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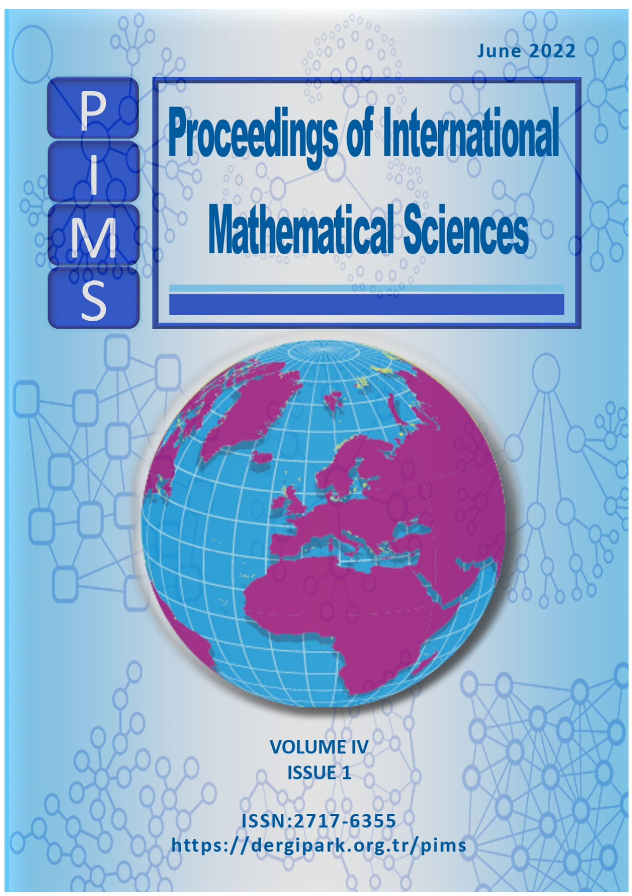

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Contents

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OPTIMAL CONTROL FOR FRACTIONAL STOCHASTIC DIFFERENTIAL SYSTEM DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH POISSON JUMPS

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Abstract. The purpose of this article is to study the optimal control problem for fractional stochastic differential system driven by fractional Brownian motion with Poisson jumps in Hilbert space. Initially, the sufficient conditions for existence of mild solution results are formulated and proved by virtue of fractional calculus, solution operator and stochastic analysis techniques. Further, we formulated and proved the existence results for optimal control of the proposed system with corresponding cost function by using Balder's theorem. Finally an example is provided to illustrate the main results.

1. Introduction

Fractional Calculus (FC) has been introduced since the end of the nineteenth century by famous mathematicians Riemann and Liouville, but the concept of non-integer calculus as a generalization of the traditional integer order calculus was mentioned already in 1695 by Leibnitz and L'Hospital. The subject of FC has become a rapidly growing area in the field of system physics, chemistry, biology, medicine and finance etc. On the other, fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes. Hence, there is a growing need to find the behavior of solution of the fractional differential equations (FDEs). For more details on FDEs, the reader may refer to the monographs [3, 4, 5, 6, 2] and references therein.

The fractional Brownian motion (fBm) is usual candidate to model phenomena due to its self-similarity of increments and long-range dependence. This fBm $B^{\mathtt{H}}$ is the continuous centered Gaussian process with covariance function described by

$$R^{\mathtt{H}}(t,s) \quad = \quad \mathbf{E}\left[B^{\mathtt{H}}(t)B^{\mathtt{H}}(s)\right] = \frac{1}{2}(t^{2\mathtt{H}} + s^{2\mathtt{H}} - |t-s|^{2\mathtt{H}})$$

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The parameter $\mathtt{H}$ characterizes all the important properties of the process, when $\mathtt{H} < \frac{1}{2}$, the increments are negatively correlated and the correlation decays more slowly the quadratically; when $\mathtt{H} > \frac{1}{2}$, the increments are positively correlated and the correlation decays so slowly that they are not summable, a situation which is commonly known as the long memory property. The fBm can be expressed as Wiener integral with respect to the standard Wiener process, i.e. the integral of a deterministic kernal with respect to a standard Brownian motion, the Hermite process of order 1 is fBm and of order 2 is the Rosenblatt process. However, there exist only a few papers in this field, for more details (see [13, 14, 15, 16] and reference therein).

On the other hand, the Poisson jumps have become very popular one in recent years, the Poisson jumps are generally based on the Poisson random measure in aspects of applications in many real life phenomena such as, finance, biology and any other field of science see [7, 8, 10]. For example Poisson jump models that are very popular in financial modeling sice Merton first derived an option pricing formula based on a stock price process generated by a mixture of a Brownian motion and a Poisson process. This mixed process is also called the jump diffusion process. The requirement for a jump component in a stock price process is intuitive, and supported by the big crashes in stock markets: The Black Monday on October 17, 1987 and the recent market crashes in the financial crisis since 2008 are two prominent examples. To model jump events, we need two quantities: jump frequency and jump size. The first one specifies how many times jumps happed in a given time period, and the second one determines low large a jump is if it occurs. It is natural and necessary to include a jump term in the stochastic differential equation. Recently, Balasubramaniam et al. [1] and Muthukumar et al. [10] have studied, respectively, fractional stochastic differential equations driven by Poisson jumps and fractional stochastic differential equations with Poisson jumps. Very recently Rihan et al. [11] extended to study the existence of solutions of fractional stochastic differential equations with Hilfer fractional derivative and Poisson jumps.

An optimal control problem (OCP) describes the path of control variables concerned with minimizing the cost functional or maximizing a payoff to the corresponding system over a set of admissible control functions. Nowadays, optimal control theory has a considerable development and have fruitful applications in many fields like science and engineering (see [17, 18]). Stochastic optimal control problem (SOCP) makes to design the time path of the controlled variables which performs the desired control task with minimum cost despite the presence of noise. SOCPs and its applications have extensive attention in the literature see [24, 25, 26, 9]. The main goal of optimal control is to find, in an open-loop control, the optimal values of the control variables for the dynamic system which maximize or minimize a given performance index. If a fractional differential equation describes the performance index and system dynamics, then an optimal control problem is known as a fractional optimal control problem. Using the fractional variational principle and lagrange multiplier technique, Agrawal [21] discussed the general formulation and solution scheme for Riemann-Liouville fractional optimal control problems. It is remarkable thathe fixed point technique, which is used to establish the existence results for abstract fractional differential equations, could be extended to address the fractional optimal control problems. Recently, Aicha Harrat et al. [19] studied the optimal controls of impulsive fractional system with

Clarke subdifferential. Very recently, using the LeraySchauder fixed point theorem, Balasubramaniam et al. [1] studied the solvability and optimal controls for impulsive fractional stochastic integrodifferential equations. Tamilalagan et al. [20] investigated the solvability and optimal controls for fractional stochastic differential equations driven by Poisson jumps in Hilbert space via analytic resolvent operators and Banach contraction mapping principle.

Motivated by the aforementioned research works, in this manuscript we drive the sufficient conditions for the existence of solutions of the following class of optimal control for fractional stochastic differential system driven by fractional Brownian motion with Poisson jumps

$$\begin{aligned} {}^{C}D_t^{\alpha}x(t) &= Ax(t)+B(t)u(t)+f(t,x(t))+\sigma(t,x(t))\frac{dw^{\mathtt{H}}(t)}{dt} \\ &+ \int_{\mathcal{U}} h(t,x(t),u)\widetilde{N}(ds,du),\ t\in]0,\tau], \\ x(0) &= x_0\in \mathrm{X}, \end{aligned} \tag{1.1}$$

where the integral $l=[0,\tau]\times\mathrm{X}\times\mathrm{U}\to\mathbb{R}\cup\{\infty\}$ is specified latter; ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha<1$, the state $x(\cdot)$ is X-valued stochastic process; Suppose $\{w^{\mathtt{H}}(t)\}_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter $\mathtt{H}\in(\frac{1}{2},1)$ defined on $(\Omega,\Im,\{\Im_t\}_{t\geq 0},\mathbb{P})$ with values in Hilbert space Y. The control function $u(\cdot)$ takes its values from a separable reflexive Hilbert space U; $A:\mathscr{D}(A)\subseteq \mathrm{X}\to\mathrm{X}$ is the infinitesimal generator of a resolvent $S_{\alpha}(t)$, $t\geq 0$ on X; $\{B(t):t\geq 0\}$ is a family of linear operator from U to X; the functions $f:[0,\tau]\times\mathrm{X}\to\mathrm{X}$, $\sigma:[0,\tau]\times\mathrm{X}\to\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ and $h:[0,\tau]\times\mathrm{X}\times\mathcal{U}\to\mathrm{X}$ are nonlinear, where $\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ be the space of all Q-Hilbert-Schmidt operators from Y into X.

Let $(\Omega,\Im,\{\Im_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space equipped with a normal filtration $(\Im_t)$, $t\in[0,a]$ and Let X, Y be real separable Hilbert spaces and $\mathcal{L}(\mathrm{Y},\mathrm{X})$ denote the space of all bounded linear operator from Y into X. Let $Q\in\mathcal{L}(\mathrm{Y},\mathrm{Y})$ be an operator defined by $Qe_n-\lambda_n e_n$ with finite trace $tr(Q)=\sum_{n=1}^{\infty}\lambda_n<\infty$ where $\lambda_n\geq 0$ $(n=1,2,...)$ are non-negative real numbers and $\{e_n\}$ $(n=1,2,...)$ is a complete orthonormal basis in $\mathcal{Y}$.

We define the infiite dimensional fractional Brownian motion on Y with covariance Q as

$$w^{\mathtt{H}}(t) = w_Q^{\mathtt{H}}(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n^{\mathtt{H}}(t)$$

where $\beta_n^{\mathtt{H}}$ are real, independent fractional Brownian motions.

In order to define Wiener integrals with respect to the Q-fractional Brownian motion, we introduce the space $\mathcal{L}_2^0=\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ of all Q-Hilbert-Schmidt operators $\psi:\mathrm{Y}\to\mathrm{X}$. We recall that $\psi\in\mathcal{L}(\mathrm{Y},\mathrm{X})$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})} = \sum_{n=1}^{\infty}\left\|\sqrt{\lambda_n}\psi e_n\right\|^2<\infty$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $<\upsilon,\psi>_{\mathcal{L}_2^0}=\sum_{n=1}^{\infty}<\upsilon e_n,\psi e_n>$ is a separable Hilbert space. Let $\phi(s);s\in[0,a]$ be a function with values in $\mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$, the Wiener integral of ϕ with respect to $w^{\mathtt{H}}$ is defined by

$$\begin{aligned}\int_0^t \phi(s)dw^{\mathtt{H}}(s) &= \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda}\phi(s)e_n d\beta_n^{\mathtt{H}} \\ &= \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda}K^*(\phi e_n)(s)d\beta_n(s) \qquad (1.2)\end{aligned}$$

where β_n is the standard Brownian motion. Let $\mathscr{C}([0,\tau],\mathcal{L}_2(\Omega,\mathrm{X}))$ be the Banach space of continuous maps from $[0,\tau]$ into $\mathcal{L}_2(\Omega,\mathrm{X})$ satisfying $\sup_{0\le t\le\tau}\mathbf{E}\left\|x(t)\right\|^2 < \infty$. Let X_2 be the closed subspace of $\mathscr{C}([0,\tau],\mathcal{L}_2(\Omega,\mathrm{X}))$ consistingof measurable, $\Im_t$-adapted, X-valued processes $x\in\mathscr{C}([0,\tau],\mathcal{L}_2(\Omega,\mathrm{X}))$ equipped with the norm

$$\|x\|_{\mathrm{X}_2} = \left(\sup_{0\le t\le\tau}\mathbf{E}\left\|x(t)\right\|^2\right)^{1/2}.$$

Suppose that $\{q(t);t\in[0,\tau]\}$ is the Poisson point process, taking its value in a measurable space $(\mathcal{U},\mathcal{B}(\mathcal{U}))$ with a σ-finite intensity measure $v(du)$. The compensating martingle measure and Poisson counting measure are defined by

$$\widetilde{N}(ds,du) = N(ds,du)-v(du)ds.$$

Let us assume that the filteration $\Im_t = \sigma\left\{N((0,s],\Lambda): s\le t, \Lambda\in\mathcal{B}(\mathcal{U})\right\}\vee N$, $t\in[0,\tau]$, produced by $q(\cdot)$ Poisson point process and is augmented, where N is the class of $\mathbb{P}$-null sets. Let $p_2([0,\tau]\times\mathcal{U};\mathrm{X})$ be the space of all predictable mappings $h:[0,\tau]\times\mathcal{U}\to\mathrm{X}$ for

$$\int_0^\tau\int_{\mathcal{U}}\mathbf{E}\left\|h(t,u)\right\|_{\mathrm{X}}^2 dtv(du) < \infty.$$

Consider the following integral cost functional

$$\jmath\{x,u\} = \mathbf{E}\left\{\int_0^\tau l(t,x^u(t),u(t))dt\right\}, \qquad (1.3)$$

Define the admissible set U_{ad}, the set of all $v(\cdot):[0,\tau]\times\Omega\to\mathrm{U}$ such that v is $\Im_t$-adapted stochastic process and $\mathbf{E}\int_0^\tau\left\|v(t)\right\|^p dt < \infty$. Clearly $\mathrm{U}_{ad}\neq\emptyset$ and $\mathrm{U}_{ad}\subset\mathcal{L}^p([0,\tau];\mathrm{U})$ $(1<p<+\infty)$ is bounded, closed and convex.

Denoted by the set of all admissible state-control pairs (x,u) by $\mathcal{A}_{ad}$, where x is the mild solution of the system (1.1) corresponding to the control $u\in\mathrm{U}_{ad}$. The main objective of this paper is to find a pair $(x^0,u^0)\in\mathcal{A}_{ad}$ such that

$$\jmath(x^0,u^0) := \inf\left\{\jmath(x,u):(x,u)\in\mathcal{A}_{ad}\right\} = \epsilon.$$

To the best of authors knowledge, up to now, no work has been reported to derive the optimal control for fractional stochastic differential system driven by fractional Brownian motion with Poisson jumps. The main contributions are summarized as follows:

(1) Fractional stochastic differential system driven by fractional Brownian motion with Poisson jumps is formulated.
(2) Fractional calculus theory is effectively used to derive the existence and uniqueness of mild solution, a set of sufficient conditions is constructed by using fixed point theorem.
(3) The existence of fractional optimal control for stochastic system is also discussed.

(4) An example is provided to illustrate the obtained theoretical results.

2. Preliminaries

In this section, we collect basic concepts, definitions and Lemmas which will be used in the sequel to obtain the main results.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f : [0, \infty) \to \mathbb{R}$ with the lower limit 0 is defined as*

$$I^{\alpha} f(t) \quad = \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the Euler gamma function.

Definition 2.2. *The Caputo fractional derivative of order $\alpha > 0$ for the function $f \in \mathscr{C}^m([0, \tau], \mathbb{R})$ is defined by*

$$^{\mathcal{C}}D_t^{\alpha} f(t) \quad = \quad \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\infty-1} f^{(m)}(s) ds, \;\; m-1 < \infty < m \in \mathbb{N}.$$

If f is an abstract function with values in X, then the integrals appearing in Definition 2.1 and Definition 2.2 are taken in the Bochner sense. Moreover, the Caputo derivative of a constant is always zero.

The two-parameter function of the Mittag-Leffler type is defined by the series expension

$$\mathrm{E}_{\alpha,\beta}(z) \quad = \quad \sum_{j=1}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\lambda} \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - z} d\lambda; \;\; \alpha, \beta > 0, \;\; z \in \mathbb{C},$$

where $\mathcal{C}$ is a contour that start and end at $-\infty$ and encircles the disc $\|\lambda\| \leq |z|^{1/2}$ counterclockwise.

Definition 2.3. [22] *A closed and linear operator A is said to be sectorial type μ if there exist $\pi/2 \leq \theta \leq \pi$, $\widetilde{M} > 0$ and $\mu \in \mathbb{R}$ such that the following conditions are satisfied: $\rho(A) \subset \sum_{(\theta,\mu)} = \{\lambda \neq \mu, |\arg(\lambda - \mu)| < \theta\}$, and $\|R(\lambda, A)\| = \left\|(\lambda - A)^{-1}\right\| \leq \frac{\widetilde{M}}{|\lambda-\mu|}$, $\lambda \in \sum_{(\theta,\mu)}$.*

Lemma 2.1. [22] *For $0 < \alpha < 2$, a linear closed densely defined operator A belongs to $A^{\alpha}(\theta_0, \mu_0)$ iff $\lambda^{\alpha} \in \rho(A)$ for each $\lambda \in \sum_{(\theta_0+\pi/2),\mu}$ and for any $\mu > \mu_0$, $\theta < \theta_0$ there is a constant $\mathcal{C}_0 = \mathcal{C}_0(\theta, \mu)$ such that*

$$\left\|\lambda^{\alpha+1} R(\lambda^{\alpha}, A)\right\| \quad \leq \quad \frac{\mathcal{C}_0}{|\lambda - \mu|}, \text{ for } \lambda \in \sum_{(\theta_0+\pi/2),\mu}.$$

Lemma 2.2. [22] *If f satisfies the uniform Holder condition with the exponent $0 < \gamma \leq 1$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$\begin{aligned} ^{\mathcal{C}}D_t^{\alpha} x(t) &= Ax(t) + f(t), \;\; 0 < \infty < 1, \;\; t \in (0, \tau], \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

is given by

$$x(t) \quad = \quad S_{\alpha}(t) x_0 + \int_0^t T_{\alpha}(t-s) F(s) ds,$$

where

$$S_\alpha(t) = \mathrm{E}_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\hat{\mathcal{B}}_\rho} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda,$$
$$T_\alpha(t) = t^{\alpha-1}\mathrm{E}_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\hat{\mathcal{B}}_\rho} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda,$$

$\hat{\mathcal{B}}_\rho$ is the Bromwich path, $T_\alpha(t)$ is called the α-resolvent family, and $S_\alpha(t)$ is the solution operator generated by A.

An operator A is said to belong to $\mathcal{D}^\alpha(\widetilde{M},\mu)$ if problem (4) with $f=0$ has a solution operator $S_\alpha(t)$ satisfying $\|S_\alpha(t)\| \le \widetilde{M}e^{\mu t}$. Denote $\mathcal{D}^\alpha(\mu) = \cup\{\mathcal{D}^\alpha(\widetilde{M},\mu) : \widetilde{M} \ge 1\}$, $\mathcal{D}^\alpha = \{\mathcal{D}^\alpha(\mu : \mu \ge 0)\}$, and $\mathcal{A}^\alpha(\theta_0,\mu_0) = \{A \in \mathcal{D}^\alpha : A$ generates analytic solution operators $S_\alpha(t)$ of type $(\theta_0,\mu_0)\}$.

If $0<\alpha<1$ and $A \in \mathcal{A}^\alpha(\theta_0,\mu_0)$, then we have $\|S_\alpha(t)\| \le \widetilde{M}e^{\mu t}$ and $\|T_\alpha(t)\| \le \mathcal{C}e^{\mu t}(1+t^{\alpha-1})$, $t>0$, $\mu>\mu_0$. If

$$M_S = \sup_{0\le t\le\tau}\|S_\alpha(t)\|,\ M_T = \sup_{0\le t\le\tau}\mathcal{C}e^{\mu t}(1+t^{1+\alpha}),$$

then, we have

$$\|S_\alpha(t)\| \le M_S,\ \|T_\alpha(t)\| \le t^{\alpha-1}M_T.$$

Lemma 2.3. [13] *If $\psi : [0,a] \to \mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ satisfies $\int_0^a \|\psi(s)\|^2_{\mathcal{L}_2^0} < \infty$ then the above sum in (1.2) is well defined as* X*-valued random variable and we have*

$$\mathbf{E}\left\|\int_0^t \psi(s)dw^{\mathtt{H}}(s)\right\|^2 \le 2\mathtt{H}t^{2\mathtt{H}-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0}ds.$$

By Lemma 2.2. a mild solution of the system (1.1) is defined as

Definition 2.4. *An $\Im_t$-adapted stochastic process $x(t) \in \mathscr{C}([0,\tau],\mathcal{L}^2(\Omega,\Im,\mathrm{X}))$ is called a mild solution of system (1.1) if for each $u(\cdot) \in \mathcal{L}^p([0,\tau];\mathrm{U})$, $x(t)$ is measurable and the following stochastic integral equation*

$$x(t) = S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)[B(s)u(s) + f(s,x(s))]ds$$
$$+ \int_0^t T_\alpha(t-s)\sigma(s,x(s))dw^{\mathtt{H}}(s)$$
$$+ \int_0^t\int_{\mathcal{U}} T_\alpha(t-s)h(s,x(s),u)\widetilde{N}(ds,du). \tag{2.2}$$

3. Existence and Uniqueness

To prove the existence and uniqueness of mild solution of the system (1.1), we impose the following hypotheses:

(H1) The functions $f : [0,\tau]\times\mathrm{X} \to \mathrm{X}$, $\sigma : [0,\tau]\times\mathrm{X} \to \mathcal{L}_2^0(\mathrm{Y},\mathrm{X})$ and $h : [0,\tau]\times\mathrm{X}\times\mathcal{U} \to \mathrm{X}$ are continuous, and satisfying linear growth and Lipschitz conditions: there are positive constants L_f, L_σ and L_h such that

$$\begin{aligned}\|f(t,x)-f(t,y)\|^2 \le L_f \|x-y\|^2,\ \|f(t,x)\|^2 \le L_f(1+\|x\|^2),\\ \|\sigma(t,x)-\sigma(t,y)\|^2 \le L_\sigma \|x-y\|^2,\ \|\sigma(t,x)\|^2 \le L_\sigma(1+\|x\|^2),\\ \int_{\mathcal{U}} \|h(t,x,u)-h(t,y,u)\|^2 v(du) \le L_h \|x-y\|^2,\\ \int_{\mathcal{U}} \|h(t,x,u)\|^2 v(du) \le L_h(1+\|x\|^2).\end{aligned}$$

(H2) The operator $B \in \mathcal{L}_\infty([0,\tau];\mathcal{L}(\mathrm{U},\mathrm{X}))$ and $\|B\|_\infty$ stand for the norm of operator B in the Banach space $\mathcal{L}_\infty([0,\tau];\mathcal{L}(\mathrm{U},\mathrm{X}))$.

(H3) The multi-valued map $\Xi(\cdot) : [0,\tau] \to 2^u/\{\emptyset\}$ has closed, convex and bounded values, $\Xi(\cdot)$ is graph measurable and $\Xi(\cdot) \subseteq \Phi$, where Φ is a bounded subset of U.

Theorem 3.1. *Assumptions* $(\mathbf{H1})-(\mathbf{H3})$ *the system (2.2) admits a unique mild solution on* $[0,\tau]$ *for each control function* $u(\cdot) \in \mathrm{U}_{ad}$ *and for each some* p *such that* $p\alpha > 1$.

Proof. Define an operator $\mathcal{G} : \mathrm{X}_2 \to \mathrm{X}_2$ as

$$\begin{aligned}(\mathcal{G}x)(t) &= S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)[B(s)u(s)+f(s,x(s))]ds\\ &+ \int_0^t T_\alpha(t-s)\sigma(s,x(s))dw^{\mathtt{H}}(s)\\ &+ \int_0^t\int_{\mathcal{U}} T_\alpha(t-s)h(s,x(s),u)\widetilde{N}(ds,du).\end{aligned}$$

To show that (2.2) is the mild solution of the system (1.1) on $[0,\tau]$, it is enough to prove that $\mathcal{G}$ has a fixed point in the space X_2. We first show that $\mathcal{G}(\mathrm{X}_2) \subset \mathrm{X}_2$. Let $x \in \mathrm{X}_2$, then we have

$$\mathbf{E}\,\|(\mathcal{G}x)(t)\|^2 \le 5\,[\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5] \tag{3.1}$$

Clearly

$$\begin{aligned}\Gamma_1 &= \|S_\alpha(t)x_0\|^2\\ &\le M_s^2\mathbf{E}\,\|x_0\|^2.\end{aligned}$$

Next, using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\Gamma_2 &= \|T_\alpha(t-s)B(s)u(s)ds\|^2\\ &\le M_T^2\,\|B\|_\infty^2 \left[\int_0^t (t-s)^{\alpha-1}\,\|u(s)\|\,ds\right]^2\\ &\le M_T^2\,\|B\|_\infty^2 \left[\left(\int_0^t (t-s)^{\frac{p(\alpha-1)}{p-1}} ds\right)^{\frac{p-1}{p}} \left(\int_0^t \|u(s)\|_{\mathrm{U}}^p\,ds\right)^{\frac{1}{p}}\right]^2\\ &\le M_T^2\,\|B\|_\infty^2\,\|u\|_{\mathcal{L}([0,\tau];\mathrm{U})}^2\,\tau^{2(\frac{p\alpha-1}{p})}(\frac{p-1}{p\alpha-1})^{\frac{2(p-1)}{p}}.\end{aligned}$$

Next, by (**H1**) and Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
\Gamma_3 &= \mathbf{E}\left\|\int_0^t T_\alpha(t-s)f(s,x(s))ds\right\|^2 \\
&\leq M_T^2\left(\int_0^t (t-s)^{\alpha-1}ds\right)\left(\int_0^t (t-s)^{\alpha-1}\mathbf{E}\left\|f(s,x(s))\right\|^2 ds\right) \\
&\leq M_T^2 L_f \frac{\tau^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}(1+\mathbf{E}\left\|x(s)\right\|^2)ds \\
&\leq M_T^2 L_f \frac{\tau^\alpha}{\alpha^2}(1+\|x\|_{\mathrm{X}_2}^2).
\end{aligned}
$$

By (**H1**) and Lemma 2.3, we have

$$
\begin{aligned}
\Gamma_4 &= \mathbf{E}\left\|\int_0^t T_\alpha(t-s)\sigma(s,x(s))dw^{\mathtt{H}}(s)\right\|^2 \\
&\leq M_T^2\left[\int_0^t (t-s)^{\alpha-1}2\mathtt{H}t^{2\mathtt{H}-1}\left(\mathbf{E}\left\|\sigma(s,x(s))\right\|_{\mathcal{L}_2^0}^2\right)ds\right] \\
&\leq M_T^2\left[2\mathtt{H}t^{2\mathtt{H}-1}\int_0^t (t-s)^{\alpha-1}\mathbf{E}\left\|\sigma(s,x(s))\right\|^2 ds\right] \\
&\leq M_T^2 L_\sigma 2\mathtt{H}t^{2\mathtt{H}-1}\frac{\tau^{2\alpha}}{\alpha^2}(1+\|x\|_{\mathrm{X}_2}^2).
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_5 &= \mathbf{E}\left\|\int_0^t T_\alpha\int_{\mathcal{U}}(t-s)h(s,x(s),u)\widetilde{N}(ds,du)\right\|^2 \\
&\leq M_T^2\left(\int_0^t (t-s)^{\alpha-1}ds\right)\left(\int_0^t\int_{\mathcal{U}}(t-s)^{\alpha-1}\mathbf{E}\left\|h(s,x(s),u)\right\|^2 v(du)ds\right) \\
&\leq M_T^2 L_h\frac{\tau^\alpha}{\alpha^2}(1+\|x\|_{\mathrm{X}_2}^2).
\end{aligned}
$$

Thus (3.1) becomes

$$
\mathbf{E}\left\|(\mathcal{G}x)(t)\right\|^2 \quad\leq\quad a+b\left\|x\right\|_{\mathrm{X}_2}^2,
$$

where a and b are suitable positive constants. Thus $\mathcal{G}$ maps X_2 into itself.

Next, we prove that $\mathcal{G}$ is a contraction. For $x,y\in\mathrm{X}_2$, the Cauchy-Schwartz inequality, and (**H1**) yield that

$$
\begin{aligned}
&\mathbf{E}\left\|(\mathcal{G}x)(t)-(\mathcal{G}y)(t)\right\|^2 \\
&\leq 3\mathbf{E}\left\|\int_0^t T_\alpha(t-s)[f(s,x(s))-f(s,y(s))]ds\right\|^2 \\
&\quad+3\mathbf{E}\left\|\int_0^t T_\alpha(t-s)[\sigma(s,x(s))-\sigma(s,y(s))]dw^{\mathtt{H}}(s)\right\|^2 \\
&\quad+3\mathbf{E}\left\|\int_0^t T_\alpha\int_{\mathcal{U}}(t-s)[h(s,x(s),u)-h(s,y(s),u)]\widetilde{N}(ds,du)\right\|^2 \\
&\leq 3M_T^2\left(L_f+L_\sigma 2\mathtt{H}t^{2\mathtt{H}-1}+L_h\right)\frac{\tau^{2\alpha}}{\alpha^2}\left\|x-y\right\|_{\mathrm{X}_2}^2.
\end{aligned}
$$

Consequently if

$$3M_T^2\left[L_f+L_\sigma 2\mathtt{H}t^{2\mathtt{H}-1}+L_h\right]\frac{\tau^{2\alpha}}{\alpha^2}<1, \tag{3.2}$$

then the operator $\mathcal{G}$ has a unique fixed point in X_2, which is a solution of the system (1.1). The extra condition on τ can be easily removed by considering the equation on intervals $[0,\tilde{\tau}]$, $[0,2\tilde{\tau}]$, ... with $\tilde{\tau}$ satisfying (3.2). □

We now obtain a priori estimate of mild solution for the system (1.1), that helps us to obtain our main results.

Lemma 3.2. *Assuming that system (2.2) is the mild solution of system (1.1) on $[0,\tau]$ corresponding to the control u. Then there exists a constant $M>0$ such that*

$$\mathbf{E}\left\|x(t)\right\|^2 \le M,\ t\in[0,\tau].$$

Proof. By (**H1**) and Holder's inequality, we obtain

$$\begin{aligned}
\mathbf{E}\left\|x(t)\right\|^2 &\le 5\mathbf{E}\left\|S_\alpha(t)x_0\right\|^2\\
&+ 5\mathbf{E}\left\|\int_0^t T_\alpha(t-s)B(s)u(s)ds\right\|^2+5\mathbf{E}\left\|\int_0^t T_\alpha(t-s)f(s,x(s))ds\right\|^2\\
&+ 5\mathbf{E}\left\|\int_0^t T_\alpha(t-s)\sigma(s,x(s))dw^{\mathtt{H}}(s)ds\right\|^2\\
&+ 5\mathbf{E}\left\|\int_0^t\int_{\mathcal{U}} T_\alpha(t-s)\sigma(s,x(s),u)\widetilde{N}(ds,du)\right\|^2\\
&\le 5M_S^2+5M_T^2\left\|B\right\|_\infty^2\left\|u\right\|^2_{\mathcal{L}^p([0,\tau];\mathrm{U})}\tau^{2(\frac{p\alpha-1}{p})}(\frac{p-1}{p\alpha-1})^{\frac{2(p-1)}{p}}\\
&+ 5M_T^2\left(L_f+L_\sigma 2\mathtt{H}t^{2\mathtt{H}-1}+L_h\right)\frac{\tau^{2\alpha}}{\alpha^2}\\
&+ 5M_T^2\left(L_f+L_\sigma 2\mathtt{H}t^{2\mathtt{H}-1}+L_h\right)\frac{\tau^{2\alpha}}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbf{E}\left\|x(s)\right\|^2ds.
\end{aligned}$$

Now using the Gronwall inequality, one can easily obtain the boundedness of x in X_2. □

4. Existence of Fractional Optimal Control

In this section, we prove the existence of fractional optimal control under the hypothesis:

(**H4**) Following conditions are imposed on the integrand

$$l:[0,\tau]\times\mathrm{X}\times\mathrm{U}\to\mathbb{R}\cup\{\infty\}$$

such that
(1) The integrand $l:[0,\tau]\times\mathrm{X}\times\mathrm{U}\to\mathbb{R}\cup\{\infty\}$ is $\Im_t$-measurabl.
(2) The integrand $l(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $\mathrm{X}\times\mathrm{U}$ for almost all $t\in[0,\tau]$.
(3) The integrand $l(t,x,\cdot)$ is convex on U for each $x\in\mathrm{X}$ and almost all $t\in[0,\tau]$.
(4) There exist constants $d\ge 0$, $e>0$, μ_0 is nonnegative and $\mu_0\in\mathcal{L}^1([0,\tau];\mathbb{R})$ such that

$$\mu_0(t)+d\mathbf{E}\left\|x\right\|^2+e\mathbf{E}\left\|u\right\|_{\mathrm{U}}^p \le l(t,x,u).$$

Theorem 4.1. *Suppose* $(\mathbf{H1})-(\mathbf{H4})$ *hold, then Lagrange problem (1.3) admits at least one optimal pair, that is, there exists an admissible state-control pair* $(x^0, u^0) \in \mathcal{A}_{ad}$ *such that*

$$\jmath(x^0, u^0) \quad := \quad \mathbf{E}\left\{\int_0^\tau l(t, x^0(t), u^0(t))dt\right\} \leq \jmath(x, u), \ \forall (x, u) \in \mathcal{A}_{ad}.$$

Proof. If $\inf\left\{l(x,u)|(x,u) \in \mathcal{A}_{ad}\right\} = +\infty$, then there is nothing to prove. Without any loss of generality, we may assume that $\inf\left\{l(x,u)|(x,u) \in \mathcal{A}_{ad}\right\} = +\infty$. Now assumption $(\mathbf{H4})$ implies that $\epsilon > -\infty$. By definition of infimum, there is a minimizing sequence of feasible pairs $(x^m, u^m) \in \mathcal{A}_{ad}$, such that $l(x^m, u^m) \to \epsilon$ as $m \to +\infty$. Since $\{u^m\} \subseteq \mathrm{U}_{ad}$, $m = 1, 2, \cdots, \{u^m\}$ is a bounded subset of the separable reflexive Banach space $\mathcal{L}^p([0,\tau]; \mathrm{U})$, there exists a subsequence, relabeled as $\{u^m\}$ and $\mathcal{L}^p([0,\tau]; \mathrm{U})$ such that $u^m \xrightarrow{w} u^0$ $(u^m \to u^0)$ weakly as $m \to +\infty$ in $\mathcal{L}^p([0,\tau]; \mathrm{U})$. Since U_{ad} is closed and convex, the Mazur lemma forces us to conclude that $u^0 \in \mathrm{U}_{ad}$.

Let $\{x^m\}$ be the sequence of solution of the system (1.1) corresponding to $\{u^m\}$, that is

$$\begin{aligned} x^m(t) = S_\alpha(t)x_0 &+ \int_0^t T_\alpha(t-s)[B(s)u^m(s) + f(s, x^m(s))]ds \\ &+ \int_0^t T_\alpha(t-s)\sigma(s, x^m(s))dw^{\mathtt{H}}(s) \\ &+ \int_0^t \int_{\mathcal{U}} T_\alpha(t-s)h(s, x^m(s), u)\widetilde{N}(ds, du). \end{aligned} \tag{4.1}$$

By Lemma 3.1, it is easy to see that there exists $\delta > 0$ such that

$$\mathbf{E}\left\|x^m\right\|^2 \quad \leq \quad \delta, \ m = 0, 1, 2, \cdots,$$

where x^0 is the mild solution of the system (1.1) corresponding to the control $u^0 \in \mathrm{U}_{ad}$ given by

$$\begin{aligned} x^0(t) = S_\alpha(t)x_0 &+ \int_0^t T_\alpha(t-s)[B(s)u^0(s) + f(s, x^0(s))]ds \\ &+ \int_0^t T_\alpha(t-s)\sigma(s, x^0(s))dw^{\mathtt{H}}(s) \\ &+ \int_0^t \int_{\mathcal{U}} T_\alpha(t-s)h(s, x^0(s), u)\widetilde{N}(ds, du). \end{aligned}$$

For all $t \in [0, \tau]$, using $(\mathbf{H4})$, the Cauchy-Schwartz inequality and the Holder inequality, we obtain

$$\begin{aligned} &\mathbf{E}\left\|x^m(t) - x^0(t)\right\|^2 \\ &\leq 4\mathbf{E}\left\|\int_0^t T_\alpha(t-s)[B(s)u^m(s) - B(s)u^0(s)]ds\right\|^2 \\ &\leq 4\mathbf{E}\left\|\int_0^t T_\alpha(t-s)[f(s, x^m(s)) - f(s, x^0(s))]ds\right\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq 4\mathbf{E}\left\|\int_0^t T_\alpha(t-s)[\sigma(s,x^m(s))-\sigma(s,x^0(s))]dw^{\mathtt{H}}(s)\right\|^2\\
&\leq 4\mathbf{E}\left\|\int_0^t\int_{\mathcal{U}} T_\alpha(t-s)[h(s,x^m(s),u)-h(s,x^0(s),u)]\widetilde{N}(ds,du)\right\|^2\\
&\leq 4M_T^2(\frac{p-1}{p\alpha-1})^{\frac{2p-2}{p}}\tau^{2\alpha-\frac{2}{p}}\left(\int_0^t\left\|B(s)u^m(s)-B(s)u^0(s)\right\|^p ds\right)^{\frac{2}{p}}\\
&+ 4M_T^2\frac{\tau^\alpha}{\alpha}(L_f+L_\sigma 2\mathtt{H}t^{2\mathtt{H}-1}+L_h)\int_0^t(t-s)^{\alpha-1}\mathbf{E}\left\|x^m(s)-x^0(s)\right\|^2 ds.
\end{aligned}$$

By applying Gronwall inequality, there exists a constant $K^*(\alpha)$ independent of u, m and t such that

$$\begin{aligned}
\mathbf{E}\left\|x^m(t)-x^0(t)\right\|^2 &\leq K^*(\alpha)\left(\int_0^\tau\left\|B(s)u^m(s)-B(s)u^0(s)\right\|^p ds\right)^{\frac{2}{p}}\\
&\leq K^*(\alpha)\left\|Bu^m-Bu^0\right\|^2_{\mathcal{L}^p([0,\tau];\mathrm{U})}. \qquad (4.2)
\end{aligned}$$

Since B is strongly continuous, we get

$$\left\|Bu^m-Bu^0\right\|^2_{\mathcal{L}^p([0,\tau];\mathrm{U})}\xrightarrow{s}0 \text{ as } m\to\infty. \qquad (4.3)$$

From (4.2) and (4.3), we conclude that

$$\mathbf{E}\left\|x^m(t)-x^0(t)\right\|^2\to 0 \text{ as } m\to\infty. \qquad (4.4)$$

This implies that $\mathbf{E}\left\|x^m-x^0\right\|^2\to 0$ in $\mathscr{C}([0,\tau];\mathcal{L}^2(\Omega,\mathrm{X}))$ as $m\to\infty$.

By (**H4**) implies the assumptions of Balder (see Theorem 2.1, [23]). Hence, by Balder's theorem, we get

$$(x,u)\to\mathbf{E}\int_0^\tau L(t,x(t),u(t))dt$$

is sequentially lower semicontinuous in the strong topology of $\mathcal{L}^1([0,\tau];\mathrm{X})$ and week topology of $\mathcal{L}^p([0,\tau];\mathrm{U})\subset\mathcal{L}^1([0,\tau];\mathrm{X})$. Hence, $\jmath$ is weakly lower semicontinuous on $\mathcal{L}^p([0,\tau];\mathrm{U})$, and since by (**H4**)(4), $\jmath>-\infty$, $\jmath$ attains its infimum at $u^0\in\mathrm{U}_{ad}$, that is,

$$\begin{aligned}
\epsilon &:= \lim_{m\to\infty}\mathbf{E}\int_0^\tau l(t,x^m(t),u^m(t))dt\\
&\geq \mathbf{E}\int_0^\tau l(t,x^0(t),u^0(t))dt\\
&= \jmath(x^0,u^0)\geq\epsilon.
\end{aligned}$$

Hence completes the proof. □

5. Application

Consider the following fractional stochastic integrodifferential system driven by Rosenblatt process with Poisson jumps

$$\begin{aligned}
{}^{C}D_t^{\frac{2}{3}}x(t,z) &= \Delta x(t,z) + \int_0^t \tilde{B}(z,s)u(s,t)ds + \int_0^t \tilde{f}(s,z)sin(x,s)ds \\
&+ \int_0^t \frac{(x(t,z))^2}{1+(x(t,z))^2}\frac{dw^{\mathtt{H}}(t)}{dt} + \int_{\mathcal{U}}(1+e^{-t})cosy(t,x,u)\widetilde{N}(dt,du), \\
x(0,z) &= x_0(z),\ z\in\Omega_1, \\
x(t,z)|_{z\in\partial\Omega} &= 0,\ t>0, \qquad (5.1)
\end{aligned}$$

Here Let $w^{\mathtt{H}}$ is a fractional Brownian motion with Hurst parameter $\mathtt{H}\in(\frac{1}{2},1)$. Let $\Omega_1\subset\mathbb{R}^3$ be abounded domain and $\partial\Omega_1\in\mathbb{C}^3$. Further let $\mathrm{X}=\mathrm{U}=\mathcal{L}^2(\Omega_1)$, $w(t)$ is a standard cylindrical Wiener process in X defined on a stochastic space $(\Omega,\Im,\mathbb{P})$. Suppose $\mathscr{D}(A)=\mathrm{X}^2(\Omega_1)\bigcap\mathrm{X}_0^1(\Omega_1)$ and for $z\in\mathscr{D}(A)$, $Az=\left(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\frac{\partial^2}{\partial z_3^2}\right)z$. The admissible control set $\mathrm{U}_{ad}:=\left\{u\in\mathrm{U}:\|u\|_{\mathcal{L}^p([0,1];\mathrm{U})}\le 1\right\}$. Define the fractional Brownian motion in $\mathbb{Y}$ by $w^{\mathtt{H}}(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n\beta^{\mathtt{H}}(t)}e_n$, where $\mathtt{H}\in(\frac{1}{2},1)$ and $\{\beta_n^{\mathtt{H}}\}_{n\in\mathbb{N}}$ is a sequence of one-dimensional fractional Brownian motions mutually independent.

The functions $f:[0,\tau]\times\mathrm{X}\to\mathrm{X}$, $\sigma:[0,\tau]\times\mathrm{X}\to\mathcal{L}_2^0(Q^{1/2}\mathrm{Y},\mathrm{X})$ and $h:[0,\tau]\times\mathrm{X}\times\mathcal{U}\to\mathrm{X}$ are defined by

$$\begin{aligned}
x(t)(z) &= x(t,z),\ x(0)(z)=x(0,z)=x_0(z), \\
(Bu)(t)(z) &= \int_0^t \tilde{B}(z,s)u(s,t)ds, \\
f(t,x(t))(z) &= f(t,x(t,z)) = \int_0^t \tilde{f}(s,z)sin(x,s)ds, \\
\sigma(t,x(t))(z) &= \sigma(t,x(t,z)) = \frac{(x(t,z))^2}{1+(x(t,z))^2}, \\
\int_{\mathcal{U}} h(t,x,u)\widetilde{N}(ds,du) &= \int_{\mathcal{U}}(1+e^{-t})cosy(t,x,u)\widetilde{N}(dt,du),
\end{aligned}$$

Thus, for $\alpha=\frac{2}{3}$ the problem (5.1) can be written as the abstract from of system (1.1) with the cost function

$$\jmath(x,u) = \mathbf{E}\left\{\int_0^1 l(t,x(t),u(t))dt\right\},$$

where $l(t,x(t),u(t))(z)=\int_{\Omega_1}|x(t,z)|^2dz+\int_{\Omega_1}|u(t,z)|^2dz$. It is easy to see that the assumptions $(\mathbf{H1})-(\mathbf{H4})$ are satisfied, there exists an optimal pair $(x^0,u^0)\in\mathcal{L}^0([0,1]\Omega_1\times\mathcal{L}^2[0,1]\times\Omega_1)$ such that $\jmath(x^0,u^0)\le\jmath(x,u)$ for all $(x,u)\in\mathcal{L}^2([0,1]\times\Omega_1\times\mathcal{L}^2([0,1]\times\Omega_1)$.

6. Conclusion

In this paper, we studied the existence of solutions and optimal control results of fractional stochastic differential system driven by fractional Brownian motion with Poisson jumps in Hilbert space. The sufficient conditions for the existence of mild solution results are formulated and proved by virtue of fractional calculus, solution operator and stochastic analysis techniques. Furthermore, the existence of optimal control of the proposed problem is presented by using Balder's theorem. The optimal control analysis for fractional stochastic differential inclusions with distributed delays, time varying delays, and impulsive effects will be our future work.

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STABILITY ANALYSIS OF TWO PREDATORS-ONE PREY MODEL WITH FEEDBACK CONTROL AND TIME FRACTIONAL DERIVATIVE

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Abstract. The interaction between prey and predator is one of the most fundamental processes in ecology. In this paper, we first consider the system incorporating a feedback control and we discuss the dynamic behavior of prey-predator interaction model that includes two competitive predators and one prey with a generalized interaction functional. The primary presumption in the model construction is the effects of feedback control and the competition between two predators on the only prey which gives a strong implication of the real-world situation. By analyzing characteristic equations, we carry out detailed discussion with respect to stability of equilibrium points of the considered model. Further, we investigate the impact of the memory measured by fractional time derivative on the temporal behavior.

1. Introduction

Mathematical modeling of the real-world phenomenon is a potent tool for predicting some ecological and biological components. The validity of this mathematical approximation depends on the model itself. The crucial component that describes the interaction between different species in a certain environment is the interaction functional. There are many types of these functionals in the literature [9,10,14,15,19]. Each one describes a specific manner of intermingling between two species. The reason for this great diversity in functionals is due to the variety of environmental conditions in the problem. Some of the factors that influence the selection of these parameters are the behavior of the prey and predator and the studied area. For the last factor, many components play a crucial role such as

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rivers (water availability), food (for the prey) and the density of prey and predator. Overall, the functional selection depends on many factors.

In the environment, the intermingling is not limited to just two populations, but interactions can be defined between more than two species in one single place. The scientists interested in this point of view have put efforts to model such complex interactions in the last few decades. We can take as an example two types of prey and one predator [5], where the predator has the capability of hunting both prey populations. Moreover, in prey–predator–super predator models the predator feeds the prey only, and the super predator feeds both prey and predator. In some models, it is studied the interaction between two predators and one prey model where two types of predators are fed the same prey. Due to the intrinsic nature of the predators, there will always be a constant struggle to capture this one prey. The predator-prey models with three species have been attracted many researchers. In [8], it is highlighted and studied the intermingling and competition between two competitive predators on one prey with a generalized class of interaction functionals in the presence of the time-fractional derivative. Fractional ordinal systems are not just an extension of traditional integer ordinal systems in mathematics but also have some merits that integer-order systems do not have, such as memory and hereditary properties [11,21]. As known, many biological systems have memory [18]. Fractional order systems compared to integer order systems can more accurately describe population patterns and reveal the relationships between prey species and predatory species [1,4].

In real situations, it is seen that one predator determines its own hunting territory. The presence of other predators in such territories is entirely unacceptable. This situation is called competition. The models in which competition is found, have also received much attention in many research papers such as.

When examining the local asymptotic stability of the equilibrium points of dynamic systems, note that the equilibrium value of the considered system is sometimes not as we would like, and maybe in some cases what we need is a smaller value. In this case, we may change the system structurally by introducing a feedback control variable [2,12], which can be implemented by employing biological control strategy. In [13], the dynamic behavior of fractional-order predator–prey model incorporating a constant prey refuge and feedback control has investigated.

In this paper, we are interested in studying the intermingling and competition between two competitive predators on one prey with a generalized class of interaction functionals in the presence of the time-fractional derivative. By summarizing all the previously mentioned components let us focus on the following incorporating feedback control time-fractional formulation with a generalized consumption functional:

$$\begin{cases} {}_0^cD_t^q x(t) = x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, & x(0) = x_0 \\ {}_0^cD_t^q y(t) = e_1 f(x)y - \mu_1 y - \beta yz, & y(0) = y_0 \\ {}_0^cD_t^q z(t) = e_2 g(x)z - \mu_2 z - \gamma yz, & z(0) = z_0 \\ {}_0^cD_t^q u(t) = -hu + mx, & u(0) = u_0 \end{cases} \tag{1.1}$$

where $0 < q < 1$, ${}_0^cD_t^q$ is the Caputo $q-$order fractional derivative. The conditions on the functionals f and g are defined as

(A_1) $f(0) = 0,\ g(0) = 0,$
(A_2) $f'(x) > 0,\, g'(x) > 0 \ for\ x > 0.$

In the system (1.1), $x(t), y(t)$ and $z(t)$ are the densities of prey, first predator and second predator populations at time t, respectively; $u(t)$ denotes the feedback control variable for prey population at time t. We assume that the prey population reproduces logistically with the increasing rate r, a is the intraspecific competition coefficient of prey population and the carrying capacity k of the space, e_1 and e_2 are respectively the conversion rate of the prey biomass into the first predator population and the diversion of the prey biomass into the second predator biomass, μ_1 and μ_2 are the mortality rates of the first and second predators, respectively, $\beta(resp., \gamma)$ is the competition rate of the first predator with the second one (resp., of the second predator with the first one). The functionals f and g are respectively the interaction functionals for the first and second predator populations with the prey population. Here all the parameters are assumed to be positive.

The rest of this paper is organized as follows. In section 2, we introduce some notations, definitions and lemmas. In section 3, we give the equilibrium points of fractional-order predator–prey model (1.1), and we discuss their stability. The concluding section of the paper is intended to highlight the biological meanings of the acquired numerical results.

2. Preliminaries

We introduce some useful definitions and lemmas in this section which are necessary for our latter study.

Definition 2.1. *[11] The $q-$order fractional integral for a function ζ is defined as*

$$ {}_0I_t^q\zeta(s) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\zeta(s)ds, \qquad q > 0$$

where $\Gamma(.)$ is the well-known Gamma function which is defined by $\Gamma(q) = \int_0^\infty e^{-t}t^{z-1}dt$.

Definition 2.2. *[11] The Caputo $q-$order fractional derivative for a function ζ is defined as*

$$ {}_0^cD_t^q\zeta(s) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}\zeta^n(s)ds,$$

where n is a positive integer, $n-1 < q < n$. Particularly, when $0 < q < 1$,

$$ {}_0^cD_t^q\zeta(s) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\zeta'(s)ds.$$

Lemma 2.1. *[11] If the Caputo $q-$order fractional derivative ${}_0^cD_t^q$ is integrable then*

$$ {}_0I_t^q\ {}_0^cD_t^q\zeta(s) = \zeta(t) - \sum_{k=0}^{n-1}\frac{\zeta^k(0)}{k!}t^k.$$

Especially, for $0 < q \leq 1$, one can obtain

$${}_0I_t^q\ {}_0^cD_t^q\zeta(s) = \zeta(t) - \zeta(0).$$

Lemma 2.2. *[3] Let $V(t)$ be a continuous function on $[0, +\infty)$ and satisfying*

$${}_0^cD_t^q V(t) \leq \theta\ V(t),$$

where $0 < q < 1$ and θ is a constant. Then

$$V(t) \leq V(0)\ E_q(\theta t^q) \qquad \forall t \geq 0.$$

Lemma 2.3. *[20] Consider the following $q-$order fractional system:*

$$\begin{cases} {}_0^cD_t^q z(t) = f(z), \\ z(0) = z_0, \end{cases} \tag{2.1}$$

where $0 < q < 1$ and $z \in R^n$. The equilibrium points of the system (2.1) can be calculated by solving the following equation: $f(z) = 0$. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial z}$ evaluated at the equilibrium points satisfy the Matignon conditions:

$$|\arg(\lambda_i)| > \frac{q\pi}{2}.$$

Theorem 2.4. *[13] The trivial equilibrium point of the system attained by the $\lambda^2 + (k-r)\lambda + cm - rk = 0$ characteristic equation is locally asymptotically stable if either of the following criteria is satisfied.*

(H_1) $\quad k \geq r$ *and* $rk < cm$,

(H_2) $\quad k < r,\ rk < cm,\ (k+r)^2 < 4cm$ *and* $0 < q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4(cm-rk)-(k-r)^2}}{r-k}\right)$.

Theorem 2.5. *[13] The predator-extinction equilibrium point of the system attained by the $\lambda^2 + \left(r - \frac{2cm}{k} + k\right)\lambda + rk - cm = 0$ characteristic equation is locally asymptotically stable if either of the following criteria is satisfied.*

(H_3) $\quad k^2 + rk - 2cm \geq 0$ *and* $rk > cm$,

(H_4) $\quad k^2 + rk - 2cm < 0,\ rk > cm,\ \left(k^2 + rk - 2cm\right)^2 < 4k^2(rk - cm)$ *and* $0 < q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4k^2(rk-cm)-(k^2+rk-2cm)^2}}{2cm-k^2-rk}\right)$.

3. Mathematical analysis and asymptotic behavior of the solution

3.1. **Equilibria of the model.** In this subsection, we determine the local behavior of the system (1.1). First, we determine the equilibria of the system (1.1), which are the solutions of the following system:

$$\begin{aligned} 0 &= x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, \\ 0 &= e_1 f(x)y - \mu_1 y - \beta yz, \\ 0 &= e_2 g(x)z - \mu_2 z - \gamma yz, \\ 0 &= -hu + mx. \end{aligned} \tag{3.1}$$

As a first remark, we deduce that the system 3.1 has the following particular cases:

(i) For the system (1.1) there always exists the trivial equilibrium point $E_0(0,0,0,0)$, which represents the extinction of the three populations.

(ii) $E_1(x_1, 0, 0, u_1)$ which implies the extinction of two types of predators, where $x_1 = \frac{k(rh-cm)}{h(ak+r)}$, $u_1 = \frac{km(rh-cm)}{h^2(ak+r)}$. This point is called the predator-free equilibrium (PFE).

(iii) Searching for the first predator-free equilibrium (FPFE) as $E_2(x_2, 0, z_2, u_2)$, we insert $y = 0$. By replacing this result in the third equation of system (3.1) we get, $x_2 = g^{-1}\left(\frac{\mu_2}{e_2}\right)$, $u_2 = \frac{m}{h} g^{-1}(\frac{\mu_2}{e_2})$ where g^{-1} is the inverse function of g, which exists since g is a bijective function from the conditions (A_1) and $(A_2\ )$. Substituting this last result into the first equation of (3.1) yields

$$z_2 = \frac{e_2 x_2 \left(r - a x_2 - \frac{r x_2}{k} - \frac{cm}{h}\right)}{\mu_2},$$

which is positive if $x_2 < \frac{k(rh-cm)}{h(ak+r)}$. Summarizing all the results, we can conclude that FPFE $E_2(x_2, 0, z_2, u_2)$ exists if $x_2 < \frac{k(rh-cm)}{h(ak+r)}$.

(iv) Seeking for the second predator-free equilibrium (SPFE) as $E_3(x_3, y_3, 0, u_3)$ by replacing $z = 0$ in (3.1). By substituting this result into the second equation of system (3.1) we get $x_3 = f^{-1}(\frac{\mu_1}{e_1})$, $u_3 = \frac{m}{h} f^{-1}(\frac{\mu_1}{e_1})$ where f^{-1} is the inverse function of f , which exists since f is a bijective function function from the conditions (A_1) and $(A_2\ )$. Taking this last result along with the first equation of (3.1), we get

$$y_3 = \frac{e_1 x_3 \left(r - a x_3 - \frac{r x_3}{k} - \frac{cm}{h}\right)}{\mu_1},$$

which is biologically relevant if $x_3 < \frac{k(rh-cm)}{h(ak+r)}$. Summarizing all the results, we can deduce that SPFE as $E_3(x_3, y_3, 0, u_3)$ exists if $x_3 < \frac{k(rh-cm)}{h(ak+r)}$.

Remark. *It is assumed that both functional f and g are increasing in x. From x_3 and x_2, if $\lim_{x\to\infty} f(x) = a$ (resp.,$\lim_{x\to\infty} g(x) = b$) then another condition on the parameters arises, $\frac{\mu_1}{e_1} < a$ (resp., $\frac{\mu_2}{e_2} < b$), which is a necessary condition for having a solution for the equation $f(x) = \frac{\mu_1}{e_1}$ (resp., $g(x) = \frac{\mu_2}{e_2}$).*

(v) Now we are in a position to seek the coexistence equilibrium point $E_4(x^*, y^*, z^*, u^*)$, which is the positive solution of the following system:

$$\begin{aligned} 0 &= x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, \\ 0 &= e_1 f(x) - \mu_1 - \beta z, \\ 0 &= e_2 g(x) - \mu_2 - \gamma y, \\ 0 &= -hu + mx. \end{aligned} \tag{3.2}$$

From $0 = e_2 g(x) - \mu_2 - \gamma y$ we obtain,

$$y^* = \frac{e_2}{\gamma} g(x) - \frac{\mu_2}{\gamma}. \tag{3.3}$$

Moreover, from $0 = e_1 f(x) - \mu_1 - \beta z$ we find that

$$z^* = \frac{e_1}{\beta} f(x) - \frac{\mu_1}{\beta}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), from the first equation, we get $F_1(x) = F_2(x)$, where

$$\begin{aligned} F_1(x) &= x(r - ax - \frac{rx}{k}) - \frac{cmx}{h}, \\ F_2(x) &= f(x)g(x)\left(\frac{e_2}{\gamma} - \frac{e_1}{\beta}\right) - \left(\frac{\mu_2}{\gamma}f(x) - \frac{\mu_1}{\beta}g(x)\right). \end{aligned} \tag{3.5}$$

Some straightforward calculations suggest that

$$F_1(0) = F_1\left(\frac{k(rh-cm)}{h(ak+r)}\right) = 0, \qquad F_1(x) = \begin{cases} >0 & for \quad x < \frac{k(rh-cm)}{h(ak+r)} \\ <0 & for \quad x > \frac{k(rh-cm)}{h(ak+r)}. \end{cases}$$

To guarantee at least one nontrivial intersection between two curves of the functionals F_1 and F_2, we introduce the following assumption:

$$F_1(\widetilde{x}) > F_2(\widetilde{x}), \qquad F_2\left(\frac{k(rh-cm)}{h(ak+r)}\right) > 0 \qquad with \qquad \widetilde{x} = \max\{x_2, x_3\},$$

which it can be rewritten as

$$\widetilde{x} < \frac{k(rh-cm)}{h(ak+r)}, \qquad r > r_\varepsilon := \frac{k\left(\frac{f(x)g(x)\left(\frac{e_2}{\gamma}-\frac{e_1}{\beta}\right)-\left(\frac{\mu_2}{\gamma}f(x)-\frac{\mu_1}{\beta}g(x)\right)}{x}+\frac{cm}{h}+ax\right)}{(k-x)}.$$

Under this condition , we get the existence of at least one nonnegative solution of system.

3.2. **Asymptotic behavior of the system (1.1).** In this part, we are interested in determining the asymptotic stability of the equilibria obtained in the previous section. For the time-fractional-order derivative, the concept of the local stability is very different from the first-order derivative, where in this case, we have an expansion of the stability region in comparison with the first-order derivative.

Let $E(x, y, z, u)$ be an equilibrium for the system (1.1). The Jacobian matrix of system (1.1) at $E(x, y, z, u)$ is expressed as

$$J(E) = \begin{pmatrix} r - 2ax - \frac{2rx}{k} - f^{'}(x)y - g^{'}(x)z & -f(x) & -g(x) & -c \\ e_1 f^{'}(x)y & e_1 f(x) - \mu_1 - \beta z & -\beta y & 0 \\ e_2 g^{'}(x)z & -\gamma z & e_2 g(x) - \mu_2 - \gamma y & 0 \\ m & 0 & 0 & -h \end{pmatrix} \tag{3.6}$$

At $E_0(0, 0, 0, 0)$, the Jacobian matrix of the system (1.1) is

$$J(E_0) = \begin{pmatrix} r & -f(0) & -g(0) & -c \\ 0 & e_1 f(0) - \mu_1 & 0 & 0 \\ 0 & 0 & e_2 g(0) - \mu_2 & 0 \\ m & 0 & 0 & -h \end{pmatrix},$$

and the characteristic equation for J(E_0) is

$$(\lambda - (e_1 f(0) - \mu_1))\,(\lambda - (e_2 g(0) - \mu_2))\,(\lambda^2 + (h-r)\lambda + cm - rh) = 0. \tag{3.7}$$

The eigenvalues of (3.7) are

$$\lambda_2 = e_1 f(0) - \mu_1, \quad \lambda_3 = e_2 g(0) - \mu_2, \quad \lambda_{1,4} = \frac{-(h-r) \pm \sqrt{\Delta_1}}{2}, \tag{3.8}$$

where $\Delta_1 = (h-r)^2 - 4(cm - rh)$.

Obviously, $\lambda_2 = e_1 f(0) - \mu_1 < 0$ *and* $\lambda_3 = e_2 g(0) - \mu_2 < 0$ are always negative. Now we discuss the eigenvalues λ_1 and λ_4 , it is clear that the cases $h > r,\ h = r$ and $h < r$ are possible, so we consider three separate cases.

Case **1.** $h > r$

(1a) $rh < cm$. If $\Delta_1 \geq 0$, we can derive from (3.8) that four eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are negative, which imply that the equilibrium point E_0 is locally asymptotically stable for all $0 < q < 1$. In fact, $|\arg(\lambda_{1,2,3,4})| = \pi > \frac{q\pi}{2}$ for all $0 < q < 1$, which satisfy the condition of Lemma 2.3. If $\Delta_1 < 0$, then λ_1 and λ_4 are complex conjugates with negative real parts, which imply $|\arg(\lambda_{1,4})| = \arctan\left(\frac{\sqrt{-\Delta_1}}{r-h}\right) + \pi > \frac{q\pi}{2}$ for all $0 < q < 1$. According to Lemma 2.3, we know that the equilibrium point E_0 is locally asymptotically stable.

(1b) $rh = cm$. From (3.7) we know that one eigenvalue must be zero and remaining three eigenvalues are negative. Then E_0 is marginally stable.

(1c) $rh > cm$. Then $\Delta_1 = (h+r)^2 - 4(cm) > 0$. From (3.8), we see that one of the eigenvalues λ_1 and λ_4 is positive and the other eigenvalue is negative. Let $\lambda_4 < 0$ and $\lambda_1 > 0$, which imply $|\arg(\lambda_4)| = \pi > \frac{q\pi}{2}$ and $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Hence E_0 is unstable.

Case **2**. $h = r$.

(2a) $rh < cm$. Then $\Delta_1 < 0$ and (3.7) has pure imaginary roots $\lambda_1 = 2\sqrt{cm - rh}i$ and $\lambda_4 = -2\sqrt{cm - rh}i$, which mean $|\arg(\lambda_{1,4})| = \frac{\pi}{2} > \frac{q\pi}{2}$ for all $0 < q < 1$. Since $\lambda_2 < 0,\ \lambda_3 < 0$ according to Lemma 2.3, we know that the equilibrium point E_0 is locally asymptotically stable.

(2b) $rh = cm$. From (3.7), we see that $\lambda_1 = \lambda_4 = 0$, λ_2 and λ_3 is negative. Then E_0 is marginally stable.

(2c) $rh > cm$. From (3.7), we know that $\lambda_1 = 2\sqrt{cm - rh}$ and $\lambda_4 = -2\sqrt{cm - rh}$, which imply $|\arg(\lambda_4)| = \pi > \frac{q\pi}{2}$ and $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Hence E_0 is unstable.

Case **3**. $h < r$.

(3a) $rh < cm$. If $\Delta_1 \geq 0$, then the two eigenvalues λ_1 and λ_4 are positive which imply $|\arg(\lambda_{1,4})| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Thus the equilibrium point E_0 is unstable. If $\Delta_1 < 0$, then λ_1 and λ_4 are complex conjugates with positive real parts. According to Lemma 2.3, we know that the equilibrium point E_0 is locally asymptotically stable if $|\arg(\lambda_{1,4})| = \arctan\left(\frac{\sqrt{-\Delta_1}}{r-h}\right) > \frac{q\pi}{2}$ is satisfied.

(3b) $rh = cm$. It is clear that (3.7) has a positive eigenvalue $\lambda_1 = r - h$, which means $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Hence E_0 is unstable.

(3c) $rh > cm$. Then $\Delta_1 = (h+r)^2 - 4(cm) > 0$. From (3.7) , we see that one of the eigenvalues λ_1 and λ_4 is positive and the other eigenvalue is negative. Thus the equilibrium point E_0 is unstable.

If $h<r$, $rh<cm$, $(h+r)^2<4(cm)$, one has $|\arg(\lambda_{1,4})|=\arctan\left(\frac{\sqrt{-\Delta_1}}{r-h}\right)<\frac{\pi}{2}$, where $\Delta_1=(h-r)^2-4(cm-rh)$, thus $q<\frac{2}{\pi}\arctan\left(\frac{\sqrt{4(cm-rh)-(h-r)^2}}{r-h}\right)<\frac{2}{\pi}\times\frac{\pi}{2}=1$.

Hence we resume the stability conditions for the equilibrium $E_0(0,0,0,0)$ by the following theorem.

Theorem 3.1. *The trivial equilibrium point $E_0(0,0,0,0)$ representing the extinction of the three populations of the system (1.1) is locally asymptotically stable if either of the following criteria is satisfied:*

(i) $h\geq r$ *and* $rh<cm$,

(ii) $h<r,\ rh<cm,\ (h+r)^2<4cm$ *and* $0<q<\frac{2}{\pi}\arctan\left(\frac{\sqrt{4(cm-rh)-(h-r)^2}}{r-h}\right)$.

At the predator-free equilibrium $E_1(x_1,0,0,u_1)$, the Jacobian matrix of the system (1.1) is

$$J(E_1)=\begin{pmatrix} r-\frac{2cm}{h} & -f\left(\frac{k(rh-cm)}{h(ak+r)}\right) & -g\left(\frac{k(rh-cm)}{h(ak+r)}\right) & -c\\ 0 & e_1f\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_1 & 0 & 0\\ 0 & 0 & e_2g\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_2 & 0\\ m & 0 & 0 & -h\end{pmatrix} \tag{3.9}$$

and the characteristic equation for $E_1(x_1,0,0,u_1)$ is

$$(\lambda-(e_1f(x_1)-\mu_1))(\lambda-(e_2g(x_1)-\mu_2))\left(\lambda^2+\left(r-\frac{2cm}{h}+h\right)\lambda+rh-cm\right)=0. \tag{3.10}$$

The Jacobian matrix (3.9) has the eigenvalues

$$\begin{aligned}\lambda_2 &= e_1f\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_1, \qquad \lambda_3=e_2g\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_2,\\ \lambda_{1,4} &= \frac{-\frac{h^2+rh-2cm}{h}\pm\sqrt{\Delta_2}}{2},\end{aligned} \tag{3.11}$$

where

$$\Delta_2=\frac{\left(h^2+rh-2cm\right)^2-4h^2(rh-cm)}{h^2}.$$

Then, we have

$$\lambda_2=\begin{cases}<0 & for \quad \frac{k(rh-cm)}{h(ak+r)}<x_3\\ >0 & for \quad \frac{k(rh-cm)}{h(ak+r)}>x_3\end{cases}$$

and

$$\lambda_3=\begin{cases}<0 & for \quad \frac{k(rh-cm)}{h(ak+r)}<x_2\\ >0 & for \quad \frac{k(rh-cm)}{h(ak+r)}>x_2.\end{cases}$$

Obviously, λ_2 and λ_3 are negative if $x < \widetilde{x} = \min\{x_2, x_3\}$. Now we discuss the eigenvalues λ_1 and λ_4, it is clear that the cases $h^2+rh-2cm > 0$, $h^2+rh-2cm = 0$ *and* $h^2+rh-2cm < 0$ are possible, respectively, so we consider three separate cases.

Case **4**. $h^2 + rh - 2cm > 0$.

$rh > cm$. If $\Delta_2 \geq 0$, we can derive from (3.11) that four eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are negative if $x < \widetilde{x} = \min\{x_2, x_3\}$, which imply that the equilibrium point E_1 is locally asymptotically stable for all $0 < q < 1$. In fact, $|\arg(\lambda_{1,2,3,4})| = \pi > \frac{q\pi}{2}$ for all $0 < q < 1$, which satisfy the condition of Lemma 2.3. If $\Delta_2 < 0$, then λ_1 and λ_4 are complex conjugates with negative real parts, which imply $|\arg(\lambda_{1,4})| = \arctan\left(\frac{h\sqrt{-\Delta_2}}{2cm-h^2-rh}\right)+\pi > \frac{q\pi}{2}$ for all $0 < q < 1$. According to Lemma 2.3, we know that the equilibrium point E_1 is locally asymptotically stable

Case **5**. $h^2 + rh - 2cm = 0$.

$rh > cm$. Then $\Delta_2 < 0$ and (3.10) has pure imaginary roots $\lambda_1 = 2\sqrt{rh-cm}i$ and $\lambda_4 = -2\sqrt{rh-cm}i$ which means that $|\arg(\lambda_{1,4})| = \frac{\pi}{2} > \frac{q\pi}{2}$ for all $0 < q < 1$. If $x < \widetilde{x} = \min\{x_2, x_3\}$ holds, then we have $\lambda_2 < 0$ and $\lambda_3 < 0$. According to Lemma 2.3, we know that the equilibrium point E_1 is locally asymptotically stable.

Case **6**. $h^2 + rh - 2cm < 0$.

$rh > cm$, If $\Delta_2 \geq 0$, then the two eigenvalues λ_1 and λ_4 are positive which imply $|\arg(\lambda_{1,4})| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Thus the equilibrium point E_1 is unstable. If $\Delta_2 < 0$ then λ_1 and λ_4 are complex conjugates with positive real parts. In addition, $x < \widetilde{x} = \min\{x_2, x_3\}$ holds, then we have $\lambda_2 < 0$ and $\lambda_3 < 0$. According to Lemma 2.3, we know that the equilibrium point E_1 is locally asymptotically stable if $|\arg(\lambda_{1,4})| = \arctan\left(\frac{h\sqrt{-\Delta_2}}{2cm-h^2-rh}\right) > \frac{q\pi}{2}$ is satisfied.

When $h^2+rh-2cm < 0$, $rh > cm$, $\left(h^2+rh-2cm\right)^2 < 4h^2(rh-cm)$, one has $|\arg(\lambda_{1,4})| = \arctan\left(\frac{\sqrt{4h^2(rh-cm)-(h^2+rh-2cm)^2}}{2cm-h^2-rh}\right) < \frac{\pi}{2}$, thus

$q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4h^2(rh-cm)-(h^2+rh-2cm)^2}}{2cm-h^2-rh}\right) < \frac{2}{\pi}\times\frac{\pi}{2} = 1.$

Hence we resume the stability conditions for the equilibrium $E_1(x_1, 0, 0, u_1)$ by the following theorem.

Theorem 3.2. *The predator-extinction equilibrium point of the system is locally asymptotically stable if either of the following criteria is satisfied:*

(i) $h^2 + rh - 2cm \geq 0$, $rh > cm$ *and* $x < \widetilde{x} = \min\{x_2, x_3\}$,

(ii) $h^2 + rh - 2cm < 0$, $rh > cm$, $\left(h^2+rh-2cm\right)^2 < 4h^2(rh-cm)$,

$0 < q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4h^2(rh-cm)-(h^2+rh-2cm)^2}}{2cm-h^2-rh}\right)$ *and* $x < \widetilde{x} = \min\{x_2, x_3\}$.

Now we analyze the linear stability of FPFE point of $E_2(x_2, 0, z_2, u_2)$. The Jacobian matrix corresponding to the equilibrium FPFE is evaluated as

$$J(E_2) = \begin{pmatrix} r - 2ax_2 - \frac{2rx_2}{k} - g'(x_2)z_2 & -f(x_2) & -g(x_2) & -c \\ 0 & e_1 f(x_2) - \mu_1 - \beta z_2 & 0 & 0 \\ e_2 g'(x_2) z_2 & -\gamma z_2 & 0 & 0 \\ m & 0 & 0 & -h \end{pmatrix}. \tag{3.12}$$

As a first look, we can deduce that $\lambda_2 = e_1 f(x_2) - \mu_1 - \beta z_2$ is an eigenvalue of the Jacobian matrix (3.12). By replacing the explicit formula of $z_2 = \frac{e_2 x_2\left(r - a x_2 - \frac{r x_2}{k} - \frac{cm}{h}\right)}{\mu_2}$ we obtain $\lambda_2 = e_1 f(x_2) - \mu_1 - \frac{\beta e_2 x_2\left(r - a x_2 - \frac{r x_2}{k} - \frac{cm}{h}\right)}{\mu_2}$. Obviously, if $e_1 f(x_2) - \mu_1 < 0$ (equivalent to $x_2 < x_3$), then $|\arg(\lambda_2)| > \frac{q\pi}{2}$. Now we presume that if $e_1 f(x_2) - \mu_1 > 0$ (equivalent to $x_2 > x_3$). Then

$$\lambda_2 = \begin{cases} > 0 & for \quad r < r_1 := \frac{k\left(\frac{(e_1 f(x_2) - \mu_1)\mu_2}{\beta e_2 x_2} + a x_2 + \frac{cm}{h}\right)}{k - x_2}, \\ < 0 & for \quad r > r_1. \end{cases}$$

Under the condition $\lambda_2 > 0$, we get $|\arg(\lambda_2| < \frac{q\pi}{2}$. This means that FPFE is an unstable equilibrium point. Besides, from $\lambda_2 > 0$, we conclude that $|\arg(\lambda_2)| < \frac{q\pi}{2}$. This means that three remaining eigenvalues of the Jacobian matrix (3.12) determine the stability (resp., instability) of this equilibrium. Note that these significant eigenvalues are the eigenvalues of the matrix

$$\widetilde{J} = \begin{pmatrix} r - 2ax_2 - \frac{2rx_2}{k} - g^{'}(x_2)z_2 & -g(x_2) & -c \\ e_2 g^{'}(x_2)z_2 & 0 & 0 \\ m & 0 & -h \end{pmatrix}. \tag{3.13}$$

To determine the nature of the eigenvalues of the reduced matrix (3.13), we define the characteristic equation of (3.13) as

$$P(\lambda) = \lambda^3 + \vartheta_1 \lambda^2 + \vartheta_2 \lambda + \vartheta_3,$$

where

$$\begin{aligned} \vartheta_1 &= h - r + 2ax_2 + \frac{2rx_2}{k} + g^{'}(x_2)z_2, \\ \vartheta_2 &= -cm - hr + 2hax_2 + \frac{2hrx_2}{k} + hg^{'}(x_2)z_2 + e_2 g^{'}(x_2)g(x_2)z_2, \\ \vartheta_3 &= he_2 g^{'}(x_2)g(x_2)z_2. \end{aligned}$$

$D(P)$ denotes the discriminant of the cubic polynomial $P(\lambda)$, as follows:

$$\begin{aligned} D(P) &= \begin{vmatrix} 1 & \vartheta_1 & \vartheta_2 & \vartheta_3 & 0 \\ 0 & 1 & \vartheta_1 & \vartheta_2 & \vartheta_3 \\ 3 & 2\vartheta_1 & \vartheta_2 & 0 & 0 \\ 0 & 3 & 2\vartheta_1 & \vartheta_2 & 0 \\ 0 & 0 & 3 & 2\vartheta_1 & \vartheta_2 \end{vmatrix} \\ &= 18\vartheta_1\vartheta_2\vartheta_3 + (\vartheta_1\vartheta_2)^2 - 4\vartheta_3(\vartheta_1)^2 - 4(\vartheta_2)^2 - 27(\vartheta_3)^2 \end{aligned}$$

Using the Routh-Hurwitz stability criterion for fractional calculus defined in [7], [16] and [17] we get the stability conditions for the nontrivial equilibrium.

Theorem 3.3. *The positive equilibrium point is asymptotically stable if either of the following criteria is satisfied:*

(i) $D(P) > 0$, $\vartheta_1 > 0$, $\vartheta_3 > 0$, $\vartheta_1\vartheta_2 - \vartheta_3 > 0$ for all $q \in (0,1)$,
(ii) $D(P) < 0$, $\vartheta_1 \geq 0$, $\vartheta_2 \geq 0$, $\vartheta_3 > 0$, $0 < q < \frac{2}{3}$,
(iii) $D(P) < 0$, $\vartheta_1 > 0$, $\vartheta_3 > 0$, $\vartheta_1\vartheta_2 = \vartheta_3$ for all $q \in (0,1)$.

Hence we resume the stability conditions for the equilibrium $E_2(x_2, 0, z_2, u_2)$ by the following theorem. Therefore,

Theorem 3.4. *For FPFE, if* $x_2 < \frac{k(rh-cm)}{h(ak+r)}$*, then we have;*
(i) *If* $x_2 > x_3$ *and* $r < r_1$*, then the FPFE is unstable,*
(ii) *For* $x_2 < x_3$ *or* $(x_2 > x_3$ *and* $r > r_1)$ *if one of the condition* $(i), (ii)$ *or* (iii) *in Theorem 3.3 holds we get the local stability of FPFE.*

To study the stability of the SPFE of $E_3(x_3, y_3, 0, u_3)$, the Jacobian matrix corresponding to the equilibrium SPFE is evaluated as

$$J(E_3) = \begin{pmatrix} r - 2ax_3 - \frac{2rx_3}{k} - f'(x_3)y_3 & -f(x_3) & -g(x_3) & -c \\ e_1 f'(x_3)y_3 & 0 & -\beta y_3 & 0 \\ 0 & 0 & e_2 g(x_3) - \mu_2 - \gamma y_3 & 0 \\ m & 0 & 0 & -h \end{pmatrix}. \tag{3.14}$$

As a first look, we can deduce that $\lambda_3 = e_2 g(x_3) - \mu_2 - \gamma y_3$ is an eigenvalue of the Jacobian matrix (3.14). By replacing the explicit formula of $y_3 = \frac{e_1 x_3\left(r - ax_3 - \frac{rx_3}{k} - \frac{cm}{h}\right)}{\mu_1}$ we obtain $\lambda_3 = e_2 g(x_3) - \mu_2 - \frac{\gamma e_1 x_3\left(r - ax_3 - \frac{rx_3}{k} - \frac{cm}{h}\right)}{\mu_1}$. Obviously, if $e_2 g(x_3) - \mu_2 < 0$ (equivalent to $x_2 > x_3$), then $|\arg(\lambda_3)| > \frac{q\pi}{2}$. Now we presume that if $e_1 f(x_2) - \mu_1 > 0$ (equivalent to $x_2 < x_3$). Then

$$\lambda_3 = \begin{cases} > 0 & for \quad r < r_2 := \frac{k\left(\frac{(e_2 g(x_3) - \mu_2)\mu_1}{\gamma e_1 x_3} + ax_3 + \frac{cm}{h}\right)}{k - x_3}, \\ < 0 & for \quad r > r_2. \end{cases}$$

Under the condition $\lambda_3 > 0$, we get $|\arg(\lambda_3)| < \frac{q\pi}{2}$. This means that FPFE is an unstable equilibrium point. Besides, from $\lambda_3 > 0$, we conclude that $|\arg(\lambda_3)| < \frac{q\pi}{2}$. This means that three remaining eigenvalues of the Jacobian matrix (3.14) determine the stability (resp., instability) of this equilibrium. Note that these significant eigenvalues are the eigenvalues of the matrix

$$\widetilde{\overline{J}} = \begin{pmatrix} r - 2ax_3 - \frac{2rx_3}{k} - f'(x_3)y_3 & -f(x_3) & -c \\ e_1 f'(x_3)y_3 & 0 & 0 \\ m & 0 & -h \end{pmatrix}. \tag{3.15}$$

To determine the nature of the eigenvalues of the reduced matrix (3.15), we define the characteristic equation of (3.15) as

$$P^*(\lambda) = \lambda^3 + \theta_1\lambda^2 + \theta_2\lambda + \theta_3,$$

where

$$\begin{aligned}\theta_1 &= h - r + 2ax_3 + \frac{2rx_3}{k} + f^{'}(x_3)y_3,\\ \theta_2 &= -cm - hr - 2hax_3 + \frac{2hrx_3}{k} + hf^{'}(x_3)y_3 + e_1 f^{'}(x_3)f(x_3)y_3,\\ \theta_3 &= he_1 f^{'}(x_3)f(x_3)y_3.\end{aligned}$$

$D(P^*)$ denotes the discriminant of the cubic polynomial
$D(P^*) = 18\theta_1\theta_2\theta_3 + (\theta_1\theta_2)^2 - 4\theta_3(\theta_1)^2 - 4(\theta_2)^2 - 27(\theta_3)^2$

With the same technics in Theorem 3.3, we get the stability conditions for the nontrivial equilibrium. Therefore

Theorem 3.5. *For SPFE if $x_2 < \frac{k(rh-cm)}{h(ak+r)}$, then we have;*
(i) If $x_2 < x_3$ and $r < r_2$, then the FPFE is unstable.
(ii) For $x_2 > x_3$ or $(x_2 < x_3$ and $r > r_2)$ if one of the conditions in $(i), (ii)$ or (iii) in Theorem 3.3 holds, we get the local stability of FPFE.

Now we are in a position to focus on studying the local behavior of the coexistence equilibrium. For this positive equilibrium point, we have that assumption for the existence of at least one non-negative solution of the system (1.1). The Jacobian matrix of the system (1.1) evaluated at the equilibrium $E_4(x^*, y^*, z^*, u^*)$ is given by

$$J(E_4) = \begin{pmatrix} r - 2ax^* - \frac{2rx^*}{k} - f^{'}(x^*)y^* - g^{'}(x^*)z^* & -f(x^*) & -g(x^*) & -c \\ e_1 f^{'}(x^*)y^* & e_1 f(x^*) - \mu_1 - \beta z^* & -\beta y^* & 0 \\ e_2 g^{'}(x^*)z^* & -\gamma z^* & e_2 g(x^*) - \mu_2 - \gamma y^* & 0 \\ m & 0 & 0 & -h \end{pmatrix}. \tag{3.16}$$

Therefore, the characteristic equation associated with Jacobian (3.16) is

$$\Delta(\lambda) = \lambda^4 + \Phi_1\lambda^3 + \Phi_2\lambda^2 + \Phi_3\lambda + \Phi_4,$$

where

$$\begin{aligned}\Phi_1 &= h - r + 2ax^* + \frac{2rx^*}{k} + g^{'}(x^*)z^* + f^{'}(x^*)y^*,\\ \Phi_2 &= cm - rh + 2hax^* + \frac{2rhx^*}{k} + hg^{'}(x^*)z^* + hf^{'}(x^*)y^* + \beta y^*\gamma z^* + e_1 f^{'}(x^*)f(x^*)y^*\\ &\quad + e_2 g^{'}(x^*)g(x^*)z^*,\\ \Phi_3 &= h\beta y^*\gamma z^* + he_1 f^{'}(x^*)f(x^*)y + he_2 g^{'}(x^*)g(x^*)z^* - r\beta y^*\gamma z^* + 2ax^*\beta y^*\gamma z^*\\ &\quad + \frac{2rx^*}{k}\beta y^*\gamma z^* + g^{'}(x^*)\left(z^*\right)^2\beta y^*\gamma + f^{'}(x^*)\left(y^*\right)^2\beta\gamma z^* - f(x^*)\beta y^* e_2 g^{'}(x^*)z^*\\ &\quad - g(x^*)e_1 f^{'}(x^*)y^*\gamma z^*,\\ \Phi_4 &= cm\beta y^*\gamma z^* - rh\beta y^*\gamma z^* + h\beta y^*\gamma z^* 2ax^* + \frac{2hrx^*}{k}\beta y^*\gamma z^* + hg^{'}(x^*)\left(z^*\right)^2\beta y^*\gamma\\ &\quad + hf^{'}(x^*)\left(y^*\right)^2\beta\gamma z^* - hf(x^*)\beta y^* e_2 g^{'}(x^*)z^* - hg(x^*)e_1 f^{'}(x^*)y^*\gamma z^*.\end{aligned}$$

and $D(\triangle)$ denotes the discriminant of the polinom $\triangle(\lambda)$ as follows,

$$
\begin{aligned}
D(\Delta) &= \begin{vmatrix} 1 & 0 & 0 & 4 & 0 & 0 & 0 \\ \Phi_1 & 1 & 0 & 3\Phi_1 & 4 & 0 & 0 \\ \Phi_2 & \Phi_1 & 1 & 2\Phi_2 & 3\Phi_1 & 4 & 0 \\ \Phi_3 & \Phi_2 & \Phi_1 & \Phi_3 & 2\Phi_2 & 3\Phi_1 & 4 \\ \Phi_4 & \Phi_3 & \Phi_2 & 0 & \Phi_3 & 2\Phi_2 & 3\Phi_1 \\ 0 & \Phi_4 & \Phi_3 & 0 & 0 & \Phi_3 & 2\Phi_2 \\ 0 & 0 & \Phi_4 & 0 & 0 & 0 & \Phi_3 \end{vmatrix} \\
&= 256\,(\Phi_4)^3 - 192\Phi_1\Phi_3\,(\Phi_4)^2 - 128\,(\Phi_4)^2\,(\Phi_2)^2 + 144\Phi_2(\Phi_3)^2\Phi_4 \\
&\quad -27(\Phi_3)^4 + 144(\Phi_1)^2\Phi_2(\Phi_4)^2 - 6(\Phi_1)^2(\Phi_3)^2\Phi_4 - 80\Phi_1(\Phi_2)^2\Phi_3\Phi_4 \\
&\quad +18\Phi_1\Phi_2(\Phi_3)^3 + 16(\Phi_2)^4\Phi_4 - 4(\Phi_2)^3(\Phi_3)^2 - 27(\Phi_1)^4(\Phi_4)^2 \\
&\quad +18(\Phi_1)^3\Phi_2\Phi_3\Phi_4 - 4(\Phi_1)^3(\Phi_3)^3 - 4(\Phi_1)^2(\Phi_2)^3\Phi_4 + (\Phi_1)^2(\Phi_2)^2(\Phi_3)^2.
\end{aligned}
$$

Using the Routh-Hurwitz stability criterion for fractional calculus, we get the stability conditions for the nontrivial positive equilibrium.

Theorem 3.6. *The positive equilibrium point $E_4(x^*, y^*, z^*, u^*)$ is asymptotically stable if either of the following criteria is satisfied:*

(*i*) $D(\Delta) > 0, \Phi_1 > 0,\ \ \Phi_3 > 0,\ \ \Phi_4 > 0,\ \ \Phi_1\Phi_2 - \Phi_3 > 0,$ $\Phi_3(\Phi_1\Phi_2 - \Phi_3) - (\Phi 1)^2\,\Phi_4 > 0$

(*ii*) $D(\Delta) < 0,\ \Phi_1 \geq 0,\ \Phi_2 \geq 0,\ \Phi_3 \geq 0,\ \Phi_4 \geq 0,\ \ 0 < q < \frac{2}{3}$.

(*iii*) $D(\Delta) < 0,\ \Phi_1 > 0,\ \Phi_3 > 0,\ \ \Phi_4 > 0,\ \Phi_1\Phi_2 = \Phi_3,\ \ \Phi_3(\Phi_1\Phi_2 - \Phi_3) = (\Phi_1)^2\,\Phi_4)$ *for all* $q \in (0, 1)$.

4. Numerical analysis of the system (1.1)

The main purpose of this section is to solve the following fractal problem numerically:

$$ {}_0^cD_t^qV(t) = P\left(t, V(t)\right). \tag{4.1} $$

By applying the fundamental theorem of fractional calculus on (1.1), we get

$$ V(t) - V(0) = \frac{1}{\Gamma(q)} \int_0^t P\left(s, V(s)\right)(t-s)^{q-1}ds. \tag{4.2} $$

Letting $t = t_n = nh$ in (4.2), we arrive at

$$ V(t_n) = V(0) + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} P\left(s, V(s)\right)(t_n - s)^{q-1}ds. $$

Now we can approximate the function $P(t, V(t))$ by the following linear approximation:

$$P\left(t, K(t)\right) \approx P\left(t_{i+1}, V_{i+1}\right) + \frac{t - t_{i+1}}{h}\left(P\left(t_{i+1}, V_{i+1}\right) - P\left(t_i, V_i\right)\right), \qquad t\epsilon[t_i, t_{i+1}] \tag{4.3}$$

with the notation $V_i = V(t_i)$.

By substituting equation (4.2) into (4.3) and applying some algebra (for more detail, see [6]) we get

$$V_n = V_0 + h^q\left(\Phi_n P(t_0, V_0) + \sum_{i=1}^{n} \Psi_{n-i} P\left(t_i, V_i\right)\right) \tag{4.4}$$

with

$$\Phi_n = \frac{(n-1)^{q+1} - n^q(n-q-1)}{\Gamma(q+2)},$$

$$\Psi_n = \begin{cases} \frac{1}{\Gamma(q+2)}, & n = 0, \\ \frac{(n-1)^q - 2n^q + (n+1)^q}{\Gamma(q+2)} & n = 1, 2, 3, ... \end{cases}$$

Using the numerical method presented in the formula (4.4) to solve the problem (4.1), we obtain the following iterative schemes:

$$\begin{aligned}
x_n &= x_0 + h^q\left(\Phi_n P_1\left(x_0, y_0, z_0, u_0\right) + \sum_{i=1}^{n} \Psi_{n-i} P_1\left(x_i, y_i, z_i, u_i\right)\right), \\
y_n &= y_0 + h^q\left(\Phi_n P_2\left(x_0, y_0, z_0, u_0\right) + \sum_{i=1}^{n} \Psi_{n-i} P_2\left(x_i, y_i, z_i, u_i\right)\right), \\
z_n &= z_0 + h^q\left(\Phi_n P_3\left(x_0, y_0, z_0, u_0\right) + \sum_{i=1}^{n} \Psi_{n-i} P_3\left(x_i, y_i, z_i, u_i\right)\right), \\
u_n &= u_0 + h^q\left(\Phi_n P_4\left(x_0, y_0, z_0, u_0\right) + \sum_{i=1}^{n} \Psi_{n-i} P_4\left(x_i, y_i, z_i, u_i\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
P_1(x, y, z, u) &= x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, \\
P_2(x, y, z, u) &= e_1 f(x)y - \mu_1 y - \beta yz, \\
P_3(x, y, z, u) &= e_2 g(x)z - \mu_2 z - \gamma yz, \\
P_4(x, y, z, u) &= -hu + mx.
\end{aligned}$$

5. Conclusion

In this research, we studied an ecological model with two predators fighting on one prey with a generalized functional response. We consider a fractional-order predator–prey model incorporating feedback control. The reason behind considering a comprehensive generalized class of functional interaction is to model the diversity in predator–prey interaction with the environment. These interactions can be affected by many factors, such as the environment and the adaptation of the three species. We analyzed the existence of different equilibrium points and some criteria were derived to ensure the asymptotical stability of these equilibrium points. In the first section, we studied the existence of the equilibria of the system (1.1), where we can have many equilibrium points next to the predator-free equilibrium. By analyzing the existence of the equilibria we obtained that these populations may have many scenarios. They include the extinction of three populations, two types of predators, the extinction of each population of predators, and finally the coexistence of the three populations. For the coexistence stage, we provided some conditions on the model parameters for the existence of this equilibrium. The theoretical results show that feedback control play important roles in adjusting coexistence of prey species and predator species. To determine which scenario will prevail, we have utilized the local asymptotic stability using the Jacobian matrix.

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FIFA/COCA-COLA WORLD RANKINGS ON THE PREDICTABILITY OF THE MENS AND WOMENS FIFA WORLD CUP: A COMPARATIVE ANALYSIS

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Abstract. Since 1992, the International Federation of Association Football (FIFA) has been ranking senior mens national soccer teams based on a variety of criteria. In 2003, FIFA extended the FIFA/Coca-Cola World Rankings into ranking senior womens national soccer teams. The FIFA/Coca-Cola World Rankings published just before the 1994 FIFA World Cup USA, 1998 FIFA World Cup France, 2002 FIFA World Cup Korea/Japan, 2006 FIFA World Cup Germany, 2010 FIFA World Cup South Africa, 2014 FIFA World Cup Brazil, 2018 FIFA World Cup Russia, 2003 FIFA World Cup USA, 2007 FIFA World Cup China, 2011 FIFA World Cup Germany, 2015 FIFA World Cup Canada, and the 2019 FIFA World Cup France were considered. These rankings were compared to the final results of those FIFA World Cups based on two different methods of displaying the teams finish and were analyzed. Of the top 16 teams in each of the Mens FIFA World Cups, 74.1% of those teams advanced to the Round of 16. Meanwhile, 83.9% of the top 12 teams in each of the Womens FIFA World Cups advanced to the Round of 16 or Quarterfinals. The Pearson correlation coefficient between the Pre-Tournament rankings and final results was calculated using both ranking methods. The Womens World Cups had higher Pearson correlation coefficients for both methods than the Mens World Cups. In addition, the Womens World Cups had higher t-values and z-scores than the Mens World Cup when tested for independence and association between the Pre-Tournament rankings and final results using both ranking methods. These findings indicate that the Womens World Cups were more predictable than Mens World Cups based on the FIFA/Coca-Cola World Rankings.

1. Introduction

In December 1992, FIFA instituted a ranking system of mens senior national soccer teams. The first iteration of the ranking system was in place from 1993 until

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1998. The system gave teams one point for a tie or draw, three points for a win, and no points for a draw in games that were acknowledged by FIFA. Over this time period, FIFA saw the need for improvement in the ranking of senior national teams. This improvement included the addition of criteria to the ranking procedure such as considering the results of games played by senior national teams over the last 8 years and including data such as game outcome (win, loss, or draw), number of goals, location of the game (home, away, or neutral), importance of the match, and strength of the region.

The weighting procedure for the importance of the match assigned a 1.0 weight for a friendly match, a 1.50 weight for a continental championship group stage or qualifying match and a FIFA World Cup qualifying match, group stage match, a 1.75 weight for a Continental Finals match or a FIFA Confederation Cup match, and a 2.0 weight for a FIFA World Cup finals match. Additionally, different regions had different weights added to their matches. For example, the Union of European Football Associations (UEFA) had a weight of 1, Confederacin Sudamericana de Ftbol or the South American Football Federation (CONMEBOL) had a weight of 0.99, the Confederation of North, Central America and Caribbean Association Football (CONCACAF) had a weight of 0.94, the Asian Football Confederation (AFC) had a weight of 0.93, and the Oceania Football Confederation (OCF) had a weight of 0.93. In the case of a negative point total, the points would be rounded up to 0 (FIFA, 2005).

A third iteration of the rankings made their debut in 2006 following the 2006 FIFA World Cup in Germany. This ranking system was based off of the match outcome which awarded 3 points for a win, 1 point for a tie, and 0 points for a loss. This varied the importance of matches from a weighted multiplier of 1 for a friendly to 4 for a FIFA World Cup match. The strength of the opponent formula was ({200-Position in rankings}/100) and the strength of the region which was based on the regions results at the last 3 FIFA World Cup. The occurrence of the game with more recent games have more of an impact on the ranking, and the average number of points won from matches in the last 12 months prior to the ranking (FIFA, 2007).

Although not relevant for any of the World Cups considered in this study, the FIFA/Coca-Cola World Ranking changed again in 2018 following the 2018 FIFA World Cup and then had an additional change made in 2022 to round decimals to the nearest hundredth to promote accuracy. The current ranking format follows the formula:

$$P = P_{\text{before}} + I(WW_e). \tag{1.1}$$

The P of the equation stands for total points. The P_{before} stands for points before a particular game. I stands for the importance of the match with a value of 5 for international friendlies played outside of the International Match Calendar (windows set aside for senior national team matches), 10 for international friendlies played within these windows, 15 for matches that happen during the group stage of Nations League matches within each region, 25 for any playoff and finals matches in these Nations Leagues and qualifying matches for the FIFA World Cup and Confederations finals, 35 for matches that occur between the group stages and quarterfinals of a Confederations Final, 40 for Confederations Final matches from the quarterfinal stage onwards and all games that happen at the FIFA Confederations Cup, 50 for FIFA World Cup matches that occur between the group stages and quarterfinals, and 60 for FIFA World Cup quarterfinals, semifinals, 3rd place, and finals matches. W stands for the outcome of the match with 1 for a win, 0.5 for a draw, and 0 for a loss. W_e stands for expected win and is defined as

$$W_e = \frac{1}{10^{(P_{\text{before,B}} - P_{\text{before,A}})/600} + 1}. \tag{1.2}$$

Additionally, this model analyzes results from a penalty shootout and other results with different weights (FIFA, 2018). The FIFA Womens World Rankings havent changed since their inception in 2003. The FIFA Womens World Rankings has the formula

$$\text{WWR}_{\text{new}} = \text{WWR}_{\text{old}} + (\text{Actual} - \text{Predicted}). \tag{1.3}$$

This is where WWR_{new} stands for the new senior national team Womens World Ranking. WWR_{old} stands for the old Womens World Ranking. The actual and predicted value come from the match outcome, goal differential, goals scores, location of the match, importance of the match, and difference in their and their opponents points before a match (FIFA). While the FIFA/Coca-Cola Mens World Rankings have had three different formats under which World Cups have been played, the FIFA/Coca-Cola Womens World Rankings have had one iteration of the rankings. However, the Womens FIFA World Cup has undergone changes including going from 16 to 24 teams and increasing the number of teams who make the knockout rounds, while the Mens World Cup has increased from 24 to 32 teams, but has not increased the number of teams who make the knockout rounds. For this reason, the predictability of both of the FIFA World Cup final results was studied based on two methods of classification of those results against the Pre-Tournament Rankings.

2. Methods

2.1. **Data Collection.** The data collected for this study include the results of the Mens 1994 FIFA World Cup USA, 1998 FIFA World Cup France, 2002 FIFA World Cup Korea/Japan, 2006 FIFA World Cup Germany, 2010 FIFA World Cup South Africa, 2014 FIFA World Cup Brazil, and the 2018 FIFA World Cup Russia. Additionally, results of the Womens 2003 FIFA World Cup USA, 2007 FIFA World Cup China, 2011 FIFA World Cup Germany, 2015 FIFA World Cup Canada, and 2019 FIFA World Cup France were also obtained. The FIFA/Coca-Cola Mens and Women's Ranking was gathered from the ranking that occurred in March before the Womens FIFA World Cup and May for the Mens FIFA World Cup. These results are all publicly available on the FIFA website (fifa.com). The organized data of both the Pre-Tournament Rankings and final results are available in the appendix.

2.2. **Analytical Procedures.** The acquired data was studied using Chi-Square tests to standardize the data, rules were adapted and implemented to try to understand how FIFA World Cups would play out based on suggested Group winners and teams that would advance to the Round of 16. This data was supplemented by results of teams ranked in the top 12 or 16 of the respective FIFA World Cup Pre-Rankings against those outside of the top 12 or 16. The effectiveness of these predictive methods was tested by running the Fisher Transformation Hypothesis Test and the Students t-test to analyze the effectiveness of using the FIFA/Coca-Cola World Rankings in predicting FIFA World Cup final results (Suzuki & Ohmori, 2008).

2.3. **Rules Analysis.** The first step in analyzing these results is to create rules to sort the data and evaluate the effectiveness of rankings between the Mens and Womens FIFA World Cup. First, the teams that qualified for the FIFA World Cup in their respective year and classification had their ranking documented and then sorted to get a list of 16 teams (Womens 2003 FIFA World Cup USA, 2007 FIFA World Cup China, and 2011 FIFA World Cup Germany), 24 teams (1994 FIFA World Cup USA Womens 2015 FIFA World Cup Canada, and 2019 FIFA World Cup France), 32 teams (all other Mens FIFA World Cups besides the 1994 FIFA World Cup USA). The game results of teams that fell within the top 16 teams qualified for the Mens FIFA World Cups and top 12 for the Womens FIFA World Cup against teams below these marks were gathered and sorted for testing. The final placement of teams was determined using two methods.

For the Mens World Cups, in Method A, the top 4 finishing teams were given a 1,2,3, or 4 based on their corresponding place. Then, if a team was eliminated in the Group Stage, they were assigned a 7, teams eliminated in the Round of 16 assigned a 6, and teams eliminated in the Quarterfinals a 5. In Method B, the top 4 teams were given a 1,2,3, or 4 based on their corresponding place. Then, if a team was eliminated in the Group Stage, they were assigned a 24 (if this was the 1994 FIFA World Cup USA) or 32 (if this was any other Mens FIFA World Cup), teams eliminated in the Round of 16 a assigned 16, and teams eliminated in the Quarterfinals assigned an 8.

For the Womens World Cups, in Method A, the top 4 finishing teams were given a 1,2,3, or 4 based on their corresponding place. Then, if a team was eliminated in the Group Stage, they were assigned a 7 (if this was the 2015 or 2019 FIFA World Cup) or 6 (if this was the 2003, 2007, or 2011 FIFA World Cup), teams eliminated in the Round of 16 are assigned a 6, and teams eliminated in the Quarterfinals are assigned a 5. In Method B, the top 4 teams were given a 1,2,3, or 4 based on their corresponding place. Then, if a team was eliminated in the Group Stage, they were assigned a 16 (if this was the 2003, 2007, or 2011 FIFA World Cup) or 24 (if this was the 2015 or 2019 FIFA World Cup), teams eliminated in the Round of 16 were assigned a 16, and teams eliminated in the Quarterfinals were assigned an 8.

3. Statistical Testing Analysis

3.1. **Fisher Exact Test.** The study used the Fisher Exact Test to test the number of top 12 or 16 teams that advanced from the Group Stage vs the number of lower ranked teams advancing from the Group Stage. A second Fisher Exact Test was run on the differences in winning percentage in games played by teams ranked in the top 12 or 16 teams of the World Cup and against those of lower ranked teams. This allowed the study to determine if the association between the differences in advancement or winning percentage was different or not. Thus, consider the population in the study to be the FIFA World Cups in which there was a ranking that was available right before the World Cup was played. For simplicity, let there be variables S and F such that there are m and n collected states in S and F that creates an mxn matrix (Hoffman, 2014). Then, to represent a specific cell in the m n matrix, let's denote this x_{ij} such that $s = i$ and $f = j$. Then the total sum of observable states is $N = \Sigma_i R_i = \Sigma_j R_j$ such that C_j is the sum of the columns and R_i is the sum of the rows. The Fisher Exact Test calculates the conditional probability that this matrix exists through the formula (Weisstein)

$$p = \frac{(R_1!R_2!\dots R_m!)(C_1!C_2!\dots C_n!)}{N!\ \prod_{i,j} x_{ij}}. \tag{3.1}$$

Fishers exact test is then paired with the Chi-Square Test so that the study has a standard measurement of association between variables.

3.2. **Chi-Square Test.** Since the measurement for over 80% of the variables used and boolean values, a Chi-Square Test is also considered. The Chi-Square test statistics is classically defined as

$$\chi^2 = \sum_{i=1}^{k} \frac{[n_i - E(n_i)]^2}{E(n_i)} = \sum_{i=1}^{k} \frac{[n_i - np_i]^2}{np_i}, \tag{3.2}$$

where n stands for total number of games or teams, n_i stands for a particular number of teams advancing or not or games won, loss, or drawn, and p_i is the probability of this event happening. Expected cell frequencies are calculated using $E(\hat{n}_{ij}) = (r_i c_j)/n$ where n stands for total number of games or teams, and r_i and c_j stand for specific row and column totals (Wackerly et al., 2012).

3.3. **Pearson Correlation Coefficient.** The rules analysis described by Methods A and B, in combination with the Fisher Exact Test and Chi-Square Test set the basis which allows the study to take into account the Pearson correlation coefficient because the graphs of our rules vs final results are not monotonic. Therefore, the study draws a line of best fit of the form $Y = \beta_0 + \beta_1 x + \varepsilon$ such that the parameter $\beta_1 = \frac{\sigma_y}{\sigma_x}\rho$ and $E(Y|X = x) = \beta_0 + \beta_1 x$. This implies that ρ is positive when, generally, as X increases Y increases and that ρ is negative when as Y decreases, X increases. Going forward, ρ can be expressed in terms of r where

$$r = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}}. \tag{3.3}$$

Here, n is the total teams that participated in the FIFA World Cups considered and X_i and Y_i are sample team rankings within the World Cups considered, and $\bar{X}$ and $\bar{Y}$ are the sample means (Wackerly et al., 2012). The study will use the Fisher Transformation when constructing the null and alternative hypotheses.

3.4. **Fisher Transformation Hypothesis Test.** Consider the null hypothesis $H_0 : \rho = 0$ with an alternative hypothesis $H_a : \rho \neq 0$ using a level of $\alpha = 0.05$. From the Fisher transformation, the study has $F(r) = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)$. The Fisher transformation will allow for $F(r)$ to follow an approximately normal distribution such that the mean$= F(\rho) = F(0) = 0$ with standard deviation $\frac{1}{\sqrt{n-3}}$, where n is the number of teams that have played in the FIFA World Cup since the Ranking system began. Using these variables, a z-score is obtained (Vrbik, 2005) such that

$$z = \frac{x}{\frac{1}{\sqrt{n-3}}} = F(r)\sqrt{n-3}. \tag{3.4}$$

To further strengthen the argument for whether the Mens or Womens World Cup is more predictable, this test is paired with a Students t-distribution.

3.5. **Student's t-test.** The t-distribution test is defined from a bivariate normal distribution of a population value (which is consistent with this studys data) with the null hypothesis. The null hypothesis is that there is no correlation between the final finish ranking through Methods A and B against the Pre-Tournament Rankings of the teams participating in the FIFA World Cup. The t-test has n-2 degrees of freedom where n stands for the total number of teams that participated in the Mens or Womens World Cup. In this studys analysis this will be 216 for Men and 96 for Women. The Students t-test statistic (Rahman, 1968) is calculated by

$$t = r\sqrt{\frac{n-2}{1-r^2}}. \tag{3.5}$$

The critical value of r (Soper et al., 1917) is determined as

$$r = \frac{t}{\sqrt{n-2+t^2}}. \tag{3.6}$$

Using the results from the statistical tests, this study will aim to define whether the Mens or Womens World Cup is more predictable based on Pre-Tournament Rankings.

4. Results

4.1. **Mens World Cup Analysis.**

4.1.1. *Tests of Association.* The Fisher Exact Test was used to determine if there was a nonrandom association between the number of teams in the Top 16 that advanced to the Round of 16 against the teams in the lower 16 based on each of the different ranking formats.

Table 1: Tests of Association

Advancement from Group Stage in FIFA World Cup

Type	FIFA World Cup 1994 and 1998	FIFA World Cup 2002 and 2006	FIFA World Cup 2010, 2014, and 2018	Total
Top 16 Ranked Teams Advancing to the Round of 16	25 (23.714)	22 (23.714)	36 (35.571)	83
Lower 16 Teams Advancing to the Round of 16	7 (8.286)	10 (8.286)	12 (12.429)	29
Total	32	32	48	112

Note: the number in parenthesis are the expected counts. From the information detailed in **Table 1**, we find the p-value to be $p = 0.679$. From this we find that there is not a statistically significant association between the number of teams that were ranked in the top 16 of the World Cup that advanced to the Round of 16 against the number of teams that were ranked in the lower 16 of the World Cup that advanced.

As a standardizing tool, we use a Chi-Squared test of association. Using the data from **Table 1**, we find the Chi-Square value to be 0.768 with degrees of freedom (*d.f.*) = 2, and a p-value (p)= 0.681. Thus, there is not a statistically significant association between the number of teams that were ranked in the top 16 of the World Cup that advanced to the Round of 16 against the number of teams that were ranked in the lower 16 of the World Cup that advanced. This is consistent with our findings from the Fisher Exact test.

Another Chi-Squared test of association was run using the win, loss, and draw data from matches played of teams ranked in the top 16 against those from outside of the top 16 based on each of the different ranking formats.

Table 2: Tests of Association				
Win/Loss/Draw Record Comparison				
Type	FIFA World Cup 1994 and 1998	FIFA World Cup 2002 and 2006	FIFA World Cup 2010, 2014, and 2018	Total
Win	32 (32.888)	39 (39.981)	67 (65.131)	138
Loss	10 (8.579)	10 (10.430)	16 (16.991)	36
Draw	9 (9.533)	13 (11.589)	18 (18.879)	40
Total	51	62	101	214

Note: The numbers in parenthesis are the expected counts. From the data in **Table 2**, we find the Chi-Squared value is 0.655 with a p-value of p= 0.957. Note, the Fisher Exact Test cannot be used since the number of matches played was over 90. Since the p-value was found to be greater than 0.05, there is not a statistically significant association between the win, loss, draw records of the teams ranked in the top 16 against those from outside of the top 16.

4.1.2. *Rules Analysis.* Using the Chi-Square test and Fisher Exact test as a baseline, the rules described were analyzed by finding the Pearson Correlation Coefficient and running the Fisher Transformation Hypothesis Test and the Students t-test to test the independence of the Pre-Tournament Rankings and the Final Results analyzed by the rules of Methods A and B.

In **Figure 1**, the Pre-Tournament Rankings and Final Results using Method A. The Pearson Correlation Coefficient (r) was computed by taking the square root of R^2 to obtain $r = 0.405$. Next, a Fisher Transformation Hypothesis Test is done using a two-tailed test to test if $r \neq 0$. Thus, in running the test, $\mathrm{F}(0.405) = \frac{1}{2}ln(2.361) = 0.430$, with $n = 216$, and $z = (0.430)\sqrt{213} = 6.270$. Hence, there is a correlation between the Pre-Tournament Rankings and Final Results using Method A. To further provide evidence of the existence of a correlation between Pre-Tournament Rankings and Final Results using Method A, a Students t-test was run and gave the following data: $t = 6.006$, $r = 0.380$, and $d.f. = 214$. Similarly, the probability of this happening by chance is found to be roughly 0, and this provides further evidence of a correlation between Pre-Tournament Rankings and Final Results using Method A.

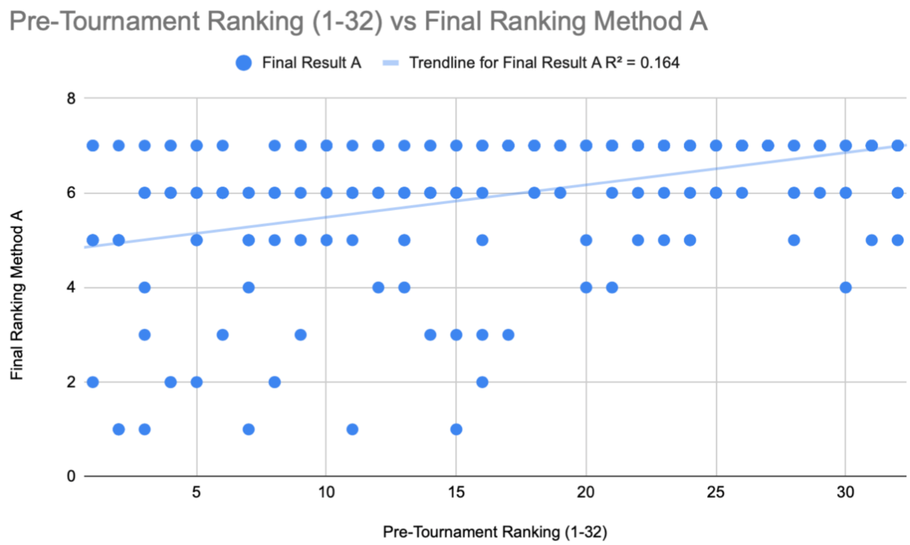

FIGURE 1.

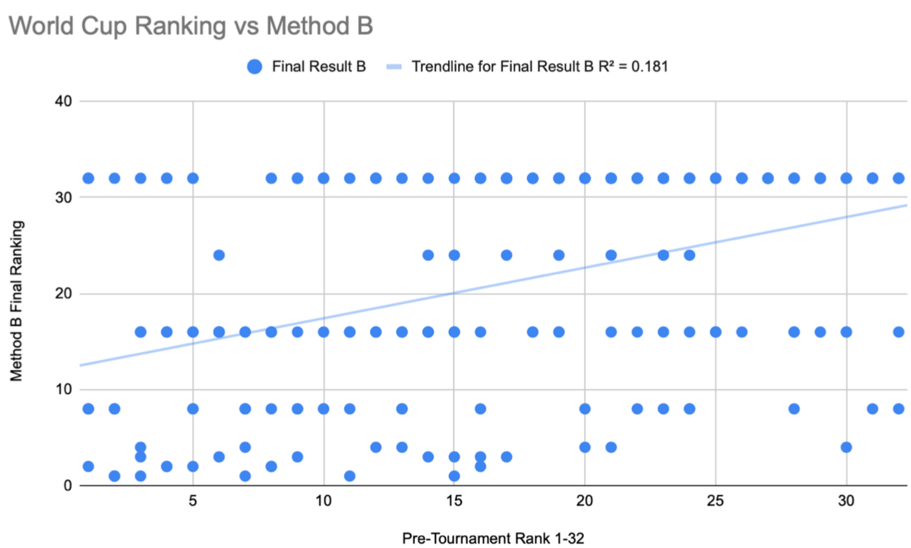

FIGURE 2.

To determine which method was better at predicting the outcome of the World Cup, the same steps were taken in regards to Method B. In **Figure 2**, the Pre-Tournament Rankings and Final Results using Method B were analyzed via a scatter plot. As before, the Pearson Correlation Coefficient (r) was computed by taking the square root of R^2 to get $r = 0.425$. The Fisher Transformation Hypothesis Test is done using a two-tailed test to test if $r \neq 0$. Thus, we find $F(0.425) = \frac{1}{2}ln(2.489) = 0.454$, $n = 216$, and $z = (0.454)\sqrt{213} = 6.631$. Hence, we find there is a correlation between the Pre-Tournament Rankings and Final Results using Method B. To further provide evidence of the existence of a correlation between Pre-Tournament Rankings and Final Results using Method B, a Students t-test was run and gave the following data: $t = 6.878, r = 0.425$, and $d.f. = 214$. Likewise the probability of this happening by chance is found to be roughly 0, and this provides evidence of a correlation between Pre-Tournament Rankings and Final Results using Method B.

4.2. **Womens World Cup Analysis.**

4.2.1. *Tests of Association.* In a similar fashion to the Mens World Cups, the Womens World Cups are analyzed first by using the Fisher Exact Test to determine if there is a nonrandom association between the number of teams in the Top 12 that advanced to the Round of 8 or 16 against the teams in the lower 12 based on each of the Womens World Cup formats.

Table 3: Tests of Association

Advancement from Group Stage in FIFA World Cup

Type	FIFA World Cup 2003, 2007, and 2011	FIFA World Cup 2015 and 2019	Total
Top 12 Ranked Teams Advancing to the Round of 8 or 16	24 (20.143)	23 (26.857)	47
Lower Ranked Teams Advancing to the Round of 8 or 16	0 (3.857)	9 (5.146)	9
Total	24	32	56

Note: The numbers in parenthesis are the expected counts. We find the two-tailed p-value to be $p = 0.007$. From this we find that there is a statistically significant association between the number of teams that were ranked in the top 12 of the World Cup that advanced to the Round of 8 or 16 against the number of teams that were ranked in the lower 12 of the World Cup that advanced.

Proceeding as we did before, we use a Chi-Squared test of association to compute the Chi-Square value. We found the value to be 8.043 with degrees of freedom $d.f. = 1$, and a p-value $(p) = 0.005$. Thus, there is a statistically significant association between the number of teams that were ranked in the top 12 of the World Cup that advanced to the Round of 8 or 16 against the number of teams that were ranked in the lower 12 of the World Cup that advanced. This is consistent with our finding using the Fisher Exact Test.

Another Chi-Squared test of association was run using the win, loss, and draw data from matches played of teams ranked in the top 12 against those from outside of the top 12 based on each of the Womens World Cup formats.

Table 4: Tests of Association			
Win/Loss/Draw Record Comparison			
Type	FIFA World Cup 2003, 2007, and 2011	FIFA World Cup 2015 and 2019	Total
Win	31 (29.494)	44 (45.506)	75
Loss	0 (1.180)	3 (1.82)	3
Draw	4 (4.326)	7 (6.674)	11
Total	35	54	89

Note: The numbers in parenthesis are the expected counts. We computed the Chi-Squared value as 2.112 with a p-value of $p = 0.348$ and $d.f. = 2$. Thus we find that there is not a statistically significant association between the win, loss, draw records of the teams ranked in the top 12 against those from outside of the top 12.

4.2.2. *Rules Analysis.* Using the Chi-Square test and Fisher Exact test as a baseline, the rules ranking the teams described were analyzed by first calculating the Pearson Correlation Coefficient and then running Fisher Transformation Hypothesis Test and the Students t-test to test the independence of the Pre-Tournament Rankings and the Final Results analyzed by the rules of Methods A and B.

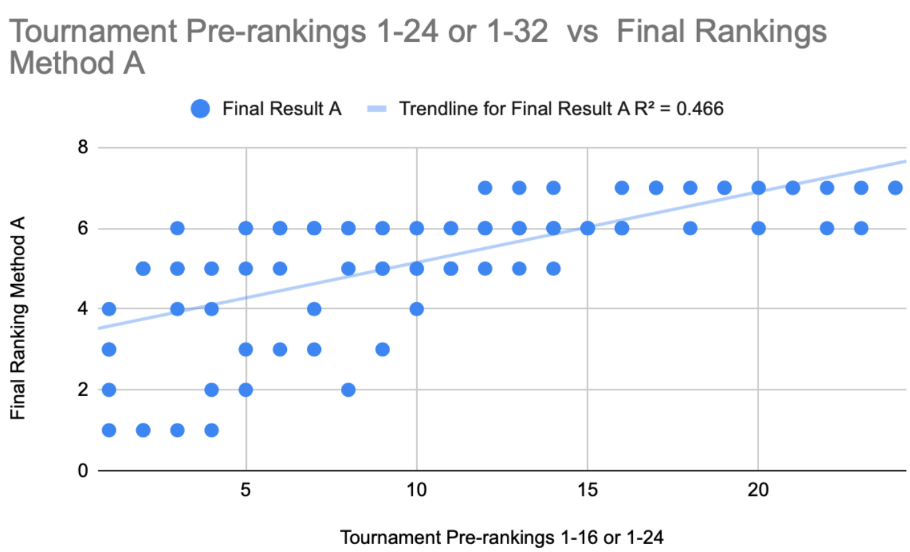

FIGURE 3.

In **Figure 3**, the Pre-Tournament Rankings and Final Results using Method A were analyzed via a scatter plot. The Pearson Correlation Coefficient (r) was computed by taking the square root of R^2 to get $r = 0.683$. Next, a Fisher Transformation Hypothesis Test is done using a two-tailed test to test if $r \neq 0$. The Fisher Hypothesis Test results in F(0.683)$= 0.835$, with $z = 8.052$, and $n = 96$.

This leaves the study with a critical value that is less than 0.05 since $z = 8.052$, which has a p-value of approximately 0. Hence, there is a correlation between the Pre-Tournament Rankings and Final Results using Method A. To further support our claim that there is a correlation between Pre-Tournament Rankings and Final Results using Method A, a Students t-test was run and gave the following data: $t = 9.062, r = 0.683$, and $d.f.$=94. Similarly the probability of this happening due to chance is found to be approximately 0, and this provides evidence of a correlation between Pre-Tournament Rankings and Final Results using Method A.

In order to discuss which method is better in analyzing the outcome of the World Cup, the same analysis was conducted using the rules described in Method B.

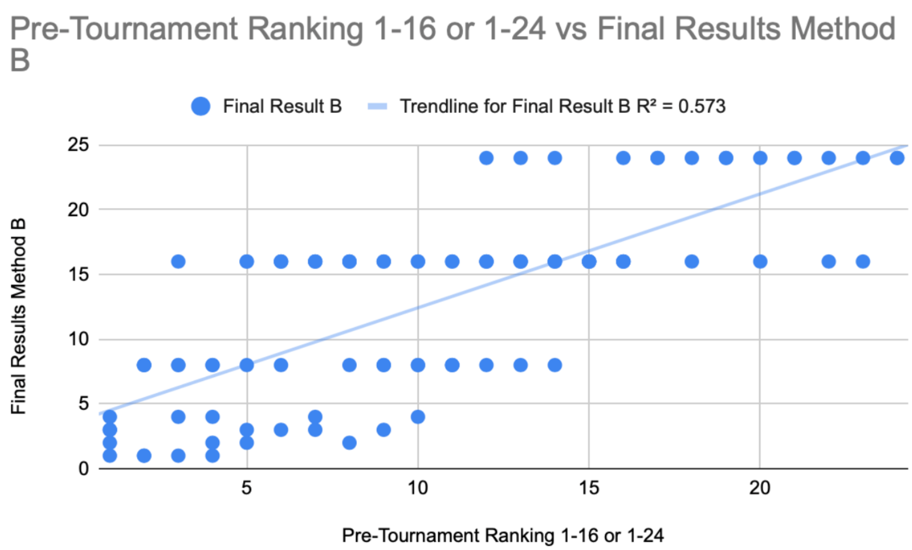

FIGURE 4.

The Pre-Tournament Rankings and Final Results using Method B were analyzed via a scatter plot. From here, the Pearson Correlation Coefficient (r) was computed by taking the square root of R^2 to get $r = 0.757$. Next, a Fisher Transformation Hypothesis Test is done using a two-tailed test to test if $r \neq 0$. The Fisher Transformation F(0.757) = 0.989, $z = 9.538$, and $n = 96$. This leaves the study with a critical value that is less than 0.05 since $z = 9.538$, which has a p-value of approximately 0. Hence, there is a correlation between the Pre-Tournament Rankings and Final Results using Method B. To further support our claim that there is a correlation between Pre-Tournament Rankings and Final Results using Method B, a Students t-test was run and gave the following data: $t = 11.232, r = 0.757$, and $d.f. = 94$. Similarly, the probability of this happening by chance is found to be roughly 0, and this provides further support of our claim of a correlation between Pre-Tournament Rankings and Final Results using Method B.

5. Discussion

Our objective was to determine which FIFA/Coca-Cola World Ranking system was better at predicting the Mens or Womens World Cup winner when changes in the ranking and FIFA World Cup format and rules based Final results ranking were taken into account. Based on **Table 1** and **Table 3**, the majority of teams advancing to the Round of 8 or 16 were teams that were in the top 12 or 16 of the Pre-Tournament Rankings. This proves to be a good indicator because each FIFA World Cup Champion has been ranked inside the top 12 or 16 in the rankings prior to each tournament. One can argue that if the FIFA/Coca-Cola World Ranking system was completely accurate at predicting the World Cup, then 100% of the teams in the top 16 or 12 of the rankings should be in the World Cup and advance to the Round of 16 or 8. However, this is not the case because each region has different allotments for teams that qualify to the World Cup, and sometimes a higher ranked team doesnt qualify for the FIFA World Cup. For that reason, teams were then re-ranked based on the qualified teams for the FIFA World Cup. Additionally, FIFA does not have a way in drawing the groups such that each of the top 12 or 16 teams do not end up in the same group. For that reason, there are instances where a group may have had three or four teams in the top 12 or 16 with only the top two advancing, while others may only have one team. This would mean that there are some groups where a team from outside of the top 16 would be guaranteed to make the Round of 16 or 8, such as in Group A of the 2018 World Cup where Uruguay and Egypt were predicted to advance, but Uruguay was the only team in the Top 16 of the Pre-Tournament Rankings, meaning the other prediction would not be as accurate and subsequently was not as Russia advanced.

For this reason, winning percentages are also taken into account when determining the validity of the FIFA/Coca-Cola World Rankings. With the top 16 teams of the Mens World Cup having an overall winning percentage of 73.8% and the top 12 teams of the Womens World Cup having a winning percentage of 90.4%, the trend is that the top 16 or 12 teams often beat teams outside of the top 16 or 12. Based on our analysis using the Fisher Transformation Hypothesis Test and the Student t-test, we found evidence that a correlation between the Pre-tournament rankings and the final outcomes of the FIFA World Cup was present using both ranking methods. This implies that the FIFA/Coca-Cola World Rankings are a reliable predictor of World Cup outcomes, to an extent.

Having shown that the FIFA/Coca-Cola World Ranking, to an extent, are a predictor of the FIFA Mens and Womens World Cup winners, it became an objective to find which World Cup it predicted better. First, from Table 1 and Table 3, the Womens World Cup has 83.9% of their top 12 teams advancing to the Round of 8 or 16 in comparison to the 74.1% of the Mens World Cup top 16 teams advancing to the Round of 16. Our analysis demonstrates that the Womens World Cup has statistically significant values from the Fisher Exact test and Chi-Square test for association, while the Mens World Cup does not. This implies that there is a statistically significant association between the number of teams that were ranked in the top 12 of the Womens World Cup that advanced to the Round of 8 or 16 against the number of teams that were ranked in the lower 12 of the Womens World Cup that advanced. Similarly, in **Table 2** and **Table 4**, Womens World Cup top 12 teams have a winning percentage of 90.4% in comparison to the 73.8% winning percentage of the top 16 Mens World Cup teams against those outside of the top 12 or 16. Furthermore, while the win, loss, and draw records did not have statistically significant results, the Womens World Cup had lower Chi-Square test for association values. This shows the Womens World Cup win, loss, and draw values for teams in the top 12 against those outside of the top 12 is more statistically significant than the match results of the teams in the top 16 of the Mens World Cup against those who are not. When doing the rules analysis, Method B has higher Pearson Correlation Coefficients in **Figure 2** and **Figure 4** when compared with their corresponding graph (0.425 vs 0.405 and 0.757 vs 0.683), but the Womens results are more statistically significant. For example, when comparing **Figure 1** and **Figure 3**, the Mens World Cup has a lower t-value (6.006 vs 9.062), z-score (6.269 vs 8.052), Pearson Correlation Coefficient (0.405 vs 0.683) than the corresponding Womens World Cup values. Moreover, in **Figure 2** and **Figure 4**, the Mens World Cup has a lower t-value (6.630 vs 11.232), z-score (6.878 vs 9.538), Pearson Correlation Coefficient (0.425 vs 0.757) than the corresponding Womens World Cup values. This demonstrates that the Womens World Cup is more predictable than the Mens World Cup.

6. Conclusion

Based on our analysis, we have established statistical justification to the claim that the Womens World Cup is more predictable than the Mens World Cup based on the Final Results using Methods A and B and the Pre-Tournament Rankings from the FIFA/Coca-Cola World Rankings. However, there are some crucial differences between the Mens and Womens World Cups outside of the ranking structure and World Cup format that this study did not consider such as the qualifying formats (this sets the field of teams that will participate in the FIFA World Cup), different playing surfaces (Womens World Cup are sometimes played on artificial turf while the Mens World Cups are not), the differences in prize money ($400 million for the Mens World Cup and $30 million for the Womens World Cup), and differences in accommodations (Womens teams are forced to share hotel accommodations while Mens teams do not). For example, all of these factors may impact the results of this study and were not taken into consideration (Prahl, 2019). For that reason, as the FIFA Womens soccer game grows worldwide and fights for equal pay such as that undertaken by the United States Womens National Team (USWNT), the results of this study may change. Like the Womens World Cup, the Mens will soon be changing as the 2026 FIFA Mens World Cup will feature 48 teams in 16 groups of three where the top two teams from each group will progress through to a 32-team knockout stage (FIFA, 2017). This would potentially weaken the results of this model as the methods of ranking and the format of the World Cups are ever changing and this model only utilized two rules for determining the final ranking. However, our results are consistent with other studies that have used similar ranking procedures and have found the process to be reliable. The results of our analysis pointed to the Womens game having less independence between the final results and pre-tournament ranking. Thus, under the current format, we conclude that the Womens World Cup is more predictable than the Mens World Cup.

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7. Appendix

Table 1: FIFA 1994 Mens World Cup

FIFA 1994 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Germany	5	8
2	2	Netherlands	5	8
3	3	Brazil	1	1
4	4	Italy	2	2
5	5	Spain	5	8
6	6	Norway	7	24
7	7	Romania	5	8

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
8	8	Argentina	6	16
10	9	Sweden	3	3
11	10	Nigeria	6	16
12	11	Switzerland	6	16
14	12	Republic of Ireland	6	16
16	13	Mexico	6	16
17	14	Colombia	7	24
19	15	Russia	7	24
23	16	United States	6	16
24	17	Cameroon	7	24
27	18	Belgium	6	16
28	19	Morocco	7	24
29	20	Bulgaria	4	4
31	21	Greece	7	24
34	22	Saudi Arabia	6	16
37	23	South Korea	7	24
43	24	Bolivia	7	24
End of Table				

FIFA 1994 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Romania, Switzerland	Romania, Switzerland
B	Brazil, Sweden	Brazil, Sweden
C	Germany, Spain	Germany, Spain
D	Argentina, Nigeria	Nigeria, (Bulgaria)
E	Italy, Norway	(Mexico), (Ireland)
F	Netherlands, (Belgium)	Netherlands, Belgium

Note: Parenthesis in the Suggested outcome by rules column is representative of an outside of the top 16 that is predicted to move on, while parenthesis around teams in Actual Outcome by Rules are results that were different than predicted results.

Table 2: FIFA 1998 Mens World Cup

FIFA 1998 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Brazil	2	2
2	2	Germany	5	8
4	3	Mexico	6	16
5	4	England	6	16

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
6	5	Argentina	5	8
7	6	Norway	6	16
8	7	Yugoslavia	6	16
9	8	Chile	6	16
10	9	Colombia	7	32
11	10	United States	7	32
12	11	Japan	7	32
13	12	Morocco	7	32
14	13	Italy	5	8
15	14	Spain	7	32
18	15	France	1	1
19	16	Croatia	3	3
20	17	South Korea	7	32
21	18	Tunisia	7	32
22	19	Romania	6	16
24	20	South Africa	7	32
25	21	Netherlands	4	4
27	22	Denmark	5	8
29	23	Paraguay	6	16
30	24	Jamaica	7	32
31	25	Austria	7	32
34	26	Saudi Arabia	7	32
35	27	Bulgaria	7	32
36	28	Belgium	7	32
41	29	Scotland	7	32
42	30	Iran	7	32
49	31	Cameroon	7	32
74	32	Nigeria	6	16
End of Table				

FIFA 1998 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Brazil, Norway	Brazil, Norway
B	Italy, Chile	Italy, Chile
C	France, (South Africa)	France, (Denmark)
D	Spain, (Paraguay)	(Nigeria), Paraguay
E	Mexico, (South Korea)	Netherlands, (Mexico)
F	Germany, Yugoslavia	Germany, Yugoslavia
G	England, Colombia	(Romania), England
H	Argentina, Japan	Argentina, (Croatia)

Table 3: FIFA 2002 Mens World Cup

FIFA 2002 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	France	7	32
2	2	Brazil	1	1
3	3	Argentina	7	32
5	4	Portugal	7	32
6	5	Italy	6	16
7	6	Mexico	6	16
8	7	Spain	5	8
11	8	Germany	2	2
12	9	England	5	8
13	10	United States	5	8
15	11	Republic of Ireland	6	16
17	12	Cameroon	7	32
18	13	Paraguay	6	16
19	14	Sweden	6	16
20	15	Denmark	6	16
21	16	Croatia	7	32
22	17	Turkey	3	3
23	18	Belgium	6	16
24	19	Uruguay	7	32
25	20	Slovenia	7	32
27	21	Nigeria	7	32
28	22	Russia	7	32
29	23	Costa Rica	7	32

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
31	24	Tunisia	7	32
32	25	Japan	6	16
34	26	Saudi Arabia	7	32
36	27	Ecuador	7	32
37	28	South Africa	7	32
38	29	Poland	7	32
40	30	South Korea	4	4
42	31	Senegal	5	8
50	32	China	7	32
End of Table				

FIFA 2002 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	France, Denmark	Denmark, (Senegal)
B	Spain, Paraguay	Spain, Paraguay
C	Brazil, (Turkey)	Brazil, Turkey
D	Portugal, United States	(South Korea), United States
E	Germany, Ireland	Germany, Ireland
F	Argentina, England	(Sweden), England
G	Mexico, Italy	Mexico, Italy
H	(Belgium),(Japan)	Belgium, Japan

Table 4: FIFA 2006 Mens World Cup

FIFA 2006 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Brazil	5	8
2	2	Czech Republic	7	32
3	3	Netherlands	6	16
4	4	Mexico	6	16
5	5	United States	7	32
5	6	Spain	6	16
7	7	Portugal	4	4
8	8	France	2	2
9	9	Argentina	5	8
10	10	England	5	8
13	11	Italy	1	1
16	12	Sweden	6	16
18	13	Japan	7	32
19	14	Germany	3	3

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
21	15	Tunisia	7	32
23	16	Iran	7	32
23	17	Croatia	7	32
26	18	Costa Rica	7	32
29	19	South Korea	7	32
29	20	Poland	7	32
32	21	Ivory Coast	7	32
33	22	Paraguay	7	32
34	23	Saudi Arabia	7	32
35	24	Switzerland	6	16
39	25	Ecuador	6	16
42	26	Australia	6	16
44	27	Serbia and Montenegro	7	32
45	28	Ukraine	5	8
47	29	Trinidad and Tobago	7	32
48	30	Ghana	6	16
57	31	Angola	7	32
61	32	Togo	7	32
End of Table				

FIFA 2006 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Germany, (Costa Rica)	Germany, (Ecuador)
B	England, Sweden	England, Sweden
C	Argentina, Netherlands	Argentina, Netherlands
D	Portugal, Mexico	Portugal, Mexico
E	Czech Republic, United States	(Italy), (Ghana)
F	Brazil, Japan	Brazil, (Australia)
G	France, (South Korea)	France, (Switzerland)
H	Spain, Tunisia	Spain, (Ukraine)

Table 5: FIFA 2010 Mens World Cup

FIFA 2010 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Brazil	5	8

Continuation of Table 12

FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
2	2	Spain	1	1
3	3	Portugal	6	16
4	4	Netherlands	2	2
5	5	Italy	7	32
6	6	Germany	3	3
7	7	Argentina	5	8
8	8	England	6	16
9	9	France	7	32
13	10	Greece	7	32
14	11	United States	6	16
15	12	Serbia	7	32
16	13	Uruguay	4	4
17	14	Mexico	6	16
18	15	Chile	6	16
19	16	Cameroon	7	32
20	17	Australia	7	32
21	18	Nigeria	7	32
24	19	Switzerland	7	32
25	20	Slovenia	7	32
27	21	Ivory Coast	7	32
30	22	Algeria	7	32
31	23	Paraguay	5	8
32	24	Ghana	5	8
34	25	Slovakia	6	16
36	26	Denmark	7	32
38	27	Honduras	7	32
45	28	Japan	6	16
47	29	South Korea	6	16
78	30	New Zealand	7	32
83	31	South Africa	7	32
105	32	North Korea	7	32

End of Table

FIFA 2010 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	France, Uruguay	Uruguay, (Mexico)
B	Argentina, Greece	Argentina, (South Korea)
C	United States, England	United States, England
D	Germany, Serbia	Germany, (Ghana)
E	Netherlands, Cameroon	Netherlands, (Japan)
F	Italy, (Paraguay)	Paraguay, (Slovakia)
G	Brazil, Portugal	Brazil, Portugal
H	Spain, Chile	Spain, Chile

Table 6: FIFA 2014 Mens World Cup

FIFA 2014 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Spain	7	32
2	2	Germany	1	1
3	3	Brazil	4	4
4	4	Portugal	7	32
5	5	Argentina	2	2
6	6	Switzerland	6	16
7	7	Uruguay	6	16
8	8	Colombia	5	8
9	9	Italy	7	32
10	10	England	7	32
11	11	Belgium	5	8
12	12	Greece	6	16
13	13	United States	6	16
14	14	Chile	6	16
15	15	Netherlands	3	3
17	16	France	5	8
18	17	Croatia	7	32
19	18	Russia	7	32
20	19	Mexico	6	16
21	20	Bosnia and Herzegovina	7	32
22	21	Algeria	6	16
23	22	Ivory Coast	7	32
26	23	Ecuador	7	32
28	24	Costa Rica	5	8
33	25	Honduras	7	32

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
37	26	Ghana	7	32
43	27	Iran	7	32
44	28	Nigeria	6	16
46	29	Japan	7	32
56	30	Cameroon	7	32
57	31	South Korea	7	32
62	32	Australia	7	32
End of Table				

FIFA 2014 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Brazil, (Croatia)	Brazil, (Mexico)
B	Spain, Chile	Chile, (Netherlands)
C	Colombia, Greece	Colombia, Greece
D	Uruguay, England	(Costa Rica), Uruguay
E	France, Switzerland	France, Switzerland
F	Argentina, (Bosnia and Herzegovina)	Argentina, (Nigeria)
G	Germany, Portugal	Germany, (United States)
H	Belgium, (Russia)	Belgium, (Algeria)

Table 7: FIFA 2018 Mens World Cup

FIFA 2018 Mens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Germany	7	32
2	2	Brazil	5	8
3	3	Belgium	3	3
4	4	Portugal	6	16
5	5	Argentina	6	16
6	6	Switzerland	6	16
7	7	France	1	1
8	8	Poland	7	32
10	9	Spain	6	16
11	10	Peru	7	32
12	11	Denmark	6	16
12	12	England	4	4
14	13	Uruguay	5	8

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
15	14	Mexico	6	16
16	15	Colombia	6	16
20	16	Croatia	2	2
21	17	Tunisia	7	32
22	18	Iceland	7	32
23	19	Costa Rica	7	32
24	20	Sweden	5	8
27	21	Senegal	7	32
34	22	Serbia	7	32
36	23	Australia	7	32
37	24	Iran	7	32
41	25	Morocco	7	32
45	26	Egypt	7	32
48	27	Nigeria	7	32
55	28	Panama	7	32
57	29	South Korea	7	32
61	30	Japan	6	16
67	31	Saudi Arabia	7	32
70	32	Russia	5	8
End of Table				

FIFA 2018 Mens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Uruguay, (Egypt)	Russia, (Uruguay)
B	Spain, Portugal	Spain, Portugal
C	France, Peru	France, (Denmark)
D	Croatia, Argentina	Croatia, Argentina
E	Brazil, Switzerland	Brazil, Switzerland
F	Germany, Mexico	(Sweden), Mexico
G	Belgium, England	Belgium, England
H	Argentina, Japan	Argentina, Japan

Table 8: FIFA 2003 Womens World Cup

FIFA 2003 Womens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	United States	3	3
2	2	Norway	5	8
3	3	Germany	1	1
4	4	China	5	8
5	5	Sweden	2	2
6	6	Brazil	5	8

Continuation of Table 12				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
7	7	North Korea	6	16
9	8	France	6	16
11	9	Russia	5	8
12	10	Canada	4	4
14	11	Japan	6	16
15	12	Australia	6	16
23	13	Nigeria	6	16
25	14	South Korea	6	16
35	15	Argentina	6	16
53	16	Ghana	6	16
End of Table				

FIFA 2003 Womens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	United States, Sweden	United States, Sweden
B	Norway, Brazil	Norway, Brazil
C	Germany, Canada	Germany, Canada
D	China, Russia	China, Russia

Table 9: FIFA 2007 Womens World Cup

FIFA 2007 Womens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	United States	3	3
2	2	Germany	1	1
3	3	Sweden	6	16
4	4	Norway	4	4
5	5	North Korea	5	8
6	6	Denmark	6	16
8	7	Brazil	3	3
9	8	Canada	6	16
10	9	Japan	6	16
11	10	China	5	8
12	11	England	5	8
15	12	Australia	5	8
23	13	New Zealand	6	16
24	14	Nigeria	6	16
29	15	Argentina	6	16
47	16	Ghana	6	16
End of Table				

FIFA 2007 Womens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Germany, Japan	Germany, (England)
B	United States, Sweden	United States, (North Korea)
C	Norway, Canada	Norway, (Australia)
D	Denmark, Brazil	Brazil, (China)

Table 10: FIFA 2011 Womens World Cup

FIFA 2011 Womens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	United States	2	2
2	2	Germany	5	8
3	3	Brazil	5	8
4	4	Japan	1	1
5	5	Sweden	3	3
6	6	Canada	6	16
7	7	France	4	4
8	8	North Korea	6	16
9	9	Norway	6	16
10	10	England	5	8
11	11	Australia	5	8
22	12	Mexico	6	16
24	13	New Zealand	6	16
27	14	Nigeria	6	16
31	15	Colombia	6	16
61	16	Equatorial Guinea	6	16
End of Table				

FIFA 2011 Womens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Germany, Canada	Germany, (France)
B	England, Japan	England, Japan
C	Sweden, United States	Sweden, United States
D	Brazil, Norway	Brazil, (Australia)

Table 11: FIFA 2015 Womens World Cup

FIFA 2015 Womens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	Germany	4	4
2	2	United States	1	1
3	3	France	5	8
4	4	Japan	2	2
5	5	Sweden	6	16
6	6	England	3	3
7	7	Brazil	6	16
8	8	Canada	5	8
10	9	Australia	5	8
11	10	Norway	6	16
12	11	Netherlands	6	16
14	12	Spain	7	24
16	13	China	5	8
17	14	New Zealand	7	24
18	15	South Korea	6	16
19	16	Switzerland	6	16
25	17	Mexico	7	24
28	18	Colombia	6	16
29	19	Thailand	7	24
33	20	Nigeria	7	24
37	21	Costa Rica	7	24
48	22	Ecuador	7	24
53	23	Cameroon	6	16
67	24	Ivory Coast	7	24
End of Table				

FIFA 2015 Womens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	Canada, Netherlands	Canada, (China)
B	Germany, Norway	Germany, Norway
C	Japan, (Switzerland)	Japan, (Cameroon)
D	United States, Sweden	United States, (Australia)
E	Brazil, Spain	Brazil, (South Korea)
F	France, England	France, England

Table 12: FIFA 2019 Womens World Cup

FIFA 2019 Womens World Cup				
FIFA Ranking	World Cup Ranking	Country	Final Result A	Final Result B
1	1	United States	1	1
2	2	Germany	5	8
3	3	England	4	4
4	4	France	5	8
5	5	Canada	6	16
6	6	Australia	6	16
7	7	Japan	6	16
8	8	Netherlands	2	2
9	9	Sweden	3	3
10	10	Brazil	6	16
12	11	Norway	5	8
13	12	Spain	6	16
14	13	South Korea	7	24
15	14	Italy	5	8
16	15	China	6	16
19	16	New Zealand	7	24
20	17	Scotland	7	24
34	18	Thailand	7	24
37	19	Argentina	7	24
38	20	Nigeria	6	16
39	21	Chile	7	24
46	22	Cameroon	6	16
49	23	South Africa	7	24
53	24	Jamaica	7	24
End of Table				

FIFA 2019 Womens World Cup Suggested Outcomes		
Group	Suggested Outcome by Rules	Actual Outcome by rules
A	France, Norway	France, Norway
B	Germany, Spain	Germany, Spain
C	Brazil, Australia	(Italy), Australia
D	England, Japan	England, Japan
E	Netherlands, Canada	Netherlands, Canada
F	United States, Sweden	United States, Sweden

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