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Two New Versions of the Pasting Lemma via Soft Mixed Structure

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Abstract

In this paper, we present two new generalizations of the pasting lemma using soft mixed structure. To do this, we introduce the notions of a (τ_1,τ_2)-g-closed soft set and a (τ_1,τ_2)-gpr-closed soft set. We establish the notions of mixed g-soft continuity and mixed gpr-soft continuity between two soft topological spaces (X,τ_1,Δ_1), (X,τ_2,Δ_1) and a soft topological space (X,τ,Δ_2). Finally we prove two new versions of the pasting lemma using the mixed g-soft continuous mapping and the mixed gpr-soft continuous mapping.

1. Introduction and motivation

"*Soft set theory"* was introduced as a general mathematical tool for dealing with encountered difficulties and problems in medical science, social science, engineering, economics etc. [1]. Many researchers have been studying some topological concepts with basic properties and some generalizations of a soft topological space via different approaches (for example, see [2]-[15]). Also some applications of the soft set theory were obtained to other sciences such as medical science, food science, insurance, investment etc. (see [16]-[26] for some examples). Recently, different decision making applications have been studied (for example, see [27]-[30]).

"*Mixed structure* has"been studied on various topological spaces such as a soft topological space, a generalized topological space etc. Using the mixed structure, some topological notions have been generalized with a new approach. For example, some mixed sets and mixed continuities were defined on a generalized topological space (resp. on a soft topological space) (see [31]-[37]).

"*Pasting lemma"* is one of the most important notions on a topological space for continuous functions. Especially, it has a significant place in algebraic topology. Recent years, some new forms of the pasting lemma have been introduced by many mathematicians (for example, see [15], [38]-[42] and the references therein).

Motivated by the above studies, we present two new version of the pasting lemma using mixed structure on a soft topological space. For this purpose, we introduce the notions of (τ_1,τ_2)-g-closed soft set, a (τ_1,τ_2)-gpr-closed soft set, mixed soft pre closure and mixed soft pre interior. We prove some topological properties of these new notions. Also we give some counter examples for necessary relationships. We define the notions of mixed g-soft continuity and mixed gpr-soft continuity between two soft topological spaces (X,τ_1,Δ_1), (X,τ_2,Δ_1) and a soft topological space (X,τ,Δ_2). Finally, we establish two new versions of the pasting lemma for mixed g-soft continuous functions and mixed gpr-soft continuous functions on a soft topological space.

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2. Preliminaries

In this section, we recall some basic concepts related to soft set theory. Throughout this paper, we assume that X is an initial universal set, Δ is a nonempty set of parameters and $\Delta_1, \Delta_2 \subseteq \Delta$.

Definition 2.1. *[1] Let $\phi : \Delta_1 \to P(X)$ be a mapping. Then a pair (ϕ, Δ_1) is called a soft set over X. $SS(X)_\Delta$ denotes the family of all soft sets on X.*

Definition 2.2. *[7] Let (ϕ, Δ_1) be a soft set over X.*
(1) (ϕ, Δ_1) is called a null soft set if $\phi(e) = \emptyset$ for all $e \in \Delta_1$. It is denoted by $\widetilde{\emptyset}$.
(2) (ϕ, Δ_1) is called an absolute soft set if $\phi(e) = X$ for all $e \in \Delta_1$. It is denoted by $\widetilde{X}$.

Definition 2.3. *[7] Let $(\phi, \Delta_1) \in SS(X)_{\Delta_1}$ and $(\varphi, \Delta_2) \in SS(X)_{\Delta_2}$.*
(1) (ϕ, Δ_1) is called a soft subset of (φ, Δ_2) if $\Delta_1 \subseteq \Delta_2$ and $\phi(e) \subseteq \varphi(e)$ for all $e \in \Delta_1$. It is denoted by

$$(\phi, \Delta_1) \widetilde{\subseteq} (\varphi, \Delta_2).$$

(2) (ϕ, Δ_1) is called soft equal to (φ, Δ_2) if $(\phi, \Delta_1) \widetilde{\subseteq} (\varphi, \Delta_2)$ and $(\varphi, \Delta_2) \widetilde{\subseteq} (\phi, \Delta_1)$. It is denoted by

$$(\phi, \Delta_1) = (\varphi, \Delta_2).$$

Definition 2.4. *[10] Let (ϕ, Δ_1), $(\varphi, \Delta_1) \in SS(X)_{\Delta_1}$.*
(1) The complement of (ϕ, Δ_1) is defined as

$$(\phi, \Delta_1)^c = (\phi^c, \Delta_1),$$

where $\phi^c(e) = (\phi(e))^c = X - \phi(e)$ for all $e \in \Delta_1$.
(2) The difference of (ϕ, Δ_1) and (φ, Δ_1) is defined as

$$(\phi, \Delta_1) - (\varphi, \Delta_1) = (\phi - \varphi, \Delta_1),$$

where $(\phi - \varphi)(e) = \phi(e) - \varphi(e)$ for all $e \in \Delta_1$.

Definition 2.5. *[14] Let J be an arbitrary index set and $\{(\phi_i, \Delta)\}_{i \in J}$ be a subfamily of $SS(X)_\Delta$.*
(1) The union of these soft sets is the soft set (φ, Δ), where

$$\varphi(e) = \bigcup_{i \in J} \phi_i(e),$$

for each $e \in \Delta$. It is denoted by $\widetilde{\bigcup_{i \in J}} (\phi_i, \Delta) = (\varphi, \Delta)$.
(2) The intersection of these soft sets is the soft set (θ, Δ), where

$$\theta(e) = \bigcap_{i \in J} \phi_i(e),$$

for each $e \in \Delta$. It is denoted by $\widetilde{\bigcap_{i \in J}} (\phi_i, \Delta) = (\theta, \Delta)$.

Definition 2.6. *[10] Let $(\phi, \Delta_1) \in SS(X)_{\Delta_1}$ and $x \in X$. The point x is called in the soft set (ϕ, Δ_1) if $x \in \phi(e)$ for all $e \in \Delta_1$. It is denoted by $x \in (\phi, \Delta_1)$.*

Definition 2.7. *[10] Let $(\phi, \Delta) \in SS(X)_\Delta$ and Y a nonempty subset of X. The sub soft set of (ϕ, Δ) over Y, denoted by $({}^Y\phi, \Delta)$, is defined by*

$${}^Y\phi(e) = Y \cap \phi(e),$$

for all $e \in \Delta$. In other words, $({}^Y\phi, \Delta) = \widetilde{Y} \cap (\phi, \Delta)$.

Definition 2.8. *[10] Let τ be the collection of soft sets over X. Then τ is called a soft topology on X if the following conditions hold:*
(t_1) $\widetilde{\emptyset}, \widetilde{X} \in \tau$.
(t_2) The intersection of any two soft sets in τ belongs to τ.
(t_3) The union of any number of soft sets in τ belongs to τ.
The triple (X, τ, Δ) is called a soft topological space over X.

Definition 2.9. *[10] The members of τ are said to be τ-soft open sets or soft open sets in X and also a soft set over X is called soft closed in X if its complement belongs to τ.*
$OS(X, \tau, \Delta)$ or $OS(X)$ denotes the set of all soft open sets over X and $CS(X, \tau, \Delta)$ or $CS(X)$ denotes the set of all soft closed sets.

Definition 2.10. *[10] Let (X,τ,Δ) be a soft topological space over X and Y a nonempty subset of X. Then*

$$\tau_Y = \left\{ ({}^Y\phi,\Delta) : (\phi,\Delta) \in \tau \right\}$$

is called the soft relative topology on Y and (Y,τ_Y,Δ) is called a soft subspace of (X,τ,Δ).

Theorem 2.11. *[10] Let (Y,τ_Y,Δ) be a soft subspace of a soft topological space (X,τ,Δ) and (ϕ,Δ) be a soft set over X. Then*
(1) (ϕ,Δ) is soft open in Y if and only if $(\phi,\Delta) = \widetilde{Y} \cap (\varphi,\Delta)$ for some $(\varphi,\Delta) \in \tau$.
(2) (ϕ,Δ) is soft closed in Y if and only if $(\phi,\Delta) = \widetilde{Y} \cap (\varphi,\Delta)$ for some soft closed set (φ,Δ) in X.

Theorem 2.12. *[10] Let (Y,τ_Y,Δ) be a soft subspace of a soft topological space (X,τ,Δ) and (ϕ,Δ) be a soft set over X. If $\widetilde{Y} \in \tau$ then $(\phi,\Delta) \in \tau$.*

Definition 2.13. *[10] Let (X,τ,Δ) be a soft topological space and $(\phi,\Delta) \in SS(X)_\Delta$. The soft closure of (ϕ,Δ) is the intersection of all soft closed super sets of (ϕ,Δ). It is denoted by $cl(\phi,\Delta)$ or $\tau - cl(\phi,\Delta)$.*

Definition 2.14. *[14] Let (X,τ,Δ) be a soft topological space and $(\phi,\Delta) \in SS(X)_\Delta$.The soft interior of (ϕ,Δ) is the union of all open soft subsets of (ϕ,Δ). It is denoted by $int(\phi,\Delta)$ or $\tau - int(\phi,\Delta)$.*

Theorem 2.15. *[43] Let (X,τ,Δ) be a soft topological space and $(\phi,\Delta),(\varphi,\Delta) \in SS(X)_\Delta$. Then*
(1) $cl\widetilde{\emptyset} = \widetilde{\emptyset}$, $cl\widetilde{X} = \widetilde{X}$, $int\widetilde{\emptyset} = \widetilde{\emptyset}$ and $int\widetilde{X} = \widetilde{X}$.
(2) $(\phi,\Delta)\widetilde{\subseteq}cl(\phi,\Delta)$ and $int(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$.
(3) $cl(cl(\phi,\Delta)) = cl(\phi,\Delta)$ and $int(int(\phi,\Delta)) = int(\phi,\Delta)$.
(4) (ϕ,Δ) is a closed soft set if and only if $(\phi,\Delta) = cl(\phi,\Delta)$.
(5) (ϕ,Δ) is a soft open set if and only if $(\phi,\Delta) = int(\phi,\Delta)$.
(6) $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ implies both $cl(\phi,\Delta)\widetilde{\subseteq}cl(\varphi,\Delta)$ and $int(\phi,\Delta)\widetilde{\subseteq}int(\varphi,\Delta)$.
(7) $cl((\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)) = cl(\phi,\Delta)\widetilde{\cup}cl(\varphi,\Delta)$ and $int((\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)) = int(\phi,\Delta)\widetilde{\cap}int(\varphi,\Delta)$.
(8) $cl((\phi,\Delta)\widetilde{\cap}(\varphi,\Delta))\widetilde{\subseteq}cl(\phi,\Delta)\widetilde{\cap}cl(\varphi,\Delta)$ and $int((\phi,\Delta)\widetilde{\cup}(\varphi,\Delta))\widetilde{\supseteq}int(\phi,\Delta)\widetilde{\cup}int(\varphi,\Delta)$.

Definition 2.16. *[14, 44] Let $SS(X)_{\Delta_1}$, $SS(Y)_{\Delta_2}$ be two families of soft sets, $u : X \to Y$ and $p : \Delta_1 \to \Delta_2$ mappings. Then the mapping $f_{pu} : SS(X)_{\Delta_1} \to SS(Y)_{\Delta_2}$ is defined as:*
(1) Let $(\phi,\Delta_1) \in SS(X)_{\Delta_1}$. The image of (ϕ,Δ_1) under f_{pu}, written as $f_{pu}(\phi,\Delta_1) = (f_{pu}(\phi), p(\Delta_1))$, is a soft set in $SS(Y)_{\Delta_2}$ such that

$$f_{pu}(\phi)(y) = \begin{cases} \bigcup\limits_{x \in p^{-1}(y)\cap\Delta_1} u(\phi(x)) & \text{if} \quad p^{-1}(y)\cap\Delta_1 \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases},$$

for all $y \in \Delta_2$.
(2) Let $(\varphi,\Delta_2) \in SS(Y)_{\Delta_2}$. The inverse image of (φ,Δ_2) under f_{pu}, written as $f_{pu}^{-1}(\varphi,\Delta_2) = (f_{pu}^{-1}(\varphi), p^{-1}(\Delta_2))$, is a soft set in $SS(X)_{\Delta_1}$ such that

$$f_{pu}^{-1}(\varphi)(x) = \begin{cases} u^{-1}(\varphi(p(x))) & \text{if} \quad p(x) \in \Delta_2 \\ \emptyset & \text{otherwise} \end{cases},$$

for all $x \in \Delta_1$.

Definition 2.17. *[15] Let $f_{pu} : SS(X)_{\Delta_1} \to SS(Y)_{\Delta_2}$ be a soft mapping and $Z \subseteq X$. Then the restriction of f_{pu} to $SS(Z)_{\Delta_1}$ is the soft mapping $f_{pu}\Big|_{SS(Z)_{\Delta_1}}$ from $SS(Z)_{\Delta_1}$ to $SS(Y)_{\Delta_2}$ which defined by the functions $p : \Delta_1 \to \Delta_2$ and $u|_Z : Z \to Y$ where $u|_Z$ is the restriction of u to Z.*

Definition 2.18. *[36] Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta) \in SS(X)_\Delta$. Then (ϕ,Δ) is said to be*
(1) (τ_1,τ_2)-semi open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_2 - cl(\tau_1 - int(\phi,\Delta))$,
(2) (τ_1,τ_2)-pre open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_1 - int(\tau_2 - cl(\phi,\Delta))$,
(3) (τ_1,τ_2)-α-open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_1 - int(\tau_2 - cl(\tau_1 - int(\phi,\Delta)))$,
(4) (τ_1,τ_2)-β-open soft if $(\phi,\Delta)\widetilde{\subseteq}\tau_2 - cl(\tau_1 - int(\tau_2 - cl(\phi,\Delta)))$,
(5) (τ_1,τ_2)-regular open soft if $(\phi,\Delta) = \tau_1 - int(\tau_2 - cl(\phi,\Delta))$.
The complement of a (τ_1,τ_2)-semi open soft set ((τ_1,τ_2)-pre open soft set, (τ_1,τ_2)-α-open soft set, (τ_1,τ_2)-β-open soft set, (τ_1,τ_2)-regular open soft set) is called a (τ_1,τ_2)-semi closed soft set ((τ_1,τ_2)-pre closed soft set, (τ_1,τ_2)-α-closed soft set, (τ_1,τ_2)-β-closed soft set, (τ_1,τ_2)-regular closed soft set).

3. Main results

In this section, we present two new versions of the pasting lemma on a soft topological space.

3.1. (τ_1,τ_2)-g-closed soft sets and a pasting lemma

In this subsection we introduce the notion of a (τ_1,τ_2)-g-closed soft set and investigate some properties of this new notion to obtain a new pasting lemma on a soft topological space.

Definition 3.1. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. Then (ϕ,Δ) is called a (τ_1,τ_2)-generalized closed soft if $\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ whenever $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ and (φ,Δ) is τ_1-soft open. It is denoted by (τ_1,τ_2)-g-closed soft. The complement of a (τ_1,τ_2)-g-closed soft set is (τ_1,τ_2)-g-open soft.*

Example 3.2. *Let $X=\{a,b,c\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi,\zeta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X}\right\}$ where (ϕ,ζ) is a soft set over X defined as*

$$(\zeta,\Delta)=\{(e_1,\{a\}),(e_2,\{b\})\}.$$

Then the soft set $(\phi,\Delta)=\{(e_1,\{a,b\}),(e_2,\{a,c\})\}$ is a (τ_1,τ_2)-g-closed soft set. Indeed, if we take $(\varphi,\Delta)=\widetilde{X}\in\tau_1$ then we have

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$$

and

$$(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

Theorem 3.3. *Let τ_1, τ_2 be two soft topologies over X such that $\tau_2\subset\tau_1$. If $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}$, (φ,Δ) is a (τ_1,τ_2)-g-closed soft set relative to (ϕ,Δ) and (ϕ,Δ) is a (τ_1,τ_2)-g-closed soft set in X, then (φ,Δ) is (τ_1,τ_2)-g-closed soft relative to $\widetilde{X}$.*

Proof. Let $(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ and (θ,Δ) is τ_1-soft open. Then, using the hypothesis $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}$, we have

$$(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\theta,\Delta)$$

and

$$\tau_{2_{(\phi,\Delta)}}-cl(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\theta,\Delta).$$

It follows that

$$(\phi,\Delta)\widetilde{\cap}\left(\tau_2-cl(\varphi,\Delta)\right)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\theta,\Delta)$$

and

$$(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cup}\left(\tau_2-cl(\varphi,\Delta)\right)^c.$$

Since (ϕ,Δ) is a (τ_1,τ_2)-g-closed soft set and $\tau_2\subset\tau_1$, then we have

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cup}\left(\tau_2-cl(\varphi,\Delta)\right)^c.$$

Therefore, we obtain

$$\tau_2-cl(\varphi,\Delta)\widetilde{\subseteq}\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cup}\left(\tau_2-cl(\varphi,\Delta)\right)^c$$

and so

$$\tau_2-cl(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta).$$

Consequently, (φ,Δ) is (τ_1,τ_2)-g-closed soft relative to $\widetilde{X}$. □

In the following theorem, we see that the union of two (τ_1,τ_2)-g-closed soft sets is a (τ_1,τ_2)-g-closed soft set.

Theorem 3.4. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$. If (ϕ,Δ) and (φ,Δ) are two (τ_1,τ_2)-g-closed soft sets then $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$ is (τ_1,τ_2)-g-closed soft.*

Proof. If $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ and (θ,Δ) is a τ_1-soft open set, then using the hypothesis, we get

$$\tau_2-cl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]=\tau_2-cl(\phi,\Delta)\widetilde{\cup}\tau_2-cl(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta).$$

Hence $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$ is (τ_1,τ_2)-g-closed soft. □

The intersection of two (τ_1,τ_2)-g-closed soft sets is generally not a (τ_1,τ_2)-g-closed soft set as seen in the following example.

Example 3.5. *Let $X=\{a,b,c\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X}\right\}$ where (ϕ,Δ) is a soft set over X defined as*

$$(\phi,\Delta)=\{(e_1,\{a\}),(e_2,\{a\})\}.$$

Then the soft sets $(\varphi,\Delta)=\{(e_1,\{a,b\}),(e_2,\{a,c\})\}$ *and* $(\theta,\Delta)=\{(e_1,\{a,c\}),(e_2,\{a,b\})\}$ *are two* (τ_1,τ_2)*-g-closed soft sets. We get*

$$(\varphi,\Delta)\widetilde{\cap}(\theta,\Delta)=\{(e_1,\{a\}),(e_2,\{a\})\}$$

and so $(\varphi,\Delta)\widetilde{\cap}(\theta,\Delta)$ *is not a* (τ_1,τ_2)*-g-closed soft set.*

Proposition 3.6. *Let* τ_1, τ_2 *be two soft topologies over* X *such that* $\tau_2\subset\tau_1$. *Let* (ϕ,Δ) *be a* (τ_1,τ_2)*-g-closed soft set and* (φ,Δ) *a* τ_2*-soft closed set. Then* $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ *is a* (τ_1,τ_2)*-g-closed soft set.*

Proof. Since (φ,Δ) is τ_2-soft closed, then $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is a τ_2-soft closed set in (ϕ,Δ) and so it is (τ_1,τ_2)-g-closed soft. From Theorem 3.3, $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is a (τ_1,τ_2)-g-closed soft set. □

Theorem 3.7. *Let* $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$ *and* (ϕ,Δ) *be a* (τ_1,τ_2)*-g-closed soft set in* X. *Then* (ϕ,Δ) *is* (τ_1,τ_2)*-g-closed soft relative to* (Y,E).

Proof. Let $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\cap}(\varphi,\Delta)$ and (φ,Δ) be a τ_1-soft open set in X. Then $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ and so by the hypothesis, we get

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

It follows that $\widetilde{Y}\widetilde{\cap}[\tau_2-cl(\phi,\Delta)]\widetilde{\subseteq}\widetilde{Y}\widetilde{\cap}(\varphi,\Delta)$. Consequently, (ϕ,Δ) is (τ_1,τ_2)-g-closed soft relative to (Y,E). □

Theorem 3.8. *Let* τ_1, τ_2 *be two soft topologies over* X *such that* $\tau_2\subset\tau_1$. *If a soft set* (ϕ,Δ) *is* (τ_1,τ_2)*-g-closed soft then* $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ *contains no nonempty* τ_2*-soft closed set.*

Proof. Let (φ,Δ) be a τ_2-soft closed set of $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$. So we get $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)^c$. Since (ϕ,Δ) is (τ_1,τ_2)-g-closed soft, we have

$$\tau_2-cl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)^c$$

or

$$(\varphi,\Delta)\widetilde{\subseteq}[\tau_2-cl(\phi,\Delta)]^c.$$

Thus we obtain

$$(\varphi,\Delta)\widetilde{\subseteq}[\tau_2-cl(\phi,\Delta)]\widetilde{\cap}[\tau_2-cl(\phi,\Delta)]^c=\widetilde{\emptyset},$$

that is, (φ,Δ) is a null soft set. □

As a consequence of Theorem 3.8, we give the following corollary.

Corollary 3.9. *Let* τ_1, τ_2 *be two soft topologies over* X *such that* $\tau_2\subset\tau_1$. *A* (τ_1,τ_2)*-g-closed soft set* (ϕ,Δ) *is* τ_2*-soft closed if and only if* $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ *is* τ_2*-soft closed.*

Proof. If (ϕ,Δ) is τ_2-soft closed, then we have $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)=\widetilde{\emptyset}$. Conversely, assume that $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ is τ_2-soft closed. But (ϕ,Δ) is (τ_1,τ_2)-g-closed soft and $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)$ is a τ_2-soft closed subset of itself. From Theorem 3.8, we have $[\tau_2-cl(\phi,\Delta)]-(\phi,\Delta)=\widetilde{\emptyset}$ and so $\tau_2-cl(\phi,\Delta)=(\phi,\Delta)$. □

We introduce the notion of mixed g-soft continuity as follows:

Definition 3.10. *Let* X,Y *be two initial universe sets,* $\Delta_1,\Delta_2\subseteq\Delta$ *two sets of parameters,* τ_1, τ_2 *two soft topologies over* X *and* τ *a soft topology over* Y. *Assume that* $u:X\to Y$, $p:\Delta_1\to\Delta_2$ *are two mappings and* $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *is a function. Then* f_{pu} *is called mixed g-soft continuous (briefly,* $(\tau_1\tau_2,\tau)$*-g-soft cts) if* $f_{pu}^{-1}(\varphi,\Delta_2)$ *is a* (τ_1,τ_2)*-g-closed soft set for every* τ*-soft closed set* (φ,Δ_2) *in* Y.

Now we present a new version of the pasting lemma in the following theorem.

Theorem 3.11. *(Pasting lemma for* (τ_1,τ_2)*-g-closed soft sets) Let* $\widetilde{X}=\widetilde{A}\widetilde{\cup}\widetilde{B}$ *be a soft topological space with two soft topologies* τ_1, τ_2 *and* Y *a soft topological space with a soft topology* τ. *Let* $f_{p_1u_1}:SS(A)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *and* $f_{p_2u_2}:SS(B)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *be two mixed g-soft continuous mappings where* $p_1=p_2:\Delta_1\to\Delta_2$, $u_1:A\to Y$ *and* $u_2:B\to Y$ *are functions. Assume that* $\widetilde{A}$, $\widetilde{B}$ *are two* (τ_1,τ_2)*-g-closed soft sets and* $\tau_2\subset\tau_1$. *If* $u_1(x)=u_2(x)$ *for every* $x\in A\cap B$, *then* $f_{p_1u_1}$ *and* $f_{p_2u_2}$ *combine to give a mixed g-soft continuous mapping* $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ *defined by the functions* $p=p_1=p_2$ *and* $u(x)=u_1(x)$ *if* $x\in A$ *and* $u(x)=u_2(x)$ *if* $x\in B$.

Proof. Let (φ,Δ_2) be a τ-soft closed set in Y. Then we can easily seen that

$$f_{pu}^{-1}(\varphi,\Delta_2)=f_{p_1u_1}^{-1}(\varphi,\Delta_2)\widetilde{\cup}f_{p_2u_2}^{-1}(\varphi,\Delta_2).$$

From the mixed g-soft continuity of $f_{p_1u_1}$, then $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-g-closed soft set in A. Since $\widetilde{A}$ is (τ_1,τ_2)-g-closed soft, by Theorem 3.3, $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-g-closed soft set relative to $\widetilde{X}$. Similarly, $f_{p_2u_2}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-g-closed soft set relative to $\widetilde{X}$. Also using Theorem 3.4, we get that $f_{pu}^{-1}(\varphi,\Delta_2)$ is (τ_1,τ_2)-g-closed soft in X. Therefore, f_{pu} is a mixed g-soft continuous mapping. □

3.2. (τ_1,τ_2)-gpr-closed soft sets and a pasting lemma

In this subsection, we define the notion of a (τ_1,τ_2)-gpr-closed soft set. To do this, we introduce the notion of a mixed soft pre closure and a mixed soft pre interior. We investigate some basic properties of these new notions.

Definition 3.12. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$.*
(1) *The mixed soft pre closure of (ϕ,Δ) is defined by*

$$\tau_1\tau_2-pcl(\phi,\Delta)=\widetilde{\cap}\left\{(\varphi,\Delta):(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta) \text{ and } (\varphi,\Delta) \text{ is } (\tau_1,\tau_2)\text{-pre closed soft}\right\}.$$

(2) *The mixed soft pre interior of (ϕ,Δ) is defined by*

$$\tau_1\tau_2-pint(\phi,\Delta)=\widetilde{\cup}\left\{(\varphi,\Delta):(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta) \text{ and } (\varphi,\Delta) \text{ is } (\tau_1,\tau_2)\text{-pre open soft}\right\}.$$

We give some properties of (τ_1,τ_2)-pre open soft sets to obtain some basic theorems related to mixed soft pre closure and mixed soft pre interior.

Theorem 3.13. *Arbitrary union of (τ_1,τ_2)-pre open soft sets is a (τ_1,τ_2)-pre open soft set.*

Proof. Let $\mathscr{A}=\{(\phi,\Delta)_i : i\in I\}$ be a collection of (τ_1,τ_2)-pre open soft sets. Then we have

$$(\phi,\Delta)_i\widetilde{\subseteq}\tau_1-int(\tau_2-cl(\phi,\Delta)_i),$$

for each $(\phi,\Delta)_i\in\mathscr{A}$. Therefore, we get

$$\widetilde{\cup}(\phi,\Delta)_i\widetilde{\subseteq}\widetilde{\cup}[\tau_1-int(\tau_2-cl(\phi,\Delta)_i)]\widetilde{\subseteq}\tau_1-int(\widetilde{\cup}[\tau_2-cl(\phi,\Delta)_i])\widetilde{\subseteq}\tau_1-int\left(\tau_2-cl\left(\widetilde{\cup}(\phi,\Delta)_i\right)\right).$$

Consequently, $\widetilde{\cup}(\phi,\Delta)_i$ is a (τ_1,τ_2)-pre open soft set. □

As a result of Theorem 3.13, we give the following corollary.

Corollary 3.14. *Arbitrary intersection of (τ_1,τ_2)-pre closed soft sets is a (τ_1,τ_2)-pre closed soft set.*

Finite intersection of (τ_1,τ_2)-pre open soft sets is not always a (τ_1,τ_2)-pre open soft set as seen in the following example.

Example 3.15. *Let $X=\{a,b,c\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\varphi,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (φ,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\{(e_1,\{a\}),(e_2,\{b,c\})\},$$

$$(\phi_2,\Delta)=\{(e_1,\{b,c\}),(e_2,\{a\})\}$$

and

$$(\varphi,\Delta)=\{(e_1,X),(e_2,\{a,b\})\}.$$

Then the soft sets $(\theta,\Delta)=\{(e_1,\{a\}),(e_2,\{a,c\})\}$ and $(\psi,\Delta)=\{(e_1,\{b\}),(e_2,\{b,c\})\}$ are two (τ_1,τ_2)-pre open soft sets. We get

$$(\theta,\Delta)\widetilde{\cap}(\psi,\Delta)=\{(e_1,\emptyset),(e_2,\{c\})\}$$

and so $(\theta,\Delta)\widetilde{\cap}(\psi,\Delta)$ is not a (τ_1,τ_2)-pre open soft set.

Now we prove the following theorems.

Theorem 3.16. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$. Then the followings hold*:
(1) *(ϕ,Δ) is (τ_1,τ_2)-pre closed soft if and only if $(\phi,\Delta)=\tau_1\tau_2-pcl(\phi,\Delta)$.*
(2) *(ϕ,Δ) is (τ_1,τ_2)-pre open soft if and only if $(\phi,\Delta)=\tau_1\tau_2-pint(\phi,\Delta)$.*
(3) *$\tau_1\tau_2-pcl\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pcl\widetilde{X}=\widetilde{X}$.*
(4) *$\tau_1\tau_2-pint\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pint\widetilde{X}=\widetilde{X}$.*
(5) *$\tau_1\tau_2-pcl\,[\tau_1\tau_2-pcl(\phi,\Delta)]=\tau_1\tau_2-pcl(\phi,\Delta)$.*
(6) *$\tau_1\tau_2-pint\,[\tau_1\tau_2-pint(\phi,\Delta)]=\tau_1\tau_2-pint(\phi,\Delta)$.*
(7) *$[\tau_1\tau_2-pcl(\phi,\Delta)]^c=\tau_1\tau_2-pint(\phi^c,\Delta)$.*
(8) *$[\tau_1\tau_2-pint(\phi,\Delta)]^c=\tau_1\tau_2-pcl(\phi^c,\Delta)$.*

Proof. (1) Let (ϕ,Δ) be a (τ_1,τ_2)-pre closed soft set. Since (ϕ,Δ) is the smallest (τ_1,τ_2)-pre closed soft set containing itself, using Definition 3.12 (1), we have $(\phi,\Delta)=\tau_1\tau_2-pcl(\phi,\Delta)$. The converse statement of the proof is clear from Corollary 3.14.
(2) Let (ϕ,Δ) be a (τ_1,τ_2)-pre open soft set. Since (ϕ,Δ) is the largest (τ_1,τ_2)-pre open soft set contained (ϕ,Δ), using Definition 3.12 (2), we have $(\phi,\Delta)=\tau_1\tau_2-pint(\phi,\Delta)$. The converse part of the proof can be easily from Theorem 3.13.
(3) Since $\widetilde{\emptyset}$ and $\widetilde{X}$ are (τ_1,τ_2)-pre closed soft sets, then using (1), we get $\tau_1\tau_2-pcl\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pcl\widetilde{X}=\widetilde{X}$.
(4) Since $\widetilde{\emptyset}$ and $\widetilde{X}$ are (τ_1,τ_2)-pre open soft sets, then using (2), we get $\tau_1\tau_2-pint\widetilde{\emptyset}=\widetilde{\emptyset}$ and $\tau_1\tau_2-pint\widetilde{X}=\widetilde{X}$.
(5) Using (1), we obtain

$$\tau_1\tau_2-pcl\left[\tau_1\tau_2-pcl(\phi,\Delta)\right]=\tau_1\tau_2-pcl(\phi,\Delta),$$

since $\tau_1\tau_2-pcl(\phi,\Delta)$ is (τ_1,τ_2)-pre closed soft.
(6) Using (2), we get

$$\tau_1\tau_2-pint\left[\tau_1\tau_2-pint(\phi,\Delta)\right]=\tau_1\tau_2-pint(\phi,\Delta),$$

since $\tau_1\tau_2-pint(\phi,\Delta)$ is (τ_1,τ_2)-pre open soft.
(7) Using Definition 2.4 (1) and Definition 3.12, we get

$$\begin{aligned}
& \left[\tau_1\tau_2-pcl(\phi,\Delta)\right]^c \\
= & \left[\widetilde{\cap}\left\{(\varphi,\Delta):(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\text{ and }(\varphi,\Delta)\text{ is }(\tau_1,\tau_2)\text{-pre closed soft}\right\}\right]^c \\
= & \widetilde{\cup}\left\{(\varphi^c,\Delta):(\varphi^c,\Delta)\widetilde{\subseteq}(\phi^c,\Delta)\text{ and }(\varphi^c,\Delta)\text{ is }(\tau_1,\tau_2)\text{-pre open soft}\right\} \\
= & \tau_1\tau_2-pint(\phi^c,\Delta).
\end{aligned}$$

(8) By the similar arguments used in the proof of (7), it can be easily proved. □

Theorem 3.17. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$. Then the followings hold:*
(1) *If $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ then $\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\varphi,\Delta)$.*
(2) *If $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ then $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$.*
(3) $\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]=\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta)$.
(4) $\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]=\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pint(\varphi,\Delta)$.
(5) $\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pcl(\varphi,\Delta)$.
(6) $\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]\widetilde{\supseteq}\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pint(\varphi,\Delta)$.

Proof. (1) Using the hypothesis, we have

$$\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\Longrightarrow\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

Since $\tau_1\tau_2-pint(\phi,\Delta)$ is the largest (τ_1,τ_2)-pre open soft set contained in (φ,Δ). Therefore, we get

$$\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\varphi,\Delta).$$

(2) Since $(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)$ and $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$, we have

$$(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)\Longrightarrow(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta).$$

Because $\tau_1\tau_2-pcl(\phi,\Delta)$ is the smallest (τ_1,τ_2)-pre closed soft set containing (ϕ,Δ), then we obtain

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta).$$

(3) We have

$$(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\text{ and }(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta).$$

By the condition (2), we get

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right],$$

$$\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]$$

and so

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right].\tag{3.1}$$

Conversely, we have

$$(\phi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta),\ (\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$$

and so

$$(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta),$$

that is, $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta)$ is a (τ_1,τ_2)-pre closed soft set containing $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$. Since $\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]$ is the smallest (τ_1,τ_2)-pre closed soft set containing $(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)$, we obtain

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta). \tag{3.2}$$

From the inequalities (3.1) and (3.2), we get

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]=\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pcl(\varphi,\Delta).$$

(4) By the similar arguments used in the proof of (3), we prove

$$\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]=\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pint(\varphi,\Delta).$$

(5) Since $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ and $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$, we get

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta),$$

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\varphi,\Delta)$$

and so

$$\tau_1\tau_2-pcl\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\cap}\tau_1\tau_2-pcl(\varphi,\Delta).$$

(6) By the similar arguments used in the proof of (5), we obtain

$$\tau_1\tau_2-pint\left[(\phi,\Delta)\widetilde{\cup}(\varphi,\Delta)\right]\widetilde{\supseteq}\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\cup}\tau_1\tau_2-pint(\varphi,\Delta).$$

□

Theorem 3.18. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$. Then the followings hold*:
(1) $\tau_1\tau_2-pcl(\phi,\Delta)=(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))$.
(2) $\tau_1\tau_2-pint(\phi,\Delta)=(\phi,\Delta)\widetilde{\cap}\tau_1-int(\tau_2-cl(\phi,\Delta))$.

Proof. (1) We have

$$\begin{aligned}
&\tau_1-cl\left[\tau_2-int\left[(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\right]\right]\\
&\widetilde{\subseteq}\tau_1-cl\left[\tau_2-int(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\right]\\
=\;&\tau_1-cl(\tau_2-int(\phi,\Delta))\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta)).
\end{aligned}$$

Therefore, $(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))$ is a (τ_1,τ_2)-pre closed soft set whence

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta)). \tag{3.3}$$

Conversely, since $\tau_1\tau_2-pcl(\phi,\Delta)$ is (τ_1,τ_2)-pre closed soft, we get

$$\begin{aligned}
\tau_1-cl(\tau_2-int(\phi,\Delta))&\widetilde{\subseteq}\tau_1-cl(\tau_2-int(\tau_1\tau_2-pcl(\phi,\Delta)))\\
&\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)
\end{aligned}$$

and so

$$(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta). \tag{3.4}$$

By the inequalities (3.3) and (3.4), we obtain

$$\tau_1\tau_2-pcl(\phi,\Delta)=(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta)).$$

(2) It is a consequence of (1). □

Proposition 3.19. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. If $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$ and $\widetilde{Y}\in\tau_2$ then we have*

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)=\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}.$$

Proof. From Theorem 3.18, we get

$$\begin{aligned}
\tau_1\tau_2-pcl_Y(\phi,\Delta) &= (\phi,\Delta)\widetilde{\cup}[\tau_1-cl_Y(\tau_2-int_Y(\phi,\Delta))]\\
&= (\phi,\Delta)\widetilde{\cup}[\tau_1-cl_Y(\tau_2-int(\phi,\Delta))]\\
&= (\phi,\Delta)\widetilde{\cup}\left[\tau_1-cl\left(\tau_2-int(\phi,\Delta)\widetilde{\cap}\widetilde{Y}\right)\right]\\
&= \left[(\phi,\Delta)\widetilde{\cup}\tau_1-cl(\tau_2-int(\phi,\Delta))\right]\widetilde{\cap}\left[(\phi,\Delta)\widetilde{\cup}\widetilde{Y}\right]\\
&= \tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}.
\end{aligned}$$

□

Proposition 3.20. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. If $\widetilde{Y}\in\tau_2$ and $\widetilde{Y}$ is a (τ_1,τ_2)-pre closed soft set then we have*

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)=\tau_1\tau_2-pcl_X(\phi,\Delta).$$

Proof. From Proposition 3.19, we have

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)=\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}.$$

Since $\widetilde{Y}$ is a (τ_1,τ_2)-pre closed soft set, we get

$$\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}.$$

Consequently, we obtain

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)=\tau_1\tau_2-pcl_X(\phi,\Delta).$$

□

We introduce the notion of a (τ_1,τ_2)-gpr-closed soft set.

Definition 3.21. *Let τ_1, τ_2 be two soft topologies over X and $(\phi,\Delta)\in SS(X)_\Delta$. Then (ϕ,Δ) is called a (τ_1,τ_2)-generalized pre regular closed soft if $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ whenever $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ and (φ,Δ) is (τ_1,τ_2)-regular open soft. It is denoted by (τ_1,τ_2)-gpr-closed soft. The complement of a (τ_1,τ_2)-gpr-closed soft set is (τ_1,τ_2)-gpr-open soft.*

Example 3.22. *Let $X=\{a,b,c,d\}$, $\Delta=\{e_1,e_2\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta)\right\}$ and $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_2,\Delta)\right\}$ where (ϕ_1,Δ) and (ϕ_2,Δ) are two soft sets over X defined as*

$$(\phi_1,\Delta)=\{(e_1,\{a,b\}),(e_2,\{c,d\})\}$$

and

$$(\phi_2,\Delta)=\{(e_1,\{c,d\}),(e_2,\{a,b\})\}.$$

Then the soft set $(\varphi,\Delta)=\{(e_1,\{a\}),(e_2,\{c\})\}$ is a (τ_1,τ_2)-gpr-closed soft set.

Now we give the following implications:

$$\begin{gathered}
(\tau_1,\tau_2)\text{-regular closed soft}\\
\Downarrow\\
(\tau_1,\tau_2)\text{-pre closed soft}\\
\Downarrow\\
(\tau_1,\tau_2)\text{-}gpr\text{-closed soft}
\end{gathered}$$

The inverse implications of these are not always true as seen in the following example.

Example 3.23. *Let* $X=\{a,b,c\}$, $\Delta=\{e\}$, $\tau_1=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta),(\phi_3,\Delta)\right\}$ *and* $\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_3,\Delta)\right\}$ *where* (ϕ_1,Δ), (ϕ_2,Δ) *and* (ϕ_3,Δ) *are soft sets over X defined as*

$$(\phi_1,\Delta)=\{(e,\{b\})\},\ (\phi_2,\Delta)=\{(e,\{c\})\}$$

and

$$(\phi_3,\Delta)=\{(e,\{b,c\})\}.$$

Then the soft set $(\varphi_1,\Delta)=\{(e,\{b,c\})\}$ *is a* (τ_1,τ_2)*-gpr-closed soft set, but it is not* (τ_1,τ_2)*-pre closed soft. Also the soft set* $(\varphi_2,\Delta)=\{(e,\{a\})\}$ *is a* (τ_1,τ_2)*-pre closed soft set, but it is not* (τ_1,τ_2)*-regular closed soft.*

Now we prove some necessary properties and theorems related to the notion of a (τ_1,τ_2)-gpr-open soft set.

Theorem 3.24. *Let X be a soft topological space with two soft topologies* τ_1*,* τ_2*.* $(\phi,\Delta)\in SS(X)_E$ *is* (τ_1,τ_2)*-gpr-open soft if and only if* $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$ *whenever* (φ,Δ) *is* (τ_1,τ_2)*-regular closed soft and* $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$*.*

Proof. Let (ϕ,Δ) be a (τ_1,τ_2)-gpr-open soft set, (φ,Δ) be a (τ_1,τ_2)-regular closed soft set and $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$. Then we have $\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)$ where $\widetilde{X}-(\varphi,\Delta)$ is (τ_1,τ_2)-regular open soft. Since $\widetilde{X}-(\phi,\Delta)$ is (τ_1,τ_2)-gpr-closed soft, then we get $\tau_1\tau_2-pcl(\widetilde{X}-(\phi,\Delta))\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)$. Hence we obtain

$$\widetilde{X}-\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)$$

and so $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$. Conversely, we suppose that (φ,Δ) is (τ_1,τ_2)-regular closed soft and $(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ implies $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$. Let $\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ where (θ,Δ) is (τ_1,τ_2)-regular open soft. Then we have $\widetilde{X}-(\theta,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ where $\widetilde{X}-(\theta,\Delta)$ is (τ_1,τ_2)-regular closed soft. By the hypothesis, we get $\widetilde{X}-(\theta,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)$, that is, $\widetilde{X}-\widetilde{\subseteq}\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Hence we obtain

$$\tau_1\tau_2-pcl(\widetilde{X}-(\phi,\Delta))\widetilde{\subseteq}(\theta,\Delta)$$

and so $\widetilde{X}-(\phi,\Delta)$ is (τ_1,τ_2)-gpr-closed soft, that is, (ϕ,Δ) is (τ_1,τ_2)-gpr-open soft. □

Theorem 3.25. *Let X be a soft topological space with two soft topologies* τ_1*,* τ_2*. If* (ϕ,Δ) *is* (τ_1,τ_2)*-gpr-closed soft and* $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)$*, then* (φ,Δ) *is* (τ_1,τ_2)*-gpr-closed soft.*

Proof. Let $(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ where (θ,Δ) is (τ_1,τ_2)-regular open soft. Then $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ implies $(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Since (ϕ,Δ) is (τ_1,τ_2)-gpr-closed soft, we get $\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Also $(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta)$ implies

$$\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\phi,\Delta).$$

Thus we obtain

$$\tau_1\tau_2-pcl(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$$

and so (φ,Δ) is (τ_1,τ_2)-gpr-closed soft. □

Theorem 3.26. *Let X be a soft topological space with two soft topologies* τ_1*,* τ_2*. If* (ϕ,Δ) *is* (τ_1,τ_2)*-gpr-open soft and* $\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$*, then* (φ,Δ) *is* (τ_1,τ_2)*-gpr-open soft.*

Proof. $\tau_1\tau_2-pint(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ implies

$$\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)\widetilde{\subseteq}\widetilde{X}-[\tau_1\tau_2-pint(\phi,\Delta)],$$

that is,

$$\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}\widetilde{X}-(\varphi,\Delta)\widetilde{\subseteq}\tau_1\tau_2-pcl(\widetilde{X}-(\phi,\Delta)).$$

Since $\widetilde{X}-(\phi,\Delta)$ is a (τ_1,τ_2)-gpr-closed soft set, from Theorem 3.25, $\widetilde{X}-(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-closed soft and so (φ,Δ) is (τ_1,τ_2)-gpr-open soft. □

The union and the intersection of two (τ_1,τ_2)-gpr-closed soft sets can not be always (τ_1,τ_2)-gpr-closed soft as seen in the following examples, respectively.

Example 3.27. *Let $X=\{a,b,c,d,e\}$, $\Delta=\{e'\}$, $\tau_1=\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta),(\phi_3,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (ϕ_3,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\left\{(e',\{a,c\})\right\},\ (\phi_2,\Delta)=\left\{(e',\{b,d\})\right\}$$

and

$$(\phi_3,\Delta)=\left\{(e',\{a,b,c,d\})\right\}.$$

Then the soft set $(\varphi,\Delta)=\{(e',\{a\})\}$ and $(\theta,\Delta)=\{(e',\{c\})\}$ are two (τ_1,τ_2)-gpr-closed soft set, but $(\varphi,\Delta)\widetilde{\cup}(\theta,\Delta)=\{(e',\{a,c\})\}$ is not (τ_1,τ_2)-gpr-closed soft.

Example 3.28. *Let $X=\{a,b,c\}$, $\Delta=\{e'\}$, $\tau_1=\tau_2=\left\{\widetilde{\emptyset},\widetilde{X},(\phi_1,\Delta),(\phi_2,\Delta),(\phi_3,\Delta)\right\}$ where (ϕ_1,Δ), (ϕ_2,Δ) and (ϕ_3,Δ) are soft sets over X defined as*

$$(\phi_1,\Delta)=\left\{(e',\{b\})\right\},\ (\phi_2,\Delta)=\left\{(e',\{c\})\right\}$$

and

$$(\phi_3,\Delta)=\left\{(e',\{b,c\})\right\}.$$

Then the soft set $(\varphi,\Delta)=\{(e',\{b,c\})\}$ and $(\theta,\Delta)=\{(e',\{a,b\})\}$ are two (τ_1,τ_2)-gpr-closed soft set, but $(\varphi,\Delta)\widetilde{\cap}(\theta,\Delta)=\{(e',\{b\})\}$ is not (τ_1,τ_2)-gpr-closed soft.

Proposition 3.29. *Let X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$. If (φ,Δ) is (τ_1,τ_2)-gpr-open soft and $\tau_1\tau_2-pint(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ then $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft.*

Proof. Since (φ,Δ) is (τ_1,τ_2)-gpr-open soft and $\tau_1\tau_2-pint(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)$ then we have

$$\tau_1\tau_2-pint(\varphi,\Delta)\widetilde{\subseteq}(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

From Theorem 3.26, $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft. □

The class of all (τ_1,τ_2)-pre-open soft sets is denoted by $PO(X,\tau_1,\tau_2)$.

Proposition 3.30. *Let X be a soft topological space with two soft topologies τ_1, τ_2, $(\phi,\Delta),(\varphi,\Delta)\in SS(X)_\Delta$ and $PO(X,\tau_1,\tau_2)$ closed under finite intersections. If (ϕ,Δ) and (φ,Δ) are two (τ_1,τ_2)-gpr-open soft sets, then $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft.*

Proof. Let us consider

$$\widetilde{X}-\left[(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right]=\left[\widetilde{X}-(\phi,\Delta)\right]\widetilde{\cup}\left[\widetilde{X}-(\varphi,\Delta)\right]\widetilde{\subseteq}(\theta,\Delta),$$

where (θ,Δ) is (τ_1,τ_2)-regular open soft. Then we have $\widetilde{X}-(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$ and $\widetilde{X}-(\varphi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Since (ϕ,Δ) and (φ,Δ) are two (τ_1,τ_2)-gpr-open soft sets, we have

$$\tau_1\tau_2-pcl\left(\widetilde{X}-(\phi,\Delta)\right)\widetilde{\subseteq}(\theta,\Delta)$$

and

$$\tau_1\tau_2-pcl\left(\widetilde{X}-(\varphi,\Delta)\right)\widetilde{\subseteq}(\theta,\Delta).$$

By the hypothesis, we find

$$\begin{aligned}&\tau_1\tau_2-pcl\left[\left(\widetilde{X}-(\phi,\Delta)\right)\widetilde{\cup}\left(\widetilde{X}-(\varphi,\Delta)\right)\right]\\&\widetilde{\subseteq}\tau_1\tau_2-pcl\left(\widetilde{X}-(\phi,\Delta)\right)\widetilde{\cup}\tau_1\tau_2-pcl\left(\widetilde{X}-(\varphi,\Delta)\right)\widetilde{\subseteq}(\theta,\Delta),\end{aligned}$$

that is,

$$\tau_1\tau_2-pcl\left[\widetilde{X}-\left((\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)\right)\right]\widetilde{\subseteq}(\theta,\Delta).$$

Consequently, $(\phi,\Delta)\widetilde{\cap}(\varphi,\Delta)$ is (τ_1,τ_2)-gpr-open soft. □

The following lemma will be used in the proof of a proposition related to a (τ_1,τ_2)-gpr-closed soft set in a soft subspace.

Lemma 3.31. *Let $\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$, X be a soft topological space with two soft topologies τ_1, τ_2 and $(\phi,\Delta)\in SS(X)_\Delta$. If $\widetilde{Y}$ is a τ_2-soft open set and $\tau_2\subset\tau_1$, then $(\phi,\Delta)\widetilde{\cap}\widetilde{Y}$ is a (τ_1,τ_2)-regular open soft set relative to $\widetilde{Y}$ for some (ϕ,Δ) which is a (τ_1,τ_2)-regular open soft set relative to $\widetilde{X}$.*

Proof. Let (ϕ,Δ) be a (τ_1,τ_2)-regular open soft set and $(\varphi,\Delta)=(\phi,\Delta)\widetilde{\cap}\widetilde{Y}$. Then we have

$$\begin{aligned}\tau_1-int\left(\tau_2-cl\left((\varphi,\Delta)\widetilde{\cap}\widetilde{Y}\right)\right) &= \tau_1-int\left(\tau_2-cl\,(\phi,\Delta)\,\widetilde{\cap}\widetilde{Y}\right)\\ &= \tau_1-int\,(\tau_2-cl\,(\phi,\Delta))\,\widetilde{\cap}\widetilde{Y}\\ &= (\phi,\Delta)\,\widetilde{\cap}\widetilde{Y}=(\varphi,\Delta).\end{aligned}$$

Hence (φ,Δ) is a (τ_1,τ_2)-regular open soft set relative to $\widetilde{Y}$. □

Proposition 3.32. *Let X be a soft topological space with two soft topologies τ_1, τ_2 such that $\tau_2\subset\tau_1$ and $(\phi,\Delta)\widetilde{\subseteq}\widetilde{Y}\widetilde{\subseteq}\widetilde{X}$. Then the followings hold*:
(1) *If $\widetilde{Y}$ is a τ_2-soft open set and (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in X then (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in Y.*
(2) *If $\widetilde{Y}$ is a τ_2-soft open set and a (τ_1,τ_2)-pre closed soft set in X and (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in Y then (ϕ,Δ) is a (τ_1,τ_2)-gpr-closed soft set in X.*

Proof. (1) Let (ϕ,Δ) be a (τ_1,τ_2)-*gpr*-closed soft set in X and $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ where (φ,Δ) is a (τ_1,τ_2)-regular open soft set in Y. By Lemma 3.31, we have $(\varphi,\Delta)=(\theta,\Delta)\widetilde{\cap}\widetilde{Y}$ where (θ,Δ) is a (τ_1,τ_2)-regular open soft set in X, that is, $(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta)$. Since (ϕ,Δ) is a (τ_1,τ_2)-*gpr*-closed soft set in X then we get

$$\tau_1\tau_2-pcl(\phi,\Delta)\widetilde{\subseteq}(\theta,\Delta),$$

which implies

$$\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\cap}\widetilde{Y}\widetilde{\subseteq}(\theta,\Delta)\widetilde{\cap}\widetilde{Y}.$$

By Lemma 3.20, we have

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta).$$

Therefore, (ϕ,Δ) is a (τ_1,τ_2)-*gpr*-closed soft set in Y.
(2) Let (ϕ,Δ) be a (τ_1,τ_2)-*gpr*-closed soft set in Y. Then $(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)$ where (φ,Δ) is a (τ_1,τ_2)-regular open soft set in X. Hence we get

$$(\phi,\Delta)=(\phi,\Delta)\widetilde{\cap}\widetilde{Y}\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\cap}\widetilde{Y},$$

where $(\varphi,\Delta)\widetilde{\cap}\widetilde{Y}$ is (τ_1,τ_2)-regular open soft in Y by Lemma 3.31. Using the hypothesis, we get

$$\tau_1\tau_2-pcl_Y(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\cap}\widetilde{Y}.$$

By Lemma 3.20, we obtain

$$\tau_1\tau_2-pcl_X(\phi,\Delta)\widetilde{\subseteq}(\varphi,\Delta)\widetilde{\cap}\widetilde{Y}\widetilde{\subseteq}(\varphi,\Delta),$$

that is, (ϕ,Δ) is a (τ_1,τ_2)-*gpr*-closed soft set in X. □

We introduce the notion of mixed *gpr*-soft continuity as follows:

Definition 3.33. *Let X,Y be two initial universe sets, $\Delta_1,\Delta_2\subseteq\Delta$ two sets of parameters, τ_1, τ_2 two soft topologies over X and τ a soft topology over Y. Assume that $u:X\to Y$, $p:\Delta_1\to\Delta_2$ are two mappings and $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ is a function. Then f_{pu} is called mixed gpr-soft continuous (briefly, $(\tau_1\tau_2,\tau)$-gpr-soft cts) if $f_{pu}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-gpr-closed soft set for every τ-soft closed set (φ,Δ_2) in Y.*

Using the concept of mixed *gpr*-soft continuity, we present a new version of the pasting lemma in the following theorem.

Theorem 3.34. ($Pasting lemma for (τ_1,τ_2)-gpr-closed soft sets*) *Let $\widetilde{X}=\widetilde{A}\widetilde{\cup}\widetilde{B}$ be a soft topological space with two soft topologies τ_1, τ_2, Y a soft topological space with a soft topology τ and the family of all (τ_1,τ_2)-gpr-open soft sets closed under finite intersections. Let $f_{p_1u_1}:SS(A)_{\Delta_1}\to SS(Y)_{\Delta_2}$ and $f_{p_2u_2}:SS(B)_{\Delta_1}\to SS(Y)_{\Delta_2}$ be two mixed gpr-soft continuous mappings where $p_1=p_2:\Delta_1\to\Delta_2$, $u_1:A\to Y$ and $u_2:B\to Y$ are functions. Suppose that $\widetilde{A}$, $\widetilde{B}$ are τ_2-soft open and (τ_1,τ_2)-pre closed soft and $\tau_2\subset\tau_1$. If $u_1(x)=u_2(x)$ for every $x\in A\cap B$, then $f_{p_1u_1}$ and $f_{p_2u_2}$ combine to give a mixed gpr-soft continuous mapping $f_{pu}:SS(X)_{\Delta_1}\to SS(Y)_{\Delta_2}$ defined by the functions $p=p_1=p_2$ and $u(x)=u_1(x)$ if $x\in A$ and $u(x)=u_2(x)$ if $x\in B$.*

Proof. Let (φ,Δ_2) be a τ-soft closed set in Y. Then we can easily seen that

$$f_{pu}^{-1}(\varphi,\Delta_2)=f_{p_1u_1}^{-1}(\varphi,\Delta_2)\widetilde{\cup}f_{p_2u_2}^{-1}(\varphi,\Delta_2).$$

Since $f_{p_1u_1}$ is mixed *gpr*-soft continuous then $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-*gpr*-closed soft set in A. Since $\widetilde{A}$ is τ_2-soft open and (τ_1,τ_2)-pre closed soft, then $f_{p_1u_1}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-*gpr*-closed soft set in $\widetilde{X}$ by Proposition 3.32 (2). Similarly, $f_{p_2u_2}^{-1}(\varphi,\Delta_2)$ is a (τ_1,τ_2)-*gpr*-closed soft set in $\widetilde{X}$. Also we get that $f_{pu}^{-1}(\varphi,\Delta_2)$ is (τ_1,τ_2)-*gpr*-closed soft in X from the hypothesis. Therefore, f_{pu} is a mixed *gpr*-soft continuous mapping. □

4. Conclusion and future work

In this paper, two new versions of the pasting lemma for mixed g-soft continuous functions and mixed gpr-soft continuous functions are presented on a soft topological space. As a future work, some applications of these pasting lemmas can be investigated to analytic continuation on a complex plane.

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Hurewicz and Poincaré Theorems for Simplicial Modules

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Abstract

We will give the simplicial analogues of Hurewicz and Poincaré theorems as an application of simplicial homology and homotopy.

1. Introduction

Let X be a topological space. Then we have a singular simplicial set. $C_*(X)$which is chain complex is obtained with its singular homology $H_*(X;\mathbb{Z})$. Any singular homology of X can be get from $S_*(X)$. So the concept of simplicial sets was defined as combinatorial models of spaces. In the following diagram, one can see relations among simplicial sets and spaces:

$$\begin{array}{ccc} \boxed{\text{Simplical sets}} & \xrightarrow{|.|} & \boxed{\text{CW- complexes}} \\ S_*\circ|.|\downarrow & & \downarrow|.|\circ S_* \\ \boxed{\text{Simplical sets}} & \xleftarrow{S_*} & \boxed{\text{Topological spaces}} \end{array}$$

In [1], J.C. Moore defined simplicial groups. Author also gave the isomorphism

$$\pi_*(|\mathscr{G}|) \cong H_*(N\mathscr{G}),$$

where $N\mathscr{G}$ is Moore chain complex of and$|\mathscr{G}|$ is geometrical realization of $\mathscr{G}$. J.W. Milnor [2] shown that a loop space is homotopy equivalent to the geometric realization of any simplicial group. Hence, the homotopy groups of any space is defined as the homology of a Moore chain complex.

The simplicial modules and simplicial algebras are developed by M. André [3] and D. Quillen [4]. They constructed ways of building simplicial resolutions of algebras and defined a homology and cohomology of commutative algebras. Also Z. Arvasi [5, 6], analyses the Higher order Peiffer elements of simplicial algebras.

In this work, firstly we will give some preliminaries for simplicial modules and their homology and homotopy. Then we will proof the main theorem called as Hurewicz Theorem and also its corollary called as Poincaré Theorem. These theorems are applications for homology and homotopy of simplicial modules.

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2. Δ-sets

For details of this section you can see [7].

A Δ-*set*, $X = \{X_n\}_{n\geq 0}$, is a sequence with the maps, $d_i : X_n \to X_{n-1}$, satisfying the Δ-identity:

$$d_i d_j = d_j d_{i+1}$$

for $i \geq j$ and $0 \leq i \leq n$.

Remark: We can write Δ-idenitity:

$$d_i : (r_0, r_1, \cdots, r_n) \mapsto (r_0, \cdots, r_{i-1}, r_{i+1}, \cdots r_n).$$

A Δ-set $M = \{M_n\}_{n\geq 0}$ is defined a Δ-module satisfying M_n is a module, and d_i is a module homomorphism. Given any category $\mathscr{C}$, a Δ-object is a sequence of objects in $\mathscr{C}$, with faces as morphisms in $\mathscr{C}$.

We have a category $\mathscr{O}^+$ with the objects which are finite order sets, morphisms which are monoton functions. We can write the objects as $n \geq 0, [n] = \{0, 1, \cdots, n\}$, and the morphisms generated by $d^i : [n-1] \to [n]$ such that

$$d^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i\,, \end{cases}$$

$0 \leq i \leq n$.

Corollary 2.1. *Δ-sets has one to one correspondence to contravarient fuctors from $\mathscr{O}^+$ to S.*

A Δ-*map* is a sequence of $f := f_n(X_n \to Y_n)$ satisfying the following commutative diagram, that is $n \geq 0$, $f_0 d_i = d_i f$.

$$\begin{array}{ccc} X_n & \xrightarrow{f} & Y_n \\ \downarrow & & \downarrow \\ X_{n-1} & \xrightarrow[f]{} & Y_{n-1} \end{array}$$

A Δ-*subset* of X is any sequece of $Y_n \subseteq X_n$ satisfying

$$d_i(Y_n) \subseteq Y_{n-1}$$

where X is a Δ-set, $0 \leq i \leq n < \infty$. Suppose X and Y are Δ-set. If there exists a bijective Δ-map between X and Y, then X is isomorphic to Y.

2.1. Geometric realization of Δ-sets

Suppose A is a Δ-set. A geometrical realization of A, $|A|$, is determined as

$$|A| = \coprod_{\substack{x \in A_n \\ n \geq 0}} (\Delta^n, x)/\sim = \coprod_{n=0}^{\infty} \Delta^n \times A_n/\sim$$

where $\sim$ is obtained by $(z, d_i x) \sim (d^i z, x)$ for $x \in A_n$, $z \in \Delta^{n-1}$ is labeled by $d_i x$.

2.2. Homology of Δ-sets

It is well known that a chain complex is a collection of $C = \{C_n\}$ with diferantial $\partial_n : C_n \to C_{n-1}$ which is satisfy $\mathrm{Im}(\partial_{n+1}) \subseteq Ker(\partial_n)$, namely $\partial_n\, \partial_{n+1}$ is trivial. Then the homology can be defined as

$$H_n(C) = Ker(\partial_n)/Im(\partial_{n+1}).$$

Proposition 2.2 ([7])**.** *For a Δ-abelian group G, G is a chain complex with ∂_* where*

$$\partial_n : \sum_{i=0}^{n} (-1)^i d_i : G_n \longrightarrow G_{n-1}.$$

Proof. We must show that $\partial_n \circ \partial_{n+1}$ is trivial.

$$\begin{aligned}
\partial_{n-1}\circ\partial_n &= \sum_{i=0}^{n-1}(-1)^i d_i \sum_{j=0}^{n}(-1)^j d_j,\\
&= \sum_{0\le i<j\le n}^{n}(-1)^{i+j}d_id_j + \sum_{0\le j<i\le n-1}^{n}(-1)^{i+j}d_id_j,\\
&= \sum_{0\le i<j\le n}^{n}(-1)^{i+j}d_id_j + \sum_{0\le j<i+1\le n}^{n}(-1)^{i+j}d_jd_{i+1},\\
&= \sum_{0\le i<j\le n}^{n}(-1)^{i+j}d_id_j + \sum_{0\le i<j<n}^{n}(-1)^{i+j-1}d_jd_i,\\
&= 0.
\end{aligned}$$

□

For a given Δ-set X , the homology $H_*(X;G)$ of X with coefficients in an abelian group G can be defined as

$$H_*(X;G) = H_*(\mathbb{Z}(X)\otimes G, \partial_*).$$

Here $\mathbb{Z}(X_n)$ is a free abelian group with generator X_n, $\mathbb{Z}(X) = \{\mathbb{Z}(X_n)\}_{n\ge 0}$.

3. Simplicial modules

Let R be a fixed commutative ring. Fore more details about simplicial modules and algebras, we refer to [5, 6], [8]-[10].

A *simplicial R-module* (shotrly simplicial module) is a Δ-module M with degeneracies and faces satisfying the following identities:

$$d_jd_i = d_{i-1}d_j, \text{ for } j<i,$$

$$s_js_i = s_{i+1}s_j, \text{ for } j\le i,$$

also

$$d_js_i = \begin{cases} s_{i-1}d_j & j<i \\ id & j=i,i+1 \\ s_id_{j-1} & j>i+1. \end{cases}$$

These are defined as simplicial identities.

A *simplicial module homomorphism* $f: M\to M'$is a sequence of module homomorphisms $f_n: M_n \to M'_n$ $(n\ge 0)$ satisfying the following commutative diagram, i.e $f_{n-1}\ d_i = d_if_n$ and $f_ns_i = s_if_{n+1}$:

$$\begin{array}{ccccc}
M_{n+1} & \xleftarrow{s_i} & M_n & \xrightarrow{d_i} & M_{n-1}\\
f_{n+1}\downarrow & & f_n\downarrow & & \downarrow f_{n-1}\\
M'_{n+1} & \xleftarrow[s_i]{} & M'_n & \xrightarrow[d_i]{} & M'_{n-1}
\end{array}$$

M is defined as *simplicial submodule* of M' if each M_n is a submodule of M'_n. A simplicial module M is said to be *isomorphic* to a simplicial module M', if a bijective simplicial module homomorphism $f: M\to M'$exists.

3.1. Geometric realization of simplicial modules

The standart n-simplex Δ^n is

$$\Delta^n = \{(r_0, r_1, \cdots r_n) \mid r_i \ge 0 \text{ and } \sum_{i=0}^{n} r_i = 1\}$$

where $d^i: \Delta^{n-1} \longrightarrow \Delta^n$ and $s^i: \Delta^{n+1} \longrightarrow \Delta^n$ are given as

$$\begin{aligned}
d^i(r_0, r_1, \cdots, r_{n-1}) &= (r_0, \cdots, r_{i-1}, 0, r_i, \cdots, r_{n-1}),\\
s^i(r_0, r_1, \cdots, t_{n+1}) &= (r_0, \cdots, r_{i-1}, r_i + r_{i+1}, \cdots, r_{n+1}),
\end{aligned}$$

where $0 \leq i \leq n$.

Suppose M is a simplicial module. Its geometric realization $|M|$ is a CW-complex such that

$$|M| = \bigsqcup_{\substack{x \in M_n \\ n \geq 0}} (\Delta^n, x)/\sim = \bigsqcup_{n=0}^{\infty} (\Delta^n \times M_n)/\sim .$$

Here (Δ^n, x) is Δ^n associated with $x \in M_n$, $\sim$ is generated by

$$(z, d_i x) \sim (d^i z, x)$$

$x \in M_n$, $z \in \Delta^{n-1}$ associated with $d_i x$,

$$(z, s_i x) \sim (s^i z, x)$$

$x \in M_n$, $z \in \Delta^{n+1}$ associated with $s_i x$.

3.2. Homotopy and fibrant simplicial modules

Suppose $f, g : M \to N$ are simplicial module homomorphisms. If we have a simplicial module homomorphism $F : M \times I \to N$ satisfying $F|_{M\times 0} = f$, $F|_{M\times 1} = g$, then we can say that f *homotopic* to g and can be written as $f \simeq g$. Suppose that X is any simplicial submodule of M, $f; g : M \to N$ are simplicial module homomorphisms satisfying$f|_X = g|_X$. If we have a homotopy $F : M \times I \to N$ satisfying $F|_{M\times 0} = f$ and $F|_{M\times 1} = g$ and $F|_{X\times I} = f$, then we say that f homotopic to g relative to X, and can be shown as $f \simeq g$ rel X.

The image of $f_{x_0} : \Delta[0] \to M$ is a simplicial submodule of M which has only element $f_{x_0}(0, 0, \cdots, 0) = S_I(x_0)$ for each dimension where M is any simplicial module and $x_0 \in M_0$. So a basepoint $*$ of M is a sequence of $\{f_{x_0}(\overbrace{0, 0 \cdots, 0}^{n+1})\}_{n\geq 0}$ correspond to $x_0 \in M_0$.

A *pointed simplicial module* is a simplicial module with basepoint. Suppose M and N are pointed simplicial modules. A *pointed simplicial module homomorphism* $f : M \to N$ is a simplicial module homomorphism which preserve the basepoints. We usually use $*$ for defining the basepoint.

For given pointed simplicial module homomorphisms $f, g : M \to N$, pointed homotopy means that f and g are homotopic rel $*$.

We should assume that there is a homotopy relation $\simeq$ on the module of simplicial module homomorphisms from M to $N$where N is a fibrant simplicial module. So, we will define fibrant simplicial module.

Given a simplicial module M, if $d_j x_k = d_k x_{j+1}$, where $j \geq k$; $k, j+1 \neq i$, then the elements $a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \in M_{n-1}$ are called matching faces w.r.t i.

If the simplicial module M provides the homotopy extension condition, then it is called *fibrant*. Suppose the elements $a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \in M_{n-1}$ are matching faces w.r.t i, we have an element $w \in M_n$ such that $d_j w = a_j$ for $j \neq i$. This condition is called homotopy extension condition.

3.3. Homotopy modules

The homotopy module $\pi_n(M)$ is defined by

$$\pi_n(M) = [S^n, M]$$

and so $\pi_n(M) = \pi_n(|M|)$ where M is a pointed fibrant simplicial module.

An element $x \in M_n$ satisfying the condition $d_i x = *$ for all $0 \leq i \leq n$, is named *spherical*. For a spherical element $x \in M_n$, the map $f_x : \Delta[n] \to M$ sends to quotient simplicial module $S^n = \Delta[n]/\partial\Delta[n]$. In contrast, a simplicial map $f : S^n \to M$ gets a spherical element $f(\sigma_n) \in M_n$, where σ_n is a nondegenerate element in S^n. So we have one to one correspondence such as

$$\boxed{\text{Spherical element in } M_n} \longleftrightarrow \boxed{\text{Simplicial module homomorphism } S^n \longrightarrow M.}$$

Theorem 3.1. *(Homotopy Addition Theorem) For pointed fibrant simplicial module M and spherical elements $x_i \in M_n$, the equation in $\pi_n(M)$*

$$[x_0]-[x_1]+[x_2]\cdots+(-1)^{n+1}[x_{n+1}]=0$$

satisfies if and only if there is $x \in K_{n+1}$ such that $d_i x = x_i$ where $0 \leq i \leq n+1$.

Proof. See [7], for details. □

Suppose M is a fibrant simplicial module. If f_x, f_y are homotopic relative to $\partial\Delta[n]$, then $x,y \in M_n$ is $x \simeq y$. So a fibrant simplicial module M is named *minimal* if it satisfies $x \simeq y \Rightarrow x = y$,

3.4. Homology of simplicial modules

For a simplicial module M, we define

$$N_nM = \bigcap_{j=1} Ker(d_j : M_n \longrightarrow M_{n-1}),$$

such that $x \in N_nM$, i.e $x \in M_n$ such that $d_jx = 1$ for $j > 0$. That is,

$$d_k(d_0x) = d_0d_{k+1}x = 1$$

for any $0 \leq k \leq n-1$.

A *chain complex* (C,∂) consists of modules and module homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

satisfying $\mathrm{Im}(\partial_{n+1}) \subseteq Ker(\partial_n)$, i.e $\partial_n \circ \partial_{n+1}$ is trivial. The *homology* $H_n(C,\partial)$ is written as $Ker(\partial_n)/\mathrm{Im}(\partial_{n+1})$.

Proposition 3.2. *Given a simplicial module M, if*

$$\partial_n =: \sum_{i=0}^{n}(-1)^i d_i : M_n \longrightarrow M_{n-1},$$

then $\partial_{n-1} \circ \partial_n = 0$, i.e M is a chain complex.

Proof. Similar to proposition 2.2. □

Remark The homology $H_*(M;A)$ of M with coefficients in a $\mathbb{Z}$-module A is defined by

$$H_*(M;A) = H_*(\mathbb{Z}(M) \otimes_{\mathbb{Z}} A, \partial_*),$$

where M is a simplicial module, $\mathbb{Z}(M) = \{\mathbb{Z}(M_n)\}_{n\geq 0}$ and $\mathbb{Z}(M_n)$ is the free $\mathbb{Z}$-module generated by M_n.

The *Moore chain complex* of simplicial R-module M, denoted NM, is the sequence of R-modules

$$\cdots \longrightarrow N_{n+1}M \xrightarrow{d_0} N_nM \xrightarrow{d_0} N_{n-1}M \longrightarrow \cdots$$

The elements in Z_nM, are called Moore *cycles* and the elements B_nM are called Moore *boundaries*.
By definition,

$$\begin{aligned} H_n(NM,d_0) &= Ker(d_0)/d_0(N_{n+1}M), \\ &= \bigcap_{j=0}^{n} Ker(d_j)/B_nM, \\ &= Z_nM/B_nM \\ &= \pi_n(M). \end{aligned}$$

So, for the simplicial module M, we can write

$$H_n(NM,d_0) \cong \pi_n(M) \cong \pi_n(|M|).$$

The significiance of this corollary is that the homology modules can be defined as the homology of chain complex.

4. Hurewicz and Poincaré theorems for simplicial modules

Suppose M is a simplicial module and $\mathbb{Z}(M)=\{\mathbb{Z}(M_n)\}_{n\geq 0}$ is a sequence of the free $\mathbb{Z}$-module generated by M_n. By using $d_i : M_n \longrightarrow M_{n-1}$, $s_i : M_n \longrightarrow M_{n+1}$, we can write

$$d_i^{\mathbb{Z}(M)} : \mathbb{Z}(M_n) \longrightarrow \mathbb{Z}(M_{n-1})$$

and the degeneracies

$$s_i^{\mathbb{Z}(M)} : \mathbb{Z}(M_n) \longrightarrow \mathbb{Z}(M_{n+1}).$$

Hence $\mathbb{Z}(M)$ is a simplicial $\mathbb{Z}$-module. The homology of M is defined by

$$H_*(M) = H_*(\mathbb{Z}(M)) \cong H_*(\mathbb{Z}(M),\partial).$$

Clearly, a simplicial module homomorphism $f : M \longrightarrow M'$ induces a simplicial $\mathbb{Z}$-module morphism $\mathbb{Z}(f) : \mathbb{Z}(M) \longrightarrow \mathbb{Z}(M')$. So we have a functor such that $M \longmapsto \mathbb{Z}(M)$, $f \longmapsto \mathbb{Z}(f)$. If $f \simeq g : M \longrightarrow M'$ (suppose that M' is fibrant), we have $\mathbb{Z}(f) \simeq \mathbb{Z}(g) : \mathbb{Z}(M) \longrightarrow \mathbb{Z}(M')$. Hence if $M \simeq M'$ with M and M' fibrant, we get $\mathbb{Z}(M) \simeq \mathbb{Z}(M')$ and so $H_*(M) \cong H_*(M')$.

As the geometric realization of any simplicial module is Δ-complex, the homology $H_*(M) = H_*(|M|)$ is the simplicial homology of the Δ-complex $|M|$. So, if $|M| \simeq |M'|$, then $H_*(M) \cong H_*(|M'|)$.

Thus the homology $H_*(M;A)$ with coefficients in A is defined by

$$H_*(M;A) = \pi_*(\mathbb{Z}(M)\otimes_{\mathbb{Z}} A) \cong H_*(\mathbb{Z}(M)\otimes_{\mathbb{Z}} A,\partial),$$

where A is a free $\mathbb{Z}$-module, $\mathbb{Z}(M)\otimes_{\mathbb{Z}} A = \{\mathbb{Z}(M)\otimes_{\mathbb{Z}} A\}_{n\geq 0}$.

As using redued homology, one can obtain a single relation on $\mathbb{Z}(M)$. Suppose $\mathbb{Z}[M]$ is the quotient $\mathbb{Z}$-module of $\mathbb{Z}(M)$ the simplicial submodule with the basepoint $*$. So the reduced integral homology $\overline{H_*}(M)$can be defined as

$$\overline{H_*}(M) = \pi_*(\mathbb{Z}[M]) \cong H_*(\mathbb{Z}[M],\partial).$$

The reuced homology with coefficients in A is defined by

$$\overline{H_*}(M;A) = \pi_*(\mathbb{Z}[M]\otimes_{\mathbb{Z}} A) \cong H_*(\mathbb{Z}[M]\otimes_{\mathbb{Z}} A,\partial).$$

The inclusion $j : M \hookrightarrow \mathbb{Z}(M)$ is a simplicial module homomorphism and the composite

$$\overline{j} : M \hookrightarrow \mathbb{Z}(M) \twoheadrightarrow \mathbb{Z}[M]$$

is pointed simplicial module homomorphism. (Note that the basepoint in M is $*$ and the basepoint in $\mathbb{Z}(M)$ is 0.) The map $\overline{j}$ induces a $\mathbb{Z}$-module homomorphism

$$h_n = \overline{j_*} : \pi_n(M) \longrightarrow \pi_n(\mathbb{Z}[M]) = \overline{H_n}(M)$$

where M is a fibrant simplicial module and $n \geq 1$, then this homomorphism is called *Hurewicz homomorphism*.

Theorem 4.1. *(Hurewicz Theorem) Suppose M is any fibrant simplicial module with $\pi_i(M) = 0$ for $i < n$ with $n \geq 2$. Then $\overline{H_i}(M) = 0$ for $i < n$ and $h_n : \pi_n(M) \longrightarrow \overline{H_n}(M)$ is an isomorphism.*

Proof. Suppose M is a minimal simplicial module. Think that

$$\overline{j} : M \longrightarrow \mathbb{Z}[M].$$

We can write $M_q = *$ for $q < n$ and $M_n = \pi_n(M)$, since M is minimal. Hence $\mathbb{Z}[M]_q = \{0\}$ for $q < n$ and $\mathbb{Z}[M]_n = \mathbb{Z}[M_n]$.

(1). h_n is onto: As the following diagram is commutative

$$\begin{array}{ccc} M_n & \hookrightarrow & \mathbb{Z}[M_n] \\ \| & & \downarrow \\ \pi_n(M) & \longrightarrow & \overline{H_n}(M) \end{array}$$

$\overline{H_n}(M)$ is generated by M_n as a $\mathbb{Z}$-module. As h_n is a $\mathbb{Z}$-module homomorphism and every generator of $\overline{H_n}(M)$ is in its image, it should be onto.

(2). $Ker(h_n) = \{0\}$: Assume that $x \in Ker(h_n)$. As x is 0 in

$$\overline{H_n}(M) = H_n(\mathbb{Z}[M], \partial),$$

we have an element $c \in \mathbb{Z}[M]_{n+1}$ such that $\partial(y) = x$ in $\mathbb{Z}[M]_n$. Let $y = \sum_{j=1}^{t} n_j y_j$ with $n_j \in \mathbb{Z}$ and $y_j \in M_{n+1}$. Then $\phi : \mathbb{Z}[M_n] \longrightarrow \pi_n(M)$ is the $\mathbb{Z}$-module homomorphism such that $\phi|_{M_n} : M_n \longrightarrow \pi_n(M) = M_n$ is the identity map seen as the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[M_{n+1}] & \longrightarrow & \mathbb{Z}[M_n] & & \\ \uparrow & & \uparrow & \searrow & \\ M_{n+1} & & M_n & = & \pi_n(M). \end{array}$$

For $y_j \in M_{n+1}$, we get

$$\begin{aligned} \phi \circ \partial(y_j) &= \phi(\sum_{i=0}^{n+1} (-1)^i d_i y_j \\ &= \sum_{i=0}^{n+1} (-1)^i \phi(d_i y_j) \\ &= \sum_{i=0}^{n+1} (-1)^i d_i y_j \ (\because d_i y_j \in M_n) \end{aligned}$$

in $\pi_n(M)$. By using Homotopy Addition Theorem, we can get $\phi(\partial(y_j)) = 0$ for each j. So

$$x = \phi\overline{j_n}(x) = \phi(\partial(y)) = 0$$

in $\pi_n(M)$ and i.e $Ker(h_n) = \{0\}$. □

Theorem 4.2. *(Poincaré Theorem) Suppose M is a connected (that is $\pi_0(|M|) = 0$) fibrant simplicial module. Then there is an isomorphism*

$$h' : \pi_1(M)/[\pi_1(M), \pi_1(M)] \longrightarrow \overline{H_1}(M)$$

induced by $h_1 : \pi_1(M) \longrightarrow \overline{H_1}(M)$.

Proof. Assume that M is a minimal simplicial module. By similar way, one can show that h' is onto. To proof that that h' is one to one, let $\phi : \mathbb{Z}[M_1] \longrightarrow \pi_1(M)/[\pi_1(M), \pi_1(M)]$ be the $\mathbb{Z}$-module homomorphism such that $\phi|_{M_1} : M_1 = \pi_1(M) \longrightarrow \pi_1(M)/[\pi_1(M), \pi_1(M)]$ is the quotient homomorphism consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}[M_2] & \longrightarrow & \mathbb{Z}[M_1] & & \\ \uparrow & & \uparrow & \searrow & \\ M_2 & & M_1 & \twoheadrightarrow & \pi_1(M)/[\pi_1(M), \pi_1(M)]. \end{array}$$

From after, one can continue the proof by same way of Hurewicz Theorem. □

5. Conclusion

By using simplicial theory, we give applications for simplicial homology and simplicial homotopy. Also, we proof the Hurewicz and Poincaré Theorems for simplicial modules.

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Examples of Almost Paracontact Metric Structures on 5-Dimensions

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Abstract

In this study, the classes of several almost paracontact metric structures on 5 dimensional nilpotent Lie algebras are determined. It is also shown that there are no $\eta-$ Einstein structures on 5 dimensional nilpotent Lie algebras.

1. Introduction

Differentiable manifolds having almost paracontact structures were introduced by [1] and after the work of [2] many authors have made contribution, see [2]-[6] and references therein. Almost paracontact metric manifolds were classified according to the covariant derivative of the structure tensor. The space of tensors having the same symmetry properties as the structure tensor is decomposed into the direct sum of twelve subspaces. Thus there are 12 basic classes and 2^{12} classes of almost paracontact metric structures. The defining relations and projections onto each subspace are given in [4] and [3].
There are six classes of non-isomorphic non-abelian nilpotent Lie algebras in five dimensions [7]. In this work, we give the explicit classes of some almost paracontact metric structures defined on 5-dimensional nilpotent Lie algebras by calculating projections onto each subspace. In addition, we show that a 5-dimensional nilpotent Lie algebra does not have the structure of an $\eta-$ Einstein manifold. For the existence of some classes of almost paracontact metric structures on 5-dimensional nilpotent Lie algebras, see [8].

2. Preliminaries

Let M^{2n+1} be an odd dimensional differentiable manifold. An ordered triple (φ,ξ,η) of an endomorphism, a vector field and a 1-form, respectively, with the following properties is called an almost paracontact structure on M

$$\varphi^2 = I-\eta\otimes\xi, \qquad \eta(\xi)=1, \varphi(\xi)=0,$$

$$\text{there is a distribution } \mathbb{D}: p\in M \longrightarrow \mathbb{D}_p = Ker\eta.$$

In this case M is called an almost paracontact manifold. If M also admits a semi-Riemannian metric g with the property that

$$g(\varphi(u),\varphi(v)) = -g(u,v)+\eta(u)\eta(v)$$

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for all $u,v\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of smooth vector fields on M, then M is called an almost paracontact metric manifold. The 2-form defined by

$$\Phi(u,v)=g(\varphi u,v)$$

for all $u,v\in\mathfrak{X}(M)$, is called the fundamental 2-form. We denote the vector fields and tangent vectors by letters u,v,w. 2^{12} classes of almost paracontact manifolds are obtained by using the covariant derivative of Φ. Consider the tensor F defined by

$$F(u,v,w)=g((\nabla_u\varphi)(v),w),$$

for all $u,v,w\in T_pM$, where T_pM is the tangent space at p and ∇ denotes the covariant derivative of g. Then F satisfies

$$F(u,v,w)=-F(u,w,v), \tag{2.1}$$

$$F(u,\varphi v,\varphi w)=F(u,v,w)+\eta(v)F(u,w,\xi)-\eta(w)F(u,v,\xi). \tag{2.2}$$

The Lee forms associated with F are

$$\theta(u)=g^{ij}F(e_i,e_j,u),\;\; \theta^*(u)=g^{ij}F(e_i,\varphi e_j,u),\;\; \omega(u)=F(\xi,\xi,u),$$

where $u\in T_pM$, $\{e_i,\xi\}$ is a basis for T_pM and g^{ij} is the inverse of the matrix g_{ij}.
Let $\mathscr{F}$ be the set of $(0,3)$ tensors over T_pM having properties (2.1), (2.2). $\mathscr{F}$ is the direct sum of four subspaces W_i, $i=1,\dots,4$ where projections F^{W_i} onto W_i are

$$F^{W_1}(u,v,w)=F(\varphi^2u,\varphi^2v,\varphi^2w),$$

$$F^{W_2}(u,v,w)=-\eta(v)F(\varphi^2u,\varphi^2w,\xi)+\eta(w)F(\varphi^2u,\varphi^2v,\xi),$$

$$F^{W_3}(u,v,w)=\eta(u)F(\xi,\varphi v,\varphi w),$$

$$F^{W_4}(u,v,w)=\eta(u)\{\eta(v)F(\xi,\xi,w)-\eta(w)F(\xi,\xi,v)\}.$$

In addition W_1 can be written as a direct sum of subspaces $\mathbb{G}_i$, $i=1,\dots,4$, W_2 is a direct sum of subspaces $\mathbb{G}_i$, $i=5,\dots,10$, and denote W_3 and W_4 as $\mathbb{G}_{11}$ and $\mathbb{G}_{12}$, respectively. Then $\mathscr{F}$ is a direct sum of twelve subspaces $\mathbb{G}_i$, $i=1,\dots,12$. An almost paracontact manifold is said to be in the class $\mathbb{G}_i\oplus\mathbb{G}_j$, etc if the tensor F is in the class $\mathbb{G}_i\oplus\mathbb{G}_j$ over T_pM for all $p\in M$. The defining relations of basic classes $\mathbb{G}_i$ of almost paracontact metric structures and projections F^i onto each $\mathbb{G}_i$ are listed below [3, 4].

$$\begin{aligned}\mathbb{G}_1:\; F(u,v,w) &= \frac{1}{2(n-1)}\{g(u,\varphi v)\theta_F(\varphi w)-g(u,\varphi w)\theta_F(\varphi v)-g(\varphi u,\varphi v)\theta_F(\varphi^2 w)\\ &\quad +g(\varphi u,\varphi w)\theta_F(\varphi^2 v)\end{aligned}$$

$$\mathbb{G}_2:\; F(\varphi u,\varphi v,w)=-F(u,v,w),\theta_F=0$$

$$\mathbb{G}_3:\; F(\xi,v,w)=F(u,\xi,w)=0,F(u,v,w)=-F(v,u,w)$$

$$\begin{aligned}\mathbb{G}_4:\; F(\xi,v,w) &= F(u,\xi,w)=0,\\ \sum_{cyc}F(u,v,w) &= 0 \text{ where } \sum_{cyc} \text{ denotes the cyclic sum over } u,v,w\end{aligned}$$

$$\mathbb{G}_5:\; F(u,v,w)=\frac{\theta_F(\xi)}{2n}\{g(\varphi u,\varphi w)\eta(v)-g(\varphi u,\varphi v)\eta(w)\}$$

$$\mathbb{G}_6: \; F(u,v,w) = -\frac{\theta_F^*(\xi)}{2n}\{g(u,\varphi w)\eta(v) - g(u,\varphi v)\eta(w)\}$$

$$\begin{aligned} \mathbb{G}_7: \; F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= -F(v,u,\xi) = -F(\varphi u,\varphi v,\xi), \;\; \theta_F^*(\xi) = 0 \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathbb{G}_8: \; F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= F(v,u,\xi) = -F(\varphi u,\varphi v,\xi), \;\; \theta_F(\xi) = 0 \end{aligned}$$

$$\begin{aligned} \mathbb{G}_9: \; F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= -F(v,u,\xi) = F(\varphi u,\varphi v,\xi) \end{aligned}$$

$$\begin{aligned} \mathbb{G}_{10}: \; F(u,v,w) &= -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\ F(u,v,\xi) &= F(v,u,\xi) = F(\varphi u,\varphi v,\xi) \end{aligned}$$

$$\mathbb{G}_{11}: \; F(u,v,w) = \eta(u)F(\xi,\varphi v),\varphi w))$$

$$\mathbb{G}_{12}: \; F(u,v,w) = \eta(u)\{\eta(v)F(\xi,\xi,w) - \eta(w)F(\xi,\xi,v)\}$$

Projections F^i onto each subspace $\mathbb{G}_i$ are

$$\begin{aligned} F^1(u,v,w) &= \frac{1}{2(n-1)}\{g(u,\varphi v)\theta_{F^1}(\varphi w) - g(u,\varphi w)\theta_{F^1}(\varphi v) \\ &\quad - g(\varphi u,\varphi v)\theta_{F^1}(\varphi^2 w) + g(\varphi u,\varphi w)\theta_{F^1}(\varphi^2 v)\}, \end{aligned}$$

$$F^2(u,v,w) = \frac{1}{2}\{F(\varphi^2 u,\varphi^2 v,\varphi^2 w) - F(\varphi u,\varphi^2 v,\varphi w)\} - F^1(u,v,w)$$

$$\begin{aligned} F^3(u,v,w) &= \frac{1}{6}\{F(\varphi^2 u,\varphi^2 v,\varphi^2 w) + F(\varphi u,\varphi^2 v,\varphi w) \\ &\quad + F(\varphi^2 v,\varphi^2 w,\varphi^2 u) + F(\varphi v,\varphi^2 w,\varphi u) \\ &\quad + F(\varphi^2 w,\varphi^2 u,\varphi^2 v) + F(\varphi w,\varphi^2 u,\varphi v)\} \end{aligned}$$

$$F^4(u,v,w) = \frac{1}{2}\{F(\varphi^2 u,\varphi^2 v,\varphi^2 w) + F(\varphi u,\varphi^2 v,\varphi w)\} - F^3(u,v,w) \tag{2.4}$$

$$F^5(u,v,w) = \frac{\theta_{F^5}(\xi)}{2n}\{\eta(v)g(\varphi u,\varphi w) - \eta(w)g(\varphi u,\varphi v)\} \tag{2.5}$$

$$F^6(u,v,w) = -\frac{\theta_{F^6}^*(\xi)}{2n}\{\eta(v)g(u,\varphi w) - \eta(w)g(u,\varphi v)\}$$

$$\begin{aligned} F^7(u,v,w) &= -\frac{1}{4}\eta(v)\{F(\varphi^2 u,\varphi^2 w,\xi) - F(\varphi u,\varphi w,\xi) \\ &\quad - F(\varphi^2 w,\varphi^2 u,\xi) + F(\varphi w,\varphi u,\xi)\} + \frac{1}{4}\eta(w)\{F(\varphi^2 u,\varphi^2 v,\xi) \\ &\quad - F(\varphi u,\varphi v,\xi) - F(\varphi^2 v,\varphi^2 u,\xi) + F(\varphi v,\varphi u,\xi)\} - F^6(u,v,w) \end{aligned}$$

$$\begin{aligned} F^8(u,v,w) &= -\frac{1}{4}\eta(v)\{F(\varphi^2u,\varphi^2w,\xi)-F(\varphi u,\varphi w,\xi) \\ &+F(\varphi^2w,\varphi^2u,\xi)-F(\varphi w,\varphi u,\xi)\}+\frac{1}{4}\eta(w)\{F(\varphi^2u,\varphi^2v,\xi) \\ &-F(\varphi u,\varphi v,\xi)+F(\varphi^2v,\varphi^2u,\xi)-F(\varphi v,\varphi u,\xi)\}-F^5(u,v,w) \end{aligned} \tag{2.6}$$

$$\begin{aligned} F^9(u,v,w) &= -\frac{1}{4}\eta(v)\{F(\varphi^2u,\varphi^2w,\xi)+F(\varphi u,\varphi w,\xi) \\ &-F(\varphi^2w,\varphi^2u,\xi)-F(\varphi w,\varphi u,\xi)\}+\frac{1}{4}\eta(w)\{F(\varphi^2u,\varphi^2v,\xi) \\ &+F(\varphi u,\varphi v,\xi)-F(\varphi^2v,\varphi^2u,\xi)-F(\varphi v,\varphi u,\xi)\} \end{aligned} \tag{2.7}$$

$$\begin{aligned} F^{10}(u,v,w) &= -\frac{1}{4}\eta(v)\{F(\varphi^2u,\varphi^2w,\xi)+F(\varphi u,\varphi w,\xi) \\ &+F(\varphi^2w,\varphi^2u,\xi)+F(\varphi w,\varphi u,\xi)\}+\frac{1}{4}\eta(w)\{F(\varphi^2u,\varphi^2v,\xi) \\ &+F(\varphi u,\varphi v,\xi)+F(\varphi^2v,\varphi^2u,\xi)+F(\varphi v,\varphi u,\xi)\} \end{aligned} \tag{2.8}$$

$$F^{11}(u,v,w)=\eta(u)F(\xi,\varphi^2v,\varphi^2w) \tag{2.9}$$

$$F^{12}(u,v,w)=\eta(u)\{\eta(v)F(\xi,\xi,\varphi^2w)-\eta(w)F(\xi,\xi,\varphi^2v)\}. \tag{2.10}$$

It is known that ξ is Killing in $\mathbb{G}_1\oplus\mathbb{G}_2\oplus\mathbb{G}_3\oplus\mathbb{G}_4\oplus\mathbb{G}_5\oplus\mathbb{G}_8\oplus\mathbb{G}_9\oplus\mathbb{G}_{11}$, that is $F^6=F^7=F^{10}=F^{12}=0$ in this case and ξ is parallel in the basic classes $\mathbb{G}_1$, $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_{11}$. Also for five dimensional manifolds, the dimension of $\mathbb{G}_3$ is zero, so $F^3=0$ [3].

A K-paracontact manifold (M,φ,η,ξ,g) is called an η-Einstein manifold if its Ricci tensor is of the form

$$Ric(u,v)=ag(u,v)+b\eta(u)\eta(v),$$

where a, b are constants. Also, the Ricci curvature in the direction of ξ satisfies

$$Ric(\xi,\xi)=-2n \tag{2.11}$$

on a K-paracontact metric manifold of dimension $2n+1$ [2].

Let G be a connected Lie group and (φ,ξ,η,g) a left invariant almost paracontact metric structure on G, that is,

$$\varphi\circ L_a=L_a\circ\varphi,\ \ L_a(\xi)=\xi,$$

where L_a is the left multiplication by $a\in G$ in G and g is left invariant. The almost paracontact metric structure on G induces an almost paracontact metric structure on the Lie algebra $\mathfrak{g}$ of G denoted by (φ,ξ,η,g).

In this study, we determine the classes of some almost paracontact metric structures on 5-dimensional nilpotent Lie algebras. We use the classification of 5 dimensional nilpotent Lie algebras in [7]. There are six non-isomorphic non-abelian algebras $\mathfrak{g}_i$ with basis $\{e_1,\dots,e_5\}$ and non-zero brackets:

$$\begin{aligned} \mathfrak{g}_1 &: [e_1,e_2]=e_5,[e_3,e_4]=e_5 \\ \mathfrak{g}_2 &: [e_1,e_2]=e_3,[e_1,e_3]=e_5,[e_2,e_4]=e_5 \\ \mathfrak{g}_3 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5,[e_2,e_3]=e_5 \\ \mathfrak{g}_4 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_1,e_4]=e_5 \\ \mathfrak{g}_5 &: [e_1,e_2]=e_4,[e_1,e_3]=e_5 \\ \mathfrak{g}_6 &: [e_1,e_2]=e_3,[e_1,e_3]=e_4,[e_2,e_3]=e_5. \end{aligned}$$

In addition, we show that a five-dimensional almost paracontact metric manifold (G,φ,ξ,η,g) can not be an η-Einstein manifold, where G is a connected Lie group with 5 dimensional nilpotent Lie algebra.

3. Classes of almost paracontact metric structures on $\mathfrak{g}_i$

Assume that (φ,ξ,η,g) is a left invariant almost paracontact metric structure on a connected Lie group G_i with corresponding Lie algebra $\mathfrak{g}_i$. Denote the corresponding almost paracontact metric structure on $\mathfrak{g}_i$ by the same quadruple.
The algebra $\mathfrak{g}_1$: Consider the basis $\{e_1,\dots,e_5\}$ with non-zero brackets

$$[e_1,e_2]=e_5,[e_3,e_4]=e_5.$$

Let g be the semi-Riemannian metric such that $\{e_1,\dots,e_5\}$ is orthonormal and $\varepsilon_i=g(e_i,e_i)=\pm1$. The nonzero covariant derivatives are evaluated in [8] by Kozsul's formula:

$$\nabla_{e_1}e_2=\frac{1}{2}e_5,\ \ \nabla_{e_1}e_5=-\frac{1}{2}\varepsilon_2\varepsilon_5e_2,$$

$$\nabla_{e_2}e_1=-\frac{1}{2}e_5,\ \ \nabla_{e_2}e_5=\frac{1}{2}\varepsilon_1\varepsilon_5e_1,$$

$$\nabla_{e_3}e_4=\frac{1}{2}e_5,\ \ \nabla_{e_3}e_5=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4,$$

$$\nabla_{e_4}e_3=-\frac{1}{2}e_5,\ \ \nabla_{e_4}e_5=\frac{1}{2}\varepsilon_3\varepsilon_5e_3,$$

$$\nabla_{e_5}e_1=-\frac{1}{2}\varepsilon_2\varepsilon_5e_2,\ \ \nabla_{e_5}e_2=\frac{1}{2}\varepsilon_1\varepsilon_5e_1,\ \ \nabla_{e_5}e_3=-\frac{1}{2}\varepsilon_4\varepsilon_5e_4,\ \ \nabla_{e_5}e_4=\frac{1}{2}\varepsilon_3\varepsilon_5e_3.$$

For each Lie algebra we consider two different almost paracontact metric structures and determine the class of the structure.

- Let (φ,ξ,η,g) be the quadruple such that
$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=-g(e_4,e_4)=-g(e_5,e_5)=1$,
$\xi=e_1$, $\eta=e^1$,
$\varphi(e_1)=0$, $\varphi(e_2)=e_4$, $\varphi(e_3)=e_5$, $\varphi(e_4)=e_2$, $\varphi(e_5)=e_3$.

$$(\varphi,\xi,\eta,g)\tag{3.1}$$

is an almost paracontact metric structure on $\mathfrak{g}_1$. Note that $\xi=e_1$ is not Killing and $\eta=e^1$ is the metric dual of $\xi=e_1$ such that $\eta(x)=g(x,e_1)$ for all vectors x. We evaluate the projections F^i and determine the class of the structure. The nonzero structure constants $F(e_i,e_j,e_k)=g((\nabla_{e_i}\varphi)(e_j),e_k)$ are given below.

$$F(e_2,e_1,e_3)=F(e_1,e_3,e_2)=-F(e_1,e_2,e_3)=-F(e_2,e_3,e_1)=1/2,$$

$$F(e_1,e_5,e_4)=F(e_5,e_1,e_4)=-F(e_1,e_4,e_5)=-F(e_5,e_4,e_1)=1/2,$$

$$F(e_3,e_5,e_2)=F(e_5,e_3,e_2)=-F(e_5,e_2,e_3)=-F(e_3,e_2,e_5)=1/2,$$

$$F(e_3,e_3,e_4)=-F(e_3,e_4,e_3)=F(e_5,e_5,e_4)=-F(e_5,e_4,e_5)=1/2.$$

By Theorem 3.4 in [3] the dimension of $\mathbb{G}_3$ is zero in 5-dimensions. Thus for each almost paracontact metric structure in 5 dimensions $F^3=0$. Since $(\nabla_{e_1}\varphi)(e_1)=0$, we have
$F(\xi,\xi,\varphi^2z)=F(e_1,e_1,\varphi^2z)=g((\nabla_{e_1}\varphi)(e_1),\varphi^2z)=0$ and $F^{12}=0$ from (2.10).
For any vector $u=\sum u_ie_i$, $\varphi^2(u)=u_2e_2+u_3e_3+u_4e_4+u_5e_5$ and from (2.9),

$$\begin{aligned}F^{11}(u,v,w)&=u_1F(e_1,v_2e_2+v_3e_3+v_4e_4+v_5e_5,w_2e_2+w_3e_3+w_4e_4+w_5e_5)\\&=\frac{1}{2}u_1\{-v_2w_3+v_3w_2-v_4w_5+v_5w_4\}\\&\neq0.\end{aligned}$$

Now since

$$F(\varphi^2u,\varphi^2w,\xi)=-\frac{1}{2}\{u_2w_3+u_5w_4\}$$

and

$$F(\varphi u,\varphi w,\xi)=-\frac{1}{2}\{u_4w_5+u_3w_2\},$$

we have

$$F(\varphi^2u,\varphi^2w,\xi)+F(\varphi u,\varphi w,\xi)=F(\varphi^2w,\varphi^2u,\xi)+F(\varphi w,\varphi u,\xi).$$

Thus from (2.8), (2.7), (2.5), (2.6) respectively, we get

$$\begin{aligned} F^{10}(u,v,w) &= \frac{1}{4}y_1\{u_2w_3+u_3w_2+u_4w_5+u_5w_4\} \\ &\quad -\frac{1}{4}w_1\{u_2v_3+u_5v_4+u_4v_5+u_3v_2\} \\ &\neq 0, \end{aligned}$$

$F^9=0$, $F^5+F^8=0$ and thus $F^5=F^8=0$. Also since $F^{W_2}=F^5+F^6+F^7+F^8+F^9+F^{10}$ and $F^5=F^8=F^9=0$, we get

$$\begin{aligned} (F^6+F^7)(u,v,w) &= F^{W_2}(u,v,w)-F^{10}(u,v,w) \\ &= \frac{1}{4}u_2v_1w_3+\frac{1}{4}u_5v_1w_4-\frac{1}{4}u_4v_1w_5 \\ &\quad -\frac{1}{4}u_3v_1w_2-\frac{1}{4}u_2v_3w_1-\frac{1}{4}u_5v_4w_1 \\ &\quad +\frac{1}{4}u_4v_5w_1+\frac{1}{4}u_3v_2w_1. \end{aligned}$$

Let $T=F^6+F^7$. The nonzero structure constants of the tensor T are

$$T(e_2,e_1,e_3)=-T(e_2,e_3,e_1)=-T(e_3,e_1,e_2)=T(e_3,e_2,e_1)=1/4,$$

$$T(e_5,e_1,e_4)=-T(e_4,e_1,e_5)=-T(e_5,e_4,e_1)=T(e_4,e_5,e_1)=1/4.$$

We show that T satisfies the defining relation (2.3) of $\mathbb{G}_7$.

$$\begin{aligned} & -\eta(v)T(u,w,\xi)+\eta(w)T(u,v,\xi) \\ =& -v_1\{-\frac{1}{4}u_2w_3-\frac{1}{4}u_5w_4+\frac{1}{4}u_4w_5+\frac{1}{4}u_3w_2\} \\ & +w_1\{-\frac{1}{4}u_2v_3-\frac{1}{4}u_5v_4+\frac{1}{4}u_4v_5+\frac{1}{4}u_3v_2\} \\ =& \ T(u,v,w), \end{aligned}$$

$$-T(v,u,\xi)=-T(v,u,e_1)=\frac{1}{4}v_2u_3+\frac{1}{4}v_5u_4-\frac{1}{4}v_4u_5-\frac{1}{4}v_3u_2=T(u,v,\xi),$$

$$\begin{aligned} T(\varphi u,\varphi v,\xi) &= T(u_4e_2+u_5e_3+u_2e_4+u_3e_5,v_4e_2+v_5e_3+v_2e_4+v_3e_5,e_1) \\ &= -\frac{1}{4}u_4v_5-\frac{1}{4}u_3v_2+\frac{1}{4}u_2v_3+\frac{1}{4}u_5v_4=-T(u,v,\xi). \end{aligned}$$

According to the basis $\{f_1,f_2,f_3,f_4,f_5\}=\{e_2,e_3,e_4,e_5,\xi=e_1\}$, since $g_{ij}=diag(1,1,-1,-1,1)$ and $g^{ij}=diag(1,1,-1,-1,1)$, we have

$$\begin{aligned} \theta^*_T(\xi) &= \theta^*_T(e_1)=g^{ij}T(f_i,\varphi f_j,\xi) \\ &= T(f_1,\varphi f_1,e_1)+T(f_2,\varphi f_2,e_1)-T(f_3,\varphi f_3,e_1)-T(f_4,\varphi f_4,e_1) \\ &= T(e_2,\varphi e_2,e_1)+T(e_3,\varphi e_3,e_1)-T(e_4,\varphi e_4,e_1)-T(e_5,\varphi e_5,e_1) \\ &= T(e_2,e_4,e_1)+T(e_3,e_5,e_1)-T(e_4,e_2,e_1)-T(e_5,e_3,e_1) \\ &= 0. \end{aligned}$$

As a result $T=F^6+F^7\in\mathbb{G}_7$, in particular $F^6=0$ and $F^7\neq 0$. In addition,

$$\begin{aligned} F(\varphi^2u,\varphi^2v,\varphi^2w) &= -\frac{1}{2}u_3v_2w_5+\frac{1}{2}u_3v_3w_4-\frac{1}{2}u_3v_4w_3+\frac{1}{2}u_3v_5w_2 \\ &\quad -\frac{1}{2}u_5v_2w_3+\frac{1}{2}u_5v_3w_2-\frac{1}{2}u_5v_4w_5+\frac{1}{2}u_5v_5w_4 \\ &= F(\varphi u,\varphi^2v,\varphi w) \end{aligned}$$

together with (2.4) implies

$$\begin{aligned} F^4(u,v,w) &= \frac{1}{2}\{F(\varphi^2u,\varphi^2u,\varphi^2w)+F(\varphi u,\varphi^2v,\varphi w)\}-F^3(u,v,w) \\ &= F(\varphi^2u,\varphi^2v,\varphi^2w) \\ &= F^{W_1}(u,v,w) \\ &= -\frac{1}{2}u_3v_2w_5+\frac{1}{2}u_3v_3w_4-\frac{1}{2}u_3v_4w_3+\frac{1}{2}u_3v_5w_2 \\ &\quad -\frac{1}{2}u_5v_2w_3+\frac{1}{2}u_5v_3w_2-\frac{1}{2}u_5v_4w_5+\frac{1}{2}u_5v_5w_4 \\ &\neq 0. \end{aligned}$$

Since $F^{W_1}=F^1+F^2+F^3+F^4=F^4$, we obtain $F^1=F^2=0$.
To sum up, since the only nonzero projections are F^4, F^7, F^{10} and F^{11}, the almost paracontact structure (3.1) belongs to the class $\mathbb{G}_4\oplus\mathbb{G}_7\oplus\mathbb{G}_{10}\oplus\mathbb{G}_{11}$.

- Consider now the almost paracontact metric structure

$$(\varphi,\xi,\eta,g) \tag{3.2}$$

defined by $g(e_1,e_1)=g(e_2,e_2)=-g(e_3,e_3)=-g(e_4,e_4)=g(e_5,e_5)$,
$\xi=e_5$, $\eta=e^5$,
$\varphi(e_1)=e_3$, $\varphi(e_2)=e_4$, $\varphi(e_3)=e_1$, $\varphi(e_4)=e_2$, $\varphi(e_5)=0$.
Note that $\xi=e_5$ is Killing [8], and thus, $F^6=F^7=F^{10}=F^{12}=0$ by Proposition 4.7 in [3]. The 1-form $\eta=e^5$ is the metric dual of $\xi=e_5$. Nonzero structure constants of F are

$$F(e_1,e_4,e_5)=-F(e_1,e_5,e_4)=-F(e_2,e_3,e_5)=F(e_2,e_5,e_3)=1/2,$$

$$-F(e_3,e_5,e_2)=F(e_3,e_2,e_5)=-F(e_4,e_1,e_5)=F(e_4,e_5,e_1)=1/2,$$

$$-F(e_5,e_1,e_4)=F(e_5,e_4,e_1)=F(e_5,e_2,e_3)=-F(e_5,e_3,e_2)=1.$$

Then by (2.9),

$$F^{11}(u,v,w)=u_5\{-v_1w_4+v_2w_3-v_3w_2+v_4w_1\}\neq 0.$$

Since

$$\begin{aligned} F(\varphi^2u,\varphi^2w,\xi) &= \frac{1}{2}\{u_1w_4-u_2w_3+u_3w_2-u_4w_1\} \\ &= F(\varphi u,\varphi w,\xi), \end{aligned}$$

from (2.7),

$$\begin{aligned} F^9(u,v,w) &= -\frac{1}{2}v_5\{u_1w_4-u_2w_3+u_3w_2-u_4w_1\} \\ &\quad +\frac{1}{2}w_5\{u_1v_4-u_2v_3+u_3v_2-u_4v_1\} \\ &\neq 0. \end{aligned}$$

Also since $F(\varphi^2u,\varphi^2w,\xi)=F(\varphi u,\varphi w,\xi)$, by (2.5) and (2.6) we have $F^5+F^8=0$ implying $F^5=F^8=0$. In addition, $F^{W_1}=F^1+F^2+F^3+F^4=0$ and thus $F^1=F^2=F^3=F^4=0$.
As a result the structure (3.2) is in $\mathbb{G}_9\oplus\mathbb{G}_{11}$.

Note that the almost paracontact structures (3.1) and (3.2) can also be considered as almost paracontact structures on other Lie algebras $\mathfrak{g}_i$, $i=1,2,\ldots,6$. By calculating projections F^i for each structure, we determine the class of two different structures (3.1) and (3.2) on each Lie algebra. We omit calculations for other Lie algebras since they are similar to those for $\mathfrak{g}_1$. We only write the class of the structures.

The algebra $\mathfrak{g}_2$:

- Let (φ,ξ,η,g) be the almost paracontact structure (3.1) on $\mathfrak{g}_2$. The class of this structure is $\mathbb{G}_1\oplus\mathbb{G}_7\oplus\mathbb{G}_{10}\oplus\mathbb{G}_{11}$.
- (3.2) considered as an almost paracontact structure on $\mathfrak{g}_2$ is in $\mathbb{G}_4\oplus\mathbb{G}_5$.

The algebra $\mathfrak{g}_3$:

- The structure (3.1) on $\mathfrak{g}_3$ belongs to $\mathbb{G}_4\oplus\mathbb{G}_5\oplus\mathbb{G}_6\oplus\mathbb{G}_7\oplus\mathbb{G}_8\oplus\mathbb{G}_{10}\oplus\mathbb{G}_{11}$.
- The structure (3.2) on $\mathfrak{g}_3$ is of type $\mathbb{G}_1\oplus\mathbb{G}_2\oplus\mathbb{G}_4\oplus\mathbb{G}_8$.

The algebra $\mathfrak{g}_4$:

- (3.1) on $\mathfrak{g}_4$ is in $\mathbb{G}_5 \oplus \mathbb{G}_6 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8 \oplus \mathbb{G}_{10} \oplus \mathbb{G}_{11}$.
- (3.2) on $\mathfrak{g}_4$ is in $\mathbb{G}_2 \oplus \mathbb{G}_4 \oplus \mathbb{G}_8 \oplus \mathbb{G}_9 \oplus \mathbb{G}_{11}$.

The algebra $\mathfrak{g}_5$:

- (3.1) on $\mathfrak{g}_5$ lies in $\mathbb{G}_{10}$.
- (3.2) on $\mathfrak{g}_5$ is in the class $\mathbb{G}_4 \oplus \mathbb{G}_5 \oplus \mathbb{G}_8$.

The algebra $\mathfrak{g}_6$:

- (3.1) on $\mathfrak{g}_6$ belongs to $\mathbb{G}_4 \oplus \mathbb{G}_7 \oplus \mathbb{G}_{10} \oplus \mathbb{G}_{11}$.
- (3.2) on $\mathfrak{g}_6$ is in $\mathbb{G}_1 \oplus \mathbb{G}_4 \oplus \mathbb{G}_8 \oplus \mathbb{G}_9 \oplus \mathbb{G}_{11}$.

Note that almost paracontact structures obtained here belong to the given direct sum properly, that is, they contain summand from each subclass, since corresponding projections are nonzero. Thus we give examples of almost paracontact metric structures which contain summands from several classes.

4. η-Einstein manifolds of 5-dimensions

It is known that paracontact structures exist only on $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$ for five dimensional nilpotent Lie algebras. In addition a vector field is Killing iff $\xi \in < e_5 >$, see [8].
We state

Proposition 4.1. *Let G be a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_i$, $i = 1, \ldots, 6$. Then a K-paracontact metric structure $(G, \varphi, \xi, \eta, g)$ is not η-Einstein.*

Proof. A five-dimensional almost paracontact metric manifold $(G, \varphi, \xi, \eta, g)$ is not an η-Einstein manifold, if the Lie algebra of the connected Lie group G is isomorphic to $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$ since there are no paracontact structures on $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$, paracontact structures exist only on $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, see [8]. Thus it is enough to check the existence of η-Einstein manifolds only on $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$. Assume that $(G, \varphi, \xi, \eta, g)$ is η-Einstein, where G is a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_1$. Since ξ is Killing, $\xi = \xi_5 e_5$.
$\eta(\xi) = 1 = g(\xi, \xi) = g(\xi_5 e_5, \xi_5 e_5) = \xi_5^2 \varepsilon_5$ implies $\xi_5^2 = 1$ and $\varepsilon_5 = +1$.
From the equation (2.11), we have

$$Ric(\xi, \xi) = \xi_5^2 Ric(e_5, e_5) = Ric(e_5, e_5) = -4.$$

On the other hand, by direct calculation

$$R_{e_5 e_m} e_5 = \nabla_{[e_5, e_m]} e_5 - \nabla_{e_5}(\nabla_{e_m} e_5) + \nabla_{e_m}(\nabla_{e_5} e_5) = -\nabla_{e_5}(\nabla_{e_m} e_5)$$

and

$$R_{e_5 e_1} e_5 = -\nabla_{e_5}(\nabla_{e_1} e_5) = -\nabla_{e_5}(-\frac{1}{2}\varepsilon_2 \varepsilon_5 e_2) = \frac{1}{2}\varepsilon_2 \varepsilon_5 (\frac{1}{2}\varepsilon_1 \varepsilon_5 e_1) = \frac{1}{4}\varepsilon_1 \varepsilon_2 e_1,$$

$$R_{e_5 e_2} e_5 = \frac{1}{4}\varepsilon_1 \varepsilon_2 e_2, \quad R_{e_5 e_3} e_5 = \frac{1}{4}\varepsilon_3 \varepsilon_4 e_3, \quad R_{e_5 e_4} e_5 = \frac{1}{4}\varepsilon_3 \varepsilon_4 e_4$$

yields

$$\begin{aligned} Ric(e_5, e_5) &= \sum_{m=1}^{5} \varepsilon_m g(R_{e_5 e_m} e_5, e_m) \\ &= \varepsilon_1 g(\frac{1}{4}\varepsilon_1 \varepsilon_2 e_1, e_1) + \varepsilon_2 g(\frac{1}{4}\varepsilon_1 \varepsilon_2 e_2, e_2) + \varepsilon_3 g(\frac{1}{4}\varepsilon_3 \varepsilon_4 e_3, e_3) + \varepsilon_4 g(\frac{1}{4}\varepsilon_3 \varepsilon_4 e_4, e_4) \\ &= \frac{1}{2}\varepsilon_1 \varepsilon_2 + \frac{1}{2}\varepsilon_3 \varepsilon_4. \end{aligned}$$

Since $\varepsilon_i = \pm 1$, $Ric(e_5, e_5) = \frac{1}{2}\varepsilon_1 \varepsilon_2 + \frac{1}{2}\varepsilon_3 \varepsilon_4 \neq 4$. Thus $(G, \varphi, \xi, \eta, g)$ can not be η-Einstein. The proof is similar for $\mathfrak{g}_2$ and $\mathfrak{g}_3$. In $\mathfrak{g}_2$, $\xi = \xi_5 e_5$ and by (2.11), $Ric(\xi, \xi) = \xi_5^2 Ric(e_5, e_5) = Ric(e_5, e_5) = -4$. By direct calculation,

$$\begin{aligned} Ric(e_5, e_5) &= \sum_{m=1}^{5} \varepsilon_m g(R_{e_5 e_m} e_5, e_m) \\ &= \frac{1}{2}\varepsilon_1 \varepsilon_3 + \frac{1}{2}\varepsilon_2 \varepsilon_4 \\ &\neq 4. \end{aligned}$$

In $\mathfrak{g}_3$, $\xi = \xi_5 e_5$ and

$$\begin{aligned} Ric(e_5,e_5) &= \sum_{m=1}^{5} \varepsilon_m g(R_{e_5 e_m} e_5, e_m) \\ &= \frac{1}{2}\varepsilon_1\varepsilon_4 + \frac{1}{2}\varepsilon_2\varepsilon_3, \end{aligned}$$

which contradicts with (2.11). □

5. Conclusion

In this manuscript new examples of almost paracontact metric structures on some five dimensional nilpotent Lie algebras are given. These examples contain summands from several classes of almost paracontact metric structures. In addition, we show that a K-paracontact metric structure (G,φ,ξ,η,g) on a connected Lie group G is not η-Einstein.

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Author's contributions

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Computation of the Solutions of Lyapunov Matrix Equations with Iterative Decreasing Dimension Method

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Abstract

The existence of a solution of continuous and discrete-time Lyapunov matrix equations was studied. Both Lyapunov matrix equations are transformed into a matrix-vector equation and the solution of the obtained new system was examined. The iterative decreasing dimension method (IDDM) was implemented for solving the generated matrix-vector equation. Computations have been done with Maple procedures that run the constituted algorithms.

1. Introduction

The systems are

$$y'(t) = Ay(t) \tag{1.1}$$

and

$$y(n+1) = Ay(n), \quad n \in \mathbb{Z}, \tag{1.2}$$

respectively differential equation system and difference equation system considered.
In the system (1.1) $y(t) = \left(y_1(t), y_2(t), \ldots, y_N(t)\right)^T$, $y_i(t)\,(i = 1, 2, \cdots, N)$ are differentiable functions. The coefficient matrix of systems is $A \in M_N(\mathbb{C})$. $M_N^P(\mathbb{C})$ and $M_N(\mathbb{C})$ respectively will denote the set of all $N \times P$ matrices and set of square matrices of size $N \times N$ that matrices elements are complex numbers.
Hurwitz stability is well known in the literature. Regarding the equation system (1.1), in order for system to be Hurwitz stable, the real part of all the eigenvalues of A must be less than zero. Another qualification of Hurwitz stability of equation system (1.1) is concerning with continuous-time Lyapunov matrix equation

$$A^*H + HA = -I \tag{1.3}$$

that has a unique solution under $H = H^* > 0$ condition where I is unit matrix and A^* is adjoint of the matrix A.
Schur stability is well known in the literature too. In accordance with the spectral criterion, all eigenvalues of the matrix A must fall into the unit disc so that the equation system (1.2) get Schur stable [1, 2]. Another occurrence for equation system (1.2) is that there exists and unique positive definite $H = H^*$ matrix satisfying the discrete-time Lyapunov matrix equation

$$A^*HA - H = -I. \tag{1.4}$$

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2. From Lyapunov matrix equations to linear algebraic equation

The *Kronecker product* of B and C denoted as $B\otimes C$ and the *Kronecker sum* of A and D, denoted by $A\oplus D$, is defined in [3] as the expression $A\oplus D = A\otimes I_S + I_N\otimes D$. The *VEC* operator is a vector valued function of the U matrix, denoted by $VEC(U)$ which represent a $N\cdot M$ dimensional vector defined in [3] as follows

$$VEC(U) = [u_{11}, u_{21}, \cdots, u_{N1}, u_{12}, \cdots, u_{NS}]^T.$$

A property of Kronecker product that, in [4] is

$$U = CXB^* \Leftrightarrow VEC(U) = (B\otimes C)VEC(X)$$

where $B\in M_S^Q(\mathbb{C})$, $C\in M_N^P(\mathbb{C})$, $D\in M_S(\mathbb{C})$, $U\in M_N^S(\mathbb{C})$ and $X\in M_P^Q(\mathbb{C})$.

2.1. Transformation for continuous-time Lyapunov matrix equation

When matrix equation (1.3) is considered,

$$\begin{aligned}
-I &= A^*HI + IHA\\
VEC(-I) &= VEC(A^*HI + IHA)\\
&= VEC(A^*HI) + VEC(IHA)\\
&= (I\otimes A^* + A^*\otimes I)VEC(H)\\
VEC(-I) &= (A^*\oplus A^*)VEC(H)
\end{aligned}$$

is obtained. $G\in M_{N^2}(\mathbb{C})$ and $G = (A^*\oplus A^*)$, $h = VEC(H)$ and $z = VEC(-I)$ is formed the

$$Gh = z \tag{2.1}$$

matrix-vector equation. This linear algebraic equation has a unique solution if G is non-singular. As well this linear algebraic equation is affair with continuous-time Lyapunov matrix equation. Let $G = (g_{ij})$, $g_{ij}\in\mathbb{C}$ and $A = (a_{ij})$, $a_{ij}\in\mathbb{C}$. The G matrix's computation algorithm entitled as *LyapunovC* is follows.

LyapunovC algorithm

$$g_{(i-1)N+k,\,(j-1)N+l} = \begin{cases} \overline{a_{kl} + a_{ij}} & i=j\,;\,k=l,\\ \overline{a_{lk}} & i=j\,;\,k\neq l,\\ \overline{a_{ji}} & i\neq j\,;\,k=l,\\ 0 & i\neq j\,;\,k\neq l,\end{cases}$$

for $i,j,k,l = 1,2,\cdots,N$.

2.2. Transformation for discrete-time Lyapunov matrix equation

If the matrix equation (1.4) is taken into account,

$$\begin{aligned}
VEC(-I) &= VEC(A^*HA - H)\\
&= VEC(A^*HA) - VEC(H)\\
&= (A^*\otimes A^*VEC(H)) - VEC(H)\\
VEC(-I) &= (A^*\otimes A^* - I)VEC(H)
\end{aligned}$$

is obtained. On this situation, the matrix vector-equation (2.1) is composed by $G = (A^*\otimes A^* - I)$, $h = VEC(H)$ and $z = VEC(-I)$. This equation is affair with discrete-time Lyapunov matrix equation and has a unique solution if G is invertible. The matrix G computation algorithm entitled as *LyapunovD* is follows.

LyapunovD algorithm

$$g_{(i-1)N+k,\,(j-1)N+l} = \begin{cases} \overline{a_{kl}\,a_{ij}} - 1 & i=j\,;\,k=l,\\ \overline{a_{lk}\,a_{ji}} & i\neq j\,;\,k\neq l,\end{cases}$$

for $i,j,k,l = 1,2,\cdots,N$.

3. Solving the $Gh = z$ linear algebraic equation

The equation (2.1) may be solved by varied methods. Iterative decreasing dimension method(IDDM) is one of them which is decreases by one dimension at every step for get to the solution without any pre-processing. This method and the algorithm that processes this method is given in detail in [5, 6]. As synopsis, framework computation of this method has given with equation (3.1) by [5, 6].

$$h = \sum_{k=1}^{N^2} \left(\prod_{l=1}^{k-1} \widehat{R}^{(l)} \right) h_0^{(k)} \tag{3.1}$$

$h_0^{(k)}$ is a special solution that $h_0^{(k)} = \begin{pmatrix} 0\cdots 0 & \dfrac{z_1^{(k)}}{g_{1s}^{(k)}} & 0\cdots 0 \end{pmatrix}^T$, where $g_{1s}^{(k)}$ which is the first non-zero elements of first row of matrix $G^{(k)}$. $G^{(k)}$ and $z^{(k)}$ are reduced matrix and vectors, of equation (2.1).

$$G^{(k)} = \begin{cases} G & \text{if } k=1, \\ G_2^{(k-1)} \widehat{R}^{(k-1)} & \text{if } k \neq 1 \end{cases} \quad ; \quad G_2^{(k-1)} = g_{ij}^{(k-1)} \quad \begin{matrix} j = 1, \cdots, N^2-k, \\ i = 2, \cdots, N^2-k; \end{matrix}$$

$$z^{(k)} = \begin{cases} z & \text{if } k=1, \\ v^{(k-1)} - G_2^{(k-1)} h_0^{(k-1)} & \text{if } k \neq 1 \end{cases} \quad ; \quad v^{(k-1)} = z_j^{(k-1)}, j = 2, \cdots, N^2-k.$$

$\widehat{R}^{(k)} \in M_{(N^2-k)}^{(N^2-k+1)}(\mathbb{C})$ are matrices which are composed of the base vectors of solution space as

$$\widehat{R}^{(k)} = \left(\begin{array}{c:c} I & 0 \\ \hdashline 0 & r^{(k)} \\ \hdashline 0 & I \end{array} \right) ; \ \widehat{r}_{s,s+j-1}^{(k)} = r_j^{(k)} = -\frac{g_{1j}^{(k)}}{g_{1s}^{(k)}}; \ \ j = 1, 2, \cdots, N^2-k-s+1.$$

In the condition of $s = N^2-k+1$, particular cases of $\widehat{R}^{(k)} = \left(\begin{array}{c} I \\ \hdashline 0 \end{array} \right)$ are evident. This method has been arranged for equation (2.1) and has been prepared for computer aided computation. The algorithm named *IDDMforLyapunov* that calculates matrix H with IDDM has been given follows.

IDDMforLyapunov algorithm

`Step 1. Settlementing of input matrix;` $G^{(1)} = G$.

`Step 2. Checking input matrix at initial situation;`

$s = min(j)$ as provided by $g_{1j}^{(1)} \neq 0$, for $j = 1, 2, \cdots, N^2$, if s is not available,there is no exist or unique solution for the G, the algorithm is terminated.

`Step 3. Establishing initial values;`

$$z_{(i-1)N+j}^{(1)} = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \text{for } i, j = 1, 2, \cdots, N,$$

$$h_{ij}^{(1)} = \begin{cases} \eta^{(1)} = \dfrac{1}{g_{1s}^{(1)}} & \text{if } N(j-1)+i = s, \\ 0 & \text{if } j \neq s, \end{cases} \quad \text{for } i, j = 1, 2, \cdots, N,$$

$$r_j^{(1)} = -\frac{g_{1,s+j}^{(1)}}{g_{1s}^{(1)}}, \qquad \text{for } j = 1, 2, \cdots, N^2-s,$$

$$\widehat{R}^{(1)} = \begin{cases} \widehat{r}_{jj}^{(1)} = 1 & \text{if } j < s, \\ \widehat{r}_{j+1,j}^{(1)} = 1 & \text{if } j \geq s, \\ \widehat{r}_{sj}^{(1)} = r_{j-s+1}^{(1)} & \text{if } j \geq s, \end{cases} \quad \text{for } j = 1, 2, \cdots, N^2-1.$$

`Step 4. Iterative computation of solution of matrix` H;

Overall iteration of substeps on hereinafter is continuing for $k = 2,3,\cdots,N^2-1$;

`Step 4.1. Dimension decreasing for vector` z `and matrix` G;

$$z_i^{(k)} = z_{i+1}^{(k-1)} - g_{i+1,s}^{(k-1)} \cdot \eta^{(k-1)},$$
$$g_{ij}^{(k)} = \begin{cases} g_{i+1,j}^{(k-1)} & \text{if } j < s, \\ g_{i+1,s}^{(k-1)} \cdot r_{j-s+1}^{(k-1)} + g_{i+1,j+1}^{(k-1)} & \text{if } j \geq s, \end{cases} \qquad \text{for } i,j = 1,2,\cdots,N^2-k+1.$$

`Step 4.2. Checking reduced matrix;`

$s = min(j)$ as provided by $g_{1,j}^{(k)} \neq 0,$ for $j = 1,2,\cdots,N^2-k+1,$ if s is not available, the algorithm is terminated.

`Step 4.3. Accumulating the solution;`

$$\eta^{(k)} = \frac{z_1^{(k)}}{g_{1s}^{(k)}}, h_{ij}^{(k)} = h_{ij}^{(k-1)} + \widehat{r}_{N(j-1)+i,s}^{(k-1)} \cdot \eta^{(k)}, \qquad \text{for } j = 1,2,\cdots,N.$$

`Step 4.4. Successive multiplication of` $\widehat{R}$;

$$r_j^{(k)} = -\frac{g_{1,s+j}^{(k)}}{g_{1s}^{(k)}}, \qquad \text{for } j = 1,2,\cdots,N^2-k-s+1,$$
$$\widehat{r}_{ij}^{(k)} = \begin{cases} \widehat{r}_{ij}^{(k-1)} & \text{if } j < s, \\ \widehat{r}_{is}^{(k-1)} \cdot r_{j-s+1}^{(k)} + \widehat{r}_{i,j+1}^{(k-1)} & \text{if } j \geq s, \end{cases} \qquad \begin{matrix} \text{for } i = 1,2,\cdots,N^2, \\ \text{for } j = 1,2,\cdots,N^2-k. \end{matrix}$$

`Step 5. Computation of IDDMforLyapunov algorithm is completed with the output solution matrix` H.

Thus, if the *IDDMforLyapunov* algorithm gives a solution, this solution require be a symmetric positive defined matrix.

4. Maple procedures

The *ComputeSystem* main procedure calls the some procedures according to the sequence. These procedures are executed the algorithms defined previous sections. This procedure takes four parameter. The first parameter named *ErrTolerance* is a small number that describes the tolerance of comparison with respect to zero in *Step 2* and *Step 4.2* in the $Gh = z$ computation. The second parameter, *TestTolerance*, is a small number used to corroborate that a unique H solution matrix was symmetrically and positively defined. Distinctly, this tolerance value was defined an acceptability limit by any one for special purpose. The third parameter, is allows the choice either continuous time system or discrete-time system computation. At last, fourth parameter is being the coefficient matrix that belong the system (1.1) or (1.2).

```
> restart
> with (Linear Algebra, Dimension, Eigenvalues)
> Continuous :: integer_1 := 0; Discrete :: integer_1 := 1;

> LyapunovC := proc(A :: Matrix) :: Matrix
local i, j, k, l, N, G; N := Dimension(A);
G := Matrix(N_1 · N_2, N_1 · N_2, datatype = complex_8);
for i from 1 to N_1 do for j from 1 to N_2 do
for k from 1 to N_1 do for l from 1 to N_2 do
if i ≠ j and k ≠ l then next fi: if i = j and k = l then
G_{(i−1)·N_1+k,(j−1)·N_2+l} := conjugate(A_{k,l} + A_{i,j}); next; fi :
if i = j then G_{(i−1)·N_1+k,(j−1)·N_2+l} := conjugate(A_{l,k});next;fi :
G_{(i−1)·N_1+k,(j−1)·N_2+l} := conjugate(A_{j,i});
od: od: od: od:return G; end proc:

> LyapunovD := proc(A :: Matrix) :: Matrix
local i, j, k, l, N, G; N := Dimension(A);
G := Matrix(N_1 · N_2, N_1 · N_2, datatype = complex_8);
for i from 1 to N_1 do for j from 1 to N_2 do
```

```
for k from 1 to N_1 do for l from 1 to N_2 do
if i = j and k = l then
G_{(i-1)·N_1+k,(j-1)·N_2+l} := conjugate(A_{i,j} · A_{k,l}) - 1 else
G_{(i-1)·N_1+k,(j-1)·N_2+l} := conjugate(A_{j,i} · A_{l,k}) fi:
od: od: od: od:return G; end proc:

> IDDMforLyapunov := proc(W :: Matrix) :: Matrix
global ConclusionSituation; local i, j, k, η, r, z, H, G, R, Temp
DimBase, DimVec, DimMat, RowNumber, CoulumnNumber,
IndexNONZERO; ConclusionSituation :=
"Exist and unique solution been computed that provide the
Lyapunov equation."; DimMat := Dimension(W);
RowNumber := DimMat_1; CoulumnNumber := DimMat_2;
DimVec := DimMat_1; DimBase := sqrt(DimVec);
z := Vector(DimVec, datatype = complex_8);
for i from 1 to DimBase do z_{DimBase·(i-1)+i} := -1 od:
G := Matrix(DimMat, datatype = complex_8);
for i from 1 to DimMat_1 do for j from 1 to DimMat_2 do
G_{i,j} := W_{i,j}; od: od:
H := Matrix(DimBase, DimBase datatype = complex_8);
R := Matrix(DimMat_1, DimMat_2 - 1, datatype = complex_8);
for IndexNONZERO from 1 to DimMat_2 do
if |G_{1,IndexNONZERO}| > ErrTolerance then break fi: od:
if IndexNONZERO > DimMat_2 then ConclusionSituation :=
"Has no unique solution so system can't provide the
 Lyapunov equation!"; return Matrix([0]); fi:
η := z_1 · G^{-1}_{1,IndexNONZERO};
for i from 1 to DimBase do for j from 1 to DimBase do
if DimBase · (i-1) + j = IndexNONZERO then H_{i,j} := η;fi:od:od:
r := Vector(DimMat_2 - IndexNONZERO, datatype = complex_8);
for j from 1 to DimMat_2 - IndexNONZERO do
r_j := -G_{1,IndexNONZERO+j} · G^{-1}_{1,IndexNONZERO} od:
for j from 1 to DimMat_2 - 1 do if j < IndexNONZERO then
R_{j,j} := 1 else R_{IndexNONZERO,j} := r_{j-IndexNONZERO+1};R_{j+1,j} := 1;fi:od:
Temp := Vector(DimMat_1, datatype = complex_8);
for k from 1 to DimMat_1 - 1 do
for i from 1 to RowNumber - 1 do z_i := z_{i+1} - G_{i+1,IndexNONZERO} · η od:
for i from 1 to RowNumber - 1 do for j from 1 to CoulumnNumber - 1 do
if j < IndexNONZERO then G_{i,j} := G_{i+1,j} else
G_{i,j} := G_{i+1,indexNONZERO} · r_{j-IndexNONZERO+1} + G_{i+1,j+1}fi:od:od:
RowNumber := RowNumber - 1;CoulumnNumber := CoulumnNumber - 1;
for IndexNONZERO from 1 to CoulumnNumber do
if |G_{1,IndexNONZERO}| > ErrTolerance then break fi: od:
if IndexNONZERO > CoulumnNumber then ConclusionSituation :=
"Has no unique solution so that system can't provide the
 Lyapunov equation!"; return Matrix([0]); fi:
η := z_1 · G^{-1}_{1,IndexNONZERO};
for i from 1 to DimBase do for j from 1 to DimBase do
H_{i,j} := H_{i,j} + R_{DimBase·(i-1)+j,IndexNONZERO} · η od: od:
r := Vector(CoulumnNumber - IndexNONZERO, datatype = complex_8);
for j from 1 to CoulumnNumber - IndexNONZERO do
r_j := -G_{1,IndexNONZERO+j} · G^{-1}_{1,IndexNONZERO} od:
for i from 1 to DimMat_1 do Temp_i := R_{i,IndexNONZERO} od:
for j from IndexNONZERO to CoulumnNumber - 1 do
for i from 1 to DimMat_1 do R_{i,j} := Temp_i · r_{j-IndexNONZERO+1} + R_{i,j+1}
od: od: od: return H; end proc:

> IsCorroborate := proc(H :: Matrix) :: boolean
global ValidationTest; local i, j, N, EigVal; N := Dimension(H);
for i from 1 to N_1 - 1 do for j from i + 1 to N_2 do
if |H_{j,i} - H_{i,j}| < TestTolerance then next fi:
```

$ValidationTest$:= "Symmetry situation is out of accepted tolerance value!";return $false$; od: od: $EigVal$:= Eigenvalues(H);
for i from 1 to N_1 do if $\Re(EigVal_i) \geq TestTolerance$ then next fi:
$ValidationTest$:= "Positivity of solution matrix is out of accepted tolerance value!";return $false$; od: $ValidationTest$:=
"Both situations that symmetry and positivity of solution matrix been in accepted tolerance range."; return $true$; end proc:

> ComputeSystem := proc
($argErrTolerance$ :: float, $argTestTolerance$ :: float,
$EqType$ :: integer_1, A :: Matrix) :: boolean
global $ErrTolerance, TestTolerance, boolResult, txtResult$; local G, H;
if $argErrTolerance < 10.^{-13}$ then $ErrTolerance := 10.^{-13}$
elif $argErrTolerance > 10.^{-4}$ then $ErrTolerance := 10.^{-4}$
else $ErrTolerance := argErrTolerance$ fi:
if $argTestTolerance < 10.^{-13}$ then $TestTolerance := 10.^{-13}$
elif $argTestTolerance > 10.^{-4}$ then $TestTolerance := 10.^{-4}$
else $TestTolerance := argTestTolerance$ fi:
if $EqType = 1$ then $G :=$ LyapunovD(A) elif $EqType = 0$ then
$G :=$ LyapunovC(A) fi: print($'G' = G$);
$H :=$ IDDMforLyapunov(G); print($ConclusionSituation$);
if $H = [0]$ then return $false$ fi: print($'H' = H$);
$boolResult$:= IsCorroborate(H);
print($ValidationTest$, tolerance value is = $TestTolerance$);
print("The related system is asymptotic stable.");
return $boolResult$; end proc:

Example 4.1.
$> A := Matrix(2, 2, [[1, -3], [2, -4]], datatype = complex_8)$

$$A := \begin{bmatrix} 1.0+0.I & -3.0+0.I \\ 2.0+0.I & -4.0+0.I \end{bmatrix}$$

$> ComputeSystem(10^{-13}, 10^{-10}, continuous, A)$

$$G = \begin{bmatrix} 2.0+0.I & 2.0+0.I & 2.0+0.I & 0.+0.I \\ -3.0+0.I & -3.0+0.I & 0.+0.I & 2.0+0.I \\ -3.0+0.I & 0+0.I & -3.0+0.I & 2.0+0.I \\ 0.+0.I & -3.0+0.I & -3.0+0.I & -8.0+0.I \end{bmatrix}$$

"Exist and unique solution been computed that provide the continuous-time Lyapunov matrix equation."

$$H = \begin{bmatrix} 1.83333333299999990+0.I & -1.16666666700000010+0.I \\ -1.16666666700000010+0.I & 1.0+0.I \end{bmatrix}$$

"Both situations that symmetry and positivity of solution matrixbeen in accepted tolerance range.",

tolerace value is $1.000000000 \cdot 10^{-10}$, *"The related system is asymptotic stable."*

$> ComputeSystem(10^{-13}, 10^{-10}, discrete, A)$

$$G = \begin{bmatrix} 0.0+0.I & 2.0+0.I & 2.0+0.I & 4.0+0.I \\ -3.0+0.I & -5.0+0.I & -6.0+0.I & -8.0+0.I \\ -3.0+0.I & -6.0+0.I & -5.0+0.I & -8.0+0.I \\ 9.0+0.I & 12.0+0.I & 12.0+0.I & 15.0+0.I \end{bmatrix}$$

"Has no unique solution so that can't provide the discrete-time Lyapunov matrix equation!"

Example 4.2.
$> A := Matrix(2, 2, [[0, 0], [0, 0]], datatype = complex_8)$

$$A := \begin{bmatrix} 0.0+0.I & 0.0+0.I \\ 0.0+0.I & 0.0+0.I \end{bmatrix}$$

$> \mathsf{ComputeSystem}(10^{-13}, 10^{-10}, discrete, A)$

$$G = \begin{bmatrix} -1.0+0.I & 0.0+0.I & 0.0+0.I & 0.0+0.I \\ 0.0+0.I & -1.0+0.I & 0.0+0.I & 0.0+0.I \\ 0.0+0.I & 0.0+0.I & -1.0+0.I & 0.0+0.I \\ 0.0+0.I & 0.0+0.I & 0.0+0.I & -1.0+0.I \end{bmatrix}$$

"Exist and unique solution been computed that provide the discrete-time Lyapunov matrix equation."

$$H = \begin{bmatrix} 1.0+0.I & 0.0+0.I \\ 0.0+0.I & 1.0+0.I \end{bmatrix}$$

"Both situations that symmetry and positivity of solution matrix been in accepted tolerance range.",

tolerace value is $1.000000000 \cdot 10^{-10}$, *"The related system is asymptotic stable."*

5. Conclusion

On discrete set of double precision computer numbers, γ the base of number system, ε_0 the minimal positive number, ε_∞ the maximal number, and ε_1 is the step of computer numbers on the interval from 1 to γ. Thus, let be $v \in [-\varepsilon_\infty, -\varepsilon_0] \cup [\varepsilon_0, \varepsilon_\infty]$, any memorizable double precision computer number is $v_{dp} = v(1+\alpha)+\beta$, $|v - v_{dp}| \leq \varepsilon_1 |v| + \varepsilon_0$, $|\alpha| \leq \varepsilon_1$, $|\beta| \leq \varepsilon_0$, $\alpha \cdot \beta = 0$ (see for example[1] and [2]). The selected tolerance values from a particular interval $[10^{-13}, 10^{-4}]$ were used in the evaluation of the inequalities. The lower bound of the interval was chosen to be a larger number than ε_1, depending on the ε_1 which determines the size of computation error. *TestTolerance* should always be larger than *ErrTolerance* so that the assessment be efficient.

DDM which is inspiration for IDDM is described as type of Schur complement domain decomposition method in [7]. Decreasing dimension method (DDM) divides a large system into two smaller systems to be solved separately. To give a general understanding of the computational quantum of DDM was used DDM and Gaussian elimination method to solve a system of n dimension linear algebraic equations in [7]. The explanation in [7] tell us that the computational quantum of the two methods are approximately the same to solve the system whose coefficient matrix is full, but the quantum of DDM is much less than that of Gaussian elimination to solve band matrix equations. IDDM have made an improvement by modifying the method in [7]. Notwithstanding DDM needed some pre-processing situations, without any pre-processing IDDM decreases the dimension of the linear systems, by one order in every step (see for example ([6]). By its very nature, IDDM performs division by a number away from 0 bound up with the *ErrTolerance* value. Thus inherently prevents the error of division by 0.

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Generalized Cylinder with Geodesic and Line of Curvature Parameterizations

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Abstract

Constructing a surface with geodesic or line of curvature parameterization is an important problem in many practical applications. The present paper aims to design a generalized cylinder that is parametrized along the geodesics and lines of curvature curves in Euclidean 3- space. The main results show that the generalized cylinder with geodesic or line of curvature parameterization is a rectifying cylinder or a right cylinder respectively.

1. Introduction

A generalized cylinder is constructed by the constant motion of a straight line called the ruling through a given curve called the base curve. The generalized cylinders are a class of developable ruled surfaces that have no singularities points and can be produced from paper or sheet metal with no distortion. For this construction, the generalized cylinder has been investigated as a basic modeling surface in various fields of science including geometric modeling, computer graphic, architectural designing and manufacturing [1]-[4].

Geodesic and line of curvature are characteristic curves that lie on the surface. The geodesic curve gives the shortest path between two given points on curved spaces. A curve is a line of curvature if its direction always points in the principal directions, i.e., the direction in which the surface bends extremaly. Geodesics and lines of curvature have been used in shape analysis, therefore, the problems of computing and visualizing them on the surface have been investigated [5]-[7]. The rulings of the generalized cylinder are geodesics and lines of curvature.

Surface parameterization is the process of mapping a surface to a planar region [8]. Extracting and transferring the geometric information from shapes or between them depends on the parameterizations that are used as coordinate systems on the shapes. Several types of parameterizations are constructed on a surface and differ by their characterizing properties. During the parameterization some geometric quantities can be lost or distorted, therefore, designing and choosing the suitable parameterizations that minimize, maximize or preserve the desired geometrical properties is an interesting problem and hot topic in many areas of applications such as computer graphic [9]-[11], geometric modeling [12], and robot motion planning [13].

A parameterization on a surface is said to be geodesic or line of curvature if the two families of parametric curves are geodesics or lines of curvature. Parametrizations of smooth surfaces by curvature line exist on non-umbilical points as orthogonal curves on the surface. Geodesic and line of curvature parameterizations mean that the shape is charted or covered by two families of lines that are characterized by special directions. Parameterizing the surface along their geodesics or lines of curvature are widely investigated in many areas of sciences such as CAGD [14]-[16], surfaces motions [17, 18], architectural design

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[19, 20], and discrete differential geometry [21]-[23].

The main goal of this paper is to design a generalized cylinder whose parametric curves are geodesics or lines of curvature in Euclidean 3-space. A generalized cylinder has two families of parametric curves, rulings, and base curves. It is well known that the rulings are geodesics and lines of curvature on a generalized cylinder. Consequently, throughout this paper, our focus lies on the family of base curves. The generalized cylinders are a class of ruled surfaces, therefore, we start from a ruled surface parametrization, then with additional three conditions called the cylindrical conditions, the generalized cylinder is defined. After that, under some geometric constraints, we obtain the resulting cylinder that is parameterized by geodesic or line of curvature base curves. The main results show that the generalized cylinder with geodesic or line of curvature parameterization is a rectifying cylinder or a right cylinder respectively. In this article, we used the same approach that was used in [24] and with the developable surface.

The rest of this paper is organized as follows: In section 2, some basic notations, facts, and definitions of the space curve, regular surface, and special curves in Euclidean 3-space are reviewed. The main results are studied in section 3, where the generalized cylinder is defined in the first subsection, then the generalized cylinder with geodesic and line of curvature parameterizations are constructed subsequently in the other two subsections respectively. Examples to illustrate the main results are presented in section 4. Finally, the conclusion is given in section 5.

2. Preliminaries

This section introduces some basic concepts on the classical differential geometry of space curves and surfaces in three-dimensional Euclidean space. More details can be found in such standard references as [25]-[27].

2.1. Curves in Euclidean 3-space

A smooth space curve in 3-dimensional Euclidean space is parameterized by a map $\gamma : I \subseteq \mathbb{R} \to E^3$, γ is called a regular curve if $\gamma' \neq 0$ for every point of an interval $I \subseteq \mathbb{R}$, and if $|\gamma'(s)| = 1$ where $|\gamma'(s)| = \sqrt{\langle \gamma'(s), \gamma'(s) \rangle}$, then γ is said to be of unit speed (or parameterized by arc-length s). For a unit speed regular curve $\gamma(s)$ in E^3, the unit tangent vector $t(s)$ of γ at $\gamma(s)$ is given by $t(s) = \gamma'(s)$. If $\gamma''(s) \neq 0$, the unit principal normal vector $n(s)$ of the curve at $\gamma(s)$ is given by $n(s) = \frac{\gamma''(s)}{\|\gamma''\|}$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of γ at $\gamma(s)$. For each point of $\gamma(s)$ where $\gamma''(s) \neq 0$, we associate the Serret-Frenet frame $\{t, n, b\}$ along the curve γ. As the parameter s traces out the curve, the Serret-Frenet frame moves along γ and satisfies the following Frenet-Serret formula :

$$\begin{aligned} t'(s) &= \kappa(s) n(s), \\ n'(s) &= -\kappa(s) t(s) + \tau b(s), \\ b'(s) &= -\tau(s) n(s), \end{aligned} \tag{2.1}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and torsion functions. When the point moves along the unit speed curve with non-vanishing curvature and torsion, the Serret-Frenet frame $\{t, n, b\}$ is drawn to the curve at each position of the moving point, this motion consists of translation with rotation and described by the following Darboux vector

$$\omega = \tau t + \kappa b$$

where the unit Darboux vector is given by

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b \tag{2.2}$$

Direction of Darboux vector is the direction of rotational axis and its magnitude gives the angular velocity of rotation. A necessary and sufficient condition that a curve be of constant slope (or general helix) is that the ratio of torsion to curvature is constant ($\frac{\tau}{\kappa} = c$). The general helix lies on a general cylinder and also known as a cylindrical helix. The circular helix (a helix on a circular cylinder) is a special helix with both of $\kappa(s) \neq 0$ and $\tau(s)$ are constants. The Darboux vector is constant for circular helix. For the cylindrical helix, the unit Darboux vector is constant as following

$$\hat{\omega} = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} t + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} b = \frac{c}{\sqrt{c^2 + 1}} t + \frac{1}{\sqrt{c^2 + 1}} b. \tag{2.3}$$

2.2. Surfaces in Euclidean 3-space

A smooth surface in 3-dimensional Euclidean space is parameterized by a map $X(u, v) : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. The variables (u, v) are called the (curvilinear) coordinates on the surface, the two families of u-curves ($v = const$), and v-curves ($u = const$), are called the parametric curves (or coordinate curves). Their directions are defined by the tangents vectors X_u and X_v respectively. The surface $X(u, v)$ is called a regular if the condition $X_u \times X_v \neq 0$ is satisfied for all points, that means the vectors X_u and X_v

do not vanish and have different directions. Consequently, the surface normal is defined at every point on the regular surface as a unit vector on the tangent plane and given by

$$N(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|}. \tag{2.4}$$

The first and second fundamental form of the parameterized regular surface are given by

$$I = Edu^2 + 2Fdudv + Gdv^2, \quad II = edu^2 + 2fdudv + gdv^2$$

where their coefficients can be calculated respectively as

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, e = \langle N, X_{uu} \rangle, f = \langle N, X_{uv} \rangle, and g = \langle N, X_{vv} \rangle.$$

The fundamental quantities I and II are important tools to describe the intrinsic and extrinsic geometry of surface. In particular, type of the parametric curves and their characteristics properties are described by the coefficients of the fundamental quantities I and II. For example, the coordinate curves are orthogonal if $F = 0$, conjugate if $f = 0$, and lines of curvature if satisfy both conditions.

Theorem 2.1. *[28] A necessary and sufficient condition for the coordinate curves of a parametrization to be lines of curvature in a neighborhood of a nonumbilical point is that $F = f = 0$.*

For a regular curve on a surface, there exists another frame $\{t(s), g(s), N(s)\}$ which is called Darboux frame. In this frame $t(s)$ is the unit tangent of the curve, $N(s)$ is the unit normal of the surface and g is a unit vector given by $g = N \times t$. The relations between Frenet frame and Darboux frame can be given by the following matrix representation

$$\begin{pmatrix} t \\ g \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{2.5}$$

A unit-speed curve on a surface is a geodesic if and only if the principal normal n to the curve and the surface normal N are parallel to each other at any point on the curve. Equivalently, a curve $\gamma(s)$ on the surface is a geodesic provided its acceleration vector $\gamma''(s)$ is always normal to the surface, i.e.

$$\gamma''(s) \times N = 0. \tag{2.6}$$

3. Generalized cylinder with geodesic and line of curvature parameterizations

This section is the main part of this paper, it consists of three subsections that are devoted to defining and covering the generalized cylinder with geodesics and lines of curvature parametrizations. A generalized cylinder has two families of parametric curves, rulings and base curves. It is well known that the rulings are geodesics and lines of curvature on a generalized cylinder. Consequently, this section is devoted to providing the necessary and sufficient conditions for the base curves to be geodesics or lines of curvature. We show that the generalized cylinder with geodesic parametrization is a rectifying cylinder, and the generalized cylinder with a line of curvature parametrization is a right cylinder. The following first subsection aims to parametrize the generalized cylinder, we start from the ruled parametrization, and with the cylindrical condition that is described by the constrains three equations that are must be satisfied, we obtain the cylindrical parametrization.

3.1. Generalized cylinder

A generalized cylinder is generated by a constant moving of a straight line on a given curve and defined by the following ruled parametrization

$$X(s,v) = \gamma(s) + vD(s), 0 \le s \le \ell, \, v \in \mathbb{R}, \text{ where } D'(s) = 0. \tag{3.1}$$

A unit regular curve $\gamma(s)$ is called a base curve, and the line passing through $\gamma(s)$ that is parallel to $D(s)$ is called the ruling. $D(s)$ is a unit director vector field that gives the direction of the ruling, $D'(s) = 0$ is the cylindrical condition which means that the ruling moves in a constant direction. The unit normal vector field (shortly surface normal) of the generalized cylinder is defined by using (2.4) as

$$N(s,v) = \frac{X_s \times X_v}{|X_s \times X_v|} = \frac{(\gamma' \times D) + v(D' \times D)}{|(\gamma' \times D) + v(D' \times D)|} = \frac{\gamma' \times D}{|\gamma' \times D|}.$$

$D(s)$ is a unit vector field that lies in the space formed by the frame $\{t,n,b\}$ and can be written using (2.5) as following

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)g(s), \text{ where } g(s) = \cos\phi(s)n(s) + \sin\phi(s)b(s).$$

Therefore $D(s)$ can be decomposed as the following [29]

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)), \tag{3.2}$$

where $\theta(s)$ and $\phi(s)$ are two scalar functions called the first and second angular functions [30]. The derivative of $D(s)$ is given by

$$D'(s) = -\sin\theta[\kappa\cos\phi + \frac{d\theta}{ds}]t + [\cos\theta(\kappa + \cos\phi\frac{d\theta}{ds}) - \sin\theta\sin\phi(s)(\frac{d\phi}{ds} + \tau)]n + [\sin\phi\cos\theta\frac{d\theta}{ds} + \sin\theta\cos\phi(\frac{d\phi}{ds} + \tau)]b.$$

Definition 3.1. *The ruled parametrization with base curve $\gamma(s)$ and a unit director vector D(s) (3.2) is defined by*

$$X(s,v) \quad = \gamma(s) + vD(s), 0 \leq s \leq L, \, v \in \mathbb{R}, \tag{3.3}$$

where

$$D(s) = \cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s)).$$

In the following theorem, we give the necessary and sufficient conditions to construct a generalized cylinder parametrization from a ruled parametrization (3.3), we call them the cylindrical conditions.

Theorem 3.2. *The ruled parametrization (3.1) is a generalized cylinder if and only if the following conditions are satisfied*

$$\begin{aligned} \kappa\cos\phi + \frac{d\theta}{ds} &= 0, \\ \cos\theta(\kappa + \cos\phi\frac{d\theta}{ds}) - \sin\theta\sin\phi(\frac{d\phi}{ds} + \tau) &= 0, \\ \sin\phi\cos\theta\frac{d\theta}{ds} + \sin\theta\cos\phi(\frac{d\phi}{ds} + \tau) &= 0. \end{aligned} \tag{3.4}$$

Definition 3.3. *The generalized cylinder with base curve $\gamma(s)$ and a unit director vector D(s) (3.2) is parameterized by*

$$X(s,v) = \gamma(s) + v[\cos\theta(s)t(s) + \sin\theta(s)(\cos\phi(s)n(s) + \sin\phi(s)b(s))], 0 \leq s \leq L, \, v \in \mathbb{R}, \tag{3.5}$$

where

$$\kappa\cos\phi + \frac{d\theta}{ds} = 0, \cos\theta(\kappa + \cos\phi\frac{d\theta}{ds}) - \sin\theta\sin\phi(\frac{d\phi}{ds} + \tau) = 0, \textit{ and } \sin\phi\cos\theta\frac{d\theta}{ds} + \sin\theta\cos\phi(\frac{d\phi}{ds} + \tau) = 0.$$

The first and second derivatives of the generalized cylinder parameterized by (3.5) are given in the following equations

$$X_s = t(s), X_{ss} = \kappa(s)n(s), X_{sv} = 0, X_v = D(s), X_{vv} = 0. \tag{3.6}$$

The inner and cross products of the tangents vectors X_s and X_v are given by

$$\begin{aligned} &\langle X_s, X_v \rangle = \cos\theta(s), \\ &X_s \times X_v = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \end{aligned}$$

By using (2.4), the unit normal of the generalized cylinder (3.5) is defined everywhere and given by the following

$$N(s,v) = -\sin\phi(s)n(s) + \cos\phi(s)b(s). \tag{3.7}$$

The main result of this paper is the following theorem which is proved in the next subsections.

Theorem 3.4. *Let $X(s,v) = \gamma(s) + vD(s), 0 \leq s \leq L, v \in \mathbb{R}$ be a generalized cylinder, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature , $D(s)$ is a unit director vector defined by (3.2) satisfying $D'(s) = 0$. Then the generalized cylinder with geodesic or line of curvature parameterization is a rectifying cylinder or a right cylinder respectively.*

3.2. Generalized cylinder with geodesic parameterization

Theorem 3.5. *All base curves of the generalized cylinder parameterized by (3.5) are geodesics if and only if the following conditions are satisfied.*

$$\cos\phi(s)=0, \qquad \frac{d\theta}{ds}=0, \qquad \cos\theta(s)\kappa(s)-\sin\theta(s)\tau=0. \tag{3.8}$$

Proof. According to (2.6), the base curves on a generalized cylinder (3.5) are geodesics if and only if their acceleration vector X_{ss} is normal to the surface, or equivalently $N(s,v)\times X_{ss}=0$. From (3.6) and (3.7), it follows that $N(s,v)\times X_{ss}=-\cos\phi\, t(s)$, the geodesic condition $N(s,v)\times X_{ss}=0$ is satisfied if and only if $\cos\phi(s)=0$ which is the first condition of (3.8). By substitution it in the cylindrical conditions (3.4), we get the other conditions of (3.8). □

Definition 3.6. *A generalized cylinder with geodesic base curves is defined by*

$$X(s,v)=\gamma(s)+v[\cos\theta(s)t(s)+\sin\theta(s)b(s)],\, 0\le s\le L,\, v\in\mathbb{R}, \tag{3.9}$$

$$\tau(s)\sin\theta(s)-\kappa(s)\cos\theta(s)=0, \textit{ and } \quad \theta'(s)=0.$$

Proposition 3.7. *[24] Suppose that $D(s)=\cos\theta(s)t(s)+\sin\theta(s)b(s)$ is a unit rectifying vector defined along a unit speed curve $\gamma(s)$ with non vanishing curvature and torsion, then $D(s)$ is a unit Darboux vector field if and only if $\kappa\cos\theta-\tau\sin\theta=0$.*

Proof. Let $D(s)=\cos\theta(s)t(s)+\sin\theta(s)b(s)$ be a unit Darboux vector. From (2.2),

$$\cos\theta=\frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \quad \sin\theta(s)=\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \quad \textit{and} \quad \cot\theta=\frac{\tau}{\kappa}.$$

This implies that $\kappa\cos\theta-\tau\sin\theta=0$, and vice versa. □

Definition 3.8. *A generalized cylinder with geodesic base curves is defined by*

$$X(s,v)=\gamma(s)+vD(s), 0\le s\le L, v\in\mathbb{R}$$

where

$$D(s)=\frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s)+\frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s), \quad D'(s)=0.$$

As discussed in (2.3), the condition for unit Darboux vector to be constant is equivalent to the base curve is a helix. As well known, the base curve and director vector are responsible to build the generalized cylinder, so the following theorem gives the conditions that can be applied on the base curve and director vector at the same time to generate a generalized cylinder with geodesic base curves.

Theorem 3.9. *Let $X(s,v)=\gamma(s)+vD(s), 0\le s\le L, v\in\mathbb{R}$ be a generalized cylinder, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature and torsion, $D(s)$ is a unit director vector defined by (3.2) satisfying D'(s) =0. Then every ruling is a geodesic and the base curves are geodesics if and only if $\gamma(s)$ is a helix and $D(s)$ is a unit Darboux vector.*

Definition 3.10. *A generalized cylinder with geodesic parameterization is defined by*

$$X(s,v)=\gamma(s)+vD(s), 0\le s\le L, v\in\mathbb{R}, \tag{3.10}$$

where

$$D(s)=\frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s)+\frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s), \quad \textit{and } \gamma(s) \textit{ is a helix.}$$

The developable ruled surface whose director vector is a unit Darboux vector has been studied by many researchers and it has been called the rectifying developable surface, (see, e.g., [31]). The generalized cylinder defined by (3.10) is a special case where the unit Darboux vector is a constant and we call it the rectifying cylinder. The base curve is a geodesic on its rectifying developable is a classical result has been stated in the classical differential geometry books [26], but according to theorem (3.9) all base curves are geodesics on their rectifying cylinder.

Corollary 3.11. *A generalized cylinder with geodesic parameterization (3.10) is a rectifying cylinder.*

Theorem 3.12. *Among all generalized cylinders parameterized by (3.5), only the rectifying cylinder (3.10) can be equipped with geodesic parameterization.*

In the above definition (3.10) we remark that for the rectifying cylinder (3.10) whose parametric curves are geodesics, the base geodesic curves have the same curvature and torsion, and differ only by the rigid motion modeled by a constant unit Darboux vector with fixed direction and fixed angular velocity. Therefore, it is interesting to end this subsection with the following result

Corollary 3.13. *The geodesic parametric curves of the the rectifying cylinder (3.10) are lines and helices.*

3.3. Generalized cylinder with line of curvature parameterization

Theorem 3.14. *All base curves of the generalized cylinder parameterized by (3.5) are lines of curvature if and only if the following conditions are satisfied*

$$\cos\theta(s)=0, \qquad \cos\phi(s)=0, \qquad \tau(s)=0. \tag{3.11}$$

Proof. By Theorem (2.1), the base curves on a generalized cylinder (3.5) are lines of curvature if and only if $F=f=0$. From (3.6) and (3.7), $f=\langle N, X_{vs}\rangle=0$ is satisfied without further condition, and $F=\langle X_s, X_v\rangle=\cos\theta$, therefore, $F=0$ if and only if $\cos\theta=0$ which is the first condition of (3.11). By substitution it in the cylindrical conditions (3.4), we get the other conditions of (3.11). □

Definition 3.15. *A generalized cylinder with line of curvature base curves is defined by*

$$X(s,v)=\gamma(s)+vb(s), 0\leq s\leq L, v\in\mathbb{R}, \quad \textit{where} \quad \tau(s)=0.$$

The plane curve ($\tau(s)=0$) has no binormal unit vector $b(s)$, therefore, the binormal of plane curve coincides with the normal vector to the plane of the curve. Without loss in generality we may assume that the unit vector $\langle 0,0,1\rangle$ is the normal to the plane of planar curve $\gamma(s)$.

Theorem 3.16. *Let $X(s,v)=\gamma(s)+vD(s), 0\leq s\leq L, v\in\mathbb{R}$ be a generalized cylinder, where $\gamma(s)$ is a unit speed regular curve with non vanishing curvature, $D(s)$ is a unit director vector defined by (3.2) satisfying D'(s) =0. Then every ruling is a line of curvature and the base curves are lines of curvature if and only if $\gamma(s)$ is a plane curve and $D(s)$ is a unit normal vector to the plane of $\gamma(s)$.*

Definition 3.17. *A generalized cylinder with line of curvature parameterization is defined by*

$$X(s,v)=\gamma(s)+vD(s), 0\leq s\leq L, v\in\mathbb{R}, \tag{3.12}$$

where

$$D(s)=\langle 0,0,1\rangle \quad \textit{and} \quad \gamma(s) \textit{ is a plane curve.}$$

The generalized cylinder whose base curve is a plane curve and the director vector is a unit normal vector to the plane of the base curve is called a right generalized cylinder [32] or shortly right cylinder.

Corollary 3.18. *A generalized cylinder with line of curvature parameterization (3.12) is a right cylinder.*

Theorem 3.19. *Among all generalized cylinders parameterized by (3.5), only the right cylinder (3.12) can be equipped with line of curvature parameterization.*

Corollary 3.20. *The line of curvature parametric curves of the the right cylinder (3.12) are lines and plane curves.*

4. Examples

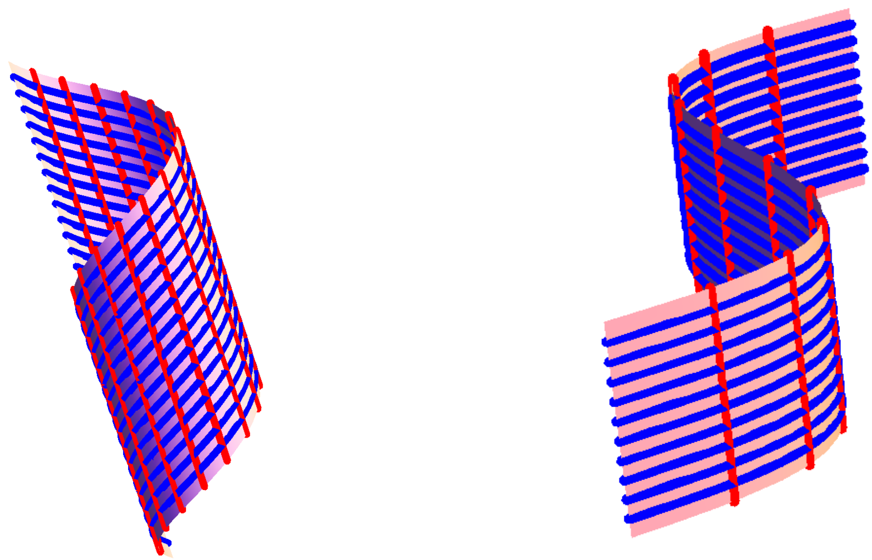
In this section, we give two examples of a generalized cylinder with geodesic and line of curvature parametrization and draw their pictures by using Mathematica. It is worth noting that the results are satisfied even the base curve is not a unit speed as shown in the second example.

Example 4.1. *Let $\gamma(s)=(\frac{\sqrt{3}}{2}\sin(s), \frac{s}{2}, \frac{\sqrt{3}}{2}\cos(s))$ be a unit speed helix curve, therefore the unit tangent and binormal vectors are given respectively by $t=\left(\frac{\sqrt{3}}{2}\cos(s), \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin(s)\right)$ and $b=(-\frac{1}{2}\cos(s), \frac{\sqrt{3}}{2}, \frac{1}{2}\sin(s))$. Their curvature and torsion are $\kappa=\frac{\sqrt{3}}{2}$ and $\tau=\frac{1}{2}$. According to definition (3.10), the generalized cylinder with geodesic parametrization is defined by*

$$X(s,v)=\gamma(s)+v[\frac{\tau(s)}{\sqrt{\kappa^2+\tau^2}}t(s)+\frac{\kappa(s)}{\sqrt{\kappa^2+\tau^2}}b(s)],\ 0\leq s\leq L, v\in\mathbb{R}.$$

By substitution $\frac{\tau}{\sqrt{\kappa^2+\tau^2}}=\frac{1}{2}$ and $\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}=\frac{\sqrt{3}}{2}$, and for $0\leq s\leq 2\pi$, $0\leq v\leq\pi$, the constructed cylinder is a rectifying cylinder with geodesic parametrization as shown in Figure 1(a).

Example 4.2. *Let $\gamma(s)=(s,\sin(s),0)$ be a plane curve. According to definition (3.12), the generalized cylinder with line of curvature parametrization can be defined by $X(s,v)=\gamma(s)+v\,(0,0,1)$, $0\leq s\leq 2\pi, 0\leq v\leq\pi/2$. The constructed cylinder is a right cylinder with line of curvature parametrization as shown in Figure 1(b).*



(a) Rectifying cylinder with geodesic parametrization

(b) Right cylinder with line of curvature parametrization

Figure 4.1: Generalized cylinder with geodesic or line of curvature parametrizations

5. Conclusion

In this paper, using a ruled parametrization (3.1), and with three conditions called the cylindrical conditions (3.4) we constructed a generalized cylinder parametrization (3.5). After that, through many geometric constraints we obtained the resulting cylinder that is parameterized by geodesics or line of curvatures. The main results asserted that the generalized cylinder with geodesic or line of curvature parametrization is a rectifying cylinder (3.10) or a right cylinder (3.12) respectively.

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Analysis of a System of Nonlinear Hadamard Type Fractional Boundary Value Problems

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Abstract

The aim of this work is to analyze the existence of positive solutions for a coupled system of Hadamard type fractional boundary value problems. By using the five functional fixed point theorem, the conditions for the existence of positive solutions are derived. Finally, to show the applicability of the main result, an illustrative example is also involved.

1. Introduction

Fractional calculus and fractional differential equations have recently gained significance due to the expansion of the application fields against real world problems in the areas of applied mathematics, engineering, physics, system control, etc. One reason for such interest is that fractional differential equations can explain more precise results with respect to integer order models, see [1]-[5]. Moreover, a lot of scientists have been studying on the existence results of positive solutions for fractional boundary value problems and the systems of fractional differential equations by means of methods of nonlinear analysis. The importance of the area of coupled systems of fractional order differential equations comes from that they can be observed in a large number of problems of applied nature. For details and examples on the topic, see [6]-[15] and the references therein.

Other than the commonly mentioned Riemann-Liouville and Caputo fractional differential equations, there is a gap in investigation of Hadamard fractional differential equations and coupled systems under different boundary conditions on an bounded/unbounded domain. One of the main speciality of Hadamard fractional derivative is that the definition contains logarithmic function of arbitrary exponent. For some recent results on boundary value problems of Hadamard fractional differential equations and coupled systems, we refer to [16]-[33].

In [23], Zhai and Wang investigated the following coupled Hadamard type fractional boundary value problems:

$$\begin{cases} {}^{H}D^{p}u(t)+f(t,v(t))=a, \quad 1<p\leq 2, \quad t\in(1,e), \\ {}^{H}D^{q}v(t)+g(t,u(t))=b, \quad 1<q\leq 2, \quad t\in(1,e), \\ u(1)=0, \quad {}^{H}D^{p-1}u(e)=\displaystyle\sum_{i=1}^{m}\mu_i {}^{H}I^{\alpha_i}v(\eta), \\ v(1)=0, \quad {}^{H}D^{q-1}v(e)=\displaystyle\sum_{j=1}^{n}\sigma_j {}^{H}I^{\beta_j}u(\xi), \end{cases}$$

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where ${}^{H}D$ denotes the Hadamard-type fractional derivative; ${}^{H}I$ is the Hadamard-type fractional integral. By the use of increasing $\varphi-(h,r)$ concave operators, the authors obtained the existence and uniqueness of solutions for Hadamard fractional differential systems.

Motivated particularly by the above mentioned papers, we are interested in investigating a coupled system of Hadamard fractional differential equations, which include both integral boundary conditions and m-point fractional integral boundary conditions:

$$\begin{cases} {}^{H}D_{1^+}^{\vartheta_1}u(t)+f_1(t,u(t),v(t))=0, \quad n-1<\vartheta_1\leq n, \quad t\in(1,e),\\ {}^{H}D_{1^+}^{\vartheta_2}v(t)+f_2(t,u(t),v(t))=0, \quad m-1<\vartheta_2\leq m, \quad t\in(1,e),\\ u(1)=u'(1)=...=u^{(n-2)}(1)=0, \qquad {}^{H}D_{1^+}^{\vartheta_1-1}u(e)=\displaystyle\int_1^e g_1(t)u(t)\frac{dt}{t}+\sum_{i=1}^{p}\lambda_i^{H}I_{1^+}^{\beta_i}u(\sigma_1^*),\\ v(1)=v'(1)=...=v^{(m-2)}(1)=0, \qquad {}^{H}D_{1^+}^{\vartheta_2-1}v(e)=\displaystyle\int_1^e g_2(t)v(t)\frac{dt}{t}+\sum_{j=1}^{q}\sigma_j^{H}I_{1^+}^{\alpha_j}v(\sigma_2^*), \end{cases} \tag{1.1}$$

where $n,m\in\mathbb{N}$, $n,m\geq 3$, ${}^{H}D_{1^+}^{\vartheta_1}$ and ${}^{H}D_{1^+}^{\vartheta_2}$ are the Hadamard-type fractional derivatives of order ϑ_1, ϑ_2, respectively. ${}^{H}I_{1^+}^{\beta_i}$ and ${}^{H}I_{1^+}^{\alpha_j}$ are the Hadamard-type fractional integrals of order $\beta_i>0$ $(i=1,2,...,p)$, $\alpha_j>0$ $(j=1,2,...,q)$, $f_1,f_2\in C([1,e]\times[0,\infty)\times[0,\infty),[0,\infty))$, $g_1,g_2\in C([1,e],(0,\infty))$ and $\lambda_i\geq 0$ $(i=1,2,...,p)$, $\sigma_j\geq 0$ $(j=1,2,...,q)$, $\sigma_1^*,\sigma_2^*\in(1,e)$ are given constants.

We deal with the analysis of existence result of positive solutions for Hadamard differential systems. We accentuate that there are a lot of studies on Riemann-Liouville or Caputo type fractional differential systems. To the best authors' knowledge, there are a little number of papers which are studied on the systems of nonlinear Hadamard fractional differential equations. Here, unlike other papers, we attempt to study new Hadamard differential systems which consist of both integral boundary conditions and m-point fractional integral boundary conditions on an bounded domain.

We prepare the following sections of this paper as follows: Section 2 includes some preliminaries. We also summarize all properties of the corresponding Green's function. We indicate the existence of positive solutions of the problem and an example illustrating our result is ensured in Section 3.

2. Preliminaries

In this section, basic concepts, notations and some lemmas about the Hadamard-type fractional calculus are demonstrated for the convenience of the readers.

Definition 2.1 ([4])**.** The Hadamard fractional derivative of fractional order $\nu>0$ for a function $k:[1,\infty)\to\mathbb{R}$ is defined as

$$ {}^{H}D_{1^+}^{\nu}k(t)=\frac{1}{\Gamma(n-\nu)}\Big(t\frac{d}{dt}\Big)^n\int_1^t\Big(\log\frac{t}{s}\Big)^{n-\nu-1}k(s)\frac{ds}{s}, \quad n-1<\nu<n,\ n=[\nu]+1,$$

where $[\nu]$ denotes the integer part of the real number ν and $\log(\cdot)=\log_e(\cdot)$.

Definition 2.2 ([4])**.** The Hadamard fractional integral of order $\nu>0$ for a function $k:[1,\infty)\to\mathbb{R}$ is defined as

$$ {}^{H}I_{1^+}^{\nu}k(t)=\frac{1}{\Gamma(\nu)}\int_1^t(\log\frac{t}{s})^{\nu-1}k(s)\frac{ds}{s}, \quad \nu>0,$$

provided the integral exists.

Lemma 2.3 ([4])**.** If $a,\nu,\mu>0$, then

$$({}^{H}I_a^{\nu}(\log\frac{t}{a})^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}(\log\frac{x}{a})^{\mu+\nu-1}, \qquad ({}^{H}D_a^{\nu}(\log\frac{t}{a})^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}(\log\frac{x}{a})^{\mu-\nu-1}.$$

in particular, $({}^{H}D_a^{\nu}(\log\frac{t}{a})^{\nu-j})(x)=0, j=1,2,...,[\nu]+1$.

Lemma 2.4 ([4])**.** Let $\nu>0$. Assume that $c\in C[1,\infty)\cap L^1[1,\infty)$, then the solution of Hadamard-type fractional differential equation ${}^{H}D_{1^+}^{\nu}c(t)=0$ can be denoted as

$$c(t)=\sum_{i=1}^{m}c_i(\log t)^{\nu-i},$$

and the following formula holds:

$$ {}^{H}I_{1^+}^{\nu}\,{}^{H}D_{1^+}^{\nu}c(t) = c(t) + \sum_{i=1}^{m} c_i(\log t)^{\nu-i}, $$

where $c_i \in \mathbb{R}, i = 1,2,...,n, n-1 < \nu < n, n = [\nu]+1$.

Lemma 2.5. *If $x, y \in C[1,e]$, then, for the functions $u, v \in C[1,e]$, the following system*

$$ \begin{cases} {}^{H}D_{1^+}^{\vartheta_1}u(t) + x(t) = 0, \quad n-1 < \vartheta_1 \le n, \quad t \in (1,e), \\ {}^{H}D_{1^+}^{\vartheta_2}v(t) + y(t) = 0, \quad m-1 < \vartheta_2 \le m, \quad t \in (1,e), \\ u(1) = u'(1) = ... = u^{(n-2)}(1) = 0, \qquad {}^{H}D_{1^+}^{\vartheta_1-1}u(e) = \displaystyle\int_1^e g_1(t)u(t)\frac{dt}{t} + \sum_{i=1}^{p} \lambda_i^{H} I_{1^+}^{\beta_i}u(\sigma_1^*), \\ v(1) = v'(1) = ... = v^{(m-2)}(1) = 0, \qquad {}^{H}D_{1^+}^{\vartheta_2-1}v(e) = \displaystyle\int_1^e g_2(t)v(t)\frac{dt}{t} + \sum_{j=1}^{q} \sigma_j^{H} I_{1^+}^{\alpha_j}v(\sigma_2^*), \end{cases} \tag{2.1} $$

can be given in the integral representations of the form

$$ u(t) = \int_1^e H_1(t,s)x(s)\frac{ds}{s}, $$

$$ v(t) = \int_1^e H_2(t,s)y(s)\frac{ds}{s}, $$

where

$$ H_1(t,s) = G_1(t,s) + G_2(t,s), \tag{2.2} $$

$$ H_2(t,s) = G_3(t,s) + G_4(t,s), \tag{2.3} $$

and

$$ G_1(t,s) = g_1(t,s) + \sum_{i=1}^{p} \frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)} g_1^{\beta_i}(\sigma_1^*, s), $$

$$ G_2(t,s) = \frac{(\log t)^{\vartheta_1-1}}{\Upsilon_1}\int_1^e G_1(t,s)g_1(t)\frac{dt}{t}, $$

$$ G_3(t,s) = g_2(t,s) + \sum_{j=1}^{q} \frac{\sigma_j(\log t)^{\vartheta_2-1}}{\Upsilon^*\Gamma(\vartheta_2+\alpha_j)} g_2^{\alpha_j}(\sigma_2^*, s), $$

$$ G_4(t,s) = \frac{(\log t)^{\vartheta_2-1}}{\Upsilon_1^*}\int_1^e G_3(t,s)g_2(t)\frac{dt}{t}, $$

with

$$ g_1(t,s) = \frac{1}{\Gamma(\vartheta_1)} \begin{cases} (\log t)^{\vartheta_1-1} - (\log \frac{t}{s})^{\vartheta_1-1}, & 1 \le s \le t \le e, \\ (\log t)^{\vartheta_1-1}, & 1 \le t \le s \le e, \end{cases} \tag{2.4} $$

$$ g_2(t,s) = \frac{1}{\Gamma(\vartheta_2)} \begin{cases} (\log t)^{\vartheta_2-1} - (\log \frac{t}{s})^{\vartheta_2-1}, & 1 \le s \le t \le e, \\ (\log t)^{\vartheta_2-1}, & 1 \le t \le s \le e, \end{cases} \tag{2.5} $$

$$ g_1^{\beta_i}(\sigma_1^*, s) = \begin{cases} (\log \sigma_1^*)^{\vartheta_1+\beta_i-1} - (\log \frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}, & 1 \le s \le \sigma_1^* \le e, \\ (\log \sigma_1^*)^{\vartheta_1+\beta_i-1}, & 1 \le \sigma_1^* \le s \le e, \end{cases} $$

$$ g_2^{\alpha_j}(\sigma_2^*, s) = \begin{cases} (\log \sigma_2^*)^{\vartheta_2+\alpha_j-1} - (\log \frac{\sigma_2^*}{s})^{\vartheta_2+\alpha_j-1}, & 1 \le s \le \sigma_2^* \le e, \\ (\log \sigma_2^*)^{\vartheta_2+\alpha_j-1}, & 1 \le \sigma_2^* \le s \le e, \end{cases} $$

where $\Upsilon = \Gamma(\vartheta_1) - \sum_{i=1}^{p} \frac{\lambda_i \Gamma(\vartheta_1)}{\Gamma(\vartheta_1+\beta_i)} (\log \sigma_1^*)^{\vartheta_1+\beta_i-1}$ *and* $\Upsilon^* = \Gamma(\vartheta_2) - \sum_{j=1}^{q} \frac{\sigma_j \Gamma(\vartheta_2)}{\Gamma(\vartheta_2+\alpha_j)} (\log \sigma_2^*)^{\vartheta_2+\alpha_j-1}$

$$\Upsilon_1 = \Upsilon - \int_1^e g_1(t)(\log t)^{\vartheta_1-1} \frac{dt}{t} > 0,$$

and

$$\Upsilon_1^* = \Upsilon^* - \int_1^e g_2(t)(\log t)^{\vartheta_2-1} \frac{dt}{t} > 0.$$

Proof. Using Lemma (2.4), the above system (2.1) can be given by

$$u(t) = -\frac{1}{\Gamma(\vartheta_1)} \int_1^t (\log \frac{t}{s})^{\vartheta_1-1} x(s) \frac{ds}{s} + c_1 (\log t)^{\vartheta_1-1} + c_2 (\log t)^{\vartheta_1-2} + ... + c_n (\log t)^{\vartheta_1-n},$$

$$v(t) = -\frac{1}{\Gamma(\vartheta_2)} \int_1^t (\log \frac{t}{s})^{\vartheta_2-1} y(s) \frac{ds}{s} + d_1 (\log t)^{\vartheta_2-1} + d_2 (\log t)^{\vartheta_2-2} + ... + d_m (\log t)^{\vartheta_2-m},$$

where $c_i, d_j \in \mathbb{R}$, $i = 1,..,n$ and $j = 1,..,m$. Using the boundary conditions, we derive $c_2 = c_3 = ... = c_n = 0$ and $d_2 = d_3 = ... = d_m = 0$. Then,

$$u(t) = -\frac{1}{\Gamma(\vartheta_1)} \int_1^t (\log \frac{t}{s})^{\vartheta_1-1} x(s) \frac{ds}{s} + c_1 (\log t)^{\vartheta_1-1}. \tag{2.6}$$

$$v(t) = -\frac{1}{\Gamma(\vartheta_2)} \int_1^t (\log \frac{t}{s})^{\vartheta_2-1} y(s) \frac{ds}{s} + d_1 (\log t)^{\vartheta_2-1}. \tag{2.7}$$

By using Lemma (2.3)

$${}^H D_{1^+}^{\vartheta_1-1} u(t) = c_1 \Gamma(\vartheta_1) - \int_1^t x(s) \frac{ds}{s},$$

$${}^H D_{1^+}^{\vartheta_2-1} v(t) = d_1 \Gamma(\vartheta_2) - \int_1^t y(s) \frac{ds}{s}.$$

Using ${}^H D_{1^+}^{\vartheta_1-1} u(e) = \int_1^e g_1(t) u(t) \frac{dt}{t} + \sum_{i=1}^{p} \lambda_i^H I_{1^+}^{\beta_i} u(\sigma_1^*)$, and ${}^H D_{1^+}^{\vartheta_2-1} v(e) = \int_1^e g_2(t) v(t) \frac{dt}{t} + \sum_{j=1}^{q} \sigma_j^H I_{1^+}^{\alpha_j} v(\sigma_2^*)$, we have

$$c_1 = \frac{1}{\Upsilon} (\int_1^e g_1(t) u(t) \frac{dt}{t} + \int_1^e x(s) \frac{ds}{s} - \sum_{i=1}^{p} \frac{\lambda_i}{\Gamma(\vartheta_1+\beta_i)} \int_1^{\sigma_1^*} (\log \frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1} x(s) \frac{ds}{s}). \tag{2.8}$$

$$d_1 = \frac{1}{\Upsilon^*} (\int_1^e g_2(t) v(t) \frac{dt}{t} + \int_1^e y(s) \frac{ds}{s} - \sum_{j=1}^{q} \frac{\sigma_j}{\Gamma(\vartheta_2+\alpha_j)} \int_1^{\sigma_2^*} (\log \frac{\sigma_2^*}{s})^{\vartheta_2+\alpha_j-1} y(s) \frac{ds}{s}). \tag{2.9}$$

Substituting (2.8) into (2.6), we get

$$\begin{aligned}
u(t) =& \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t} - \frac{1}{\Gamma(\vartheta_1)}\int_1^t (\log\frac{t}{s})^{\vartheta_1-1}x(s)\frac{ds}{s}\\
&-\sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
=&\frac{(\log t)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \frac{(\Gamma(\vartheta_1)-\Upsilon)(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t}\\
&-\frac{1}{\Gamma(\vartheta_1)}\int_1^t(\log\frac{t}{s})^{\vartheta_1-1}x(s)\frac{ds}{s} - \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
=&\frac{(\log t)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^e(\log\sigma_1^*)^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
&+\frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t} - \frac{1}{\Gamma(\vartheta_1)}\int_1^t(\log\frac{t}{s})^{\vartheta_1-1}x(s)\frac{ds}{s}\\
&-\sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\
=&\int_1^e g_1(t,s)x(s)\frac{ds}{s} + \sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^e g_1^{\beta_i}(\sigma_1^*,s)x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t}\\
=&\int_1^e G_1(t,s)x(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_1-1}}{\Upsilon}\int_1^e g_1(t)u(t)\frac{dt}{t}.
\end{aligned}$$

Similarly, substituting (2.9) into (2.7), we get

$$v(t) = \int_1^e G_2(t,s)y(s)\frac{ds}{s} + \frac{(\log t)^{\vartheta_2-1}}{\Upsilon^*}\int_1^e g_2(t)v(t)\frac{dt}{t}.$$

Furthermore,

$$\begin{aligned}
\int_1^e g_1(t)u(t)\frac{dt}{t} =& \int_1^e g_1(t)\int_1^e G_1(t,s)x(s)\frac{ds}{s}\frac{dt}{t}\\
&+\frac{1}{\Upsilon}\int_1^e g_1(t)(\log t)^{\vartheta_1-1}\frac{dt}{t}\int_1^e g_1(t)u(t)\frac{dt}{t},
\end{aligned}$$

and

$$\begin{aligned}
\int_1^e g_2(t)v(t)\frac{dt}{t} =& \int_1^e g_2(t)\int_1^e G_3(t,s)y(s)\frac{ds}{s}\frac{dt}{t}\\
&+\frac{1}{\Upsilon^*}\int_1^e g_2(t)(\log t)^{\vartheta_2-1}\frac{dt}{t}\int_1^e g_2(t)v(t)\frac{dt}{t},
\end{aligned}$$

which provide

$$\int_1^e g_1(t)u(t)\frac{dt}{t} = \frac{\Upsilon}{\Upsilon_1}\int_1^e g_1(t)\int_1^e G_1(t,s)x(s)\frac{ds}{s}\frac{dt}{t},$$

$$\int_1^e g_2(t)v(t)\frac{dt}{t} = \frac{\Upsilon^*}{\Upsilon_1^*}\int_1^e g_2(t)\int_1^e G_3(t,s)y(s)\frac{ds}{s}\frac{dt}{t}.$$

Then,

$$\begin{aligned}
u(t) =& \int_1^e G_1(t,s)x(s)\frac{ds}{s} + \int_1^e G_2(t,s)x(s)\frac{ds}{s}\\
=& \int_1^e H_1(t,s)x(s)\frac{ds}{s},
\end{aligned}$$

$$\begin{aligned}
v(t) =& \int_1^e G_3(t,s)y(s)\frac{ds}{s} + \int_1^e G_4(t,s)y(s)\frac{ds}{s}\\
=& \int_1^e H_2(t,s)y(s)\frac{ds}{s}.
\end{aligned}$$

The proof is completed. □

Lemma 2.6. *The functions $g_i(t,s)$, (i=1,2) given by (2.4) and (2.5) satisfy*

(i) $g_i(t,s)$ are continuous functions and $g_i(t,s) \geq 0$ for any $t,s \in [1,e]$, $i = 1,2$.

(ii)$g_i(t,s) \leq g_i(e,s)$ for any $t,s \in [1,e]$, $i = 1,2$.

(iii)$g_1(t,s) \geq (\frac{1}{4})^{\vartheta_1 - 1} g_1(e,s)$ and $g_2(t,s) \geq (\frac{1}{4})^{\vartheta_2 - 1} g_2(e,s)$ for any $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$ and $s \in [1,e]$.

Proof. To show (i), it is easy to check that the functions $g_i(t,s)$, (i=1,2) are continuous functions. Next, for $1 \leq s \leq t \leq e$, we have

$$\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1 - 1} - (\log \frac{t}{s})^{\vartheta_1 - 1}) \\
&= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1 - 1} - (\log t)^{\vartheta_1 - 1}(1 - \frac{\log s}{\log t})^{\vartheta_1 - 1}) \\
&\geq \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1 - 1}(1 - (1 - \log s)^{\vartheta_1 - 1})) \\
&\geq 0.
\end{aligned}$$

For $1 \leq t \leq s \leq e$, $g_1(t,s) = \dfrac{1}{\Gamma(\vartheta_1)}(\log t)^{\vartheta_1 - 1} \geq 0$. Using a similar proof, we obtain $g_2(t,s) \geq 0$ for any $t,s \in [1,e]$. To prove (ii), for $1 \leq s \leq t \leq e$, we get

$$\begin{aligned}
g_{1t}(t,s) &= \frac{(\vartheta_1 - 1)(\log t)^{\vartheta_1 - 2}\frac{1}{t} - (\vartheta_1 - 1)(\log \frac{t}{s})^{\vartheta_1 - 2}\frac{1}{t}}{\Gamma(\vartheta_1)} \\
&\geq \frac{(\vartheta_1 - 1)(\log t)^{\vartheta_1 - 2}\left[1 - (1 - \log s)^{\vartheta_1 - 2})\right]}{\Gamma(\vartheta_1)t} \geq 0.
\end{aligned}$$

Then, $g_{1t}(t,s)$ is increasing on $[s,e]$ according to t. That is, $g_1(t,s) \leq g_1(e,s)$ is obtained. It is easy to see that $g_1(t,s) \leq g_1(e,s)$ when $1 \leq t \leq s \leq e$. Thus, $g_1(t,s) \leq g_1(e,s)$ for any $t,s \in [1,e]$. Similarly, we have $g_2(t,s) \leq g_2(e,s)$ for any $t,s \in [1,e]$. To demonstrate (iii), for $1 \leq s \leq t \leq e$ and $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$,

$$\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1 - 1} - (\log \frac{t}{s})^{\vartheta_1 - 1}) \\
&= \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1 - 1} - (\log t)^{\vartheta_1 - 1}(1 - \frac{\log s}{\log t})^{\vartheta_1 - 1}) \\
&\geq \frac{1}{\Gamma(\vartheta_1)}((\log t)^{\vartheta_1 - 1}(1 - (1 - \log s)^{\vartheta_1 - 1})) \\
&\geq (\frac{1}{4})^{\vartheta_1 - 1} g_1(e,s).
\end{aligned}$$

It is clear that for $1 \leq t \leq s \leq e$ and $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$, $g_1(t,s) \geq (\frac{1}{4})^{\vartheta_1 - 1} g_1(e,s)$. In a similar manner, we get $g_2(t,s) \geq (\frac{1}{4})^{\vartheta_2 - 1} g_2(e,s)$ for any $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$ and $s \in [1,e]$. The proof is completed. □

Lemma 2.7. *Let $K_1(s) = g_1(e,s) + \sum_{i=1}^{p} \frac{\lambda_i}{\Upsilon\Gamma(\vartheta_1 + \beta_i)} g_1^{\beta_i}(\sigma_1^*, s)$, $K_2(s) = g_2(e,s) + \sum_{j=1}^{q} \frac{\sigma_j}{\Upsilon^*\Gamma(\vartheta_2 + \alpha_j)} g_2^{\alpha_j}(\sigma_2^*, s)$, for $s \in [1,e]$ and $\varpi_1 = 1 + \frac{1}{\Upsilon_1}\int_1^e g_1(t)\frac{dt}{t}$, $\varpi_2 = 1 + \frac{1}{\Upsilon_1^*}\int_1^e g_2(t)\frac{dt}{t}$. Then, the functions $H_i(t,s)$, $i = 1,2$ defined by (2.2) and (2.3) ensure the following properties:*

(i) $H_i(t,s)$ are continuous and $H_i(t,s) \geq 0$, for $(t,s) \in [1,e] \times [1,e]$, $i = 1,2$;

(ii)$H_1(t,s) \leq K_1(s)\varpi_1$, for $(t,s) \in [1,e] \times [1,e]$;

(iii)$\min_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} H_1(t,s) \geq (\frac{1}{4})^{2\vartheta_1 - 2} K_1(s)\varpi_1$, for $s \in [1,e]$;

(iv) $H_2(t,s) \leq K_2(s)\varpi_2$, for $(t,s) \in [1,e] \times [1,e]$;

(v) $\min_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} H_2(t,s) \geq (\frac{1}{4})^{2\vartheta_2 - 2} K_2(s)\varpi_2$, for $s \in [1,e]$.

Proof. We can evidently see that (i) holds. To show (ii), for $(t,s)\in[1,e]\times[1,e]$, we have,

$$\begin{aligned}
H_1(t,s) &= G_1(t,s)+G_2(t,s)\\
&= g_1(t,s)+\sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\\
&\quad+\frac{(\log t)^{\vartheta_1-1}}{\Upsilon_1}\int_1^e G_1(t,s)g_1(t)\frac{dt}{t}\\
&\le g_1(e,s)+\sum_{i=1}^{p}\frac{\lambda_i}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\\
&\quad+\frac{1}{\Upsilon_1}\int_1^e\Big(g_1(e,s)+\sum_{i=1}^{p}\frac{\lambda_i}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\Big)g_1(t)\frac{dt}{t}\\
&= K_1(s)\varpi_1.
\end{aligned}$$

To prove (iii), for $(t,s)\in[1,e]\times[1,e]$, we get,

$$\begin{aligned}
\min_{t\in[e^{\frac14},e^{\frac34}]}H_1(t,s) &= \min_{t\in[e^{\frac14},e^{\frac34}]}\Big[g_1(t,s)+\sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\\
&\quad+\frac{(\log t)^{\vartheta_1-1}}{\Upsilon_1}\int_1^e G_1(t,s)g_1(t)\frac{dt}{t}\Big]\\
&\ge(\frac14)^{\vartheta_1-1}g_1(e,s)+\sum_{i=1}^{p}\frac{\lambda_i(\frac14)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\\
&\quad+\frac{(\frac14)^{\vartheta_1-1}}{\Upsilon_1}\int_1^e\Big(g_1(t,s)+\sum_{i=1}^{p}\frac{\lambda_i(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\Big)g_1(t)\frac{dt}{t}\\
&\ge(\frac14)^{\vartheta_1-1}K_1(s)+\frac{(\frac14)^{\vartheta_1-1}}{\Upsilon_1}\int_1^e\Big((\frac14)^{\vartheta_1-1}g_1(e,s)+\sum_{i=1}^{p}\frac{\lambda_i(\frac14)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}g_1^{\beta_i}(\sigma_1^*,s)\Big)g_1(t)\frac{dt}{t}\\
&\ge(\frac14)^{\vartheta_1-1}K_1(s)+\frac{(\frac14)^{2\vartheta_1-2}}{\Upsilon_1}K_1(s)\int_1^e g_1(t)\frac{dt}{t}\\
&\ge(\frac14)^{2\vartheta_1-2}K_1(s)\varpi_1.
\end{aligned}$$

The proofs of the parts (iv) and (v) can be shown similar to the proofs above (ii) and (iii).
The proof is completed. □

We deal with the Banach space $E=C[1,e]\times C[1,e]$ with the norm $\|(u,v)\|_E=\|u\|+\|v\|$ for $(u,v)\in E$ and $\|u\|=\max_{t\in[1,e]}|u(t)|$. We introduce the cone $P\subset E$,

$$P=\left\{(u,v)\in E: u(t)\ge0, v(t)\ge0, \forall t\in[1,e],\ \min_{t\in[e^{\frac14},e^{\frac34}]}(u(t)+v(t))\ge\Psi\|(u,v)\|\right\}, \tag{2.10}$$

where $\Psi=min\left\{(\frac14)^{2\vartheta_1-2},(\frac14)^{2\vartheta_2-2}\right\}$. Define the operator $F:P\to E$ by

$$F(u,v)(t)=(F_1(u,v)(t),F_2(u,v)(t)), \text{for all } t\in[1,e], \tag{2.11}$$

with $F_1,F_2:P\to C[1,e]$ are given by

$$F_1(u,v)(t)=\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s}, \tag{2.12}$$

$$F_2(u,v)(t)=\int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}.$$

Lemma 2.8. *Consider that (u,v) is a positive solution of the system (1.1) if and only if (u,v) is a fixed point of the operator F.*

Proof. It is obvious that a positive solution of the system (1.1) is a fixed point of the operator F.
In fact, if $u(t) = F_1(u,v)(t)$, by applying the operator ${}^HD_{1^+}^{\vartheta_1}$ on both sides of (2.12), after some arrangement, for $x(s) = f_1(s,u(s),v(s), s \in [1,e]$ in Lemma (2.5), we get

$$\begin{aligned}{}^HD_{1^+}^{\vartheta_1}F_1(u,v)(t) =& \frac{{}^HD_{1^+}^{\vartheta_1}(\log t)^{\vartheta_1-1}}{\Gamma(\vartheta_1)}\int_1^e x(s)\frac{ds}{s} + \sum_{i=1}^{p}\frac{\lambda_i^H D_{1^+}^{\vartheta_1}(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^e(\log\sigma_1^*)^{\vartheta_1+\beta_i-1}x(s)\frac{ds}{s}\\ &-({}^HD_{1^+}^{\vartheta_1}\,{}^HI_{1^+}^{\vartheta_1}x)(t) - \sum_{i=1}^{p}\frac{\lambda_i^H D_{1^+}^{\vartheta_1}(\log t)^{\vartheta_1-1}}{\Upsilon\Gamma(\vartheta_1+\beta_i)}\int_1^{\sigma_1^*}(\log\frac{\sigma_1^*}{s})^{\vartheta_1+\beta_i-1}k(s)\frac{ds}{s}\\ &+\frac{{}^HD_{1^+}^{\vartheta_1}(\log t)^{\vartheta_1-1}}{\Upsilon_1}\Big(\int_1^e g_1(t)\int_1^e G_1(t,s)x(s)\frac{ds}{s}\frac{dt}{t}\Big).\end{aligned}$$

Applying Lemma (2.3), we have

$${}^HD_{1^+}^{\vartheta_1}F_1(u,v)(t) = -x(t),$$

which implies that the system (1.1) is satisfied. Then by a direct computation, it follows that u satisfies the boundary conditions of (1.1). Similarly, we obtain that $v(t) = F_2(u,v)(t)$ is a solution of the system (1.1). The proof is completed. □

Lemma 2.9. $F : \mathrm{P} \to \mathrm{P}$ *is a completely continuous operator.*

Proof. Let us indicate that $F(P) \subset P$. The continuity of H_1,H_2, f_1, f_2, it follows that F is continuous. Lemma (2.7) and the nonnegativity of f_1 and f_1 ensure that $F_1(u,v)(t) \geq 0$, $F_2(u,v)(t) \geq 0$ for $t \in [1,e]$. Also, for $(u,v) \in P$

$$\begin{aligned}\|F_1(u,v)\| &\leq \varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s},\\ \|F_2(u,v)\| &\leq \varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s},\end{aligned}$$

and

$$\begin{aligned}\min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]} F_1(u,v)(t) &\geq (\frac{1}{4})^{2\vartheta_1-2}\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s}\\ &\geq (\frac{1}{4})^{2\vartheta_1-2}\|F_1(u,v)\|.\end{aligned}$$

Similary, we get $\min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]} F_2(u,v)(t) \geq (\frac{1}{4})^{2\vartheta_2-2}\|F_2(u,v)\|$. Hence,

$$\begin{aligned}\min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]}\{F_1(u,v)(t)+F_2(u,v)(t)\} &\geq (\frac{1}{4})^{2\vartheta_1-2}\|F_1(u,v)\| + (\frac{1}{4})^{2\vartheta_2-2}\|F_2(u,v)\|\\ &\geq \Psi[\|F_1(u,v)\| + \|F_2(u,v)\|]\\ &= \Psi\|F(u,v)\|,\end{aligned}$$

so $F : P \to P$. Moreover, we can use the Arzela–Ascoli theorem, we obtain that F is a completely continuous operator. The proof is completed. □

Let Φ, Λ, θ be nonnegative continuous convex functionals on P and κ, ψ be nonnegative continuous concave functionals on P. Then for nonnegative real numbers k,s,d,l and h, we define the following convex sets:

$$\begin{aligned}&P(\Phi,h) = \{\vartheta \in P : \Phi(\vartheta) < h\},\\ &P(\Phi,\kappa,s,h) = \{\vartheta \in P : s \leq \kappa(\vartheta), \Phi(\vartheta) \leq h\},\\ &Q(\Phi,\Lambda,l,h) = \{\vartheta \in P : \Lambda(\vartheta) \leq l, \Phi(\vartheta) \leq h\},\\ &P(\Phi,\theta,\kappa,s,d,h) = \{\vartheta \in P : s \leq \kappa(\vartheta), \theta(\vartheta) \leq d, \Phi(\vartheta) \leq h\},\\ &Q(\Phi,\Lambda,\psi,k,l,h) = \{\vartheta \in P : k \leq \psi(\vartheta), \Lambda(\vartheta) \leq l, \Phi(\vartheta) \leq h\}.\end{aligned}$$

In ensuring positive solutions of (1.1), the following theorem will be essential.

Lemma 2.10. *[see [34]] Let P be a cone in a real Banach space E. Assume there exist $h > 0$ and $\mathrm{M} > 0$, nonnegative, continuous, concave functionals κ and ψ on P, and nonnegative, continuous, convex functionals Φ, Λ, and θ on P, satisfying*

$$\kappa(\vartheta) \le \Lambda(\vartheta) \ \text{ and } \ \|\vartheta\| \le \mathrm{M}\Phi(\vartheta)$$

for all $\vartheta \in \overline{P(\Phi, h)}$. If

$$S : \overline{P(\Phi, h)} \to \overline{P(\Phi, h)}$$

is completely continuous and there exist nonnegative numbers k, l, d, s with $0 < l < s$ such that:

(i) $\{\vartheta \in P(\Phi, \theta, \kappa, s, d, h) : \kappa(\vartheta) > s\} \ne \emptyset$ and $\kappa(S\vartheta) > s$ for $\vartheta \in P(\Phi, \theta, \kappa, s, d, h)$,

(ii) $\{\vartheta \in \mathrm{Q}(\Phi, \Lambda, \psi, k, l, h) : \Lambda(\vartheta) < l\} \ne \emptyset$ and $\Lambda(S\vartheta) < l$ for $\vartheta \in \mathrm{Q}(\Phi, \Lambda, \psi, k, l, h)$,

(iii) $\kappa(S\vartheta) > s$ for $\vartheta \in P(\Phi, \kappa, s, h)$ with $\theta(S\vartheta) > d$,

(vi) $\Lambda(S\vartheta) < l$ for $\vartheta \in \mathrm{Q}(\Phi, \Lambda, l, h)$ with $\psi(S\vartheta) < k$.

Then, S has at least three fixed points $\vartheta_1, \vartheta_2, \vartheta_3 \in \overline{P(\Phi, h)}$ satisfying;

$$\Lambda(\vartheta_1) < l, s < \kappa(\vartheta_2),$$

and

$$l < \Lambda(\vartheta_3) \ \text{ with } \ \kappa(\vartheta_3) < s.$$

For the readers convenience, let us denote

$$\begin{aligned} W &= \min\Big\{\Big[\varpi_1 \int_1^e K_1(s)\frac{ds}{s}\Big]^{-1}, \Big[\varpi_2 \int_1^e K_2(s)\frac{ds}{s}\Big]^{-1}\Big\}, \\ V &= \max\Big\{\Big[(\frac{1}{4})^{2\vartheta_1 - 2}\varpi_1 \int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_1(s)\frac{ds}{s}\Big]^{-1}, \Big[(\frac{1}{4})^{2\vartheta_2 - 2}\varpi_2 \int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_2(s)\frac{ds}{s}\Big]^{-1}\Big\}. \end{aligned}$$

Now, we introduce the nonnegative continuous concave functionals ξ, ψ and the nonnegative continuous convex functionals β, θ, σ on P by

$$\begin{gathered} \xi(u,v) = \psi(u,v) = \min_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} (u(t) + v(t)), \quad \theta(u,v) = \max_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} (u(t) + v(t)), \\ \beta(u,v) = \sigma(u,v) = \max_{t \in [1,e]} (u(t) + v(t)). \end{gathered}$$

3. Main result

Theorem 3.1. *Assume that there exist constants $0 < \ell < \kappa < \frac{\kappa}{\Psi} < h$ such that $\kappa V < hW$. If f_i, $i = 1,2$ satisfy the following conditions:*

(M_1) $f_i(t,u,v) < \dfrac{\ell W}{2}$ for $t \in [1,e]$, $(u+v) \in [0,\ell]$,

(M_2) $f_i(t,u,v) > \dfrac{\kappa V}{2}$ for $t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]$, $(u+v) \in [\kappa, \frac{\kappa}{\Psi}]$,

(M_3) $f_i(t,u,v) \le \dfrac{hW}{2}$ for $t \in [1,e]$, $(u+v) \in [0,h]$.

Then the problem (1.1) has at least three positive solutions (u_i, v_i) $(i = 1,2,3)$ such that $\beta(u_1,v_1) < \ell$, $\kappa < \xi(u_2,v_2)$, $\ell < \beta(u_3,v_3)$ with $\xi(u_1,v_1) < \kappa$.

Proof. We introduce P and F as above equations (2.10) and (2.11). For any $(u,v) \in P$,

$$\begin{gathered} \xi(u,v) \le \beta(u,v), \\ \|(u,v)\| \le \frac{1}{\Psi} \min_{t \in [e^{\frac{1}{4}}, e^{\frac{3}{4}}]} (u(t) + v(t)) \le \frac{1}{\Psi} \max_{t \in [1,e]} (u(t) + v(t)) = \frac{1}{\Psi}\sigma(u,v). \end{gathered}$$

Next, we denote that the operator F ensures all conditions in Lemma (2.10). According to Lemma (2.9), F is completely continuous. As a beginning, we prove that $F : \overline{P(\sigma, h)} \to \overline{P(\sigma, h)}$. If $(u,v) \in \overline{P(\sigma, h)}$, then $\sigma(u,v) \le h$, $0 \le \|u\| + \|v\| \le h$. With respect to (M_3), we obtain that,

$$\begin{aligned}
\sigma(F(u,v)) &= \max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\leq \varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s} + \varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&\leq \frac{hW}{2}\varpi_1\int_1^e K_1(s)\frac{ds}{s} + \frac{hW}{2}\varpi_2\int_1^e K_2(s)\frac{ds}{s}\\
&\leq \frac{h}{2}+\frac{h}{2} = h.
\end{aligned}$$

Hence, we ensure $F:\overline{P(\sigma,h)}\to\overline{P(\sigma,h)}$.
To verify condition (i) of Lemma (2.10), by choosing, $(\frac{\kappa\Psi+\kappa}{4\Psi},\frac{\kappa\Psi+\kappa}{4\Psi})$, we get that $(\frac{\kappa\Psi+\kappa}{4\Psi},\frac{\kappa\Psi+\kappa}{4\Psi})\in P(\sigma,\theta,\xi,\kappa,\frac{\kappa}{\Psi},h)$ and $\xi(u,v)>\kappa$. Thus, $\left\{(u,v)\in P(\sigma,\theta,\xi,\kappa,\frac{\kappa}{\Psi},h):\xi(u,v)>\kappa\right\}\neq\emptyset$. Let $(u,v)\in P(\sigma,\theta,\xi,\kappa,\frac{\kappa}{\Psi},h)$, then $(u(t)+v(t))\in[\kappa,\frac{\kappa}{\Psi}]$ for any $t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]$. By $(M2)$, we obtain

$$\begin{aligned}
\xi(F(u,v)) &= \min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\geq (\frac{1}{4})^{2\vartheta_1-2}\varpi_1\int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_1(s)f_1(s,u(s),v(s))\frac{ds}{s} + (\frac{1}{4})^{2\vartheta_2-2}\varpi_2\int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&> \frac{\kappa V}{2}(\frac{1}{4})^{2\vartheta_1-2}\varpi_1\int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_1(s)\frac{ds}{s} + \frac{\kappa V}{2}(\frac{1}{4})^{2\vartheta_2-2}\varpi_2\int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_2(s)\frac{ds}{s}\\
&> \frac{\kappa}{2}+\frac{\kappa}{2} = \kappa.
\end{aligned}$$

Then, the condition (i) of Lemma (2.10) is satisfied. Now, we demonstrate that the condition (ii) of Lemma (2.10) is fulfilled. Let $(\frac{\Psi\ell+\ell}{4},\frac{\Psi\ell+\ell}{4})$, then $(\frac{\Psi\ell+\ell}{4},\frac{\Psi\ell+\ell}{4})\in Q(\sigma,\beta,\psi,\Psi\ell,\ell,h)$ and $\beta(u,v)<\ell$. Hence, $\{(u,v)\in Q(\sigma,\beta,\psi,\Psi\ell,\ell,h):\beta(u,v)<\ell\}\neq\emptyset$. Let $(u,v)\in Q(\sigma,\beta,\psi,\Psi\ell,\ell,h)$, then $(u(t)+v(t))\in[0,\ell]$ for any $t\in[1,e]$. By $(M1)$, we obtain

$$\begin{aligned}
\beta(F(u,v)) &= \max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\leq \varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s} + \varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&< \frac{\ell W}{2}\varpi_1\int_1^e K_1(s)\frac{ds}{s} + \frac{\ell W}{2}\varpi_2\int_1^e K_2(s)\frac{ds}{s}\\
&< \frac{\ell}{2}+\frac{\ell}{2} = \ell.
\end{aligned}$$

Now, we can show that the condition (iii) of Lemma (2.10) is satisfied. Let $(u,v)\in P(\sigma,\xi,\kappa,h)$ with $\theta(F(u,v))>\frac{\kappa}{\Psi}$. Then, we have,

$$\begin{aligned}
\xi(F(u,v)) &= \min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\geq (\frac{1}{4})^{2\vartheta_1-2}\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s} + (\frac{1}{4})^{2\vartheta_2-2}\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\\
&\geq \Psi\Big[\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s} + \varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\geq \Psi\max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&\geq \Psi\max_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big]\\
&= \Psi\theta(F(u,v)) = \kappa.
\end{aligned}$$

Finally, we can verify that the condition (iv) of Lemma (2.10) ensures. Let $(u,v)\in Q(\sigma,\beta,\ell,h)$ with $\psi(F(u,v))<\Psi\ell$,

$$
\begin{aligned}
\beta(F(u,v)) &= \max_{t\in[1,e]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big] \\
&\leq \frac{1}{\Psi}\Big[\Psi\varpi_1\int_1^e K_1(s)f_1(s,u(s),v(s))\frac{ds}{s} + \Psi\varpi_2\int_1^e K_2(s)f_2(s,u(s),v(s))\frac{ds}{s}\Big] \\
&\leq \frac{1}{\Psi}\min_{t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]}\Big[\int_1^e H_1(t,s)f_1(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f_2(s,u(s),v(s))\frac{ds}{s}\Big] \\
&= \frac{1}{\Psi}\psi(F(u,v)) < \ell.
\end{aligned}
$$

Because the conditions of Lemma (2.10) are satisfied, the system (1.1) has at least three positive solutions (u_i,v_i) $(i=1,2,3)$ such that $\beta(u_1,v_1)<\ell$, $\kappa<\xi(u_2,v_2)$, $\ell<\beta(u_3,v_3)$ with $\xi(u_1,v_1)<\kappa$. The proof is completed. □

Example 3.2. *Consider the system of Hadamard fractional differential equations*

$$
\begin{cases}
{}^HD_{1^+}^{\frac{5}{2}}u(t) + f_1(t,u(t),v(t)) = 0, & t\in(1,e), \\
{}^HD_{1^+}^{\frac{5}{2}}v(t) + f_2(t,u(t),v(t)) = 0, & t\in(1,e), \\
u(1)=u'(1)=0, \quad {}^HD_{1^+}^{\frac{3}{2}}u(e) = \int_1^e u(t)\frac{dt}{t} + \frac{1}{2}\ {}^HI_{1^+}^{\frac{1}{2}}u(e^{\frac{1}{2}}) + {}^HI_{1^+}^{\frac{3}{2}}u(e^{\frac{1}{2}}), \\
v(1)=v'(1)=0, \quad {}^HD_{1^+}^{\frac{3}{2}}v(e) = \int_1^e v(t)\frac{dt}{t} + \frac{3}{2}\ {}^HI_{1^+}^{\frac{3}{2}}u(e^{\frac{1}{3}}) + 2\,{}^HI_{1^+}^{\frac{7}{2}}u(e^{\frac{1}{3}}),
\end{cases}
\tag{3.1}
$$

in which $\vartheta_1=\vartheta_2=\frac{5}{2}$, $n=m=3$, $p=q=2$, $\lambda_1=\frac{1}{2},\lambda_2=1$, $\sigma_1=\frac{3}{2},\sigma_2=2$, $\sigma_1^*=e^{\frac{1}{2}},\sigma_2^*=e^{\frac{1}{3}},\beta_1=\frac{1}{2},\beta_2=\frac{3}{2}$, $\alpha_1=\frac{3}{2},\alpha_2=\frac{7}{2}$, $g_1(t)=g_2(t)=1$ *for* $t\in[1,e]$,

$$
f_1(t,u,v) = \begin{cases}
\frac{t}{10} + \frac{(u+v)}{4}, & t\in[1,e], \quad (u+v)\in[0,4], \\
\frac{t}{10} + 170(u+v) - 679, & t\in[1,e], \quad (u+v)\in[4,6], \\
\frac{t}{10} + \frac{10(u+v)+270694}{794}, & t\in[1,e], \quad (u+v)\in[6,\infty),
\end{cases}
$$

$$
f_2(t,u,v) = \begin{cases}
\frac{t}{20} + \frac{(u+v)}{4}, & t\in[1,e], \quad (u+v)\in[0,4], \\
\frac{t}{20} + 170(u+v) - 679, & t\in[1,e], \quad (u+v)\in[4,6], \\
\frac{t}{20} + \frac{(u+v)+135371}{397}, & t\in[1,e], \quad (u+v)\in[6,\infty).
\end{cases}
$$

By direct calculation, we get $\Psi = 0,015625$,

$$
\begin{aligned}
W &= \min\Big\{\Big[\varpi_1\int_1^e K_1(s)\frac{ds}{s}\Big]^{-1},\Big[\varpi_2\int_1^e K_2(s)\frac{ds}{s}\Big]^{-1}\Big\} \approx \min\big\{0.8842, 1.0435\big\} = 0.8842, \\
V &= \max\Big\{\Big[(\frac{1}{4})^{2\vartheta_1-2}\varpi_1\int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_1(s)\frac{ds}{s}\Big]^{-1},\Big[(\frac{1}{4})^{2\vartheta_2-2}\varpi_2\int_{e^{\frac{1}{4}}}^{e^{\frac{3}{4}}} K_2(s)\frac{ds}{s}\Big]^{-1}\Big\} \\
&\approx \max\big\{106.383, 111.1111\big\} = 111.1111.
\end{aligned}
$$

Choosing the constants as $\ell=4,\kappa=6,h=800$, *then* $0<\ell<\kappa<\frac{\kappa}{\Psi}<h$ *such that* $\kappa V<hW$. *Then,* f_i, $i=1,2$ *satisfy the following conditions:*

(M_1) $f_i(t,u,v) < \dfrac{\ell W}{2} \approx 1.7684$ *for* $t\in[1,e]$, $(u+v)\in[0,4]$,

(M_2) $f_i(t,u,v) > \dfrac{\kappa V}{2} \approx 333.3333$ *for* $t\in[e^{\frac{1}{4}},e^{\frac{3}{4}}]$, $(u+v)\in[6,384]$,

(M_3) $f_i(t,u,v) \leq \dfrac{hW}{2} \approx 353.63$ *for* $t\in[1,e]$, $(u+v)\in[0,800]$.

Then, all the hypotheses of Theorem (3.1) *are satisfied. Thus, the system of fractional differential equations* (3.1) *has at least three positive solutions.*

4. Conclusion

In our main result, it is obtained positive solutions for Hadamard differential systems. By using the five functionals fixed point theorem, the conditions for the existence of positive solutions are derived. There are a little number of papers which are studied on the systems of nonlinear Hadamard fractional differential equations. Here, unlike other papers, we attempt to study new Hadamard differential systems which consist of both integral boundary conditions and m-point fractional integral boundary conditions on an bounded domain.

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B-Riesz Transforms Generated by Generalized Translate Operator on HM^{p}_{q,Δ_ν} Hardy-Morrey Spaces

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Abstract

We study the decomposition of Hardy-Morrey spaces via atoms and molecules, which have similar properties of $H^{p}_{\Delta_\nu}(\mathbb{R}^{n}_{+})$ Hardy spaces. Then we introduce the HM^{p}_{q,Δ_ν} boundedness of B-Riesz transforms generated by a generalized translate operator that is associated to the Laplace Bessel operator for $0 < p \leq 1 < q \leq \infty$ with $p \neq q$ through atomic decomposition and molecular characterization.

1. Introduction

The notion of classical Hardy-Morrey space HM^{p}_{q} originates from Jia and Wang [1, 2]. Since then, this theory received continuous development and now is increasingly mature; see, for example [3]-[5].

It is well known that the classical Hardy-Morrey space generalizes both Morrey (M^{p}_{q} , $q > 1$) and Hardy (H^{p} , $p \leq 1$) spaces [6]. It plays important roles in several fields of harmonic analysis and PDEs. Also, these spaces are important because they have close relations with L^{p} spaces, Hardy spaces and BMO^{-1} spaces, and etc.

In recent years, studies in the classical theory of Hardy-Morrey spaces related to some operators have gained great interest and importance. Therefore, our study focused on these spaces. Similar results in other function spaces can be developed in this spaces. These results can be seen in decomposition of Hardy-Morrey spaces, decomposition of Hardy-Morrey spaces with weighted, and decomposition of weighted Hardy-Morrey spaces with variable exponent in [1],[3]-[5].

In this paper, our main purpose is to prove that some properties of Hardy-Morrey spaces, and Hardy-Morrey characterization of the operators depend on conditions via atoms can be obtained. For example, the boundedness of an singular integral operators can be often proved by estimating Ta when a is an atom. While it is generally not true that atoms are mapped into atoms, for many convolution type operators Ta is a function enjoying many of the properties of atoms. Such functions were called molecules. Moreover, classical Hardy spaces and Hardy-Morrey spaces have molecular characterizations that are completely analogous to their atomic characterizations.

We define Hardy-Morrey spaces called HM^{p}_{q,Δ_ν} Hardy-Morrey spaces which was similar with Hardy spaces associated to the following Laplace-Bessel differential operator [7]

$$\Delta_\nu := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}, \quad \nu > 0.$$

The main conclusion of this article is to prove that the B-Riesz transformation defined in (4.1) is a bounded operator from Hardy-Morrey spaces HM^{p}_{q,Δ_ν} to Hardy-Morrey spaces HM^{p}_{q,Δ_ν}. Here $R^{(k)}_{\nu}$,B-Riesz transform related to Laplace-Bessel

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differential operator Δ_ν. This operator has been studied by many mathematicians on weighted Lebesgue spaces (see [8]-[11]). Even though the boundedness of B-Riesz transform is well known for $1 < p < \infty$ on Lebesgue spaces, we cannot say for $0 < p < 1$ on Lebesgue spaces. But these transformations are bounded in Hardy spaces for $0 < p < 1$ (see [7]). Therefore, in this study, a new characterization of the B-Riesz transform obtained by generalized translation has been obtained for $0 < p \leq 1$ in Hardy-Morrey spaces HM^p_{q,Δ_ν}.
We investigate the Hardy-Morrey spaces characterizing boundedness properties of related Riesz transforms called B-Riesz transforms. These operators give us the most popular examples of Calderon-Zygmund singular integral operators. Also these transforms are related to generalized translate operator. Furthermore, they present some applications especially in the area of partial differential equations. To characterize the boundedness of these transforms, we apply the atomic decomposition. By using this decomposition we give the molecular characterizations for HM^p_{q,Δ_ν} Hardy-Morrey spaces. We follow the ideas in [7] to obtain the boundedness of high order B-Riesz transforms on HM^p_{q,Δ_ν} Hardy-Morrey spaces at the end of Section 4 as an application of our main result. For this reason, we pass by other characterizations of HM^p_{q,Δ_ν} Hardy-Morrey spaces.
The remainder of this paper is structured as follows. The HM^p_{q,Δ_ν} Hardy-Morrey spaces are introduced, also their atomic decompositions are given in Section 2. In Section 3, we will give appropriate definition of molecule is given. We will show that each such molecule has an atomic decomposition. As an application, we present the B-Riesz transforms and give its boundedness properties on HM^p_{q,Δ_ν} Hardy-Morrey spaces extending the results in [7].
Throughout this paper, we denote dyadic cubes with Q or J. Moreover, C indicates constant depending on n, ν, p, q.

2. Preliminaries

Let $\mathbb{R}^n$ be the n dimensional Euclidean space and $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $E(x,t) = \{y \in \mathbb{R}^n_+ \ ; \ |x-y| < t\}$ and $E(x,t)^c = \mathbb{R}^n_+ \backslash E(x,t)$. Let us take a measurable set E on $\mathbb{R}^n_+$, we can define

$$|E|_\nu = \int_E x_n^\nu dx,$$

where $\nu > 0$. Denoting $|E(0,r)|_\nu = \omega(n,\nu) r^{n+\nu}$, where

$$\omega(n,\nu) = \int_{E(0,1)} x_n^\nu dx = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{n+\nu-2}{2}\right)}.$$

The generalized translate operator T^y is defined by

$$T^y f(x) = c_\nu \int_0^\pi \ldots \int_0^\pi f\left(x' - y', (x_n, y_n)_\theta\right) \, d\nu(\theta), \tag{2.1}$$

where $c_\nu = \dfrac{\pi^{-\frac{1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$, $(x_n, y_n)_\theta = \sqrt{x_n^2 - 2x_n y_n \cos\theta + y_n^2}$, $d\nu(\theta) = \sin^{\nu-1}\theta \ \ d\theta$ [9, 10, 12, 13]. Note that the generalized translate operator is closely connected with Δ_ν-Laplace-Bessel differential operator denoted by

$$\Delta_\nu = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + B_{x_n}, \qquad B_{x_n} = \frac{\partial^2}{\partial x_n^2} + \frac{\nu}{x_n} \frac{\partial}{\partial x_n}, \quad \nu > 0.$$

The B_{x_n}-convolution operator related to T^y is defined by

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y) T^y g(x) y_n^\gamma dy.$$

Let $L^p_\nu = L^p_\nu(\mathbb{R}^n_+)$ be the space of measurable functions with a finite norm

$$||f||_{L^p_\nu} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\nu dx \right)^{1/p}$$

is denoted by $L^p_\nu \equiv L^p_\nu(\mathbb{R}^n_+)$, $1 \leq p < \infty$. We denote by $\mathscr{S}'_+ = \mathscr{S}'_+\left(\mathbb{R}^n_+\right)$ the topological dual of $\mathscr{S}_+$ is the collection of all tempered distributions on $\mathbb{R}^n_+$.
First, let's start by giving the definition of Morrey space [14, 15].

Definition 2.1. *For p and q satisfying $0 < q \leq p < \infty$, the homogeneous Morrey spaces M^p_q are defined as*

$$M^p_q = \left\{ f \in L^q_{loc} : ||f||_{M^p_q} = \sup_{x \in \mathbb{R}^n, R > 0} |B(x,R)|^{\frac{1}{p} - \frac{1}{q}} ||f||_{L^q(B(x,R))} < \infty \right\},$$

where $B(x,R)$ is the closed ball of $\mathbb{R}^n$ with center x and radius R.

Let $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. The set

$$Q_{jk} = \left\{x \in \mathbb{R}^n : 2^{-j}k_i \le x_i \le 2^{-j}k_{i+1},\ i = 1,2,\dots n\right\},$$

is called a dyadic cube. We remark that

$$||f||_{M_q^p} \approx \sup_{J:dyadic} |J|^{\frac{1}{p}-\frac{1}{q}} ||f||_{L^q(J)}.$$

We now introduce the Hardy-Morrey spaces that we work mainly on and give their decomposition results.
The HM^p_{q,Δ_ν} Hardy-Morrey spaces is given by the set of all distributions $f \in \mathscr{S}_+ \setminus \mathscr{P}$ with the quasi-norm

$$||f||_{HM^p_{q,\Delta_\nu}} = \Big|\Big| \sup_{t>0} |\phi_t \otimes f| \Big|\Big|_{M^p_{q,\nu}}$$

is finite. Here $\phi \in \mathscr{S}_+(\mathbb{R}^n_+)$ satisfies $\int \phi(x) x_n^\nu dx = 1$. Also, $\mathscr{P}$ indicates the set of polynomials.
For the Hardy-Morrey space, if $1 < p \le q \le \infty$, it is obvious that $HM^p_{q,\Delta_\nu} = M^p_{q,\Delta_\nu}$ since the Hardy-Littlewood maximal operator associated with the Laplace-Bessel differential operator Δ_ν is bounded on $M^p_{q,\nu}$. Moreover, the HM^p_{q,Δ_ν} Hardy-Morrey spaces cover Hardy spaces for $0 < p \le 1$. In general, $H^p_{\Delta_\nu} = HM^p_{p,\Delta_\nu} \subset HM^p_{q,\Delta_\nu}$ for $p \le q \le \infty$ and $HM^p_{1,\Delta_\nu} \ne M^p_{1,\Delta_\nu}$. Here , the Hardy spaces $H^p_{\Delta_\nu}$ are defined by

$$H^p_{\Delta_\nu} = \left\{ ||f||_{H^p_{\Delta_\nu}} = \Big\| \sup_{t>0} \phi_t \otimes f \Big\|_{L_p} < \infty \right\}$$

[2].
Now, let us start with to give the definition of (p,q,s)-atoms.

Definition 2.2. *Let $0 < p \le 1 < q \le \infty$ with $p \ne q$ and $s \in \mathbb{N} \cup \{0\}$. For a dyadic cube Q, a function a_Q is called a (p,q,s)-atom of HM^p_{q,Δ_ν} if the following properties are satisfied:*

(i) a_Q *be supported on a cube Q, namely, supp $a_Q \subset Q$,*
(ii) $\|a_Q\|_{L_{q,\nu}} \le |Q|_\nu^{\frac{1}{q}-\frac{1}{p}}$,
(iii) $\int_{\mathbb{R}^n_+} a_Q(x) x^\alpha x_n^\nu dx = 0$ *for all* $s \ge [(n+k+\nu)\left(\frac{1}{p}-1\right)], 1 \le k \le n$, *with* $|\alpha| \le s$.

Also, we introduce atomic decomposition theorem in HM^p_{q,Δ_ν} space is as follows:

Theorem 2.3. *Let $0 < p \le 1 < q \le \infty$ with $p \ne q$, $\{a_Q : Q \text{ dyadic}\}$ be a collection of (p,q,s)-atoms and $\{\lambda_Q : Q \text{ dyadic}\}$ be a sequence of scalars with*

$$||\lambda||_{p,q} = \left\{ \sup_J \left(\frac{1}{|J|_\nu}\right)^{1-p/q} \sum_{Q \subset J} |Q|_\nu^{1-p/q} |\lambda_Q|^p \right\}^{1/p} < \infty.$$

Then the sum

$$f = \sum_Q \lambda_Q a_Q \tag{2.2}$$

converges in $\mathscr{S}'_+ \setminus \mathscr{P}$ and $f \in HM^p_{q,\Delta_\nu}$ with $||f||_{HM^p_{q,\Delta_\nu}} \le C||\lambda||_{p,q}$, where $C = C(n,p,q,\nu)$. Conversely, $\forall f \in HM^p_{q,\Delta_\nu}$ has atomic decomposition (2.2) in $\mathscr{S}'_+ \setminus \mathscr{P}$. Here a_Q are (p,q,s)-atoms and $\lambda = \{\lambda_Q\}$ satisfies $||\lambda||_{p,q} \le C||f||_{HM^p_{q,\Delta_\nu}}$, where $C > 0$ independent of f.

Proof. The proof of Theorem 2.3 can be found in [1, 16], so we omit it here. □

3. Molecular characterizations of HM^p_{q,Δ_ν}

Next, we continue to give the notion of molecule related to HM^p_{q,Δ_ν}. The following definition for molecule is modified from the corresponding definition of molecule from [2].

Definition 3.1. *Let $0 < p \le 1 < q \le \infty$ with $p \ne q$, $s = [(n+k+\nu)\left(\frac{1}{p}-1\right)]$ and $\varepsilon > (n+k+\nu)\left(\frac{1}{p}-\frac{1}{2}\right), 1 \le k \le n$. A measurable function $m_Q(x)$ is called a (p,q,s,ε)-molecule for a dyadic cube Q if and only if*

(i) $\left(\int_{\mathbb{R}^n_+} |m_Q(x)|^2 (1+|x-x_Q|_\nu/\ell_Q)^{2s} x_n^\nu dx\right)^{1/2} \le |Q|_\nu^{1/2-1/p}$, *(this means that ℓ_Q is large)*
(ii) $\int_{\mathbb{R}^n_+} m_Q(x) x^\alpha x_n^\nu dx = 0, |\alpha| \le s$.

Similar to the atomic decomposition of HM^p_{q,Δ_ν} Hardy-Morrey space, the decomposition in terms of molecule is given as follows:

Theorem 3.2. *Let $0 < p \leq 1 < q \leq \infty$ with $p \neq q$ and $\varepsilon > (n+k+\nu)\left(\frac{1}{p}-\frac{1}{2}\right)$. There is exists a sequence of scalars $\{\lambda_Q : Q \text{ dyadic}\}$, a collection of (p,q,s,ε)-molecules $\{m_Q : Q \text{ dyadic}\}$ for HM^p_{q,Δ_ν}, the series*

$$f = \sum_Q \lambda_Q m_Q \tag{3.1}$$

converges in $\mathscr{S}'_+ \setminus \mathscr{P}$ and $f \in HM^p_{q,\Delta_\nu}$ with

$$||f||_{HM^p_{q,\Delta_\nu}} \leq C||\lambda||_{p,q},$$

where $C > 0$ independent of f.

Proof. The proof of this theorem has a similar technique to those of [2, 17, 18]. Let us start with consider the sets

$$E_0 = \{x \in \mathbb{R}^n_+ : |x| \leq \sigma\}$$

$$E_j = \{x \in \mathbb{R}^n_+ : 2^{j-1}\sigma \leq |x| < 2^j\sigma\}, \quad j = 1,2,\ldots,$$

where $\sigma^{(n+k+\nu)\left(\frac{1}{p}-\frac{1}{2}\right)} = ||\lambda||^{-1}_{p,2}$. Set $m_j = m\chi_{E_j}$, where χ_{E_j} is the characteristic function of E_j. For all $j = 1,2,\ldots$, α a multi-index such that $|\alpha| \leq s$, let φ^α_j be the function on E_j (the restriction to E_j of a polynomial of degree at most s). If $P_j = \varphi_j\chi_j$ then

$$\int_{\mathbb{R}^n_+} (m_j - P_j)x^\alpha x_n^\nu dx = 0, |\alpha| \leq s.$$

Since $m = \sum_{j=0}^\infty m_j = \sum(m_j - P_j) + \sum P_j$, to show both $\sum(m_j - P_j)$ and $\sum P_j$ in HM^p_{q,Δ_ν}, it suffices to verify that

(i) each $(m_j - P_j)$ is a multiple of a (p,q,s)-atom with coefficients sum appropriately,
(ii) the sum $\sum P_j$ can be written as an infinite liner combination of (p,∞,s)-atom with coefficients sum appropriately.

For a dyadic cube Q, we define $E_0 = Q$ and for all $j \geq 1$, $Q_j = 2^jQ$ and $E_j = Q_j - Q_{j-1}$. For $j \geq 0$, let $\{\varphi^\alpha_{E_j} : |\alpha| \leq s\}$ (or $\{\Phi^\alpha_{E_j} : |\alpha| \leq s\}$, respectively) be the Gram-Schmidt orthonormalization of monomials $\{x^\alpha : |\alpha| \leq s\}$(or the dual basis of monomials $\{\Phi^\alpha_{E_j} : |\alpha| \leq s\}$, respectively) on E_j according to the weight $1/|E_j|_\nu$. We consider the function $\varphi^\alpha_{E_j}$ to be defined on $\mathbb{R}^n$, having the value zero outside E_j. (namely, if $x \notin E_j$, then we set $\Phi^\alpha_{E_j}(x) = 0$.) By homogeneity and the uniqueness of Gram-Schmidt orthogonalization process (see [18]), we obtain the following estimate

$$|\varphi^\alpha_{E_j}(x)| \leq C, \text{ for } x \in E_j, \tag{3.2}$$

and

$$|\Phi^\alpha_{E_j}(x)| \leq C(2^j\sigma)^{-|\alpha|}, \tag{3.3}$$

where C depends on s. Let m_Q be a molecule function. We set $m_{E_j}(x) = m_Q(x)\chi_{E_j}(x)$ and

$$P_{E_j}(x) = P_{E_j}(m_Q)(x) = \sum_{|\alpha|\leq s} a^\alpha_{E_j}\varphi^\alpha_{E_j}(x) = \sum_{|\alpha|\leq s} m^\alpha_{E_j}\Phi^\alpha_{E_j}(x), \tag{3.4}$$

where

$$a^\alpha_{E_j} = \int m_{E_j}(x)\varphi^\alpha_{E_j}(x)x_n^\nu \frac{dx}{|E_j|_\nu}, m^\alpha_{E_j} = \int m_{E_j}(x)x^\alpha x_n^\nu \frac{dx}{|E_j|_\nu}.$$

From [19], we obtain

$$\begin{aligned} \int (m_{E_j} - P_{E_j})x^\alpha x_n^\nu dx &= 0, \text{for all } |\alpha| \leq s, \\ ||m_{E_j} - P_{E_j}||_{L^2_\nu(E_j)} &\leq C||m_{E_j}||_{L^2_\nu(E_j)} \end{aligned} \tag{3.5}$$

We may write a decomposition of the molecule $m_Q(x)$ as folllows

$$m_Q(x) = \sum_{j=0}^\infty (m_{E_j} - P_{E_j})(x) + \sum_{j=0}^\infty P_{E_j}(x). \tag{3.6}$$

By the equality (3.4), and the cancellation properties of the molecule, we get

$$\sum_{j=0}^{\infty} P_{E_j}(x) = \sum_{j=0}^{\infty}\sum_{|\alpha|\le s}\left(\frac{\Phi^{\alpha}_{E_{j+1}}}{|E_{j+1}|_v} - \frac{\Phi^{\alpha}_{E_j}}{|E_j|_v}\right)E_{Q_{j,\alpha}}, \tag{3.7}$$

here

$$E_{Q_{0,\alpha}} = \sum_{j\ge 0}\int m_{E_j}(x)x^{\alpha}x_n^{v}dx = \int m(x)x^{\alpha}x_n^{v}dx = 0,$$

$$E_{Q_{j,\alpha}} = \sum_{i=j}^{\infty}\int m_{E_j}(x)x^{\alpha}x_n^{v}dx = \int_{|x|\ge 2^j\sigma} m_Q(x)x^{\alpha}x_n^{v}dx, \ \text{ for all } \ j\ge 1.$$

By using (3.6) and (3.7), we may write

$$m_Q(x) = \sum_{j=0}^{\infty} t_{Q_j}a_{Q_j}(x) + \sum_{j\ge 0}\sum_{|\alpha|\le s}\delta_{Q_{j,\alpha}}b_{Q_{j,\alpha}}(x), \tag{3.8}$$

where for each $j\ge 0$

$$t_{Q_j} = ||m_{E_j} - P_{E_j}||_{L^2_v(E_j)}|Q_j|_v^{\frac{1}{p}-\frac{1}{2}}, \ \ a_{Q_j}(x) = \frac{(m_{E_j}-P_{E_j})(x)}{||m_{E_j}-P_{E_j}||_{L^2_v(E_j)}}|Q_j|_v^{\frac{1}{p}-\frac{1}{2}},$$

and

$$\lambda_{Q_{j,\alpha}} = E_{Q_{j,\alpha}}|Q_j|_v^{\frac{1}{p}-1}(2^j\sigma)^{-|\alpha|}, \ b_{Q_{j,\alpha}}(x) = \left(\frac{\Phi^{\alpha}_{E_{j+1}}}{|E_{j+1}|_v} - \frac{\Phi^{\alpha}_{E_j}}{|E_j|_v}\right)|Q_j|_v^{1-\frac{1}{p}}(2^j\sigma)^{|\alpha|}.$$

From the inequalities (3.2), (3.3) and (3.5), it can be easily seen that a_{Q_j} and $b_{Q_{j,\alpha}}$ are supported in a cube Q_j and they are $(p,q,2)$-atoms and (p,q,∞)-atoms respectively. For simplicity, we now just consider the sum (3.1) is finite. Then by (3.8), we obtain

$$f = \sum_{Q,j}\lambda_Q t_{Q_j}a_{Q_j}(x) + \sum_{Q,j}\lambda_Q\sum_{|\alpha|\le s}\delta_{Q_{j,\alpha}}b_{Q_{j,\alpha}}(x) \tag{3.9}$$

in $\mathscr{S}'(\mathbb{R}^n_+)$. Let J be a fixed dyadic cube. We consider the following equality

$$\sum_{Q_j\subset J}|\lambda_Q t_{Q_j}|^p|Q_j|_v^{1-p/q} = \sum_{Q\subset J}|\lambda_Q|^p\sum_{j:Q_j\subset J}|t_{Q_j}|^p|Q_j|_v^{1-p/q}.$$

By the Hölder's inequality, (3.5) and $\varepsilon > (n+k+v)\left(\frac{1}{p}-\frac{1}{2}\right)$, we find that

$$\sum_{j:Q_j\subset J}|t_{Q_j}|^p|Q_j|_v^{1-p/q} \le C|Q|_v^{1-p/q}. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\left|\left|\sum_{Q,j}\lambda_Q t_{Q_j}\right|\right|_{HM^p_{q,\Delta_v}} \le C||\lambda||_{p,q}. \tag{3.11}$$

From an argument similar to that used in above (3.9)-(3.11), it also follows that

$$\left|\left|\sum_{Q}\sum_{j\ge 0}\sum_{Q,j}\lambda_Q t_{Q_j}\right|\right|_{HM^p_{q,\Delta_v}} \le C||\lambda||_{p,q}. \tag{3.12}$$

Combining the inequalities (3.11) and (3.12), we end of the proof if the sum (3.1) is finite. Also, this sum converges in the sense of distributions. □

With the above theorem, we are ready to give the following section which offers an important estimates for Hardy-Morrey spaces related to Laplace-Bessel operator used in the proof of our main result.

4. The B-Riesz transform on Hardy-Morrey spaces HM^p_{q,Δ_ν}

In this section, we restrict ourselves to the high order B-Riesz transforms and give its boundedness properties on Hardy-Morrey spaces. We recall the high order B-Riesz transform.

Definition 4.1. ([8, 9]) *Let $1 \leq p < \infty$ and $f \in L^p_\nu$. B-Riesz transform of f with high order is defined*

$$\begin{aligned} R^{(k)}_\nu(f)(x) &= C_{k,\nu}\big[\,p.v\,\big(\frac{P_k(y)}{|y|^{n+k+\nu}}\otimes f\big)\big](x),\ 1\leq k\leq n,\\ &= C_{k,\nu}\big[\,p.v\,\big(K\otimes f\big)\big](x)\\ &= C_{k,\nu}\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|}\frac{P_k(y)}{|y|^{n+k+\nu}}T^y f(x)y_n^\nu dy, \end{aligned} \tag{4.1}$$

where $C_{k,\nu} = 2^{\frac{n+\nu}{2}}\Gamma(\frac{n+k+\nu}{2})\big[\Gamma(\frac{k}{2})\big]^{-1}$ and $P_k(y) = P_k(y_1,y_2,\ldots,y_{n-1},y_n^2)$ is a homogeneous polynomial of degree k which holds $\triangle_\nu P_k(y) = 0$ on $\mathbb{R}^n_+$. Also, the following two conditions are satisfied for this polynomial:

$$\int_{S_+} P_k(\theta)(\theta^{'})^\nu d\theta = 0 \tag{4.2}$$

and

$$\sup_{\theta\in S_+}|P_k(\theta)| = M < \infty, \tag{4.3}$$

here $S_+ = \{y\in\mathbb{R}^n_+ : |y| = 1\}$ and $\theta = \frac{y}{|y|}$. Also, here T^y denotes the generalized translate operator given in (2.1).

Before establishing the B-Riesz transform characterization of $HM^p_{q,\Delta_\nu}(\mathbb{R}^n_+)$, we first introduce some background on this kernel of this transform.
Let $R^{(k)}_\nu f := K\otimes f$ be defined as in (4.1). There exists a bounded distribution function $K(x)$ with $|F_\nu[K(x)]| \leq C$. We give the following equality

$$F_\nu[R^{(k)}_\nu f](x) = i^k P_k(x)|x|^{-k}F_\nu(f)(x)$$

for all $f\in L^2_\nu$. Here, for any $f\in\mathscr{S}(\mathbb{R}^n_+)$, we use $F_\nu f$ to denote its Fourier-Bessel transform, which is defined by setting

$$F_\nu f(x) \;= \int_{\mathbb{R}^n_+} f(y)\,e^{-i(x'y')}\,j_{\frac{\nu-1}{2}}(x_n y_n)\,y_n^\nu dy, \quad \text{for all } x\in\mathbb{R}^n_+,$$

where $(x'y') = x_1y_1+\ldots+x_{n-1}y_{n-1}$, j_ν, $(\nu > -1/2)$ is Bessel function and $C_{n,\nu} = (2\pi)^{n-1}2^{\nu-1}\Gamma^2((\nu+1)/2) = \frac{2}{\pi}\omega(2,\nu)$. This transform is also associated with Laplace-Bessel differential operator.
Moreover, $K(x)$ satisfies the following Hörmander's condition,

$$\int\limits_{|x|\geq A_1|y|} |T^y K(x) - K(x)|x_n^\nu dx \leq A_2, \tag{4.4}$$

for some $A_1, A_2 < \infty$ (See more detail [20]). So, we conclude that property (4.4) and the L^2_ν-boundedness of $R^{(k)}_\nu f$ maps HM^p_{q,Δ_ν} to itself for $0 < p \leq 1 < q \leq \infty$ with $p\neq q$.
However, we make stronger assumption on kernel, that is $K\in C^\infty(\mathbb{R}^n_+\setminus\{0\})$ satisfies for all $|\alpha|\leq s$ and $x\neq 0$,

$$|D^\alpha_\nu T^y K(x)| \leq AM|x|^{-n-k-\nu-|\alpha|}.$$

We also have the following L^p_ν and $H^p_{\Delta_\nu}$ boundedness of high order B-Riesz transform.

Theorem 4.2. ([8, 21]) *Let P_k be the characteristic of the singular integral (4.1) satisfiying the conditions (4.2) and (4.3). Then there exists a constant $C > 0$ such that for all $1 < p < \infty$ and $\nu > 0$*

$$\|R^{(k)}_\nu(f)\|_{L^p_\nu} \leq CM\|f\|_{L^p_\nu},$$

where C is a constant independent of f and P_k is a homogeneous polynomial of degree k.

Theorem 4.3. ([7]) *Let $R^{(k)}_\nu f := K\otimes f$ and $0 < p \leq 1$. Then there exists a constant $C^*_{n,p,\nu}$ such that for all $f\in H^p_{\Delta_\nu}$*

$$\|K\otimes f\|_{H^p_{\Delta_\nu,at}} \leq C^*_{n,p,\nu}\|f\|_{H^p_{\Delta_\nu,at}} \qquad \nu > 0.$$

The following main theorem demonstrate B-Riesz characterization of HM^p_{q,Δ_ν} Hardy-Morrey spaces.

Theorem 4.4. *Let $0 < p \leq 1 < q \leq \infty$ with $p \neq q$. Then B-Riesz transform can be extended to the bounded transform on Hardy-Morrey spaces HM^p_{q,Δ_ν}.*

Proof. In order to prove this theorem, it is sufficient to show $R^{(k)}_\nu(f)$ is a (p,q,s,ε)-molecule whenever f is a (p,q,s)-atom. We prove this theorem by following the similar strategy used in [7]. Let us take the function supported in the upper half ball $B(0,1)$ with $\int \varphi(x)x^n_\nu dx$ on $\mathbb{R}^n_+$. We define $K^{(t)} = \varphi_t \otimes K$. Then the function $K^{(t)}$ satisfies the following inequalities

$$\sup_{t>0} F_\nu(K^{(t)})(x) \leq C||F_\nu \varphi_t||_{L^\infty_\nu}$$

and

$$\sup_{t>0}(K^{(t)})(x) \leq C_\varphi M|x|^{-n-k-\nu-|\alpha|}, \quad |\alpha| \leq s\ .$$

For a dyadic cube Q, $m_Q(x)$ be a (p,q,s)-molecule and a_Q be a (p,q,s)-atom of HM^p_{q,Δ_ν}. Finally, the proof rests on the checking that $m_Q(x) = R^{(k)}_\nu(a_Q)(x)$ satisfies the moment and size condition. Namely,

(i) $\left(\int_{\mathbb{R}^n_+} |R^{(k)}_\nu a_Q(x)|^2(1+|x-x_Q|_\nu/\sigma)^{2s}x^\nu_n dx\right)^{1/2} \leq |Q|^{1/2-1/p}_\nu$,

(ii) $\int_{\mathbb{R}^n_+} R^{(k)}_\nu a_Q(x)x^\alpha x^\nu_n dx = 0, \ \ |\alpha| \leq s.$

So, we omit the details and leave it to the reader. □

5. Conclusion

In this study, the decomposition of Hardy-Morrey spaces related to the Laplace-Bessel differential operator are introduced in terms of atoms and molecules. Also, we give the HM^p_{q,Δ_ν} boundedness of higher order B-Riesz transforms for $0 < q \leq p < \infty$ by using this atomic decomposition and molecular characterization. We follow the similar approach for developing the atomic decomposition and molecular characterization as classical Hardy-Morrey spaces. The interesting of our result depends on the existence of the different differential operator.

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Author's contributions

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Approximation by Modified Bivariate Bernstein-Durrmeyer and GBS Bivariate Bernstein-Durrmeyer Operators on a Triangular Region

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Abstract

In this paper, the approximation properties and the rate of convergence of modified bivariate Bernstein-Durrmeyer Operators on a triangular region are examined. Furthermore, definitions and some properties of modulus of continuity for functions of two variables are given. Voronovskaya and Grüss Voronovskaja type theorems are used to determine the order of approximation. The GBS (Generalized Boolean Sum) operator of Bivariate Bernstein-Durrmeyer type on a triangular region is studied. Lastly, some numerical examples are given and related graphs are plotted for comparison.

1. Introduction

Classical approximation theory, including polynomial approximation is a fundamental research area in applied mathematics. Development in approximation theory plays an important role in numerical solution of partial differential equations, image processing as well as in data sciences and many other disciplines. For example, radial basis functions and shift-invariant spaces are widely used for geometric modeling in aerospace and automobile industries [1]. In this paper we intend to study the approximation properties of functions of two variables by means of Bernstein-Durrmeyer operator in a triangular domain. Several studies have been conducted on the classical Bernstein operators, as well as using two variables.

From literature , Kingsley [2] proposed the Bernstein operator of two-variables. Pop [3] added some features to the Bernstein operators, defined by Kingsley. Stancu [4] defined two variables of Bernstein operators on the triangular region. Pop and Farcas [5] researched the approximation features of the Bernstein-Kantorovich operators on the triangular region. In [6], authors examined the weighted approximation features of two-variables by Bernstein -Stancu-Chlodowsky polynomials in a triangular region. In 1992, Zhou [7] defined the two variables of Bernstein-Durrmeyer polynomials and obtained the rate of convergence of the functions in L_p spaces.

Some generalization of these polynomials in the one-dimensional case may be found in [8]-[18].

In the light of these studies, we defined the new generalized operator that we think will get better results.

Let $V := \left\{(u,v) \in \mathbb{R}^2 : -1 \leq v \leq 1, -1 \leq u \text{ and } u+v \leq 0\right\}$ and $h \in C(V)$, we will examine the Bernstein-Durrmeyer operator of two variables on a triangular region as

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$$H_n(h;u.v)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\varphi_{n,k,l}(u,v)\frac{(n+1)(n+2)}{16}\int_{-1}^{1}\int_{-1}^{-t}\varphi_{n,k,l}(s,t)h(s,t)dsdt$$

in which

$$\varphi_{n,k,l}(u,v)=\binom{n}{k}\binom{n-k}{l}\left(\frac{1+u}{2}\right)^k\left(\frac{1+v}{2}\right)^l\left(1-\frac{1+u}{2}-\frac{1+v}{2}\right)^{n-k-l}$$

In this paper, the approximation features and the speed of approximation of Modified Bivariate Bernstein-Durrmeyer Operators on a Triangular Region will be examined. Furthermore, definitions and some features of moduli of continuity of two variables function are given.We examine the order of approximation by Voronovskaya type theorem and Grüss Voronovskaja type theorem. The GBS (Generalized Boolean Sum) operators of Bivariate Bernstein-Durrmeyer type on a triangular region will be studied. Lastly some numerical examples and the graphics will be drawn.

2. Preliminary results

Theorem 2.1. *For* $e_{i,j}=s^it^j,(i,j)\in\mathbb{N}^0\times\mathbb{N}^0,\mathbb{N}^0=\mathbb{N}\cup\{0\}$, *we have*

i) $H_n(e_{0,0};u,v)=1.$

ii) $H_n(e_{1,0};u,v)=u-\dfrac{3u+1}{n+3}.$

iii) $H_n(e_{0,1};u,v)=v-\dfrac{3v+1}{n+3}.$

iv) $H_n(e_{2,0};u,v)=u^2-\dfrac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)}.$

v) $H_n(e_{0,2};u,v)=v^2-\dfrac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)}.$

vi) $H_n(e_{1,1};u,v)=uv-\dfrac{(8n+3)uv-3n(u+v)-2n}{(n+3)(n+4)}.$

vii) $$H_n(e_{4,0};u,v)=u^4-\frac{(24n^3+108n^2+348n+360)}{(n+3)(n+4)(n+5)(n+6)}u^4+\frac{(-4n^3+12n^2-8n)}{(n+3)(n+4)(n+5)(n+6)}u^3+\frac{(12n^3+12n^2-24n)}{(n+3)(n+4)(n+5)(n+6)}u^2+\frac{(-24n^2-48n)}{(n+3)(n+4)(n+5)(n+6)}u+\frac{(12n^2+60n+72)}{(n+3)(n+4)(n+5)(n+6)}.$$

viii) $$H_n(e_{0,4};u,v)=v^4-\frac{(24n^3+108n^2+348n+360)}{(n+3)(n+4)(n+5)(n+6)}v^4+\frac{(-4n^3+12n^2-8n)}{(n+3)(n+4)(n+5)(n+6)}v^3+\frac{(12n^3+12n^2-24n)}{(n+3)(n+4)(n+5)(n+6)}v^2+\frac{(-24n^2-48n)}{(n+3)(n+4)(n+5)(n+6)}v+\frac{(12n^2+60n+72)}{(n+3)(n+4)(n+5)(n+6)}.$$

Theorem 2.2. *For* $k_{i,j}=(s-u)^i(t-v)^j,(i,j)\in\mathbb{N}^0\times\mathbb{N}^0$, *we have*

i) $H_n(k_{0,0};u,v)=1.$

ii) $H_n(k_{1,0};u,v)=-\dfrac{3u+1}{n+3}.$

iii) $H_n(k_{0,1};u,v)=-\dfrac{3v+1}{n+3}.$

iv) $H_n(k_{2,0};u,v)=-\dfrac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)}.$

v) $H_n(k_{0,2};u,v)=-\dfrac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)}.$

vi) $$H_n(k_{4,0};u,v)=\frac{12n^2-588n-936}{(n+3)(n+4)(n+5)(n+6)}u^4+\frac{144n+480}{(n+3)(n+4)(n+5)(n+6)}u^3+\frac{-24n^2+312n+720}{(n+3)(n+4)(n+5)(n+6)}u^2+\frac{144n+288}{(n+3)(n+4)(n+5)(n+6)}u.$$

vii) $$H_n(h_{0,4};u,v)=\frac{12n^2-588n-936}{(n+3)(n+4)(n+5)(n+6)}v^4+\frac{144n+480}{(n+3)(n+4)(n+5)(n+6)}v^3+\frac{-24n^2+312n+720}{(n+3)(n+4)(n+5)(n+6)}v^2+\frac{144n+288}{(n+3)(n+4)(n+5)(n+6)}v.$$

Theorem 2.3. *For the bivariate operators* $H_n(f;u,v)$, *we have*

i) $\lim\limits_{n\longrightarrow\infty} nH_n((s-u);u,v) = -(3u+1)$.
ii) $\lim\limits_{n\longrightarrow\infty} nH_n((t-v);u,v) = -(3v+1)$.
iii) $\lim\limits_{n\longrightarrow\infty} nH_n((s-u)^2;u,v) = -(8u^2+2u-2)$.
iv) $\lim\limits_{n\longrightarrow\infty} nH_n((t-v)^2;u,v) = -(8v^2+2v-2)$.
v) $\lim\limits_{n\longrightarrow\infty} nH_n((s-u)(t-v);u,v) = -2uv+4(u+v)+2$.

Theorem 2.4. *From Theorem 2.1, we get*

$$H_n((s-u)^2;u,v) \leq \frac{3}{n}.$$

$$H_n((t-v)^2;u,v) \leq \frac{3}{n}.$$

Proof. For all $u \in [-1,1]$, we write

$$H_n((s-u)^2;u,v) = -\frac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)}.$$

$$H_n((t-v)^2;u,v) = -\frac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)}.$$

If we take the max values of the equations we have obtained, we get $u = \frac{-n}{8n+12}$ and $v = \frac{-n}{8n+12}$. From here

$$\begin{aligned} H_n((s-u)^2;u,v) &= -\frac{8nu^2+2nu-2n+12u^2-4}{(n+3)(n+4)} \\ &= \frac{136n^3+652n^2+1056n+576}{64n^4+640n^3+2112n^2+3456n+1728} \leq \frac{3}{n} \end{aligned}$$

and

$$\begin{aligned} H_n((t-v)^2;u,v) &= -\frac{8nv^2+2nv-2n+12v^2-4}{(n+3)(n+4)} \\ &= \frac{136n^3+652n^2+1056n+576}{64n^4+640n^3+2112n^2+3456n+1728} \leq \frac{3}{n} \end{aligned}$$

are obtained and proof is completed. □

3. Main results

Basic convergence theorem.

Theorem 3.1. *Let* $V := \{(u,v) : v \leq 1, -1 \leq u \text{ and } u+v \leq 0\}$ *and* $h \in C(V) := \{h : V \to R,\ f \text{ is continuous}\}$; $H_n(h;u,v) : C(V) \to C(R)$ *be linear positive operators.If*
i) $\lim\limits_{n\longrightarrow\infty} H_n(1;u,v) = 1$
ii) $\lim\limits_{n\longrightarrow\infty} H_n(s;u,v) = u$
iii) $\lim\limits_{n\longrightarrow\infty} H_n(t;u,v) = v$
iv) $\lim\limits_{n\longrightarrow\infty} H_n((s^2+t^2;u,v) = u^2+v^2$
H_n *converges to* h*, for* $h \in C(V)$.

3.1. Degree of approximation by $H_n(h;u,v)$

For $h \in C(V)$, the complete moduli of continuity for the two-variable functions is defined as:

$$\omega(h,\delta) = \max_{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2} \leq \delta} |h(u_1,v_1)-h(u_2,v_2)|$$

here $(u_1,v_1),(u_2,v_2) \in V$.
In addition, partial continuity moduli according to u and v are defined as ;

$$\omega^{(1)}(h,\delta) = \max_{|u_1-u_2| \leq \delta} |h(u_1,v)-h(u_2,v)|$$

here $(u_1,v),(u_2,v) \in V$.

$$\omega^{(2)}(h,\delta) = \max_{|v_1-v_2| \leq \delta} |h(u,v_1)-h(u,v_2)|$$

here $(u,v_1),(u,v_2) \in V$.
It is seen that they provide the characteristics of the continuity modulus. In what follows, $\omega(h,\delta) \leq (1+\lambda)\ \omega(h,\delta)$ and $\lim\limits_{\delta\to 0}\omega(h,\delta)=0$.

Theorem 3.2. *Let $h \in C(V)$, we have*

$$\|H_n(h;u,v)-h(u,v)\|_{C(V)} \leq 3\omega\left(h,\frac{1}{\sqrt{n}}\right)$$

Proof. From the well-known features of modulus of continuity, we have

$$|h(s,t)-h(u,v)| \ \leq\ \omega(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2+(t-v)^2\right)^{\frac{1}{2}}\right)$$

Using Cauchy-Schwartz inequality and Theorem 2.2, we obtain

$$\begin{aligned}
|H_n(h;u,v)-h(u,v)| &= |H_n(h(s,t)-h(u,v);u,v)| \\
&\leq H_n(|h(s,t)-h(u,v)|;u,v) \\
&\leq H_n(\omega(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2+(t-v)^2\right)^{\frac{1}{2}}\right);u,v) \\
&\leq \omega(h,\delta)\left(1+\frac{1}{\delta}\left(H_n((s-u)^2+(t-v)^2);u,v\right)^{\frac{1}{2}}\right) \\
&= \omega(h,\delta)\left(1+\frac{1}{\delta}\left(H_n\left((s-u)^2;u,v\right)+H_n\left((t-v)^2;u,v\right)\right)^{\frac{1}{2}}\right) \\
&= \omega(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{(-8n-12)u^2-2nu+2n-4}{(n+3)(n+4)}\right.\right. \\
&\qquad \left.\left.+\frac{(-8n-12)v^2-2nv+2n-4}{(n+3)(n+4)}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

Moreover, if we calculate maximum value of the square root and $\delta=\frac{1}{\sqrt{n}}$, then we obtain

$$=\omega(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{4}{n}\right)^{\frac{1}{2}}\right)=\omega\left(f,\frac{1}{\sqrt{n}}\right)\left(1+\frac{1}{\delta}\left(\frac{4}{n}\right)^{\frac{1}{2}}\right)=3\omega\left(f,\frac{1}{\sqrt{n}}\right)$$

□

Theorem 3.3. *Let $h \in C(V)$, then the following inequality holds.*

$$\|H_n(h;u,v)-h(u,v)\|_{C(V)} \leq \left(1+\sqrt{3}\right)\left(\omega^{(1)}\left(h,\frac{1}{\sqrt{n}}\right)+\omega^{(2)}\left(h,\frac{1}{\sqrt{n}}\right)\right)$$

Proof. From the well-known features of modulus of continuity, we have

$$|h(s,t)-h(u,v)| \leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2\right)^{\frac{1}{2}}\right)+\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left((t-v)^2\right)^{\frac{1}{2}}\right)$$

Using Cauchy-Schwartz inequality and Theorem 2.2, we obtain

$$\begin{aligned}
|H_n(h;u,v)-h(u,v)| &= |H_n(h(s,t)-h(u,v);u,v)| \\
&\leq H_n(|h(s,t)-h(u,v)|;u,v) \\
&\leq H_n\left(\omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left((s-u)^2\right)^{\frac{1}{2}}\right)\right. \\
&\quad \left.+\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left((t-v)^2\right)^{\frac{1}{2}}\right);u,v\right) \\
&\leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left(H_n\left((s-u)^2;u,v\right)\right)^{\frac{1}{2}}\right) \\
&\quad +\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left(H_n\left((t-v)^2;u,v\right)\right)^{\frac{1}{2}}\right) \\
&\leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{(-8n-12)u^2-2nu+2n-4}{(n+3)(n+4)}\right)^{\frac{1}{2}}\right) \\
&\quad +\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{(-8n-12)v^2-2nv+2n-4}{(n+3)(n+4)}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

Moreover, if we calculate maximum value of the square root and $\delta=\frac{1}{\sqrt{n}}$, then we obtain

$$\begin{aligned}
&\leq \omega^{(1)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right)+\omega^{(2)}(h,\delta)\left(1+\frac{1}{\delta}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right) \\
&= \omega^{(1)}\left(h,\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{n}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right)+\omega^{(2)}\left(h,\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{n}\left(\frac{3}{n}\right)^{\frac{1}{2}}\right) \\
&= \left(1+\sqrt{3}\right)\left(\omega^{(1)}\left(h,\frac{1}{\sqrt{n}}\right)+\omega^{(2)}\left(h,\frac{1}{\sqrt{n}}\right)\right)
\end{aligned}$$

□

3.2. The Voronovskaja-type result

Theorem 3.4. *For* $\forall\, h\in C^2(V)$*, we have*

$$\begin{aligned}
\lim_{n\to\infty} n.\left(H_n(h;u,v)-h(u,v)\right) &= (-3u-1)\,h_u(u,v)+(-3v-1)\,h_v(u,v) \\
&\quad +\left(-4u^2-u+1\right)h_{uu}(u,v)+(-4uv+8(u+v)+4)\,h_{uv}(u,v) \\
&\quad +\left(-4v^2-v+1\right)h_{vv}(u,v)
\end{aligned}$$

uniformly in $(u,v)\in V$.

Proof. If we apply Taylor's formula to $h\in C^2(V)$,

$$\begin{aligned}
h(s,t) &= h(u,v)+h_u(u,v)(s-u)+h_v(u,v)(t-v) \\
&\quad +\frac{1}{2}\left\{h_{uu}(u,v)(s-u)^2+2h_{uv}(u,v)(s-u)(t-v)+h_{vv}(u,v)(t-v)^2\right\} \\
&\quad +\mho(s,t)\left((s-u)^2+(t-v)^2\right)
\end{aligned}$$

here $\mho(.,.;u,v)=\mho(.,.)\in C(V)$ represents the remainder of the Taylor formula. $\mho(.,.)\in C(V)$ is defined in this way

$$\mho(s,t;u,v)=\begin{cases} \dfrac{h(s,t)-h(u,v)-h_u(s-u)-h_v(t-v)-\frac{1}{2}\left\{h_{uu}(s-u)^2+2h_{uv}(s-u)(t-v)+h_{vv}(t-v)^2\right\}}{\sqrt{(s-u)^4+(t-v)^4}} & ,(s,t)\neq(u,v) \\ 0 & ,(s,t)=(u,v) \end{cases}$$

Then, H_n is a linear-positive operator, we write

$$\begin{aligned} H_n\left(h(s,t);u,v\right) &= h(u,v)+h_u(u,v)H_n\left((s-u);u,v\right)+h_v(u,v)H_n\left((t-v);u,v\right) \\ &+\frac{1}{2}\left\{h_{u\,u}(u,v)H_n\left((s-u)^2;u,v\right)+2h_{uv}(u,v)H_n\left((s-u)(t-v);u,v\right)\right. \\ &\left.+\,h_{vv}(u,v)H_n\left((t-v)^2;u,v\right)\right\}+H_n\left(\mho(s,t)\left((s-u)^2+(t-v)^2\right);u,v\right) \end{aligned}$$

Now, let us use the Cauchy-Schwarz inequality in the last term of the last equation,

$$\begin{aligned} & \left|H_n\left(\mho(s,t)\left((s-u)^2+(t-v)^2\right);u,v\right)\right| \\ \leq & \left|H_n\left(\mho(s,t)\sqrt{(s-u)^4+(t-v)^4};u,v\right)\right| \\ \leq & \left\{H_n\left(\mho^2(s,t);u,v\right)\right\}^{\frac{1}{2}}\left\{H_n\left((s-u)^4+(t-v)^4;u,v\right)\right\}^{\frac{1}{2}} \\ = & \left\{H_n\left(\mho^2(s,t);u,v\right)\right\}^{\frac{1}{2}}\left\{H_n\left((s-u)^4;u,v\right)+H_n\left((t-v)^4;u,v\right)\right\}^{\frac{1}{2}} \end{aligned}$$

Since $\mho(.,.;u,v)\in C(V)$ and $\mho(s,t;u,v)\to 0$ as $(s,t)\to(u,v)$ applying Theorem 3.1

$$\lim_{n\to\infty} H_n\left(\mho^2(s,t);u,v\right)=\mho^2(s,t)=0$$

as a result

$$\lim_{n\to\infty} n.\left(H_n\left(\mho(s,t)\sqrt{(s-u)^4+(t-v)^4};u,v\right)\right)=0$$

Then applying Theorem 2.3 and last equality, we have

$$\begin{aligned} \lim_{n\to\infty} n.\left(H_n\left(h(s,t);u,v\right)-h(u,v)\right) &= (-3u-1)\,h_u(u,v)+(-3v-1)\,h_v(u,v)+\left(-4u^2-u+1\right)h_{u\,u}(u,v) \\ &+(-4uv+8(u+v)+4)\,h_{uv}(u,v)+\left(-4v^2-v+1\right)h_{vv}(u,v) \end{aligned}$$

Thus, the proof is completed. □

3.3. The Grüss Voronovskaja-type result

Theorem 3.5. *Let $h''\in C^2(V)\,, w''\in C^2(V)$ then we write*

$$\begin{aligned} \lim_{n\to\infty} n\left\{H_n\left(hw;u,v\right)-h(u,v)w(u,v)\right\} &= (2-2u-8u^2)h_u(u,v)w_u(u,v) \\ &+(4(u+v)-2uv+2)\left[h_u(u,v)w_v(u,v)+h_v(u,v)w_u(u,v)\right] \\ &+(2-2v-8v^2)h_v(u,v)w_v(u,v) \end{aligned}$$

Proof. In this study, we examine $n\left\{H_n\left(hw;u,v\right)-h(u,v)w(u,v)\right\}$

$$\begin{aligned} &= n\left\{H_n\left(hw;u,v\right)-h(u,v)w(u,v)-\left[h(u,v)w_u(u,v)+h_u(u,v)w(u,v)\right]H_n\left((s-u);u,v\right)\right. \\ &-\left[h(u,v)w_v(u,v)+h_v(u,v)w(u,v)\right]H_n\left((t-v);u,v\right) \\ &-\tfrac{1}{2}\left[h(u,v)w_{u\,u}(u,v)+2h_u^{'}(u,v)w_u(u,v)+h_{u\,u}(u,v)w(u,v)\right]H_n\left((s-u)^2;u,v\right) \\ &-\left[h(u,v)w_{uv}(u,v)+h_u(u,v)w_v(u,v)+h_v(u,v)w_u(u,v)+h_{uv}(u,v)w(u,v)\right]H_n\left((t-v)(s-u);u,v\right) \\ &-\tfrac{1}{2}\left[h(u,v)w_{vv}(u,v)+2h_v^{'}(u,v)w_v(u,v)+h_{vv}(u,v)w(u,v)\right]H_n\left((t-v)^2;u,v\right) \\ &-w(u,v)\left[H_n\left(h;u,v\right)-h(u,v)-h_u(u,v)H_n\left((s-u);u,v\right)-h_v(u,v)H_n\left((t-v);u,v\right)\right. \\ &\left.-\tfrac{1}{2}h_{u\,u}(u,v)H_n\left((s-u)^2;u,v\right)-h_{uv}(u,v)H_n\left((t-v)(s-u);u,v\right)-\tfrac{1}{2}h_{vv}(u,v)H_n\left((t-v)^2;u,v\right)\right] \\ &-H_n\left(h;u,v\right)\left[H_n\left(w;u,v\right)-w(u,v)-w_u(u,v)H_n\left((s-u);u,v\right)-w_v(u,v)H_n\left((t-v);u,v\right)\right. \\ &\left.-\tfrac{1}{2}w_{u\,u}(u,v)H_n\left((s-u)^2;u,v\right)-w_{uv}(u,v)H_n\left((s-u)(t-v);u,v\right)-\tfrac{1}{2}w_{vv}(u,v)H_n\left((s-u)^2;u,v\right)\right] \\ &-w_u(u,v)H_n\left((s-u);u,v\right)\left[H_n\left(h;u,v\right)-h(u,v)\right]-w_v(u,v)H_n\left((t-v);u,v\right)\left[H_n\left(h;u,v\right)-h(u,v)\right] \\ &-w_{u\,u}(u,v)\frac{H_n\left((s-u)^2;u,v\right)}{2}\left[H_n\left(h;u,v\right)-h(u,v)\right]-w_{vv}(u,v)\frac{H_n\left((t-v)^2;u,v\right)}{2}\left[H_n\left(h;u,v\right)-h(u,v)\right] \\ &-w_{uv}(u,v)H_n\left((t-v)(s-u);u,v\right)\left[H_n\left(h;u,v\right)-h(u,v)\right]+h_u(u,v)w_u(u,v)H_n\left((s-u)^2;u,v\right) \\ &+h_u(u,v)w_v(u,v)H_n\left((t-v)(s-u);u,v\right)+h_v(u,v)w_u(u,v)H_n\left((t-v)(s-u);u,v\right) \\ &\left.+h_v(u,v)w_v(u,v)H_n\left((t-v)^2;u,v\right)\right\} \end{aligned}$$

Then, applying Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 3.4, we have

$$\begin{aligned} \lim_{n\to\infty} n\left\{H_n\left(hw;u,v\right)-h(u,v)w(u,v)\right\} &= (2-2u-8u^2)h_u(u,v)w_u(u,v) \\ &+(4(u+v)-2uv+2)\left[h_u(u,v)w_v(u,v)+h_v(u,v)w_u(u,v)\right] \\ &+(2-2v-8v^2)h_v(u,v)w_v(u,v) \end{aligned}$$

the proof is completed. □

4. GBS of Bivariate Bernstein-Durrmeyer Operators on a Triangular Region

In [19, 20], the author has defined B-continuous and B-differentiable functions. The expression of Generalized Boolean Sum (GBS) operators was first defined by Badea in [21, 22]. Dobrescu and Matei [23], introduced the approximation features of the two-variable Bernstein GBS operators. Recently, some researchers have made different researches on GBS in the approximation theory [24]-[27]. In this study, we examined the uniform approximation of B-continuous functions using bivariate Bernstein-Durrmeyer GBS operators on a triangular region.
Let $\nabla_{(u,v)}h\,[u_0,v_0;u,v]$ be mixed difference and Φ and Λ be compact real spacing of h defined by

$$\nabla_{(u,v)}h\,[u_0,v_0;u,v]=h(u,v)-h(u,v_0)-h(u_0,v)+h(u_0,v_0)$$

The function $h:\Phi\times\Lambda\to\mathbb{R}$ is called B-continuous function for $(u_0,v_0)\in\Phi\times\Lambda$.

$$\lim_{(u,v)\to(u_0,v_0)}\nabla_{(u,v)}h\,[u_0,v_0;u,v]=0$$

for each $(u,v)\in\Phi\times\Lambda$. Let $C_b(V)$ indicate the space of whole B-continuous functions on V. Here, $C(V)\subset C_b(V)$ [19, 20].

The GBS (Generalized Boolean Sum) associated with $H_n\,(h;u,v)$ defined as

$$E_n(h;u,v)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\varphi_{n,k,l}(u,v)\frac{(n+1)(n+2)}{16}\int_{-1}^{1}\int_{-1}^{-t}\varphi_{n,k,l}(s,t)\,(h(u,t)+h(s,v)-h(s,t))\,dsdt \tag{4.1}$$

for every $h\in C_b(V)$ at each point $(u,v)\in V$. It is clear that $E_n(h;u,v)$ is a linear and positive operator.

4.1. Approximation by GBS operator $E_n(h;u,v)$

The mixed modulus of smoothness of $h\in C_b(V)$ is defined by

$$\omega_{mixed}\,(h;\delta_1,\delta_2):=\sup\{|\nabla h\,[(s,t);(u,v)]|:|u-s|<\delta_1,|v-t|<\delta_2\}$$

for all $(u,v),(s,t)\in V$ and for any $\delta_1,\delta_2\in\mathbb{R}^+$.
The features of mixed moduli of continuity ;

$$\omega_{mixed}\,(h;\lambda_1\delta_1,\lambda_1\delta_2)\le(1+\lambda_1)\,(1+\lambda_2)\,\omega_{mixed}\,(h;\delta_1,\delta_2)$$

we can write,

$$\begin{aligned}|\nabla h[(s,t);(u,v)]| &\le \omega_{mixed}\,(h;|s-u|,|t-v|)\\ &\le \left(1+\frac{|s-u|}{\delta_1}\right)\left(1+\frac{|t-u|}{\delta_2}\right)\omega_{mixed}\,(h;\delta_1,\delta_2)\end{aligned}$$

Theorem 4.1. *For $\forall\, h\in C_b\,(V)$ at all point $(u,v)\in V$, the $E_n(h;u,v)$ operator provides the following disparity*

$$|E_n(h;u,v)-h(u,v)|\le 8\;\omega_{mixed}\,(h;\delta_1(n),\delta_2(n))$$

Proof. From the well-known features of mixed moduli of continuity and by the definition of mixed difference, we have

$$\nabla_{(u,v)}h\,[(t,s);u,v]=h(u,t)+h(s,v)-h(s,t)$$

and

$$E_n(h;u,v)-h(u,v)=-H_n(\nabla_{(u,v)}h\,[(s,t);(u,v)]\,;u,v)$$

Then using Cauchy-Schwarz inequality, we have,

$$\begin{aligned}|E_n(h;u,v)-h(u,v)| &\le H_n\left|(\nabla_{(u,v)}h\,[(s,t);u,v]\,;u,v)\right|\\ &\le \Big(H_n(e_{00})+\delta_1^{-1}\sqrt{H_n\left((s-u)^2;u,v\right)}+\delta_2^{-1}\sqrt{H_n\left((t-v)^2;u,v\right)}\\ &\quad+\delta_1^{-1}\delta_2^{-1}\sqrt{H_n\left((s-u)^2;u,v\right)H_n\left((t-v)^2;u,v\right)}\Big)\,\omega_{mixed}\,(h;\delta_1(n),\delta_2(n))\end{aligned}$$

Then, applying Theorem 2.1 and Theorem 2.4

$$
\begin{aligned}
|E_n(h;u,v)-h(u,v)| &\leq H_n\left|(\nabla_{(u,v)}h[(s,t);u,v];u,v)\right| \\
&\leq \left(1+\delta_1^{-1}\sqrt{\frac{3}{n}}+\delta_2^{-1}\sqrt{\frac{3}{n}}+\delta_1^{-1}\delta_2^{-1}\sqrt{\frac{3}{n}\frac{3}{n}}\right)\omega_{mixed}(h;\delta_1(n),\delta_2(n))
\end{aligned}
$$

Therefore, taking $\delta_1 = n^{-\frac{1}{2}}$ and $\delta_2 = n^{-\frac{1}{2}}$, We achieve the desired result

$$
|E_n(h;u,v)-h(u,v)| \leq 8\ \omega_{mixed}(h;\delta_1(n),\delta_2(n))
$$

□

4.2. Approximation for the $E_n(h;u,v)$ operators with functions in Lipschitz class

The Lipschitz class $Lip_\beta(\mu,\eta)$ with $\mu,\eta \in (0,1]$ for $h \in C_b(V)$ B-continuous functions is defined as

$$
Lip_\beta(\mu,\eta) = \left\{h \in C_b(V) : \left|\nabla_{(u,v)}h[(s,t);(u,v)]\right| \leq \beta\ |s-u|^\mu |t-v|^\eta\right\} \tag{4.2}
$$

here $(s,t),(u,v) \in V$.

Theorem 4.2. *For $h \in Lip_\beta(\mu,\eta)$, we have*

$$
|E_n(h;u,v)-h(u,v)| \leq \beta\ \Psi_n(u)^{\frac{\mu}{2}}\Psi_n(v)^{\frac{\eta}{2}}
$$

where $\Psi_n(u) = H_n\left((s-u)^2;u,v\right)$ and $\Psi_n(v) = H_n\left((t-v)^2;u,v\right)$

Proof. From (4.1) and (4.2), we may write

$$
\begin{aligned}
|E_n(h;u,v)-h(u,v)| &\leq H_n\left(\left|\nabla_{(u,v)}h[(s,t);(u,v)]\right|;u,v\right) \\
&\leq \beta H_n\left(|s-u|^\mu |t-v|^\eta;u,v\right) \\
&= \beta H_n\left(|s-u|^\mu;u,v\right) H_n\left(|t-v|^\eta;u,v\right)
\end{aligned}
$$

Applying the Hölder's inequality with $(p_1,q_1) = \left(\frac{2}{\mu},\frac{2}{2-\mu}\right)$ and $(p_2,q_2) = \left(\frac{2}{\eta},\frac{2}{2-\eta}\right)$, we get

$$
\begin{aligned}
|E_n(h;u,v)-h(u,v)| &\leq \beta\left(H_n\left((s-u)^2;u,v\right)^{\frac{\mu}{2}} H_n\left(e_{0,0};u,v\right)^{\frac{2-\mu}{2}}\right. \\
&\quad \left.\times H_n\left((t-v)^2;u,v\right)^{\frac{\eta}{2}} H_n\left(e_{0,0};u,v\right)^{\frac{2-\eta}{2}}\right) \\
&\leq \beta\Psi_n(u)^{\frac{\mu}{2}}\Psi_n(v)^{\frac{\eta}{2}}
\end{aligned}
$$

the proof is completed. □

For $(u,v) = (0.05,-0.05)$ in Table 1, we calculated the error in the approximation of $H_n(h;u,v)$ operator and $E_n(h;u,v)$ GBS operator at certain n values. Here $h : V \to R$; $h(u,v) = \left|u^2v^2\right|$

n	$\lvert H_n(h;u,v)-h(u,v)\rvert$	$\lvert E_n(h;u,v)-h(u,v)\rvert$
10	0.008858399150	0.008704518631
25	0.003920956488	0.003831667796
50	0.002201491056	0.002140252626
100	0.001415762625	0.001310766823

Table 1: Error bounds at different n values for $H_n(h;u,v)$ and $E_n(h;u,v)$ GBS operator .

Example 4.3. *The convergence of $H_n(h;u,v)$ operator for n=1 (brown),n=5 (yellow),n=10 (green), n=20 (red) to the function $h(u,v) = \left|u^2v^2\right|$ (blue) is pictorial as shown in Figure 4.1.*

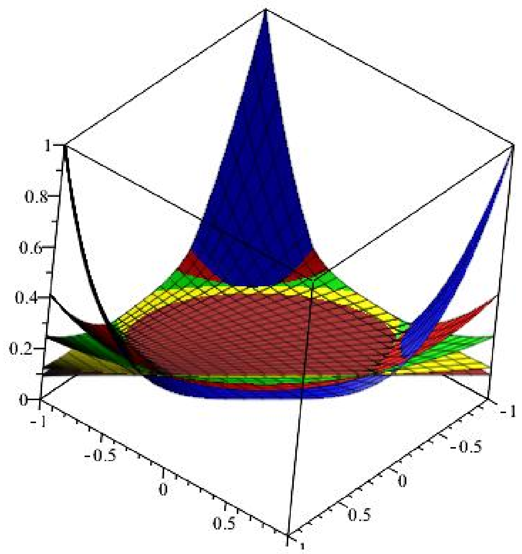

Figure 4.1: The convergence of the $H_n(h;u,v)$ operator to the function $h(u,v)$.

Example 4.4. *The convergence of $E_n(h;u,v)$ GBS operator for n=1 (brown),n=5 (yellow), n=10 (green), n=20 (red) to the function $h(u,v) = \left|u^2v^2\right|$ (blue) is pictorial as shown in Figure 4.2.*

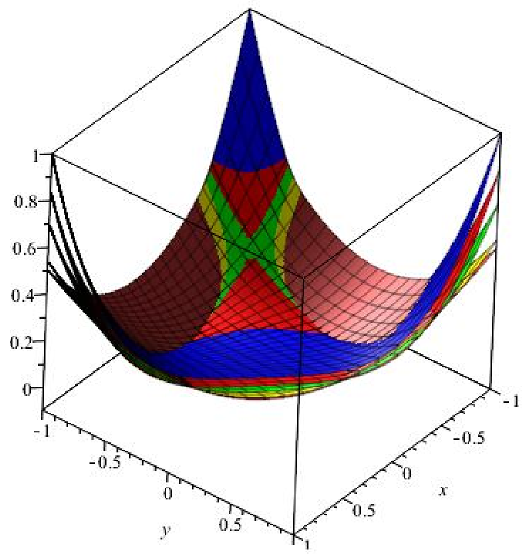

Figure 4.2: The convergence of the $E_n(h;u,v)$ operator to the function $h(u,v)$.

Example 4.5. *For $n = 50$,The convergence of $H_n(h;u,v)$ operator (green) and $E_n(h;u,v)$ GBS operator (red) to the function $h(u,v) = \left|u^2v^2\right|$ (blue) is pictorial as shown in Figure 4.3.*

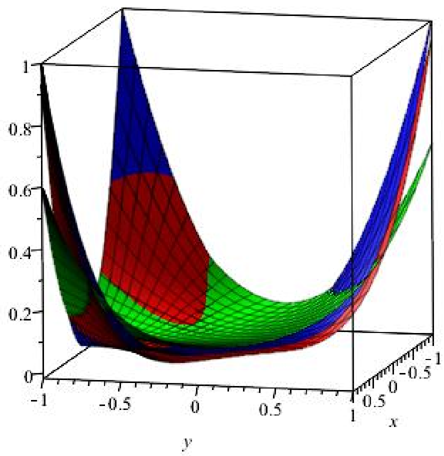

Figure 4.3: The convergence of $H_n(h;u,v)$ operator and $E_n(h;u,v)$ GBS operator to the function $h(u,v) = \left|u^2v^2\right|$

(u,v)	$\lvert H_n(h;u,v)-h(u,v)\rvert$	$\lvert E_n(h;u,v)-h(u,v)\rvert$
$(1,-1)$	0.8495540691	0.7663879599
$(0.9,-1)$	0.6861251394	0.6206321071
$(0.9,-0.9)$	0.5459929013	0.5080926891
$(0.8,-0.9)$	0.4291121293	0.4016625079
$(0.8,-1)$	0.5397625418	0.4901649946

Table 2: Error bounds at different (u,v) points for $H_n(h;u,v)$ and $E_n(h;u,v)$ GBS operator .

In table 4.2, we have computed the error in the approximation of $H_n(h;u,v)$ operator and $E_n(h;u,v)$ GBS operator at certain (u,v) points for $n = 200$. It was observed that the convergence rate of $E_n(h;u,v)$ GBS operator to the function $h(u,v)$ is much better than $H_n(h;u,v)$ operator.

5. Conclusion

We proved that bivariate Bernstein-Durrmeyer type operators and GBS form of these operators in a triangular region are better than the classical Bernstein-Durrmeyer type operators.

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