UJMA

Universal Journal of Mathematics and Applications

VOLUME V
ISSUE I

ISSN 2619-9653

http://dergipark.gov.tr/ujma

VOLUME V ISSUE I
ISSN 2619-9653

March 2022
http://dergipark.gov.tr/ujma

UNIVERSAL JOURNAL OF MATHEMATICS AND APPLICATIONS

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Universal Journal of Mathematics and Applications, 5 (1) (2022) 1-9
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1028304

Stability Result for a Kirchhoff Beam Equation with Variable Exponent and Time Delay

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Article Info

Keywords: *Kirchhoff beam equation, stability, time delay, variable exponents.*
2010 AMS: *35B35, 35B40,35L05.*
Received: *25 November 2021*
Accepted: *22 January 2022*
Available online: *07 March 2022*

Abstract

This paper is concerned with a stability result for a Kirchhoff beam equation with variable exponents and time delay. The exponential and polynomial stability results are proved based on Komornik's inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. For $(x,t) \in \Omega \times \mathbb{R}^+$ we consider a Kirchhoff beam equation with variable exponents and time delay given by

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t)\,|u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)\,|u_t|^{m(x)-2}(x,t-\tau) = 0, \tag{1.1}$$

with Dirichlet boundary condition

$$u(x,t) = 0 \text{ in } \partial\Omega \times [0,\infty), \tag{1.2}$$

and initial data

$$\begin{aligned} u(x,0) &= u_0(x) \text{ in } \Omega, &(1.3)\\ u_t(x,0) &= u_1(x) \text{ in } \Omega, &(1.4)\\ u_t(x,t-\tau) &= f_0(x,t-\tau) \text{ in } \Omega \times (0,\tau), &(1.5)\end{aligned}$$

where $\tau > 0$ is time delay term, μ_1 is a positive constant, μ_2 is a real number. $M(s)$ is a positive C^1-function like $M(s) = a + bs^\gamma$ for $s \geq 0$, specially we take $a, b = 1$ and $\gamma > 0$. The functions u_0, u_1, f_0 are the initial data to be specified later.
The variable exponent $m(\cdot)$ is a measurable function on $\overline{\Omega}$ satisfying

$$2 \leq m^- \leq m(x) \leq m^+ \leq m^*, \text{ where } m^- = ess\inf_{x\in\Omega} m(x), \quad m^+ = ess\sup_{x\in\Omega} m(x) \text{ e } m^* = \frac{2(n-2)}{n-4} \text{ if } n \geq 5. \tag{1.6}$$

The problems with variable exponents arise in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [7,8,26]. Time delay often appears in many practical problems such as thermal, economic phenomena, biological, chemical

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and physical [12].
One of the first mathematical analysis of beam equation for $\Omega=(0,L)\subset\mathbb{R}$, $L>0$,

$$u_{tt}+u_{xxxx}-M\left(\int_0^L |u_x|^2\,dx\right)u_{xx}=0 \tag{1.7}$$

was done by Ball (1973) [6]. Tucsnak (1996) [29] extended (1.7) for the beam equation in $\Omega\subset\mathbb{R}^n$

$$u_{tt}+\Delta^2 u+M(||\nabla u||^2)(-\Delta u)=0, \tag{1.8}$$

where

$$||\nabla u||^2=:\int_\Omega |\nabla u|^2\,dx.$$

The problem (1.8) is called of Kirchhoff type in reason of the one-dimensional nonlinear equation (1.9) proposed by Kirchhoff [13] (1883),

$$\frac{\partial^2 u}{\partial t^2}-\left(\frac{\tau_0}{m}+\frac{k}{2mL}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.9}$$

where τ_0 is the initial tension, m the mass of the string and k the Young's modulus of the material of the string. This model, in connection with some problems in nonlinear elasticity, describes small vibrations of a stretched string of the length L when only the transverse component of the tension is considered.
This kind of problem (1.9) is obtained from the model (1.10), first proposed by Woinowsky-Krieger [30] (1950), for the transverse motion of an extensible beam of the length L whose ends are attached at a fixed distance

$$\frac{\partial^2 u}{\partial t^2}+\frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4}+\left(\frac{H}{\rho}+\frac{EA}{2\rho L}\int_0^L|\frac{\partial u}{\partial x}|^2 dx\right)\left(-\frac{\partial^2 u}{\partial x^2}\right)=0, \tag{1.10}$$

where L, E, I, ρ, H and A denote, respectively, the length of the beam in the rest position, the Young's modulus, the cross-sectional moment of inertia, the mass density, the tension in the rest position, and the cross-sectional area.
Antontsev et. al. [3], considered the nonlinear plate (or beam) Petrovsky equation with variable exponents as follows:

$$u_{tt}+\Delta^2 u-\Delta u_t+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u. \tag{1.11}$$

The authors, by using the Banach contraction mapping principle, obtained the local weak solutions. Also, they showed that the solution is global if $p(\cdot)\geq q(\cdot)$. Moreover, the authors proved that a solution with negative initial energy and $p(\cdot)<q(\cdot)$ blows up in finite time.
In [4], Antontsev et al., considered the Timoshenko-type equation with variable exponents as follows:

$$u_{tt}+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u. \tag{1.12}$$

The authors proved the local existence of the solution. Moreover, they investigated the nonexistence of solutions for negative initial energy.
In [5], Antontsev et al., studied the nonlinear $p(x)$-Laplacian equation with time delay and variable exponents as follows:

$$u_{tt}-\Delta_{p(x)}u+\mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t)+\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau)=bu|u|^{q(x)-2}. \tag{1.13}$$

The authors proved the blow up of solutions. Then, by applying an integral inequality due to Komornik, they obtained the decay result.
There are few results on Kirchhoff beam equation with delay. In [10] was considered the following nonlinear viscoelastic Kirchhoff beam equation with a time delay term in the internal feedback, given by

$$u_{tt}+\Delta^2 u-divF(\nabla u)-\sigma(t)\int_0^t g(t-s)\Delta^2 u(s)ds+\mu_1 u_t|u_t|^{m-1}(x,t)+\mu_2 u_t|u_t|^{m-1}(x,t-\tau)=0,$$

where $\Omega\subset\mathbb{R}^n,(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The function $u=u(x,t)$ is the transverse displacement, and $\sigma(t)$ and $g(t)$ are positive functions defined on $\mathbb{R}^+$. μ_1,μ_2 are positive constants and $\tau>0$ represents the time delay. Under suitable assumptions, the authors established the general rates of energy decay by using the energy perturbation method.
Kafini and Messaoudi [16], studied the equation with variable exponents and delay term as follows:

$$u_{tt}-\Delta u+\mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t)+\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau)=bu|u|^{p(x)-2}. \tag{1.14}$$

They established the decay estimates and global nonexistence results for the equation (1.14).
Santos et al. [27], investigated the existence and the decay of the beam equation as follows:

$$u_{tt}+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u-\int_0^t h(t-s)\Delta u(s)\,ds+\alpha u_t=0, \tag{1.15}$$

in a non-cylindrical domain. Recently, some other authors investigate hyperbolic type equations (see [11, 21–25, 28]).
Our aim in this work is to prove the stability of solutions for the Kirchhoff beam equation with the delay term $(\mu_2 u_t(x,t-\tau))$ and variable exponents which make the problem more different than from those considered in the literature. This manuscript extends the result of [16] to Kirchhoff beam equation.
The paper is organized as follows: In Section 2, the definition of the variable exponents Sobolev and Lebesgue spaces are stated. In Section 3, we obtain the stability result.

2. Preliminaries

In this section, we present some material needed for the statement and proof of our results. In what follows, we present some properties related to $W^{1,p(\cdot)}(\Omega)$ Sobolev spaces with variable exponents, see [2,9,14]. The spaces $L^{p(\cdot)}(\Omega)$ are special cases of the generalized Orlicz Spaces originated by Nakano [19] and developed by Musielak [17] and Orlicz [18]. The study of these spaces have been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations.
Let $p:\Omega\to[1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space with variable exponent $p(\cdot)$ by:

$$L^{p(\cdot)}(\Omega)=\left\{u:\Omega\to R;\ \text{measurable in }\Omega\ :\int_\Omega |u|^{p(\cdot)}dx<\infty\right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u}{\lambda}\right|^{p(x)}dx\le 1\right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space (see [8]).
The relation between the modular $\int_\Omega |f|^{p(x)}dx$ and the norm follows from

$$\min(\|f\|_{p(x)}^{p^-},\|f\|_{p(x)}^{p^+})\le\int_\Omega |f|^{p(x)}dx\le\max(\|f\|_{p(x)}^{p^-},\|f\|_{p(x)}^{p^+}).$$

In the case $p(x)=const>1$, these inequalities transform into equalities. For all $f\in L^{p(x)}(\Omega)$, $g\in L^{p'(x)}(\Omega)$ with

$$p(x)\in(1,\infty),\quad p'(x)=\frac{p(x)}{p(x)-1}$$

the generalized Hölder inequality holds, that is,

Lemma 2.1. *[1] (Hölder's inequality) Let $p,q,s\ge 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(y)}=\frac{1}{p(y)}+\frac{1}{q(y)}\ ,\text{ for a.e. } y\in\Omega,$$

satisfies. If $f\in L^{p(\cdot)}(\Omega)$ and $g\in L^{q(\cdot)}(\Omega)$, then, $fg\in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)}\le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega):\ \nabla u\ \text{ exists and }\ |\nabla u|\in L^{p(\cdot)}(\Omega)\right\}.$$

Variable exponent Sobolev space with respect to the norm:

$$\|u\|_{1,p(\cdot)}=\|u\|_{p(\cdot)}+\|\nabla u\|_{p(\cdot)}$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u\in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm:

$$\|u\|_{1,p(\cdot)}=\|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, similar to the usual Sobolev spaces, where $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$.
We also assume that:

$$|m(x)-m(y)|\le-\frac{B}{\log|x-y|}\ \text{ for all } x,y\in\Omega, \tag{2.1}$$

$B>0$ and $0<\delta<1$ with $|x-y|<\delta$. (log-Hölder condition)

Lemma 2.2. *[2] (Poincarė inequality) Suppose that $p(\cdot)$ satisfies (2.1) and let Ω be a bounded domain of R^n. Then,*

$$\|u\|_{p(\cdot)}\le c\|\nabla u\|_{p(\cdot)}\ \textit{for all } u\in W_0^{1,p(\cdot)}(\Omega),$$

where $c=c\left(p^-,p^+,|\Omega|\right)>0$.

Remark 2.3. *We denote by c various positive constants which may be different at different occurrences. Also, throughout this paper, we use the embedding*

$$H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^p(\Omega)$$

which implies

$$\|u\|_p\le C\|\nabla u\|\le C\|\Delta u\|,$$

where $2\le p<\infty$ $(n=1,2)$, $2\le p\le\frac{2n}{n-2}$ $(n\ge 3)$.

3. Stability of solutions

In this section, we get the stability results for the problem (3.1)-(3.7), with the exponent $m(\cdot)$.
Similar to the work of [20], we introduce a new variable

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\ x\in\Omega,\ \rho\in(0,1),\ t>0;$$

hence, we have

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Consequently, problem (1.1)- (1.5) is transformed to:

$$u_{tt}+\Delta^2u-M\left(\|\nabla u\|^2\right)\Delta u+\mu_1u_t(x,t)|u_t(x,t)|^{m(x)-2}+\mu_2z(x,1,t)|z(x,1,t)|^{m(x)-2}=0,\ \text{in}\ \ \Omega\times(0,\infty), \tag{3.1}$$

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0\ \text{ in }\ \Omega\times(0,1)\times(0,\infty), \tag{3.2}$$

with Dirichlet boundary condition

$$u(x,t)=0\ \text{ on }\ \partial\Omega\times[0,\infty), \tag{3.3}$$

$$z(x,\rho,t)=0\ \text{ on }\ \partial\Omega\times[0,1)\times[0,\infty), \tag{3.4}$$

and initial data

$$u(x,0)=u_0(x)\ \text{ in }\ \Omega, \tag{3.5}$$

$$u_t(x,0)=u_1(x)\ \text{ in }\ \Omega, \tag{3.6}$$

$$z(x,\rho,0)=f_0(x,-\rho\tau)\ \text{ in }\ \Omega\times(0,1). \tag{3.7}$$

Similar to [16] we can define the strong solution as follows:

Definition 3.1. *Fix $T>0$. We call (u,z) a strong solution of (3.1)-(3.7) if*

$$u\in W^{2,\infty}([0,T);L^2(\Omega))\cap W^{1,\infty}([0,T);H_0^2(\Omega))\cap L^\infty([0,T);H^2(\Omega)\cap H_0^2(\Omega)),$$

$$u_t\in L^{m(\dot{})}(\Omega)\times[0,T)),$$

$$z\in W^{1,\infty}([0,1]\times[0,T);L^2(\Omega))\cap L^\infty([0,1];L^{m(\dot{})(\Omega)\cap[0,T)}),$$

and (u,z) satisfies the initial data and (3.1) in the following sense

$$\int_\Omega[u_{tt}+\Delta^2u-M\left(\|\nabla u\|^2\right)\Delta u+\mu_1u_t(x,t)|u_t(x,t)|^{m(x)-2}+\mu_2z(x,1,t)|z(x,1,t)|^{m(x)-2}]v\,dx=0,$$

$$\int_\Omega[\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)]w\,dx=0,$$

for a.e. $t\in[0,t)$ and for $(v,w)\in H_0^1(\Omega)\cap L^2(\Omega)$.

In order to state our main result, we define the "modified" energy functional of (3.1) is given by

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\int_0^1\int_\Omega\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho, \tag{3.8}$$

for $t\geq 0$, where ξ is a continuous function yields

$$\tau|\mu_2|(m(x)-1)<\xi(x)<\tau(\mu_1m(x)-|\mu_2|),\ x\in\overline{\Omega}. \tag{3.9}$$

The following lemma gives that, $E(t)$ is decreasing under the condition $\mu_1>|\mu_2|$.

Lemma 3.2. *Let (u,z) be a solution of (3.1)-(3.7). Then, there exists some $C_0>0$ such that*

$$E'(t)\leq -C_0\int_\Omega\left(|u_t|^{m(x)}+|z(x,1,t)|^{m(x)}\right)dx\leq 0. \tag{3.10}$$

Proof. Multiplying (3.1) by u_t, integrating over Ω, then, multiplying (3.2) by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2}z$ and integrating over $\Omega\times(0,1)$, summing up, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\int_0^1\int_\Omega\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right]\\&=-\mu_1\int_\Omega|u_t|^{m(x)}dx-\frac{1}{\tau}\int_\Omega\int_0^1\xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx-\mu_2\int_\Omega u_tz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.\end{aligned} \tag{3.11}$$

The last two terms of the right-hand side of (3.11) can be estimated as follows:

$$-\frac{1}{\tau}\int_{\Omega}\int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2} zz_{\rho}(x,\rho,t)\,d\rho dx = -\frac{1}{\tau}\int_{\Omega}\int_0^1 \frac{\partial}{\partial\rho}\left(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\right)d\rho dx$$
$$= \frac{1}{\tau}\int_{\Omega}\frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)} - |z(x,1,t)|^{m(x)}\right)dx$$
$$= \int_{\Omega}\frac{\xi(x)}{\tau m(x)}|u_t|^{m(x)}\,dx - \int_{\Omega}\frac{\xi(x)}{\tau m(x)}|z(x,1,t)|^{m(x)}.$$

By using Young's inequality, $q=\frac{m(x)}{m(x)-1}$ and $q'=m(x)$ for the last term, we get

$$|u_t|\,|z(x,1,t)|^{m(x)-1} \le \frac{1}{m(x)}|u_t|^{m(x)} + \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}.$$

Consequently, we obtain

$$-\mu_2\int_{\Omega} u_t z|z(x,1,t)|^{m(x)-2}dx \le |\mu_2|\left(\int_{\Omega}\frac{1}{m(x)}|u_t(t)|^{m(x)}dx + \int_{\Omega}\frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}dx\right).$$

Thus,

$$\frac{dE(t)}{dt} \le -\int_{\Omega}\left[\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)}\right)\right]|u_t(t)|^{m(x)}dx - \int_{\Omega}\left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)}\right)|z(x,1,t)|^{m(x)}dx.$$

As a result, for all $x\in\overline{\Omega}$, the relation (3.9) satisfies

$$f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)}\right) > 0, \text{ and } f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. Hence, if we define

$$C_0(x) = \min\{f_1(x),\ f_2(x)\} > 0 \text{ for any } x\in\overline{\Omega},$$

and take $C_0(x) = \inf_{\overline{\Omega}} C_0(x)$, so $C_0(x) \ge C_0 > 0$. Therefore,

$$E'(t) \le -C_0\left[\int_{\Omega}|u_t(t)|^{m(x)}dx + \int_{\Omega}|z(x,1,t)|^{m(x)}dx\right] \le 0.$$

□

We need the following lemmas before obtain our stability results.

Lemma 3.3. *(Komornik, [15]) Let $E: R^+ \to R^+$ be a nonincreasing function and suppose that there are constants $\sigma,\ \omega > 0$ such that*

$$\int_s^{\infty} E^{1+\sigma}(t)\,dt \le \frac{1}{\Omega}E^{\sigma}(0)E(s) = cE(s),\ \forall s>0.$$

Then, we have

$$\begin{cases} E(t) \le cE(0)(1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ E(t) \le cE(0)e^{-\omega t} & \text{if } \sigma = 0. \end{cases}$$

for all $t\ge 0$.

Lemma 3.4. *[16] The functional*

$$F(t) = \tau\int_0^1\int_{\Omega} e^{-\rho\tau}\xi(x)|z(x,\rho,t)|^{m(x)}dx d\rho$$

satisfies

$$F'(t) \le \int_{\Omega}\xi(x)|u_t|^{m(x)}dx - \tau e^{-\tau}\int_0^1\int_{\Omega}\xi(x)|z(x,\rho,t)|^{m(x)}dx d\rho$$

along the solution of (3.1)-(3.7).

Theorem 3.5. *Assume that conditions (1.6) and (2.1) are satisfied. Then, there exist two constants $c,\ \alpha > 0$ independent of t such that any global solution of (3.1)-(3.7) satisfies,*

$$\begin{cases} E(t) \le ce^{-\alpha t} & \text{if } m(\cdot) = 2, \\ E(t) \le cE(0)(1+t)^{2/(m^+-2)} & \text{if } m^+ > 2. \end{cases}$$

Proof. We multiply the equation (3.1) by $uE^q(t)$, for $q>0$ to be specified later, and integrate over $\Omega\times(s,T)$, $s<T$, to have

$$\int_s^T E^q(t)\int_\Omega\left[\ uu_{tt}+u\Delta^2u-u\Delta u-\|\nabla u\|^{2\gamma}u\Delta u+\mu_1uu_t|u_t|^{m(x)-2}+\mu_2uz(x,1,t)|z(x,1,t)|^{m(x)-2}\ \right]dxdt=0,$$

which implies that

$$\int_s^T E^q(t)\int_\Omega\left(\tfrac{d}{dt}(uu_t)-u_t^2+|\Delta u|^2+|\nabla u|^2+\|\nabla u\|^{2\gamma}|\nabla u|^2+\mu_1uu_t(x,t)|u_t(x,t)|^{m(x)-2}+\mu_2uz(x,1,t)|z(x,1,t)|^{m(x)-2}\right)dxdt=0. \tag{3.12}$$

Recalling the definition of $E(t)$, given in (3.8) adding and subtracting some terms and using the relation

$$\frac{d}{dt}\left(E^q(t)\int_\Omega uu_tdx\right)=qE^{q-1}(t)E'(t)\int_\Omega uu_tdx+E^q(t)\frac{d}{dt}\int_\Omega uu_tdx,$$

the equation (3.12) satisfies

$$\begin{aligned}2\int_s^T E^{q+1}(t)dt=&-\int_s^T\frac{d}{dt}\left(E^q(t)\int_\Omega uu_tdx\right)dt+q\int_s^T E^{q-1}(t)E'(t)\int_\Omega uu_tdxdt\\&-\frac{\gamma}{\gamma+1}\int_s^T E^q\int_\Omega\|\nabla u\|^{2\gamma}\left|\nabla u^2\right|dxdt+2\int_s^T E^q(t)\int_\Omega u_t^2dxdt-\mu_1\int_s^T E^q(t)\int_\Omega uu_t|u_t|^{m(x)-2}dxdt\\&-\mu_2\int_s^T E^q(t)\int_\Omega uz(x,1,t)|z(x,1,t)|^{m(x)-2}dxdt+2\int_s^T E^q(t)\int_0^1\int_\Omega\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt.\end{aligned} \tag{3.13}$$

Next, we estimate the parts of the right side in inequality (3.13), respectively.
The first term is estimated as follows:

$$\begin{aligned}\left|-\int_s^T\frac{d}{dt}\left(E^q(t)\int_\Omega uu_tdx\right)dt\right|&=\left|E^q(s)\int_\Omega uu_t(x,s)dx-E^q(T)\int_\Omega uu_t(x,T)dx\right|\\&\le\frac12E^q(s)\left[\int_\Omega u^2(x,s)dx+\int_\Omega u_t^2(x,s)dx\right]+\frac12E^q(T)\left[\int_\Omega u^2(x,T)dx+\int_\Omega u_t^2(x,T)dx\right]\\&\le\frac12E^q(s)\left[C_p\|\Delta u(s)\|_2^2+2E(s)\right]+\frac12E^q(T)\left[C_p\|\Delta u(T)\|_2^2+2E(T)\right]\\&\le E^q(s)\left[C_pE(s)+E(s)\right]+E^q(T)\left[C_pE(T)+E(T)\right],\end{aligned}$$

where C_p is the Poincare's constant. Because of $E(t)$ is nonincreasing, we infer that

$$\left|-\int_s^T\frac{d}{dt}\left(E^q(t)\int_\Omega uu_tdx\right)dt\right|\le cE^{q+1}(s)\le cE^q(0)E(s)\le cE(s). \tag{3.14}$$

In similar way, we handle the term

$$\begin{aligned}\left|q\int_s^T E^{q-1}(t)E'(t)\int_\Omega uu_tdxdt\right|&\le-q\int_s^T E^{q-1}(t)E'(t)\left[C_pE(T)+E(T)\right]dt\\&\le-c\int_s^T E^q(t)E'(t)\le cE^{q+1}(s)\le cE(s).\end{aligned} \tag{3.15}$$

We estimate the other term as follows:

$$\begin{aligned}\left|-\frac{\gamma}{\gamma+1}\int_s^T E^q\int_\Omega\|\nabla u\|^{2\gamma}\left|\nabla u^2\right|dxdt\right|&=\left|-2\gamma\int_s^T E^q\left(\frac{\|\nabla u\|^{2\gamma}}{2(\gamma+1)}\int_\Omega\left|\nabla u^2\right|dx\right)dt\right|\\&=\left|-2\gamma\int_s^T E^q\left(\frac{\|\nabla u\|^{2(\gamma+1)}}{2(\gamma+1)}\right)dt\right|\\&\le\left|-2\gamma\int_s^T E^q(E(t))dt\right|\\&\le C^*\int_s^T E^{q+1}(t)dt\\&\le C^*E(s)\end{aligned} \tag{3.16}$$

where C^* is a generic constant.
To treat the other term, we set

$$\Omega_+=\{x\in\Omega,\ |u_t(x,t)|\ge1\}\text{ and }\Omega_-=\{x\in\Omega,\ |u_t(x,t)|<1\}.$$

Then, by using the Hölder's and Young's inequalities, we get

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega u_t^2 dxdt\right| &= \left|\int_s^T E^q(t)\left[\int_{\Omega_+} u_t^2 dx+\int_{\Omega_-} u_t^2 dx\right]dt\right|\\ &\le c\int_s^T E^q(t)\left[\left(\int_{\Omega_+}|u_t|^{m^-}dx\right)^{2/m^-}+\left(\int_{\Omega_-}|u_t|^{m^+}dx\right)^{2/m^+}\right]dt\\ &\le c\int_s^T E^q(t)\left[\left(\int_{\Omega}|u_t|^{m(x)}dx\right)^{2/m^-}+\left(\int_{\Omega}|u_t|^{m(x)}dx\right)^{2/m^+}\right]dt\\ &\le c\int_s^T E^q(t)\left[\left(-E'(t)\right)^{2/m^-}+\left(-E'(t)\right)^{2/m^+}\right]dt\\ &\le c\varepsilon\int_s^T [E(t)]^{qm^-/(m^--2)}dt+c(\varepsilon)\int_s^T\left(-E'(t)\right)dt+c\varepsilon\int_s^T E(t)^{q+1}dt+c(\varepsilon)\int_s^T\left(-E'(t)\right)^{2(q+1)/m^+}dt.\end{aligned}$$

For $m^->2$ and the choice of $q=m^+/2-1$ will give $\frac{qm^-}{m^--2}=q+1+\frac{m^+-m^-}{m^--2}$.
Therefore,

$$\begin{aligned}\left|\int_s^T E^q(t)\int_\Omega u_t^2 dxdt\right| &\le c\varepsilon\int_s^T E(t)^{q+1}dt+c\varepsilon[E(0)]^{\frac{m^+-m^-}{m^--2}}\int_s^T[E(t)]^{q+1}dt+c(\varepsilon)E(s)\\ &\le c\varepsilon\int_s^T E(t)^{q+1}dt+c(\varepsilon)E(s).\end{aligned} \tag{3.17}$$

For the case $m^-=2$ and the choice of $q=m^+/2-1$ will give the similar result.
For the other term, utilizing Young's inequality we conclude

$$\begin{aligned}\left|-\mu_1\int_s^T E^q(t)\int_\Omega u|u_t|^{m(x)-1}dxdt\right| &\le \varepsilon\int_s^T E^q(t)\int_\Omega|u(t)|^{m(x)}dxdt+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dxdt\\ &\le \varepsilon\int_s^T E^q(t)\left[\int_{\Omega_+}|u(t)|^{m^-}dx+\int_{\Omega_-}|u(t)|^{m^+}dx\right]dt+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dxdt,\end{aligned}$$

where we have used Young's inequality with

$$p(x)=\frac{m(x)}{m(x)-1},\ p'(x)=m(x)$$

and hence

$$c_\varepsilon(x)=(m(x)-1)\,m(x)^{m(x)/(1-m(x))}\,\varepsilon^{1/(1-m(x))}.$$

By using the embeddings $H_0^2(\Omega)\hookrightarrow L^{m^-}(\Omega)$ and $H_0^2(\Omega)\hookrightarrow L^{m^+}(\Omega)$, we obtain

$$\begin{aligned}\left|-\mu_1\int_s^T E^q(t)\int_\Omega u|u_t|^{m(x)-1}dxdt\right| &\le \varepsilon\int_s^T E^q(t)\left[c\|\Delta u(s)\|_2^{m^-}+c\|\Delta u(s)\|_2^{m^+}\right]dt+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dxdt\\ &\le \varepsilon\int_s^T E^q(t)\left[cE^{(m^--2)/2}(0)E(t)+cE^{(m^+-2)/2}(0)E(t)\right]dt+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dxdt\\ &\le c\varepsilon\int_s^T E^{q+1}(t)dt+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dxdt.\end{aligned} \tag{3.18}$$

The next term of (3.13) can be estimated in a similar attitude to get

$$\begin{aligned}\left|-\mu_2\int_s^T E^q(t)\int_\Omega u|z(x,1,t)|^{m(x)-1}dxdt\right| &\le \varepsilon\int_s^T E^q(t)\left[c\|\Delta u(s)\|_2^{m^-}+c\|\Delta u(s)\|_2^{m^+}\right]dt+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}dxdt\\ &\le c\varepsilon\int_s^T E^{q+1}(t)dt+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}dxdt.\end{aligned} \tag{3.19}$$

For the last term of (3.13), from Lemma 3.4, we get

$$\begin{aligned}2\int_s^T E^q(t)\int_0^1\int_\Omega\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt &\le \frac{2}{m^-}\int_s^T E^q(t)\int_0^1\int_\Omega\xi(x)|z(x,\rho,t)|^{m(x)}dxd\rho dt\\ &\le -\frac{2\tau}{m^-}\int_s^T E^q(t)\frac{d}{dt}\left(\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)|z|^{m(x)}dxd\rho\right)dt\\ &+\frac{2}{m^-}\int_s^T E^q(t)\int_\Omega\xi(x)|u_t|^{m(x)}dxdt\\ &\le -\frac{2\tau}{m^-}\left[E^q(t)\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)|z|^{m(x)}dxd\rho\right]_{t=s}^{t=T}+\frac{2}{m^-}\int_s^T E^q(t)\int_\Omega\xi(x)|u_t|^{m(x)}dxdt.\end{aligned}$$

As $\xi(x)$ is bounded, by (3.8), we obtain

$$2\int_s^T E^q(t)\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt \le \frac{2\tau e^{-\tau}}{m^-}E^q(s)E(s)+\frac{2c}{m^-}E^{q+1}(T)$$
$$\le \frac{2\tau e^{-\tau}}{m^-}E^q(0)E(s)+\frac{2c}{m^-}E^q(T)E(s)\le cE(s), \tag{3.20}$$

for some $c>0$.
By combining (3.13)-(3.20), we conclude that

$$\int_s^T E^{q+1}(t)dt \le \varepsilon\int_s^T E^{q+1}(t)dt + cE(s) + c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}dxdt. \tag{3.21}$$

Choosing ε so small such that

$$\int_s^T E^{q+1}(t)dt \le cE(s) + c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}dxdt.$$

Once ε is fixed, then $c_\varepsilon(x)\le M$, since $m(x)$ is bounded. Therefore, we infer that

$$\int_s^T E^{q+1}(t)dt \le cE(s) + cM\int_s^T E^q(t)\int_\Omega |z(x,1,t)|^{m(x)}dxdt$$
$$\le cE(s) - C_0M\int_s^T E^q(t)E'(t)dt$$
$$\le cE(s) + \frac{C_0M}{q+1}\left[E^{q+1}(s)-E^{q+1}(T)\right]\le cE(s). \tag{3.22}$$

By taking $T\to\infty$, we obtain

$$\int_s^\infty E^{q+1}(t)dt \le cE(s).$$

Thus, Komornik's Lemma (with $\sigma = q = m^+/2-1$) implies the desired result. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 5 (1) (2022) 10-14
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1055172

Dwell Time for the Hurwitz Stability of Switched Linear Differential Equation Systems

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Article Info

Keywords: *Differential equation systems, Hurwitz stability, Switched system, Dwell time*
2010 AMS: *34D20, 93D05.*
Received: *8 January 2022*
Accepted: *21 February 2022*
Available online: *7 March 2022*

Abstract

In this paper, the problem of dwell time for the Hurwitz stability of switched linear systems is considered. Dwell time is determined based on the solution of Lyapunov matrix equation for the Hurwitz stability of switched linear differential systems. A numerical example illustrating the efficiency of theorem has been given.

1. Introduction

Switched differential systems are used in mathematical modeling of many fields such as automotive engineering, motor engine control, constrained robotics, networked control systems [1]- [5]. Stability analysis of switched systems is a basic issue. Therefore, it is important to study the stability of the switched systems. There are roughly two different types of studies on the stability of the switched systems in the literature. The first of these studies is to examine the stability of the switched systems under the given switching signals and the other is to determine the switching signal that will ensure the stability of the switched systems [6]. In this study, we have focused on the second method for linear switched systems and we have aimed to give a method that determines the dwell for the system to be Hurwitz stable.
Let us consider the linear switched systems described by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \sigma \in S, t \in [0,\infty) \tag{1.1}$$

where $\{A_p \in \mathbb{C}^{m\times m}, p \in \mathscr{P}\}$ is matrix family for $\mathscr{P} = \{1,2,\ldots,N\}$, S is the set of the functions $\sigma : [0,\infty) \to \mathscr{P}$, σ denotes the switching signals. The amount of time passed between the consecutive switching events is called dwell time of system (1.1). If each subsystems are stable then there exists a minimum dwell time that guarantees stability of system (1.1). For system (1.1), the dwell time method refer to allowing certain set of switching signals; namely,

$$S = S_{dwell[\tau]} = \{\sigma(t) | t \in [t_k, t_{k+1}), t_{k+1} - t_k \geq \tau\}$$

and finding the dwell time, the infimum of the numbers τ for which the switched system is Hurwitz stable [7], [8].
This paper is organized as follows. In section 2, preliminary is given. In section 3, the dwell time for Hurwitz stability is determined. Finally, in section 4, a numerical example is given.

2. Preliminaries

Let $A \in \mathbb{C}^{m\times m}$ and $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$, $x_i(t) (i = 1,2,...,m)$ be a differentiable function and consider the following differential equation system:

$$\dot{x}(t) = Ax(t), t \in [0,\infty). \tag{2.1}$$

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Differential equation system (2.1) is stable if for any positive number ε there exists $\delta = \delta(\varepsilon)$ such that $\|x(t)\| \leq \varepsilon$ for $t \in [0,\infty)$ whenever for $\|x(0)\| \leq \delta$. Further, system (2.1) is Hurwitz stable (asymptotically stable) if it is stable and $\|x(t)\| \to 0$ with increase t to infinity [9]- [13]. Lyapunov theorem, a criterion for Hurwitz stability, is as follows.

The matrix A is Hurwitz stable if and only if there is a solution $X = X^* > 0$ of the Lyapunov matrix equation $A^*X + XA = -I$ where I is unit matrix, A^* is adjoint of the matrix A, the matrix $X = \int_0^\infty e^{tA^*}e^{tA}dt$ is positive definite solution of Lyapunov matrix equation [9]- [13].

Theorem 2.1. *Following inequalitiy*

$$\|e^{At}\| \leq \sqrt{\|X^{-1}\|\|X\|}e^{\frac{-t}{2\|X\|}} \tag{2.2}$$

is valid for the Hurwitz stable matrix A where the matrix X is solution of Lyapunov matrix equation [14], [15].

3. Determination of Dwell Time for Hurwitz Stability

Let us give the following theorem, which gives the upper bound of the solution of the system (1.1) to use determining the dwell time for Hurwitz stability.

Theorem 3.1. *If the matrix $A_p(p = 1,2,\ldots,N)$ are the Hurwitz stable matrix, then the following equation is provided for the solution of the linear switched system* (1.1)

$$\|x(t)\| \leq (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\frac{-1}{2\|X_{\sigma_{n+1}}\|}(t-t_n)} e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}(t_k-t_{k-1})]}\|x_0\|$$

where $t \in [t_n,t_{n+1})$ and X_{σ_k} is solution of Lyapunov matrix equation $A^*_{\sigma_k}X_{\sigma_k} + X_{\sigma_k}A_{\sigma_k} = -I$.

Proof. Let us consider the switched system (1.1), where $A_p(p = 1,2,\ldots,N)$ are the Hurwitz stable matrix. The solution of system (1.1) which is given with the initial value $x(0) = x_0$ is expressed as

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)}e^{A_{\sigma_n}(t_n-t_{n-1})}\ldots e^{A_{\sigma_1}(t_1-t_0)}x_0, t \in [t_n,t_{n+1})$$

or

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)}(\prod_{k=1}^{n} e^{A_{\sigma_k}(t_k-t_{k-1})})x_0, t \in [t_n,t_{n+1}). \tag{3.1}$$

By taking the norm of solution (3.1) and applying the triangle inequality, the following inequality is obtained

$$\|x(t)\| = \|e^{A_{\sigma_{n+1}}(t-t_n)}(\prod_{k=1}^{n} e^{A_{\sigma_k}(t_k-t_{k-1})})x_0\|$$
$$\|x(t)\| \leq \|e^{A_{\sigma_{n+1}}(t-t_n)}\|\prod_{k=1}^{n}\|e^{A_{\sigma_k}(t_k-t_{k-1})}\|\|x_0\|.$$

If we use inequality (2.2) we obtain the upper bound of the solution as:

$$\|x(t)\| \leq \sqrt{\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|}e^{-\frac{(t-t_n)}{2\|X_{\sigma_{n+1}}\|}}\prod_{k=1}^{n}\sqrt{\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|}e^{-\frac{(t_k-t_{k-1})}{2\|X_{\sigma_k}\|}}\|x_0\|$$
$$= (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}}e^{-\frac{(t-t_n)}{2\|X_{\sigma_{n+1}}\|}}\prod_{k=1}^{n}(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)^{\frac{1}{2}}e^{-\frac{(t_k-t_{k-1})}{2\|X_{\sigma_k}\|}}\|x_0\|$$
$$\|x(t)\| \leq (\|X^{-1}_{\sigma_{n+1}}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\frac{-1}{2\|X_{\sigma_{n+1}}\|}(t-t_n)} e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}(t_k-t_{k-1})]}\|x_0\|.$$

□

Let us denote by $\mathscr{F}$ the set of all transitions between subsystems and give the following theorem, which gives the dwell time for Hurwitz stability.

Theorem 3.2. *Let $A_p(p \in \mathscr{P})$ be Hurwitz stable matrices. Then, the switched system* (1.1) *is Hurwitz stable for dwell time*

$$\tau > \max_{n\in\mathscr{F}} \frac{\alpha(n)}{\beta(n)} \tag{3.2}$$

where $\alpha(n) = \sum_{k=1}^{n} ln(\|X^{-1}_{\sigma_k}\|\|X_{\sigma_k}\|)^{\frac{1}{2}}$ and $\beta(n) = \sum_{k=1}^{n}\frac{1}{2\|X_{\sigma_k}\|}$.

Proof. The upper bound of the solution of the switched differential equation system (1.1) from Theorem 3.1.

$$\|x(t)\| \leq (\|X_{\sigma_{n+1}}^{-1}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\frac{-1}{2\|X_{\sigma_{n+1}}\|}(t-t_n)} e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X_{\sigma_k}^{-1}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}(t_k-t_{k-1})]}\|x_0\|.$$

Consider that $\|X_{\sigma_{n+1}}\| > 1$, $e^{\frac{-(t-t_n)}{2\|X_{\sigma_{n+1}}\|}} < 1$ and $\tau \leq t_k - t_{k-1}$, the above inequality can be written as

$$\|x(t)\| \leq \max_{i\in\mathscr{P}}(\|X_{\sigma_{n+1}}^{-1}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\sum_{k=1}^{n}[\frac{1}{2}ln(\|X_{\sigma_k}^{-1}\|\|X_{\sigma_k}\|)-\frac{1}{2\|X_{\sigma_k}\|}\tau]}\|x_0\|$$
$$= \max_{i\in\mathscr{P}}(\|X_{\sigma_{n+1}}^{-1}\|\|X_{\sigma_{n+1}}\|)^{\frac{1}{2}} e^{\alpha(n)-\tau\beta(n)}\|x_0\|$$

where $\alpha(n) = \sum_{k=1}^{n} ln(\|X_{\sigma_k}^{-1}\|\|X_{\sigma_k}\|)^{\frac{1}{2}}$ and $\beta(n) = \sum_{k=1}^{n} \frac{1}{2\|X_{\sigma_k}\|}$.
Since $\alpha(n) - \tau\beta(n) < 0$, it is seen that $\|x(t)\| \to 0$ as $t \to \infty$. Thus, it is shown that the system (1.1) is Hurwitz stable for dwell time in (3.2). □

4. A Numerical Example

Example 4.1. *Let us consider the following system consisting three Hurwitz stable subsystems:*
$A_1 = \begin{pmatrix} -0.1 & -1 \\ 0.3 & -0.5 \end{pmatrix}, A_2 = \begin{pmatrix} -0.7 & -0.4 \\ 2 & -0.6 \end{pmatrix}, A_3 = \begin{pmatrix} -0.01 & 3 \\ -0.5 & -0.4 \end{pmatrix}$

$$\dot{x}(t) = A_i x(t), x(0) = [10,-12]^T, t \geq 0; i \in \{1,2,3\}. \tag{4.1}$$

Let $\mathscr{D}$ be the switching graph of the system (4.1) given in Figure 4.1.

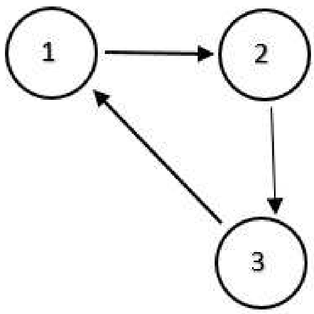

Figure 4.1: Switching graphs of the Cauchy problem consisting of three subsystems with A_1, A_2 and A_3.

For the graph $\mathscr{D}$, the minimum dwell time calculated in Theorem 3.2. is obtained as $\tau > 4.844453$. For this minimum dwell time and switched for graph $\mathscr{D}$, the solution curves given in the graph below are obtained.

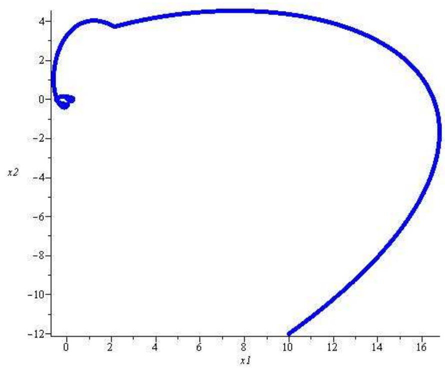

Figure 4.2: State trajectory under constant dwell time ($\tau = 4.85$) condition.

5. Conclusion

In this paper, a new minimum dwell time which make differential equation systems (1.1) Hurwitz stable, is obtained depending on solution of the Lyapunov matrix equation. The effect of the new minimum dwell time is illustrated with an example.

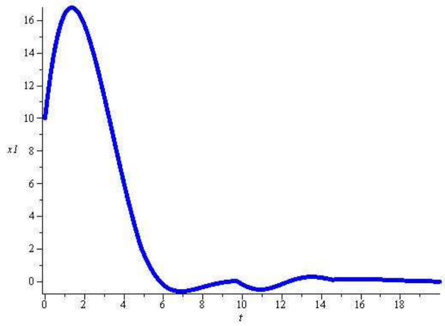

Figure 4.3: State trajectory x_1 of system (4.1).

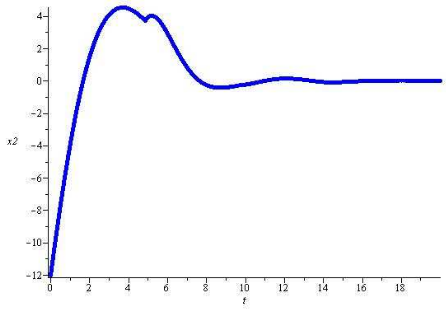

Figure 4.4: State trajectory x_2 of system (4.1).

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 5 (1) (2022) 15-23
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1010490

Forecasting COVID-19 Disease Cases Using the SARIMA-NNAR Hybrid Model

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Article Info

Keywords: *COVID-19, Time series, SARIMA, NNAR, Forecasting, R Programming*
2010 AMS: *62M10, 62M20, 62P10*
Received: *15 October 2021*
Accepted: *25 February 2022*
Available online: *7 March 2022*

Abstract

Background: COVID-19 is a new disease that is associated with high morbidity that has spread around the world. Credible estimating is crucial for control and prevention. Nowadays, hybrid models have become popular, and these models have been widely implemented. Better estimation accuracy may be attained using time-series models. Thus, our aim is to forecast the number of COVID-19 cases with time-series models. Objective: Using time-series models to predict deaths due to COVID-19. Design: SARIMA, NNAR, and SARIMA-NNAR hybrid time series models were used using the COVID-19 information of the Republic of Turkey Health Ministry. Participants: We analyzed data on COVID-19 in Turkey from March 11, 2020 to February 22, 2021. Main Measures: Daily numbers of COVID-19 confirmed cases and deaths. Materials and methods: We fitted a seasonal autoregressive integrated moving average (SARIMA)–neural network nonlinear autoregressive (NNAR) hybrid model with COVID-19 monthly cases from March 11, 2020, to February 22, 2021, in Turkey. Additionally, a SARIMA model, an NNAR model, and a SARIMA–NNAR hybrid model were established for comparison and estimation. Results The RMSE, MAE, and MAPE values of the NNAR model were obtained the lowest in the training set and the validation set. Thus, the NNAR model demonstrates excellent performance whether in fitting or forecasting compared with other models. Conclusions The NNAR model that fits this study is the most suitable for estimating the number of deaths due to COVID-19. Hence, it will facilitate the prevention and control of COVID-19.

1. Introduction

The latest threat to global health is the ongoing outbreak of respiratory disease that was recently given the name Coronavirus Disease 2019 (Covid-19). Covid-19 was recognized in December 2019 [1]. It was rapidly shown to be caused by a novel coronavirus that is structurally related to the virus that causes severe acute respiratory syndrome (SARS). As in two preceding instances of emergence of coronavirus disease in the past 18 years [2] — SARS (2002 and 2003) and the Middle East respiratory syndrome (MERS) (2012 to the present) — the Covid-19 outbreak has posed critical challenges for the public health, research, and medical communities [3].

Public health groups are monitoring the pandemic and posting updates on their websites. These groups have also issued recommendations for preventing and treating the illness. As of now, researchers know that the new coronavirus is spread through droplets released into the air when an infected person coughs or sneezes. The droplets generally do not travel more than 30 centimeters, and they fall to the ground (or onto surfaces) in a few seconds.

Due to the high infectivity of the virus and the lack of immunity in the human population, the epidemic grows exponentially without intervention, and thus can greatly stress the public health system and bring enormous the disruption to economy and society. Thus, a crucial

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task facing every country is to reduce the transmission rate and flatten the (infection) curve. Additionally, each country is at a different stage of the epidemic and it is essential for countries to understand their own pattern of virus growth, as such information is critical for important policy decisions such as extending lockdown or reopening. To (at least partially) answer these questions, a natural step is to analyze the trajectory of the infection curve of COVID-19 since the initial outbreak in each country.

The virus can exacerbate through the respiratory tract and enter into a person's lungs. This causes damage to the air sacs or alveoli, which can fill with fluid. This progression then constraints a person's ability to take in oxygen. Continuous oxygen deprivation can damage many of the body's organs, causing kidney failure, heart attacks, and other life-threatening conditions. People who have pre-existing conditions such as cancer, diabetes, high blood pressure, kidney or liver disease, including but not limited to asthma are at most risk of COVID-19 pneumonia. People over the age of 65 years are more prone to the intense effects of this disease. The disease has turned into a widespread pandemic where the cases and deaths seem to surge rapidly day by day.

Time series analysis is a scientific quantitative prediction of the future trend of diseases based on historical data and time variables. It is a quantitative analysis method that does not consider the influence of complex factors. In this study, Seasonal Auto-Regressive Moving Average (SARIMA) and Neural Network Auto-Regressive (NNAR) were used were applied on the predicting COVID-19 disease cases.

A number of authors have published papers on models for the occurrence of Coronavirus pandemic [4]- [10]. In [4], the trajectory of the cumulative confirmed cases and deaths of COVID-19 (in log scale) via a piecewise linear trend model is given a model. Peipei and et al [5] integrated the most updated COVID-19 epidemiological data before June 16, 2020, into the Logistic model to fit the cap of epidemic trend. A machine learning-based time series prediction model to derive the epidemic curve and predict the trend of the epidemic is produced in [5]. In [6], the Bayesian structural time series (BSTS) models to investigate the temporal dynamics of COVID-19 in the top five affected countries around the world in the time window March 1, 2020, to June 29, 2020, are used. In the study of He et al. [7], an analysis was made between deaths due to COVID-19 and environmental conditions (including ambient pollutants and meteoroidal parameters). Ribeiro et al. [8] developed efficient short-term forecasting models for forecasting the number of future cases by using an autoregressive integrated moving average (ARIMA), cubist regression (CUBIST), random forest (RF), ridge regression (RIDGE), support vector regression (SVR) and stacking-ensemble learning models for evaluating in the task of time series forecasting with one, three, and six days ahead of the COVID-19 cumulative confirmed cases in ten Brazilian states with a high daily incidence. Topaç et al. [9] analyzed the cognitions, feelings, and thoughts of early childhood children who stayed at home during the quarantine process due to coronavirus with multiple correspondence analyses. In [10], the risks of the effects of quarantine status related to the COVID-19 pandemic on the cognition and behavior of children staying at home are evaluated.

The Covid-19 outbreak is a stark reminder of the ongoing challenge of emerging and reemerging infectious pathogens and the need for constant surveillance, prompt diagnosis, and robust research to understand the basic biology of new organisms and our susceptibilities to them, as well as to develop effective countermeasures.

The purpose of this study was to compare the time series models to forecast the COVID-19 incidence epidemic and it is the prediction of deaths due to Covid-19 in Turkey. For better forecasting performance, a comparison of time-series models to forecast infectious disease was studied. The results from this study will be helpful to predict future COVID-19 incidence epidemics and optimize COVID-19 control and intervention using the predictions as reference information.

2. Methods

2.1. Dataset

Data were taken in COVID-19 Information Page of Republic of Turkey Health Ministry. The dataset comprises of both confirmed and death cases only as variables refer to daily cases and covers the period from March 11, 2020, up until February 22, 2021 [11].

The R programming language [12] has been used to carry out the all analyses involved in the present investigation.

2.2. Time-Series Approaches

SARIMA:
Autoregressive Integrated Moving Average, or ARIMA, is one of the most widely used forecasting methods for univariate time series data forecasting. Although the method can handle data with a trend, it does not support time series with a seasonal component. An extension to ARIMA that supports the direct modeling of the seasonal component of the series is called SARIMA. In this tutorial, you will discover the Seasonal Autoregressive Integrated Moving Average, or SARIMA, method for time series forecasting with univariate data containing trends and seasonality. Seasonal Autoregressive Integrated Moving Average, SARIMA or Seasonal ARIMA, is an extension of ARIMA that explicitly supports univariate time series data with a seasonal component. It adds three new hyper parameters to specify the auto regression (AR), differencing (I) and moving average (MA) for the seasonal component of the series, as well as an additional parameter for the period of the seasonality.

For a seasonal series with s periods per year, the $SARIMA(p,d,q)(P,D,Q)s$ models are used. Thus, having a B^s operator such that $B^sX_t = X_{t-s}$ and since the seasonal difference can be written as $(X_t - X_{t-s}) = (1-B^s)X_t$, a SARIMA model with (p,d,q) non-seasonal order terms and (P,D,Q) seasonal order terms. A time series $\{Z_t : 1,2,\cdots,k\}$ is generated by $SARIMA(p,d,q)(P,D,Q)s$ process of Box and Jenkins time series model if

$$\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D Z_t = \theta_q(B)\Theta_Q(B^s)\varepsilon_t,$$

where p,d,q,P,D,Q are integers, s is the season length;

$$\begin{aligned}
\phi_p(B) &= 1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p,\\
\Phi_P(B^s) &= 1-\Theta_s B^s-\Theta_{2s}B^{2s}-\cdots-\Theta_{Ps}B^{Ps},\\
\phi_q(B) &= 1-\phi_1 B-\phi_2 B^2-\cdots-\phi_q B^q,\\
\Phi_Q(B^s) &= 1-\Theta_s B^s-\Theta_{2s}B^{2s}-\cdots-\Theta_{Qs}B^{Qs},
\end{aligned}$$

are polynomials in B, where B is the backward shift operator, ε_t is the estimated residual at time t,d is the number of regular differences, D is the number of seasonal differences, Z_t denotes the observed value at time $t(t=1,2,\cdots,k)$.

Fitting a SARIMA model to data involves the following four-steps iterative cycles: a) identify the $SARIMA(p,d,q)(P,D,Q)$'s structure; (b) estimate unknown parameters; (c) perform goodness-fit tests on the estimated residuals; (d) forecast future outcomes based on the known data. The fitting of SARIMA models is a challenging task.

NNAR:

Neural networks are used for complex non-linear forecasting. With time series data, lagged values of the time series can be used as inputs to a neural network, just as we used lagged values in a linear autoregression model [13]. We call this a neural network autoregression or NNAR model. NNAR is generally describes by $NNAR(p,k)$ where p is lagged inputs and k a number of hidden layers. Also, $NNAR(p,P,k)$ is the general denotation of seasonal NNAR. For example, a $NNAR(9,5)$ model is a neural network with last nine observations $(y_{t-1},y_{t-2},\dots,y_{t-9})$ used as inputs for forecasting the output y_t and with five neurons in the hidden layer. A $NNAR(p,0)$ model is equivalent to an $ARIMA(p,0,0)$ model, but without the restrictions on the parameters to ensure stationary. With seasonal data, it is useful to also add the last observed values from the same season as inputs. For example, an $NNAR(3,1,2)_{12}$ model has inputs y_{t-1},y_{t-2},y_{t-3} and y_{t-12}, and two neurons in the hidden layer. More generally, an $NNAR(p,P,k)_m$ model has inputs $(y_{t-1},y_{t-2},\dots,y_{t-p},y_{t-m},y_{t-2m},y_{t-Pm})$ and k neurons in the hidden layer. A $NNAR(p,P,0)_m$ model is equivalent to an $SARIMA(p,0,0)(P,0,0)_m$ model but without restrictions on the parameters that ensure stationary.

The NNAR model is a feedforward neural network which involves a linear combination function an d a activation function. The formations of these function are defined as,

$$\begin{aligned}
net_j &= \sum_i w_{ij}y_{ij}\\
f(y) &= \frac{1}{1+e^{-y}}.
\end{aligned}$$

Hybrid Model:
A time series can be considered as comprising a linear autocorrelation structure and a non-linear component. The SARIMA model and the NNAR are methodologies that predict future values using historically observed data, and are suitable for linear and non-linear problems, respectively. We used the hybrid model combining the SARIMA model (linear approach) and the NNAR (non-linear approach) for this study.

In the first place, a SARIMA model was fitted. Subsequently, its residual series were inputted to the NNAR model. The nonlinear relationships that the residuals may contain can be mined adequately by neural networks. The final combined forecasting values of the time series were the sum of predictions from the SARIMA model and the adjusted residuals from NNAR model. The structure of the hybrid model is shown in Figure 2.1.

2.3. Forecast evaluation methods

The performance of the model is related to the similarity in the forecast values for the test data and observed values. Three different forecast consistency measures are used for comparing the performances obtained for the SARIMA, NNAR and SARIMA-NNAR models: mean square error(MSE), mean absolute error(MAE) and mean absolute percentage error(MAPE). The smaller the MSE, MAE, and MAPE of the prediction model, the better its prediction accuracy. In the other words, the prediction model with the smallest MSE, Mae, and MAPE can be selected as the optimum models.

$$\begin{aligned}
MSE &= \frac{1}{n}\sum_{t=1}^{n}\left(y_t-(\hat{y})^t\right)^2,\\
MAE &= \frac{1}{n}\sum_{t=1}^{n}\left|y_t-(\hat{y})^t\right|,\\
MAPE &= \frac{1}{n}\sum_{t=1}^{n}\frac{|y_t-(\hat{y})^t|}{y_t}.
\end{aligned}$$

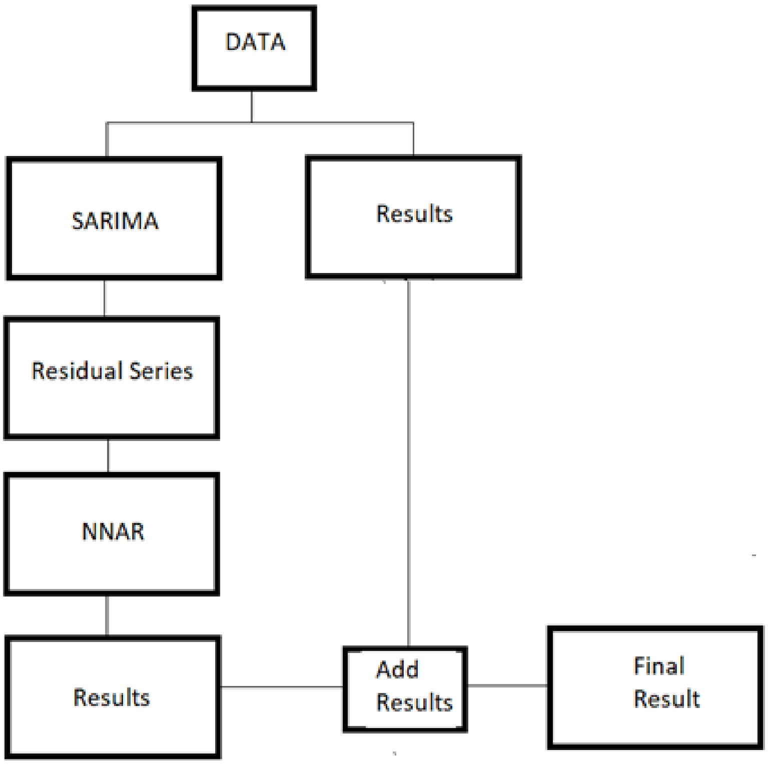

Figure 2.1: Structure of Hybrid Model

3. Results

In Turkey, 35608 people died due to the COVID-19, until April 17, 2021. Precautions are the most important means of reducing these deaths. If the death rates can be predicted forward, the governments can move faster in taking action. Covid-19 cases in Turkey increased in April 2020. The data used in the analysis are from March 11, 2020 to February 22, 2021. However, the precautions taken have led to a decrease in the cases. When the precautions relaxed, it increased again. The time-series distribution of those who died from COVID-19 on a daily basis is given in Figure 3.1. This chart also includes ACF and PACF charts. According to these values, p and q values are estimated in the time series model. Since the ACF chart is showing a slow decline, it can speak of a trend in time series.

In Figure 3, the *decomposition()* function is divided into time series components to examine whether there is seasonality. The components of the series are given in Figure 3.2.

Time series analysis was first applied to the time series of those who died due to COVID-19. In the decomposition, only the trend is clearly separated, but we do not have any information for the p and q values. The ADF test states that the time series is not stationary ($ADF = -13.737, p = 0.01$). For this reason, the most suitable model has been determined with the *auto.arima()* function. In this case, $SARIMA(2,1,2)(2,0,1)_7$ model was obtained as the most suitable model with *auto.arima()* function of forecast library [14]. The ACC_C value of the model is 1784.77 and the BIC value is 1813.88. The residuals of this model, ACF, and Normality analysis graph are given in Figure 4. Whether the residuals of the created model show autocorrelation was examined by Ljung-Box test and it was found that the residues did not show autocorrelation ($\chi^2 = 14.476, df = 24, p = 0.9352$).

Best performing NNAR model:
Due to seasonality, $p = 1$ was set. Again, the model of the forecast library was obtained automatically with the *nnetar* function. The model obtained by the function is the $NNAR(2,1,2)_7$ model. In addition, *nnar* models with different p and q values from 1 to 10 were obtained. Obtained models are listed in Table 1. Here, there are 9 models with the lowest RMSE value. Considering three performance indices in both training and validation sets, $NNAR(10,1,7)_7$ was chosen as the most suitable model. Here, 70% of the data is divided into training data set and 30% as test data set. The graph of residuals, ACF, and Normality analysis of this model is given in Figure 3.4. Whether the residuals of the created model show autocorrelation were examined by Ljung-Box test and it was found that the residues did not show autocorrelation($\chi^2 = 14.775, df = 24, p = 0.9272$).

Best performing SARIMA - NNAR model:
The SARIMA-NNAR model was obtained from the *forecastHybrid* library with the *hybridModel* function. Here, the SARIMA model is the model obtained by *auto.arima* function and the NNAR model is determined as $NNAR(3,1,2)_7$. The graph of residuals, ACF, and Normality analysis of this model is given in Figure 6. Whether the residuals of the created model show autocorrelation were examined by Ljung-Box test and it was found that the residues did not show autocorrelation ($\chi^2 = 10.602, df = 24, p = 0.9559$).

The performances of the SARIMA, NNAR, and SARIMA-NNAR models tested in the study are given in Table 2. According to the table, the

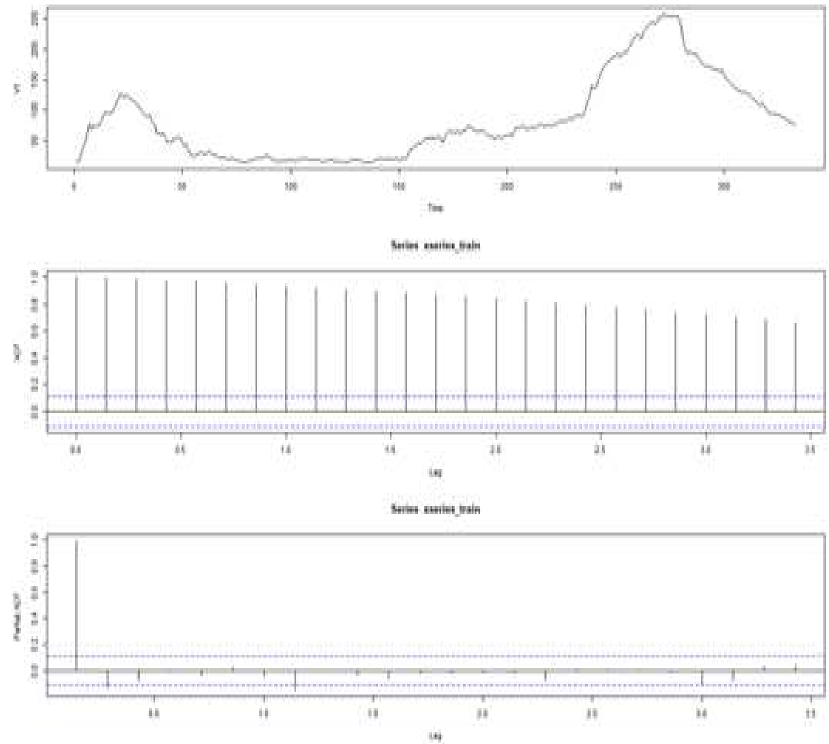

Figure 3.1: Time series and ACF-PACF plot

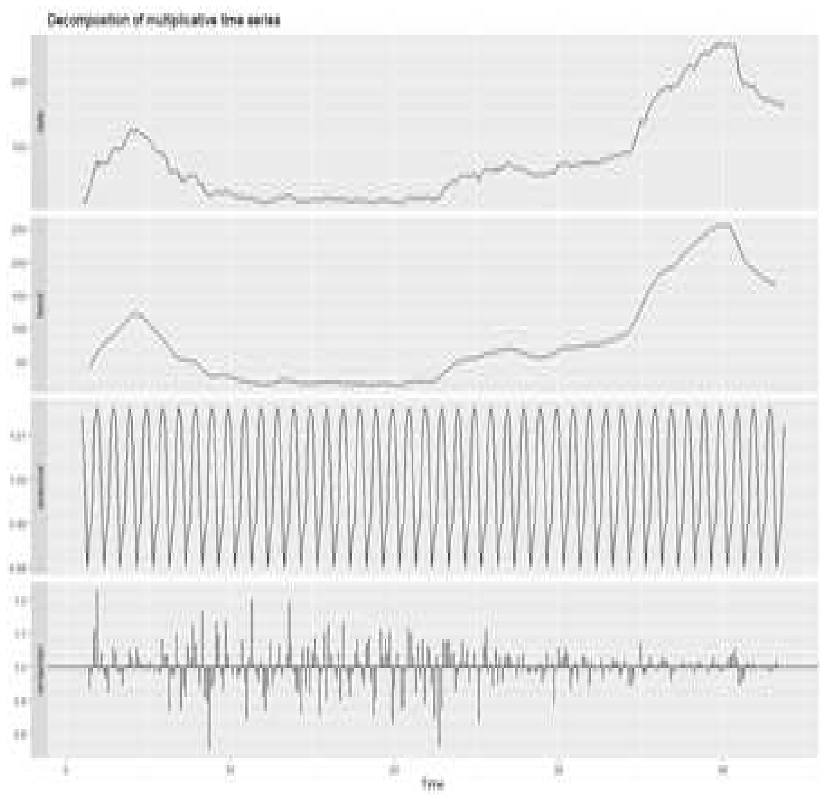

Figure 3.2

RMSE, MAE, and MAPE values of the NNAR model have the lowest values in the training set and the validation set.

The training and testing time series of the model of the three models obtained in the study is shown in Figure 3.6. It seems that all models fit well in terms of the training set. Since it would be misleading to look at it visually, the best model was obtained as the NNAR model according to the comparison criteria given in Table 3. The forecast for the next 24 days according to the NNAR model is given in Figure 3.7. Accordingly, it is seen that the number of deaths from COVID 19 will increase in the coming days.

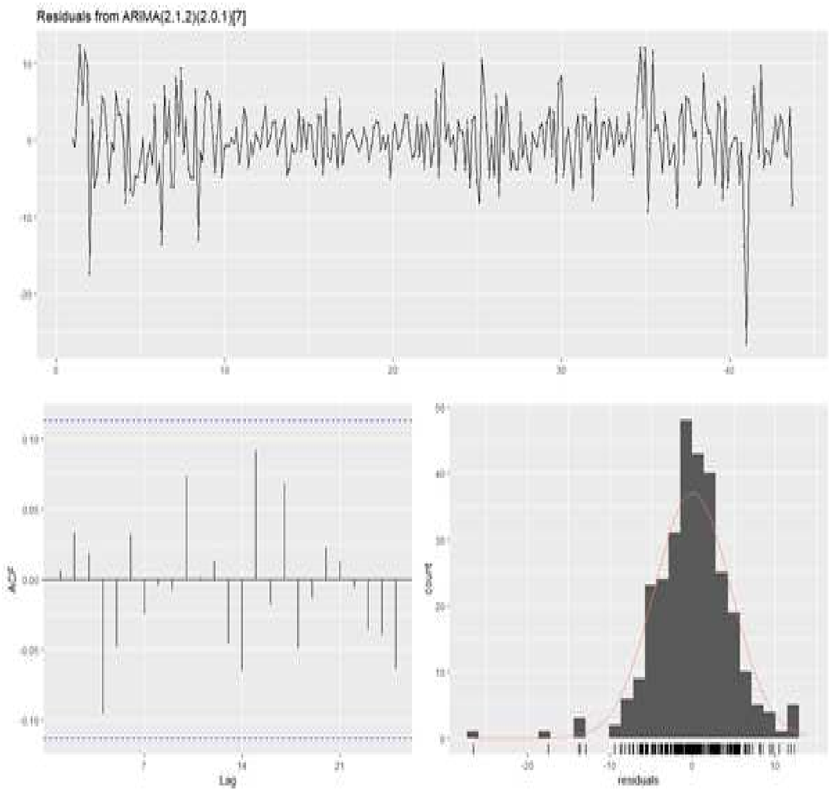

Figure 3.3: Residual plot, corresponding ACF plot and histogram from $ARIMA(2,1,2)(2,0,1)_7$

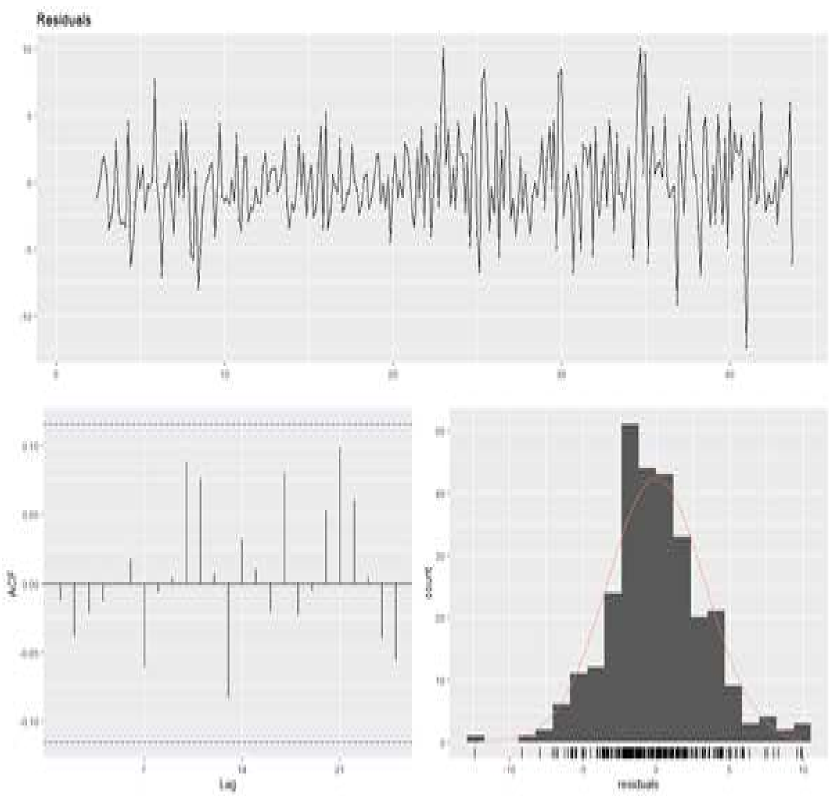

Figure 3.4: Residual plot, corresponding ACF plot and histogram from $NNAR(10,1,7)_7$

4. Discussion

In this study, in Turkey, to estimate the number of deaths from COVID-19, SARIMA, NNAR, and SARIMA-NNAR models are used. It has been determined that the model that gives the best estimation result among these models and obtains the best result according to the comparison criteria of the models is NNAR. While developing the NNAR model, various simulations were made to try to predict the best model. In Figure 3.6, in the comparison of the three models, it was not observed that there was no serious difference between the models.

	RMSE	MAE	MAPE	RMSE	MAE	MAPE
$(8,1,5)_7$	3.723869	2.868839	5.960852	0.3500508	0.24699	0.24328
$(8,1,6)_7$	3.612052	2.793796	5.86323	0.2381594	0.16410	0.15624
$(8,1,7)_7$	3.559917	2.775637	5.850855	0.1829261	0.13529	6 0.13222
$(9,1,5)_7$	3.602116	2.775739	5.853346	0.1111011	0.08418	0.08819
$(9,1,6)_7$	3.486982	2.682023	5.732168	0.0888780	0.06401	0.06719
$(10,1,4)_7$	3.705423	2.829739	5.903758	0.03699127	0.02864	0.03179
$(10,1,5)_7$	3.495667	2.661528	5.67319	0.03167837	0.02561	0.02743
$(10,1,6)_7$	3.383682	2.58536	5.569271	0.02092535	0.01722	0.01825
$(10,1,7)_7$	3.2764	2.498271	5.460094	0.02042849	0.01772	0.01863

Table 1

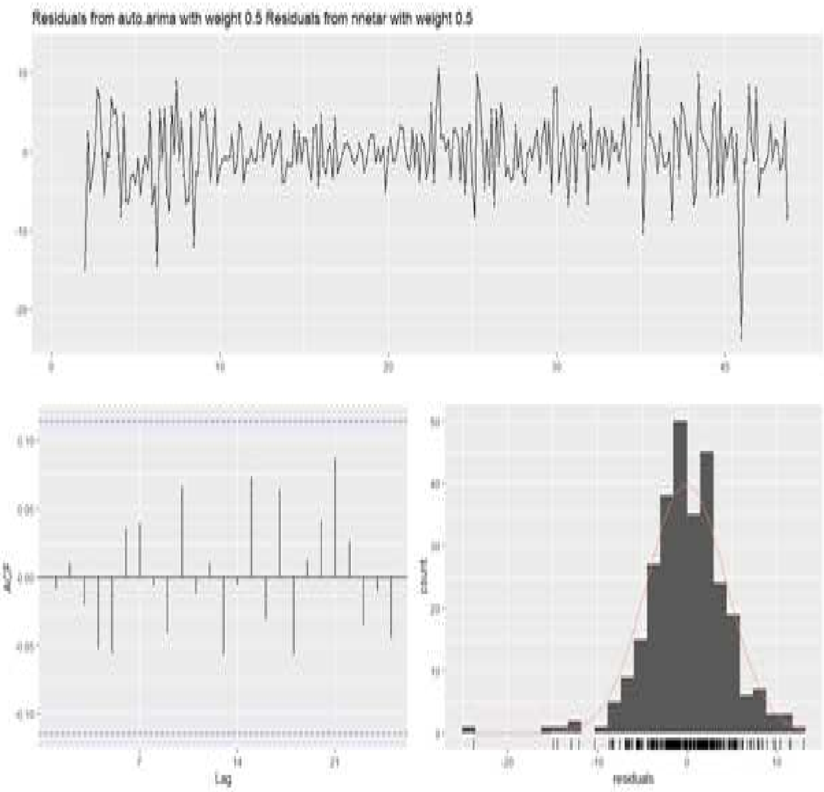

Figure 3.5: Residual plot, corresponding ACF plot and histogram from SARIMA-NNAR

	RMSE	MAE	MAPE	RMSE	MAE	MAPE
SARIMA	4.6364	3.3918	6.9269	2.1686	1.5680	1.4772
NNAR	3.2639	2.4890	5.4078	0.0094	0.0073	0.0075
SARIMA-NNAR	4.4262	3.3961	6.7221	2.0836	1.5336	1.5143

Table 2: Accuracy of training set and 52 weeks forecasting in validation set.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Forecast	74.75	73.22	69.94	68.91	67.47	66.21	65.11	63.64	62.25	61.05	60.10	59.36
Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Forecast	58.53	57.77	56.99	56.23	55.63	55.04	54.46	53.89	53.30	52.73	52.15	51.57

Table 3: Twenty four days forecast

However, the most suitable model for the result was determined according to the comparison criteria in Table 2. The second-best model was determined as the SARIMA-NNAR model. Using the NNAR model, it is possible to predict the number of deaths from COVID-19 in the future and take necessary measures accordingly. Thanks to these estimates and measures, the number of deaths can be reduced.

However, there are some limitations to this study. The first of these limitations is where the original data is taken. These data were obtained from the COVID-19 Information Page of the Republic of Turkey Health Ministry. This data can be included the possibility of false reporting and negligent reporting. The quality of the data can affect the build process and performance of the model to some extent. However, in order to obtain a better model, the best model was tried to be obtained by examining the p and q values from 1 to 10 instead of using an automatic

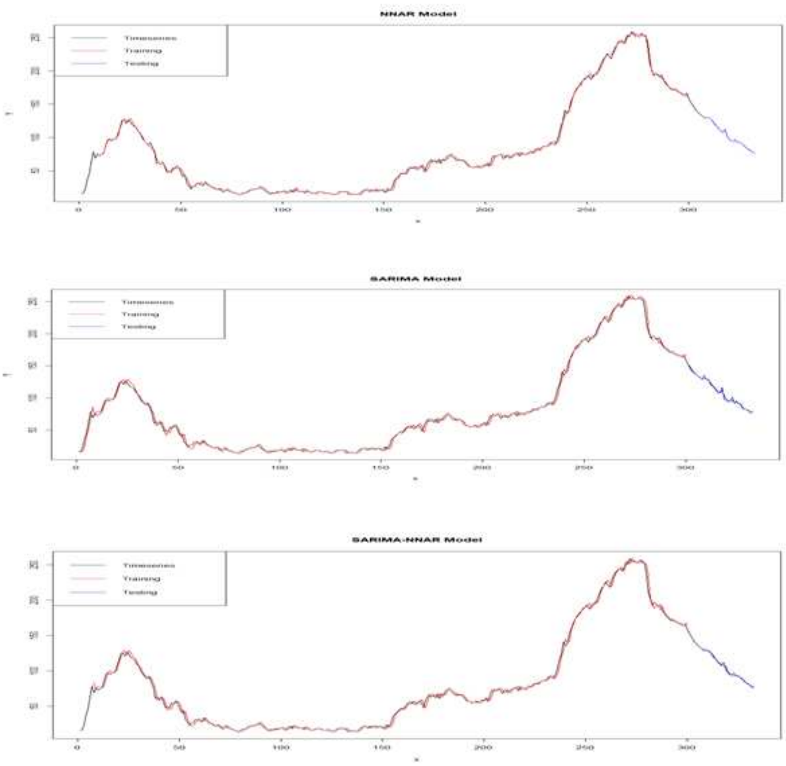

Figure 3.6: Fitted and 52 weeks forecasting time series plot of three models.

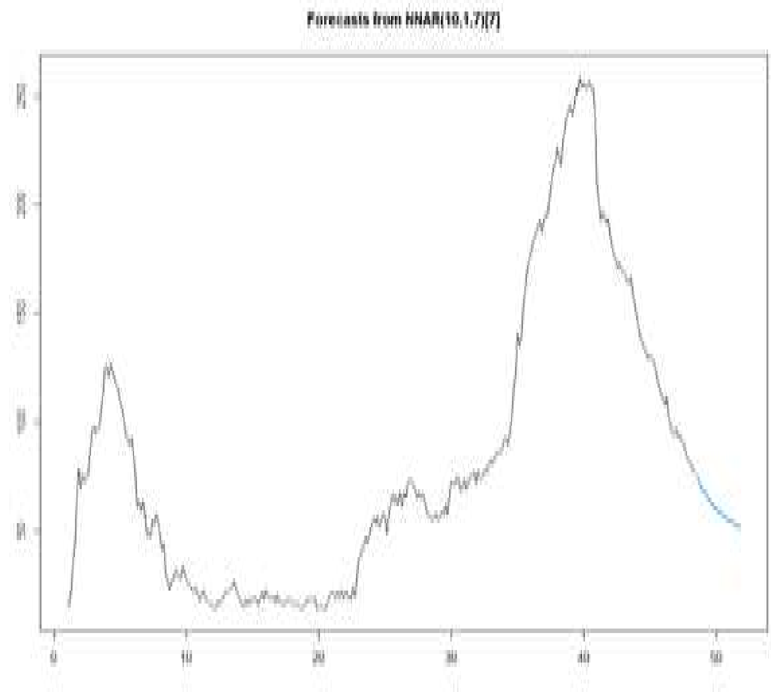

Figure 3.7: Prediction with $NNAR(10,1,7)_7$ model

model in the NNAR model. This model could also be achieved with SARIMA. In general, this process has not been tried, as the NNAR model gives better results than the SARIMA model.

5. Conclusion

A SARIMA, NNAR and SARIMA-NNAR models were developed to forecast COVID-19 disease cases in Turkey. In this study, from March 11, 2020, to February 22, 2021, in Turkey to estimate the number of cases of deaths from COVID-19 different models were tested and determined to be the most appropriate model NNAR model. This model is the model that gives the most appropriate result according to the comparison criteria. The NNAR model is the most suitable model for estimating the number of deaths and will give managers an idea to prevent and control the increase in deaths.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

All data generated or analyzed during this study are included in this published article. The dataset (the per day number of COVID-19 cases from March 11, 2020 to February 22, 2021) used and/or analyzed during the current study were obtained from web-site of Republic of Turkey of Health Ministry [11].

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 5 (1) (2022) 24-31
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1067101

Some Curvature Tensor Relations on Nearly Cosymplectic Manifolds with Tanaka-Webster Connection

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Article Info

Keywords: *Curvature tensors, nearly cosymplectic manifolds, Tanaka-Webster connection.*
2010 AMS: *53B25, 53C25.*
Received: *3 February 2022*
Accepted: *7 March 2022*
Available online: *8 March 2022*

Abstract

In this article, some curvature properties with respect to Tanaka-Webster connection on nearly cosymplectic manifolds have been studied.

1. Introduction

In 1959 Liberman [18] and in 1967 Blair [6] described cosymplectic manifolds similar to Kähler manifolds as odd-dimensional, respectively. Later in 1970, nearly Kähler's structure manifolds (M,J,g) were introduced by Gray as almost Hermitian manifolds. According to the Levi-Civita connection, the covariant derivative of the almost complex structure is skew symmetric operator. The covariant derivative operator also satisfies

$$(\nabla_X J)X = 0,$$

for every vector field X on M [17]. Blair defined an almost contact manifold with Killing structure tensors the following year, which is a nearly cosymplectic manifold [5]. Furthermore, $(2m+1)-$dimensional manifold M is a normal almost contact metric structure (J,ξ,η,g) with cosymplectic structure in which both the fundamental $2-$form Φ and $1-$form η are closed ([4, 6, 11]). Nearly cosymplectic manifolds are defined in the same way as cosymplectic (also known as coKähler) manifolds. By the way, almost contact metric structure (φ,ξ,η,g) that provides the condition

$$(\nabla_X \varphi)X = 0, \tag{1.1}$$

is called a nearly cosymplectic structure. Also a smooth manifold M which endowed with almost contact metric structure (φ,ξ,η,g) (1.1) is said to be nearly cosymplectic manifolds.

On the other hand, Tanno first explored a generalized Tanaka-Webster connection for contact metric manifolds through the canonical connection [22]. If the associated CR structure is integrable this connection coincides with the Tanaka-Webster connection. Many authors have studied the Tanaka-Webster connection later. In relation to the generalized Tanaka-Webster connection Ghosh, Prakasha, Ünal and Montano have done important studies on this connection in various structures [13, 19–21]. We, on the other hand, tried to define some curvature tensors on nearly cosymplectic structures according to the tanaka webster connection, taking into account some previous studies. In this study, after the introduction section, the definition and basic curvature properties of the nearly cosymplectic manifolds are given. In later section, according to the conditions, Riemannian curvature tensor, Ricci tensor and scalar curvature tensor of a nearly cosymplectic manifold satisfying Tanaka-Webster connection have been obtained. In section 3, Weyl projective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, $M-$ projective curvature tensor, pseudo projective and quasi conformal curvature tensor with respect to the generalized Tanaka-Webster connection on nearly cosymplectic manifolds have been defined and finally new studies that can be done depending on these curvature tensor definitions are mentioned.

Throughout this study R is the Riemannian curvature tensor, S is the Ricci tensor, Q is the Ricci operator, r is the scaler curvature tensor.

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2. Nearly Cosymplectic Manifolds

We present some information and curvature properties of nearly cosymplectic manifolds in this section.
Let (M,φ,ξ,η,g) be an $n=(2m+1)-$ dimensional almost contact Riemannian manifold, where φ is a $(1,1)-$tensor field, ξ is the structure vector field, η is a $1-$form and g is the Riemannian metric. This (φ,ξ,η,g)-structure satisfies the following conditions [4],

$$\varphi\xi=0,\quad \eta(\varphi X)=0,\quad \eta(\xi)=1, \tag{2.1}$$

$$\varphi^2X=-X+\eta(X)\xi,\quad \eta(X)=g(X,\xi), \tag{2.2}$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.3}$$

for any vector fields X and Y on M.
From the above definition, φ is skew-symmetric operator according to g, in order that the bilinear form $\Phi:=g(.,\varphi.)$ defines a $2-$form on M called fundamental $2-$form. An almost contact metric manifold satisfying $d\eta=2\Phi$ is called a contact metric manifold. Under the circumstances, η is a contact form, i.e., $\eta\wedge(d\eta)^n\neq 0$ everywhere on M [12].
A nearly cosymplectic manifold with the (M,φ,ξ,η,g) form is an almost contact metric manifold such that

$$(\nabla_X\varphi)Y+(\nabla_Y\varphi)X=0, \tag{2.4}$$

for all vector fields X,Y. It is clear that this condition is equivalent to $(\nabla_X\varphi)X=0$.
It is known that in a nearly cosymplectic manifold with the Reeb vector field ξ is Killing and satisfies the $\nabla_\xi\xi=0$ and $\nabla_\xi\eta=0$ conditions. Also the tensor field H of type $(1,1)$ defined by

$$\nabla_X\xi=HX, \tag{2.5}$$

is skew symmetric and anti-commutative with φ. Also H providing $H\xi=0$, $\eta\circ H=0$ features and the following formulas hold ([3, 12, 14, 15, 25]):

$$(\nabla_\xi\varphi)X=\varphi HX=\frac{1}{3}(\nabla_\xi\varphi)X,$$

$$g((\nabla_X\varphi)Y,HZ)=\eta(Y)g(H^2X,\varphi Z)-\eta(X)g(H^2Y,\varphi Z), \tag{2.6}$$

$$(\nabla_XH)Y=g(H^2X,Y)\xi-\eta(Y)H^2X, \tag{2.7}$$

$$tr(H^2)=constant, \tag{2.8}$$

$$R(Y,Z)\xi=\eta(Y)H^2Z-\eta(Z)H^2Y, \tag{2.9}$$

$$S(\xi,Z)=-\eta(Z)tr(H^2), \tag{2.10}$$

$$S(\varphi Y,Z)=S(Y,\varphi Z),\quad \varphi Q=Q\varphi, \tag{2.11}$$

$$S(\varphi Y,\varphi Z)=S(Y,Z)+\eta(Y)\eta(Z)tr(H^2). \tag{2.12}$$

3. Properties of Nearly Cosymplectic Manifolds Satisfying Tanaka Webster Connection

We associate $\widetilde{\nabla}$ with the quantities with respect to the generalized Tanaka-Webster connection throughout this paper.
The generalized Tanaka-Webster connection $\widetilde{\nabla}$ associated to the Levi-Civita connection ∇ is given by [24]

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi Y \tag{3.1}$$

for any vector fields X, Y on M.
Using (2.3) and (2.4), the above equation yields,

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)HX + g(\nabla_X \xi, Y)\xi - \eta(X)\phi Y \tag{3.2}$$

By taking $Y = \xi$ in (3.2) and using (1.1) and (2.3) we obtain

$$\widetilde{\nabla}_X \xi = 0 \tag{3.3}$$

We now calculate the Riemannian curvature tensor $\widetilde{R}$ using (3.2) as follows:

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z\eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi \\ &\quad - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX \\ &\quad + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi\end{aligned} \tag{3.4}$$

Using (2.5) and taking $Z = \xi$ in (3.4) we get

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi + \eta(X)H^2Y - \eta(Y)H^2X. \tag{3.5}$$

On contracting (3.4), we obtain the Ricci tensor $\widetilde{S}$ of a nearly cosymplectic manifold with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ as

$$\widetilde{S}(Y,Z) = S(Y,Z) + 2g(HY,\phi Z) - \eta(Y)(div\phi)(Z) + g(Z,HY)tr(H) - \eta(Z)\eta(Y)tr(H^2) - \eta(Y)\eta(Z)tr(\phi H) + 2g(HZ,HY) \tag{3.6}$$

This gives

$$\widetilde{Q}Y = QY - tr(H^2)Y. \tag{3.7}$$

Contracting with respect to Y and Z in (3.6) , we get

$$\widetilde{r} = r - tr(H^2)(2m+1), \tag{3.8}$$

where $\widetilde{r}$ and r are the scalar curvatures with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ respectively.

4. Some Curvature Tensors On Nearly Cosymplectic Manifolds with Respect to the Generalized Tanaka Webster Connection

The exploration of curvature tensors and curvature properties has an important place in the literature within the scope of the study of structures on Riemannian manifolds. The properties provided by a curvature tensor give us important information about the structure of the manifold.
Until this time, curvature tensors have been defined by many mathematicians and their properties have been studied. Some of these curvature tensors are; weyl protective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor $M-$projective curvature tensor ... etc.
In this study, we tried to define Weyl protective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor $M-$ projective curvature tensor, pseudo projective curvature tensor and quasi conformal curvature tensor with respect to the generalized Tanaka-Webster connection on nearly cosymplectic manifolds, based on the curvature tensors previously defined.
In the $n-$dimensional space Vn, the Weyl projective curvature tensor is given by [12].

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2m}\{S(Y,Z)X - S(X,Z)Y\}.$$

where g is the associated Riemannian metric, R, S are Riemannian curvature tensor, Ricci tensor respectively.
Based on this definition, we give the Weyl curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka-Webster connection is as follows.

Definition 4.1. *On a nearly cosymlectic manifold M of dimension $n > 2$, the Weyl projective curvature tensor $\widetilde{W}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{W}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{2m}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y] \tag{4.1}$$

for all vector fields X,Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$are the Riemannian curvature tensor and Ricci tensor respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.2. *On a nearly cosymlectic manifold M of dimension $n > 2$, the Weyl projective curvature tensor $\widetilde{W}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{W}(X,Y)Z = {} & R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z \\
& - \eta(Y)g(HX,\phi Z)\xi + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX \\
& + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi - \frac{1}{2m}[S(Y,Z)X + 2g(HY,\phi Z)X - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X \\
& - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X + 2g(HZ,HY)XS(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY - g(Z,HX)tr(H)Y \\
& + \eta(Z)\eta(X)tr(H^2)Y + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y]
\end{aligned}$$

Proof. Using (3.4) and (3.6) in (4.1), we have the equation above. □

In the $n-$dimensional space Vn, the concircular curvature tensor is given by [12].

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2m(2m+1)}\{g(Y,Z)X - g(X,Z)Y\}$$

where g is the associated Riemannian metric, R, S and r are Riemannian curvature tensor, Ricci tensor and scalar curvature tensor respectively. Based on this definition, we give the concircular curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka-Webster connection is as follows.

Definition 4.3. *The concircular curvature tensor [26] $\widetilde{C}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{C}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{\widetilde{r}}{2m(2m+1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{4.2}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$ and $\widetilde{r}$are the Riemannian curvature tensor, scalar curvature tensor respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.4. *In a nearly cosymlectic manifold M, the concircular curvature $\widetilde{W}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{C}(X,Y)Z = {} & R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z \\
& + \eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi \\
& + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX \\
& + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi - \frac{r - tr(H^2)(2m+1)}{2m(2m+1)}\{g(Y,Z)X - g(X,Z)Y\}
\end{aligned}$$

Proof. Using (3.4) and (3.8) in (4.2), we have the equation above. □

In the $n-$dimensional space Vn, the conharmonic curvature tensor is given by [12].

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{(2m-1)}\{S(Y,Z)X - gSX,Z)Y + g(Y,Z)QX - g(X,Z)QY\}$$

where g is the associated Riemannian metric, R, S and Q are Riemannian curvature tensor, Ricci tensor and the Ricci operator respectively. Based on this definition, we give the conharmonic curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka Webster connection is as follows.

Definition 4.5. *On a nearly cosymlectic manifold M of dimension $n>2$, the conharmonic curvature tensor $\widetilde{K}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{K}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{1}{(2m-1)}[\widetilde{S}(Y,Z)X-\widetilde{S}(X,Z)Y+g(Y,Z)\widetilde{Q}X-g(X,Z)\widetilde{Q}Y] \tag{4.3}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$and $\widetilde{Q}$are the Riemannian curvature tensor, Ricci tensor and Ricci operator, respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.6. *On a nearly cosymlectic manifold M of dimension $n>2$, the conharmonic curvature tensor $\widetilde{K}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by [1]*

$$\begin{aligned}\widetilde{K}(X,Y)Z&=R(X,Y)Z-g(Z,HX)HY-g(H^2Y,Z)\eta(X)\xi-2g(Y,HX)\phi Z+\eta(X)\eta(Z)\phi HY+g(H^2X,Z)\eta(Y)\xi-\eta(Y)(\nabla_X\phi)Z\\&-\eta(Y)g(HX,\phi Z)\xi+g(Z,HY)HX+\eta(Z)\eta(X)H^2Y-\eta(Z)\eta(Y)H^2X-\eta(Y)\eta(Z)\phi HX+\eta(X)(\nabla_Y\phi)Z\\&+\eta(X)g(HY,\phi Z)\xi-\frac{1}{(2m-1)}[S(Y,Z)X+2g(HY,\phi Z)X-\eta(Y)div(\phi)ZX+g(Z,HY)tr(H)X-\eta(Z)\eta(Y)tr(H^2)X\\&-\eta(Y)\eta(Z)tr(\phi H)X+2g(HZ,HY)X-S(X,Z)Y-2g(HX,\phi Z)Y+\eta(X)div(\phi)ZY-g(Z,HX)tr(H)Y+\eta(Z)\eta(X)tr(H^2)Y\\&+\eta(X)\eta(Z)tr(\phi H)Y-2g(HZ,HX)Y+g(Y,Z)QX-g(Y,Z)tr(H^2)X-g(X,Z)QY+g(X,Z)tr(H^2)Y].\end{aligned}$$

Proof. Using (3.4) ,(3.6) and (3.7) in (4.3), we have the equation above. □

In the $n-$dimensional space Vn, the conformal curvature tensor is given by [12].

$$V(X,Y)Z=R(X,Y)Z-\frac{1}{2m-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]+\frac{r}{(2m)(2m-1)}[g(Y,Z)X-g(X,Z)Y]$$

where g is the associated Riemannian metric, R, S, r and Q are Riemannian curvature tensor, Ricci tensor, scalar curvature tensor and the Ricci operator respectively.
Based on this definition, we give the conformal curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka Webster connection is as follows.

Definition 4.7. *On a nearly cosymlectic manifold M of dimension $n>2$, the conformal curvature tensor $\widetilde{V}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{V}(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{1}{2m-1}[\widetilde{S}(Y,Z)X-\widetilde{S}(X,Z)Y+g(Y,Z)\widetilde{Q}X-g(X,Z)\widetilde{Q}Y]+\frac{\widetilde{r}}{(2m)(2m-1)}[g(Y,Z)X-g(X,Z)Y] \tag{4.4}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$, $\widetilde{Q}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.8. *On a nearly cosymlectic manifold M of dimension $n>2$, the conformal curvature tensor $\widetilde{V}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}\widetilde{V}(X,Y)Z&=R(X,Y)Z-g(Z,HX)HY-g(H^2Y,Z)\eta(X)\xi-2g(Y,HX)\phi Z+\eta(X)\eta(Z)\phi HY+g(H^2X,Z)\eta(Y)\xi-\eta(Y)(\nabla_X\phi)Z\\&-\eta(Y)g(HX,\phi Z)\xi+g(Z,HY)HX+\eta(Z)\eta(X)H^2Y\\&-\eta(Z)\eta(Y)H^2X-\eta(Y)\eta(Z)\phi HX+\eta(X)(\nabla_Y\phi)Z+\eta(X)g(HY,\phi Z)\xi\\&-\frac{1}{2m-1}[S(Y,Z)X+2g(HY,\phi Z)X-\eta(Y)(div\phi)(Z)X+g(Z,HY)tr(H)X\\&-\eta(Z)\eta(Y)tr(H^2)X-\eta(Y)\eta(Z)tr(\phi H)X+2g(HZ,HY)X\\&-S(X,Z)Y-2g(HX,\phi Z)Y+\eta(X)div(\phi)ZY-g(Z,HX)tr(H)Y\\&+\eta(Z)\eta(X)tr(H^2)Y+\eta(X)\eta(Z)tr(\phi H)Y-2g(HZ,HX)Y\\&+g(Y,Z)QX-g(Y,Z)tr(H^2)X-g(X,Z)QY+g(X,Z)tr(H^2)Y]+\frac{r-tr(H^2)(2m+1)}{(2m)(2m-1)}[g(Y,Z)X-g(X,Z)Y]\end{aligned}$$

Proof. Using (3.4), (3.6), (3.7) and (3.8) in (4.4), we have the equation above. □

In 1971 on an $n-$dimensional Riemannian manifold, G. P. Pokhariyal and R.S. Mishra [22] defined $M-$projective curvature tensor field as

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{4m}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

It has been further studied in [7–9].

Definition 4.9. *On a nearly cosymlectic manifold M of dimension $n > 2$, the $M-$projective curvature tensor $\widetilde{M}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{M}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{4m}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y] \tag{4.5}$$

for all vector fields X,Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and Ricci operator respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.10. *On a nearly cosymlectic manifold M of dimension $n > 2$, the $M-$projective curvature tensor $\widetilde{M}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{M}(X,Y)Z = {} & R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY \\
& + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y \\
& - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi \\
& - \frac{1}{4m}[S(Y,Z)X + 2g(HY,\phi Z)X - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X \\
& - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X + 2g(HZ,HY)X - S(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY \\
& - g(Z,HX)tr(H)Y + \eta(Z)\eta(X)tr(H^2)Y \\
& + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y + g(Y,Z)QX - g(Y,Z)tr(H^2)X - g(X,Z)QY + g(X,Z)tr(H^2)Y]
\end{aligned}$$

Proof. Using (3.4), (3.6) and (3.7) in (4.5), we have the equation above. □

In 2002 on a $n-$dimensional $n > 2$ Riemannian manifold, Prasad [12] defined pseudo-projective curvature tensor P as

$$P(X,Y)Z = \alpha R(X,Y)Z + \beta\,[S(Y,Z)X - S(X,Z)Y] - \frac{r}{2m+1}\left(\frac{\alpha}{2m} + \beta\right)[g(Y,Z)X - g(X,Z)Y]$$

where α, β are non-zero constants, g is the associated Riemannian metric, R, S and r are Riemannian curvature tensor, Ricci tensor and scalar curvature tensor respectively.
Based on this definition, we give the quasi conformal curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka Webster connection is as follows.

Definition 4.11. *On a nenarly cosymlectic manifold M of dimension $n > 2$, the pseuso-projective curvature tensor $\widetilde{P}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{P}(X,Y)Z = \alpha\widetilde{R}(X,Y)Z + \beta\left[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y\right] - \frac{\widetilde{r}}{2m+1}\left(\frac{\alpha}{2m} + \beta\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.6}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$, $\widetilde{Q}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.12. *On a nenarly cosymlectic manifold M of dimension $n > 2$, the pseuso-projective curvature tensor $\widetilde{P}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{P}(X,Y)Z = {} & \alpha[R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi \\
& + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi] - \beta[S(Y,Z)X + 2g(HY,\phi Z)X \\
& - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X \\
& + 2g(HZ,HY)X - S(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY - g(Z,HX)tr(H)Y \\
& + \eta(Z)\eta(X)tr(H^2)Y + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y] - \frac{r - tr(H^2)(2m+1)}{2m+1}\left(\frac{\alpha}{2m} + \beta\right)[g(Y,Z)X - g(X,Z)Y]
\end{aligned}$$

where α and β are constants such that $\alpha = 1$ and $\beta = -\frac{1}{2m}$ then the definition of pseudo projective curvature tensor takes the form

$$\widetilde{P}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{(n-1)}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y] = \widetilde{W}(X,Y)Z$$

where $\widetilde{W}$ is the Weyl projective curvature tensor of a nearly cosymlectic manifold.

Proof. Using (3.4), (3.6), (3.7) and (3.8) in (4.6), we have the equation above. □

Quasi-conformal curvature tensor has introduced by K. Yano and S. Sawaki in 1968 for or a $n-$dimensional Riemannian manifold and the quasi-conformal curvature tensor Q is given by [12].

$$Q(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{2m+1}\left(\frac{a}{2m} + 2b\right)[g(Y,Z)X - g(X,Z)Y]$$

where a and b are two scalars, and r is the scalar curvature of the manifold.
Based on this definition, we give the quasi conformal curvature tensor for a nearly cosypmlectic manifold with respect to the generalized Tanaka-Webster connection is as follows.

Definition 4.13. *On a nearly cosymlectic manifold M of dimension $n > 2$, the quasi conformal curvature tensor $\widetilde{Q}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is defined by*

$$\widetilde{Q}(X,Y)Z = a\widetilde{R}(X,Y)Z + b\left[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y\right] - \frac{\widetilde{r}}{2m+1}\left(\frac{a}{2m} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \tag{4.7}$$

for all vector fields X, Y and Z on M, where $\widetilde{R}$, $\widetilde{S}$, $\widetilde{Q}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with respect to the connection $\widetilde{\nabla}$.

Theorem 4.14. *On a nearly cosymlectic manifold M of dimension $n > 2$, the quasi conformal curvature tensor $\widetilde{Q}$ with respect to the generalized Tanaka-Webster connection $\widetilde{\nabla}$ is given by*

$$\begin{aligned}
\widetilde{Q}(X,Y)Z = {} & a[R(X,Y)Z - g(Z,HX)HY - g(H^2Y,Z)\eta(X)\xi - 2g(Y,HX)\phi Z + \eta(X)\eta(Z)\phi HY + g(H^2X,Z)\eta(Y)\xi - \eta(Y)(\nabla_X\phi)Z - \eta(Y)g(HX,\phi Z)\xi \\
& + g(Z,HY)HX + \eta(Z)\eta(X)H^2Y - \eta(Z)\eta(Y)H^2X - \eta(Y)\eta(Z)\phi HX + \eta(X)(\nabla_Y\phi)Z + \eta(X)g(HY,\phi Z)\xi] + b[S(Y,Z)X + 2g(HY,\phi Z)X \\
& - \eta(Y)(div\phi)(Z)X + g(Z,HY)tr(H)X - \eta(Z)\eta(Y)tr(H^2)X - \eta(Y)\eta(Z)tr(\phi H)X + 2g(HZ,HY)X - S(X,Z)Y - 2g(HX,\phi Z)Y + \eta(X)div(\phi)ZY \\
& - g(Z,HX)tr(H)Y + \eta(Z)\eta(X)tr(H^2)Y + \eta(X)\eta(Z)tr(\phi H)Y - 2g(HZ,HX)Y + g(Y,Z)QX - g(Y,Z)tr(H^2)X - g(X,Z)QY + g(X,Z)tr(H^2)Y] \\
& - \frac{r - tr(H^2)(2m+1)}{2m+1}\left(\frac{a}{2m} + 2b\right)[g(Y,Z)X - g(X,Z)Y].
\end{aligned}$$

From the definition of quasi-conformal curvature tensor, if we take $a = 1$ and $b = -\frac{1}{2m-1}$, then the above equality takes the form as

$$\widetilde{Q}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{1}{2m-1}[\widetilde{S}(Y,Z)X - \widetilde{S}(X,Z)Y + g(Y,Z)\widetilde{Q}X - g(X,Z)\widetilde{Q}Y] + \frac{\widetilde{r}}{(2m)(2m-1)}[g(Y,Z)X - g(X,Z)Y] = \widetilde{V}(X,Y)Z \tag{4.8}$$

where $\widetilde{V}$ is the conformal curvature tensor.

Proof. Using (3.4), (3.6), (3.7) and (3.8) in (4.7), we have the equation above. □

5. Conclusion

In this study, various curvature tensors have been defined in nearly cosyplectic structures with respect to the generalized Tanaka-Webster connection. Starting from these new connection curvature tensor definitions, conditions of being flat, conditions of being symmetri such as Ricci symmetric, semi-symmetric, conditions of being recurrent can be examined for each curvature tensor.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Universal Journal of Mathematics and Applications, 5 (1) (2022) 32-41
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1070029

Spinor Equations of Successor Curves

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Article Info

Keywords: *Euclidean space, spinors, successor curves*
2010 AMS: *15A66, 53A04, 53Z05.*
Received: *8 February 2022*
Accepted: *15 March 2022*
Available online: *23 March 2022*

Abstract

The aim of this study is to give spinor representation of successor curves in three-dimensional Euclidean space. In this study, the spinor representations of a curve with arc-length parameter and a successor curve of this curve with some arc-length parameter in three dimensional Euclidean space have been studied. For this, first of all, the curve with unit speed and its successor curve have been corresponded to two different spinors. Then, using the relationships between these curves, the relationships between the spinors corresponding to these curves have been given. Therefore, geometric interpretations of these curves and corresponding spinors have been made. In addition, different spinor equations of the mates and derivatives of these spinors have been examined and geometric interpretations of these spinor equations have been given. Then, spinor equations have been obtained in case the successor curves are helices. Consequently, two examples have been given.

1. Introduction

Curve theory is one of the most studied areas of differential geometry, with studies in different dimensions and spaces. Curve and curvature studies by Newton and Leibniz form the basis of curve theory. After giving the definition of curvature by Euler in the 18th century, space curves were defined by Monge. Later, in the 19th century, the equations, known today as Serret-Frenet frame and formulas, were studied by Serret and Frenet at different times. In general, the geometric locus of different positions taken by a moving point in space during motion is a curve. In this motion, the parameter range of the curve represents the time elapsed during the motion. In addition, the parameter of the curve has an important place in characterizing the curve. That is, the parameter of a given curve has many differences depending on whether the curve is parameterized in terms of arc length. The parametrization of the curve in terms of arc length provides great convenience in the characterization of the curve. On the other hand, considering any two curves in space, various special curves are defined by establishing some different relations between the Frenet vectors at the opposite points of these curves such as, in 1850, Bertrand curve pair was defined by establishing a special relationship between the normal vectors of any two curves. Another special curve pair is the Mannheim curve pair, which was given by A. Mannheim in 1878 and obtained by establishing a relationship between the normal of one of the two curves and the binormal of the other [23]. Another is the involute evolute curves obtained by establishing a special relationship between the tangents of any two curves [14]. There are a lot of studies about curve theory [4, 17]. The curves that form the basis of this study are the successor curves. The definition of successor curves as "Consider a unit speed curve (α) in three-dimensional Euclidean space. If the normal vector field of a curve (β) with the same arc-length parameter as the curve (α) is the tangent vector field of the curve (α), the curve (β) is called a successor curve of the curve (α). Every Frenet curve has a family of successor curves" was given by Menninger where both the curve (α) and the curve (β) are unit speed curves with the same arc-length parameter. Therefore, there are relationships between Frenet frames and Frenet curvatures of both curves [13]. Later, Masal obtained relations depending on the ground vectors of the successor curves by defining the successor planes and investigating the geometric meanings of the successor curvatures [13]. Before from that, the predecessor transformation, as opposed to the successor transformation, although not well defined in general, was given by Bilinski [2].

Spinors are physical structures used in many fields of applied sciences. It is used in physics, especially in quantum mechanics, applications of spinor theory, electron spin and theory of relativity. The wave function of a particle with a spin of $1/2$ is called a spinor. Also, the application of spinors in electromagnetic theory is very important. A spin structure in four-dimensional space is an extension of spinors to

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obtain Dirac spinors in physics [5, 11, 22]. Spinors are column vectors and act on Pauli spin matrices that are $2x2$ dimensional complex matrices. In quantum theory, spin is expressed in Pauli matrices. Cartan [3] first studied spinors geometrically. In that study by Cartan the spinor representations of basic geometric definitions were given. In Cartan's study, it was emphasized that the set of isotropic vectors in the vector space $\mathbb{C}^3$ creates a two-dimensional surface in the space $\mathbb{C}^2$. In that study of Cartan, it is seen that each isotropic vector in the complex vector space $\mathbb{C}^3$ corresponds to a vector with two components in the space $\mathbb{C}^2$. Cartan named these two-dimensional vectors as spinors [3]. The spinor algebra with two complex components acting on Pauli matrices representing $SU(2)$, a group of $2x2$ dimensional unitary matrices, gives a different representation of rotations in three-dimensional real vector space. Using this information, Vivarelli established a relationship between spinors and quaternions and obtained the quaternion representation of rotations in three-dimensional Euclidean space with spinors [21]. Then, the spinor representation of curves in three-dimensional Euclidean space was given by Torres del Castillo and Barrales [18]. They expressed the Frenet vectors and curvatures of curves in terms of spinors in that study [18]. Therefore, that study greatly has contributed to this our study. Based on this study, a spinor formulation of the Darboux frame of a curve on a surface of three-dimensional Euclidean space and the relationship between Frenet and Darboux frames was given by Kişi and Tosun [10]. After that, in another study, the spinor formulation of Bishop's frame of curves in three-dimensional Euclidean space was obtained [20]. In addition, Erişir and Kardağ obtained spinor equations of involute-evolute curve pairs in three-dimensional Euclidean space [7]. In another study, spinor formulation of Bertrand curves was given [8]. Moreover, the spinor formulations of some special curves in three-dimensional Minkowski space were obtained based on these studies in three-dimensional Euclidean space [1, 6, 9].
In this study, a curve (α) with unit speed and a successor curve (β) with the same arc-length parameter of the curve (α) have been considered. In addition, two spinor have been corresponded to the curve (α) and the successor curve (β) of the curve (α). After that, considering the relationships between these curves, the relationships between the spinors corresponding to these curves have been obtained. In addition, the geometric interpretations have been given for the angles between these spinors. Then, the spinor equations have been obtained for the mates and derivatives of these spinors corresponding to the curve (α) and the successor curve (β), and the geometric interpretations of these equations have been made. After that, considering that the successor curve (β) is helix, some theorems and results have been obtained for the spinor equations of this curve. Consequently, two examples have been given.

2. Preliminaries

In this section, firstly, the basic definitions and theorems about successor curves have been given. Then, the basic definitions and theorems about spinors introduced by Cartan [3], which is a fundamental study for spinors, have been mentioned. In addition, the spinor equations given by Torres del Castillo and Barrales have been expressed [18]. Consequently, the spinor formulation of curves in three-dimensional Euclidean space is given.

Definition 2.1. *Consider that the curves $\alpha : I \to \mathbb{E}^3$ and $\beta : I \to \mathbb{E}^3$ have the same arc-length parameter. In that case, if the tangent vector field of the curve (α) is the normal vector field of the curve (β), then the curve (β) is defined as a successor curve of the curve (α). Each Frenet curve has a family of successor curves [13].*

Theorem 2.2. *Let $\alpha, \beta : I \to \mathbb{E}^3$ be two curves which have same arc-length parameter and the curve (β) be the successor curve of the curve (α). In that case, we consider that the Frenet system of the curve (α) is $F = \{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}, \kappa, \tau\}$ and the successor system of the successor curve (β) is $F_1 = \{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1, \kappa_1, \tau_1\}$. Therefore, there are the relationships between of these systems as*

$$\begin{pmatrix} \boldsymbol{T}_1 \\ \boldsymbol{N}_1 \\ \boldsymbol{B}_1 \end{pmatrix} = \begin{pmatrix} 0 & -\cos\vartheta & \sin\vartheta \\ 1 & 0 & 0 \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} \tag{2.1}$$

and

$$\begin{pmatrix} \kappa_1 \\ \tau_1 \end{pmatrix} = \kappa \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \tag{2.2}$$

where the angle ϑ is the angle between the binormal vectors $\boldsymbol{B}$ and $\boldsymbol{B}_1$. Moreover, for the torsion τ of the curve (α) the equation

$$\vartheta(s) = \vartheta_0 + \int \tau(s)ds$$

is hold where $\vartheta_0 = constant \in \mathbb{R}$ [13].

Theorem 2.3. *Suppose that the curve (β) is successor curve of the curve (α) and the vector $\boldsymbol{D}_1$ is Darboux vector of the successor curve (β). Therefore, Darboux vector is*

$$\boldsymbol{D}_1 = \kappa \boldsymbol{B} \tag{2.3}$$

where κ and $\boldsymbol{B}$ are the curvature and binormal vector field of the curve (α), respectively [13].

Theorem 2.4. *Consider that the curve (β) is successor curve of the curve (α) and the successor system of the successor curve (β) is $F_1 = \{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1, \kappa_1, \tau_1\}$. If the angle of intersection of tangent vector field $\boldsymbol{T}_1$ with a constant vector is always constant, then the successor curve (β) is defined helix [12, 13].*

Theorem 2.5. *Let the curve (β) be successor curve of the curve (α). In that case, if the successor curve (β) is helix, then the ratio of the curvatures of (β) $\frac{\kappa_1}{\tau_1}$ is constant [12].*

Theorem 2.6. *Consider that the curve (β) is successor curve of the curve (α). The successor curve (β) is helix, then the necessary and sufficient condition is that the curve (α) is planar curve [13].*

Spinors construct a vector space usually built on complex numbers with the aid of a linear group representation of the spin group. Cartan [3] expressed the spinors over the complex numbers geometrically. Therefore, Cartan gave that a vector γ with two complex components corresponds to an isotropic vector $\boldsymbol{x}=(x_1,\,x_2,\,x_3)$ in three dimensional complex vector space $\mathbb{C}^3$. In that case, the isotropic vectors in $\mathbb{C}^3$ form a surface with two dimensional in $\mathbb{C}^2$. Now, we consider that this surface is written by the parameters γ_1 and γ_2, therefore, $x_1={\gamma_1}^2-{\gamma_2}^2$, $x_2=i({\gamma_1}^2+{\gamma_2}^2)$, $x_3=-2\gamma_1\gamma_2$ and $\gamma_1=\pm\sqrt{\frac{x_1-ix_2}{2}}$, $\gamma_2=\pm\sqrt{\frac{-x_1-ix_2}{2}}$ [3]. Cartan called these complex vectors mentioned here as spinors

$$\gamma=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$$

[3]. With the aid of the study [3], in [18] it was matched the isotropic vector $\boldsymbol{a}+i\boldsymbol{b}\in\mathbb{C}^3$ with spinor $\gamma=(\gamma_1,\gamma_2)$ where $\boldsymbol{a},\,\boldsymbol{b}\in\mathbb{R}^3$. In that case, considering the Pauli matrices (P_1,P_2,P_3), the $2x2$ dimensional complex symmetric matrices $\boldsymbol{\sigma}$, can be created as

$$\sigma_1=CP_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_2=CP_2=\begin{pmatrix}i&0\\0&i\end{pmatrix},\quad \sigma_3=CP_3=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$$

where $C=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ [15, 16, 18]. In that case, in [18] the spinor equations are given by

$$\begin{aligned}\boldsymbol{a}+i\boldsymbol{b}&=\gamma^t\sigma\gamma,\\ \boldsymbol{c}&=-\hat{\gamma}^t\sigma\gamma\end{aligned}$$

where $\boldsymbol{a}+i\boldsymbol{b}$ is the isotropic vector in the space $\mathbb{C}^3$, $\boldsymbol{c}\in\mathbb{R}^3$ and the spinor mate $\hat{\gamma}$ of the spinor γ is

$$\hat{\gamma}=-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\overline{\gamma}=-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\overline{\gamma_1}\\ \overline{\gamma_2}\end{pmatrix}=\begin{pmatrix}-\overline{\gamma_2}\\ \overline{\gamma_1}\end{pmatrix}.$$

For the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ we know that these vectors have the same length $\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=\|\boldsymbol{c}\|=\overline{\gamma}^t\gamma$ and are orthogonal to each other. In addition to that, the triads $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$, $\{\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}\}$ and $\{\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}\}$ correspond to different spinors [18].

Proposition 2.7. *Let two arbitrary spinors be γ and ψ. Then, the following statements hold;*

$$\begin{array}{ll} i) & \overline{\psi^t\sigma\gamma}=-\hat{\psi}^t\sigma\hat{\gamma}\\ ii) & \widehat{\lambda\psi+\mu\gamma}=\overline{\lambda}\hat{\psi}+\overline{\mu}\hat{\gamma}\\ iii) & \hat{\hat{\gamma}}=-\gamma\\ iv) & \psi^t\sigma\gamma=\gamma^t\sigma\psi\end{array}$$

where $\lambda,\,\mu\in\mathbb{C}$ and "$-$" is complex conjugate [18].

Now, let a curve parameterized by arc-length be $\alpha:I\to\mathbb{E}^3$, $(I\subseteq\mathbb{R})$. Therefore, $\|\alpha'(s)\|=1$ where s is the arc-length parameter of (α). In addition to that, consider that the Frenet frame of this curve $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ and the spinor ξ represents Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$. Therefore, from equation (2.3) the following equations can be written as

$$\begin{aligned}\boldsymbol{N}+i\boldsymbol{B}&=\xi^t\sigma\xi=(\xi_1^2-\xi_2^2,\;i(\xi_1^2+\xi_2^2),\;-2\xi_1\xi_2),\\ \boldsymbol{T}&=-\widehat{\xi}^t\sigma\xi=(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2,\;i(\xi_1\overline{\xi_2}-\overline{\xi_1}\xi_2),\;|\xi_1|^2-|\xi_2|^2)\end{aligned}$$

with $\overline{\xi}^t\xi=1$ [18]. Moreover, the following theorem can be given.

Theorem 2.8. *If the spinor ξ with two complex components represents Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of a curve (α) parameterized by its arc-length s, the Frenet curvatures are equivalent to the single spinor equation*

$$\frac{d\xi}{ds}=\frac{1}{2}(-i\tau\xi+\kappa\hat{\xi})\tag{2.4}$$

where κ and τ are the curvature and torsion of (α), respectively [18].

3. Main Theorems and Proofs

3.1. Spinor Representation of Successor Curves

Let $\alpha:I\to\mathbb{E}^3$ be a curve with unit speed which has the arc-length parameter s and Frenet frame of this curve be $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$. Therefore, we know that if the spinor corresponds to Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α), then the spinor equations of these Frenet vectors can be written

$$\begin{aligned}\boldsymbol{N}+i\boldsymbol{B}&=\xi^t\sigma\xi,\\ \boldsymbol{T}&=-\hat{\xi}^t\sigma\xi\end{aligned}$$

[18]. In addition, the vectors of Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ can be written as

$$\begin{aligned}\boldsymbol{T} &= -\hat{\xi}^t\sigma\xi,\\ \boldsymbol{N} &= \frac{1}{2}(\xi^t\sigma\xi-\hat{\xi}^t\sigma\hat{\xi}),\\ \boldsymbol{B} &= \frac{-i}{2}(\xi^t\sigma\xi+\hat{\xi}^t\sigma\hat{\xi})\end{aligned} \tag{3.1}$$

in terms of spinors, separately [7].

Now, consider that the curve (β), which has the same arc-length-parameter s with the curve (α), is a successor curve of the curve (α). Moreover, suppose that Frenet frame of the successor curve (β) is $\{\boldsymbol{T}_1,\boldsymbol{N}_1,\boldsymbol{B}_1\}$ and the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) where we know that the different spinors correspond to Frenet frames $\{\boldsymbol{T}_1,\boldsymbol{N}_1,\boldsymbol{B}_1\}$, $\{\boldsymbol{N}_1,\boldsymbol{B}_1,\boldsymbol{T}_1\}$ and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$. In that case, we can write

$$\begin{aligned}\boldsymbol{B}_1+i\boldsymbol{T}_1 &= \phi^t\sigma\phi,\\ \boldsymbol{N}_1 &= -\hat{\phi}^t\sigma\phi\end{aligned} \tag{3.2}$$

and give the following theorem.

Theorem 3.1. *Suppose that the curve (β) is a successor curve of the curve (α) and the curves (α) and (β) have the same arc-length parameter s. In that case, if the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), then the spinor equation of the curvatures $\{\kappa_1,\tau_1\}$ of the successor curve (β) is*

$$\frac{d\phi}{ds}=\frac{\tau_1-i\kappa_1}{2}\hat{\phi}. \tag{3.3}$$

Proof. Let ϕ be the spinor corresponding to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) of the curve (α). Therefore, since the spinor pair $\{\phi,\hat{\phi}\}$ forms a basis in the spinor space, the spinor $\frac{d\phi}{ds}$ can be written

$$\frac{d\phi}{ds}=f\phi+g\hat{\phi} \tag{3.4}$$

where f and g are two arbitrary complex functions. On the other hand, if we take derivative of the complex vector $\boldsymbol{B}_1+i\boldsymbol{T}_1$ in the equation (3.2) in terms of the arc-length parameter $s\in I$, we get

$$\frac{d\boldsymbol{B}_1}{ds}+i\frac{d\boldsymbol{T}_1}{ds}=\frac{d\phi}{ds}^t\sigma\phi+\phi^t\sigma\frac{d\phi}{ds}$$

and with the aid of the equation (3.4) we have

$$(-\tau_1+i\kappa_1)\boldsymbol{N}_1=f\phi^t\sigma\phi+g\hat{\phi}^t\sigma\phi+f\phi^t\sigma\phi+g\phi^t\sigma\hat{\phi}.$$

In addition to that, if we use the option *iv*) in Proposition 2.7, we have

$$(-\tau_1+i\kappa_1)\,\boldsymbol{N}_1=2f(\phi^t\sigma\phi)+2g(\hat{\phi}^t\sigma\phi).$$

Therefore, we obtain $f=0$ and $g=\frac{\tau_1-i\kappa_1}{2}$. As a result, we get $\frac{d\phi}{ds}=\left(\frac{\tau_1-i\kappa_1}{2}\right)\hat{\phi}$. Consequently, the proof is completed by expressing the curvature κ_1 and torsion τ_1 of the successor curve (β) as a single spinor equation. □

Theorem 3.2. *Let (α) and (β) be two curves which have the same arc-length parameter in Euclidean space $\mathbb{E}^3$ and the curve (β) be the successor curve of the curve (α). Therefore, the spinor equations of the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) are*

$$\begin{aligned}\boldsymbol{B}_1 &= \frac{1}{2}(\phi^t\sigma\phi-\hat{\phi}^t\sigma\hat{\phi}),\\ \boldsymbol{T}_1 &= \frac{-i}{2}(\phi^t\sigma\phi+\hat{\phi}^t\sigma\hat{\phi}),\\ \boldsymbol{N}_1 &= -\hat{\phi}^t\sigma\phi.\end{aligned} \tag{3.5}$$

Proof. Consider that the curve (β) is the successor curve of the curve (α) and the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β). Therefore, considering the spinor equation $\boldsymbol{B}_1+i\boldsymbol{T}_1=\phi^t\sigma\phi$ in the equation (3.2) we see that

$$\begin{aligned}\boldsymbol{T}_1 &= \mathrm{Im}(\phi^t\sigma\phi),\\ \boldsymbol{B}_1 &= \mathrm{Re}(\phi^t\sigma\phi).\end{aligned}$$

In that case, we obtain $\boldsymbol{T}_1=\frac{-i}{2}(\phi^t\sigma\phi-\overline{\phi^t\sigma\phi})$, $\boldsymbol{B}_1=\frac{1}{2}(\phi^t\sigma\phi+\overline{\phi^t\sigma\phi})$ and, consequently, considering the option *i*) in Proposition 2.7 we have

$$\begin{aligned}\boldsymbol{T}_1 &= \frac{-i}{2}(\phi^t\sigma\phi+\hat{\phi}^t\sigma\hat{\phi}),\\ \boldsymbol{B}_1 &= \frac{1}{2}(\phi^t\sigma\phi-\hat{\phi}^t\sigma\hat{\phi}).\end{aligned}$$

We also know the spinor equation of the normal vector field $\boldsymbol{N}_1=-\hat{\phi}^t\sigma\phi$ in the equation (3.2). □

In addition to Theorem 3.2, we see that Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) can be written

$$\begin{aligned}
\boldsymbol{B}_1 &= \frac{1}{2}\left(\phi_1^2-\phi_2^2-\overline{\phi_2}^2+\overline{\phi_1}^2, i(\phi_1^2+\phi_2^2-\overline{\phi_1}^2-\overline{\phi_2}^2), -2\phi_1\phi_2-2\overline{\phi_1}\,\overline{\phi_2}\right),\\
\boldsymbol{T}_1 &= \frac{-i}{2}\left(\phi_1^2-\phi_2^2+\overline{\phi_2}^2-\overline{\phi_1}^2, i(\phi_1^2+\phi_2^2+\overline{\phi_1}^2+\overline{\phi_2}^2), -2\phi_1\phi_2+2\overline{\phi_1}\,\overline{\phi_2}\right),\\
\boldsymbol{N}_1 &= \left(\phi_1\overline{\phi_2}+\overline{\phi_1}\phi_2, i(\phi_1\overline{\phi_2}-\overline{\phi_1}\phi_2), |\phi_1|^2-|\phi_2|^2\right)
\end{aligned} \tag{3.6}$$

in terms of the components of the spinor $\phi=\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$ with easy calculations.

Now, we give the relationship between the spinors ξ and ϕ with following theorem.

Theorem 3.3. *Suppose that (α) and (β) have the same arc-length parameter in Euclidean space $\mathbb{E}^3$ and the curve (β) is the successor curve of the curve (α). In addition, the spinor pair (ξ,ϕ) corresponds to the Frenet frames $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the curves (α,β), respectively. Therefore, there is the relationship between these spinors ξ and ϕ as*

$$\xi=\pm e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\phi \tag{3.7}$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$.

Proof. Let ξ be the spinor corresponding to the Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α) and ϕ be the spinor corresponding to the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β). In that case, we can write for the complex vector $\boldsymbol{B}_1+i\boldsymbol{T}_1\in\mathbb{C}^3$

$$\boldsymbol{B}_1+i\boldsymbol{T}_1=-ie^{i\vartheta}(\boldsymbol{N}+i\boldsymbol{B})$$

and

$$\xi^t\sigma\xi=e^{i\left(\frac{\pi}{2}-\vartheta\right)}\phi^t\sigma\phi \tag{3.8}$$

with the aid of the equation (2.1). Therefore, we obtain that

$$\begin{aligned}
\xi_1 &= \pm\, e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\phi_1,\\
\xi_2 &= \pm\, e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\phi_2
\end{aligned}$$

and, consequently

$$\xi=\pm\, e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\phi$$

where the spinors ξ and $-\xi$ correspond to the same Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ and, similarly, the spinors ϕ and $-\phi$ correspond to the same Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$. □

Now, we give a geometric interpretation of spinor representations of the successor curves with following corollary.

Corollary 3.4. *Consider that the curve (β) is the successor curve of the curve (α) and the spinor pair (ξ,ϕ) corresponds to the Frenet frames $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the curves (α,β), respectively. Therefore, the angle between the spinors ξ and ϕ is $\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)$ where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$.*

In addition to that, similar to Theorem 3.3 we can give a relationship between the mates of spinors ξ and ϕ with following corollary.

Corollary 3.5. *Suppose that the spinors ξ and ϕ correspond to the Frenet frames $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α) and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), respectively. In that case, there is the relationship between the mates of spinors ξ and ϕ as*

$$\hat{\phi}=\pm e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\hat{\xi}$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$.

Proof. Let ξ and ϕ be two spinors corresponding to the Frenet frames of the (α) and the successor curve (β). In that case, if the operation of spinor mate is applied to both sides of the equation (3.7), we get $\hat{\xi}=\pm\left(\widehat{e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\phi}\right)$. We know *ii)* in Proposition 2.7, therefore, we have $\hat{\xi}=\pm\overline{e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}}\hat{\phi}$ and, consequently,

$$\hat{\phi}=\pm e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\hat{\xi}.$$

□

Therefore, we can obtain a geometric interpretation of spinor representations of the successor curves below.

Corollary 3.6. *Suppose that the spinors ξ and ϕ correspond to the Frenet frames $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α) and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), respectively. In this case, while the spinor ϕ rotates at an angle $\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)$ to the spinor ξ, the spinor $\hat{\phi}$ makes a rotation in the **negative** direction with the **same** angle to the spinor $\hat{\xi}$.*

Corollary 3.7. *There is the relationship between the derivative spinors $\frac{d\xi}{ds}$ and $\frac{d\phi}{ds}$ that*

$$\frac{d\xi}{ds}=\pm e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}\right)}\left(\frac{d\phi}{ds}-\frac{i}{2}\frac{d\vartheta}{ds}\phi\right)$$

where $\frac{d\vartheta}{ds}=\tau$ is the torsion of the curve (α).

3.2. Some Applications

In this section, firstly, we give the spinor equations of the Darboux vector of the successor curve (β). After that, we assume that the successor curve (β) of the curve (α) is helix and we give the spinor equations in that case. Therefore, we can express the following theorems and corollaries.

Theorem 3.8. *Let the curve (β) be the successor curve of the curve (α) and (α), (β) be the curves which have the same arc-length parameter s. If we consider that the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) and Darboux vector of the successor curve (β) is $\boldsymbol{D}_1$, then the spinor equation of Darboux vector of the successor curve is*

$$\boldsymbol{D}_1 = \frac{\kappa}{2}\left[e^{-i\vartheta}\phi^t\sigma\phi - e^{i\vartheta}\hat{\phi}^t\sigma\hat{\phi}\right]$$

where ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ of the curve (α) and $\boldsymbol{B}_1$ of the successor curve (β).

Proof. Suppose that the successor curve of the curve (α) is (β), the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) and Darboux vector of the successor curve (β) is $\boldsymbol{D}_1$. On the other hand, we know that Darboux vector of the successor curve is $\boldsymbol{D}_1 = \tau_1\boldsymbol{T}_1 + \kappa_1\boldsymbol{B}_1$. Therefore, considering the equations (2.2) and (3.6) we have

$$\boldsymbol{D}_1 = \frac{1}{2}\left[(\kappa_1 - i\tau_1)\phi^t\sigma\phi - (\kappa_1 + i\tau_1)\hat{\phi}^t\sigma\hat{\phi}\right]$$

and consequently,

$$\boldsymbol{D}_1 = \frac{\kappa}{2}\left[e^{-i\vartheta}\phi^t\sigma\phi - e^{i\vartheta}\hat{\phi}^t\sigma\hat{\phi}\right]$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$. □

We can give the following corollary with the aid of the equations (2.3) and (3.1), and Theorem 3.3.

Corollary 3.9. *Consider that the curve (β) is the successor curve of the curve (α) and (α, β) have the same arc-length parameter s. Now, we assume that the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) and Darboux vector of the successor curve (β) is $\boldsymbol{D}_1$. In that case, Darboux vector of the successor curve (β) can be written as*

$$\boldsymbol{D}_1 = -\frac{i\kappa}{2}\left(\xi^t\sigma\xi + \hat{\xi}^t\sigma\hat{\xi}\right)$$

in terms of the spinor ξ corresponding to the curve (α).

Now, we consider that the successor curve (β) of the curve (α) is helix and we give the spinor equations in that case with following theorems and corollaries.

Theorem 3.10. *Suppose that the curve (β) is the successor curve of the curve (α) and the successor curve is helix. In addition, the spinor ϕ corresponds to the Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β). Therefore, the spinor $\frac{d\phi}{ds}$ is*

$$\frac{d\phi}{ds} = \frac{\tau_1}{2\cos\theta}e^{-i\theta}\hat{\phi} \tag{3.9}$$

where s is the arc-length parameter of both the curve (α) and the successor curve (β), τ_1 is the torsion of the successor curve (β), and $\theta = \arccos\langle\boldsymbol{T}_1, \boldsymbol{U}\rangle = constant$.

Proof. Let the successor curve (β) of the curve (α) be helix and ϕ be corresponds to the Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β). In that case, if the equation $\frac{\kappa_1}{\tau_1} = \tan\theta = constant$ is considered since the successor curve is helix, the equation (3.3) can be written as

$$\frac{d\phi}{ds} = \frac{\tau_1 - i\tan\theta\,\tau_1}{2}\hat{\phi} = \frac{\tau_1}{2\cos\theta}(\cos\theta - i\sin\theta)\hat{\phi}$$

and consequently,

$$\frac{d\phi}{ds} = \frac{\tau_1}{2\cos\theta}e^{-i\theta}\hat{\phi}.$$

□

We know that if the successor curve (β) of the curve (α) is helix, then the curve (α) is planar curve and $\tau = 0$ in Theorem 2.6. Therefore with the aid of the equation (2.4) we can give the following corollary.

Corollary 3.11. *Consider that the curve (β) is the successor curve of the curve (α) and the successor curve is helix. In addition, the spinor ξ corresponds to the Frenet frame $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the curve (α). In that case, the spinor $\frac{d\xi}{ds}$ can be written that*

$$\frac{d\xi}{ds} = \frac{\kappa}{2}\hat{\xi}$$

where s is the arc-length parameter of both the curve (α) and the successor curve (β) and κ is the curvature of the curve (α).

Corollary 3.12. *Consider that the spinors ξ and ϕ correspond to the Frenet frames $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α) and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), selected as helix, respectively. Therefore, there is the relationship between the spinors $\frac{d\xi}{ds}$ and $\frac{d\phi}{ds}$ as*

$$\frac{d\phi}{ds}=\pm\frac{\sin\vartheta}{\cos\theta}e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}-\theta\right)}\frac{d\xi}{ds}$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$ and $\theta=\arccos\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=constant$.

Corollary 3.13. *Suppose that the curve (β) is the successor curve of the curve (α) and the spinor ϕ corresponds to the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), respectively. In that case, the successor curve (β) is helix with constant angle θ and axis $\boldsymbol{U}$, then the necessary and sufficient condition is that the constant vector $\boldsymbol{U}$ can be written*

$$\boldsymbol{U}=\frac{1}{2}\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi+e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right). \tag{3.10}$$

Proof. $(\Rightarrow)$: Let the successor curve (β) of the curve (α) be helix. In that case, there is a constant vector $\boldsymbol{U}$ making a constant angle θ with tangent vector $\boldsymbol{T}_1$ at all points of the curve (β) and it can be written $\boldsymbol{U}=\cos\theta\boldsymbol{T}_1+\sin\theta\boldsymbol{B}_1$. On the other hand, consider that the spinor ϕ corresponds to the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β). Therefore, if we use the equation (3.5) in the equation $\boldsymbol{U}$, then we get

$$\boldsymbol{U}=\frac{1}{2}\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi+e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right).$$

Now, if we take the derivative of the vector $\boldsymbol{U}$ with respect to the arc-length parameter s, then we have

$$\boldsymbol{U}'=\frac{1}{2}\left[e^{i\left(\theta-\frac{\pi}{2}\right)}\left(\left(\frac{d\phi}{ds}\right)^t\sigma\phi+\phi^t\sigma\left(\frac{d\phi}{ds}\right)\right)+e^{-i\left(\theta-\frac{\pi}{2}\right)}\left(\left(\widehat{\frac{d\phi}{ds}}\right)^t\sigma\hat{\phi}+\hat{\phi}^t\sigma\left(\widehat{\frac{d\phi}{ds}}\right)\right)\right].$$

and with the aid of the equation (3.9) we get

$$\begin{aligned}\boldsymbol{U}'&=\frac{i}{2}\left[-\frac{\tau_1}{2\cos\theta}\left(\hat{\phi}^t\sigma\phi+\phi^t\sigma\hat{\phi}\right)+\frac{\tau_1}{2\cos\theta}\left(\hat{\phi}^t\sigma\phi+\phi^t\sigma\hat{\phi}\right)\right],\\ \boldsymbol{U}'&=\boldsymbol{0}\end{aligned}$$

and consequently, the vector $\boldsymbol{U}$ is constant.

$(\Leftarrow)$: Let the successor curve of the curve (α) be (β). Moreover, we suppose that a constant vector $\boldsymbol{U}$ as

$$\boldsymbol{U}=\frac{1}{2}\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi+e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right).$$

Therefore, with the aid of the equation (3.6) we can obtain

$$\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=-\frac{1}{4}\left[\begin{array}{l}\left(e^{i\theta}\left(\phi_1^2-\phi_2^2\right)+e^{-i\theta}\left(-\overline{\phi_1}^2+\overline{\phi_2}^2\right)\right)\left(\phi_1^2-\phi_2^2+\overline{\phi_2}^2-\overline{\phi_1}^2\right)\\ -\left(e^{i\theta}\left(\phi_1^2+\phi_2^2\right)+e^{-i\theta}\left(\overline{\phi_1}^2+\overline{\phi_2}^2\right)\right)\left(\phi_1^2+\phi_2^2+\overline{\phi_1}^2+\overline{\phi_2}^2\right)\\ +\left(-2\phi_1\phi_2+\overline{\phi_1}\,\overline{\phi_2}\right)\left(-2e^{i\theta}\phi_1\phi_2+2e^{-i\theta}\overline{\phi_1}\,\overline{\phi_2}\right)\end{array}\right]$$

and

$$\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=\frac{1}{2}\left(|\phi_1|^2+|\phi_2|^2\right)\left(e^{i\theta}+e^{-i\theta}\right)=\cos\theta\left(|\phi_1|^2+|\phi_2|^2\right).$$

We know that the spinor ϕ corresponds to the unit vectors, Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$, therefore, $\overline{\phi}^t\phi=|\phi_1|^2+|\phi_2|^2=1$ and $\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=\cos\theta$. On the other hand, since the vector $\boldsymbol{U}$ is constant as per the theorem, $\boldsymbol{U}'=\boldsymbol{0}$ is hold. Therefore, we obtain

$$\boldsymbol{U}'=\frac{1}{2}\left[i\theta'\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi-e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right)+\hat{\phi}^t\sigma\phi\left((\tau_1-i\kappa_1)e^{i\left(\theta-\frac{\pi}{2}\right)}-(\tau_1+i\kappa_1)e^{-i\left(\theta-\frac{\pi}{2}\right)}\right)\right]$$

and if we make necessary adjustments in last equation, then we get

$$\boldsymbol{U}'=\boldsymbol{0}=i\left[\theta'\left(\cos(\theta-\frac{\pi}{2})\boldsymbol{B}_1-\sin(\theta-\frac{\pi}{2})\boldsymbol{T}_1\right)-\left(\tau_1\sin(\theta-\frac{\pi}{2})+\kappa_1\cos(\theta-\frac{\pi}{2})\right)\boldsymbol{N}_1\right].$$

As a result, we have

$$\begin{aligned}\theta'\sin(\theta-\frac{\pi}{2})&=0,\\ \theta'\cos(\theta-\frac{\pi}{2})&=0,\end{aligned}$$

and $\theta'=0$ and $\theta=constant$. Consequently, $\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=cos\theta=constant$ and the successor curve (β) is helix. $\square$

Corollary 3.14. *Let the curve (β), selected as helix, be the successor curve of the curve (α). In addition, the spinor ξ corresponds to the Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α). Therefore, the spinor equation of axis $\boldsymbol{U}$ of the helix successor curve can be written*

$$\boldsymbol{U}=\frac{1}{2}\left[-e^{i(\theta+\vartheta)}\xi^t\sigma\xi+e^{-i(\theta+\vartheta)}\hat{\xi}^t\sigma\hat{\xi}\right].$$

Proof. Consider that the successor curve (β) of the curve (α) is helix. Therefore, there is the constant angle θ such as $\langle \boldsymbol{T}_1, \boldsymbol{U} \rangle = \cos\theta = constant$. If we use the equations (3.8) and (3.10) we get easily

$$\boldsymbol{U} = \frac{1}{2}\left[-e^{i(\theta+\vartheta)}\xi^t\sigma\xi + e^{-i(\theta+\vartheta)}\hat{\xi}^t\sigma\hat{\xi}\right]$$

where the successor curve (β) is helix, therefore, the curve (α) is planar curve. In that case, $\tau = 0 = \vartheta' = 0$ and the angles θ, ϑ are constant angles. □

Now, we give two examples.

Example 3.15. *Let $\alpha : I \to \mathbb{E}^3$ be curve with arc-length parameter s such as*

$$\alpha(s) = \left(\frac{2}{\sqrt{5}}\cos s, \frac{2}{\sqrt{5}}\sin s, \frac{s}{\sqrt{5}}\right).$$

Therefore, we obtain that Frenet apparatus $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}, \kappa, \tau\}$ of (α) are

$$\boldsymbol{T}(s) = \frac{1}{\sqrt{5}}(-2\sin s, 2\cos s, 1),$$
$$\boldsymbol{N}(s) = (-\cos s, -\sin s, 0),$$
$$\boldsymbol{B}(s) = \frac{1}{\sqrt{5}}(\sin s, -\cos s, 2)$$

and

$$\kappa = \frac{2}{\sqrt{5}}, \quad \tau = \frac{1}{\sqrt{5}}.$$

Now, we assume that the spinor ξ corresponds to Frenet frame $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the curve (α). In that case, we obtain

$$\xi_1 = \pm\sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}\, ie^{\frac{-is}{2}},$$
$$\xi_2 = \pm\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}\, e^{\frac{is}{2}}$$

where s is the arc-length parameter of the curve (α). Moreover, we get

$$\frac{d\xi}{ds} = \frac{1}{2\sqrt{5}}(-i\xi + 2\hat{\xi}).$$

On the other hand, consider that the curve (β) be a representation of the family of successor curves of the curve (α). In that case, if we take Frenet frame $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$ of the successor curve (β) and the relationship $\boldsymbol{T} = \boldsymbol{N}_1$, we have

$$\boldsymbol{T}_1 = \begin{pmatrix} \cos\left(\frac{1}{\sqrt{5}}s + c\right)\cos s + \frac{1}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s + c\right)\sin s, \\ \cos\left(\frac{1}{\sqrt{5}}s + c\right)\sin s - \frac{1}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s + c\right)\cos s, \\ \frac{2}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s + c\right) \end{pmatrix},$$
$$\boldsymbol{N}_1 = \frac{1}{\sqrt{5}}(-2\sin s, 2\cos s, 1),$$
$$\boldsymbol{B}_1 = \begin{pmatrix} -\sin\left(\frac{1}{\sqrt{5}}s + c\right)\cos s + \frac{1}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s + c\right)\sin s, \\ -\sin\left(\frac{1}{\sqrt{5}}s + c\right)\sin s - \frac{1}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s + c\right)\cos s, \\ \frac{2}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s + c\right) \end{pmatrix}$$

where $\tau = \frac{d\vartheta}{ds} = \frac{1}{\sqrt{5}}$ and as a result, $\vartheta = \frac{1}{\sqrt{5}}s + c$, and $c = constant$. In addition to that, the curvatures κ_1 and τ_1 of the successor curve (β) can be obtained

$$\kappa_1 = \frac{2}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s + c\right),$$
$$\tau_1 = \frac{2}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s + c\right).$$

Therefore, if we assume that the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$, then the spinor components of ϕ can be given

$$\phi_1 = \pm\sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}\, e^{i\left(\frac{\pi}{4} + \frac{\vartheta - s}{2}\right)},$$
$$\phi_2 = \pm\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}\, e^{i\left(\frac{\pi}{4} - \frac{\vartheta + s}{2}\right)}$$

where $\vartheta = \frac{1}{\sqrt{5}}s + c$ and consequently, we have

$$\frac{d\phi}{ds} = \frac{1}{2\sqrt{5}}\left(\sin\left(\frac{1}{\sqrt{5}}s+c\right) - 2i\cos\left(\frac{1}{\sqrt{5}}s+c\right)\right)\hat{\phi}.$$

Example 3.16. *Consider that the helix $\beta(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ be a successor curve of the curve (α). In that case, the successor system $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1, \kappa_1, \tau_1\}$ can be calculated as*

$$\begin{aligned}
\boldsymbol{T}_1 &= \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right),\\
\boldsymbol{N}_1 &= \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right),\\
\boldsymbol{B}_1 &= \frac{1}{\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 1\right)
\end{aligned}$$

and

$$\kappa_1 = \frac{1}{2}, \quad \tau_1 = \frac{1}{2}$$

where we can take $\theta = \frac{\pi}{4}$ since $\frac{\tau_1}{\kappa_1} = \cot\theta = 1$. Moreover, the torsion of the curve (α) is $\tau = \vartheta' = 0$ as per Theorem 2.6 and the curvature of the curve (α) is $\kappa = \frac{1}{\sqrt{2}}$ since $\kappa_1 = \kappa\cos\vartheta$.
On the other hand, let ϕ be the spinor corresponding to Frenet frame of the successor curve (β). In that case, we obtain that the components of this spinor are

$$\begin{aligned}
\phi_1 &= \pm\sqrt{\frac{1+i}{2\sqrt{2}}}e^{-i\frac{s}{2\sqrt{2}}},\\
\phi_2 &= \pm\sqrt{\frac{1+i}{2\sqrt{2}}}e^{i\frac{s}{2\sqrt{2}}}.
\end{aligned}$$

Now, we give the curve (α). We know that $\boldsymbol{T} = \boldsymbol{N}_1$ therefore,

$$\boldsymbol{T} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

and, as a result, we can take

$$\alpha(s) = \sqrt{2}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right).$$

In that case, we have

$$\begin{aligned}
\boldsymbol{N} &= \left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 0\right),\\
\boldsymbol{B} &= (0,0,1).
\end{aligned}$$

Consequently, if we assume the spinor ξ corresponding to Frenet frame of the curve (α), then we have

$$\begin{aligned}
\xi_1 &= \pm\frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{4}-\frac{s}{2\sqrt{2}}\right)},\\
\xi_2 &= \pm\frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{4}+\frac{s}{\sqrt{2}}\right)}.
\end{aligned}$$

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

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