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A Mathematical Modelling Approach for a Past-Dependent Prey-Predator System

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Abstract

A memory dependent prey-predator model incorporating Allee effect in prey is analysed. For a small and high values of memory rate, the dynamical changes in the prey and predator densities are demonstrated. The equilibria of the proposed model and the local stability analysis corresponding to each equilibrium are presented. The variables of prey and predator species with respect to memory rate are investigated and the existence of the Hopf bifurcation is shown. The analytical part of this paper is supported with detailed numerical simulations.

1. Backround

Mathematical modelling of complex interactions among species through the systems of ordinary differential equations has attracted a great attention by many researchers in the last few decades. These models trace their roots back to a seminal work of Lotka and Volterra and since then numerous papers have been published with various functional responses [1, 2, 3]. The functional response refers to the intake rate of a predator as a function of prey density. The most common functional responses are known as Holling type I-IV functional responses, where type I is the most simplest linear functional form and type II is the most frequently used functional form [4, 5]. Here consumed rate of prey monotonically decrease with its density. Type III response is rather similar to type II in terms of saturation at high prey density, yet a super-linear increase can be observed at lower densities. Apart from these functional forms, some specific interaction functions have been also studied. These include Beddington–DeAngelis interaction functional, Hassel–Varley interaction functional and Ratio-dependent interaction functional [6]. Here we concentrate on Holling type III functional response [7].

There are many social and characteristic factors that affect the interactions of species. Among these, Allee effect and intrascpecific competition have been thoroughly investigated. Allee effect was introduced by Allee to model different circumstances associated with aggregation and related cooperative characteristics [8]. This has been found particularly important to understand the relationship between species' suvival probability and population size [8, 9]. On the other hand, population dynamics may also be influenced by the intraspecific competition in real world scenario [10, 11, 12]. In fact, the resource for survival of species are always limited and thus competition among species of a population is more likely to occur. In this paper we will consider Allee effect in prey population and intraspecific competition in predator population.

In many models of ecology, it is usually assumed that the growth rate of both prey and predator species is respectively associated with the predator and prey density in the present time. However, there has been a great interest in the role of memory in population dynamics, through considering nonlocal addition of past influence or using fractional differential equations. In reality, the predator density not only depends on the current density of prey but also depends on the past density of prey [5, 13, 14]. One way to incorporate memory through ordinary differential equation is to turn a non-local memory function into another differential equation. In this context, an exponential probability density function will be considered.

This paper is organised as in the following order. Firstly, in Sec. 2, the mathematical model is described. The dimensionless version of the proposed model and its steady states are respectively presented in Sec. 2.1 and Sec. 2.2. The local stability analysis for each steady state is given in Sec. 3. Theoretical work presented in these sections will be supported with extensive numerical simulations.

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2. Mathematical Model

The model with memory effects that we will consider in this paper can be written as

$$\begin{aligned}\frac{dP}{dT} &= \alpha P(T)\left(1-\frac{P(T)}{K}\right)(P(T)-\kappa)-\frac{\mu P(T)^2 Q(T)}{\beta+P(T)^2},\\ \frac{dQ}{dT} &= \frac{\nu\mu M(T)^2 Q(T)}{\beta+M(T)^2}-\eta Q(T)-\gamma Q(T)^2,\\ \frac{dM}{dT} &= \frac{1}{\omega}(P(T)-M(T)).\end{aligned}\tag{2.1}$$

with initial conditions

$$P_0 = P(T)\geq 0,\ Q_0 = Q(T)\geq 0,\ M_0 = M(T)\geq 0.$$

Here parameter α is the strength of growth rate of prey, K stands for the carrying capacity for prey species and κ is for the survival threshold for a strong Allee effect. Besides, μ represents the consumption ratio or predator capture rate of the prey, β is the constant for half saturation and ν in the second equation for predator represents the conversion efficacy of prey species to predator species. The parameter η is associated with natural death of predator species and parameter γ is the strength of intraspecific competition.

Now we demonstrate how to derive the last equation in the system of (2.1), providing a memory contribution in the system. Since the density of predator is associated with past and present density of prey species, a nonlocal term can be considered with a continuous density function, represented with $\mathscr{A}$ [5]. This function has a task by weighting past moments in the system. Therefore, the prey density in the second equation of system (2.1) can be substituted with the expression

$$M(T)=\int_{-\infty}^{T} P(s)\mathscr{A}(T-s)ds,\tag{2.2}$$

with a probability density presented by $\mathscr{A}(t)=exp(-t/\omega)/\omega$, satisfying

$$\int_0^\infty \mathscr{A}(t)dt = 1,\quad \text{with}\quad \mathscr{A}:[0,\infty)\to\mathbb{R}.$$

Note that ω measures the past influence. Here, smaller $1/\omega$ (larger ω) implies the existence of past influence for larger time interval. For details of this approach, we refer the reader to papers [5, 13, 14]. Differentiating the statement given by Eq. (2.2), one obtains the last differential equation in the system of Eq. (2.1).

2.1. Dimensionless model

To reduce the number of parameters, new dimensionless variables can be introduced as $P=K\tilde{P},\quad Q=K^2\alpha\tilde{Q}/\mu,\quad M=K\tilde{M},\quad t=\tilde{t}/K\alpha$, and new parameters as

$$\tilde{\kappa}=\frac{\kappa}{K},\quad \tilde{\beta}=\frac{\beta}{K^2},\quad \tilde{\nu}=\frac{\nu\mu}{\alpha K},\tilde{\eta}=\frac{\eta}{\alpha K},\quad \tilde{\gamma}=\frac{\gamma K}{\alpha},\quad \tilde{\omega}'=\frac{1}{\omega\alpha K},$$

the nondimensional model can be expressed by

$$\begin{aligned}\frac{dP}{dT} &= P(T)(1-P(T))(P(T)-\kappa)-\frac{P(T)^2 Q(T)}{\beta+P(T)^2},\\ \frac{dQ}{dT} &= \frac{\nu M(T)^2 Q(T)}{\beta+M(T)^2}-\eta Q(T)-\gamma Q(T)^2,\\ \frac{dM}{dT} &= \omega'(P(T)-M(T)),\end{aligned}\tag{2.3}$$

with non-negative initial conditions

$$P_0 = P(T)\geq 0,\ Q_0 = Q(T)\geq 0,\ M_0 = M(T)\geq 0.$$

2.2. Steady States

Depending on the parameter space, the model has four possible equilibria : one extinction, two predator free and one coexistence state:

- The extinction steady state $S_{000}=(0,0,0)$ is trivial and thus always exists in the system.
- There exist two predator free states that are given by $S_{101}=(P_s,0,M_s)$ and $S'_{101}=(P'_s,0,M'_s)$. Here, considering $Q=0$ in Eq. (2.3) leads to $P^2-(\kappa+1)P+\kappa$ and thus $P_*=\{\kappa,1\}$ and $M_*=P_*$.
- There exists one positive non-zero state $S_{111}=(P^s,Q^s,M^s)$ that has the relation

$$(1-P)(P-\kappa)-\frac{PQ}{\beta_1+P^2}=0.$$

  Here Q^s and M^s can be written in terms of P^s as

$$Q^s=\frac{[(1-P^s)(P^s-\kappa)](\beta+P^{s2})}{P^s}\quad\text{and}\quad M^s=P^s.$$

3. Local Stability Analysis

The local stability for the system (2.3)-(2.3) can be performed through linearisation argument around a steady state $S=(P^s,Q^s,M^s)$, that is

$$\begin{aligned} P &= P^s+\bar{P},\\ Q &= Q^s+\bar{Q},\\ M &= M^s+\bar{M}, \end{aligned} \tag{3.1}$$

leading to a Jacobian matrix $D=(d_{ij}),\ i,j=\{1,2,3\}$ for which

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} P\\ Q\\ M\end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ d_{31} & d_{23} & d_{33}\end{bmatrix}\Bigg|_S \begin{bmatrix}\bar{P}\\ \bar{Q}\\ \bar{M}\end{bmatrix}, \tag{3.2}$$

where $\bar{\cdot}$ stands for the perturbed states and components of D matrix are given as

$$\begin{aligned} d_{11} &= \kappa(2P-1)+(2-3P)P-\frac{2\beta PQ}{(\beta+P^2)^2} \quad d_{12}=-\frac{P^2}{\beta+P^2}, \quad d_{13}=0, \quad d_{21}=0,\\ d_{22} &= \frac{\nu M^2}{\beta+M^2}-\eta-2\gamma Q, \quad d_{23}=\frac{2\beta_2\nu MQ}{(\beta+M^2)^2}, \quad d_{31}=\omega', \quad d_{32}=0, \quad d_{33}=-\omega'. \end{aligned}$$

3.1. Stability of S_{000} state

The characteristic polynomial corresponding the trivial extinction state $S_{000}=(0,0,0)$ is found by Det $\left[\varphi I_3 - D|_{S_{000}}\right]=0$, leading to

$$(\varphi+\kappa)(\varphi+\eta)\left(\varphi+\omega'\right)=0, \tag{3.3}$$

for which I_3 represents a 3×3 identity matrix. Here all roots of Eq. (3.3) is found to be negative and thus trivial extinction state is always a stable node.

3.2. Stability of S_{101} and S'_{101} states

As stated in Sec. 2.2, there are two predator free axial states given by $S_{101}=(P_s,0,M_s)$ and $S'_{101}=(P'_s,0,M'_s)$. The roots of the characteristic polynomial corresponding to these steady states is found using

$$\left(\varphi-\kappa(2P_*-1)-(2-3P_*)P_*\right)\left(\varphi+\omega'\right)\left(\varphi-\frac{\nu M_*^2}{\beta+M_*^2}+\eta\right). \tag{3.4}$$

Thus eigenvalues are obtained as

$$\begin{aligned} \varphi_1 &= \kappa(2P_s-1)+(2-3P_s)P_s,\\ \varphi_2 &= -\omega',\\ \varphi_3 &= \frac{\nu {M_s}^2}{\beta+{M_s}^2}-\eta, \end{aligned}$$

Therefore the roots of this polynomial is found to be negative or have negative real part satisfying the conditions

$$\kappa(2P_s-1)+(2-3P_s)P_s<0 \quad \text{and} \quad \frac{\nu {M_s}^2}{\beta+{M_s}^2}<\eta.$$

3.3. Stability of S_{111} state

The characteristic equation at the positive coexisting state $S_{111}=(P^s,Q^s,M^s)$ can be computed using Det $\left[\varphi I_3 - D|_{S_{111}}\right]=0$, leading to

$$(\varphi-d_{11})(\varphi-d_{22})(\varphi-d_{33})-d_{12}d_{23}d_{31}=0. \tag{3.5}$$

Therefore the local stability for the positive steady state can be determined solving Eq. (3.5).
In Fig. 3.1, the temporal dynamics the prey and predator densities and their corresponding phase diagram are demonstrated with respect to various values of memory rate ω'. Fig. 3.1(a,c,e), the dynamics of both species exhibit periodic oscillations, thus leading to unstable behaviour. The frequency of oscillations in Fig. 3.1(a) is larger compared with the frequency of oscillations in Fig. 3.1(c), where the rate of memory is increased from $\omega'=0.1$ to $\omega'=0.3$. Other parameters are $\kappa=0.1$, $\beta=0.5$, $\nu=1.8$, $\eta=0.7$ and $\gamma=0.4$. In both Fig. 3.1(a) and Fig. 3.1(c), the simulations are initiated using the coexisting state obtained with these parameters. Hence the dynamics stays at their equilibrium for about $t=500$ before oscillating. This can be more evident from Fig. 3.1(e), that the systems stays at the steady state for a longer time, e.g. till $t\in[0\ \ 1300]$, when increasing the memory rate from $\omega'=0.3$ to $\omega'=0.4$. Lastly, in 3.1(g), the system appears to be stable, yet it may be unstable at very large times. The corresponding phase diagrams for all these four cases are presented in 3.1(b,d,f,h), where the initial point of the trajectory is given by a yellow diamond sign; and end point of the trajectory is given by a magenta diamond sign.

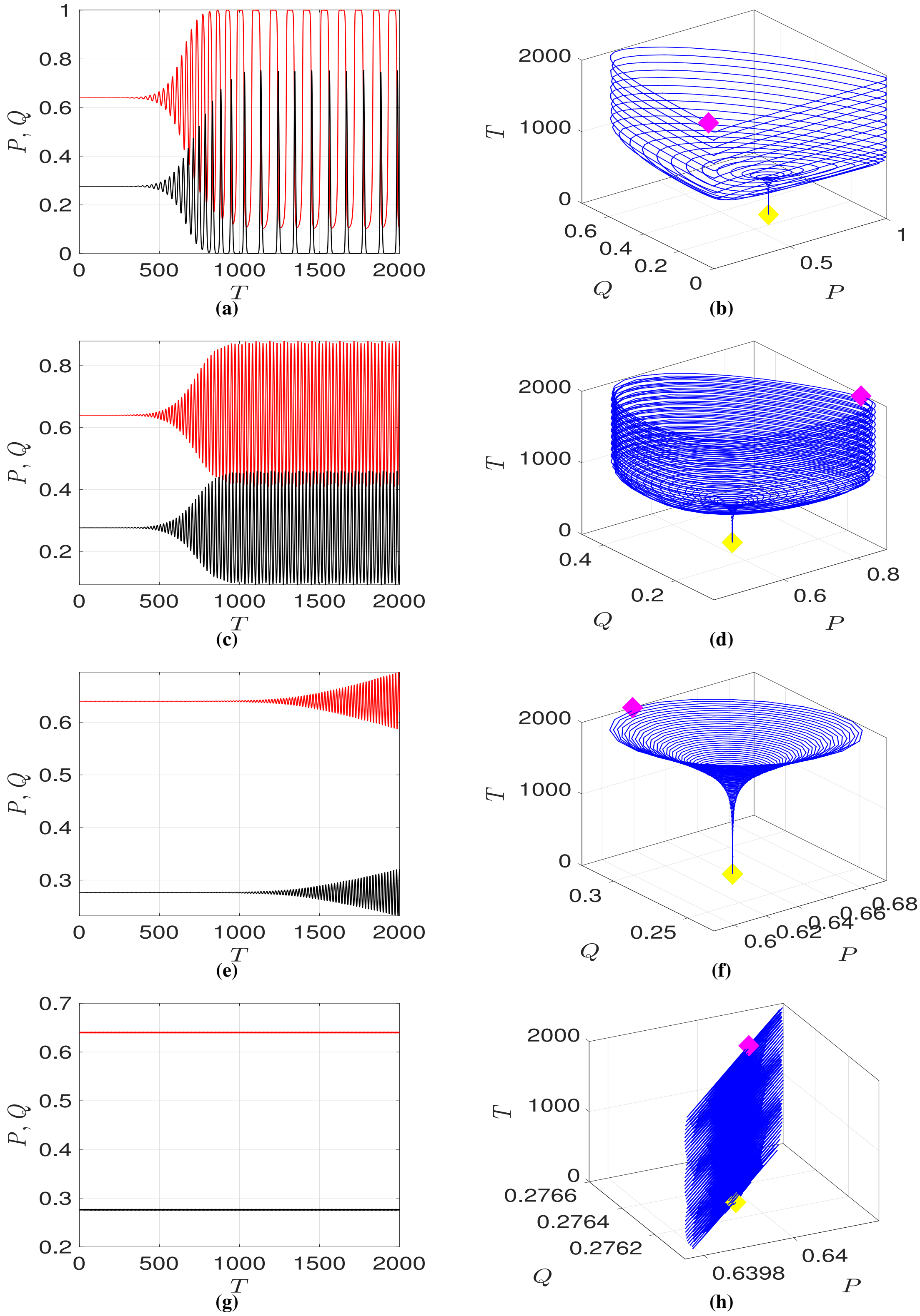

Figure 3.1: Densities of prey (P) and predator (Q) species with various memory rate, where $\omega' = 0.1$ (a), $\omega' = 0.3$ (c), $\omega' = 0.4$ (e) and $\omega' = 0.5$ (g). The red and black lines respectively stand for prey and predator species. The corresponding phase portraits are respectively given in (b,d,f,h). The initial and end points of the phase trajectories are given by yellow and magenta diamonds. Other parameters are $\kappa = 0.1$, $\beta = 0.5$, $\nu = 1.8$, $\eta = 0.7$ and $\gamma = 0.4$. The initial condition is given by the steady state corresponding to these parameters.

Figure 3.2 demonstrates dynamics of prey and predator for two different memory rate a smaller intraspecific competition strength, compared with Fig. 3.1. As seen a small decrease in the intraspecific competition may lead to extinction of both species, where parameter γ is reduced from $\gamma = 0.4$ to $\gamma = 0.38$. Increasing memory rate from $\omega' = 0.1$ to $\omega' = 0.4$, unstable dynamics with periodic oscillations appears again. Other parameter are same as Fig. 3.2.

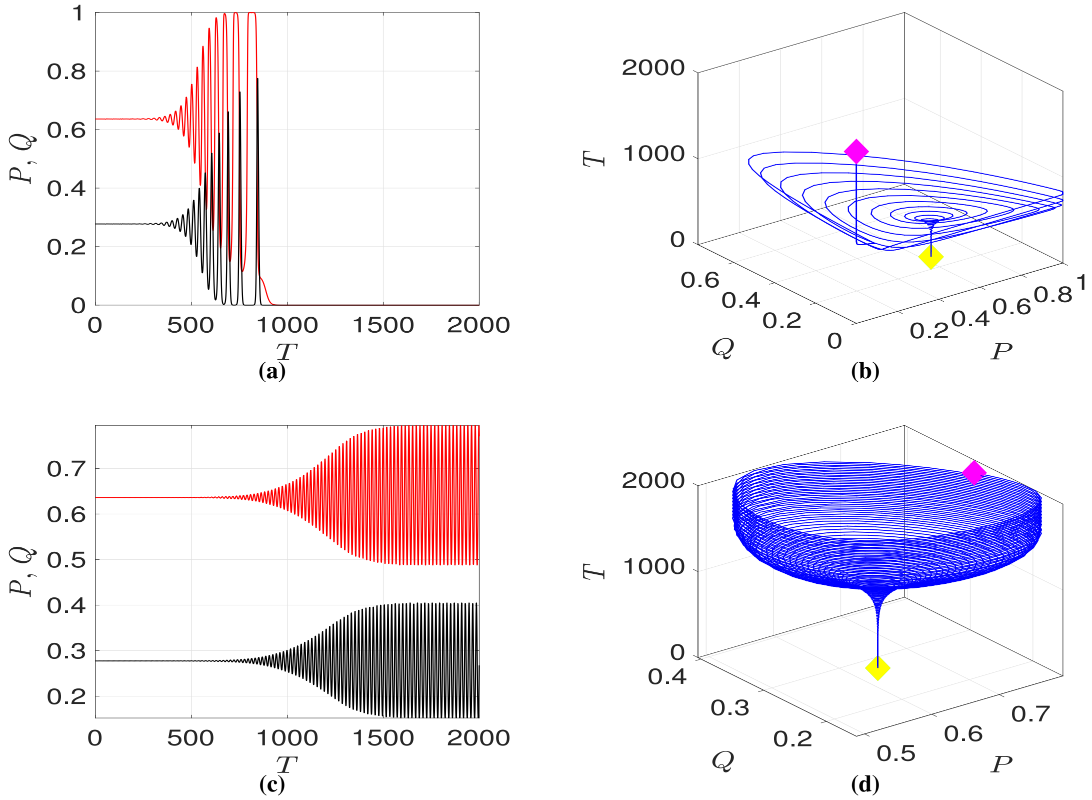

Figure 3.2: Densities of prey (P) and predator (Q) species with memory rates $\omega' = 0.1$ (a) and $\omega' = 0.4$ (c) for a smaller intraspecific competition $\gamma = 0.38$ (compared to Fig. 3.1). The red and black lines respectively stand for prey and predator species. The corresponding phase portraits are respectively given in (b,d). The initial and end points of the phase trajectories are given by yellow and magenta diamonds. Other parameters are $\kappa = 0.1$, $\beta = 0.5$, $\nu = 1.8$, $\eta = 0.7$. The initial condition is given by the steady state corresponding to these parameters.

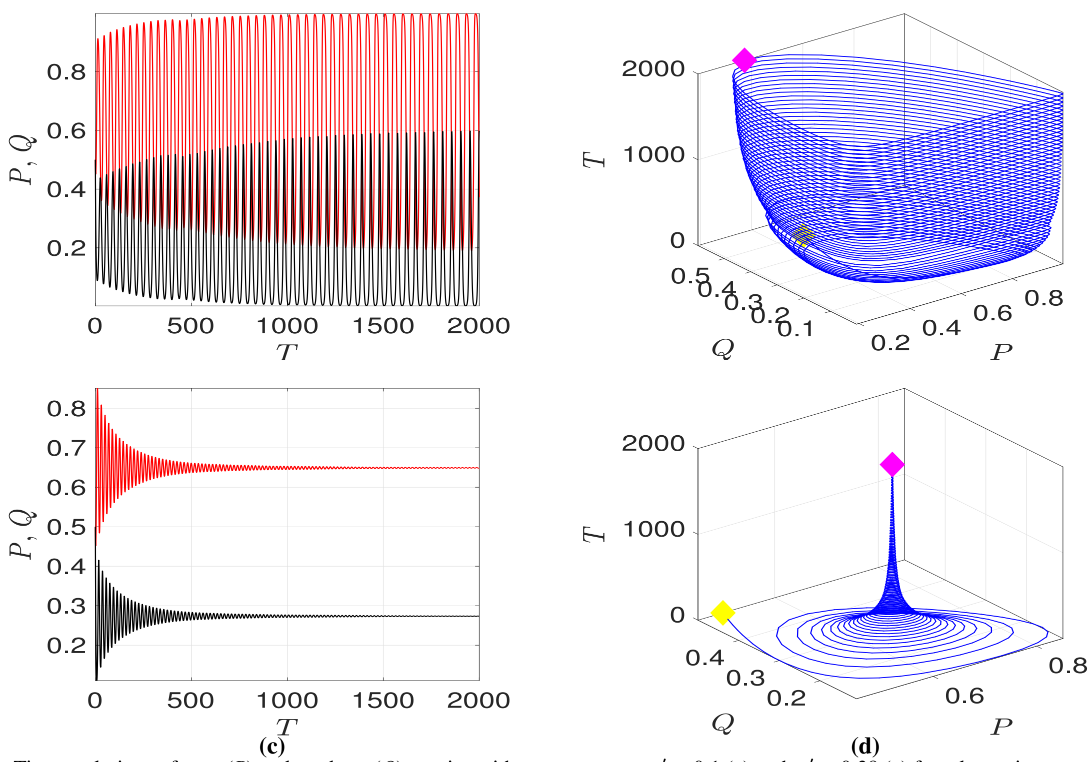

Figure 3.3: Time evolutions of prey (P) and predator (Q) species with memory rates $\omega' = 0.1$ (a) and $\omega' = 0.38$ (c) for a larger intraspecific competition $\gamma = 0.45$ (compared to Fig. 3.1). The red and black lines respectively stand for prey and predator species. The corresponding phase portraits are respectively given in (b,d). The initial and end points of the phase trajectories are given by yellow and magenta diamonds. Other parameters are $\kappa = 0.1$, $\beta = 0.5$, $\nu = 1.8$, $\eta = 0.7$. The initial condition is chosen as $(P_0, Q_0, M_0) = (0.5, 0.5, 0.5)$.

Compared to Fig. 3.1, the role of a larger intraspecific competition ($\gamma = 0.45$) is presented in Fig. 3.3. As seen, smaller memory rate $\omega' = 0.1$ lead to an unstable system. However, increasing memory rate to $\omega' = 0.38$, the system stability is changed from unstable to stable. The initial conditions of the system is now chosen as $(P_0, Q_0, M_0) = (0.5, 0.5, 0.5)$. Figure 3.4 shows bifurcation diagrams and the existence of Hopf bifurcation occuring with respect to memory rate ω' for various intraspecific competition rates. Here it is chosen as $\gamma = 0.15$ (a), $\gamma = 0.2$ (b), $\gamma = 0.38$ (c) and $\gamma = 0.49$ (d), respectively. Here, the green and magenta circles represent the maximum and minimum points of the prey and predator dynamics at each memory rate. As seen in Fig. 3.4(a), for $\gamma = 0.2$, both prey and predator dynamics are always unstable for $\omega' \in [0, 1]$ regardless of the memory rate. In Fig. 3.4(b), a small region, e.g. $\omega' \in [0.9, 1]$, system becomes stable, yet unstable dynamics with periodic oscillations can be seen for smaller memory constant. Increasing intraspecific competition to $\gamma = 0.38$, the stable region is obtain for a larger interval of ω' and unstable dynamics are obtained for small values of memory rate. Lastly, in Fig. 3.4(d), for both very small and large values of memory rate ω', stable dynamics is observed. Only in a small region with $\omega' \in [0.09, 0.23]$, unstable dynamics where oscillations with small amplitude are obtained.

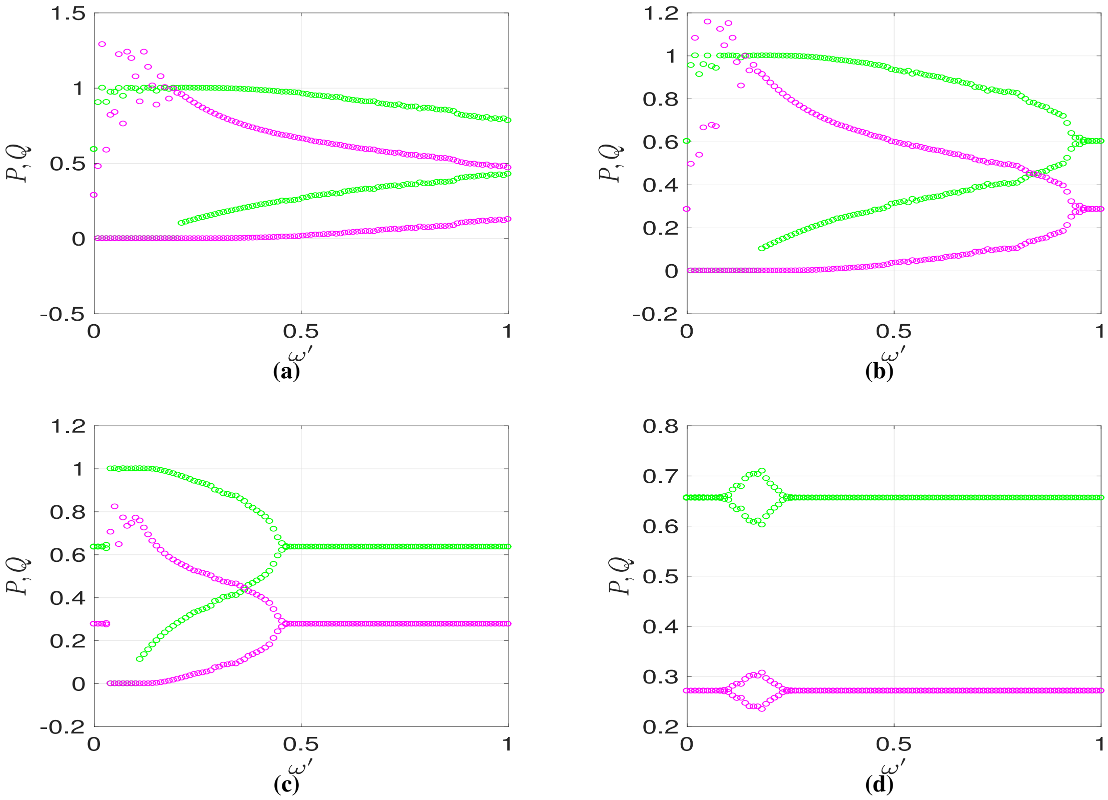

Figure 3.4: Densities of prey (P) and predator (Q) populations as a function of memory rate ω' for various intraspecific competition rates, where $\gamma = 0.15$ (a), $\gamma = 0.2$ (b), $\gamma = 0.38$ (c) and $\gamma = 0.49$. The green and magenta circles respectively stand for the dynamics of prey and predator species. The corresponding phase portraits are respectively given in (b,d). Other parameters are $\kappa = 0.1$, $\beta = 0.5$, $\nu = 1.8$, $\eta = 0.7$. The initial condition is chosen as $(P_0, Q_0, M_0) = (0.5, 0.5, 0.5)$.

Simulations presented in this section demonstrate that larger values of ω' has a stabilising role in the dynamics of both prey and predator populations. This also implies that smaller values of ω' account for the past influence for a larger time interval. The intraspecific competition parameter γ has also a significant impact on the dynamics, where depending on the memory parameter ω', the change in γ may induce extinction, stability or instability of species densities.

4. Summary

In prey predator interactions, predator population may expose to Allee effect, and this leads to an influence on the survival probability of each individual. The existence of Allee effect has been experimentally demonstrated in various species, including for example mammals such as suricates [15], and marine invertebrates such as gastropod [16]. Furthermore infraspecific competition is known to exist among species and occurs very common in nature due to limited sources and difficulties in mate finding. Therefore the inclusion of memory term in a model with Allee effect and intraspecific competition provide an additional degree of realism.

In this paper, a mathematical system comprising Allee effect with fading memory is considered. In fact, the fading memory is crucial in real prey-predator dynamics in ecology as the density of predator is highly dependent not only on the density of prey at the present time but also on the density of prey in the past. In our model, we also consider intraspecific competition as an additional degree of realism in the mathematical modeling. In this paper, we concentrate on the role of memory rate ω' and intraspecific competition γ [5, 17]. Besides, Holling type III functional response, that assumes that the predator species effectively seeks for prey species, is considered. The most important finding of this paper is the dramatic change in the dynamics with ω' and γ. In this context, for a fixed memory rate, larger intraspecific competition lead to stability of the system. Furthermore, the memory rate ω' refers to the measure of the past influence. Namely, larger ω' stands for the past effect for a larger time interval. The system tends to be unstable with larger ω'.

This work can be extended in a couple of direction. The more straightforward extension would be to inclusion of local and non-local delay terms, as switching from one state to another state is not immediate and require some time delay [18]. This could be in the growth rate of prey or in the maturation of predator. Besides, the random fluctuations may occur in nature as a result of climate change or other factors including some short term diseases. Thus another extension would be to consider environmental noise terms in the parameters of the model to capture more realistic dynamics of prey-predator model investigated here [19, 20]. Furthermore, wide spectrum of numerical techniques can be applied to approximate the solutions of the model [21, 22]. The local stability analysis of the system is extensively studied. Following the ideas in [23], the global stability of the model can be analysed through LaSalle's Invariance Principle-Lyapunov's direct method.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

The authors constructed and analysed the model and performed the numerical simulations, the authors has also written the original manuscript and gave final approval of the current version and any revised version to be submitted to the journal. All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Differential Equations of Rectifying Curves and Focal Curves in $\mathbb{E}^n$

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Abstract

In this present paper, rectifying curves are re-characterized in a shorter and simpler way using harmonic curvatures and some relations between rectifying curves and focal curves are found in terms of their harmonic curvature functions in $n-$dimensional Euclidean space. Then, a rectifying Salkowski curve, which is the focal curve of a given space curve is investigated. Finally, some figures related to the theory are given in the case $n = 3$.

1. Introduction

Kim and et al. consider a space curve in which the relationship between torsion and curvature is a non-constant linear function, [1]. Then, Chen characterize a special curve whose position vector always lies in its rectifying plane, [2, 3]. In other words, the position vector of a rectifying curve α with Frenet vector $\{T,N,B\}$ can be stated by

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s) \tag{1.1}$$

for $\lambda(s)$ and $\mu(s)$ differentiable functions. The most known characterization of the rectifying curve is that the ratio of torsion to curvature is a non-constant linear function in terms of its arc-length parameter s. The authors prove that the centrode of a unit speed curve with non-zero constant curvature (or non-constant curvature) and non-constant torsion (or non-zero constant torsion) is a rectifying curve . Then, Chen obtain that a curve on a cone in $\mathbb{E}^3$ is a geodesic if and only if it is either a rectifying curve or an open portion of a ruling, [4]. Furthermore, Cambie et al. generalize rectifying curves in an arbitrary dimensional Euclidean space, [5]. In addition to these, in Minkowski space, rectifying curve is similar to in Euclidean space, [6, 7].

In 1975, authors introduced the functions of harmonic curvature, [8]. The authors generalize inclined curves thanks to the harmonic curvature in E^3 to E^n and then give a characterization for the inclined curves in E^n. This subject has been studied by many authors since then and it also has many geometric interpretations. For example, Camci et al. investigate the relations between the harmonic curvatures of a non-degenerate curve and the focal curvatures of tangent indicatrix of the curve and they give that harmonic curvature of the curve is focal curvature of the tangent indicatrix [9]. Kaya et al. give a new definition of helix strip. They study the harmonic curvatures functions of a strip by using harmonic curvature functions and give some characterizations of the strips's harmonic curvature functions and total curvature functions of a strip [10]. The authors look in a non-generated curve for a generalized helix using these curvatures in [11]. Then, Gök et al. define a new kind of helix called V_n-slant helix by using a similar approach of harmonic curvature functions in n-dimensional Euclidean space and Minkowski space, [12, 13]. Harmonic curvatures are also studied in the Lorentz-Minkowski space [14, 15, 16]. As it can be easily seen when these studies are examined, it has been very useful to use harmonic curvature functions when characterizing curves. The existing characterizations have been made very short and simple through the harmonic curvature, especially when working in high-dimensional spaces. Many geometric concepts such as helices, slant helices, strips and some other special curves have been defined by using their harmonic curvature functions.

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On the other hand, Vargas defined focal curve of α which is the centers of its osculating hyperspheres of the curve, [17]. The centers of the osculating hyperspheres of the curve are well defined only for the points of the curve where all curvatures are non-zero. Öztürk and Arslan characterized focal curves and their Darboux vectors. They have shown that if the ratios of the curvatures of a curve γ are constant, then the ratios of the curvatures of the focal curve C_γ are constant, [18]. Furthermore, Öztürk et al. studied the focal representation of $k-$slant helices in $\mathbb{E}^{m+1}$, [19].

In this study, rectifying curves with their harmonic curvature functions are re-characterized in $n-$dimensional Euclidean space. Then, some relations between rectifying curves and focal curves are investigated. Also, a necessary condition for the focal curve of any space curve to be a rectifying curve is given. Finally, the rectifying Salkowski curve whose focal curve is a rectifying curve is investigated.

2. Basic Concepts and Notations

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve in $\mathbb{E}^n$. Let $\{T, N, B_1, B_2, ..., B_{n-2}\}$ be the moving Serret-Frenet frame along the unit speed curve α. Then the Frenet formulas are given as follows

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ \vdots \\ B_{n-4}' \\ B_{n-3}' \\ B_{n-2}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ \vdots \\ B_{n-4} \\ B_{n-3} \\ B_{n-2} \end{bmatrix}$$

where all k_i curvatures denotes the i^{th} curvature function of the curve and positive, [20, 21].

Definition 2.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve. Harmonic curvatures of α is defined by*

$$H_i : I \subset \mathbb{R} \to \mathbb{R}, \quad i = 0, 1, 2, ..., n-2, \tag{2.1}$$

$$H_i(s) = \begin{cases} 0 & , \quad i = 0 \\ \dfrac{k_1(s)}{k_2(s)} & , \quad i = 1 \\ \dfrac{1}{k_{i+1}(s)} \{V_1[H_{i-1}] + H_{i-2}k_i\} & , \quad i = 2, 3, ..., n-1 \end{cases} \tag{2.2}$$

in the paper, [8].

Definition 2.2. *A curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ is a rectifying curve if the orthogonal complement of $N(s)$ contains a fixed point for all $s \in I$, [5].*

Definition 2.3. *The center of the osculating hypersphere of the curve α at a point lies in the hyperplane normal to the curve α at that point. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a curve with Frenet vectors $\{T, N, B_1, B_2, ..., B_{n-2}\}$ and k_i curvature functions. Then the focal curve of the curve α is written as follows:*

$$C_\alpha = \alpha(s) + c_1(s)N(s) + c_2(s)B_1(s) + c_3(s)B_2(s) + ... + c_{n-1}(s)B_{n-2}(s) \tag{2.3}$$

where $c_1, c_2, ..., c_{n-1}$ smooth functions called focal curvatures of the curve α. Moreover, the function c_1 never vanishes and $c_1(s) = \dfrac{1}{k_1(s)}$. Then, the focal curvature functions of the curve α have defined as

$$c_i(s) = \begin{cases} 0 & , \quad i = 0 \\ \dfrac{1}{k_1(s)} & , \quad i = 1 \\ \dfrac{1}{k_i(s)} \{c'_{i-1}(s) + c_{i-2}(s)k_{i-1}(s)\} & , \quad i = 2, 3, ..., n-1 \end{cases} \tag{2.4}$$

in the paper, [17].

Theorem 2.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve with Frenet vectors $\{T, N, B_1, ..., B_{n-2}\}$ and C_α be its focal curve with the Frenet vectors $\{\overline{T}, \overline{N}, \overline{B}_1, ..., \overline{B}_{n-2}\}$ in $\mathbb{E}^n$. Then, k_i and K_i, denotes the i^{th} curvature functions of the curve α and the curve C_α, respectively. There are following relationship between the Frenet frames and curvatures of the curves.*

$$\begin{aligned} \overline{T} &= B_{n-2}, \quad \overline{N} = B_{n-3}, \quad \overline{B}_1 = B_{n-4} \quad , \ldots, \quad \overline{B}_{n-3} = N, \quad \overline{B}_{n-2} = T, \\ K_i &= \frac{k_{n-i+1}}{c'_{n-1} + k_{n-1}c_{n-2}}, \; i \in \{1, 2, ..., n-1\} \end{aligned} \tag{2.5}$$

where c_i, $i \in \{1, 2, ..., n-1\}$ are the focal curvatures of the curve, [17].

Definition 2.5. *A curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ is the Salkowski curve if and only if it has the constant curvature but non-constant torsion with an explicit parametrization, [22, 23].*

3. Rectifying Curves in $n-$Dimensional Euclidean Space

In this first subsection, re-characterization of rectifying curve according to harmonic curvature functions is given with similar idea defined by Özdamar and Hacısalihoğlu in [8]. In the next subsection, we will look for the answer to the following question

"When does the focal curve of a given curve become rectifying curve?".

3.1. Rectifying Curves with Harmonic Curvature Functions

Let α be an arc-length parametrized rectifying curve in $\mathbb{E}^n$ as

$$\alpha(s) = \lambda(s)T(s) + \mu_1(s)B_1(s) + ... + \mu_{n-2}(s)B_{n-2}(s) \tag{3.1}$$

with $\lambda, \mu_1, ..., \mu_{n-2}$ real valued functions. If we take the derivative of α, get following equation

$$\begin{aligned}\alpha'(s) &= \lambda'(s)T(s) + \lambda(s)k_1(s)N(s) + \mu_1'(s)B_1(s) + \mu_1(s)(-k_2(s)N(s) + k_3(s)B_2(s)) + ... \\ &\quad + \mu_{n-2}'(s)B_{n-2}(s) + \mu_{n-2}(s)(-k_{n-1}(s)B_{n-3}(s)).\end{aligned}$$

Also, if we make the necessary arrangements, we have

$$\begin{aligned}T(s) &= \lambda'(s)T(s) + (\lambda(s)k_1(s) - \mu_1(s)k_2(s))N(s) + (\mu_1'(s) - \mu_2(s)k_3(s))B_1(s) + (\mu_1(s)k_3(s) + \mu_2'(s) - \mu_3(s)k_4(s))B_2(s) \\ &\quad + (\mu_2(s)k_4(s) + \mu_3'(s) - \mu_4(s)k_5(s))B_3(s) + ... + (\mu_{n-2}'(s) + \mu_{n-3}(s)k_{n-1}(s))B_{n-2}(s).\end{aligned}$$

So, we can write following equations as

$$\begin{cases} i. & \lambda'(s) = 1 \\ ii. & \lambda(s)k_1(s) - \mu_1(s)k_2(s) = 0 \\ iii. & \lambda(s)k_1(s) - \mu_1(s)k_2(s) = 0 \\ iv. & \mu_{i-1}(s)k_{i+1}(s) + \mu_i'(s) - \mu_{i+1}(s)k_{i+2}(s) = 0, \quad i \in \{2,3,...,n-3\} \\ v. & \mu_{n-2}'(s) + \mu_{n-3}(s)k_{n-1}(s) = 0. \end{cases} \tag{3.2}$$

We will try to determine λ and μ_i functions with the help of the harmonic curvature functions defined by the following definitions. In fact, we want to emphasize the similarity of the previously described $\mu_{i,k}$ functions in [5] and harmonic curvature functions.

Definition 3.1. *Let α be parameterized by an arc-length parameter curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. Then, we define the harmonic curvature of rectifying curve α in terms of the curvatures using the similar idea given in the paper [8].*

$$H_i(s) = \begin{cases} 0 & , \quad i = 0 \\ (s+c)\dfrac{k_1(s)}{k_2(s)} & , \quad i = 1 \\ \dfrac{1}{k_{i+1}(s)}\left\{H_{i-1}'(s) + H_{i-2}(s)k_i(s)\right\} & , \quad i = 2,3,...,n-2 \end{cases} \tag{3.3}$$

where c is a real constant.

Definition 3.2. *Let α be an arc-lengthed regular curve in $\mathbb{E}^n$ with focal curvatures $\{c_1, c_2, ..., c_{n-1}\}$. Then the harmonic curvature functions of α in terms of the focal curvatures as follows:*

$$H_i(s) = \begin{cases} 0 & , \quad i = 0 \\ \dfrac{c_1'(s)}{c_1(s)c_2(s)} & , \quad i = 1 \\ \dfrac{2c_i(s)c_{i+1}(s)}{\delta_i(s)}\left\{\dfrac{2c_{i-1}(s)c_i(s)}{\delta_{i-1}(s)}H_{i-2}(s) + H_{i-1}'(s)\right\} & , \quad i = 2,3,...,n-2 \end{cases}$$

where $\delta_i(s) = \left(\sum\limits_{j=1}^{i} c_j^2(s)\right)'$, [18].

Corollary 3.3. *Let α be an arc-lengthed rectifying curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. Then, following equalities are obtained from equation (3.2) according to harmonic curvatures in equation (3.3).*

i. $\lambda(s) = s + c$

ii. $\mu_1(s) = \lambda(s)\dfrac{k_1(s)}{k_2(s)} = H_1(s)$

iii. $\mu_2(s) = \dfrac{\mu_1'(s)}{k_3(s)} = \dfrac{1}{k_3(s)}H_1'(s) = H_2(s)$

iv. $\mu_i(s) = \dfrac{1}{k_{i+1}(s)}\left\{\mu_{i-1}'(s) + \mu_{i-2}(s)k_i(s)\right\}$

$\mu_i(s) = \dfrac{1}{k_{i+1}(s)}\left\{H_{i-1}'(s) + H_{i-2}(s)k_i(s)\right\} = H_i(s)$

In the following Corollary, we will reconstract the Theorem 4.1 given in [5] in terms of the harmonic curvature functions.

Corollary 3.4. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. Then α is congruent to a rectifying curve if and only if*

$$H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0 \tag{3.4}$$

where H_i are harmonic curvature functions.

Proof. Assume that α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. If α is a rectifying curve, we have the following equation according to item (v) in equation (3.2)

$$\mu'_{n-2}(s) + \mu_{n-3}(s)k_{n-1}(s) = 0. \tag{3.5}$$

Also, from the above Corollary, we have $\mu_i(s) = H_i(s)$. If this equation is substituted in the above equation, we can easily write that

$$H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0. \tag{3.6}$$

Conversely, assume that equation (3.4) is provided. Then, we can see that α is congruent to a rectifying curve. □

Corollary 3.5. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures $\{k_1, k_2, ..., k_{n-1}\}$. The position vector of the rectifying curve α satisfies*

$$\alpha(s) = (s+c)T(s) + H_1(s)B_1(s) + ... + H_{n-2}(s)B_{n-2}(s) \tag{3.7}$$

for H_i differentiable harmonic curvature functions.

Now we give a relationship between Corollary 3.4 and Theorem 4.1 in [5] with the following Corollary. The first two items are our results and the third item is the characterization of being a rectifying curve in study [5]. In other words, these theories are compatible.

Corollary 3.6. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures. Then the following equations are equivalent*
i) *α is a rectifying curve.*
ii) $H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0.$
iii) $k_{n-1}(s) \sum_{m=0}^{n-4} \mu_{n-3,m}(s) \frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right) + \sum_{m=0}^{n-3}\left(\mu_{n-2,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right)\right)' = 0$

The authors gave a new approach on helices in $\mathbb{E}^n$ with harmonic curvature functions in [24]. With the help of this idea we give a relation between rectifying curve and harmonic curvature functions in the following theorem.

Theorem 3.7. *Let α be an arc-length parameterized curve in $\mathbb{E}^n$ with non-zero curvatures. Then, $\sum_{i=1}^{n-2} H_i^2(s)$ is non-zero constant where $H_{n-2}(s) \neq 0$ if and only if the curve α is a rectifying curve.*

Proof. Let $H_1^2(s) + H_2^2(s) + ... + H_{n-2}^2(s)$ be a non-zero constant. From the equation (3.3), we have that

$$k_{i+1}(s)H_i(s) = H'_{i-1}(s) + k_i(s)H_{i-2}(s),\ 2 \leq i \leq n-2 \tag{3.8}$$

If we write $i+1$ instead of i in equation (3.8), we get

$$H'_i(s) = k_{i+2}(s)H_{i+1}(s) - k_{i+1}(s)H_{i-1}(s),\ 1 \leq i \leq n-3. \tag{3.9}$$

For $i = 1$,

$$H'_1(s) = k_3(s)H_2(s). \tag{3.10}$$

We know that $H_1^2 + H_2^2 + ... + H_{n-2}^2$ is constant. So we can see that

$$H_1(s)H'_1(s) + H_2(s)H'_2(s) + ... + H_{n-2}(s)H'_{n-2}(s) = 0$$

and

$$H_{n-2}(s)H'_{n-2}(s) = -H_1(s)H'_1(s) - H_2(s)H'_2(s) - ... - H_{n-3}(s)H'_{n-3}(s). \tag{3.11}$$

If we multiply $H_i(s)$ and $H_1(s)$ both sides of the equation (3.9) and equation (3.10), respectively, we get

$$H_i(s)H'_i(s) = k_{i+2}(s)H_i(s)H_{i+1}(s) - k_{i+1}(s)H_{i-1}(s)H_i(s) \tag{3.12}$$

and

$$H_1(s)H'_1(s) = k_3(s)H_1(s)H_2(s). \tag{3.13}$$

Hence, from the equations (3.11), (3.12) and (3.13) we can easily show that

$$H_{n-2}(s)H'_{n-2}(s) = -k_{n-1}(s)H_{n-3}(s)H_{n-2}(s).$$

Since $H_{n-2}(s) \neq 0$, we have

$$H'_{n-2}(s) + k_{n-1}(s)H_{n-3}(s) = 0.$$

So, from the Corollary 3.2, the curve α is a rectifying curve.
Conversely, assume that α is a rectifying curve. From the Corollary 3.2, we know that the equality

$$H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0$$

is provided. Moreover, for $H_{n-2} \neq 0$, we can write

$$H_{n-2}(s)H'_{n-2}(s) = -k_{n-1}(s)H_{n-2}(s)H_{n-3}(s).$$

From the equations (3.13) and (3.12) , we obtain

$$H_1(s)H'_1(s) = k_3(s)H_1(s)H_2(s)$$

and

$$\begin{aligned}
\text{for } i &= n-3,\ H_{n-3}H'_{n-3} = k_{n-1}H_{n-3}H_{n-2} - k_{n-2}H_{n-4}H_{n-3},\\
\text{for } i &= n-4,\ H_{n-4}H'_{n-4} = k_{n-2}H_{n-4}H_{n-3} - k_{n-3}H_{n-5}H_{n-4},\\
\text{for } i &= n-5,\ H_{n-5}H'_{n-5} = k_{n-3}H_{n-5}H_{n-4} - k_{n-4}H_{n-6}H_{n-5},\\
&\vdots\\
\text{for } i &= 2,\ H_2H'_2 = k_4H_2H_3 - k_3H_1H_2.
\end{aligned}$$

Then it is easy to see that

$$H_1(s)H'_1(s) + H_2(s)H'_2(s) + \ldots + H_{n-3}(s)H'_{n-3}(s) + H_{n-2}(s)H'_{n-2}(s) = 0 \tag{3.14}$$

and

$$H_1^2(s) + H_2^2(s) + \ldots + H_{n-2}^2(s)$$

is a non-zero constant. □

Special Case for $n = 3$

In this part, we will verify the general theory for $n = 3$ because of the fact that the following characterizations are given in previous works [2] and [3]. Then, considering the definition of harmonic curvature functions of rectifying curves we show that the theory of paper is right for $n = 3$.

Let α be an arc-length parameterized rectifying curve in $\mathbb{E}^3$ as follows

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s) \tag{3.15}$$

with λ, μ real functions.
If we take the derivative of α, then we have

$$\alpha'(s) = \lambda'(s)T(s) + \lambda(s)k_1(s)N(s) + \mu'(s)B(s) + \mu(s)(-k_2(s)N(s))$$

and if the necessary arrangements are made, it is available

$$T(s) = \lambda'(s)T(s) + (\lambda(s)k_1(s) - \mu(s)k_2(s))N(s) + \mu'(s)B(s)$$

So, we can easily obtain the following equations from the above equality.

i) $\lambda'(s) = 1$
ii) $\lambda(s)k_1(s) - \mu(s)k_2(s) = 0$
iii) $\mu'(s) = 0$

We will try to determine λ and μ functions with the help of the harmonic curvature of the curve α given in the equation (3.3). Then, the functions
i) $\lambda(s) = s + c$
ii) $\mu(s) = \lambda(s)\dfrac{k_1(s)}{k_2(s)} = H_1(s)$
are easily obtained.

Corollary 3.8. *Let α be an arc-length parametrized curve in $\mathbb{E}^3$ with non-zero curvatures. Then α is congruent to a rectifying curve if and only if*

$$H'_1(s) = 0 \tag{3.16}$$

where H_1 is the 1^{th} harmonic curvature functions of the curve.

Corollary 3.9. *Let α be an arc-length parametrized curve in $\mathbb{E}^3$ with non-zero curvatures. If the curve α is a rectifying, then the position vector of the curve satisfies*

$$\alpha(s) = (s+c)T(s) + H_1(s)B(s) \tag{3.17}$$

where H_1 is 1^{th} harmonic curvature functions of the curve.

3.2. Rectifying Curves and Focal Curves

In this subsection, some relations between rectifying curve and focal curve are given in $n-$dimensional Euclidean space.

Theorem 3.10. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a curve with $\{T, N, B_1, ..., B_{n-2}\}$ and C_α be focal curve of α with $\{\overline{T}, \overline{N}, \overline{B}_1, ..., \overline{B}_{n-2}\}$. λ_i, $i \in \{1, 2, ..., n-1\}$ denotes the i^{th} function of the position vector of α and c_i, $i \in \{1, 2, ..., n-1\}$ denotes the i^{th} focal curvature of the curve α. Then, the focal curve C_α of the curve α is a rectifying curve if and only if following equation is satisfied*

$$\lambda_{n-1} = -c_{n-2}. \tag{3.18}$$

Proof. Let α be an arbitrary curve and C_α be focal curve of the curve α. Then the curve C_α can be written as follows

$$C_\alpha = \lambda_1 T + \lambda_2 N + \lambda_3 B_1 + ... + \lambda_n B_{n-2} + c_1 N + c_2 B_1 + c_3 B_2 + ... + c_{n-1} B_{n-2}.$$

If we rearrange the C_α by using $\{\overline{T}, \overline{N}, \overline{B}_1, \overline{B}_2, ..., \overline{B}_{n-2}\}$ from the equation (2.5), we get

$$\begin{aligned} C_\alpha &= \lambda_1 \overline{B}_{n-2} + \lambda_2 \overline{B}_{n-3} + ... + \lambda_n \overline{T} + c_1 \overline{B}_{n-3} + c_2 \overline{B}_{n-4} + ... + c_{n-1} \overline{T}, \\ C_\alpha &= (\lambda_n + c_{n-1})\overline{T} + (\lambda_{n-1} + c_{n-2})\overline{N} + ... + (\lambda_2 + c_1)\overline{B}_{n-3} + \lambda_1 \overline{B}_{n-2}. \end{aligned}$$

Since, C_α is a rectifying curve, following equality is available

$$\lambda_{n-1} + c_{n-2} = 0.$$

Conversely, assume that equation (3.18) is provided. Then we can easily see that C_α is a rectifying curve. □

In the following part, we will give the properties of rectifying curve with the focal curve in the 3−dimensional Euclidean space according to the Frenet apparatus $\{T, N, B, k_1, k_2\}$.

Corollary 3.11. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary curve with $\{T, N, B, k_1, k_2\}$ and C_α be focal curve of α with $\{\overline{T}, \overline{N}, \overline{B}, \overline{k}_1, \overline{k}_2\}$ in the $3-$dimensional Euclidean space. $\lambda_1, \lambda_2, \lambda_3$ denotes the functions of the position vector of α and c_1, c_2 denotes functions of the focal curvature of the curve α. Then, the focal curve C_α of α is a rectifying curve if and only if following equality holds*

$$\lambda_2 = -c_1. \tag{3.19}$$

Corollary 3.12. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a curve with $\{T, N, B\}$ and C_α be focal curve of α with $\{\overline{T}, \overline{N}, \overline{B}\}$İf the curve α is a rectifying curve, the focal curve C_α can not be rectifying curve.*

Proof. Let α be an arbitrary curve with $\{T, N, B\}$. We can write α as

$$\alpha = \lambda_1 T + \lambda_2 N + \lambda_3 B.$$

If the curve α is a rectifying, then $\lambda_2 = 0$. But from above theorem, we know that C_α focal curve of α is a rectifying curve if and only if $\lambda_2 = -c_1 = -\dfrac{1}{k_1}$. Consequently, C_α can not be a rectifying curve.

□

Salkowski curves are defined as curves with constant curvature but non-constant torsion with an explicit parametrization. In the following two Corollaries, we give a rectifying curve which is a focal curve of a given Salkowski space curve. For this purpose, we will define the torsion of the given Salkowski curve in Euclidean 3−space.

Corollary 3.13. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary Salkowski curve with $\{T, N, B\}$. If the focal curve C_α of α is a rectifying curve, the torsion of α is equal to $k_2(s) = \dfrac{1}{\sqrt{\frac{2s}{\lambda_1 k_1} + c}}$, $(\frac{2s}{\lambda_1 k_1} + c) > 0$.*

Proof. Since α is an arbitrary Salkowski curve, $k_1(s)$ is a constant function. Assume that the curve C_α be a rectifying curve. Then, the theory of focal curves and Theorem 3.3 give that the position vector of the curve α is

$$\alpha(s) = \lambda_1(s)T(s) + \lambda_2 N(s) + \lambda_3(s)B(s) \tag{3.20}$$

where $\lambda_2 = -c_1 = -\frac{1}{k_1}$ is a constant function. Differentiating the equation (3.20) with respect to s, we obtain

$$T(s) = (\lambda_1'(s) + 1)T(s) + (\lambda_1(s)k_1 - \lambda_3(s)k_2(s))N(s) + \left(\lambda_3'(s) - \frac{k_2(s)}{k_1}\right)B(s).$$

Then, the equality gives us the following system

$$\left.\begin{array}{c} \lambda_1'(s) = 0 \\ \lambda_1(s)k_1 - \lambda_3(s)k_2(s) = 0 \\ \lambda_3'(s) - \frac{k_2(s)}{k_1} = 0 \end{array}\right\} \tag{3.21}$$

If we consider the equation (3.21), we can easily find the following differential equation

$$\left(\frac{\lambda_1 k_1}{k_2(s)}\right)' - \frac{k_2(s)}{k_1} = 0 \tag{3.22}$$

and then the solition of the equation (3.22) is given by

$$k_2(s) = \frac{1}{\sqrt{\frac{2s}{\lambda_1 k_1} + c}}, \quad \left(\frac{2s}{\lambda_1 k_1} + c\right) > 0.$$

□

Corollary 3.14. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary Salkowski curve with the Frenet frame $\{T, N, B\}$. From the above Corollary, we can write rectifying focal curve such as*

$$C_\alpha(s) = \lambda_1 T(s) + \left(\sqrt{(2s + \lambda_1 k_1 c)\lambda_1 k_1} + c_2\right) B(s)$$

where λ_1, k_1, c, c_2 are constant functions.

Example 3.15. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary Salkowski curve and C_α be focal curve of the curve α and coefficient functions of the curve C_α be as follows;*

$$k_1 = 1, \quad k_2 = \frac{1}{\sqrt{2s}}, \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = \sqrt{2s}, \quad c_1 = 1, \quad c_2 = 0.$$

So, C_α focal curve of α is a rectifying curve such as

$$C_\alpha(s) = (f_1(s), f_2(s), f_3(s))$$

where

$$\begin{aligned} f_1(s) &= \frac{4\sqrt{s}\cos 2\sqrt{s} - \sin 2\sqrt{s} + 2\sqrt{s}}{2\sqrt{2}} \\ f_2(s) &= \frac{1}{2}\cos 2\sqrt{s} + 2\sqrt{s}\sin 2\sqrt{s} \\ f_3(s) &= \frac{-4\sqrt{s}\cos 2\sqrt{s} + \sin 2\sqrt{s} + 2\sqrt{s}}{2\sqrt{2}}. \end{aligned}$$

The figure of the rectifying focal curve C_α as follows,

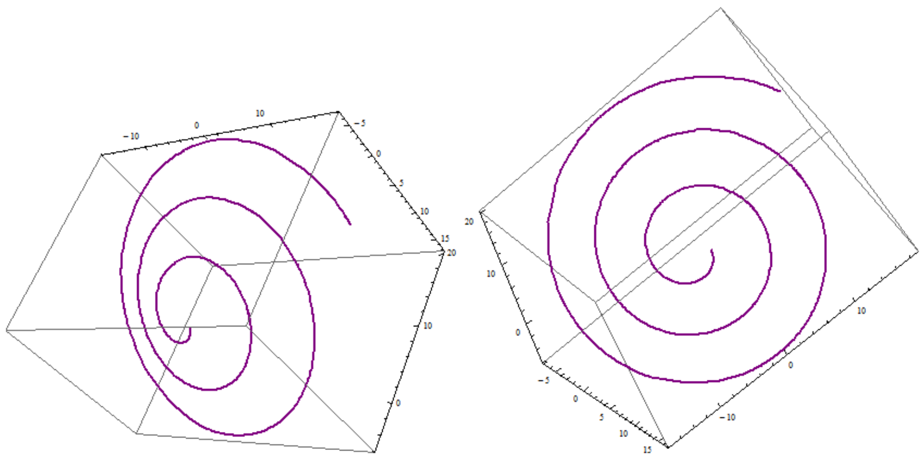

Figure 1. The focal curve C_α

4. Conclusion

Harmonic curvature functions used in several previous studies. In this study, by using harmonic curvature functions a new approach on rectifying curve is given. Characterizing rectifying curves in 3 and 4-dimensional space is easy, but calculations in $n-$dimensional space are not so easy. Harmonic curvatures have given us convenience in our operations and simplicity in characterizations. Authors in [5] characterized rectifying curve in an arbitrary dimensional Euclidean space as

$$k_{n-1}(s)\sum_{m=0}^{n-4}\mu_{n-3,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right) + \sum_{m=0}^{n-3}\left(\mu_{n-2,m}(s)\frac{\partial^m}{\partial s^m}\left(\frac{k_1(s)}{k_2(s)}\right)\right)' = 0.$$

We have shown that the μ_i coefficients in the author's work correspond to harmonic curvatures in minor adjustments. Hence, we prove this theory for rectifying curve more simply associating with harmonic curvature functions such as

$$H'_{n-2}(s) + H_{n-3}(s)k_{n-1}(s) = 0.$$

Also, we give the relationship between rectifying curve and the focal curve in $n-$dimensional Euclidean space. And give necessary and sufficient conditions in which the focal curve of any space curve is a rectifying curve. Subsequently, we examine the these theories for special case $n = 3$. In general, our aim in this study is to examine rectifying curves and focal curves from a different perspective using harmonic curvatures.

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Correlation Coefficients of Fermatean Fuzzy Sets with a Medical Application

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Abstract

The FFS is an influential extension of the available IFS and PFS, whose benefit is to better exhaustively characterize ambiguous information. For FFSs, the correlation between them is usually evaluated by the correlation coefficient. To reflect the perspective of professionals, in this paper, a new correlation coefficient of FFSs is proposed and investigated. The correlation coefficient is very important and frequently used in every field from engineering to economics, from technology to science. In this paper, we propose a new correlation coefficient and weighted correlation coefficient formularization to evaluate the affair between two FFSs. A numerical example of diagnosis has been gotten to represent the efficiency of the presented approximation. Outcomes calculated by the presented approximation are compared with the available indices.

1. Introduction

In the studies in the literature, there are different applications such as aggregation operators and information measures to solve decision-making (DM) problems. Except that these solution methods, another method for choosing the best alternative is the correlation coefficients (KK), which are used to measure the level of dependency between two sets. KKs are used to measure how strong a relationship is between two variables. A KK is a bivariate statistic when it summarizes the relationship between two variables, and its a multivariate statistic when you have more than two variables. Therefore, there is a very wide field of study, from science to economics, from engineering to medicine. Although existing probabilistic methods have successful results, they also have limitations. For example, probabilistic techniques are in accordance with a mass collection of data, which is random, to acquire the necessary confidence level. However, on a great scale, the complex system has massive fuzzy ambiguity owing to which it is tough to acquire the complete possibility of the events. Hence, outcomes according to probability theory do not all the time ensure beneficial information to the professionals owing to the limitation of being able to operate only quantitative information. Furthermore, occasionally there is inadequate data to accurately operate the statistics of parameters, in real-world practices. As a natural consequence of these limitations, the outcomes in accordance with probability theory do not all the time ensure beneficial information to the professionals and then the probabilistic approximation is insufficient to account for such built-in uncertainties in the data. There are many possibilities to overcome these difficulties. One of the most successful results of these possibilities for handling uncertainties and impreciseness in DM is methods based on fuzzy set (FS) theory.

In [2], the correlation for fuzzy information according to the classic statistics is served and ensured a formula for KK of FSs. In [3] The KK of fuzzy information by utilization of a mathematical programming approximation according to the standard definition of KKs has been investigated. Based on the results were obtained through FS theory, more comprehensive and more accurate results were obtained with intuitionistic fuzzy set IFS [1]. The theory of IFS considers non-membership degree (ND) together with membership degree (MD) and requires that their sum be 1 or less than 1. The KKs derived based on IFS have been operated in numerous different fields as DM, cluster analysis, image processing, pattern recognition, etc [4]-[8]. Many DM problems related to PF information have entered the literature thanks to PFS [9, 10], [11]-[16], which was introduced to overcome the limitation in IFS.

Medical DM is a complicated condition that largely depends on the knowledge, experience, and judgment of the physician. For medical DM, it is not enough for the physician to follow the current disease process and current treatment alternatives. It also needs to be aware of

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other variables and use this information for medical DM. Such a situation for the physician makes it necessary to consider complex causal models that can be defined with uncertain, imprecise, and incomplete information. For such cases, decision-making mechanisms derived from theories such as FS, IFS, PFS, or Neutrosophic Space(NS) [17] appear as structures that offer powerful and comprehensive solutions [9, 10], [18]-[29].KKs derived according to the FS, IFS, and PFS theories are available in the literature [26]–[42].

The Fermatean fuzzy set (FFS)was initiated by Senepati and Yager [43]. In the FFS, the MD and ND accomplish the property $0 \le m_A^3 + n_A^3 \le 1$. The FFS, which is included in the literature as a new concept, gives better results than the IFS and PFS in defining uncertainties. For example $0.9+0.6>1$, $0.9^2+0.6^2>1$ and $0.9^3+0.6^3<1$. In [43], some properties, score and accuracy functions of FFSs are served. Further, the TOPSIS method, which is frequently used in Multi Criteria Decision Making(MCDM) problems, has been applied to FFS. In addition, Senepathy and Yager [43], the TOPSIS technique, which is continually utilized in MCDM problems, has been applied to FFS. As a continuation of this work, Senapati ve Yager [44] investigated several new operations, subtraction, division, and Fermatean arithmetic mean operations over FFSs and employed Fermatean fuzzy weighted product model to solve MCDM problems. In [45], new aggregation operators belonging to FFS have been defined, and properties related to these operators have been examined. In study of Donghai and et al [46], the notion of Fermatean fuzzy linguistic term sets is offered. Operations, score, and accuracy functions belonging to these sets were given. In [47], a new similarity measure related to Fermatean fuzzy linguistic term sets is constructed. The new measurement is a combination of Euclidean distance measure and cosine similarity measure. Kirisci [48] defined fermatean fuzzy soft sets and gave the measure of entropy based on fermatean fuzzy soft sets. In [49], a new hesitant fuzzy set which is called the fermatean hesitant fuzzy set has been given and investigated some properties. In [50], the ELECTRE I method is defined with Fermatean fuzzy sets according to the group DM process in which more than one individual interacts at the same time. In [27], a decision support algorithm in accordance with the Fermatean fuzzy soft set concept is presented to maximize the effectiveness of anti-virus masks. In [51], vairous fermatean fuzzy reference relations (consistent, incomplete, consistent incomplete, acceptable incomplete) have been defined. An additive consistency based on a priority vector has been given. In addition, a model is presented to obtain missing decisions in incomplete fermatean fuzzy preference relations.

It is the aim of this study to give the KK and weighted KK formulation to measure the relationship between two FFS. The effectiveness of the proposed technique will be demonstrated by giving a numerical example of a medical diagnosis. The results of this technique will be compared with previously known techniques.

2. Preliminaries

It will be regarded as the $\mathfrak{X} = \{x_1, x_2, \cdots, x_n\}$ initial universe throughout the work.

The Intuitionistic fuzzy set(IFS) in $\mathfrak{X}$ is defined:

$$N = \{(x, \zeta_N(x), \eta_N(x)) | x \in \mathfrak{X}\}.$$

In this definition, $\zeta_N(x), \eta_N(x) : \mathfrak{X} \to [0,1]$ is said to be MD and ND, with $\zeta_N(x) + \eta_N(x) \le 1$.

The Pythagorean fuzzy set(PFS) is characterized as,

$$N = \{(x, \zeta_N(x), \eta_N(x)) | x \in \mathfrak{X}\},$$

if $\zeta_N(x), \eta_N(x) : \mathfrak{X} \to [0,1]$ are MD and ND of element of the $x \in \mathfrak{X}$, with $\zeta_N^2(x) + \eta_N^2(x) \le 1$.

Fermatean fuzzy set (FFS) is given as

$$N = \{(x, \zeta_N(x), \eta_N(x)) | x \in \mathfrak{X}\},$$

if $\zeta_F(x), \eta_F(x) : \mathfrak{X} \to [0,1]$ are MD and ND of element of the $x \in \mathfrak{X}$, with $\zeta_N^3(x) + \eta_N^3(x) \le 1$.

Principle of recognition is defined as: In discourse universe $\mathfrak{X}$, let it be assumed that there are m patterns defined by FFS $\mathfrak{N}_k$ $(k = 1, 2, \cdots, m)$. Again, let's suppose that there is an model to be identified with FFS $\mathfrak{P}$ in $\mathfrak{X}$.
The relationship index degree between FFSs $\mathfrak{N}_k$ and $\mathfrak{P}$ is described as

$$R(\mathfrak{N}_{k0}, \mathfrak{P}) = \max_{1 \le k \le m} \{R(\mathfrak{N}_k, \mathfrak{P})\}.$$

In this case, it is decided that sample $\mathfrak{P}$ belongs to $\mathfrak{N}_{k0}$.

The set

$$IE(N) = \sum_{i=1}^{n} \left[\zeta_N^2(x_i) + \eta_N^2(x_i)\right]$$

is called informational intuitionistic energy of two IFS N. Hence, the correlation and KK of IFSs can be given as

$$\begin{aligned} C_I(N,M) &= \sum_{i=1}^{n}\left[\zeta_N(x_i).\zeta_M(x_i)+\eta_N(x_i).\eta_M(x_i)\right] \\ \mathfrak{C}_I(N,M) &= \frac{C_I(N,m)}{\sqrt{IE(N).IE(M)}} \\ &= \frac{\sum_{i=1}^{n}\left[\zeta_N(x_i).\zeta_M(x_i)+\eta_N(x_i).\eta_M(x_i)\right]}{\sqrt{\sum_{i=1}^{n}\left[\zeta_N^2(x_i)+\eta_N^2(x_i)\right]}.\sqrt{\sum_{i=1}^{n}\left[\zeta_M^2(x_i)+\eta_M^2(x_i)\right]}} \\ \mathfrak{D}_I(N,M) &= \frac{C_I(N,m)}{\max\{IE(N).IE(M)\}} \\ &= \frac{\sum_{i=1}^{n}\left[\zeta_N(x_i).\zeta_M(x_i)+\eta_N(x_i).\eta_M(x_i)\right]}{\max\left[\sum_{i=1}^{n}\left[\zeta_N^2(x_i)+\eta_N^2(x_i)\right].\sum_{i=1}^{n}\left[\zeta_M^2(x_i)+\eta_M^2(x_i)\right]\right]} \end{aligned}$$

The set

$$IE_P(N)=\sum_{i=1}^{n}\left[\zeta_N^4(x_i)+\eta_N^4(x_i)+\theta_N^4(x_i)\right]$$

is called informational energies of PFS N [37].

For FFSs N and M, correlation and KK are defined as:

$$\begin{aligned} C_P(N,M) &= \sum_{i=1}^{n}\left[\zeta_N^2(x_i).\zeta_M^2(x_i)+\eta_N^2(x_i).\eta_M^2(x_i)+\theta_N^2(x_i).\theta_M^2(x_i)\right] \\ \mathfrak{C}_P(N,M) &= \frac{C_I(N,M)}{\sqrt{IE(N).IE(M)}} \\ &= \frac{\sum_{i=1}^{n}\left[\zeta_N^2(x_i).\zeta_M^2(x_i)+\eta_N^2(x_i).\eta_M^2(x_i)+\theta_N^2(x_i).\theta_M^2(x_i)\right]}{\sqrt{\sum_{i=1}^{n}\left[\zeta_N^4(x_i)+\eta_N^4(x_i)+\theta_N^4(x_i)\right]}.\sqrt{\sum_{i=1}^{n}\left[\zeta_M^4(x_i)+\eta_M^4(x_i)+\theta_M^4(x_i)\right]}} \\ \mathfrak{D}_P(N,M) &= \frac{C_I(N,M)}{\max\{IE(N).IE(M)\}} \\ &= \frac{\sum_{i=1}^{n}\left[\zeta_N(x_i).\zeta_M(x_i)+\eta_N(x_i).\eta_M(x_i)+\theta_N^2(x_i).\theta_M^2(x_i)\right]}{\max\left[\sum_{i=1}^{n}\left[\zeta_N^4(x_i)+\eta_N^4(x_i)+\theta_N^4(x_i)\right].\sum_{i=1}^{n}\left[\zeta_{MB}^4(x_i)+\eta_M^4(x_i)+\theta_M^4(x_i)\right]\right]}. \end{aligned}$$

3. New Correlation Coefficients

Let $\mathfrak{N}=\{x_i,\zeta_{\mathfrak{N}}(x_i,\eta_{\mathfrak{N}}(x_i)|x_i\in\mathfrak{X}\}$ be a FFS, where $\zeta_{\mathfrak{N}}(x_i),\eta_{\mathfrak{N}}(x_i)\in[0,1]$ and $\zeta_{\mathfrak{N}}^3(x_i+\eta_{\mathfrak{N}}^3(x_i)|x_i\leq 1$ for each $x_i\in\mathfrak{X}$, we define the informational energy of the FFS $\mathfrak{N}$ as

$$IE(\mathfrak{N})=\sum_{i=1}^{n}\left(\zeta_{\mathfrak{N}}^6(x_i)+\eta_{\mathfrak{N}}^6(x_i)+\theta_{\mathfrak{N}}^6(x_i)\right). \tag{3.1}$$

Suppose that two FFV's $\mathfrak{N}=\{x_i,\zeta_{\mathfrak{N}}(x_i,\eta_{\mathfrak{N}}(x_i)|x_i\in\mathfrak{X}\}$ and $\mathfrak{M}=\{x_i,\zeta_{\mathfrak{M}}(x_i,\eta_{\mathfrak{M}}(x_i)|x_i\in\mathfrak{X}\}$ in X, where $\zeta_{\mathfrak{N}}(x_i,\eta_{\mathfrak{N}}(x_i),\zeta_{\mathfrak{N}}(x_i,\eta_{\mathfrak{N}}(x_i)\in[0,1]$ for each $x_i\in\mathfrak{X}$. Hence, the correlation of the FFVs $\mathfrak{N}$, $\mathfrak{M}$ is defined:

$$C(\mathfrak{N},\mathfrak{M})=\sum_{i=1}^{n}\left(\zeta_{\mathfrak{N}}^3(x_i)\zeta_{\mathfrak{M}}^3(x_i)+\eta_{\mathfrak{N}}^3(x_i)\eta_{\mathfrak{M}}^3(x_i)+\theta_{\mathfrak{N}}^3(x_i)\theta_{\mathfrak{M}}^3(x_i)\right).$$

For the correlation of FFSs, the conditions
(1) $C(\mathfrak{N},\mathfrak{N})=IE(\mathfrak{N})$
(2) $C(\mathfrak{N},\mathfrak{M})=C(\mathfrak{M},\mathfrak{N})$
are hold.

Definition 3.1. *Choose two FFSs $\mathfrak{N}$ and $\mathfrak{M}$ on X. Then the KK between $\mathfrak{N},\mathfrak{M}$ is defined by*

$$\begin{aligned} \mathfrak{C}(\mathfrak{N},\mathfrak{M}) &= \frac{C(\mathfrak{N},\mathfrak{M})}{[IE(\mathfrak{N}).IE(\mathfrak{M})]^{1/2}} \\ &= \frac{\sum_{i=1}^{n}\left(\zeta_{\mathfrak{N}}^3(x_i)\zeta_{\mathfrak{M}}^3(x_i)+\eta_{\mathfrak{N}}^3(x_i)\eta_{\mathfrak{M}}^3(x_i)+\theta_{\mathfrak{N}}^3(x_i)\theta_{\mathfrak{M}}^3(x_i)\right)}{\sqrt{\sum_{i=1}^{n}\left(\zeta_{\mathfrak{N}}^6(x_i)+\eta_{\mathfrak{N}}^6(x_i)+\theta_{\mathfrak{N}}^6(x_i)\right)}.\sqrt{\sum_{i=1}^{n}\left(\zeta_{\mathfrak{M}}^6(x_i)+\eta_{\mathfrak{M}}^6(x_i)+\theta_{\mathfrak{M}}^6(x_i)\right)}} \end{aligned} \tag{3.2}$$

Theorem 3.2. *For any two FFSs $\mathfrak{N},\mathfrak{M}$ in X, the KK of FFSs satisfies the following conditions:*

(P1) $\mathfrak{C}(\mathfrak{N},\mathfrak{M})=\mathfrak{C}(\mathfrak{M},\mathfrak{N})$,
(P2) If $\mathfrak{N}=\mathfrak{M}$, *then* $\mathfrak{C}(\mathfrak{N},\mathfrak{M})=1$,

(P3) $0 \leq \mathfrak{C}(\mathfrak{N},\mathfrak{M}) \leq 1$.

Proof. We only proved condition (P2). Obviously, $\mathfrak{C}(\mathfrak{N},\mathfrak{M}) \geq 0$.

$$\begin{aligned}
C(\mathfrak{N},\mathfrak{M}) &= \sum_{i=1}^{n}\left(\zeta^3_{\mathfrak{N}}(x_i)\zeta^3_{\mathfrak{M}}(x_i)+\eta^3_{\mathfrak{N}}(x_i)\eta^3_{\mathfrak{M}}(x_i)+\theta^3_{\mathfrak{N}}(x_i)\theta^3_{\mathfrak{M}}(x_i)\right)\\
&= \left(\zeta^3_{\mathfrak{N}}(x_1)\zeta^3_{\mathfrak{M}}(x_1)+\eta^3_{\mathfrak{N}}(x_1)\eta^3_{\mathfrak{M}}(x_1)+\theta^3_{\mathfrak{N}}(x_1)\theta^3_{\mathfrak{M}}(x_1)\right)\\
&+ \left(\zeta^3_{\mathfrak{N}}(x_2)\zeta^3_{\mathfrak{M}}(x_2)+\eta^3_{\mathfrak{N}}(x_2)\eta^3_{\mathfrak{M}}(x_2)+\theta^3_{\mathfrak{N}}(x_2)\theta^3_{\mathfrak{M}}(x_2)\right)\\
&+ \cdots+\left(\zeta^3_{\mathfrak{N}}(x_n)\zeta^3_{\mathfrak{M}}(x_n)+\eta^3_{\mathfrak{N}}(x_n)\eta^3_{\mathfrak{M}}(x_n)+\theta^3_{\mathfrak{N}}(x_n)\theta^3_{\mathfrak{M}}(x_n)\right).
\end{aligned}$$

Using the Cauchyâ€“Schwarz inequality, for $(\zeta_1+\cdots+\zeta_n)\in R^n$ and $(\eta_1+\cdots+\eta_n)\in R^n$,

$$(\zeta_1\eta_1+\zeta_2\eta_2+\cdots+\zeta_n\eta_n)^2 \leq \left(\zeta_1^2+\cdots+\zeta_n^2\right).\left(\eta_1^2+\cdots+\eta_n^2\right).$$

Then,

$$\begin{aligned}
\left[C(\mathfrak{N},\mathfrak{M})\right]^2 &\leq \Big[(\zeta^6_{\mathfrak{N}}(x_1)+\eta^6_{\mathfrak{N}}(x_1)+\theta^6_{\mathfrak{N}}(x_1))+(\zeta^6_{\mathfrak{N}}(x_2)+\eta^6_{\mathfrak{N}}(x_2)+\theta^6_{\mathfrak{N}}(x_2))\\
&+ \cdots+(\zeta^6_{\mathfrak{N}}(x_n)+\eta^6_{\mathfrak{N}}(x_n)+\theta^6_{\mathfrak{N}}(x_n))\Big]\times\Big[(\zeta^6_{\mathfrak{M}}(x_1)+\eta^6_{\mathfrak{M}}(x_1)+\theta^6_{\mathfrak{M}}(x_1))+(\zeta^6_{\mathfrak{M}}(x_2)+\eta^6_{\mathfrak{M}}(x_2)+\theta^6_{\mathfrak{M}}(x_2))\\
&+ \cdots+(\zeta^6_{\mathfrak{M}}(x_n)+\eta^6_{\mathfrak{M}}(x_n)+\theta^6_{\mathfrak{M}}(x_n))\Big]\\
&= \sum_{i=1}^{n}(\zeta^6_{\mathfrak{N}}(x_i)+\eta^6_{\mathfrak{N}}(x_i)+\theta^6_{\mathfrak{N}}(x_i))\times\sum_{i=1}^{n}(\zeta^6_{\mathfrak{M}}(x_i)+\eta^6_{\mathfrak{M}}(x_i)+\theta^6_{\mathfrak{M}}(x_i))\\
&= IE(\mathfrak{N}).IE(\mathfrak{M}).
\end{aligned}$$

Therefore, $[C(\mathfrak{N},\mathfrak{M})]^3 \leq IE(\mathfrak{N}).IE(\mathfrak{M})$ and $\mathfrak{C}(\mathfrak{N},\mathfrak{M}) \leq 1$. □

Example 3.3. *Take two FFSs* $\mathfrak{N}=\{(x_1,0.8,0.6),(x_2,0.5,0.9),(x_3,0.6,0.6)\}$ *and* $\mathfrak{M}=\{(x_1,0.6,0.7),(x_2,0.8,0.6),(x_3,0.7,0.5)\}$ *in* $\mathfrak{X}$*. By Equation 3.1, the informational energies of* $\mathfrak{N}$ *and* $\mathfrak{M}$*:*

$$\begin{aligned}
IE(\mathfrak{N}) &= \sum_{i=1}^{n}\left(\zeta^6_{\mathfrak{N}}(x_i)+\eta^6_{\mathfrak{N}}(x_i)+\theta^6_{\mathfrak{N}}(x_i)\right)\\
&= (0.8^6+0.6^6+0.64^2)+(0.5^6+0.9^6+0.527^2)+(0.6^6+0.6^6+0.828^2)\\
&= 2.3221
\end{aligned}$$

$$\begin{aligned}
IE(\mathfrak{M}) &= \sum_{i=1}^{n}\left(\zeta^6_{\mathfrak{N}}(x_i)+\eta^6_{\mathfrak{N}}(x_i)+\theta^6_{\mathfrak{N}}(x_i)\right)\\
&= (0.6^6+0.7^6+0.441^2)+(0.8^6+0.6^6+0.2722)+(0.7^6+0.5^6+0.532^2)\\
&= 1.1579
\end{aligned}$$

By using the Equation 3.2, the correlation between the FFSs $\mathfrak{N},\mathfrak{M}$ is written as

$$\begin{aligned}
C(\mathfrak{N},\mathfrak{M}) &= \sum_{i=1}^{n}\left(\zeta^3_{\mathfrak{N}}(x_i)\zeta^3_{\mathfrak{M}}(x_i)+\eta^3_{\mathfrak{N}}(x_i)\eta^3_{\mathfrak{M}}(x_i)+\theta^3_{\mathfrak{N}}(x_i)\theta^3_{\mathfrak{M}}(x_i)\right)\\
&= 0.8^30.6^3+0.6^30.7^3+(0.272)(0.441)+0.5^30.8^3+0.9^30.6^3+(0.146)(0.272)+0.6^30.7^3+0.6^30.5^3+(0.568)(0.532)\\
&= 0.96148
\end{aligned}$$

Hence,the KK between the FFSs $\mathfrak{N},\mathfrak{M}$ is given by

$$\begin{aligned}
\mathfrak{C}(\mathfrak{N},\mathfrak{M}) &= \frac{C(\mathfrak{N},\mathfrak{M})}{[IE(\mathfrak{N}).IE(\mathfrak{M})]^{1/2}}\\
&= \frac{0.96148}{[(2.3221).(1.1579)]^{1/2}} = 0.58627
\end{aligned}$$

Definition 3.4. *For* $\mathfrak{N}$ *and* $\mathfrak{M}$*, the definition of KK as:*

$$\begin{aligned}
\mathfrak{D}(\mathfrak{N},\mathfrak{M}) &= \frac{C(\mathfrak{N},\mathfrak{M})}{\max[IE(\mathfrak{N}).IE(\mathfrak{M})]}\\
&= \frac{\sum_{i=1}^{n}\left(\zeta^3_{\mathfrak{N}}(x_i)\zeta^3_{\mathfrak{M}}(x_i)+\eta^3_{\mathfrak{N}}(x_i)\eta^3_{\mathfrak{M}}(x_i)+\theta^3_{\mathfrak{N}}(x_i)\theta^3_{\mathfrak{M}}(x_i)\right)}{\max\left[\sum_{i=1}^{n}\left(\zeta^6_{\mathfrak{N}}(x_i)+\eta^6_{\mathfrak{N}}(x_i)+\theta^6_{\mathfrak{N}}(x_i)\right),\sum_{i=1}^{n}\left(\zeta^6_{\mathfrak{M}}(x_i)+\eta^6_{\mathfrak{M}}(x_i)+\theta^6_{\mathfrak{M}}(x_i)\right)\right]}
\end{aligned}$$

Theorem 3.5. *For any two FFSs $\mathfrak{N},\mathfrak{M}$ $\mathfrak{D}(\mathfrak{N},\mathfrak{M})$ satisfies the following conditions:*

(P1) $\mathfrak{D}(\mathfrak{N},\mathfrak{M}) = \mathfrak{D}(\mathfrak{M},\mathfrak{N})$,
(P2) $\mathfrak{N} = \mathfrak{M}$ *iff* $\mathfrak{D}(\mathfrak{N},\mathfrak{M}) = 1$,
(P3) $0 \le \mathfrak{D}(\mathfrak{N},\mathfrak{M}) \le 1$.

Proof. We only proved condition (P2). It is clear that $\mathfrak{D}(\mathfrak{N},\mathfrak{M}) \ge 0$. Since from Theorem 3.2, $\left[C(\mathfrak{N},\mathfrak{M})\right]^3 \le IE(\mathfrak{N}).IE(\mathfrak{M})$. Therefore, $C(\mathfrak{N},\mathfrak{M}) \le \max[IE(\mathfrak{N}).IE(\mathfrak{M})]$, thus $\mathfrak{D}(\mathfrak{N},\mathfrak{M}) \le 1$. □

It is possible to define these KKs in a different way. Weights will be used for these new definitions. Because, in numerous real-life practices, the distinct sets can have got diverse weights. Therefore, weight ω_i of every element $x_i \in \mathfrak{X}$ must be considered in new definitions. The KKs to be defined by weights will be called weighted KKs. FFor these definitions, choose the weight vector as ω. Further, Consider that satisfy the $\sum_{i=1}^n \omega_i = 1$ condition for $\omega_i \ge 1$. Therefore, $C_\omega(\mathfrak{N},\mathfrak{M})$, $\mathfrak{C}_\omega(\mathfrak{N},\mathfrak{M})$ are defined as follows:

$$\begin{aligned} \mathfrak{C}_\omega(\mathfrak{N},\mathfrak{M}) &= \frac{C_\omega(\mathfrak{N},\mathfrak{M})}{[IE_\omega(\mathfrak{N}).IE_\omega(\mathfrak{M})]^{1/3}} \quad (3.3)\\ &= \frac{\sum_{i=1}^n \omega_i\left(\zeta^3_{\mathfrak{N}}(x_i)\zeta^3_{\mathfrak{M}}(x_i)+\eta^3_{\mathfrak{N}}(x_i)\eta^3_{\mathfrak{M}}(x_i)+\theta^3_{\mathfrak{N}}(x_i)\theta^3_{\mathfrak{M}}(x_i)\right)}{\sqrt[3]{\sum_{i=1}^n \omega_i\left(\zeta^6_{\mathfrak{N}}(x_i)+\eta^6_{\mathfrak{N}}(x_i)+\theta^6_{\mathfrak{N}}(x_i)\right)}.\sqrt[3]{\sum_{i=1}^n \omega_i\left(\zeta^6_{\mathfrak{M}}(x_i)+\eta^6_{\mathfrak{M}}(x_i)+\theta^6_{\mathfrak{M}}(x_i)\right)}}\\ \mathfrak{D}_\omega(\mathfrak{N},\mathfrak{M}) &= \frac{C\omega(\mathfrak{N},\mathfrak{M})}{\max\left[IE\omega(\mathfrak{N}).IE\omega(\mathfrak{M})\right]} \quad (3.4)\\ &= \frac{\sum_{i=1}^n \omega_i\left(\zeta^3_{\mathfrak{N}}(x_i)\zeta^3_{\mathfrak{M}}(x_i)+\eta^3_{\mathfrak{N}}(x_i)\eta^3_{\mathfrak{M}}(x_i)+\theta^3_{\mathfrak{N}}(x_i)\theta^3_{\mathfrak{M}}(x_i)\right)}{\max\left[\sum_{i=1}^n \omega_i\left(\zeta^6_{\mathfrak{N}}(x_i)+\eta^6_{\mathfrak{N}}(x_i)+\theta^6_{\mathfrak{N}}(x_i)\right),\sum_{i=1}^n \omega_i\left(\zeta^6_{\mathfrak{M}}(x_i)+\eta^6_{\mathfrak{M}}(x_i)+\theta^6_{\mathfrak{M}}(x_i)\right)\right]} \end{aligned}$$

The following theorems are proved as similar to Theorem 3.2 and Theorem 3.5:

Theorem 3.6. *Take a weight vector of x_i as ω and it is considered to satisfy the $\sum_{i=1}^n \omega_i = 1$ condition for $\omega_i \ge 1$. The weighted KK between the FFSs $\mathfrak{N},\mathfrak{M}$ defined by Equation 3.3 satisfies:*

(P1) $\mathfrak{C}_\omega(\mathfrak{N},\mathfrak{M}) = \mathfrak{C}_\omega(\mathfrak{M},\mathfrak{N})$,
(P2) $0 \le \mathfrak{C}_\omega(\mathfrak{N},\mathfrak{M}) \le 1$;
(P3) $\mathfrak{C}_\omega(\mathfrak{N},\mathfrak{M}) = 1$ *iff* $\mathfrak{N} = \mathfrak{M}$

Theorem 3.7. *The weighted KK between the FFSs $\mathfrak{N},\mathfrak{M}$ defined by Equation 3.4 satisfies:*

(P1) $\mathfrak{D}_\omega(\mathfrak{N},\mathfrak{M}) = \mathfrak{D}_\omega(\mathfrak{M},\mathfrak{N})$,
(P2) $0 \le \mathfrak{D}_\omega(\mathfrak{N},\mathfrak{M}) \le 1$;
(P3) $\mathfrak{D}_\omega(\mathfrak{N},\mathfrak{M}) = 1$ *iff* $\mathfrak{N} = \mathfrak{M}$

4. Application

Consider the infectious diseases as influenza A(H1N1), Crimean-Congo Hemorrhagic Fever(CCHF), Hepatitis C, norovirus, sandfly fever and denote the set $D = \{h_1,h_2,h_3,h_4,h_5\}$. Let's consider the basic symptoms of these diseases as vomiting, headache, anorexia, temperature, nausea and and show the set of symptoms as $S = \{b_1,b_2,b_3,b_4,b_5\}$.

In order to describe a patient by symptoms, the FFS can be given as follows:

$$\mathfrak{N} = \{(b_1,0.8,0.5),(b_2,0.6,0.4),(b_3,0.3,0.9),(b_4,0.5,0.6),(b_5,0.2,0.8)\}$$

and the relationship between symptoms and diseases are given in the following sets as FFSs:

$$\begin{aligned} h_1 &= \{(b_1,0.6,0.1),(b_2,0.2,0.7),(b_3,0.1,0.8),(b_4,0.5,0.2),(b_5,0.1,0.8)\}\\ h_2 &= \{(b_1,0.7,0.2),(b_2,0.3,0.7),(b_3,0.1,0.9),(b_4,0.7,0.2),(b_5,0.1,0.7)\}\\ h_3 &= \{(b_1,0.3,0.3),(b_2,0.7,0.2),(b_3,0.2,0.7),(b_4,0.3,0.8),(b_5,0.1,0.9)\}\\ h_4 &= \{(b_1,0.1,0.7),(b_2,0.2,0.5),(b_3,0.7,0.2),(b_4,0.4,0.8),(b_5,0.3,0.8)\}\\ h_5 &= \{(b_1,0.1,0.8),(b_2,0.1,0.7),(b_3,0.2,0.9),(b_4,0.3,0.7),(b_5,0.7,0.2)\} \end{aligned}$$

The developed KK $\mathfrak{C}$ and KK $\mathfrak{D}$ were used and they were calculated as follows:

$$\begin{aligned} &\mathfrak{C}(\mathfrak{N},h_1) = 0.2308, \quad \mathfrak{C}(\mathfrak{N},h_2) = 0.24653, \quad \mathfrak{C}(\mathfrak{N},h_3) = 0.2007,\\ &\mathfrak{C}(\mathfrak{N},h_4) = 0.191, \quad \mathfrak{C}(\mathfrak{N},h_5) = 0.211 \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{D}(\mathfrak{N},h_1) = 0.7417, \quad \mathfrak{D}(\mathfrak{N},h_2) = 0.767, \quad \mathfrak{D}(\mathfrak{N},h_3) = 0.6456,\\ &\mathfrak{D}(\mathfrak{N},h_4) = 0.6133, \quad \mathfrak{D}(\mathfrak{N},h_5) = 0.6721. \end{aligned}$$

Choose $\omega = \{0.20, 0.15, 0.13, 0.27, 0.25\}$. Then,

$$\mathfrak{C}_\omega(\mathfrak{N}, h_1) = 0.6136, \quad \mathfrak{C}_\omega(\mathfrak{N}, h_2) = 0.652, \quad \mathfrak{C}_\omega(\mathfrak{N}, h_3) = 0.52115$$
$$\mathfrak{C}_\omega(\mathfrak{N}, h_4) = 0.585, \quad \mathfrak{C}_\omega(\mathfrak{N}, h_5) = 0.6243$$

and

$$\mathfrak{D}_\omega(\mathfrak{N}, h_1) = 0.7424, \quad \mathfrak{D}_\omega(\mathfrak{N}, h_2) = 0.8, \quad \mathfrak{D}_\omega(\mathfrak{N}, h_3) = 0.656$$
$$\mathfrak{D}_\omega(\mathfrak{N}, h_4) = 0.614, \quad \mathfrak{D}_\omega(\mathfrak{N}, h_5) = 0.6721$$

The recognition principle showed us that the h_2 pattern is the most desired pattern. When the results of all KK indices were compared, it was seen that each result was the same.

4.1. Comparison

The KK based on the IFS defined by Xu [8] is given as follows for the two IFSs M and N:

$$\mathfrak{C}_{Xu}(N, M) = \frac{\sum_{i=1}^{n} [\zeta_N(x_i)\zeta_M(x_i) + \eta_N(x_i)\eta_M(x_i) + \theta_N(x_i)\theta_M(x_i)]}{\max\left[(\sum_{i=1}^{n}(\zeta_N^2(x_i) + \eta_N^2(x_i) + \theta_N(x_i)))^{1/2}, (\sum_{i=1}^{n}(\zeta_M^2(x_i) + \eta_M^2(x_i) + \theta_M(x_i)))^{1/2}\right]}.$$

For $\{(b_1, 0.1, 0.1), (b_2, q, 0), (b_3, 0, 1)\}$, choose the three patients h_1, h_2, h_3. These patients shown in form the IFS as follows:

$$\begin{aligned} h_1 &= \{(b_1, 0.35, 0.55), (b_2, 0.65, 0.15), (b_3, 0.3, 0.3)\} \\ h_2 &= \{(b_1, 0.55, 0.35), (b_2, 0.65, 0.2), (b_3, 0.4, 0.3)\} \\ h_3 &= \{(b_1, 0.55, 0.35), (b_2, 0.65, 0.2), (b_3, 0.4, 0.3)\}. \end{aligned}$$

Then $\mathfrak{C}_{Xu}(h_1, N) = \mathfrak{C}_{Xu}(h_2, N) = \mathfrak{C}_{Xu}(h_3, N) = 0.4102$. These results showed that Xu's formula cannot classify h_1, h_2, h_3 with N. When the suggested method was used with the same values, $\mathfrak{C}(h_1, N) = 0.4735$ $\mathfrak{C}(h_2, N) = 0.3976$, $\mathfrak{C}(h_3, N) = 0.3933$ were found.

Now, consider the KK of PFS defined by Garg [37]. Taking the values from the illustrative example used by Garg, the resulting values are as follows: $\mathfrak{C}_{Garg}(h_1, N) = 0.6726$, $\mathfrak{C}_{Garg}(h_2, N) = 0.332$, $\mathfrak{C}_{Garg}(h_3, N) = 0.8943$.When the suggested method was used with the same values, $\mathfrak{C}(h_1, N) = 0.987$ $\mathfrak{C}(h_2, N) = 0.237$, $\mathfrak{C}(h_3, N) = 0.285$ were found.

5. Discussion

First of all, let's explain the advantages of the presented technique and the differences with other techniques. As is known, FFSs can investigate the problems with imprecise and incomplete information more effectively than of IFSs. Since the sets of Pythagorean and intuitionistic MDs are not as extensive as the sets of fermatean MDs [43], it is clear that FFSs will have many comprehensive possibilities for identifying and resolving uncertainties than IFS and PFS.

IFS is a successful generalization of FS theory in dealing with uncertainty and uncertainty, which is characterized by $MD + ND \leq 1$. However, there are cases where the sum of the MDs and NDs will be greater than 1. In this case, the IFS technique will be insufficient to solve this problem. To solve this inadequacy, PFS which is initiated by Yager has emerged. PFS is a natural generalization of FS theory, with successful results. However, the sum of the squares of MD and ND of DMR of a particular attribute may also be greater than 1, in which case it will not be an appropriate solution method in PFS.

There are KKs obtained with IFSs and PFSs in the literature, and there are algorithms defined using these KKs. As mentioned earlier, some cases cannot be symbolized by IFSs and PFSs, hence appropriate results may not be obtained from their corresponding algorithms. The KKs obtained with IFSs and PFSs are a specific situation of the KK of FFSs. Then, the suggested KK is more generalized than existing ones and is appropriate for solving real-life problems more accurately.

6. Conclusion

This study is dedicated to defining a KK for FFS. This study has extended the constraint conditions of $MD + ND \leq 1$ for IFS and the $MD^2 + ND^2 \leq 1$ for PFS to the FFS KK theory. The numeric example has been served that represents that The offered KK can easily operate the conditions where the present KKs in the IFS and PFS frameworks fail. The offered KK in FFS has been improved by taking the MD, ND, and their hesitation degree between MD and ND. Furthermore, weighted KKs have been described to handle cases where elements in a set are correlative. The medical diagnosis model is shown that the correlation coefficient given in the study is easy to use and optimum results can be obtained. From the illustrative example study, it has been accomplished that the offered KK in the FFS framework can conveniently operate the real-life DM problem with their objectives.

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Quasi-Rational and Rational Solutions to the Defocusing Nonlinear Schrödinger Equation

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Abstract

Quasi-rational solutions to the defocusing nonlinear Schrödinger equation (dNLS) in terms of wronskians and Fredholm determinants of order $2N$ depending on $2N-2$ real parameters are given. We get families of quasi-rational solutions to the dNLS equation expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables x and t. We present also rational solutions as a quotient of determinants involving certain particular polynomials.

1. Introduction

We consider the one dimensional defocusing nonlinear Schrödinger equation (dNLS) which can be written in the form

$$iv_t + v_{xx} - 2|v|^2 v = 0. \tag{1.1}$$

Nakemura and Hirota presented solutions to this equation in terms of wronskians in 1985 [1] using bilinear method. They constructed rational solutions by using a connection with a Bäcklund transformation for the classical Boussinesq system (BS)

$$\left\{ \begin{array}{l} u_t = ((1+u)v + a^2 v_{xx})_x, \\ v_t = (u + \frac{1}{2}v^2)_x. \end{array} \right. \tag{1.2}$$

Hone [2] constructed rational solutions in terms of determinant by using Crum dressing method in 1997. In 1999, Barran and Kovalyov presented slowly oscillatory decaying solutions in terms of determinants [3].
Clarkson presented rational solutions and rational-oscillatory solutions expressed in terms of special polynomials associated with rational solutions of the fourth Painlevé equation in [4]. Lenells considered in 2015 solutions of the dNLS equation on the halfline [5] whose Dirichlet and Neumann boundary values become periodic for sufficiently large t. In the same year, Prinari et al. [6] derived novel dark-bright soliton solutions with nonzero boundary conditions obtained within the framework of the inverse scattering transform.

Here we present solutions to the defocusing nonlinear Schrödinger equation (dNLS) of order N depending on $2N-2$ real parameters in terms of wronskians and Fredholm determinants. Families of quasi-rational solutions to the dNLS equation are obtained. These quasi rational solutions can be expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables x and t.
We present also rational solutions as a quotient of determinants using certain particular polynomials.

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2. Different representations of quasi-rational solutions to the dNLS equation

2.1. Quasi-rational solutions of the dNLS equation in terms of Fredholm determinant

We have to define the following notations.
The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of the parameters λ_ν, $1 \le \nu \le 2N$; they are defined by the formulas:

$$\begin{aligned} &\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \\ &x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1,3. \end{aligned} \tag{2.1}$$

The parameters $-1 < \lambda_\nu < 1$, $\nu = 1,\dots,2N$, are real numbers such that

$$\begin{aligned} &-1 < \lambda_{N+1} < \lambda_{N+2} < \dots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 1, \\ &\lambda_{N+j} = -\lambda_j, \quad j = 1,\dots,N. \end{aligned} \tag{2.2}$$

The condition (2.2) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1,\dots,N. \tag{2.3}$$

Complex numbers e_ν $1 \le \nu \le 2N$ are defined in the following way:

$$\begin{aligned} e_j &= i\sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} - \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1}, \\ e_{j+N} &= i\sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} + \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1}, \\ &1 \le j \le N-1. \end{aligned} \tag{2.4}$$

ε, a_ν, b_ν, $\nu = 1\dots 2N$ are arbitrary real numbers.
Let I be the unit matrix, and

$$\varepsilon_j = j \quad 1 \le j \le N, \quad \varepsilon_j = N + j, \quad N+1 \le j \le 2N. \tag{2.5}$$

Let's consider the matrix $D_r = (d_{jk}^{(r)})_{1 \le j,k \le 2N}$ defined by:

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{2.6}$$

Using all the previous notations, the solution to the dNLS equation can be written as

Theorem 2.1. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} e^{2it - i\varphi}{}_{\{x = i\tilde{x}, t = -\tilde{t}\}}, \tag{2.7}$$

is a solution to the defocusing dNLS equation depending on $2N-1$ real parameters a_j, b_j, ε, $1 \le j \le N-1$ with the matrix $D_r = (d_{jk}^{(r)})_{1 \le j,k \le 2N}$ defined by

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

where κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in (2.1), (2.2) and (2.4).

Proof. It is a consequence of the previous works of the author [7, 8, 9] with the change of variables defined by $\{x = i\tilde{x}, t = -\tilde{t}\}$. □

2.2. Wronskian representation

For this, we need to define the following notations :

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N, \quad r = 1,3, \tag{2.8}$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \le \nu \le 2N. \tag{2.9}$$

The functions $\phi_{r,\nu}$ are defined by

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N, \quad r = 1,3. \tag{2.10}$$

We denote $W_r(y)$ the wronskian of the functions $\phi_{r,1},\dots,\phi_{r,2N}$ defined by

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_{r,\nu})_{\nu,\mu \in [1,\dots,2N]}]. \tag{2.11}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\mu \in [1,\dots,2N]}$ defined in (2.6). Then we have the following statement [8]:

Theorem 2.2.

$$\det(I+D_r)=k_r(0)\times W_r(\phi_{r,1},\ldots,\phi_{r,2N})(0), \tag{2.12}$$

where

$$k_r(y)=\frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_{r,\nu})}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu-\gamma_\mu)}.$$

With these notations, we have the following result

Theorem 2.3. *The function v defined by*

$$v(\tilde{x},\tilde{t})=\frac{W_3(\phi_{3,1},\ldots,\phi_{3,2N})(0)}{W_1(\phi_{1,1},\ldots,\phi_{1,2N})(0)}e^{2it-i\varphi}{}_{\{x=i\tilde{x},t=-\tilde{t}\}}.$$

is a solution to the defocusing dNLS equation depending on $2N-1$ *real parameters* a_j, b_j, ε, $1\leq j\leq N-1$ *with* ϕ_ν^r *defined in (2.10)*

$$\phi_{r,\nu}=\sin(\kappa_\nu x/2+i\delta_\nu t-ix_{r,\nu}/2+\gamma_\nu y-ie_\nu/2),\quad 1\leq\nu\leq N,$$
$$\phi_{r,\nu}=\cos(\kappa_\nu x/2+i\delta_\nu t-ix_{r,\nu}/2+\gamma_\nu y-ie_\nu/2),\quad N+1\leq\nu\leq 2N,\quad r=1,3.$$

κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν *being defined in (2.1), (2.2) and (2.4).*

Proof. It is a consequence of [8] with the change of variables defined by $\{x=i\tilde{x},t=-\tilde{t}\}$. ☐

We can give another representation of the solutions to the dNLS equation depending only on terms γ_ν, $1\leq\nu\leq 2N$. From the relations (2.1), we can express the terms κ_ν, δ_ν and $x_{r,\nu}$ in function of γ_ν, for $1\leq\nu\leq 2N$ and we obtain:

$$\begin{aligned}
&\kappa_j=\frac{4\gamma_j}{(1+\gamma_j^2)},\quad \delta_j=\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2},\quad x_{r,j}=(r-1)\ln\frac{\gamma_j-i}{\gamma_j+i},\quad 1\leq j\leq N,\\
&\kappa_j=\frac{4\gamma_j}{(1+\gamma_j^2)},\quad \delta_j=-\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2},\quad x_{r,j}=(r-1)\ln\frac{\gamma_j+i}{\gamma_j-i},\quad N+1\leq j\leq 2N.
\end{aligned} \tag{2.13}$$

We have the following new representation

Theorem 2.4. *The function v defined by*

$$v(\tilde{x},\tilde{t})=\frac{\det[(\partial_y^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\mu\in[1,\ldots,2N]}]}{\det[(\partial_y^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\mu\in[1,\ldots,2N]}]}e^{2it-i\varphi}{}_{\{x=i\tilde{x},t=-\tilde{t}\}} \tag{2.14}$$

is a solution to the defocusing dNLS equation (1.1) depending on $2N-1$ *real parameters* a_j, b_j, ε, $1\leq j\leq N-1$. *The functions* $\tilde{\phi}_{r,\nu}$ *are defined by*

$$\begin{aligned}
&\tilde{\phi}_{r,j}(y)=\sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x+i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t-i\frac{(r-1)}{2}\ln\frac{\gamma_j-i}{\gamma_j+i}+\gamma_j y-ie_j\right),\\
&\tilde{\phi}_{r,N+j}(y)=\cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x-i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t+i\frac{(r-1)}{2}\ln\frac{\gamma_j-i}{\gamma_j+i}+\frac{1}{\gamma_j}y-ie_{N+j}\right),\\
&\text{where } \gamma_j=\sqrt{\frac{1-\lambda_j}{1+\lambda_j}},\ 1\leq j\leq N.\\
&\lambda_j \text{ is an arbitrary real parameter such that } 0<\lambda_j<1,\ \lambda_{N+j}=-\lambda_j,\ 1\leq j\leq N.\\
&\text{The terms } e_\nu \text{ are defined by } (2.4), \text{where } a_j \text{ and } b_j \text{ are arbitrary real numbers}, 1\leq j\leq N-1.
\end{aligned} \tag{2.15}$$

Proof. We have to make the following change of variables defined by $\{x=i\tilde{x},t=-\tilde{t}\}$ in the previous works [8, 10, 11, 12]. ☐

Remark 2.5. *In the formula (2.14), the determinants* $\det[(\partial_y^{\mu-1}f_\nu(0))_{\nu,\mu\in[1,\ldots,2N]}]$ *are the wronskians of the functions* $f_1,\ldots,f_{2N}$ *evaluated in* $y=0$. *In particular* $\partial_y^0 f_\nu$ *means* f_ν.

2.3. Families of quasi-rational solutions of dNLS equation in terms of a quotient of two determinants

The following notations are used:

$$X_\nu=\kappa_\nu x/2+i\delta_\nu t-ix_{3,\nu}/2-ie_\nu/2,$$

$$Y_\nu=\kappa_\nu x/2+i\delta_\nu t-ix_{1,\nu}/2-ie_\nu/2,$$

for $1\leq\nu\leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (2.1).
Parameters e_ν are defined by (2.4).
Below the following functions are used :

$$\begin{aligned}
&\varphi_{4j+1,k}=\gamma_k^{4j-1}\sin X_k,\quad \varphi_{4j+2,k}=\gamma_k^{4j}\cos X_k,\\
&\varphi_{4j+3,k}=-\gamma_k^{4j+1}\sin X_k,\quad \varphi_{4j+4,k}=-\gamma_k^{4j+2}\cos X_k,
\end{aligned} \tag{2.16}$$

for $1 \le k \le N$, and

$$\begin{array}{ll} \varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k}, \end{array} \tag{2.17}$$

for $1 \le k \le N$.
We define the functions $\psi_{j,k}$ for $1 \le j \le 2N$, $1 \le k \le 2N$ in the same way, the term X_k is only replaced by Y_k.

$$\begin{array}{ll} \psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \end{array} \tag{2.18}$$

for $1 \le k \le N$, and

$$\begin{array}{ll} \psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{array} \tag{2.19}$$

for $1 \le k \le N$.
Then we get the following result

Theorem 2.6. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi}{}_{\{x=i\tilde{x},t=-\tilde{t}\}} \tag{2.20}$$

is a quasi-rational solution of the defocusing dNLS equation (1.1) depending on $2N-2$ real parameters a_j, b_j, $1 \le j \le N-1$, where

$$\begin{array}{l} n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \le j \le 2N \quad n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \le j \le 2N \quad n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ d_{j1} = \psi_{j,1}(x,t,0),\ 1 \le j \le 2N \quad d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \le j \le 2N \quad d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ 2 \le k \le N,\ 1 \le j \le 2N. \end{array}$$

The functions φ and ψ are defined in (2.16), (2.17), (2.18), (2.19).

Proof: It is also a consequence of the previous work [10] with the following change of variables defined by $\{x = i\tilde{x}, t = -\tilde{t}\}$. □

We don't give examples of solutions in terms of Fredholm determinants, wronskians or quasi-rational solutions because these types of solutions have been already explicitly constructed by the author until order 13 in the case of the focusing equation and it is easy to deduce these in the defocusing case. These results can be found from the previous published works. We do not give all the references; for the first orders in [13], until last orders (13) in [14].

3. Structure of the multi-parametric quasi-rational solutions to the dNLS equation

Here we present a result which states the structure of the quasi-rational solutions of the dNLS equation. In this section we use the notations defined in the previous sections. The functions φ and ψ are defined in (2.16), (2.17), (2.18), (2.19).
The structure of the quasi rational solutions to the dNLS equation is given by the following theorem

Theorem 3.1. *The function v defined by*

$$v(\tilde{x},\tilde{t}) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi}\{x = i\tilde{x}, t = -\tilde{t}\} \tag{3.1}$$

is a quasi-rational solution of the defocusing dNLS equation (1.1) quotient of two polynomials of degrees $N(N+1)$ in x and t depending on $2N-2$ real parameters a_j and b_j, $1 \le j \le N-1$.

Proof. It is sufficient to realize the following change of variables defined by $\{x = i\tilde{x}, t = -\tilde{t}\}$ in [11, 12]. □

4. Rational solutions of order k to the dNLS equation

4.1. Expression of the rational solutions of order k

We consider the polynomials $p_n(x,t)$ defined by

$$\left\{ \begin{array}{l} p_n(x,t) = \sum_{k=0}^{n} \dfrac{(-x)^k}{k!} \dfrac{t^{\left(\frac{n-k}{2}\right)}}{\left(\frac{n-k}{2}\right)!} \left(1-(n-k)+2\left[\dfrac{n-k}{2}\right]\right), \quad n \ge 0, \\ p_n(x,t) = 0, \quad n < 0, \end{array} \right. \tag{4.1}$$

where $[x]$ is the greater integer less or equal to x.
We denote $W_{n,k}(x,t)$ the following determinants

$$W_{n,k}(x,t)=\begin{vmatrix} p_n & p_{n-1} & \dots & p_k \\ -p_{n-1} & -p_{n-2} & \dots & -p_{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-k}p_k & (-1)^{n-k}p_{k-1} & \dots & (-1)^{n-k}p_{2k-n} \end{vmatrix} \tag{4.2}$$

We define the function v_k by

$$v_k(x,t)=\frac{W_{2k+1,k}(x,t)}{W_{2k+1,k+1}(x,t)}.$$

We will call this function a function of order k and with these notations we have the following result

Theorem 4.1. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{W_{2k+1,k}(x,t)}{W_{2k+1,k+1}(x,t)} \tag{4.3}$$

is a rational to the (dNLS) equation

$$iv_t+v_{xx}-2|v|^2v=0.$$

Proof. It is well known that $v=\dfrac{G}{F}$, where F and G are polynomials, is a solution to the dNLS equation if G and F verify the two following equations:

$$(iD_t+D_x^2)G\cdot F=0 \tag{4.4}$$
$$D_x^2F\cdot F+2\overline{G}G=0, \tag{4.5}$$

where D is the bilinear differential Hirota operator.
We have to verify (4.4) for $G=W_{2k+1,k}(x,t)$ and $F=W_{2k+1,k+1}(x,t)$. We denote C_l and $\tilde{C}_l$ the following columns :

$$C_l=\begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^{k+1}p_{l-k-1} \end{pmatrix}, \quad \tilde{C}_l=\begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^{k}p_{l-k} \end{pmatrix}. \tag{4.6}$$

With these notations, $W_{2k+1,k}(x,t)$ and $W_{2k+1,k+1}(x,t)$ can be written as

$$W_{2k+1,k}(x,t)=|C_{2k+1},\dots,C_k| \quad \text{and} \quad W_{2k+1,k+1}(x,t)=|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|. \tag{4.7}$$

We denote A the expression $A=(iD_t+D_x^2)W_{2k+1,k}(x,t)\cdot W_{2k+1,k+1}(x,t)$. We have to evaluate A.
The polynomials p_k verify $\partial_x(p_k)=-p_{k-1}$ and $\partial_t(p_k)=ip_{k-2}$.
So A can be written as

$$\begin{aligned} A \;=\; & |C_{2k+1},\dots,C_{k+2},C_k,C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|-|C_{2k+1},\dots,C_{k+1},C_{k-2}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}| \\ & -|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|+|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}| \\ & +|C_{2k+1},\dots,C_{k+2},C_k,C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|+|C_{2k+1},\dots,C_{k+1},C_{k-2}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}| \\ & -2|C_{2k+1},\dots,C_{k+1},C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_k|+|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k| \\ & +|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}|. \end{aligned}$$

A can be reduced to

$$\begin{aligned} A \;=\; & 2(|C_{2k+1},\dots,C_{k+2},C_k,C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|+|C_{2k+1},\dots,C_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}| \\ & -|C_{2k+1},\dots,C_{k+1},C_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|). \end{aligned}$$

A can be rewritten as the following determinant of order $2k+3$

$$A=\begin{vmatrix} C_{2k+1} & \dots & C_{k+2} & C_{k+1} & C_k & 0 & \dots & 0 & -C_{k-1} \\ 0 & \dots & 0 & -\tilde{C}_{k+1} & -\tilde{C}_k & \tilde{C}_{2k+1} & \dots & \tilde{C}_{k+2} & \tilde{C}_{k-1} \end{vmatrix} \tag{4.8}$$

We denote by $\mathscr{L}$ the rows and by $\mathscr{C}$ the columns of this determinant of order $2k+3$.

We combine the lines of the previous determinant in the following way:
We replace $\mathscr{L}_{k+2+j}$ by $\mathscr{L}_{k+2+j}+\mathscr{L}_j$ for $1\leq j\leq k+1$, then we obtain the following determinant

$$A=\begin{vmatrix} p_{2k+1} & p_{2k} & \dots & p_{k+1} & p_k & 0 & \dots & 0 & -p_{k-1}\\ -p_{2k} & -p_{2k-1} & \dots & -p_k & -p_{k-1} & 0 & \dots & 0 & p_{k-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{k+1}p_k & (-1)^{k+1}p_{k-1} & \dots & \dots & 0 & 0 & \dots & 0 & 0\\ p_{2k+1} & p_{2k} & \dots & 0 & 0 & p_{2k+1} & \dots & p_{k+2} & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^k p_{k+1} & (-1)^k p_k & \dots & 0 & 0 & (-1)^k p_{k+1} & \dots & (-1)^k p_2 & 0 \end{vmatrix} \tag{4.9}$$

Then replacing $\mathscr{C}_j$ by $\mathscr{C}_j-\mathscr{C}_{k+2+j}$ for $1\leq j\leq k+1$, when we obtain the following determinant

$$A=\begin{vmatrix} p_{2k+1} & p_{2k} & \dots & p_{k+1} & p_k & 0 & \dots & 0 & -p_{k-1}\\ -p_{2k} & -p_{2k-1} & \dots & -p_k & -p_{k-1} & 0 & \dots & 0 & p_{k-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{k+1}p_k & (-1)^{k+1}p_{k-1} & \dots & \dots & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & p_{2k+1} & \dots & p_{k+2} & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & p_k & \dots & p_2 & 0 \end{vmatrix} \tag{4.10}$$

This last determinant is clearly equal to 0, which proves that:

$$A=(iD_t+D_x^2)W_{2k+1,k}(x,t)\cdot W_{2k+1,k+1}(x,t)=0.$$

The relation (4.5) can be proven with the same type of arguments.
We give a sketch of the proof.
We denote B the expression $B=D_x^2F\cdot F+2\overline{G}G$. We have to evaluate B.
The polynomials p_k verify $\partial_x(p_k)=-p_{k-1}$.
So B can be written as

$$\begin{aligned} B \;=\; & 2(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|+|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_{k-1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|\\ & -|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|+|C_{2k+1},\dots,C_k|\times|\overline{C_{2k+1}},\dots,\overline{C_k}|). \end{aligned}$$

The determinant $\overline{G}=|\overline{C_{2k+1}},\dots,\overline{C_k}|$ is equal to $|C^*_{2k+1},\dots,C^*_{k+2}|$, where C^*_l is defined by:

$$C^*_l=\begin{pmatrix} p_l\\ -p_{1-l}\\ \vdots\\ (-1)^{k-1}p_{1-k+1}\end{pmatrix}. \tag{4.11}$$

The product $G\times\overline{G}$ can be written as $G\times(G[k+1,k+1])\,[k+2,k+2]$, where $G[i,j]$ means that $G[i,j]$ is obtained from G by deleting the row i and the column j.
We denote $\widehat{C}_l$

$$\widehat{C}_l=\begin{pmatrix} p_l\\ -p_{1-l}\\ \vdots\\ (-1)^{k-1}p_{1-k+1}\\ (-1)^{k+1}p_{1-k-1}\end{pmatrix}. \tag{4.12}$$

Using the Jacobi identity, we can write $G\times\overline{G}$ as

$$\begin{aligned} G\times(G[k+1,k+1])G[k+2,k+2]-G[k+1,k+2]G[k+2,k+1] \;=\; & |\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_k|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|\\ & -|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+1}|\times|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|. \end{aligned}$$

So, B can be rewritten as the sum

$$\begin{aligned} B \;=\; & 2|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+1}|\times(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|+|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+2},\tilde{C}_{k-1}|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_k|)\\ & -2|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|\times(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_{k+1}|). \end{aligned}$$

But the sums

$$|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+1},\tilde{C}_k|+|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+3},\tilde{C}_{k+2},\tilde{C}_{k-1}|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_k|)$$

and

$$(|\tilde{C}_{2k+1},\dots,\tilde{C}_{k+2},\tilde{C}_k|+|\widehat{C}_{2k+1},\dots,\widehat{C}_{k+2},\widehat{C}_{k+1}|)$$

are equal to 0 which proves that $B=0$.
Then we get the relation (4.5).
So we get the result. □

4.2. Some examples of rational solutions to the dNLS equation

In this section we will give some explicit examples of rational solutions to the dNLS equation. We recall that k means the order of the solution defined by

$$v_k(x,t) = \frac{W_{2k+1,k}(x,t)}{W_{2k+1,k+1}(x,t)}.$$

4.2.1. Rational solutions of order 1 to the dNLS equation

Example 4.2. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = -2\,\frac{x\left(-x^2+6\,it\right)}{-x^4+12\,t^2} \tag{4.13}$$

is a rational solution to the (dNLS) equation.

4.2.2. Rational solutions of order 2 to the dNLS equation

Example 4.3. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = -3\,\frac{-x^8+16\,ix^6t+120\,x^4t^2-720\,t^4}{x\left(-x^8+72\,x^4t^2+2160\,t^4\right)} \tag{4.14}$$

is a rational solution to the (dNLS) equation.

4.2.3. Rational solutions of order 3 to the dNLS equation

Example 4.4. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.15}$$

with

$$n(x,t) = 4\,(-x^{14}+30\,itx^{12}+540\,t^2x^{10}-4200\,it^3x^8-10800\,t^4x^6+151200\,it^5x^4-504000\,t^6x^2+3024000\,it^7)x$$

and

$$d(x,t) = x^{16}-240\,t^2x^{12}-7200\,t^4x^8-2016000\,t^6x^4+6048000\,t^8$$

is a rational solution to the (dNLS) equation.

4.2.4. Rational solutions of order 4 to the dNLS equation

Example 4.5. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.16}$$

with

$$\begin{aligned} n(x,t) \;=\;& -5\,x^{24}+240\,itx^{22}+7560\,t^2x^{20}-134400\,it^3x^{18}-1436400\,t^4x^{16}+12096000\,it^5x^{14}+98784000\,t^6x^{12} \\ & +677376000\,it^7x^{10}-1905120000\,t^8x^8+71124480000\,it^9x^6+533433600000\,t^{10}x^4-1066867200000\,t^{12} \end{aligned}$$

and

$$d(x,t) = (x^{24}-600\,t^2x^{20}+25200\,t^4x^{16}-14112000\,t^6x^{12}-4021920000\,t^8x^8+106686720000\,t^{10}x^4+1066867200000\,t^{12})x$$

is a rational solution to the (dNLS) equation.

4.2.5. Rational solutions of order 5 to the dNLS equation

Example 4.6. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.17}$$

with

$$\begin{aligned} n(x,t) \;=\;& -6\,(-x^{34}+70\,itx^{32}+3360\,t^2x^{30}-100800\,ix^{28}t^3-2116800\,t^4x^{26}+33022080\,it^5x^{24}+423360000\,t^6x^{22} \\ & -3217536000\,ix^{20}t^7-1778112000\,t^8x^{18}+522764928000\,it^9x^{16}+2782389657600\,t^{10}x^{14} \\ & +39431411712000\,it^{11}x^{12}+1552163751936000\,t^{12}x^{10}-11435109396480000\,ix^8t^{13} \\ & -14195308216320000\,t^{14}x^6+198734315028480000\,it^{15}x^4 \\ & -248417893785600000\,t^{16}x^2+1490507362713600000\,it^{17})x \end{aligned}$$

and

$$\begin{aligned} d(x,t) &= -x^{36}+1260t^2x^{32}-302400t^4x^{28}+76204800t^6x^{24}+30939148800t^8x^{20}+12943232870400t^{10}x^{16} \\ &\quad -1623857227776000t^{12}x^{12}-21292962324480000t^{14}x^8-223576104470400000t^{16}x^4 \\ &\quad +298101472542720000t^{18} \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.6. Rational solutions of order 6 to the dNLS equation

Example 4.7. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)}$$

with

$$\begin{aligned} n(x,t) &= 7x^{48}-672itx^{46}-45360t^2x^{44}+2016000it^3x^{42}+67102560t^4x^{40}-1717148160it^5x^{38}-35611349760t^6x^{36} \\ &\quad +580375756800it^7x^{34}+6847687123200t^8x^{32}-82242658713600it^9x^{30}-1292786998272000t^{10}x^{28} \\ &\quad -2839061643264000it^{11}x^{26}-158869157787648000t^{12}x^{24}+2004377520144384000it^{13}x^{22} \\ &\quad -1042758950954434560000t^{14}x^{20}+251136579215189606400 0it^{15}x^{18}+22959387883325988864000t^{16}x^{16} \\ &\quad -142130012484808212480000it^{17}x^{14}-1281924569969568645120000t^{18}x^{12} \\ &\quad -3781319401921409187840000it^{19}x^{10}-4253984327161585336320000t^{20}x^8 \\ &\quad -15881541488069918588928 0000it^{21}x^6-119111561160524389416960 0000t^{22}x^4 \\ &\quad +11911156116052438941696 00000t^{24} \end{aligned}$$

and

$$\begin{aligned} d(x,t) &= (-x^{48}+2352t^2x^{44}-1481760t^4x^{40}+516499200t^6x^{36}+79481606400t^8x^{32}+125617211596800t^{10}x^{28} \\ &\quad +52451663859302400t^{12}x^{24}-2576448441262080000t^{14}x^{20}+42792466383507456000 0t^{16}x^{16} \\ &\quad -1548005174737639833600 00t^{18}x^{12}-1748860223388651749376 0000t^{20}x^8 \\ &\quad +2382231223210487788339 20000t^{22}x^4+1191115611605243894169 600000t^{24})x \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.7. Rational solutions of order 7 to the dNLS equation

Example 4.8. *The function $v_k(x,t)$ defined by*

$$v_k(x,t)=\frac{n(x,t)}{d(x,t)} \tag{4.18}$$

with

$$\begin{aligned} n(x,t) &= 8(-x^{62}+126itx^{60}+11340t^2x^{58}-693000ix^{56}t^3-32916240t^4x^{54}+1236422880it^5x^{52}+38294182080t^6x^{50} \\ &\quad -981414403200it^7x^{48}-20719094457600t^8x^{46}+373342708569600it^9x^{44}+6234317431372800t^{10}x^{42} \\ &\quad -78116020651468800ix^{40}t^{11}-380937010696704000t^{12}x^{38}+7441864983641088000it^{13}x^{36} \\ &\quad +23450962773792153600 0t^{14}x^{34}-1049152892936782233600 0it^{15}x^{32}+288726381993467658240 00t^{16}x^{30} \\ &\quad -674072893110830653440 0000it^{17}x^{28}-169474893181970199183360000t^{18}x^{26} \\ &\quad +2400831552640985128304640000it^{19}x^{24}+29802589444030637579304960000t^{20}x^{22} \\ &\quad -228707538154566157550223360000it^{21}x^{20}+2302924801111261090283520 00000t^{22}x^{18} \\ &\quad +11716238554181895101585817600000it^{23}x^{16}+9113325209167385962862346 2400000t^{24}x^{14} \\ &\quad -235200751536287015684171366400000it^{25}x^{12}+13016193545913179761829058969600000t^{26}x^{10} \\ &\quad -939489035466174392258002944000000 00it^{27}x^8-6764321055356455624257621196800000 0t^{28}x^6 \\ &\quad +94700494774990378739606696755200 00000it^{29}x^4-631336631833269191597377978368000000t^{30}x^2 \\ &\quad +3788019790999615149584267870208000000it^{31})x \end{aligned}$$

and

$$\begin{aligned} d(x,t) &= x^{64}-4032t^2x^{60}+5201280t^4x^{56}-3353011200t^6x^{52}+613601049600t^8x^{48}-737286926745600t^{10}x^{44} \\ &\quad -6096479298674688 00t^{12}x^{40}-164001698405056512000t^{14}x^{36}+436233208152604262400000t^{16}x^{32} \\ &\quad -5711758289010198577152 0000t^{18}x^{28}+1031547791333346060533760 0000t^{20}x^{24} \\ &\quad +1960527679738492460256460800000t^{22}x^{20}+327241269096776028738551808000000t^{24}x^{16} \\ &\quad -2459753111038711136093680435200000 0t^{26}x^{12}-1803818948095054833135365652480000 00t^{28}x^8 \\ &\quad -10101386109332307065558047653888000000t^{30}x^4+757603958199923029916853574041600000 0t^{32} \end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.8. Rational solutions of order 8 to the dNLS equation

Example 4.9. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.19}$$

with

$$\begin{aligned}
n(x,t) \quad = \quad & -9x^{80} + 1440\,itx^{78} + 166320t^2x^{76} - 13305600\,it^3x^{74} - 846568800t^4x^{72} + 43445445120\,it^5x^{70} \\
& +1870141996800t^6x^{68} - 68163843686400\,it^7x^{66} - 2125554631200000t^8x^{64} \\
& +57307521503232000\,it^9x^{62} + 1363297604917248000t^{10}x^{60} - 28253298553159680000\,it^{11}x^{58} \\
& -484086300466728960000t^{12}x^{56} + 7566415631881666560000\,it^{13}x^{54} + 133152327700676444160000t^{14}x^{52} \\
& -2058963324486458277888000\,it^{15}x^{50} - 22021666720077485260800000t^{16}x^{48} \\
& -139857830692567689461760000\,it^{17}x^{46} - 3066455043451203128524800000t^{18}x^{44} \\
& -43732810958030201621053440000\,it^{19}x^{42} - 465799874464596133601633894400000t^{20}x^{40} \\
& +1425504528307712192388739891200000\,it^{21}x^{38} + 31263949644881871172712634777600000t^{22}x^{36} \\
& -50901829685576593714265142067200000\,it^{23}x^{34} - 493251555973550706551886918451200000t^{24}x^{32} \\
& +3939679652581176245055907524771840000\,it^{25}x^{30} \\
& +68488489929343657277817910355558400000t^{26}x^{28} \\
& -2294663852713834896871972008886272000000\,it^{27}x^{26} \\
& +14778780673321768689451169942077440000000t^{28}x^{24} \\
& -94720108545103542927669850282131456000000\,it^{29}x^{22} \\
& +332149098425336720044569760349395353600000t^{30}x^{20} \\
& -1987110701786406181360979053528901222400000\,it^{31}x^{18} \\
& -16487495609030007436640859884216333107200000t^{32}x^{16} \\
& +74772946571466528047991786308993286144000000\,it^{33}x^{14} \\
& +698164988101331604721721353272733728768000000t^{34}x^{12} \\
& +1170257503865089166009742458819058440601600000\,it^{35}x^{10} \\
& +2681840113024162672105659801460342259712000000t^{36}x^{8} \\
& +27306008423518747206893990705778030280704000000\,it^{37}x^{6} \\
& +20479506317639060405170493029333522710528000000t^{38}x^{4} \\
& -12287703790583436243102295817600113626316800000t^{40}
\end{aligned}$$

and

$$\begin{aligned}
d(x,t) \quad = \quad & (x^{80} - 6480t^2x^{76} + 14968800t^4x^{72} - 17603308800t^6x^{68} + 10318053715200t^8x^{64} - 6006932976844800t^{10}x^{60} \\
& -2425558108925952000t^{12}x^{56} - 356811818824581120000t^{14}x^{52} + 1771127741654469918720000t^{16}x^{48} \\
& +7598918248410742916382720000t^{18}x^{44} - 3390923311730068095298437120000t^{20}x^{40} + \\
& 916481140720063998978215116800000t^{22}x^{36} + 10661196076862440946653200384000000t^{24}x^{32} \\
& +64248472376758454748787784024064000000t^{26}x^{28} \\
& +91228216925170585739079613195223040000000t^{28}x^{24} \\
& -363842989551198731535819544517935104000000t^{30}x^{20} \\
& +3225081783326530607549942721580892160000000t^{32}x^{16} \\
& -79278808172678680389646014474193207296000000t^{34}x^{12} \\
& -495734081498405827268015902694184478310400000000t^{36}x^{8} \\
& +409590126352781208103409860586670454210560000000t^{38}x^{4} \\
& +12287703790583436243102295817600113626316800000t^{40})x
\end{aligned}$$

is a rational solution to the (dNLS) equation.

4.2.9. Rational solutions of order 9 to the dNLS equation

Example 4.10. *The function $v_k(x,t)$ defined by*

$$v_k(x,t) = \frac{n(x,t)}{d(x,t)} \tag{4.20}$$

with

$$\begin{aligned}
n(x,t) \quad = \quad & -10\,(-x^{98}+198\,itx^{96}+28512\,t^{2}x^{94}-2882880\,ix^{92}t^{3}-235414080\,t^{4}x^{90}+15723227520\,it^{5}x^{88}+892747215360\,t^{6}x^{86}\\
& -43550538071040\,it^{7}x^{84}-1851377348620800\,t^{8}x^{82}+69126330181708800\,it^{9}x^{80}+2290609252566835200\,t^{10}x^{78}\\
& -67587198729187737600\,it^{11}x^{76}-1769295226634333798400\,t^{12}x^{74}+41464108043171573760000\,it^{13}x^{72}\\
& +896769265742927216640000\,t^{14}x^{70}-17596890565184340393984000\,it^{15}x^{68}\\
& -294155011471335642980352000\,t^{16}x^{66}+3342933346282600754331648000\,it^{17}x^{64}\\
& -24509684290061469981081600000\,t^{18}x^{62}-2433287180223360105672867840000\,it^{19}x^{60}\\
& -130937651100257873779296829440000\,t^{20}x^{58}+3075516842912088157390236549120000\,it^{21}x^{56}\\
& -62774936194714235337761850654720000\,t^{22}x^{54}+642613916951522230503969883422720000\,it^{23}x^{52}\\
& +2711537814255492085577714208079872000000\,t^{24}x^{50}-78014081861464809452654641620713472000000\,it^{25}x^{48}\\
& -1593441818517645683457723068376416256000000\,t^{26}x^{46}+24885149560584639176620849366107684864000000\,it^{27}x^{44}\\
& +386625555598655080628036924868824924160000000\,t^{28}x^{42}\\
& -5038144976069943692864084422006941941760000000\,it^{29}x^{40}\\
& -28927990798127747807146732998684384952320000000\,t^{30}x^{38}\\
& +114495548645121617338289433808403913768960000000\,it^{31}x^{36}\\
& -1966672495435151207116333588567548714024960000000\,t^{32}x^{34}\\
& -19007340765009740342410339347299858833735680000000\,it^{33}x^{32}\\
& -243813871772100040030824399553967097674465280000000\,t^{34}x^{30}\\
& -2039857021478025692575568254747153812066140160000000\,it^{35}x^{28}\\
& -9235475349273617458141231733749180574359093248000000\,t^{36}x^{26}\\
& +117850883813438704463234535957236559983570583552000000\,it^{37}x^{24}\\
& +1397864071492793101760818073933692790817737932800000000\,t^{38}x^{22}\\
& -10957892109314458638488314003177389319262306304000000000\,it^{39}x^{20}\\
& +10510341802238027300957583024620038551196076605440000000\,t^{40}x^{18}\\
& +240979190910346887273875131969580258513181663559680000000\,it^{41}x^{16}\\
& +186573741348846203370304005095500602643393944748032000000\,t^{42}x^{14}\\
& -1010056424796354150633560689524104917383858471567360000000\,it^{43}x^{12}\\
& +1358944910391950715328384491014745524218195611746304000000\,t^{44}x^{10}\\
& -9713302439679175155131997163624383620361280026574848000000\,it^{45}x^{8}\\
& -45412842575123416309708038687075040302987802721648640000000\,t^{46}x^{6}\\
& +63577979605172782833591254161905056424182923810308096000000\,it^{47}x^{4}\\
& -264908248354886595139963559007937735100762182542950400000000\,t^{48}x^{2}\\
& +158944949012931957083978135404762641060457309525770240000000\,it^{49})x
\end{aligned}$$

and

$$\begin{aligned}
d(x,t) \quad = \quad & -x^{100}+9900\,t^{2}x^{96}-37540800\,t^{4}x^{92}+75243168000\,t^{6}x^{88}-86477751360000\,t^{8}x^{84}+69030822212352000\,t^{10}x^{80}\\
& -18841861512714240000\,t^{12}x^{76}+3213319037194506240000\,t^{14}x^{72}+800128288418889830400000\,t^{16}x^{68}\\
& -71696063588052183920640000000\,t^{18}x^{64}-1223508256241822655062999040000000\,t^{20}x^{60}\\
& +16323950793276454578355961856000000\,t^{22}x^{56}-886344533834954585654182312673280000000\,t^{24}x^{52}\\
& +5976929116686443410537623060480000000000\,t^{26}x^{48}\\
& -90376185192441396225618162676137984000000000\,t^{28}x^{44}\\
& -297859518059409014875858745094586761216000000000\,t^{30}x^{40}\\
& +1826538360765959122221072907192800116736000000000\,t^{32}x^{36}\\
& +73238562239110143318755110276731600654827520000000000\,t^{34}x^{32}\\
& -5972791989999472523782344122456920795014758400000000000\,t^{36}x^{28}\\
& +77433614769321662507215500261562498051316121600000000000\,t^{38}x^{24}\\
& +872923144660058559307277530405151995358292738048000000000\,t^{40}x^{20}\\
& +8150655465742049960079770120159034404289323728896000000000\,t^{42}x^{16}\\
& -40424310625583344063565867770691797239402021362073600000000\,t^{44}x^{12}\\
& -189220177396347567957116827862812667929115844673536000000000\,t^{46}x^{8}\\
& -662270620887216487849908897519844337751905456357376000000000\,t^{48}x^{4}\\
& +317889898025863914167956270809525282120914619051540480000000000\,t^{50}
\end{aligned}$$

is a rational solution to the (dNLS) equation.

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 60 terms and the denominator 31 terms with big coefficients. It will be relevant to study in detail the structure of these solutions.

5. Conclusion

Different representations of quasi-rational solutions to the defocusing nonlinear Schrödinger equation have been given. First quasi rational solutions in terms of wronskians of order $2N$ depending on $2N-2$ real parameters have been presented. Another representation in terms of Fredholm determinants are given depending on $2N-2$ real parameters. These solutions give families of quasi-rational solutions to the dNLS equation expressed as a quotient of two polynomials of degree $N(N+1)$ in the variables x and t depending on $2N-2$ real parameters. Rational solutions as a quotient of determinants have been also given using certain particular polynomials and some explicit expressions are given for some orders. It will be relevant to study the structure of these last solutions.

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Approximating Fixed Points of Generalized $\alpha-$Nonexpansive Mappings by the New Iteration Process

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Abstract

In this paper we introduce a new iteration process for approximation of fixed points. We numerically compare convergence behavior of our iteration process with other iteration process like M-iteration process. We also prove weak and strong convergence theorems for generalized $\alpha-$nonexpansive mappings by using new iteration process. Furthermore we give an example for generalized $\alpha-$nonexpansive mapping but does not satisfy (C) *condition*.

1. Introduction and Preliminaries

Let be X be a real Banach space and K be a nonempty subset of X, and $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of $T : K \to K$ if $x = Tx$. We denote $F(T)$ the set of all fixed points of T. A mapping $T : K \to K$ is called *nonexpansive* if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in K$. T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx-p\| \le \|x-p\|$ for all $x \in K$ and $p \in F(T)$. In the last 60 years, many iteration processes have been developed regarding the fixed point approach. Recently, with the development of iteration processes, a faster approach to the fixed point has gained importance. Some of well-known iteration processes are Mann iterative scheme [1], Ishikawa [2], Noor [3], S-iteration process [4], Abbas and Nazir [5], Picard-S [6], Thakur et al. [7] and so on.

In 2018, Ullah and Arshad [8] introduced the following iteration process called M-iteration process : for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1-a_n)x_n + a_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\} \in [0,1]$.

Motivated by above, in this paper, we introduce new iteration scheme:for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = T((1-b_n)x_n + b_n T x_n), \\ y_n = T z_n, \\ x_{n+1} = T((1-a_n)T x_n + a_n T y_n), \forall n \in \mathbb{N}, \end{cases} , \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\} \in [0,1]$.

In order to show numerically that our new iteration process (1.2) have a good speed of convergence comparatively to (1.1), we consider the following example.

Example 1.1. *Let us define a function $T : [0,10) \to [0,10)$ by $T(x) = \sqrt{2x+3}$. Then clearly T is a contraction map. Let $a_n = 0.70, b_n = 0.30$ for all n. Set the stop parameter to $\|x_n - 3\| \le 10^{-6}$, 3 is the fixed point of T. The iterative values for initial value $x_1 = 4$ are given in Table 1. The efficiency of new iteration process is clear. We can see that our new iteration process (1.2) have a good speed of convergence comparatively to (1.1) iteration process.*

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Table 1: Sequences generated by M-iteration and New iteration processes for mapping T of Example 1.1.

	M-iteration	New iteration
x_1	4	4
x_2	3.083577194937360	3.037893699789630
x_3	3.007388352660220	3.001521367442330
x_4	3.000656421483590	3.000061224295530
x_5	3.000058346040820	3.000002464079130
x_6	3.000005186294710	3.000000099171560
x_7	3.000000461003820	3.000000003991350
x_8	3.000000040978120	3.000000000160640
x_9	3.000000003642500	3.000000000006470
x_{10}	3.000000000323780	3.000000000000260
x_{11}	3.000000000028780	3.000000000000010
x_{12}	3.000000000002560	3.000000000000000
x_{13}	3.000000000000230	3.000000000000000
x_{14}	3.000000000000020	3.000000000000000
x_{15}	3.000000000000000	3.000000000000000

In the recent years, several generalizations of nonexpansive mappings and related fixed point have have been studied by many authors (see [7], [8], [9], [10], [12], [14], [15], [16], [17], [20]). In 2008, Suzuki [17] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called (C) *condition*. Let K be a nonempty convex subset of a Banach space X, a mapping $T: K \to K$ is satisfy *condition* (C) if for all $x, y \in K$, $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ implies $\|Tx - Ty\| \leq \|x - y\|$. Suzuki [17] showed that the mapping satisfying *condition* (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. The mapping satisfy *condition* (C) is called Suzuki generalized nonexpansive mapping. In 2011, Aoyama and Kohsaka [9] introduced the class of $\alpha-$nonexpansive mappings in the setting of Banach spaces and obtained some fixed point results for such mappings. A mapping $T: K \to K$ is called a $\alpha-$nonexpansive mapping if there exists an $\alpha \in [0,1)$ such that for each $x, y \in K$,

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

In [14], authors introduced the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings. A mapping $T: K \to K$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0,1)$ and for each $x, y \in K$, $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ implies

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|.$$

In 2019, Şahin [15] studied the M-iteration process in hyperbolic spaces and proved some strong and Δ-convergence theorems of this iteration process for generalized nonexpansive mappings. In 2021, Ullah et al. [20] introduced some convergence results for generalized α-nonexpansive mappings using M-iteration process in the framework of Banach spaces.

Inspired and motivated by these facts, we consider generalized α-nonexpansive mappings which properly contains, the α-nonexpansive mappings. Also we give an example for generalized $\alpha-$nonexpansive mapping but does not satisfy (C) *condition*. Further we prove some convergence theorems of new iterative process (1.2) to fixed point for generalized α-nonexpansive mappings in a Banach space.

The following definitions will be needed in proving our main results.

A Banach space X is said to be uniformly convex [11] if for each $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that $\|\frac{(x+y)}{2}\| \leq 1 - \delta$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| > \varepsilon$.

Recall that a Banach space X is said to satisfy *Opial's condition* [13] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$.

In what follows, we give some definitions and lemmas to be used in main results:

Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to K is defined by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of $\{x_n\}$ relative to K is the set

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one-point.

Lemma 1.2. *[18] Suppose that X is a uniformly convex Banach space and $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequence of X such that $\limsup_{n\to\infty}\|x_n\| \leq \xi$, $\limsup_{n\to\infty}\|y_n\| \leq \xi$ and $\lim_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = \xi$ hold for $\xi \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Let $\{u_n\}$ in K be a given sequence. $T: K \to K$ with the nonempty fixed point set $F(T)$ in K is said to satisfy *Condition (I)* [19] with respect to the $\{u_n\}$ if there is a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|u_n - Tu_n\| \geq f(d(u_n, F(T)))$ for all $n \geq 1$.

Now we give the following well-known facts about generalized α-nonexpansive mapping, which can be found in [14].

Lemma 1.3. *(1) If T is Suzuki generalized nonexpansive mapping then T is a generalized α -nonexpansive mapping.*

(2) If T is a generalized α -nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.

(3) If T is a generalized α -nonexpansive mapping, then $F(T)$ is closed. Moreover if X is strictly convex and K is convex, then $F(T)$ is also convex.

(4) If T is a generalized α -nonexpansive mapping, then for each $x,y\in K$,

$$\|x-Ty\|\leq(\frac{3+\alpha}{1-\alpha})\|Tx-x\|+\|x-y\|.$$

(5) If X has Opial property, T is a generalized α -nonexpansive mapping, $\{x_n\}$ converges weakly to a point $x*$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, then $x*\in F(T)$.That is, $I-T$ is demiclosed at zero, where I is the identity mapping on X.

Now we give an example where T is a generalized α-nonexpansive mapping but does not satisfy condition (C).

Example 1.4. *Let $K=[0,2]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define a mapping $T:K\to K$ by*

$$Tx=\begin{cases}0, & x\neq 2,\\ 1, & x=2.\end{cases}$$

For $x\in(1,1.33]$ and $y=2$, then we have $\frac{1}{2}|x-Tx|\leq|x-y|$ and $|Tx-Ty|=1>2-x=|x-y|$. Thus T does not satisfy Suzuki's condition (C). However, T is a generalized $\alpha-$nonexpansive mapping with $\alpha\geq\frac{1}{3}$.

2. Weak and Strong Convergence Theorems of New Iteration Process for Generalized $\alpha-$Nonexpansive Mapping

In this section, we prove weak and strong convergence theorems of new iterative scheme defined by (1.2) for generalized $\alpha-$nonexpansive mapping in a uniformly convex Banach space.

Lemma 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X , T be a generalized $\alpha-$nonexpansive mapping with $F(T)\neq\emptyset$. For arbitrary chosen $x_1\in K$, let $\{x_n\}$ be a sequence generated by (1.2) with $\{a_n\}$ and $\{b_n\}$ real sequences in $[0,1]$, then $\lim_{n\to\infty}\|x_n-p\|$ exists for any $p\in F(T)$.*

Proof. For any $p\in F(T)$, using (1.2), we have,

$$\begin{aligned}\|z_n-p\| &= \|T((1-b_n)x_n+b_nTx_n)-p\| \\ &\leq \|(1-b_n)(x_n-p)+b_n(Tx_n-p)\| \\ &\leq (1-b_n)\|x_n-p\|+b_n\|x_n-p\|=\|x_n-p\|.\end{aligned}\tag{2.1}$$

Using (1.2) and (2.1), we get

$$\|y_n-p\| = \|Tz_n-p\|\leq\|z_n-p\|\leq\|x_n-p\|\tag{2.2}$$

By using (1.2) and (2.2) , we get

$$\begin{aligned}\|x_{n+1}-p\| &= \|T((1-a_n)Tx_n+a_nTy_n)-p\| \\ &\leq \|(1-a_n)(Tx_n-p)+a_n(Ty_n-p)\| \\ &\leq (1-a_n)\|x_n-p\|+a_n\|y_n-p\| \\ &\leq (1-a_n)\|x_n-p\|+a_n\|x_n-p\|=\|x_n-p\|\end{aligned}\tag{2.3}$$

This implies that $\{\|x_n-p\|\}$ is bounded and non-increasing for all $p\in F(T)$. It follows that $\lim_{n\to\infty}\|x_n-p\|$ exists. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X , T be a generalized $\alpha-$nonexpansive mapping. For arbitrary chosen $x_1\in K$, let $\{x_n\}$ be a sequence in K defined by (1.2) with the real sequences $\{a_n\}$ in $(0,1]$ and $\{b_n\}$ in $[k,m]$ for some $k,m\in(0,1)$, then $F(T)\neq\emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.*

Proof. Suppose $F(T)\neq\emptyset$ and by Lemma 2.1, $\lim_{n\to\infty}\|x_n-p\|$ exists. Put $\lim_{n\to\infty}\|x_n-p\|=\xi$. From (2.1) and (2.2) we have

$$\limsup_{n\to\infty}\|z_n-p\|\leq\limsup_{n\to\infty}\|x_n-p\|\leq\xi,$$

and

$$\limsup_{n\to\infty}\|y_n-p\|\leq\limsup_{n\to\infty}\|x_n-p\|\leq\xi,$$

and also we have

$$\limsup_{n\to\infty}\|Tx_n-p\|\leq\limsup_{n\to\infty}\|x_n-p\|\leq\xi.$$

On the other hand,

$$\begin{aligned} \|x_{n+1}-p\| &= \|T((1-a_n)Tx_n+a_nTy_n)-p\| \le \|(1-a_n)(x_n-p)+a_n(Ty_n-p)\| \\ &\le (1-a_n)\|x_n-p\|+a_n\|Ty_n-p\| \le (1-a_n)\|x_n-p\|+a_n\|y_n-p\|. \end{aligned}$$

$$\|x_{n+1}-p\|-\|x_n-p\| \le \frac{\|x_{n+1}-p\|-\|x_n-p\|}{a_n} \le \|y_n-p\|-\|x_n-p\|.$$

So we can get $\|x_{n+1}-p\| \le \|y_n-p\|$. Therefore $\xi \le \liminf_{n\to\infty}\|y_n-p\|$. Thus we have $\lim_{n\to\infty}\|y_n-p\| = \xi$. Also,

$$\begin{aligned} \xi = \lim_{n\to\infty}\|y_n-p\| &= \lim_{n\to\infty}\|Tz_n-p\| \\ &\le \lim_{n\to\infty}\|T(T((1-b_n)x_n+b_nTx_n))-p\| \\ &\le \lim_{n\to\infty}\|T((1-b_n)x_n-p+b_nTx_n)-p\| \\ &\le \lim_{n\to\infty}\|(1-b_n)(x_n-p)+b_n(Tx_n-p)\| \\ &\le \lim_{n\to\infty}(1-b_n)\|x_n-p\|+\lim_{n\to\infty}b_n\|Tx_n-p\| \le \xi. \end{aligned}$$

Hence we have $\lim_{n\to\infty}\|(1-b_n)(x_n-p)+b_n(Tx_n-p)\| = \xi$. Thus by Lemma 1.2, we have $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$.
Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$. Let $p \in A(K,\{x_n\})$. By Lemma 1.3 (4), we have

$$\begin{aligned} r(Tp,\{x_n\}) &= \limsup_{n\to\infty}\|x_n-Tp\| \\ &\le \limsup_{n\to\infty}((\frac{3+\alpha}{1-\alpha})\|Tx_n-x_n\|+\|x_n-p\|+\|p-Tp\| \\ &= \limsup_{n\to\infty}\|x_n-p\| = r(p,\{x_n\}) \end{aligned}$$

This implies that $Tp = p \in A(K,\{x_n\})$. Since X is a uniformly Banach space, $A(K,\{x_n\})$ is consists of a unique element.Thus, we have $Tp = p$. This completes the proof. □

In the next result, we prove strong convergence theorems as follows.

Theorem 2.3. *Let X be a real uniformly convex Banach space and K be a nonempty compact convex subset of X and T be a generalized $\alpha-$nonexpansive mapping on K and $F(T) \neq \emptyset$. Assume that $p \in F(T)$ is a fixed point of T and let $\{x_n\}$ be as in Theorem 2.2. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. $F(T) \neq \emptyset$, so by Theorem 2.2, we have $\lim_{n\to\infty}\|Tx_n-x_n\| = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p$ as $k \to \infty$ for $p \in K$. Then for $(\frac{3+\alpha}{1-\alpha}) \ge 1$ we have

$$\|x_{n_k}-Tp\| \le (\frac{3+\alpha}{1-\alpha})\|Tx_{n_k}-x_{n_k}\|+\|x_{n_k}-p\| \text{ for all } k \ge 0.$$

Letting $k \to \infty$, we get $Tp = p$, $p \in F(T)$. $\lim_{n\to\infty}\|x_n-p\|$ exists for every $p \in F(T)$, so $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 2.4. *Let the conditions of Theorem 2.2 be satisfied. Also if T satisfies condition (I), then $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T.*

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n-p\|$ exists and so $\lim_{n\to\infty}d(x_n,p)$ exists for all $p \in F(T)$. Also by Theorem 2.2, $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$. It follows from condition (I) that $\lim_{n\to\infty}f(d(x_n,F(T)) \le \lim_{n\to\infty}\|x_n-Tx_n\|$. That is, $\lim_{n\to\infty}f(d(x_n,F(T)) = 0$. Since $f:[0,\infty)\to[0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty}d(x_n,F(T)) = 0$. So, all the assumptions of Theorem 2.5 in [20] are satisfied. The rest of the proof is similar to the proof of Theorem 2.5 in [20] and therefore it is omitted. Thus, we can easily see that $\{x_n\}$ strongly converges to an element of $F(T)$. □

Finally, we prove the weak convergence of the iterative scheme (1.2) for generalized $\alpha-$nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.5. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let T be a generalized $\alpha-$nonexpansive mapping on K with $F(T) \neq \emptyset$. Assume that $p \in F(T)$ is a fixed point of T and let $\{x_n\}$ be as in Theorem 2.2. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, it follows from Theorem 2.2 that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n-x_n\| = 0$. Let v_1, v_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By $\lim_{n\to\infty}\|x_n-Tx_n\|$ and $I-T$ is demiclosed with respect to zero , therefore we obtain $Tv_1 = v_1$. Again in the same manner, we can $Tv_2 = v_2$. Next we prove the uniqueness. By Lemma 2.1, $\lim_{n\to\infty}\|x_n-v_1\|$ and $\lim_{n\to\infty}\|x_n-v_2\|$ exist. For suppose that $v_1 \neq v_2$, then by the Opial's condition, we have

$$\begin{aligned} \lim_{n\to\infty}\|x_n-v_1\| &= \lim_{j\to\infty}\|x_{n_j}-v_1\| < \lim_{j\to\infty}\|x_{n_j}-v_2\| = \lim_{n\to\infty}\|x_n-v_2\| \\ &= \lim_{k\to\infty}\|x_{n_k}-v_2\| < \lim_{k\to\infty}\|x_{n_k}-v_1\| = \lim_{n\to\infty}\|x_n-v_1\| \end{aligned}$$

which is a contradiction. So, $v_1 = v_2$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

3. Conclusions

We introduce a new iteration process to approximate fixed points of a new type of nonexpansive mappings. We noticed from Table 1 that our new iteration process is faster than M-iteration process for contraction mapping. We also illustrated an example of a mapping that is generalized $\alpha-$nonexpansive mapping but not Suzuki's generalized nonexpansive mapping.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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