x^2\right)^{-1}[Q(u)\, u_x] - \partial_x\left(1-\partial_x^2\right)^{-1}\left[\frac{Q'(u)}{2}u_x^2 + P(u) + Q(0)\, u\right]. \tag{3.3}$$

If we apply the operator Λ^s, then multiply by $2\Lambda^s u$ on both sides of (3.3) and finally integrate with respect to the variable x over $\Re$, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Re (\Lambda^s u)^2 dx &= -2\int_\Re \Lambda^s u \Lambda^s [(Q(u) - Q(0))\, u_x]\, dx + 2\int_\Re \Lambda^s u \Lambda^s \left(1-\partial_x^2\right)^{-1}[Q(u)\, u_x]\, dx \\ &\quad -2\int_\Re \Lambda^s u \Lambda^s \partial_x \left(1-\partial_x^2\right)^{-1}\left[\frac{Q'(u)}{2}u_x^2 + P(u) + Q(0)\, u\right] = I_1 + I_2 + I_3. \end{aligned} \tag{3.4}$$

We now estimate I_1, I_2, I_3. By using Lemma 2.1, Lemma 2.3 with $G(u) = Q(u) - Q(0)$, Remark 3.1 and Cauchy-Schwartz inequality as well as (3.1), we obtain

$$\begin{aligned}
I_1 &= -2\int_{\Re} \Lambda^s u \Lambda^s \left[(Q(u) - Q(0))\, u_x\right] dx \\
&= -2\int_{\Re} \Lambda^s u \left[\Lambda^s \left[(Q(u) - Q(0))\, u_x\right] - (Q(u) - Q(0))\, \Lambda^s u_x\right] dx \\
&\quad -2\int_{\Re} (Q(u) - Q(0))\, \Lambda^s u \Lambda^s u_x dx \\
&= -2\int_{\Re} \Lambda^s u \left[\Lambda^s, (Q(u) - Q(0))\right] u_x dx - \int_{\Re} Q'(u)\, u_x \left(\Lambda^s u\right)^2 dx \\
&\le 2K\|u\|_s \left[\|\partial_x (Q(u) - Q(0))\|_{L^\infty} \left\|\Lambda^{s-1} u_x\right\|_{L^2} + \|\Lambda^s (Q(u) - Q(0))\|_{L^2} \|u_x\|_{L^\infty}\right] \\
&\quad + \|u\|_s^2 \|Q'(u)\, u_x\|_{L^\infty} \\
&\le K\|u\|_s \left[\|Q'(u)\, u_x\|_{L^\infty} \|u\|_s + \|(Q(u) - Q(0))\|_s \|u_x\|_{L^\infty}\right] + \|u\|_s^2 \|Q'(u)\, u_x\|_{L^\infty} \\
&\le K\|u\|_s^2 \left(2a_1 \|u_x\|_{L^\infty} + \tilde{G}(\|u_0\|_1)\, \|u_x\|_{L^\infty}\right) \\
&\le K\|u\|_s^2 \|u_x\|_{L^\infty}.
\end{aligned} \tag{3.5}$$

By using Lemma 2.3 with $G(u) = G_1(u) - G_1(0)$ and Remark 3.1, Cauchy-Schwartz inequality and Sobolev embedding ($H^s \hookrightarrow H^{s-1}$), we obtain

$$\begin{aligned}
I_2 &= 2\int_{\Re} \Lambda^s u \Lambda^s \left(1 - \partial_x^2\right)^{-1} \left[Q(u)\, u_x\right] dx \\
&= 2\int_{\Re} \Lambda^s u \Lambda^s \partial_x \left(1 - \partial_x^2\right)^{-1} \left[G_1(u) - G_1(0)\right] dx \quad \text{(where } G_1'(u) = Q(u)\text{)} \\
&\le K\|u\|_s \|G_1(u) - G_1(0)\|_{s-1} \\
&\le K\tilde{G}(\|u_0\|_1)\, \|u\|_s \|u\|_{s-1} \\
&\le K\|u\|_s^2.
\end{aligned} \tag{3.6}$$

By using Lemma 2.2, Lemma 2.3 with $G(u) = Q'(u) - Q'(0)$ and Remark 3.1, Cauchy-Schwartz inequality, Sobolev embedding ($H^s \hookrightarrow H^{s-1}$) and (3.1), we obtain

$$\begin{aligned}
I_3 &= -2\int_{\Re} \Lambda^s u \Lambda^s \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\frac{Q'(u)}{2} u_x^2 + P(u) + Q(0)\, u\right] \\
&\le 2\|u\|_s \left\|\frac{Q'(u)}{2} u_x^2 + P(u) + Q(0)\, u\right\|_{s-1} \\
&\le K\|u\|_s \left[\left\|\left(\frac{Q'(u) - Q'(0) + Q'(0)}{2}\right) u_x^2\right\|_{s-1} + \|P(u)\|_{s-1} + |Q(0)|\, \|u\|_{s-1}\right] \\
&\le K\|u\|_s \left[\left\|(Q'(u) - Q'(0))\, u_x^2\right\|_{s-1} + |Q'(0)| \left\|u_x^2\right\|_{s-1} + \|u\|_{s-1} + |Q(0)|\, \|u\|_{s-1}\right] \\
&\le K\|u\|_s \left[K\left(\|Q'(u) - Q'(0)\|_{L^\infty} \left\|u_x^2\right\|_{s-1} + \left\|u_x^2\right\|_{L^\infty} \|Q'(u) - Q'(0)\|_{s-1}\right) + K\left(\|u\|_s + \|u\|_s \|u_x\|_{L^\infty}\right)\right] \\
&\le K\|u\|_s \left[\left(\sup_{|z| \le \|u_0\|_1} |Q'(z)|\right) \|u_x\|_{L^\infty} \|u\|_s + \tilde{G}(\|u_0\|_1) \left\|u_x^2\right\|_{L^\infty} \|u\|_s + \left(1 + \|u_x\|_{L^\infty}\right) \|u\|_s\right] \\
&\le K\|u\|_s^2 \left[a_1 \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + 1 + \|u_x\|_{L^\infty}\right] \\
&\le K\|u\|_s^2 \left(1 + \|u_x\|_{L^\infty}\right)^2.
\end{aligned} \tag{3.7}$$

Combining (3.5)-(3.7) with (3.4), we get

$$\frac{d}{dt} \|u\|_s^2 \le K\|u\|_s^2 \left(1 + \|u_x\|_{L^\infty}\right)^2. \tag{3.8}$$

When we apply Gronwall's inequality to (3.8), we obtain

$$\|u\|_s^2 \leq e^{K\int_0^t \left(\|u_x\|_{L^\infty}+1\right)^2 d\tau} \|u_0\|_s^2 . \tag{3.9}$$

If the solution to (2.2) blows up in finite time, in other words,

$$\lim_{t\to T} \sup_{0\leq\tau\leq t} \|u\|_s = +\infty, \tag{3.10}$$

then from (3.9), we have

$$\lim_{t\to T} \sup_{0\leq\tau\leq t} \|u_x(\tau,x)\|_{L^\infty} = +\infty. \tag{3.11}$$

If (3.11) is valid, since $\|u\|_{L^\infty} \leq \|u\|_{s-1}$ with $s > \frac{3}{2}$, we have (3.10). When the maximal existence time $T < \infty$, if

$$\int_0^T \left(\|u_x(t,x)\|_{L^\infty} + 1\right)^2 dt < +\infty,$$

from (3.9), we know that $\|u\|_s < \infty$ which contradicts with the fact that T is the maximal existence time. We get the same result for $\Gamma \neq Q(0)$. We complete the proof of Theorem 3.1.

Theorem 3.2. *Let $P, Q \in C^{n+3}$, $n \geq 3$, $P(0) = 0$. Given $u_0 \in H^s$, $3 \leq s \leq n$. If $Q'(u) \geq a_2 > 0$, then the corresponding $u(t,x)$ of (2.1) blows up in finite time $T < \infty$ if and only if*

$$\lim_{t\to T} \inf_{0\leq\tau\leq t} \inf_{x\in\Re} u_x(\tau,x) = -\infty. \tag{3.12}$$

Proof. If (3.12) is valid, then the corresponding solution $u(t,x)$ of (2.1) blows up in finite time $T < \infty$ since $\|u\|_{L^\infty} \leq \|u\|_{s-1}$ with $s > \frac{3}{2}$. We prove (3.12) by contradiction. Assume that (3.12) is invalid, then there exists $J > 0$ such that $\inf_{x\in\Re} u_x(t,x) > -J$, then we make inference that the solution will not blow up in finite time. Let's take the differentiate of (2.1) with respect to x, so we get

$$u_{tx} + Q'(u)u_x^2 + Q(u)u_{xx} - \Gamma u_{xx} = \partial_x f * [Q(u)u_x] - \partial_x^2 f * \left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right]. \tag{3.13}$$

Since $\partial_x^2 (f*v) = f*v - v$ and $\partial_x (f*v) = f*v_x$, we have

$$\begin{aligned} u_{tx} + Q'(u)u_x^2 + Q(u)u_{xx} - \Gamma u_{xx} &= f*\left[Q'(u)u_x^2 + Q(u)u_{xx}\right] - f*\left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right] \\ &+ \frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u. \end{aligned} \tag{3.14}$$

From Lemma 2.4, we define

$$J(t) = u_x(t,\Theta(t)) = \sup_{x\in\Re}[u_x(t,x)]$$

and

$$j(t) = u_x(t,\theta(t)) = \inf_{x\in\Re}[u_x(t,x)].$$

Since we deal with a maximum, $u_{xx}(t,\Theta(t)) = 0$ for all $t \in [0,T)$, it follows that a.e. on $[0,T)$

$$\begin{aligned} J'(t) &= -\frac{Q'(u(t,\Theta(t)))}{2}J^2(t) + f*\left[Q'(u)u_x^2\right](t,\Theta(t)) + P(u(t,\Theta(t))) + \Gamma u(t,\Theta(t)) \\ &\quad - f*\left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right](t,\Theta(t)). \end{aligned} \tag{3.15}$$

By Young's inequality and $f(x) = \frac{1}{2}e^{-|x|}$, we have

$$\|f*v\|_{L^\infty} \leq \|f\|_{L^\infty}\|v\|_{L^1} \leq \frac{1}{2}\|v\|_{L^1}$$

and

$$\|f * v\|_{L^\infty} \leq \|f\|_{L^1} \|v\|_{L^\infty} \leq \|v\|_{L^\infty}.$$

By using these inequalities, (3.1) and Remark 3.1, we obtain

$$\begin{aligned} \left\|f * \left(Q'(u)\, u_x^2\right)\right\|_{L^\infty} &\leq \|f\|_{L^\infty} \left\|Q'(u)\, u_x^2\right\|_{L^1} \\ &\leq \frac{1}{2} \|Q'(u)\|_{L^\infty} \|u_x\|_{L^2}^2 \\ &\leq \frac{a_1}{2} \|u\|_1^2 = \frac{a_1}{2} \|u_0\|_1^2. \end{aligned} \tag{3.16}$$

Similarly, we have

$$\|f * P(u)\|_{L^\infty} \leq \|P(u)\|_{L^\infty} \leq \sup_{|z| \leq \|u_0\|_1} |P(z)| \tag{3.17}$$

and

$$\|f * u\|_{L^\infty} \leq \|u\|_{L^\infty} \leq \|u_0\|_1. \tag{3.18}$$

Using (3.16)-(3.18) and the assumption in lemma, it then follows from (3.15) that a.e. on $[0, T)$,

$$J'(t) \leq -\frac{a_2}{2} J^2(t) + A, \tag{3.19}$$

where

$$A = 2\left(\sup_{|z| \leq \|u_0\|_1} |P(z)| + \frac{3}{8} a_1 \|u_0\|_1^2 + \|u_0\|_1\right) \tag{3.20}$$

If $J(t) > \sqrt{\frac{2A}{a_2}}$, then $J'(t) < 0$ and $J(t)$ is decreasing. Otherwise, $J(t) \leq \sqrt{\frac{2A}{a_2}}$. Thus we obtain that

$$-J < j(t) \leq u_x \leq J(t) \leq \max\left\{J(0), \sqrt{\frac{2A}{a_2}}\right\}, \qquad t \in [0, T).$$

From this inequality, we obtain the fact that u_x, that is, the slope of solution of (2.1) is bounded. When Theorem 3.1 is applied, the solution of (2.1) will not blow up in finite time. We finish the proof of Theorem 3.2.

Now, we present the following blow up result.

Theorem 3.3. *Assume that $P, Q \in C^{n+3}$, $n \geq 2$, $P(0) = 0$, $u_0 \in H^s$, $\frac{3}{2} < s \leq n$, $Q'(u) \geq a_2 > 0$. If there exists a point $x_0 \in \Re$ such that $u_0'(x_0) < -\sqrt{\frac{2A}{a_2}}$, then the corresponding solution $u(t, x)$ of (2.1) blows up in finite time $T < \infty$ and*

$$T < \frac{1}{\sqrt{2Aa_2}} \ln\left(\frac{\sqrt{\frac{a_2}{2}}\, u_0'(x_0) - \sqrt{A}}{\sqrt{\frac{a_2}{2}}\, u_0'(x_0) + \sqrt{A}}\right),$$

where

$$A = 2\left(\sup_{|z| \leq \|u_0\|_1} |P(z)| + \frac{3}{8} a_1 \|u_0\|_1^2 + \|u_0\|_1\right).$$

Proof. By Theorem 2.1- Theorem 2.2 and a simple density argument, we only need to prove that theorem provides for $s = 3$. Let T be maximal existence time of the solution $u \in C([0, T); H^s) \cap C^1\left([0, T); H^{s-1}\right)$ of (2.1). Differentiating (2.1) with respect to x, since $\partial_x^2 (f * v) = (f * v - v)$ and $\partial_x (f * v) = f * v_x$, we have

$$\begin{aligned} u_{tx} + Q'(u)\, u_x^2 + Q(u)\, u_{xx} - \Gamma u_{xx} &= f * \left[Q'(u)\, u_x^2 + Q(u)\, u_{xx}\right] - f * \left[\frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u\right] \\ &+ \frac{Q'(u)}{2} u_x^2 + P(u) + \Gamma u. \end{aligned} \tag{3.21}$$

Now define $j(t) = \inf_{x\in R}[u_x(t,x)] = u_x(t,\theta(t))$ by Lemma 2.4 and let $\theta(t) \in \Re$ be a point where this infimum is attained. For $x = \theta(t)$, since $u_{xx}(t,\theta(t)) = 0$, we have

$$\begin{aligned} j'(t) &= -\frac{Q'(u(t,\theta(t)))}{2} j^2(t) + f * \left[Q'(u)u_x^2\right](t,\theta(t)) + P(u(t,\theta(t))) + \Gamma u(t,\theta(t)) \\ &\quad - f * \left[\frac{Q'(u)}{2}u_x^2 + P(u) + \Gamma u\right](t,\theta(t)). \end{aligned} \tag{3.22}$$

Using (3.16)-(3.18) and the assumption in lemma, it then follows from (3.22) that a.e. on $[0,T)$,

$$j'(t) \le -\frac{a_2}{2} j^2(t) + A, \tag{3.23}$$

where

$$A = 2\left(\sup_{|z|\le\|u_0\|_1} |P(z)| + \frac{3}{8}a_1\|u_0\|_1^2 + \|u_0\|_1\right).$$

Note that if $j(0) \le -\sqrt{\frac{2}{a_2}A}$, then $j(t) \le -\sqrt{\frac{2}{a_2}A}$, fol all $t \in [0,T)$. By (3.23), we get

$$\frac{\sqrt{\frac{a_2}{2}}j(0) + \sqrt{A}}{\sqrt{\frac{a_2}{2}}j(0) - \sqrt{A}} e^{\sqrt{2a_2A}t} - 1 \le \frac{2\sqrt{A}}{\sqrt{\frac{a_2}{2}}j(t) - \sqrt{A}} \le 0.$$

Due to $0 < \frac{\sqrt{\frac{a_2}{2}}j(0)+\sqrt{A}}{\sqrt{\frac{a_2}{2}}j(0)-\sqrt{A}} < 1$, there exists

$$0 < T < \frac{1}{\sqrt{2Aa_2}} \ln\left(\frac{\sqrt{\frac{a_2}{2}}j(0) - \sqrt{A}}{\sqrt{\frac{a_2}{2}}j(0) + \sqrt{A}}\right)$$

such that $\lim_{t\to T} j(t) = -\infty$. For this reason, the solution u does not exist globally in time. Thus, the proof of Theorem 3.3 is completed.

4. Conclusion

In this study, we investigated the blow up of solutions of the Cauchy problem (2.1) (or (2.2)), which we obtained by taking $\beta = 1$ in (1.2).

Our main results can be summarised as follows:

1. We give the precise blow up scenario for solutions of the Cauchy problem (2.1), see Theorem 3.2.

2. We also give a blow up result of solutions of (2.1), see Theorem 3.3.

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Approximation Properties of The Nonlinear Jain Operators

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Abstract
We define the nonlinear Jain operators of max-product type. We studied approximation properties of these operators.

Keywords: *Nonlinear max-product operators; max-product Jain operators; degree of approximation.*

AMS Subject Classification (2020): *Primary: 41A30 ; Secondary: 41A25; 41A29.*

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1. Introduction

The main topic in the classical approximation theory is approximating a continuous function $f : [a, b] \to R$ with more elementary functions such as polynomials, trigonometric functions, etc.. The well-known Korovkin's theorem, which gives a simple proof of Weierstrass theorem, is based on the approximation of functions by linear and positive operators. The underlying algebraic structure of these mentioned operators is linear over R and they are also linear operators. In 2006, Bede et.al [4] asked whether they could change the underlying algebraic structure to more general structures. In this sense they presented nonlinear Shepard-type operators by replacing the operations sum and product by max and product. They proved Weierstrass-type uniform approximation theorem and obtained error estimates in terms of the modulus of continuity. Following this paper Bede et. al. [5] defined and studied pseudo linear approximation operators. Based upon these studies, there appeared an open problem in the book of S.Gal [10] in which the max-product type Bernstein operators were introduced. Related to this open problem, a nonlinear modification of the classical Bernstein operators were first studied by Bede and Gal [3] (see also [2]). The idea behind these studies were also applied to other well-known approximating operators. Several authors introduced the nonlinear versions of the stated operators and studied order of approximation [3,4,12]. Also see [6] for the collected papers.

The nonlinear Favard-Szasz-Mirakjan operators of max-product kind is introduced in [2] as (here $\bigvee$ means

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maximum)

$$F_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n})}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}$$

whose order of pointwise approximation is obtained as $\omega_1(f; \sqrt{x}/\sqrt{n})$. In [7], the authors dealed with the same operator in order to obtain the same order of approximation but by a simpler method. They also presented some shape preserving properties of the operators.

In 1972, Jain [11] introduced the following operators to generalize classical Szász-Mirakyan operators : for $\lambda > 0$ and $0 \leq \beta < 1$,

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} \omega_\beta(k, nx) f\left(\frac{k}{n}\right), f \in C[0,\lambda], n \in \mathbb{N}$$

where the basis function is

$$\omega_\beta(k,x) = x(x+k\beta)^{k-1} \frac{e^{-(x+k\beta)}}{k!}; k = 0,1,2,\ldots,$$

and

$$\sum_{k=0}^{\infty} \omega_\beta(k,x) = 1.$$

It is easy to see that for $\beta = 0$, the operator reduces to the classical Szász-Mirakyan operators. Farcas [9] proved a Voronovskaja type result for Jain's operators. Doğru et. al. [8] investigated a modification of the Jain operators preserving the linear functions. Recently, Özarslan [12] introduced the Stancu type generalization of Jain's operators and investigated the weighted approximation properties and Olgun et. al. [13] introduced a generalization of Jain's operators based on a function ρ. Also, Bernstein and generalizations of Jain operators were studied by many authors (see [14]-[21].) The aim of this study is to introduce the nonlinear Jain operators of max-product type and estimate the rate of pointwise convergence of the operators. The non-truncated Jain operators are defined by

$$T_{n,\beta}^{(M)}(f;x) = \frac{\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x) f(\frac{k}{n})}{\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x)}, n \in \mathbb{N} \tag{1.1}$$

where $W_{n,k,\beta}(x) = (nx+k\beta)^{k-1} \dfrac{e^{-(nx+k\beta)}}{k!}$ and $f : [0,\lambda] \to \mathbb{R}_+$ is considered as a bounded function on $[0,\lambda]$, $\lambda > 0$.

2. Preliminaries

Here, it is emphasized some general notations about the nonlinear operators of max-product kind. Over the set of positive reals, $\mathbb{R}_+$, we deal with the operations $\bigvee$ (maximum) and $\cdot$ (product). Then $(\mathbb{R}_+, \bigvee, \cdot)$ has a semiring structure and it is called as Max-Product algebra.

Let $I \subset \mathbb{R}$ be a bounded or unbounded interval, and

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\}.$$

A discrete max-product type approximation operator $L_n : CB_+(I) \to CB_+(I)$, has a general form

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) \cdot f(x_i),$$

or

$$L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) \cdot f(x_i)$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(\cdot, x_i) \in CB_+(I)$ and $x_i \in I$, for all $i = \{0,1,2,\cdots\}$. These operators are nonlinear, positive operators and satisfy a a pseudo-linearity condition of the form

$$L_n(\alpha \cdot \ f \vee \beta \cdot \ g)(x) = \alpha \cdot \ L_n(f)(x) \vee \beta \cdot \ L_n(g)(x), \forall \alpha, \beta \in \mathbb{R}_+, f, g : I \to \mathbb{R}_+.$$

In order to give some properties of the operators L_n, we present the following auxiliary Lemma.

Lemma 2.1. ([2]) Let $I \subset \mathbb{R}$ be a bounded or unbounded interval,

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties :

(i) (Monotonicity)

$f, g \in CB_+(I)$ satisfy $f \le g$ then $L_n(f) \le L_n(g)$ for all $n \in \mathbb{N}$;

(ii) (Subadditivity)

$L_n(f + g) \le L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.

Then for all $f, g \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|L_n(f)(x) - L_n(g)(x)| \le L_n(|f - g|)(x).$$

Remark 2.1. Max-product for Jain operators defined by (4) verify the conditions in Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x), f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \le g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \le L_n(g)(x)$.

Furthermore, the Jain operators of max-product type is positive homogenous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \ge 0$.

Corollary 2.2. ([2]) Let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the conditions (i)-(ii) in Lemma 1 and in addition be a positive homogenous operator. Then for all $f \in CB_+(I)$,$n \in \mathbb{N}$ and $x \in I$ we have

$$|f(x) - L_n(f)(x)| \le \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega(f;\delta) + f(x) \cdot \ |L_n(e_0)(x) - 1| \,,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$ for all $t \in I, x \in I$.

$$\omega(f;\delta) = \max_{\substack{x,y \in I \\ |x-y| \le \delta}} |f(x) - f(y)|$$

is the first modulus of continuity. If I is unbounded then we suppose that there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \bigcup \{+\infty\}$, for any $x \in I, n \in \mathbb{N}$.

Corollary 2.3. ([2]) Suppose that in addition to the conditions in Corollary 2.2, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$|f(x) - L_n(f)(x)| \le \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega(f;\delta).$$

3. Construction of the Operators and Auxiliary Results

Since $T^{(M)}_{n,\beta}(f)(0) - f(0) = 0$ for all n, throughout the paper we may suppose that $x > 0$. We need the following notations and Lemmas for the proof the main results.

For each $k, j \in \{1, 2, ..., \}$ and $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$,$j = 0, x \in \left[0, \dfrac{a+\beta}{n}\right] = \left[0, \dfrac{e^\beta}{n}\right]$, $a = e^\beta - \beta, 0 \le \beta < 1$, let us denote

$$M_{k,n,j}(x) := \frac{W_{n,k,\beta}\left(x\right)\left|\frac{k}{n} - x\right|}{W_{n,j,\beta}\left(x\right)}, m_{k,n,j}(x) := \frac{W_{n,k,\beta}\left(x\right)}{W_{n,j,\beta}\left(x\right)}.$$

where $W_{n,k,\beta}$ is defined as in the operators (1.1). It is clear that if $k \ge j+1$ then

$$M_{k,n,j}(x) = \frac{W_{n,k,\beta}(x)\left(\frac{k}{n} - x\right)}{W_{n,j,\beta}(x)}$$

and if $k \leq j$ then

$$M_{k,n,j}(x) = \frac{W_{n,k,\beta}(x)\left(x - \frac{k}{n}\right)}{W_{n,j,\beta}(x)}.$$

Lemma 3.1. Denoting $W_{n,k,\beta}(x) = (nx + k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}$, we have

$$\bigvee_{k=0}^{\infty} W_{n,k,\beta}(x) = W_{n,j,\beta}(x), \text{ for all } x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right],$$

where $a = e^{\beta} - \beta,\ j = 1, 2, ..., x \in \left[0, \frac{a+\beta}{n}\right] = \left[0, \frac{e^{\beta}}{n}\right]$.

Proof. Firstly, we show that for fixed $n \in \mathbb{N}$ and $0 \leq k$ we have

$$0 \leq W_{n,k+1,\beta}(x) \leq W_{n,k,\beta}(x) \text{ if and only if } x \in \left[0, \frac{a(k+1)+\beta}{n}\right].$$

Indeed, writing the the above inequality explicitly, we have

$$0 \leq (nx + (k+1)\beta)^{k}\ \frac{e^{-(nx+(k+1)\beta)}}{(k+1)!} \leq (nx + k\beta)^{k-1}\ \frac{e^{-(nx+k\beta)}}{k!}.$$

If $x = 0$, this inequality is true. For $x > 0$, after simplifications it becomes

$$\begin{aligned}
\left(\frac{nx + (k+1)\beta}{nx + k\beta}\right)^{k} &\leq \frac{e^{\beta}(k+1)}{nx + k\beta} \\
(nx + k\beta)\left(\frac{nx + (k+1)\beta}{nx + k\beta}\right)^{k} &\leq e^{\beta}(k+1) \\
(nx + k\beta)\left(1 + \frac{\beta}{nx + k\beta}\right)^{k} &\leq e^{\beta}(k+1) \\
nx &\leq \frac{1}{\left(1 + \frac{\beta}{nx+k\beta}\right)^{k}} e^{\beta}(k+1) - k\beta \\
x &\leq \frac{e^{\beta}(k+1)}{n} - \frac{k\beta}{n} \\
&= \frac{e^{\beta}(k+1) - k\beta}{n} = \frac{\left(e^{\beta} - \beta\right)(k+1) + \beta}{n} \\
&= \frac{a(k+1)+\beta}{n},
\end{aligned}$$

where $a = \left(e^{\beta} - \beta\right),\ 0 \leq \beta < 1$. Then

$$0 \leq x \leq \frac{a(k+1)+\beta}{n}, a = e^{\beta} - \beta.$$

By taking $k = 0, 1, 2, ...$ in the inequality just proved above, we get

$$\begin{aligned}
W_{n,1,\beta}(x) &\leq W_{n,0,\beta}(x), \text{ if and only if } x \in \left[0, \frac{a+\beta}{n}\right], \\
W_{n,2,\beta}(x) &\leq W_{n,1,\beta}(x), \text{ if and only if } x \in \left[0, \frac{2a+\beta}{n}\right], \\
&\vdots \\
W_{n,k+1,\beta}(x) &\leq W_{n,k,\beta}(x), \text{ if and only if } x \in \left[0, \frac{a(k+1)+\beta}{n}\right].
\end{aligned}$$

From the above inequalities, we obtain,

$$\begin{aligned}
\text{if } x &\in \left[0, \frac{a+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x) \le W_{n,0,\beta}(x), \text{ for all } k = 0,1,... \\
\text{if } x &\in \left[\frac{a+\beta}{n}, \frac{2a+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x) \le W_{n,1,\beta}(x), \text{ for all } k = 0,1,... \\
\text{if } x &\in \left[\frac{2a+\beta}{n}, \frac{3a+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x) \le W_{n,2,\beta}(x), \text{ for all } k = 0,1,...
\end{aligned}$$

and proceeding in the same manner,

$$\text{if } x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right] \text{ then } W_{n,k,\beta}(x) \le W_{n,j,\beta}(x), \text{for all } \ k = 0,1,2,...$$

then we have

$$0 \le W_{n,k+1,\beta}(x) \le W_{n,k,\beta}(x) \text{ if and only if } x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right].$$

Lemma 3.2. For all $k, j \in \{1,2,...,\}$, and $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$,$j=0, x \in \left[0, \frac{a+\beta}{n}\right] = \left[0, \frac{e^{\beta}}{n}\right]$, we have

$$m_{k,n,j}(x) \le 1.$$

Proof. We have two cases: 1) $k \ge j$ and 2) $k < j$.

Let $k \ge j$. Since the function $g(x) = \dfrac{1}{x}$ is nonincreasing on $\left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$ it follows

$$\begin{aligned}
\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} &= \frac{W_{n,k,\beta}(x)}{W_{n,k+1,\beta}(x)} = \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k+1)\beta)^{k}\dfrac{e^{-(nx+(k+1)\beta)}}{(k+1)!}} \\
&= \frac{(nx+k\beta)^{k} e^{\beta}(k+1)}{(nx+(k+1)\beta)^{k}(nx+k\beta)}, x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right] \\
&\ge 1,
\end{aligned}$$

which implies

$$m_{j,n,j}(x) \ge m_{j+1,n,j}(x) \ge m_{j+2,n,j}(x) \ge ...$$

We now turn to the case $k \le j$

$$\begin{aligned}
\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} &= \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k-1)\beta)^{k-2}\dfrac{e^{-(nx+(k-1)\beta)}}{(k-1)!}} \\
&= \frac{(nx+k\beta)^{k-2}}{(nx+(k-1)\beta)^{k-2}}, x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right] \\
&\ge 1, \\
\text{where } \frac{(nx+k\beta)^{k-2}}{(nx+(k-1)\beta)^{k-2}} &= \left(1+\frac{\beta}{nx+(k-1)\beta}\right)^{k-2} \ge 1 \text{ and } \frac{(nx+k\beta)}{e^{\beta}(k-1)} \ge 1.
\end{aligned}$$

which implies

$$m_{j,n,j}(x) \ge m_{j-1,n,j}(x) \ge m_{j-2,n,j}(x) \ge ...$$

Since $m_{j,n,j}(x) = 1$, the proof of the lemma is complete.

Lemma 3.3. Let $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$,

(i) If $k \ge (j+1)$ is such that

$$k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta k j} \ge aj,$$

then

$$M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$$

where $a_1 = -\beta^2 + 2e^\beta + 2\beta - 1, a_2 = -2a\beta - 2a - ae^\beta, a_3 = -\beta^2 + 2e^\beta + \beta - \beta e^\beta$.

(ii) If $k \leq j$ is such that

$$k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} \leq aj,$$

then

$$M_{k,n,j}(x) \geq M_{k-1,n,j}(x).$$

where $a_4 = 2\beta - \beta^2 + a + 1, a_5 = -2\beta a$.

Proof. (i) We observe that

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} &= \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k+1)\beta)^k\dfrac{e^{-(nx+(k+1)\beta)}}{(k+1)!}}\frac{\left(\frac{k}{n}-x\right)}{\left(\frac{k+1}{n}-x\right)} \\
&= \left(1-\frac{\beta}{nx+(k+1)\beta}\right)^{k-1}\frac{e^\beta(k+1)}{nx+(k+1)\beta}\frac{(k-nx)}{(k+1-nx)} \\
&\geq \frac{(k+1)}{nx+(k+1)\beta}\frac{(k-nx)}{(k+1-nx)}\left(1-\frac{\beta}{nx+(k+1)\beta}\right)^{k-1}e^\beta \\
&\geq \frac{(k+1)}{(j+1)a+(k+1)\beta}\frac{(k-(j+1)a)}{(k+1-ja)},
\end{aligned}$$

$x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$. Then, since the condition

$$k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj} \geq aj,$$

where $a_1 = -\beta^2 + 2e^\beta + 2\beta - 1, a_2 = -2a\beta - 2a - ae^\beta, a_3 = -\beta^2 + 2e^\beta + \beta - \beta e^\beta$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} \geq 1.$$

(ii) We observe that

$$\begin{aligned}
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} &= \frac{(nx+k\beta)^{k-1}\dfrac{e^{-(nx+k\beta)}}{k!}}{(nx+(k-1)\beta)^k\dfrac{e^{-(nx+(k-1)\beta)}}{(k-1)!}}\frac{\left(x-\frac{k}{n}\right)}{\left(x-\frac{k-1}{n}\right)} \\
&= \left(1+\frac{\beta}{nx+(k-1)\beta}\right)^{k}\frac{nx+k\beta}{e^\beta k}\frac{(nx-k)}{(nx-k+1)} \\
&\geq \frac{ja+\beta+k\beta}{k}\frac{ja+\beta-k}{ja+\beta-k+1}
\end{aligned}$$

Then, since the condition

$$k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} \leq aj,$$

where $a_4 = 2\beta - \beta^2 + a + 1, a_5 = -2\beta a$, we obtain

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} \geq 1,$$

which proves the lemma.

4. Approximation Result

For the function $f \in CB_+(I)$, we obtain the degree of approximation by using the Shisha-Mond Theorem given in [1],[2].

Theorem 4.1. If $f : [0,\lambda] \to \mathbb{R}_+$ is a bounded and continuous function on $[0,\lambda]$, $\lambda > a+1, a = e^\beta - \beta, 0 \leq \beta < 1$, then we get the following estimate

$$\left|T_{n,\beta}^{(M)}(f)(x) - f(x)\right| \leq 6\lambda\omega_1\left(f, \frac{1}{\sqrt{n}}\right), \text{for all } n \in \mathbb{N}, x \in [0,\lambda],$$

where

$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,\lambda], |x-y| \leq \delta\}.$$

Proof. Since $T_n^{(M)}(e_0)(x) = 1$ and using the Shisha-Mond Theorem, we have

$$\left|T_n^{(M)}(f)(x) - f(x)\right| \leq \left(1 + \frac{1}{\delta_n} T_n^{(M)}\left(\varphi_x\right)(x)\right)\omega_1(f,\delta_n)$$

where $\left(\varphi_x\right)(t) = |t-x|$. Hence, it is sufficient to estimate the following term

$$E_n(x) := T_n^{(M)}\left(\varphi_x\right)(x) = \frac{\bigvee\limits_{k=0}^{\infty} W_{n,k,\beta}(x)\left|\frac{k}{n} - x\right|}{\bigvee\limits_{k=0}^{\infty} W_{n,k,\beta}(x)}$$

Let $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$ and $j \in \{1,2,...,\}$ is arbitrarily fixed. By Lemma 3.1 we get

$$E_n(x) = \max_{k=0,1,2,\ldots}\left\{M_{k,n,j}(x)\right\}, x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right].$$

For $j = 0$, we get

$$M_{k,n,0}(x) = nx(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{k!}\left|\frac{k}{n} - x\right|, k \geq 0$$

If $k = 0$, then we have

$$M_{0,n,0}(x) = x = \sqrt{x}\sqrt{x} \leq \sqrt{x}\sqrt{\frac{a+\beta}{n}} = \sqrt{\frac{e^\beta x}{n}} \leq \sqrt{\frac{e^\beta\lambda}{n}}$$

If $k = 1$ then

$$\begin{aligned} M_{k,n,0}(x) &= nx(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{k!}\left|\frac{k}{n} - x\right|, x \in \left[0, \frac{e^\beta}{n}\right] \\ &= nx(nx+\beta)^0\frac{e^{-\beta}}{1!}\left|\frac{1}{n} - x\right| \\ &\leq xe^{-\beta} = \sqrt{x}\sqrt{x}e^{-\beta} \\ &\leq \sqrt{\frac{xe^\beta}{n}} \leq \sqrt{\frac{e^\beta\lambda}{n}}. \end{aligned}$$

If $k \geq 2$ then

$$\begin{aligned} M_{k,n,0}(x) &= nx(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{k!}\left|\frac{k}{n} - x\right|, x \in \left[0, \frac{e^\beta}{n}\right] \\ &\leq x(nx+k\beta)^{k-1}\frac{e^{-k\beta}}{(k-1)!} \\ &\leq x \\ &\leq \sqrt{\frac{e^\beta\lambda}{n}}. \end{aligned}$$

So, we obtain an upper estimate for each $M_{k,n,j}(x)$ where $j \in \{1, 2, ..., \}$ is fixed, $x \in \left[\frac{aj+\beta}{n}, \frac{a(j+1)+\beta}{n}\right]$ and $k = 1, ..., .$ Actually, we will prove that

$$M_{k,n,j}(x) \leq \max\left\{\frac{\sqrt{\max\{a_4, a_5\}} + 2a}{\sqrt{n}}, \sqrt{\frac{e^{\beta}\lambda}{n}}, \frac{\sqrt{\max\{a_1, a_2\}}}{\sqrt{n}}\right\},$$

for all $x \in [0, \lambda], n \in \mathbb{N}$.

The proof of the inequality (2) will be investigated by the following cases:

1) $k \geq (j+1)$ and 2)$k \leq j$.

Case 1) Subcase a) Initially, let take

$$k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj} < aj,$$

then we get

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)\left(\frac{k}{n} - x\right) \\ &\leq \left(\frac{k}{n} - x\right) \leq \left(\frac{k}{n} - \frac{ja+\beta}{n}\right) \\ &\leq \frac{k}{n} - \frac{k}{n} + \frac{\sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj}}{n} \\ &\leq \frac{\sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj}}{n} \\ &\leq \frac{\sqrt{a_1 + a_2 j}}{n} \leq \sqrt{\max\{a_1, a_2\}}\frac{1}{\sqrt{n}}. \end{aligned}$$

Subcase b) Now let $k - \sqrt{\beta k^2 + a_1 k + a_2 j + a_3 - a\beta kj} \geq aj$.

Since the function $g(x) = x - \sqrt{\beta x^2 + a_1 x + a_2 j + a_3 - a\beta xj}$ is nondecreasing, it follows that there exists $\overline{k} \in \{2, 3, ..., \}$,of maximum value, such that $\overline{k} - \sqrt{\beta \overline{k}^2 + a_1 \overline{k} + a_2 j + a_3 - a\beta \overline{k} j} < aj$. Then for $k_1 = \overline{k} + a$ we get $k_1 - \sqrt{\beta k_1^2 + a_1 k_1 + a_2 j + a_3 - a\beta k_1 j} \geq aj$,

$$\begin{aligned} M_{\overline{k}+a,n,j}(x) &= m_{\overline{k}+a,n,j}(x)\left|\frac{\overline{k}+a}{n} - x\right| \\ &\leq \left(\frac{\overline{k}+a}{n} - \frac{\overline{k} - \sqrt{\beta \overline{k}^2 + a_1 \overline{k} + a_2 j + a_3 - a\beta \overline{k} j}}{n}\right) \\ &\leq \sqrt{\max\{a_1, a_2\}}\frac{1}{\sqrt{n}}. \end{aligned}$$

The last above inequality follows from the fact that

$\overline{k} - \sqrt{\beta \overline{k}^2 + a_1 \overline{k} + a_2 j + a_3 - a\beta \overline{k} j} < aj$ necessarily implies $k < 3aj$. Also, we have $k_1 \geq (j+1)$. Indeed, this is a consequence of the fact that g is nondecreasing and because is easy to see that $g(j) < j$. By Lemma 3.3, (i) it follows that $M_{\overline{k}+1,n,j}(x) \geq M_{\overline{k}+2,n,j}(x) \geq ...$

Hence, we get $M_{k,n,j}(x) \leq \sqrt{\max\{a_1, a_2\}}\frac{1}{\sqrt{n}}$ for any $\overline{k} \in \{\overline{k}+1, \overline{k}+2, ..., \}$.

Case 2) Subcase a) Firstly, let $k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} > aj$. Then we get,

$$\begin{aligned} M_{k,n,j}(x) &= m_{k,n,j}(x)\left(x - \frac{k}{n}\right) \\ &\leq \frac{a(j+1)+\beta}{n} - \frac{k}{n} \\ &\leq \frac{k + \sqrt{\beta k^2 + a_4 k + a_5 j - \beta^2 - a\beta kj} + \beta}{n} - \frac{k}{n} \\ &\leq \frac{\sqrt{\max\{a_4, a_5\}} + \beta}{\sqrt{n}}. \end{aligned}$$

Subcase b) Suppose now that $k+\sqrt{\beta k^2+a_4k+a_5j-\beta^2-a\beta kj}\leq aj$. Let $\widetilde{k}\in\{1,2,...,\}$ be the minimum value such that

$$\widetilde{k}+\sqrt{\beta\widetilde{k}^2+a_4\widetilde{k}+a_5j-\beta^2-a\beta\widetilde{k}j}>aj.$$

Then $k_2=\widetilde{k}-a$ satisfies $k_2+\sqrt{\beta k_2^2+a_4k_2+a_5j-\beta^2-a\beta k_2j}\leq aj$ and

$$\begin{aligned}
M_{\widetilde{k}-a,n,j}(x) &= m_{\widetilde{k}-a,n,j}(x)\left(x-\frac{\widetilde{k}-a}{n}\right)\\
&\leq \frac{a(j+1)+\beta}{n}-\frac{\widetilde{k}-a}{n}\\
&\leq \frac{\widetilde{k}+\sqrt{\beta\widetilde{k}^2+a_4\widetilde{k}+a_5j-\beta^2-a\beta\widetilde{k}j}+a}{n}-\frac{\widetilde{k}-a}{n}\\
&\leq \frac{\sqrt{\max\{a_4,a_5\}}+2a}{\sqrt{n}}.
\end{aligned}$$

For the last inequality we used the obvious relationship $k_2=\widetilde{k}-a$,

$$k_2+\sqrt{\beta k_2^2+a_4k_2+a_5j-\beta^2-a\beta k_2j}\leq aj$$

which implies $\widetilde{k}\leq(j+1)$ and $k_2\leq j$.

By Lemma 3.2, (ii) it follows that

$$M_{\widetilde{k}-a,n,j}(x)\geq M_{\widetilde{k}-2a,n,j}(x)\geq M_{\widetilde{k}-3a,n,j}(x)\geq...\geq M_{0,n,j}(x).$$

We thus obtain $M_{k,n,j}(x)\leq\dfrac{\sqrt{\max\{a_4,a_5\}}+2a}{\sqrt{n}}$for any $k\leq j$ and $x\in\left[\frac{aj+\beta}{n},\frac{a(j+1)+\beta}{n}\right]$.

Collecting all the above estimates we have the proof of case (2). Thus, the proof is completed.

5. Conclusion

In this study, we introduced the nonlinear Jain operators of max-product type. We also estimate the rate of pointwise convergence of these operators.

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Why Flc-Frame is Better than Frenet Frame on Polynomial Space Curves?

Mustafa Dede

Abstract
It is well known that the binormal and normal vectors of Frenet frame rotate around the tangent vector. That is why the Frenet frame is not suitable for some applications such as tube surfaces. However, there is not enough information about why the vectors of the Frenet frame rotate around the tangent vector. In this paper we will deal with this problem. Moreover we show the advantages of Flc-frame over the Frenet frame.

Keywords: *Frenet frame; space curve; adapted frame; polynomial curve.*

AMS Subject Classification (2020): *Primary: 53A04 ; Secondary: 68U05.*

1. Introduction

Recently, the study of the frames along a space curve has arisen some engineering applications [18, 22, 24]. For intance in [19], the authors investigated the Mannheim curves with a new frame called modified orthogonal frame. Despite the fact that Bishop frame (rotation minimizing frame) is more suitable for engineering applications [9], this frame can not be computed analytically. Therefore a number of approximation methods have been proposed for RMF computation [4]. In this paper we will compare the frames which can be computed analytically on polynomial space curves.

The Frenet frame is the most known frame along a space curve [4, 5, 23]. Let $\alpha(t)$ be a regular space curve. The Frenet frame is defined as follows,

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \mathbf{n} = \mathbf{b} \times \mathbf{t}. \tag{1.1}$$

The well-known Frenet formulas are given by,

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \|\alpha'(t)\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{1.2}$$

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where the curvature κ and the torsion τ of the curve are given by

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}. \tag{1.3}$$

The Frenet frame has inflection points and two type of singular points [5, 11].

Definition 1.1. Let $\alpha(t) : I \to \mathbb{R}^3$ be a space curve. A point $t_0 \in I$ is said to be singular point of order 0 of the curve if $\alpha'(t_0)$ vanishes.

We say that $t_1 \in I$ is a singular point of order 1 if $\alpha''(t_1)$ vanishes.

Definition 1.2. Let $\alpha(t) : I \to \mathbb{R}^3$ be a space curve. A point $t_2 \in I$ is called inflection point if $\alpha'(t_2) \wedge \alpha''(t_2)$ vanishes, namely curvature is zero [16].

Apart from Frenet frame we can define more frame along a space curve [1, 25]. Recently, Dede [15] introduced a new frame along a polynomial space curve, called as Flc-frame. The computation of Flc-frame is easier than the both Frenet and Bishop frames [1, 2]. Moreover they showed that to have a inflection point on Flc frame is less possible than Frenet frame. Discussion of the Flc-frame and its application to the tube surfaces can be found in [15].

Let $\alpha(t)$ be a polynomial space curve of degree n. The Flc-frame is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{D}_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\|\alpha' \wedge \alpha^{(n)}\|}, \mathbf{D}_2 = \mathbf{D}_1 \wedge \mathbf{t}. \tag{1.4}$$

Where the prime $'$ indicates the differentiation with respect to t [15]. If the order of derivative exceeds three, we replace prime by the superscript (n), such as $\alpha'''' = \alpha^{(4)}$. The new vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ are called as binormal-like vector and normal-like vector, respectively.

The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{D}_2' \\ \mathbf{D}_1' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{D}_2 \\ \mathbf{D}_1 \end{bmatrix}. \tag{1.5}$$

The curvatures of the Flc-frame are given by

$$d_1 = \frac{\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \rangle}{\|\alpha'\|^3 \|\alpha' \wedge \alpha^{(n)}\|}, d_2 = \frac{\det[\alpha'', \alpha', \alpha^{(n)}]}{\|\alpha'\|^2 \|\alpha' \wedge \alpha^{(n)}\|}, \tag{1.6}$$

and

$$d_3 = \frac{\det[\alpha', \alpha'', \alpha^{(n)}] \langle \alpha', \alpha^{(n)} \rangle}{\|\alpha'\|^2 \|\alpha' \wedge \alpha^{(n)}\|^2}. \tag{1.7}$$

Corollary 1.1. *If the degree of polynomial space curve is two, then the Flc-frame coincides with the Frenet frame with curvatures $d_1 = \kappa, d_2 = 0$ and $d_3 = \tau = 0$.*

2. Flc-frame vs Frenet frame

There are three main drawbacks associated with the Frenet frame. In this chapter we discuss the drawbacks of the Frenet frame. Moreover we explain why the Flc-frame is better than Frenet frame from points of these drawbacks. As an application we consider tube surfaces.

- Singular point of order 1.

One of the most important advantages of the Flc-frame over the Frenet frame is that when the second derivative of the curve vanishes the Frenet frame behaves erratically. This is why the rotation minimizing frame (RMF) is widely used in surface modeling such as tube (pipe) surfaces.

Theorem 2.1. *Let $\alpha(t)$ be a polynomial space curve of degree n. The Flc-frame doesn't have singular point of order 1.*

Proof. From point of Definition 1.1, since the n-th derivative of polynomial space curve of degree n never vanishes, the Flc-frame doesn't have singular point of order 1. □

Here's an example about this case.

Example 2.1. Assume that a curve $\alpha(t)$ is given by

$$\alpha(t) = (t, \frac{t^4}{12} - \frac{t^3}{6}, (t-1)^3).$$

It follows that $\alpha''(t) = (0, t^2 - t, 6t - 6)$ therefore the point $t = 1$ is a singular point of order 1. When $t = 1$ the

Figure 1. The normal and binormal vectors of Frenet frame (Left) and the normal-like and binormal-like vectors of Flc-frame (Right) along the curve $t \in (-2, 2)$.

binormal vectors of the Frenet frame suddenly exhibits 180 degree rotation (highlighted by an arrow in Figure 1). The Figure 1 compares the behaviour of the binormal (black) and the normal (red) vectors of the Frenet frame with the binormal-like (black) and the normal-like (red) vectors of the Flc-frame.

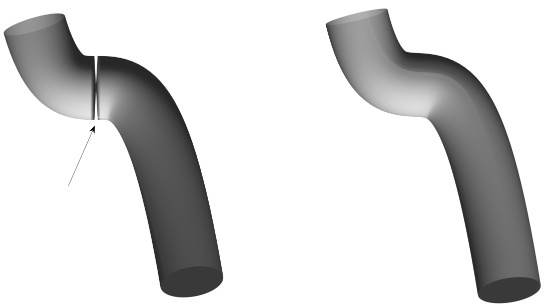


Figure 2. The tube surfaces generated by Frenet frame (Left) and the Flc-frame (Right) $t \in (-2, 2)$, $v \in (-4, 4)$.

As a result, the sudden rotation of normal and binormal vectors of the Frenet frame causes deformation in tube surface. The tube (pipe) surfaces with radius $r = 0.8$ generated by the Frenet frame and Flc-frame are illustrated in Figure 2.

- Inflection points; at the points where the curvature κ is zero, namely $\|\alpha' \wedge \alpha''\| = 0$.

In the case of Flc-frame it corresponds to $\|\alpha' \wedge \alpha^n\| = 0$. Dede [15] showed that to have a inflection point on Flc-frame is less possible than the Frenet frame. However since Flc frame permit analytical computation, it has inflection points when $\|\alpha'(t_2) \wedge \alpha^n(t_2)\| = 0$ at the point t_2.

Example 2.2. In this example, we would like to deal with the inflection points. Let us consider a curve given by

$$\alpha(t) = (t^3, t^3, t^2 - 2t). \tag{2.1}$$

By using the derivatives of the curve, we have

$$\alpha'(t) \wedge \alpha''(t) = (-6t^2 + 12t, 6t^2 - 12t, 0),$$

and

$$\alpha'(t) \wedge \alpha^n(t) = (12 - 12t, 12t - 12, 0).$$

Observe that the Frenet frame has two inflection points at $t = 0$ and $t = 2$ whereas the Flc frame has one at the point $t = 1$.

Note that it is all about the degree of a curve. Since the degree of $\alpha'(t) \wedge \alpha^n(t)$ is less than $\alpha'(t) \wedge \alpha''(t)$, it has fewer possible roots. The Figure 3 compares the behaviour of the vectors of Frenet frame with the Flc-frame. Similar

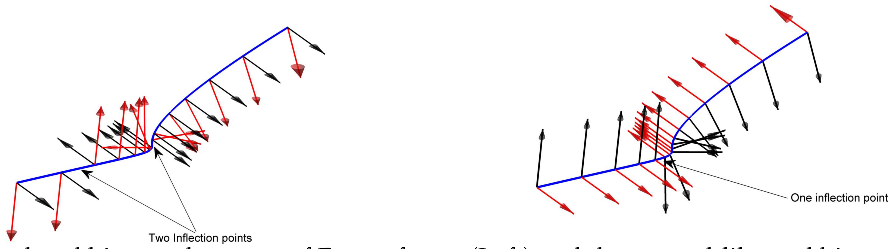


Figure 3. The normal and binormal vectors of Frenet frame (Left) and the normal-like and binormal-like vectors of Flc-frame (Right) along the curve $t \in (-2, 2)$.

to the case of singular point of order 1, the vectors of the Frenet frame suddenly exhibits 180 degree rotation at the inflection points.

Figure 4. The tube surfaces generated by Frenet frame (left) and the Flc-frame (right) $t \in (-2, 2)$, $v \in (-4, 4)$.

The tube (pipe) surfaces with radius $r = 1.9$ generated by the Frenet frame and Flc-frame are illustrated in Figure 4.

The solution of this problem is not hard. The following theorem and algorithm demonstrate a good way to solve this problem.

Theorem 2.2. *Let $\alpha(t)$ be a polynomial space curve of degree 3. The Flc-frame has just one inflection point which never coincidences with the inflection points of the Frenet frame.*

Proof. **:** Three-dimensional cubic polynomial curve is of the form

$$\alpha(t) = (\sum_{i=0}^{3} a_i t^i, \sum_{i=0}^{3} b_i t^i, \sum_{i=0}^{3} c_i t^i),$$

which we represent by its polynomial coefficients, a_i,b_i and c_i.

The inflection point of the Flc-frame is

$$\alpha' \wedge \alpha^n = \begin{cases} 12(b_2c_3 - b_3c_2)t + 6(b_1c_3 - b_3c_1) = 0 \\ 12(-a_2c_3 - a_3c_2)t + 6(a_3c_1 - a_1c_3) = 0 \\ 12(a_2b_3 - a_3b_2)t + 6(a_1b_3 - a_3b_1) = 0 \end{cases}$$

The solution of the above system of equations is obtained as

$$t = \frac{b_3c_1 - b_1c_3}{2b_2c_3 - 2b_3c_2}, c_3 = \frac{a_1b_3c_2 - a_2b_3c_1 - a_3b_1c_2 + a_3b_2c_1}{b_2a_1 - a_2b_1}.$$

□

The nice result is that this point is not the inflection point of the Frenet frame. For matlab program, we can write an easy algorithm to construct tube surface as follows

$$\begin{array}{cccc} & begin & & \\ & for\ i = 1\ to\ j\ do & & \\ if \quad \|\alpha' \wedge \alpha^n\| > 0 & \text{then } Flc - frame & \text{(use Flc-frame for tube)} \\ else & Frenet \text{ frame} & \text{(use Frenet frame for tube)} \\ break & & \text{(to stop to use of Frenet frame)} \\ & end & & \\ & for\ k = i+1\ to\ j\ do & & \\ continue & \text{use } -D_1 \text{ and } -D_2 & \text{(to avoid 180 degree rotation)} \\ & end & & \\ & end & & \end{array}$$

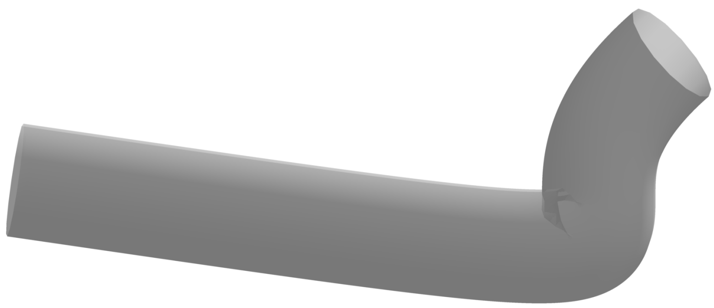


Figure 5. The tube surface generated by the above algorithm $t \in (-2, 2)$, $v \in (-4, 4)$.

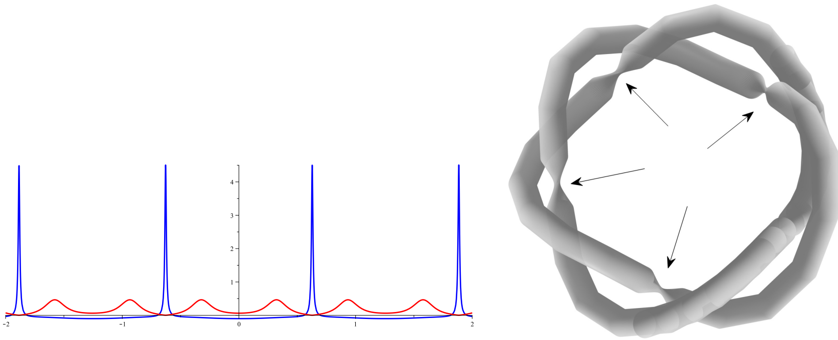
With this algorithm, the tube surface generated by the curve given in Equation 2.1 is shown in Figure 5. The following case is the most interesting. Because currently there is not exact description for this error. Let's begin with the most convenient one.

- At the points where the curvature of curve is small and the absolute value of the torsion is large.

Sometimes, despite the fact that where the Frenet frame doesn't have singular point of order 1 and inflection point, interestingly the normal and binormal vectors still exhibit rotation around the tangent vector, but not 180 degree. There are some instances in the literature to explain why the Frenet frame behaves badly. In this section we focus on this problem, and review some recently published comments that are used to explain the unpredictable behavior of the Frenet frame.

In [8] the authors have tried to explain what causes abnormal behavior of the normal and binormal vectors of the Frenet frame. They realized that the small curvature and large absolute value of torsion produce so much twisting in the tube. The Figure 6 shows that this is a highly convincing explanation. Let us consider a curve given by

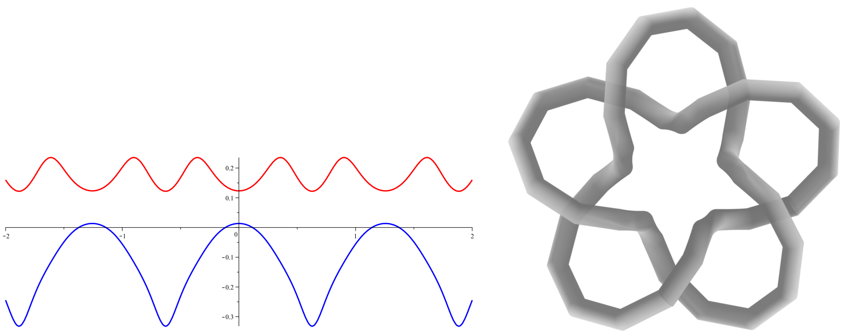
$$\alpha(t) = (8 + \cos(5t)\cos(2t), (8 + \cos(5t))\sin(2t), 5\sin(5t)). \tag{2.2}$$



(a) The curvature (Red) and the torsion (Blue) of the curve $\alpha(t)$ $t \in (-2, 2)$.

(b) The tube surface generated by Frenet frame of the curve $\alpha(t)$ $t \in (-2, 2)$, $v \in (-4, 4)$.

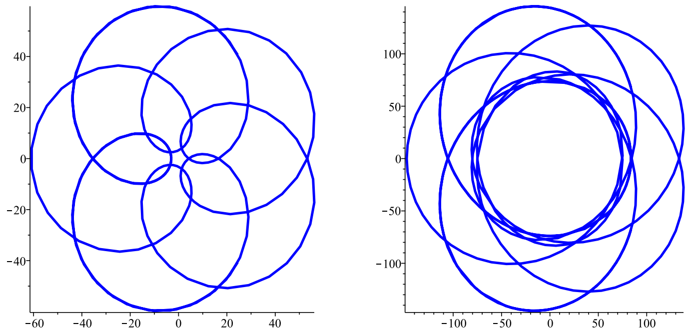
Figure 6.



(a) the curvature (Red) and the torsion (Blue) of the curve $\beta(t)$ $t \in (-2, 2)$.

(b) The tube surface generated by Frenet frame of the curve $\beta(t)$ $t \in (-2, 2)$, $v \in (-4, 4)$.

Figure 7.



(a) The second derivative of the curve $\alpha(t)$.

(b) The second derivative of the curve $\beta(t)$.

Figure 8. The top view from z-axis

In [8] the authors pointed out that at the plot of the tube shows that this increased twisting occurs in four different places where simultaneously the curvature is small and the absolute value of the torsion is large. However the small curvature and large torsion is a relative concept. For instance, let us consider a curve given by

$$\beta(t) = (8 + 3\cos(5t)\cos(2t), (8 + 3\cos(5t))\sin(2t), 5\sin(5t)). \tag{2.3}$$

The Figure 7 shows that despite the fact that the graph of the curvature and torsion is similar, the tube surface generated by the curve $\beta(t)$ doesn't have any deformation on it.

In addition, a different approach has been given for this case in [14]. The author claimed that when the the second derivative of the curve becomes very small, the Frenet frame behaves erratically which causes twisting in the tube.

Now let us plot the graph of the second derivative of the curves given in (2.2) and (2.3). In Figure 8, observe that the curve $\alpha''(t)$ approaches to zero at the four points, $\beta''(t)$ is not. Note that this explanation also shows that why the Flc-frame is better than the Frenet frame? Because we use highest order derivative instead of second order derivatives of the curve to construct the Flc-frame. The following example shows these advantages.

Example 2.3. Let us consider a curve given by

$$\alpha(t) = (t, t^3 - t^2 + 3t, t^3). \tag{2.4}$$

Figure 9. The tube surfaces generated by Frenet frame (left) and the Flc-frame (right) $t \in (-2, 2)$, $v \in (-4, 4)$.

It is easy to see that although this curve doesn't have neither singular point of order 1 nor inflection point, the Figure 9 shows that the tube surface generated by the Frenet frame is deformed.

3. Conclusion

In this paper we investigated three drawbacks of the Frenet frame and compared the Frenet frame with a new frame called as Flc-frame. Moreover, we tried to explain what causes the last drawback of the Frenet frame. Where as the Frenet frame, the Flc-frame has just one drawback, for which we constructed an easy algorithm. As a result, this new frame not only decreases the singular points, but it also decreases the undesirable rotation around the tangent vector of the curve which is a advantage in computer graphics and related fields.

- Whereas the Frenet frame, the Flc-frame does not have singular point of order 1.
- To have a inflection point on Flc frame is less possible than the Frenet frame.
- Whereas the Frenet frame, the normal and binormal vectors of the Flc-frame does not exhibit rotation around the tangent vector.

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Author's contributions

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Domain of Jordan Totient Matrix in the Space of Almost Convergent Sequences

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Abstract
In this paper, the notion of almost convergence is used to obtain a space as the domain of a regular matrix. After defining a new type of core for complex-valued sequences, certain inclusion theorems are proved.

Keywords: *Jordan totient function; regular matrix; almost convergence.*

AMS Subject Classification (2020): *Primary: 46A45 ; Secondary: 40A05; 46A35.*

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1. Introduction and preliminaries

The classical summability theory concerns with the generalization of the concept of convergence for series or sequences by assigning a limit for non-convergent series or sequences. For this purpose, infinite special matrices are used.

One of the fundamental subject of summability is the study of the theory of sequence spaces. By a sequence space, we mean any subspace of ω consisting all sequences with real or complex terms. We use the classical sequence spaces

$$\begin{aligned} c_0 &= \{x = (x_j) \in \omega : \lim_j x_j = 0\}, \\ c &= \{x = (x_j) \in \omega : \lim_j x_j \text{ exists }\}, \\ \ell_\infty &= \{x = (x_j) \in \omega : \sup_j |x_j| < \infty\}, \end{aligned}$$

$$cs = \left\{x = (x_j) \in \omega : \left(\sum_{i=1}^{j} x_i\right) \in c\right\}$$

and

$$bs = \left\{x = (x_j) \in \omega : \left(\sum_{i=1}^{j} x_i\right) \in \ell_\infty\right\}.$$

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In the theory of sequence spaces, the concept of Banach limit has rised as a fascinating application of the famous Hahn–Banach extension theorem. The Banach limit is known as extension of limit functional on c to the space ℓ_∞. This notion has used by Lorentz [1] to introduce a new type of convergence called almost convergence. The spaces f and f_0 of almost convergent and almost convergent to zero are given by

$$f = \left\{ x = (x_j) \in \ell_\infty : \lim_{i\to\infty} \sum_{p=0}^{i} \frac{x_{j+p}}{i+1} = \mathcal{A} \text{ uniformly in } j \right\}$$

and

$$f_0 = \left\{ x = (x_j) \in \ell_\infty : \lim_{i\to\infty} \sum_{p=0}^{i} \frac{x_{j+p}}{i+1} = 0 \text{ uniformly in } j \right\}.$$

A Banach limit $\mathcal{L}$ defined on ℓ_∞ is a non-negative linear functional such that $\mathcal{L}(\mathcal{P}x) = \mathcal{L}x$ and $\mathcal{L}(e) = 1$, where $\mathcal{P} : \omega \longrightarrow \omega$, $\mathcal{P}_j(x) = x_{j+1}$ is the shift operator. A sequence $x = (x_j)$ is said to be almost convergent to the generalized limit $\mathcal{A}$ if all Banach limits of x are coincide and are equal to $\mathcal{A}$. It is denoted by $f - \lim x_j = \mathcal{A}$. If $\mathcal{P}^p$ is the p-times composition of $\mathcal{P}$ with itself, we use the notation

$$a_{ij}(x) = \frac{1}{i+1} \sum_{p=0}^{i} (\mathcal{P}^p x)_j \text{ for all } i, j \in \mathbb{N}.$$

It is proved by Lorentz [1] that $f - \lim x_j = \mathcal{A}$ if and only if $\lim_{i\to\infty} a_{ij}(x) = \mathcal{A}$ uniformly in j. It is a known fact that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. See the papers [2–14] for more on almost convergence and Banach limit.

Given any sequence spaces $\mathcal{X}$ and $\mathcal{Y}$, an infinite matrix $S = (s_{ij})$ is considered as a matrix mapping from $\mathcal{X}$ into $\mathcal{Y}$ if the sequence $Sx = \{(Sx)_i\} = \left(\sum_j s_{ij} x_j\right) \in \mathcal{Y}$ for every $x = (x_j) \in \mathcal{X}$. By $(\mathcal{X} : \mathcal{Y})$, we denote the class of all such matrices. It is said that S regularly maps $\mathcal{X}$ into $\mathcal{Y}$ if $S \in (\mathcal{X} : \mathcal{Y})$ and $\lim_j (Sx)_j = \lim_j x_j$ for all $x \in \mathcal{X}$. This is denoted by $S \in (\mathcal{X} : \mathcal{Y})_{reg}$.

By f_S, we mean the domain of an infinite matrix S in the space f; that is

$$f_S = \{x = (x_j) \in \omega : Sx \in f\}.$$

For more on matrix domains and new sequence spaces, see [15–25]

Let $x = (x_j) \in \omega$ and C_j be the least convex closed region in complex plane containing $x_j, x_{j+1}, x_{j+2}, \ldots$ for each $j \in \mathbb{N} = \{1, 2, ...\}$. The Knopp Core or $\mathcal{K} - core$ of $x = (x_j)$ is defined as the intersection of all C_j ([26]). If $x \in \ell_\infty$, we have that

$$\mathcal{K} - core(x) = \bigcap_{z\in\mathbb{C}} \left\{ \tilde{z} \in \mathbb{C} : |\tilde{z} - z| \leq \limsup_j |x_j - z| \right\}$$

([27]).

Knopp Core Theorem [26, p. 138] states that $\mathcal{K} - core(Sx) \subseteq \mathcal{K} - core(x)$ for all real valued sequences x and a positive matrix $S \in (c : c)_{reg}$.

Statistical convergence is another generalization of usual convergence. It is defined by the aid of natural density of a subset in $\mathbb{N}$. The natural density of a set N is

$$\delta(N) = \lim_j \frac{1}{j} |\{i \leq j : i \in N\}|$$

provided that the limit exists. Here $||$ gives the cardinality of the set written inside it. It is said that a sequence $x = (x_j)$ is statistically convergent to $\mathcal{D}$ if for every $\varepsilon > 0$ the natural density of the set

$$\{j \in \mathbb{N} : |x_j - \mathcal{D}| \geq \varepsilon\}$$

equals zero. It is denoted by $st - \lim x = \mathcal{D}$ ([28]). By st_0 and st, the spaces of all statistically null and statistically convergent sequences are denoted.

The notion of the statistical core or $st - core$ of a statistically bounded sequence x is defined by Fridy and Orhan [29] as

$$st - core(x) = \bigcap_{z\in\mathbb{C}} \left\{ \tilde{z} \in \mathbb{C} : |\tilde{z} - z| \leq st - \limsup_j |x_j - z| \right\}.$$

For some papers on core theorems, see [30–34].

The Jordan's function $J_r : \mathbb{N} \to \mathbb{N}$ of order r is an arithmetic function, where r is a positive integer. The value $J_r(n)$ equals to the number of r-tuples of positive integers all less than or equal to n that form a coprime $(r+1)$-tuples together with n.

In a recent paper, İlkhan et al. [35] define a new matrix $\Upsilon^r = (\upsilon^r_{nk})$ as

$$\upsilon^r_{nk} = \begin{cases} \frac{J_r(k)}{n^r} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

for each $r \in \mathbb{N}$. It is also observed that this special transformation is regular; that is a limit preserving mapping c into c.

The inverse $(\Upsilon^r)^{-1} = ((\upsilon^r_{nk})^{-1})$ is computed as

$$(\upsilon^r_{nk})^{-1} = \begin{cases} \frac{\mu(\frac{n}{k})}{J_r(n)} k^r & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

Here and what follows μ is the Mobius function. By using usual matrix product, the Υ^r-transform of a sequence $x = (x_j) \in \omega$ is the sequence

$$y = \Upsilon^r x = ((\Upsilon^r x)_j) = \left(\frac{1}{j^r} \sum_{d|j} J_r(d) x_d \right).$$

In this study, it is aimed to introduce and study on a new sequence space $\widehat{f}(\Upsilon^r)$ as the domain of Υ^r in the space f. Further, *Jordan Totient Core* (Υ^r-core) of a sequence is defined and characterization of matrices satisfying $\Upsilon^r - core(Sx) \subseteq \mathcal{K} - core(x)$ and $\Upsilon^r - core(Sx) \subseteq st - core(x)$ with $x \in \ell_\infty$ are given.

2. Domain of Υ^r in the space f and Jordan Totient Core

In this section, we introduce the space $\widehat{f}(\Upsilon^r)$ consisting of all sequences whose Υ^r-transforms are in f. That is,

$$\widehat{f}(\Upsilon^r) = \left\{ x = (x_j) \in \ell_\infty : \lim_{i\to\infty} \sum_{p=0}^{i} \frac{(\Upsilon^r x)_{j+p}}{i+1} = \mathcal{A} \text{ uniformly in } j \right\}.$$

One can prove that the spaces $\widehat{f}(\Upsilon^r)$ and f are linearly isomorphic.

The β-dual of a space $\mathcal{X}$ consists of all sequences $a = (a_j) \in \omega$ such that $xa = (x_j a_j) \in cs$ for all $x = (x_j) \in \mathcal{X}$. In order to determine the $\beta-$dual of the space $\widehat{f}(\Upsilon^r)$, we need the following result.

Lemma 2.1. *[36] $S = (s_{ij}) \in (f : c)$ if and only if*

$$\sup_{i\in\mathbb{N}} \sum_j |s_{ij}| < \infty, \tag{2.1}$$

$$\lim_{i\to\infty} s_{ij} = s_j \in \mathbb{C} \text{ for each } j \in \mathbb{N}, \tag{2.2}$$

$$\lim_{i\to\infty} \sum_j s_{ij} = s \in \mathbb{C}, \tag{2.3}$$

$$\lim_{i\to\infty} \sum_j \left|\Delta(s_{ij} - s_j)\right| = 0. \tag{2.4}$$

Theorem 2.1. *The $\beta-$dual of the sequence space $\widehat{f}(\Upsilon^r)$ is the intersection of the following sets*

$$\begin{aligned}
\mathfrak{B}_1 &= \Big\{t=(t_j)\in\omega : \sup_{i\in\mathbb{N}}\sum_{j=1}^{i}\Big|\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}jt_d\Big|<\infty\Big\},\\
\mathfrak{B}_2 &= \Big\{t=(t_j)\in\omega : \lim_{i\to\infty}\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d \text{ exists}\Big\},\\
\mathfrak{B}_3 &= \Big\{t=(t_j)\in\omega : \lim_{i\to\infty}\sum_{j=1}^{i}\Big[\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d\Big] \text{ exists}\Big\},\\
\mathfrak{B}_4 &= \Big\{t=(t_j)\in\omega : \lim_{i\to\infty}\sum_{j}\Big|\Delta\Big[\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d-\alpha_j\Big]\Big|=0\Big\}.
\end{aligned}$$

Proof. Given any $t=(t_j)\in\omega$, the equality

$$\begin{aligned}
\sum_{j=1}^{i}t_jx_j &= \sum_{j=1}^{i}t_j\Big(\sum_{d|j}\frac{\mu(\frac{j}{d})}{J_r(j)}d^r y_d\Big)\\
&= \sum_{j=1}^{i}\Big(\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d\Big)y_j\\
&= B_i(y); \qquad (i\in\mathbb{N}) \qquad (2.5)
\end{aligned}$$

holds, where the matrix $B=(b_{ji})$ is defined by

$$b_{ji}=\begin{cases}\displaystyle\sum_{d=j,j|d}^{i}\frac{\mu(\frac{d}{j})}{J_r(d)}j^r t_d & 1\le j\le i,\\ 0 & \text{otherwise}\end{cases} \qquad (2.6)$$

for all $j,i\in\mathbb{N}$. It follows from (2.5) that $tx=(t_jx_j)\in cs$ whenever $x=(x_j)\in c$ if and only if $By\in c$ whenever $y=(y_j)\in f$. That is, $t=(t_j)\in\{\widehat{f}(\Upsilon^r)\}^\beta$ if and only if $B\in(f:c)$. Hence the result is obtained by using Lemma 2.1. □

Now, we define Jordan totient core or Υ^r-core of a complex valued sequence.

Definition 2.1. Let C_j be the least closed convex hull containing $(\Upsilon^r x)_j,(\Upsilon^r x)_{j+1},\ldots$. Then, Υ^r-core of x is the intersection of all C_j, i.e.,

$$\Upsilon^r-core(x)=\bigcap_{j=1}^{\infty}C_j.$$

The following result is immediate since the Υ^r-core of x is the $\mathcal{K}-core$ of the sequence $\Upsilon^r x$.

Theorem 2.2. *For any $x\in\ell_\infty$, we have*

$$\Upsilon^r-core(x)=\bigcap_{z\in\mathbb{C}}\Big\{\tilde{z}\in\mathbb{C}:|\tilde{z}-z|\le\limsup_j|(\Upsilon^r x)_j-z|\Big\}.$$

Recently, İlkhan et al. [37] introduced the following spaces by the aid of Jordan totient function.

$$c_0(\Upsilon^r)=\left\{x=(x_j)\in\omega:\lim_j\left(\frac{1}{j^r}\sum_{d|j}J_r(d)x_d\right)=0\right\}$$

and

$$c(\Upsilon^r)=\left\{x=(x_j)\in\omega:\lim_j\left(\frac{1}{j^r}\sum_{d|j}J_r(d)x_d\right)\text{ exists}\right\}.$$

In order to give the necessary and sufficient conditions for an infinite matrix $S = (s_{ij})$ be in the classes $(c : c(\Upsilon^r))_{reg}$ and $(st(S) \cap \ell_\infty : c(\Upsilon^r))_{reg}$, we firstly have some auxiliary results.

Lemma 2.2. *$S = (s_{ij}) \in (\ell_\infty : c(\Upsilon^r))$ if and only if*

$$\sup_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| < \infty, \tag{2.7}$$

$$\lim_i \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} = \gamma_j \quad \text{for each } j, \tag{2.8}$$

$$\lim_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} - \gamma_j \right| = 0. \tag{2.9}$$

Lemma 2.3. *$S = (s_{ij}) \in (c : c(\Upsilon^r))_{reg}$ if and only if (2.7) and (2.8) hold with $\gamma_j = 0$ for each j and*

$$\lim_i \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} = 1. \tag{2.10}$$

Lemma 2.4. *$S = (s_{ij}) \in (st \cap \ell_\infty : c(\Upsilon^r))_{reg}$ if and only if $S \in (c : c(\Upsilon^r))_{reg}$ and*

$$\lim_i \sum_{j \in N, \delta(N)=0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| = 0. \tag{2.11}$$

Proof. It is a known fact that $c \subset st \cap \ell_\infty$ holds. So we have $S \in (c : c(\Upsilon^r))_{reg}$. Now let $\delta(N) = 0$ and $x \in \ell_\infty$. Define a sequence $\tilde{x} = (\tilde{x}_j)$ as $\tilde{x}_j = x_j$ if $j \in N$ and $\tilde{x}_j = 0$ otherwise. Clearly $\tilde{x} \in st_0$. Hence we have $S\tilde{x} \in c_0(\Upsilon^r)$. Further the equality

$$\sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \tilde{x}_j = \sum_{j \in N} \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} x_j$$

yields that $\hat{S} = (\hat{s}_{ij}) \in (\ell_\infty : c(\Upsilon^r))$, where

$$\hat{s}_{ij} = \begin{cases} \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} & , \quad \text{if } j \in N \\ 0 & , \quad \text{if } j \notin N. \end{cases}$$

Thus we deduce (2.11) from Lemma 2.2.

Conversely, choose a sequence $x \in st \cap \ell_\infty$ with $st - \lim x = \mathcal{D}$. Given any $\varepsilon > 0$, we have $\delta(N) = \delta(\{j : |x_j - \mathcal{D}| \geq \varepsilon\}) = 0$. By letting $i \to \infty$ in the following equality

$$\sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} x_j = \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (x_j - \mathcal{D}) + \mathcal{D} \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij}, \tag{2.12}$$

the inequality

$$\left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (x_j - \mathcal{D}) \right| \leq \|x\| \sum_{j \in N} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| + \varepsilon \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right|,$$

and (2.10) with (2.11) yield that

$$\lim_i \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} x_j = \mathcal{D}.$$

This means that $S \in (st \cap \ell_\infty : c(\Upsilon^r))_{reg}$. $\square$

Lemma 2.5. *[30] Let $S = (s_{ij})$ be a matrix satisfying the conditions $\sum_j |s_{ij}| < \infty$ and $\lim_i s_{ij} = 0$. Then we have*

$$\limsup_i \sum_j s_{ij}x_j = \limsup_i \sum_j |s_{ij}|$$

for some $x \in \ell_\infty$ with $\|x\| \le 1$.

Now, we are ready to give our main theorems.

Theorem 2.3. *Let $S \in (c, c(\Upsilon^r))_{reg}$ and $x \in \ell_\infty$. The inclusion $\Upsilon^r - core(Sx) \subseteq \mathcal{K} - core(x)$ holds if and only if*

$$\lim_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| = 1. \tag{2.13}$$

Proof. By combining Lemma 2.3 and Lemma 2.5 we obtain the equality

$$\left\{ \tilde{w} \in \mathbb{C} : |\tilde{w}| \le \limsup_i \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}x_j \right\} = \left\{ \tilde{w} \in \mathbb{C} : |\tilde{w}| \le \limsup_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| \right\}$$

for some $x = (x_j) \in \ell_\infty$ with $\|x\| \le 1$. Since the inclusions

$$\Upsilon^r - core(Sx) \subseteq \mathcal{K} - core(x) \subseteq \{\tilde{w} \in \mathbb{C} : |\tilde{w}| \le 1\}$$

hold, (2.13) follows from the inclusion

$$\left\{ \tilde{w} \in \mathbb{C} : |\tilde{w}| \le \limsup_i \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| \right\} \subseteq \{\tilde{w} \in \mathbb{C} : |\tilde{w}| \le 1\}.$$

Now, let $\tilde{w} \in \Upsilon^r - core(Sx)$. We have

$$\begin{aligned} |\tilde{w} - w| &\le \limsup_i |(\Upsilon^r(Sx))_i - w| \\ &= \limsup_i \left| w - \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}x_j \right| \\ &\le \limsup_i \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}(w - x_j) \right| + \limsup_i |w| \left| 1 - \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| \\ &= \limsup_i \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}(w - x_j) \right| \end{aligned} \tag{2.14}$$

for any $w \in \mathbb{C}$. Put $\limsup_j |x_j - w| = l$. Given any $\varepsilon > 0$ there exists j_0 such that $|x_j - w| \le l + \varepsilon$ for $j \ge j_0$. Hence, it follows that

$$\begin{aligned} \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}(w - x_j) \right| &= \left| \sum_{j<j_0} \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}(w - x_j) + \sum_{j \ge j_0} \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij}(w - x_j) \right| \\ &\le \sup_j |w - x_j| \sum_{j<j_0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| + (l + \varepsilon) \sum_{j \ge j_0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| \\ &\le \sup_j |w - x_j| \sum_{j<j_0} \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right| + (l + \varepsilon) \sum_j \left| \frac{1}{i^r} \sum_{j|i} J_r(j)s_{ij} \right|. \end{aligned} \tag{2.15}$$

Hence (2.14) and (2.15) yield that

$$|\tilde{w} - w| \leq \limsup_i \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| \leq l + \varepsilon.$$

This implies that $\tilde{w} \in \mathcal{K} - core(x)$. Hence the desired inclusion holds. □

Theorem 2.4. *Let $S \in (st \cap \ell_\infty : c(\Upsilon^r))_{reg}$ and $x \in \ell_\infty$. The inclusion $\Upsilon^r - core(Sx) \subseteq st - core(x)$ holds if and only if (2.13) holds.*

Proof. Since $st - core(x) \subseteq \mathcal{K} - core(x)$ holds, the inclusion $\Upsilon^r - core(Sx) \subseteq st - core(x)$ implies (2.13) by Theorem 2.3.

Now, let $\tilde{w} \in \Upsilon^r - core(Sx)$. Similarly we have inequality (2.14). Put $st - \limsup |x_j - w| = \hat{l}$. Given any $\varepsilon > 0$, we have $\delta(\tilde{N}) = \delta(\{j : |x_j - w| > \hat{l} + \varepsilon\}) = 0$ (see [38]). Hence it follows that

$$\begin{aligned}
\left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| &= \left| \sum_{j \in \tilde{N}} \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) + \sum_{j \notin \tilde{N}} \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| \\
&\leq \sup_j |w - x_j| \sum_{j \in \tilde{N}} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| + (\hat{l} + \varepsilon) \sum_{j \notin \tilde{N}} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| \\
&\leq \sup_j |w - x_j| \sum_{j \in \tilde{N}} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right| + (\hat{l} + \varepsilon) \sum_{j} \left| \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} \right|.
\end{aligned}$$

Consequently, by (2.11) and (2.13), we have

$$\limsup_i \left| \sum_j \frac{1}{i^r} \sum_{j|i} J_r(j) s_{ij} (w - x_j) \right| \leq \hat{l} + \varepsilon. \tag{2.16}$$

If we combine (2.14) with (2.16), we deduce that

$$|\tilde{w} - w| \leq st - \limsup_j |x_j - w|.$$

This implies that $\tilde{w} \in st - core(x)$. Hence the desired inclusion holds. □

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Some Results on $\mathcal{W}_8$-Curvature Tensor in α-Cosymplectic Manifolds

Selahattin Beyendi

Abstract
The object of this paper is to study $\mathcal{W}_8$ curvature tensors in α-cosymplectic manifolds.

Keywords: *$\mathcal{W}_8$-curvature tensor; α-cosymplectic manifold; η-Ricci soliton.*

AMS Subject Classification (2020): *Primary: 53C25; Secondary: 53D15.*

1. Introduction

The geometry of contact Riemannian manifolds and related issues have received great attention in recent years. One of the most important of these is the almost cosymplectic manifolds presented by Goldberg and Yano [11] in 1969. A special variant of almost contact manifolds was presented by Kenmotsu [15] in 1972. Afterwards Kim and Pak in [16] described a new class of manifolds known as almost α-cosymplectic manifolds by combining almost cosymplectic and almost α-Kenmotsu manifolds, where α is a real number. Almost cosymplectic manifolds have been studied by many mathematicians in literature ([1], [2], [3], [7], [10], [16], [17], [20], [21], [28]) and many others. On the other hand, many different kinds of almost contact structures are defined in the literature. Pokhariyal and Mishra [23] have presented new tensor fields. In 1982, $\mathcal{W}$-curvature tensor have been studied by Pokhariyal [22]. Pokhariyal, described the curvature tensor $\mathcal{W}_8$ in this work. Many authors have worked on $\mathcal{W}$ curvature tensors ([4], [19], [25], [27], [29]). Ingalahalli et al. [13] have been studied the $\mathcal{W}_8$-curvature tensor on Kenmotsu manifolds. Also, Ruganzu et al. [26] have been studied the $\mathcal{W}_8$ curvature tensor on para Kenmotsu manifolds. By the motivations of all these studies, we, authors, in the present manuscript, are going to study the $\mathcal{W}_8$ curvature tensor on α-cosymplectic manifolds.

This manuscript has been structured as follows: After a brief presentation of α-cosymplectic manifolds we examine the cases $\xi-\mathcal{W}_8$ flat, $\varphi-\mathcal{W}_8$ semisymmetric, $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$, $\mathcal{W}_8.\mathcal{R} = 0$, $\mathcal{W}_8.\mathcal{W}_8 = 0$, $\mathcal{W}_8$-Ricci pseudosymmetric, $\mathcal{W}_8.\mathcal{Q} = 0$. Also, we examine η-Ricci solitons on α-cosymplectic manifolds satisfying $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$ and $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$.

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2. Preliminaries

Let $(M^n, \varphi, \xi, \eta, g)$ be an n-dimensional $(n = 2m + 1)$ almost contact metric manifold, in which ξ is the structure vector field, φ is a $(1, 1)$-tensor field, g is the Riemannian metric and η is a 1-form. The (φ, ξ, η, g) structure satisfies the following conditions [6].

$$\varphi\xi = 0, \quad \eta(\varphi\xi) = 0, \quad \eta(\xi) = 1,$$

$$\varphi^2 X_1 = -X_1 + \eta(X_1)\xi, \quad g(X_1, \xi) = \eta(X_1),$$

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2),$$

for any $X_1, X_2 \in \chi(M)$; in which $\chi(M)$ represents the collection of all smooth vector fields of M.

If moreover

$$\nabla_{X_1}\xi = -\alpha\varphi^2 X_1,$$

$$(\nabla_{X_1}\eta)X_2 = \alpha[g(X_1, X_2) - \eta(X_1)\eta(X_2)],$$

in which ∇ indicates the Riemannian connection and α is a real number, in that case $(M^n, \varphi, \xi, \eta, g)$ is known a α-cosymplectic manifold [16].

Then, it is also well known that [21]

$$\mathcal{R}(X_1, X_2)\xi = \alpha^2[\eta(X_1)X_2 - \eta(X_2)X_1], \tag{2.1}$$

$$\mathcal{R}ic(X_1, \xi) = -\alpha^2(n-1)\eta(X_1), \tag{2.2}$$

$$\mathcal{R}ic(\xi, \xi) = -\alpha^2(n-1), \tag{2.3}$$

$$\mathcal{Q}\xi = -\alpha^2(n-1)\xi \tag{2.4}$$

for all $X_1, X_2 \in \chi(M)$, in which $\mathcal{R}$, $\mathcal{R}ic$, $\mathcal{Q}$ indicates the curvature tensor, Ricci tensor and Ricci operator $g(\mathcal{Q}X_1, X_2) = \mathcal{R}ic(X_1, X_2)$ on M. Using (2.1), one can easily conclude that

$$\mathcal{R}(\xi, X_1)X_2 = \alpha^2[\eta(X_2)X_1 - g(X_1, X_2)\xi] \tag{2.5}$$

$$\mathcal{R}(X_1, \xi)\xi = \alpha^2[\eta(X_1)\xi - X_1]. \tag{2.6}$$

An α-cosymplectic manifold is known to be an η-Einstein manifold if Ricci tensor $\mathcal{R}ic$ satisfies condition

$$\mathcal{R}ic(X_1, X_2) = \lambda_1 g(X_1, X_2) + \lambda_2\eta(X_1)\eta(X_2) \tag{2.7}$$

in which λ_1, λ_2 are certain scalars. The manifold is known as Einstein when $\lambda_2 = 0$ in eq. (2.7).

On the other hand, η-Ricci solitons on α-cosymplectic manifolds have the following properties [30]:

$$\mathcal{R}ic(X_1, X_2) = -(\alpha + \lambda)g(X_1, X_2) + (\alpha - \mu)\eta(X_1)\eta(X_2), \tag{2.8}$$

$$\mathcal{R}ic(X_1, \xi) = -(\lambda + \mu)\eta(X_1) \tag{2.9}$$

for all $X_1, X_2 \in \chi(M)$.

3. $\xi - \mathcal{W}_8$-flat α-Cosymplectic Manifolds

In this part, we consider $\xi - \mathcal{W}_8$-flat in α-cosymplectic manifolds.

Definition 3.1. An α-cosymplectic manifold is known to be $\xi - \mathcal{W}_8$-flat if

$$\mathcal{W}_8(X_1, X_2)\xi = 0 \tag{3.1}$$

for all $X_1, X_2 \in \chi(M)$. $\mathcal{W}_8$-curvature tensor [22] is defined as

$$\mathcal{W}_8(X_1, X_2)X_3 = \mathcal{R}(X_1, X_2)X_3 + \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)X_3 - \mathcal{R}ic(X_2, X_3)X_1] \tag{3.2}$$

in which $\mathcal{R}ic$ and $\mathcal{R}$ are Ricci tensor and the curvature tensor of the manifold, respectively. By using of (3.1) in (3.2), we get

$$\mathcal{R}(X_1, X_2)\xi + \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)\xi - \mathcal{R}ic(X_2, \xi)X_1] = 0. \tag{3.3}$$

By virtue of (2.1), (2.2) in (3.3) and on simplification, we have

$$\alpha^2[\eta(X_1)X_2 - \eta(X_2)X_1] + \frac{1}{n-1}[\mathcal{R}ic(X_1, X_2)\xi + \alpha^2(n-1)\eta(X_2)X_1] = 0. \tag{3.4}$$

When the inner product is taken with ξ in eq. (3.4) and on simplification, one has

$$\mathcal{R}ic(X_1, X_2) = -\alpha^2(n-1)\eta(X_1)\eta(X_2).$$

In conclusion, one has the theorem given below:

Theorem 3.1. *Let M be an α-cosymplectic manifold satisfying $\xi - \mathcal{W}_8$-flat condition, then the manifold is a special kind of η-Einstein manifold.*

4. $\varphi - \mathcal{W}_8$-semisymmetric condition in α-cosymplectic manifolds

At this part, we examine $\varphi - \mathcal{W}_8$-semisymmetric condition in α cosymplectic manifolds.

Definition 4.1. An α-cosymplectic manifold is known to be $\varphi - \mathcal{W}_8$-semisymmetric if

$$\mathcal{W}_8(X_1, X_2).\varphi = 0 \tag{4.1}$$

for all $X_1, X_2 \in \chi(M)$.

At this time, eq. (4.1) becomes

$$(\mathcal{W}_8(X_1, X_2).\varphi)X_3 = \mathcal{W}_8(X_1, X_2)\varphi X_3 - \varphi\mathcal{W}_8(X_1, X_2)X_3 = 0. \tag{4.2}$$

Making use of (3.2) in (4.2), we obtain

$$\mathcal{R}(X_1, X_2)\varphi X_3 - \varphi\mathcal{R}(X_1, X_2)X_3 + \frac{1}{n-1}[\mathcal{R}ic(X_2, X_3)\varphi X_1 - \mathcal{R}ic(X_2, \varphi X_3)X_1] = 0. \tag{4.3}$$

By using $X_1 = \xi$ in (4.3) and with the help of (2.2), (2.5) equations and on simplification, we get

$$\alpha^2 g(X_2, \varphi X_3)\xi + \alpha\eta(X_3)\varphi X_2 + \frac{1}{n-1}\mathcal{R}ic(X_2, \varphi X_3)\xi = 0. \tag{4.4}$$

When X_3 by φX_3 is replaced in eq. (4.4), one has

$$\alpha^2 g(X_2, X_3)\xi = -\frac{1}{n-1}\mathcal{R}ic(X_2, X_3)\xi. \tag{4.5}$$

By taking inner product with ξ in (4.5), we obtain

$$\mathcal{R}ic(X_2, X_3) = -\alpha^2(n-1)g(X_2, X_3).$$

In conclusion, one has the theorem given below:

Theorem 4.1. *Let M be an α-cosymplectic manifold satisfying $\varphi - \mathcal{W}_8$-semisymmetric condition, then the manifold is an Einstein manifold.*

5. α-cosymplectic manifolds satisfying $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$ condition

At this part, we examine α-cosymplectic manifold satisfying $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$. Then, we get

$$\begin{aligned}&\mathcal{R}(\xi, X_1)\mathcal{W}_8(X_2, X_3)X_4 - \mathcal{W}_8(\mathcal{R}(\xi, X_1)X_2, X_3)X_4\\ &-\mathcal{W}_8(X_2, \mathcal{R}(\xi, X_1)X_3)X_4 - \mathcal{W}_8(X_2, X_3)\mathcal{R}(\xi, X_1)X_4 = 0.\end{aligned} \tag{5.1}$$

By using (2.5) in (5.1), we have

$$\begin{aligned}&\alpha^2\eta(\mathcal{W}_8(X_2, X_3)X_4)X_1 - \alpha^2 g(X_1, \mathcal{W}_8(X_2, X_3)X_4)\xi\\ &-\alpha^2\eta(X_2)\mathcal{W}_8(X_1, X_3)X_4 + \alpha^2 g(X_1, X_2)\mathcal{W}_8(\xi, X_3)X_4\\ &-\alpha^2\eta(X_3)\mathcal{W}_8(X_2, X_1)X_4 + \alpha^2 g(X_1, X_3)\mathcal{W}_8(X_2, \xi)X_4\\ &-\alpha^2\eta(X_4)\mathcal{W}_8(X_2, X_3)X_1 + \alpha^2 g(X_1, X_4)\mathcal{W}_8(X_2, X_3)\xi = 0.\end{aligned} \tag{5.2}$$

By using inner product with ξ in (5.2) and the aid of (3.2) and on simplification, we obtain

$$\begin{aligned}&\alpha^2\eta(\mathcal{W}_8(X_2, X_3)X_4)\eta(X_1) - \alpha^2 g(X_1, \mathcal{W}_8(X_2, X_3)X_4)\\ &-\alpha^2\eta(X_2)\eta(\mathcal{W}_8(X_1, X_3)X_4) + \alpha^2 g(X_1, X_2)\eta(\mathcal{W}_8(\xi, X_3)X_4)\\ &-\alpha^2\eta(X_3)\eta(\mathcal{W}_8(X_2, X_1)X_4) + \alpha^2 g(X_1, X_3)\eta(\mathcal{W}_8(X_2, \xi)X_4)\\ &-\alpha^2\eta(X_4)\eta(\mathcal{W}_8(X_2, X_3)X_1) + \alpha^2 g(X_1, X_4)\eta(\mathcal{W}_8(X_2, X_3)\xi) = 0.\end{aligned} \tag{5.3}$$

By using (2.1), (2.2) and (2.6) in (5.3), we get

$$\begin{aligned}&-\alpha^2 g(X_1, \mathcal{R}(X_2, X_3)X_4) - \alpha^4 g(X_1, X_2)g(X_3, X_4) + \alpha^4 g(X_1, X_3)g(X_2, X_4)\\ &-\alpha^4 g(X_2, X_1)\eta(X_4)\eta(X_3) + \alpha^4 g(X_1, X_4)\eta(X_2)\eta(X_3)\\ &-\tfrac{1}{n-1}\alpha^2[\mathcal{R}ic(X_2, X_1)\eta(X_3)\eta(X_4) - \mathcal{R}ic(X_1, X_4)\eta(X_3)\eta(X_2)] = 0.\end{aligned} \tag{5.4}$$

It is assumed that $\{e_i : i = 1, 2, ..., n\}$ is an orthonormal frame field at any point of the manifold. Then contracting $X_1 = X_2 = e_i$ in (5.4), we have

$$\mathcal{R}ic(X_3, X_4) = \alpha^2(1-n)g(X_3, X_4) - [\frac{scal}{n-1} + n\alpha^2]\eta(X_3)\eta(X_4).$$

In conclusion, one has the theorem given below:

Theorem 5.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{R}(\xi, X_1).\mathcal{W}_8 = 0$ condition, then the manifold is an η-Einstein manifold.*

6. α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{R} = 0$ condition

At this part, we examine α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{R} = 0$ condition. Then, we have

$$\begin{aligned}&\mathcal{W}_8(\xi, X_4)\mathcal{R}(X_1, X_2)X_3 - \mathcal{R}(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3\\ &-\mathcal{R}(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 = 0.\end{aligned} \tag{6.1}$$

If $X_3 = \xi$ is used in eq. (6.1), then one gets

$$\begin{aligned}&\mathcal{W}_8(\xi, X_4)\mathcal{R}(X_1, X_2)\xi - \mathcal{R}(\mathcal{W}_8(\xi, X_4)X_1, X_2)\xi\\ &-\mathcal{R}(X_1, \mathcal{W}_8(\xi, X_4)X_2)\xi - \mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)\xi = 0.\end{aligned} \tag{6.2}$$

By taking (2.1) in (6.2) and making the necessary simplifications, we get

$$-\alpha^2\eta(\mathcal{W}_8(\xi, X_4)X_1)X_2 + \alpha^2\eta(\mathcal{W}_8(\xi, X_4)X_2)X_1 - \mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)\xi = 0. \tag{6.3}$$

If eqs. (2.2), (2.5) and (3.2) is used in eq. (6.3), then one obtains

$$\begin{aligned}&\alpha^4[g(X_1, X_4)X_2 - g(X_4, X_2)X_1 + \eta(X_4)\eta(X_1)X_2 - \eta(X_4)\eta(X_2)X_1]\\ &\quad -\alpha^2\mathcal{R}(X_1, X_2)X_4 + \tfrac{\alpha^2}{n-1}[\mathcal{R}ic(X_4, X_1)X_2 - \mathcal{R}ic(X_4, X_2)X_1] = 0.\end{aligned} \tag{6.4}$$

Putting $X_2 = \xi$ in (6.4) and the aid of (2.2), (2.5), we have

$$\alpha^4 \eta(X_4)\eta(X_1)\xi + \frac{\alpha^2}{n-1}\mathcal{R}ic(X_4, X_1)\xi = 0. \tag{6.5}$$

When the inner product is taken with ξ in eq. (6.5), one has

$$\mathcal{R}ic(X_1, X_4) = -\alpha^2(n-1)\eta(X_1)\eta(X_4).$$

In conclusion, one has the theorem given below:

Theorem 6.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{R} = 0$ condition, then the manifold is a special kind of η-Einstein manifold.*

7. α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{W}_8 = 0$ condition

At this part, we examine α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{W}_8 = 0$ condition. Then, we have

$$\begin{aligned} &\mathcal{W}_8(\xi, X_4)\mathcal{W}_8(X_1, X_2)X_3 - \mathcal{W}_8(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 \\ &-\mathcal{W}_8(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \mathcal{W}_8(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 = 0. \end{aligned} \tag{7.1}$$

By using (3.2) in (7.1), we obtain

$$\begin{aligned} &\mathcal{R}(\xi, X_4)\mathcal{W}_8(X_1, X_2)X_3 + \tfrac{1}{n-1}[\mathcal{R}ic(\xi, X_4)\mathcal{W}_8(X_1, X_2)X_3 - \mathcal{R}ic(X_4, \mathcal{W}_8(X_1, X_2)X_3)\xi] \\ &-\mathcal{R}(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 - \tfrac{1}{n-1}[\mathcal{R}ic(\mathcal{W}_8(\xi, X_4)X_1, X_2)X_3 - \mathcal{R}ic(X_2, X_3)\mathcal{W}_8(\xi, X_4)X_1] \\ &-\mathcal{R}(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \tfrac{1}{n-1}[\mathcal{R}ic(X_1, \mathcal{W}_8(\xi, X_4)X_2)X_3 - \mathcal{R}ic(\mathcal{W}_8(\xi, X_4)X_2, X_3)X_1] \\ &-\mathcal{R}(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 - \tfrac{1}{n-1}[\mathcal{R}ic(X_1, X_2)\mathcal{W}_8(\xi, X_4)X_3 - \mathcal{R}ic(X_2, \mathcal{W}_8(\xi, X_4)X_3)X_1] = 0. \end{aligned} \tag{7.2}$$

Putting $X_2 = X_3 = \xi$ in (7.2) and with the help of (2.2), (2.3), (2.6), (3.2) equations and on simplification, we get

$$\mathcal{R}ic(X_1, X_4)\xi = -\alpha^2(n-1)g(X_1, X_4)\xi. \tag{7.3}$$

When the inner product is taken with ξ in eq. (7.3), one has

$$\mathcal{R}ic(X_1, X_4) = -\alpha^2(n-1)g(X_1, X_4).$$

In conclusion, one has the theorem given below:

Theorem 7.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{W}_8 = 0$ condition, then the manifold is an Einstein manifold.*

8. α-cosymplectic manifolds satisfying $\mathcal{W}_8$-Ricci pseudosymmetric condition

At this part, we examine $\mathcal{W}_8$-Ricci pseudosymmetric α-cosymplectic manifolds.
The notion of Ricci pseudosymmetric manifold has been presented by Deszcz ([8], [9]). In the case of Riemannian, the geometric comment of Ricci pseudosymmetric manifolds has been presented by [14]. A Riemannian manifold (M, g) is known Ricci pseudosymmetric ([8], [9], [12], [24]) if the tensor $\mathcal{R}.\mathcal{R}ic$ and the Tachibana tensor $\mathcal{Q}(g, \mathcal{R}ic)$ are linearly dependent, where

$$(\mathcal{R}(X_1, X_2).\mathcal{R}ic)(X_3, X_4) = -\mathcal{R}ic(\mathcal{R}(X_1, X_2)X_3, X_4) - \mathcal{R}ic(X_3, \mathcal{R}(X_1, X_2)X_4),$$

$$\mathcal{Q}(g, \mathcal{R}ic)(X_3, X_4; X_1, X_2) = -\mathcal{R}ic((X_1 \wedge_g X_2)X_3, X_4) - \mathcal{R}ic(X_3, (X_1 \wedge_g X_2)X_4)$$

and

$$(X_1 \wedge_g X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2$$

for all $X_1, X_2, X_3, X_4 \in \chi(M)$. $\mathcal{R}$ indicates the curvature tensor of M.
An α-cosymplectic manifold is known to be $\mathcal{W}_8$-Ricci pseudosymmetric if its curvature tensor satisfies

$$(\mathcal{W}_8(X_1, X_2).\mathcal{R}ic)(X_3, X_4) = L_{\mathcal{R}ic}\mathcal{Q}(g, \mathcal{R}ic)(X_3, X_4; X_1, X_2), \tag{8.1}$$

holds on $X_{4\mathcal{R}ic} = \{x \in M : \mathcal{R}ic \neq \frac{scal}{n} g \quad at \quad x\}$, in which $L_{\mathcal{R}ic}$ is some function on $X_{4\mathcal{R}ic}$. By using eq. (8.1), one obtains

$$\begin{aligned}&\mathcal{R}ic(\mathcal{W}_8(X_1, X_2)X_3, X_4) + \mathcal{R}ic(X_3, \mathcal{W}_8(X_1, X_2)X_4)\\&= L_{\mathcal{R}ic}[g(X_2, X_3)\mathcal{R}ic(X_1, X_4) - g(X_1, X_3)\mathcal{R}ic(X_2, X_4)\\&+g(X_2, X_4)\mathcal{R}ic(X_1, X_3) - g(X_1, X_4)\mathcal{R}ic(X_2, X_3)].\end{aligned} \tag{8.2}$$

Taking into account of $X_3 = \xi$ in (8.2) and by using the eqs. (2.1), (2.2), (3.2) and making the necessary simplifications, we get

$$\begin{aligned}&2\alpha^2[\mathcal{R}ic(X_2, X_4)\eta(X_1) - \mathcal{R}ic(X_1, X_2)\eta(X_4)]\\&+\alpha^4(n-1)[g(X_4, X_2)\eta(X_1) + g(X_1, X_4)\eta(X_2)]\\&= L_{\mathcal{R}ic}[\mathcal{R}ic(X_2, X_4)\eta(X_1) - \mathcal{R}ic(X_1, X_4)\eta(X_2)\\&-\alpha^2(n-1)(g(X_1, X_4)\eta(X_2) + g(X_2, X_4)\eta(X_1))].\end{aligned} \tag{8.3}$$

Putting $X_2 = \xi$ in (8.3) and by using (2.2) and making the necessary simplifications, we obtain

$$\mathcal{R}ic(X_1, X_4) = \frac{(n-1)(\alpha^4 - \alpha^2 L_{\mathcal{R}ic)}}{L_{\mathcal{R}ic}} g(X_1, X_4) - \frac{\alpha^4(n-1)}{L_{\mathcal{R}ic}}\eta(X_1)\eta(X_4).$$

In conclusion, one has the theorem given below:

Theorem 8.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8$-Ricci pseudosymmetric condition, then the manifold is an η-Einstein manifold.*

9. α-cosymplectic manifolds satisfying $\mathcal{W}_8.\mathcal{Q} = 0$ condition

At this part, we examine α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{Q} = 0$ condition. Then, we obtain

$$\mathcal{W}_8(X_1, X_2)\mathcal{Q}X_3 - \mathcal{Q}(\mathcal{W}_8(X_1, X_2)X_3) = 0. \tag{9.1}$$

When $X_2 = \xi$ is put in eq. (9.1), one gets

$$\mathcal{W}_8(X_1, \xi)\mathcal{Q}X_3 - \mathcal{Q}(\mathcal{W}_8(X_1, \xi)X_3) = 0. \tag{9.2}$$

If eq. (3.2) is used in eq. (9.2), then one gets

$$\begin{aligned}&\mathcal{R}(X_1, \xi)\mathcal{Q}X_3 + \tfrac{1}{n-1}(\mathcal{R}ic(X_1, \xi)\mathcal{Q}X_3 - \mathcal{R}ic(\xi, \mathcal{Q}X_3)X_1)\\&-\mathcal{Q}[\mathcal{R}(X_1, \xi)X_3 + \tfrac{1}{n-1}(\mathcal{R}ic(X_1, \xi)X_3 - \mathcal{R}ic(\xi, X_3)X_1)] = 0.\end{aligned} \tag{9.3}$$

Taking into account of (2.2), (2.4), (2.5) in eq. (9.3) and making the necessary simplifications, we get

$$\alpha^2\mathcal{R}ic(X_1, X_3)\xi + \alpha^4(n-1)g(X_1, X_3)\xi = 0. \tag{9.4}$$

Taking inner product with ξ in (9.4) and making the necessary simplifications, we obtain

$$\mathcal{R}ic(X_1, X_3) = -\alpha^2(n-1)g(X_1, X_3).$$

In conclusion, one has the theorem given below:

Theorem 9.1. *Let M be an α-cosymplectic manifold satisfying $\mathcal{W}_8.\mathcal{Q} = 0$ condition, then the manifold is an Einstein manifold.*

10. η-Ricci solitons on α-cosymplectic manifolds satisfying $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$

At this part, we examine η-Ricci solitons on α-cosymplectic manifold satisfying $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$. The condition that must be satisfied by $\mathcal{R}ic$ is [5]:

$$\mathcal{R}ic(\mathcal{W}_8(\xi, X_1)X_2, X_3) + \mathcal{R}ic(X_2, \mathcal{W}_8(\xi, X_1)X_3) = 0 \tag{10.1}$$

for all X_1, X_2 and $X_3 \in \chi(M)$. By using (2.5), (2.8), (2.9) and (3.2) in (10.1), we have

$$\begin{aligned}&[\tfrac{(\alpha+\lambda)(2\alpha+\lambda-\mu)}{n-1} - \alpha^2(\alpha-\mu)][g(X_1,X_2)\eta(X_3) + g(X_1,X_3)\eta(X_2)]\\ &+\tfrac{2(\alpha+\lambda)(\lambda+\mu)}{n-1} g(X_2,X_3)\eta(X_1) + 2\alpha^2(\alpha-\mu)\eta(X_1)\eta(X_2)\eta(X_3) = 0\end{aligned} \tag{10.2}$$

for all X_1, X_2 and $X_3 \in \chi(M)$. Putting $X_1 = \xi$ in (10.2), we obtain

$$(2\alpha+\lambda-\mu)\eta(X_2)\eta(X_3) + (\lambda+\mu)g(X_2,X_3) = 0$$

for all $X_2, X_3 \in \chi(M)$. But $\alpha+\lambda = n-1$, so $2\alpha+\lambda-\mu = 0$ and $\lambda = -\mu$ and we can state:

Theorem 10.1. *If (φ, ξ, η, g) is an almost contact metric structure on the n-dimensional α-cosymplectic manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\mathcal{W}_8(\xi, X_1).\mathcal{R}ic = 0$, then $\lambda = -\alpha + n - 1$ and $\mu = \alpha - n + 1$.*

11. η-Ricci solitons on α-cosymplectic manifolds satisfying $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$

At this part, we examine η-Ricci solitons on α-cosymplectic manifold satisfying $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$. The condition to be satisfied by $\mathcal{R}ic$ is [5]:

$$\begin{aligned}&\mathcal{R}ic(X_1, \mathcal{W}_8(X_2,X_3)X_4)\xi - \mathcal{R}ic(\xi, \mathcal{W}_8(X_2,X_3)X_4)X_1\\ &+\mathcal{R}ic(X_1,X_2)\mathcal{W}_8(\xi,X_3)X_4 - \mathcal{R}ic(\xi,X_2)\mathcal{W}_8(X_1,X_3)X_4\\ &+\mathcal{R}ic(X_1,X_3)\mathcal{W}_8(X_2,\xi)X_4 - \mathcal{R}ic(\xi,X_3)\mathcal{W}_8(X_2,X_1)X_4\\ &+\mathcal{R}ic(X_1,X_4)\mathcal{W}_8(X_2,X_3)\xi - \mathcal{R}ic(\xi,X_4)\mathcal{W}_8(X_2,X_3)X_1 = 0\end{aligned} \tag{11.1}$$

for all X_1, X_2, X_3 and $X_4 \in \chi(M)$. By taking an inner product with ξ in (11.1), we get

$$\begin{aligned}&\mathcal{R}ic(X_1, \mathcal{W}_8(X_2,X_3)X_4) - \mathcal{R}ic(\xi, \mathcal{W}_8(X_2,X_3)X_4)\eta(X_1)\\ &+\mathcal{R}ic(X_1,X_2)\eta(\mathcal{W}_8(\xi,X_3)X_4) - \mathcal{R}ic(\xi,X_2)\eta(\mathcal{W}_8(X_1,X_3)X_4)\\ &+\mathcal{R}ic(X_1,X_3)\eta(\mathcal{W}_8(X_2,\xi)X_4) - \mathcal{R}ic(\xi,X_3)\eta(\mathcal{W}_8(X_2,X_1)X_4)\\ &+\mathcal{R}ic(X_1,X_4)\eta(\mathcal{W}_8(X_2,X_3)\xi) - \mathcal{R}ic(\xi,X_4)\eta(\mathcal{W}_8(X_2,X_3)X_1) = 0\end{aligned} \tag{11.2}$$

for all X_1, X_2, X_3 and $X_4 \in \chi(M)$.
By taking $X_2 = X_3 = \xi$ in (11.2) and by virtue (2.5), (2.8), (2.9) and (3.2) and making the necessary simplifications, we have

$$(\alpha+\lambda)[g(X_1,X_4) - \eta(X_1)\eta(X_4)] = 0 \tag{11.3}$$

or

$$(\alpha+\lambda)g(\varphi X_1, \varphi X_4) = 0$$

for all $X_1, X_4 \in \chi(M)$. But $\alpha+\mu = n-1$, so $(\alpha+\lambda) = 0$ and we can state:

Theorem 11.1. *If (φ, ξ, η, g) is an almost contact metric structure on the n-dimensional α-cosymplectic manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$, then $\lambda = -\alpha$ and $\mu = -\alpha + n - 1$.*

By using (11.3) from (2.8), we have

$$\mathcal{R}ic(X_1, X_2) = -(\lambda+\mu)\eta(X_1)\eta(X_2).$$

So, one has the corollary given below:

Corollary 11.1. *If (φ, ξ, η, g) is an almost contact metric structure on the n-dimensional α-cosymplectic manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\mathcal{R}ic(\xi, X_1).\mathcal{W}_8 = 0$, then the manifold is a special type of η-Einstein manifold.*

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