VOLUME IV ISSUE 1
ISSN: 2667-7660

April 2022
http://dergipark.gov.tr/mjm

MALTEPE JOURNAL OF MATHEMATICS

Editorial Board

Pratulananda DAS
Jadavpur University, Kolkata, West Bengal, India
pratulananda@yahoo.co.in

Mohammad Kazim KHAN
Kent State University, USA
kazim@math.kent.edu

Bipan HAZARIKA
Rajiv Gandhi University, India
bh rgu@yahoo.co.in

Ayhan ESİ
Malatya Turgut Ozal University, Malatya, Turkey
aesi23@hotmail.com

İbrahim ÇANAK
Ege University, Izmir, Turkey
icanak@gmail.com

Hacer ŞENGÜL KANDEMİR
Harran University, Şanlıurfa, Turkey
hacer.sengul@hotmail.com

Emrah Evren KARA
Düzce University, Düzce, Turkey
eevrenkara@duzce.edu.tr

Fuat USTA
Düzce University, Düzce, Turkey
fuatusta@duzce.edu.tr

Osman MUCUK
Erciyes University, Kayseri, Turkey
mucuk@erciyes.edu.tr

Contents

Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume IV Issue 1 (2022), Pages 1-8. Doi: https://doi.org/ 10.47087/mjm.1037337

SOME RESULTS FOR (s,m)-CONVEX FUNCTION IN THE SECOND SENSE

MELTEM SERTBAŞ* AND İLKNUR MİHYAZ**
*DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, KARADENIZ TECHNICAL UNIVERSITY, 61080, TRABZON, TURKEY, ORCID ID: 0000-0001-9606-951X
**INSTITUTE OF NATURAL SCIENCES, KARADENIZ TECHNICAL UNIVERSITY, TRABZON, TURKEY, ORCID ID: 0000-0002-8585-7390

Abstract. In this paper, it is given some properties for an (s,m)-convex function defined on $[0,d]$, $d>0$ in the first sense and the second sense with $m\in(0,1)$. Also, some integral inequalities are examined for any non positive (s,m)-convex function in the second sense with any measure space.

1. Introduction

Convex functions, like differentiable functions, have a important role in many fields of pure and applied mathematics. It connects concepts from topology, algebra, geometry and analysis, and is an important tool in optimization, mathematical programming and game theory [3].

In recent years, after Miheşan [14] defined (s,m)-convex functions in the first sense, several investigations have emerged resulting in applications in mathematics, as it can be seen in [1, 2, 12, 4, 5, 10, 7, 6, 8, 9, 13].

Definition 1.1. *A function $f:[0,d]\to\mathbb{R}$ is called an (s,m)-convex function in the first sense, where $(s,m)\in[0,1]$ and $d>0$, if for all $x,y\in[0,d]$ and $t\in[0,1]$*

$$f\left(tx+m\left(1-t\right)y\right)\leq t^{s}f(x)+m(1-t^{s})f(y).$$

Moreover, Eftekhari [15] introduced (s,m)-convex functions in the second sense in 2014 as follows:

Definition 1.2. *$f:[0,d]\to\mathbb{R}$, $d>0$ is called to be an (s,m)- convex in the second sense function for some $(s,m)\in(0,1]^2$ if*

$$f\left(tx+m\left(1-t\right)y\right)\leq t^{s}f\left(x\right)+m(1-t)^{s}f\left(y\right)$$

for any $x,y\in[0,d]$.

2020 *Mathematics Subject Classification.* Primary:26D15; Secondaries: 26D10; 39B62 .
Key words and phrases. (s,m)-convex function; Integral inequalities; Measure space.

Submitted on December 16th, 2021. Published on March 18th, 2022
Communicated by Hacer SENGUL KANDEMIR, and Canan Hazar GULEC.

Example 1.1. *Let $s,m \in (0,1]$, $p \in [1,+\infty)$ and $f:[0,+\infty) \to \mathbb{R}$ defined by $f(x) = x^p + c$, $c \leq 0$, then f is an (s,m)-convex function in the second sense. Indeed, for all $x,y \in [0,+\infty)$, $t \in [0,1]$ and $(s,m) \in [0,1]$ we have*

$$\begin{aligned} f(tx+m(1-t)y) \quad &= (tx+m(1-t)y))^p + c \leq ty^p + m^p(1-t)y^p + c \\ &\leq t^s x^p + m(1-t)^s y^p + (t^s + m(1-t)^s)c \\ &\leq t^s f(x) + m(1-t)^s f(y). \end{aligned}$$

We note that if a nonnegative function is convex and starshaped, then it is an (s,m)-convex function in the second sense function for all $(s,m) \in (0,1]^2$. This function class is an extension of s-convex functions in the second sense that are $(s,1)$- convex functions in the second sense [12]. Dragomir and Fitzpatrick proved that a s-convex functions in the second sense f is Riemann integrable if $f(c) = 0$ for any point c in domain of the function f in [17]. Also, when f is Lebesgue integrable on $[a,b]$ they give the Hermite-Hadamard type inequality for a s-convex functions in the second sense f on $[a,b]$ as the following inequality

$$2^{s-1} f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

However, there is not any result for integrability of (s,m)- convex functions in the second sense with $m \in (0,1)$, and so researchers like [18, 19, 20] have to stipulate integrability.

In this paper, we deal with some properties and some inequalities for (s,m)-convex functions in the second sense with $m \in (0,1)$.

2. Some Properties

Let's first recall the well known H. Lebesgue Theorem ([21] p.257).

Theorem 2.1 (H. Lebesgue). *Let f be a real-valued increasing function on $[a,b]$. Then the derivative f' exists and is nonnegative in $(a,b)\backslash E$ where E is a null set in $(\mathbb{R}, \mathfrak{M}_L, \mu_L)$ contained in (a,b). Further more f' is $\mathfrak{M}_L$ measurable and μ_L-integrable on $(a,b)\backslash E$ with*

$$\int_{[a,b]} f' d\mu_L \leq f(b) - f(a).$$

Theorem 2.2. *If $f:[0,d] \to \mathbb{R}$, $d > 0$ is an (s,m)- convex in the first sense or the second sense function for $m \in (0,1)$, then the derivative f' exists and*

$$sf(x) \leq xf'(x)$$

is hold for all $x \in (a,b)\backslash E$, $[a,b] \subset (0,d]$ where E is a null set in $(\mathbb{R}, \mathfrak{M}_L, \mu_L)$ contained in (a,b).

Proof. Let $f:[0,d] \to \mathbb{R}$, $d > 0$ be an (s,m)- convex in the first or the second sense function for $m \in (0,1)$. In this case,

$$f(0) \leq mf(0),$$

and so it is obtained $f(0)\le 0$. Also for all $0<x\le y$ we have

$$f(x)=f\left(\frac{x}{y}y+m\left(1-\frac{x}{y}\right)0\right)\le\left(\frac{x}{y}\right)^s f(y)+m\left(1-\left(\frac{x}{y}\right)^s\right)f(0)\le\left(\frac{x}{y}\right)^s f(y)$$

or

$$f(x)=f\left(\frac{x}{y}y+m\left(1-\frac{x}{y}\right)0\right)\le\left(\frac{x}{y}\right)^s f(y)+m\left(1-\frac{x}{y}\right)^s f(0)\le\left(\frac{x}{y}\right)^s f(y)$$

i.e., $\frac{f(x)}{x^s}\le\frac{f(y)}{y^s}$, $0<x\le y\le d$. This means that the function $g(x)=\frac{f(x)}{x^s}$ is monotone increasing function on $[a,d]$, $a>0$. Since the functions $h(x)=x^s$ and $g(x)=\frac{f(x)}{x^s}$ are differentiable, according to H. Lebesgue Theorem we gain that the derivative f' exists and

$$sf(x)\le xf'(x)$$

is satisfied for all $x\in(a,b)\backslash E$, $(a,b]\subset(0,d]$ where E is a null set in $(\mathbb{R},\mathfrak{M}_L,\mu_L)$ contained in (a,b). □

Corollary 2.3. *If $f:[0,d]\to\mathbb{R}$, $d>0$ is an (s,m)- convex in the first sense or the second sense function for $m\in(0,1)$, f is Riemann integrable on $[a,d]$, $a>0$.*

Corollary 2.4. *If $f:[0,d]\to\mathbb{R}$, $d>0$ is a nonnegative (s,m)- convex in the first sense or second sense function for $m\in(0,1)$, then f is continuous at the zero, $f(0)=0$ and monotone increasing, and so Riemann integrable on $[0,d]$.*

Corollary 2.5. *If $f:[0,d]\to\mathbb{R}$, $d>0$ is a nonnegative (s,m)- convex in the first sense or second sense function for $m\in(0,1)$ and is the derivative of a function on $(0,d)$, then f is continuous on $[0,d)$.*

Proof. This result is taken from the fact that the derivative function has points of discontinuity only if it has points of the second type discontinuity. □

Theorem 2.6. *Let $f:[0,d]\to\mathbb{R}$, $d>0$ be a nonnegative (s,m)- convex in the first sense or second sense function for $m\in(0,1)$ and continuous on any subinterval $[0,c]$, $c\le d$. Then, the limit $\lim_{x\to 0}\frac{f(x)}{x^s}$ exists.*

Proof. Suppose that $f:[0,d]\to\mathbb{R}$, $d>0$ be a nonnegative (s,m)- convex in the first sense or second sense function for $m\in(0,1)$ and continuous on any subinterval $[0,c]$, $0<c\le d$. Therefore $g:[0,c]\to\mathbb{R}$ defined as $g(x)=x^{1-s}f(x)$ is continuous on $[0,c]$ and for all $n\in\mathbb{N}$ and all $x\in[0,c]$

$$g(\frac{1}{n}x)=\left(\frac{1}{n}x\right)^{1-s}f\left(\frac{1}{n}x\right)\le x^{1-s}\left(\frac{1}{n}\right)^{1-s}\left(\frac{1}{n}x\right)^s f(x)=\frac{1}{n}g(x)$$

is satisfied. According to Theorem 6 in [24], $g(x)$ is differentiable at $x=0$. This means that the limit $\lim_{x\to 0}\frac{f(x)}{x^s}$ exists. □

Theorem 2.7. *If $f:[0,d]\to\mathbb{R}$, $d>0$ is a negative valued (s,m)- convex in the first sense or the second sense function for $m\in(0,1)$, f is a starshaped function on $[0,d]$.*

Proof. Under the assumption of theorem, for all $x\in[0,d]$, $f(x)<0$. Now, we suppose that the function is not starshaped. From here, there exist two point $x_0\in[0,d]$ and $t_0\in(0,1)$

$$t_0f(x_0)<f(t_0x_0).$$

Because f is an (s,m)- convex in the first sense and second sense function for $m \in (0,1)$,

$$t_0 f(x_0) < f(t_0 x_0) \leq t_0^s f(x_0)$$

is hold. However, since f is a negative valued function, for $t_0 \in (0,1)$

$$t_0^s < t_0$$

is obtained. This is a contradiction. Therefore, it is gained that the function f is starshaped on $[0,d]$. □

Corollary 2.8. *If $f : [0,d] \to \mathbb{R}$, $d > 0$ is a negative valued (s,m)- convex in the first sense and second sense function for $m \in (0,1)$, there exists a point $c \in [0,d]$ such that $f\chi_{[0,c]}$ is a non positive starshaped function on $[0,c]$ and $f\chi_{[c,d]}$ is a nonnegative monotone increasing function on $[c,d]$, where χ_A is the characteristic function of the subset A of $\mathbb{R}$.*

3. Some Inequalities

Theorem 3.1. *Let $f : [0,d] \to \mathbb{R}$, $d > 0$ be an (s,m)-convex function in the second sense and Riemann integrable on $[a,b]$, $0 \leq a \leq mb \leq b \leq d$. Then*

$$2^{s-1} f\left(m\frac{a+b}{2}\right) \leq \frac{m}{b-a}\int_a^b f(x)\,dx \leq m\frac{(b-ma)\,f(b)+(mb-a)\,f(a)}{(s+1)(b-a)}.$$

Proof. Because f is an (s,m)-convex function in the second sense, for all $x, y \in [a,b]$ we have

$$f\left(m\frac{x+y}{2}\right) \leq m\frac{f(x)+f(y)}{2^s}.$$

If $x = ta + (1-t)b$ and $y = tb + (1-t)a$ are chosen, then we get

$$f\left(m\frac{a+b}{2}\right) \leq \frac{m}{2^s}\left(f(ta+(1-t)b) + f(tb+(1-t)a)\right).$$

We obtain by integrating the last inequality

$$f\left(m\frac{a+b}{2}\right) \leq \frac{m}{2^{(s-1)}}\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Since $a \leq mb$, and $f : [0,d] \to \mathbb{R}$, $d > 0$ is an (s,m)-convex function in the second sense

$$\begin{aligned}
&\int_a^b f(y)\,dy = \int_a^{mb} f(x)\,dx + \int_{mb}^b f(x)\,dx \\
&= (mb-a)\int_0^1 f(ta+m(1-t)b)\,dt + (b-mb)\int_0^1 f(tb+m(1-t)b)\,dt \\
&\leq (mb-a)\int_0^1 \left(t^s f(a) + m(1-t)^s f(b)\right)dt + (b-mb)\int_0^1 \left(t^s + m(1-t)^s\right) f(b)\,dt \\
&= \tfrac{(b-ma)f(b)+(mb-a)f(a)}{s+1}
\end{aligned}$$

we have

$$f\left(m\frac{a+b}{2}\right) \leq m\frac{2^{1-s}}{b-a}\int_a^b f(x)\,dx \leq m2^{1-s}\frac{(b-ma)\,f(b)+(mb-a)\,f(a)}{(s+1)(b-a)}.$$

$\square$

Remark. *If we take $m=1$ and $s=1$ in Theorem 3.1, then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(b)+f(a)}{2}$$

is famous Hermite-Hadamard inequality.

Corollary 3.2. *If $f:[0,d]\to\mathbb{R}$, $d>0$ is an (s,m)-convex function in the second sense and the derivative function f' is Riemann integrable on $[a,b]$, $0\leq a\leq mb\leq b\leq d$, then*

$$\int_a^b f(x)\,dx\leq\min\{\frac{bf(b)-af(a)}{s+1},\frac{(b-ma)\,f(b)+(mb-a)\,f(a)}{s+1}\}.$$

Theorem 3.3. *Let $f:[0,d]\to\mathbb{R}$ be a differentiable on $[0,d]$ and $|f'|$ is an (s,m)-convex function in the second sense in $[0,d]$ for $m\in(0,1)$, then for all $x\in[a,b]$, $[a,b]\subset[0,d]$*

$$\left|f(mx)-\frac{1}{b-a}\int_a^b f(y)\,dy\right|\leq\frac{|f'(b)|}{b-a}\left[\frac{(m(s+1)+1)\left((mx-a)^2+(b-mx)^2\right)}{(s+1)(s+2)}\right].$$

Proof. In this case, we use the equality given by Cerone and Dragomir in [22], and so

$$\begin{aligned}
&\left|f(mx)-\frac{1}{b-a}\int_a^b f(y)dy\right|\\
=&\left|\frac{(mx-a)^2}{b-a}\int_0^1 tf'(tmx+(1-t)\,a)dt-\frac{(b-mx)^2}{b-a}\int_0^1 tf'(tmx+(1-t)\,b)dt\right|\\
\leq&\frac{(mx-a)^2}{b-a}\int_0^1 t|f'(tmx+(1-t)\,a)|dt+\frac{(b-mx)^2}{b-a}\int_0^1 t|f'(tmx+(1-t)\,b)|dt\\
\leq&\frac{(mx-a)^2}{b-a}\int_0^1 t\,(mt^s+(1-t)^s)|f'(b)|dt+\frac{(b-mx)^2}{b-a}\int_0^1 t\,(mt^s+(1-t)^s)|f'(b)|dt\\
=&\frac{|f'(b)|}{b-a}\left[\frac{(m(s+1)+1)\left((mx-a)^2+(b-mx)^2\right)}{(s+1)(s+2)}\right]
\end{aligned}$$

is obtained. $\square$

Remark. *If it is chosen as $m=1$ in Theorem 3.3, it is obtained the inequality given Alomari et. al. in [23].*

Theorem 3.4. *If $f:[0,d]\to\mathbb{R}$, $d>0$ is an (s,m)-convex function in the second sense for any $m\in(0,1)$ then the following inequality is hold*

$$f\left(m\sum_{k=1}^{n}t_kx_k\right)\leqslant m\sum_{k=1}^{n}t_k^sf\left(x_k\right) \tag{3.1}$$

where $\sum\limits_{k=1}^{n}t_k\leqslant 1,\ t_k\in[0,1]$ *and* $x_k\in[a,b]$.

Proof. It can be proved by using the mathematical induction method as in [25]. First of all, since $f:[0,d]\to\mathbb{R},\ d>0$ is an (s,m)-convex function in the second sense with $m\in(0,1)$ for $n=1$, $t\in[0,1]$ and $x\in[a,b]$

$$f(mtx)=f(mtx+(1-t)0)\leq(1-t)^sf(0)+mt^sf(x)\leq mt^sf(x).$$

Now, for the next step of induction we consider that the equation 3.1 is true for $n-1$. In this case, if $\sum\limits_{k=1}^{n}t_k\leqslant 1$, then $\frac{t_k}{1-t_n}\leq 1$, $1\leq k\leq n-1$ is hold and

$$\begin{aligned}
f\left(m\sum_{k=1}^{n}t_kx_k\right) &= f\left(m(1-t_n)\sum_{k=1}^{n-1}\frac{t_k}{1-t_n}x_k+mt_nx_n\right)\\
&\leq (1-t_n)^sf\left(m\sum_{k=1}^{n-1}\frac{t_k}{1-t_n}x_k\right)+mt_n^sf(x_n)\\
&\leq m\sum_{k=1}^{n-1}t_k^sf(x_k)+mt_n^sf(x_n)=m\sum_{k=1}^{n}t_k^sf\left(x_k\right).
\end{aligned}$$

This conclusion completes the proof of the theorem. □

Theorem 3.5. *Suppose that* (X,Σ,μ) *is a finite measure space and* $h:X\to[0,+\infty)$ *is a* μ*-integrable function such that* $h\left(x\right)\leqslant\frac{1}{\mu(X)}$ *a.e. . If* $f:[0,d]\to\mathbb{R}$, $d>0$ *is a non positive continuous* (s,m)*-convex function in the second senses for any* $m\in(0,1)$ *and* $g:X\to[0,d]$ *is a* μ*-integrable function, then we have*

$$f\left(m\int_E h\left(x\right)g\left(x\right)d\mu\left(x\right)\right)\leqslant m\int_E h\left(x\right)f\left(g\left(x\right)\right)d\mu\left(x\right)$$

for any $E\in\Sigma$.

Proof. Let $I=\bigcup\limits_{k=1}^{n}I_{n_k}$ be any partition of disjoint intervals I_{n_k} for $n\in\mathbb{N}$. Because g is an μ-integrable function, the set $E_{n_k}:=g^{-1}\left(I_{n_k}\right)\cap E$ is in Σ for any set $E\in\Sigma$ and $E=\bigcup\limits_{k=1}^{n}E_{n_k}$. Choosing any point x_{n_k} in each set E_{n_k}. Since h is a positive valued function and $\int\limits_X h\left(x\right)d\mu\left(x\right)\leqslant 1$, the linear combination

$$\sum_{k=1}^{n}\mu\left(E_{n_k}\right)h\left(x_{n_k}\right)g\left(x_{n_k}\right)$$

is in $[0,d]$ for large enough $n\in\mathbb{N}$. Because f is a non positive (s,m)-convex function for any $m\in(0,1)$ on $[0,d]$, the following inequality is satisfied by using the previous theorem

$$\begin{aligned} f\left(m\sum_{k=1}^{n}\mu\left(E_{n_k}\right)h\left(x_{n_k}\right)g\left(x_{n_k}\right)\right) &\leq m\sum_{k=1}^{n}\mu^s\left(E_{n_k}\right)h^s\left(x_{n_k}\right)f\left(g\left(x_{n_k}\right)\right)\\ &\leq m\sum_{k=1}^{n}\mu\left(E_{n_k}\right)h\left(x_{n_k}\right)f\left(g\left(x_{n_k}\right)\right)\end{aligned}$$

The proof of the theorem is completed under the continuity assumption of the function f. □

References

[1] M.K Bakula, M.E. Özdemir and J. Pečarić, *Hadamard type inequalities for m-convex and (α,m)-convex functions*, J. Inequal. Pure Appl. Math. **9**, 4 (2008), Article ID 96.

[2] M.K. Bakula, J. Pečarić and M. Ribićzić, *Companion inequalities to Jensen's for inequality for m-convex and (α,m)-convex functions*, J. Inequal. Pure Appl. Math., **7**, 5 (2006), Article ID 194.

[3] J.M. Borwein and Jon D. Vanderwerff, *Convex functions: constructions, characterizations and counterexamples*, Vol. 172. Cambridge: Cambridge University Press, 2010.

[4] M. Kunt and İ. İşcan, *Hermite- Hadamard type inequalities for harmonically (α,m)-convex functions by using fractional integrals*, Konuralp J. Math., **5**, 1 (2017), 201–213.

[5] M.A. Latif, S.S. Dragomir and E. Momoniat, *On Hermite-Hadamard type integral inequalities for n-times differentiable m- and (α,m)-logarithmically convex functions*, Filomat, **30**, 11 (2016), 3101–3114.

[6] M.E. Özdemir, E. Set and M.Z. Sarıkaya, *Some new Hadamard type inequalities for co-ordinatedm-convex and (α,m)-convex functions*, Hacet. J. Math. Stat., **40**, 2 (2011), 219–229.

[7] M. E. Özdemir, H. Kavurmacıand E. Set, *Ostrowski's Type Inequalities for (α,m)-Convex Functions*, Kyungpook Math. J., **50** (2010), 371–378.

[8] E. Set, M.E. Özdemir, M.Z. Sarıkaya and F. Karakoç, *Hermite-Hadamard type inequalitiesfor (α,m)-convex functions via fractional integrals*, Moroccan J. Pure Appl. Anal., **3** 1 (2017), 15–21.

[9] E. Set, M. Sardari, M.E. Özdemir and J. Rooin, *On generalization sof the Hadamard inequality for (α,m)-convex functions*, Kyungpook Math. J., **52** (2012), 307–317.

[10] S. Özcan, *Hermite-Hadamard type inequalities for m-convex and (α,m)-convex functions*, J. Inequal. Appl., **175** (2020), 1–10.

[11] M. Klaričić Bakula, J. Pečarić and M. Ribićić, *Hadamard type inequalities for m-convex and $(s;m)$- convex functions*, J. Inequal. Pure Appl. Math. **5** (4) (2008), Art 96.

[12] M. Bracamonte, J. Gim Zénez and M.J. Vivas-Cortez, *Hermite-Hadamard-Fejer type inequalities for strongly (s,m)-convex functions with modulus c in second sense*, Appl. Math. Inf. Sci., **10**, 6 (2016), 2045–2053.

[13] Z. Yang, Y. Li and T. Du, *A generalization of Simpson type inequality via differentiable functions using (s,m)-convex functions*, Italian J. Pure Appl. Math., **35** (2015) , 327–338.

[14] V. G. Miheşan, *A generalization of the convexity*, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca (Romania), 1993.

[15] N. Eftekhari, *Some remarks on $(s;m)$-convexity in the second sense*, J. Math. Inequal., **8** 2 (2014), 489–495.

[16] W.W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Rau-men*, Publ. Inst. Math. (Beograd) (N.S.), **23** (1978), 13–20.

[17] S.S. Dragomir and S. Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstr. Math., **32**, 4 (1999), 687–696.

[18] T. Lian, W. Tang and R. Zhou, *Fractional Hermite-Hadamard inequalities for (s,m)-convex or s-concave functions*, J. Inequal. Appl., **240** (2018).

[19] M. Vivas-Cortez, *Fejer type inequalities for (s,m)- convex functions in second sense* Appl Math Inform Sci, **10** (2016), 1–8.

[20] M. Vivas-Cortez, *Ostrowski and Jensen-type inequalities via (s,m)-convex functions in the second sense*, Bol. Soc. Mat. Mex., **26** (2020), 287–302.

[21] J. Yeh, *Real Analysis*, World Scientific Publishing Co. Pte. Ltd., USA, 2006.
[22] P. Cerone and S.S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions*, Demonstratio Math., **37**, 2 (2004), 299–308.
[23] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett., **23** (2010) 1071–1076.
[24] A.M. Bruckner and E. Ostrow, *Some function classes related to the class of convex functions*, Pacific J. Math., **12** (1962), 1203–1215.
[25] Z. Pavić and M. A. Ardıç, *The most important inequalities of m-convex functions*, Turk J Math., **41** (2017), 625–635.

Meltem Sertbaş,
Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, TURKEY, ORCID ID: 0000-0001-9606-951X
Email address: m.erolsertbas@gmail.com

İlknur Mİhyaz,
Institute of Natural Sciences ,Karadeniz Technical University, Trabzon, Turkey, ORCID ID: 0000-0002-8585-7390
Email address: mihyazilknur123@gmail.com

Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume IV Issue 1 (2022), Pages 9-14. Doi: https://doi.org/ 10.47087/mjm.1092559

ON $\rho-$ STATISTICAL CONVERGENCE IN TOPOLOGICAL GROUPS

HACER ŞENGÜL KANDEMIR
FACULTY OF EDUCATION, HARRAN UNIVERSITY, OSMANBEY CAMPUS 63190, ŞANLIURFA, TURKEY, ORCID ID: 0000-0003-4453-0786

Abstract. In this study, by using definition of ρ-statistical convergence which was defined by Cakalli [5], we give some inclusion relations between the concepts of ρ-statistical convergence and statistical convergence in topological groups.

1. Introduction

In 1951, Steinhaus [29] and Fast [14] introduced the notion of statistical convergence and later in 1959, Schoenberg [28] reintroduced independently. Caserta et al. [4], Cakalli ([6],[7]), Cinar et al. [8], Colak [9], Connor [10], Et et al. ([11],[12],[13]), Fridy [15], Gadjiev and Orhan [16], Isik and Akbas ([17],[18]), Kolk [19], Mursaleen [20], Salat [21], Sengul et al. ([22]-[27]), Aral et al. ([1],[2],[3]) and many others investigated some arguments related to this notion.

The opinion of statistical convergence depends on the density of subsets of the natural set $\mathbb{N}$. We say that the $\delta(E)$ is the density of a subset E of $\mathbb{N}$ if the following limit exists such that

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c)=1-\delta(E)$.

We say that the sequence $x=(x_k)$ is statistically convergent to ℓ if for every $\varepsilon>0$,

$$\delta\left(\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\}\right)=0.$$

In this case we write $S-\lim x_k=\ell$ or $x_k\to\ell\,(S)$. Equivalently,

2020 *Mathematics Subject Classification.* Primary: 40A05 ; Secondaries: 40C05, 46A45.
Key words and phrases. Topological groups; statistical convergence; ρ-statistical convergence.

Submitted on March 24th, 2022. Published on April 17th, 2022
Communicated by Huseyin CAKALLI, Ibrahim CANAK and Sefa Anıl SEZER.

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n:|x_k-\ell|\ge\varepsilon\}\right|=0.$$

S will denote the set of all statistically convergent sequences.

If x is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that x_k satisfies P for "almost all k ", and we abbreviate this by "$a.a.k.$"

2. Main Results

In this section we give the main results of this article. Now we begin a new definition.

Definition 2.1. *Let X be an abelian topological Hausdorf group. A sequence $(x(k))$ of points in $\mathbb{R}$, the set of real numbers, is called $\rho-$statistically convergent in topological groups to ℓ ($S_\rho(X)-$convergent to ℓ) if there is a real number ℓ for each neighbourhood U of 0 such that*

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\le n:x(k)-\ell\notin U\}|=0$$

for each $\varepsilon>0$, where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$, and $\Delta x(n)=x(n+1)-x(n)$ for each positive integer n. In this case we write $S_\rho(X)-\lim x(k)=\ell$ or $x(k)\to\ell\,(S_\rho(X))$. We denote the set of all $\rho-$statistically convergent in topological groups sequences by $S_\rho(X)$. If $\rho=(\rho_n)=n$, $\rho-$statistically convergent in topological groups is coincide statistical convergence in topological groups.

Definition 2.2. *Let X be an abelian topological Hausdorf group. A sequence $x=(x(k))$ of points in $\mathbb{R}$, the set of real numbers, is called $S_\rho(X)$-Cauchy sequence in topological groups if there is a subsequence $(x(k'(n)))$ of x such that $k'(n)\leqslant n$ for each n, $\lim_{n\to\infty}x(k'(n))=\ell$ and for each neighbourhood U of 0*

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\le n:x(k)-x\left(k'\left(n\right)\right)\notin U\}|=0,$$

where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ and $\Delta x(n)=x(n+1)-x(n)$ for each positive integer n.

Theorem 2.1. *If x is $\rho-$statistically convergent in topological groups, then $S_\rho(X)-\lim x(k)=\ell$ is unique.*

Proof. Suppose that $(x\left(k\right))$ has two different $\rho-$statistical in topological groups limits ℓ_1, ℓ_2 say. Since X is a Hausdorff space there exists a neighbourhood U of 0 such that $\ell_1-\ell_2\notin U$. Then we may choose a neighbourhood W of 0 such that $W+W\subset U$. Write $z\left(k\right)=\ell_1-\ell_2$ for all $k\in\mathbb{N}$. Therefore for all $n\in\mathbb{N}$,

$$\{k\le n:z\left(k\right)\notin U\}\subset\{k\le n:\ell_1-x\left(k\right)\notin W\}\cup\{k\le n:x\left(k\right)-\ell_2\notin W\}.$$

Now it follows from this inclusion that, for all $n\in\mathbb{N}$,

$$\left|\{k\le n:z\left(k\right)\notin U\}\right|\le\left|\{k\in I_r:\ell_1-x\left(k\right)\notin W\}\right|+\left|\{k\le n:x\left(k\right)-\ell_2\notin W\}\right|.$$

Since $S_\rho(X) - \lim x(k) = \ell_1$ and $S_\rho(X) - \lim x(k) = \ell_2$ we get

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : z(k) \notin U\}\right| \quad &\le \quad \lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : \ell_1 - x(k) \notin W\}\right| \\ &\quad + \lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : x(k) - \ell_2 \notin W\}\right|.\end{aligned}$$

This contradiction shows that $\ell_1 = \ell_2$. □

Theorem 2.2. *If* $\lim_{k\to\infty} x(k) = \ell$ *and* $S_\rho(X) - \lim y(k) = 0$, *then*

$$S_\rho(X) - \lim (x(k) + y(k)) = \lim_{k\to\infty} x(k).$$

Proof. Let U be any neighborhood of 0. Then we may choose a symetric neighbourhood W of 0 such that $W + W \subset U$. Since $\lim_{k\to\infty} x(k) = \ell$ there exists an integer k_0 such that $k \ge k_0$ implies that $x(k) - \ell \in W$. Hence

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : x(k) - \ell \notin W\}\right| \le \lim_{n\to\infty} \frac{k_0}{\rho_n} = 0$$

and by the assumption that $S_\rho(X) - \lim y(k) = 0$ we have

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : y(k) \notin W\}\right| = 0.$$

Now we have

$$\{k \le n : (x(k) - \ell) + y(k) \notin U\} \subset \{k \le n : x(k) - \ell \notin W\} \cup \{k \le n : y(k) \notin W\}.$$

Hence

$$\frac{1}{\rho_n} \left|\{k \le n : (x(k) - \ell) + y(k) \notin U\}\right| \le \frac{1}{\rho_n} \left|\{k \le n : x(k) - \ell \notin W\}\right| + \frac{1}{\rho_n} \left|\{k \le n : y(k) \notin W\}\right|$$

It follows from the above inequality that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : (x(k) - \ell) + y(k) \notin U\}\right| = 0.$$

Thus $S_\rho(X) - \lim (x(k) + y(k)) = \lim_{k\to\infty} x(k)$. □

Theorem 2.3. *If a sequence* $x(k)$ *is* ρ*-statistically convergent to* ℓ, *then there are sequences* $y(k)$ *and* $z(k)$ *such that* $\lim_{k\to\infty} y(k) = \ell$, $x = y + z$ *and* $\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : x(k) \ne y(k)\}\right| = 0$ *and* z *is a* ρ*-statistically null sequence.*

Proof. Let (V_j) be a nested base of neighborhoods of 0. Take $n_0 = 0$ and choose an increasing sequence (n_j) of positive integers such that

$$\frac{1}{\rho_n} \left|\{k \le n : x(k) - \ell \notin V_j\}\right| < \frac{1}{j} \text{ for } n > n_j.$$

Let us define sequences $y = y(k)$ and $z = z(k)$ in the following way. Write $z(k) = 0$ and $y(k) = x(k)$ if $n_0 < k \le n_1$ and suppose that $n_j < n_{j+1}$ for $j \ge 1$. $z(k) = 0$ and $y(k) = x(k)$ if $x(k) - \ell \in V_j$, $y(k) = \ell$ and $z(k) = x(k) - \ell$ if $x(k) - \ell \notin V_j$. Firstly, we prove that $\lim_{k\to\infty} y(k) = \ell$. Let V be any neighborhood of 0. We may choose a positive integer j such that $V_j \subset V$. Then $y(k) - \ell = x(k) - \ell \in V_j$ and so $y(k) - \ell \in V$ for $k > n_j$. If $x(k) - \ell \notin V_j$, then $y(k) - \ell = \ell - \ell = 0 \in V$. Hence $\lim_{k\to\infty} y(k) = \ell$. Finally we show that $z = z(k)$ is a statistically null sequence. It is enough to show that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \le n : z(k) \ne 0\}\right| = 0.$$

For any $n \in \mathbb{N}$ any neighborhood V of 0, we have

$$|\{k \leq n : z(k) \notin V\}| \leq |\{k \leq n : z(k) \neq 0\}|.$$

If $j \in \mathbb{N}$ such that $V_j \subset V$ and $\varepsilon > 0$, we are going to show that

$$\frac{1}{\rho_n}|\{k \leq n : z(k) \neq 0\}| < \varepsilon.$$

If $n_p < n \leq n_{p+1}$, then

$$\{k \leq n : z(k) \neq 0\} \subset \{k \leq n : x(k) - \ell \notin V_p\}.$$

If $p > j$ and $n_p < n \leq n_{p+1}$, then

$$\frac{1}{\rho_n}|\{k \leq n : z(k) \neq 0\}| \leq \frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin V_p\}| < \frac{1}{p} < \frac{1}{j} < \varepsilon.$$

Thus, the proof is completed. $\square$

Theorem 2.4. *The sequence x is $S_\rho(X)-$convergent if and only if x is $S_\rho(X)-$Cauchy sequence.*

Proof. Assume that x is $S_\rho(X)-$convergent. Since X is a Hausdorff space there exists a neighbourhood U of 0. Then we may choose a neighbourhood Y of 0 such that $Y + Y \subset U$. We can write

$$|\{k \leq n : x(k) - x(k'(n)) \notin U\} \subset \{k \leq n : x(k) - \ell \notin Y\} \cup \{k \leq n : \ell - x(k'(n)) \notin Y\}.$$

Now it follows from this inclusion that, for all $n \in \mathbb{N}$,

$$\frac{1}{\rho_n}|\{k \leq n : x(k) - x(k'(n)) \notin U\}| \leq \frac{1}{\rho_n}|\{k \in I_r : x(k) - \ell \notin Y\}| + \frac{1}{\rho_n}|\{k \leq n : \ell - x(k'(n)) \notin Y\}|.$$

Since $S_\rho(X) - \lim x(k) = \ell$, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n : x(k) - x(k'(n)) \notin U\}| &\leq \lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin Y\}| \\ &\quad + \lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n : \ell - x(k'(n)) \notin Y\}|.\end{aligned}$$

The proof to the contrary is obvious. $\square$

Theorem 2.5. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty, \Delta\rho_n = O(1)$. If $\frac{\rho_n}{n} \geq 1$ for all $n \in \mathbb{N}$, then $S(X) \subset S_\rho(X)$.*

Proof. If $S(X) - \lim x(k) = \ell$, then for every $\varepsilon > 0$ we have

$$\begin{aligned}\frac{1}{n}|\{k &\leq n : x(k) - \ell \notin U\}| = \frac{\rho_n}{n}\frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin U\}| \\ &\geqslant \frac{1}{\rho_n}|\{k \leq n : x(k) - \ell \notin U\}|.\end{aligned}$$

This proves the proof. $\square$

Theorem 2.6. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$. If $\liminf_{n\to\infty} \frac{\rho_n}{\tau_n} > 0$, then $S_\rho(X) \subset S_\tau(X)$.*

Proof. If $S_\rho(X) - \lim x(k) = \ell$, then for every $\varepsilon > 0$ we can write

$$\frac{1}{\tau_n}|\{k \le n : x(k) - \ell \notin U\}| \le \frac{\rho_n}{\tau_n}\frac{1}{\rho_n}|\{k \le n : x(k) - \ell \notin U\}|.$$

This is enough for proof. □

The following result is obtained from Theorem 2.5 and Theorem 2.6.

Corollary 2.7. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences such that $\rho_n \leqslant \tau_n$ and $n < \tau_n$ for all $n \in \mathbb{N}$. If $\liminf_{n\to\infty} \frac{\rho_n}{\tau_n} > 0$, then $S(X) \subset S_\rho(X) \subset S_\tau(X)$.*

References

[1] N. D. Aral, and M. Et, *Generalized difference sequence spaces of fractional order defined by Orlicz functions*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69(1)** (2020) 941–951.

[2] N. D. Aral, and H. Şengül Kandemir, *I-Lacunary Statistical Convergence of order β of Difference Sequences of Fractional Order*, Facta Universitatis(NIS) Ser. Math. Inform. **36(1)** (2021) 43–55.

[3] N. D. Aral, and S. Gunal, *On $M_{\lambda_{m,n}}$-statistical convergence*, Journal of Mathematics (2020), Article ID 9716593, 8 pp.

[4] A. Caserta, Di M. Giuseppe, and L. D. R. Kočinac, *Statistical convergence in function spaces*, Abstr. Appl. Anal. (2011), Art. ID 420419, 11 pp.

[5] H. Cakalli, *A variation on statistical ward continuity*, Bull. Malays. Math. Sci. Soc. **40** (2017) 1701–1710.

[6] H. Cakalli, *Lacunary statistical convergence in topological groups*, Indian J. Pure Appl. Math. **26(2)** (1995) 113–119.

[7] H. Cakalli, *A study on statistical convergence*, Funct. Anal. Approx. Comput. **1(2)** (2009) 19–24.

[8] M. Cinar, M. Karakas, and M. Et, *On pointwise and uniform statistical convergence of order α for sequences of functions*, Fixed Point Theory Appl. **2013(33)** (2013) 11 pp.

[9] R. Colak, *Statistical convergence of order α*, Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub, **2010** (2010) 121–129.

[10] J. S. Connor, *The Statistical and strong $p-$Cesàro convergence of sequences*, Analysis **8** (1988) 47–63.

[11] M. Et, R. Colak, and Y. Altin, *Strongly almost summable sequences of order α*, Kuwait J. Sci. **41(2)** (2014) 35–47.

[12] M. Et, S. A. Mohiuddine, and A. Alotaibi, *On $\lambda-$statistical convergence and strongly $\lambda-$summable functions of order α*, J. Inequal. Appl. **2013(469)** 2013 8 pp.

[13] M. Et, H. Altinok, and R. Colak, *On lambda-statistical convergence of difference sequences of fuzzy numbers*, Information Sciences, **176(15)** (2006) 2268-2278.

[14] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951) 241-244.

[15] J. Fridy, *On statistical convergence*, Analysis **5** (1985) 301–313.

[16] A. D. Gadjiev, and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32(1)** (2002) 129-138.

[17] M. Isik, and K. E. Akbas, *On $\lambda-$statistical convergence of order α in probability*, J. Inequal. Spec. Funct. **8(4)** (2017) 57–64.

[18] M. Isik, and K. E. Akbas, *On asymptotically lacunary statistical equivalent sequences of order α in probability*, ITM Web of Conferences **13** (2017) 01024.

[19] E. Kolk, *The statistical convergence in Banach spaces*, Acta Comment. Univ. Tartu **928** (1991) 41–52.

[20] M. Mursaleen, *$\lambda-$statistical convergence*, Math. Slovaca, **50(1)** (2000) 111–115.

[21] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980) 139–150.

[22] H. Sengul, and M. Et, *On $I-$lacunary statistical convergence of order α of sequences of sets*, Filomat **31(8)** (2017) 2403–2412.

[23] H. Sengul, *On Wijsman $I-$lacunary statistical equivalence of order (η, μ)*, J. Inequal. Spec. Funct. **9(2)** (2018) 92–101.

[24] H. Sengul, and M. Et, *$f-$lacunary statistical convergence and strong $f-$lacunary summability of order α*, Filomat **32(13)** (2018) 4513–4521.
[25] H. Sengul, and M. Et, *On $(\lambda, I)-$statistical convergence of order α of sequences of function*, Proc. Nat. Acad. Sci. India Sect. A **88(2)** (2018) 181–186.
[26] M. Et, M. Çınar, and H. Sengul, *On Δ^m-asymptotically deferred statistical equivalent sequences of order α*, Filomat **33(7)** (2019) 1999–2007.
[27] H. Sengul, and O. Koyun, *On $(\lambda, A)-$statistical convergence of order α*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **68(2)** (2019) 2094–2103.
[28] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959) 361–375.
[29] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951) 73–74.

Hacer Şengül Kandemir,
Faculty of Education, Harran University, Osmanbey Campus 63190, Şanliurfa, Turkey,
ORCID ID: 0000-0003-4453-0786
Email address: hacer.sengul@hotmail.com

Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume IV Issue 1 (2022), Pages 15-23. Doi: https://doi.org/ 10.47087/mjm.1092599

ρ-STATISTICAL CONVERGENCE DEFINED BY A MODULUS FUNCTION OF ORDER (α, β)

NAZLIM DENIZ ARAL
DEPARTMENT OF MATHEMATICS, BITLIS EREN UNIVERSITY, 13000, BITLIS, TURKEY, ORCID ID: 0000-0002-8984-2620

Abstract. The concept of strong $w[\rho, f, q]$ −summability of order (α, β) for sequences of complex (or real) numbers is introduced in this work. We also give some inclusion relations between the sets of ρ-statistical convergence of order (α, β), strong $w_\alpha^\beta[\rho, f, q]$ −summability and strong $w_\alpha^\beta(\rho, q)$ −summability.

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [28] and Fast [13] and later in 1959, Schoenberg [27] reintroduced independently. Afterwards there has appeared much research with some arguments related of this concept (see Caserta et al. [3], Connor [4], Çakallı ([5],[6]), Çolak [7], Et et al. ([8],[9],[10]), Fridy [14], Gadjiev and Orhan [15], Kolk [17], Salat [26], Şengül et al.([2],[29],[30],[31],[32],[33],[34]) and many others).

The statistical convergence order α was introduced by Çolak [7] as follows:

The sequence $x = (x_k)$ is said to be statistically convergent of order α to L if there is a complex number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

Let $0 < \alpha \le \beta \le 1$. Then the (α, β)−density of the subset E of $\mathbb{N}$ is defined by

$$\delta_\alpha^\beta(E) = \lim_n \frac{1}{n^\alpha} |\{k \le n : k \in E\}|^\beta$$

if the limit exists (finite or infinite), where $|\{k \le n : k \in E\}|^\beta$ denotes the βth power of number of elements of E not exceeding n.

If $x = (x_k)$ is a sequence such that satisfies property $P(k)$ for all k except a set of (α, β)−density zero, then we say that x_k satisfies $P(k)$ for "almost all k according to β" and we denote this by "$a.a.k \ \ (\alpha, \beta)$".

Throughout this study, we shall denote the space of sequences of real number by w.

2020 *Mathematics Subject Classification.* Primary 40A05, 40C05; Secondary 46A45.
Key words and phrases. Cesàro summability; modulus function; statistical convergence.

Submitted on March 24th, 2022. Published on May 6th, 2022
Communicated by Huseyin CAKALLI, Ibrahim CANAK and Sefa Anıl SEZER.

Let $0<\beta\leq 1, 0<\alpha\leq 1$, $\alpha\leq\beta$ and $x=(x_k)\in w$. Then we say the sequence $x=(x_k)$ is statistically convergent of order (α,β) if there is a complex number L such that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{k\leq n:|x_k-L|\geq\varepsilon\}\right|^{\beta}=0$$

i.e. for $a.a.k$ (α,β) $|x_k-L|<\varepsilon$ for every $\varepsilon>0$, in that case a sequence x is said to be statistically convergent of order (α,β) to L. This limit is denoted by $S_{\alpha}^{\beta}-\lim x_k=L$ ([29]).

Let $0<\alpha\leqslant 1$. A sequence (x_k) of points in $\mathbb{R}$, the set of real numbers, is called $\rho-$statistically convergent of order α to an element L of $\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}|\{k\leq n:|x_k-L|\geq\varepsilon\}|=0$$

for each $\varepsilon>0$, where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ and $\Delta\rho_n=\rho_{n+1}-x_n$ for each positive integer n. In this case we write $st_{\rho}^{\alpha}-\lim x_k=L$. If $\rho=(\rho_n)=n$ and $\alpha=1$, then $\rho-$statistically convergent of order α is coincide statistical convergence ([5]).

Here and in what follows we suppose that the sequence $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ where $0<\alpha\leqslant 1$ and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer n.

The notion of a modulus function was given by Nakano [21]. Following Maddox [19] and Ruckle [25], we recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,
ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,
iii) f is increasing,
iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$.

Altın [1], Et ([11], [12]), Gaur and Mursaleen [20], Işık [16], Nuray and Savaş [22], Pehlivan and Fisher [23] and some others have been studied with some sequence spaces defined by modulus function.

The following inequality will be used frequently throught the paper:

$$|a_k+b_k|^{p_k}\leq A\left(|a_k|^{p_k}+|b_k|^{p_k}\right)\tag{1.1}$$

where $a_k,b_k\in\mathbb{C}$, $0<p_k\leq\sup p_k=B$, $A=\max\left(1,2^{B-1}\right)$ ([18]).

2. Main Results

In this section we first give the sets of strongly $w_{\alpha}^{\beta}(\rho,q)-$summable sequences and strongly $w_{\alpha}^{\beta}[\rho,f,q]-$summable sequences with respect to the modulus function f. Secondly we state and prove the theorems on some inclusion relations between the $S_{\alpha}^{\beta}(\rho)-$ statistical convergence, strong $w_{\alpha}^{\beta}[\rho,f,q]-$summability and strong $w_{\alpha}^{\beta}(\rho,q)-$summability.

Definition 2.1. *Let $0<\alpha\leq\beta\leq 1$ be given. A sequence $x=(x_k)$ is said to be $S_{\alpha}^{\beta}(\rho)-$statistically convergent (or ρ- statistically convergent sequences of order (α,β)) if there is a real number L such that*

$$\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}\left|\{k\leqslant n:|x_k-L|\geq\varepsilon\}\right|^{\beta}=0,$$

where ρ_n^α *denotes the* α*th power* $(\rho_n)^\alpha$ *of* ρ_n, *that is* $\rho^\alpha = (\rho_n^\alpha) = (\rho_1^\alpha, \rho_2^\alpha, ..., \rho_n^\alpha, ...)$ *and* $|\{k \leq n : k \in E\}|^\beta$ *denotes the* β*th power of number of elements of* E *not exceeding* n. *In the present case this convergence is indicated by* $S_\alpha^\beta(\rho) - \lim x_k = L$. $S_\alpha^\beta(\rho)$ *will indicate the set of all* $S_\alpha^\beta(\rho)-$*statistically convergent sequences.*

Definition 2.2. *Let* $0 < \alpha \leq \beta \leq 1$ *and* q *be a positive real number. A sequence* $x = (x_k)$ *is said to be strongly* $N_\alpha^\beta(\rho, q)-$*summable (or strongly* $N(\rho, q)-$*summable of order* (α, β)*) if there is a real number* L *such that*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \left(\sum_{k=1}^n |x_k - L|^q \right)^\beta = 0.$$

We denote it by $N_\alpha^\beta(\rho, q) - \lim x_k = L$. $N_\alpha^\beta(\rho, q)$ *will denote the set of all strongly* $N(\rho, q)-$*summable sequences of order* (α, β). *If* $\alpha = \beta = 1$, *then we will write* $N(\rho, q)$ *in the place of* $N_\alpha^\beta(\rho, q)$. *If* $L = 0$, *then we will write* $w_{\alpha,0}^\beta(\rho, q)$ *in the place of* $w_\alpha^\beta(\rho, q)$. $N_{\alpha,0}^\beta(\rho, q)$ *will denote the set of all strongly* $N_\rho(q)-$*summable sequences of order* (α, β) *to* 0.

Definition 2.3. *Let* f *be a modulus function,* $q = (q_k)$ *be a sequence of strictly positive real numbers and* $0 < \alpha \leq \beta \leq 1$ *be real numbers. A sequence* $x = (x_k)$ *is said to be strongly* $w_\alpha^\beta[\rho, f, q]-$*summable of order* (α, β) *if there is a real number* L *such that*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \left(\sum_{k=1}^n [f(|x_k - L|)]^{q_k} \right)^\beta = 0.$$

In this case, we write $w_\alpha^\beta[\rho, f, q] - \lim x_k = L$. *We denote the set of all strongly* $w_\alpha^\beta[\rho, f, q]-$*summable sequences of order* (α, β) *by* $w_\alpha^\beta[\rho, f, q]$. *In the special case* $q_k = q$, *for all* $k \in \mathbb{N}$ *and* $f(x) = x$ *we will denote* $N_\alpha^\beta(\rho, q)$ *in the place of* $w_\alpha^\beta[\rho, f, q]$. $w_{\alpha,0}^\beta[\rho, f, q]$ *will denote the set of all strongly* $w[\rho, f, q]-$*summable sequences of order* (α, β) *to* 0.

In the following theorems, we shall assume that the sequence $q = (q_k)$ is bounded and $0 < d = \inf_k q_k \leq q_k \leq \sup_k q_k = D < \infty$.

Theorem 2.1. *The class of sequences* $w_{\alpha,0}^\beta[\rho, f, q]$ *is linear space.*

Proof. Omitted. □

Theorem 2.2. *The space* $w_{\alpha,0}^\beta[\rho, f, q]$ *is paranormed by*

$$g(x) = \sup_n \left\{ \frac{1}{\rho_n^\alpha} \left(\sum_{k=1}^n [f(|x_k|)]^{q_k} \right)^\beta \right\}^{\frac{1}{H}}$$

where $0 < \alpha \leq \beta \leq 1$ *and* H=$\max(1, D)$.

Proof. Clearly $g(0) = 0$ and $g(x) = g(-x)$. Let $x, y \in w_{\alpha,0}^\beta[\rho, f, q]$ be two sequences. Since $\frac{q_k}{\frac{H}{\beta}} \leq 1$ and $\frac{H}{\beta} \geq 1$, using the Minkowski's inequality and definition of f, we have

$$
\begin{aligned}
\left\{\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k+y_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} &\leq \left\{\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k\right|\right)+f\left(\left|y_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} \\
&= \frac{1}{\rho_n^{\frac{\alpha}{H}}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k\right|\right)+f\left(\left|y_k\right|\right)\right]^{q_k}\right)^{\frac{1}{\frac{H}{\beta}}} \\
&\leq \frac{1}{\rho_n^{\frac{\alpha}{H}}}\left\{\left(\sum_{k=1}^{n}\left[f\left(\left|x_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} \\
&\quad +\frac{1}{\rho_n^{\frac{\alpha}{H}}}\left\{\left(\sum_{k=1}^{n}\left[f\left(\left|y_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}}.
\end{aligned}
$$

Hence, we have $g(x+y) \leq g(x)+g(y)$ for $x, y \in w_{\alpha,0}^{\beta}[\rho, f, q]$. Let μ be complex number. By defnition of f we have

$$
g(\mu x)=\sup_n\left\{\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|\mu x_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} \leq K^{\frac{D}{\frac{H}{\beta}}} g(x)
$$

where $[\mu]$ denotes the integer part of μ, and $K=1+[|\mu|]$. Now, let $\mu \rightarrow 0$ for any fixed x with $g(x) \neq 0$. By definition of f, for $|\mu|<1$ and $0<\alpha \leq \beta \leq 1$, we have

$$
\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|\mu x_k\right|\right)\right]^{q_k}\right)^{\beta}<\varepsilon \quad \text{for} \quad n>N(\varepsilon). \tag{2.1}
$$

Also, for $1 \leq n \leq N$, taking μ small enough, since f is continuous we have

$$
\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|\mu x_k\right|\right)\right]^{q_k}\right)^{\beta}<\varepsilon. \tag{2.2}
$$

Therefore, (2.1) and (2.2) imply that $g(\mu x) \rightarrow 0$ as $\mu \rightarrow 0$. □

Proposition 2.3. *([24]) Let f be a modulus and $0<\delta<1$. Then for each $\|u\| \geq \delta$, we have $f(\|u\|) \leq 2 f(1) \delta^{-1}\|u\|$.*

Theorem 2.4. *If $0<\alpha=\beta \leq 1$, $q>1$ and $\liminf_{u \rightarrow \infty} \frac{f(u)}{u}>0$, then $w_{\alpha}^{\beta}[\rho, f, q]=w_{\alpha}^{\beta}(\rho, q)$.*

Proof. If $\liminf_{u \rightarrow \infty} \frac{f(u)}{u}>0$ then there exists a number $c>0$ such that $f(u)>c u$ for $u>0$. Let $x \in w_{\alpha}^{\beta}[\rho, f, q]$, then

$$
\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q}\right)^{\beta} \geq \frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[c\left|x_k-L\right|\right]^{q}\right)^{\beta}=\frac{c^{q \alpha \beta}}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left|x_k-L\right|^{q}\right)^{\beta}.
$$

This means that $w_{\alpha}^{\beta}[\rho, f, q] \subseteq w_{\alpha}^{\beta}(\rho, q)$.

Let $x \in w_{\alpha}^{\beta}(\rho, q)$. Thus we have

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}|x_k-L|^q\right)^{\beta}\to 0 \text{ as } n\to\infty.$$

Let $\varepsilon>0$, $\beta=\alpha$ and choose δ with $0<\delta<1$ such that $cu<f(u)<\varepsilon$ for every u with $0\le u\le\delta$. Therefore, by Proposition 1, we have

$$\begin{aligned}\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}[f(|x_k-L|)]^q\right)^{\beta} &= \frac{1}{\rho_n^{\alpha}}\left(\sum_{\substack{k=1\\|x_k-L|\le\delta}}^{n}[f(|x_k-L|)]^q\right)^{\beta}+\frac{1}{\rho_n^{\alpha}}\left(\sum_{\substack{k=1\\|x_k-L|>\delta}}^{n}[f(|x_k-L|)]^q\right)^{\beta}\\ &\le \frac{1}{\rho_n^{\alpha}}\varepsilon^{q\beta}n^{\beta}+\frac{1}{\rho_n^{\alpha}}\left(\sum_{\substack{k=1\\|x_k-L|>\delta}}^{n}\left[2f(1)\,\delta^{-1}|x_k-L|\right]^q\right)^{\beta}\\ &\le \frac{1}{\rho_n^{\alpha}}\varepsilon^{q\alpha}n^{\beta}+\frac{2^{q\beta}f(1)^{q\beta}}{\rho_n^{\alpha}\delta^{q\beta}}\left(\sum_{k=1}^{n}|x_k-L|^q\right)^{\beta}.\end{aligned}$$

This gives $x\in w_{\alpha}^{\beta}[\rho,f,q]$. □

Example 2.1. *We now give an example to show that $w_{\alpha}^{\beta}[\rho,f,q]\neq w_{\alpha}^{\beta}(\rho,q)$ in this case $\liminf_{u\to\infty}\frac{f(u)}{u}=0$. Consider the sequence $f(x)=\frac{x}{1+x}$ of modulus function. Define $x=(x_k)$ by*

$$x_k=\begin{cases}k, & \text{if } k=m^3\\ 0, & \text{if } k\neq m^3.\end{cases}$$

Then we have, for $L=0$, $q=1$, $(\rho_n)=(n)$ and $\alpha=\beta$

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}[f(|x_k|)]^q\right)^{\beta}\leqslant\frac{1}{n^{\alpha}}n^{\frac{1}{3}\beta}\to 0 \text{ as } n\to\infty$$

and so $x\in w_{\alpha}^{\beta}[\theta,f,q]$. But

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}|x_k|^q\right)^{\beta}=\frac{1}{n^{\alpha}}\left(1+2^3+3^3+\cdots+\left[\sqrt[3]{n}\right]\right)^{\beta}$$

$$\geqslant\frac{1}{n^{\alpha}}\left[\frac{(\sqrt[3]{n}-1)(\sqrt[3]{n})}{2}\right]^{2\beta}=\frac{1}{n^{\alpha}}\frac{\left(n^{4/3}-2n+n^{2/3}\right)^{\beta}}{4^{\beta}}\to\infty \text{ as } n\to\infty$$

and so $x\notin w_{\alpha}^{\beta}(p)$.

Theorem 2.5. *Let $0<\alpha\le\beta\le 1$ and $\liminf q_k>0$. If a sequence is convergent to L, then it is strongly $w_{\alpha}^{\beta}[\rho,f,q]$ −summable of order (α,β) to L.*

Proof. We assume that $x_k\to L$. Since f be a modulus function, we have $f(|x_k-L|)\to 0$. Since $\liminf q_k>0$, we have $[f(|x_k-L|)]^{q_k}\to 0$. Hence $w_{\alpha}^{\beta}[\rho,f,q]-\lim x_k=L$. □

Theorem 2.6. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1]$ be real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, f be a modulus function , then $w_{\alpha_1}^{\beta_2}[\rho, f, q] \subset S_{\alpha_2}^{\beta_1}(\rho)$.*

Proof. Let $x \in w_{\alpha_1}^{\beta_2}[\rho, f, q]$ and let $\varepsilon > 0$ be given. Let $\sum_1$ and $\sum_2$ denote the sums over $k \leqslant n$ with $|x_k - L| \geq \varepsilon$ and $k \leqslant n$ with $|x_k - L| < \varepsilon$ respectively. Since $\rho_n^{\alpha_1} \leq \rho_n^{\alpha_2}$ for each n we have

$$\begin{aligned}
\frac{1}{\rho_n^{\alpha_1}} \left(\sum_{k=1}^{n} [f(|x_k - L|)]^{q_k} \right)^{\beta_2} &= \frac{1}{\rho_n^{\alpha_1}} \left[\sum\nolimits_1 [f(|x_k - L|)]^{q_k} + \sum\nolimits_2 [f(|x_k - L|)]^{q_k} \right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}} \left[\sum\nolimits_1 [f(|x_k - L|)]^{q_k} + \sum\nolimits_2 [f(|x_k - L|)]^{q_k} \right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}} \left[\sum\nolimits_1 [f(\varepsilon)]^{q_k} \right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}} \left[\sum\nolimits_1 \min([f(\varepsilon)]^d, [f(\varepsilon)]^D) \right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}} |\{k \leqslant n : |x_k - L| \geq \varepsilon\}|^{\beta_1} \left[\min([f(\varepsilon)]^d, [f(\varepsilon)]^D) \right]^{\beta_1}.
\end{aligned}$$

We get $x \in S_{\alpha_2}^{\beta_1}(\rho)$. □

Theorem 2.7. *If f is a bounded modulus function and $\lim_{n\to\infty} \frac{\rho_n^{\beta_2}}{\rho_n^{\alpha_1}} = 1$ then $S_{\alpha_1}^{\beta_2}(\rho) \subset w_{\alpha_2}^{\beta_1}[\rho, f, q]$.*

Proof. Let $x \in S_{\alpha_1}^{\beta_2}(\rho)$. Suppose that f be bounded. Therefore $f(x) \leq R$, for a positive integer R and all $x \geq 0$. Then for each $\varepsilon > 0$ we can write

$$\begin{aligned}
\frac{1}{\rho_n^{\alpha_2}} \left(\sum_{k=1}^{n} [f(|x_k - L|)]^{q_k} \right)^{\beta_1} &\leq \frac{1}{\rho_n^{\alpha_1}} \left(\sum_{k=1}^{n} [f(|x_k - L|)]^{q_k} \right)^{\beta_1} \\
&= \frac{1}{\rho_n^{\alpha_1}} \left(\sum\nolimits_1 [f(|x_k - L|)]^{q_k} + \sum\nolimits_2 [f(|x_k - L|)]^{q_k} \right)^{\beta_1} \\
&\leq \frac{1}{\rho_n^{\alpha_1}} \left(\sum\nolimits_1 \max(R^d, R^D) + \sum\nolimits_2 [f(\varepsilon)]^{q_k} \right)^{\beta_1} \\
&\leq \left(\max(R^d, R^D) \right)^{\beta_2} \frac{1}{\rho_n^{\alpha_1}} |\{k \leqslant n : f(|x_k - L|) \geq \varepsilon\}|^{\beta_2} \\
&\quad + \frac{\rho_n^{\beta_2}}{\rho_n^{\alpha_1}} \left(\max\left(f(\varepsilon)^d, \ f(\varepsilon)^D \right) \right)^{\beta_2}.
\end{aligned}$$

Hence $x \in w_{\alpha_2}^{\beta_1}[\rho, f, q]$. □

Theorem 2.8. *Let f be a modulus function. If $\lim q_k > 0$, then $w_\alpha^\beta[\rho, f, q] - \lim x_k = L$ uniquely.*

Proof. Let $\lim q_k = t > 0$. Suppose that $w_\alpha^\beta[\rho, f, q] - \lim x_k = L_1$ and $w_\alpha^\beta[\rho, f, q] - \lim x_k = L_2$. Then

$$\lim_n \frac{1}{\rho_n^\alpha} \left(\sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^\beta = 0,$$

and

$$\lim_n \frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L_2\right|\right)\right]^{q_k}\right)^{\beta}=0.$$

By definition of f and using (1.1), we may write

$$\begin{aligned}\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|L_1-L_2\right|\right)\right]^{q_k}\right)^{\beta} &\leq \frac{A}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L_1\right|\right)\right]^{q_k}+\sum_{k=1}^{n}\left[f\left(\left|x_k-L_2\right|\right)\right]^{q_k}\right)^{\beta}\\ &\leq \frac{A}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L_1\right|\right)\right]^{q_k}\right)^{\beta}+\frac{A}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L_2\right|\right)\right]^{q_k}\right)^{\beta}\end{aligned}$$

where $\sup_k q_k=D$, $0<\beta\leq\alpha\leq 1$ and $A=\max\left(1,2^{D-1}\right)$. Hence

$$\lim_n \frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|L_1-L_2\right|\right)\right]^{q_k}\right)^{\beta}=0.$$

Since $\lim_{k\to\infty} q_k=t$ we have $L_1-L_2=0$. Hence the limit is unique. □

Theorem 2.9. *Let $\rho=(\rho_n)$ and $\tau=(\tau_n)$ be two sequences such that $\rho_n\leqslant\tau_n$ for all $n\in\mathbb{N}$ and let $\alpha_1,\alpha_2,\beta_1$ and β_2 be such that $0<\alpha_1\leq\alpha_2\leq\beta_1\leq\beta_2\leq 1$,*

(i) If

$$\lim\inf_{n\to\infty}\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}>0 \tag{2.3}$$

then $w_{\alpha_2}^{\beta_2}[\tau,f,q]\subset w_{\alpha_1}^{\beta_1}[\rho,f,q]$,

(ii) If

$$\lim\sup_{n\to\infty}\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}<\infty \tag{2.4}$$

then $w_{\alpha_1}^{\beta_2}[\rho,f,q]\subset w_{\alpha_2}^{\beta_1}[\tau,f,q]$.

Proof. (i) Let $x\in w_{\alpha_2}^{\beta_2}[\tau,f,q]$. We have

$$\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q_k}\right)^{\beta_2}\geq\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q_k}\right)^{\beta_1}.$$

Thus if $x\in w_{\alpha_2}^{\beta_2}[\tau,f,q]$, then $x\in w_{\alpha_1}^{\beta_1}[\rho,f,q]$.

(ii) Let $x=(x_k)\in w_{\alpha_1}^{\beta_2}[\rho,f,q]$ and (2.4) holds. Now, since $\rho_n\leq\tau_n$ for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q_k}\right)^{\beta_1} &\leq \frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q_k}\right)^{\beta_2}\\ &= \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q_k}\right)^{\beta_2}\end{aligned}$$

for every $n\in\mathbb{N}$. Therefore $w_{\alpha_1}^{\beta_2}[\rho,f,q]\subset w_{\alpha_2}^{\beta_1}[\tau,f,q]$. □

References

[1] Y. Altın, Properties of some sets of sequences defined by a modulus function, *Acta Math. Sci. Ser. B Engl. Ed.* **29**(2) (2009), 427–434.

[2] N. D. Aral and H. Şengül Kandemir, I-Lacunary statistical Convergence of order β of difference sequences of fractional order, *Facta Universitatis (NIS) Ser. Math. Inform.* **36**(1) (2021), 43–55.

[3] A. Caserta, Di M. Giuseppe and L. D. R. Kočinac, Statistical convergence in function spaces, *Abstr. Appl. Anal.* 2011, Art. ID 420419, 11 pp.

[4] J. S. Connor, The Statistical and Strong p-Cesaro Convergence of Sequences, *Analysis*, 8, pp. (1988), 47-63.

[5] H. Çakallı, H. Şengül Kandemir and M. Et, ρ-statistical convergence of order beta, *American Institute of Physics.*, https://doi.org/10.1063/1.5136141.

[6] H. Çakallı, A study on statistical convergence, *Funct. Anal. Approx. Comput.* **1**(2) (2009), 19–24.

[7] R. Çolak, Statistical convergence of order α, *Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub*, 2010: 121–129.

[8] M. Et, Generalized Cesàro difference sequence spaces of non-absolute type involving lacunary sequences, *Appl. Math. Comput.* **219**(17) (2013), 9372–9376.

[9] M. Et, M. Çınar and H. Şengül, On Δ^m-asymptotically deferred statistical equivalent sequences of order α, *Filomat*, **33**(7) (2019), 1999–2007.

[10] M. Et and H. Şengül, Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α, *Filomat*, **28**(8) (2014), 1593–1602.

[11] M. Et, Strongly almost summable difference sequences of order m defined by a modulus, *Studia Sci. Math. Hungar.* **40**(4) (2003), 463–476.

[12] M. Et, Spaces of Cesàro difference sequences of order r defined by a modulus function in a locally convex space, *Taiwanese J. Math.* **10**(4) (2006), 865-879.

[13] H. Fast, Sur La Convergence Statistique, *Colloq. Math.*, 2, pp. (1951), 241–244.

[14] J. Fridy, On Statistical Convergence, *Analysis*, 5, pp. (1985), 301-313.

[15] A. D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* **32**(1) (2002), 129-138.

[16] M. Işık, Strongly almost $(w, \lambda, q)-$summable sequences, *Math. Slovaca* **61**(5) (2011), 779–788.

[17] E. Kolk, The statistical convergence in Banach spaces, *Acta Comment. Univ. Tartu*, **928** (1991), 41-52.

[18] I. J. Maddox, Elements of Functional Analysis, Cambridge University Press, 1970.

[19] I. J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Camb. Philos. Soc,* 1986, 100:161-166.

[20] A. K. Gaur and M. Mursaleen, Difference sequence spaces defined by a sequence of moduli, *Demonstratio Math.* **31**(2) (1998), 275–278.

[21] H. Nakano, Modulared sequence spaces, *Proc. Japan Acad.* **27** (1951), 508–512.

[22] F. Nuray and E. Savaş, Some new sequence spaces defined by a modulus function, *Indian J. Pure Appl. Math.* **24**(11) (1993), 657–663.

[23] S. Pehlivan and B. Fisher, Lacunary strong convergence with respect to a sequence of modulus functions, *Comment. Math. Univ. Carolin.* **36**(1) (1995), 69-76.

[24] S. Pehlivan and B. Fisher, Some sequence spaces defined by a modulus, *Mathematica Slovaca*, **45**(3) (1995), 275–280.

[25] W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.* **25** (1973), 973–978.

[26] T. Salat, On Statistically Convergent Sequences of Real Numbers, *Math. Slovaca.* **30** (1980), 139-150.

[27] I. J. Schoenberg, The Integrability of Certain Functions and Related Summability Methods, *Amer. Math. Monthly* **66** (1959), 361–375.

[28] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* **2** (1951) 73-74.

[29] H. Şengül, Some Cesàro summability spaces defined by a modulus function of order (α, β), *Commun. Fac .Sci. Univ. Ank. Series A1* **66**(2) (2017), 80–90.

[30] H. Şengül, On Wijsman $I-$lacunary statistical equivalence of order (η, μ), *J. Inequal. Spec. Funct.* **9**(2) (2018), 92–101.

[31] H. Şengül and M. Et, $f-$lacunary statistical convergence and strong $f-$lacunary summability of order α, *Filomat* **32**(13) (2018), 4513–4521.

[32] H. Şengül and M. Et, On $(\lambda, I)-$statistical convergence of order α of sequences of function, *Proc. Nat. Acad. Sci. India Sect. A* **88**(2) (2018), 181–186.

[33] H. Şengül and Ö. Koyun, On $(\lambda, A)-$statistical convergence of order α, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.* **68**(2) (2019), 2094–2103.

[34] H. Şengül and M. Et, On lacunary statistical convergence of order α, *Acta Math. Sci. Ser. B Engl. Ed.* **34**(2) (2014), 473–482.

Nazlim Deniz Aral
Department of Mathematics; Bitlis Eren University 13000; Bitlis; TURKEY, ORCID ID: 0000-0002-8984-2620
Email address: ndaral@beu.edu.tr

Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume IV Issue 1 (2022), Pages 24-32. Doi: https://doi.org/10.47087/mjm.1089672

PAIR DIFFERENCE CORDIALITY OF MIRROR GRAPH,SHADOW GRAPH AND SPLITTING GRAPH OF CERTAIN GRAPHS

R. PONRAJ*, A. GAYATHRI**, AND S. SOMASUNDARAM***
*DEPARTMENT OF MATHEMATICS,
SRI PARAMAKALYANI COLLEGE,
ALWARKURICHI–627 412, INDIA.
ORCID ID: 0000-0001-7593-7429
**RESEARCH SCHOLOR,REG.NO:20124012092023,
DEPARTMENT OF MATHEMATICS,
MANONMANIAM SUNDARANAR UNIVERSITY,ABHISHEKAPATI,
TIRUNELVELI–627 012, INDIA.
ORCID ID: 0000-0002-1004-0175
***DEPARTMENT OF MATHEMATICS,
MANONMANIAM SUNDARANAR UNIVERSITY,ABHISHEKAPATI,
TIRUNELVELI–627 012, INDIA.ORCID ID: 0000-0001-6980-7468

Abstract. In this paper we dicuss the pair difference cordility of Mirror graph, Splitting graph, Shadow graph of some graphs.

1. Introduction

We consider only finite, undirected and simple graphs. The origin of graph labeling is graceful labeling and introduced this concept by Rosa.A [15].Afterwards many labeling was defined and few of them are harmonious labeling[7], cordial labeling [1], magic labeling [16], mean labeling [19]. Cordial analogous labeling was studied in [2,3,4,5,10,11,12,13,14,17,18]. The notion of pair diference cordial labeling of a graph has been introduced and studied some properties of pair difference cordial labeling in [9].The pair difference cordial labeling behavior of several graphs like path, cycle, star etc have been investigated in [9].In this paper we dicuss the pair difference cordility of Mirror graph,Splitting graph,Shadow graph of some graphs.Term not defined here follow from Harary[8].

1991 *Mathematics Subject Classification.* Primary: 05C78 .

Key words and phrases. Path, Cycle, Ladder, Mirror graph, Splitting graph, Shadow graph.

Submitted on March 18th, 2022. Published on May 15th, 2022
Communicated by Hacer SENGUL KANDEMIR.

2. Preliminaries

Definition 2.1. [6]. *For a bipartite graph G with partite sets X and Y ,let G' be a copy of G and $X^{'}$ and $Y^{'}$ be copies of X and Y. The mirror graph $M^{'}(G)$, of a graph G as the disjoint union of G and $G^{'}$ with additional edges joining each vertex Y to its corresponding vertex in $Y^{'}$.*

Definition 2.2. [6].

The splitting graph of $G, S^{'}(G)$, is obtained from G by adding for each vertex v of G a new vertex $v^{'}$ so that $v^{'}$ is adjacent to every vertex that is adjacent to v.

Definition 2.3. [6].

The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of $G, G^{'}$ and $G^{''}$ and joining each vertex $u^{'}$ in $G^{'}$ to the neighbours of the corresponding vertex $u^{''}$ in $G^{''}$.

Definition 2.4. [6].

The ladder L_n is the product graph $P_n \times K_2$.

Theorem 2.1. [9].

If G is a (p, q) pair difference cordial graph then

$$q \leq \begin{cases} 2p-3 & \text{if } p \text{ is even} \\ 2p-1 & \text{if } p \text{ is odd} \end{cases}$$

Theorem 2.2. [9].

The path P_n is pair difference cordial for all values of n except $n \neq 3$.

Corollary 2.3. [9].

The cycle C_n is pair difference cordial if and only if $n > 3$.

Theorem 2.4. [9].

The ladder L_n is pair difference cordial for all values of n.

3. Mirror Graphs

Theorem 3.1. *The mirror graph of the path P_n is pair difference cordial.*

Proof. Since $M'(P_n) \cong L_n$,the proof follows from theorem 2.8.

□

Theorem 3.2. *$M^{'}(K_{1,n})$ is pair difference cordial if and only if $n \leq 2$.*

Proof. Let $V(M^{'}(K_{1,n})) = \{x, y, x_i, y_i : 1 \leq i \leq n\}$ and
$E(M^{'}(K_{1,n})) = \{xx_i, yy_i, y_ix_i, xy : 1 \leq i \leq n\}$.Since $S^{'}(K_{1,1}) \cong C_4$. By corollary 2.7, $M^{'}(K_{1,1})$ is pair difference cordial. A pair difference cordial labeling of $M^{'}(K_{1,2})$ is shown in Table 1.

Suppose f is a pair difference cordial labeling of $M^{'}(K_{1,n}), n \geq 3$. Obviously $\Delta_{f_1} \leq 4$. Then $\Delta^c_{f_1} \geq q - 4$. This implies that $\Delta^c_{f_1} \geq 3n + 1 - 4 = 3n - 3$. Hence $\Delta^c_{f_1} - \Delta_{f_1} \geq 3n - 7 > 1$, a contradiction.

□

n	x	x_1	x_2	y	y_1	y_2
2	-2	-1	-3	2	1	3

TABLE 1.

Theorem 3.3. *$M^{'}(S(K_{1,n}))$ is pair difference cordial if and only if $n \leq 2$.*

Proof. Let (X_1, Y_1) be bipartition of the first copy of $S(K_{1,n})$ where $X_1 = \{x, y_i : 1 \leq i \leq n\}, Y_1 = \{x_i : 1 \leq i \leq n\}$ and (X_2, Y_2) be bipartition of the second copy of $S(K_{1,n})$ where $X_2 = \{x^{'}, y_i^{'} : 1 \leq i \leq n\}, Y_2 = \{x_i^{'} : 1 \leq i \leq n\}$. Therefore $E(M^{'}(S(K_{1,n}))) = \{xx_i, x^{'}x_i^{'}, y_ix_i, y_i^{'}x_i^{'} : 1 \leq i \leq n\} \cup \{xx^{'}, x_ix_i^{'}, y_iy_i^{'} : 1 \leq i \leq n\} \cup \{xy, x^{'}y^{'}\}$. Clearly there are $4n + 2$ vertices and $6n + 1$ edges in the mirror graph of $S(K_{1,n})$. Since $M^{'}(S(K_{1,1})) \cong L_3$, by theorem 2.8, $M^{'}(S(K_{1,1}))$ is pair difference cordial. A pair difference cordial labeling of $M^{'}(S(K_{1,2}))$ is given in Table 2.

n	x	x_1	x_2	y_1	y_2	$x^{'}$	$x_1^{'}$	$x_2^{'}$	$y_1^{'}$	$y_2^{'}$
2	5	2	4	1	3	-5	-2	-4	-1	-3

TABLE 2.

Suppose f is a pair difference cordial labeling of $M^{'}(S(K_{1,n})), n \geq 3$. Obviously $\Delta_{f_1} \leq 2n+2$. Then $\Delta_{f_1}^c \geq q-2n-2$. This implies that $\Delta_{f_1}^c \geq 6n+1-2n-2 = 4n-1$. Hence $\Delta_{f_1}^c - \Delta_{f_1} \geq 2n - 3 > 1$, a contradiction. □

Theorem 3.4. *$M^{'}(B_{n,n})$ is pair difference cordial if and only if $n \leq 2$.*

Proof. Let (X_1, Y_1) be bipartition of the first copy of $B_{n,n}$ where $X_1 = \{x, y_i : 1 \leq i \leq n\}, Y_1 = \{y, x_i : 1 \leq i \leq n\}$ and (X_2, Y_2) be bipartition of the second copy of $B_{n,n}$ where $X_2 = \{x^{'}, y_i^{'} : 1 \leq i \leq n\}, Y_2 = \{y^{'}, x_i^{'} : 1 \leq i \leq n\}$. Therefore $E(M^{'}(B_{n,n})) = \{xx_i, x^{'}x_i^{'}, yy_i, y^{'}y_i^{'} : 1 \leq i \leq n\} \cup \{xx^{'}, yy^{'}, x_ix_i^{'}, y_iy_i^{'} : 1 \leq i \leq n\}$. Obviously $M^{'}(B_{n,n})$ has $4n + 2$ vertices and $6n + 1$ edges. Since $M^{'}(B_{1,1}) \cong L_4$,by theorem 2.8, $M^{'}(B_{1,1})$ is pair difference cordial. A pair difference cordial labeling of $M^{'}(B_{1,2})$ is shown in Table 3.

n	x	x_1	x_2	y	y_1	y_2	$x^{'}$	$x_1^{'}$	$x_2^{'}$	$y^{'}$	$y_1^{'}$	$y_2^{'}$
2	2	1	3	-2	-1	-3	5	4	6	-5	-4	-6

TABLE 3.

Suppose f is a pair difference cordial labeling of $M^{'}(B_{n,n}), n \geq 3$. Obviously $\Delta_{f_1} \leq 8$. Then $\Delta_{f_1}^c \geq q - 8$. This implies that $\Delta_{f_1}^c \geq 6n + 4 - 8 = 6n - 4$. Hence $\Delta_{f_1}^c - \Delta_{f_1} \geq 6n - 12 > 1$, a contradiction. □

4. SHADOW GRAPHS

Theorem 4.1. *Let G be a (p, q) graph with $q \geq p$.Then $D_2(G)$ is not pair difference cordial.*

Proof. Suppose G is a pair difference cordial graph with $q \geq p$. Obviously $|V(D_2(G))| = 2p$ and $|E(D_2(G))| = 4q$. By theorem 2.5, $4q \leq 2(2p) - 3$. This implies that $4q \leq 4q - 3$, a contradiction. □

Theorem 4.2. *$D_2(P_n)$ is pair difference cordial for all values of n.*

Proof. Let $V(D_2(P_n)) = \{x_i, y_i : 1 \leq i \leq n\}$, $E(D_2(P_n)) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_{i+1}, y_i x_{i+1} : 1 \leq i \leq n-1\}$. Clearly $D_2(P_n)$ has $2n$ vertices and $4n-4$ edges.
Define $f : V(D_2(P_n)) \to \{\pm 1, \pm 2, \pm 3, \cdots \pm n\}$ by

$$f(x_i) = i, \qquad 1 \leq i \leq n,$$
$$f(y_i) = -i, \qquad 1 \leq i \leq n.$$

This vertex labeling yields that $D_2(P_n)$ is pair difference cordial for all values of n, since $\Delta_{f_1} = 2n - 2 = \Delta_{f_1^c}$. □

Theorem 4.3. *$D_2(C_n)$ is not pair difference cordial for all values of n.*

Proof. Let C_n be the first copy of the cycle $x_1 x_2 \cdots x_n x_1$ and $y_1 y_2 \cdots y_n y_1$ be the second copy of the cycle C_n. The maximum number of the edges with the labels 1 among the vertex labels $1, 2, \cdots, n$ is $n-1$. Also the maximum number of the edges with the labels 1 among the vertex labels $-1, -2, \cdots, -n$ is $n-1$. Therefore $\Delta_{f_1} \leq 2n-2$. This implies that $\Delta^c_{f_1} \geq 4n - (2n-2) = 2n+2$. Hence $\Delta_{f_1} - \Delta^c_{f_1} \geq 2n + 2 - (2n-2) = 4 > 1$, a contradiction. □

Theorem 4.4. *$D_2(K_n)$ is pair difference cordial if and only if $n \leq 2$.*

Proof. Clearly $|V(D_2(K_n))| = 2n$ and $|E(D_2(K_n))| = n(n-1) + 2\binom{n}{2}$. Suppose $D_2(K_n)$ is a pair difference cordial. By theorem 2.5 ,$n(n-1) + 2\binom{n}{2} \leq 2(2n) - 3$, which implies that $2n^2 - 6n + 3 \leq 0$. It gives that $n \leq 2$. Hence $D_2(K_n), n > 3$ is not pair difference cordial. Obviously $D_2(K_1)$ is pair difference cordial. Since $K_2 \cong P_2$, by theorem 2.6, $D_2(K_2)$ is pair difference cordial. □

Theorem 4.5. *$D_2(K_{1,n})$ is pair difference cordial if and only if $n \leq 2$.*

Proof. Clearly $|V(D_2(K_{1,n}))| = 2n+2$ and $|E(D_2(K_{1,n}))| = 4n$. Suppose $D_2(K_{1,n})$ is a pair difference cordial . Obviously $\Delta_{f_1} \leq 2n+1$. Let u be the central vertex of $K_{1,n}$ and u' be the corresponding shadow vertex . Hence $d(u) = d(u') = 2n$ in $D_2(K_{1,n})$. Now $\Delta f_1 \geq 2n - 2 + 2n - 2 \geq 4n - 4$. Hence $\Delta_{f_1} - \Delta_{f_1^c} \geq 2n - 3$. This implies $n \leq 2$. Since $D_2(K_{1,1}) \cong C_4$, by corollary 2.7, $D_2(K_{1,1})$ is pair difference cordial. The labeling f defined by $f(u) = 2, f(u') = -2, f(u_1) = -1, f(u_2) = -3, f(u'_1) = 1, f(u'_2) = 3$ is a pair difference cordial labeling of $D_2(K_{1,2})$. □

Theorem 4.6. *$D_2(P_n \odot K_1)$ is not pair difference cordial for all values of n.*

Proof. Let $V(D_2(P_n \odot K_1)) = \{x_i, x'_i, y_i, y'_i : 1 \leq i \leq n\}$. There are $4n$ vertices and $8n-4$ edges.
Suppose $D_2(P_n \odot K_1)$ is pair difference cordial for all values of n. The maximum

number of the edges with the labels 1 among the vertex labels $1, 2, \cdots, n$ is $n-1$ and the maximum number of the edges with the labels 1 among the vertex labels $-1, -2, \cdots, -n$ is $n-1$. Therefore $\Delta_{f_1} \leq 2n - 2 + 2 = 2n$. This implies that $\Delta_{f_1}^c \geq 8n - 4 - 2n = 6n - 4$. Hence $\Delta_{f_1} - \Delta_{f_1}^c \geq 6n - 4 - 2n = 4n - 4 > 1$, a contradiction.

□

5. Spilitting Graphs

Theorem 5.1. *$S^{'}(P_n)$ is pair difference cordial for all n.*

Proof. Let $V(S^{'}(P_n)) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(S^{'}(P_n)) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_{i+1}, y_i x_{i+1} : 1 \leq i \leq n-1\}$. There are two cases arises.

Case 1. $n \leq 5$.
A pair difference cordial labeling for this case given in Table 4.

n	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5
2	1	2				-1	-2			
3	1	-1	3			-2	2	-3		
4	1	2	-1	-2		3	4	-3	-4	
5	1	2	5	-2	-4	3	4	-1	-5	-3

Table 4.

Case 2. $n > 5$.
There are four cases arises.
Subcase 1. $n \equiv 0 \pmod 4$.

Assign the labels $1, 5, 9, \cdots, n-3$ to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-2}{2}}$ respectively and assign the labels $2, 6, 10, \cdots, n-2$ respectively to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n}{2}}$. Now we assign the labels $3, 7, 11, \cdots, n-1$ respectively to the vertices $y_1, y_3, y_5, \cdots,$ $y_{\frac{n-2}{2}}$ and assign the labels $4, 8, 12, \cdots, n$ to the vertices $y_2, y_4, y_6, \cdots, y_{\frac{n}{2}}$ respectively.

Next we assign the labels $-1, -5, -9, \cdots, -(n-3)$ respectively to the vertices $x_{\frac{n+2}{2}}, x_{\frac{n+6}{2}}, x_{\frac{n+10}{2}}, \cdots, x_{n-1}$ and we assign the labels $-2, -6, -10, \cdots, -(n-2)$ respectively to the vertices $x_{\frac{n+4}{2}}, x_{\frac{n+8}{2}}, x_{\frac{n+12}{2}}, \cdots, x_n$. Lastly assign the labels $-3, -7, -11, \cdots, -(n-1)$ respectively to the vertices $y_{\frac{n+2}{2}}, y_{\frac{n+6}{2}}, y_{\frac{n+10}{2}}, \cdots, y_{n-1}$ and assign the labels $-4, -8, -12, \cdots, -n$ to the vertices $y_{\frac{n+4}{2}}, y_{\frac{n+8}{2}}, y_{\frac{n+12}{2}}, \cdots, y_n$ respectively.

Subcase 2. $n \equiv 1 \pmod 4$.

Assign the labels $1, 5, 9, \cdots, n-4$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-3}{2}}$ and assign the labels $2, 6, 10, \cdots, n-3$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n-1}{2}}$ respectively. Now we assign the labels $3, 7, 11, \cdots, n-2$ to the vertices $y_1, y_3, y_5, \cdots, y_{\frac{n-3}{2}}$ respectively and assign the labels $4, 8, 12, \cdots, n-1$ respectively to the vertices

$y_2, y_4, y_6, \cdots, y_{\frac{n-1}{2}}$ and assign the label n to the vertex $y_{\frac{n+1}{2}}$.

Now we assign the labels $-1, -3, -5, \cdots, -(\frac{n+1}{2})$ respectively to the vertices $x_{\frac{n+1}{2}}, x_{\frac{n+5}{2}}, x_{\frac{n+9}{2}}, \cdots, x_n$ and we assign the labels $-(\frac{n+3}{2}), -(\frac{n+7}{2}), -(\frac{n+11}{2}), \cdots,$ $-(n-1)$ respectively to the vertices $y_n, y_{n-2}, y_{n-4}, \cdots, y_{\frac{n+5}{2}}$. Next assign the labels $-2, -4, -6, \cdots, -(\frac{n-1}{2})$ respectively to the vertices $y_{\frac{n+3}{2}}, y_{\frac{n+7}{2}}, y_{\frac{n+11}{2}}, \cdots, y_{n-1}$ and assign the labels $-(\frac{n+5}{2}), -(\frac{n+9}{2}), -(\frac{n+13}{2}), \cdots, -(n)$ to the vertices $x_{n-1}, x_{n-3}, x_{n-5},$ $\cdots, x_{\frac{n+3}{2}}$ respectively.

Subcase 3. $n \equiv 2 \pmod 4$.
as in case 1 assign the labels to the vertices $x_i, y_i (1 \leq i \leq n-2)$. Finally we assign the labels $(n-1), n, -n, -(n-1)$ to the vertices $x_{n-1}, x_n, y_{n-1}, y_n$.

Subcase 4. $n \equiv 3 \pmod 4$.

Assign the labels $1, 5, 9, \cdots, n-2$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-1}{2}}$ and we assign the labels $2, 6, 10, \cdots, n-5$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n-3}{2}}$ respectively. Now we assign the labels $3, 7, 11, \cdots, n-4$ to the vertices $y_1, y_3, y_5, \cdots, y_{\frac{n-5}{2}}$ respectively and assign the labels $4, 8, 12, \cdots, n-3$ respectively to the vertices $y_2, y_4, y_6, \cdots, y_{\frac{n-3}{2}}$.

Next we assign the labels $-1, -3, -5, \cdots, -(\frac{n-1}{2})$ respectively to the vertices $x_{\frac{n+1}{2}}, x_{\frac{n+5}{2}}, x_{\frac{n+9}{2}}, \cdots, x_{n-1}$ and we assign the labels $-(\frac{n+3}{2}), -(\frac{n+7}{2}), -(\frac{n+11}{2}), \cdots,$ $-(n)$ respectively to the vertices $x_n, x_{n-2}, x_{n-4}, \cdots, x_{\frac{n+3}{2}}$. Next assign the labels $-2, -4, -6, \cdots, -(\frac{n+1}{2})$ respectively to the vertices $y_{\frac{n+3}{2}}, y_{\frac{n+7}{2}}, y_{\frac{n+11}{2}}, \cdots, y_n$ and assign the labels $-(\frac{n+5}{2}), -(\frac{n+9}{2}), -(\frac{n+13}{2}), \cdots, -(n-1)$ to the vertices $y_{n-1}, y_{n-3},$ $y_{n-5}, \cdots, y_{\frac{n+5}{2}}$ respectively.

Finally assign the labels $n-1, n$ to the vertices $y_{\frac{n-1}{2},} y_{\frac{n+1}{2}}$ respectively.

□

Theorem 5.2. *$S^{'}(P_n \odot K_1)$ is pair difference cordial.*

Proof. Let $V(S^{'}(P_n \odot K_1)) = \{x_i, x_i^{'}, y_i, y_i^{'} : 1 \leq i \leq n\}$ and $E(S^{'}(P_n \odot K_1)) =$ $\{x_i x_{i+1}^{'}, x_{i+1} x_i^{'} : 1 \leq i \leq n-1\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_i x_i^{'}, x_i y_i^{'} : 1 \leq i \leq n\}$. There are $4n$ vertices and $6n-3$ edges.There are two cases arises.

Case 1.n is even.

Assign the labels $1, 5, 9, \cdots, (2n-3)$ to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n}{2}}$ respectively and we assign the labels $-1, -5, -9, \cdots, -(2n-3)$ respectively $x_{\frac{n+2}{2}}, x_{\frac{n+4}{2}}, x_{\frac{n+6}{2}}, \cdots,$ x_n. Next assign the labels $4, 8, 12, \cdots, 2n$ to the vertices $x_1^{'}, x_2^{'}, x_3^{'}, \cdots, x_{\frac{n}{2}}^{'}$ respectively and we assign the labels $-4, -8, -12, \cdots, -(2n)$ respectively $x_{\frac{n+2}{2}}^{'}, x_{\frac{n+4}{2}}^{'}, x_{\frac{n+6}{2}}^{'},$ $\cdots, x_n^{'}$.

Now we assign the labels $3, 7, 11, \cdots, (2n-1)$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n}{2}}$ respectively and we assign the labels $-3, -7, -11, \cdots, -(2n-1)$ respectively $y_{\frac{n+2}{2}}, y_{\frac{n+4}{2}}, y_{\frac{n+6}{2}}, \cdots, y_n$. Next assign the labels $2, 6, 10, \cdots, (2n-2)$ to the vertices $y_1^{'}, y_2^{'}, y_3^{'}, \cdots, y_{\frac{n}{2}}^{'}$ respectively and assign the labels $-2, -6, -10, \cdots, -(2n-2)$ respectively $y_{\frac{n+2}{2}}^{'}, y_{\frac{n+4}{2}}^{'}, y_{\frac{n+6}{2}}^{'}, \cdots, y_n^{'}$.

Clearly $\Delta_{f_1} = 3n-2, \Delta_{f_1^c} = 3n-1$. This vertex labeling gives that $S^{'}(P_n \odot K_1)$ is pair difference cordial for all even values of n.

Case 2. n is odd.

As in case 1 , assign the labels to the vertices $x_i, y_i, x_i^{'}, y_i{'} (1 \leq i \leq n-1)$. Finally we assign the labels $2n-1, -(2n-1), 2n, -2n$ to the vertices $x_n^{'}, x_n, y_n, y_n^{'}$.

Clearly $\Delta_{f_1} = 3n-2, \Delta_{f_1^c} = 3n-1$. This vertex labeling gives that $S^{'}(P_n \odot K_1)$ is pair difference cordial for all odd values of n.

□

Theorem 5.3. *$S^{'}(K_n)$ is pair difference cordial if and only if $n \leq 3$.*

Proof. Clearly $\left|V(S^{'}(K_n))\right| = 2n$ and $\left|E(S^{'}(K_n))\right| = \frac{3n(n-1)}{2}$.
Case 1. $n \leq 3$.
Obviously $S^{'}(K_1)$ is pair difference cordial.Since $S^{'}(K_n) \cong C_4$, then $S^{'}(K_2)$ is pair difference cordial. By theorem 5.2, $S^{'}(K_3)$ is pair difference cordial.

Case 2. $n > 3$.
Suppose $S^{'}(K_n)$ is pair difference cordial. By theorem 2.5 ,

$$\frac{3n(n-1)}{2} \leq 2(2n) - 3,$$
$$\Rightarrow 3n^2 - 3n \leq 4(2n) - 6,$$
$$\Rightarrow 3n^2 - 11n \leq -6,$$
$$\Rightarrow -3n^2 + 11n \geq 6, \text{a contradiction}$$

□

Theorem 5.4. *$S^{'}(K_{1,n})$ is pair difference cordial if and only if $n \leq 3$.*

Proof. Let $V(S^{'}(K_{1,n})) = \{x, y, x_i, y_i : 1 \leq i \leq n\}$ and $E(S^{'}(K_{1,n})) = \{xx_i, yy_i, y_ix : 1 \leq i \leq n\}$.Since $S^{'}(K_{1,1}) \cong P_4$. By theorem 2.6, $S^{'}(K_{1,1})$ is pair difference cordial. A pair difference cordial labeling of $S^{'}(K_{1,2})$ and $S^{'}(K_{1,3})$ is shown in Table 5.

n	x	x_1	x_2	x_3	y	y_1	y_2	y_3
2	-1	-2	-3		2	1	3	
3	-1	-2	-3	-4	2	1	3	4

TABLE 5.

Suppose f is a pair difference cordial labeling of $S^{'}(K_{1,n}), n > 3$. Obviously $\Delta_{f_1} \leq 4$. Then $\Delta_{f_1}^c \geq q - 4$. This implies that $\Delta_{f_1}^c \geq 3n - 4$. Hence $\Delta_{f_1}^c - \Delta_{f_1} \geq 3n - 8 > 1$, a contradiction.

□

6. Conclusions

In this paper, we have studied about the pair difference cordility of Mirror graph,Splitting graph,Shadow graph of some graphs.Investigation of the pair difference cordility of Mirror graph,Splitting graph,Shadow graph of some special graphs are the open problems.

References

[1] I.Cahit., Cordial Graphs : A weaker version of Graceful and Harmonious graphs, *Ars combin.*, 23 (1987), 201–207.

[2] I.Cahit., On cordial and 3-equitable labellings of graphs, *Util.Math.*, 37 (1990), 189–198.

[3] I.Cahit., Some Totally modular cordial graphs, *Discussion.Math.Graph Theory*, 22 (2002), 247–258.

[4] Ebrahim Salehi.,Seth Churchman.,Tahj Hill.,Jim Jordan., Product Cordial Sets of Trees, *Congressus Numerantium*, 220 (2014), 183–193.

[5] Ebrahim Salehi., Product Cordial Sets of Long Grids, *Ars combin.*, 107 (2012), 339–351.

[6] J.A. Gallian., A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics.*, 19, (2016).

[7] R.L.Graham., and N.J,A.Solane., On additive bases and harmonious graphs, *SIAM J.Alg.Disrete Methods.*, 1, (1980)382–404.

[8] F.Harary., Graph theory, *Addision wesley*, New Delhi, 1969.

[9] R.Ponraj., A.Gayathri., and S.Soma Sundaram., Pair difference cordial labeling of graphs, *J.Math. Comp.Sci.* Vol.11(1), (2015), 189–197.

[10] U.M.Prajapati, U.M., and S.J.Gajjar., Cordial labeling for complement graphs,*Mathematics Today* Vol.30, (2015), 99–118.

[11] U.M.Prajapati., and S.J.Gajjar., Some results on prime cordial labeling of generalized prism graph $Y_{m,n}$,*Ultra Scientist* Vol.27(3)A , (2015), 189–204.

[12] U.M.Prajapati., Prakruti D.Shah., Some Edge Product Cordial graphs in the context of Duplication of some Graph Elements, *Open Journal of Discrete Mathematics* , (6), (2016), 248–258.

[13] U.M.Prajapati., N.B.Patel., Edge Product Cordial labeling of Some Cycle Related Graphs, *Open Journal of Discrete Mathematics* , (6), (2016), 268–278.

[14] U.M.Prajapati., N.B.Patel., Edge Product Cordial labeling of Some Graphs, *Journal of Applied Mathematics and Computer Mechanics Discrete Mathematics* ,(Vol18)(1), (2019), 69–76.

[15] A. Rosa., On certain Valuations of the vertices of a graph,Theory of Graphs, *International Symposium,Rome.*, July, (1967) 349–345.

[16] J.Sedlacek., Problem 27, *in Theory Graphs and its Applications, Proc.Symposium Smolenice.*, June, (1963) 163–167.

[17] M.A.Seoud., E.F.Helmi., On Product cordial graphs, *Ars.Combin*, Vol.101, (2011), 519–529.

[18] M.A.Seoud., M.S.Salman., Some results and examples on difference cordial graphs, *Turkish Journal of Mathematics*, 40, (2016), 417–427.

[19] S.Somasundaram., and R.Ponraj., Mean Labeling of Graphs, *Natl.Acad.sci.Let.*, 26, (2003),210–213.

R. Ponraj

Department of Mathematics,, Sri Paramakalyani College,, Alwarkurichi–627 412, India., Orcid Id: 0000-0001-7593-7429

Email address: `ponrajmaths@gmail.com`

A. Gayathri
Research Scholor,Reg.No:20124012092023,, Department of Mathematics,, Manonmaniam Sundaranar University,Abhishekapati,, Tirunelveli–627 012, India., Orcid Id: 0000-0002-1004-0175
Email address: gayugayathria555@gmail.com

S. Somasundaram
Department of Mathematics,, Manonmaniam Sundaranar University,Abhishekapati,, Tirunelveli–627 012, India., Orcid Id: 0000-0001-6980-7468
Email address: somutvl@gmail.com