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ON GLM TYPE INTEGRAL EQUATION FOR SINGULAR STURM-LIOUVILLE OPERATOR WHICH HAS DISCONTINUOUS COEFFICIENT

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Abstract. In this study, we derive Gelfand-Levitan-Marchenko type main integral equation of the inverse problem for singular Sturm-Liouville equation which has discontinuous coefficient. Then we prove the unique solvability of the main integral equation.

1. Introduction

We consider boundary value problem L as follows:

$$-y'' + \left[\frac{A}{x} + q(x)\right] y = \lambda^2 \rho(x) y, \quad x \in I = (0,d) \cup (d,\pi), \tag{1}$$

$$U(y) := y(0) = 0, V(y) := y(\pi) = 0 \tag{2}$$

where λ is spectral parameter, $A \in \mathbb{R}^+$, $\rho(x) = \begin{cases} 1, & 0 \leq x \leq d \\ \alpha^2, & d < x \leq \pi \end{cases}$, $\alpha \in R$, $\alpha \neq 1$, $\alpha > 0$, $d \in \left(\frac{\pi}{2}, \pi\right)$, $q(x)$ is a real valued bounded function and $q(x) \in L_2(0,\pi)$.

Boundary value problems with discontinuous coefficient often appear in applied mathematics, geophysics, mechanics, electromagnetics, elasticity and other branches of engineering and physics. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to the geosciences. For example, torodial vibrations and free vibrations of the earth, reconstructing the

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discontinuous material properties of a nonabsorbing media, as a rule leads to direct and inverse problems or the Sturm-Liouville equation which has discontinuous coefficient. (see [1]- [7]) Discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [6]. After reducing corresponding mathematical model we come to boundary value problem L where $q(x)$ must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium [7], [1]. Boundary value problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth [2]. Here, the main discontinuity is cased by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behaviour of solutions for such nonlinear equations. We also note that inverse problem considered here appears in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators.

Sturm-Liouville operators with singular potential were studied in [8]- [10]. In [10], Sturm-Liouville operators generated by the differential expression $-y''+q(x)\,y$ were considered. Here $q(x)$ is a distribution of first order, i.e., $\int q(x)\,dx \in L_2[0,\pi]$. The minimal and maximal operators corresponding to potentials of this type on a finite interval were constructed in [8]. All self-adjoint extensions of the minimal operator were described and the asymptotics of the eigenvalues of these extensions were found there.

The authors in [11]- [14] study asymptotics of eigenvalue, eigenfunctions and normalizing numbers and solve the inverse spectral problems of recovering the singular potential $q \in W_2^{-1}(0,1)$ of Sturm-Liouville operators by two spectra. The reconstruction algorithm is presented and necessary and sufficient conditions on two sequences to be spectral data Sturm-Liouville operators under consideration are given. Unlike these studies, the proposed method in our work is more practical and more feasible.

In this study, we derive the Gelfand-Levitan-Marchenko type main integral equation of the inverse problem for singular Sturm-Liouville equation which has discontinuous coefficient. Then we prove the unique solvability of the main integral equation.

In [15] and [16] , we defined $y_1(x) = y(x)\,, y_2(x) = (\Gamma y)(x) = y'(x) - u(x)\,y(x)\,,$ $u(x) = A\ln x$ and got the expression of left hand side of the equation (1) as follows

$$\ell(y) = -\left[(\Gamma y)(x)\right]' - u(x)(\Gamma y)(x) - u^2(x)\,y + q(x)\,y = \lambda^2\rho(x)y, \tag{3}$$

then the equation (1) reduced to the system;

$$\begin{cases} y_1' - y_2 = u(x)\,y_1 \\ y_2' + \lambda^2\rho(x)y_1 = -u(x)\,y_2 - u^2(x)\,y_1 + q(x)\,y_1 \end{cases} \tag{4}$$

with the boundary conditions

$$y_1(0) = 0, y_1(\pi) = 0. \tag{5}$$

Matrix form of system (4)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -\lambda^2\rho(x) - u^2 + q & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{6}$$

or $y' = Ay$ such that $A = \begin{pmatrix} u(x) & 1 \\ -\lambda^2\rho(x) - u^2(x) + q(x) & -u(x) \end{pmatrix}$, $y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

$x = 0$ is a regular-singular end point for equation (4) and Theorem 2 in [17] (see Remark 1-2, p.56) extends to interval $[0, \pi]$. For this reason, by [17], there exists only one solution of the system (2) which satisfies the initial conditions $y_1(\xi) = \upsilon_1, y_2(\xi) = \upsilon_2$ for each $\xi \in [0, \pi], \upsilon = (\upsilon_1, \upsilon_2)^T \in C^2$, especially the initial conditions $y_1(0) = 1, y_2(0) = i\lambda$.

Definition 1. *The first component of the solution of the system (4) which satisfies the initial conditions $y_1(\xi) = \upsilon_1, y_2(\xi) = (\Gamma y)(\xi) = \upsilon_2$ is called the solution of the equation (1) which satisfies these same initial conditions.*

It was obtained in [3] by the successive approximations method that (see [18], [19]) the following theorem is true.

Theorem 1. *[3] For each solution of system (6) satisfying the initial conditions $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i\lambda \end{pmatrix}$ the following expression is true:*

$$\begin{cases} y_1 = e^{i\lambda x} + \int\limits_{-x}^{x} K_{11}(x,t) e^{i\lambda t} dt \\ y_2 = i\lambda e^{i\lambda x} + b(x) e^{i\lambda x} + \int\limits_{-x}^{x} K_{21}(x,t) e^{i\lambda t} dt + i\lambda \int\limits_{-x}^{x} K_{22}(x,t) e^{i\lambda t} dt \end{cases}, \quad x < d$$

$$\begin{cases} y_1 = \alpha^+ e^{i\lambda\mu^+(x)} + \alpha^- e^{i\lambda\mu^-(x)} + \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{11}(x,t) e^{i\lambda t} dt \\ y_2 = i\lambda\alpha \left(\alpha^+ e^{i\lambda\mu^+(x)} - \alpha^- e^{i\lambda\mu^-(x)}\right) \\ \qquad + b(x) \left[\alpha^+ e^{i\lambda\mu^+(x)} + \alpha^- e^{i\lambda\mu^-(x)}\right] \\ \quad + \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{21}(x,t) e^{i\lambda t} dt + i\lambda\alpha \int\limits_{-\mu^+(x)}^{\mu^+(x)} K_{22}(x,t) e^{i\lambda t} dt \end{cases}, \quad x > d$$

where

$$b(x) = -\frac{1}{2}\int_0^x \left[u^2(s) - q(s)\right] e^{-\frac{1}{2}\int_s^x u(t)dt} ds,$$

$$K_{11}(x,x) = \frac{\alpha^+}{2} u(x),$$

$$K_{21}(x,x) = b'(x) - \frac{1}{2}\int_0^x \left[u^2(s) - q(s)\right] K_{11}(s,s)\,ds - \frac{1}{2}\int_0^x u(s) K_{21}(s,s)\,ds,$$

$$K_{22}(x,x) = -\frac{\alpha^+}{2}\left[u(x) + 2b(x)\right],$$

$$K_{11}(x, 2d - x + 0) - K_{11}(x, 2d - x - 0) = \frac{\alpha^-}{2} u(x),$$

$$\frac{\partial K_{ij}(x,.)}{\partial x}, \frac{\partial K_{ij}(x,.)}{\partial t} \in L_2(0,\pi), i,j = 1,2,$$

$$\alpha^{\pm}(x) = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{\rho(x)}}\right), \quad \mu^{\pm}(x) = \pm x\sqrt{\rho(x)} + d\left(1 \pm \sqrt{\rho(x)}\right).$$

2. The Main Equation of the Inverse Problem

Assume that $s(x,\lambda)$ is solution of the equation (1) with initial condition

$$s(0,\lambda) = \begin{pmatrix} 0 \\ i\lambda \end{pmatrix}.$$

We have

$$s(x,\lambda) = s_0(x,\lambda) + \int_{\mu^-(x)}^{\mu^+(x)} K_{11}(x,t)\sin\lambda t dt,$$

where

$$s_0(x,\lambda) = \alpha^+(x)\sin\lambda\mu^+(x) + \alpha^-(x)\sin\lambda\mu^-(x).$$

Also, let us define α_n, α_n^0 and $\Phi_N(x,t)$ as follows:

$$\alpha_n = \int_0^\pi \rho(x)\, s^2(x,\lambda_n)\,dx,$$

$$\alpha_n^0 = \int_0^\pi \rho(x)\, s_0^2\left(x,\lambda_n^0\right)dx,$$

$$\Phi_N(x,t) = \Phi_{N_1}(x,t) + \Phi_{N_2}(x,t) + \Phi_{N_3}(x,t) + \Phi_{N_4}(x,t),$$

$$\Phi_N(x,t) = \sum_{n=0}^{N} \left(\frac{s(x,\lambda_n)\, s(t,\lambda_n)}{\alpha_n} - \frac{s_0\left(x,\lambda_n^0\right) s_0\left(t,\lambda_n^0\right)}{\alpha_n^0} \right),$$

$$\Phi_{N_1}(x,t) = \sum_{n=0}^{N} \left(\frac{s_0(x,\lambda_n)\, s_0(t,\lambda_n)}{\alpha_n} - \frac{s_0\left(x,\lambda_n^0\right) s_0\left(t,\lambda_n^0\right)}{\alpha_n^0} \right),$$

$$\Phi_{N_2}(x,t) = \int_0^{\mu^+(x)} K_{11}(x,\xi) \sum_{n=0}^{N} \frac{s_0\left(x,\lambda_n^0\right) \sin \lambda_n^0 \xi}{\alpha_n^0} d\xi,$$

$$\Phi_{N_3}(x,t) = \int_0^{\mu^+(x)} K_{11}(x,\xi) \sum_{n=0}^{N} \left(\frac{s_0(x,\lambda_n) \sin \lambda_n \xi}{\alpha_n} - \frac{s_0\left(x,\lambda_n^0\right) \sin \lambda_n^0 \xi}{\alpha_n^0} \right) d\xi,$$

$$\Phi_{N_4}(x,t) = \int_0^{\mu^+(x)} K_{11}(x,\xi) \sum_{n=0}^{N} \frac{s(x,\lambda_n) \sin \lambda_n \xi}{\alpha_n} d\xi.$$

Here, using

$$s_0(\xi,\lambda) = \begin{cases} \sin \lambda_n \xi, & \xi \le d \\ \frac{1}{2}\left(1+\frac{1}{\alpha}\right) \sin \lambda \mu^+(\xi) + \frac{1}{2}\left(1-\frac{1}{\alpha}\right) \sin \lambda \mu^+(\xi), & \xi > d \end{cases},$$

we have

$$s_0(\xi,\lambda) = \alpha^+ \sin \lambda \mu^+(\xi) + \alpha^- s_0 \lambda \left(2d - \mu^+(\xi)\right), \quad \xi > d.$$

Also, since $2d - \mu^+(\xi) < d$ we obtain

$$\sin \lambda \mu^+(\xi) = \frac{1}{\alpha^+} s_0(\xi,\lambda) - \frac{\alpha^-}{\alpha^+} s_0\left(2d - \mu^+(\xi), \lambda\right).$$

Substituting $\mu^+(\xi) \to \xi$, we get

$$\sin \lambda_n \xi = \begin{cases} s_0(\xi,\lambda), & \xi \le d \\ \frac{1}{\alpha^+} s_0(\xi,\lambda) - \frac{\alpha^-}{\alpha^+} s_0(2d - \mu^+(\xi), \lambda), & \xi > d \end{cases}. \tag{7}$$

Now, define $F_0(x,t)$ and $F(x,t)$ as follows:

$$F_0(x,t) = \sum_{n=0}^{\infty} \left[\frac{s_0(t,\lambda_n) \sin \lambda_n x}{\alpha_n} - \frac{s_0\left(t,\lambda_n^0\right) \sin \lambda_n^0 x}{\alpha_n^0} \right] \tag{8}$$

and

$$F(x,t) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right) F_0(\mu^+(x), t) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right) F_0(\mu^-(x), t). \tag{9}$$

We can write

$$F(x,t)=\sum_{n=0}^{\infty}\left[\frac{s_0\left(x,\lambda_n\right)s_0\left(t,\lambda_n\right)}{\alpha_n}-\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}\right]. \tag{10}$$

Let $f\in AC[0,\pi]$, using Theorem 6 in [4],

$$f(x)=\sum_{n=0}^{\infty}\int_0^{\pi}f(t)\rho\left(t\right)\frac{s\left(x,\lambda_n\right)s\left(t,\lambda_n\right)}{\alpha_n}dt \tag{11}$$

and

$$f(x)=\sum_{n=0}^{\infty}\int_0^{\pi}f(t)\rho\left(t\right)\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}dt, \tag{12}$$

we get

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\int_0^{\pi}f(t)\rho\left(t\right)\Phi_N(x,t)dt$$

$$\le\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^{\pi}f(t)\rho\left(t\right)\sum_{n=0}^{\infty}\frac{s\left(x,\lambda_n\right)s\left(t,\lambda_n\right)}{\alpha_n}dt-f(x)\right|,$$

$$\lim_{N\to\infty}\max_{0\le x\le\pi}\left|\int_0^{\pi}f(t)\rho\left(t\right)\sum_{n=0}^{\infty}\frac{s_0\left(x,\lambda_n^0\right)s_0\left(t,\lambda_n^0\right)}{\alpha_n^0}dt-f(x)\right|=0.$$

Furthermore, uniformly for $x\in[0,\pi]$,

$$\lim_{N\to\infty}\int_0^{\pi}f(t)\rho\left(t\right)\Phi_{N_1}(x,t)dt=\int_0^{\pi}f(t)\rho\left(t\right)F(x,t)dt. \tag{13}$$

Similarly, we have

$$\lim_{N\to\infty}\int_0^{\pi}f(t)\rho\left(t\right)\Phi_{N_2}(x,t)dt=\int_0^{d}f(t)K_{11}(x,t)dt+$$

$$\frac{1}{\alpha^+}\int_d^{x}f(t)K_{11}(x,\mu^+\left(t\right))dt-\frac{\alpha^-}{\alpha^+}\int_d^{x}f(t)K_{11}(x,\mu^+\left(2d-t\right))dt.$$

Because for $2d-t>\mu^+\left(x\right)$, $K(x,\mu^+\left(2d-t\right)\equiv 0$, we have

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_2}(x,t)dt=$$

$$\int_0^{x} f(t)K_{11}(x,\mu^+(t))\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}dt+\int_0^{x} f(t)K_{11}(x,\mu^+(2d-t))\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}dt$$

uniformly in $x\in[0,\pi]$.

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_3}(x,t)dt=\int_0^{\pi}\rho(t)\,f(t)\left(\int_0^{\mu^+(x)}K_{11}(x,\xi)\,F_0(\xi,t)d\xi\right)dt.$$

Using residue theorem, we get

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_4}(x,t)dt=$$

$$=2\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\frac{1}{2\pi i}\oint_{\Gamma_n}\left(\frac{\lambda}{\Delta(\lambda)}\int_0^{\mu^+(x)}K_{11}(x,\xi)\sin\lambda\xi\right)d\lambda dt$$

here $\Gamma_n=\{\lambda:|\lambda|=N\}$.

$$s(x,\lambda)=O\left(e^{|\operatorname{Im}\lambda|\left(\mu^+(\pi)-\mu^+(x)\right)}\right)$$

and

$$|\Delta(\lambda)|\geq C_\delta\,|\lambda|\,e^{\left|\operatorname{Im}\lambda\mu^+(x)\right|},\quad \lambda\in G_\delta$$

where $C_\delta>0,\ \ G_\delta=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta\right\}$, for all $\lambda\in G_\delta$, we get

$$\left|\frac{\lambda}{\Delta(\lambda)}\right|\leq\widetilde{C_\delta}e^{-|\operatorname{Im}\lambda|\left(\mu^+(x)-\mu^+(t)\right)}$$

where $\widetilde{C_\delta}>0$ is a constant. Using $\mu(t)<\mu^+(x)$, we have

$$\lim_{|\lambda|\to\infty}\max_{0\leq x\leq\pi}\frac{\lambda}{\Delta(\lambda)}=0.$$

By the way, due to Riemann-Lebesgue lemma, we can write

$$\lim_{N\to\infty}\int_0^{\pi} f(t)\rho(t)\,\Phi_{N_4}(x,t)dt=0.$$

If we use the last equations we obtain

$$\int_0^{\pi} f(t)\rho(t)\,F(x,t)dt + \int_0^{x} f(t)K_{11}(x,\mu^+(t))\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}dt +$$

$$\int_0^{x} f(t)K_{11}(x,\mu^+(2d-t))\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}dt + \int_0^{\pi} f(t)\rho(t)\int_0^{\mu^+(x)} K_{11}(x,\xi)\,F_0(\xi,t)d\xi dt = 0.$$

Since $f \in AC[0,\pi]$ is arbitrary, the following theorem could be proved:

Theorem 2. *For every fix $x \in (0,\pi)$, the kernel function $K_{11}(x,t)$ of the integral representation of the solution $\varphi(x,\lambda)$ satisfies the following linear-functional integral equation.*

$$\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}K_{11}(x,\mu^+(t)) + \frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{11}(x,\mu^+(2d-t)) + F(x,t) +$$

$$+\int_0^{\mu^+(x)} K_{11}(x,\xi)\,F_0(\xi,t)d\xi dt = 0, \tag{14}$$

where the functions $F_0(x,t)$ and $F(x,t)$ are defined by the formulas (8) and (9) respectively.

Theorem 3. *For every fix $x \in (0,\pi)$, the equation (14) has a unique solution $K_{11}(x,t)$, which belongs to $L_2(0,\pi)$.*

Proof. For $x \leq d$, equation (14) is written as follows:

$$K_{11}(x,t) + F(x,t) + \int_0^{x} K_{11}(x,\xi)\,F_0(\xi,t)d\xi = 0 \tag{15}$$

which is a Fredholm integral equation and equivalent to the equation of type

$$(I+B)\,f = g \tag{16}$$

where I is the unit operator, B is a compact operator in the space $L_2(0,\pi)$, $f, g \in L_2(0,\pi)$. Let us prove that in the case $x > d$ the equation (14) is also equivalent to an equation of type (16).

If $x > d$, the equation (14) can be written as

$$L_x K_{11}(x,.) + M_x K_{11}(x,.) = -F(x,.),$$

where

$$(L_x f)(t) = \frac{2}{1+\sqrt{\rho(t)}} f\left(\mu^+(t)\right) + \frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}} f(2d-t), \quad 0 < t < x, \tag{17}$$

$$(M_x f)(t) = \int_0^{\mu^+(x)} f(\xi) F_0(\xi,t) d\xi, \quad 0 < t < x. \tag{18}$$

It was shown in [5] that the operator L_x has a bounded inverse in the space $L_2(0,\pi)$ and

$$L_x^{-1} f(t) = \begin{cases} f(t) - \frac{1-\alpha}{2} f\left(\frac{-t+\alpha d+d}{\alpha}\right), & t < d \\ \frac{1+\alpha}{2} f\left(\frac{t+\alpha d-d}{\alpha}\right), & t > d \end{cases} \tag{19}$$

Therefore the equation (14) is equivalent to the equation

$$K_{11}(x,.) + L_x^{-1} M_x K_{11}(x,.) = -L_x^{-1} F(x,.). \tag{20}$$

Because L_x^{-1} is a bounded and M_x is a compact operator in $L_2(0,\pi)$, then the operator $B_x = L_x^{-1} M_x$ is compact in $L_2(0,\pi)$. The right hand side of (20) also belongs to $L_2(0,\pi)$, since M_x is invertible in $L_2(0,\pi)$. Consequently, the equation (20) is a Fredholm integral equation type (16) and it is sufficient to prove that the homogeneous equation

$$L_x K_{11}(x,.) + \int_0^{\mu^+(x)} K_{11}(x,\xi) F_0(\xi,t) d\xi = 0 \tag{21}$$

has only trivial solution $K_{11}(x,t) = 0$. Let $K(t) := K_{11}(x,t)$ be solution of equation (21). Then

$$\int_0^x \rho(t) [L_x K(t)]^2 dt + \int_0^x \rho(t) L_x K(t) dt \int_0^{\mu^+(x)} K(\xi) F_0(\xi,t) d\xi dt = 0. \tag{22}$$

Using for $\xi < \mu^-(x)$, $K(2d-\xi) = 0$ and the formulas (7) and (8), we have

$$L_x K(t) = \int_0^{\mu^+(x)} K(\xi) F_0(\xi,t) d\xi = \int_0^x \rho(\xi) L_x K(\xi) F(\xi,t) d\xi.$$

Therefore (20) can be written as

$$\int_0^x \rho(t) [L_x K(t)]^2 dt + \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^x \rho(t) s_0(t,\lambda_n) L_x K(t) dt \right)^2 \tag{23}$$

$$- \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left(\int_0^x \rho(t) s_0(t,\lambda_n^0) L_x K(t) dt \right)^2 = 0. \tag{24}$$

Now if we use Parseval's equality [4],

$$\int_0^x \rho(t) f^2(t)\, dt + \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left(\int_0^x \rho(t) f(t) s_0\left(t, \lambda_n^0\right) dt \right)^2,$$

for

$$f(t) = \begin{cases} L_x K(t), & 0 < t < x, \\ 0, & t > x \end{cases}$$

which belongs to $L_2(0,\pi)$, we have

$$\int_0^x \rho(t) (L_x K(t)) s_0(t, \lambda_n)\, dt = 0, \quad n \geq 0.$$

Since the system of function $\{s_0(t,\lambda_n)\}_{n\geq 0}$ is complete in $L_2(0,\pi)$ by the theorem in [3], we get $L_x K(t) = 0$. Since the operator L_x has inverse in the space $L_2(0,\pi)$, we obtain $K(t) \equiv K(x,.)$. It means that, the theorem is proved. □

Using Theorem 1 and the fact that the functions $\{s_0(t,\lambda_n)\}_{n\geq 0}$ is a Riesz basis of the space $L_2(0,\pi)$ (see [3]), we get the following theorem:

Theorem 4. *The spectral data* $\left\{\lambda_n^2, \alpha_n\right\}_{n\geq 0}$ *uniquely determines the boundary value problem* L.

The integral equation (14) is called main integral equation of GLM (Gelfand-Levitan-Marchenko) type for the problem L.

3. Properties of the Functions $F_0(x,t)$, $F(x,t)$, $K_{11}(x,t)$.

Lemma 1. *Denote*

$$B(x) = \sum_{n=0}^{\infty} \left(\frac{\sin \lambda_n x}{\alpha_n} - \frac{\sin \lambda_n^0 x}{\alpha_n^0} \right). \tag{25}$$

Then, $B(x) \in W_2^1(0,2\pi), F_0(x,x) \in W_2^1(0,2\pi), F(x,x) \in W_2^1(0,2\pi)$.

Proof.

$$\sum_{n=0}^{\infty} \left(\frac{\sin \lambda_n x}{\alpha_n} - \frac{\sin \lambda_n^0 x}{\alpha_n^0} \right) = \sum_{n=0}^{\infty} \left(\frac{\sin \lambda_n x - \sin \lambda_n^0 x}{\alpha_n} + \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \sin \lambda_n^0 x \right) \tag{26}$$

If we denote $\varepsilon_n := \lambda_n - \lambda_n^0$ and using asymptotic formulas of λ_n as follows:(see [3])

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, d_n \text{ is a bounded squence, } \{k_n\} \in \ell_2, \tag{27}$$

then

$$\sin \lambda_n x - \sin \lambda_n^0 x = \varepsilon_n x \cos \lambda_n^0 x + (\sin \varepsilon_n x - \varepsilon_n x) \cos \lambda_n^0 x - 2 \sin^2 \frac{\varepsilon_n x}{2} \sin \lambda_n^0 x. \tag{28}$$

$$B(x) = B_1(x) + B_2(x),$$

where

$$B_1(x) = \sum_{n=0}^{\infty} \frac{d_n x \cos \lambda_n^0 x}{\alpha_n^0 \lambda_n^0} \tag{29}$$

$$\begin{aligned} B_2(x) = & \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \sin \lambda_n^0 x + \left(\frac{\sin \lambda_0 x - \sin \lambda_0^0 x}{\alpha_n^0} \right) \\ & - \sum_{n=1}^{\infty} \frac{k_n x \cos \lambda_n^0 x}{\alpha_n^0 n} - \sum_{n=1}^{\infty} \frac{\cos \lambda_n^0 x}{\alpha_n^0 n} \left(\sin \varepsilon_n x - \varepsilon_n x \right) \\ & - 2 \sum_{n=1}^{\infty} \frac{\sin \lambda_n^0 x}{\alpha_n^0} \sin^2 \frac{\varepsilon_n x}{2}. \end{aligned} \tag{30}$$

Using

$$\alpha_n^0 = \int_0^{\pi} \rho(x) s_0^2 \left(x, \lambda_n^0 \right) dx, \tag{31}$$

where

$$s_0(x, \lambda_n) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) \sin \lambda \mu^+(x) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) \sin \lambda \mu^+(x) \tag{32}$$

and asymptotic behaviour of α_n we obtain $B_1(x), B_2(x) \in W_2^1(0, 2\pi)$ i.e., $B(x) \in W_2^1(0, 2\pi)$.

It is easy to verify that

$$\begin{aligned} F_0(x,t) = & \frac{1}{4} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) \left[B\left(x - \mu^+(t) \right) + B\left(x + \mu^+(t) \right) \right] + \\ & \frac{1}{4} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) \left[B\left(x - \mu^-(t) \right) + B\left(x + \mu^-(t) \right) \right]. \end{aligned} \tag{33}$$

So, $F_0(x,x) \in W_2^1(0, 2\pi)$ and by formula (9) we have $F(x,x) \in W_2^1(0, 2\pi)$. □

Now using the main integral equation (14), the formulas (15), (16), (18), (33) and (9) we obtain the following theorem.

Theorem 5. *The kernel function $K(x,t)$ of the main integral equation and the functions $F_0(x,t), F(x,t)$ satisfy the following relations:*

$$\rho(t) \frac{\partial^2 F_0(x,t)}{\partial t^2} = \rho(x) \frac{\partial^2 F_0(x,t)}{\partial x^2}, \ \rho(t) \frac{\partial^2 F(x,t)}{\partial t^2} = \rho(x) \frac{\partial^2 F(x,t)}{\partial x^2}, \tag{34}$$

$$F_0(x,t) \mid_{t=0} = B(x),$$

$$F(x,t) \mid_{t=0} = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) B\left(\mu^+(t) \right) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) B\left(\mu^+(t) \right), \tag{35}$$

$$\frac{\partial F_0(x,t)}{\partial t}\mid_{t=0}=0, \frac{\partial F(x,t)}{\partial t}\mid_{t=0}=0, \tag{36}$$

$$\frac{\partial F_0(\mu^{\pm}(x),t)}{\partial x}=\pm\rho(x)\frac{\partial F_0(\xi,t)}{\partial \xi}\mid_{t=\mu^{\pm}(x)}, \tag{37}$$

$$\frac{\partial K(x,0)}{\partial x}=0, \tag{38}$$

$$\frac{\sqrt{\rho(x)}-1}{\sqrt{\rho(x)}+1}K_{11}(x,\mu^{+}(x))=\frac{d}{dx}\left[K_{11}(x,\mu^{-}(x)+0)-K_{11}(x,\mu^{-}(x)-0)\right]. \tag{39}$$

4. Solution of the Inverse Problem

In this section the following theorem has been proved for the necessary and sufficient condition for solvability of the inverse problem with respect to the spectral data.

The following asymptotic relations were obtained in [3]:

Let $\left\{\lambda_n^2,\alpha_n\right\}_{n\geq 0}$ to be the spectral data for a certain boundary value problem $L=L(q(x),A)$ with $q(x)\in L_2(0,\pi)$, then

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{k_n}{n},(k_n)\in\ell_2, \tag{40}$$

$$\alpha_n=\alpha_n^0+\frac{t_n}{n},(t_n)\in\ell_2, \tag{41}$$

where λ_n^0 are zeros of the characteristic function $\Delta_0(\lambda)=s_0(\pi,\lambda),(d_n)$ is the bounded sequence

$$d_n=\frac{\alpha^{+}\sin\lambda_n^0\mu^{+}(\pi)-\alpha^{-}\sin\lambda_n^0\mu^{-}(\pi)}{\dot{\Delta}_0\left(\lambda_n^0\right)}$$

$$\alpha_n^0=\int_0^{\pi}\rho(x)\,s_0^2(x,\lambda_n)\,dx.$$

Let real numbers $\{\lambda_n,\alpha_n\}_{n\geq 0}$ be given. We construct function $F_0(x,t)$, $F(x,t)$ by the formulas (8) and (9) of the section 2 and consider the main integral equation (14). Let the function $K_{11}(x,t)$ is the solution of (14). We construct the function $\varphi(x,\lambda)$ by the formula

$$s(x,\lambda)=s_0(x,\lambda)+\int_0^{\mu^{+}(x)}K_{11}(x,t)\sin\lambda t dt. \tag{42}$$

To prove the theorem we need some lemmas.

Lemma 2. *The following relations hold:*

$$-s''(x,\lambda)+\left[\frac{A}{x}+q(x)\right]s(x,\lambda)=\lambda\rho(x)s(x,\lambda) \tag{43}$$

$$s(0,\lambda)=0,\ \ s(\pi,\lambda)=0. \tag{44}$$

Proof. Assume that $B(x)\in W_2^2(0,\pi)$, where $B(x)$ is defined in equation (25). Differentiating the identity

$$G(x,t):=\frac{2}{1+\sqrt{\rho(t)}}K_{11}\left(x,\mu^+(x)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}+$$

$$F(x,t)+\int_0^{\mu^+(x)}K_{11}(x,\xi)F_0(\xi,t)d\xi=0, 0<t<x \tag{45}$$

we calculate

$$G_t(x,t):=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}\frac{\partial K_{11}(x,\mu^+(t))}{\partial t}-\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial K_{11}(x,\mu^+(2d-t))}{\partial t}$$

$$+F_t(x,t)+\int_0^{\mu^+(x)}K_{11}(x,\xi)\frac{\partial F_0(\xi,t)}{\partial t}d\xi=0,$$

$$G_{tt}(x,t):=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}\frac{\partial^2 K_{11}(x,\mu^+(t))}{\partial t^2}+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2 K_{11}(x,\mu^+(2d-t))}{\partial t^2}$$

$$+\frac{\partial^2 F(x,t)}{\partial t^2}+\int_0^{\mu^+(x)}K_{11}(x,\xi)\frac{\partial^2 F_0(\xi,t)}{\partial t^2}d\xi=0, \tag{46}$$

$$G_x(x,t):=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}\frac{\partial K_{11}(x,\mu^+(t))}{\partial x}+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial K_{11}(x,\mu^+(2d-t))}{\partial x}$$

$$+\frac{\partial F(x,t)}{\partial x}+\sqrt{\rho(x)}K_{11}\left(x,\mu^+(x)\right)F_0(\mu^+(x),t)+\int_0^{\mu^+(x)}K_{11}(x,\xi)F_0(\xi,t)d\xi$$

$$+\sqrt{\rho(x)}\left[K_{11}\left(x,\mu^-(x)+0\right)-K_{11}\left(x,\mu^-(x)-0\right)\right]F_0(\mu^-(x),t)=0,$$

$$G_{xx}(x,t):=\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2 K_{11}(x,\mu^+(t))}{\partial x^2}+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2 K_{11}(x,\mu^+(2d-t))}{\partial x^2}$$

$$+\frac{\partial^2 F(x,t)}{\partial x^2}+\int_0^{\mu^+(x)}\frac{\partial^2 K_{11}(x,\xi)}{\partial x^2}F_0(\xi,t)d\xi+\sqrt{\rho(x)}F_0(\mu^+(x),t)\frac{\partial K_{11}(x,\xi)}{\partial x}\ |_{\xi=\mu^+(x)}$$

$$+\sqrt{\rho(x)}F_0(\mu^+(x),t)\left[\frac{\partial K_{11}(x,\xi)}{\partial x}\ |_{\xi=\mu^+(x)+0}-\frac{\partial K_{11}(x,\xi)}{\partial x}\ |_{\xi=\mu^+(x)-0}\right]$$

$$+\sqrt{\rho(x)}F_0(\mu^+(x),t)\frac{d}{dx}K_{11}\left(x,\mu^+(x)\right)+\sqrt{\rho(x)}K_{11}\left(x,\mu^+(x)\right)\frac{\partial F_0(\mu^+(x),t)}{\partial x}$$
$$+\sqrt{\rho(x)}\left[K_{11}\left(x,\mu^-(x)+0\right)-K_{11}\left(x,\mu^-(x)-0\right)\right]\frac{\partial F_0(\mu^-(x),t)}{\partial x}$$
$$+\sqrt{\rho(x)}F_0(\mu^-(x),t)\frac{d}{dx}\left[K_{11}\left(x,\mu^-(x)+0\right)-K_{11}\left(x,\mu^-(x)-0\right)\right]=0. \quad (47)$$

Using (34) we can write the last equation as follows:

$$G_{tt}(x,t):=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}K_{tt}\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{tt}\left(x,\mu^+(2d-t)\right)$$
$$+\frac{\partial^2}{\partial t^2}F(x,t)+\rho(t)\int_0^{\mu^+(x)}K_{11}(x,\xi)\frac{\partial^2 F_0(\xi,t)}{\partial \xi^2}d\xi=0. \quad (48)$$

Then using the formula(15) we have

$$\frac{1}{\rho(t)}\frac{\partial^2}{\partial t^2}G(x,t):=\frac{2\rho(t)}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}K_{11}\left(x,\mu^+(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{1}{\rho(t)}$$
$$\times\frac{\partial^2}{\partial t^2}K_{11}\left(x,\mu^+(2d-t)\right)+\frac{1}{\rho(t)}\frac{\partial^2}{\partial t^2}F(x,t)+\int_0^{\mu^+(x)}K_{11}(x,\xi)\frac{\partial^2 F_0(\xi,t)}{\partial \xi^2}d\xi=0. \quad (49)$$

By integrating in parts we obtain

$$\int_0^{\mu^+(x)}K_{11}(x,\xi)\tfrac{\partial^2 F_0(\xi,t)}{\partial \xi^2}d\xi=\left[K_{11}(x,\mu^-(x)+0)-K_{11}(x,\mu^-(x)-0)\right]\frac{\partial F_0(\xi,t)}{\partial \xi}\Big|_{\xi=\mu^-(x)}$$
$$+K_{11}\left(x,\mu^+(x)\right)\frac{\partial}{\partial \xi}F_0(\xi,t)\Big|_{\xi=\mu^+(x)}-F_0\left(x,\mu^-(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial \xi}\Big|_{\xi=\mu^-(x)-0}$$
$$+F_0(x,0)\frac{\partial K_{11}(x,\xi)}{\partial \xi}\Big|_{\xi=0}-F_0\left(x,\mu^+(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial \xi}\Big|_{\xi=\mu^+(x)-0}$$
$$+F_0\left(x,\mu^-(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial \xi}\Big|_{\xi=\mu^-(x)+0}+\int_0^{\mu^+(x)}\frac{\partial^2 K_{11}(x,\xi)}{\partial^2 \xi}F_0(\xi,t)d\xi. \quad (50)$$

Therefore

$$G_{tt}(x,t)=\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}K_{11}(x,\mu^+(t))+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2}{\partial t^2}K_{11}(x,2d-t)+$$
$$\frac{1}{\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}F(x,t)$$
$$+\left[K_{11}\left(x,\mu^-(x)-0\right)-K_{11}\left(x,\mu^-(x)+0\right)\right]\frac{\partial}{\partial \xi}F_0(\xi,t)\Big|_{\xi=\mu^-(x)}$$

$$+K_{11}\left(x,\mu^{+}(x)\right)\frac{\partial}{\partial\xi}F_0(\xi,t)\,|_{\xi=\mu^{+}(x)}\,-F_0\left(x,\mu^{-}(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=\mu^{-}(x)-0}$$
$$+F_0(x,0)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=0}\,-F_0\left(x,\mu^{+}(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=\mu^{+}(x)-0}$$
$$+F_0\left(x,\mu^{-}(x)\right)\frac{\partial K_{11}(x,\xi)}{\partial\xi}\,|_{\xi=\mu^{-}(x)+0}+\int\limits_0^{\mu^{+}(x)}\frac{\partial^2 K_{11}(x,\xi)}{\partial^2\xi}F_0(\xi,t)d\xi. \tag{51}$$

It follows from (45), (46), and (50), the identity

$$G_{xx}(x,t)-\rho(x)\,G_{tt}(x,t)-\left[\frac{A}{x}+q(x)\right]G(x,t)\equiv 0.$$

Using the identity acording to formulas (9),(16)- (22), we get

$$\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial x^2}K_{11}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2}{\partial x^2}K_{11}(x,2d-t)$$
$$-\rho(x)\left[\frac{2}{1+\sqrt{\rho(t)}}\frac{\partial^2}{\partial t^2}K_{11}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}\frac{\partial^2}{\partial t^2}K_{11}(2d-t)\right]$$
$$-\left[\frac{A}{x}+q(x)\right]\left[\frac{2}{1+\sqrt{\rho(t)}}K_{11}\left(x,\mu^{+}(t)\right)+\frac{1-\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{11}(2d-t)\right]$$
$$\int\limits_0^{\mu^{+}(x)}\left\{K_{11xx}(x,t)-\rho(x)\,K_{11tt}(x,t)-\left[\frac{A}{x}+q(x)\right]K_{11}(x,t)\right\}F_0(\xi,t)d\xi=0. \tag{52}$$

By the Theorem 3 in the first section, the equation (52) has only trivial solution i.e.,

$$K_{11xx}(x,t)-\rho(x)\,K_{11tt}(x,t)-\left[\frac{A}{x}+q(x)\right]K_{11}(x,t)\equiv 0,\quad 0<t<x. \tag{53}$$

Now differentiating equation (42) twice, we have

$$s'(x,\lambda)=s_0'(x,\lambda)+\int\limits_0^{\mu^{+}(x)}(K_{11})_x(x,t)\sin\lambda t dt+\sqrt{\rho(x)}K_{11}\left(x,\mu^{+}(x)\right)\sin\lambda\mu^{+}(x)$$
$$+\sqrt{\rho(x)}\left[K_{11}\left(x,\mu^{-}(x)+0\right)-K_{11}\left(x,\mu^{-}(x)-0\right)\right]\sin\lambda\mu^{-}(x)$$

$$s''(x,\lambda)=s_0''(x,\lambda)+\int\limits_0^{\mu^{+}(x)}(K_{11})_{xx}(x,t)\sin\lambda t dt$$

$$+\sqrt{\rho(x)}\sin\lambda\mu^{+}(x)\left[\frac{\partial}{\partial x}K_{11}\left(x,\mu^{-}(x)\right)\right]_{t=\mu^{-}(x)}$$
$$+\sqrt{\rho(x)}\sin\lambda\mu^{-}(x)\left[\frac{\partial}{\partial x}K_{11}\left(x,\mu^{-}(x)+0\right)|_{t=\mu^{-}(x)+0}\right.$$
$$\left.-\frac{\partial}{\partial x}K_{11}\left(x,\mu^{-}(x)-0\right)|_{t=\mu^{-}(x)-0}\right]$$
$$+\sqrt{\rho(x)}\sin\lambda\mu^{+}(x)\frac{\partial}{\partial x}\left[K_{11}\left(x,\mu^{-}(x)\right)\right]$$
$$+\sqrt{\rho(x)}\sin\lambda\mu^{-}(x)\frac{\partial}{\partial x}\left[K_{11}\left(x,\mu^{-}(x)+0\right)-K_{11}\left(x,\mu^{-}(x)-0\right)\right]$$
$$+\lambda\rho(x)K_{11}\left(x,\mu^{+}(x)\right)\cos\lambda\mu^{+}(x)$$
$$+\lambda\rho(x)\left[K_{11}\left(x,\mu^{-}(x)+0\right)-K_{11}\left(x,\mu^{-}(x)-0\right)\right]\cos\lambda\mu^{-}(x)$$

□

Lemma 3. *For each function $g(x)\in L_2(0,\pi)$ the following relation holds:*

$$\int_0^{\pi}\rho(x)g^2(x)\,dx=\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\left(\int_0^{\pi}\rho(t)s(t,\lambda_n)\,dt\right)^2 \tag{54}$$

Proof. Using the formulas (7), (8), (9) of the previous section it is easy to transform solution

$$s(x,\lambda)=s_0(x,\lambda)+\int_0^{x}\rho(x)s(x,t)s_0(t,\lambda)\,dt \tag{55}$$

and the main integral equation (14) form the previous section to the form

$$w(x,t)+F(x,t)+\int_0^{x}\rho(\xi)w(x,\xi)F(\xi,t)\,d\xi=0, \tag{56}$$

where

$$w(x,t)=\frac{2\sqrt{\rho(t)}}{1+\sqrt{\rho(t)}}K_{11}(x,\mu^{+}(t))+\frac{2\sqrt{\rho(2d-t)}}{1+\sqrt{\rho(2d-t)}}K_{11}(x,\mu^{+}(2d-t)).$$

Solving the equation (55) with respect to $s_0(x,\lambda)$ we obtain
For $x<d$

$$s_0(x,\lambda)=s(x,\lambda)+\int_0^{x}\rho(t)H(x,t)s(x,\lambda)\,dt. \tag{57}$$

By the standart method (see [20]) it can be proved that

$$H(x,t) = F(x,t) + \int_0^t \rho(\xi)\, w(t,\xi)\, F(x,\xi)\, d\xi, 0 \le t \le x. \tag{58}$$

Denote $Q(\lambda) = \int_0^\pi \rho(t)\, g(t)\, \varphi(t,\lambda)\, dt$. Then using (55) we have

$$Q(\lambda) = \int_0^\pi \rho(t)\, h(t)\, s_0(t,\lambda)\, dt$$

where

$$h(t) = g(t) + \int_t^\pi \rho(\xi)\, g(\xi)\, w(t,\xi)\, d\xi. \tag{59}$$

By the similar way, using the formula (57) we obtain

$$g(t) = h(t) + \int_t^\pi \rho(\xi)\, h(\xi)\, H(\xi,t)\, d\xi. \tag{60}$$

Now according to equation (59) we have

$$\begin{aligned} &\int_0^\pi \rho(t)\, h(t)\, F(x,t)\, dt \\ &= \int_0^x \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, W(\xi)\, F(x,\xi)\, d\xi\right] dt \\ &+ \int_x^\pi \rho(t)\, g(t) \left[F(x,t) + \int_0^t \rho(\xi)\, W(\xi)\, F(x,\xi)\, d\xi\right] dt. \end{aligned} \tag{61}$$

Consequently, by the formulas (56) and (58) we obtain

$$\int_0^\pi \rho(t)\, h(t)\, F(x,t)\, dt = \int_0^\pi \rho(t)\, g(t)\, H(x,t)\, dt - \int_x^\pi \rho(t)\, g(t)\, W(x,t)\, dt \tag{62}$$

From the Parseval equality we have

$$\int_0^\pi \rho(t)\, h^2(t)\, dt + \int_0^\pi \rho(t)\, \rho(x)\, h(t)\, h(x)\, F(x,t)\, dxdt$$

$$= \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^{\pi} h(t)\, g(t)\, s_0(t, \lambda_n)\, \varphi(t, \lambda_n)\, dt \right)^2 = \sum_{n=0}^{\infty} \frac{Q(\lambda_n)^2}{\alpha_n}.$$

Using (61) we get

$$\sum_{n=0}^{\infty} \frac{Q(\lambda_n)^2}{\alpha_n} = \int_0^{\pi} \rho(t)\, h^2(t)\, dt + \int_0^{\pi} \rho(t)\, g(t) \left[\int_t^{\pi} \rho(x)\, h(x)\, H(x,t)\, dx \right] dt$$
$$- \int_0^{\pi} \rho(x)\, h(x) \left[\int_x^{\pi} \rho(t)\, g(t)\, W(t,x)\, dx \right] dt.$$

Finally, from (59) and (60) we get

$$\sum_{n=0}^{\infty} \frac{Q(\lambda_n)^2}{\alpha_n} = \int_0^{\pi} \rho(t)\, h^2(t)\, dt + \int_0^{\pi} \rho(t)\, g(t)\, (g(t) - h(t))\, dt$$
$$- \int_0^{\pi} \rho(x)\, g(x)\, (g(x) - h(x))\, dx = \int_0^{\pi} \rho(t)\, g^2(t)\, dt.$$

The lemma is proved. □

Corollary 1. *For arbitrary functions $f, g \in L_2(0,\pi)$, the following relation holds:*

$$\int_0^{\pi} \rho(x)\, f(x)\, g(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \int_0^{\pi} f(t)\, s(t, \lambda_n)\, dt \int_0^{\pi} g(t)\, s(t, \lambda_n)\, dt. \tag{63}$$

Lemma 4. *The following relations hold:*

$$\int_0^{\pi} s(t, \lambda_k)\, s(t, \lambda_n)\, dt = \begin{cases} 0, n \neq k \\ \alpha_n, n = k \end{cases} \tag{64}$$

Proof. Let $f(x) \in W_2^2(0,\pi)$, consider the series

$$f^*(x) = \sum_{n=0}^{\infty} c_n s(x, \lambda_n), \tag{65}$$

where

$$c_n = \frac{1}{\alpha_n} \int_0^{\pi} f(x)\, s(x, \lambda_n)\, dx. \tag{66}$$

Using Lemma 1 and integrating by parts we calculate:

$$c_n = \frac{1}{\alpha_n \lambda_n^2} \Big(hf(0) - f'(0) + s(\pi, \lambda_n)\, f'(\pi) - \varphi(\pi, \lambda_n)\, f(\pi)$$

$$+\int_0^\pi s(x,\lambda_n)\left[-f''(x)+g(x)\right]dx\Bigg)$$

From the asymptotic formulas for the $\varphi(x,\lambda)$ and λ_n in [3], we get

$$c_n = O\left(\frac{1}{n^2}\right), s(x,\lambda_n) = O(1)$$

uniformly for $x \in [0,\pi]$. Therefore the series (64) converges absolutely and uniformly on $[0,\pi]$. Using (62) and (65) we obtain

$$\int_0^\pi \rho(x) f(x) g(x)\,dx = \int_0^\pi g(t)\sum_{n=0}^{\infty} c_n s(x,\lambda_n)\,dt = \int_0^\pi g(t) f^*(t)\,dt.$$

Since $g(x)$ is arbitrary, we get

$$f^*(x) = f(x) = \sum_{n=0}^{\infty} c_n s(x,\lambda_n). \tag{67}$$

Now, for fix $k \geq 0$ and take $f(x) = \varphi(x,\lambda_k)$, then since (66)

$$s(x,\lambda_k) = \sum_{n=0}^{\infty} c_{n_k} s(x,\lambda_n), c_{n_k} = \frac{1}{\alpha_n}\int_0^\pi s(x,\lambda_k) s(x,\lambda_n)\,dx.$$

Moreover, the system $\{s_0(x,\lambda_n)\}_{n\geq 0}$ is minimal in $L_2(0,\pi)$, (see theorem 2 of the previous section), and consequently, in view of (42) the system $\{\varphi(x,\lambda_n)\}_{n\geq 0}$ is also minimal in $L_2(0,\pi)$. Therefore $c_{n_k} = \delta_{n_k}$ (δ_{n_k} is a Kronecker symbol). The lemma is proved. □

Now we can give the algorithm to construct the problem $L(q(x))$ using the spectral data $\{\lambda_n,\alpha_n\}_{n\geq 0}$ as follows:

1- Use formulas (8) and (9), to construct the functions $F_0(x,t)$ and $F(x,t)$.

2- Construct the function $K(x,t)$ as the unique solution of the main integral equation.

3- Calculate the function $q(x)$ and coefficient A by the formulas in Theorem 1.

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References

[1] Shepelsky, D. G., The inverse problem of reconstruction of te medium's conductivity in a class of discontinuous and increasing functions, *Adv. Soviet Math.*, 19 (1994), 209-231.

[2] Anderssen, R. S., The effect of discontinuities in density and shear velocity on the asypmtotic overtone sturcture of toritonal eigenfrequencies of the Earth, Geophys, *J. R. Astr. Soc.*, 50 (1997), 303-309.

[3] Amirov, R. Kh., Topsakal, N., On Sturm-Liouville operators with Coulomb potential which have discontinuity conditions inside an interval, *Integral Transforms Spec. Funct.*, 19(12) (2008), 923-937. http://dx.doi.org/10.1080/10652460802420386

[4] Adiloğlu, A., Nabiev, Amirov, R. Kh., On the boundary value problem for the Sturm-Liouville equation with the discontinuous coefficients, *, Mathematical methods in the Applied Sciences*, 36 (2013). http://dx.doi.org/1685-1700.10.1002/mma.2714

[5] Akhmedova, E.N., Hüseyin, H.M., On inverse problem for the Sturm-Liouville operator with the discontinuous coefficients*, Proc. of Saratov University, New ser., Ser.Math., Mech., and Inf.*, 10(1) (2010), 3-9.

[6] Litvinenko, O. N., Soshnikov, V. I., The Theory of Heterogeneous Lines and Their Applications in Radio Engineering*, Radio*, Moscow (in Russian) 1964.

[7] Krueger, R. J., Inverse problems for nonabsorbing media with discontinuous material properties, *J. Math. Phys.*, 23(3) (1982), 396-404.

[8] Savchuk, A.M., Shkalikov, A.A., Sturm-Liouville operator with singular potentials*, Mathematical Notes*, 66(6) (1999), 741-753. https://doi.org/10.1007/BF02674332

[9] Savchuk, A.M., Shkalikov, A.A., Trace formula for Sturm-Liouville operator with singular potentials*, Mathematical Notes*, 69(3) (2001), 427-442. https://doi.org/10.4213/mzm515

[10] Savchuk, A.M., On the eigenvalues and eigenfunctions of the Sturm-Liouville operator with a singular potential*, Mathematical Notes*, 69(2) (2001), 277-285. https://doi.org/10.4213/mzm502

[11] Hryniv, R., Mykityuk, Y., Inverse spectral problems for Sturm-Liouville operators with singular potentials*, Inverse Problems*, 19(3) (2003), 665-684. http://dx.doi.org/10.1088/0266-5611/19/3/312

[12] Hryniv, R., Mykityuk, Y., Transformation operators for Sturm-Liouville operators with singular potentials*, Math. Phys. Anal. and Geometry*, 7(2) (2004), 119-149. http://dx.doi.org/10.1023/B:MPAG.0000024658.58535

[13] Hryniv, R., Mykityuk, Y., Eigenvalue asymptotics for Sturm-Liouville operators with singular potentials*,arXivpreprint math/0407252.*

[14] Hryniv, R., Mykityuk, Y., Inverse spectral problems for Sturm-Liouville operators with singular potentials, II. Reconstruction by two spectra*, North-Holland Mathematics Studies*, 197 (2004), 97-114.

[15] Amirov, R. Kh., Topsakal, N., A representation for solutions of Sturm-Liouville equations with Coulomb Potential inside finite interval*, Journal of Cumhuriyet University Natural Sciences*, 28(2) (2007), 11-38.

[16] Topsakal, N., Amirov, R. Kh., Inverse problem for Sturm-Liouville operators with Coulomb potential which have discontinuity conditions inside an interval. *Math. Phys. Anal. Geom.* 13(1) (2010), 29–46. http://dx.doi.org/10.1007/s11040-009-9066-y

[17] Naimark, M. A., Linear Differential Operators, Moscow, Nauka, (in Russian) 1967.

[18] Marchenko, V. A., Sturm-Liouville Operators and Their Applications, Naukova Dumka, Kiev, Birkhauser, Basel, 1986.

[19] Levitan, B. M., Inverse Sturm-Louville Problems, Nauka, Moscow, 1984. English transl.:VNU Sci. Press, Utrecht, 1987.
[20] Yurko, V. A., Inverse Spectral Problems of Differential Operators and Their Applications, Gordon and Breach, New York, 2000.

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DIRECTION CURVES OF GENERALIZED BERTRAND CURVES AND INVOLUTE-EVOLUTE CURVES IN E^4

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Abstract. In this study, we define (1,3)-Bertrand-direction curve and (1,3)-Bertrand-donor curve in the 4-dimensional Euclidean space E^4. We introduce necessary and sufficient conditions for a special Frenet curve to have a (1,3)-Bertrand-direction curve. We introduce the relations between Frenet vectors and curvatures of these direction curves. Furthermore, we investigate whether (1,3)-evolute-donor curves in E^4 exist and show that there is no (1,3)-evolute-donor curve in E^4.

1. Introduction

Associated curves are the most interesting subject of curve theory. Such curves have a special property between their Frenet apparatus. Bertrand curves are one of the most famous type of such curve pairs. These curves were first discovered by J. Bertrand in 1850 [1]. In the 3-dimensional Euclidean space E^3, a curve $\alpha(s)$ is called Bertrand curve if there exists a curve γ different from α with the same principal normal line as α. Bertrand partner curves are important and fascinating examples of offset curves used in computer-aided design [13]. The classical characterization for the Bertrand curve is that a curve $\alpha(s)$ is a Bertrand curve if and only if its curvature functions $\kappa(s)$, $\tau(s)$ satisfy the condition $a\kappa(s) + b\tau(s) = 1$, where a, b are real constant numbers. And, the parametric form of the Bertrand mate of $\alpha(s)$ is defined by $\gamma(s) = \alpha(s) + \lambda N(s)$, where $\lambda \neq 0$ is constant and $N(s)$ is unit principal normal line of α [17]. It is interesting that for $n \geq 4$, there exists no Bertrand curves in this form. This fact was proved by Matsuda and Yorozu [12]. Considering this fact, in the same paper, they have defined a new type of associated curves called (1,3)-Bertrand curves in E^4.

Moreover, another well-known type of associated curve pairs is involute-evolute curve couple. These curves were first studied by Huygens in his work [8]. Classically,

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an evolute of a given curve is defined as the locus of the centers of curvatures of the curve, which is the envelope of the normal of reference curve. Fuchs defined an involute of a given curve as a curve for which all tangents of reference curve are normal [3]. In the same study, equation of enveloping curve of the family of normal planes for space curve has been also defined. Gere and Zupnik studied involute-evolute curves by considering a curve composed of two arcs with common evolute [6]. Fukunaga and Takahashi defined evolutes and involutes of fronts in the plane and introduced some properties of these curves [4,5]. Later, Yu, Pei and Cui considered evolutes of fronts on Euclidean 2-sphere [18]. Özyılmaz and Yılmaz studied involute-evolute of W-curves in Euclidean 4-space E^4 [16]. Li and Sun studied evolutes of fronts in the Minkowski Plane [9].

Recently, Hanif and Hou have defined generalized involute and evolute curves in E^4[7]. They have obtained necessary and sufficient conditions for a curve to have a generalized involute or evolute curve. Another study of generalized involute-evolute curves has been given by Öztürk, Arslan and Bulca [15]. They have given characterization of involute curves of order k of a given curve in E^n and also introduced some results on these type of curves in E^3 and E^4.

Furthermore, Choi and Kim have defined a new type of associated curves in E^3 called principal normal (binormal) direction-curve and principal normal (binormal) donor-curve [2]. Similarly, Macit and Düldül have defined W-direction curve and W-donor curve in E^3, where W is unit Darboux vector of the reference curve [10]. Later, the author has defined Bertrand direction curves, Mannheim direction curves and involute-evolute direction curves in E^3 and introduced relations between those curves and some special curves such as helices and slant helices [14].

In this study, first, we define (1,3)-Bertrand-direction curves and introduce the relations between the Frenet apparatus of these curves. We show that a curve with non-constant first curvature κ does not have (1,3)-Bertrand-direction curve. Later, we give that no C^∞-special Frenet curve in E^4 is an (1,3)-evolute-donor curve.

2. Preliminaries

Let $\alpha : I \to E^4$ be a regular curve, i.e., $\|\alpha'(t)\| \neq 0$, where I is subset of real numbers set $\mathbb{R}$ and $\|\alpha'(t)\|$ denotes the norm of tangent vector $\alpha'(t)$ in the Euclidean 4-space E^4. This norm is defined by $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$ where $\langle x, x\rangle$ is the Euclidean inner(dot) product and $x = (x_1, x_2, x_3, x_4)$ is a vector in E^4. The curve $\alpha(t)$ is called unit speed if $\|\alpha'(t)\| = 1$. The parameter of a unit speed curve is represented by s and called arc-length parameter. The curve $\alpha(s)$ is called special Frenet curve if there exist differentiable functions $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ on I and differentiable orthonormal frame field $\{T, N, B_1, B_2\}$ along $\alpha(s)$ such that:

i) Following Frenet formulas hold

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B_1, \\ B_1' &= -\tau N + \sigma B_2, \\ B_2' &= -\sigma B_1. \end{aligned} \tag{1}$$

ii) The orthonormal frame field $\{T, N, B_1, B_2\}$ has positive orientation.

iii) The functions $\kappa(s)$, $\tau(s)$ are positive and the function $\sigma(s)$ does not vanish. The unit vector fields T, N, B_1 and B_2 are called tangent, principal normal, first binormal and second binormal of $\alpha(s)$ and the functions $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ are called first, second and third curvatures of $\alpha(s)$, respectively [11].

If we take $T = n_1$, $N = n_2$, $B_1 = n_3$, $B_2 = n_4$, the term "special" means that the vector field n_{i+1}, $(1 \le i \le 3)$ is inductively defined by the vector fields n_i and n_{i-1} and the positive functions κ and τ [12]. For this, the Frenet apparatus of a special Frenet curve have been determined by the following steps:

(1) $\alpha'(s) = T(s)$
(2) $\kappa(s) = \|T'(s)\| > 0$, $N(s) = \frac{1}{\kappa(s)} T'(s)$.
(3) $\tau(s) = \|N'(s) + \kappa(s)T(s)\| > 0$, $B_1(s) = \frac{1}{\tau(s)}\left(N'(s) + \kappa(s)T(s)\right)$
(4) $B_2(s) = \varepsilon \frac{1}{\left\|{B_1}'(s)+\tau(s)N(s)\right\|}\left({B_1}'(s) + \tau(s)N(s)\right)$, where $\varepsilon = \pm 1$ is chosen as the frame $\{T, N, B_1, B_2\}$ has positive orientation and $\sigma(s) = \langle B_1'(s), B_2(s)\rangle$ does not vanish.

All these 4 steps should be checked that the curve $\alpha(s)$ is a special Frenet curve [11].

The plane spanned by the vectors T, B_1 is called the Frenet (0,2)-plane and the plane spanned by the vectors N, B_2 is called the Frenet (1,3)-normal plane of α[7,12]

Definition 1. *([12]) A C^∞-special Frenet curve $\alpha : I \to E^4$ is called a (1,3)-Bertrand curve if there exits another C^∞-special Frenet curve $\beta : J \to E^4$ and a C^∞-mapping $\varphi : I \to J$ such that the Frenet (1,3)-normal planes of α and β at the corresponding points coincide. The parametric representation of β is $\beta(\varphi(s)) = \alpha(s) + zN(s) + tB_2(s)$, where z, t are constant real numbers.*

Theorem 1. *([12]) If $n \ge 4$, then no C^∞-special Frenet curve in E^n is a Bertrand curve.*

Definition 2. *([7]) Let $\alpha(s)$ and $\gamma(\bar{s})$ be two regular curves in E^4 such that $\bar{s} = f(s)$ is the arc-length parameter of $\gamma(\bar{s})$. If the Frenet (0,2)-plane of α and Frenet (1,3)-plane of γ at the corresponding points coincide, then α is called (1,3)-evolute curve of γ and γ is called (0,2)-involute curve of α. The (0,2)-involute curve γ has the parametric form $\gamma(s) = \alpha(s) + (c - s)T(s) + kB_1(s)$, where c, k are real constants.*

Let $I \subset \mathbb{R}$ be an open interval. For a unit speed special Frenet curve $\alpha : I \to E^4$, let define a vector valued function $X(s)$ as follows

$$X(s) = p(s)T(s) + l(s)N(s) + r(s)B_1(s) + n(s)B_2(s), \tag{2}$$

where p, l, r and n are differentiable scalar functions of s. Let $X(s)$ be unit, i.e.,

$$p^2(s) + l^2(s) + r^2(s) + n^2(s) = 1, \tag{3}$$

holds. Then the definitions of X-donor curve and X-direction curve in E^4 are given as follows.

Definition 3. *Let α be a special Frenet curve in E^4 and $X(s)$ be a unit vector valued function as given in* (2)*. The integral curve $\gamma : I \to E^4$ of $X(s)$ is called an X-direction curve of α. The curve α having γ as an X-direction curve is called the X-donor curve of γ in E^4.*

3. (1,3)-Bertrand-Direction Curves in E^4

In this section, we define (1,3)-Bertrand-direction curves and (1,3)-Bertrand-donor curves for special Frenet curves and introduce necessary and sufficient conditions for these curve pairs.

Definition 4. *Let $\alpha = \alpha(s)$ be a special Frenet curve in E^4 with arc-length parameter s and $X(s)$ be a unit vector field as given in* (2)*. Let special Frenet curve $\beta(\bar{s}) : I \to E^4$ be an X-direction curve of α. The Frenet frames and curvatures of α and β be denoted by $\{T, N, B_1, B_2\}$, κ, τ, σ and $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\}$, $\bar{\kappa}$, $\bar{\tau}$, $\bar{\sigma}$, respectively, and let any Frenet vector of α does not coincide with any Frenet vector of β. If β is a (1,3)-Bertrand partner curve of α, then β is called (1,3)-Bertrand-direction curve of α and α is said to be (1,3)-Bertrand-donor curve of β.*

From Definition 4, it is clear that at the corresponding points of the curves, the planes spanned by $\{N, B_2\}$ and $\{\bar{N}, \bar{B}_2\}$ coincide. Then, we have,

$$sp\{N, B_2\} = sp\{\bar{N}, \bar{B}_2\}, \;\; sp\{T, B_1\} = sp\{\bar{T}, \bar{B}_1\}, \tag{4}$$

Moreover, since β is an integral curve of $X(s)$, we have $\frac{d\beta}{ds} = X(s)$. Also, since $X(s)$ is unit, the arc-length parameter $\bar{s}$ of β is obtained as

$$\bar{s} = \int_0^s \left\| \frac{d\beta}{ds} \right\| ds = \int_0^s ds = s \tag{5}$$

i.e., arc-length parameters of (1,3)-Bertrand-direction curves α and β are same. Thus, hereafter we will use prime for both curves to show the derivative with respect to s.

Theorem 2. *The special Frenet curve $\alpha : I \to E^4$ is a (1,3)-Bertrand-donor curve if and only if there exist non-zero constants r, μ, λ, p such that*

$$p^2 + r^2 = 1, \;\; \lambda^2 + \mu^2 = 1, \tag{6}$$

$$p\kappa - r\tau = \frac{\lambda}{\mu} r\sigma, \tag{7}$$

$$(p^2 - \lambda^2)\kappa - pr\tau \neq 0. \tag{8}$$

Proof. Let $X(s) = p(s)T(s) + l(s)N(s) + r(s)B_1(s) + n(s)B_2(s)$ be a unit vector valued function and the special Frenet curve $\beta : I \to E^4$ be integral curve of $X(s)$ and also be a (1,3)-Bertrand-direction curve of α, where $p(s)$, $l(s)$, $r(s)$ and $n(s)$ are smooth scalar functions of arc-length parameter s. Then, we have

$$\bar{T}(s) = p(s)T(s) + l(s)N(s) + r(s)B_1(s) + n(s)B_2(s). \tag{9}$$

From (4), it follows $\bar{T} \perp sp\{N, B_2\}$. Then, multiplying (9) with N and B_2, we have $l(s) = 0$, $n(s) = 0$, respectively, and (9) becomes

$$\bar{T}(s) = p(s)T(s) + r(s)B_1(s), \tag{10}$$

and from (10), it follows $p^2(s) + r^2(s) = 1$, since $\bar{T}$ is unit. Differentiating (10) with respect to s and using Frenet formulas (1), we get

$$\bar{\kappa}\bar{N} = p'T + (p\kappa - r\tau)N + r'B_1 + r\sigma B_2. \tag{11}$$

Multiplying (11) with T and B_1 and considering (4), we get $p' = 0$, $r' = 0$, respectively, i.e., p and r are constants. If p or r is zero, then Frenet vectors of α and β coincide. It follows that p and r are non-zero constants. Then, from (10), we get $p^2 + r^2 = 1$ and we have first equality in (6).

Now, (11) becomes

$$\bar{\kappa}\bar{N} = (p\kappa - r\tau)N + r\sigma B_2, \tag{12}$$

which gives

$$\bar{\kappa} = \sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}. \tag{13}$$

Let define

$$\lambda = \frac{p\kappa - r\tau}{\sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}}, \quad \mu = \frac{r\sigma}{\sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}}. \tag{14}$$

Then, (12) becomes

$$\bar{N} = \lambda N + \mu B_2, \quad \lambda^2 + \mu^2 = 1. \tag{15}$$

By Definition 4, any Frenet vector of α does not coincide with any Frenet vector of β. Thus, we have that $\lambda \neq 0$, $\mu \neq 0$. Differentiating the first equation in (15) with respect to s and considering Frenet formulas (1), it follows

$$-\bar{\kappa}\bar{T} + \bar{\tau}\bar{B}_1 = -\lambda\kappa T + \lambda' N + (\lambda\tau - \mu\sigma)B_1 + \mu' B_2. \tag{16}$$

Multiplying (16) with N and B_2, we get $\lambda' = 0$, $\mu' = 0$, respectively, i.e., λ, μ are real non-zero constants. So, we have $\lambda^2 + \mu^2 = 1$, which is the second equality in (6).

Moreover, from (13) and (14), we have

$$\bar{\kappa} = \frac{p\kappa - r\tau}{\lambda} = \frac{r\sigma}{\mu}. \tag{17}$$

Then, (17) gives us $p\kappa - r\tau = \frac{\lambda}{\mu} r\sigma$ and we obtain (7).

Now, writing (10) and (17) in (16), it follows

$$\lambda\bar{\tau}\bar{B}_1 = \left((p^2-\lambda^2)\kappa - pr\tau\right) T + \left(pr\kappa + (\lambda^2 - r^2)\tau - \lambda\mu\sigma\right) B_1. \tag{18}$$

From (7), we have

$$\sigma = \frac{\mu(p\kappa - r\tau)}{\lambda r}. \tag{19}$$

Writing (19) in (18) and using (6), equality (18) becomes

$$\bar{\tau}\bar{B}_1 = A\left(T - \frac{p}{r}B_1\right), \tag{20}$$

where $A = \frac{(p^2-\lambda^2)\kappa - pr\tau}{\lambda}$. Since $\bar{B}_1 \neq 0$, we get $A \neq 0$, i.e., $(p^2-\lambda^2)\kappa - pr\tau \neq 0$. Then we have (8).

Conversely, assume that relations (6), (7) and (8) hold for some non-zero constants $r,\ \mu,\ \lambda,\ p$ and α be a special Frenet curve with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\kappa,\ \tau,\ \sigma$. Let define a vector valued function

$$X(s) = pT(s) + rB_1(s), \tag{21}$$

and let $\beta : I \to E^4$ be an integral curve of $X(s)$. We will show that β is a (1,3)-Bertrand-direction curve of α. Differentiating (21) with respect to s gives

$$\bar{\kappa}\bar{N} = (p\kappa - r\tau)N + r\sigma B_2. \tag{22}$$

Writing (7) in (22), it follows

$$\bar{\kappa}\bar{N} = r\sigma\left(\frac{\lambda}{\mu}N + B_2\right). \tag{23}$$

From (23), it follows,

$$\bar{\kappa} = \varepsilon_1 \frac{r\sigma}{\mu}, \tag{24}$$

where $\varepsilon_1 = \pm 1$ such that $\bar{\kappa} > 0$. Writing (24) in (23) gives

$$\bar{N} = \varepsilon_1\left(\lambda N + \mu B_2\right). \tag{25}$$

Differentiating (25) with respect to s gives

$$\bar{N}' = \varepsilon_1\left(-\lambda\kappa T + (\lambda\tau - \mu\sigma)B_1\right). \tag{26}$$

Using (21), (24) and (26), we have

$$\bar{N}' + \bar{\kappa}\bar{T} = \frac{\varepsilon_1}{\mu}\left((pr\sigma - \lambda\mu\kappa)T + (r^2\sigma + \lambda\mu\tau - \mu^2\sigma)B_1\right). \tag{27}$$

Writing (7) in (27) and using (6), (27) becomes

$$\bar{N}' + \bar{\kappa}\bar{T} = \varepsilon_1 \frac{(p^2-\lambda^2)\kappa - pr\tau}{\lambda}\left(T - \frac{p}{r}B_1\right). \tag{28}$$

From (28) and (8), we have

$$\bar{\tau} = \left\|\bar{N}' + \bar{\kappa}\bar{T}\right\| = \varepsilon_2 \frac{(p^2-\lambda^2)\kappa - pr\tau}{\lambda r} \neq 0, \tag{29}$$

where $\varepsilon_2 = \pm 1$ such that $\bar{\tau} > 0$. Then,

$$\bar{B}_1 = \frac{1}{\bar{\tau}}\left(\bar{N}' + \bar{\kappa}\bar{T}\right) = \frac{\varepsilon_1}{\varepsilon_2}\left(rT - pB_1\right). \tag{30}$$

Considering (21), (25) and (30), we can define the unit vector $\bar{B}_2$ as

$$\bar{B}_2 = \frac{1}{\varepsilon_2}\left(\mu N - \lambda B_2\right),$$

that is

$$\bar{B}_2 = \frac{1}{\varepsilon_2\sqrt{(p\kappa - r\tau)^2 + (r\sigma)^2}}\left(r\sigma N - (p\kappa - r\tau)B_2\right), \tag{31}$$

and we have $\det(\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2) = 1$. Using (30) and (31), it follows

$$\bar{\sigma} = \left\langle \bar{B}_1', \bar{B}_2 \right\rangle = \varepsilon_1\left(\mu(r\kappa + p\tau) + p\lambda\sigma\right). \tag{32}$$

If we assume that $\bar{\sigma} = 0$, then we have $\mu(r\kappa + p\tau) = -p\lambda\sigma$. Multiplying that with r, we get $\mu(r^2\kappa + pr\tau) = -pr\lambda\sigma$. Since $r^2 = 1 - p^2$, the last equality becomes $\mu\left(-p(p\kappa - r\tau) + \kappa\right) = -pr\lambda\sigma$. Using (7), it follows $\mu\kappa = 0$, which is a contradiction since $\mu \neq 0$ and α is a special Frenet curve. Then, $\bar{\sigma} \neq 0$, i.e., β is a special Frenet curve. Moreover, since $r, \ \mu, \ \lambda, \ p$ are non-zero constants, from the equalities (21), (25), (30) and (31), it follows that no Frenet vectors of α and β coincide. Furthermore, since we obtain $sp\{N, B_2\} = sp\left\{\bar{N}, \bar{B}_2\right\}$, we have that β is (1,3)-Bertrand-direction curve of α.

□

Moreover, since α is a (1,3)-Bertrand curve, by Definition 1, its (1,3)-Bertrand partner curve β has the parametric form $\beta(s) = \alpha(s) + zN(s) + tB_2(s)$ where $z, \ t$ are constant real numbers. Differentiating that with respect to s and using the equality $\bar{T} = pT + rB_1$, we have $pT + rB_1 = (1 - z\kappa)T + (z\tau - t\sigma)B_1$ which gives that $\kappa z = 1 - p$. If $z = 0$, we get $p = 1$. But this is a contradiction since $p^2 + r^2 = 1$ and $r \neq 0$. Then, $\kappa = (1-p)/z$ is a non-zero positive constant and we have the followings.

Corollary 1. *No C^∞-special Frenet curve in E^4 with non-constant first curvature κ is a (1,3)-Bertrand-donor curve.*

Corollary 2. *If the special Frenet curve $\alpha : I \to E^4$ is a (1,3)-Bertrand-donor curve, then there exists a linear relation $c_1\tau + c_2\sigma = \kappa$ where c_1, c_2, $\kappa \neq 0$ are constants and κ, τ, σ are Frenet curvatures of α.*

Corollary 3. *Let β be (1,3)-Bertrand-direction curve of α. Then the relations between Frenet apparatus are given as follows*

$$\bar{T} = pT + rB_1, \ \bar{N} = \varepsilon_1\left(\lambda N + \mu B_2\right), \ \bar{B}_1 = \frac{\varepsilon_1}{\varepsilon_2}\left(rT - pB_1\right), \ \bar{B}_2 = \frac{1}{\varepsilon_2}\left(\mu N - \lambda B_2\right), \tag{33}$$

$$\bar{\kappa}=\varepsilon_1\frac{r\sigma}{\mu}>0,\ \ \bar{\tau}=\varepsilon_2\frac{(p^2-\lambda^2)\kappa-pr\tau}{\lambda r}>0,\ \ \bar{\sigma}=\varepsilon_1\left(\mu(r\kappa+p\tau)+p\lambda\sigma\right), \quad (34)$$

where $r,\ \mu,\ \lambda,\ p$ are non-zero real constants and $\varepsilon_1=\pm1$, $\varepsilon_2=\pm1$.

Since we have $p^2+r^2=1,\ \ \lambda^2+\mu^2=1$, from (33) we also have,

$$T=p\bar{T}+\frac{\varepsilon_1}{\varepsilon_2}r\bar{B}_1, N=\varepsilon_1\lambda\bar{N}+\varepsilon_2\mu\bar{B}_2, B_1=r\bar{T}-\frac{\varepsilon_1}{\varepsilon_2}p\bar{B}_1, B_2=\varepsilon_1\mu\bar{N}-\varepsilon_2\lambda\bar{B}_2. \quad (35)$$

Example 1. *Let consider unit speed special Frenet curve $\alpha(s)$ given by*

$$\alpha(s)=\frac{1}{\sqrt{2}}\left[\frac{1}{2}\sin 2s,\ -\frac{1}{2}\cos 2s,\ \ \frac{1}{3}\sin 3s,\ -\frac{1}{3}\cos 3s\right]. \quad (36)$$

The Frenet vectors of $\alpha(s)$ are obtained as

$$T(s)=\frac{1}{\sqrt{2}}\left(\cos 2s,\ \sin 2s,\ \cos 3s,\ \sin 3s\right), \quad (37)$$

$$N(s)=\frac{1}{\sqrt{13}}\left(-2\sin 2s,\ 2\cos 2s,\ -3\sin 3s,\ 3\cos 3s\right), \quad (38)$$

$$B_1(s)=\frac{1}{\sqrt{2}}\left(\cos 2s,\ \sin 2s,\ -\cos 3s,\ -\sin 3s\right), \quad (39)$$

$$B_2(s)=\frac{1}{\sqrt{13}}\left(-3\sin 2s,\ 3\cos 2s,\ 2\sin 3s,\ -2\cos 3s\right), \quad (40)$$

respectively. Then the curvatures are

$$\kappa=\frac{\sqrt{26}}{2},\ \ \ \tau=\frac{5\sqrt{26}}{26},\ \ \sigma=\frac{6\sqrt{26}}{13}. \quad (41)$$

For real constants

$$r=\frac{1}{3},\ p=\frac{2\sqrt{2}}{3},\ \lambda=\frac{5+26\sqrt{2}}{\sqrt{\left(5+26\sqrt{2}\right)^2+144}},\ \mu=\frac{12}{\sqrt{\left(5+26\sqrt{2}\right)^2+144}}, \quad (42)$$

the conditions (6), (7) and (8) hold. Then $\alpha(s)$ is a (1,3)-Bertrand-donor curve. From (33), (1,3)-Bertrand-direction curve β of $\alpha(s)$ is obtained as

$$\begin{aligned}\beta(s)=\tfrac{1}{3\sqrt{2}}\Big(&\tfrac{2\sqrt{2}+1}{2}\sin 2s+c_1,\ -\tfrac{2\sqrt{2}+1}{2}\cos 2s+c_2,\\ &+\tfrac{2\sqrt{2}-1}{3}\sin 3s+c_3,\ -\tfrac{2\sqrt{2}-1}{3}\cos 3s+c_4\Big)\end{aligned} \quad (43)$$

where c_i; $(1\leq i\leq 4)$ are integration constants.

4. Generalized Involute-Evolute-Direction Curves in E^4

In this section, we will consider a new type of curve pairs. In ref. [7], the authors defined (1,3)-evolute curve and (0,2)-involute curve in E^4 as given in Definition 2. Now, we will show that similar definitions for (1,3)-evolute curve and (0,2)-involute curve in E^4 as direction curves don't exist, i.e., there are no (0,2)-involute-direction curves and (1,3)-evolute-donor curves. For this purpose, let assume the converse, i.e., suppose that (0,2)-involute-direction curves and (1,3)-evolute-donor curves exist. Let $\alpha = \alpha(s)$ be a special Frenet curve in E^4 with arc-length parameter s and $X(s)$ be a unit vector field in the form Eq. (2). Let the special Frenet curve $\gamma(\bar{s}) : I \to E^4$ be an X-direction curve of α. The Frenet vectors and curvatures of α and γ be denoted by $\{T, N, B_1, B_2\}$, κ, τ, σ and $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\}$, $\bar{\kappa}$, $\bar{\tau}$, $\bar{\sigma}$, respectively and let any Frenet vector of α does not coincide with any Frenet vector of γ. By the assumption, let γ be a (0,2)-involute curve of α. Since also γ is direction curve of α let we call γ as (0,2)-involute-direction curve of α and α as (1,3)-evolute-donor curve of γ. Then, the Frenet planes spanned by $\{T, B_1\}$ and $\{\bar{N}, \bar{B}_2\}$ coincide and we have,

$$sp\{T, B_1\} = sp\{\bar{N}, \bar{B}_2\}, \;\; sp\{N, B_2\} = sp\{\bar{T}, \bar{B}_1\}. \tag{44}$$

Similar to the (1,3)-Bertrand-direction curves, since γ is an integral curve of $X(s)$ and $X(s)$ is unit, for the arc-length parameter $\bar{s}$ of γ we have $\bar{s} = \int_0^s \left\| \frac{d\gamma}{ds} \right\| ds = \int_0^s ds = s$. Then, hereafter the prime will show the derivative with respect to s.

Theorem 3. *No C^∞-special Frenet curve in E^4 is a (1,3)-evolute-donor curve.*

Proof. First, we will show that if such curves exist, then the special Frenet curve $\alpha : I \to E^4$ is a (1,3)-evolute-donor curve if and only if there exist non-zero constants $b,\ d,\ x_1,\ x_2$ such that

$$b^2 + d^2 = 1, \;\; x_1^2 + x_2^2 = 1, \tag{45}$$

$$d\sigma - b\tau = \frac{x_2}{x_1} b\kappa. \tag{46}$$

$$(d^2 - x_2^2)\kappa - x_1 x_2 \tau \neq 0. \tag{47}$$

For this purpose, let define a unit vector valued function $X(s)$ as $X(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s)$ where $a(s)$, $b(s)$, $c(s)$ and $d(s)$ are differentiable scalar functions of arc-length parameter s. Let the special Frenet curve $\gamma : I \to E^4$ be integral curve of $X(s)$ and also be (0,2)-involute-direction curve of $\alpha(s)$. Then, we have

$$\bar{T}(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s). \tag{48}$$

By assumption, $\bar{T} \perp sp\{T, B_1\}$. Then, taking the inner product of (48) with T and B_1, we have $a(s) = 0,\ c(s) = 0$, respectively, and (48) becomes

$$\bar{T}(s) = b(s)N + d(s)B_2, \;\; b^2(s) + d^2(s) = 1. \tag{49}$$

Now, differentiating the first equation in (49) with respect to s, it follows

$$\bar{\kappa}\bar{N} = -b\kappa T + b'N + (b\tau - d\sigma)B_1 + d'B_2. \tag{50}$$

Taking the inner product of (50) with N and B_2 and considering (44), we get $b' = 0,\ d' = 0$, respectively, i.e., $b,\ d$ are non-zero constants. Also, we have $b^2 + d^2 = 1$, the first equality in (45).

Now, (50) becomes

$$\bar{\kappa}\bar{N} = -b\kappa T + (b\tau - d\sigma)B_1. \tag{51}$$

From (51), it follows

$$\bar{\kappa} = \sqrt{(b\kappa)^2 + (b\tau - d\sigma)^2}. \tag{52}$$

Let define

$$x_1 = \frac{-b\kappa}{\sqrt{(b\kappa)^2 + (b\tau - d\sigma)^2}}, \quad x_2 = \frac{b\tau - d\sigma}{\sqrt{(b\kappa)^2 + (b\tau - d\sigma)^2}}. \tag{53}$$

Then, (51) becomes

$$\bar{N} = x_1 T + x_2 B_1, \quad x_1^2 + x_2^2 = 1. \tag{54}$$

Since, any Frenet vector of α does not coincide with any Frenet vector of γ, we have $x_1 \neq 0,\ x_2 \neq 0$. Differentiating the first equation in (54) with respect to s, we get

$$-\bar{\kappa}\bar{T} + \bar{\tau}\bar{B}_1 = x_1'T + (x_1\kappa - x_2\tau)N + x_2'B_1 + x_2\sigma B_2. \tag{55}$$

Taking the inner product of (55) with T and B_1, we get $x_1' = 0,\ x_2' = 0$, respectively, i.e., $x_1,\ x_2$ are non-zero real constants. Then, from (54), we have the second equality in (45).

Moreover, from (52) and (53), it follows

$$x_1\bar{\kappa} = -b\kappa, \quad x_2\bar{\kappa} = b\tau - d\sigma, \tag{56}$$

which gives us $d\sigma - b\tau = \frac{x_2}{x_1}b\kappa$, we get (46).

Now, writing (49) and (56) in (55) gives

$$\bar{\tau}\bar{B}_1 = \frac{(d^2 - x_2^2)\kappa - x_1x_2\tau}{x_1}N + \frac{-bd\kappa + x_1x_2\sigma}{x_1}B_2. \tag{57}$$

From (46), we get

$$\sigma x_1 d = x_1 b\tau + x_2 b\kappa. \tag{58}$$

Writing (58) in (57) and using (46), we have,

$$\bar{\tau}\bar{B}_1 = \zeta\left(N - \frac{b}{d}B_2\right), \tag{59}$$

where

$$\zeta = \frac{(d^2 - x_2^2)\kappa - x_1x_2\tau}{x_1}. \tag{60}$$

Since $\bar{B}_1 \neq 0$, it should be $(d^2 - x_2^2)\kappa - x_1x_2\tau \neq 0$. Then we have (47).

Conversely, assume that relations (45), (46) and (47) hold for some non-zero constants $b,\ d,\ x_1,\ x_2$ and α be a special Frenet curve with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\kappa,\ \tau,\ \sigma$. Let define a vector valued function

$$X(s) = bN(s) + dB_2(s), \tag{61}$$

and let $\gamma : I \to E^4$ be an integral curve of $X(s)$. We will show that γ is a (0,2)-involute-direction curve of α. Since $\bar{T}(s) = X(s)$, differentiating (61) with respect to s gives

$$\bar{\kappa}\bar{N} = -b\kappa T + (b\tau - d\sigma)B_1. \tag{62}$$

Writing (46) in (62), we have

$$\bar{\kappa}\bar{N} = -b\kappa\left(T + \frac{x_2}{x_1}B_1\right). \tag{63}$$

From (63), it follows

$$\bar{\kappa} = \xi_1\frac{b\kappa}{x_1}, \tag{64}$$

where $\xi_1 = \pm 1$ such that $\bar{\kappa} > 0$. Writing (64) in (63) gives

$$\bar{N} = -\xi_1\left(x_1T + x_2B_1\right). \tag{65}$$

By differentiating (65) with respect to s, we get

$$\bar{N}' = -\xi_1\left((x_1\kappa - x_2\tau)N + x_2\sigma B_2\right). \tag{66}$$

Using (61), (64) and (66), we have

$$\bar{N}' + \bar{\kappa}\bar{T} = \frac{\xi_1}{x_1}\big((x_1x_2\tau + (x_2^2 - d^2)\kappa)N + (bd\kappa - x_1x_2\sigma)B_2\big). \tag{67}$$

Writing (46) in (67) and using (45), (67) becomes

$$\bar{N}' + \bar{\kappa}\bar{T} = \xi_1\frac{(x_2^2 - d^2)\kappa + x_1x_2\tau}{x_1}\left(N - \frac{b}{d}B_2\right). \tag{68}$$

From (68) and (47), we have

$$\bar{\tau} = \left\|\bar{N}' + \bar{\kappa}\bar{T}\right\| = \xi_2\frac{(x_2^2 - d^2)\kappa + x_1x_2\tau}{x_1d} \neq 0, \tag{69}$$

where $\xi_2 = \pm 1$ such that $\bar{\tau} > 0$. Then, we get

$$\bar{B}_1 = \frac{1}{\bar{\tau}}\left(\bar{N}' + \bar{\kappa}\bar{T}\right) = \frac{\xi_1}{\xi_2}\left(dN - bB_2\right). \tag{70}$$

Considering (61), (65) and (70), we can define a unit vector

$$\bar{B}_2 = \frac{1}{\xi_2}\left(-x_2T + x_1B_1\right), \tag{71}$$

and the necessary condition $\det(\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2) = 1$ for the Frenet frame holds. Using (70) and (71), we obtain

$$\bar{\sigma} = \left\langle \bar{B}_1', \bar{B}_2\right\rangle = \xi_1\left(dx_2\kappa + x_1(d\tau + b\sigma)\right). \tag{72}$$

If we assume that $\bar{\sigma} = 0$, then we have $x_1(d\tau + b\sigma) = -dx_2\kappa$. Multiplying that with b, we get $x_1(bd\tau + b^2\sigma) = -bdx_2\kappa$. Since $b^2 = 1 - d^2$, the last equality becomes $x_1(-d(d\sigma - b\tau) + \sigma) = -bdx_2\kappa$. Using (46), it follows $x_1\sigma = 0$, which is a contradiction since $x_1 \neq 0$ and α is a special Frenet curve. Then, $\bar{\sigma} \neq 0$, i.e., γ is a special Frenet curve. Consequently, since $b,\ d,\ x_1,\ x_2$ are non-zero constants, from (61), (65), (70) and (71), we get $sp\{T, B_1\} = sp\{\bar{N}, \bar{B}_2\}$ and no Frenet vectors of α and γ coincide. So, we have that γ is (0,2)-involute-direction curve of α.

Furthermore, from Definition 2, the parametric form of γ is $\gamma(s) = \alpha(s) + (c - s)T(s) + kB_1(s)$ where $c,\ k$ are real constants. Differentiating that with respect to s and using the equality $\bar{T} = bN + dB_2$, we have

$$bN + dB_2 = ((c - s)\kappa - k\tau)\, N + k\sigma B_2$$

which gives that

$$\kappa(c - s) = b + k\tau, \quad k\sigma = d. \tag{73}$$

From (45)-(47) and (73), we have that if the special Frenet curve $\alpha : I \to E^4$ is a (1,3)-evolute-donor curve then there exists a linear relation

$$c_3\kappa + c_4\tau = \sigma \tag{74}$$

where c_3, c_4, σ are non-zero constants and κ, τ, σ are Frenet curvatures of α. From (74), we have that if κ (or respectively τ) is constant, then τ (or respectively κ) must be constant. But considering (73), it follows if the first curvature κ (or respectively τ) is constant, then τ (or respectively κ) is always non-constant which is a contradiction and that finishes the proof. □

5. Conclusions

There is no Bertrand curves in E^4 given by the classical definition that Bertrand curves have common principal normal lines. Then, a new type of Bertrand curves have been introduced in [12] and called (1,3)-Bertrand curves. We considered this definition with integral curves and define (1,3)-Bertrand-direction curves and (1,3)-Bertrand-donor curves. Necessary and sufficient conditions for a curve to be a (1,3)-Bertrand-donor curve have been introduced. Moreover, we investigated whether (1,3)-evolute-donor curves in E^4 exist and show that there is no (1,3)-evolute-donor curve in E^4.

Declaration of Competing Interests The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Bertrand, J., Mémoire sur la théorie des courbes à double courbure, Comptes Rendus 36, *Journal de Mathématiques Pures et Appliquées.*, 15 (1850), 332-350.

[2] Choi, J.H., Kim, Y.H., Associated curves of a Frenet curve and their applications, *Applied Mathematics and Computation*, 218 (2012), 9116-9124. https://doi.org/10.1016/j.amc.2012.02.064

[3] Fuchs, D., Evolutes and involutes of spatial curves, *American Mathematical Monthly*, 120(3) (2013), 217-231. https://doi.org/10.4169/amer.math.monthly.120.03.217

[4] Fukunaga, T., Takahashi, M., Evolutes and involutes of frontals in the euclidean plane, *Demonstratio Mathematica*, 48(2) (2015), 147-166. https://doi.org/10.1515/dema-2015-0015

[5] Fukunaga, T., Takahashi, M., Involutes of fronts in the Euclidean plane, *Beitrage zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 57(3) (2016), 637-653. https://doi.org/10.1007/s13366-015-0275-1

[6] Gere, B.H., Zupnik, D., On the construction of curves of constant width, *Studies in Applied Mathematics*, 22(1-4) (1943), 31-36.

[7] Hanif, M., Hou, Z.H., Generalized involute and evolute curve-couple in Euclidean space, *Int. J. Open Problems Compt. Math.*, 11(2) (2018), 28-39.

[8] Huygens, C., Horologium oscillatorium sive de motu pendulorum ad horologia aptato, *Demonstrationes Geometricae*, 1673.

[9] Li, Y., Sun, G.Y., Evolutes of fronts in the Minkowski Plane, *Mathematical Methods in the Applied Science*, 42(16) 2018, 5416-5426. https://doi.org/10.1002/mma.5402

[10] Macit, N., Düldül, M., Some new associated curves of a Frenet curve in E^3 and E^4, *Turk J Math.*, 38 (2014), 1023-1037. https://doi.org/10.3906/mat-1401-85

[11] Matsuda, H., Yorozu, S., On generalized Mannheim curves in Euclidean 4-space, *Nihonkai Math. J.*, 20 (2009), 33-56.

[12] Matsuda, H., Yorozu, S., Notes on Bertrand curves, *Yokohama Mathematical Journal*, 50 (2003), 41-58.

[13] Nutbourne, A.W., Martin, R.R., Differential Geometry Applied to Design of Curves and Surfaces, Ellis Horwood, Chichester, UK, 1988.

[14] Önder, M., Construction of curve pairs and their applications, *Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 91(1) 2021, 21-28. https://doi.org/10.1007/s40010-019-00643-2

[15] Öztürk, G., Arslan, K., Bulca, B., A Characterization of involutes and evolutes of a given curve in E^n. *Kyungpook Math. J.*, 58 (2018), 117-135.

[16] Özyılmaz, E., Yılmaz, S., Involute-evolute curve couples in the Euclidean 4-space, *Int. J. Open Problems Compt. Math.*, 2(2) (2009), 168-174.

[17] Struik, D.J., Lectures on Classical Differential Geometry, 2^{nd} ed. Addison Wesley, Dover, 1988.

[18] Yu, H., Pei, D., Cui, X., Evolutes of fronts on Euclidean 2-sphere, *J. Nonlinear Sci. Appl.*, 8 (2015), 678-686. http://dx.doi.org/10.22436/jnsa.008.05.20

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A STUDY ON SET-CORDIAL GRAPHS

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ABSTRACT. For a non-empty ground set X, finite or infinite, the *set-valuation* or *set-labeling* of a given graph G is an injective function $f : V(G) \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of the set X. In this paper, we introduce a new type of set-labeling, called set-cordial labeling and study the characteristics of graphs which admit the set-cordial labeling.

1. INTRODUCTION

For all terms and definitions, not defined specifically in this paper, we refer to [11] and for further terminology on graph classes, we refer to [3]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

After the introduction of the notion of β-valuations of graphs in [8], studies on graph labeling problems have emerged as a major research area. It is estimated that more than two thousand research articles have been published since then. Interested readers may refer to [6] for a detailed literature and for further investigation on graph labeling problems.

As an extension of the number valuation of graphs, the notion of *set-indexers* of graphs has been introduced in [1] as an injective set-valued function $f : V(G) \to P(X)$ such that the induced function $f^* : E(G) \to P(X) - \{\emptyset\}$, defined by $f^*(uv) = f(u) * f(v)$ is also injective, where X is a non-empty set, $P(X)$ is the power set of X and $*$ is a binary operation between the elements of $P(X)$. Note that in the literature, $*$ is the symmetric difference of two sets. In [1], it is proved that every graph admits a set-indexer.

In this paper, a set-labeling of a graph G is an injective function $f : V(G) \to P(X)$. Motivated by the studies on the number valuations and set-valuations of

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graphs, mentioned above, in this paper, we introduce a particular type of set-labeling called set-cordial labeling and study the characteristics of graphs which admit this type of labeling.

2. Set-Cordial Graphs

We define the notion of the set-cordial labeling of a graph as follows:

Definition 1. Let X be a non-empty set and $f : V(G) \to P(X)$ be a set-labeling defined on a graph G. Then, f is said to be a *strict set-cordial labeling* or simply, a *set-cordial labeling* of G if $|f(v_i)| - |f(v_j)| = \pm 1$ for all $v_iv_j \in E(G)$. A graph which admits a set-cordial labeling is called a *set-cordial graph.*

Definition 2. The minimum cardinality of a ground set X with respect to which a given graph G admits a set-cordial labeling is called *the set-cordiality index* of G, denoted by $\varsigma(G)$.

An illustration of set-cordial graphs is provided in Figure 1.

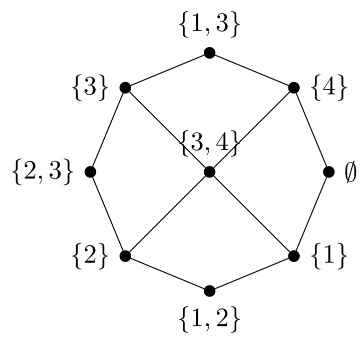

FIGURE 1. An illustration to a set-cordial graph.

In Figure 1, it can be noticed that the set-cordial index of the graph G is 4 as the minimal ground set is $X = \{1, 2, 3, 4\}$.

Next, we discuss the admissibility of set-cordial labeling by certain fundamental graph classes. In order to consider set-cordial labelings on paths on n vertices, we first show that the hypercube graph Q_n contains a Hamiltonian path.

Lemma 1. *Every hypercube graph Q_n contains a Hamiltonian path. Furthermore, if $n \geq 2$, then Q_n has a Hamiltonian cycle.*

Proof. We first observe that $Q_1 = K_2$ and hence Q_2 itself is a Hamiltonian path. For any positive integer $n \geq 2$, let $v_1 - v_2 - \ldots - v_{2^{n-1}}$ be the list of vertices in a Hamiltonian path in Q_{n-1}. Then, the list of vertices

$$(v_1, 0), (v_2, 0), \ldots, (v_{2^{n-1}}, 0), (v_{2^{n-1}}, 1), (v_{2^{n-1}-1}, 1), \ldots, (v_2, 1), (v_1, 1)$$

is a Hamiltonian path in Q_n, and the list of vertices

$$(v_1,0),(v_2,0),\ldots,(v_{2^{n-1}},0),(v_{2^{n-1}},1),(v_{2^{n-1}-1},1),\ldots,(v_2,1),(v_1,1),(v_1,0)$$

is a Hamiltonian cycle in Q_n as required. □

Recall that a connected bipartite graph G with bipartition (X,Y), is called *Hamilton-laceable* (see [9]), if it has a $u-v$ Hamiltonian path for all pairs of vertices $u \in X$ and $v \in Y$. The hypercube Q_n is a bipartite Cayley graph on the Abelian group $\mathbb{Z}_2^n = \prod_n \mathbb{Z}_2$ with the natural generating set $S = \{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,0,\ldots,0,1)\}$. It is proved in [4] that a connected bipartite Cayley graph on an Abelian group is Hamiltonian laceable.

In view of the above-mentioned concepts, the following theorem discusses the admissibility of set-cordial labeling by a path and the corresponding set-cordiality index.

Theorem 1. *Every path P_n is set-cordial. Furthermore, $\varsigma(P_n) = \lceil \log_2 n \rceil$.*

Proof. Let P_n denotes a path of order n, whose vertices are consecutively named by $v_1, v_2, \ldots, v_n$. Let $X = \{x_1, x_2, \ldots, x_{n-1}\}$ be the ground set for labeling. Start labeling the vertex v_1 by the empty set $\emptyset$. For $2 \le i \le n$, label vertices v_i by the set $\{x_1, x_2, \ldots, x_{i-1}\}$. Clearly, $f(v_{i+1}) - f(v_i) = \{x_{i-1}\}$ for $0 \le i \le n-1$. Therefore, f is a set-cordial labeling of P_n.

Let $k = \lceil \log_2 n \rceil$. Then $n \le 2^k < 2n$. By Lemma 1, let $v_1, v_2, \ldots, v_{2^k}$ be the list of vertices in a Hamiltonian path in the hypercube Q_k. Let $X = \{1,2,3,\ldots,k\}$. We can identify each vertex of Q_k with a unique element in $P(X)$ and hence we identify the path P_n with the subpath $v_1 - v_2 - \ldots - v_n$ in Q_k. Thus, the set-labeling on P_n given by $f(v_i) = v_i$ is a set-cordial labeling on P_n. Since $n > 2^{k-1}$, there is no set-labeling on P_n that uses a ground set with fewer than k elements. Hence, $\varsigma(Pn) = k = \lceil \log_2 n \rceil$. This completes the proof. □

Theorem 2. *A graph G admits a set-cordial labeling if and only if G is bipartite.*

Proof. Let G be a bipartite graph with bipartition (X,Y). Choose the set $\mathbb{N}$ of natural numbers as the ground set for labeling. For any positive integer k, assign distinct k-element subsets of $\mathbb{N}$ to distinct vertices in X and distinct $(k+1)$-element subsets of $\mathbb{N}$ to distinct vertices in X. Clearly, this labeling is a set-cordial labeling of G.

Conversely, assume that G is a set-cordial graph and let $f : V(G) \to P(A)$ be a set-cordial labeling on G. Let X and Y be the partite sets of G defined by

$$\begin{aligned} X &= \{v \in V(G) : |f(v)| \text{ is even; } \textit{and}\} \\ Y &= \{v \in V(G) : |f(v)| \text{ is odd}\}. \end{aligned}$$

Let $u, v \in X$. Since $f(u)$ and $f(v)$ have an even number of elements, $|f(u)| - |f(v)|$ is even. Thus, $|f(u)| - |f(v)| \neq \pm 1$. Hence, X is an independent set. A similar argument shows that Y is also an independent set. Since $V(G) = X \cup Y$, G is bipartite, completing the proof. □

The following theorem characterises the cycles which admit set-cordial labeling.

Theorem 3. *A cycle C_n admits a set-cordial labeling if and only if n is even. Furthermore, $\varsigma(P_n) = \lceil \log_2 n \rceil$.*

Proof. First part of the theorem is an immediate consequence of Theorem 2. Hence, we shall now determine the set-cordiality index of cycles. By Lemma 1, let $v_1, v_2, \ldots, v_{2^k}$ be the list of vertices in a Hamiltonian path in the hypercube Q_k. Let $X = \{1, 2, 3, \ldots, k\}$.

Let $n = 2m, m \in \mathbb{N}_0$ and $k = \lceil \log_2 n \rceil$. Then, by Lemma 1, we have a list of vertices

$$(v_1, 0), (v_2, 0), \ldots, (v_{2^{k-1}}, 0), (v_{2^{k-1}}, 1), (v_{2^{k-1}-1}, 1), \ldots, (v_2, 1), (v_1, 1), (v_1, 0),$$

which are in a Hamiltonian path in Q_k, where $v_1, v_2, \ldots, v_{2^{k-1}}$ are the vertices in the Hamiltonian path in the hypercube Q_{k-1}. Also, we can identify a cycle of length $n = 2m$ in Q_k, whose vertices are

$$(v_1, 0), (v_2, 0), \ldots, (v_{m-1}, 0), (v_m, 0), (v_m, 1), (v_{m-1}, 1), \ldots, (v_2, 1), (v_1, 1), (v_1, 0).$$

Now, let $X = \{1, 2, 3, \ldots, k\}$. As explained in the proof of Theorem 1, we can identify each vertex of Q_k with a unique element in $P(X)$ and hence we identify the cycle C_n with the sub-cycle in Q_k. Thus, the set-labeling on C_n given by $f(v_i, j) = (v_i, j)$ is a set-cordial labeling on C_n. Since $n > 2^{k-1}$, in this case also, we have no set-labeling on C_n that uses a ground set with fewer than k elements. Hence, $\varsigma(Cn) = k = \lceil \log_2 n \rceil$, completing the proof. □

In view of Theorem 2, we notice that graphs consisting of odd cycles will not admit set-cordial labelings. Therefore, the fundamental graph classes like wheel graphs, friendship graphs and helm graphs do not admit a set-cordial labeling. Also, we note that a complete graph K_n admits a set-cordial labeling if and only if $n \leq 2$.

Suppose that a and b are positive integers such that $a \leq b$. Let $\alpha = \alpha(a, b)$ be the smallest positive integer such that

$$a \leq \binom{2\alpha}{\alpha} \text{ and } b \leq \binom{2\alpha}{\alpha - 1} + \binom{2\alpha}{\alpha + 1}.$$

Similarly, define $\beta = \beta(a, b)$ as the smallest positive integer such that

$$a \leq \binom{2\beta + 1}{\beta + 1} \text{ and } b \leq \binom{2\beta + 1}{\beta} + \binom{2\beta + 1}{\beta + 2}.$$

Using the above notations,the set-cordiality index of a complete bipartite graph is determined in the following theorem.

Theorem 4. *A complete bipartite graph $K_{a,b}$, where $a \leq b$, admits a set-cordial labeling. Furthermore,* $\varsigma(K_{a,b}) = \min\{2\alpha, 2\beta + 1\}$.

Proof. Let (A, B) be the bipartition of $K_{a,b}$ such that $|A| = a \leq |B| = b$. Assume that f is a set-cordial labeling of $K_{a,b}$ with respect to the minimal ground set X. Then, $f(v_i)$ can be an empty set, a single set or a 2-element set. We try to label the vertices of A by singleton subsets of the ground set X and label one vertex of B with empty set and other vertices by 2-element subset of X. This labeling is possible only when $b - 1$ is less than or equal to the number of 2-element subsets of the set $\bigcup\limits_{v \in A} f(v)$. If this condition holds, then f is a set-cordial labeling which yields the minimum ground set $\bigcup\limits_{v \in A} f(v)$. If this condition does not hold, we cannot label the vertices in A by singleton subsets of X and as a result, the vertices of B must be labeled by singleton subsets of X. In this case, $a - 1$ will be less than the number 2-element combinations of the set $\bigcup\limits_{v \in B} f(v)$ and f will be a set-cordial labeling of $K_{a,b}$.

Now, we shall determine the set-cordiality number of $K_{a,b}$. Here, the following two cases are to be addressed.

Case-1: Let n be even, say $n = 2m, m \in \mathbb{N}_0$. Let X be a set containing $n = 2m$ elements, and let $f : V(K_{a,b}) \to P(X)$ be a set-cordial labeling on $K_{a,b}$. Suppose that there exists a vertex u_0 in one partite set of $K_{a,b}$ such that $|f(u_0)| = k$, and there exist vertices v_0 and w_0 in the other partite set of $K_{a,b}$ such that $|f(v_0)| = k-1$ and $|f(w_0)| = k + 1$. Since $|f(u)| - |f(v_0)| = \pm 1$ and $|f(u)| - |f(w_0)| = \pm 1$ for all u in the first partite set, we have $|f(u)| = k$, for all u in the first partite set. Since $|f(v)| - |f(u_0)| = \pm 1$, for all v in the second partite set, we have either $|f(v)| = k-1$ or $|f(v)| = k + 1$. Since $a \leq b$ and

$$\binom{2m}{k} \leq \binom{2m}{k-1} + \binom{2m}{k+1},$$

we have

$$a \leq \binom{2m}{k} \text{ and } b \leq \binom{2m}{k-1} + \binom{2m}{k+1}.$$

Since for all $1 \leq k \leq 2m - 1$,

$$\binom{2m}{k} \leq \binom{2m}{m},$$

we have $k = m$. Thus, $\varsigma(K_{a,b}) = 2m = 2\alpha$.

Case-2: Let n be odd, say $n = 2m + 1, m \in \mathbb{N}_0$. Let X be a set containing $n = 2m+1$ elements and let $f : V(K_{a,b}) \to P(X)$ be a set-cordial labeling on $K_{a,b}$. An argument similar to that in the above paragraph shows that we have $|f(u)| = k$, for all u in one partite set, and either $|f(v)| = k - 1$ or $|f(v)| = k + 1$ for all v in the other partite set. Since $a \leq b$ and

$$\binom{2m+1}{k} \leq \binom{2m+1}{k-1} + \binom{2m+1}{k+1},$$

we have

$$a \leq \binom{2m+1}{k} \text{ and } b \leq \binom{2m+1}{k-1} + \binom{2m+1}{k+1}.$$

Since for all $1 \leq k \leq 2m$,

$$\binom{2m+1}{k} \leq \binom{2m+1}{m+1},$$

we have $k = m+1$. Thus, $\varsigma(K_{a,b}) = 2m+1 = 2\beta+1$. From, the above two cases, we have $\varsigma(K_{a,b}) = \min\{2\alpha, 2\beta+1\}$, completing the proof. □

Figure 2 illustrates a set-cordial labeling of a complete bipartite graph $K_{6,7}$, with respect to the ground set $X = \{1, 2, 3, 4\}$.

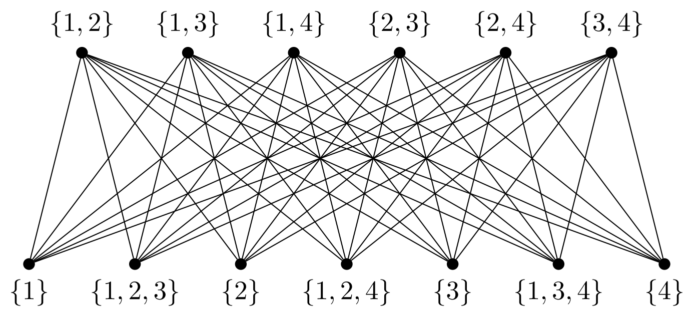

FIGURE 2. An illustration to a set-cordial labeling of $K_{6,7}$.

3. Glutting Number of a Graph

As a consequence of Theorem 2, non-bipartite graphs do not admit a set-cordial labeling. But, by the removal of certain edges from the graph will make the graph set-cordial. Hence, we have the following notion:

Definition 3. The *glutting number* of a graph G, denoted by $\xi(G)$, is the minimum number of edges of G to be removed so that the reduced graph admits a set-cordial labeling.

In view of Theorem 2, we note that the glutting number of a bipartite graph is 0. Therefore, $\xi(P_n) = 0$.
The following discusses the glutting number of a cycle C_n.

Proposition 1.

$$\xi(C_n) = \begin{cases} 0; & \textit{if } n \textit{ is even} \\ 1; & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. The proof is straight forward from Theorem 3. □

We shall now discuss the glutting number of certain fundamental graph classes. Recall that a wheel graph is defined by $W_{1,n} = K_1 + C_n$. The following result discusses the glutting number of a wheel graph.

Proposition 2. $\xi(W_{1,n}) = \begin{cases} \frac{n}{2}; & \textit{if } n \textit{ is even;} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. Note that every edge incident on the central vertex of $W_{1,n}$ is contained in exactly two triangles of $W_{1,n}$. So, removal of any such edge will result in the removal of two triangles in W_n. Also, there are n triangles in $W_{1,n}$. Here, we have to address the following two cases:

Case-1: Let n be even. Then, we need to remove $\frac{n}{2}$ edges incident on the central vertex to make the graph triangle free. Then, the reduced graph has girth 4 and has no odd cycles. Hence, in this case, $\xi(W_{1,n}) = \frac{n}{2}$.

Case-2: Let n be odd. Then, the outer cycle C_n is an odd cycle and hence one edge, say e, must be removed from C_n. Now, there exist $n-1$ triangles in the graph $W_n - e$. Since, $n-1$ is even, we need to remove $\frac{n-1}{2}$ edges from $W_n - e$ to make it triangle free. After the removal of this much edges, the reduced graph has girth 4 and has no odd cycles (see Figure 3, for example). Therefore, in this case, $\xi(W_n) = 1 + \frac{n-1}{2} = \frac{n+1}{2}$. □

A *helm graph* $H_{1,n}$ is the graph obtained from a wheel graph $W_{1,n}$ by attaching one pendant edge to each vertex of the outer cycle C_n of W_n. Then, we have

Proposition 3. $\xi(H_{1,n}) = \begin{cases} \frac{n}{2}; & \textit{if } n \textit{ is even;} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. The proof is exactly as in the proof of Theorem 2. □

A *closed helm* $CH_{1,n}$ is the graph obtained from a helm graph H_n by joining the pendant vertices of H_n so as to form an outer cycle of length n. Then, we have

Proposition 4. $\xi(H_n) = \begin{cases} \frac{n}{2}; & \textit{if } n \textit{ is even;} \\ \frac{n+3}{2} & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. In CH_n, the central vertex is contained in all triangles. Hence, the only thing to be noted here is that if n is odd, we need to remove one edge each from inner and outer cycles. Then, the proof is exactly as in the proof of Theorem 2. □

The following theorem determines the glutting number of a complete graph K_n.

Theorem 5. $\xi(K_n) = \begin{cases} \frac{1}{4}n(n-2); & \textit{if } n \textit{ is even;} \\ \frac{1}{4}(n-1)^2; & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. Consider the complete graph K_n. Let G be a spanning subgraph of K_n such that $\xi(K_m) = |E(K_n)| - |E(G)|$. By Theorem 2, G is bipartite. Since $|E(K_n)| - |E(G)|$ is a minimum among all bipartite spanning subgraphs G of K_n, G is a complete bipartite spanning subgraph of K_n. Let A and B be partite vertex sets of G such that $|A| = k$ and $|B| = m - k$. Since A and B are independent sets in G, we have

$$\begin{aligned} \xi(K_n) &= \frac{1}{2}k(k-1) + \frac{1}{2}(n-k)(n-k-1) \\ &= \frac{n^2}{4} - \frac{n}{2} + \left(k - \frac{n}{2}\right)^2 \end{aligned}$$

Here, we have to address the following cases:

Case-1: Let n even. Thus, there exists a positive integer m such that $n = 2m$. Then, $\xi(K_n) = n^2 - n + (k-n)^2$. This value is a minimum when $k = m$. Thus, $\xi(K_n) = m^2 - m = \frac{n^2-n}{4}$.

Case-2: Suppose n is odd. Let m be the positive integer such that $n = 2m+1$. Then, $\xi(K_n) = n^2 - \frac{1}{4} + (k - n - \frac{1}{2})^2$. This value is a minimum when either $k = m$ or $k = m+1$. Thus $\xi(K_n) = m^2 = \frac{(n-1)^2}{4}$. This completes the proof. □

4. Some Variations of Set-Cordial Labeling

Definition 4. Let X be a non-empty set and $f : V(G) \to X$ be a set-labeling defined on a graph G. Then, f is said to be a *weakly set-cordial labeling* of G if $||f(v_i)| - |f(v_j)|| \leq 1$ for all $v_iv_j \in E(G)$. A graph which admits a set-cordial labeling is called a *weakly set-cordial graph.*

Theorem 6. *Every graph G admits a weakly set-cordial labeling.*

Proof. If G is bipartite, the theorem follows by Theorem 2. So, let G be a non-bipartite graph. Let I and be a maximal independent of G. Then, it is possible to choose the ground set X, sufficiently large, in such a way that

(i) all vertices in $G - I$ can be labeled by distinct singleton subsets of X,
(ii) one vertex of I is labeled by the empty set and other vertices can be labeled by distinct 2-element subsets of X.

Clearly, this labeling will be a set-cordial labeling of G, completing the proof. □

Observation 7. It can be noted that the glutting number of G is equal to the number of edges uv in G having $||f(u)| - |f(v)|| = 0$, with respect to a weakly set-cordial labeling f.

Figure 3 depicts a weakly set-cordial labeling of a wheel graph. The dashed lines represent the edges uv with $||f(u)| - |f(v)|| = 0$.

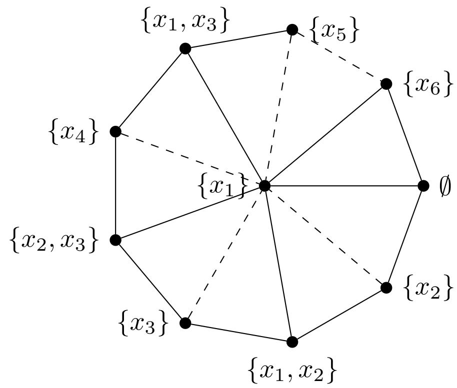

FIGURE 3. A weakly set-cordial labeling of a wheel graph.

5. Conclusion

In this article, we have introduced a particular type of set-labeling, called set-cordial labeling, of graphs and discussed certain properties of graphs which admits this type type of labeling. A couple of new graph parameters, related to the set-cordial labeling have also been introduced. These graph parameters seem to be promising for further studies. The set-cordial labeling of the operations, products and certain derived graphs of given set-cordial graphs can also be studied in detail. The newly introduced parameters can also be studied. All these facts highlight the wide scope for further research in this area.

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References

[1] Acharya, B. D., Set-Valuations and Their Applications, MRI Lecture Notes in Applied Mathematics, The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1983.

[2] Bondy, J. A., Murty, U. S. R., Graph Theory, Springer, New York, 2008.

[3] Brandstädt, A., Le, V. B., Spinrad, J. P., Graph Classes: A Survey, SIAM, Philadelphia, 1999.
[4] Chen, C. C., Quimpo, N. F., On Strongly Hamiltonian Abelian Group Graphs, In: McAvaney K.L., Combinatorial Mathematics VIII, Lecture Notes in Mathematics, vol 884, Springer, Berlin, Heidelberg, 1981.
[5] Deo, N., Graph Theory with Application to Engineering and Computer Science, Prentice Hall of India, New Delhi, 1974.
[6] Gallian, J. A., A dynamic survey of graph labeling, *Electron. J. Combin.*, 2020, # DS-6.
[7] Harary, F., Graph Theory, Narosa Publications, New Delhi, 2001.
[8] Rosa, A., On Certain Valuation of the Vertices of a Graph, Theory of Graphs, Gordon and Breach., Philadelphia, 1967.
[9] Simmons, G. J., Almost All n-Dimensional Rectangular Lattices are Hamilton-Laceable, Proc. 9th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Utilitas Mathematica Publishing, 1978, 649-661.
[10] Weisstein, E. W., CRC Concise Encyclopedia of Mathematics, CRC Press, Boca Raton, 2011.
[11] West, D. B., Introduction to Graph Theory, Pearson Education, Delhi, 2001.
[12] Information System on Graph Classes and Their Inclusions, `http://www.graphclasses.org/smallgraphs`.

http://communications.science.ankara.edu.tr

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THE COMPLEMENTARY NABLA BENNETT-LEINDLER TYPE INEQUALITIES

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Abstract. We aim to find the complements of the Bennett-Leindler type inequalities in nabla time scale calculus by changing the exponent from $0 < \zeta < 1$ to $\zeta > 1$. Different from the literature, the directions of the new inequalities, where $\zeta > 1$, are the same as that of the previous nabla Bennett-Leindler type inequalities obtained for $0 < \zeta < 1$. By these settings, we not only complement existing nabla Bennett-Leindler type inequalities but also generalize them by involving more exponents. The dual results for the delta approach and the special cases for the discrete and continuous ones are obtained as well. Some of our results are novel even in the special cases.

1. Introduction

The theory of inequalities containing series or integrals has been shown to be of great importance due to their effective usage in differential equations and in their applications after the celebrated discrete and continuous inequalities of Hardy have been obtained. In 1920, when Hardy [24] tried to find a simple and elementary proof of Hilbert's inequality [32]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m c_n}{m+n} \le \pi\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty} c_n^2\right)^{1/2},$$

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where $a_m, c_n \geq 0$ and $\sum_{m=1}^{\infty} a_m^2$ and $\sum_{n=1}^{\infty} c_n^2$ are convergent, he showed the following pioneering discrete inequality

$$\sum_{j=1}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{m} c(i)\right)^{\zeta} \leq \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\sum_{j=1}^{\infty} c^{\zeta}(j), \quad c(j)\geq 0,\ \zeta>1 \tag{1}$$

and pioneering continuous inequality for a nonnegative function Γ and for a real constant $\zeta > 1$, as

$$\int_0^{\infty}\left(\frac{1}{t}\int_0^t \Gamma(s)ds\right)^{\zeta} dt \leq \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\int_0^{\infty}\Gamma^{\zeta}(t)dt, \tag{2}$$

where $\int_0^{\infty}\Gamma^{\zeta}(t)dt < \infty$. In fact, Hardy only stated inequality (2) in [24] but did not prove it. Later in 1925, the proof of inequality (2), which depends on the calculus of variations, was shown by Hardy in [25].

The constant $\left(\frac{\zeta}{\zeta-1}\right)^{\zeta}$ that appears in the above inequalities also has been found as the best possible one, since if it is replaced by a smaller constant then inequalities (1) and (2) are not fulfilled anymore for the involved sequences and functions, respectively.

Hardy et al. [26, Theroem 330] developed inequality (2) and derived the following integral inequality for a nonnegative function Γ as

$$\int_0^{\infty}\frac{\Psi^{\zeta}(t)}{t^{\theta}}dt \leq \left|\frac{\zeta}{\theta-1}\right|^{\zeta}\int_0^{\infty}\frac{\Gamma^{\zeta}(t)}{t^{\theta-\zeta}}dt, \quad \zeta>1, \tag{3}$$

where $\Psi(t) = \begin{cases} \int_0^t \Gamma(s)ds, & \text{if } \theta > 1, \\ \int_t^{\infty} \Gamma(s)ds, & \text{if } \theta < 1. \end{cases}$

The exhibition of the results containing the improvements, generalizations and applications of the discrete and continuous Hardy inequalities can be found in the books [7, 26, 32, 33, 38] and references therein.

Since various generalizations and numerous variants of the discrete Hardy inequality (1) exist in the literature, all of which can not be covered here, we only focus on the extensions which have been established by Copson [15, Theorem 1.1, Theorem 2.1]. We refer these inequalities as Hardy-Copson type inequalities. The discrete Hardy inequality (1) or Copson's discrete inequalities were generalized in [9, 14, 34–37] and references therein.

The investigation of the reverse Hardy-Copson inequalities, which are called Bennett-Leindler inequalities, were started almost at the same time with the original inequalities.

The first reverse discrete Hardy-Copson inequalities were obtained by Hardy and Littlewood [23] in 1927 for $0 < \zeta < 1$ without finding the best possible constants. Then Copson [15], Bennett [10] and Leindler [35] established discrete Bennett-Leindler inequalities by means of the following: Assume that the sequences z and h are nonnegative. If $0 < \zeta < 1$, then

$$\sum_{m=1}^{\infty} \frac{z(m)}{\left[\overline{G}(m)\right]^{\theta}} \left(\sum_{j=m}^{\infty} h(j)z(j)\right)^{\zeta} \geq \zeta^{\zeta} \sum_{m=1}^{\infty} z(m)h^{\zeta}(m) \left[\overline{G}(m)\right]^{\zeta-\theta}, \quad 0 \leq \theta < 1, \tag{4}$$

where $\overline{G}(m) = \sum_{j=1}^{m} z(j)$ and

$$\sum_{m=1}^{\infty} \frac{z(m)}{\left[\overline{G}(m)\right]^{\theta}} \left(\sum_{j=m}^{\infty} h(j)z(j)\right)^{\zeta} \geq \left(\frac{\zeta}{1-\theta}\right)^{\zeta} \sum_{m=1}^{\infty} z(m)h^{\zeta}(m) \left[\overline{G}(m)\right]^{\zeta-\theta}, \quad \theta < 0 \tag{5}$$

and for $0 < L \leq \dfrac{z(m)}{z(m+1)}$,

$$\sum_{m=1}^{\infty} \frac{z(m)}{\left[\overline{G}(m)\right]^{\theta}} \left(\sum_{j=1}^{m} h(j)z(j)\right)^{\zeta} \geq \left(\frac{L\zeta}{\theta-1}\right)^{\zeta} \sum_{m=1}^{\infty} z(m)h^{\zeta}(m) \left[\overline{G}(m)\right]^{\zeta-\theta}, \quad \theta > 1. \tag{6}$$

There are some results in [36] about the reverse discrete Hardy-Copson inequalities different than the above ones and in [19] about finding conditions on the sequence $z(m)$ for $0 < \zeta < 1$ to obtain best possible constant.

The following results are interesting due to the fact that in contrast to the literature, discrete Bennett-Leindler inequalities were obtained for $\zeta > 1$, which is the same interval as for the Hardy-Copson inequalities. In 1986, Renaud [45] established the following discrete Bennett-Leindler inequality for the nonnegative and nonincreasing sequence $h(m)$ whenever $\zeta > 1$ as

$$\sum_{m=1}^{\infty} \frac{1}{m^{\zeta}} \left(\sum_{j=1}^{m} h(j)\right)^{\zeta} \geq Z(\zeta) \sum_{m=1}^{\infty} h^{\zeta}(m), \tag{7}$$

where $Z(\zeta)$ is Riemann-Zeta function.

Similar to the discrete Hardy inequality (1), the continuous versions (2) or (3) have attracted many mathematicians' interests and expansions of these continuous inequalities have appeared in the literature. The first continuous refinements were obtained by Copson [16, Theorem 1, Theorem 3] and after these results many papers were devoted to continuous analogues and continuous improvements of the discrete Hardy-Copson inequalities, see [8, 27, 39, 41, 42].

The first continuous Bennett-Leindler inequality, which is the reverse version of the continuous Hardy-Copson inequality (3), when $\theta = \zeta$, was established in [26, Theorem 337] for $0 < \zeta < 1$ and for $\overline{H}(t) = \int_t^\infty h(s)ds$ as

$$\int_0^\infty \frac{\overline{H}^\zeta(t)}{t^\zeta} dt \geq \left(\frac{\zeta}{1-\zeta}\right)^\zeta \int_0^\infty h^\zeta(t)dt, \quad h(t) \geq 0. \tag{8}$$

Then Copson derived continuous analogues of the discrete Bennett-Leindler inequalities (5) and (6), which are called continous Bennett-Leindler inequalities, in [16, Theorem 4, Theorem 2], respectively, for $z(t) \geq 0$ and $h(t) \geq 0$ and $\overline{G}(t) = \int_0^t z(s)ds$, $H(t) = \int_0^t z(s)h(s)ds$, $\overline{H}(t) = \int_t^\infty z(s)h(s)ds$ in the following manners: If $0 < \zeta \leq 1$, $\theta < 1$ then

$$\int_0^b \frac{z(t)}{[\overline{G}(t)]^\theta} [\overline{H}(t)]^\zeta dt \geq \left(\frac{\zeta}{1-\theta}\right)^\zeta \int_0^b z(t)[\overline{G}(t)]^{\zeta-\theta} h^\zeta(t)dt, \quad 0 < b \leq \infty. \tag{9}$$

If $0 < \zeta \leq 1 < \theta$, $a > 0$, then

$$\int_a^\infty \frac{z(t)}{[\overline{G}(t)]^\theta} [H(t)]^\zeta dt \geq \left(\frac{\zeta}{\theta-1}\right)^\zeta \int_a^\infty z(t)[\overline{G}(t)]^{\zeta-\theta} h^\zeta(t)dt. \tag{10}$$

Unlike the above classical results, for $\zeta > 1$, the continuous counterpart of the discrete Bennett-Leindler inequality (7) was obtained in [45] as follows: Let $\zeta > 1$ and for nonnegative and decreasing function h, we have

$$\int_0^\infty \frac{1}{t^\zeta} \left[\int_0^t h(s)ds\right]^\zeta dt \geq \frac{\zeta}{\zeta-1} \int_0^\infty h^\zeta(t)dt. \tag{11}$$

Following the development of the time scale concept [6, 12, 13, 20, 21], the analysis of dynamic inequalities have become a popular research area and most classical inequalities have been extended to an arbitrary time scale. The surveys [1, 46] and the monograph [3] can be used to see these extended dynamic inequalities for delta approach. Although the nabla dynamic inequalities are less attractive compared to the delta ones, some of the nabla dynamic inequalities can be found in [5, 11, 22, 40, 43].

The growing interest to Hardy-Copson type inequalities take place in the time scale calculus as well and delta unifications of these inequalities are established in the book [4] and in the articles [2, 18, 44, 47, 48, 50–54] whereas their nabla counterparts and extensions can be seen in [29–31] for $\zeta > 1$.

In the delta time scale calculus, the reverse Hardy-Copson type inequalities, which are called delta Bennett-Leindler inequalities, can be found in [17,47,49,54,55] for $0 < \zeta < 1$. These results are unifications of discrete and continuous Bennett-Leindler inequalities mentioned above except the ones in [45]. In addition to delta calculus, the above discrete and continuous Bennett-Leindler inequalities can be unified by nabla calculus and the previous reverse Hardy-Copson type inequalities can be obtained for the nabla case, see [28] for $0 < \zeta < 1$. Then these inequalities are called nabla Bennett-Leindler inequalities.

For our further purposes, we will show the nabla Bennett-Leindler inequalities established for $0 < \zeta < 1$ in [28] and use them in the sequel. As is customary, ρ denotes the backward jump operator and $f^\rho(t) = (f \circ \rho)(t) = f(\rho(t))$.

The following theorem presented in [28, Theorem 3.1] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.1] for $0 < \zeta < 1$.

Theorem 1. *[28] For nonnegative functions z and h, let us define the functions $G(t) = \int_t^\infty z(s)\nabla s$ and $H(t) = \int_a^t z(s)h(s)\nabla s$. If $\theta \leq 0 < \zeta < 1$, then*

$$\int_a^\infty \frac{z(t)}{[G^\rho(t)]^\theta}[H(t)]^\zeta \nabla t \geq \left(\frac{\zeta}{1-\theta}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[G^\rho(t)]^{\zeta-\theta}\nabla t. \tag{12}$$

The following theorem presented in [28, Theorem 3.9] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.3] for $0 < \zeta < 1$.

Theorem 2. *[28] For nonnegative functions z and h, let us define the functions $\overline{G}(t) = \int_a^t z(s)\nabla s$ and $\overline{H}(t) = \int_t^\infty z(s)h(s)\nabla s$. If $\theta \leq 0 < \zeta < 1$, then*

$$\int_a^\infty \frac{z(t)}{[\overline{G}(t)]^\theta}[\overline{H}^\rho(t)]^\zeta \nabla t \geq \left(\frac{\zeta}{1-\theta}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[\overline{G}(t)]^{\zeta-\theta}\nabla t. \tag{13}$$

The following theorem presented in [28, Theorem 3.12] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.4] for $0 < \zeta < 1$.

Theorem 3. *[28] For nonnegative functions z and h, let us define the functions $\overline{G}(t) = \int_a^t z(s)\nabla s$ and $H(t) = \int_a^t z(s)h(s)\nabla s$. For $L = \inf_{t\in\mathbb{T}} \frac{\overline{G}^\rho(t)}{\overline{G}(t)} > 0$, if $0 < \zeta < 1 < \theta$, then*

$$\int_a^\infty \frac{z(t)}{[\overline{G}(t)]^\theta}[H(t)]^\zeta \nabla t \geq \left(\frac{\zeta L^\theta}{\theta-1}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[\overline{G}(t)]^{\zeta-\theta}\nabla t. \tag{14}$$

The following theorem presented in [28, Theorem 3.4] asserts a nabla analogue of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.2] for $0 < \zeta < 1$.

Theorem 4. *[28] For nonnegative functions z and h, let us define the functions $G(t) = \int_t^\infty z(s)\nabla s$ and $\overline{H}(t) = \int_t^\infty z(s)h(s)\nabla s$. If $0 < \zeta < 1 < \theta$, then*

$$\int_a^\infty \frac{z(t)}{[G^\rho(t)]^\theta}[\overline{H}^\rho(t)]^\zeta \nabla t \geq \left(\frac{\zeta}{\theta-1}\right)^\zeta \int_a^\infty z(t)h^\zeta(t)[G^\rho(t)]^{\zeta-\theta}\nabla t. \tag{15}$$

Although delta and nabla Bennett-Leindler type inequalities for the case $0 < \zeta < 1$ have been deeply analyzed, the case $\zeta > 1$ has been investigated neither via nabla and delta approaches nor for continuous and discrete cases. Hence the main aim of this article is to complement aforementioned Bennett-Leindler type inequalities obtained for $0 < \zeta < 1$ to the case $\zeta > 1$ by using nabla and delta time scale calculi without changing the directions of the inequalities derived for $0 < \zeta < 1$. We preserve the directions of the known inequalities since otherwise we obtain the reverse Bennett-Leindler type inequalities, which are called Hardy-Copson type inequalities and have already been established for the case $\zeta > 1$ in delta [53] and nabla settings [29]. Our results are inspired by the papers [28] and [55] which contain nabla and delta Bennett-Leindler type inequalities for the case $0 < \zeta < 1$. We notice that the cases $\theta \leq 0$ and $\theta > 1$ were considered in [28] and [55] while the case $0 \leq \theta < 1$ was not investigated therein. By taking account of another constant $\eta \geq 0$, we not only generalize the nabla and delta Bennett-Leindler type inequalities presented in [28] and [55] for $\eta \geq 0$, but also complement them from the case $0 < \zeta < 1$ to the case $\zeta > 1$. Furthermore novel discrete and continuous Bennett-Leindler type inequalites, which are complementary and generalized inequalities of inequalities (4)-(11), are established for $\zeta > 1$ and $\eta \geq 0$.

The organization of this paper can be seen as follows. The nabla time scale calculus and its main properties are introduced in Section 2. The delta version can be obtained similarly. The contribution of Section 3, which includes the main result, is to extend the recently developed nabla and delta results, which were established for $0 < \zeta < 1$ and presented in [28, 55], to the case $\zeta > 1$ by using the properties of nabla and delta derivatives and integrals. Then the special cases of the nabla and delta $\zeta > 1$ type inequalities, which are continuous and discrete inequalities, are stated.

2. Preliminaries

This section is devoted to present the main definitions and theorems of the nabla time scale calculus. The fundamental theories of the delta and nabla calculi can be found in [6, 12].

If $\mathbb{T} \neq \emptyset$ is a closed subset of $\mathbb{R}$, then $\mathbb{T}$ is called a time scale. If $t > \inf \mathbb{T}$, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := sup\,\{\tau < t : \tau \in \mathbb{T}\}$. The backward graininess function $\nu : \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\nu(t) := t - \rho(t)$, for $t \in \mathbb{T}$.

The ∇-derivative of $\Gamma : \mathbb{T} \to \mathbb{R}$ at the point $t \in \mathbb{T}_\kappa = \mathbb{T}/[\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$ denoted by $\Gamma^\nabla(t)$ is the number enjoying the property that for all $\epsilon > 0$, there exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_\kappa$ such that

$$|\Gamma(s) - \Gamma(\rho(t)) - \Gamma^\nabla(t)(s - \rho(t))| \leq \epsilon |s - \rho(t)|$$

for all $s \in V$.

The nabla derivative satisfies the following.

Lemma 1. *[6, 12] Let $\Lambda : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$.*

(1) *If* Λ *is continuous at a left scattered point* t, *then* Λ *is nabla differentiable at* t *with* $\Lambda^{\nabla}(t) = \dfrac{\Lambda(t) - \Lambda(\rho(t))}{\nu(t)}$.

(2) Λ *is nabla differentiable at a left dense point* t *if and only if the limit* $\Lambda^{\nabla}(t) = \lim\limits_{s \to t} \dfrac{\Lambda(t) - \Lambda(s)}{t - s}$ *exists as a finite number.*

(3) *If* Λ *is nabla differentiable at* t, *then* $\Lambda^{\rho}(t) = \Lambda(t) - \nu(t)\Lambda^{\nabla}(t)$.

A function $\Gamma : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at each left-dense points in $\mathbb{T}$ and $\lim\limits_{s \to t^+} \Gamma(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T}, \mathbb{R})$ denotes the class of real, ld-continuous functions defined on a time scale $\mathbb{T}$.

If $\Gamma \in C_{ld}(\mathbb{T}, \mathbb{R})$, then there exists a function $\overline{\Gamma}(t)$ such that $\overline{\Gamma}^{\nabla}(t) = \Gamma(t)$ and the nabla integral of Γ is defined by $\displaystyle\int_a^b \Gamma(s)\nabla s = \overline{\Gamma}(b) - \overline{\Gamma}(a)$.

Some of the properties of the nabla integral are gathered next.

Lemma 2. *[6, 12] Let* $t_1, t_2, t_3 \in \mathbb{T}$ *with* $t_1 < t_3 < t_2$ *and* $a, b \in \mathbb{R}$. *If* $\Lambda, \Gamma : \mathbb{T} \to \mathbb{R}$ *are ld-continuous, then*

1) $\displaystyle\int_{t_1}^{t_2} [a\Lambda(s) + b\Gamma(s)]\nabla s = a\int_{t_1}^{t_2} \Lambda(s)\nabla(s) + b\int_{t_1}^{t_2} \Gamma(s)\nabla s.$

2) $\displaystyle\int_{t_1}^{t_1} \Lambda(s)\nabla(s) = 0.$

3) $\displaystyle\int_{t_1}^{t_3} \Lambda(s)\nabla s + \int_{t_3}^{t_2} \Lambda(s)\nabla s = \int_{t_1}^{t_2} \Lambda(s)\nabla s = -\int_{t_2}^{t_1} \Lambda(s)\nabla s.$

4) *integration by parts formula holds:*

$$\int_{t_1}^{t_2} \Lambda(s)\Gamma^{\nabla}(s)\nabla s = \Lambda(t_2)\Gamma(t_2) - \Lambda(t_1)\Gamma(t_1) - \int_{t_1}^{t_2} \Lambda^{\nabla}(s)\Gamma(\rho(s))\nabla s.$$

Lemma 3 (Hölder's inequality)**.** *[40] Let* $t_1, t_2 \in \mathbb{T}$. *For* $\Lambda, \Gamma \in C_{ld}([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ *and for constants* $\kappa, \varpi > 1$ *with* $\dfrac{1}{\kappa} + \dfrac{1}{\varpi} = 1$, *Hölder's inequality*

$$\int_{t_1}^{t_2} |\Lambda(s)\Gamma(s)|\nabla s \le \left[\int_{t_1}^{t_2} |\Lambda(s)|^{\kappa}\nabla s\right]^{1/\kappa} \left[\int_{t_1}^{t_2} |\Gamma(s)|^{\varpi}\nabla s\right]^{1/\varpi} \quad \textit{holds true.}$$

If $0 < \kappa < 1$ *or* $\kappa < 0$ *with* $\dfrac{1}{\kappa} + \dfrac{1}{\varpi} = 1$, *then the reversed Hölder's inequality*

$$\int_{t_1}^{t_2} |\Lambda(s)\Gamma(s)|\nabla s \ge \left[\int_{t_1}^{t_2} |\Lambda(s)|^{\kappa}\nabla s\right]^{1/\kappa} \left[\int_{t_1}^{t_2} |\Gamma(s)|^{\varpi}\nabla s\right]^{1/\varpi} \tag{16}$$

is satisfied.

Lemma 4 (Chain rule for the nabla derivative)**.** *[22] If $\Lambda:\mathbb{R}\to\mathbb{R}$ is continuously differentiable and $\Gamma:\mathbb{T}\to\mathbb{R}$ is nabla differentiable, then $\Lambda\circ\Gamma$ is nabla differentiable and*

$$(\Lambda\circ\Gamma)^{\nabla}(s)=\Gamma^{\nabla}(s)\left[\int_0^1 \Lambda'(\Gamma(\rho(s))+h\nu(s)\Gamma^{\nabla}(s))dh\right].$$

3. Bennett-Leindler type inequalities

In the sequel, we will obtain several Bennett-Leindler type inequalities for non-negative, ld-continuous, ∇-differentiable and locally nabla integrable functions z and h and for the functions $G, H, \overline{G}$ and $\overline{H}$ defined in Theorem 1-Theorem 4.

The next theorem, which is proven for $\zeta>1,\ \eta\geq 0$ and $\eta+\theta\leq 0$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0<\zeta<1,\ \eta=0$ and $\theta\leq 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality obtained by Saker et al. [55, Remark 2] or Kayar et al. [28, Remark 3.3].
(b) The continuous inequality obtained by Saker et al. [55, Remark 1] or Kayar et al. [28, Remark 3.2].
(c) The delta counterpart of the nabla inequality (12) in Theorem 1 obtained by Saker et al. [55, Theorem 2.1].
(d) The nabla inequality (12) in Theorem 1 obtained by Kayar et al. [28, Theorem 3.1].

Theorem 5. *Let the functions z, h, G and H be defined as in Theorem 1. For a constant $L_1>0$, assume that $\dfrac{G^{\rho}(t)}{G(t)}\leq L_1$ for $t\in(a,\infty)_{\mathbb{T}}$. Let $\zeta>1,\ \eta\geq 0$ be real constants. If $\eta+\theta\leq 0$, then we have*

(1)

$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t\geq\frac{L_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\int_a^{\infty}\frac{z(t)h(t)[H^{\rho}(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}\nabla t, \tag{17}$$

$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t\geq\left[\frac{L_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty}\frac{z(t)h^{1/\zeta}(t)[H^{\rho}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{18}$$

(2)

$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}}\nabla t\geq\frac{\eta+\zeta}{1-\eta-\theta}\int_a^{\infty}\frac{z(t)h(t)[H^{\rho}(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}\nabla t, \tag{19}$$

$$\int_a^{\infty}\frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}}\nabla t\geq\left[\frac{L_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty}\frac{z(t)h^{1/\zeta}(t)[H^{\rho}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^{\rho}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{20}$$

Proof. The same methodology used in the proof of [28, Theorem 3.1] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.2) in the proof of [28, Theorem 3.1] as
$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t = \int_a^\infty -u(t)\left[H^{\eta+\zeta}(t)\right]^\nabla \nabla t, \tag{21}$$
where $u(t) = -\int_t^\infty \frac{z(s)}{[G(s)]^{\eta+\theta}}\nabla s$. Observe that since $\eta+\zeta>1$,
$$[H^{\eta+\zeta}(t)]^\nabla \geq (\eta+\zeta)z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}, \tag{22}$$
which is different than (3.3) in the proof of [28, Theorem 3.1]. In our case, when $\eta+\theta\leq 0$, since
$$\left[G^{1-\eta-\theta}(t)\right]^\nabla \geq -(1-\eta-\theta)\frac{z(t)}{[G^\rho(t)]^{\eta+\theta}} \geq -(1-\eta-\theta)\frac{z(t)}{L_1^{\eta+\theta}[G(t)]^{\eta+\theta}},$$
using (22) and
$$-u(t) = \int_t^\infty \frac{z(s)\nabla s}{[G(s)]^{\eta+\theta}} \geq \int_t^\infty \frac{-L_1^{\eta+\theta}\left[G^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{1-\eta-\theta} = \frac{L_1^{\eta+\theta}[G(t)]^{1-\eta-\theta}}{1-\eta-\theta}$$
in (21) implies the desired result (17). In order to obtain inequality (18), we apply reversed Hölder inequality (16) to inequality (17) with the constants $\frac{1}{\zeta}<1$ and $\frac{1}{1-\zeta}<0$.

(2) When the above process is repeated for the left hand side of inequality (19) with $u(t) = -\int_t^\infty \frac{z(s)}{[G^\rho(s)]^{\eta+\theta}}\nabla s$, the desired results can be obtained.

□

Remark 1. *The nabla Bennett-Leindler type inequalities* (17)-(20) *obtained for* $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta\leq 0$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.1] for* $0<\zeta<1,\ \eta=0$ *and* $\theta\leq 0$.

Corollary 1. *From inequalities* (17)-(20) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* G^ρ, G, H^ρ, H *presented in Theorem 1 by* G, G^σ, H, H^σ*, repectively, where*
$$G(t) = \int_t^\infty z(s)\Delta s \quad \text{and} \quad H(t) = \int_a^t z(s)h(s)\Delta s \tag{23}$$
and $\sigma:\mathbb{T}\to\mathbb{T}$ *denotes the forward jump operator defined by* $\sigma(t) := inf\,\{\tau>t:\tau\in\mathbb{T}\}$ *with* $f^\sigma(t) = (f\circ\sigma)(t) = f(\sigma(t))$.

Let z and h be nonnegative functions and G and H be defined as in (23)*. For a constant $M_1 > 0$, assume that $\frac{G(t)}{G^\sigma(t)} \leq M_1$ for $t \in (a,\infty)_\mathbb{T}$. In this case for $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta \leq 0$, nabla Bennett-Leindler type inequalities* (17)*-*(20) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (18) *and* (20) *can be written as follows*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (17)*-*(20) *obtained for $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta \leq 0$ are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.1] for $0 < \zeta < 1,\ \eta = 0$ and $\theta \leq 0$.*

Remark 2. *If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_1 = 1$ in* (17)*-*(20)*. Hence inequalities* (17) *and* (19) *as well as inequalities* (18) *and* (20) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (17) *and* (18) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[H(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta \leq 0$ and the functions G and H are defined as

$$G(t) = \int_t^\infty z(s)ds \quad \text{and} \quad H(t) = \int_a^t z(s)h(s)ds. \tag{24}$$

For the continuous case, when $0 < \zeta < 1,\ \eta = 0$ and $\theta \leq 0$, the first Bennett-Leindler type inequalities were established in [55, Remark 1] and [28, Remark 3.2] for the given aforementioned functions G and H. These inequalities are extended to the cases $\zeta > 1,\ \eta \geq 0$ and $\eta+\theta \leq 0$ by the above novel continuous Bennett-Leindler type inequalities.

Remark 3. *If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t-1$ in* (17)-(20).

Using $\int_t^\infty z(s)\nabla s = \sum_{k=t+1}^{\infty} z(k)$, *we have* $G^\rho(t) = G(t-1) = \sum_{k=t}^{\infty} z(k)$, *where* $G(t) = \sum_{k=t+1}^{\infty} z(k)$. *Moreover* $H(t) = \sum_{k=a+1}^{t} z(k)h(k)$. *For a constant* $L_1 > 0$, *let us assume that* $\frac{G(t-1)}{G(t)} \le L_1$. *For* $a = 0,\ \zeta > 1,\ \eta \ge 0$ *and* $\eta + \theta \le 0$, *in the set of natural numbers, inequalities* (17)-(20) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (18) *and* (20) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \ge \left[\frac{L_1^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \ge \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t-1)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

For the discrete case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \le 0$, *the first Bennett-Leindler type inequalities were established in [55, Remark 2] and [28, Remark 3.3] for the given aforementioned series* G *and* H. *These inequalities are extended to the cases* $\zeta > 1,\ \eta \ge 0$ *and* $\eta + \theta \le 0$ *by the above novel discrete Bennett-Leindler type inequalities.*

The next theorem, which is proven for $\zeta > 1,\ \eta \ge 0$ and $0 \le \eta + \theta < 1$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0$ and $\theta \le 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality obtained by Saker et al. [55, Remark 2] or Kayar et al. [28, Remark 3.3].
(b) The continuous inequality obtained by Saker et al. [55, Remark 1] or Kayar et al. [28, Remark 3.2].
(c) The delta counterpart of the nabla inequality (12) in Theorem 1 obtained by Saker et al. [55, Theorem 2.1].
(d) The nabla inequality (12) in Theorem 1 obtained by Kayar et al. [28, Theorem 3.1].

Theorem 6. *Let the fuctions* z, h, G *and* H *be defined as in Theorem 1. For a constant* $L_2 > 0$, *let us assume that* $1 \le \frac{G^\rho(t)}{G(t)} \le \frac{1}{L_2}$ *for* $t \in (a,\infty)_{\mathbb{T}}$. *Let* $\zeta > 1,\ \eta \ge 0$ *be real constants. If* $0 \le \eta + \theta < 1$, *then we have*

(1)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \frac{L_2^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}\nabla t, \tag{25}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_2(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\rho(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{26}$$

(2)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t, \tag{27}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{28}$$

Proof. The same methodology used in the proofs of [28, Theorem 3.1] and Theorem 6 works for the proof of this theorem except that for $0 \leq \eta+\theta < 1$, we have

$$\left[G^{1-\eta-\theta}(t)\right]^\nabla \geq -(1-\eta-\theta)\frac{z(t)}{[G(t)]^{\eta+\theta}}.$$

□

Remark 4. *The nabla Bennett-Leindler type inequalities* (25)-(28) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.1] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Corollary 2. *From inequalities* (25)-(28) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* G^ρ, G, H^ρ, H *presented in Theorem 1 by* G, G^σ, H, H^σ *defined in* (23)*, repectively.*

Let z *and* h *be nonnegative functions and* G *and* H *be defined as in* (23)*. For a constant* $M_2 > 0$*, let us assume that* $1 \leq \dfrac{G(t)}{G^\sigma(t)} \leq \dfrac{1}{M_2}$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$*, the nabla Bennett-Leindler type inequalities* (25)-(28) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (26) *and* (28) *can be written as follows*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \geq \left[M_2\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (25)-(28) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.1] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Remark 5. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_2 = 1$ *in* (25)-(28). *Hence inequalities* (25) *and* (27) *as well as inequalities* (26) *and* (28) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (25) *and* (26) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} dt \geq \frac{\eta+\zeta}{1-\eta-\theta} \int_a^\infty \frac{z(t)h(t)[H(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}} dt$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}} dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *and the functions* G *and* H *are defined as in* (24).

For the continuous case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$, *the first Bennett-Leindler type inequalities were established in [55, Remark 1] and [28, Remark 3.2] for the given aforementioned functions* G *and* H. *These inequalities are extended to the cases* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *by the above novel continuous Bennett-Leindler type inequalities.*

Remark 6. *If the time scale is the set of natural numbers, then for all* $t \in \mathbb{N}$, *the backward jump operator results in* $\rho(t) = t - 1$ *in* (25)-(28). *Suppose that the series* G *and* H *are defined as in Remark 3. For a constant* $L_2 > 0$, *let us assume that* $1 \leq \dfrac{G(t-1)}{G(t)} \leq \dfrac{1}{L_2}$. *For* $a = 0,\ \zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$, *in the set of natural numbers, inequalities* (25)-(28) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (26) *and* (28) *can be written as follows*

$$\sum_{t=1}^\infty \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \geq \left[L_2 \frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^\infty \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t-1)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^\infty \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^\infty \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

For the discrete case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$, *the first Bennett-Leindler type inequalities were established in [55, Remark 2] and [28, Remark 3.3] for the given aforementioned series* G *and* H. *These inequalities are extended to the* *cases* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *by the above novel discrete Bennett-Leindler type inequalities.*

The next theorem, which is proven for $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta \leq 0$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0$ and $\eta + \theta \leq 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality (5) obtained by Copson [15, Theorem 2.3] and by Bennett [10, Corollary 1] or Leindler [35, Proposition 6].
(b) The continuous inequality (8) obtained by Hardy et al. [26, Theorem 337] and the continuous inequality (9) obtained by Copson [16, Theorem 4].
(c) The delta counterpart of the nabla inequality (13) in Theorem 2 obtained by Saker et al. [55, Theorem 2.3].
(d) The nabla inequality (13) in Theorem 2 obtained by Kayar et al. [28, Theorem 3.9].

Theorem 7. *Let the fuctions* $z, h, \overline{G}$ *and* $\overline{H}$ *be defined as in Theorem 2. For a constant* $L_3 > 0$*, assume that* $\dfrac{\overline{G}(t)}{\overline{G}^{\rho}(t)} \leq L_3$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. Let* $\zeta > 1,\ \eta \geq 0$ *be real constants. If* $\eta + \theta \leq 0$*, then we have*

(1)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \geq \frac{L_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta} \int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}} \nabla t, \tag{29}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \geq \left[\frac{L_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \nabla t. \tag{30}$$

(2)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta} \int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}} \nabla t, \tag{31}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \geq \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}} \nabla t. \tag{32}$$

Proof. The same methodology used in the proof of [28, Theorem 3.9] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.11) in the proof of [28, Theorem 3.9] as

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t = \int_a^{\infty} u^{\rho}(t) \left\{ -\left[\overline{H}^{\eta+\zeta}(t)\right]^{\nabla} \right\} \nabla t, \tag{33}$$

where $u(t)=\int_a^t \frac{z(s)}{[\overline{G}^\rho(s)]^{\eta+\theta}}\nabla s$. Observe that since $\eta+\zeta>1$,

$$-[\overline{H}^{\eta+\zeta}(t)]^\nabla \geq (\eta+\zeta)z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}, \tag{34}$$

which is different than (3.12) in the proof of [28, Theorem 3.9]. In our case, when $\eta+\theta\leq 0$, since

$$\left[\overline{G}^{1-\eta-\theta}(t)\right]^\nabla \leq (1-\eta-\theta)\frac{z(t)}{[\overline{G}(t)]^{\eta+\theta}} \leq (1-\eta-\theta)\frac{z(t)}{L_3^{\eta+\theta}[\overline{G}^\rho(t)]^{\eta+\theta}},$$

using (34) and

$$u^\rho(t)=\int_a^{\rho(t)} \frac{z(s)\nabla s}{[\overline{G}^\rho(s)]^{\eta+\theta}} \geq \int_a^{\rho(t)} \frac{L_3^{\eta+\theta}\left[\overline{G}^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{1-\eta-\theta} = \frac{L_3^{\eta+\theta}[\overline{G}^\rho(t)]^{1-\eta-\theta}}{1-\eta-\theta}$$

in (33) implies the desired result (29). In order to obtain inequality (30), we apply reversed Hölder inequality (16) to inequality (29) with the constants $\frac{1}{\zeta}<1$ and $\frac{1}{1-\zeta}<0$.

(2) When the above process is repeated for the left hand side of inequality (31) with $u(t)=\int_a^t \frac{z(s)}{[\overline{G}(s)]^{\eta+\theta}}\nabla s$, the desired results can be obtained.

□

Remark 7. *The nabla Bennett-Leindler type inequalities* (29)-(32) *obtained for* $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta\leq 0$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.9] for* $0<\zeta<1,\ \eta=0$ *and* $\theta\leq 0$.

Corollary 3. *From inequalities* (29)-(32) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $\overline{G}^\rho, \overline{G}, \overline{H}^\rho, \overline{H}$ *presented in Theorem 2 by* $\overline{G}, \overline{G}^\sigma, \overline{H}, \overline{H}^\sigma$ *defined as*

$$\overline{G}(t)=\int_a^t z(s)\Delta s \quad \text{and} \quad \overline{H}(t)=\int_t^\infty z(s)h(s)\Delta s, \tag{35}$$

respectively.

Let z *and* h *be nonnegative functions and* $\overline{G}$ *and* $\overline{H}$ *be defined as in* (35). *For a constant* $M_3>0$, *let us assume that* $\frac{\overline{G}^\sigma(t)}{\overline{G}(t)}\leq M_3$ *for* $t\in(a,\infty)_\mathbb{T}$. *In this case for* $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta\leq 0$, *the nabla Bennett-Leindler type inequalities* (29)-(32) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (30) *and* (32) *can be written as follows*

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (29)-(32) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta+\theta \leq 0$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.3] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Remark 8. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_3 = 1$ *in* (29)-(32). *Hence inequalities* (29) *and* (31) *as well as inequalities* (30) *and* (32) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (29) *and* (30) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $\eta+\theta \leq 0$ *and the functions* $\overline{G}$ *and* $\overline{H}$ *are defined as*

$$\overline{G}(t) = \int_a^t z(s)ds \quad \textit{and} \quad \overline{H}(t) = \int_t^\infty z(s)h(s)ds. \tag{36}$$

These novel inequalities complement and generalize the continuous inequality (8) *obtained by Hardy et al. [26, Theorem 337] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta = \zeta$ *and the continuous inequality* (9) *obtained by Copson [16, Theorem 4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta < 1$ *to the cases* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta \leq 0$.

Remark 9. *If the time scale is the set of natural numbers, then for all* $t \in \mathbb{N}$, *the backward jump operator results in* $\rho(t) = t-1$ *in* (29)-(32).

Using $\overline{G}(t) = \int_a^t z(s)\nabla s = \sum_{k=a+1}^{t} z(k)$, *we have* $\overline{G}^\rho(t) = \overline{G}(t-1) = \sum_{k=a+1}^{t-1} z(k)$. *Moreover* $\overline{H}(t) = \sum_{k=t+1}^{\infty} z(k)f(k)$. *For a constant* $L_3 > 0$, *let us assume that* $\frac{\overline{G}(t)}{\overline{G}(t-1)} \leq L_3$. *For* $a = 0,\ \zeta > 1,\ \eta \geq 0$, *and* $\eta+\theta \leq 0$, *in the set of natural numbers, inequalities* (29)-(32) *become novel discrete Bennett-Leindler type* *inequalities, two of which obtained from* (30) *and* (32) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \geq \left[\frac{L_3^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t-1)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \geq \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

The discrete Bennett-Leindler type inequality (5) *obtained by Copson [15, Theorem 2.3] and by Bennett [10, Corollary 1] or Leindler [35, Proposition 6] for* $0<\zeta<1,\ \eta=0,\ \theta<0$ *is complemented and generalized to the cases* $\zeta>1,\ \eta\geq 0,\ \eta+\theta\leq 0$ *by Theorem 7 and particularly by this remark.*

The next theorem, which is proven for $\zeta>1,\ \eta\geq 0$ and $0\leq\eta+\theta<1$, provides complements and generalizations of some of the abovementioned Bennett-Leindler type inequalities given for $0<\zeta<1,\ \eta\geq 0$ and $\eta+\theta\leq 0$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality (4) obtained by Copson [15, Theorem 2.3].
(b) The continuous inequality (8) obtained by Hardy et al. [26, Theorem 337] and the continuous inequality (9) obtained by Copson [16, Theorem 4].
(c) The delta analogue of the inequality (13) in Theorem 2 obtained by Saker et al. [55, Theorem 2.3].
(d) The nabla inequality (13) in Theorem 2 obtained by Kayar et al. [28, Theorem 3.9].

Theorem 8. *Let the functions* $z, h, \overline{G}$ *and* $\overline{H}$ *be defined as in Theorem 2. For a constant* $L_4>0$*, let us assume that* $1\leq\dfrac{\overline{G}(t)}{\overline{G}^{\rho}(t)}\leq\dfrac{1}{L_4}$ *for* $t\in(a,\infty)_{\mathbb{T}}$*. Let* $0<\zeta<1,\ \eta\geq 0$ *be real constants. If* $0\leq\eta+\theta<1$*, then we have*

(1)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \geq \frac{L_4^{\eta+\theta}(\eta+\zeta)}{1-\eta-\theta}\int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}}\nabla t, \tag{37}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_4(\eta+\zeta)}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{38}$$

(2)

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-1}}\nabla t, \tag{39}$$

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta}\int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{40}$$

Proof. The same methodology used in the proofs of [28, Theorem 3.9] and Theorem 7 works for the proof of this theorem except that for $0 \leq \eta + \theta < 1$, we have

$$\left[\overline{G}^{1-\eta-\theta}(t)\right]^{\nabla} \leq (1-\eta-\theta)\frac{z(t)}{[\overline{G}^{\rho}(t)]^{\eta+\theta}}.$$

□

Remark 10. *The nabla Bennett-Leindler type inequalities* (37)-(40) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.9] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Corollary 4. *From inequalities* (37)-(40) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $\overline{G}^{\rho}, \overline{G}, \overline{H}^{\rho}, \overline{H}$ *presented in Theorem 2 by* $\overline{G}, \overline{G}^{\sigma}, \overline{H}, \overline{H}^{\sigma}$ *defined in* (35)*, repectively.*

Let z *and* h *be nonnegative functions and* $\overline{G}$ *and* $\overline{H}$ *be defined as in* (35)*. For a constant* $M_4 > 0$*, let us assume that* $1 \leq \dfrac{\overline{G}^{\sigma}(t)}{\overline{G}(t)} \leq \dfrac{1}{M_4}$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$*, the nabla Bennett-Leindler type inequalities* (37)-(40) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (38) *and* (40) *can be written as follows*

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}}\Delta t \geq \left[M_4\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (37)-(40) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.3] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta \leq 0$.

Remark 11. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$*, the backward jump operator results in* $\rho(t) = t$ *and* $L_4 = 1$ *in* (37)-(40)*. Hence inequalities* (37) *and* (39) *as well as inequalities* (38) *and* (40) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (37) *and* (38) *reduce to the following inequalities as*

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{1-\eta-\theta}\int_a^{\infty} \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$ *and the functions* $\overline{G}$ *and* $\overline{H}$ *are defined as in* (36).

These novel inequalities complement and generalize the continuous inequality (8) *obtained by Hardy et al. [26, Theorem 337] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta = \zeta$ *and the continuous inequality* (9) *obtained by Copson [16, Theorem 4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta < 1$ *to the cases* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta + \theta < 1$.

Remark 12. *If the time scale is the set of natural numbers, then for all* $t \in \mathbb{N}$, *the backward jump operator results in* $\rho(t) = t-1$ *in* (37)-(40). *Suppose that the series* $\overline{G}$ *and* $\overline{H}$ *are defined as in Remark 9. For a constant* $L_4 > 0$, *let us assume that* $\frac{\overline{G}(t)}{\overline{G}(t-1)} \leq \frac{1}{L_4}$. *For* $a = 0,\ \zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$, *in the set of natural numbers, inequalities* (37)-(40) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (38) *and* (40) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \geq \left[L_4\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \geq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t-1)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively.

The discrete Bennett-Leindler type inequality (4) *obtained by Copson [15, Theorem 2.3] for* $0 < \zeta < 1,\ \eta = 0,\ 0 \leq \theta < 1$ *is complemented and generalized to the case* $\zeta > 1,\ \eta \geq 0,\ 0 \leq \eta+\theta < 1$ *by Theorem 8 and particularly by this remark.*

The next theorem, which is proven for $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta > 1$, provides complements and generalizations of some of the previous Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0,\ \theta > 1$ or $\zeta > 1,\ \eta = 0,\ \theta = \zeta$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequality (6) obtained by Copson [15, Theorem 1.3] and Bennett [10, Corollary 3] or Leindler [35, Proposition 7] as well as the discrete inequality (7) obtained by Renaud [45, Theorem 1].
(b) The continuous inequality (10) obtained by Copson [16, Theorem 2] and the continuous inequality (11) obtained by Renaud in [45, Theorem 3].
(c) The delta counterpart of the nabla inequality (14) in Theorem 3 obtained by Saker et al. [55, Theorem 2.4].
(d) The nabla inequality (14) in Theorem 3 obtained by Kayar et al. [28, Theorem 3.12].

Theorem 9. *Suppose that the functions $z, h, \overline{G}$ and H are defined as in Theorem 3 and the constant L_4 is defined as in Theorem 8. Let $\zeta > 1,\ \eta \geq 0$ be real numbers. If $\eta + \theta > 1$, then we have*

(1)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \geq \frac{L_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}\nabla t, \tag{41}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{42}$$

(2)

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}^\rho(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}\nabla t, \tag{43}$$

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}^\rho(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H^\rho(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^\rho(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{44}$$

Proof. The same methodology used in the proof of [28, Theorem 3.12] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.16) in the proof of [28, Theorem 3.12] as

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t = \int_a^\infty -u(t)\left[H^{\eta+\zeta}(t)\right]^\nabla \nabla t, \tag{45}$$

where $u(t) = \displaystyle\int_t^\infty \frac{z(s)}{[\overline{G}(s)]^{\eta+\theta}}\nabla s$. In our case, when $\eta+\theta > 1$, since

$$\left[\overline{G}^{1-\eta-\theta}(t)\right]^\nabla \geq -(\eta+\theta-1)\frac{z(t)}{[\overline{G}^\rho(t)]^{\eta+\theta}} \geq -(\eta+\theta-1)\frac{z(t)}{L_4^{\eta+\theta}[\overline{G}(t)]^{\eta+\theta}},$$

using (22) and

$$-u(t) = \int_t^\infty \frac{z(s)\nabla s}{[\overline{G}(s)]^{\eta+\theta}} \geq \int_t^\infty \frac{-L_4^{\eta+\theta}\left[\overline{G}^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{\eta+\theta-1} = \frac{L_4^{\eta+\theta}[\overline{G}(t)]^{1-\eta-\theta}}{\eta+\theta-1}$$

in (45) implies the desired result (41). In order to obtain inequality (42), we apply reversed Hölder inequality (16) to inequality (41) with the constants $\dfrac{1}{\zeta} < 1$ and $\dfrac{1}{1-\zeta} < 0$.

(2) When the above process is repeated for the left hand side of inequality (43) with $u(t) = \displaystyle\int_t^\infty \frac{z(s)}{[\overline{G}^\rho(s)]^{\eta+\theta}}\nabla s$, the desired results can be obtained.

□

Remark 13. *The nabla Bennett-Leindler type inequalities* (41)-(44) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.12] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Corollary 5. *From inequalities* (41)-(44) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $\overline{G}^{\rho}, \overline{G}, H^{\rho}, H$ *presented in Theorem 3 by* $\overline{G}, \overline{G}^{\sigma}, H, H^{\sigma}$ *defined in* (35) *and* (23), *repectively.*

Let z *and* h *be nonnegative functions and* $\overline{G}$ *and* H *be defined as in* (35) *and* (23), *repectively, and the constant* M_4 *be defined as in Corollary 4. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$, *the nabla Bennett-Leindler type inequalities* (41)-(44) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (42) *and* (44) *can be written as follows*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively.

The delta variants of the nabla Bennett-Leindler type inequalities (41)-(44) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Remark 14. *If the time scale is the set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_4 = 1$ *in* (41)-(44). *Hence inequalities* (41) *and* (43) *as well as inequalities* (42) *and* (44) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (41) *and* (42) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[H(t)]^{\eta+\zeta-1}}{[\overline{G}(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[H(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt.$$

respectively, where $\zeta < 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *and the functions* $\overline{G}$ *and* H *are defined as in* (36) *and* (24), *respectively.*

These novel inequalities complement and generalize the continuous inequality (10) *obtained by Copson [16, Theorem 2] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$ *and the continuous inequality* (11) *obtained by Renaud in [45, Theorem 3] for* $\zeta > 1,\ \eta = 0$ *and* $\theta = \zeta$ *to the cases* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$.

Remark 15. *If the time scale is the set of natural numbers, then for all* $t \in \mathbb{N}$, *the backward jump operator results in* $\rho(t) = t - 1$ *in* (41)-(44). *Let the constant* L_4 *be defined as in Remark 12. For* $a = 0,\ \zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$, *in the set of natural numbers, inequalities* (41)-(44) *become novel discrete Bennett-Leindler type inequalities, two of which obtained from* (42) *and* (44) *can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \geq \left[\frac{L_4^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \geq \left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[H(t-1)]^{\eta+\zeta-\frac{1}{\zeta}}}{[\overline{G}(t-1)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively, where the series $\overline{G}$ *and* H *are defined as in Remark 9 and Remark 3, respectively.*

The discrete Bennett-Leindler type inequality (6) *obtained by Copson [15, Theorem 1.3] and Bennett [10, Corollary 3] or Leindler [35, Proposition 7] for* $0 < \zeta < 1,\ \eta = 0,\ \theta > 1$ *as well as the discrete inequality* (7) *obtained by Renaud [45, Theorem 1] for* $\zeta > 1,\ \eta = 0,\ \theta = \zeta$ *are complemented and generalized to the cases* $\zeta > 1,\ \eta \geq 0,\ \eta + \theta > 1$ *by Theorem 9 and particularly by this remark.*

The next theorem, which is proven for $\zeta > 1,\ \eta \geq 0$ and $\eta + \theta > 1$, provides complements and generalizations of some of the previous Bennett-Leindler type inequalities given for $0 < \zeta < 1,\ \eta = 0$ and $\theta > 1$. These previous Bennett-Leindler type inequalities are listed as follows:

(a) The discrete inequalities obtained by Saker et al. [55, Remark 4] and by Kayar et al. [28, Remark 3.8].
(b) The continuous inequalities obtained by Saker et al. [55, Remark 3] and by Kayar et al. [28, Remark 3.7].
(c) The delta counterpart of the nabla inequality (15) in Theorem 4 obtained by Saker et al. [55, Theorem 2.2].
(d) The nabla inequality (15) in Theorem 4 obtained by Kayar et al. [28, Theorem 3.4].

Theorem 10. *Suppose that the functions* z, h, G *and* $\overline{H}$ *are defined as in Theorem 4 and the constant* L_2 *is defined as in Theorem 6. Let* $\zeta > 1,\ \eta \geq 0$ *be real numbers. If* $\eta + \theta > 1$, *then we have*

(1)

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \frac{L_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[G^\rho(t)]^{\eta+\theta-1}}\nabla t, \tag{46}$$

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\rho(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{47}$$

(2)

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[G^\rho(t)]^{\eta+\theta-1}}\nabla t, \tag{48}$$

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \geq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta}\int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\nabla t. \tag{49}$$

Proof. The same methodology used in the proof of [28, Theorem 3.4] works for the proof of this theorem except some steps.

(1) We start by the following equation similar to (3.7) in the proof of [28, Theorem 3.4] as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t = \int_a^\infty u^\rho(t)\left\{-\left[\overline{H}^{\eta+\zeta}(t)\right]^\nabla\right\}\nabla t, \tag{50}$$

where $u(t) = \int_a^t \frac{z(s)}{[G^\rho(s)]^{\eta+\theta}}\nabla s$. In our case, when $\eta+\theta>1$, since

$$\left[G^{1-\eta-\theta}(t)\right]^\nabla \leq (\eta+\theta-1)\frac{z(t)}{[G(t)]^{\eta+\theta}} \leq (\eta+\theta-1)\frac{z(t)}{L_2^{\eta+\theta}[G^\rho(t)]^{\eta+\theta}},$$

using (34) and

$$u^\rho(t) = \int_a^{\rho(t)} \frac{z(s)\nabla s}{[G^\rho(s)]^{\eta+\theta}} \geq \int_a^{\rho(t)} \frac{L_2^{\eta+\theta}\left[G^{1-\eta-\theta}(s)\right]^\nabla \nabla s}{\eta+\theta-1} = \frac{L_2^{\eta+\theta}[G^\rho(t)]^{1-\eta-\theta}}{\eta+\theta-1}$$

in (50) implies the desired result (46). In order to obtain inequality (47), we apply reversed Hölder inequality (16) to inequality (46) with the constants $\frac{1}{\zeta}<1$ and $\frac{1}{1-\zeta}<0$.

(2) When the above process is repeated for the left hand side of inequality (48) with $u(t) = \int_a^t \frac{z(s)}{[G(s)]^{\eta+\theta}}\nabla s$, the desired results can be obtained.

□

Remark 16. *The nabla Bennett-Leindler type inequalities* (46)-(49) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the nabla Bennett-Leindler type inequalities given in [28, Theorem 3.4] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Corollary 6. *From inequalities* (46)-(49) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $G^\rho, G, \overline{H}^\rho, \overline{H}$ *presented in Theorem 4 by* $G, G^\sigma, \overline{H}, \overline{H}^\sigma$ *defined in* (23) *and* (35), *repectively.*

Let z *and* h *be nonnegative functions and* $\overline{H}$ *be defined as in* (23) *and* (35), *repectively, and the constant* M_2 *be defined as in Corollary 2. In this case for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$, *the nabla Bennett-Leindler type inequalities* (46)-(49) *become novel delta Bennett-Leindler type inequalities, two of which obtained from* (47) *and* (49) *can be written as follows*

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \geq \left[\frac{M_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}^\sigma(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G^\sigma(t)]^{\eta+\theta-\frac{1}{\zeta}}}\Delta t,$$

respectively. The delta variants of the nabla Bennett-Leindler type inequalities (46)-(49) *obtained for* $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *are complements and generalizations of the delta Bennett-Leindler type inequalities given in [55, Theorem 2.2] for* $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$.

Remark 17. *If the time scale is set of real numbers, then for all* $t \in \mathbb{R}$, *the backward jump operator results in* $\rho(t) = t$ *and* $L_2 = 1$ *in* (46)-(49). *Hence inequalities* (46) *and* (48) *as well as inequalities* (47) *and* (49) *coincide and their delta versions become exactly the same inequalities as them. Therefore together with their coincident inequalities, inequalities* (46) *and* (47) *reduce to the following inequalities as*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \frac{\eta+\zeta}{\eta+\theta-1}\int_a^\infty \frac{z(t)h(t)[\overline{H}(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}}dt$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}dt \geq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{1/\zeta} \int_a^\infty \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}}dt,$$

respectively, where $\zeta > 1,\ \eta \geq 0$ *and* $\eta + \theta > 1$ *and the functions* G *and* $\overline{H}$ *are defined as in* (24) *and* (36), *repectively.*

For the continuous case, when $0 < \zeta < 1,\ \eta = 0$ *and* $\theta > 1$, *the first Bennett-Leindler type inequalities were established in [55, Remark 3] and [28, Remark 3.7] for the given aforementioned functions* G *and* $\overline{H}$. *By this remark, these inequalities* *are extended to the cases $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ by the above novel continuous Bennett-Leindler type inequalities.*

Remark 18. *If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (46)-(49). Let the constant L_2 be defined as in Remark 5. For $a = 0$, $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$, in the set of natural numbers, inequalities (46)-(49) become novel discrete Bennett-Leindler type inequalities, two of which obtained from (47) and (49) can be written as follows*

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \geq \left[\frac{L_2^{\eta+\theta}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t-1)]^{\eta+\theta-\frac{1}{\zeta}}}$$

and

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \geq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{1/\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{1/\zeta}(t)[\overline{H}(t)]^{\eta+\zeta-\frac{1}{\zeta}}}{[G(t)]^{\eta+\theta-\frac{1}{\zeta}}},$$

respectively, where the series $\overline{H}$ and G are defined as in Remark 9 and Remark 3, respectively.

For the discrete case, when $0 < \zeta < 1$, $\eta = 0$ and $\theta > 1$, the first Bennett-Leindler type inequalities were established in [55, Remark 4] and [28, Remark 3.8] for the given aforementioned series G and $\overline{H}$. By this remark, these inequalities are extended to the cases $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ by the above novel discrete Bennett-Leindler type inequalities.

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References

[1] Agarwal, R., Bohner, M., Peterson, A., Inequalities on time scales: A survey, *Math. Inequal. Appl.* 4(4) (2001), 535–557. https://doi.org/dx.doi.org/10.7153/mia-04-48

[2] Agarwal, R. P., Mahmoud, R. R., Saker, S., Tunç, C., New generalizations of Németh-Mohapatra type inequalities on time scales, *Acta Math. Hungar.* 152(2) (2017), 383-403. https://doi.org/10.1007/s10474-017-0718-2

[3] Agarwal, R., O'Regan, D. and Saker, S., Dynamic Inequalities on Time Scales, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-11002-8

[4] Agarwal, R., O'Regan, D., Saker, S., Hardy Type Inequalities on Time Scales, Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-44299-0

[5] Anderson, D. R., Time-scale integral inequalities, *J. Inequal. Pure Appl. Math.*, 6(3) Article 66 (2005), 1-15.

[6] Atici, F. M., Guseinov, G. S., On Green's functions and positive solutions for boundary value problems on time scales, *J. Comput. Appl. Math.*, 141(1-2) (2002), 75-99. https://doi.org/10.1016/S0377-0427(01)00437-X

[7] Balinsky, A. A., Evans, W. D., Lewis, R. T., The Analysis and Geometry of Hardy's Inequality, Springer International Publishing, Switzerland, 2015. https://doi.org/10.1007/978-3-319-22870-9

[8] Beesack, P. R., Hardy's inequality and its extensions, *Pacific J. Math.*, 11(1) (1961), 39-61. http://projecteuclid.org/euclid.pjm/1103037533

[9] Bennett, G., Some elementary inequalities, *Quart. J. Math. Oxford Ser. (2)*, 38(152) (1987), 401-425. https://doi.org/10.1093/qmath/38.4.401

[10] Bennett, G., Some elementary inequalities II., *Quart. J. Math.*, 39(4) (1988), 385-400. https://doi.org/10.1093/qmath/39.4.385

[11] Bohner, M., Mahmoud, R., Saker, S. H, Discrete, continuous, delta, nabla, and diamond-alpha Opial inequalities, *Math. Inequal. Appl.,* 18(3) (2015), 923-940. https://doi.org/10.7153/mia-18-69

[12] Bohner, M., Peterson, A., Dynamic Equations on Time Scales, An Introduction With Applications, Birkhäuser Boston, Inc., Boston, MA, 2001. https://doi.org/10.1007/978-1-4612-0201-1

[13] Bohner, M., Peterson, A., Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, Inc., Boston, MA, 2003. https://doi.org/10.1007/978-0-8176-8230-9

[14] Chu, Y.-M., Xu, Q., Zhang, X.-M., A note on Hardy's inequality, *J. Inequal. Appl.*, 2014(271) (2014), 1-10. https://doi.org/10.1186/1029-242X-2014-271

[15] Copson, E. T., Note on series of positive terms, *J. London Math. Soc.*, 3(1) (1928), 49-51. https://doi.org/10.1112/jlms/s1-3.1.49

[16] Copson, E. T., Some integral inequalities, *Proc. Roy. Soc. Edinburgh Sect. A*, 75(2) (1976), 157-164. https://doi.org/10.1017/S0308210500017868

[17] El-Deeb, A. A., Elsennary, H. A., Khan, Z. A., Some reverse inequalities of Hardy type on time scales, *Adv. Difference Equ.,* 2020(402) (2020), 1-18. https://doi.org/10.1186/s13662-020-02857-w

[18] El-Deeb, A. A., Elsennary, H. A., Dumitru, B., Some new Hardy-type inequalities on time scales, *Adv. Difference Equ.*, 2020(441) (2020), 1-22. https://doi.org/10.1186/s13662-020-02883-8

[19] Gao, P., Zhao, H. Y., On Copson's inequalities for $0 < p < 1$, *J. Inequal. Appl.,* 2020(72) (2020), 1-13. https://doi.org/10.1186/s13660-020-02339-3

[20] Guseinov, G. S., Kaymakçalan, B., Basics of Riemann delta and nabla integration on time scales, *J. Difference Equ. Appl.,* 8(11) (2002), 1001-1017. https://doi.org/10.1080/10236190290015272

[21] Gürses, M., Guseinov, G. S., Silindir, B., Integrable equations on time scales, *J. Math. Phys.*, 46(11) 113510 (2005), 1-22. https://doi.org/10.1063/1.2116380

[22] Güvenilir, A. F., Kaymakçalan, B., Pelen, N. N., Constantin's inequality for nabla and diamond-alpha derivative, *J. Inequal. Appl.*, 2015(167) (2015), 1-17. https://doi.org/10.1186/s13660-015-0681-9

[23] Hardy, G. H., Littlewood, J. E., Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, *Journal für die reine und angewandte Mathematik*, 157 (1927), 141-158. https://doi.org/10.1515/crll.1927.157.141

[24] Hardy, G. H., Note on a theorem of Hilbert, *Math. Z.*, 6(3-4) (1920), 314-317. https://doi.org/10.1007/BF01199965

[25] Hardy, G. H., Notes on some points in the integral calculus, LX. An inequality between integrals, *Messenger Math.*, 54(3) (1925), 150-156.

[26] Hardy, G. H., Littlewood and Pólya, G., Inequalities, Cambridge University Press, London, 1934.

[27] Iddrisu, M. M., Okpoti, A. C., Gbolagade, A. K., Some proofs of the classical integral Hardy inequality, *Korean J. Math.*, 22(3) 2014, 407-417. https://doi.org/10.11568/kjm.2014.22.3.407

[28] Kayar, Z., Kaymakçalan, B., Pelen, N. N., Bennett-Leindler type inequalities for time scale nabla calculus, *Mediterr. J. Math.*, 18(14) (2021). https://doi.org/10.1007/s00009-020-01674-5

[29] Kayar, Z., Kaymakçalan, B., Hardy-Copson type inequalities for nabla time scale calculus, *Turk. J. Math.*, 45(2) (2021), 1040-1064. https://doi.org/10.3906/mat-2011-38

[30] Kayar, Z., Kaymakçalan, B., Some extended nabla and delta Hardy-Copson type inequalities with applications in oscillation theory, *Bull. Iran. Math. Soc.*, Accepted. https://doi.org/10.1007/s41980-021-00651-2.

[31] Kayar, Z., Kaymakçalan, B., Complements of nabla and delta Hardy-Copson type inequalities and their applications, Submitted.

[32] Kufner, A., Maligranda, L., Persson, L. E., The Hardy Inequality. About Its History and Some Related Results, Vydavatelský Servis, Pilsen, 2007.

[33] Kufner, A., Persson, L. E., Samko, N., Weighted Inequalities of Hardy Type, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. https://doi.org/10.1142/10052

[34] Lefèvre, P., A short direct proof of the discrete Hardy inequality, *Arch. Math. (Basel).*, 114(2) (2020), 195-198. https://doi.org/10.1007/s00013-019-01395-6

[35] Leindler, L., Some inequalities pertaining to Bennett's results, *Acta Sci. Math. (Szeged).,* 58(1-4) (1993), 261-279.

[36] Leindler, L., Further sharpening of inequalities of Hardy and Littlewood, *Acta Sci. Math.,* 54(3–4) (1990), 285–289.

[37] Liao, Z.-W., Discrete Hardy-type inequalities, *Adv. Nonlinear Stud.,* 15(4) (2015), 805-834. https://doi.org/10.1515/ans-2015-0404

[38] Masmoudi, N., About the Hardy Inequality, in: An Invitation to Mathematics. From Competitions to Research, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-19533-4_11

[39] Nikolidakis, E. N., A sharp integral Hardy type inequality and applications to Muckenhoupt weights on $\mathbb{R}$, *Ann. Acad. Sci. Fenn. Math.,* 39(2) (2014), 887-896. https://doi.org/10.5186/aasfm.2014.3947

[40] Özkan, U. M., Sarikaya, M. Z., Yildirim, H., Extensions of certain integral inequalities on time scales, *Appl. Math. Lett.*, 21(10) (2008), 993-1000. https://doi.org/10.1016/j.aml.2007.06.008

[41] Pachpatte, B. G., On Some Generalizations of Hardy's Integral Inequality, *J. Math. Anal. Appl.,* 234(1) (1999), 15-30. https://doi.org/10.1006/jmaa.1999.6294

[42] Pečarić, J., Hanjš, Ž., On some generalizations of inequalities given by B. G. Pachpatte, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)*, 45(1) (1999), 103-114.

[43] Pelen, N. N., Hardy-Sobolev-Mazya inequality for nabla time scale calculus, *Eskişehir Technical University Journal of Science and Technology B - Theoretical Sciences*, 7(2) (2019), 133-145. https://doi.org/10.20290/estubtdb.609525

[44] Řehák, P., Hardy inequality on time scales and its application to half-linear dynamic equations, *J. Inequal. Appl.,* 2005(5) (2005), 495–507. https://doi.org/10.1155/JIA.2005.495

[45] Renaud, P., A reversed Hardy inequality, *Bull. Austral. Math. Soc.,* 34 (1986), 225-232. https://doi.org/10.1017/S0004972700010091

[46] Saker, S. H., Dynamic inequalities on time scales: A survey, *J. Fractional Calc. & Appl.,* 3(S)(2) (2012), 1-36.

[47] Saker, S. H., Hardy–Leindler Type Inequalities on Time Scales, *Appl. Math. Inf. Sci.,* 8(6) (2014), 2975-2981. https://doi.org/10.12785/amis/080635

[48] Saker, S. H., Mahmoud, R. R., A connection between weighted Hardy's inequality and half-linear dynamic equations, *Adv. Difference Equ.,* 2014(129) (2019), 1-15. https://doi.org/10.1186/s13662-019-2072-x

[49] Saker, S. H., Mahmoud, R. R., Peterson, A., Some Bennett-Copson type inequalities on time scales, *J. Math. Inequal.,* 10(2) (2016), 471-489. https://doi.org/10.7153/jmi-10-37

[50] Saker, S. H., Mahmoud, R. R., Osman, M. M., Agarwal, R. P., Some new generalized forms of Hardy's type inequality on time scales, *Math. Inequal. Appl.,* 20(2) (2017), 459-481. https://doi.org/10.7153/mia-20-31

[51] Saker, S. H., O'Regan, D., Agarwal, R. P., Dynamic inequalities of Hardy and Copson type on time scales, *Analysis,* 34(4) (2014), 391-402. https://doi.org/10.1515/anly-2012-1234

[52] Saker, S. H., O'Regan, D., Agarwal, R. P., Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales, *Math. Nachr.,* 287(5-6) (2014), 686-698. https://doi.org/10.1002/mana.201300010

[53] Saker, S. H., Osman, M. M., O'Regan, D., Agarwal, R. P., Inequalities of Hardy type and generalizations on time scales, *Analysis,* 38(1) (2018), 47–62. https://doi.org/10.1515/anly-2017-0006

[54] Saker, S. H., Mahmoud, R. R., Peterson, A., A unified approach to Copson and Beesack type inequalities on time scales, *Math. Inequal. Appl.,* 21(4) (2018), 985-1002. https://doi.org/10.7153/mia-2018-21-67

[55] Saker, S. H., O'Regan, D., Agarwal, R. P., Converses of Copson's inequalities on time scales, *Math. Inequal. Appl.,* 18(1) (2015), 241-254. https://doi.org/10.7153/mia-18-18

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WEAK SUBGRADIENT METHOD WITH PATH BASED TARGET LEVEL ALGORITHM FOR NONCONVEX OPTIMIZATION

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Abstract. We study a new version of the weak subgradient method, recently developed by Dinc Yalcin and Kasimbeyli for solving nonsmooth, nonconvex problems. This method is based on the concept of using any weak subgradient of the objective of the problem at the currently generated point with a version of the dynamic stepsize in order to produce a new point at each iteration. The target value needed in the dynamic stepsize is defined using a path based target level (PBTL) algorithm to ensure the optimal value of the problem is reached. We analyze the convergence and give an estimate of the convergence rate of the proposed method. Furthermore, we demonstrate the performance of the proposed method on nonsmooth, nonconvex test problems, and give the computational results by comparing them with the approximately optimal solutions.

1. Introduction

In this paper, we focus on nonsmooth problems where the objective function is lower locally Lipschitz but not necessarily convex or smooth. Many real-world application such as control theory, machine learning, optimal shape design are nonsmooth optimization problems.

In nonsmooth convex optimization, a subgradient defines the normal vectors of the supporting hyperplane to the graph of the function at the relevant point. Thus, in nonsmooth convex optimization, the projected subgradient methods are well known and the fundamentals of these methods have been investigated by Polyak [50], Ermoliev [23], Shor [53]. The main purpose of a projected subgradient method

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is to generate a new point by using a subgradient of the function at the current point and a positive stepsize parameter. The projection is not computationally expensive if the constraint set is easy for example box constraints. For the convergence analysis, the selection of the stepsize parameter is significant. The classical stepsize types are a (fixed positive) constant, diminishing, and dynamic stepsize. With the dynamic stepsize, the target value is an estimate of the optimal value of the problem and it can be defined as a constant or it can be updated throughout the projected subgradient method. The constant target value may be greater or lower than the optimal value. Alternatively, the target value may be calculated by a path based target level (PBTL) algorithm, which guarantees that the target value will converge to the optimal value [14, 27, 45, 56].

When the function is nonsmooth and nonconvex, various definitions of subgradients are used such as Clarke's subgradient [18] and weak subgradient [3, 4]. Clarke's subgradient is used in nonsmooth, nonconvex (unconstrained or only box constrained) optimization problems, and employed in various methods such as bundle-type methods (see, e.g., [24, 29, 30, 36, 41]), gradient sampling algorithm (see, e.g., [16, 19, 39]), variable metric method (see, e.g., [55]), trust region method (see, e.g., [1, 21, 31, 52]), cutting planes (see, e.g., [25]), proximal algorithms (see, e.g., [9, 11, 12, 48]), quasi-Newton method (see, e.g., [20, 40]). In these methods, the descent directions are usually computed by solving a subproblem which may be quadratic.

Besides subgradient based methods, smoothing methods are also proposed in literature to solve some class of nonsmooth optimization problems. In these methods, the nonsmooth function is approximated by a smooth function, then the smooth function is optimized. The nonsmooth function may be convex (see, e.g., [8, 10, 13, 47, 54]), convex composite(see, e.g., [15]), or nonconvex (see, e.g., [10, 17]).

In addition to these methods, for solving nonsmooth, nonconvex optimization problems, the weak subgradient method [22] is the first to use weak subgradients which have vector and scalar parts, corresponding the supporting conic surfaces to the graph of the function at the relevant point. The weak subgradient method is a generalization of projected subgradient methods, and a convergence analysis of it is investigated with various stepsize parameters: constant and diminishing as well as three types of dynamic.

The aim of this paper is to propose a new version of the weak subgradient method that uses a stepsize parameter computed with PBTL algorithm. Then, the convergence properties and the convergence rate of the proposed method are also investigated. We approximately compute the weak subgradient of the function at the relevant point with the algorithm using the theorem [22, Theorem 2.8] which establish the relation between the directional derivative and weak subdifferential. Additionally, we test the performance of the method on nonsmooth, nonconvex test problems from the literature.

The rest of the paper is organized as follows. Section 2 gives the main properties of the weak subdifferentials and the algorithm for the approximate computing of the weak subgradient is presented. In section 3, we give the convergence properties and convergence rate of the weak subgradient method with PBTL algorithm. Section 4 gives the computational results. In section 5 we draw some conclusions.

2. Preliminaries

In this section, we explain the weak subdifferential and the approximate computing of the weak subgradient.

2.1. **Weak Subdifferentials.**

In this section, we give the definition of the weak subdifferentials and some properties related to this study (see [3, 4, 22, 33, 34]).

Definition 1. *Let $f : \mathbb{S} \to \mathbb{R}$ and $\bar{x} \in \mathbb{S}$. A pair $(v, c) \in \mathbb{R}^n \times \mathbb{R}_+$ is called a weak subgradient of f at $\bar{x}$ on $\mathbb{S}$ if*

$$f(x) \geq f(\bar{x}) + \langle v, x - \overline{x}\rangle - c\|x - \bar{x}\|, \quad \forall x \in \mathbb{S}. \tag{1}$$

The set

$$\partial^w_{\mathbb{S}} f(\bar{x}) = \{(v, c) \in \mathbb{R}^n \times \mathbb{R}_+ : f(x) \geq f(\bar{x}) + \langle v, x - \bar{x}\rangle - c\|x - \bar{x}\|, \quad \forall x \in \mathbb{S}\}$$

of all weak subgradients of f at $\bar{x}$ is called the weak subdifferential of f at $\bar{x}$ on $\mathbb{S}$.

As a result of the definition of the weak subgradient, a continuous (superlinear) and concave function is obtained as follows

$$g(x) = f(\bar{x}) + \langle v, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

where $x \in \mathbb{S}, g(\bar{x}) = f(\bar{x})$, and $(v, c) \in \partial^w_{\mathbb{S}} f(\bar{x})$. In addition, the hypograph of this function $g(x)$ is a cone and thus supports the epigraph of the function $f(x)$ at the point $(\bar{x}, f(\bar{x}))$.

Assumption 1. *Let $\mathbb{S} \subseteq \mathbb{R}^n$ be starshaped at $\bar{x} \in \mathbb{S}$, and let $f : \mathbb{S} \to \mathbb{R}$ be a given function. Suppose that f has a directional derivative at $\bar{x}$ in every direction $x - \bar{x}$ with arbitrary $x \in \mathbb{S}$ and*

$$f(x) - f(\bar{x}) \geq f'(\bar{x}; x - \bar{x}) \quad \text{for all } x \in \mathbb{S} - \{\bar{x}\}. \tag{2}$$

When Assumption 1 holds, the following equation

$$f'(\bar{x}; h) = \max\{\langle v, h\rangle - c\,\|h\| : \ (v, c) \in \partial^w_{\mathbb{S}} f(\bar{x}), \|v\| + c \leq M\}, \ \forall h \in \mathbb{R}^n$$

explains the relation between the weak subdifferential $\partial^w_{\mathbb{S}} f(\bar{x})$ and the directional derivative $f'(\bar{x}; h)$ (see [22, Theorem 2.8]), where M is a positive number. The relation plays an important role in the approximation of the weak subgradients.

In addition, it is known that the weak subdifferential of a function is convex and closed (see [33, Theorem 2.4]), and also compact (see [22, Theorem 2.9]). The property of compactness is handled by limiting the scalar part of weak subgradient c with an upper bound L and thus the norm of the vector part of the weak subgradient v is also bounded with an uper bound D. It means that $\partial^w_{\mathbb{S}_L} f(\bar{x})$ is nonempty for

$c \le L$ and with the number $D > 0$, $\|v\| \le D$ for all $(v, c) \in \partial^w_{\mathbb{S}_L} f(\bar{x})$. This property of the weak subgradient is essential for both the approximation of the weak subgradients and the convergence analysis of the weak subgradient method.

2.2. **Approximation of Weak Subgradients.** Dinc Yalcin and Kasimbeyli [22] presented an algorithm which makes use of the relation between the directional derivative and weak subgradients, and also the compactness property of the weak subdifferential and, in addition, utilizes the discrete gradient method given by [6]. The algorithm numerically computes the weak subgradient of a function at a given point. Note that the approximation is computed more properly when the value of L which is the upper limit of the scalar part of the weak subgradient c is defined large enough. In addition, throughout this work Assumption 1 holds. We briefly explain the method.

Let us consider the set $G = \{e = (e_1, e_2, ..., e_n) \in \mathbb{R}^n : |e_j| = 1, j = \bar{1,n}\}$ and generate the n vectors $e^j(\alpha) = (\alpha e_1, \alpha^2 e_2, ..., \alpha^j e_j, 0, ..., 0)$, $j = \bar{1,n}$ where $e = (e_1, e_2, ..., e_n) \in G$ and $\alpha \in (0, 1]$ is a fixed number. Then, the equation $f'(\bar{x}; e^j(\alpha)) = \langle \bar{v}, e^j(\alpha) \rangle - \bar{c} \left\| e^j(\alpha) \right\|$ is constructed by the relation between the directional derivative and the weak subdifferential. In addition, with using the compactness of the weak subdifferential, the set $V_{\bar{c}} = \{v \in \mathbb{R}^n : (v, \bar{c})\}$ is obtained for the particular $\bar{c} \le L$. Thus, the weak subgradient $(\bar{v}, \bar{c})$ exists, where $\bar{v} \in V_{\bar{c}}$. Note that L may be defined as the lower Lipschitz constant.

Due to the compactness of the weak subdifferential and the relation with the directional derivative, a weak subgradient $(\bar{v}, \bar{c})$ that satisfies the equation $f'(\bar{x}; e^j(\alpha)) = \langle \bar{v}, e^j(\alpha) \rangle - \bar{c} \left\| e^j(\alpha) \right\|$ exists, where $\bar{v} \in V_{\bar{c}}$ defined as $V_{\bar{c}} = \{v \in \mathbb{R}^n : (v, \bar{c})\}$ for the particular $\bar{c}$. Note that $\bar{c}$ can be taken less or equal to the lower Lipschitz constant L.

Let take any $e \in G$, and let define $\lambda > 0, \alpha > 0$ and given any $\bar{c}$ and generate the points where the zeroth point is the current point $x^0 = \bar{x}$ and the others are obtained as $x^j = x^0 + \lambda e^j(\alpha)$, $\quad j = \bar{1,n}$. Furthermore, the points are easily generated by $x^j = x^{j-1} + (0, \ldots, 0, \lambda \alpha^j e_j, 0, \ldots)$ for every $j = \bar{1,n}$. After that, the vector $v(e, \alpha, \lambda) \in \mathbb{R}^n$ with the coordinates

$$v_j(e, \alpha, \lambda) = \frac{f(x^j) - f(x^{j-1})}{\lambda \alpha^j e_j} + \frac{\bar{c}}{e_j}, \quad j = \bar{1,n}$$

is defined and with the given numbers, we can state the set $W(e, \alpha) = \{(w, \bar{c}) \in \mathbb{R}^n \times C : \exists (\lambda_k \to +0, k \to +\infty), w = \lim_{k \to \infty} v(e, \alpha, \lambda_k)\}$. Finally, the set $W(e, \alpha)$ is a subset of weak subdifferential, $W(e, \alpha) \subset \partial^w_{\mathbb{S}_L} f(\bar{x}) \quad \forall \alpha \in (0, \alpha_0]$ (see [22, Proposition 3.5], also see [22, Proposition 3.1], [22, Proposition 3.3], [22, Corollary 3.4] for more details).

By using the construction given above, Algorithm 1 is constructed in [22] as follows.

Algorithm 1 Approximate computing of the weak subgradient $(v, c) \in \partial^w_{\mathbb{S}_L} f(\bar{x})$.

Let $e \in G = \{e = (e_1, e_2, ..., e_n) \in \mathbb{R}^n : |e_j| = 1, j = \overline{1,n}\}$ and $\lambda > 0, \alpha \in (0, 1]$, $\bar{x} \in \mathbb{S}$, and $L > 0$ sufficient large.
Define $e^j(\alpha) = (e_1\alpha, e_2\alpha^2, ..., e_j\alpha^j, 0, ..., 0), j = \overline{1,n}$.
Choose a number $0 < c < L$.
Let $x^0 = \bar{x}$.
$j \leftarrow 1$.
while $j \leq n$ **do**
    $x^j = x^0 + \lambda e^j(\alpha)$,
    $v_j = \frac{f(x^j) - f(x^{j-1})}{\lambda \alpha^j e_j} + \frac{c}{e_j}$,
    $j \leftarrow j + 1$.
end while

3. Weak Subgradient Method with Path Based Target Level (PBTL) Algorithm

In this paper, we focus on the following box constrained nonsmooth optimization problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{S} \end{array} \tag{3}$$

where $f : \mathbb{S} \to \mathbb{R}$ is a lower locally Lipschitz function not necessarily convex and smooth. $\mathbb{S} \subset \mathbb{R}^n$ defines the box constraints $\mathbb{S} = \{x \in \mathbb{R}^n : l \leq x \leq u\}$, where l and u shows the lower and upper bounds, respectively.

We present the weak subgradient method with the PBTL algorithm for solving Problem (3). The process of weak subgradient method at every iteration k is as follows:

$$x_{k+1} = P_{\mathbb{S}}(x_k - \alpha_k v_k). \tag{4}$$

Here, $P_{\mathbb{S}}$ denotes projection on the set $\mathbb{S}$, $(v_k, c_k) \in \partial^w_{\mathbb{S}_L} f(x_k)$ is the weak subgradient and the parameter α_k is a positive stepsize. Since the set consists of box constraints, the projection is simple.

Some notations is used through this section. x^* and f^* denote a critical point and the critical value of the problem (3) in the sense of weak subdifferential, respectively. We assume that positive numbers D and L exists satisfying

$$\|v_k\| \leq D, \tag{5}$$

$$c_k \leq L, \tag{6}$$

for all $(v_k, c_k) \in \partial^w_{\mathbb{S}_L} f(x_k)$ for all $x_k \in \mathbb{S}$. The diameter of $\mathbb{S}$ is denoted by the notion $d_{\mathbb{S}} = diam(\mathbb{S}) = \max_{x_1, x_2 \in \mathbb{S}} \|x_1 - x_2\|$. Then

$$\|x_k - x^*\| \leq d_{\mathbb{S}}, \tag{7}$$

where $\|\cdot\|$ is the Euclidean norm.

The dynamic stepsize is generally defined as

$$\alpha_k = \gamma_k \frac{f(x_k) - f_k^{lev} - c_k d_{\mathbb{S}}}{\|v_k\|^2}, 0 < \underline{\gamma} \leq \gamma_k \leq \bar{\gamma} < 2, \tag{8}$$

where the target value f_k^{lev} is an estimate of f^*. The convergence analysis for the various selections of f_k^{lev} is given in [22]. When these selections of f^{lev} are defined constantly, greater or lower value of f^{lev} than the optimal value f^* occurs. In this circumstances, the convergence depends on f^{lev} and the difference $(f^* - f^{lev})$, respectively. When f_k^{lev} is updated during the algorithm with the procedure $f_k^{lev} = \min_k\{f(x_k)\} - \delta_k$ and the parameter δ_k is computed, regardless of whether or not the current iteration is better than f_k^{lev}, the upper limit of δ_k has an impact on the convergence.

In this paper, we analyze the weak subgradient method with a new dynamic stepsize (8), where f_k^{lev} is defined by the PBTL algorithm given in [14, 27, 45, 56] to ensure $f_k^{lev} \to f^*$. The pseudocode is given in Algorithm 2.

The algorithm decreases the δ_l parameter only in Steps 14-16 if the length of the path σ_k travelled by iterates for all $k < k_{l+1}$ exceeds the prescribed upper bound R; otherwise, the parameter remains the same. Decreasing δ_l means increasing the target level f_k^{lev}. σ_k is reset when a new point is generated with sufficient descent of the objective function.

We begin the convergence analysis with the following lemma without proof which gives a general inequality between the generated points and the critical point that is true for all stepsizes (also, see e.g. [2, 26, 32, 37, 38, 45, 46, 51] for other subgradient methods) This lemma is essential for the subsequent convergence analysis.

Lemma 1. [22, Lemma 2] Let $\{x_k\}$ be the sequence generated by the weak subgradient method. Then for all $k \geq 0$, we have

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - 2\alpha_k[f(x_k) - f^* - c_k\|x^* - x_k\|] + \alpha_k^2\|v_k\|^2.$$

We start with a lemma which explains that if δ_l is nondiminishing, then the target values f_k^{lev} are updated infinitely through iterations which means $\inf_{k\geq 0} f(x_k) = -\infty$. The lemma holds true regardless of whether the computation of the weak subgradient is exact or approximate.

Lemma 2. *Algorithm 2 generates infinitely many values of l which means $l \to \infty$. Thus we have either $\inf_{k\geq 0} f(x_k) = -\infty$ or $\lim_{l\to\infty} \delta_l = 0$ for the sequence $\{x_k\}$ generated by the weak subgradient method with the PBTL algorithm.*

Proof. Assume that l takes only a finite number of values, let $\bar{l} > 0$ be the upper bound of l. In this case, we have

$$\sigma_k + \alpha_k\|v_k\| \leq \sigma_k + \alpha_k D = \sigma_{k+1} \leq R$$

Algorithm 2 Weak subgradient method with PBTL algorithm

1: Select a starting initial solution $x_0 \in \mathbb{X}$, and $\delta_0 > 0, R > 0$, let the incumbent solution be $x^{best} = x_0$, and $\sigma_0 = 0, f_{-1}^{rec} = \infty$.
2: Define tolerance tol, and let iteration counter $k = 0$ and $l = 0, k_l = 0$.
3: **while** $\delta_l > tol$ **do**
4:     Calculate $f(x_k)$.
5:     **if** $f(x_k) < f_{k-1}^{rec}$, **then**
6:         Set $f_k^{rec} = f(x_k)$, $x^{best} = x_k$
7:     **else**
8:         Set $f_k^{rec} = f_{k-1}^{rec}$.
9:     **end if**
10:     **if** $f(x_k) < f_{k_l}^{rec} - \frac{\delta_l}{2}$, **then**
11:         Set $k_{l+1} = k, \sigma_k = 0, \delta_{l+1} = \delta_l, l = l + 1$,
12:         Go to 17.
13:     **end if**
14:     **if** $\sigma_k > R$, **then**
15:         Set $k_{l+1} = k, \sigma_k = 0, \delta_{l+1} = \frac{\delta_l}{2}, l = l + 1$.
16:     **end if**
17:     Set $f_k^{lev} = f_{k_l}^{rec} - \delta_l$.
18:     Compute a weak subgradient $(v_k, c_k) \in \partial_{\mathbb{S}_L}^w f(x_k)$ of f at x_k via Algorithm 1 in Sect. 2.2.
19:     Calculate x_{k+1} via (4) and (8).
20:     $\sigma_{k+1} = \sigma_k + \alpha_k \|v_k\|$.
21:     $k \leftarrow k + 1$.
22: **end while**

from (5) and Step 20 for all $k \geq k_{\bar{l}}$. This would mean that $\lim_{k\to\infty} \alpha_k = 0$, which is impossible. Since for all $k \geq k_{\bar{l}}$, from Step 17, we have

$$f(x_k) - f_k^{lev} \geq \delta_{\bar{l}}. \tag{9}$$

Furthermore, c_k is chosen less than $\frac{f(x_k) - f_k^{lev}}{d_{\mathbb{S}}}$ since the stepize is a positive parameter. Thus, with (9) and the way of choosing the vale of c_k, we have

$$\alpha_k = \gamma_k \frac{f(x_k) - f_k^{lev} - c_k d_{\mathbb{S}}}{\|v_k\|^2} > 0.$$

This implies that for all $k \geq k_{\bar{l}}$, the stepsize α_k is bounded below with a positive value which means $\lim_{k\to\infty} \alpha_k > 0$. As a consequence l cannot be finite: $l \to \infty$.

Since l goes to infinite, there should be a limit $\delta = \lim_{l\to\infty} \delta_l$. If $\delta = 0$, then $\lim_{l\to\infty} \delta_l = 0$. Otherwise, let l_0 is large enough so that for all $l \geq l_0$, we have $\delta_l = \delta$ from 10-13 and 14-16 and

$$f_{k_{l+1}}^{rec} - f_{k_l}^{rec} \leq -\frac{\delta}{2},$$

implying that $\inf_{k\geq 0} f(x_k) = -\infty$. □

Remark 1. *Algorithm 2 is terminated when* δ_l *is less than* tol. *According to Lemma 2, if the function* f *goes to negative infinity, then while* l *goes to infinity,* δ_l *has a limit point* δ. *In this case, Algorithm 2 runs infinite iterations since the stopping condition* $\delta_l \leq tol$ *cannot be hold. Therefore, another termination rule such as a time limit or an iteration limit may be used to prevent this situation.*

The convergence property of the weak subgradient method with the PBTL algorithm is given in the following proposition.

Proposition 1. *For the sequence* $\{x_k\}$ *generated by the weak subgradient method with the PBTL algorithm, we have*

- (a) *if* $\lim_{l\to\infty} \delta_l > 0$, *then*
$$\inf_{k\geq 0} f(x_k) = -\infty,$$
- (b) *if* $\lim_{l\to\infty} \delta_l = 0$, *then*
$$\inf_{k\geq 0} f(x_k) = f^*.$$

Proof. If $\lim_{l\to\infty} \delta_l > 0$, according to Lemma 2, we have $\inf_{k\geq 0} f(x_k) = -\infty$. Thus the proof is completed for part (a).

Now, we prove part (b).

Let ψ be the set of l given by

$$\psi = \Big\{ l | \delta_l = \frac{\delta_{l-1}}{2}, l \geq 1 \Big\}.$$

We obtain

$$\sigma_k = \sigma_{k-1} + \alpha_{k-1}\|v_{k-1}\| = \sum_{j=k_l}^{k-1} \alpha_j \|v_j\|$$

from Steps 10-16 and 20 . When the length of the path becomes greater than the upper value $\sum_{j=k_l}^{k-1} \alpha_j D > \sum_{j=k_l}^{k-1} \alpha_j \|v_j\| > R$ at Steps 14-16, k_{l+1} becomes equal to iteration number $k_{l+1} = k$ where $l+1 \in \psi$. Thus, the sum gives

$$\sum_{j=k_{l-1}}^{k-1} \alpha_j > \frac{R}{D} \quad \forall l \in \psi,$$

and, since the cardinality of ψ is infinite, we have the inequality,

$$\sum_{j=0}^{\infty} \alpha_j \geq \sum_{l\in\psi} \sum_{k_{l-1}}^{k-1} \alpha_j > \sum_{l\in\psi} \frac{R}{D} = \infty. \tag{10}$$

Now, assume to contrary that there exists some $\varepsilon > 0$

$$\inf_{k\geq 0} f(x_k) > f^* + \varepsilon,$$

$$\inf_{k \geq 0} f(x_k) - \varepsilon > f^*.$$

Since $\lim_{l\to\infty} \delta_l = 0$, let $\bar{l}$ be large enough so that there exits some ε such that $\delta_l \leq \varepsilon$ for all $l \leq \bar{l}$. Thereby for all $k \leq k_{\bar{l}}$ we obtain,

$$f_k^{lev} = f_{k_l}^{rec} - \delta_l \geq \inf_{k \geq 0} f(x_k) - \varepsilon > f^*. \tag{11}$$

By using the inequality obtained in (11) and by Lemma 1, and in addition, with assumption (5), the diameter of $\mathbb{S}$ given in (7), the dynamic stepsize (8), and finally using the fact that $\gamma_k < 2, \gamma_k^2 \leq \gamma_k$, $0 < \underline{\gamma} < \gamma_k \leq \bar{\gamma} < 2$, the following inequality is obtained

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \underline{\gamma}(2 - \underline{\gamma})\frac{(f(x_k) - f_k^{lev} - c_k d_{\mathbb{S}})^2}{D^2}. \tag{12}$$

By summing these inequalities over $k \geq k_{\bar{l}}$, we have

$$\|x_{k+1} - x^*\|^2 \leq \|x_{k_{\bar{l}}} - x^*\| - \frac{\underline{\gamma}(2 - \underline{\gamma})}{D^2} \sum_{k=k_{\bar{l}}}^{\infty} (f(x_k) - f_k^{lev} - c_k d_{\mathbb{S}})^2. \tag{13}$$

Due to (8) and (10), the last term $\sum_{k=k_{\bar{l}}}^{\infty}(f(x_k) - f_k^{lev} - c_k d_{\mathbb{S}})^2$ of the inequality (13) goes to infinity. Then, the relation cannot hold true. Thus, we obtain the contradiction. □

Now, we give a convergence rate analysis.

Proposition 2. *If the weak subgradient method with the PBTL algorithm terminates after a finite number of K iterations, then K is the largest positive integer such that*

$$\sum_{k=0}^{K-1} (\delta_k - Ld_{\mathbb{S}})^2 \leq \frac{D^2}{\underline{\gamma}(2 - \bar{\gamma})} \|x_0 - x^*\|^2$$

and we have

$$\inf_{0 \leq k \leq K} f(x_k) \leq f^* + \delta_0.$$

Proof. Assume to the contrary that

$$f(x_k) \geq f^* + \delta_0 \tag{14}$$

for all $k = 0, ..., K$.

Since $f_k^{lev} = \min_{0 \geq j \geq k} f(x_j) - \delta_k$ and $\delta_k \leq \delta_0$ for all $k \geq 0$, with (14) we have

$$f_k^{lev} \geq f^* - \delta_0 \geq f^* - \delta_k \geq f^* \tag{15}$$

for all $k = 0, ..., K$.

Hereby, by using the inequality $f_k^{lev} \geq f^*$ obtained in (15) and by Lemma 1, and in addition, with the diameter of $\mathbb{S}$ given in (7), the definition of the dynamic stepsize (8),$0 < \underline{\gamma} < \gamma_k \leq \bar{\gamma} < 2$ (similar to the Proposition 1), we get the inequality (12).

When we combine the inequality (12) using the fact $f(x_k) - f_k^{lev} \geq \delta_k \forall k$ and $c_k \leq L$ given in (6), for all $k \leq K$ the following inequality

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \underline{\gamma}(2 - \underline{\gamma})\frac{(\delta_k - Ld_{\mathbb{S}})^2}{D^2}$$

is obtained.

By summing these inequalities over $k = 0, ...K$, we have

$$\|x_{K+1} - x^*\|^2 \leq \|x_0 - x^*\|^2 - \frac{\underline{\gamma}(2 - \underline{\gamma})}{D^2}\sum_{k=0}^{K}(\delta_k - Ld_{\mathbb{S}})^2.$$

The last relation cannot hold for sufficiently large K because of the compactness of the set $\mathbb{S}$. Thus, it implies

$$\sum_{k=0}^{K-1}(\delta_k - Ld_{\mathbb{S}})^2 \leq \frac{D^2}{\underline{\gamma}(2 - \bar{\gamma})}\|x_0 - x^*\|^2.$$

□

Remark 2. *Let the Assumption 1 hold true. Then, there exists a weak subgradient* $(v_k, c_k) \in W(e, \alpha) \subset \partial^w_{\mathbb{S}_L} f(x_k)$ *and thus, we have*

$$f(x^*) - f(x_k) \geq f'(x_k; x^* - x_k) = \langle v_k, x^* - x_k\rangle - c_k \|x^* - x_k\|$$

for all $k \geq 0$, *which plays an important role in proving Lemma* 1. *Since Lemma* 1 *is essential to prove the results on the propositions of convergence analysis and convergence rate, all the results of this section are valid if the weak subgradient is computed via Algorithm 1.*

4. Computational Results

In this section, we verify the performance and analyze the efficiency of the weak subgradient method with the PBTL algorithm by solving completely 49 nonsmooth, nonconvex test problems, of which 19 are small scale, P-SS, (P1-P19) with 2 to 10 decision variables and 15+15 are large scale (P20-P34), with 50, P-LS-50, (shown as P20-50 to P34-50) and 200, P-LS-200 (shown as P20-200 to P34-200) decision variables, respectively. Table 1 shows the properties of the test problems, including the names given in the literature and references to where they were taken from, the variable numbers, n, and the optimal values of the problems, f^*. Note that the optimal values of some problems are approximate, P12 and P13 are the L_1 version of the Rosenbrock and Wood functions, respectively, and P21 is the nonsmooth version of the Brown function.

Table 1: Nonsmooth nonconvex test problems

Small Scale Problem	**n**	f^*	**Large Scale Problem**	**n**	f^*
P1 Crescent [35]	2	0	P20 Active faces [28]	any	0
P2 Mifflin 2 [44]	2	-1	P21 Brown function [29]	any	0
P3 WF [42]	2	0	P22 Chained crescent I [29]	any	0
P4 SPIRAL [42]	2	0	P23 Chained crescent II [29]	any	0
P5 EVD52 [42]	3	3.5991193	P24 Problem 6 in [43]	any	0
P6 PBC3 [42]	3	0.0042021427	P25 Problem 17 in [43]	any	0
P7 Bard [42]	3	0.050816327	P26 Problem 19 in [43]	any	0
P8 Polak 6 [49]	4	-44	P27 Problem 20 in [43]	any	0
P9 El-Attar [42]	6	0.5598131	P28 Problem 22 in [43]	any	0
P10 Gill [42]	10	9.7857721	P29 Problem 24 in [43]	any	0
P11 Problem 1 [5]	2	2	P30 DC Maxl [5]	any	0
P12 Rosenbrock [5]	2	0	P31 DC Maxlq [7]	any	0
P13 Wood [5]	4	0	P32 Problem 6 in [7]	any	0
P14 EXP [42]	5	0.00012237125	P33 Problem 7 in [7]	any	0
P15 Kow.-Osb. [42]	4	0.0080843684	P34 Chained Mifflin 2 [29]	50	-34.795
P16 OET5 [42]	4	0.0026359735		200	-140.86
P17 OET6 [42]	4	0.0020160753			
P18 PBC1 [42]	5	0.022340496			
P19 EVD61 [42]	6	0.034904926			

The constraint set is $\mathbb{S} = \{x | x_i \in [-5, 5] \quad i = 1, ..., n\}$ in the problems, however if any component of the optimal solution is not in this interval, then the constraint set is updated as $[x_i^* - 5, x_i^* + 5] \quad i \in \{1, ..., n\}$. In addition, the starting points of the problems needed in the algorithm are the same in reference to the corresponding problems.

We code the weak subgradient method with the PBTL algorithm in the Python programming language and carry out numerical experiments on MacBook Pro with 2.5GHz Intel Core i7 processor and with 16GB 1600 MHz DDR3 RAM. The algorithm is terminated if δ_k becomes less than $tol = 0.001$ or the CPU (s) time reaches $3600s$ for all test problems. δ_0 is defined as $|f(x_0)|$. The prescribed upper bound R is defined as $100, 5000$ and 50 for P-SS, P-LS-50 and P-LS-200, respectively. For P7, R is defined as 10000. The parameters α and λ is set as 1 and 0.001 for the approximate computing of the weak subgradient via Algorithm 1, respectively. The upper bound $\bar{c}_k$ of the scaler parameter c_k of the weak subgradient is $\bar{c}_k = \frac{f(x_k) - f_k^{lev}}{d_{\mathbb{S}}}$ to ensure the positiveness of the stepsize. The scaler parameter c_k is defined $c_k = \bar{c}_k * 0.5$ to compute the vector part v_k of the subgradient in Algorithm 1.

The computational results, the CPU times, and iteration numbers of nonsmooth, nonconvex test problems obtained via weak subgradient method with the PBTL algorithm are given in Table 2, where the following notations are used:

- $WSM - Path$: The weak subgradient method with the PBTL algorithm.
- $WSM - Dyn$: The weak subgradient method with dynamic stepsize with dynamic f_k^{lev} from [22].
- f_{wsa}^{path}: The best value of the objective function, computed using $WSM - Path$.
- f_{wsa}^{dyn}: The best value of the objective function, computed using $WSM - Dyn$.
- $iter$: The number of iterations at which the weak subgradient method with the PBTL algorithm is terminated.

Table 2 compares the results with the (approximate) optimal solutions obtained so far and the results obtained by $WSM - Dyn$. The better results are shown in bold. The results show that $WSM - Path$ outperforms $WSM - Dyn$ in 29 out of 49 test problems and two algorithms find the same value in 7 out of 49 test problems.

$$\frac{f - f^*}{1 + |f^*|} \leq \varepsilon. \tag{16}$$

We evaluate the results with the evaluation criteria (16) given above, where f is the results obtained by the relevant method (f_{wsm}^{dyn} or f_{wsm}^{path} in this paper). When the evaluation criteria of each result is less than ε, the results is accepted as successful. The successful percentage is computed by the total number of successful results over the total number of the problems. We take ε as $10^{-2}, 10^{-3}$, and 5×10^{-5}. We summarize the results in Table 3.

If we take the $\varepsilon = 10^{-2}$, then $WSA - Path$ reaches the optimal value with $\%95, \%60$ and $\%40$ percentages for P-SS, P-LS-50, and P-LS-200, respectively. Similar, If we take the $\varepsilon = 10^{-3}$, then $\%95, \%46$ and $\%33$ percentages are obtained. Last, if we take the $\varepsilon = 5 \times 10^{-5}$, then $\%68, \%40$ and $\%27$ percentages are observed. Additional, $WSA - Path$ finds better solution for P14 (EXP). Moreover, $WSA - Path$ outperforms the successful percentage of $WSM - Dyn$.

5. Conclusion

In this paper, we propose a new version of the weak subgradient method with the PBTL algorithm ($WSA - Path$). A weak subgradient of the current point with a version of dynamic stepsize is used to produce a new solution at each iteration, where the weak subgradient is computed with Algorithm 1 using the theorem about the directional derivative and weak subdifferential. The target level in the dynamic stepsize is computed with the PBTL algorithm. Then, the difference with the PBTL algorithm compared to the other dynamic stepsizes is the method of defining the target level to ensure $f_k^{lev} \to f^*$. We give the convergence analysis and converge rate of the method. Furthermore, we show the tests performed using the method

Table 2: Computational results for nonsmooth test problems for test problems

Prob.	f^*	f_{wsm}^{dyn}	$WSM - Path$	
			f_{wsm}^{path}	CPU (s)
P1	0	0	0	86.19
P2	-1	-1	-1	80.41
P3	0	**0**	0.00000169	81.30
P4	0	0	0	1.14
P5	3.5991193	3.59984305	**3.59973074**	123.51
P6	0.0042021427	0.00421077	**0.00420479**	429.55
P7	0.050816327	0.0508552	**0.050829**	232.20
P8	-44	-43.99	-43.99	215.58
P9	0.5598131	0.56171104	**0.55993735**	1859.47
P10	9.7857721	9.813723	**9.79246244**	2739.11
P11	2	2	2	118.39
P12	0	0.00015433	**0**	16.50
P13	0	0.0090316	**0**	0.04
P14	0.00012237125	-0.0024076	**-6**	552.60
P15	0.0080843684	0.00815057	**0.00810742**	114.80
P16	0.0026359735	0.00325996	**0.0026544**	367.68
P17	0.0020160753	0.00317971	**0.00209686**	353.22
P18	0.022340496	0.11826176	**0.02251701**	155.31
P19	0.034904926	**0.03578041**	0.07816864	12.48
P20-50	0	0.004235249	**0**	90.75
P21-50	0	0.01909278	**0**	1656.90
P22-50	0	0.045976722	**0**	3606.05
P23-50	0	0.0048727756	**0**	3636.80
P24-50	0	0.004071199	**0.00300322**	3544.41
P25-50	0	**0**	0.87361276	200, 47
P26-50	0	**0.002417618**	0.18014279	4106, .27
P27-50	0	**0.0073205**	0.103582595	552.50
P28-50	0	**0.000680983**	0.00068109	0.004
P29-50	0	0.012916485	**0.00928192**	3019.05
P30-50	0	2.575630571	**0**	1054.63
P31-50	0	1	1	906.04
P32-50	0	0.028024848	**0.02395819**	3548.23
P33-50	0	**0**	0.07654164	3684.00
P34-50	-34.795	-34.70324	**-34.774069**	2517.02
P20-200	0	**0.01170317**	0.76033843	27.00
P21-200	0	0.096967	**0**	3600.04

Prob.	f^*	f_{wsm}^{dyn}	$WSM-Path$	
			f_{wsm}^{path}	CPU (s)
P22-200	0	0.662850133	**0**	1211.46
P23-200	0	0.08312041	**0**	3600.07
P24-200	0	0.0093181	**0.00489712**	1698.22
P25-200	0	**0**	0.83829569	544.62
P26-200	0	**0.00852129**	0.0300995	398.30
P27-200	0	0.505147266	**0.2388156**	1018.13
P28-200	0	0	0	0.04
P29-200	0	**0.01205072**	0.02550528	1037.98
P30-200	0	9.54184555	**0.0787**	3600.05
P31-200	0	1	1	152.23
P32-200	0	**0.0061375149**	0.09293225	3600.28
P33-200	0	1.170935921	**0.32948383**	3600.28
P34-200	-140.86	-139.8939	$\mathbf{-140.75363}$	3196.39

Table 3: Success percentage of $WSM-Path$ for nonsmooth test problems versus the optimal value and $WSM-Dyn$

Type of Prob.	**Criteria** $\frac{f-f*}{1+\lvert f*\rvert}$	f_{wsm}^{dyn}	f_{wsm}^{path}
P-SS	$< 5\times 10^{-5}$	63%	68%
	$< 10^{-3}$	74%	95%
	$< 10^{-2}$	95%	95%
P-LS-50	$< 5\times 10^{-5}$	14%	40%
	$< 10^{-3}$	20%	46%
	$< 10^{-2}$	60%	60%
P-LS-200	$< 5\times 10^{-5}$	14%	27%
	$< 10^{-3}$	14%	33%
	$< 10^{-2}$	34%	40%

on nonsmooth, nonconvex optimization problems. The performance of $WSA-Path$ over the (approximate) optimal values and $WSM-Dyn$ is shown by the computational experiments. Besides $WSM-Path$ shows good performance in reaching the optimal values, it also outperforms $WSM-Dyn$ in 29 out of 49 test problems and the two algorithms find the same value from 7 out of 49 test problems. We intend to investigate the ways of weakening Assumption 1 as a part of our future work. Additionally, we would like to solve other nonsmooth optimization problems, such as those found in machine learning problems.

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References

[1] Akbari, Z., Yousefpour, R., Peyghami, M. R., A new nonsmooth trust region algorithm for locally Lipschitz unconstrained optimization problems, *Journal of Optimization Theory and Applications, 164* (3) (2015), 733–754, https://dx.doi.org/10.1007/s10957-014-0534-6.

[2] Allen, E., Helgason, R., Kennington, J., Shetty, B., A generalization of Polyak's convergence result for subgradient optimization, *Mathematical Programming, 37* (3) (1987), 309–317.

[3] Azimov, A., Gasimov, R., On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, *International Journal of Applied Mathematics, 1* (2) (1999), 171–192.

[4] Azimov, A., Gasimov, R., Stability and duality of nonconvex problems via augmented Lagrangian, *Cybernetics and Systems Analysis, 38* (3) (2002), 412–421, https://dx.doi.org/10.1023/A:1020316811823.

[5] Bagirov, A., A method for minimization of quasidifferentiable functions, *Optimization Methods and Software, 17* (1) (2002), 31–60, https://dx.doi.org/10.1080/10556780290027837.

[6] Bagirov, A. M., A method for minimizing convex functions based on continuous approximations to the subdifferential, *Optimization Methods and Software, 9* (1998), 1–17, https://dx.doi.org/10.1080/10556789808805683.

[7] Bagirov, A. M., Ugon, J., Codifferential method for minimizing nonsmooth DC functions, *Journal of Global Optimization, 50* (1) (2011), 3–22, https://dx.doi.org/10.1007/s10898-010-9569-x.

[8] Beck, A., Teboulle, M., Smoothing and first order methods: A unified framework, *SIAM Journal on Optimization, 22* (2) (2012), 557–580, https://dx.doi.org/10.1137/100818327.

[9] Bolte, J., Sabach, S., Teboulle, M., Proximal alternating linearized minimization for nonconvex and nonsmooth problems, *Mathematical Programming, 146* (1-2) (2014), 459–494, https://dx.doi.org/10.1007/s10107-013-0701-9.

[10] Borges, P., Sagastizábal, C., Solodov, M., A regularized smoothing method for fully parameterized convex problems with applications to convex and nonconvex two-stage stochastic programming, *Mathematical Programming* (2020), 1–33, https://dx.doi.org/10.1007/s10107-020-01582-2.

[11] Boţ, R. I., Csetnek, E. R., An inertial Tsengs type proximal algorithm for nonsmooth and nonconvex optimization problems, *Journal of Optimization Theory and Applications, 171* (2) (2016), 600–616, https://dx.doi.org/10.1007/s10957-015-0730-z.

[12] Bot, R. I., Csetnek, E. R., Nguyen, D. K., A proximal minimization algorithm for structured nonconvex and nonsmooth problems, *SIAM Journal on Optimization, 29* (2) (2019), 1300–1328, https://dx.doi.org/10.1137/18M1190689.

[13] Boţ, R. I., Hendrich, C., A variable smoothing algorithm for solving convex optimization problems, *TOP, 23* (1) (2015), 124–150, https://dx.doi.org/10.1007/s11750-014-0326-z.

[14] Brannlund, U., On Relaxation Methods for Nonsmooth Convex Optimization, PhD thesis, Stockholm: Department of Mathematics, Royal Institute of Technology, 1993.

[15] Burke, J. V., Hoheisel, T., Epi-convergent smoothing with applications to convex composite functions, *SIAM Journal on Optimization, 23* (3) (2013), 1457–1479, https://dx.doi.org/10.1137/120889812.

[16] Burke, J. V., Lewis, A. S., Overton, M. L., A robust gradient sampling algorithm for nonsmooth, nonconvex optimization, *SIAM Journal on Optimization, 15* (3) (2005), 751–779, https://dx.doi.org/10.1137/030601296.

[17] Chen, X., Smoothing methods for nonsmooth, nonconvex minimization, *Mathematical Programming, 134* (1) (2012), 71–99, https://dx.doi.org/10.1007/s10107-012-0569-0.

[18] Clarke, F. H., Optimization and Nonsmooth Analysis, SIAM, 1990.

[19] Curtis, F. E., Que, X., An adaptive gradient sampling algorithm for nonsmooth optimization, *Optimization Methods and Software, 28* (6) (2013), 1302–1324, https://dx.doi.org/10.1080/10556788.2012.714781.

[20] Curtis, F. E., Que, X., A quasi-Newton algorithm for nonconvex, nonsmooth optimization with global convergence guarantees, *Mathematical Programming Computation, 7* (4) (2015), 399–428, https://dx.doi.org/10.1007/s12532-015-0086-2.

[21] Dennis, J. E., Li, S. B. B., Tapia, R. A., A unified approach to global convergence of trust region methods for nonsmooth optimization, *Mathematical Programming, 68* (1-3) (1995), 319–346, https://dx.doi.org/10.1007/BF01585770.

[22] Dinc Yalcin, G., Kasimbeyli, R., Weak subgradient method for solving nonsmooth nonconvex optimization problems, *Optimization, 70* (7) (2021), 1513–1553, https://dx.doi.org/10.1080/02331934.2020.1745205.

[23] Ermoliev, Y. M., Methods of solution of nonlinear extremal problems, *Cybernetics, 2* (4) (1966), 1–14, https://dx.doi.org/10.1007/BF01071403.

[24] Fuduli, A., Gaudioso, M., Giallombardo, G., A DC piecewise affine model and a bundling technique in nonconvex nonsmooth minimization, *Optimization Methods and Software, 19* (1) (2004), 89–102, https://dx.doi.org/10.1080/10556780410001648112.

[25] Fuduli, A., Gaudioso, M., Giallombardo, G., Minimizing nonconvex nonsmooth functions via cutting planes and proximity control, *SIAM Journal on Optimization, 14* (3) (2004), 743–756, https://dx.doi.org/10.1137/S1052623402411459.

[26] Gasimov, R. N., Augmented Lagrangian duality and nondifferentiable optimization methods in nonconvex programming, *Journal of Global Optimization, 24* (2) (2002), 187–203, https://dx.doi.org/10.1023/A:1020261001771.

[27] Goffin, J. L., Kiwiel, K. C., Convergence of a simple subgradient level method, *Mathematical Programming, 85* (1) (1999), 207–211, https://dx.doi.org/10.1007/s101070050053.

[28] Grothey, A., Decomposition Methods for Nonlinear Nonconvex Optimization Problems, PhD thesis, Citeseer, 2001.

[29] Haarala, M., Miettinen, K., Mäkelä, M. M., New limited memory bundle method for large-scale nonsmooth optimization, *Optimization Methods and Software, 19* (6) (2004), 673–692, https://dx.doi.org/10.1080/10556780410001689225.

[30] Haarala, N., Miettinen, K., Mäkelä, M. M., Globally convergent limited memory bundle method for large-scale nonsmooth optimization, *Mathematical Programming, 109* (1) (2007), 181–205, https://dx.doi.org/10.1007/s10107-006-0728-2.

[31] Hoseini, N., Nobakhtian, S., A new trust region method for nonsmooth nonconvex optimization, *Optimization, 67* (8) (2018), 1265–1286, https://dx.doi.org/10.1080/02331934.2018.1470175.

[32] Hu, Y., Yang, X., Sim, C.-K., Inexact subgradient methods for quasi-convex optimization problems, *European Journal of Operational Research, 240* (2) (2015), 315–327, https://dx.doi.org/10.1016/j.ejor.2014.05.017.

[33] Kasimbeyli, R., Mammadov, M., On weak subdifferentials, directional derivatives, and radial epiderivatives for nonconvex functions, *SIAM Journal on Optimization, 20* (2) (2009), 841–855, https://dx.doi.org/10.1137/080738106.

[34] Kasimbeyli, R., Mammadov, M., Optimality conditions in nonconvex optimization via weak subdifferentials, *Nonlinear Analysis: Theory, Methods & Applications, 74* (7) (2011), 2534–2547, https://dx.doi.org/10.1016/j.na.2010.12.008.

[35] Kiwiel, K. C., Methods of Decent for Nondifferentiable Optimization, Lecture Notes in Mathematics, No. 1133, Amer Chemical Soc 1155 16TH St, NW, Washington, DC 20036, 1985.

[36] Kiwiel, K. C., Restricted step and Levenberg–Marquardt techniques in proximal bundle methods for nonconvex nondifferentiable optimization, *SIAM Journal on Optimization, 6* (1) (1996), 227–249, https://dx.doi.org/10.1137/0806013.

[37] Kiwiel, K. C., Convergence and efficiency of subgradient methods for quasiconvex minimization, *Mathematical Programming, 90* (1) (2001), 1–25, https://dx.doi.org/10.1007/s101070100198.

[38] Kiwiel, K. C., Convergence of approximate and incremental subgradient methods for convex optimization, *SIAM Journal on Optimization, 14* (3) (2004), 807–840, https://dx.doi.org/10.1137/S1052623400376366.

[39] Kiwiel, K. C., Convergence of the gradient sampling algorithm for nonsmooth nonconvex optimization, *SIAM Journal on Optimization, 18* (2) (2007), 379–388, https://dx.doi.org/10.1137/050639673.

[40] Lewis, A. S., Overton, M. L., Nonsmooth optimization via quasi-Newton methods, *Mathematical Programming, 141* (1-2) (2013), 135–163, https://dx.doi.org/10.1007/s10107-012-0514-2.

[41] Lukšan, L., Vlček, J., A bundle-Newton method for nonsmooth unconstrained minimization, *Mathematical Programming, 83* (1-3) (1998), 373–391, https://dx.doi.org/10.1007/BF02680566.

[42] Lukšan, L., Vlček, J., NDA: Algorithms for Nondifferentiable Optimization, Technical Report 797, Institute of Computer Science, Academy of Sciences of the Czech Rebuplic, Prague, 2000.

[43] Lukšan, L., Vlček, J., Ramešová, N., UFO 2002, Interactive System for Universal Functional Optimization, Technická Zpráva 883, 2002.

[44] Mäkelä, M. M., N. P., Nonsmooth Optimization: Analysis and Algorithms with Applications to Optimal Control, World Scientific, 1992.

[45] Nedic, A., Bertsekas, D. P., Incremental subgradient methods for nondifferentiable optimization, *SIAM Journal on Optimization, 12* (1) (2001), 109–138, https://dx.doi.org/10.1137/S1052623499362111.

[46] Nedić, A., Bertsekas, D. P., The effect of deterministic noise in subgradient methods, *Mathematical Programming, 125* (1) (2010), 75–99, https://dx.doi.org/10.1007/s10107-008-0262-5.

[47] Nesterov, Y., Smooth minimization of non-smooth functions, *Mathematical Programming, 103* (1) (2005), 127–152, https://dx.doi.org/10.1007/s10107-004-0552-5.

[48] Pock, T., Sabach, S., Inertial proximal alternating linearized minimization (ipalm) for nonconvex and nonsmooth problems, *SIAM Journal on Imaging Sciences, 9* (4) (2016), 1756–1787, https://dx.doi.org/10.1137/16M1064064.

[49] Polak, E., Mayne, D. Q., Higgins, J. E., Superlinearly convergent algorithm for minmax problems, *Journal of Optimization Theory and Applications, 69* (3) (1991), 407–439, https://dx.doi.org/10.1007/BF00940683.

[50] Polyak, B. T., A general method for solving extremal problems, *Doklady Akademii Nauk, Russian Academy of Sciences, 174* (1) (1967), 33–36.

[51] Polyak, B. T., Minimization of unsmooth functionals, *USSR Computational Mathematics and Mathematical Physics, 9* (3) (1969), 14–29, https://dx.doi.org/10.1016/0041-5553(69)90061-5.

[52] Qi, L., Sun, J., A trust region algorithm for minimization of locally Lipschitzian functions, *Mathematical Programming, 66* (1-3) (1994), 25–43, https://dx.doi.org/10.1007/BF01581136.

[53] Shor, N. Z., Minimization Methods for Non-differentiable Functions (K. Kiwiel and A. Ruszcynski, translate), Heidelberg:Springer-Verlag Berlin, 1985.

[54] Tran-Dinh, Q., Adaptive smoothing algorithms for nonsmooth composite convex minimization, *Computational Optimization and Applications, 66* (3) (2017), 425–451, https://dx.doi.org/10.1007/s10589-016-9873-6.

[55] Vlček, J., Lukšan, L., Globally convergent variable metric method for nonconvex nondifferentiable unconstrained minimization, *Journal of Optimization Theory and Applications, 111* (2) (2001), 407–430, https://dx.doi.org/10.1023/A:1011990503369.

[56] Zhang, P., Bao, G., Path-based incremental target level algorithm on riemannian manifolds, *Optimization, 69* (4) (2020), 799–819, https://dx.doi.org/10.1080/02331934.2019.1671840.

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QUATERNIONIC BERTRAND CURVES ACCORDING TO TYPE 2-QUATERNIONIC FRAME IN $\mathbb{R}^4$

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Abstract. In this paper, we give some characterizations of quaternionic Bertrand curves whose torsion is non-zero but bitorsion is zero in $\mathbb{R}^4$ according to Type 2-Quaternionic Frame. One of the most important points in working on quaternionic curves is that given a curve in $\mathbb{R}^4$, the curve in $\mathbb{R}^3$ associated with this curve is determined individually. So, we obtain some relationships between quaternionic Bertrand curve $\alpha^{(4)}$ in $\mathbb{R}^4$and its associated spatial quaternionic curve α in $\mathbb{R}^3$. Also, we support some theorems in the paper by means of an example.

1. Introduction

Bertrand curve was introduced by *Bertrand* in 1850 (see [1]). When a curve is given, if there exists a second curve whose principal normal is the principal normal of that curve, then the first curve is called Bertrand curve and the second curve is called the Bertrand mate of the first curve. The most important properties of Bertrand curves in Euclidean 3-space are that the distance between corresponding points is constant and there is a linear relation between the curvature functions of the first curve, that is, for $\lambda, \mu \in \mathbb{R}$, $\lambda\kappa + \mu\tau = 1$, where κ is curvature and τ is the torsion of the first curve. Also, the absolute value of the real number λ in this linear relation is equal to the distance between corresponding points of Bertrand curves. The Bertrand curves in Euclidean 3-space were extended by L. R. Pears into Riemannian $n-$space and gave general results for Bertrand curves [13]. If these general results were applied to Euclidean $n-$space, then either torsion k_2 or bitorsion k_3 of the curve vanishes. In other words, Bertrand curves in $\mathbb{R}^n$ $(n > 3)$ are degenerate curves. Also, for $n > 3$, some studies about Bertrand curves in Euclidean $n-$space and Lorentzian $n-$space were made in [3], [5], [15].

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Bharathi and Nagaraj introduced spatial quaternionic curve in $\mathbb{R}^3$ and quaternionic curve in $\mathbb{R}^4$. By using the quaternionic multiplication, they obtained the Serret-Frenet equations of the curve in $\mathbb{R}^3$ and then they formed the Serret-Frenet formulae of a quaternionic curve in $\mathbb{R}^4$ by means of the Frenet vectors and curvature functions of the spatial quaternionic curve in $\mathbb{R}^3$ [2]. After then by using these quaternionic frames defined by Bharathi and Nagaraj, a lot of paper about quaternionic curves were made in $\mathbb{R}^3$ and $\mathbb{R}^4$ ([4], [6], [7], [9], [10], [11], [12], [14], [16], [17], [18]).

Kahraman Aksoyak introduced a new quaternionic frame in $\mathbb{R}^4$. This new type of quaternionic frame was called Type 2-Quaternionic Frame [8].

In this paper, we investigate quaternionic Bertrand curves whose torsion is nonzero but bitorsion is zero in $\mathbb{R}^4$ according to Type 2-Quaternionic Frame. One of the most important points in working on quaternionic curves is that given a curve in $\mathbb{R}^4$, the curve in $\mathbb{R}^3$ associated with this curve is determined individually. Hence we obtain some relationships between quaternionic Bertrand curve $\alpha^{(4)}$ in $\mathbb{R}^4$ and spatial quaternionic curve α in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$. For example, we obtain that quaternionic curve $\alpha^{(4)}$ in $\mathbb{R}^4$ is a quaternionic Bertrand curve if and only if the curve α in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$ is a spatial quaternionic Bertrand curve. Also, we show that result: if $\left(\alpha^{(4)}, \beta^{(4)}\right)$ is a quaternionic Bertrand curve couple then (α, β) is a spatial quaternionic Bertrand curve couple, where α and β are curves in $\mathbb{R}^3$ associated with quaternionic curves $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively. And then we give an example about these results.

2. Preliminaries

The quaternion was defined by Hamilton. A real quaternion is as:

$$q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$$

where $q_i \in \mathbb{R}$ for $0 \leq i \leq 3$ and e_1, e_2, e_3 are unit vectors in usual three dimensional real vector space. Any quaternion q can be divided into two parts such that the scalar part denoted by S_q and the vectorial part denoted by V_q, that is, for $S_q = q_0$ and $V_q = q_1e_1 + q_2e_2 + q_3e_3$ we can express any real quaternion as $q = S_q + V_q$.

If $q = S_q + V_q$ and $q' = S_{q'} + V_{q'}$ are any two quaternions, then equality, addition, the multiplication by a real scalar c and the conjugate of q denoted by γq are as:

$$\begin{aligned}
\text{equality} &: \quad q = q' \text{ if and only if } S_q = S_{q'} \text{ and } V_q = V_{q'} \\
\text{addition} &: \quad q + q' = (S_q + S_{q'}) + (V_q + V_{q'}) \\
\text{multiplication by a real scalar} &: \quad cq = cS_q + cV_q \\
\text{conjugate} &: \quad \gamma q = S_q - V_q.
\end{aligned}$$

Let us denote the set of quaternions by H. H is a real vector space with above addition and scalar multiplication. A basis of this vector space is $\{1,\ e_1,\ e_2,\ e_3\}$. Hence, we can think of any quaternion q as an element (q_0, q_1, q_2, q_3) of $\mathbb{R}^4$. Even a quaternion whose scalar part is zero (it is called spatial quaternion) can be considered as a ordered triple (q_1, q_2, q_3) of $\mathbb{R}^3$.

The product of two quaternions is defined by means of the multiplication rule between the units e_1, e_2, e_3 are given by:

$$e_1e_1 = e_2e_2 = e_3e_3 = e_1e_2e_3 = -1. \tag{1}$$

So, by using (1), quaternionic multiplication is obtained as:

$$q \times q' = S_qS_{q'} - \langle V_q, V_{q'}\rangle + S_qV_{q'} + S_{q'}V_q + V_q \wedge V_{q'} \text{ for every } q,\ q' \in H, \tag{2}$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and cross products in $\mathbb{R}^3$, respectively. Also, H is a real algebra and it is called quaternion algebra.

Now, by using (2) the symetric, non-degenerate, bilinear form h on H is given by :

$$h : H \times H \to \mathbb{R},$$
$$h(q, q') = \frac{1}{2}(q \times \gamma q' + q' \times \gamma q) \text{ for } q,\ q' \in H \tag{3}$$

and the norm of any q real quaternion is defined by

$$\|q\|^2 = h(q, q) = q \times \gamma q = S_q^2 + \langle V_q, V_q\rangle .$$

So the mapping given by (3) is called the quaternion inner product [2].

We note that a quaternionic curve in $\mathbb{R}^4$ is denoted by $\alpha^{(4)}$ and the spatial quaternionic curve in $\mathbb{R}^3$ associated with $\alpha^{(4)}$ in $\mathbb{R}^4$ is denoted by α.

Bharathi and Nagaraj introduced the Serret-Frenet formulas for spatial quaternionic curves in $\mathbb{R}^3$ and quaternionic curves in $\mathbb{R}^4$ follow as:

Theorem 1. *(see [2]) Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and S be the set of spatial quaternionic curve*

$$\alpha : I \subset \mathbb{R} \longrightarrow S,$$
$$s \longrightarrow \alpha(s) = \alpha_1(s)e_1 + \alpha_2(s)e_2 + \alpha_3(s)e_3$$

be an arc-lenghted curve. Then the Frenet equations of α are as follows:

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $t = \alpha^{'}$ is unit tangent, n is unit principal normal, $b = t \times n$ is binormal, where $\times$ denotes the quaternion product. $k = \|t'\|$ is the principal curvature and r is the torsion of the curve γ.

Theorem 2. *(see [2]) Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and*

$$\alpha^{(4)} : I \subset \mathbb{R} \longrightarrow Q,$$
$$s \longrightarrow \alpha^{(4)}(s) = \alpha_0^{(4)}(s)+\alpha_1^{(4)}(s)e_1+\alpha_2^{(4)}(s)e_2+\alpha_3^{(4)}(s)e_3$$

be an arc-length curve in $\mathbb{R}^4$. *Then Frenet equations of* $\alpha^{(4)}$ *are given by*

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (K-r) \\ 0 & 0 & -(K-r) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix},$$

where $T = \frac{d\alpha^{(4)}}{ds}$, N_1, N_2, N_3 *are the Frenet vectors of the curve* $\alpha^{(4)}$ *and* $K = \left\| T' \right\|$ *is the principal curvature,* k *is the torsion and* $(K-r)$ *is the bitorsion of the curve* $\alpha^{(4)}$. *There exists following relations between the Frenet vectors of* $\alpha^{(4)}$ *and the Frenet vectors of* α

$$N_1(s) = t(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = b(s) \times T(s).$$

Type 2-Quaternionic Frame which is introduced by Kahraman Aksoyak in [8] is given as:

Theorem 3. *(see [8]) Let* $I = [0,1]$ *denote the unit interval in the real line* $\mathbb{R}$ *and*

$$\alpha^{(4)} : I \subset \mathbb{R} \longrightarrow Q,$$
$$s \longrightarrow \alpha^{(4)}(s) = \alpha_0^{(4)}(s) + \alpha_1^{(4)}(s)e_1 + \alpha_2^{(4)}(s)e_2 + \alpha_3^{(4)}(s)e_3$$

be an arc-length curve in $\mathbb{R}^4$. *Then Frenet equations of* $\alpha^{(4)}$ *are given by*

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & -r & 0 \\ 0 & r & 0 & (K-k) \\ 0 & 0 & -(K-k) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{4}$$

where $T = \frac{d\alpha^{(4)}}{ds}$, N_1, N_2, N_3 *are the Frenet vectors of the curve* $\alpha^{(4)}$ *and* $K = \left\| T' \right\|$ *is the principal curvature,* $-r$ *is the torsion and* $(K-k)$ *is the bitorsion of the curve* $\alpha^{(4)}$. *There exists following relations between the Frenet vectors of* $\alpha^{(4)}$ *and the Frenet vectors of* α

$$N_1(s) = b(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = t(s) \times T(s).$$

3. Characterizations of Quaternionic Bertrand Curve

In this section, we consider the quaternionic curve whose the torsion $(-r)$ is non-zero and bitorsion $(K-k)$ is zero according to Type 2-Quaternionic Frame in $\mathbb{R}^4$ given by (4) and obtain various characterizations for cases where such curves are quaternionic Bertrand curves. Also, we give some relationships between quaternionic Bertrand curves in $\mathbb{R}^4$ and spatial quaternionic curves in $\mathbb{R}^3$ which are related to these curves and discuss some theorems in this section on an example.

Definition 1. *Let* $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ *and* $\beta^{(4)} : \bar{I} \subset \mathbb{R} \to \mathbb{E}^4$ *be quaternionic curves given by the arc-length parameter* s *and* $\bar{s}$, *respectively.*
$\{T(s),\ N_1(s),\ N_2(s),\ N_3(s)\}$ *and* $\{\bar{T}(\bar{s}),\ \bar{N}_1(\bar{s}),\ \bar{N}_2(\bar{s}),\ \bar{N}_3(\bar{s})\}$ *are Frenet vectors*

of these curves. If the principal normal vectors $N_1(s)$ and $\bar{N}_1(\bar{s})$ of the curves $\alpha^{(4)}$ and $\beta^{(4)}$ are linearly dependent, then these curves are called quaternionic Bertrand curves. Let $(\alpha^{(4)}, \beta^{(4)})$ be quaternionic Bertrand curve couple, where $\alpha^{(4)}$ is a quaternionic Bertrand curve and $\beta^{(4)}$ is quaternionic Bertrand mate of $\alpha^{(4)}$.

Theorem 4. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ and $\beta^{(4)} : \bar{I} \subset \mathbb{R} \to \mathbb{E}^4$ be quaternionic curves with arc-length parameter s and $\bar{s}$, respectively. If $(\alpha^{(4)}, \beta^{(4)})$ is a quaternionic Bertrand curve couple, then the distance at corresponding points is constant, that is*

$$d(\alpha^{(4)}(s), \beta^{(4)}(\bar{s})) = const., \text{ for all } s \in I.$$

Proof. We assume that $\alpha^{(4)}$ is a quaternionic Bertrand curve and $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$. From Definition (1), we can write

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda(s) N(s),$$

where $\lambda : I \to \mathbb{R}$ is a differentiable function. If we take the derivative of the above equation with respect to s and use the equations of Type 2-Quaternionic Frame given by (4), we get

$$\bar{T}(\bar{s}) = \frac{ds}{d\bar{s}} \left[(1 - \lambda(s) K(s)) T(s) + \lambda'(s) N_1(s) - \lambda(s) r(s) N_2(s) \right]. \tag{5}$$

Since $h(\bar{T}(\bar{s}), \bar{N}_1(\bar{s})) = 0$ and $h(N_1(s), \bar{N}_1(\bar{s})) = \pm 1$,

$$\lambda'(s) = 0$$

and we have that λ is a constant function on I. □

Theorem 5. *The measure of the angle between the tangent vector fields of quaternionic Bertrand curve couple $\left(\alpha^{(4)}, \beta^{(4)}\right)$ is constant, that is*

$$h\left(T(s), \bar{T}(\bar{s})\right) = \cos\phi_0 = const. \tag{6}$$

Proof. If we derivative $h\left(T(s), \bar{T}(\bar{s})\right)$ and use the equations of Type 2-Quaternionic Frame, we obtain following equality:

$$\begin{aligned} \frac{dh\left(T(s), \bar{T}(\bar{s})\right)}{ds} &= h\left(\frac{dT(s)}{ds}, \bar{T}(\bar{s})\right) + h\left(T(s), \frac{\bar{T}(\bar{s})}{d\bar{s}} \frac{d\bar{s}}{ds}\right) \\ &= h\left(K(s) N_1(s), \bar{T}(\bar{s})\right) + h\left(T(s), \bar{K}(\bar{s}) \bar{N}_1(\bar{s}) \frac{d\bar{s}}{ds}\right). \end{aligned}$$

Since $\bar{N}_1(\bar{s}) = \pm N_1(s)$, we find

$$\frac{dh\left(T(s), \bar{T}(\bar{s})\right)}{ds} = 0$$

which implies that $h\left(T(s), \bar{T}(\bar{s})\right)$ is constant. □

Theorem 6. *Let* $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ *be a quaternionic curve with arc-length parameter* s *whose torsion is non-zero and bitorsion is zero. Then* $\alpha^{(4)}$ *is a quaternionic Bertrand curve if and only if*

$$\lambda K + \mu r = 1,$$

where λ *and* μ *are real numbers,* K *is the principal curvature,* $-r$ *is the torsion of the curve* $\alpha^{(4)}$.

Proof. We suppose that $\alpha^{(4)}$ is a quaternionic Bertrand curve such that $r \neq 0$ and $K - k = 0$. Then there exists a quaternionic Bertrand mate of $\alpha^{(4)}$ denoted by $\beta^{(4)}$. $\beta^{(4)}$ can be expressed as:

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N_1(s), \tag{7}$$

where λ is non-zero real number. Since the angle between the tangent vector fields of $\alpha^{(4)}$ and $\beta^{(4)}$ is constant, from (5) and (6), the tangent vector of $\beta^{(4)}$ can be written as:

$$\bar{T}(\bar{s}) = \cos\phi_0 T(s) + \sin\phi_0 N_2(s)$$

in here

$$\cos\phi_0 = (1 - \lambda K(s)) \frac{ds}{d\bar{s}}, \tag{8}$$

$$\sin\phi_0 = -\lambda r(s) \frac{ds}{d\bar{s}}. \tag{9}$$

Since λ and $r(s)$ are non-zero, $\sin\phi_0$ is non-zero. If we take as $-\lambda\frac{\cos\phi_0}{\sin\phi_0} = \mu$ and ratio the equations given by (8) and (9) side by side, we find

$$\lambda K + \mu r = 1.$$

Conversely, let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic curve whose the curvatures K and $-r$ hold the relation $\lambda K + \mu r = 1$ for λ and μ real numbers. Let define a quaternionic curve by using λ real number as:

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N_1(s).$$

It is clearly shown that the principal normal lines of $\alpha^{(4)}$ and $\beta^{(4)}$ are linearly dependent. □

Theorem 7. *Let* $\left(\alpha^{(4)}, \beta^{(4)}\right)$ *be a quaternionic Bertrand curve couple, then the product of torsions* $r(s)$ *and* $\bar{r}(\bar{s})$ *at the corresponding points of the curves* α *and* β *is a constant, where* α *and* β *are spatial quaternionic curves in* $\mathbb{R}^3$ *related to quaternionic curves* $\alpha^{(4)}$ *and* $\beta^{(4)}$ *in* $\mathbb{R}^4$*, respectively.*

Proof. Let consider that $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$. Then we have

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N_1(s).$$

If we displace the position vectors $\alpha^{(4)}(s)$ and $\beta^{(4)}(s)$, we get

$$\alpha^{(4)}(s) = \beta^{(4)}(s) - \lambda \bar{N}_1(\bar{s}). \tag{10}$$

By differentiating (10) with respect to s and using the equations of Type 2-Quaternionic Frame, we obtain

$$T(s) = \left[(1+\lambda\bar{K}\,(\bar{s}))\bar{T}(\bar{s}) + \lambda\bar{r}\,(\bar{s})\,\bar{N}_2\,(\bar{s})\right]\frac{d\bar{s}}{ds}.$$

So, we can rewrite

$$T(s) = \cos\phi_0\bar{T}(\bar{s}) - \sin\phi_0\bar{N}_2\,(\bar{s})\,,$$

where

$$\cos\phi_0 = (1+\lambda\bar{K}\,(\bar{s}))\frac{d\bar{s}}{ds}, \tag{11}$$

$$\sin\phi_0 = -\lambda\bar{r}\,(\bar{s})\,\frac{d\bar{s}}{ds}. \tag{12}$$

Multiplying the equations (9) and (12) side by side, we find

$$r\bar{r} = \frac{\sin^2\phi_0}{\lambda^2} = const.$$

□

Theorem 8. *Let* $\left(\alpha^{(4)},\beta^{(4)}\right)$ *be a quaternionic Bertrand curve couple. Then the curvatures* $K(s)$, $-r(s)$ *and* $\bar{K}\,(\bar{s})\,,\,-\bar{r}\,(\bar{s})$ *of the curves* $\alpha^{(4)}$ *and* $\beta^{(4)}$, *respectively, satisfy the following equation*

$$\lambda\left(K+\bar{K}\right) + \mu\,(r-\bar{r}) = 0. \tag{13}$$

Proof. We assume that $\left(\alpha^{(4)},\beta^{(4)}\right)$ is a quaternionic Bertrand curve couple. Then if we ratio the equations given by (8) and (9) side by side, we find

$$\frac{\cos\phi_0}{\sin\phi_0} = \frac{1-\lambda K}{-\lambda r}$$

and similarly if we proportion the equation (11) to equation (12),

$$\frac{\cos\phi_0}{\sin\phi_0} = \frac{1+\lambda\bar{K}}{-\lambda\bar{r}}.$$

If we take as $-\frac{\cos\phi_0}{\sin\phi_0}\lambda = \mu$, we have

$$\lambda K + \mu r = 1 \tag{14}$$

and

$$\lambda\bar{K} - \mu\bar{r} = -1. \tag{15}$$

From (14) and (15), we obtain

$$\lambda\left(K+\bar{K}\right) + \mu\,(r-\bar{r}) = 0.$$

□

Theorem 9. *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ be a quaternionic curve whose the torsion is non-zero and bitorsion is zero and $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a spatial quaternionic curve associated with $\alpha^{(4)}$ quaternionic curve. Then α is a spatial quaternionic Bertrand curve if and only if $\alpha^{(4)}$ is a quaternionic Bertrand curve.*

Proof. We assume that $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is a spatial quaternionic Bertrand curve. Then there are λ, μ are real constants such that the curvatures $k(s)$ and $r(s)$ of α satisfy

$$\lambda k + \mu r = 1. \tag{16}$$

Since the bitorsion of the quaternionic curve $\alpha^{(4)}$ vanishes, we have

$$K = k. \tag{17}$$

From (16) and (17), we get

$$\lambda K + \mu r = 1.$$

From Theorem (6), the above equality says that $\alpha^{(4)}(s)$ is a quaternionic Bertrand curve.

Conversely it is clearly shown that if $\alpha^{(4)}$ is a quaternionic Bertrand curve whose the bitorsion vanishes, then α is a spatial quaternionic Bertrand curve. □

Theorem 10. *If $\left(\alpha^{(4)}, \beta^{(4)}\right)$ is a quaternionic Bertrand curve couple then (α, β) is a spatial quaternionic Bertrand curve couple, where α and β are curves in $\mathbb{R}^3$ associated with quaternionic curves $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively.*

Proof. We consider that $\alpha^{(4)}$ is a quaternionic Bertrand curve and $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$. Then from Definition (1), $N_1(s)$ and $\bar{N}_1(\bar{s})$ are linearly dependent. On the other hand, from Theorem (9), we know that if $\alpha^{(4)}$ and $\beta^{(4)}$ are Bertrand curve then the curves α and β in $\mathbb{R}^3$ which are associated with $\alpha^{(4)}$ and $\beta^{(4)}$ in $\mathbb{R}^4$, respectively are Bertrand curves, too. Now, we show that β is quaternionic Bertrand mate of α.

From Type 2- Quaternionic Frame, the binormal $\bar{b}$ of β is written as:

$$\bar{b} = \bar{N}_1 \times \gamma \bar{T} \tag{18}$$

Since $\beta^{(4)}$ is a quaternionic Bertrand mate of $\alpha^{(4)}$, we have $\bar{N}_1 = N_1$ and $\bar{T} = \cos\phi_0 T + \sin\phi_0 N_2$. So we can rewrite (18) following as:

$$\begin{aligned} \bar{b} &= N_1 \times \gamma\left(\cos\phi_0 T + \sin\phi_0 N_2\right) \\ &= \cos\phi_0\left(N_1 \times \gamma T\right) + \sin\phi_0\left(N_1 \times \gamma N_2\right). \end{aligned}$$

In last equality, if we use $N_1 \times \gamma T = b$ and $N_2 = n \times T$, we obtain

$$\bar{b} = \cos\phi_0 b + \sin\phi_0 t. \tag{19}$$

Differentiating (19), we find

$$-\bar{r}\bar{n}\frac{d\bar{s}}{ds} = \left(-\cos\phi_0 r + \sin\phi_0 k\right) n$$

and it implies that $\bar{n} = \pm n$. Hence β is a Bertrand mate of α. $\square$

Now, we will see an application of some theorems in the paper by means of following example.

Example 1. *Let $\alpha^{(4)}(s) = \left(\cos \frac{s}{\sqrt{3}}, \sin \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$ be a quaternionic curve in $\mathbb{R}^4$ which is given by arc-length parameter s. The Frenet vectors and the curvatures of the curve $\alpha^{(4)}$ in $\mathbb{R}^4$ are as:*

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{3}}\left(-\sin \frac{s}{\sqrt{3}}, \cos \frac{s}{\sqrt{3}}, 1, 1\right), \\ N_1(s) &= \left(-\cos \frac{s}{\sqrt{3}}, -\sin \frac{s}{\sqrt{3}}, 0, 0\right), \\ N_2(s) &= \frac{1}{\sqrt{6}}\left(-2\sin \frac{s}{\sqrt{3}}, 2\cos \frac{s}{\sqrt{3}}, -1, -1\right), \\ N_3(s) &= \frac{1}{\sqrt{2}}(0, 0, -1, 1) \end{aligned}$$

and

$$k_1 = K = \frac{1}{3},\ k_2 = -r = -\frac{\sqrt{2}}{3},\ k_3 = K - k = 0.$$

By using the definition of Type-2 Quaternionic Frame, the curve α in $\mathbb{R}^3$ which is associated with $\alpha^{(4)}$ is obtained as:

$$\alpha(s) = \frac{1}{\sqrt{2}}\left(2\frac{s}{\sqrt{3}}, -\cos \frac{s}{\sqrt{3}} - \sin \frac{s}{\sqrt{3}}, \cos \frac{s}{\sqrt{3}} - \sin \frac{s}{\sqrt{3}}\right).$$

The Frenet vectors and the curvatures of α are computed as:

$$\begin{aligned} t(s) &= \frac{1}{\sqrt{6}}\left(2, -\cos \frac{s}{\sqrt{3}} + \sin \frac{s}{\sqrt{3}}, -\cos \frac{s}{\sqrt{3}} - \sin \frac{s}{\sqrt{3}}\right), \\ n(s) &= \frac{1}{\sqrt{2}}\left(0, \cos \frac{s}{\sqrt{3}} + \sin \frac{s}{\sqrt{3}}, -\cos \frac{s}{\sqrt{3}} + \sin \frac{s}{\sqrt{3}}\right), \\ b(s) &= \frac{1}{\sqrt{3}}\left(1, \cos \frac{s}{\sqrt{3}} - \sin \frac{s}{\sqrt{3}}, \cos \frac{s}{\sqrt{3}} + \sin \frac{s}{\sqrt{3}}\right) \end{aligned}$$

and

$$k = \frac{1}{3},\ \ r = \frac{\sqrt{2}}{3}.$$

From the definition of Type-2 Quaternionic Frame, there exists following relations between Frenet vectors of $\alpha^{(4)}$ in $\mathbb{R}^4$ and α in $\mathbb{R}^3$:

$$N_1(s) = b(s) \times T(s),\ N_2(s) = n(s) \times T(s),\ N_3(s) = t(s) \times T(s).$$

$\alpha^{(4)}$ is a quaternionic curve whose torsion is non zero and bitorsion is zero and we can see that the curvatures of $\alpha^{(4)}$ hold $\lambda K + \mu r = 1$, for $\lambda = -2$ and $\mu = \frac{5}{\sqrt{2}}$.

So it is a quaternionic Bertrand curve. From Theorem (9), we know that if $\alpha^{(4)}$ is a quaternionic Bertrand curve then α is a spatial quaternionic Bertrand curve. We can easily see that $\lambda k+\mu r=1$, for $\lambda=-2$ and $\mu=\frac{5}{\sqrt{2}}$. Since $\alpha^{(4)}$ is a quaternionic Bertrand curve, we can determine the quaternionic Bertrand mate of it as:

$$\begin{aligned}\beta^{(4)}(s) &= \alpha^{(4)}(s)-2N_1(s)\\ &= \left(3\cos\frac{s}{\sqrt{3}},3\sin\frac{s}{\sqrt{3}},\frac{s}{\sqrt{3}},\frac{s}{\sqrt{3}}\right),\end{aligned}$$

where $\bar{s}=\varphi(s)=\int\left\|\frac{d\beta^{(4)}(s)}{ds}\right\|ds=\frac{\sqrt{11}}{\sqrt{3}}s$ and $\bar{s}$ is arc-length parameter of $\beta^{(4)}$. Now by using Type-2 Quaternionic Frame, we can determine the Frenet vectors and the curvatures of the curve $\beta^{(4)}$ as follows:

$$\beta^{(4)}(\bar{s})=\left(3\cos\frac{\bar{s}}{\sqrt{11}},3\sin\frac{\bar{s}}{\sqrt{11}},\frac{\bar{s}}{\sqrt{11}},\frac{\bar{s}}{\sqrt{11}}\right),$$

$$\begin{aligned}\bar{T}(\bar{s}) &= \frac{1}{\sqrt{11}}\left(-3\sin\frac{\bar{s}}{\sqrt{11}},3\cos\frac{\bar{s}}{\sqrt{11}},1,1\right),\\ \bar{N}_1(\bar{s}) &= \left(-\cos\frac{\bar{s}}{\sqrt{11}},-\sin\frac{\bar{s}}{\sqrt{11}},0,0\right),\\ \bar{N}_2(\bar{s}) &= \frac{1}{\sqrt{22}}\left(-2\sin\frac{\bar{s}}{\sqrt{11}},2\cos\frac{\bar{s}}{\sqrt{11}},-3,-3\right),\\ \bar{N}_3(\bar{s}) &= \frac{1}{\sqrt{22}}(0,0,-11,11)\end{aligned}$$

and

$$\bar{k}_1=\bar{K}=\frac{3}{11},\ \bar{k}_2=-\bar{r}=-\frac{\sqrt{2}}{11},\ \bar{k}_3=\bar{K}-\bar{k}=0.$$

The curve β which is associated with $\beta^{(4)}$ is found as:

$$\beta(\bar{s})=\frac{1}{\sqrt{2}}\left(2\frac{\bar{s}}{\sqrt{11}},3\left(-\sin\frac{\bar{s}}{\sqrt{11}}-\cos\frac{\bar{s}}{\sqrt{11}}\right),3\left(-\sin\frac{\bar{s}}{\sqrt{11}}+\cos\frac{\bar{s}}{\sqrt{11}}\right)\right).$$

The Frenet vectors and the curvatures of β are found as:

$$\begin{aligned}\bar{t}(\bar{s}) &= \frac{1}{\sqrt{22}}\left(2,3\left(-\cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}\right),3\left(-\cos\frac{\bar{s}}{\sqrt{11}}-\sin\frac{\bar{s}}{\sqrt{11}}\right)\right),\\ \bar{n}(\bar{s}) &= \frac{1}{\sqrt{2}}\left(0,\cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}},-\cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}\right),\\ \bar{b}(\bar{s}) &= \frac{1}{\sqrt{11}}\left(3,\cos\frac{\bar{s}}{\sqrt{11}}-\sin\frac{\bar{s}}{\sqrt{11}},\cos\frac{\bar{s}}{\sqrt{11}}+\sin\frac{\bar{s}}{\sqrt{11}}\right)\end{aligned}$$

and

$$\bar{k}=\frac{3}{11},\ \bar{r}=\frac{\sqrt{2}}{11}.$$

From the definition of Type-2 Quaternionic Frame, there exists following relations between Frenet vectors of $\beta^{(4)}$ *in* $\mathbb{R}^4$ *and* β *in* $\mathbb{R}^3$

$$\bar{N}_1(\bar{s}) = \bar{b}(\bar{s}) \times \bar{T}(\bar{s}),\ \bar{N}_2(\bar{s}) = \bar{n}(\bar{s}) \times \bar{T}(\bar{s}),\ \bar{N}_3(\bar{s}) = \bar{t}(\bar{s}) \times \bar{T}(\bar{s}).$$

Since $\beta^{(4)}$ *is a quaternionic Bertrand curve,* β *is a spatial quaternionic Bertrand curve and the curvatures of* β *satisfy* $\bar{\lambda}\bar{k} + \bar{\mu}\bar{r} = 1$, *for* $\bar{\lambda} = 2$ *and* $\bar{\mu} = \frac{5}{\sqrt{2}}$ *real numbers. From Theorem (10), we know that* β *is Bertrand mate of* α*. In fact* $\bar{n} = n$ *and* $\beta(s) = \alpha(s) - 2n(s)$.

Also, in this example, we can see that the equation (13) in Theorem (8) holds for $\lambda = -2$, $\mu = \frac{5}{\sqrt{2}}$, $K = \frac{1}{3}$, $r = \frac{\sqrt{2}}{3}$, $\bar{K} = \frac{3}{11}$, $\bar{r} = \frac{\sqrt{2}}{11}$, *that is* $\lambda\left(K + \bar{K}\right) + \mu(r - \bar{r}) = 0$.

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References

[1] Bertrand, J. M., Mémoire sur la théorie des courbes á double courbure, *Comptes Rendus*, 15 (1850), 332-350. https://doi.org/10.24033/bsmf.387

[2] Bharathi, K., Nagaraj, M., Quaternion valued function of a real Serret-Frenet formulae, *Indian J. Pure Appl. Math.*, 18(6) (1987), 507-511.

[3] Ekmekci, N., İlarslan, K., On Bertrand curves and their characterization, *Differ. Geom. Dyn. Syst.*, 3 (2001), 17-24.

[4] Gök, İ., Okuyucu, O. Z., Kahraman, F., Hacısalihoğlu H. H., On the quaternionic B_2- slant helices in the Euclidean space $\mathbb{E}^4$, *Adv. Appl. Clifford Algebr.*, 21 (2011), 707-719. https://doi.org/ 10.1007/s00006-011-0284-6

[5] Görgülü, A., Özdamar, E., A Generalization of the Bertrand curves as general inclined curves in $\mathbb{E}^n$, *Commun. Fac. Sci. Univ. Ankara, Ser.* A_1, 35 (1986), 53-60. https://doi.org/10.1501/Commua1-0000000254.

[6] Güngör, M. A., Tosun, M., Some characterizations of quaternionic rectifying curves, *Differ. Geom. Dyn. Syst.,* 13 (2011), 89-100.

[7] Irmak, Y., Bertrand Curves and Geometric Applications in Four Dimensional Euclidean Space, MSc thesis, Ankara University, Institute of Science, 2018.

[8] Kahraman Aksoyak, F., A new type of quaternionic Frame in $\mathbb{R}^4$, *Int. J. Geom. Methods Mod. Phys.,* 16(6) (2019), 1950084 (11 pages). https://doi.org/10.1142/S0219887819500841.

[9] Karadağ, M., Sivridağ, A. İ., Quaternion valued functions of a single real variable and inclined curves, *Erciyes Univ. J. Inst. Sci. Technol.,* 13 (1997), 23-36.

[10] Keçilioğlu, O., İlarslan, K., Quaternionic Bertrand curves in Euclidean 4-space, *Bull. Math. Anal. Appl.*, 5(3) (2013), 27-38.

[11] Önder, M., Quaternionic Salkowski curves and quaternionic similar curves, *Proc. Natl. Acad. Sci. India, Sect. A Phys. Sci.*, 90(3) (2020), 447-456. https://doi.org/10.1007/s40010-019-00601-y

[12] Öztürk, G., Kişi, İ., Büyükkütük, S., Constant ratio quaternionic curves in Euclidean spaces, *Adv. Appl. Clifford Algebr.,* 27(2) (2017), 1659-1673. https://doi.org/10.1007/s00006-016-0716-4

[13] Pears, L. R., Bertrand curves in Riemannian space, *J. London Math. Soc.* 1-10(2) (1935), 180-183. https://doi.org/10.1112/jlms/s1-10.2.180

[14] Şenyurt, S., Cevahir, C., Altun, Y., On spatial quaternionic involute curve a new view, *Adv. Appl. Clifford Algebr.,* 27(2) (2017), 1815-1824. https://doi.org/10.1007/s00006-016-0669-7
[15] Tanrıöver, N., Bertrand curves in $n-$dimensional Euclidean space, *Journal of Karadeniz University, Faculty of Arts and Sciences, Series of Mathematics-Physics,* 9 (1986), 61-62.
[16] Yıldız, Ö. G., İçer, Ö., A note on evolution of quaternionic curves in the Euclidean space $\mathbb{R}^4$, *Konuralp J. Math.*, 7(2) (2019), 462-469.
[17] Yoon, D. W., On the quaternionic general helices in Euclidean 4-space, *Honam Mathematical J.,* 34(3) (2012), 381-390. https://doi.org/10.5831/HMJ.2012.34.3.381
[18] Yoon, D. W., Tunçer Y., Karacan, M. K., Generalized Mannheim quaternionic curves in Euclidean 4-space, *Appl. Math. Sci. (Ruse),* 7 (2013), 6583-6592. https://doi.org/6583-6592. 10.12988/ams.2013.310560

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APPROXIMATION BY SZÁSZ-MIRAKJAN-DURRMEYER OPERATORS BASED ON SHAPE PARAMETER λ

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Abstract. In this work, we study several approximation properties of Szász-Mirakjan-Durrmeyer operators with shape parameter $\lambda \in [-1,1]$. Firstly, we obtain some preliminaries results such as moments and central moments. Next, we estimate the order of convergence in terms of the usual modulus of continuity, for the functions belong to Lipschitz type class and Peetre's K-functional, respectively. Also, we prove a Korovkin type approximation theorem on weighted spaces and derive a Voronovskaya type asymptotic theorem for these operators. Finally, we show the comparison of the convergence of these newly defined operators to certain functions with some graphics and an error of approximation table.

1. Introduction

One of the famous linear positive operators in the theory of approximation, Szász [29] and Mirakjan [18] introduced following operators

$$S_m(\mu;y)=\sum_{j=0}^{\infty} s_{m,j}(y)\mu\left(\frac{j}{m}\right), \tag{1}$$

where $m\in\mathbb{N}$, $y\geq 0$, $\mu\in C[0,\infty)$ and Szász-Mirakjan basis functions $s_{m,j}(y)$ are defined as below:

$$s_{m,j}(y)=e^{-my}\frac{(my)^j}{j!}. \tag{2}$$

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Keywords. Modulus of continuity, Lipschitz type class, Voronovskaya type asymptotic theorem, Szász-Mirakjan-Durrmeyer operators, weighted approximation.

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In 1985, Mazhar and Totik [17] proposed Durrmeyer type integral modifications of operators (1) as follows:

$$D_m(\mu;y)=m\sum_{j=0}^{\infty}s_{m,j}(y)\int_0^{\infty}s_{m,j}(t)\mu(t)\,dt,\ y\in[0,\infty), \tag{3}$$

where $s_{m,j}(y)$ given as in (2).

Recently, some various approximation properties of operators (3) have been introduced by several authors. We refer the readers some papers on this direction [1, 3, 11–15].

A short time ago, the Bézier basis with shape parameter $\lambda\in[-1,1]$ which is presented by Ye et al. [30], has attracted attention by some authors. Firstly, Cai et al. [7] introduced $\lambda-$Bernstein operators and obtained various approximation theorems, namely, Korovkin type convergence, local approximation and Voronovskaya-type asymptotic. Acu et al. [2] proposed the Kantorovich type $\lambda-$Bernstein operators and established some approximation features such as order of convergence, in connection with the Ditzian-Totik modulus of smoothness and Grüss-Voronovskaya type theorems. In 2019, Qi et al. [25] introduced a new generalization of Szász-Mirakjan operators based on shape parameter $\lambda\in[-1,1]$ as below:

$$S_{m,\lambda}(\mu;y)=\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\mu\left(\frac{j}{m}\right), \tag{4}$$

where Szász-Mirakjan basis functions $\widetilde{s}_{m,j}(\lambda;y)$ with shape parameter $\lambda\in[-1,1]$:

$$\begin{aligned}\widetilde{s}_{m,0}(\lambda;y)&=s_{m,0}(y)-\frac{\lambda}{m+1}s_{m+1,1}(y);\\ \widetilde{s}_{m,i}(\lambda;y)&=s_{m,i}(y)+\lambda\left(\frac{m-2i+1}{m^2-1}s_{m+1,i}(y)\right.\\ &\left.\quad-\frac{m-2i-1}{m^2-1}s_{m+1,i+1}(y)\right)\ (i=1,2,...,\infty,\ y\in[0,\infty)).\end{aligned} \tag{5}$$

For the operators defined by (4), they studied some theorems such as Korovkin type convergence, local approximation, Lipschitz type convergence, Voronovskaja and Grüss-Voronovskaja type. Also, we refer some recent works based on shape parameter $\lambda\in[-1,1]$, see details: [5, 6, 8, 19–24, 26–28].

Motivated by all above-mentioned papers, we construct the following $\lambda-$Szász-Mirakjan-Durrmeyer operators as:

$$D_{m,\lambda}(\mu;y)=m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty}s_{m,j}(t)\mu(t)\,dt,\quad y\in[0,\infty), \tag{6}$$

where $\widetilde{s}_{m,j}(\lambda;y)$ $(j=0,1,..\infty)$ given in (5) and $\lambda\in[-1,1]$.

This work is organized as follows: In Sect. 2, we compute some preliminaries results such as moments and central moments. Then, in Sect. 3, we obtain the order of convergence in respect of the usual modulus of continuity, for the functions belong to Lipschitz class and Peetre's K-functional, respectively. Next, In Sect. 4, we prove a Korovkin type convergence theorem on weighted spaces also in Sect. 5, we establish a Voronovskaya type asymptotic theorem. Finally, with the aid of Maple software, we present the comparison of the convergence of operators (6) to certain functions with some graphics and error of approximation table.

2. Preliminaries

Lemma 1. *[25]. For the $\lambda-$Szász-Mirakjan operators $S_{m,\lambda}(\mu; y)$, following results are satisfied:*

$$\begin{aligned}
S_{m,\lambda}(1; y) &= 1;\\
S_{m,\lambda}(t; y) &= y + \left[\frac{1 - e^{-(m+1)y} - 2y}{m(m-1)}\right]\lambda;\\
S_{m,\lambda}(t^2; y) &= y^2 + \frac{y}{m} + \left[\frac{2y + e^{-(m+1)y} - 1 - 4(m+1)y^2}{m^2(m-1)}\right]\lambda;\\
S_{m,\lambda}(t^3; y) &= y^3 + \frac{3y^2}{m} + \frac{y}{m^2}\\
&\quad + \left[\frac{1 - e^{-(m+1)y} - 2y + 3(m-3)(m+1)y^2 - 6(m+1)y^3}{m^3(m-1)}\right]\lambda;\\
S_{m,\lambda}(t^4; y) &= y^4 + \frac{6y^3}{m} + \frac{7y^2}{m^2} + \frac{y}{m^3}\\
&\quad + \left[\frac{e^{-(m+1)y} - 1 + 2my + 2(3m-11)(m+1)y^2}{m^4(m-1)}\right.\\
&\quad \left. + \frac{4(m-8)(m+1)^2y^3 - 8(m+1)^3y^4}{m^4(m-1)}\right]\lambda.
\end{aligned}$$

Lemma 2. *For the operators defined by (6), we obtain the following moments*

$$D_{m,\lambda}(1; y) = 1; \tag{7}$$

$$D_{m,\lambda}(t; y) = y + \frac{1}{m} + \left[\frac{1 - e^{-(m+1)y} - 2y}{m(m-1)}\right]\lambda; \tag{8}$$

$$D_{m,\lambda}(t^2; y) = y^2 + \frac{4y}{m} + \frac{2}{m^2} + \left[\frac{1 - e^{-(m+1)y} - 2y - 2(m+1)y^2}{m^2(m-1)}\right]2\lambda; \tag{9}$$

$$\begin{aligned}
D_{m,\lambda}(t^3; y) &= y^3 + \frac{9y^2}{m} + \frac{18y}{m^2} + \frac{6}{m^3}\\
&\quad + \left[\frac{2 - 2e^{-(m+1)y} - 4y + (m-11)(m+1)y^2 - 2(m+1)y^3}{m^3(m-1)}\right]3\lambda;
\end{aligned} \tag{10}$$

$$D_{m,\lambda}(t^4;y) = y^4 + \frac{16y^3}{m} + \frac{72y^2}{m^2} + \frac{96y}{m^3} + \frac{24}{m^4} + \left[\frac{24 - 24e^{-(m+1)y} + y(m-25) + 18(m-7)(m+1)y^2}{m^4(m-1)} - \frac{2(m^2-7m-23)(m+1)y^3 + 4(m+1)^3y^4}{m^4(m-1)}\right]2\lambda. \tag{11}$$

Proof. In view of the following relation

$$\int_0^\infty s_{m,j}(t)t^u dt = m^{-(u+1)}\frac{\Gamma(j+u+1)}{\Gamma(j+1)},$$

it is easy to get $\sum_{j=0}^{\infty} \widetilde{s}_{m,j}(\lambda;y) = 1$, hence we find (7).

Now, with the help of Lemma 1, we will compute the expressions (8) and (9).

$$\begin{aligned} D_{m,\lambda}(t;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^\infty e^{-mt}\frac{(mt)^j}{j!}tdt \\ &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{1}{m^2}\frac{\Gamma(j+2)}{\Gamma(j+1)} \\ &= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j+1}{m} \\ &= S_{m,\lambda}(t;y) + \frac{1}{m}S_{m,\lambda}(1;y) \\ &= y + \frac{1}{m} + \left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda. \\ D_{m,\lambda}(t^2;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^\infty e^{-mt}\frac{(mt)^j}{j!}tdt \\ &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{1}{m^3}\frac{\Gamma(j+3)}{\Gamma(j+1)} \\ &= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{(j+2)(j+1)}{m^2} \\ &= S_{m,\lambda}(t^2;y) + \frac{3}{m}S_{m,\lambda}(t;y) + \frac{2}{m^2}S_{m,\lambda}(1;y) \\ &= y^2 + \frac{4y}{m} + \frac{2}{m^2} + \left[\frac{1-e^{-(m+1)y}-2y-2(m+1)y^2}{m^2(m-1)}\right]2\lambda. \end{aligned}$$

Similarly, from Lemma 1, we can get expressions (10) and (11) by simple computation, thus we have omitted details. □

Corollary 1. *As a consequence of Lemma 2, we arrive the following relations:*

$$
\begin{aligned}
(i)\ D_{m,\lambda}(t-y;y) &= \frac{1}{m} + \left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda \\
&\leq \frac{m+e^{-(m+1)y}+2y}{m(m-1)} := \alpha_m(y); \\
(ii)\ D_{m,\lambda}((t-y)^2;y) &= \frac{2y}{m} + \frac{2}{m^2} \\
&+ \left[\frac{1+(m-1)ye^{-(m+1)y}+(m+2)y+(2m-2(m+1)^2)y^2}{m^2(m-1)}\right]2\lambda \\
&\leq \frac{2y}{m} + \frac{2}{m^2} \\
&+ \frac{2+2(m-1)ye^{-(m+1)y}+2(m+2)y+2(2m-2(m+1)^2)y^2}{m^2(m-1)} \\
&:= \beta_m(y); \\
(iii)\ D_{m,\lambda}((t-y)^4;y) &= \frac{12y^2}{m^2} + \frac{48y}{m^3} + \frac{24}{m^4} \\
&+ \left(\frac{24-24e^{-(m+1)y}+y(m-25)+18(m-7)(m+1)y^2}{m^4(m-1)}\right. \\
&- \frac{2(m^2-7m-23)(m+1)y^3+4(m+1)^3y^4}{m^4(m-1)} \\
&+ \frac{12ye^{-(m+1)y}-12y+24y^2-6(m-11)(m+1)y^3+12(m+1)y^4}{m^3(m-1)} \\
&+ \frac{6y^2(1-e^{-(m+1)y})-12y^3-12(m+1)^2y^4}{m^2(m-1)} \\
&+ \left.\frac{2y^3(1-e^{-(m+1)y})+4y^4}{m(m-1)}\right)2\lambda.
\end{aligned}
$$

3. Direct Theorems of $D_{m,\lambda}$ Operators

In this section, we discuss the order of convergence in connection with the usual modulus of continuity, for the function belong to Lipschitz type class and Peetre's K-functional, respectively. Let the space $C_B[0,\infty)$ denotes the all continuous and bounded functions on $[0,\infty)$ and it has the sup-norm for a function μ as below:

$$\|\mu\|_{[0,\infty)} = \sup_{y\in[0,\infty)} |\mu(y)|.$$

The Peetre's K-functional is defined as

$$K_2(\mu,\eta)=\inf_{\nu\in C^2[0,\infty)}\left\{\|\mu-\nu\|+\eta\,\|\nu''\|\right\},$$

where $\eta>0$ and $C_B^2[0,\infty)=\{\nu\in C_B[0,\infty):\nu',\nu''\in C_B[0,\infty)\}$.

From [9], there exists an absolute constant $C>0$ such that

$$K_2(\mu;\eta)\le C\omega_2(\mu;\sqrt{\eta}),\qquad \eta>0, \tag{12}$$

where

$$\omega_2(\mu;\eta)=\sup_{0<z\le\eta}\ \sup_{y\in[0,\infty)}|\mu(y+2z)-2\mu(y+z)+\mu(y)|,$$

is the second order modulus of smoothness of the function $\mu\in C_B[0,\infty)$. Also, we define the usual modulus of continuity of $\mu\in C_B[0,\infty)$ as follows

$$\omega(\mu;\eta):=\sup_{0<\alpha\le\eta}\ \sup_{y\in[0,\infty)}|\mu(y+z)-\mu(y)|.$$

Since $\eta>0$, $\omega(\mu;\eta)$ has some useful properties see details in [4].

Further, we give an elements of Lipschitz type continuous function with $Lip_L(\zeta)$, where $L>0$ and $0<\zeta\le 1$. If the following expression

$$|\mu(t)-\mu(y)|\le L\,|t-y|^{\zeta},\qquad (t,y\in\mathbb{R}),$$

holds, then one can say a function μ belongs to $Lip_L(\zeta)$.

Theorem 1. *Let $\mu\in C_B[0,\infty)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, the following inequality is satisfied:*

$$|D_{m,\lambda}(\mu;y)-\mu(y)|\le 2\omega(\mu;\sqrt{\beta_m(y)}),$$

where $\beta_m(y)$ given as in Corollary 1.

Proof. Using the well-known property of modulus of continuity $|\mu(t)-\mu(y)|\le\left(1+\frac{|t-y|}{\delta}\right)\omega(\mu;\delta)$ and operating $D_{m,\lambda}(.;y)$, we arrive

$$|D_{m,\lambda}(\mu;y)-\mu(y)|\le\left(1+\frac{1}{\delta}D_{m,\lambda}(|t-y|\,;y)\right)\omega(\mu;\delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality and by Corollary 1, we get

$$\begin{aligned}|D_{m,\lambda}(\mu;y)-\mu(y)|&\le\left(1+\frac{1}{\delta}\sqrt{D_{m,\lambda}((t-y)^2;y)}\right)\omega(\mu;\delta)\\&\le\left(1+\frac{1}{\delta}\sqrt{\beta_m(y)}\right)\omega(\mu;\delta).\end{aligned}$$

Choosing $\delta=\sqrt{\beta_m(y)}$, thus we have the proof of this theorem. □

Theorem 2. *Let $\mu\in Lip_L(\zeta)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, we obtain*

$$|D_{m,\lambda}(\mu;y)-\mu(y)|\le L(\beta_m(y))^{\frac{\zeta}{2}}.$$

Proof. Taking into consideration the linearity and monotonicity properties of the operators (6), it gives

$$\begin{aligned}&|D_{m,\lambda}(\mu;y)-\mu(y)| \leq D_{m,\lambda}(|\mu(t)-\mu(y)|\,;y)\\ &\leq m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty} e^{-mt}\frac{(mt)^j}{j!}\,|\mu\,(t)-\mu(y)|\,dt\\ &\leq L\left(m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty} e^{-mt}\frac{(mt)^j}{j!}\,|t-y|^{\zeta}\,dt\right).\end{aligned}$$

Utilizing the Hölder's inequality with $p_1=\frac{2}{\zeta}$ and $p_2=\frac{2}{2-\zeta}$, from Corollary 1 and Lemma 2, we arrive

$$\begin{aligned}|D_{m,\lambda}(\mu;y)-\mu(y)| &\leq L\left\{m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_0^{\infty} e^{-mt}\frac{(mt)^j}{j!}\,(t-y)^2\,dt\right\}^{\frac{\zeta}{2}}\\ &\cdot\left\{\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\right\}^{\frac{2-\zeta}{2}}\\ &= L\left\{D_{m,\lambda}((t-y)^2;y)\right\}^{\frac{\zeta}{2}}\left\{D_{m,\lambda}(1;y)\right\}^{\frac{2-\zeta}{2}}\\ &\leq L(\beta_m(y))^{\frac{\zeta}{2}}.\end{aligned}$$

Hence, we obtain the required sequel. □

Theorem 3. *For all $\mu\in C_B[0,\infty)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$, the following inequality holds:*

$$|D_{m,\lambda}(\mu;y)-\mu(y)|\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y)+(\alpha_m(y))^2})+\omega(\mu;|\alpha_m(y)|),$$

where $C>0$ is a constant, $\alpha_m(y)$, $\beta_m(y)$ defined as in Corollary 1.

Proof. We denote $\gamma_{m,\lambda}(y):=y+\frac{1}{m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda$, it is obvious that $\gamma_{m,\lambda}(y)\in[0,\infty)$ for sufficently large m. We define the following auxiliary operators:

$$\widehat{D}_{m,\lambda}(\mu;y)=D_{m,\lambda}(\mu;y)-\mu(\gamma_{m,\lambda}(y))+\mu(y). \tag{13}$$

From (7) and (8), we find

$$\widehat{D}_{m,\lambda}(t-y;y)=0.$$

Using Taylor's formula, one has

$$\xi(t)=\xi(y)+(t-y)\xi'(y)+\int_y^t(t-u)\xi''(u)du,\quad (\xi\in C_B^2[0,\infty)). \tag{14}$$

Operating $\widehat{D}_{m,\lambda}(.;y)$ to (14), it gives

$$\widehat{D}_{m,\lambda}(\xi;y)-\xi(y)=\widehat{D}_{m,\lambda}((t-y)\xi'(y);y)+\widehat{D}_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)$$
$$=\xi'(y)\widehat{D}_{m,\lambda}(t-y;y)+D_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)$$
$$-\int_y^{\gamma_{m,\lambda}(y)}(\gamma_{m,\lambda}(y)-u)\xi''(u)du$$
$$=D_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)-\int_y^{\gamma_{m,\lambda}(y)}(\gamma_{m,\lambda}(y)-u)\xi''(u)du.$$

Taking Lemma 2 and (13) into the account, we get

$$\left|\widehat{D}_{m,\lambda}(\xi;y)-\xi(y)\right|\leq\left|D_{m,\lambda}(\int_y^t (t-u)\xi''(u)du;y)\right|$$
$$+\left|\int_y^{\gamma_{m,\lambda}(y)}(\gamma_{m,\lambda}(y)-u)\xi''(u)du\right|$$
$$\leq D_{m,\lambda}(\int_y^t (t-u)\left|\xi''(u)\right|du;y)$$
$$+\int_y^{\gamma_{m,\lambda}(y)}\left(\gamma_{m,\lambda}(y)-u\right)\left|\xi''(u)\right|du$$
$$\leq\left\|\xi''\right\|\left\{D_{m,\lambda}((t-y)^2;y)+\left(\gamma_{m,\lambda}(y)-y\right)^2\right\}$$
$$\leq\left\{\beta_m(y)+(\alpha_m(y))^2\right\}\left\|\xi''\right\|.$$

From (7), (8) and (13), it deduces the following

$$\left|\widehat{D}_{m,\lambda}(\mu;y)\right|\leq|D_{m,\lambda}(\mu;y)|+2\|\mu\|\leq\|\mu\|D_{m,\lambda}(1;y)+2\|\mu\|\leq 3\|\mu\|.$$

Also by (14) and using above relation, we get

$$|D_{m,\lambda}(\mu;y)-\mu(y)|\leq\left|\widehat{D}_{m,\lambda}(\mu-\xi;y)-(\mu-\xi)(y)\right|$$

$$+\left|\widehat{D}_{m,\lambda}(\xi;y)-\xi(y)\right|+|\mu(y)-\mu(\alpha_{m,\lambda}(y))|$$
$$\leq 4\|\mu-\xi\|+\left\{\beta_m(y)+(\gamma_{m,\lambda}(y))^2\right\}\left\|\xi''\right\|+\omega\left(\mu;|\alpha_m(y)|\right).$$

Hence, if we take the infimum on the right hand side over all $\xi \in C_B^2[0,\infty)$ and by (12), we arrive

$$|D_{m,\lambda}(\mu;y)-\mu(y)| \leq 4K_2(\mu;\frac{\left\{\beta_m(y)+(\alpha_m(y))^2\right\}}{4})+\omega(\mu;|\alpha_m(y)|)$$
$$\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\beta_m(y)+(\alpha_m(y))^2})+\omega(\mu;|\alpha_m(y)|).$$

Thus, the proof is completed. □

4. Weighted Approximation

In this section, we will establish the Korovkin type convergence theorem on weighted spaces. Let $B_{y^2}[0,\infty)$ be the space of all functions κ verifying the condition $|\kappa(y)| \leq M_\kappa(1+y^2)$, $y \in [0,\infty)$ with constant M_κ, which depend only on κ. We denote with $C_{y^2}[0,\infty)$ the set of all continuous functions belonging to $B_{y^2}[0,\infty)$ and it is endowed with the norm $\|\kappa\|_{y^2} = \sup\limits_{y\in[0,\infty)} \frac{|\kappa(y)|}{1+y^2}$ and also we define $C^*_{y^2}[0,\infty) := \{\kappa : \ \kappa \in C_{y^2}[0,\infty),\ \lim\limits_{y\to\infty} \frac{|\kappa(y)|}{1+y^2} < \infty\}$.

Theorem 4. *For all $\mu \in C^*_{y^2}[0,\infty)$, we arrive*

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(\mu;y)-\mu(y)|}{1+y^2}=0.$$

Proof. Considering to the Korovkin type convergence theorem presented by Gadzhiev [10], we want to show that operators (3) verifies the following condition:

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(t^s;y)-y^s|}{1+y^2}=0, \qquad s=0,1,2. \tag{15}$$

By (7), the first condition in (15) is clear for $s=0$.

For $s=1$, using (8), we have

$$\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(t;y)-y|}{1+y^2} \leq \left|\frac{m-1+\lambda}{m(m-1)}\right|\sup_{y\in[0,\infty)}\frac{1}{1+y^2}+\left|\frac{3\lambda}{m(m-1)}\right|\sup_{y\in[0,\infty)}\frac{y}{1+y^2}.$$

Hence,

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{|D_{m,\lambda}(t;y)-y|}{1+y^2}=0.$$

Similarly for $s=2$, using (9), we get

$$\sup_{y\in[0,\infty)}\frac{\left|D_{m,\lambda}(t^2;y)-y^2\right|}{1+y^2}\leq\left|\frac{2((m-1)+\lambda)}{m^2(m-1)}\right|\sup_{y\in[0,\infty)}\frac{1}{1+y^2}$$
$$+\left|\frac{4m(m-1)-6\lambda}{m^2(m-1)}\right|\sup_{y\in[0,\infty)}\frac{y}{1+y^2}+\left|\frac{4(m+1)\lambda}{m^2(m-1)}\right|\sup_{y\in[0,\infty)}\frac{y^2}{1+y^2}.$$

It follows

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\frac{\left|D_{m,\lambda}(t^2;y)-y^2\right|}{1+y^2}=0.$$

This gives the proof of this theorem. □

5. Voronovskaya Type Asymptotic Theorem

In this section, we will prove Voronovskaya type asymptotic theorem. Firstly we consider the following lemma, which we will use in the proof of our main theorem.

Lemma 3. *Let $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, the following results are satisfied:*

$$(i)\lim_{m\to\infty}mD_{m,\lambda}(t-y;y)=1,$$
$$(ii)\lim_{m\to\infty}mD_{m,\lambda}((t-y)^2;y)=2y(1-2y),$$
$$(iii)\lim_{m\to\infty}m^2D_{m,\lambda}((t-y)^4;y)=4y^2(1-y)(2y+3).$$

Theorem 5. *Let $\mu\in C_{y^2}[0,\infty)$ such that $\mu',\mu''\in C_{y^2}[0,\infty)$ and $\lambda\in[-1,1]$, then we have for any $y\in[0,\infty)$ that*

$$\lim_{m\to\infty}m\left[D_{m,\lambda}(\mu;y)-\mu(y)\right]=\mu'(y)+y(1-2y)\mu''(y).$$

Proof. Suppose that $\mu,\mu',\mu''\in C_{y^2}[0,\infty)$ and $y\in[0,\infty)$. Using Taylor's expansion formula, we find

$$\mu(t)=\mu(y)+(t-y)\mu'(y)+\frac{1}{2}(t-y)^2\mu''(y)+(t-y)^2\Delta(t;y). \tag{16}$$

In (16), $\Delta(t;y)$ is a Peano of the remainder term and by the fact that $\Delta(.;y)\in C^*_{y^2}[0,\infty)$, we arrive $\lim_{t\to y}\Delta(t;y)=0$.

After operating $D_{m,\lambda}(.;y)$ to (16), then

$$D_{m,\lambda}(\mu;y)-\mu(y)=D_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}D_{m,\lambda}((t-y)^2;y)\mu''(y)$$
$$+D_{m,\lambda}((t-y)^2\Delta(t;y);y).$$

If we take the limit of the both sides of above expression as $m\to\infty$, hence

$$\lim_{m\to\infty} m(D_{m,\lambda}(\mu;y)-\mu(y))$$
$$= \lim_{m\to\infty} m\left(D_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}D_{m,\lambda}((t-y)^2;y)\mu''(y)+D_{m,\lambda}((t-y)^2\Delta(t;y);y)\right). \tag{17}$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality to the last term on the right hand side of the above relation, it gives

$$\lim_{m\to\infty} mD_{m,\lambda}((t-y)^2\Delta(t;y);y) \le \sqrt{\lim_{m\to\infty} D_{m,\lambda}(\Delta^2(t;y);y)}\sqrt{\lim_{m\to\infty} m^2 D_{m,\lambda}((t-y)^4;y)}.$$

Since $\Delta(t;y)\in C_{y^2}[0,\infty)$, then from Theorem 4, $\lim_{t\to y}\Delta(t;y)=0$. It becomes

$$\lim_{m\to\infty} D_{m,\lambda}(\Delta^2(t;y);y)=\Delta^2(y;y)=0. \tag{18}$$

Combining (17)-(18) and by Lemma 3 (iii), one has

$$\lim_{m\to\infty} mD_{m,\lambda}((t-y)^2\Delta(t;y);y)=0.$$

Hence, we obtain the following desired sequel

$$\lim_{m\to\infty} m\left[D_{m,\lambda}(\mu;y)-\mu(y)\right]=\mu'(y)+y(1-2y)\mu''(y).$$

□

6. Graphical and Numerical Analysis

In this section, with the aid of Maple software, we present some graphics and an error of approximation table to see the convergence of operators (6) to certain functions with the different values of m and λ parameters.

In Figure 1, we show the convergence of operators (6) to the function $\mu(y) = ysin(y)/2$ (black) for $\lambda = 1$, $m = 10$ (red), $m = 30$ (green) and $m = 75$ (blue). In Figure 2, we show the convergence of operators (6) to the function $\mu(y) = ysin(y)/2$ (black) for $\lambda = -1$, $m = 10$ (red), $m = 30$ (green) and $m = 75$ (blue). It is obvious from Figure 1 and Figure 2 that, as the values of m increases than the convergence of operators (6) to the functions $\mu(y)$ becomes better. In Figure 3, we compare the convergence of operators (3) (green) and operators (6) (red) with the function $\mu(y) = 1-\sin(\pi y)$ (black) for $\lambda = 1$ and $m = 10$. It is clear from Figure 3 that, operators (6) has better approximation than operators (3). Also, in Table 1, we present an error of approximation of operators (6) to function $\mu(y) = ysin(y)/2$ for the certain values of m and $\lambda \in [-1,1]$. We can check from Table 1 that, as the value of m increases than the error of approximation of operators (6) to $\mu(y)$ is decreases. One the other hand, for $\lambda > 0$, the absolute difference between operators (6) and $\mu(y)$ is smaller than between operators (3) and $\mu(y)$.

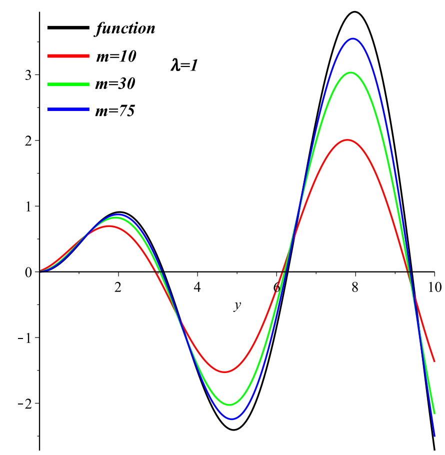

FIGURE 1. The convergence of operators $D_{m,\lambda}(\mu; y)$ to the function $\mu(y) = ysin(y)/2$ for $\lambda = 1$ and $m = 10, 30, 75$

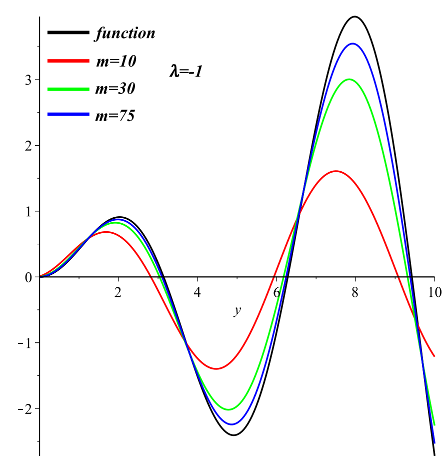

FIGURE 2. The convergence of operators $D_{m,\lambda}(\mu; y)$ to the function $\mu(y) = ysin(y)/2$ for $\lambda = -1$ and $m = 10, 30, 75$

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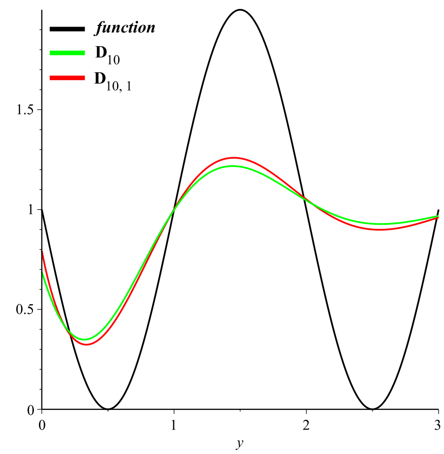

FIGURE 3. The convergence of operators $D_{m,\lambda}(\mu;y)$ and $D_m(\mu;y)$ to the function $\mu(y) = 1 - \sin(\pi y)$ for $\lambda = 1$ and $m = 10$

TABLE 1. Error of approximation $D_{m,\lambda}(\mu;y)$ operators to $\mu(y) = ysin(y)/2$ for $m = 10, 30, 75, 150$

λ	$\lvert\mu(y) - D_{m,\lambda}(\mu;y)\rvert$			
	$m = 10$	$m = 30$	$m = 75$	$m = 150$
-1	0.0779267654	0.0274289801	0.0110996263	0.0055687488
-0.75	0.0778118774	0.0274227761	0.0110991971	0.0055686940
0	0.0774672138	0.0274041639	0.0110979093	0.0055685292
0.75	0.0771225502	0.0273855517	0.0110966215	0.0055683644
1	0.0770076622	0.0273793477	0.0110961923	0.0055683096

REFERENCES

[1] Acar, T., Ulusoy, G., Approximation by modified Szász-Durrmeyer operators, *Period Math. Hung.,* 72(1) (2016), 64–75. https://doi.org/10.1007/s10998-015-0091-2

[2] Acu, A. M., Manav, N., Sofonea, D. F., Approximation properties of λ-Kantorovich operators, *J. Inequal. Appl.*, 2018(1) (2018), 202. https://doi.org/10.1186/s13660-018-1795-7

[3] Alotaibi, A., Özger, F., Mohiuddine, S. A., Alghamdi, M. A., Approximation of functions by a class of Durrmeyer–Stancu type operators which includes Euler's beta function, *Adv. Differ. Equ.,* 2021(1) (2021). https://doi.org/10.1186/s13662-020-03164-0

[4] Altomare, F., Campiti, M., Korovkin-type approximation theory and its applications, Walter de Gruyter, 1994. https://doi.org/10.1515/9783110884586

[5] Aslan, R., Some approximation results on λ-Szasz-Mirakjan-Kantorovich operators, *FUJMA*, 4(3) (2021), 150–158. https://doi.org/10.33401/fujma.903140

[6] Cai, Q. -B., Aslan, R., On a new construction of generalized q-Bernstein polynomials based on shape parameter λ, *Symmetry*, 13(10) (2021), 1919. https://doi.org/10.3390/sym13101919

[7] Cai, Q. -B., Lian, B. Y., Zhou, G., Approximation properties of λ-Bernstein operators, *J. Inequal. Appl.*, 2018(1) (2018), 61. https://doi.org/10.1186/s13660-018-1653-7

[8] Cai, Q. -B., Zhou, G., Li, J., Statistical approximation properties of λ-Bernstein operators based on $q-$integers, *Open Math.*, 17(1) (2019), 487–498. https://doi.org/10.1515/math-2019-0039

[9] Devore, R. A., Lorentz, G. G., Constructive Approximation, Springer, Berlin Heidelberg, 1993. https://doi.org/10.1007/978-3-662-02888-9

[10] Gadzhiev, A. D., The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P.P. Korovkin, *Dokl. Akad. Nauk.*, 218(5) (1974), 1001–1004.

[11] Gupta, M. K., Beniwal, M. S., Goel, P., Rate of convergence for Szász–Mirakyan–Durrmeyer operators with derivatives of bounded variation, *Appl. Math. comput.*, 199(2) (2008), 828–832. https://doi.org/10.1016/j.amc.2007.10.036

[12] Gupta, V., Simultaneous approximation by Szász-Durrmeyer operators, *Math. Stud.*, 64(1-4) (1995), 27—36.

[13] Gupta, V., Noor, M. A., Beniwal, M. S., Rate of convergence in simultaneous approximation for Szász–Mirakyan–Durrmeyer operators, *J. Math. Anal. Appl.*, 322(2) (2006), 964–970. https://doi.org/10.1016/j.jmaa.2005.09.063

[14] Gupta, V., Pant, R. P., Rate of convergence for the modified Szász–Mirakyan operators on functions of bounded variation, *J. Math. Anal. Appl.*, 233(2) (1999), 476–483. https://doi.org/10.1006/jmaa.1999.6289

[15] Içöz, G., Mohapatra, R. N., Weighted approximation properties of stancu type modification of q-szász-durrmeyer operators, *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.*, 65(1) (2016), 87–104. https://doi.org/10.1501/commua10000000746

[16] Korovkin, P. P., On convergence of linear positive operators in the space of continuous functions, *Dokl. Akad. Nauk. SSSR.*, 90(953) (1953), 961–964.

[17] Mazhar, S., Totik, V., Approximation by modified Szász operators, *Acta Sci. Math.*, 49(1-4) (1985), 257–269.

[18] Mirakjan, G. M., Approximation of continuous functions with the aid of polynomials, *In Dokl. Acad. Nauk. SSSR.*, 31 (1941), 201–205.

[19] Mursaleen, M., Al-Abied, A. A. H., Salman, M. A., Approximation by Stancu-Chlodowsky type λ-Bernstein operators, *J. Appl. Anal.*, 26(1) (2020), 97–110. https://doi.org/10.1515/jaa-2020-2009

[20] Mursaleen, M., Al-Abied, A. A. H., Salman, M. A., Chlodowsky type (λ, q)-Bernstein-Stancu operators, *Azerb. J. Math.*, 10(1) (2020), 75–101.

[21] Özger, F., Applications of generalized weighted statistical convergence to approximation theorems for functions of one and two variables, *Numer. Funct. Anal. Optim.*, 41(16) (2020), 1990–2006. https://doi.org/10.1080/01630563.2020.1868503

[22] Özger, F., Weighted statistical approximation properties of univariate and bivariate λ Kantorovich operators, *Filomat*, 33(11) (2019), 3473–3486. https://doi.org/10.2298/fil1911473o

[23] Özger, F., On new Bézier bases with Schurer polynomials and corresponding results in approximation theory, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 69(1) (2020), 376–393. https://doi.org/10.31801/cfsuasmas.510382

[24] Özger, F., Demirci, K., Yıldız, S., Approximation by Kantorovich variant of λ-Schurer operators and related numerical results, In: Topics in Contemporary Mathematical Analysis and Applications, pp. 77-94, CRC Press, Boca Raton, 2020. https://doi.org/10.1201/9781003081197-3

[25] Qi, Q., Guo, D., Yang, G., Approximation properties of λ-Szász-Mirakian operators, *Int. J. Eng. Res.*, 12(5) (2019), 662–669.

[26] Rahman, S., Mursaleen, M., Acu, A. M., Approximation properties of λ-Bernstein-Kantorovich operators with shifted knots, *Math. Meth. Appl. Sci.*, 42(11) (2019), 4042–4053. https://doi.org/10.1002/mma.5632

[27] Srivastava, H. M., Ansari, K. J., Özger, F., Ödemiş Özger, Z., A link between approximation theory and summability methods via four-dimensional infinite matrices. *Mathematics*, 9(16) (2021), 1895. https://doi.org/10.3390/math9161895

[28] Srivastava, H. M., Özger, F., Mohiuddine, S. A., Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ, *Symmetry*, 11(3) (2019), 316. https://doi.org/10.3390/sym11030316

[29] Szász, O., Generalization of the Bernstein polynomials to the infinite interval, *J. Res. Nat. Bur. Stand.*, 45(3) (1950), 239–245. https://doi.org/10.6028/jres.045.024

[30] Ye, Z., Long, X., Zeng, X. M., Adjustment algorithms for Bézier curve and surface, *In: International Conference on 5th Computer Science and Education*, (2010), 1712–1716. https://doi.org/10.1109/iccse.2010.5593563

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ON CERTAIN BIHYPERNOMIALS RELATED TO PELL AND PELL-LUCAS NUMBERS

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Abstract. The bihyperbolic numbers are extension of hyperbolic numbers to four dimensions. In this paper we introduce the concept of Pell and Pell-Lucas bihypernomials as a generalization of bihyperbolic Pell and Pell-Lucas numbers, respectively.

1. Introduction

Let consider Pell and Pell-Lucas numbers which belong to the family of the Fibonacci type numbers, for details see [14]. We recall that the nth Pell number P_n is defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$. The nth Pell-Lucas number Q_n is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$ with $Q_0 = Q_1 = 2$.

For the nth Pell number and the nth Pell-Lucas number the explicit formulas have the form

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

The above equations are named as Binet type formulas for Pell and Pell-Lucas numbers, respectively. For other properties of P_n and Q_n see [5,6,9]. In [7] Horadam and Mahon introduced Pell and Pell-Lucas polynomials and next their properties were studied among others in [4].

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Let x be any variable quantity. Polynomials $P_n(x)$ and $Q_n(x)$ defined as follows

$$P_n(x) = 2x \cdot P_{n-1}(x) + P_{n-2}(x) \text{ for } n \geq 2 \text{ with } P_0(x) = 0, P_1(x) = 1$$

$$Q_n(x) = 2x \cdot Q_{n-1}(x) + Q_{n-2}(x) \text{ for } n \geq 2 \text{ with } Q_0(x) = 2, Q_1(x) = 2x$$

generalize Pell and Pell-Lucas numbers and they are called as Pell polynomials and Pell-Lucas polynomials, respectively. Clearly $P_n(1) = P_n$ and $Q_n(1) = Q_n$.

Let

$$\alpha(x) = x + \sqrt{x^2+1}, \quad \beta(x) = x - \sqrt{x^2+1} \tag{1}$$

be roots of the characteristic equation for the Pell and Pell-Lucas polynomials. Then solving this equation we have

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{2}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x), \tag{3}$$

respectively.

We recall selected identities for Pell and Pell-Lucas polynomials, which will be used in the next part of this paper.

Theorem 1. [7] *Let n be an integer. Then*

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x) = 2x \cdot P_n(x) + 2P_{n-1}(x), \text{ for } n \geq 1, \tag{4}$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2+1)P_n(x), \text{ for } n \geq 1, \tag{5}$$

$$\sum_{l=1}^{n-1} P_l(x) = \frac{P_n(x) + P_{n-1}(x) - 1}{2x}, \text{ for } n \geq 2, \tag{6}$$

$$\sum_{l=1}^{n-1} Q_l(x) = \frac{Q_n(x) + Q_{n-1}(x) - 2 - 2x}{2x}, \text{ for } n \geq 2. \tag{7}$$

For Pell numbers and Pell polynomials we can find different generalizations given by the kth order linear recurrence relations, $k \geq 2$. One of the fundamental generalization of Pell polynomials is Horadam polynomials which describe a wide family of polynomials defined by linear recurrence relations of order two. Some properties of the Horadam polynomials can be found in [8]. Horadam polynomials play an important role in the theory of hypercomplex numbers, for details see [12–14]. In this paper we will use Pell and Pell-Lucas polynomials in the theory of bihyperbolic numbers.

Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and j_1, j_2, $j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1, \; j_1j_2 = j_2j_1 = j_3, \; j_1j_3 = j_3j_1 = j_2, \; j_2j_3 = j_3j_2 = j_1. \tag{8}$$

From the definition of bihyperbolic numbers follows that their multiplication can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients.

Since the addition and multiplication on $\mathbb{H}_2$ are commutative and associative, so $(\mathbb{H}_2, +, \cdot)$ is a commutative ring.

Note that bihyperbolic numbers are a generalization of hyperbolic numbers. For the definition of hyperbolic numbers and their properties see [10, 11]. For the algebraic properties of bihyperbolic numbers see [1].

A special kind of bihyperbolic numbers, namely bihyperbolic Pell numbers, were introduced in [2] in the following way.

The nth bihyperbolic Pell number BhP_n is defined as

$$BhP_n = P_n + j_1 P_{n+1} + j_2 P_{n+2} + j_3 P_{n+3}. \tag{9}$$

By analogy

$$BhQ_n = Q_n + j_1 Q_{n+1} + j_2 Q_{n+2} + j_3 Q_{n+3} \tag{10}$$

is the nth bihyperbolic Pell-Lucas number. Note that some combinatorial properties of bihyperbolic Pell numbers we can find in [3].

Based on definitions of BhP_n and BhQ_n we introduce Pell and Pell-Lucas bihypernomials.

For $n \geq 0$ Pell and Pell-Lucas bihypernomials are defined by

$$BhP_n(x) = P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x) \tag{11}$$

and

$$BhQ_n(x) = Q_n(x) + j_1 Q_{n+1}(x) + j_2 Q_{n+2}(x) + j_3 Q_{n+3}(x), \tag{12}$$

respectively. Note that $BhP_n(1) = BhP_n$ and $BhQ_n(1) = BhQ_n$.

2. Main results

In this section we will give some identities for Pell bihypernomials and Pell-Lucas bihypernomials.

Theorem 2. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$BhP_n(x) = 2x \cdot BhP_{n-1}(x) + BhP_{n-2}(x) \text{ for } n \geq 2 \tag{13}$$

with $BhP_0(x) = j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2 + 1)$
and $BhP_1(x) = 1 + j_1 \cdot 2x + j_2 \cdot (4x^2 + 1) + j_3 \cdot (8x^3 + 4x)$.

Proof. If $n = 2$ we have

$$\begin{aligned} BhP_2(x) &= P_2(x) + j_1 P_3(x) + j_2 P_4(x) + j_3 P_5(x) \\ &= 2x + j_1 \cdot (4x^2+1) + j_2 \cdot (8x^3+4x) + j_3 \cdot (16x^4+12x^2+1) \\ &= 2x \cdot (1 + j_1 \cdot 2x + j_2 \cdot (4x^2+1) + j_3 \cdot (8x^3+4x)) \\ &\quad + j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2+1) \\ &= 2x \cdot BhP_1(x) + BhP_0(x). \end{aligned}$$

Let $n \geq 3$. By the definition of $P_n(x)$ we obtain

$$\begin{aligned} BhP_n(x) &= P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x) \\ &= (2x \cdot P_{n-1}(x) + P_{n-2}(x)) + j_1(2x \cdot P_n(x) + P_{n-1}(x)) \\ &\quad + j_2(2x \cdot P_{n+1}(x) + P_n(x)) + j_3(2x \cdot P_{n+2}(x) + P_{n+1}(x)) \\ &= 2x\,(P_{n-1}(x) + j_1 P_n(x) + j_2 P_{n+1}(x) + j_3 P_{n+2}(x)) \\ &\quad + P_{n-2}(x) + j_1 P_{n-1}(x) + j_2 P_n(x) + j_3 P_{n+1}(x) \\ &= 2x \cdot BhP_{n-1}(x) + BhP_{n-2}(x), \end{aligned}$$

which ends the proof. □

Using the same method we can prove the next result.

Theorem 3. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$BhQ_n(x) = 2x \cdot BhQ_{n-1}(x) + BhQ_{n-2}(x) \text{ for } n \geq 2$$

with $BhQ_0(x) = 2 + j_1 \cdot 2x + j_2 \cdot (4x^2+2) + j_3 \cdot (8x^3+6x)$
and $BhQ_1(x) = 2x + j_1 \cdot (4x^2+2) + j_2 \cdot (8x^3+6x) + j_3 \cdot (16x^4+16x^2+2)$.

Note that some identities for $BhP_n(x)$ and $BhQ_n(x)$ can be found based on identities for Pell and Pell-Lucas polynomials mentioned in the introduction of this paper.

Theorem 4. *Let $n \geq 1$ be an integer. Then*

$$BhP_{n+1}(x) + BhP_{n-1}(x) = BhQ_n(x) = 2x \cdot BhP_n(x) + 2BhP_{n-1}(x).$$

Proof. Using (4) we have

$$\begin{aligned} &BhP_{n+1}(x) + BhP_{n-1}(x) \\ &= P_{n+1}(x) + j_1 P_{n+2}(x) + j_2 P_{n+3}(x) + j_3 P_{n+4}(x) \\ &\quad + P_{n-1}(x) + j_1 P_n(x) + j_2 P_{n+1}(x) + j_3 P_{n+2}(x) \\ &= (P_{n+1}(x) + P_{n-1}(x)) + j_1(P_{n+2}(x) + P_n(x)) \\ &\quad + j_2(P_{n+3}(x) + P_{n+1}(x)) + j_3(P_{n+4}(x) + P_{n+2}(x)) \\ &= Q_n(x) + j_1 Q_{n+1}(x) + j_2 Q_{n+2}(x) + j_3 Q_{n+3}(x) \\ &= BhQ_n(x). \end{aligned}$$

On the other hand

$$\begin{aligned}
&2x \cdot BhP_n(x) + 2BhP_{n-1}(x)\\
&= 2x \cdot (P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x))\\
&\ + 2\,(P_{n-1}(x) + j_1 P_n(x) + j_2 P_{n+1}(x) + j_3 P_{n+2}(x))\\
&= (2x \cdot P_n(x) + 2P_{n-1}(x)) + j_1(2x \cdot P_{n+1}(x) + 2P_n(x))\\
&\ + j_2(2x \cdot P_{n+2}(x) + 2P_{n+1}(x)) + j_3(2x \cdot P_{n+3}(x) + 2P_{n+2}(x))\\
&= Q_n(x) + j_1 Q_{n+1}(x) + j_2 Q_{n+2}(x) + j_3 Q_{n+3}(x)\\
&= BhQ_n(x).
\end{aligned}$$

□

Theorem 5. *Let $n \geq 1$ be an integer. Then*

$$BhQ_{n+1}(x) + BhQ_{n-1}(x) = 4(x^2+1)BhP_n(x).$$

Theorem 6. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} BhP_l(x) = \frac{BhP_n(x) + BhP_{n-1}(x) - BhP_0(x) - BhP_1(x)}{2x}.$$

Proof. For an integer $n \geq 2$ we have

$$\begin{aligned}
&\sum_{l=1}^{n-1} BhP_l(x) = BhP_1(x) + BhP_2(x) + \cdots + BhP_{n-1}(x)\\
&= P_1(x) + j_1 P_2(x) + j_2 P_3(x) + j_3 P_4(x)\\
&\ + P_2(x) + j_1 P_3(x) + j_2 P_4(x) + j_3 P_5(x) + \cdots\\
&\ + P_{n-1}(x) + j_1 P_n(x) + j_2 P_{n+1}(x) + j_3 P_{n+2}(x)\\
&= P_1(x) + P_2(x) + \cdots + P_{n-1}(x)\\
&\ + j_1(P_2(x) + P_3(x) + \cdots + P_n(x) + P_1(x) - P_1(x))\\
&\ + j_2(P_3(x) + P_4(x) + \cdots + P_{n+1}(x) + P_1(x) + P_2(x) - P_1(x) - P_2(x))\\
&\ + j_3(P_4(x) + P_5(x) + \cdots + P_{n+2}(x) + P_1(x) + P_2(x) + P_3(x)\\
&\ - P_1(x) - P_2(x) - P_3(x)).
\end{aligned}$$

Using (6) we obtain

$$\begin{aligned}
\sum_{l=1}^{n-1} BhP_l(x) &= \frac{P_n(x)+P_{n-1}(x)-1}{2x} \\
&+ j_1\left(\frac{P_{n+1}(x)+P_n(x)-1}{2x} - P_1(x)\right) \\
&+ j_2\left(\frac{P_{n+2}(x)+P_{n+1}(x)-1}{2x} - P_1(x) - P_2(x)\right) \\
&+ j_3\left(\frac{P_{n+3}(x)+P_{n+2}(x)-1}{2x} - P_1(x) - P_2(x) - P_3(x)\right) \\
&= \frac{P_n(x)+P_{n-1}(x)-1}{2x} \\
&+ j_1\frac{P_{n+1}(x)+P_n(x)-1-2x}{2x} \\
&+ j_2\frac{P_{n+2}(x)+P_{n+1}(x)-1-2x-4x^2}{2x} \\
&+ j_3\frac{P_{n+3}(x)+P_{n+2}(x)-1-2x-4x^2-2x(4x^2+1)}{2x} \\
&= \frac{P_n(x)+j_1P_{n+1}(x)+j_2P_{n+2}(x)+j_3P_{n+3}(x)}{2x} \\
&+ \frac{P_{n-1}(x)+j_1P_n(x)+j_2P_{n+1}(x)+j_3P_{n+2}(x)}{2x} \\
&+ \frac{-(0+1)-j_1(1+2x)-j_2(2x+(4x^2+1))-j_3((4x^2+1)+(8x^3+4x))}{2x} \\
&= \frac{BhP_n(x)+BhP_{n-1}(x)-BhP_0(x)-BhP_1(x)}{2x}.
\end{aligned}$$

Thus the Theorem is proved. □

Theorem 7. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} BhQ_l(x) = \frac{BhQ_n(x)+BhQ_{n-1}(x)-BhQ_0(x)-BhQ_1(x)}{2x}.$$

Now we give Binet type formulas for Pell and Pell-Lucas bihypernomials.

Theorem 8. *(Binet type formula for Pell bihypernomials) Let $n \geq 0$ be an integer. Then*

$$\begin{aligned}
BhP_n(x) &= \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(1+j_1\alpha(x)+j_2\alpha^2(x)+j_3\alpha^3(x)\right) \\
&- \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(1+j_1\beta(x)+j_2\beta^2(x)+j_3\beta^3(x)\right),
\end{aligned} \tag{14}$$

where $\alpha(x)$, $\beta(x)$ *are given by* (1).

Proof. Using (2) and (11) we obtain

$$\begin{aligned} BhP_n(x) &= P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x) \\ &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + j_1 \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} \\ &+ j_2 \frac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)} + j_3 \frac{\alpha^{n+3}(x) - \beta^{n+3}(x)}{\alpha(x) - \beta(x)} \end{aligned}$$

and by simple calculations the result follows. □

In the same way, using (3) and (12), we obtain the next theorem.

Theorem 9. *(Binet type formula for Pell-Lucas bihypernomials) Let* $n \geq 0$ *be an integer. Then*

$$\begin{aligned} BhQ_n(x) &= \alpha^n(x) \left(1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x)\right) \\ &+ \beta^n(x) \left(1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x)\right), \end{aligned} \tag{15}$$

where $\alpha(x)$, $\beta(x)$ *are given by* (1).

Using Binet type formulas for Pell and Pell-Lucas bihypernomials we can obtain Catalan type identity, Cassini type identity and d'Ocagne type identity for Pell and Pell-Lucas bihypernomials.

For simplicity of notation let

$$\begin{aligned} \hat{\alpha}(x) &= 1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x), \\ \hat{\beta}(x) &= 1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x). \end{aligned}$$

Consequently we can write (14) and (15) as

$$BhP_n(x) = \frac{\alpha^n(x)\hat{\alpha}(x) - \beta^n(x)\hat{\beta}(x)}{\alpha(x) - \beta(x)} \tag{16}$$

and

$$BhQ_n(x) = \alpha^n(x)\hat{\alpha}(x) + \beta^n(x)\hat{\beta}(x). \tag{17}$$

Moreover,

$$\begin{aligned} \alpha(x) \cdot \beta(x) &= -1, \\ \alpha(x) + \beta(x) &= 2x, \\ \alpha(x) - \beta(x) &= 2\sqrt{x^2 + 1}, \\ \alpha^3(x) + \beta^3(x) &= (\alpha(x) + \beta(x))^3 - 3\alpha(x)\beta(x)(\alpha(x) + \beta(x)) = 8x^3 + 6x, \end{aligned}$$

and

$$\begin{aligned}
&\hat{\alpha}(x)\cdot\hat{\beta}(x)=\hat{\beta}(x)\cdot\hat{\alpha}(x)\\
&=1+j_1\beta(x)+j_2\beta^2(x)+j_3\beta^3(x)+j_1\alpha(x)-1-j_3\beta(x)-j_2\beta^2(x)\\
&\ +j_2\alpha^2(x)-j_3\alpha(x)+1+j_1\beta(x)+j_3\alpha^3(x)-j_2\alpha^2(x)+j_1\alpha(x)-1\\
&=j_1\left(2\alpha(x)+2\beta(x)\right)+j_3\left(\alpha^3(x)+\beta^3(x)-\alpha(x)-\beta(x)\right)\\
&=j_1\cdot 4x+j_3\left(8x^3+4x\right).
\end{aligned}$$

Theorem 10. *(Catalan type identity for Pell bihypernomials) Let $n\geq 0$, $r\geq 0$ be integers such that $n\geq r$. Then*

$$\begin{aligned}
&BhP_{n-r}(x)\cdot BhP_{n+r}(x)-\left(BhP_n(x)\right)^2\\
&=\frac{(-1)^{n-r+1}((x+\sqrt{x^2+1})^r-(x-\sqrt{x^2+1})^r)^2}{4x^2+4}\left(j_1\cdot 4x+j_3\left(8x^3+4x\right)\right).
\end{aligned}$$

Proof. By formula (16) we have

$$\begin{aligned}
&BhP_{n-r}(x)\cdot BhP_{n+r}(x)-\left(BhP_n(x)\right)^2\\
&=-\frac{\alpha^{n-r}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\frac{\beta^{n+r}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)-\frac{\beta^{n-r}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)\frac{\alpha^{n+r}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\\
&+\frac{\alpha^{n}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\frac{\beta^{n}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)+\frac{\beta^{n}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)\frac{\alpha^{n}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\\
&=\frac{\alpha^n(x)\beta^n(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)\frac{2\alpha^r(x)\beta^r(x)-(\beta^r(x))^2-(\alpha^r(x))^2}{(\alpha(x)\beta(x))^r}\\
&=\frac{(\alpha(x)\beta(x))^n}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)(-1)\frac{(\alpha^r(x)-\beta^r(x))^2}{(\alpha(x)\beta(x))^r}\\
&=\frac{(-1)^{n-r+1}(\alpha^r(x)-\beta^r(x))^2}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x),
\end{aligned}$$

so the result follows. □

Theorem 11. *(Catalan type identity for Pell-Lucas bihypernomials) Let $n\geq 0$, $r\geq 0$ be integers such that $n\geq r$. Then*

$$\begin{aligned}
&BhQ_{n-r}(x)\cdot BhQ_{n+r}(x)-\left(BhQ_n(x)\right)^2\\
&=(-1)^{n-r}(\alpha^r(x)-\beta^r(x))^2\cdot\hat{\alpha}(x)\hat{\beta}(x)\\
&=(-1)^{n-r}((x+\sqrt{x^2+1})^r-(x-\sqrt{x^2+1})^r)^2\left(j_1\cdot 4x+j_3\left(8x^3+4x\right)\right).
\end{aligned}$$

Note that for $r=1$ we get the Cassini type identities for Pell and Pell-Lucas bihypernomials.

Corollary 1. *(Cassini type identity for Pell bihypernomials) Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}&BhP_{n-1}(x) \cdot BhP_{n+1}(x) - (BhP_n(x))^2 \\ &= (-1)^n \left(j_1 \cdot 4x + j_3 \left(8x^3 + 4x\right)\right).\end{aligned}$$

Corollary 2. *(Cassini type identity for Pell-Lucas bihypernomials) Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}&BhQ_{n-1}(x) \cdot BhQ_{n+1}(x) - (BhQ_n(x))^2 \\ &= (-1)^{n-1}(4x^2 + 4) \left(j_1 \cdot 4x + j_3 \left(8x^3 + 4x\right)\right).\end{aligned}$$

Theorem 12. *(d'Ocagne type identity for Pell bihypernomials) Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned}&BhP_m(x) \cdot BhP_{n+1}(x) - BhP_{m+1}(x) \cdot BhP_n(x) \\ &= \frac{(-1)^n \left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x).\end{aligned}$$

Proof. By formula (16) we have

$$\begin{aligned}&BhP_m(x) \cdot BhP_{n+1}(x) - BhP_{m+1}(x) \cdot BhP_n(x) \\ &= \frac{\alpha^{m+n+1}(x)}{(\alpha(x) - \beta(x))^2}\hat{\alpha}^2(x) - \frac{\alpha^m(x)\beta^{n+1}(x)}{(\alpha(x) - \beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x) \\ &- \frac{\alpha^{n+1}(x)\beta^m(x)}{(\alpha(x) - \beta(x))^2}\hat{\beta}(x)\hat{\alpha}(x) + \frac{\beta^{m+n+1}(x)}{(\alpha(x) - \beta(x))^2}\hat{\beta}^2(x) \\ &- \frac{\alpha^{m+1+n}(x)}{(\alpha(x) - \beta(x))^2}\hat{\alpha}^2(x) + \frac{\alpha^{m+1}(x)\beta^n(x)}{(\alpha(x) - \beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x) \\ &+ \frac{\alpha^n(x)\beta^{m+1}(x)}{(\alpha(x) - \beta(x))^2}\hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^{m+1+n}(x)}{(\alpha(x) - \beta(x))^2}\hat{\beta}^2(x) \\ &= \left(\frac{\alpha^m(x)\beta^n(x)(\alpha(x) - \beta(x))}{(\alpha(x) - \beta(x))^2} - \frac{\alpha^n(x)\beta^m(x)(\alpha(x) - \beta(x))}{(\alpha(x) - \beta(x))^2}\right)\hat{\alpha}(x)\hat{\beta}(x) \\ &= \frac{(\alpha(x)\beta(x))^n}{\alpha(x) - \beta(x)}\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)\hat{\alpha}(x)\hat{\beta}(x) \\ &= \frac{(-1)^n \left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)}{\alpha(x) - \beta(x)}\hat{\alpha}(x)\hat{\beta}(x),\end{aligned}$$

so the result follows. □

Theorem 13. *(d'Ocagne type identity for Pell-Lucas bihypernomials) Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned} &BhQ_m(x) \cdot BhQ_{n+1}(x) - BhQ_{m+1}(x) \cdot BhQ_n(x) \\ &= (-1)^{n+1}(\alpha(x) - \beta(x)) \left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right) \hat{\alpha}(x)\hat{\beta}(x). \end{aligned}$$

Theorem 14. *Let $m \geq 0$, $n \geq 0$ be integers. Then*

$$\begin{aligned} &BhP_m(x) \cdot BhQ_n(x) - BhQ_m(x) \cdot BhP_n(x) \\ &= \frac{2(-1)^n(\alpha^{m-n}(x) - \beta^{m-n}(x))}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x). \end{aligned}$$

Proof. By formulas (16) and (17) we have

$$\begin{aligned} &BhP_m(x) \cdot BhQ_n(x) - BhQ_m(x) \cdot BhP_n(x) \\ &= \frac{\alpha^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}^2(x) + \frac{\alpha^m(x)\beta^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x) \\ &- \frac{\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^m(x)\beta^n(x)}{\alpha(x) - \beta(x)} \hat{\beta}^2(x) \\ &- \frac{\alpha^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}^2(x) + \frac{\alpha^m(x)\beta^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x) \\ &- \frac{\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\beta}(x)\hat{\alpha}(x) + \frac{\beta^n(x)\beta^m(x)}{\alpha(x) - \beta(x)} \hat{\beta}^2(x) \\ &= \frac{2\alpha^m(x)\beta^n(x) - 2\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x) \\ &= \frac{2(-1)^n(\alpha^{m-n}(x) - \beta^{m-n}(x))}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x), \end{aligned}$$

so the result follows. □

Theorem 15. *The generating function for the Pell bihypernomial sequence $\{BhP_n(x)\}$ is*

$$G(t) = \frac{j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2 + 1) + (1 + j_2 + j_3 \cdot 2x)t}{1 - 2xt - t^2}.$$

Proof. Suppose that the generating function of the Pell bihypernomials sequence $\{BhP_n(x)\}$ has the form $G(t) = \sum_{n=0}^{\infty} BhP_n(x)t^n$. Then

$$G(t) = BhP_0(x) + BhP_1(x)t + BhP_2(x)t^2 + \cdots$$

Multiply the above equality on both sides by $-2xt$ and then by $-t^2$ we obtain

$$-G(t) \cdot (2x)t = -BhP_0(x) \cdot (2x)t - BhP_1(x) \cdot (2x)t^2 - BhP_2(x) \cdot (2x)t^3 - \cdots$$

$$-G(t)t^2 = -BhP_0(x)t^2 - BhP_1(x)t^3 - BhP_2(x)t^4 - \cdots$$

By adding these three equalities above, we will get

$$G(t)(1-2xt-t^2) = BhP_0(x) + (BhP_1(x) - BhP_0(x)\cdot(2x))t$$

since $BhP_n(x) = 2x \cdot BhP_{n-1}(x) + BhP_{n-2}(x)$ (see (13)) and the coefficient of t^n, for $n \geq 2$, are equal to zero. Moreover, $BhP_0(x) = j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2+1)$, $BhP_1(x) - BhP_0(x) \cdot (2x) = 1 + j_2 + j_3 \cdot 2x$. □

Using the same method we give the generating function $g(t)$ for Pell-Lucas bihypernomials.

Theorem 16. *The generating function for the Pell-Lucas bihypernomials sequence* $\{BhQ_n(x)\}$ *is*

$$g(t) = \frac{2 + j_1 \cdot 2x + j_2 \cdot (4x^2+2) + j_3 \cdot (8x^3+6x)}{1-2xt-t^2} + \frac{(-2x + 2j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2+2))t}{1-2xt-t^2}.$$

At the end we will give the matrix generator of Pell and Pell-Lucas bihypernomials.

Theorem 17. *Let* $n \geq 0$ *be an integer. Then*

$$\begin{bmatrix} BhP_{n+2}(x) & BhP_{n+1}(x) \\ BhP_{n+1}(x) & BhP_n(x) \end{bmatrix} = \begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. (by induction on n)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now suppose that for any $n \geq 0$ holds

$$\begin{bmatrix} BhP_{n+2}(x) & BhP_{n+1}(x) \\ BhP_{n+1}(x) & BhP_n(x) \end{bmatrix} = \begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

We shall show that

$$\begin{bmatrix} BhP_{n+3}(x) & BhP_{n+2}(x) \\ BhP_{n+2}(x) & BhP_{n+1}(x) \end{bmatrix} = \begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^{n+1}.$$

Using induction's hypothesis we have

$$\begin{aligned} &\begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} BhP_{n+2}(x) & BhP_{n+1}(x) \\ BhP_{n+1}(x) & BhP_n(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x \cdot BhP_{n+2}(x) + BhP_{n+1}(x) & BhP_{n+2}(x) \\ 2x \cdot BhP_{n+1}(x) + BhP_n(x) & BhP_{n+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} BhP_{n+3}(x) & BhP_{n+2}(x) \\ BhP_{n+2}(x) & BhP_{n+1}(x) \end{bmatrix}, \end{aligned}$$

which ends the proof. □

Theorem 18. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} BhQ_{n+2}(x) & BhQ_{n+1}(x) \\ BhQ_{n+1}(x) & BhQ_n(x) \end{bmatrix} = \begin{bmatrix} BhQ_2(x) & BhQ_1(x) \\ BhQ_1(x) & BhQ_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Using algebraic operations and matrix algebra, properties of these bihypernomials can be found.

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References

[1] Bilgin, M., Ersoy, S., Algebraic properties of bihyperbolic numbers, *Adv. Appl. Clifford Algebr.*, 30(13) (2020). https://doi.org/10.1007/s00006-019-1036-2

[2] Bród, D., Szynal-Liana, A., Włoch, I., Bihyperbolic numbers of the Fibonacci type and their idempotent representation, *Comment. Math. Univ. Carolin.*, 62(4) (2021), 409-416. http://dx.doi.org/10.14712/1213-7243.2021.033

[3] Bród, D., Szynal-Liana, A., Włoch, I., On some combinatorial properties of bihyperbolic numbers of the Fibonacci type, *Math. Methods Appl. Sci.*, 44(6) (2021), 4607-4615. https://doi.org/10.1002/mma.7054

[4] Halici, S., On the Pell polynomials, *Appl. Math. Sci. (Ruse)*, 5(37) (2011), 1833–1838.

[5] Horadam, A. F., Minmax Sequences for Pell Numbers, In: Bergum G.E., Philippou A.N., Horadam A.F. (eds) Applications of Fibonacci Numbers, Springer, Dordrecht, 1996.

[6] Horadam, A. F., Pell identities, *Fibonacci Quart.*, 9(3) (1971), 245–263.

[7] Horadam, A. F., Mahon, Bro J. M., Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, 23(1) (1985), 7–20.

[8] Horzum, T., Kocer, E. G., On some properties of Horadam polynomials, *Int. Math. Forum*, 25(4) (2009), 1243–1252.

[9] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[10] Rochon, D., Shapiro, M., On algebraic properties of bicomplex and hyperbolic numbers, *An. Univ. Oradea Fasc. Mat.*, 11 (2004), 71–110.

[11] Sobczyk, G., The hyperbolic number plane, *College Math. J.*, 26(4) (1995). https://doi.org/10.1080/07468342.1995.11973712

[12] Szynal-Liana, A., Włoch, I., On Pell and Pell-Lucas hybrid numbers, *Comment. Math. Prace Mat.*, 58(1-2) (2018), 11–17. https://doi.org/10.14708/cm.v58i1-2.6364

[13] Szynal-Liana, A., Włoch, I., The Pell quaternions and the Pell octonions, *Adv. Appl. Clifford Algebr.*, 26 (2016), 435–440. https://doi.org/10.1007/s00006-015-0570-9

[14] Szynal-Liana, A., Włoch, I., Hypercomplex Numbers of the Fibonacci Type, Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów, 2019.

http://communications.science.ankara.edu.tr

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SKEW *ABC* ENERGY OF DIGRAPHS

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Abstract. In this paper, skew ABC matrix and its energy are introduced for digraphs. Firstly, some fundamental spectral features of the skew ABC matrix of digraphs are established. Then some upper and lower bounds are presented for the skew ABC energy of digraphs. Further extremal digraphs are determined attaining these bounds.

1. Introduction and Preliminaries

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. An edge joining vertices v_i and v_j is denoted by $v_iv_j \in E(G)$ and degree of a vertex v_i is denoted by d_i. The atom-bond connectivity index ABC of G is introduced by Estrada et al. [6] as

$$ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}},$$

which is a significant predictive index in the studies about the heat of formation in alkanes (see [8]- [6]), for further information about mathematical and chemical applications about atom-bond connectivity index, also see ([9]- [11]- [13]- [15]- [27]). The concept of graph energy is defined as sum of the absolute values of the eigenvalues of a graph by Gutman [16]. The energy of a graph has been widely studied by many mathematicians and chemists, as it has close links with chemistry (see [17]). So, several kinds of graph energy are introduced and examined such as Laplacian energy, Randić energy, distance energy, Zagreb energy, etc.

Estrada [7] defined the generalized ABC matrix $S_\alpha(G) = \left(s_{ij}^{\alpha}\right)$ of order n, where the $(i,j)-th$ entry is $\left(\frac{d_i+d_j-2}{d_id_j}\right)^{\alpha}$, if $v_iv_j \in E(G)$ and 0, otherwise. If $\alpha = \frac{1}{2}$, the

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generalized *ABC* matrix is called as *ABC* matrix of a graph and will be denoted by $\Omega(G)$. Let ς_i be the eigenvalues of $\Omega(G)$ (also called *ABC* eigenvalues of G). *ABC* energy of a graph is defined by $E\Omega(G) = \sum_{i=1}^{n} |\varsigma_i|$. As $\Omega(G)$ is a real symmetric matrix, the *ABC* eigenvalues of G are real numbers. Recently, some bounds have presented for the *ABC* eigenvalues and *ABC* energy of graphs by Chen [5] and Ghorbani et al. [12].

Let $\vec{G}$ be a digraph with vertex set $V(\vec{G}) = \{v_1, v_2, ..., v_n\}$ and arc set $\Gamma(\vec{G})$. An arc from v_i to v_j is denoted by $v_i \to v_j$. Throughout this paper, all the digraphs are simple and do not have loops and if there is an arc from v_i to v_j, then there is not an arc from v_j to v_i. Hence, a digraph $\vec{G}$ without orientation gives the underlying graph G is simple.

Graph energy concept is extended to digraphs in [22]. Then the skew Laplacian energy of a digraph is defined by Adiga et al. [3] and new definitions are proposed for the skew Laplacian energy (see [2]- [4]). The skew energy of a digraph is defined by Adiga et al. [1] as $ES(\vec{G}) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i are the eigenvalues of the skew adjacency matrix $S(\vec{G})$ of order n. Let $S(\vec{G})=(s_{ij})$, where the $(i,j)-th$ entry is 1, if $v_i \to v_j$; -1, if $v_j \to v_i$ and 0, otherwise. Since λ_i $(1 \le i \le n)$ are purely imaginary numbers, the singular values of $S(\vec{G})$ equal to the absolute values of λ_i. For recent studies about kinds of skew energy, also see the survey in [21] and the references therein.

The Randić index is introduced as "branching index" by $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$ in [24]. The general Randić index of a graph is defined by $R_\gamma(G) = \sum_{uv \in E(G)} (d_u d_v)^\gamma$ in [20]. R_{-1} is called as modified second Zagreb index. The skew Randić energy of a digraph is introduced by Gu et al. [14] and some bounds are presented for this energy kind. Inspired by the studies of the skew energy kinds of graphs, we will introduce skew *ABC* matrix of a digraph and its energy.

Skew *ABC* matrix of a simple digraph $\vec{G}$ is $\Omega_s=\Omega_s(\vec{G})=(b_{ij})$ of order n and we define the $(i,j)-th$ entry of Ω_s as

$$b_{ij} = \begin{cases} \left(\frac{d_i+d_j-2}{d_i d_j}\right)^{\frac{1}{2}} & \text{if } v_i \to v_j \\ -\left(\frac{d_i+d_j-2}{d_i d_j}\right)^{\frac{1}{2}} & \text{if } v_j \to v_i \\ 0 & \text{otherwise,} \end{cases}$$

where d_i and d_j are the degrees of the corresponding vertices in the underlying graph G. The skew *ABC* matrix of a simple digraph can be considered as a weighted skew adjacency matrix with $\left(\frac{d_i+d_j-2}{d_i d_j}\right)^{\frac{1}{2}}$ weights.

Let $\{\vartheta_1, \vartheta_2, ..., \vartheta_n\}$ be eigenvalues of the skew ABC matrix of $\vec{G}$, namely be skew ABC eigenvalues. Since $\Omega_s(\vec{G})$ is a skew symmetric matrix, the skew ABC eigenvalues are purely imaginary numbers. We can define skew ABC energy of a digraph as

$$E\Omega_s(\vec{G}) = \sum_{j=1}^{n} |\vartheta_j| .$$

This paper is only concerned with the mathematical aspects of the skew ABC energy of digraphs. The rest of the paper is composed of two sections. In the next section, the spectral features of the skew ABC matrix of digraphs are presented. In the last section, some upper and lower bounds are obtained for the skew energy and the extremal digraphs are determined attaining these bounds.

2. Skew ABC Eigenvalues

In this section we consider some fundamental spectral properties of the skew ABC matrix of digraphs.

Proposition 1. *Let $\vec{G}$ be a digraph of order n with no isolated vertices. If $\phi(\vec{G};\vartheta) = \det(\vartheta I_n - \Omega_S) = c_0\vartheta^n + c_1\vartheta^{n-1} + ... + c_n$ is the characteristic polynomial of $\Omega_s(\vec{G})$, then*
(i) $c_0 = 1$, $c_1 = 0$,
(ii) $c_2 = n - 2R_{-1}(G)$,
(iii) $c_j = 0$, *for all odd* j.

Proof. (i) Let $tr(.)$ stands for trace of a matrix. Obviously we have $c_0 = 1$ and $c_1 = \sum_{j=1}^{n}\vartheta_j = tr(\Omega_s) = 0$.

(ii) c_2 equals to the sum of the determinants of all 2×2 principal submatrices of $\Omega_s(\vec{G})$, thus

$$\begin{aligned} c_2 &= \sum_{j<k} \det \begin{pmatrix} 0 & b_{jk} \\ b_{kj} & 0 \end{pmatrix} = \sum_{j<k} -b_{jk}b_{kj} = \sum_{j<k} (b_{jk})^2 = \sum_{v_jv_k \in E(G)} \frac{d_j + d_k - 2}{d_jd_k} \\ &= \sum_{v_jv_k \in E(G)} \frac{d_j + d_k}{d_jd_k} - 2\sum_{v_jv_k \in E(G)} \frac{1}{d_jd_k} \\ &= n - 2R_{-1}(G), \end{aligned}$$

where $R_{-1}(G) = \sum_{uv \in E(G)} \frac{1}{d_ud_v}$.

(iii) Let j be odd. c_j equals to the sum of the determinants of all $j \times j$ principal submatrices of $\Omega_s(\vec{G})$ is 0 as a principal submatrix of a skew symmetric matrix is skew symmetric. □

Proposition 2. *Let $\vec{G}$ be a digraph of order $n(\geq 3)$ with no isolated vertices and $\{i\vartheta_1, i\vartheta_2, ..., i\vartheta_n\}$ be the skew ABC eigenvalues of $\vec{G}$ such that $\vartheta_1 \geq \vartheta_2 \geq ... \geq \vartheta_n$. Then*

(i) $\vartheta_j = -\vartheta_{n+1-j}$ for all $1 \leq j \leq n$. If n is even, then $\vartheta_{\frac{n}{2}} \geq 0$ and if n is odd, then $\vartheta_{\frac{n+1}{2}} = 0$.

(ii) $\sum_{j=1}^{n} |\vartheta_j|^2 = 2\left(n - 2R_{-1}(G)\right).$

Proof. (i) The proof is clear.

(ii) Obviously we have

$$\sum_{j=1}^{n} (i\vartheta_j)^2 = tr((\Omega_s)^2) = \sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk}b_{kj} = -\sum_{j=1}^{n}\sum_{k=1}^{n} (b_{jk})^2 = -2\left(n - 2R_{-1}(G)\right),$$

which completes the proof. □

From Proposition 1 and Proposition 2, we also have

$$\sum_{1\leq i<j\leq n} \vartheta_i\vartheta_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n}\vartheta_i\right)^2 - \sum_{i=1}^{n}\vartheta_i^2\right] = 2R_{-1}(G) - n.$$

$Sp\Big(\Omega_s(\vec{G})\Big)$ denotes the skew ABC spectrum of $\vec{G}$ which is a multiset consist of eigenvalues (with multiplicities) of $\Omega_s(\vec{G})$. Also, $Sp(\Omega(G))$ is the ABC spectrum of the underlying graph G.

Example 1. *Let $\vec{C_4}$ be a directed cycle of order 4 with the arc set $\{(1,2), (2,3), (3,4), (4,1)\}$. The skew ABC spectrum of $\vec{G}$ is $Sp\Big(\Omega_s(\vec{C_4})\Big) = \{-\frac{1}{2}i\sqrt{2}, -\frac{1}{2}\sqrt{2}, \frac{1}{2}i\sqrt{2}, \frac{1}{2}\sqrt{2}\}$ and the skew ABC energy of $\vec{C_4}$ is $E\Omega_s(\vec{C_4}) = 2\sqrt{2}$. Consider the underlying graph C_4. The ABC spectrum of C_4 is $Sp(\Omega(C_4)) = \{-\sqrt{2}, 0^{(2)}, \sqrt{2}\}$. Hence, $E\Omega_s(\vec{C_4}) = E\Omega(C_4)$.*

Example 2. *Let $\vec{K}_{p,q}$ $(p, q \neq 1)$ be a complete bipartite digraph in which the vertex set is a disjoint union $A \cup B$ with $|A| = p$ and $|B| = q$. Consider the elementary orientation that is, orienting all the edges from A to B and writing the elements of A firstly, form the matrix $\Omega_s(\vec{K}_{p,q}) = \begin{pmatrix} 0_p & \beta J_{p\times q} \\ -\beta J_{q\times p} & 0_q \end{pmatrix}$, where $\beta = \sqrt{\frac{p+q-2}{pq}}$ and J_n is the order n matrix with all entries are 1.*

$$\det(\vartheta I_{p+q} - \Omega_s(\vec{K}_{p,q})) = \det\begin{pmatrix} \vartheta I_p & -\beta J_{p\times q} \\ \beta J_{q\times p} & \vartheta I_q \end{pmatrix}.$$

Since ϑI_p is nonsingular, then

$$\det(\vartheta I_{p+q} - \Omega_s(\vec{K}_{p,q})) \quad = \quad \det(\vartheta I_p)\det\left(\vartheta I_q + \beta J_{q\times p}(\vartheta I_p)^{-1}\beta J_{p\times q}\right)$$

$$= \det(\vartheta I_p)\det\left(\vartheta I_q + \beta J_{q\times p}\frac{1}{\vartheta}I_p\beta J_{p\times q}\right),$$

(see [23]). Recall $J_{q\times p}J_{p\times q} = pJ_q$, *thus*

$$\begin{aligned}\det(\vartheta I_{p+q} - \Omega_s(\vec{K}_{p,q})) &= \vartheta^p \det\left(\vartheta I_q + \frac{\beta^2}{\vartheta}pJ_q\right) \\ &= \vartheta^{p-q}\det\left(\vartheta^2 I_q + \beta^2 pJ_q\right).\end{aligned}$$

$\beta^2 pJ_q$ *has eigenvalues* $\beta^2 pq$ *of multiplicity* 1 *and* 0 *of multiplicity* $q-1$, *since* $Sp(J_q) = \{q, 0^{(q-1)}\}$. *Then*

$$\phi\left(\Omega_s(\vec{K}_{p,q});\vartheta\right) = \vartheta^{p+q-2}\left(\vartheta^2 + \beta^2 pq\right),$$

and $\Omega_s(\vec{K}_{p,q})$ *has eigenvalues* $-\beta\sqrt{pq}i$, $\beta\sqrt{pq}i$ *and* 0 *of multiplicity* $p+q-2$, *i.e.,* $\sqrt{p+q-2}i$, $-\sqrt{p+q-2}i$ *and* 0 *of multiplicity* $p+q-2$, *hence*

$$E\Omega_s(\vec{K}_{p,q}) = 2\sqrt{p+q-2},$$

and $Sp\left(\Omega_s(\vec{K}_{p,q})\right) = \left\{-\sqrt{p+q-2}i,\ 0^{(n-2)}, \sqrt{p+q-2}i\right\}$. *It is seen that there is an orientation such that* $Sp\left(\Omega_s(\vec{K}_{p,q})\right) = iSp(\Omega(K_{p,q}))$. *Orienting all the edges from* B *to* A *and writing the elements of* B *firstly, form the matrix* $\Omega_s\left(\vec{K}_{p,q}\right) = \begin{pmatrix} 0_q & \beta J_{q\times p} \\ -\beta J_{p\times q} & 0_p\end{pmatrix}$. *Obviously, carrying out the process above gives the same skew* ABC *eigenvalues.*

The relationship between the skew spectrum of a digraph and spectrum of its underlying graph is firstly analyzed in [25]. By Example 2, it is concluded that there is an orientation such that $Sp\left(\Omega_s(\vec{K}_{p,q})\right) = iSp\left(\Omega\left(K_{p,q}\right)\right)$. An analogous relation that can be seen in Theorem 1, exists between the skew ABC spectrum and ABC spectrum.

Lemma 1 ([25])**.** *If* $A = \begin{pmatrix} 0 & Y \\ Y^T & 0\end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & Y \\ -Y^T & 0\end{pmatrix}$ *are two real matrices, then* $Sp(B) = iSp(A)$.

Theorem 1. *G is a bipartite graph if and only if there is an orientation such that* $Sp\left(\Omega_s(\vec{G})\right) = iSp(\Omega(G))$.

Proof. If G is bipartite, then by suitable labelling the vertices, the ABC matrix of G takes the form $\Omega(G) = \begin{pmatrix} 0 & Y \\ Y^T & 0\end{pmatrix}$. Let $\vec{G}$ be an orientation such that the skew ABC matrix is of the form $\Omega_s(\vec{G}) = \begin{pmatrix} 0 & Y \\ -Y^T & 0\end{pmatrix}$. By Lemma 1, the proof is obvious.

Conversely, assume that $Sp\Big(\Omega_s(\vec{G})\Big){=}iSp(\Omega(G))$ for some orientation. As $\Omega_s(\vec{G})$ is a real skew symmetric matrix, $Sp\Big(\Omega_s(\vec{G})\Big)$ has only pure imaginary eigenvalues, thus the skew ABC eigenvalues are symmetric with respect to the real axis. Hence, $Sp\Big(\Omega_s(\vec{G})\Big) = -iSp(\Omega(G))$ is symmetric about the imaginary axis. So, G is bipartite. □

3. Bounds for the Skew ABC Energy

In this section, we intend to obtain bounds for the skew ABC energy of digraphs by using the mathematical inequalities and properties of the skew ABC eigenvalues and examine the equality case of these bounds. In recent studies, many bounds are presented for $R_{-1}(G)$. Using these bounds, one can also obtain different bounds for the skew ABC energy of digraphs by combining the bounds will be presented in this section. Now, we consider the bounds for $R_{-1}(G)$ in [19] and [26]. Throughout this section, it is assumed that $\{i\vartheta_1, i\vartheta_2, ..., i\vartheta_n\}$ be the skew ABC eigenvalues of $\vec{G}$ with $\vartheta_1 \geq \vartheta_2 \geq ... \geq \vartheta_n$. Moreover K_n denotes the complete graph of order n and $G = \left(\frac{n}{2}\right) K_2$ stands for the vertex-disjoint union of $\frac{n}{2}$ copies of K_2.

Theorem 2 ([26])**.** *If G is a graph of order $n(\geq 2)$ with no isolated vertices with maximum vertex degree Δ and minimum vertex degree δ, then*

$$\frac{n}{2\Delta} \leq R_{-1}(G) \leq \frac{n}{2\delta}, \tag{1}$$

with equality if and only if G is regular.

Theorem 3 ([19])**.** *If G is a graph of order n with no isolated vertices, then*

$$\frac{n}{2(n-1)} \leq R_{-1}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{2}$$

Equality in lower bound holds if and only if $G = K_n$. Equality in upper bound holds if and only if either (i) $G = \left(\frac{n}{2}\right) K_2$ when n is even or (ii) $G = K_{1,2} \cup \frac{n-3}{2} K_2$ when n is odd.

Initially, we can give the following upper bound involving $R_{-1}(G)$ and n for the skew ABC energy of digraphs.

Theorem 4. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices, then*

$$E\Omega_s(\vec{G}) \leq \sqrt{2n\,(n - 2R_{-1}(G))}. \tag{3}$$

with equality if $|\vartheta_i| = |\vartheta_j|$ for all $1 \leq i \neq j \leq n$.

Proof. Applying Cauchy-Schwarz inequality and using Proposition 2 yields

$$E\Omega_s(\vec{G}) \quad = \quad \sum_{i=1}^{n} |\vartheta_i| \leq \sqrt{\sum_{i=1}^{n} |\vartheta_i|^2}\sqrt{n} \tag{4}$$

$$= \sqrt{2n\left(n-2R_{-1}\left(G\right)\right)}.$$

Equality case is obvious from the equality in (4). □

Using the lower bound of (1) in (3), we can obtain a new upper bound in terms of n and Δ as follows.

Corollary 1. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ and $\Delta\,(\geq 1)$ is the maximum vertex degree of the underlying graph G, then*

$$E\Omega_s\big(\vec{G}\big) \leq n\sqrt{2\left(1-\frac{1}{\Delta}\right)}, \tag{5}$$

with equality if and only if n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K_2}$.

Proof. From (1) and (3), clearly we get $E\Omega_s\big(\vec{G}\big) \leq \sqrt{2n\left(n-\frac{n}{\Delta}\right)}$, so the proof is obvious. We will focus on the equality case. Equality holds in (5) if and only if equality holds in (4), namely $|\vartheta_i| = |\vartheta_j|$ for all $1 \leq i \neq j \leq n$ and G is regular. Thus $\vartheta_1{=}\vartheta_2 = ... = \vartheta_n = 0$ that is, $\Omega_s\big(\vec{G}\big) = 0$ and we have n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K_2}$, for an arbitrary orientation. □

The following bound presents a relationship between the skew ABC energy of a digraph and ABC energy of complete graph K_n.

Corollary 2. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices, then*

$$E\Omega_s\big(\vec{G}\big) \leq \left(\frac{n}{2\sqrt{n-1}}\right)E\Omega\left(K_n\right). \tag{6}$$

Proof. If $G = K_n$, then K_n has two distinct ABC eigenvalues such that $\sqrt{2n-4}$ of multiplicity 1 and $-\frac{\sqrt{2n-4}}{n-1}$ of multiplicity $n-1$ (see Proposition 3.1, [5]). Then $E\Omega\left(K_n\right) = 2\sqrt{2n-4}$. Using this fact with (2) and (3)

$$\begin{aligned}
E\Omega_s\big(\vec{G}\big) &\leq \sqrt{2n\left(n-2R_{-1}\left(G\right)\right)} \\
&\leq \sqrt{2n\left(\frac{n^2-2n}{n-1}\right)} \\
&= \frac{n}{\sqrt{n-1}}\sqrt{2n-4} \\
&= \left(\frac{n}{2\sqrt{n-1}}\right)E\Omega\left(K_n\right)
\end{aligned}$$

yields the result. □

The following theorem presents a new upper and lower bound in terms of $\det\left(\Omega_s\big(\vec{G}\big)\right)$, $R_{-1}\left(G\right)$ and n.

Theorem 5. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices and $p = det\Big(\Omega_s(\vec{G})\Big)$, then*

$$\sqrt{2\left(n-2R_{-1}\left(G\right)\right)+n\left(n-1\right)p^{\frac{2}{n}}} \leq E\Omega_s\big(\vec{G}\big) \leq \sqrt{2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)+np^{\frac{2}{n}}}, \tag{7}$$

with equality if and only if n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K}_2$.

Proof. Recall the arithmetic-geometric mean inequality in [18], where $x_1, x_2, ..., x_n$ are non-negative numbers and

$$\begin{aligned} n\left[\frac{1}{n}\sum_{j=1}^{n}x_j - \left(\prod_{j=1}^{n}x_j\right)^{\frac{1}{n}}\right] &\leq n\sum_{j=1}^{n}x_j - \left(\sum_{j=1}^{n}\sqrt{x_j}\right)^2 \\ &\leq n\left(n-1\right)\left[\frac{1}{n}\sum_{j=1}^{n}x_j - \left(\prod_{j=1}^{n}x_j\right)^{\frac{1}{n}}\right], \end{aligned} \tag{8}$$

with equality if and only if $x_1 = x_2 = ... = x_n$. Choosing $x_j = |\vartheta_j|^2$ in (8) yields

$$nK \leq n\sum_{j=1}^{n}|\vartheta_j|^2 - \left(\sum_{j=1}^{n}|\vartheta_j|\right)^2 \leq n\left(n-1\right)K,$$

where $K = \frac{1}{n}\sum_{j=1}^{n}|\vartheta_j|^2 - \left(\prod_{j=1}^{n}|\vartheta_j|^2\right)^{\frac{1}{n}}$. Hence

$$nK \leq 2n\left(n-2R_{-1}\left(G\right)\right) - \Big(E\Omega_s\big(\vec{G}\big)\Big)^2 \leq n\left(n-1\right)K. \tag{9}$$

From Proposition 2, we have $K = \frac{1}{n}\left[2\left(n-2R_{-1}\left(G\right)\right)\right] - p^{\frac{2}{n}}$, where $p = det\Big(\Omega_s\big(\vec{G}\big)\Big)$. From the left hand side of (9), we obtain

$$\Big(E\Omega_s\big(\vec{G}\big)\Big)^2 \leq 2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)+np^{\frac{2}{n}},$$

i.e.,

$$E\Omega_s\big(\vec{G}\big) \leq \sqrt{2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right)+np^{\frac{2}{n}}}.$$

From the right hand side of (9)

$$2n\left(n-2R_{-1}\left(G\right)\right) - n\left(n-1\right)K \leq \Big(E\Omega_s\big(\vec{G}\big)\Big)^2.$$

As $n\left(n-1\right)K = 2\left(n-1\right)\left(n-2R_{-1}\left(G\right)\right) - n\left(n-1\right)p^{\frac{2}{n}}$, we have

$$E\Omega_s\big(\vec{G}\big) \geq \sqrt{2\left(n-2R_{-1}\left(G\right)\right)+n\left(n-1\right)p^{\frac{2}{n}}}.$$

Note that if n is odd, then $p = 0$. Consequently, we have

$$\sqrt{2\left(n - 2R_{-1}\left(G\right)\right)} \leq E\Omega_s\big(\overrightarrow{G}\big) \leq \sqrt{2\left(n-1\right)\left(n - 2R_{-1}\left(G\right)\right)}.$$

The equality holds in (7) if and only if $|\vartheta_1|^2 = |\vartheta_2|^2 = ... = |\vartheta_n|^2$. Thus $\vartheta_1 = \vartheta_2 = ... = \vartheta_n = 0$. So, $\Omega_s\big(\overrightarrow{G}\big) = 0$ and we have n is even and $\overrightarrow{G} = \left(\frac{n}{2}\right)\vec{K}_2$ for an arbitrary orientation. ☐

Lemma 2 ([10])**.** *If $x_1, x_2, ..., x_n \geq 0$ and $r_1, r_2, ..., r_n \geq 0$ such that $\sum_{j=1}^{n} r_j = 1$, then*

$$\sum_{j=1}^{n} x_j r_j - \prod_{j=1}^{n} x_j^{r_j} \geq nr\left(\frac{1}{n}\sum_{j=1}^{n} x_j - \prod_{j=1}^{n} x_j^{\frac{1}{n}}\right), \tag{10}$$

where $r = \min\{r_1, r_2, ..., r_n\}$. Equality holds if and only if $x_1 = x_2 = ... = x_n$.

Finally, we give a new lower bound involving $\det\Big(\Omega_s\big(\overrightarrow{G}\big)\Big)$, $|\vartheta_1|$ and n.

Theorem 6. *If $\vec{G}$ is a digraph of order $n(\geq 3)$ with no isolated vertices and $p = det\Big(\Omega_s\big(\vec{G}\big)\Big)$, then*

$$E\Omega_s\big(\vec{G}\big) \geq |\vartheta_1| + 2\left(n-1\right)\left[\frac{p^{\frac{2n-1}{2n(n-1)}}}{|\vartheta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}p^{\frac{1}{n}}\right] \tag{11}$$

with equality if and only if n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K}_2$.

Proof. Setting $x_j = |\vartheta_j|$ for $j = 1, 2, ..., n$, $r_1 = \frac{1}{2n}$, $r_j = \frac{2n-1}{2n(n-1)}$ for $j = 2, ..., n$ and $r = \frac{1}{2n}$ in (10), we obtain

$$\begin{aligned} &\left(\frac{|\vartheta_1|}{2n} + \frac{2n-1}{2n\left(n-1\right)}\sum_{j=2}^{n}|\vartheta_j|\right) - |\vartheta_1|^{\frac{1}{2n}}\prod_{j=2}^{n}|\vartheta_j|^{\frac{2n-1}{2n(n-1)}} \\ \geq\ & n\frac{1}{2n}\left(\frac{1}{n}\sum_{j=1}^{n}|\vartheta_j| - \prod_{j=1}^{n}|\vartheta_j|^{\frac{1}{n}}\right) \\ =\ & \frac{1}{2n}E\Omega_s\big(\vec{G}\big) - \frac{1}{2}p^{\frac{1}{n}}. \end{aligned}$$

Note that $|\vartheta_1|^{\frac{1}{2n}}\prod_{j=2}^{n}|\vartheta_j|^{\frac{2n-1}{2n(n-1)}} = |\vartheta_1|^{-\frac{1}{2(n-1)}}\prod_{j=1}^{n}|\vartheta_j|^{\frac{2n-1}{2n(n-1)}} = \frac{p^{\frac{2n-1}{2n(n-1)}}}{|\vartheta_1|^{\frac{1}{2(n-1)}}}$ and $\sum_{j=2}^{n}|\vartheta_j| = E\Omega_s\big(\vec{G}\big) - |\vartheta_1|$, thus

$$\left[\frac{1}{2n} - \frac{2n-1}{2n\left(n-1\right)}\right]|\vartheta_1| + \left[\frac{2n-1}{2n\left(n-1\right)} - \frac{1}{2n}\right]E\Omega_s\big(\vec{G}\big) \geq \frac{p^{\frac{2n-1}{2n(n-1)}}}{|\vartheta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}p^{\frac{1}{n}},$$

then

$$-\frac{1}{2(n-1)}|\vartheta_1| + \frac{1}{2(n-1)}E\Omega_s(\vec{G}) \geq \frac{p^{\frac{2n-1}{2n(n-1)}}}{|\vartheta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}p^{\frac{1}{n}}.$$

Hence, we have

$$E\Omega_s(\vec{G}) \geq |\vartheta_1| + 2(n-1)\left[\frac{p^{\frac{2n-1}{2n(n-1)}}}{|\vartheta_1|^{\frac{1}{2(n-1)}}} - \frac{1}{2}p^{\frac{1}{n}}\right].$$

If n is odd, then $E\Omega_s\left(\vec{G}\right) \geq |\vartheta_1|$. The equality holds in (11) if and only if $|\vartheta_1| = |\vartheta_2|=...=|\vartheta_n|$, then $\vartheta_1=\vartheta_2 = ... = \vartheta_n = 0$. So, we have $p = 0$ and $\Omega_s\left(\vec{G}\right) = 0$, that is, n is even and $\vec{G} = \left(\frac{n}{2}\right)\vec{K}_2$ for an arbitrary orientation. □

Conclusion

In recent studies, the *ABC* matrix and *ABC* energy of graphs have introduced. This paper expands these concepts to skew *ABC* matrix and skew *ABC* energy of digraphs. The skew *ABC* matrix of a digraph is defined and its spectral features are established. Further, some upper and lower bounds for the skew *ABC* energy of digraphs are presented with extremal digraphs attaining these bounds.

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References

[1] Adiga, C., Balakrishnan, R., So, W., The skew energy of a digraph, *Linear Algebra Appl.*, 432 (2010), 1825–1835. https://doi:10.1016/j.laa.2009.11.034

[2] Adiga, C., Khoshbakht, Z., On some inequalities for the skew Laplacian energy of digraphs, *J. Inequal. Pure and Appl. Math.*, 10(3) (2009), Art. 80, 6 pp.

[3] Adiga, C., Smitha, M., On the skew Laplacian energy of a digraph, *Int. Math. Forum*, 4 (39)(2009), 1907–1914.

[4] Cai, Q., Li, X., Song, J., New skew Laplacian energy of simple digraphs, *Trans. Comb.*, 2 (2013), 27–37.

[5] Chen, X., On *ABC* eigenvalues and *ABC* energy, *Linear Algebra Appl.*, 544 (2018), 141–157. https://doi.org/10.1016/j.laa.2018.01.011

[6] Estrada, E., Torres, L., Rodríguez, L., Gutman, I., An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes, *Indian J. Chem.*, 37A (1998), 849–855.
[7] Estrada, E., The *ABC* matrix, *Journal of Mathematical Chemistry*, 55 (2017), 1021–1033. https://doi.org/10.1007/s10910-016-0725-5
[8] Estrada, E., Atom–bond connectivity and the energetic of branched alkanes, *Chem.Phys.Lett.*, 463 (2008), 422–425. https://doi.org/10.1016/J.CPLETT.2008.08.074
[9] Furtula, B., Graovac, A., Vukičević, D., Atom-bond connectivity index of trees, *Discrete Applied Mathematics*, 157 (2008), 2828–2835. https://doi.org/10.1016/j.dam.2009.03.004
[10] Furuichi, S., On refined young inequalities and reverse inequalities, *J. Math. Inequal.*, 5 (2011), 21–31. dx.doi.org/10.7153/jmi-05-03
[11] Gan, L., Hou, H., Liu, B., Some results on atom–bond connectivity index of graphs, *MATCH Commun. Math. Comput. Chem.*, 66 (2011), 669–680.
[12] Ghorbani, M., Li, X., Hakimi-Nezhaad, M., Wang, J., Bounds on the *ABC* spectral radius and *ABC* energy of graphs, *Linear Algebra Appl.*, 598 (2020), 145–164. https://doi.org/10.1016/j.laa.2020.03.043
[13] Graovac, A., Ghorbani, M., A new version of atom-bond connectivity index, *Acta Chimica Slovenica*, 57(3) (2010), 609–612.
[14] Gu, R., Huang, X., Li, F., Skew Randić matrix and skew Randić energy, *Trans. Combin.*, 5(1) (2016), 1–14.
[15] Gutman, I., Tošović, J. Radenković, S., Marković, S., On atom-bond connectivity index and its chemical applicability, *Indian J. Chem.* 51A (2012), 690–694.
[16] Gutman, I., The energy of a graph, *Berlin Mathematics-Statistics Forschungszentrum*, 103 (1978), 1–22.
[17] Gutman, I., Polansky, O.E., Mathematical Concepts in Organic Chemistry, Springer-Verlag, Berlin, 1986.
[18] Kober, H., On the arithmetic and geometric means and the Hölder inequality, *Proc. Amer. Math. Soc.*, 59 (1958), 452–459.
[19] Li, X., Yang, Y., Sharp bounds for the general Randić index, *MATCH Commun. Math. Comput. Chem.*, 51 (2004), 155–166.
[20] Li, X., Gutman, I., Mathematical Aspects of Randić-type Molecular Structure Descriptors, Univ. Kragujevac, Kragujevac, 2006.
[21] Li, X., Lian, H., A survey on the skew energy of oriented graphs (2015). arXiv:1304.5707v6
[22] Peña, I., Rada, J., Energy of digraphs, *Lin. Multilin. Algebra*, 56 (2008), 565–579. https://doi.org/10.1080/03081080701482943
[23] Powell, P.D., Calculating determinants of block matrices, (2011). arXiv:1112.4379
[24] Randić, M., On characterization of molecular branching, *J. Amer. Chem. Soc.*, 97 (1975), 6609–6615.
[25] Shader, B., So, W., Skew spectra of oriented graphs, *Elec. J. Combin.*, 16 (2009), 1–6.
[26] Shi, L., Bounds on Randić indices, *Discrete Math.*, 309(16) (2009), 5238–5241. https://doi.org/10.1016/j.disc.2009.03.036
[27] Xing, R., Zhou, B., Du, Z., Further results on atom-bond connectivity index of trees, *Discrete Applied Mathematics*, 158 (2009), 1536–1545. https://doi.org/10.1016/j.dam.2010.05.015

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INDEPENDENCE COMPLEXES OF STRONGLY ORDERABLE GRAPHS

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Abstract. We prove that for any finite strongly orderable (generalized strongly chordal) graph G, the independence complex $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to a wedge of spheres of dimension at least $\mathrm{bp}(G)-1$, where $\mathrm{bp}(G)$ is the biclique vertex partition number of G. In particular, we show that if G is a chordal bipartite graph, then $\mathrm{Ind}(G)$ is either contractible or homotopy equivalent to a sphere of dimension at least $\mathrm{bp}(G)-1$.

1. Introduction

An independent set in a graph is a subset of its vertices which are pairwise non-adjacent. The independence complex $\mathrm{Ind}(G)$ of a graph G is an abstract simplicial complex whose simplices correspond to independent sets of G. The topology of independence complexes of graphs has been the central subject of many papers (see, for instance [8, 9, 12, 13]). In the present work, we are mainly concerned with the homotopy type of independence complexes of strongly orderable graphs.

The class of strongly orderable graphs is firstly introduced by Dahlhaus [5] under the name "generalized strongly chordal graphs" as it constitutes a generalization of strongly chordal graphs and chordal bipartite graphs. Dragan [6] also provided vertex and edge elimination ordering characterizations of strongly orderable graphs. In our study, we benefit one of these characterizations of strongly orderable graphs, described in terms of quasi-simple vertex elimination schemes.

It turns out that the biclique vertex partititon number has a role to play in determining the homotopy type of independence complexes of strongly orderable graphs (and possibly of many other classes). Our main result is the following.

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Theorem 1. *If G is a strongly orderable graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a wedge* $\bigvee S^{d_i}$ *of spheres, where* $d_i \geq \mathrm{bp}(G) - 1$ *for each* i*.*

Denote by $\gamma(G)$, the domination number of the graph G. A simple observation shows that $\mathrm{bp}(G) = \gamma(G)$ for every graph G which does not contain C_4 as a subgraph (see, [7]). Moreover, we show that this is also the case for C_4-free graphs. This naturally helps unifying several of earlier results regarding to the homotopy type of independence complexes of graphs. Recall from [12] that the independence complex of a chordal graph G is either contractible or homotopy equivalent to a wedge of spheres of dimension at least $\gamma(G) - 1$. Since $\mathrm{bp}(G) = \gamma(G)$ for every chordal graph G, it can be said that the independence complexes of strongly orderable graphs and chordal graphs have similar topological structure.

As Theorem 1 generalizes the current characterization for homotopy type of independence complexes of strongly chordal graphs, it is further possible to achieve an improvement in the case of bipartite graphs, since we have the advantage that any bipartite graph which is strongly orderable is a chordal bipartite graph.

Theorem 2. *If G is a chordal bipartite graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere of dimension at least* $\mathrm{bp}(G) - 1$*.*

Theorem 2 also generalizes a result from [14]. In their seminal paper [14], Nagel and Reiner introduce some classes of graphs parametrized from shifted-skew shaped diagrams and determine the homotopy type of the independence complexes corresponding to these graphs via rectangular decompositions. As bipartite graphs related to such diagrams constitute a subclass of chordal bipartite graphs, we are also able to determine the homotopy type of their independence complexes.

Our paper is structured as follows: Section 2 provides the necessary background on graphs and simplicial complexes. In the subsequent section, we recall the structural properties of strongly orderable graphs and provide the characterization on the homotopy type of their independence complexes. In Section 4, we describe the bipartite graphs associated to shifted-skew diagrams from [14] and decide the homotopy type of their independence complexes.

2. Preliminaries

We start with recalling some basic notions from graph theory.

2.1. **Graphs.** All the graphs we study on are simple, i.e., do not have any loops or multiple edges. By writing $V(G)$ and $E(G)$, we mean the vertex set and the edge set of G, respectively. The edge $e := uv$ is contained in $E(G)$ if and only if u and v are adjacent in G. If $S \subset V(G)$, the graph induced by S is written $G[S]$. A graph G is said to be *H-free*, if it does not contain any induced subgraph isomorphic to H. We abbreviate $G[V \backslash S]$ to $G - S$, and write $G - x$ whenever $S = \{x\}$. The open and closed neighborhood of a vertex v are $N_G(v) = \{u \in V(G) :\ uv \in E(G)\}$ and

$N_G[v] = N_G(v) \cup \{v\}$, respectively. The cardinality of the set $N_G(v)$ is called the *degree* of the vertex v in G and denoted by $\deg_G(v)$.

A *bipartite* graph $G = (X, Y, E)$ is a graph with the vertex set $X \cup Y$ such that each of its edges is between a vertex of X and a vertex of Y. A bipartite graph $G = (X, Y, E)$ is called *convex* on Y if the vertices of Y can be ordered in such a way that the neighbours of any vertex $v \in X$ are consecutive. A bipartite graph G is called *convex bipartite* if it is convex on X or Y. If G is both convex on X and Y, then it is called *biconvex* or *doubly convex*.

Throughout, C_k denotes the cycle graph on $k \geq 3$ vertices and $K_{m,n}$ denotes the complete bipartite graph, for any $m, n \geq 1$. In particular, the complete bipartite graph $K_{1,n}$ is called a *star*. A graph is called *chordal* if it is C_k-free for $k \geq 4$. A bipartite graph is called *chordal bipartite* if it is C_k-free for $k \geq 6$.

A *biclique* in a graph G is a complete bipartite subgraph of G which is not necessarily induced. A set $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ of bicliques of a graph G is a *biclique vertex partition* of G of size k, if each vertex of G belongs to exactly one biclique in $\mathcal{B}$. Biclique vertex-partition number of a graph G, denoted by $\mathrm{bp}(G)$, is the smallest integer k such that G admits a biclique vertex-partition of size k.

A subset $S \subseteq V(G)$ is called a *dominating set* of G, if each vertex of G is either in S or adjacent to a vertex in S. The minimum size of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G.

We also use the notation $[n] := \{1, 2, \ldots, n\}$ throughout, for any integer $n \geq 1$.

Let $G = (X, Y, E)$ be a bipartite graph with $|X| = m$ and $|Y| = n$. Then the $m \times n$ matrix $A(G) = [a_{ij}]$ is called the *bipartite adjacency (biadjacency) matrix* of G, where

$$a_{ij} = \begin{cases} 1, & x_i y_j \in E(G) \\ 0, & \textit{otherwise}. \end{cases}$$

Now let $A(G)$ be the biadjacency matrix of G indexed by any ordering of X and convex ordering $Y = [y_1 < y_2 < \ldots < y_n]$. It is clear that $A(G)$ has consecutive 1's in each row (i.e, no induced submatrix [1 0 1]). Therefore, one may observe that if $A(G)$ is a biadjacency matrix of a convex bipartite graph G, then columns (or rows) of $A(G)$ can be permuted so that all the 1's in each row (or each column) appears consecutively in the resulting matrix.

2.2. **Simplicial Complexes.** An *(abstract) simplicial complex* Δ on a finite set of vertices V is a collection of subsets of V such that

(i) $\{v\} \in \Delta$ for every $v \in V$
(ii) if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of Δ are called *faces*. The dimension of a face $\sigma \in \Delta$ is $\dim(\sigma) := |\sigma| - 1$ and the dimension of Δ is $\dim(\Delta) := \max\{\dim(\sigma) : \sigma \in \Delta\}$. The *join* of two complexes Δ_1 and Δ_2 is defined by $\Delta_1 * \Delta_2 = \{\tau \cup \sigma : \tau \in \Delta_1, \sigma \in \Delta_2\}$.

In particular, the join of a simplicial complex Δ and zero-dimensional sphere $S^0 = \{\emptyset, \{a\}, \{b\}\}$ is called the *suspension* of Δ and denoted by $\Sigma\Delta = S^0 * \Delta$.

Similarly, the join of Δ and the simplicial complex $\{\emptyset, \{a\}\}$ is called the cone of Δ with apex a. A topological space is called *contractible* if its identity map is homotopic to a constant map. Note that a simplicial complex is contractible if it is a cone of another simplicial complex. It is also well-known that the suspension of a k-dimensional sphere is homotopy equivalent to $k+1$-dimensional sphere, that is, $\Sigma S^k \simeq S^{k+1}$.

One can associate to a graph G, the simplicial complex $\mathrm{Ind}(G)$, namely the *independence complex* of G, whose faces are independent sets of G.

We now provide some well-known facts from combinatorial topology, for which we use as a tool while computing the homotopy type of given graphs.

Theorem 3. *[9, 13] Let G be a simple graph. If $N_G(u) \subseteq N_G(v)$ for some distinct vertices $u, v \in V(G)$, then the homotopy equivalence $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$ holds.*

Theorem 4. *[13] If v and u are distinct vertices with $N_G[v] \subseteq N_G[u]$, then $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - u) \vee \Sigma(\mathrm{Ind}(G - N_G[u])$.*

Note that a vertex v is called *simplicial* in G if $N_G[v]$ induces a complete graph in G. If v is a simplicial vertex in the graph G, then for any $u \in N_G(v)$, we have $N_G[v] \subseteq N_G[u]$. Since v remains simplicial in the graph $G - u$, applying Theorem 4 repeatedly for each neighbor of v leads us to the following property (see also [1]).

Corollary 1. *[9] If v is a simplicial vertex of the graph G, then*

$$\mathrm{Ind}(G) \simeq \bigvee_{u \in N_G(v)} \Sigma(\mathrm{Ind}(G - N_G[u]).$$

3. Homotopy Type of Strongly Orderable Graphs

In this section, we determine the homotopy type of independence complexes of strongly orderable graphs. We start with describing strongly orderable graphs.

Definition 1. *[6] Let $\sigma : v_1, v_2, \ldots, v_n$ be an ordering of the vertices of a graph G. Then σ is called a* simplicial ordering *of G, if $i < j$, $i < k$ and $v_i v_j, v_i v_k \in E(G)$ implies that $v_i v_k \in E(G)$. On the other hand, σ is called a* strong ordering *of G, if $v_i v_j, v_i v_k, v_j v_l \in E(G)$, $i < l$ and $j < k$ implies that $v_j v_k \in E(G)$. The ordering σ is called a* strong simplicial ordering *of G if it is both strong and simplicial.*

Chordal graphs are well-known to be the class of graphs admitting a simplicial ordering [11], while graphs admitting a strong simplicial ordering are known as strongly chordal graphs, which is introduced by Farber [10].

A graph G is called a *strongly orderable* if G has a strong ordering of its vertices. Thus by definition, the class of strongly orderable graphs is a natural generalization of strongly chordal graphs. A strong simplicial ordering of a strongly chordal graph G is known to have further properties. A vertex v of G is called *simple* if $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$ for any $x, y \in N_G(v)$, that is, the closed neighborhoods of neighbors of v are linearly ordered under inclusion. Then an ordering

$\sigma : v_1, v_2, \ldots, v_n$ is called a *simple elimination ordering* if v_i is a simple vertex in the graph $G_i := G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for each $i \in [n]$. Farber [10] showed that a graph is strongly chordal if and only if it has a simple elimination ordering of its vertices. Dragan [6] gave a similar characterization for strongly orderable graphs.

Definition 2. *[6] Any two vertices u and v are said to be* comparable *in the graph G, if there holds $N_G(v)\backslash\{u\} \subseteq N_G(u)\backslash\{v\}$ or $N_G(u)\backslash\{v\} \subseteq N_G(v)\backslash\{u\}$, otherwise they are* noncomparable. *A vertex $w \in E(G)$ is called* quasi-simple *if for every $u, v \in N_G(w)$, the vertices u and v are comparable. An ordering $v_1, v_2, \ldots, v_n$ of the vertices of a graph G is called a* quasi-simple elimination ordering *if for each $i \in [n]$, the vertex v_i is a quasi-simple vertex in $G_i := G[\{v_i, v_{i+1}, \ldots, v_n\}]$.*

Theorem 5. *[6] A graph G is strongly orderable if and only if G has a quasi-simple elimination ordering.*

Now our task is to investigate the homotopy type of the independence complexes of strongly orderable graphs. In order to do that, we first need a structural property, namely hereditary property of strongly orderable graphs. We show that being a strongly orderable graph is closed under taking induced subgraphs.

Lemma 1. *If G is strongly orderable, then so is $G - x$ for any vertex x of G.*

Proof. Let $\alpha : v_1, v_2, \ldots, v_n$ be a strong ordering of the vertices of G. We show that removal of any vertex x from G still preserves the ordering. The case when $x = v_1$ is clear. Thus we assume that $x = v_i$ for some $i \in \{2, 3, \ldots, n\}$ and consider the graph $G^i := G - v_i$. We claim that the ordering $\beta : v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ is a strong ordering of $G - v_i$. For every $s \in \{2, 3, \ldots, n\}\backslash\{i\}$, we need to verify that the vertex v_s is quasi-simple in the graph $G^i_s := G^i\backslash\{v_1, \ldots, v_{s-1}\}$. Firstly, if $s > i$, then it is straightforward because of the strong ordering α of G. Now let $s < i$ and assume on the contrary that v_s is not quasi-simple. Then for some $v_k, v_l \in N_{G^i_s}(v_s)$ with $s < k < l$, the vertices v_k and v_l must be noncomparable in G^i_s, while they are comparable in G_s. Therefore the set $N_{G^i_s}(v_k)\backslash N_{G_s}(v_l)$ must contain a vertex v_r with $r > s$ and $r \neq i$. However, this is a contradiction, since $N_{G_s}(v_k)\backslash\{v_l\} \subseteq N_{G_s}(v_l)\backslash\{v_k\}$ because of the ordering α of G. □

Lemma 2. *[4] Let G be a graph. If $N_G(u) \subseteq N_G(v)$ for some distinct vertices $u, v \in V(G)$, then $\text{bp}(G) \leq \text{bp}(G - v)$ holds.*

Proof. Let $\{B_1, B_2, \ldots, B_k\}$ be a biclique vertex partitioning of $G - v$. Assume without loss of generality that $u \in B_1$. Then observe that $\{B_1 \cup \{v\}, B_2, \ldots, B_k\}$ is a biclique vertex partitioning of G. □

Remark 1. *The inequality $\text{bp}(G) \geq \text{bp}(G - v)$ is not true in general. For the graph G in Figure 1, we have $\text{bp}(G) = 2 < \text{bp}(G - v) = n + 1$ while $N_G(u_i) \subseteq N_G(v)$ for each $i \in [n]$.*

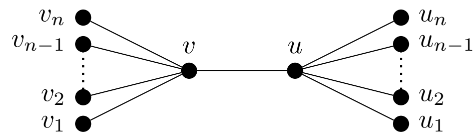


FIGURE 1. A graph G such that $\mathrm{bp}(G) = 2 < \mathrm{bp}(G - v) = n + 1$

Proof of Theorem 1. We use induction on the number of vertices. The theorem is trivial if G has fewer than 3 vertices. Let G be a strongly orderable graph and $\sigma : v_1, v_2, \ldots, v_n$ be a quasi-simple elimination ordering of the vertices of G. Suppose that the theorem is true for the graphs with fewer than n vertices. Since v_1 is quasi-simple in G, all of its neighbors are comparable. First assume that v_1 have two neighbors v_i, v_j (with $i < j$) which are not adjacent to each other. In this case, we must have $N_G(v_i) \subseteq N_G(v_j)$, since v_i and v_j are comparable in G. Then it follows that $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-y)$ by Theorem 3. Therefore, if $\mathrm{Ind}(G-y)$ is contractible, then $\mathrm{Ind}(G)$ is also contractible. If $\mathrm{Ind}(G-y)$ is not contractible, then by induction hypothesis, $\mathrm{Ind}(G-y) \simeq \bigvee S^{d_i}$, where $d_i \geq \mathrm{bp}(G-y) - 1$ for each i. Since $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G-y)$ and $bp(G-y) - 1 \geq bp(G) - 1$ by Lemma 2, we obtain that $\mathrm{Ind}(G) \simeq \bigvee S^{d_i}$, where $d_i \geq \mathrm{bp}(G) - 1$. Thus we may further assume that all the neighbors of v_1 are pairwise adjacent. Notice that the vertex v_1 is simplicial in such a case. Following Theorem 4, we have

$$\mathrm{Ind}(G) \simeq \bigvee_{u \in N_G(v_1)} \Sigma(\mathrm{Ind}(G - N_G[u])) \qquad (*)$$

Recall that $G - N_G[u]$ is a strongly orderable graph for each $u \in N_G(v_1)$, by Lemma 1. By the induction hypothesis, we know that for each $u \in N_G(v_1)$, the complex $\mathrm{Ind}(G - N_G[u])$ is either contractible or homotopy equivalent to a wedge sum $\bigvee S^{d_u}$ of spheres, where $d_u \geq \mathrm{bp}(G - N_G[u]) - 1$. Now, if $\mathrm{Ind}(G - N_G[u])$ is contractible for each $u \in N_G(v_1)$, then so is $\mathrm{Ind}(G)$. Therefore, we let u be an arbitrary neighbor of v_1 such that $\mathrm{Ind}(G - N_G[u]) \simeq \bigvee S^{d_u}$. Then it follows that $\Sigma(\mathrm{Ind}(G - N_G[u])) \simeq \bigvee S^{d_u+1}$. For any biclique vertex partition $\{B_1, B_2, \ldots, B_k\}$ of $G - N_G[u]$, observe that the collection $\{N_G[u], B_1, B_2, \ldots, B_k\}$ is a biclique partiton of G, which implies that $\mathrm{bp}(G) \leq \mathrm{bp}(G - N_G[u]) + 1$. Thus we have $d_u + 1 \geq \mathrm{bp}(G - N_G[u]) \geq \mathrm{bp}(G) - 1$. Hence the theorem follows from $(*)$. □

As every strongly chordal graph is strongly orderable, we have the following.

Corollary 2. *If G is a strongly chordal graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a wedge* $\bigvee S^{d_i}$ *of spheres, where* $d_i \geq \mathrm{bp}(G) - 1$ *for each* i.

Note that Corollary 2 is well-known since every strongly chordal graph is a chordal graph and the biclique vertex partition number coincides with the domination number on C_4-free graphs. Although it is not hard to see, we include its proof for the completeness.

Proposition 1. *If G is a C_4-free graph, then* $\text{bp}(G) = \gamma(G)$.

Proof. If S is a dominating set of G, then it is clear that $V(G)$ can be partitioned into stars each of which has its center from S, thus $\text{bp}(G) \leq \gamma(G)$.

Conversely, let $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ be a biclique partition of G. We claim that for each $i \in [k]$, the subgraph B_i has at least one vertex v_i which is adjacent all other vertices in B_i so that $\{v_i : i \in [k]\}$ is a dominating set in G. Let $B_i \in \mathcal{B}$ be an arbitrary biclique of G, with the partitioning $X_i \cup Y_i$. Note that X_i and Y_i need not to be independent. The claim is trivial if $|X_i| = 1$ or $|Y_i| = 1$. Thus we let $\min\{|X_i|, |Y_i|\} \geq 2$ and assume on the contrary that there is no such vertex in B_i. This forces that there exist some vertices $x_{i_1}, x_{i_2} \in X_i$ and $y_{i_1}, y_{i_2} \in Y_i$ such that x_{i_1} and x_{i_2} (resp. y_{i_1} and y_{i_2}) are nonadjacent in G. This is a contradiction, since the set $\{x_{i_1} x_{i_2}, y_{i_1}, y_{i_2}\}$ induces a C_4 in G. This completes the proof. □

Since chordal graphs are C_4-free, Proposition 1 allows us interpret the homotopy type of independence complexes of chordal graphs in terms of biclique vertex partition number, when they are homotopy equivalent to a wedge sum of spheres. Recall from [12] that if the complex $\text{Ind}(G)$ of a chordal graph G is homotopy equivalent to a wedge of spheres, then each of the spheres has dimension at least $\gamma(G) - 1$. Hence, Proposition 1 helps us unify the results for chordal and strongly orderable graphs.

Remark 2. *It is known that chordal graphs are vertex-decomposable, since they are codismantlable [3]. Therefore, the homotopy type of independence complexes of chordal graphs can also be inferred from vertex-decomposability [2]. However, unlike the class of chordal graphs, strongly orderable graphs are not vertex-decomposable. C_4 is an easy example of chordal bipartite (thus a strongly orderable) graph which is not vertex decomposable.*

Strongly orderable bipartite graphs coincide with the class of chordal bipartite graphs. Any quasi-simple vertex turns out to be a weak simplicial vertex in a chordal bipartite graph. A vertex x in G is said to be a *weak simplicial* if for any $u, v \in N_G(x)$, either $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$ holds [15]. This leads to a refinement of our main result on chordal bipartite graphs.

Lemma 3. *Every connected chordal bipartite graph with more than one edge has a pair x, y of vertices such that $N_G(x) \subseteq N_G(y)$.*

Proof. Let G be a chordal bipartite graph with more than one edge and let v be a weak simplicial vertex of G. First assume that $\deg_G(v) = 1$ and let $N_G(v) = \{w\}$. Then for every $u \in N_G(w) \backslash \{v\}$, we have $N_G(v) \subseteq N_G(u)$. If $\deg_G(v) \geq 2$, then any two neighbors of u form such a pair. □

Theorem 6. *If G is a chordal bipartite graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere of dimension at least* $\mathrm{bp}(G) - 1$.

Proof. Once again, we use the induction on the number of the vertices. Let G be a bipartite graph. We may assume that G has a component with more than one edge, since otherwise the claim is clear. Let H be such component of G. By Lemma 3, H has a pair of vertices x, y such that $N_G(x) \subseteq N_G(y)$. It follows that $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$, by Theorem 3. By induction hypothesis, the subcomplex $\mathrm{Ind}(G - v)$ is either contractible or homotopy equivalent to a sphere of dimension at least $\mathrm{bp}(G - v) - 1$. If $\mathrm{Ind}(G - v)$ is contractible, then so is $\mathrm{Ind}(G)$. Assume further that $\mathrm{Ind}(G - v)$ is homotopy equivalent to a sphere of dimension at least $\mathrm{bp}(G - v) - 1$. Since $\mathrm{bp}(G - v) \geq \mathrm{bp}(G)$ by Lemma 2 and $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - v)$, the complex $\mathrm{Ind}(G)$ is homotopy equivalent to a sphere of dimension of at least $bp(G) - 1$. □

We also have the following corollary, since every convex bipartite graph is a chordal bipartite graph.

Corollary 3. *If $G = (X, Y, E)$ is a convex bipartite graph, then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere.*

4. Bipartite Graphs Related to Shifted-Skew Diagrams

In [14], Nagel and Reiner introduced graph classes associated to shifted-skew shaped diagrams. They also compute the homotopy type of such constructed graphs. In the case of bipartite graphs, our results from previous section generalize the mentioned classification. We first provide the necessary background about these diagrams and then conclude the homotopy type of independence complexes of bipartite graphs corresponding such diagrams. For more detailed description of shifted-skew shapes, we refer to [14].

Definition 3. *[14] A* shifted diagram *is an interpretation of the lattice points $\{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i < j\}$ by replacing each point with unit squares/cells where the first coordinate i (row index) increases from top to the bottom and the second coordinate j (column index) increases from left to the right, as in matrices.*

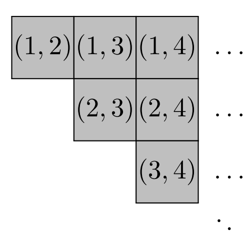
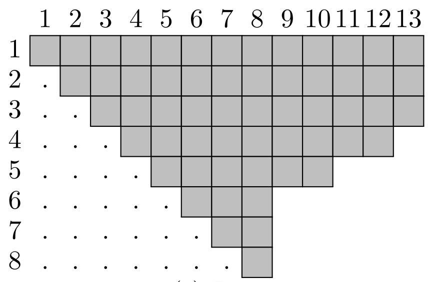


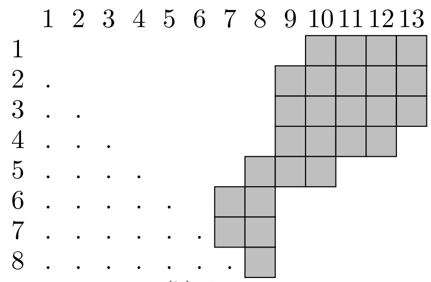
Figure 2. A shifted diagram

A *shifted Ferrers diagram* D_λ with respect to the strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ (where $\lambda_1 > \lambda_2 > \ldots > \lambda_t > 0$) is a finite shifted diagram consisting of λ_i cells in the row i. For instance, given partition $\lambda = (13, 12, 11, 9, 6, 3, 2, 1)$, corresponding diagram is depicted in the Figure 3-(a).

Now let λ and μ are such partitions with $\lambda \subseteq \mu$, that is $\mu_i \leq \lambda_i$ for all i, and possibly μ has less number of parts than λ has. Then one can form the *shifted skew diagram* $D := D_{\lambda/\mu}$ by removing the diagram D_μ from the diagram D_λ. An example with partitions $\lambda = (13, 12, 11, 9, 6, 3, 2, 1)$ and $\mu = (9, 7, 6, 5, 3, 1)$ given below (compare Figure 3-(a) with Figure 3-(b)).



(a) D_λ



(b) $D_{\lambda/\mu}$

FIGURE 3. A shifted Ferrers diagram D_λ and shifted skew diagram $D_{\lambda/\mu}$.

For any shifted skew diagram and linearly ordered subsets $X = \{x_1 < x_2 < \ldots < x_m\}$ and $Y = \{y_1 < y_2 < \ldots < y_n\}$ of positive integers, let $D_{X,Y}$ denote the diagram consisting of cells in the position (i, j) whenever the cell (x_i, x_j) is present in D, i.e., we restrict the diagram D to the rows indexed by X and columns indexed by Y. For instance, if we set $X = \{x_1, x_2, x_3, x_4, x_5\} = \{1, 3, 5, 6, 7\}$ and $Y = \{y_1, y_2, y_3, y_4\} = \{8, 9, 11, 13\}$ for the diagram $D := D_{\lambda/\mu}$ in Figure 3, the corresponding diagram $D_{X,Y}$ is drawn as in the Figure 4.

Given shifted-skew diagram $D_{X,Y}$, Nagel and Reiner [14] define the bipartite graph $G(D_{X,Y}) = (X, Y; E)$ on the vertex $X \cup Y = \{x_1, x_2, \ldots x_m\} \cup \{y_1, y_2, \ldots, y_n\}$

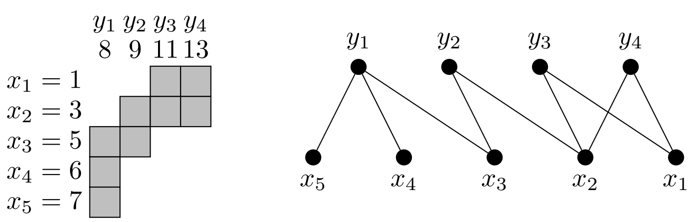


FIGURE 4. Shifted skew diagram $D_{X,Y}$ and the bipartite graph $G(D_{X,Y})$.

such that $(x_i, y_j) \in E(G)$ if the cell (i, j) is present in $D_{X,Y}$. One may observe that there is a one-to-one correspondence between the biadjacency matrix and the diagram of the graph $G(D_{X,Y})$, such that the cells in the diagram corresponds to 1's in the matrix (see, Figure 5).

$$A(G(D_{X,Y})) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 5. Biadjacency matrix of $G(D_{X,Y})$

Consequently, we deduce that all the 1's in each row (and each column) are consecutive, which in turn implies that the graph $G(D_{X,Y})$ is a doubly convex graph. Note that this fact is independent from the choice of the sets X and Y. In fact, the bipartite graph $G(D)$ corresponding to the diagram $D := D_{\lambda/\mu}$ is clearly a convex bipartite graph. Therefore, the choice of the sets X and Y will determine an induced subgraph $G(D_{X,Y})$ of $G(D)$ which is again convex bipartite. Hence the following fact is an immediate consequence of Corollary 3.

Corollary 4. *[14] Let $G(D_{X,Y})$ is the bipartite graph associated to a shifted-skew diagram $D_{X,Y}$. Then the complex $\mathrm{Ind}(G(D_{X,Y}))$ is either contractible or homotopy equivalent to a sphere.*

The same argument can be further applied to a bipartite graph $G(D)$ parametrized (in the similar fashion) from any diagram D whose cells in each row (or each column) appear consecutively, since $G(D)$ is a convex bipartite graph.

5. Conclusion

In our study, we characterize the homotopy type of the independence complexes of strongly orderable graphs. We further refine the mentioned characterization in the case of chordal bipartite graphs. These characterizations extend several known results and unify them in terms of biclique vertex partitions. There is, however, a natural question which arises in this context: "For which classes of graphs, the biclique vertex partitions is also relevant to the topology of independence complexes?".

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References

[1] Adamaszek, M., Splittings of independence complexes and the powers of cycles, *J. Comb. Theory Ser. A.*, 119(5) (2012), 1031–1047. https://doi.org/10.1016/j.jcta.2012.01.009

[2] Björner, A., Wachs, M., Shellable nonpure complexes and posets. I, *Trans. Amer. Math. Soc.*, 348(4) (1996), 1299–1327. https://doi.org/10.1090/s0002-9947-96-01534-6

[3] Bıyıkoğlu, T., Civan, Y., Vertex decomposable graphs, codismantlability, Cohen–Macaulayness and Castelnuovo–Mumford regularity, *Electron. J. Combin.*, 21(1) (2014). https://doi.org/10.37236/2387

[4] Civan, Y., Deniz, Z., Yetim, M. A., Chordal bipartite graphs, biclique vertex partitions and Castelnuovo-Mumford regularity of 1-subdivison graphs, Manuscript in preparation (2021).

[5] Dahlhaus, E., Generalized strongly chordal graphs, Technical Report, Basser Department of Computer Science, University of Sydney, 1993, 15 pp.

[6] Dragan, F. F., Strongly orderable graphs A common generalization of strongly chordal and chordal bipartite graphs, *Discrete Appl. Math.*, 99 (1-3) (2000), 427–442. https://doi.org/10.1016/S0166-218X(99)00149-3

[7] Duginov, O., Partitioning the vertex set of a bipartite graph into complete bipartite subgraphs, *Discrete Math. Theor. Comput. Sci.*, 16(3) (2014), 203–214. https://doi.org/10.46298/dmtcs.2090

[8] Ehrenborg, R., Hetyei, G., The topology of the independence complex, *European J. Combin.*, 27(6) (2006), 906–923. https://doi.org/10.1016/j.ejc.2005.04.010

[9] Engström, A., Complexes of directed trees and independence complexes, *Discrete Math.*, 309 (2009), 3299–3309. https://doi.org/10.1016/j.disc.2008.09.033

[10] Farber, M., Characterizations of strongly chordal graphs, *Discrete Math.*, 43(2-3) (1983),173–189. https://doi.org/10.1016/0012-365X(83)90154-1

[11] Golumbic, M. C., Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.

[12] Kawamura, K., Independence complexes of chordal graphs, *Discrete Math.*, 310 (2009), 2204–2211. https://doi.org/10.1016/j.disc.2010.04.021

[13] Marietti, M., Testa, D., A uniform approach to complexes arising from forests, *Electron. J. Comb.*, 15 (2008). https://doi.org/10.37236/825

[14] Nagel, U., Reiner, V., Betti numbers of monomial ideals and shifted skew shapes, *Electron. J. Comb.*, 16(2) (2009). https://doi.org/10.37236/69

[15] Uehara, R., Linear time algorithms on chordal bipartite and strongly chordal graphs, In: Automata, Languages and Programming, Lecture Notes in Computer Science, Volume 2380, Springer, 2002, pp. 993–1004. https://doi.org/10.1007/3-540-45465-9_85

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QUANTUM ANALOG OF SOME TRAPEZOID AND MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper a new quantum analog of Hermite-Hadamard inequality is presented, and based on it, two new quantum trapezoid and midpoint identities are obtained. Moreover, the quantum analog of some trapezoid and midpoint type inequalities are established.

1. Introduction

A function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval J, if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in J$ and $t \in [0, 1]$.

One of the most useful inequalities for convex functions is Hermite-Hadamard's inequality, due to its geometrical importance and applications, which is described as follows:

Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval of real numbers and $a, b \in J$ with $a < b$. Then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Hermite-Hadamard's inequality is investigated for several classes of functions in a number of papers and different types of inequalities have been obtained from it. For more details, see [16, 18, 21, 22, 28] and references therein.

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In recent years Tariboon and Ntouyas in [31], generalized the classic quantum derivative and integral. Also, in [23], the authors by using the notions of left and right quantum derivative and integral have derived a similar generalization of classic quantum derivative and integral. In many research papers, the quantum analogue of Hermite-Hadamard type inequalities has been obtained via the generalized form of quantum integral, which were given in [31]. For more information in this regard, the reader is refer to [1]- [6], [8]- [15], [17], [19], [24]- [27], [29]- [36]

In this paper, we use the notions of left and right quantum derivatives and integrals together to introduce a new quantum analogue of Hermite-Hadamard inequality and based on it we obtain two new quantum trapezoid and midpoint type identities. In addition by using these new identities, we establish quantum analogue of some trapezoid and midpoint inequalities. We get the results of the trapezoid and midpoint inequalities as a special case when $q \to 1$. The idea and techniques of this paper may help the interested reader for further research in this area.

2. Preliminaries

In this section, we recall some previously known concepts.

In [31], Tariboon and Ntouyas introduced the concepts of quantum derivative and definite quantum integral for the functions of defined on an arbitrary finite intervals as follows:

Definition 1. *[31] A function $f(t)$ defined on $[a,b]$ is called quantum differentiable on $(a,b]$ with the following expression:*

$$ {}_aD_q f(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)} \in \mathbb{R}, \quad t \neq a, \tag{2}$$

and quantum differentiable on $t = a$, if the following limit exists:

$$ {}_aD_q f(a) = \lim_{t \to a^+} {}_aD_q f(t) ,$$

for any $a < b$.

Clearly, if $a = 0$ in (2), then ${}_0D_q f(t) = D_q f(t)$ where $D_q f(t)$ is familiar quantum derivative of the function f defined by

$$D_q f(t) := \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \; D_q f(0) = \lim_{t \to 0} D_q f(t). \; [7,20] \tag{3}$$

Definition 2. *[31] Let a function f be defined on $[a,b]$. Then the quantum integral of f on $[a,b]$ is defined by*

$$\int_a^b f(t) \; {}_a d_q t = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n) a). \tag{4}$$

If the series in the right-hand side of (4) converges, then f is said to be quantum integrable on $[a,b]$. Also, for any $c \in (a,b)$

$$\int_c^b f(t) \ {}_a d_q t = \int_a^b f(t) \ {}_a d_q t - \int_a^c f(t) \ {}_a d_q t . \tag{5}$$

Clearly, if $a = 0$ in (5), then

$$\int_0^b f(t) \ {}_0 d_q t = \int_0^b f(t) \ d_q t ,$$

where $\int_0^b f(t) \ d_q t$ is well-known Jackson integral of f on $[0,b]$. For more details, see [7, 20].

In [23], the authors have denoted (2) and (4) respectively as left quantum derivative and definite left quantum integral and it has been written in this wise:

$${}_a D_q f(t) = {}_{a^+} D_q f(t),$$

$$\int_a^b f(t) \ {}_a d_q t = \int_a^b f(t) \ {}_{a^+} d_q t .$$

We use these notations in the rest of the paper.

Lemma 1. *[31] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then we have*

$$\lim_{q \to 1^-} {}_{a^+} D_q f(t) = \frac{df(t)}{dt}. \tag{6}$$

Lemma 2. *[31] Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\, dt$ is exist, then we have*

$$\lim_{q \to 1^-} \int_a^b f(t) \ {}_{a^+} d_q t = \int_a^b f(t) \ dt. \tag{7}$$

Recently, Kunt et al. [23], presented the notions of right quantum derivative and right definite quantum integral as follows:

Definition 3. *[23] A function $f(t)$ defined on $[a,b]$ is called the right quantum differentiable on $[a,b)$ with the following expression:*

$${}_{b^-} D_q f(t) = \frac{f(t) - f(qt + (1-q)b)}{(1-q)(t-b)} \in \mathbb{R}, \quad t \neq b, \tag{8}$$

and quantum differentiable on $t = b$, if the following limit exists:

$${}_{b^-} D_q f(b) = \lim_{t \to b^-} {}_{b^-} D_q f(t) ,$$

for any $a < b$.

Definition 4. *[23] Let a function f be defined on $[a,b]$. Then the right quantum integral of f on $[a,b]$ is defined by*

$$\int_a^b f(t)\ {}_{b^-}d_qt = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n) b). \tag{9}$$

If the series in right hand side of (9) converges, then f is said to be right quantum integrable on $[a,b]$. Also, for any $c \in (a,b)$

$$\int_a^c f(t)\ {}_{b^-}d_qt = \int_a^b f(t)\ {}_{b^-}d_qt - \int_c^b f(t)\ {}_{b^-}d_qt\ . \tag{10}$$

Lemma 3. *[23] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then we have*

$$\lim_{q\to 1^-}\ {}_{b^-}D_q f(t) = \frac{df(t)}{dt}. \tag{11}$$

Lemma 4. *[23] Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\ dt$ is exist, then we have*

$$\lim_{q\to 1^-}\int_a^b f(t)\ {}_{b^-}d_qt = \int_a^b f(t)\ dt. \tag{12}$$

3. Auxiliary Results

In this section, we describe some auxiliary lemmas which are used in the obtaining of main results.

Lemma 5. *Let $0 < q < 1$ be a constant, then the following equality holds:*

$$\int_0^1 |1-qt|\, t\ {}_{0^+}d_qt = \frac{1}{(1+q)(1+q+q^2)}. \tag{13}$$

Proof. By using the definition of q-integral, we have

$$\begin{aligned}\int_0^1 |1-qt|\, t\ {}_{0^+}d_qt &= \int_0^1 (1-qt)\, t\ {}_{0^+}d_qt \\ &= \int_0^1 t\ {}_{0^+}d_qt - \int_0^1 qt^2\ {}_{0^+}d_qt \\ &= \frac{1}{1+q} - \frac{q}{1+q+q^2} \\ &= \frac{1}{(1+q)(1+q+q^2)}.\end{aligned}$$

The proof is completed. □

Lemma 6. *Let $0 < q < 1$ be a constant, then the following equality holds:*

$$\int_0^1 |1-qt|\,(1-t)\ {}_{0^+}d_qt = \frac{q}{1+q+q^2}. \tag{14}$$

Proof. By using Lemma (5) and the definition of q-integral, we have

$$\begin{aligned}\int_0^1 |1-qt|\,(1-t)\ \ {}_{0^+}d_qt &= \int_0^1 |1-qt|\ \ {}_{0^+}d_qt\ - \int_0^1 |1-qt|\,t\ \ {}_{0^+}d_qt \\ &= \int_0^1 (1-qt)\ \ {}_{0^+}d_qt\ - \frac{1}{(1+q)(1+q+q^2)} \\ &= \frac{1}{1+q} - \frac{1}{(1+q)(1+q+q^2)} \\ &= \frac{q}{1+q+q^2}.\end{aligned}$$

The proof is completed. □

Lemma 7. *Let $0<q<1$ be a constant, then the following equality holds:*

$$\int_0^1 t\,(1-t)\ \ {}_{0^+}d_qt\ = \frac{q^2}{(1+q)(1+q+q^2)}. \tag{15}$$

Proof. By using the definition of q-integral, we have

$$\begin{aligned}\int_0^1 t\,(1-t)\ \ {}_{0^+}d_qt &= \int_0^1 t\ \ {}_{0^+}d_qt\ - \int_0^1 t^2\ \ {}_{0^+}d_qt \\ &= \frac{1}{1+q} - \frac{1}{1+q+q^2} \\ &= \frac{q^2}{(1+q)(1+q+q^2)}.\end{aligned}$$

The proof is completed. □

Lemma 8. *Let $0<q<1$ be a constant, then the following equality holds:*

$$\int_0^1 t^p\ \ {}_{0^+}d_qt\ = \frac{1-q}{1-q^{p+1}}. \tag{16}$$

Proof. By using the definition of q-integral, we have

$$\begin{aligned}\int_0^1 t^p\ \ {}_{0^+}d_qt &= (1-q)\sum_{n=0}^{\infty} q^n\,(q^n)^p \\ &= (1-q)\sum_{n=0}^{\infty} \left(q^{p+1}\right)^n \\ &= \frac{1-q}{1-q^{p+1}}.\end{aligned}$$

The proof is completed. □

4. Main Results

In this section, we use Definition 2 together with Definition 4 of the quantum integrals to present a new quantum form of the Hermite-Hadamard inequality.

Let a function f be defined on $[a,b]\subset\mathbb{R}$, then from (4) and (9), we can write

$$\begin{aligned}&\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\\ =&\ \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[f\left(q^n\left(\frac{b-a}{2}\right)+a\right)+f\left(q^n\left(\frac{a-b}{2}\right)+b\right)\right] \quad (17)\end{aligned}$$

For shortness we write the left-hand side of (17), as follows:

$$\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ := \int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\ .$$

If the series in the right-hand side of (17) converges or f is left quantum integrable on $\left[a,\frac{a+b}{2}\right]$ and right quantum integrable on $\left[\frac{a+b}{2},b\right]$, then $\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt$ is exist.

Lemma 9. *Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\ dt$ converges, then we have*

$$\lim_{q\to1^-}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ =\int_a^b f(t)\ dt. \quad (18)$$

Proof. By using (17) and lemma 2 and lemma 4, we get

$$\begin{aligned}\lim_{q\to1^-}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ &=\lim_{q\to1^-}\left[\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\ \right]\\ &=\lim_{q\to1^-}\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\lim_{q\to1^-}\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\\ &=\int_a^{\frac{a+b}{2}} f(t)\,dt+\int_{\frac{a+b}{2}}^b f(t)\,dt\\ &=\int_a^b f(t)\,dt.\end{aligned}$$

The proof is completed. □

Lemma 10. *Let $\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt$ be exist. Then we have*

$$\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ =\int_a^b f(a+b-t)\ {}_{\frac{a+b}{2}}d_qt\ . \quad (19)$$

Proof. By direct computing from (17), we get

$$\int_a^b f(a+b-t)\ {}_{\frac{a+b}{2}}d_qt$$

$$= \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[\begin{array}{c} f\left(a+b-\left(q^n\left(\frac{b-a}{2}\right)+a\right)\right) \\ +f\left(a+b-\left(q^n\left(\frac{a-b}{2}\right)+b\right)\right)\end{array}\right]$$

$$= \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[f\left(b-q^n\left(\frac{b-a}{2}\right)\right)+f\left(a-q^n\left(\frac{a-b}{2}\right)\right)\right]$$

$$= \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[f\left(q^n\left(\frac{a-b}{2}\right)+b\right)+f\left(q^n\left(\frac{b-a}{2}\right)+a\right)\right]$$

$$= \int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ .$$

This complete the proof. □

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $0<q<1$. Then we have*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \leq\frac{f(a)+f(b)}{2}. \tag{20}$$

Proof. Clearly $\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt$ is exist. By using (17), we have

$$\int_0^1 f(ta+(1-t)b)\ {}_{\frac{0+1}{2}}d_qt$$

$$= \frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty}q^n\left[\begin{array}{c} f\left(\left[q^n\left(\frac{1-0}{2}\right)+0\right]a+\left[1-\left(q^n\left(\frac{1-0}{2}\right)+0\right)\right]b\right) \\ +f\left(\left[q^n\left(\frac{0-1}{2}\right)+1\right]a+\left[1-\left(q^n\left(\frac{0-1}{2}\right)+1\right)\right]b\right)\end{array}\right]$$

$$= \frac{1}{b-a}\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[f\left(\frac{q^n(a-b)}{2}+b\right)+f\left(\frac{q^n(b-a)}{2}+a\right)\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ , \tag{21}$$

and

$$\int_0^1 f(tb+(1-t)a)\ {}_{\frac{0+1}{2}}d_qt$$

$$= \frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty}q^n\left[\begin{array}{c} f\left(\left[q^n\left(\frac{1-0}{2}\right)+0\right]b+\left[1-\left(q^n\left(\frac{1-0}{2}\right)+0\right)\right]a\right)+ \\ f\left(\left[q^n\left(\frac{0-1}{2}\right)+1\right]b+\left[1-\left(q^n\left(\frac{0-1}{2}\right)+1\right)\right]a\right)\end{array}\right]$$

$$= \frac{1}{b-a}\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n\left[f\left(\frac{q^n(b-a)}{2}+a\right)+f\left(\frac{q^n(a-b)}{2}+b\right)\right]$$

$$= \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ . \tag{22}$$

Again by applying (17), we get

$$\int_0^1 f\left(\frac{a+b}{2}\right)\ {}_{\frac{0+1}{2}}d_qt\quad = \quad f\left(\frac{a+b}{2}\right)\int_0^1 1\ {}_{\frac{0+1}{2}}d_qt$$

$$
\begin{aligned}
&= f\left(\frac{a+b}{2}\right)\frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty}q^n[1+1] \\
&= f\left(\frac{a+b}{2}\right), \qquad (23)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 \frac{f(a)+f(b)}{2} \; {}_{\frac{0+1}{2}}d_qt &= \frac{f(a)+f(b)}{2}\int_0^1 1 \; {}_{\frac{0+1}{2}}d_qt \\
&= \frac{f(a)+f(b)}{2}\frac{(1-q)(1-0)}{2}\sum_{n=0}^{\infty}q^n[1+1] \\
&= \frac{f(a)+f(b)}{2}. \qquad (24)
\end{aligned}
$$

Since f is convex on $[a,b]$, then we can write

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}[f(ta+(1-t)b)+f(tb+(1-t)a)] \le \frac{f(a)+f(b)}{2},$$

for all $t \in [0,1]$, and

$$
\begin{aligned}
&\int_0^1 f\left(\frac{a+b}{2}\right) \; {}_{\frac{0+1}{2}}d_qt \\
\le &\int_0^1 \left(\frac{1}{2}[f(ta+(1-t)b)+f(tb+(1-t)a)]\right) \; {}_{\frac{0+1}{2}}d_qt \\
\le &\int_0^1 \frac{f(a)+f(b)}{2} \; {}_{\frac{0+1}{2}}d_qt \; .
\end{aligned}
$$

Therefore by using (21),(22), (23) and (24), we get

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t) \; {}_{\frac{a+b}{2}}d_qt \; \le \frac{f(a)+f(b)}{2}.$$

This complete the proof. □

Remark 1. *If $q \to 1$, then by using Lemma 9, the inequality (20)reduce to (1).*

Lemma 11. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function. If ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, then the following identity holds:*

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t) \; {}_{\frac{a+b}{2}}d_qt \; - \frac{f(a)+f(b)}{2} \\
= &\frac{b-a}{4}\int_0^1 (1-qt)\left(\begin{array}{c} {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ - {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{array}\right) {}_{0^+}d_qt \; . \qquad (25)
\end{aligned}
$$

Proof. Since ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)$ are left quantum integrable on $[0,1]$, using the linearity of left quantum integral, then we have

$$\begin{aligned}
&\frac{b-a}{4}\int_0^1(1-qt)\begin{pmatrix} {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right) \\ -\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\end{pmatrix}\ {}_{0^+}d_qt \\
&= \frac{b-a}{4}\left[\int_0^1(1-qt)\ {}_{a^+}D_qf\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_qt\right. \\
&\left. - \int_0^1(1-qt)_{b^-}\ D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\ {}_{0^+}d_qt\ \right] \\
&= \frac{b-a}{4}\left[M_1-M_2\right].
\end{aligned} \tag{26}$$

Since f is continuous on $[a,b]$, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)-\sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right) \\
= \quad & f\left(\frac{a+b}{2}\right)-f(a),
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)-\sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right) \\
= \quad & f\left(\frac{a+b}{2}\right)-f(b).
\end{aligned} \tag{28}$$

Using (27), we achieve

$$\begin{aligned}
M_1 &= \int_0^1(1-qt)\ {}_{a^+}D_qf\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_qt \\
&= \int_0^1 {}_{a^+}D_qf\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_qt \\
&- q\int_0^1 t\ {}_{a^+}D_qf\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_qt \\
&= \int_0^1\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,a\right)}{(1-q)\left(\frac{a+b}{2}-a\right)t}\right]\ {}_{0^+}d_qt \\
&- q\int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,a\right)}{(1-q)\left(\frac{a+b}{2}-a\right)t}\right]\ {}_{0^+}d_qt \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right)\Bigg] \\
&-\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)\right. \\
&\left.-\sum_{n=0}^{\infty} q^{n} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right)\right] \\
&=\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(a)\right] \\
&-\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)\right. \\
&\left.-\frac{1}{q}\sum_{n=1}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)\right] \\
&=\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(a)\right] \\
&-\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)\right. \\
&\left.-\frac{1}{q}\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)+\frac{1}{q} f\left(\frac{a+b}{2}\right)\right] \\
&=\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(a)\right]-\frac{2}{b-a} f\left(\frac{a+b}{2}\right) \\
&-\frac{2q}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)\right] \\
&=\frac{4}{(b-a)^{2}}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^{n} f\left(q^{n}\left(\frac{a+b}{2}\right)+\left(1-q^{n}\right)a\right)\right] \\
&-\frac{4}{b-a}\frac{f(a)}{2} \\
&=\frac{4}{(b-a)^{2}}\int_{a}^{\frac{a+b}{2}} f(t)\ {}_{a^{+}}d_{q}t\ -\frac{4}{b-a}\frac{f(a)}{2}. \qquad (29)
\end{aligned}
$$

Similarly, using (28) 3, we get

$$
M_{2} = \int_{0}^{1}(1-qt)\ {}_{b^{-}}D_{q} f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)b\right)\ {}_{0^{+}}d_{q}t
$$

$$
\begin{aligned}
&= \int_0^1 {}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,b\right) \;{}_{0^+}d_q t \\
&\quad -q\int_0^1 t\; {}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)\,b\right) \;{}_{0^+}d_q t \\
&= \int_0^1 \left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,b\right)}{(1-q)\left(\frac{a+b}{2}-b\right)t}\right] \;{}_{0^+}d_q t \\
&\quad -q\int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,b\right)}{(1-q)\left(\frac{a+b}{2}-b\right)t}\right] \;{}_{0^+}d_q t \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty} f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\sum_{n=0}^{\infty} f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right)\right] \\
&\quad +\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\sum_{n=0}^{\infty} q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right)\right] \\
&= -\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right] \\
&\quad +\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right] \\
&= -\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right] \\
&\quad +\frac{2q}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right. \\
&\quad \left. -\frac{1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right] \\
&= -\frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right]+\frac{2}{b-a}f\left(\frac{a+b}{2}\right) \\
&\quad +\frac{2q}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,b\right)\right]
\end{aligned}
$$

$$= \frac{4}{b-a}\frac{f(b)}{2}$$
$$-\frac{4}{(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty}q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right]$$
$$= \frac{4}{b-a}\frac{f(b)}{2}-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} f(t)\ \ {}_{b^-}d_qt\ . \tag{30}$$

Combining (26), (29) and (30) , we obtain

$$\frac{b-a}{4}[M_1-M_2]$$
$$=\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(t)\ \ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^{b} f(t)\ \ {}_{b^-}d_qt\right)-\frac{f(a)+f(b)}{2}$$
$$=\frac{1}{b-a}\int_a^b f(t)\ \ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2},$$

which gives (25). This complete the proof. □

Remark 2. *If f is differentiable on $[a,b]$ and $q\to 1$, then the identity* (25) *reduce to*

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}$$
$$=\frac{b-a}{4}\int_0^1(1-t)\begin{pmatrix} f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -f'\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}dt.$$

See also [21, Lemma 1, for $x=\frac{a+b}{2}$].

Next, we present quantum analogue of some trapezoid type inequalities as follows:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$. If $|{}_{a^+}D_qf|$ and $|{}_{b^-}D_qf|$ are convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ \ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2}\right|$$
$$\leq\ \frac{b-a}{4}\left[\begin{array}{c}\frac{\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|+\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} \\ +\frac{q\left(\left|{}_{a^+}D_qf(a)\right|+\left|{}_{b^-}D_qf(b)\right|\right)}{1+q+q^2}\end{array}\right]. \tag{31}$$

Proof. By using Lemma (11) and convexity of $|{}_{a^+}D_qf|$ and $|{}_{b^-}D_qf|$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\ \ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2}\right|$$

$$
\begin{aligned}
&= \left| \frac{b-a}{4} \int_0^1 (1-qt) \left(\begin{array}{c} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right) \end{array} \right) {}_{0^+}d_q t \right| \\
&\leq \frac{b-a}{4} \left[\begin{array}{c} \int_0^1 |(1-qt)| \left|{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\right| {}_{0^+}d_q t \\ + \int_0^1 |(1-qt)| \left| {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right) \right| {}_{0^+}d_q t \end{array} \right] \\
&\leq \frac{b-a}{4} \left[\begin{array}{c} \left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right| \int_0^1 |(1-qt)|\, t \ {}_{0^+}d_q t \\ + \left|{}_{a^+}D_q f(a)\right| \int_0^1 |(1-qt)|\,(1-t)_{0^+}d_q t \\ + \left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right| \int_0^1 |(1-qt)|\, t \ {}_{0^+}d_q t \\ + \left|{}_{b^-}D_q f(b)\right| \int_0^1 |(1-qt)|\,(1-t)_{0^+}d_q t \end{array} \right].
\end{aligned}
$$

Applying Lemma 5 and Lemma 6, we have

$$
\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t) \ {}_{\frac{a+b}{2}}d_q t \ - \frac{f(a)+f(b)}{2} \right| \\
&\leq \frac{b-a}{4} \left[\begin{array}{c} \frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} + \frac{q\left|{}_{a^+}D_q f(a)\right|}{1+q+q^2} \\ + \frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} + \frac{\left|{}_{b^-}D_q f(b)\right|}{1+q+q^2} \end{array} \right] \\
&= \frac{b-a}{4} \left[\begin{array}{c} \frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right| + \left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|}{(1+q)(1+q+q^2)} \\ + \frac{q\left(\left|{}_{a^+}D_q f(a)\right| + \left|{}_{b^-}D_q f(b)\right|\right)}{1+q+q^2} \end{array} \right].
\end{aligned}
$$

This complete the proof. □

Remark 3. *If f is differentiable on $[a,b]$ and $q \to 1$, then the inequality* (31) *reduce to*

$$
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{12} \left(|f'(a)| + \left| f'(\frac{a+b}{2}) \right| + |f'(b)| \right).
$$

See also [21, Corollary 2].

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)$ and*
${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex on $[a,b]$ where $r > 0$, then the following inequality holds:

$$
\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t) \ {}_{\frac{a+b}{2}}d_q t \ - \frac{f(a)+f(b)}{2} \right| \\
&\leq \frac{b-a}{4} \left(\frac{1}{1+q} \right)^{1-\frac{1}{r}} \left[\begin{array}{c} \left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)} + \frac{q\left|{}_{a^+}D_q f(a)\right|^r}{1+q+q^2} \right)^{\frac{1}{r}} + \\ \left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)} + \frac{q\left|{}_{b^-}D_q f(a)\right|^r}{1+q+q^2} \right)^{\frac{1}{r}} \end{array} \right]. \qquad (32)
\end{aligned}
$$

Proof. Since $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex functions, so from Lemma 11 and using the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2}\right| \\
=\ &\left|\frac{b-a}{4}\int_0^1 (1-qt)\begin{pmatrix} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}\ {}_{0^+}d_qt\right| \\
\leq\ &\frac{b-a}{4}\begin{bmatrix} \int_0^1 |(1-qt)|\left|{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|\ {}_{0^+}d_qt \\ +\int_0^1 |(1-qt)|\left|\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\ \right|\ {}_{0^+}d_qt\end{bmatrix} \\
\leq\ &\frac{b-a}{4}\begin{bmatrix} \left(\int_0^1 |(1-qt)|\ \ {}_{0^+}d_qt\ \right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 |(1-qt)|\left|{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^r\ {}_{0^+}d_qt\ \right)^{\frac{1}{r}} \\ +\left(\int_0^1 |(1-qt)|\ \ {}_{0^+}d_qt\ \right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 |(1-qt)|\left|\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\ \right|\ {}_{0^+}d_qt\ \right)^{\frac{1}{r}}\end{bmatrix} \\
\leq\ &\frac{b-a}{4}\left(\int_0^1 |(1-qt)|\ \ {}_{0^+}d_qt\ \right)^{1-\frac{1}{r}} \\
&\times\begin{bmatrix} \begin{pmatrix} \left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r \int_0^1 |(1-qt)|\,t\ \ {}_{0^+}d_qt \\ +\left|{}_{a^+}D_q f(a)\right|^r \int_0^1 |(1-qt)|\,(1-t)_{0^+}d_qt\end{pmatrix}^{\frac{1}{r}} \\ +\begin{pmatrix} \left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r \int_0^1 |(1-qt)|\,t\ \ {}_{0^+}d_qt \\ +\left|{}_{b^-}D_q f(b)\right|^r \int_0^1 |(1-qt)|\,(1-t)_{0^+}d_qt\end{pmatrix}^{\frac{1}{r}}\end{bmatrix}.
\end{aligned}$$

Applying Lemma 5 and Lemma 6, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2}\right| \\
\leq\ &\frac{b-a}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\begin{bmatrix} \left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)}+\frac{q\left|{}_{a^+}D_q f(a)\right|^r}{1+q+q^2}\right)^{\frac{1}{r}} \\ +\left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r}{(1+q)(1+q+q^2)}+\frac{q\left|{}_{b^-}D_q f(a)\right|^r}{1+q+q^2}\right)^{\frac{1}{r}}\end{bmatrix}.
\end{aligned}$$

This complete the proof. □

Remark 4. *If f is differentiable on $[a, b]$ and $q \to 1$, then the inequality* (32) *reduce to*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \left(\frac{b-a}{8}\right)\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\begin{array}{c}\left(\left|f'(\frac{a+b}{2})\right|^r+2\left|f'(a)\right|^r\right)^{\frac{1}{r}}\\ +\left(\left|f'(\frac{a+b}{2})\right|^r+2\left|f'(b)\right|^r\right)^{\frac{1}{r}}\end{array}\right].$$

See also [21, Corollary 4].

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$ are convex on $[a,b]$ where $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{b-a}{4}\left(S_q(p)\right)^{\frac{1}{p}}\left[\left(\begin{array}{c}\left(\frac{\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_qf(a)\right|^r}{(1+q)}\right)^{\frac{1}{r}}\\ +\left(\frac{\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_qf(a)\right|^r}{(1+q)}\right)^{\frac{1}{r}}\end{array}\right)\right], \tag{33}$$

where

$$S_q(p)=\int_0^1(1-qt)^p\ {}_{0^+}d_qt\ ,$$

is fulfilled.

Proof. From Lemma 11 and using Holder's inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ -\frac{f(a)+f(b)}{2}\right|$$

$$= \left|\frac{b-a}{4}\int_0^1(1-qt)\left(\begin{array}{c}{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\\ -\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\end{array}\right)\ {}_{0^+}d_qt\ \right|$$

$$\leq \frac{b-a}{4}\left[\int_0^1\left|(1-qt)\right|\left|\ {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ \right|\ {}_{0^+}d_qt\right.$$

$$\left.\int_0^1\left|(1-qt)\right|\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\right|\ {}_{0^+}d_qt\ \right]$$

$$\leq \frac{b-a}{4}\left(\int_0^1(1-qt)^p\ {}_{0^+}d_qt\ \right)^{\frac{1}{p}}$$

$$\times\left[\begin{array}{c}\left(\int_0^1\left|\ {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ \right|^r\ {}_{0^+}d_qt\ \right)^{\frac{1}{r}}\\ +\left(\int_0^1\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\right|^r\ {}_{0^+}d_qt\ \right)^{\frac{1}{r}}\end{array}\right].$$

By applying the convexity of $|_{a^+}D_q f|^r$ and $|_{b^-}D_q f|^r$, we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t \ -\frac{f(a)+f(b)}{2}\right| \\ \leq\ & \frac{b-a}{4}(S(p))^{\frac{1}{p}}\left[\begin{array}{c}\left(\begin{array}{c}\left|_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_q t \\ +\left|_{a^+}D_q f(a)\right|^r\int_0^1(1-t)\ {}_{0^+}d_q t\end{array}\right)^{\frac{1}{r}} \\ +\left(\begin{array}{c}\left|_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_q t \\ +\left|_{b^-}D_q f(b)\right|^r\int_0^1(1-t)\ {}_{0^+}d_q t\end{array}\right)^{\frac{1}{r}}\end{array}\right], \end{aligned}$$

Also,

$$\int_0^1 t\ {}_{0^+}d_q t\ =\frac{1}{1+q}, \qquad \int_0^1 (1-t)\ {}_{0^+}d_q t\ =\frac{q}{1+q}.$$

Therefore

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t \ -\frac{f(a)+f(b)}{2}\right| \\ \leq\ & \frac{b-a}{4}(S_q(p))^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|_{a^+}D_q f(a)\right|^r}{1+q}\right)^{\frac{1}{r}} \\ +\left(\frac{\left|_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|_{b^-}D_q f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}\end{array}\right]. \end{aligned}$$

This complete the proof. □

Remark 5. *If f is differentiable on $[a,b]$ and $q\to 1$, then $S_1(p)=\frac{1}{p+1}$ and the inequality (33) reduce to*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{r}}\left[\begin{array}{c}\left[|f'(a)|^r+\left|f'(\frac{a+b}{2})\right|^r\right]^{\frac{1}{r}} \\ +\left[|f'(b)|^r+\left|f'(\frac{a+b}{2})\right|^r\right]^{\frac{1}{r}}\end{array}\right]. \end{aligned}$$

See also [21, Corollary 3].

Lemma 12. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function. If ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, then the following identity holds:*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t$$

$$= \frac{b-a}{4}\int_0^1 qt\begin{pmatrix} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\end{pmatrix}\ {}_{0^+}d_q t\ . \tag{34}$$

Proof. By using the similar proving argument as in Lemma 11, we have

$$\begin{aligned}
&\frac{b-a}{4}\int_0^1 qt\begin{pmatrix} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\end{pmatrix}\ {}_{0^+}d_q t \\
&= \frac{q(b-a)}{4}\begin{bmatrix} \int_0^1 t\ {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_q t \\ -\int_0^1 t\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\ {}_{0^+}d_q t\end{bmatrix} \\
&= \frac{q(b-a)}{4}\left[K_1-K_2\right].
\end{aligned} \tag{35}$$

Also,

$$\begin{aligned}
K_1 &= \int_0^1 t\ {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ {}_{0^+}d_q t \\
&= \int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)\,a\right)}{(1-q)\left(\frac{a+b}{2}-a\right)t}\right]\ {}_{0^+}d_q t \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right. \\
&\left.-\sum_{n=0}^{\infty} q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)a\right)\right] \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right. \\
&\left.-\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right] \\
&= \frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right. \\
&\left.-\frac{1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{q(b-a)}f\left(\frac{a+b}{2}\right) \\
&- \frac{2}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)\,a\right)\right] \\
&= \frac{2}{q(b-a)}f\left(\frac{a+b}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{4}{q(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)\right] \\
&= \frac{2}{q(b-a)} f\left(\frac{a+b}{2}\right) - \frac{4}{q(b-a)^2}\int_a^{\frac{a+b}{2}} f(t) \ {}_{a^+}d_q t \ , \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &= \int_0^1 t \ {}_{b^-}D_q f\left(t\left(\tfrac{a+b}{2}\right)+(1-t)b\right) \ {}_{0^+}d_q t \\
&= \int_0^1 t\left[\frac{f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)-f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)b\right)}{(1-q)\left(\frac{a+b}{2}-b\right)t}\right] \ {}_{0^+}d_q t \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right. \\
&\left. -\sum_{n=0}^{\infty} q^n f\left(q^{n+1}\left(\frac{a+b}{2}\right)+\left(1-q^{n+1}\right)b\right)\right] \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right. \\
&\left. -\frac{1}{q}\sum_{n=1}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\
&= -\frac{2}{b-a}\left[\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)\right. \\
&\left. -\frac{1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)a\right)+\frac{1}{q}f\left(\frac{a+b}{2}\right)\right] \\
&= -\frac{2}{q(b-a)} f\left(\frac{a+b}{2}\right) \\
&- \frac{2}{b-a}\left[\frac{q-1}{q}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\
&= -\frac{2}{q(b-a)} f\left(\frac{a+b}{2}\right) \\
&+ \frac{4}{q(b-a)^2}\left[\frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n f\left(q^n\left(\frac{a+b}{2}\right)+(1-q^n)b\right)\right] \\
&= -\frac{2}{q(b-a)} f\left(\frac{a+b}{2}\right) + \frac{4}{q(b-a)^2}\int_{\frac{a+b}{2}}^{b} f(t) \ {}_{b^-}d_q t \ . \qquad (37)
\end{aligned}
$$

Combining (35), (36) and (37), we get

$$\begin{aligned}&\frac{q(b-a)}{4}[K_1-K_2]\\&=f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ +\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\ \right)\\&=f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ ,\end{aligned}$$

which leads to the (34).This complete the proof. □

Remark 6. *If f is differentiable on $[a,b]$ and $q\to 1$, then the identity (34) reduce to*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&=\frac{b-a}{4}\int_0^1 t\begin{pmatrix} f'\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\\ -f'\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}dt.\end{aligned}$$

See also [18, Lemma 2.1].

Next, we establish quantum analogue of some midpoint type inequalities as follows:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$. If $|{}_{a^+}D_qf|$ and $|{}_{b^-}D_qf|$ are convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\&\le\ \frac{b-a}{4}\left(\frac{q}{1+q+q^2}\right)\begin{bmatrix}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|+\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|\\+\frac{q^2}{1+q}\left(\left|{}_{a^+}D_qf(a)\right|+\left|{}_{b^-}D_qf(b)\right|\right)\end{bmatrix}.\end{aligned}\tag{38}$$

Proof. From Lemma 12 and using convexity of $|{}_{a^+}D_qf|$ and $|{}_{b^-}D_qf|$, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_qt\ \right|\\&=\ \left|\frac{b-a}{4}\int_0^1 qt\begin{pmatrix}{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\\-\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{pmatrix}\ {}_{0^+}d_qt\ \right|\\&\le\ \frac{q(b-a)}{4}\begin{bmatrix}\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|\int_0^1 t^2\ {}_{0^+}d_qt\\+\left|{}_{a^+}D_qf(a)\right|\int_0^1(t-t^2)\ {}_{0^+}d_qt\\+\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|\int_0^1 t^2\ {}_{0^+}d_qt\\+\left|{}_{b^-}D_qf(b)\right|\int_0^1(t-t^2)\ {}_{0^+}d_qt\end{bmatrix}.\end{aligned}$$

Applying Lemma 7, we get

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t) \ {}_{\frac{a+b}{2}}d_q t \right| \\
\leq\ & \frac{b-a}{4}\left(\frac{q}{1+q+q^2}\right)\left[\begin{array}{c} \left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right| + \left|{}_{b-}D_q f\left(\frac{a+b}{2}\right)\right| \\ +\frac{q^2}{1+q}\left(\left|{}_{a^+}D_q f(a)\right| + \left|{}_{b^-}D_q f(b)\right|\right) \end{array}\right].
\end{aligned}$$

This complete the proof. □

Remark 7. *If f is differentiable on $[a,b]$ and $q \to 1$, then the inequality (38) reduce to*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\
\leq\ & \frac{b-a}{12}\left(2\left|f'(\frac{a+b}{2})\right| + \frac{|f'(a)|+|f'(b)|}{2}\right),
\end{aligned}$$

See also [18, Theorem 2.1 for $s = m = 1$].

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)a\right)$ and*
${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)b\right)$ are left quantum integrable on $[0,1]$. If $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex on $[a,b]$ where $r > 0$, then the following inequality holds:

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t) \ {}_{\frac{a+b}{2}}d_q t \right| \\
\leq\ & \frac{q(b-a)}{4(1+q)}\left(\frac{1}{(1+q+q^2)}\right)^{\frac{1}{r}} \\
&\times\left[\begin{array}{c} \left(\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r(1+q) + \left|{}_{a^+}D_q f(a)\right|^r q^2\right)^{\frac{1}{r}} \\ +\left(\left|{}_{b-}D_q f\left(\frac{a+b}{2}\right)\right|^r(1+q) + \left|{}_{b^-}D_q f(b)\right|^r q^2\right)^{\frac{1}{r}} \end{array}\right]. \qquad (39)
\end{aligned}$$

Proof. Since $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex functions, so from Lemma 12 and using the power mean inequality, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t) \ {}_{\frac{a+b}{2}}d_q t \right| \\
\leq\ & \left| \frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c} {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)a\right) \\ -\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)b\right) \end{array}\right) \ {}_{0^+}d_q t \right|
\end{aligned}$$

$$\leq \frac{q(b-a)}{4}\left[\begin{array}{c} \left(\int_0^1 t \ {}_{0^+}d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 t\left| {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^r {}_{0^+}d_qt\right)^{\frac{1}{r}} \\ +\left(\int_0^1 t \ {}_{0^+}d_qt\right)^{1-\frac{1}{r}} \\ \times\left(\int_0^1 t\left| {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|^r {}_{0^+}d_qt\right)^{\frac{1}{r}} \end{array}\right]$$

$$\leq \frac{q(b-a)}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left[\begin{array}{c} \left(\begin{array}{c} \left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t^2 \ {}_{0^+}d_qt \\ +\left|{}_{a^+}D_qf(a)\right|^r\int_0^1 t(1-t) \ {}_{0^+}d_qt \end{array}\right)^{\frac{1}{r}} \\ +\left(\begin{array}{c} \left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t^2 \ {}_{0^+}d_qt \\ +\left|{}_{b^-}D_qf(b)\right|^r\int_0^1 t(1-t) \ {}_{0^+}d_qt \end{array}\right)^{\frac{1}{r}} \end{array}\right].$$

Applying Lemma 7, we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t) \ {}_{\frac{a+b}{2}}d_qt\right|$$

$$\leq \frac{q(b-a)}{4}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left[\begin{array}{c} \left(\begin{array}{c} \left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r\frac{1}{1+q+q^2} \\ +\left|{}_{a^+}D_qf(a)\right|^r\frac{q^2}{(1+q)(1+q+q^2)} \end{array}\right)^{\frac{1}{r}} \\ +\left(\begin{array}{c} \left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r\frac{1}{1+q+q^2} \\ +\left|{}_{b^-}D_qf(b)\right|^r\frac{q^2}{(1+q)(1+q+q^2)} \end{array}\right)^{\frac{1}{r}} \end{array}\right]$$

$$= \frac{q(b-a)}{4(1+q)}\left(\frac{1}{(1+q+q^2)}\right)^{\frac{1}{r}}$$

$$\times\left[\begin{array}{c} \left(\left|{}_{a^+}D_qf\left(\frac{a+b}{2}\right)\right|^r(1+q)+\left|{}_{a^+}D_qf(a)\right|^r q^2\right)^{\frac{1}{r}} \\ +\left(\left|{}_{b^-}D_qf\left(\frac{a+b}{2}\right)\right|^r(1+q)+\left|{}_{b^-}D_qf(b)\right|^r q^2\right)^{\frac{1}{r}} \end{array}\right].$$

This complete the proof. □

Remark 8. *If f is differentiable on $[a,b]$ and $q\to 1$, then the inequality (39) reduce to*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t) \ dt\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\begin{array}{c} \left(2\left|f'\left(\frac{a+b}{2}\right)\right|^r+\left|f'(a)\right|^r\right)^{\frac{1}{r}} \\ +\left(2\left|f'\left(\frac{a+b}{2}\right)\right|^r+\left|f'(b)\right|^r\right)^{\frac{1}{r}} \end{array}\right].$$

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and*

${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)$ *are left quantum integrable on* $[0,1]$*. If* $\left|{}_{a^+}D_q f\right|^r$ *and* $\left|{}_{b^-}D_q f\right|^r$ *are convex on* $[a,b]$ *where* $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$*, then the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ \right| \leq \frac{q(b-a)}{4}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_q f(a)\right|^r}{1+q}\right)^{\frac{1}{r}}\\+\left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_q f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}\end{array}\right]. \quad (40)$$

is true.

Proof. From Lemma 12 and using Holder's inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ \right|\\ =&\left|\frac{b-a}{4}\int_0^1 qt\left(\begin{array}{c}{}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\\-\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\end{array}\right)\ {}_{0^+}d_q t\ \right|\\ \leq&\frac{q(b-a)}{4}\left[\begin{array}{c}\left(\int_0^1 t^p\ {}_{0^+}d_q t\ \right)^{\frac{1}{p}}\\ \times\left(\int_0^1\left|\ {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,a\right)\ \right|^r\ {}_{0^+}d_q t\ \right)^{\frac{1}{r}}\\+\left(\int_0^1 t^p\ {}_{0^+}d_q t\ \right)^{\frac{1}{p}}\\ \times\left(\int_0^1\left|{}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\,b\right)\right|^r\ {}_{0^+}d_q t\ \right)^{\frac{1}{r}}\end{array}\right].\end{aligned}$$

Applying Lemma 8 and convexity of $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$, we get

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a+b}{2}}d_q t\ \right|\\ \leq&\frac{q(b-a)}{4}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\begin{array}{c}\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_q t\\+\left|{}_{a^+}D_q f\left(a\right)\right|^r\int_0^1(1-t)\ {}_{0^+}d_q t\end{array}\right)^{\frac{1}{r}}\\ \left(\begin{array}{c}\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r\int_0^1 t\ {}_{0^+}d_q t\\+\left|{}_{b^-}D_q f\left(b\right)\right|^r\int_0^1(1-t)\ {}_{0^+}d_q t\end{array}\right)^{\frac{1}{r}}\end{array}\right]\\ \leq&\frac{q(b-a)}{4}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{\left|{}_{a^+}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{a^+}D_q f(a)\right|^r}{1+q}\right)^{\frac{1}{r}}\\+\left(\frac{\left|{}_{b^-}D_q f\left(\frac{a+b}{2}\right)\right|^r+q\left|{}_{b^-}D_q f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}\end{array}\right].\end{aligned}$$

This complete the proof. □

Remark 9. *If f is differentiable on $[a,b]$ and $q \to 1$, then the inequality* (40) *reduce to*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq \quad & \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\begin{array}{c} \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^r + \left|f'(a)\right|^r}{2}\right)^{\frac{1}{r}} \\ + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^r + \left|f'(b)\right|^r}{2}\right)^{\frac{1}{r}} \end{array} \right]. \end{aligned}$$

5. Conclusions

We have introduced a new quantum analogue of Hermite-Hadamard inequality and based on it we obtained two new quantum trapezoid and midpoint type identities. In [21] and [18], respectively by taking $x = \frac{a+b}{2}$ and $s = m = 1$, trapezoid and midpoint type inequalities for convex functions have been presented. We have established quantum analogs of some of these inequalities by using the new quantum trapezoid and midpoint type identities. For $q \to 1$ the obtained results give refinement of some trapezoid and midpoint type inequalities in [18, 21]. The idea and techniques of this paper may help the interested researcher in this field for further research.

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References

[1] Ali, M. A., Abbas, M., Budak, H., Agarwal, P., Murtaza, G., Chu, Y. M., New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions, *Adv. Differ. Equ.,* 2021(64) (2021), 1-21. https://doi.org/10.1186/s13662-021-03226-x

[2] Ali, M. A., Alp, N., Budak, H., Chu, Y. M., Zhang, Z., On some new quantum midpoint-type inequalities for twice quantum differentiable convex functions, *Open Math.,* 19(1) (2021), 427-439. https://doi.org/10.1515/math-2021-0015

[3] Ali, M. A., Budak, H., Abbas, M., Chu, Y. M., Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives, *Adv. Differ. Equ.,* 2021(7) (2021), 1-12. https://doi.org/10.1186/s13662-020-03163-1

[4] Ali, M. A., Budak, H., Akkurt, A., Chu, Y. M., Quantum Ostrowski-type inequalities for twice quantum differentiable functions in quantum calculus, *Open Math.,* 19(1) (2021), 440-449. https://doi.org/10.1515/math-2021-0020

[5] Ali, M. A., Budak, H., Zhang, Z., Yildirim, H., Some new Simpson's type inequalities for coordinated convex functions in quantum calculus, *Math. Methods Appl. Sci.,* 44(6) (2021), 4515-4540. https://doi.org/10.1002/mma.7048

[6] Ali, M. A., Chu, Y. M., Budak, H., Akkurt, A., Yildirim, H., Zahid, M. A., Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables, *Adv. Differ. Equ.,* 2021(25) (2021), 1-26. https://doi.org/10.1186/s13662-020-03195-7

[7] Annaby, M. H., Mansour, Z. S., q-Fractional Calculus and Equations, Springer, Heidelberg, 2012.

[8] Alp, N., Sarikaya, M. Z., Kunt, M., Iscan, I., q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud Univ.-Sci.,* 30(2) (2018), 193-203. https://doi.org/10.1016/j.jksus.2016.09.007

[9] Awan, M. U., Talib, S., Kashuri, A., Noor, M. A., Chu, Y. M., Estimates of quantum bounds pertaining to new q-integral identity with applications, *Adv. Differ. Equ.,* 2020(424) (2020), 1-15. https://doi.org/10.1186/s13662-020-02878-5

[10] Awan, M. U., Talib, S., Kashuri, A., Noor, M. A., Noor, K. I., Chu, Y. M., A new q-integral identity and estimation of its bounds involving generalized exponentially μ-preinvex functions, *Adv. Differ. Equ.,* 2020(575) (2020), 1-12. https://doi.org/10.1186/s13662-020-03036-7

[11] Awan, M. U., Talib, S., Noor, M. A., Noor, K. I., Chu, Y. M, On post quantum integral inequalities, *J. Math. Inequal.,* 15(2) (2021), 629-654. https://doi.org/10.7153/jmi-2021-15-46

[12] Budak, H., Ali, M. A., Tarhanaci, M., Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions, *J. Optim. Theory. Appl.,* 186(3) (2020), 899-910. https://doi.org/10.1007/s10957-020-01726-6

[13] Budak, H., Ali, M. A., Tunc, T., Quantum Ostrowski-type integral inequalities for functions of two variables, *Math. Methods Appl. Sci.,* 44(7) (2021), 5857-5872. https://doi.org/10.1002/mma.7153

[14] Budak, H., Erden, S., Ali, M. A., Simpson and Newton type inequalities for convex functions via newly defined quantum integrals, *Math. Methods Appl. Sci.,* 44(1) (2021), 378-390. https://doi.org/10.1002/mma.6742

[15] Budak, H., Khan, S., Ali, M. A., Chu, Y. M., Refinements of quantum Hermite-Hadamard-type inequalities, *Open Math.,* 19(1) (2021), 724-734. https://doi.org/10.1515/math-2021-0029

[16] Dragomir, S. S., Agarwal, R., Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.,* 11(5) (1998), 91-95. https://doi.org/10.1016/S0893-9659(98)00086-X

[17] Du, T. S., Luo, C. Y., Yu, B., Certain quantum estimates on the parameterized integral inequalities and their applications, *J. Math. Inequal.,* 15(1) (2021), 201-228. https://doi.org/10.7153/jmi-2021-15-16

[18] Eftekhari, N., Some remarks on (s, m)-convexity in the second sense, *J. Math. Inequal.,* 8(3) (2014), 489-495. https://doi.org/10.7153/jmi-08-36

[19] Erden, S., Iftikhar, S., Delavar, M. R., Kumam, P., Thounthong, P., Kumam, W., On generalizations of some inequalities for convex functions via quantum integrals, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM*, 114(110) (2020), 1-15. https://doi.org/10.1007/s13398-020-00841-3

[20] Kac, V., Pokman C., Quantum Calculus, Springer, New York, 2001.

[21] Kavurmaci, H., Avci, M., Özdemir, M. E., New inequalities of Hermite-Hadamard type for convex functions with applications, *J. Inequal.Appl.,* 2011(86) (2011), 1-11. https://doi.org/10.1186/1029-242X-2011-86

[22] Kirmaci, U. S., Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.,* 147(1) (2004), 137-146. https://doi.org/10.1016/S0096-3003(02)00657-4
[23] Kunt, M., Baidar, A., Sanli, Z., Left-Right quantum derivatives and definite integrals, (2020), https://www.researchgate.net/publication/343213377 (Preprint).
[24] Li, Y. X., Ali, M. A., Budak, H., Abbas, M., Chu, Y. M., A new generalization of some quantum integral inequalities for quantum differentiable convex functions, *Adv. Differ. Equ.,* 2021(225) (2021), 1-15. https://doi.org/10.1186/s13662-021-03382-0
[25] Khan, M. A., Mohammad, N., Nwaeze, E. R., Chu, Y. M., Quantum Hermite-Hadamard inequality by means of a Green function, *Adv. Differ. Equ.,* 2020(99) (2020), 1-20. https://doi.org/10.1186/s13662-020-02559-3
[26] Noor, M. A., Noor, K. I., Awan, M. U., Some quantum estimates for Hermite–Hadamard inequalities, *Appl. Math. Comput.,* 251 (2015), 675-679. https://doi.org/10.1016/j.amc.2014.11.090
[27] Prabseang, J., Nonlaopon, K., Ntouyas, S. K., On the refinement of quantum Hermite-Hadamard inequalities for continuous convex functions, *J. Math. Inequal.,* 14(3) (2020), 875-885. https://doi.org/10.7153/jmi-2020-14-57
[28] Pearce, C. E. M., Pečarić, J., Inequalities for differentiable mappings with application to special means and quadrature formulae, *Appl. Math. Lett.,* 13(2) (2000), 51-55. https://doi.org/10.1016/S0893-9659(99)00164-0
[29] Rashid, S., Butt, S. I., Kanwal, S., Ahmad, H., Wang, M. K., Quantum integral inequalities with respect to Raina's function via coordinated generalized-convex functions with applications, *J. Funct. Spaces*, Article ID 6631474 (2021). https://doi.org/10.1155/2021/6631474
[30] Sudsutad, W., Ntouyas, S. K., Tariboon, J., Quantum integral inequalities for convex functions, *J. Math. Inequal.,* 9(3) (2015), 781-793. https://doi.org/10.7153/jmi-09-64
[31] Tariboon, J., Ntouyas, S. K., Quantum calculus on finite intervals and applications to impulsive difference equations, *Adv. Differ. Equ.,* 2013(282) (2013), 1-19. https://doi.org/10.1186/1687-1847-2013-282
[32] Vivas-Cortez, M., Ali, M. A., Kashuri, A., Sial, B. I., Zhang, Z., Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus, *Symmetry,* 12(9) (2020), 1-28. https://doi.org/10.3390/sym12091476
[33] Vivas-Cortez, M., Kashuri, A., Liko, R., Hernández, J. E. H., Some new q-integral inequalities using generalized quantum Montgomery identity via preinvex functions, *Symmetry,* 12(4) (2020), 1-15. https://doi.org/10.3390/sym12040553
[34] You, X., Ali, M. A., Erden, S., Budak, H., Chu, Y. M., On some new midpoint inequalities for the functions of two variables via quantum calculus, *J. Inequal. Appl.,* 2021(142) (2021), 1-23. https://doi.org/10.1186/s13660-021-02678-9
[35] You, X., Kara, H., Budak, H., Kalsoom, H., Quantum inequalities of Hermite–Hadamard type for-convex functions, *J. Math.,* Article ID 6634614 (2021). https://doi.org/10.1155/2021/6634614
[36] Zhou, S. S., Rashid, S., Noor, M. A., Noor, K. I., Safdar, F., Chu, Y. M., New Hermite-Hadamard type inequalities for exponentially convex functions and applications, *AIMS Math.,* 5(6) (2020), 6874-6901. https://doi.org/10.3934/math.2020441

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SOME FIXED POINT THEOREMS ON ORTHOGONAL METRIC SPACES VIA EXTENSIONS OF ORTHOGONAL CONTRACTIONS

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Abstract. Orthogonal metric space is a considerable generalization of a usual metric space obtained by establishing a perpendicular relation on a set. Very recently, the notions of orthogonality of the set and orthogonality of the metric space are described and notable fixed point theorems are given in orthogonal metric spaces. Some fixed point theorems for the generalizations of contraction principle via altering distance functions on orthogonal metric spaces are presented and proved in this paper. Furthermore, an example is presented to clarify these theorems.

1. Introduction and Preliminaries

The well-known theorem on the presence and uniqueness of a fixed point of exact self maps defined on certain metric spaces were stated by Stefan Banach [3] in 1992: Every self mapping h on a complete metric space (Ω, ρ) satisfying the condition

$$\rho(hx, hy) \leq \lambda\rho(x, y), \text{ for all } x, y \in \Omega, \lambda \in (0, 1) \tag{1}$$

has a unique fixed point.
This gracious theorem has been used to show the presence and uniqueness of the solution of differantial equation

$$y'(x) = F(x; y); y(x_0) = y_0 \tag{2}$$

where F is a continuously differantiable function.
Consequently, after the Banach Contraction Principle on complete metric space, many researchers have investigated for anymore fixed point results and reported

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new fixed point theorems intended by the use of two very influential directions, assembled or apart (See [2], [4], [5], [6], [7], [8], [9], [10], [11], [16], [17], [18]).
One of them is involved with the attempts to generalize the contractive conditions on the maps and thus, soften them; the other with to attempts to generalize the space on which these contractions are described.
Among the results in the first direction, Khan et al. [15] installed fixed point theorems in complete and compact metric spaces by using altering distance functions in 1984. Then, Alber and Guerre-Delabriere [1] presented anohter generalization of the contraction principle in Hilbert spaces in 1997. In 2001, the results of [1] were shown to be valuable in complete metric spaces by Rhoades [19]. On the other hand, among the results in the second direction, Gordji et al. [13] introduced the concepts orthogonality of the set and orthogonality of the metric space in 2017. In the mentioned paper, a generalization of Banach fixed point theorem is presented in this exciting defined construction and also, acquired results in the mentioned paper is implemented to indicate the presence of a solution of an ordinary differential equation. In this paper considerable fixed point theorems on orthogonal metric spaces via orthogonal contractions are introduced inspired by [13], [15] and [19]. Furthermore, an example is presented to illustrate these theorems.

The main difference between studies in orthogonal metric spaces and studies in general metric spaces is that instead of a contraction condition provided by any two elements of the space, it is sufficient to provide a contraction condition given only for orthogonally related elements. Another important point is that orthogonal complete metric spaces do not have to be complete metric spaces. So the results of this paper, not only generalize the analogous fixed point theorems but are relatively simpler and more natural than the related ones.

In the sequel, respectively, $\mathbb{Z}, \mathbb{R}, \mathbb{R}^+$ denote integers, real numbers and positive real numbers.

Definition 1. *([13]) Let Ω be a non-empty set and $\perp \subseteq \Omega \times \Omega$ be a binary relation. $(\Omega, \perp)$ is called orthogonal set if $\perp$ satisfies the following condition*

$$\exists k_0 \in \Omega; (\forall l \in \Omega, l \perp k_0) \vee (\forall l \in \Omega, k_0 \perp l). \tag{3}$$

And also this k_0 element is named orthogonal element.

Example 1. *([12]) Let $\Omega = \mathbb{Z}$ ($\mathbb{Z}$ is integer numbers) and define $a \perp b$ if there exists $\gamma \in \mathbb{Z}$ such that $a = \gamma b$. It is effortless to see that $0 \perp b$ for all $b \in \mathbb{Z}$. On account of this $(\Omega, \perp)$ is an orthogonal set.*

This k_0 element does not have to be unique. For example;

Example 2. *([12]) Let $\Omega = [0, \infty)$, define $k \perp l$ if $kl \in \{k, l\}$, then by setting $k_0 = 0$ or $k_0 = 1$, $(\Omega, \perp)$ is an orthogonal set.*

Definition 2. *([13]) A sequence $\{k_n\}$ is named orthogonal sequence if*

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{4}$$

In the same way, a Cauchy sequence $\{k_n\}$ *is named to be an orthogonally Cauchy sequence if*

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{5}$$

Definition 3. *([13]) Let* $(\Omega, \perp)$ *be an orthogonal set,* ρ *be a usual metric on* Ω*. Afterwards* $(\Omega, \perp, \rho)$ *is named an orthogonal metric space.*

Definition 4. *([13]) An orthogonal metric space* $(\Omega, \perp, \rho)$ *is named to be a complete orthogonal metric space if every orthogonally Cauchy sequence converges in* Ω*.*

Definition 5. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal metric space and a function* $h : \Omega \to \Omega$ *is named to be orthogonally continuous at* k *if for each orthogonal sequence* $\{k_n\}$ *converging to* k *implies* $hk_n \to hk$ *as* $n \to \infty$*. Also* h *is orthogonal continuous on* Ω *if* h *is orthogonal continuous in each* $k \in \Omega$*.*

Definition 6. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal metric space and* $\lambda \in \mathbb{R}$*,* $0 < \lambda < 1$*. A function* $h : \Omega \to \Omega$ *is named to be orthogonal contraction with Lipschitz constant* λ *if*

$$\rho(hk, hl) \leq \lambda \rho(k, l) \tag{6}$$

for all $k, l \in \Omega$ *whenever* $k \perp l$*.*

Definition 7. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal metric space and a function* $h : \Omega \to \Omega$ *is named orthogonal preserving if* $hk \perp hl$ *whenever* $k \perp l$*.*

Remark 1. *The authors of [12] gave an example which shows the orthogonal continuity and orthogonal contraction are weaker than the classic continuity and classic contraction in classic metric spaces.*

Theorem 1. *([13]) Let* $(\Omega, \perp, \rho)$ *be an orthogonal complete metric space and* $0 < \lambda < 1$ *and let* $h : \Omega \to \Omega$ *be orthogonal continuous , orthogonal contraction (with Lipschitz constant* λ *) and orthogonal preserving. Afterwards* h *has a unique fixed point* $k^* \in \Omega$ *and* $\lim_{n\to\infty} h^n(k) = k^*$ *for all* $k \in \Omega$*.*

Definition 8. *([15]) Let* $\psi : [0, \infty) \to [0, \infty)$ *be a function which satisfies*
(i) $\psi(s)$ *is continuous and nondecreasing,*
(ii) $\psi(s) = 0 \iff s = 0$
properties. Then ψ *is named altering distance function. And* Ψ *is denoted as the set of altering distance functions* ψ*.*

And in [14], notable fixed point theorems on orthogonal metric spaces via altering distance functions are presented by Bilgili Gungor and Turkoglu.

2. Main Results

Firstly, in the following theorem, by giving a contraction condition that will generalize the previous works using alterne distance functions is presented and proven.

Theorem 2. *Let $(\Omega, \perp, \rho)$ be an orthogonal complete metric space, $h : \Omega \to \Omega$ be a self map, $\eta, \kappa \in \Psi$ and η is a sub-additive function. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\eta(\rho(hk, hl)) \leq \eta(\rho(k, l)) - \kappa(\rho(k, l)) \tag{7}$$

for all $k, l \in \Omega$ where $k \perp l$ and $k \neq l$. Under this circumstance, there exists a point $k^* \in \Omega$ so that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n k_0\}$ converges to this point. Also, $k^* \in \Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^* \in \Omega$.

Proof. Because of $(\Omega, \perp)$ is an orthogonal set,

$$\exists k_0 \in \Omega; (\forall l \in \Omega, l \perp k_0) \vee (\forall l \in \Omega, k_0 \perp l). \tag{8}$$

And from h is a self mapping on Ω, for any orthogonal element $k_0 \in \Omega$, $k_1 \in \Omega$ can be chosen as $k_1 = h(k_0)$. Thus,

$$\begin{aligned} & k_0 \perp hk_0 \vee hk_0 \perp k_0 \\ \Rightarrow\ & k_0 \perp k_1 \vee k_1 \perp k_0. \end{aligned} \tag{9}$$

Then, if it continues similarly

$$k_1 = hk_0, k_2 = hk_1 = h^2 k_0, ..., k_n = hk_{n-1} = h^n k_0 \tag{10}$$

so $\{h^n k_0\}$ is an iteration sequence.
If any $n \in \mathbb{N}, k_n = k_{n+1}$ then $k_n = hk_n$ and so h has a fixed point. Suppose that $k_n \neq k_{n+1}$ for all $n \in \mathbb{N}$.
Since h is orthogonal preserving, $\{h^n k_0\}$ is an orthogonal sequence and by using inequality (7)

$$\begin{aligned} \eta(\rho(k_{n+1}, k_n)) & = \eta(\rho(hk_n, hk_{n-1})) \\ & \leq \eta(\rho(k_n, k_{n-1})) - \kappa(\rho(k_n, k_{n-1})). \end{aligned} \tag{11}$$

Using the monotone property of $\eta \in \Psi$, $\{\rho(k_{n+1}, k_n)\}$ is a sequence of decreasing nonnegative real numbers. Thus there is a $\theta \geq 0$ and $\lim_{n\to\infty} \rho(k_{n+1}, k_n) = \theta$. It can be shown that $\theta = 0$. Assume, on the contrary, that $\theta > 0$. At that rate, by taking the limit $n \to \infty$ in inequality (11) and using η, κ are continuous functions,

$$\eta(\theta) \leq \eta(\theta) - \kappa(\theta) \tag{12}$$

is obtained. This is a contradiction. Therefore $\theta = 0$. Now it can be proved that $\{k_n\}$ is an orthogonal Cauchy sequence. If $\{k_n\}$ is not an orthogonal Cauchy sequence, there exists $\epsilon > 0$ and suitable subsequences $\{r(n)\}$ and $\{s(n)\}$ of $\mathbb{N}$ satisfying $r(n) > s(n) > n$ for which

$$\rho(x_{r(n)}, x_{s(n)}) \geq \epsilon \tag{13}$$

and where $r(n)$ is selected as the least integer satisfying (13), that is

$$\rho(k_{r(n)-1}, k_{s(n)}) < \epsilon. \tag{14}$$

By (13),(14) and triangular inequality of ρ, it can be easily derived that

$$\varepsilon \leq \rho(k_{r(n)}, k_{s(n)}) \leq \rho(k_{r(n)}, k_{r(n)-1}) + \rho(k_{r(n)-1}, k_{s(n)}) < \rho(k_{r(n)}, k_{r(n)-1}) + \epsilon. \tag{15}$$

Letting $n \to \infty$, by using $\lim_{n\to\infty} \rho(k_{n+1}, k_n) = \theta$

$$\lim_{n\to\infty} \rho(k_{r(n)}, k_{s(n)}) = \epsilon \tag{16}$$

is obtained. Also, for each $n \in \mathbb{N}$, by using the triangular inequality of ρ,

$$\begin{aligned} &\rho(k_{r(n)}, k_{s(n)}) - \rho(k_{r(n)}, k_{r(n)+1}) - \rho(k_{s(n)+1}, k_{s(n)}) \\ &\leq \rho(k_{r(n)+1}, k_{s(n)+1}) \\ &\leq \rho(k_{r(n)}, k_{r(n)+1}) + \rho(k_{r(n)}, k_{s(n)}) + \rho(k_{s(n)+1}, k_{s(n)}). \end{aligned} \tag{17}$$

Passing to the limit when $n \to \infty$ in the last inequality

$$\rho(k_{r(n)+1}, k_{s(n)+1}) = \epsilon. \tag{18}$$

By using the inequality (7),

$$\begin{aligned} \eta(\rho(k_{r(n)+1}, k_{s(n)+1})) &= \eta(\rho(hk_{r(n)}, hk_{s(n)})) \\ &\leq \eta(\rho(k_{r(n)}, k_{s(n)})) - \kappa(\rho(k_{r(n)}, k_{s(n)})). \end{aligned} \tag{19}$$

Passing to the limit when $n \to \infty$ in the last inequality

$$\eta(\epsilon) \leq \eta(\epsilon) - \kappa(\epsilon). \tag{20}$$

It is a contradiction. Therefore $\{k_n\}$ is a orthogonal Cauchy sequence. By the orthogonal completeness of Ω, there exists $k^* \in \Omega$ so that $\{k_n\} = \{h^n k_0\}$ converges to this point.

Now it can be shown that k^* is a fixed point of h when h is orthogonally continuous at $k^* \in \Omega$. Assume that h is orthogonally continuous at $k^* \in \Omega$. Thus,

$$k^* = \lim_{n\to\infty} k_{n+1} = \lim_{n\to\infty} hk_n = hk^*. \tag{21}$$

so $k^* \in \Omega$ is a fixed point of h.

Now the uniqueness of the fixed point can be shown. Suppose that there exist two distinct fixed points k^* and l^*.Then,

(i) If $k^* \perp l^* \vee l^* \perp k^*$, by using the inequality (7)

$$\begin{aligned} \eta(\rho(k^*, l^*)) &= \eta(\rho(hk^*, hl^*)) \\ &\leq \eta(\rho(k^*, l^*)) - \kappa(\rho(k^*, l^*)) \end{aligned} \tag{22}$$

This is a contradiction and $k^* \in \Omega$ is a unique fixed point of h.

(ii) If not, for the chosen orthogonal element $k_0 \in \Omega$,

$$[(k_0 \perp k^*) \wedge (k_0 \perp l^*)] \vee [(k^* \perp k_0) \wedge (l^* \perp k_0)] \tag{23}$$

and since h is orthogonal preserving,

$$[(hk_n \perp k^*) \wedge (hk_n \perp l^*)] \vee [(k^* \perp hk_n) \wedge (l^* \perp hk_n)] \tag{24}$$

is obtained. Now, by using the triangular inequality of ρ, ψ is nondecreasing sub-additive function and the inequality (7)

$$\begin{aligned}\eta(\rho(k^*,l^*)) &= \eta(\rho(hk^*,hl^*)) \\ &\leq \eta(\rho(hk^*,hk_{n+1})+\rho(hk_{n+1},hl^*)) \\ &\leq \eta(\rho(hk^*,h(hk_n)))+\eta(\rho(h(hk_n),hl^*)) \\ &\leq \eta(\rho(k^*,hk_n))-\kappa(\rho(k^*,hk_n))+\eta(\rho(hk_n,l^*))-\kappa(\rho(hk_n,l^*)).\end{aligned} \tag{25}$$

and taking limit $n \to \infty$, $k^* = l^*$. Thus, $k^* \in \Omega$ is a unique fixed point of h. □

If assumed to be $\kappa(s) = (1-\lambda)\eta(s)$, for all $s > 0$ where $0 < \lambda < 1$ in Theorem 2, the following Corollary is obtained.

Corollary 1. *Let $(\Omega, \perp, \rho)$ be an orthogonal complete metric space, $\lambda \in \mathbb{R}, 0 < \lambda < 1$, $h : \Omega \to \Omega$ be a self map , $\eta \in \Psi$ be a sub-additive function. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\eta(\rho(hk,hl)) \leq \lambda\eta(\rho(k,l)) \tag{26}$$

for all $k, l \in \Omega$ whenever $k \perp l$ and $k \neq l$. Under this circumstance, there exists a point $k^* \in \Omega$ such that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n k_0\}$ converges to this point. Also, $k^* \in \Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^* \in \Omega$.

If assume $\eta(s) = s$, for all $s > 0$ in Theorem 2, the following Corollary is gotten.

Corollary 2. *Let $(\Omega, \perp, \rho)$ be an orthogonal complete metric space, $h : \Omega \to \Omega$ be a self map, $\kappa \in \Psi$. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\rho(hk,hl) \leq \rho(k,l) - \kappa(\rho(k,l)) \tag{27}$$

for all $k, l \in \Omega$ where $k \perp l$ and $k \neq l$. Under this circumstance, there exists a point $k^* \in \Omega$ such that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n k_0\}$ converges to this point. Also, $k^* \in \Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^* \in \Omega$.

If assume $\eta(s) = s$ and $\kappa(s) = (1-\lambda)\eta(s)$, for all $s > 0$ where $0 < \lambda < 1$ in Theorem 2, the following Corollary which is the main result of [12] is obtained.

Corollary 3. *Let $(\Omega, \perp, \rho)$ be an orthogonal complete metric space, $\lambda \in \mathbb{R}, 0 < \lambda < 1$, $h : \Omega \to \Omega$ be a self map. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\rho(hk,hl) \leq \lambda\rho(k,l)) \tag{28}$$

for all $k, l \in \Omega$ where $k \perp l$ and $k \neq l$. Under this circumstance, there exists a point $k^* \in \Omega$ such that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n k_0\}$ converges to this point. Also, $k^* \in \Omega$ is a unique fixed point of h if h is orthogonal continuous at $k^* \in \Omega$.

Example 3. *Let $\Omega = [0, 1)$ be a set and define $\rho : \Omega \times \Omega \to \Omega$ such that $\rho(k, l) = \mid k - l \mid$. Also, let the binary relation $\perp$ on Ω such that $k \perp l \iff kl \in \{k, l\}$. Then, $(\Omega, \perp)$ is an orthogonal set and ρ is a metric on Ω. So $(\Omega, \perp, \rho)$ is an orthogonal metric space. In this space, any orthogonal Cauchy sequence is convergent. Indeed, suppose that (k_n) is an arbitrary orthogonal Cauchy sequence in Ω. Then*

$$\begin{aligned} & k_n.k_{n+1} = k_n \vee k_{n+1}.k_n = k_{n+1} \\ \Rightarrow \quad & k_n = 0, k_{n+1} \in [0, 1) \vee k_{n+1} = 0, k_n \in [0, 1) \end{aligned} \tag{29}$$

and for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $n \geq n_0$ we have

$$\mid k_n - k_{n+1} \mid < \epsilon \tag{30}$$

is provided. So, for any $\epsilon > 0$ and for all $n \in \mathbb{N}$, that is $n \geq n_0$, $\mid k_n - 0 \mid < \epsilon$ that is $\{k_n\}$ is convergent to $0 \in \Omega$. So $(\Omega, \perp, \rho)$ is a complete orthogonal metric space. Remark that, (Ω, ρ) is not a complete sub-metric space of $(\mathbb{R}, \rho)$ because of Ω is not a closed subset of $(\mathbb{R}, \rho)$.
Let $\eta : [0, \infty) \to [0, \infty)$ be defined as $\eta(s) = s$ and let $\kappa : [0, \infty) \to [0, \infty)$ be defined as $\kappa(s) = \frac{s^2}{9}$. Also let $h : \Omega \to \Omega$ be defined as

$$h(k) = \begin{cases} k - \frac{k^2}{3} & , 0 \leq k \leq \frac{1}{2}, \\ \frac{k}{2} & , \frac{1}{2} < k < 1. \end{cases} \tag{31}$$

In this case, one can see that $\eta, \kappa \in \Psi$, η is a sub-additive function. Also h is orthogonal preserving mapping. Indeed,

$$\begin{aligned} k \perp l \quad & \Rightarrow kl = k \vee kl = l \\ & \Rightarrow k = 0, l \in [0, 1) \vee l = 0, k \in [0, 1) \\ & \Rightarrow hk = 0 \vee hl = 0 \\ & \Rightarrow hk \perp hl \vee hl \perp hk. \end{aligned} \tag{32}$$

On the other hand, h is orthogonal continuous at $0 \in \Omega$. Indeed, assume that $\{k_n\}$ is an orthogonal sequence and $k_n \to 0$. In this case,

$$\begin{aligned} & k_n.k_{n+1} = k_n \vee k_{n+1}.k_n = k_{n+1} \\ \Rightarrow \quad & k_n = 0, k_{n+1} \in [0, 1) \vee k_{n+1} = 0, k_n \in [0, 1) \end{aligned} \tag{33}$$

and for any $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$, for all $n \in \mathbb{N}$ that is $n > n_0$, $\mid k_n - 0 \mid < \epsilon$ is obtained. So, for all $n \in \mathbb{N}$ that is $n > n_0$, $k_n \in [0, \frac{1}{2}]$. Thus, from the definition of h, for the same $n_0 \in \mathbb{N}$ that is $n > n_0$, $\mid h(k_n) - h(0) \mid < \epsilon$ that is $h(k_n) \to h(0) = 0$. Now, it can be shown that h is a self mapping satisfying the inequality (7) for all $k, l \in \Omega$ where $k \perp l$ and $k \neq l$.
Assume that $k, l \in \Omega$ two element of Ω, $k \perp l$ and $k \neq l$. In this case

$$kl = k \vee kl = l \Rightarrow k = 0, l \in [0, 1) \vee l = 0, k \in [0, 1). \tag{34}$$

Without loss of generality, assume that $k = 0, l \in [0, 1)$.
Case I: If $k = 0, l \in (0, \frac{1}{2}]$, *then* $hk = 0, hl = l - \frac{l^2}{3}$. *And*

$$\begin{aligned} \eta(\rho(hk, hl)) &=\mid 0 - (l - \tfrac{l^2}{3}) \mid = l - \tfrac{l^2}{3} \leq l - \tfrac{l^2}{9} \\ &=\mid 0 - l \mid - \mid 0 - \tfrac{l^2}{9} \mid = \eta(\rho(k, l)) - \kappa(\rho(k, l)). \end{aligned} \tag{35}$$

Case II: If $k = 0, l \in (\frac{1}{2}, 1)$, *then* $hk = 0, hl = \frac{l}{2}$. *And*

$$\eta(\rho(hk, hl)) =\mid 0 - \tfrac{l}{2} \mid = \tfrac{l}{2} \leq l - \tfrac{l^2}{9} =\mid 0 - l \mid - \mid 0 - \tfrac{l^2}{9} \mid = \eta(\rho(k, l)) - \kappa(\rho(k, l)). \tag{36}$$

Consequently, h *is a self mapping satisfying the inequality (7) for all* $k, l \in \Omega$ *whenever* $k \perp l$ *and* $k \neq l$. *Thus, all hypothesis of Theorem 2 satisfy and so, it is obvious that* h *has a unique fixed point* $0 \in \Omega$.

3. Conclusion

In the first part of this study, as a result of a comprehensive literature review, the developments related to the existence of fixed points for mappings that provide the appropriate contraction conditions from the beginning of the fixed point theory studies are mentioned, and then the general subject of this study is emphasized.

In this paper, some fixed point theorems in orthogonal complete metric spaces are presented by employing altering distance functions. The results of this paper, not only generalize the analogous fixed point theorems but are relatively simpler and more natural than the related ones. The results of this paper are actually three-fold: a relatively more general contraction condition is used, the continuity of the involved mapping is weakened to orthogonal continuity, the comparability conditions used by previous authors between elements are replaced by orthogonal relatedness.

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References

[1] Alber, Y. I., Guerre-Delabriere, S., Principle of Weakly Contractive Maps in Hilbert Spaces, *In: New Results in Operator Theory and Its Applications*, (pp. 7-22) Birkhuser, Basel, 1997.

[2] Altun, I., Damjanovic, B., Djoric, D., Fixed point and common fixed point theorems on ordered cone metric spaces, *Applied Mathematics Letters*, 23(3) (2010), 310-316. https://doi.org/10.1016/j.aml.2009.09.016

[3] Banach, S., Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fund. Math.*, 3 (1922), 133-181.

[4] Ciric, L., Damjanovic, B., Jleli, M., Samet, B., Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications, *Fixed Point Theory and Applications*, 2012(51) (2012), 1-13. https://doi.org/10.1186/1687-1812-2012-51

[5] Debnath, P., Srivastava, H. M., New extensions of Kannan's and Reich's fixed point theorems for multivalued maps using Wardowski's technique with application to integral equations, *Symmetry*, 12(7) (2020), 1090. https://doi.org/10.3390/sym12071090

[6] Debnath, P., Srivastava, H. M., Global optimization and common best proximity points for some multivalued contractive pairs of mappings, *Axioms*, 9(3) (2020), 102. https://doi.org/10.3390/axioms9030102

[7] Debnath, P., De La Sen, M., Contractive inequalities for some asymptotically regular set-valued mappings and their fixed points, *Symmetry*, 12(3) (2020), 411. https://doi.org/10.3390/sym12030411

[8] Debnath, P., Set-valued Meir-Keeler, Geraghty and Edelstein type fixed point results in b-metric spaces, *Rendiconti del Circolo Matematico di Palermo Series 2*, 70(3) (2021), 1389-1398. https://doi.org/10.1007/s12215-020-00561-y

[9] Debnath, P., Neog, M., Radenovic, S., Set valued Reich type G-contractions in a complete metric space with graph, *Rendiconti del Circolo Matematico di Palermo Series 2*, 69(3) (2020), 917-924. https://doi.org/10.1007/s12215-019-00446-9

[10] Debnath, P., Choudhury, B. S., Neog, M., Fixed set of set valued mappings with set valued domain in terms of start set on a metric space with a graph, *Fixed Point Theory and Applications*, 2017(1) (2016), 1-8. https://doi.org/10.1186/s13663-017-0598-8

[11] Debnath, P., Optimization through best proximity points for multivalued F-contractions, *Miskolc Mathematical Notes*, 22(1) (2021), 143-151. https://doi.org/10.18514/MMN.2021.3355

[12] Eshaghi Gordji, M., Habibi, H., Fixed point theory in generalized orthogonal metric space, *Journal of Linear and Topological Algebra* , 6(3) (2017), 251-260.

[13] Gordji, M. E., Ramezani, M., De La Sen, M., Cho, Y. J., On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory*, 18(2) (2017), 569-578. https://doi.org/10.24193/fpt-ro.2017.2.45

[14] Gungor, N. B., Turkoglu, D., Fixed point theorems on orthogonal metric spaces via altering distance functions, *In AIP Conference Proceedings*, 2183(1) (2019), p. 040011. AIP Publishing. https://doi.org/10.1063/1.5136131

[15] Khan, M. S., Swaleh, M., Sessa, S., Fixed point theorems by altering distances between the points, *Bulletin of the Australian Mathematical Society*, 30(1) (1984), 1-9. https://doi.org/10.1017/S0004972700001659

[16] Mehmood, M., Aydi, H., Ali, M. U., Shoaib, A., De La Sen, M., Solutions of integral equations via fixed-point results on orthogonal gauge structure, *Mathematical Problems in Engineering*, 2021 (2021). https://doi.org/10.1155/2021/8387262

[17] Mehmood M., Isik H., Uddin F., Shoaib A., New fixed point theorems for orthogonal contractions in incomplete metric spaces, *Carpathian Math.Publ.*, 2021(13) (2021), 405-412. https://doi.org/10.15330/cmp.13.2.405-412

[18] Neog, M., Debnath, P., Radenovic, S., New extension of some common fixed point theorems in complete metric spaces, *Fixed Point Theory*, 20(2) (2019), 567-580.

[19] Rhoades, B. E., Some theorems on weakly contractive maps, *Nonlinear Analysis: Theory, Methods and Applications*, 47(4) (2001), 2683-2693. https://doi.org/10.1016/S0362-546X(01)00388-1

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ON THE WELL-COVEREDNESS OF SQUARE GRAPHS

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ABSTRACT. The *square* of a graph G is obtained from G by putting an edge between two distinct vertices whenever their distance in G is 2. A graph is *well-covered* if every maximal independent set in the graph is of the same size. In this paper, we investigate the graphs whose squares are well-covered. We first provide a characterization of the trees whose squares are well-covered. Afterwards, we show that a bipartite graph G and its square are well-covered if and only if every component of G is K_1 or $K_{r,r}$ for some $r \geq 1$. Moreover, we obtain a characterization of the graphs whose squares are well-covered in the case $\alpha(G) = \alpha(G^2) + k$ for $k \in \{0, 1\}$.

1. INTRODUCTION

A set of vertices in a graph is *independent* if no two vertices in the set are adjacent. If every maximal independent set of vertices has the same cardinality, then the graph is called *well-covered*. These graphs have been introduced by Plummer in [11] and many researches have been done related to them. Most of the research on well-covered graphs appearing in literature has focused on certain subclasses of well-covered graphs such as well-covered line graphs [4], very well covered graphs [6] and well-covered graphs that are 3-regular [3].

The *square* of a graph G, denoted by G^2, is the graph whose vertex set is the same as G, and where two vertices are adjacent in G^2 if and only if their distance is at most 2 in G. Particularly, a graph G is called *square-stable* if it satisfies $\alpha(G) = \alpha(G^2)$ where $\alpha(G)$ denotes the size of a maximum independent set in G. Levit and Mandrescu showed in [8] that every square-stable graph is well-covered, and well-covered trees are exactly the square-stable trees. On the other hand, König–Egerváry square-stable graphs have been studied in [9]. In addition, it has

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been proved in [10] that G^2 is a König–Egerváry graph if and only if G is a square-stable König–Egerváry graph.

In this paper, we study the graphs whose squares are well-covered. We first present some observations for certain graph classes; cycles, paths, P_4-free graphs, and P_5-free graphs. Later, we consider trees, and we define a family $\mathcal{T}$ of trees (see Section 3). Our first result is that the square of a tree is well-covered if and only if the tree is a member of $\mathcal{T}$. We also extend this result to the bipartite graphs that are well-covered. We show that a bipartite graph G and its square are well-covered if and only if every component of G is K_1 or $K_{r,r}$ for some $r \geq 1$. Finally, we consider the graphs satisfying $\alpha(G) = \alpha(G^2) + k$ for $k \in \{0, 1\}$. For the case $k = 0$, we prove that G^2 is well-covered if and only if every component of G is a complete graph. By using this result, we also provide a characterization of the graphs whose squares are well-covered in the case $\alpha(G) = \alpha(G^2) + 1$.

The paper is structured as follows. We start in Section 2 with some definitions and preliminary results on square graphs. In Section 3, we present a characterization of trees whose squares are well-covered, also we extend it to well-covered bipartite graphs. Section 4 is devoted to the square of graphs satisfying $\alpha(G) = \alpha(G^2)+k$ for $k \in \{0, 1\}$. We finish the paper with Section 5 in which we discuss the results that we obtain.

2. Preliminaries

All graphs in this paper are assumed to be simple i.e. finite and undirected, with no loops or multiple edges. We refer to [14] for terminology and notation not defined here. Given a graph $G = (V, E)$ and a subset of vertices S, $G[S]$ denotes the subgraph of G induced by S, and $G - S = G[V - S]$. We denote $G - S$ by $G - v$ when S consists of a single vertex v. For a vertex v, the *open neighbourhood* of v in a subgraph H is denoted by $N_H(v)$ while the *closed neighbourhood* of v is $N_H(v) \cup \{v\}$, denoted by $N_H[v]$. We omit the subscript H whenever there is no ambiguity on H. For a subset $S \subseteq V$, $N_H(S)$ (resp. $N_H[S]$) is the union of the open (resp. closed) neighbourhoods of the vertices in S. We use the notation $[k]$ to denote the set of integers $1, 2, \ldots, k$.

A connected graph with no cycles is called a *tree*. We denote by K_n, C_n and P_n, the complete graph, the cycle and the path on n vertices, respectively. Also, we denote by $K_{r,s}$, the complete bipartite for any $r, s \geq 1$. A star S_k is the complete bipartite graph $K_{1,k}$. The complete bipartite graph $K_{1,3}$ is also known as the *claw*. A subset $S \subset V(G)$ is called a *clique* of G if $G[S]$ is isomorphic to a complete graph. We denote by $d_G(u, v)$ the distance (i.e., the length of the shortest path) between vertices u and v in G.

We say that G is F-free if no induced subgraph of G is isomorphic to F. The degree of a vertex x, the maximum and the minimum degrees of a graph G are denoted by $d_G(x)$, $\Delta(G)$ and $\delta(G)$, respectively. A *leaf* is a vertex with degree one while an *isolated* vertex is a vertex with degree zero. An edge of a graph is said to be *pendant* if one of its vertices is a leaf vertex. If a vertex is adjacent to every other vertex in G, then it is called a *full vertex*. In a graph G, a vertex v is called *simplicial* if its neighbourhood $N_G(v)$ induces a complete graph in G.

A *matching* is a set of edges of G having pairwise no common endvertex. A *perfect matching* of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching.

We start with some known results and observations on the well-coveredness of square graphs.

Theorem 1. *[2] In a graph G, an independent set S is maximum if and only if every independent set disjoint from S can be matched into S.*

Observation 1. *The following properties can be easily obtained.*

(i) *The only paths whose squares are well-covered are P_1, P_2, P_3, P_6.*

(ii) *The only cycles whose squares are well-covered are $C_3, C_4, \ldots, C_8, C_{10}$.*

Since the square of a P_4-free graph is a complete graph, the following holds.

Observation 2. *The squares of P_4-free graphs are well-covered.*

For a graph G, a subset $S \subset V(G)$ is called a *dominating set* of G if any vertex which is not in S is adjacent to a vertex in S. A set S of vertices is said to *dominate* another set T if every vertex in T is adjacent to at least one vertex in S.

Theorem 2. *[1] Every connected P_5-free graph has either a dominating clique or a dominating P_3.*

By using Theorem 2, we shall show that the P_5-free graphs whose squares are well-covered are complete graphs.

Proposition 1. *Let G be a P_5-free graph. Then, G^2 is well-covered if and only if G^2 is a complete graph.*

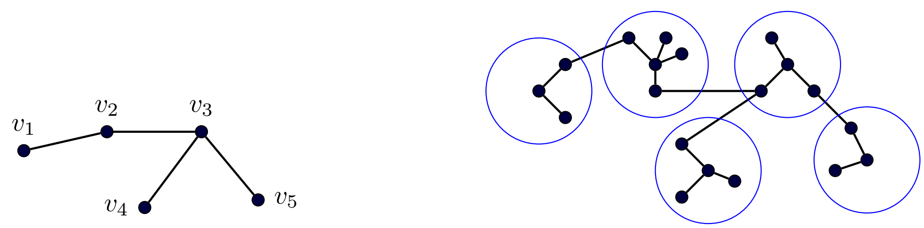
Proof. The sufficiency is clear since complete graphs are well-covered. Thus, we suppose that G^2 is well-covered. Since G is P_5-free, each pair of vertices in G is at distance at most 3. By Theorem 2, we deduce that G has a vertex v which is at distance at most 2 from each vertex of G, and so v is a full vertex in G^2. It follows that G^2 is a complete graph since G^2 is well-covered. □

3. The Square of Bipartite Graphs

In this section, we first consider the square of trees and provide a characterization of those which are well-covered. Later, we extend this result to the bipartite graphs that are well-covered.

For a tree T, we define a class $\mathcal{C}(T)$ of trees as follows. Any member of the class $\mathcal{C}(T)$ is a tree obtained from T by replacing each vertex v with a star S_k for $k \geq 2$, and where if two vertices $u, v \in V(T)$ are adjacent, then we add precisely one edge between two leaf vertices of the corresponding stars so that each star has a pendant edge in the resulting graph. If a graph G is in $\mathcal{C}(T)$, we denote it by $G \cong T(S_{k_1}, S_{k_2}, \ldots, S_{k_n})$ for some stars $\{S_{k_i}\}_{i\in[n]}$ and the tree T with $n = |T|$. For instance, $P_6 = P_2(S_2, S_2)$, $S_k = P_1(S_k)$, and the graph G depicted in Figure 1(b) is $G = T(S_2, S_4, S_3, S_3, S_2)$ for the tree T depicted in Figure 1(a).

Let $\mathcal{T}$ stand for the family of all trees that belong to a class $\mathcal{C}(T)$ for some tree T.



(a) T (b) G

FIGURE 1. A tree T and a member G of $\mathcal{C}(T)$.

Notice that if $G \cong T(S_{k_1}, S_{k_2}, \ldots, S_{k_n})$ for some stars $\{S_{k_i}\}_{i\in[n]}$ and a tree T with $|T| = n$, then we have $\alpha(G^2) = |T| = n$ by taking the centres of all stars, where the equality holds because each star corresponds to a clique in the square of G.

Proposition 2. *[7] If G is a well-covered graph and I is an independent set of vertices in G, then $G - N_G[I]$ must also be well-covered. In particular, $\alpha(G) = \alpha(G - N_G[S]) + |S|$.*

A vertex v of a graph G is called *shedding* if for every independent set S in $G - N_G[v]$, there is a vertex $u \in N_G(v)$ so that $S \cup \{u\}$ is independent. In other words, v is a shedding vertex if there is no independent set $I \subset V(G - N_G[v])$ which dominates $N_G(v)$.

Lemma 1. *For a tree T, the square of every graph in $\mathcal{C}(T)$ is well-covered.*

Proof. Given a tree T with n vertices, suppose that $G \in \mathcal{C}(T)$ and $G \cong T(S_{k_1}, S_{k_2}, \ldots, S_{k_n})$ for some star $\{S_{k_i}\}_{i\in[n]}$. Let H_i be the subgraph induced by the vertices of S_{k_i} in G for $i \in [n]$. By the definition of G, each H_i has at least one vertex that is a leaf in G. Thus the center of each star S_{k_i} is a shedding vertex in G, also in G^2. This implies that each maximal independent set of G^2 contains a vertex of H_i for each $i \in [n]$. Moreover, any maximal independent set of G^2 cannot contain two vertices of H_i for $i \in [n]$ since each H_i induces a clique in G^2. Hence G^2 is a well-covered graph. □

We next give a complete characterization of the trees whose squares are well-covered.

Theorem 3. *The square of a tree is well-covered if and only if the tree belongs to* $\mathcal{T}$.

Proof. The sufficiency has been proved in Lemma 1. So we assume that G is a tree, and G^2 is well-covered. The claim follows when G has at most three vertices, so let $|G| \geq 4$. Consider a leaf vertex v_1 in G, let w_1 be its unique neighbour. Set $G = G_1$, we similarly pick a leaf vertex v_i in $G_i = G_{i-1} - N_{G_{i-1}}[w_{i-1}]$ for $2 \leq i \leq p$ so that $G_p - N_{G_p}[w_p]$ is an edgeless graph where w_i is the unique neighbour of v_i in the graph G_i for each $i \in [p]$. Obviously, each $N_G[w_i]$ induces a star S_{k_i} in G with $k_i = d_{G_i}(w_i)$ for $i \in [p]$. We write $S = \{v_1, v_2 \ldots, v_p\}$, $T = \{w_1, w_2, \ldots, w_p\}$ and $H = G_p - N_{G_p}[w_p]$. Clearly, S is an independent set in G^2, also H is an edgeless graph. On the other hand, $V(H)$ does not need to be an independent set in G^2 while $V(H)$ is an independent set in G. That is, some pair of vertices in $V(H)$ may have a common neighbour in $G - V(H)$. For $u, v, w \in V(H)$, if each pair of u, v, w has a common neighbour c_j in G for $j \in [3]$, then $c_1 = c_2 = c_3$ since G has no cycle. Also, in such a case, we deduce that the vertices u, v, w induce a clique in G^2. Therefore, $V(H)$ induces a graph in G^2 whose each component is a clique or an isolated vertex.

Let R be a maximal subset of $V(H)$ containing no pair having a common neighbour in G. Thus, $S \cup R$ is a maximal independent set in G^2, and it follows that $\alpha(G^2) = |R| + p$ since G^2 is well-covered. In particular, at most $|R|$ vertices of H can be contained in any maximal independent set of G^2 since $V(H)$ induces a graph in G^2 whose each component is a clique or an isolated vertex. On the other hand, let H_i be the subgraph induced by the vertices of $N_{G_i}[w_i]$ in G for $i \in [p]$. Obviously, for each $i \in [p]$, the graph H_i is a star of size at least 2, and so $V(H_i)$ induces a complete graph in G^2. Therefore, any maximal independent set of G^2 contains at most one vertex from each $V(H_i)$ for $i \in [p]$. If there exists a maximal independent set L of G^2, and $\ell \in [p]$ such that L contains no vertex in $V(H_\ell)$, then we deduce that $|L| < |S \cup R|$, since any maximal independent set of G^2 contains at most $|R|$ vertices of H and contains at most one vertex from each $V(H_i)$ for $i \in [p]$. However, this contradicts that G^2 is well-covered. Hence any maximal independent set of G^2 contains exactly one vertex from each H_i for $i \in [p]$.

We now claim that T is an independent set in G^2. Indeed, if there exist $w_i, w_j \in T$ with $i < j$ having a common neighbour z in G, then z is adjacent to all vertices of $V(H_i) \cup V(H_j)$ in G^2. Extending of z into a maximal independent set in G^2 gives a set that does not contain any vertex from H_j. This is a contradiction with the fact that any maximal independent set of G^2 contains exactly one vertex from each H_i for $i \in [p]$. Thus, T is an independent set in G^2. Moreover, T is a dominating set in G^2 since every vertex of H is adjacent to some vertices of $N_{G_j}(w_j)$ for $w_j \in T$ in G. In this manner, T is a maximal independent set in G^2. Since G^2 is well-covered and $|S| = |T|$, we infer that H has no vertex, i.e., $V(H) = \emptyset$. Hence $\alpha(G^2) = |S| = |T| = p$.

Next, we claim that each $w_j \in T$ for $j \in [p]$ is a shedding vertex in G. Assume for a contradiction that there is a vertex $w_j \in T$ such that all vertices of $N_G(w_j)$ are dominated by an independent set $A \subset V(G - N_G[w_j])$ in G. Suppose that A is a minimal set with respect to this property. Then, the path between any pair of vertices in A is of size 5 in G since the graph G is a tree. This implies that A is an independent set in G^2 as well. Clearly, A dominates w_j in G^2 due to $d_{G^2}(a, w_j) = 2$ for every $a \in A$. The extension of A into a maximal independent set in G^2 gives a set that does not contain any vertex in $V(H_j)$. This is a contradiction with the fact that any maximal independent set of G^2 contains exactly one vertex from each H_i for $i \in [p]$. We then conclude that each w_j for $j \in [p]$ is a shedding vertex in G.

Finally we claim that for each $i \in [p]$, H_i consists of at least three vertices. Assume to the contrary that there exists $i \in [p]$ such that $N_{G_i}[w_i]$ induces a K_2 in G, so $N_{G_i}[w_i] = \{w_i, v_i\}$. Note that w_i cannot be adjacent to a leaf vertex of a star H_k for $k \in [p] \setminus \{i\}$ since T is an independent set in G^2. It then follows from the connectivity of G that there exists a star H_j induced by $N_{G_j}[w_j]$ for $j \in [p] \setminus \{i\}$ such that v_i is adjacent to a leaf vertex z of the star H_j. This implies that z is adjacent to all vertices of $V(H_i) \cup V(H_j)$ in G^2. Hence $(T - \{w_i, w_j\}) \cup \{z\}$ is a maximal independent set in G^2, a contradiction.

Consequently, we have a tree L such that $G = L(S_{k_1}, \ldots, S_{k_p})$ where $S_{k_i} = H_i$ for $i \in [p]$, and L is the graph on the vertex set $T = \{w_1, w_2, \ldots, w_p\}$ such that two vertices are adjacent in L if they are at distance 3 in G. Observe that the centres of two stars S_{k_i} and S_{k_j} have no common neighbour in G since T is an independent set in G^2. Also, each S_{k_i} has a pendant edge in G since L is connected and each w_i is a shedding vertex in G. Hence, G belongs to $\mathcal{C}(L)$. ☐

We now turn our attention to the square of bipartite graphs. Let us first give a useful result on well-covered bipartite graphs as follows.

Theorem 4. *[12,13] Let G be a connected bipartite graph. Then G is well-covered if and only if G has a perfect matching M such that for every edge $uv \in M$, $N_G[\{u, v\}]$ induces a complete bipartite graph.*

Lemma 2. *Let G be a connected bipartite graph with at least 2 vertices. If G and G^2 are well-covered, then $G = K_{r,r}$ for $r \geq 1$.*

Proof. Suppose that G is a connected bipartite graph with a bipartition I_1 and I_2 where $|I_i| \geq 1$ for $i \in \{1, 2\}$. Assume that G and G^2 are well-covered. By Theorem 4, G has a perfect matching M, and clearly $|I_1| = |I_2| = r$ for $r \in \mathbb{N}$. Let $I_1 = \{x_1, x_2, \ldots, x_r\}$, $I_2 = \{y_1, y_2, \ldots, y_r\}$, and $M = \{x_1y_1, x_2y_2, \ldots, x_ry_r\}$. It follows from Theorem 4 that for every edge $x_iy_i \in M$, $N_G[\{x_i, y_i\}]$ induces a complete bipartite graph.

Assume for a contradiction that $G \neq K_{r,r}$. Then, there exist $i, j \in [r]$ with $i \neq j$ such that $x_iy_j \notin E(G)$ and $x_jy_i \in E(G)$ since G is connected. We may assume, without loss of generality, that $i = 1$ and $j = 2$. Recall that for every edge $x_ky_k \in M$, $N_G[\{x_k, y_k\}]$ induces a complete bipartite graph. Therefore, every vertex in $N_G(x_1)$ is adjacent to each vertex of $N_G(y_1)$. Similarly, every vertex in $N_G(y_2)$ is adjacent to each vertex of $N_G(x_2)$. This implies that $N_G(x_1)$ is complete to $N_G(y_1)$, also $N_G(y_2)$ is complete to $N_G(x_2)$. Thus, $N_G(x_1) \subseteq N_G(x_2)$ and $N_G(y_2) \subseteq N_G(y_1)$. Consequently, y_1 is adjacent to all vertices of $N_G[\{x_1, y_2\}]$ in the graph G^2.

Consider the graph $G^2 - N_{G^2}[y_1]$, clearly it is a well-covered graph by Proposition 2. Since y_1 is adjacent to all vertices of $N_G[\{x_1, y_2\}]$ in the graph G^2, we deduce that none of x_1, y_2 is adjacent to a vertex of the graph $G^2 - N_{G^2}[y_1]$, i.e., they are isolated vertices in $G^2 - N_{G^2}[y_1]$. Then for a maximal independent set S in $G^2 - N_{G^2}[y_1]$, we observe that $S \cup \{y_1\}$ and $S \cup \{x_1, y_2\}$ are two maximal independent sets in G^2 with different sizes, contradicting to the well-coveredness of G^2. Hence, $G = K_{r,r}$ for $r \geq 1$. □

The following is an immediate consequence of Lemma 2 together with the fact that the square of $K_{r,r}$ for $r \in \mathbb{N}$ is a complete graph.

Theorem 5. *A bipartite graph G and its square are well-covered if and only if every component of G is either K_1 or $K_{r,r}$ for some $r \geq 1$*

It was shown in [12] that a tree with at least two vertices is well-covered if and only if it has a perfect matching consisting of pendant edges. Then we have the following by Theorems 3 and 5.

Corollary 1. *A tree T and its square are well-covered if and only if T is either K_1 or K_2.*

4. Square-Stable and Well-Covered Graphs

Recall that a graph G is square-stable if it satisfies $\alpha(G) = \alpha(G^2)$. It was shown in [8] that square-stable graphs are well-covered. However, the square of a square-stable graph does not need to be well-covered; e.g., the square of P_4 is not well-covered.

In this section, we investigate the squares of square-stable graphs as well as the squares of graphs satisfying $\alpha(G) = \alpha(G^2) + 1$.

Theorem 6. *[8] Any square-stable graph is well-covered.*

In what follows, we state our first result in this section, which is the characterization of the square-stable graphs whose squares are well-covered.

Theorem 7. *Let G be a square-stable graph. Then, G^2 is well-covered if and only if every component of G is a complete graph.*

Proof. The sufficiency is clear since any complete graph is well-covered. Thus, we suppose that G^2 is well-covered. Note that every component of a well-covered graph is also well-covered. In addition, every component of a square-stable graph is also square-stable. Let H be a component of G. Then, H is square-stable, and H^2 is well-covered, so $\alpha(H) = \alpha(H^2) = k$. By contradiction suppose that H is not a complete graph. Then, there exist $u, v, w \in V(H)$ such that u, v are non-adjacent and $u, v \in N_H(w)$. Consider the graph $H^2 - N_{H^2}[w]$, it is clearly well-covered by Proposition 2, and $\alpha(H^2 - N_{H^2}[w]) = k - 1$. Also, if S is an independent set in $H^2 - N_{H^2}[w]$, then S is independent in $H - N_{H^2}[w]$ as well. Thus, $\alpha(H - N_{H^2}[w]) \geq k - 1$. Notice that $H - N_{H^2}[w]$ has neither u, v nor their neighbours in H. However, $S \cup \{u, v\}$ induces an independent set in H, and so $\alpha(H) \geq |S \cup \{u, v\}| = k + 1$, contradicting that $\alpha(H) = \alpha(H^2) = k$. Hence, H is a complete graph. □

A graph G is called *almost well-covered*, which is introduced in [5], if any maximal independent set is of size $\alpha(G)$ or $\alpha(G) - 1$.

Unlike square-stable graphs, we now consider the graphs satisfying $\alpha(G) = \alpha(G^2) + 1$.

Proposition 3. *If G is a graph with $\alpha(G) = \alpha(G^2) + 1$, then G is either well-covered or almost well-covered.*

Proof. Assume for a contradiction that G has a maximal independent set T such that $|T| \leq \alpha(G) - 2$. Let $\alpha(G) = k$, we pick a maximum independent set S in G^2, and so $|S| = k - 1$. Obviously, S is an independent set in G as well. Also, $d_G(u, v) \geq 3$ for every $u, v \in S$.

First we assume that S is maximal in G. Then, every vertex of $T - S$ has a neighbour in $S - T$. Notice that a vertex of $T - S$ cannot have more than one neighbour in $S - T$ since $d_G(u, v) \geq 3$ for every $u, v \in S$. We also note that $|S| \geq |T| + 1$, and let $R := N_G(T - S) \cap S$. It then follows that $T \cup (S - R)$ is a maximal independent set including T in the graph G with $|T \cup (S - R)| \geq \alpha(G) - 1$, contradicting to the maximality of T.

We now assume that S is not maximal in G. Then, there exists a vertex $u \in V(G) - S$ such that $S \cup \{u\}$ is an independent set in G. In fact, $S \cup \{u\}$ is a maximum independent set in G since $\alpha(G) = \alpha(G^2) + 1$. We write $S' := S \cup \{u\}$, and clearly $|S'| = \alpha(G)$. Similarly as before, consider T and S', if $u \in T \cap S'$, then it turns out the previous case. Thus, we further assume that $u \notin T$, i.e., $u \in S' - T$. Let $R = N_G(T - S') \cap S'$. It then follows that $T - S'$ has at most $|T - S'| + 1$ neighbours in S' since $d_G(x, y) \geq 3$ for every $x, y \in S' - u$. In particular $|R| \leq |T - S'| + 1$. We therefore deduce that $T \cup (S' - R)$ is a maximal independent set including T in the graph G with $|T \cup (S' - R)| \geq \alpha(G) - 1$, contradicting to the maximality of T. Consequently, G has no maximal independent set of size at most $\alpha(G) - 2$. This completes the proof. □

We next deal with the graphs satisfying $\alpha(G) = \alpha(G^2) + 1$ under the assumption that G^2 is well-covered. We manage to characterize those graphs in Theorem 8 with a series of lemmas.

Lemma 3. *Let G be a graph with $\alpha(G) = \alpha(G^2) + 1$. If G^2 is well-covered, then G is claw-free.*

Proof. Let G^2 be a well-covered graph. Assume for a contradiction that G contains a claw. Let $\{x, y, z\}$ be an independent set in G, and suppose that w is adjacent to all vertices of $\{x, y, z\}$ in the graph G. Pick a maximal independent set S containing w in G^2, clearly S is also maximum in G^2 due to the well-coveredness of G^2. Thus $\alpha(G^2) = |S|$ and $\alpha(G) = |S|+1$. On the other hand, $G^2 - N_{G^2}[w]$ is well-covered by Proposition 2, and $\alpha(G^2 - N_{G^2}[w]) = |S| - 1$. Notice that $S - w$ is an independent set in $G - N_{G^2}[w]$. It then follows that $(S - w) \cup \{x, y, x\}$ is independent set in G. However, this contradicts that $\alpha(G) = \alpha(G^2) + 1$. Hence, G is claw-free. □

Lemma 4. *Let G be a graph with $\alpha(G) = \alpha(G^2) + 1$. Suppose that G^2 is well-covered. If v is a non-simplicial vertex in G, then every component of $G - N_{G^2}[v]$ is a complete graph.*

Proof. Suppose that v is a non-simplicial vertex in G, and let $\alpha(G^2) = k$. Then v has two non-adjacent neighbours x, y in G. Consider a maximum independent set S containing v in G^2, obviously it is an independent set in G as well. Also $\alpha(G) = |S| + 1 = k + 1$ since $\alpha(G) = \alpha(G^2) + 1$. Note that $S' = (S - v) \cup \{x, y\}$ is an independent set in G. Since $\alpha(G) = \alpha(G^2) + 1$, we then deduce that S' is a maximum independent set in G. Consider now the graph $H = G - N_{G^2}[v]$, if there exists an independent set T larger than $S - v$ in H, then $T \cup \{x, y\}$ would be an independent set in G of size at least $|S|+2 = \alpha(G)+1$, a contradiction. Therefore, $S - v$ is a maximum independent set in H, and so $\alpha(H) = |S| - 1 = k - 1$.

Now we claim that H^2 is a well-covered graph. Obviously, $S-v$ is an independent set in H^2, so $\alpha(H^2) \geq k - 1$. If H^2 has a maximal independent set T larger than $S - v$, then T would be an independent set in H as well, which contradicts the fact that $S-v$ is a maximum independent set in H. Hence, $\alpha(H^2) = |S|-1 = k-1$. This implies that H is square-stable, and so H is well-covered by Theorem 6. It remains to show that H^2 has no maximal independent set smaller than $S-v$. Assume to the contrary that H^2 has a maximal independent set T with $|T| \leq k - 2$. Obviously, T is not independent set in $G^2 - N_{G^2}[v]$ since otherwise $T \cup \{v\}$ would be a maximal independent set in G^2, contradicting that G^2 is well-covered with $\alpha(G^2) = k = |S|$. This implies that some vertices of T are adjacent in $G^2 - N_{G^2}[v]$ while they are non-adjacent in H^2. Then there exists $p, q \in T$ and a vertex $z \in N_{G^2}(v)$ which is at distance 2 from v in G such that z is a common neighbour of p and q (see Figure 2). However, we observe that w, z, p, q induce a claw in G where w is a common neighbour of v and z in G, a contradiction by Lemma 3. Consequently, H is square-stable, and H^2 is well-covered. Hence, the result follows from Theorem 7. □

We now ready to prove our second main result in this section.

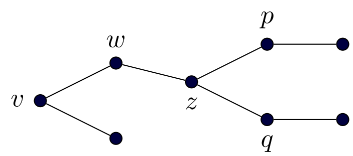


FIGURE 2. An illustration of the vertices v, w, z, p, q in the graph G.

Theorem 8. *Let G be a graph with $\alpha(G) = \alpha(G^2) + 1$, and $k \in \mathbb{N}$. Then, G^2 is well-covered if and only if for every non-simplicial vertex v, the graph $G - N_{G^2}[v]$ consists of k complete graphs such that any pair of such components has no common neighbour in G.*

Proof. The sufficiency follows from Lemmas 3 and 4. So we assume that for every non-simplicial vertex v, the graph $G - N_{G^2}[v]$ consists of k complete graphs such that any two such components have no common neighbour in G. Let $C_1, C_2, \ldots, C_k$ be the components of $G - N_{G^2}[v]$ for a non-simplicial vertex v in G where each C_i is a complete graph. Let x, y be two non-adjacent neighbours of v in G, and let u_i be a vertex of C_i for $i \in [k]$. Note that $d_G(u_i, u_j) \geq 3$ for any pair $u_i, u_j \in I$ with $i \neq j$ since no pair of the components $C_1, C_2, \ldots, C_k$ has a common neighbour in G. Consider the set $\{x, y, u_1, u_2, \ldots, u_k\}$, it is an independent set G. Thus, $\alpha(G) \geq k + 2$. On the other hand, any maximal independent set containing v in G^2 can have at most one vertex from each C_i. Thus, such a maximal independent set has at most $k + 1$ vertices, and so $\alpha(G^2) \leq k + 1$. By combining $\alpha(G) \geq k + 2$ and $\alpha(G^2) \leq k + 1$ together with the fact that $\alpha(G) = \alpha(G^2) + 1$, we deduce that $\alpha(G) = k + 2 = \alpha(G^2) + 1$. This also implies that $\{x, y, u_1, u_2, \ldots, u_k\}$ is a maximum independent set in G.

It only remains to show that G^2 has no maximal independent set of size less than $k + 1$. Assume to the contrary that there exists such a maximal independent set I of size r in G^2 with $r \leq k$. Clearly, $d_G(u, v) \geq 3$ for each pair $u, v \in I$. If a vertex $v \in I$ is a non-simplicial vertex, then $G - N_{G^2}[v]$ consists of k complete graphs. Also each vertex in $I - v$ belongs to a component of $G - N_{G^2}[v]$. Since $|I - v| = r - 1 < k$, the set I does not contain any vertex of some component of $G - N_{G^2}[v]$, a contradiction with the maximality of I in G^2. We further suppose that all vertices of I are simplicial in G. Clearly I is an independent set in G. However, it is not maximal in G by Proposition 3, since $\alpha(G) = k + 2 \geq |S| + 2$. Then, there exists an independent set $T \subset V(G) - I$ such that $I \cup T$ is a maximal independent set in G. Let u be a vertex in T. Recall that I is a maximal independent set in G^2, and so u is at distance 2 from some vertices of I. It follows that there exist $w \in I$ and $z \in V(G) - (I \cup T)$ such that z is a common neighbour of u and w in G, and so z is a non-simplicial vertex in G. By assumption, the graph $G - N_{G^2}[z]$ consists of k complete graphs such that any two of such components have no common neighbour in G. Similarly as before, let $D_1, D_2, \dots, D_k$ be the components of $G - N_{G^2}[z]$ where each D_i is a complete graph. Let u_i be a vertex of D_i for $i \in [k]$.

Notice that every vertex in $I - w$ is at distance at least 2 from z in G since I is a maximal independent set in G^2. By the same reason, for a vertex $x \in (I - w) \cap N_{G^2}[z]$, we have $d_G(x, s) \geq 3$ for every $s \in I$. We then deduce that $x \notin N_G[z]$. Also, $x \notin N_G(u)$ since $S \cup T$ is an independent set in G. Thus, if there exists a vertex $x \in (I - w) \cap N_{G^2}[z]$ such that x is not adjacent to any D_i, then this gives a contradiction since the set $\{u, w, x, u_1, u_2, \dots, u_k\}$ would be an independent set in G of size $\alpha(G) + 1$. Thus, every vertex $x \in (I - w) \cap N_{G^2}[z]$ is adjacent to a D_i. However, if there is such a vertex x, then x would have two non-adjacent neighbours; one is from $N_G(z)$, and the other is from a C_i for $i \in [k]$, it contradicts that all vertices of I are simplicial in G. Consequently, we deduce that $(I - w) \cap N_{G^2}[z] = \emptyset$. Thus every vertex of $I - w$ comes from the D_i's. But, this again contradicts that I is maximal in G since $|I - w| = r - 1 < k$. Hence, G^2 is well-covered. □

5. Conclusion

In this paper, we studied the graphs whose squares are well-covered. After we introduced some basic observations on those graphs, we exhibited an infinite family $\mathcal{T}$ of trees. We provided a characterization of the trees whose square well-covered which is based on the family $\mathcal{T}$. Also, we extended this result into bipartite graphs that are well-covered.

In the second part, we were interested in the graphs satisfying $\alpha(G) = \alpha(G^2) + k$ for $k \in \{0, 1\}$ where the case $\alpha(G) = \alpha(G^2)$ is also known as the square-stable graphs. Levit and Mandrescu showed in [8] that every square-stable graph is well-covered, and well-covered trees are exactly the square-stable trees. By using this result, we first proved that for the case $k = 0$, G^2 is well-covered if and only if every component of G is a complete graph. Moreover, the graphs for the case $k = 1$ have been characterized. In fact, we showed that if G^2 is well-covered, and v is a non-simplicial vertex in G, then every component of $G - N_{G^2}[v]$ is a complete graph. We conjecture that indeed $G - N_{G^2}[v]$ consists of a unique complete graph. That is, we believe that $\alpha(G^2) = 2$ when G^2 is a connected well-covered graph with $\alpha(G) = \alpha(G^2) + 1$.

Declaration of Competing Interests The author declares that there is no a competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

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References

[1] Bacsó, G., Tuza, Z., Dominating cliques in P_5-free graphs, *Periodica Mathematica Hungarica, 21* (4) (1990), 303–308, https://dx.doi.org/10.1007/bf02352694.

[2] Berge, C., Some Common Properties for Regularizable Graphs, Edge-Critical Graphs and B-graphs, Springer, 1981, https://dx.doi.org/10.1007/3-540-10704-5_10.

[3] Campbell, S., Ellingham, M., Royle, G., A characterization of well-covered cubic graphs, *Journal of Combinatorial Computing, 13* (1993), 193–212.

[4] Demange, M., Ekim, T., Efficient recognition of equimatchable graphs, *Information Processing Letters, 114* (1-2) (2014), 66–71, https://dx.doi.org/10.1016/j.ipl.2013.08.002.

[5] Ekim, T., Gozupek, D., Hujdurovic, A., Milanic, M., On almost well-covered graphs of girth at least 6., *Discrete Mathematics and Theoretical Computer Science, 20* (2) (2018), 1i–1i, https://dx.doi.org/10.23638/DMTCS-20-2-17.

[6] Favaron, O., Very well covered graphs, *Discrete Mathematics, 42* (2-3) (1982), 177–187, https://dx.doi.org/10.1016/0012-365X(82)90215-1.

[7] Finbow, A., Hartnell, B., Nowakowski, R. J., A characterization of well covered graphs of girth 5 or greater, *Journal of Combinatorial Theory, Series B, 57* (1) (1993), 44–68, https://dx.doi.org/10.1006/jctb.1993.1005.

[8] Levit, V. E., Mandrescu, E., Square-stable and well-covered graphs, *Acta Universitatis Apulensis, 10* (2005), 297–308.

[9] Levit, V. E., Mandrescu, E., On König–Egerváry graphs and square-stable graphs, *Acta Univ.Apulensis, Special Issue* (2009), 425–435.

[10] Levit, V. E., Mandrescu, E., When is G^2 a König–Egerváry Graph?, *Graphs and Combinatorics, 29* (5) (2013), 1453–1458, https://dx.doi.org/10.1007/s00373-012-1196-5.

[11] Plummer, M. D., Some covering concepts in graphs, *Journal of Combinatorial Theory, 8* (1) (1970), 91–98, https://dx.doi.org/10.1016/S0021-9800(70)80011-4.

[12] Ravindra, G., Well-covered graphs, *Journal of Combinatorics and System Sciences, 2* (1) (1977), 20–21.

[13] Staples, J. A. W., On Some Subclasses of Well-Covered Graphs, PhD thesis, Vanderbilt University, 1975.

[14] West, D. B., Introduction to Graph Theory, vol. 2, Prentice Hall Upper Saddle River, 2001.

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NOTES ON THE SECOND-ORDER TANGENT BUNDLES WITH THE DEFORMED SASAKI METRIC

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Abstract. The paper deals with the second-order tangent bundle T^2M with the deformed Sasaki metric $\overline{g}$ over an $n-$dimensional Riemannian manifold (M, g). We calculate all Riemannian curvature tensor fields of the deformed Sasaki metric $\overline{g}$ and search Einstein property of T^2M. Also the weakly symmetry properties of the deformed Sasaki metric are presented.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold and T^2M be its second-order tangent bundle. Second-order tangent bundles are of importance in differantial geometry. The geometry of the second-order tangent bundle T^2M over an $n-$dimensional manifold M which is the equivalent classes of curves with the same aceleration vector fields on M was studied in [9–12]. Dodson and Radivoiovici proved that a second-order tangent bundle T^2M of finite $n-$dimensional M becomes a vector bundle over M if and only if M has a linear connection in [6]. The lifts of tensor fields and connections given on M to its second-order tangent bundle T^2M were developped in [12]. In [7], Ishikawa defined a Sasaki-type lift metric in T^2M of a Riemanian manifold and investigated some of its properties. Moreover, in [3], the geometry of a second-order tangent bundle with a Sasaki-type metric was studied in detail. All forms of Riemannian curvature tensor of Sasaki metric on T^2M were computed and some curvature properties were examined in [8].

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In the present paper, motivated by the above works we study some geometric properties of the second-order tangent bundle T^2M equipped with the deformed Sasaki metric $\overline{g}$. We introduce the deformed Sasaki metric on T^2M over M and obtain the global results. Throughout this paper, all geometric objects assumed to be differentiable of class C^∞.

2. Preliminaries

Let M be an $n-$dimensional Riemannian manifold endowed with a linear connection ∇ and T^2M second-order tangent bundle be $3n-$dimensional manifold. T^2M has a natural bundle structure over M, $\pi_2 : T^2M \longrightarrow M$ denoting the canonical projection. If the canonical projection is denoted by $\pi_{12} : T^2M \longrightarrow TM$, then T^2M has a bundle structure over the tangent bundle TM with projection π_{12}. Let $\left(U, x^i\right)$ be a coordinate neighborhood of M and f be a curve in U which locally expressed as $x^i = f^i(t)$. If we take a 2−jet $j^2 f$ belonging to $\pi_2^{-1}(U)$ and put

$$x^i = f^i(0),\ y^i = \frac{df^i}{dt}(0),\ z^i = \frac{d^2 f^i}{dt^2}(0),$$

then the 2−jet $j^2 f$ is expressed in a unique by the set $\left(x^i, y^i, z^i\right)$. Thus a system of coordinates $\left(x^i, y^i, z^i\right)$ is introduced in the open set ${\pi_2}^{-1}(U)$ of T^2M from $\left(U, x^i\right)$. The coordinates $\left(x^i, y^i, z^i\right)$ in ${\pi_2}^{-1}(U)$ are called the induced coordinates. On putting

$$\xi^i = x^i, \xi^{\overline{i}} = y^i, \xi^{\overline{\overline{i}}} = z^i,$$

the induced coordinates $\left(x^i, y^i, z^i\right)$ are denoted by $\{\xi^A\}$. The indices $A, B, C, \ldots$ take values $\{1,\ 2, \ldots,\ n;\ n+1,\ n+2, \ldots, 2n;\ 2n+1,\ 2n+2,\ \ldots, 3n\}$.

Let $X = X^i \frac{\partial}{\partial X^i}$ be the local expression in U of a vector field X on M. Then the vector fields X^{H_0}, X^{H_1} and X^{H_2} on T^2M are given, with respect to the induced coordinates $\{\xi^A\}$, by [4]

$$X^{H_0} = X^j \partial_j - u^s \Gamma^j_{sh} X^h \partial_{\overline{j}} - C^j_h X^h \partial_{\overline{\overline{j}}}, \tag{1}$$

$$X^{H_1} = X^j \partial_{\overline{j}} - 2u^s \Gamma^j_{sh} X^h \partial_{\overline{\overline{j}}} \tag{2}$$

ve

$$X^{H_2} = X^j \partial_{\overline{\overline{j}}} \tag{3}$$

with respect to the natural frame $\{\partial_A\} = \left\{\ \partial_i,\ \partial_{\overline{i}},\ \partial_{\overline{\overline{i}}}\right\}$ in T^2M, where $C^j_h = z^m \Gamma^j_{hm} + u^s u^r (\partial_h \Gamma^j_{sr} + \Gamma^j_{hm} \Gamma^m_{sr} - 2\Gamma^j_{sm} \Gamma^m_{hr})$, Γ^j_{sr} are the coefficients of the Levi-Civita connection ∇ on M and $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\overline{i}} = \frac{\partial}{\partial y^i}$, $\partial_{\overline{\overline{i}}} = \frac{\partial}{\partial z^i}$. For the Lie bracket on T^2M in terms of the $\lambda-$lifts of vector fields X, Y on M, we have the following formulas:

$$\begin{aligned}
\left[X^{H_0}, Y^{H_0}\right] &= [X,Y]^{H_0} - (R(X,Y)u)^{H_1} - (R(X,Y)\omega)^{H_2},\\
\left[X^{H_0}, Y^{H_\mu}\right] &= (\nabla_X Y)^{H_\mu}, \quad \mu = 1,2\ \left[X^{H_\mu}, Y^{H_\alpha}\right] = 0, \quad \mu, \alpha = 1,2
\end{aligned}$$

where R is the curvature tensor field of the connection ∇ on M defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ [5].

With the connection ∇ of g on M, we can introduce a frame field in each induced coordinate neighborhood ${\pi_2}^{-1}(U)$ of T^2M. In each coordinate neighborhood (U, x^i), using $X_i = \frac{\partial}{\partial x^i}$, from (1)-(3) we have

$$E_i = (X_i)^{H_0} = \left(\frac{\partial}{\partial x^i}\right)^{H_0} = \partial_i - u^s \Gamma_{is}^k \partial_{\overline{k}} - C_i^k \partial_{\overline{\overline{k}}},$$

$$E_{\overline{i}} = (X_i)^{H_1} = \left(\frac{\partial}{\partial x^i}\right)^{H_1} = \partial_{\overline{i}} - 2u^s \Gamma_{is}^k \partial_{\overline{\overline{k}}},$$

$$E_{\overline{\overline{i}}} = (X_i)^{H_2} = \left(\frac{\partial}{\partial x^i}\right)^{H_2} = \partial_{\overline{\overline{i}}}$$

with respect to the natural frame $\{\partial_A\}$ in T^2M [4].

3. The Deformed Sasaki Metric and Its Levi-Civita Connection

Definition 1. *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle. The deformed Sasaki metric on the second-order tangent bundle over (M, g) is defined by the identities:*

$$\begin{cases} \overline{g}\left(X^{H_0}, Y^{H_0}\right) = fg(X,Y), \\ \overline{g}\left(X^{H_a}, Y^{H_b}\right) = 0, \quad a \neq b \\ \overline{g}\left(X^{H_1}, Y^{H_1}\right) = \overline{g}\left(X^{H_2}, Y^{H_2}\right) = g(X,Y), \end{cases}$$

for all vector fields X and Y on M, where f is a positive smooth function on (M, g).

The deformed Sasaki metric $\overline{g}$ and its inverse have components

$$\overline{g}_{\beta\gamma} = \begin{pmatrix} fg_{ij} & 0 & 0 \\ 0 & g_{ij} & 0 \\ 0 & 0 & g_{ij} \end{pmatrix} \text{ and } \overline{g}^{\gamma\alpha} = \begin{pmatrix} \frac{1}{f} g^{jk} & 0 & 0 \\ 0 & g^{jk} & 0 \\ 0 & 0 & g^{jk} \end{pmatrix}$$

with recpect to the adepted frame $\{E_\beta\}$. In adapted frame, the followings satisfy

$$[E_\beta, E_\gamma] = {\Omega_{\beta\gamma}}^{\varepsilon} E_\varepsilon,$$

$${\Omega_{ij}}^{\overline{k}} = u^p {R_{jip}}^k, \quad {\Omega_{i\overline{j}}}^{\overline{k}} = \Gamma_{ij}^k,$$

$${\Omega_{ij}}^{\overline{\overline{k}}} = \omega^s {R_{jis}}^k, \quad {\Omega_{i\overline{\overline{j}}}}^{\overline{\overline{k}}} = \Gamma_{ij}^k.$$

Using the formula

$$\begin{aligned} \overline{\Gamma}_{\beta\gamma}^{\varepsilon} &= \frac{1}{2} \overline{g}^{\varepsilon\alpha} \left(E_\beta \overline{g}_{\alpha\gamma} + E_\gamma \overline{g}_{\alpha\beta} - E_\alpha \overline{g}_{\beta\gamma}\right) \\ &\quad + \frac{1}{2}\left({\Omega_{\beta\gamma}}^{\varepsilon} + {\Omega_{\cdot\,\beta\gamma}}^{\varepsilon} + {\Omega_{\cdot\,\gamma\beta}}^{\varepsilon}\right), \end{aligned}$$

where $\Omega_{.\ \beta\gamma}^{\ \varepsilon} = \overline{g}^{\alpha\varepsilon}\overline{g}_{\delta\gamma}\Omega_{\alpha\beta}{}^{\delta}$, the non-zero components $\overline{\Gamma}^{\varepsilon}_{\beta\gamma}$ are given by

$$\overline{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{.}{}^{k}g_{ij}\right),$$

$$\overline{\Gamma}^{\overline{k}}_{ij} = \frac{1}{2}u^p R_{jip}{}^{k},\ \overline{\Gamma}^{\overline{\overline{k}}}_{ij} = \frac{1}{2}\omega^s R_{jis}{}^{k},$$

$$\overline{\Gamma}^{k}_{\overline{i}j} = \frac{1}{2f}u^p R_{pij}{}^{k},\ \overline{\Gamma}^{k}_{i\overline{j}} = \frac{1}{2f}u^p R_{pji}{}^{k},\ \overline{\Gamma}^{\overline{k}}_{i\overline{j}} = \Gamma^{k}_{ij},$$

$$\overline{\Gamma}^{k}_{\overline{\overline{i}}j} = \frac{1}{2f}\omega^s R_{sij}{}^{k},\ \overline{\Gamma}^{k}_{i\overline{\overline{j}}} = \frac{1}{2f}\omega^s R_{sji}{}^{k},\ \overline{\Gamma}^{\overline{\overline{k}}}_{i\overline{\overline{j}}} = \Gamma^{k}_{ij},$$

with respect to the adapted frame $\{E_\beta\}$.

Proposition 1. *Let (M,g) be a Riemannian manifold and $\overline{\nabla}$ be a Levi-Civita connection of $\left(T^2M,\overline{g}\right)$. Then we have*

1) $\overline{\nabla}_{E_i}E_j = \left\{\Gamma^k_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{.}{}^{k}g_{ij}\right)\right\}E_k + \frac{1}{2}u^p R^k_{jip}E_{\overline{k}} + \frac{1}{2}\omega^s R^k_{jis}E_{\overline{\overline{k}}}$,
2) $\overline{\nabla}_{E_{\overline{i}}}E_j = \frac{1}{2f}u^p R^k_{pij}E_k$,
3) $\overline{\nabla}_{E_i}E_{\overline{j}} = \frac{1}{2f}u^p R^k_{pji}E_k + \Gamma^k_{ij}E_{\overline{k}}$,
4) $\overline{\nabla}_{E_{\overline{\overline{i}}}}E_j = \frac{1}{2f}\omega^s R_{sij}{}^{k}E_k$,
5) $\overline{\nabla}_{E_i}E_{\overline{\overline{j}}} = \frac{1}{2f}\omega^s R^k_{sji}E_k + \Gamma^k_{ij}E_{\overline{\overline{k}}}$,
6) $\overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}} = 0$, $\overline{\nabla}_{E_{\overline{i}}}E_{\overline{\overline{j}}} = 0$, $\overline{\nabla}_{E_{\overline{\overline{i}}}}E_{\overline{j}} = 0$, $\overline{\nabla}_{E_{\overline{\overline{i}}}}E_{\overline{\overline{j}}} = 0$

with respect to the adapted frame $\{E_\beta\}$.

Proof. 1) By applying

$$\overline{\Gamma}^{\varepsilon}_{\beta\gamma} = \frac{1}{2}\overline{g}^{\varepsilon\alpha}\left(E_\beta\overline{g}_{\alpha\gamma} + E_\gamma\overline{g}_{\alpha\beta} - E_\alpha\overline{g}_{\beta\gamma}\right) + \frac{1}{2}\left(\Omega_{\beta\gamma}{}^{\varepsilon} + \Omega_{.\ \beta\gamma}^{\ \varepsilon} + \Omega_{.\ \gamma\beta}^{\ \varepsilon}\right)$$

and direct calculation we get

$$\overline{\nabla}_{E_i}E_j = \overline{\Gamma}^{K}_{ij}E_K = \overline{\Gamma}^{k}_{ij}E_k + \overline{\Gamma}^{\overline{k}}_{ij}E_{\overline{k}} + \overline{\Gamma}^{\overline{\overline{k}}}_{ij}E_{\overline{\overline{k}}}$$

$$\begin{aligned}
\overline{\Gamma}^{k}_{ij} &= \frac{1}{2}\overline{g}^{sk}\left(E_i\overline{g}_{sj} + E_j\overline{g}_{si} - E_s\overline{g}_{ij}\right) + \frac{1}{2}\left(\Omega_{ij}{}^{k} + \Omega_{.\ ij}^{\ k} + \Omega_{.\ ji}^{\ k}\right) \\
&= \frac{1}{2}\overline{g}^{sk}\left(\partial_i\left(fg_{sj}\right) + \partial_j\left(fg_{si}\right) - \partial_s\left(fg_{ij}\right)\right) \\
&= \frac{1}{2}g^{sk}\left(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\right) + \frac{1}{2f}g^{sk}\left(f_i g_{sj} + f_j g_{si} - f_s g_{ij}\right) \\
&= \Gamma^k_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{.}{}^{k}g_{ij}\right)\overline{\Gamma}^{\overline{k}}_{ij} \\
&= \frac{1}{2}\overline{g}^{\overline{s}\overline{k}}\left(E_i\overline{g}_{\overline{s}j} + E_j\overline{g}_{\overline{s}i} - E_{\overline{s}}\overline{g}_{ij}\right) + \frac{1}{2}\left(\Omega_{ij}{}^{\overline{k}} + \Omega_{.\ ij}^{\ \overline{k}} + \Omega_{.\ ji}^{\ \overline{k}}\right)
\end{aligned}$$

$$= \frac{1}{2}u^p R^k_{jip}\ \overline{\Gamma}^{\overline{\overline{k}}}_{ij} = \frac{1}{2}\overline{g}^{\overline{\overline{s}}\overline{\overline{k}}}\left(E_i\overline{g}_{\overline{\overline{s}}j} + E_j\overline{g}_{\overline{\overline{s}}i} - E_{\overline{\overline{s}}}\overline{g}_{ij}\right)$$
$$+\frac{1}{2}\left(\Omega_{ij}{}^{\overline{\overline{k}}} + \Omega_{\cdot}{}^{\overline{\overline{k}}}{}_{ij} + \Omega_{\cdot}{}^{\overline{\overline{k}}}{}_{ji}\right)$$
$$= \frac{1}{2}\omega^s R^k_{jis}.$$

Thus we obtain

$$\overline{\nabla}_{E_i}E_j = \left\{\Gamma^k_{ij} + \frac{1}{2f}\left(f_i\delta^k_j + f_j\delta^k_i - f_{\cdot}{}^k g_{ij}\right)\right\}E_k + \frac{1}{2}u^p R^k_{jip}E_{\overline{k}} + \frac{1}{2}\omega^s R^k_{jis}E_{\overline{\overline{k}}}.$$

The rest can be proven by following the same method in the proof of *1)*. We omit them to avoid repeat. □

Proposition 2. *The Levi-Civita connection $\overline{\nabla}$ of the deformed Sasaki metric $\overline{g}$ on T^2M is given as following*

$$\overline{\nabla}_{X^{H_0}}Y^{H_0} = \left(\nabla_X Y + A_f\left(X,Y\right)\right)^{H_0} + \frac{1}{2}\left(R\left(Y,X\right)u\right)^{H_1} + \frac{1}{2}\left(R\left(Y,X\right)\omega\right)^{H_2},$$

$$\overline{\nabla}_{X^{H_1}}Y^{H_0} = \frac{1}{2f}\left(R\left(u,X\right)Y\right)^{H_0},$$
$$\overline{\nabla}_{X^{H_0}}Y^{H_1} = \frac{1}{2f}\left(R\left(u,Y\right)X\right)^{H_0} + \left(\nabla_X Y\right)^{H_1},$$
$$\overline{\nabla}_{X^{H_2}}Y^{H_0} = \frac{1}{2f}\left(R\left(\omega,X\right)Y\right)^{H_0},$$
$$\overline{\nabla}_{X^{H_0}}Y^{H_2} = \frac{1}{2f}\left(R\left(\omega,Y\right)X\right)^{H_0} + \left(\nabla_X Y\right)^{H_2},$$

$$\overline{\nabla}_{X^{H_1}}Y^{H_1} = 0,\quad \overline{\nabla}_{X^{H_2}}Y^{H_1} = 0,\quad \overline{\nabla}_{X^{H_1}}Y^{H_2} = 0,\quad \overline{\nabla}_{X^{H_2}}Y^{H_2} = 0$$

for all vector fields X, Y on M, where

$$A_f\left(X,Y\right) = \frac{1}{2f}\left(X\left(f\right)Y - Y\left(f\right)X - g\left(X,Y\right)\circ\left(df\right)^*\right).$$

4. Riemannian Curvature Tensors of the Deformed Sasaki Metric

Let F be a smooth bundle endomorphism of T^2M. Then we have the lifts of F:

$$F^{H_0}\left(u\right) = \sum u^i F(\partial_i)^{H_0},\quad F^{H_1}\left(u\right) = \sum u^i F(\partial_i)^{H_1},$$
$$F^{H_2}\left(u\right) = \sum u^i F(\partial_i)^{H_2}, F^{H_0}\left(\omega\right) = \sum \omega^i F(\partial_i)^{H_0},$$
$$F^{H_1}\left(\omega\right) = \sum \omega^i F(\partial_i)^{H_1},\quad F^{H_2}\left(\omega\right) = \sum \omega^i F(\partial_i)^{H_2}.$$

Morever, the following expressions are obtained by direct standard calculations

$$\overline{\nabla}_{X^{H_0}}u^i = X^{H_0}\left(u^i\right) = -u^s\Gamma^i_{sh}X^h,$$
$$\overline{\nabla}_{X^{H_1}}u^i = X^i,\quad \overline{\nabla}_{X^{H_2}}u^i = 0,\quad \overline{\nabla}_{X^{H_2}}\omega^i = X^i,$$

$$\overline{\nabla}_{X^{H_0}}\omega^i = -C^i_h X^h, \quad \overline{\nabla}_{X^{H_1}}\omega^i = -2u^s\Gamma^i_{sh}X^h.$$

Lemma 1. *Let (M,g) be a Riemannian manifold and $\overline{\nabla}$ be a Levi-Civita connection of $\left(T^2M,\overline{g}\right)$. Let $F : T^2M \longrightarrow T^2M$ be a smooth endomorphism, then*

$$\begin{aligned}
\overline{\nabla}_{X^{H_0}}F^{H_0}(u) &= \left(\nabla_X F(u) + A_f(X, F(u))\right)^{H_0} \\
&\quad + \frac{1}{2}\left(R(F(u), X)u\right)^{H_1} + \frac{1}{2}\left(R(F(u), X)\omega\right)^{H_2}, \\
\overline{\nabla}_{X^{H_0}}F^{H_1}(u) &= \frac{1}{2f}\left(R(u, F(u))X\right)^{H_0} + \left(\nabla_X F(u)\right)^{H_1}, \\
\overline{\nabla}_{X^{H_0}}F^{H_2}(u) &= \frac{1}{2f}\left(R(\omega, F(u))X\right)^{H_0} + \left(\nabla_X F(u)\right)^{H_2}, \\
\overline{\nabla}_{X^{H_1}}F^{H_0}(u) &= \left(F(X)\right)^{H_0} + \frac{1}{2f}\left(R(u, X)F(u)\right)^{H_0}, \\
\overline{\nabla}_{X^{H_1}}F^{H_1}(u) &= \left(F(X)\right)^{H_1}, \quad \overline{\nabla}_{X^{H_1}}F^{H_2}(u) = \left(F(X)\right)^{H_2}, \\
\overline{\nabla}_{X^{H_2}}F^{H_0}(u) &= 0, \quad \overline{\nabla}_{X^{H_2}}F^{H_1}(u) = 0, \quad \overline{\nabla}_{X^{H_2}}F^{H_2}(u) = 0, \\
\overline{\nabla}_{X^{H_0}}F^{H_0}(\omega) &= \left(\nabla_X F(\omega) + A_f(X, F(u))\right)^{H_0} \\
&\quad + \frac{1}{2}\left(R(F(\omega), X)u\right)^{H_1} + \frac{1}{2}\left(R(F(\omega), X)\omega\right)^{H_2}, \\
\overline{\nabla}_{X^{H_0}}F^{H_1}(\omega) &= \frac{1}{2f}\left(R(u, F(\omega)X\right)^{H_0} + \left(\nabla_X F(\omega)\right)^{H_1}, \\
\overline{\nabla}_{X^{H_0}}F^{H_2}(\omega) &= \frac{1}{2f}\left(R(\omega, F(\omega)X\right)^{H_0} + \left(\nabla_X F(\omega)\right)^{H_2}, \\
\overline{\nabla}_{X^{H_1}}F^{H_0}(\omega) &= \frac{1}{2f}\left(R(u, X)F(\omega)\right)^{H_0}, \\
\overline{\nabla}_{X^{H_1}}F^{H_1}(\omega) &= 0, \quad \overline{\nabla}_{X^{H_1}}F^{H_2}(\omega) = 0, \\
\overline{\nabla}_{X^{H_2}}F^{H_0}(\omega) &= \left(F(X)\right)^{H_0} + \frac{1}{2f}\left(R(\omega, X)F(\omega)\right)^{H_0}, \\
\overline{\nabla}_{X^{H_2}}F^{H_1}(\omega) &= \left(F(X)\right)^{H_1}, \quad \overline{\nabla}_{X^{H_2}}F^{H_2}(\omega) = \left(F(X)\right)^{H_2}
\end{aligned}$$

for any vector field X on M and $u, \omega \in T^2M$.

Proposition 3. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki metric $\overline{g}$. The curvature tensor $\overline{R}$ of the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ on T^2M is given by the following formulas:*

$$\begin{aligned}
&1)\overline{R}\left(X^{H_0}, Y^{H_0}\right)Z^{H_0} \\
= &\left\{R(X,Y)Z + (\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) + A_f(X, A_f(Y,Z))\right. \\
&-A_f(Y, A_f(X,Z)) - \frac{1}{2f}R(u, R(X,Y)u)Z - \frac{1}{2f}R(\omega, R(X,Y)\omega)Z
\end{aligned}$$

$$+\frac{1}{4f}R\left(u,R\left(Z,Y\right)u\right)X+\frac{1}{4f}R\left(u,R\left(Z,X\right)u\right)Y$$

$$+\frac{1}{4f}R\left(\omega,R\left(Z,Y\right)\omega\right)X+\frac{1}{4f}R\left(\omega,R\left(Z,X\right)\omega\right)Y\Big\}^{H_0}$$

$$+\left\{\frac{1}{2}\nabla_Z R\left(X,Y\right)u+\frac{1}{2}R\left(Y,A_f\left(X,Z\right)\right)u-\frac{1}{2}R\left(X,A_f\left(Y,Z\right)\right)u\right\}^{H_1}$$

$$+\left\{\frac{1}{2}\nabla_Z R\left(X,Y\right)\omega+\frac{1}{2}R\left(Y,A_f\left(X,Z\right)\right)\omega-\frac{1}{2}R\left(X,A_f\left(Y,Z\right)\right)\omega\right\}^{H_2},$$

$$2)\overline{R}\left(X^{H_1},Y^{H_0}\right)Z^{H_0}$$

$$=\Big\{-\left(\nabla_Y\frac{1}{2f}\right)R(u,X)Z-\frac{1}{2f}(\nabla_Y R)(u,X)Z$$

$$+\frac{1}{2f}R(u,X)(A_f(Y,Z))-\frac{1}{2f}A_f(Y,R(u,X)Z)\Big\}^{H_0}$$

$$+\left\{-\frac{1}{2}R\left(Y,Z\right)X+\frac{1}{4f}R\left(Y,R\left(u,X\right)Z\right)u\right\}^{H_1}+\left\{\frac{1}{4f}R\left(Y,R\left(u,X\right)Z\right)\omega\right\}^{H_2},$$

$$3)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_1}$$

$$=\Big\{(\nabla_X\frac{1}{2f})R\left(u,Z\right)Y+\frac{1}{2f}(\nabla_X R)\left(u,Z\right)Y-(\nabla_Y\frac{1}{2f})R\left(u,Z\right)X$$

$$-\frac{1}{2f}(\nabla_Y R)\left(u,Z\right)X+\frac{1}{2f}A_f\left(X,R\left(u,Z\right)Y\right)-\frac{1}{2f}A_f\left(Y,R\left(u,Z\right)X\right)\Big\}^{H_0}$$

$$+\left\{R\left(X,Y\right)Z+\frac{1}{4f}\left[R\left(R\left(u,Z\right)Y,X\right)u-R\left(R\left(u,Z\right)X,Y\right)u\right]\right\}^{H_1}$$

$$+\frac{1}{4f}\{[R\left(R\left(u,Z\right)Y,X\right)\omega-R\left(R\left(u,Z\right)X,Y\right)\omega]\}^{H_2},$$

$$4)\overline{R}\left(X^{H_1},Y^{H_1}\right)Z^{H_0}=\left\{\frac{1}{f}R\left(X,Y\right)Z+\frac{1}{4f^2}\left[R\left(u,X\right)R\left(u,Y\right)Z-R\left(u,Y\right)R\left(u,X\right)Z\right]\right\}^{H_0},$$

$$5)\overline{R}\left(X^{H_0},Y^{H_1}\right)Z^{H_1}=\left\{-\frac{1}{2f}R\left(Y,Z\right)X-\frac{1}{4f^2}R\left(u,Y\right)R\left(u,Z\right)X\right\}^{H_0},$$

$$6)\overline{R}\left(X^{H_2},Y^{H_0}\right)Z^{H_0}$$

$$=\Big\{-\frac{1}{2f}\left(\nabla_Y R\right)\left(\omega,X,Z\right)-(\nabla_Y\frac{1}{2f})R\left(\omega,X\right)Z+\frac{1}{2f}R\left(\omega,X\right)A_f\left(Y,Z\right)$$

$$-\frac{1}{2f}A_f\left(Y,R\left(\omega,Z\right)X\right)\Big\}^{H_0}+\left\{\frac{1}{4f}R\left(Y,R\left(\omega,X\right)Z\right)u\right\}^{H_1}$$

$$+\left\{\frac{1}{2}R(Z,Y)X+\frac{1}{4f}R(Y,R(\omega,X)Z)\omega\right\}^{H_2},$$

$$7)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_2}$$
$$=\left\{\frac{1}{2f}\nabla_X R(\omega,Z)Y-\frac{1}{2f}\nabla_Y R(\omega,Z)X+(\nabla_X\frac{1}{2f})R(\omega,Z)Y\right.$$
$$\left.+(\nabla_Y\frac{1}{2f})R(\omega,Z)X+\frac{1}{2f}A_f(X,R(\omega,Z)Y)-\frac{1}{2f}A_f(Y,R(\omega,Z)X)\right\}^{H_0}$$
$$+\frac{1}{4f}\left\{R(R(\omega,Z)Y,X)u-R(R(\omega,Z)X,Y)u\right\}^{H_1}$$
$$+\left\{R(X,Y)Z-\frac{1}{4f}\left[R(R(\omega,Z)Y,X)\omega+R(R(\omega,Z)X,Y)\omega\right]\right\}^{H_2},$$

$$8)\overline{R}\left(X^{H_2},Y^{H_2}\right)Z^{H_0}$$
$$=\left\{\frac{1}{f}R(X,Y)Z+\frac{1}{4f^2}\left[R(\omega,X)R(\omega,Y)Z-R(\omega,Y)R(\omega,X)Z\right]\right\}^{H_0},$$

$$9)\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_2}=\left\{-\frac{1}{2f}R(Y,Z)X-\frac{1}{4f^2}R(\omega,Y)R(\omega,Z)X\right\}^{H_0},$$

$$10)\overline{R}\left(X^{H_1},Y^{H_0}\right)Z^{H_2}=\left\{\frac{1}{4f^2}R(u,X)R(\omega,Z)Y\right\}^{H_0},$$

$$11)\overline{R}\left(X^{H_1},Y^{H_2}\right)Z^{H_0}=\frac{1}{4f^2}\left\{R(u,X)R(\omega,Y)Z-R(\omega,Y)R(u,X)Z\right\}^{H_0},$$

$$12)\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_1}=\left\{-\frac{1}{4f^2}R(\omega,Y)R(u,Z)X\right\}^{H_0},$$

$$13)\overline{R}\left(X^{H_2},Y^{H_0}\right)Z^{H_1}=\left\{\frac{1}{4f^2}R(\omega,X)R(u,Z)Y\right\}^{H_0},$$

$$14)\overline{R}\left(X^{H_1},Y^{H_1}\right)Z^{H_1}=0,\overline{R}\left(X^{H_1},Y^{H_1}\right)Z^{H_2}=0\ ,\overline{R}\left(X^{H_1},Y^{H_2}\right)Z^{H_2}=0,$$

$$15)\overline{R}\left(X^{H_1},Y^{H_2}\right)Z^{H_1}=0,\overline{R}\left(X^{H_2},Y^{H_2}\right)Z^{H_1}=0,\overline{R}\left(X^{H_2},Y^{H_2}\right)Z^{H_2}=0$$

for all vector fields X,Y on M.

Proposition 4. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle with the deformed Sasaki metric $\overline{g}$. The sectional curvature tensor $\overline{K}$ on T^2M satisfies*

$$1)\overline{K}\left(X^{H_0},Y^{H_0}\right)=\frac{1}{f}K(X,Y)+\frac{1}{f}\{g(\nabla_X A_f(Y,Y),X)-g(\nabla_Y A_f(X,Y),X)$$
$$+g(A_f(X,A_f(Y,Y),X))-g(A_f(Y,A_f(X,Y),X))\}$$

$$-\frac{3}{4f^2}\left[\left|R(X,Y)u\right|^2+\left|R(X,Y)\omega\right|^2\right],$$

$$2)\overline{K}\left(X^{H_1},Y^{H_0}\right) = \frac{1}{4f}\left|R(u,X)Y\right|^2,$$

$$3)\overline{K}\left(X^{H_2},Y^{H_0}\right) = \frac{1}{4f}\left|R(\omega,X)Y\right|^2,$$

$$4)\overline{K}\left(X^{H_1},Y^{H_1}\right) = 0,\overline{K}\left(X^{H_1},Y^{H_2}\right)=0,\overline{K}\left(X^{H_2},Y^{H_2}\right)=0$$

for all orthonormal vector fields X,Y on M.

Proposition 5. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki Metric $\overline{g}$. The scalar curvature $\overline{S}$ of T^2M with the metric $\overline{g}$ given by*

$$\begin{aligned}\overline{S} &= \frac{1}{f^2}S+\frac{1}{f^2}\{g\left(\nabla_{X_i}A_f\left(X_j,X_j\right),X_i\right)-g\left(\nabla_{X_j}A_f\left(X_i,X_j\right),X_i\right)\\ &\quad+g\left(A_f\left(X_i,A_f\left(X_j,X_j\right),X_i\right)\right)-g\left(A_f\left(X_j,A_f\left(X_i,X_j\right),X_i\right)\right)\}\\ &\quad-\frac{3-2f\sqrt{f}}{4f^3}\sum_{i,j=1}^{m}\left|R\left(X_i,X_j\right)u\right|^2+\left|R\left(X_i,X_j\right)\omega\right|^2,\end{aligned}$$

where $\{X_1,\ldots,X_m\}$ is a local orthonormal frame for M and S is the scalar curvature of M.

Proof. The set $\{Y_1,\ldots,Y_{3m}\}$ is an orthonormal basis on T^2M with $\frac{1}{\sqrt{f}}X_i^{H_0}=Y_i$, $X_i^{H_1}=Y_{m+i}$ and $X_i^{H_2}=Y_{2m+i}$ we get

$$\begin{aligned}\overline{S} &= \sum_{i,j=1}^{3m}\overline{K}\left(Y_i,Y_j\right)=\sum_{i,j=1}^{m}\overline{K}\left({X_i}^{H_0},{X_j}^{H_0}\right)+\overline{K}\left({X_i}^{H_1},{X_j}^{H_1}\right)\\ &\quad+\overline{K}\left({X_i}^{H_2},{X_j}^{H_2}\right)+2\overline{K}\left({X_i}^{H_0},{X_j}^{H_1}\right)+2\overline{K}\left({X_i}^{H_0},{X_j}^{H_2}\right)+2\overline{K}\left({X_i}^{H_1},{X_j}^{H_2}\right)\\ &= \sum_{i,j=1}^{m}\{\frac{1}{f^2}K\left(X_i,X_j\right)+\frac{1}{f^2}\{g\left(\nabla_{X_i}A_f\left(X_j,X_j\right),X_i\right)-g\left(\nabla_{X_j}A_f\left(X_i,X_j\right),X_i\right)\\ &\quad+g\left(A_f\left(X_i,A_f\left(X_j,X_j\right),X_i\right)\right)-g\left(A_f\left(X_j,A_f\left(X_i,X_j\right),X_i\right)\right)\\ &\quad-\frac{3}{4f^3}\left[\left|R\left(X_i,X_j\right)u\right|^2+\left|R\left(X_i,X_j\right)\omega\right|^2\right]\}\\ &\quad+2\frac{1}{4f\sqrt{f}}\left|R\left(u,X_j\right)X_i\right|^2+2\frac{1}{4f\sqrt{f}}\left|R\left(\omega,X_j\right)X_i\right|^2.\end{aligned}$$

Standard calculations give that

$$\begin{aligned}\overline{S} &= \sum_{i,j=1}^{m}\frac{1}{f^2}K(X_i,X_j)+\frac{1}{f^2}\{g\left(\nabla_{X_i}A_f\left(X_j,X_j\right),X_i\right)-g\left(\nabla_{X_j}A_f\left(X_i,X_j\right),X_i\right)\\ &\quad+g\left(A_f\left(X_i,A_f\left(X_j,X_j\right),X_i\right)\right)-g\left(A_f\left(X_j,A_f\left(X_i,X_j\right),X_i\right)\right)\}\end{aligned}$$

$$-\frac{3-2f\sqrt{f}}{4f^3}\sum_{i,j=1}^{m}\left|R\left(X_i,X_j\right)u\right|^2+\left|R\left(X_i,X_j\right)\omega\right|^2.$$

□

Let Ric_g and $Ric_{\overline{g}}$ denote Ricci tensors of (M,g) and $\left(T^2M,\overline{g}\right)$, respectively. We can write

$$\begin{aligned} Ric_{\overline{g}}\left(X^{H_a},Y^{H_b}\right) &= \sum_{i=1}^{m}\overline{g}\left(\overline{R}\left(X^{H_a},\frac{1}{\sqrt{f}}X_i{}^{H_0}\right)\frac{1}{\sqrt{f}}X_i{}^{H_0},Y^{H_b}\right) \\ &\quad+\sum_{i=1}^{m}\overline{g}\left(\overline{R}\left(X^{H_a},X_i{}^{H_1}\right)X_i{}^{H_1},Y^{H_b}\right) \\ &\quad+\sum_{i=1}^{m}\overline{g}\left(\overline{R}\left(X^{H_a},X_i{}^{H_2}\right)X_i{}^{H_2},Y^{H_b}\right)\end{aligned}$$

for orthonormal basis $\frac{1}{\sqrt{f}}X_i{}^{H_0}=Y_i$, $X_i{}^{H_1}=Y_{m+i}$ and $X_i{}^{H_2}=Y_{2m+i}$, where $a,b=0,1,2$.

After a straightforward computation, the components of the Ricci tensor $Ric_{\overline{g}}$ of the deformed Sasaki metric $\overline{g}$ are characterized by

$$\begin{aligned} & Ric_{\overline{g}}\left(X^{H_0},Y^{H_0}\right) \qquad (4)\\ =\; & Ric_g\left(X,Y\right) \\ & +\sum_{i=1}^{m}g\left(\left(\nabla_X A_f\right)\left(X_i,X_i\right),Y\right)-\sum_{i=1}^{m}g\left(\left(\nabla_{X_i}A_f\right)\left(X,X_i\right),Y\right) \\ & +\sum_{i=1}^{m}g\left(A_f\left(X,A_f\left(X_i,X_i\right)\right),Y\right)-\sum_{i=1}^{m}g\left(A_f\left(X_i,A_f\left(X,X_i\right)\right),Y\right) \\ & +\frac{3}{4f}\sum_{i=1}^{m}g\left(R\left(X_i,X\right)u,R\left(X_i,Y\right)u\right)+\frac{3}{4f}\sum_{i=1}^{m}g\left(R\left(X_i,X\right)\omega,R\left(X_i,Y\right)\omega\right) \\ & -\frac{1}{4f}\sum_{i=1}^{m}g\left(R\left(u,X_i\right)R\left(u,X_i\right)X,Y\right)-\frac{1}{4f}\sum_{i=1}^{m}g\left(R\left(\omega,X_i\right)R\left(\omega,X_i\right)X,Y\right),\end{aligned}$$

$$\begin{aligned} Ric_{\overline{g}}\left(X^{H_0},Y^{H_1}\right) &= \frac{1}{2f}\sum_{i=1}^{m}g\left(\nabla_{X_i}R\left(X,X_i\right)u,Y\right) \qquad (5)\\ &\quad+\frac{1}{2f}\sum_{i=1}^{m}g\left(R\left(X_i,A_f\left(X,X_i\right)\right)u,Y\right) \\ &\quad-\frac{1}{2f}\sum_{i=1}^{m}g\left(R\left(X,A_f\left(X_i,X_i\right)\right)u,Y\right),\end{aligned}$$

$$
\begin{aligned}
Ric_{\overline{g}}\left(X^{H_0}, Y^{H_2}\right) &= \frac{1}{2f}\sum_{i=1}^{m} g\left(\nabla_{X_i} R\left(X, X_i\right)\omega,\ Y\right) \\
&\quad + \frac{1}{2f}\sum_{i=1}^{m} g\left(R\left(X_i, A_f\left(X, X_i\right)\right)\omega, Y\right) \\
&\quad - \frac{1}{2f}\sum_{i=1}^{m} g\left(R\left(X, A_f\left(X_i, X_i\right)\right)\omega, Y\right),
\end{aligned} \tag{6}
$$

$$Ric_{\overline{g}}\left(X^{H_1}, Y^{H_1}\right) = \frac{1}{4f^2}\sum_{i=1}^{m} g\left(R\left(u, X\right)X_i, R\left(u, Y\right)X_i\right),$$

$$Ric_{\overline{g}}\left(X^{H_2}, Y^{H_1}\right) = \frac{1}{4f^2}\sum_{i=1}^{m} g\left(R\left(\omega, X\right)X_i, R\left(u, Y\right)X_i\right),$$

$$Ric_{\overline{g}}\left(X^{H_2}, Y^{H_2}\right) = \frac{1}{4f^2}\sum_{i=1}^{m} g\left(R\left(\omega, X\right)X_i, R\left(\omega, Y\right)X_i\right),$$

for all vector field X, Y, Z on M and $u, \omega \in T^2M$.

Theorem 1. *Let (M, g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki metric $\overline{g}$. $\left(T^2M, \overline{g}\right)$ is an Einstein manifold if and only if (M, g) is flat and*

$$
\begin{aligned}
&\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j} A_f\right)\left(X_i, X_i\right), X_j\right) - \sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i} A_f\right)\left(X_j, X_i\right), X_j\right) \\
&+ \sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right) - \sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right) \\
= \;& 0.
\end{aligned}
$$

Proof. The set $\{Y_1, \ldots, Y_{3m}\}$ be an local orthonormal basis on T^2M with $\frac{1}{\sqrt{f}} X_i{}^{H_0} = Y_i$, $X_i{}^{H_1} = Y_{m+i}$ and $X_i{}^{H_2} = Y_{2m+i}$. At first, suppose that $\left(T^2M, \overline{g}\right)$ is an Einstein manifold. Then it must be

$$Ric_{\overline{g}}\left(\overline{X}, \overline{Y}\right) = \lambda \overline{g}\left(\overline{X}, \overline{Y}\right)$$

for all vector field $\overline{X}$, $\overline{Y}$ on T^2M, where λ is a constant. If $X = Y = X_j$ is put into (4), (5) and (6), it follows that

$$
\begin{aligned}
\lambda &= Ric_g\left(X_j, X_j\right) \\
&\quad + \sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j} A_f\right)\left(X_i, X_i\right), X_j\right) \\
&\quad - \sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i} A_f\right)\left(X_j, X_i\right), X_j\right)
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right) \\
&-\sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right) \\
&+\frac{1}{2f}\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right) u\right|^2+\frac{1}{2f}\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right)\omega\right|^2 \\
=\;& \frac{1}{4f^2}\sum_{i,j=1}^{m}\left|R\left(\omega, X_j\right) X_i\right|^2 \\
=\;& \frac{1}{4f^2}\sum_{i,j=1}^{m}\left|R\left(u, X_j\right) X_i\right|^2 .
\end{aligned}
$$

Restricting the last identity to the zero section of T^2M, it follows

$$
\begin{aligned}
Ric_g\left(X_j, X_j\right) =\;& -\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j} A_f\right)\left(X_i, X_i\right), X_j\right)+\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i} A_f\right)\left(X_j, X_i\right), X_j\right) \\
&-\sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right) \\
&+\sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right)
\end{aligned}
$$

and we obtain $\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right) u\right|^2=0$ and $\sum_{i,j=1}^{m}\left|R\left(X_i, X_j\right)\omega\right|^2=0$. Replacing u and ω by X_k in the last identiy we see that

$$
\sum_{i,j,k=1}^{m}\left|R\left(X_i, X_j\right) X_k\right|^2=0 .
$$

Thus $R\left(X_i, X_j\right) X_k=0$ for all $i, j, k=1 \ldots m$ and we deduce $R=0$. (M, g) is flat. If we reconsider the equation (7), we obtain

$$
\begin{aligned}
&\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_j} A_f\right)\left(X_i, X_i\right), X_j\right)-\sum_{i,j=1}^{m} g\left(\left(\nabla_{X_i} A_f\right)\left(X_j, X_i\right), X_j\right) \\
&+\sum_{i,j=1}^{m} g\left(A_f\left(X_j, A_f\left(X_i, X_i\right)\right), X_j\right)-\sum_{i,j=1}^{m} g\left(A_f\left(X_i, A_f\left(X_j, X_i\right)\right), X_j\right) \\
=\;& 0 .
\end{aligned}
$$

Conversely, we assume that $R=0$ and the equation above, then Ricci formulas become $Ric_{\overline{g}}=\lambda\overline{g}$ and $\lambda=0$. Thus T^2M is an Einstein manifold. $\square$

5. Weakly Symmetry Properties of the Deformed Sasaki Metric

The Riemannian manifold (M,g) is called weakly symmetric if there exist two $1-$forms $\alpha_1,\ \alpha_2$ and a vector field A, all on M, such that:

$$\begin{aligned} & (\nabla_W R)(X,Y,Z) \\ = \ & \alpha_1(W)R(X,Y)Z+\alpha_2(X)R(W,Y)Z \\ & +\alpha_2(Y)R(X,W)Z+\alpha_2(Z)R(X,Y)W+g(R(X,Y)Z,W)(\alpha_2)\#, \end{aligned} \tag{8}$$

where $A=(\alpha_2)^{\#}$ and $\alpha_i g^{ij}=\alpha^j=\alpha\#$, that is, A is the $g-$dual vector field of the $1-$form α_2. In [1], Bejan and Crasmareanu proved that the weakly symmetry property of the Sasaki metric on the tangent bundle over base manifold, generalising the result obtained in [2]. The weakly symmetry property of Sasaki metric on the second-order tangent bundle T^2M proved in [8]. In this section, we consider the result for the second-order tangent bundle with the deformed Sasaki metric $(T^2M,\overline{g})$.

Theorem 2. *Let (M,g) be a Riemannian manifold and T^2M be its second-order tangent bundle equipped with the deformed Sasaki metric $\overline{g}$. $\left(T^2M,\overline{g}\right)$ is weakly symmetric if and only if the base manifold (M,g) is flat and*

$$(\nabla_X A_f)(Y,Z)-(\nabla_Y A_f)(X,Z)+A_f(X,A_f(Y,Z))-A_f(Y,A_f(X,Z))=0,$$

where $A_f(X,Y)=\frac{1}{2f}\left(X(f)Y-Y(f)X-g(X,Y)\circ(df)^*\right)$ is a $(1,2)-$tensor field. On account of this, $\left(T^2M,\overline{g}\right)$ is flat.

Proof. In the proof, we apply the method used in [1]. If $R=0$ then $\overline{R}=0$ and so we have (8) as null equality. Primarily we take into account the condition (8) for $W^{H_0},\ X^{H_0},\ Y^{H_2}$ and Z^{H_2} and we obtain

$$\begin{aligned} & \alpha_1\left(W^{H_0}\right)\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_2}+\alpha_2\left(X^{H_0}\right)\overline{R}\left(W^{H_0},Y^{H_2}\right)Z^{H_2} \\ & +\alpha_2\left(Y^{H_2}\right)\overline{R}\left(X^{H_0},W^{H_0}\right)Z^{H_0}+\alpha_2\left(Z^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_2}\right)W^{H_0} \\ & +\overline{g}\left(\overline{R}\left(X^{H_0},Y^{H_2}\right)Z^{H_2},W^{H_0}\right)(a_2)^{\#} \\ = \ & -\overline{\nabla}_{W^{H_0}}\left[-\frac{1}{2f}R(Y,Z)X-\frac{1}{4f^2}R(\omega,Y)R(\omega,Z)X\right]^{H_0} \\ & -\overline{R}\left(\begin{array}{c}(\nabla_W X)^{H_0}+(A_f(W,X))^{H_0}+\left(\frac{1}{2}R(X,W)u\right)^{H_1} \\ +\left(\frac{1}{2}R(X,W)\omega\right)^{H_2},Y^{H_2}\end{array}\right)Z^{H_2} \\ & -\overline{R}\left(X^{H_0},\left(\frac{1}{2f}R(\omega,Y)W\right)^{H_0}+(\nabla_W Y)^{H_2}\right)Z^{H_2} \\ & -\overline{R}\left(X^{H_0},Y^{H_2}\right)\left(\frac{1}{2f}(R(\omega,Z)W)^{H_0}+(\nabla_W Z)^{H_2}\right). \end{aligned} \tag{9}$$

Thereafter we consider the H_2 part of both sides of the above equation and we get

$$\begin{aligned}
&\alpha_2\left(Y^{H_2}\right)\left(R(X,W)Z-\frac{1}{4f}R(R(\omega,Z)W,X)\omega-\frac{1}{4f}R(R(\omega,Z)X,W)\omega\right)\\
&+\alpha_2\left(Z^{H_2}\right)\left(-\frac{1}{2}R(W,X)Y-\frac{1}{4f}R(X,R(\omega,Y)W)\omega\right)\\
&-\frac{1}{f}g\left(\frac{1}{2f}R(Y,Z)X+\frac{1}{4f^2}R(\omega,Y)R(\omega,Z)X,W\right)\alpha_2{}^{\#}\\
=\;&-\frac{1}{4f}R(R(Y,Z)X,W)\omega-\frac{1}{8f^2}R(R(\omega,Y)R(\omega,Z)X,W)\omega\\
&-\frac{1}{2f}\left(\begin{array}{c}R(X,R(\omega,Y)W)Z+\frac{1}{4f}R(R(\omega,Z)R(\omega,Y)W,X)\omega\\+\frac{1}{4f}R(R(\omega,Z)X,R(\omega,Y)W)\omega\end{array}\right)\\
&-\frac{1}{2f}\left(-\frac{1}{2}R(R(\omega,Z)W,X)Y+\frac{1}{4f}R(R(\omega,Y)R(\omega,Z)W,X)\omega\right). \qquad (10)
\end{aligned}$$

By setting $Y=\omega$ and $Z=\omega$ respectively we get

$$\begin{aligned}
&\alpha_2\left(\omega^{H_2}\right)\left(R(X,W)Z+\frac{1}{4f}\left(R(R(\omega,Z)W,X)\omega-R(R(\omega,Z)X,W)\omega\right)\right)\\
&+\alpha_2\left(Z^{H_2}\right)\left(-\frac{1}{2}R(W,X)\omega\right)-\frac{1}{f}g\left(\frac{1}{2f}R(\omega,Z)X,W\right)\alpha_2{}^{\#}\\
=\;&\frac{1}{4f}R(R(\omega,Z)X,W)\omega+\frac{1}{4f}R(R(\omega,Z)W,X)\omega \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_2\left(Y^{H_2}\right)(R(X,W)\omega)\\
&+\alpha_2\left(\omega^{H_2}\right)\left(-\frac{1}{2}R(W,X)Y-\frac{1}{4f}R(X,R(\omega,Y)W)\omega\right)\\
&-\frac{1}{f}g\left(\frac{1}{2f}R(Y,\omega)X,W\right)\alpha_2{}^{\#}\\
=\;&\frac{1}{4f}R(R(Y,\omega)X,W)\omega-\frac{1}{2f}R(X,R(\omega,Y)W)\omega. \qquad (12)
\end{aligned}$$

Now we replace Y by Z in (12) equation

$$\begin{aligned}
&\alpha_2\left(Z^{H_2}\right)R(X,W)\omega\\
&+\alpha_2\left(\omega^{H_2}\right)\left(-\frac{1}{2}R(W,X)Z-\frac{1}{4f}R(X,R(\omega,Z)W)\omega\right)\\
&-\frac{1}{f}g\left(\frac{1}{2f}R(Z,\omega)X,W\right)\alpha_2{}^{\#}\\
=\;&\frac{1}{4f}R(R(Z,\omega)X,W)\omega-\frac{1}{2f}R(X,R(\omega,Z)W)\omega. \qquad (13)
\end{aligned}$$

And by adding (11) and (13) we produce

$$\begin{aligned} &\frac{3}{2}\alpha_2\left(Z^{H_2}\right)R\left(X,W\right)\omega \\ &+\alpha_2\left(\omega^{H_2}\right)\left(\frac{3}{2}R\left(X,W\right)Z+\frac{1}{2f}R\left(R\left(\omega,Z\right)W,X\right)\omega-\frac{1}{4f}R\left(R\left(\omega,Z\right)X,W\right)\omega\right) \\ =\ &\frac{3}{4f}R\left(R\left(\omega,Z\right)W,X\right)\omega. \end{aligned} \tag{14}$$

The equation (14) with $Z=\omega$ we obtain that:

$$\alpha_2\left(\omega^{H_2}\right)R\left(X,W\right)\omega=0. \tag{15}$$

If $\alpha_2\left(\omega^{H_2}\right)\neq 0$, then we have result. Suppose now that $\alpha_2\left(\omega^{H_2}\right)=0$ then $\left((\alpha_2)^{\#}\right)^{H_2}=0$.

Returning to equation (11) it results

$$\alpha_2\left(Z^{H_2}\right)\left(-\frac{1}{2}R\left(W,X\right)\omega\right)=\frac{1}{4f}R\left(R\left(\omega,Z\right)X,W\right)\omega+\frac{1}{4f}R\left(R\left(\omega,Z\right)W,X\right)\omega.$$

By setting now $W=X$ we get

$$R\left(R\left(\omega,Z\right)X,X\right)\omega=0.$$

And we take the inner product with Z, it follows that:

$$g\left(R\left(\omega,Z\right)X,R\left(\omega,Z\right)X\right)=0.$$

Thus

$$R\left(\omega,Z\right)X=0.$$

Now the inner product with an arbitary Y gives

$$g\left(R\left(X,Y\right)\omega,Z\right)=0.$$

For Z being an arbitary vector field we get $R\left(X,Y\right)\omega=0$, for every $X,\ Y$ and ω. Hence, we have $R=0$. In the case the Riemannian curvature tensor reduce to

$$\begin{aligned} \overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0} &= \{\left(\nabla_X A_f\right)\left(Y,Z\right)-\left(\nabla_Y A_f\right)\left(X,Z\right) \\ &\quad +A_f\left(X,A_f\left(Y,Z\right)\right)-A_f\left(Y,A_f(X,Z)\right)\}^{H_0} \end{aligned}$$

and also Levi-Civita connection is

$$\overline{\nabla}_{X^{H_0}}Y^{H_0}=\left(\nabla_X Y+A_f\left(X,Y\right)\right)^{H_0}.$$

Next we again consider the equation (8) for $X^{H_0},\ Y^{H_0},\ Z^{H_0},\ W^{H_2}$

$$\begin{aligned} &\alpha_1\left(W^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0}+\alpha_2\left(X^{H_0}\right)\overline{R}\left(W^{H_2},Y^{H_0}\right)Z^{H_0} \\ &+\alpha_2\left(Y^{H_0}\right)\overline{R}\left(X^{H_0},W^{H_2}\right)Z^{H_0}+\alpha_2\left(Z^{H_0}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)W^{H_2} \\ &+\overline{g}\left(R\left(X^{H_0},Y^{H_0}\right)Z^{H_0},W^{H_2}\right)\left(\alpha_2\right)\# \end{aligned}$$

$$= -\overline{\nabla}_{W^{H_2}}\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0} - \overline{R}\left(\overline{\nabla}_{W^{H_2}}X^{H_0},Y^{H_0}\right)Z^{H_0}$$
$$-\overline{R}\left(X^{H_0},\overline{\nabla}_{W^{H_2}}Y^{H_0}\right)Z^{H_0} - \overline{R}\left(X^{H_0},Y^{H_0}\right)\overline{\nabla}_{W^{H_2}}Z^{H_0}.$$

Hence, we get

$$\alpha_1\left(W^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0} + g\left(\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0},W^{H_2}\right)(\alpha_2)\# = 0,$$
$$\alpha_1\left(W^{H_2}\right)\overline{R}\left(X^{H_0},Y^{H_0}\right)Z^{H_0} = 0,$$
$$\alpha_1\left(W^{H_2}\right)\left[(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) + A_f(X,A_f(Y,Z)) - A_f(Y,A_f(X,Z))\right]^{H_0} = 0.$$

Since α_1 is arbitrary, we find

$$(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) + A_f(X,A_f(Y,Z)) - A_f(Y,A_f(X,Z)) = 0.$$

The proof is complete. □

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References

[1] Bejan, C. L., Crasmareanu, M., Weakly-symmetry of the Sasakian lifts on tangent bundles, *Publ. Math. Debrecen*, 83(1-2) (2013), 63-69.

[2] Binh, T. Q.,Tamassy, L., On recurrence or pseudo-symmetry of the Sasakian metric on the tangent bundle of a Riemannian manifold, *Indian J. Pure Appl. Math.*, 35(4) (2004), 555-560.

[3] Gezer, A., Magden, A., Geometry of the second-order tangent bundles of Riemannian manifolds, *Chin. Ann. Math. Ser. B*, 38(4) (2017), 985-998. DOI:10.1007/s11401-017-1107-4.

[4] De Leon M., Vazquez, E., On the geometry of the tangent bundles of order 2, *Analele Universitatii Bucuresti Matematica*, 34 (1985), 40-48.

[5] Djaa, M., Gancarzewicz, J., The geometry of tangent bundles of order r, *Boletin Academia, Galega de Ciencias*, 4 (1985), 147-165.

[6] Dodson, C. T. J., Radivoiovici, M. S., Tangent and frame bundles order two, *Analele Stiintifice ale Universitatii Al. I. Cuza*, 28 (1982), 63-71.

[7] Ishikawa, S., On Riemannian metrics of tangent bundles of order 2 of Riemannian manifolds, *Tensor (N.S.)*, 34(2) (1980), 173-178.

[8] Magden, A., Gezer, A., Karaca, K., Some problems concerning with Sasaki metric on the second-order tangent bundles, *Int. Electron. J. Geom.*, 13(2) (2020), 75–86. DOI:10.36890/iejg.750905.

[9] Morimoto, A., Liftings of tensor fields and connections to tangent bundles of higher order, *Nagoya Math. J.*, 40 (1970), 99-120.

[10] Tani, M., Tensor fields and connections in cross-sections in the tangent bundles of order 2, *Kodai Math. Sem. Rep.*, 21 (1969), 310-325.

[11] Yano, K., Ako, M., On certain operators associated with tensor field, *Kodai Math. Sem. Rep.*, 20 (1968), 414-436.

[12] Yano, K., Ishihara, S., Tangent and Cotangent Bundles: Differential Geometry, Marcel Dekker, Inc., 420 p, New York, 1973.

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EXPLICIT FORMULAS FOR EXPONENTIAL OF 2×2 SPLIT-COMPLEX MATRICES

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Abstract. Split-complex (hyperbolic) numbers are ordered pairs of real numbers, written in the form $x + jy$ with $j^2 = 1$, used to describe the geometry of the Lorentzian plane. Since a null split-complex number does not have an inverse, some methods to calculate the exponential of complex matrices are not valid for split-complex matrices. In this paper, we examined the exponential of a 2×2 split-complex matrix in three cases : i. $\Delta = 0$, ii. $\Delta \neq 0$ and Δ is not null split-complex number, iii. $\Delta \neq 0$ and Δ is a null split-complex number where $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

1. Introduction

The exponential of a matrix could be computed in many ways: series, matrix decomposition, differential equations and, polynomial methods. The matrix exponential gives a connection between any matrix Lie algebra and the corresponding Lie group. The matrix exponential does not satisfy some properties of the number exponential since matrix multiplication is not commutative. For example, the property $e^{a+b} = e^a e^b$ is not true for matrix exponential. The equality $e^{A+B} = e^A e^B$ is only true in the case the matrices A and B commute. Detailed information on the exponential matrix can be found in many sources. In this study, especially references [2], [26], [16], [3] and, [4] were used.

The purpose of this article is to determine the exponential of split-complex number matrices and to give useful formulas by classifying them. The formulas of calculating the matrix exponential for 2×2 complex numbers can be found in Bernstein's study [4]. Standard methods can be used to calculate the exponential of a matrix defined on a field such as complex numbers. However, for a set of numbers defined

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on a ring such as split-complex numbers, some problems are encountered in using these methods. The methods used in the literature for complex numbers can be useless, since some elements do not have an inverse in the ring of split-complex numbers. Diagonalization and finding eigenvalues and eigenvectors of a matrix defined on a ring is not as easy as in matrices defined over a field. There are many studies on this subject in the literature [18], [21], [15]. In particular, exponential of a matrix defined on the ring of quaternions can be found in Casey's paper [17]. Casey constructed a transformation from quaternionic square matrices to complex square matrices to compute the exponential of the quaternionic matrix. Also, Ablamowicz computed matrix exponential of a real, complex, and quaternionic matrix, using an isomorphism between matrix algebras and orthogonal Clifford algebras [1]. The exponential of a matrix defined in the rings of split and hyperbolic split quaternions can be found in the references [7], [8] and [20]. In this paper, exponential of a matrix defined over the split-complex numbers is studied. In the first part, some basic information and definitions about split-complex numbers are given. In the second part, the cases where some methods and formulas are insufficient are determined. In the last chapter, it is examined with the help of isomorphism between split complex matrices and real matrices.

2. Preliminaries

The set of split-complex numbers is defined as follows:

$$\mathbb{P} = \{\mathbf{z} = z_1 + jz_2 : z_1, z_2 \in \mathbb{R}\}$$

where the split-complex unit j satisfies $j^2 = 1$ and $j \neq 1$. In the literature, these numbers are also called double, spacetime, hyperbolic or perplex numbers [10], [28], [27], [5], [11], [25], [24], [9], [12]. For any $\mathbf{z} = z_1 + jz_2 \in \mathbb{P}$ we define the real part of $\mathbf{z}$ as $\mathrm{Re}(\mathbf{z}) = z_1$ and the split-complex part of $\mathbf{z}$ as $\mathrm{Im}(\mathbf{z}) = z_2$. The conjugate of $\mathbf{z}$ is denoted by $\overline{\mathbf{z}}$ and it is $\overline{\mathbf{z}} = z_1 - jz_2$. The inner product of $\mathbf{z} = z_1 + jz_2$ and $\mathbf{w} = w_1 + jw_2$ is defined as

$$\langle\ ,\ \rangle : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$$
$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathrm{Re}\,(\overline{\mathbf{z}}\mathbf{w}) = w_1 z_1 - w_2 z_2.$$

This product is a nondegenerate, symmetrical bilinear form, known as the scalar product in the Lorentzian plane. Since this scalar product is not positively defined, we will need to classify the split-complex numbers, as in the Lorentzian plane [6], [23], [13], [22], [14]. We will call a split-complex number $\mathbf{z} = z_1 + jz_2$ spacelike, timelike, or null, according to $\langle \mathbf{z}, \mathbf{z} \rangle = \overline{\mathbf{z}}\mathbf{z} > 0$, < 0 or $= 0$, respectively. So, for a split-complex number $\mathbf{z} = z_1 + jz_2$, we can call $\mathbf{z}$ spacelike, timelike or null according to $|z_1| > |z_2|$, $|z_1| < |z_2|$ and $z_1 = \pm z_2$, respectively. Null split-complex numbers have no inverse. In this paper, the set of null split-complex numbers is denoted by $\mathbb{P}_0$. Norm of $\mathbf{z} = z_1 + jz_2$ is defined as

$$|\mathbf{z}| = \mathbf{z}\overline{\mathbf{z}} = \sqrt{|z_1^2 - z_2^2|}.$$

The square root of the split-complex number $\mathbf{z} = z_1 + jz_2$ is found by

$$\sqrt{z_1 + jz_2} = \frac{\sqrt{z_1 + z_2} + \sqrt{z_1 - z_2}}{2} + \frac{\sqrt{z_1 + z_2} - \sqrt{z_1 - z_2}}{2} j. \tag{1}$$

A necessary and sufficient condition for the square root of a non-null split-complex number to be defined is that this number is spacelike. Moreover, for the null split-complex number $\mathbf{z} = x + yj$, we have $x = \pm y$ and obtain

$$\begin{aligned} &\text{If } x = y, \quad && \sqrt{x + yj} = \frac{\sqrt{2x}}{2}(1 + j) \\ &\text{If } x = -y, \quad && \sqrt{x + yj} = \frac{\sqrt{2x}}{2}(1 - j). \end{aligned}$$

In this paper, by a split-complex matrix, we mean simply a matrix with split-complex number entries. We denote the set of $m \times n$ split-complex matrices with $\mathbb{M}_{m\times n}(\mathbb{P})$. We may write $\mathbf{X} = X_1 + jX_2$ for any $\mathbf{X} \in \mathbb{M}_{n\times n}(\mathbb{P})$ where $X_1, X_2 \in \mathbb{M}_{n\times n}(\mathbb{R})$. There exists a ring isomorphism between $\mathbb{M}_{n\times n}(\mathbb{P})$ and the algebra of the matrices of the form

$$\left\{ \begin{bmatrix} X_1 & X_2 \\ X_2 & X_1 \end{bmatrix} \in \mathbb{M}_{2n\times 2n}(\mathbb{R}) : X_1, X_2 \in \mathbb{M}_{n\times n}(\mathbb{R}) \right\}$$

for $\mathbf{X} = X_1 + jX_2 \in \mathbb{M}_{n\times n}(\mathbb{P})$. In this study, split-complex numbers and matrices will be shown in bold small and bold big letters, respectively.

According to the fundamental theorem of the set of split-complex numbers, every polynomial of the n-th degree has a split-complex root of n^2. For example, a second-order polynomial defined in the split-complex number has exactly 4 split-complex roots, and the polynomial can be factored in two different ways. Since the roots of polynomial $P(\mathbf{z}) = \mathbf{z}^2 - 4$ are $\mathbf{z}_1 = 2$, $\mathbf{z}_2 = -2$, $\mathbf{z}_3 = 2j$, $\mathbf{z}_4 = -2j$ we can factorize two kinds as

$$P(\mathbf{z}) = (\mathbf{z} - 2)(\mathbf{z} + 2) = (\mathbf{z} - 2j)(\mathbf{z} + 2j).$$

Also, the characteristic polynomial of a 2×2 split-complex matrix has 4 roots and can be factored into 2 different forms. Thus, a 2×2 matrix has 2 sets of eigenvalues and eigenvectors. Detailed information on this subject can be found in Poodiack's and LeClair's excellent article [19]. For example, characteristic polynomial of

$$\mathbf{A} = \begin{bmatrix} 3 - j & 1 + 2j \\ 2 - 2j & j \end{bmatrix}$$

is $P(\mathbf{z}) = \mathbf{z}^2 - 3\mathbf{z} + j + 1$, and it can be written as

$$P(\mathbf{z}) = \left(\mathbf{z} - \frac{5 - j}{2}\right)\left(\mathbf{z} - \frac{1 + j}{2}\right) = (\mathbf{z} - 1 - j)(\mathbf{z} - 2 + j).$$

Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{P})$$

be given. A 2×2 matrix $\mathbf{A}$ has two different eigenvalues set $S_1 = \{\lambda_1, \lambda_2\}$ and $S_2 = \{\mu_1, \mu_2\}$ where

$$\boldsymbol{\lambda}_{1,2} = \frac{\mathrm{tr}\mathbf{A}}{2} \pm \frac{\sqrt{\Delta}}{2} \text{ and } \mu_{1,2} = \frac{\mathrm{tr}\mathbf{A}}{2} \pm \frac{\sqrt{\Delta}}{2} j$$

and $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4 \det \mathbf{A}$. Here, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are called primary roots of $\mathbf{A}$.

3. Exponential of a Split-Complex Matrix

In this section, we will examine exponential of a 2 by 2 split-complex matrix. First, let's give the exponential of an upper triangular 2×2 matrix. These formulas are given in Bernstein's article for complex matrices [3]. The following lemmas can be proved similarly. However, as stated in Lemma 1, if the split-complex number $\mathbf{a}_{11} - \mathbf{a}_{22}$ is null, the formulas for the complex numbers will not work. For example, exponential of the matrix

$$\mathbf{A} = \begin{bmatrix} 2+3j & 2+j \\ \mathbf{0} & 1+2j \end{bmatrix}$$

cannot be found by this method, since $j+1$ is not an inverse.

Lemma 1. *Let* $\mathbf{A} = [\mathbf{a}_{ij}]$ *be an upper triangular* 2×2 *split-complex matrix with* $\mathbf{a}_{21} = 0$.

i. *If* $\mathbf{a}_{11} = \mathbf{a}_{22}$*, then we have*

$$e^{\mathbf{A}} = e^{\mathbf{a}_{11}} \begin{bmatrix} 1 & \mathbf{a}_{12} \\ \mathbf{0} & 1 \end{bmatrix};$$

ii. *If* $\mathbf{a}_{11} \neq \mathbf{a}_{22}$ *and* $\mathbf{a}_{11} - \mathbf{a}_{22}$ *are not null, then we have*

$$e^{\mathbf{A}} = \begin{bmatrix} e^{\mathbf{a}_{11}} & \frac{\mathbf{a}_{12}\left(e^{\mathbf{a}_{11}} - e^{\mathbf{a}_{22}}\right)}{\mathbf{a}_{11} - \mathbf{a}_{22}} \\ \mathbf{0} & e^{\mathbf{a}_{22}} \end{bmatrix}.$$

Proof. Both formulas can be proved by induction, similar to complex numbers. □

In the above theorem, a solution is not given when $\mathbf{a}_{11} - \mathbf{a}_{22}$ is null. We will use a different method for such matrices in the next section. In the most general case, we will calculate the exponential of matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{P})$$

with diagonalization, and in which cases this method will fail for split-complex numbers.

We will examine the exponential of a 2×2 split-complex matrix $\mathbf{A}$ in three cases

i. $\Delta = 0$,

ii. $\Delta \neq 0$ and Δ is not null split-complex number,

iii. $\Delta \neq 0$ and Δ is a null split-complex number

where $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

3.1. **Case 1 : $\Delta = 0$.** In the case $\Delta = 0$ for the matrix $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$, we encounter 4 special cases that will change the result.
i. $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}$, $\mathbf{a}_{21}$ are not null
ii. $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{21}$ are null, $\mathbf{a}_{12}$ is not null
iii. $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}$ are null, $\mathbf{a}_{21}$ is not null
iv. $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ are null

Lemma 2 and Theorem 1 can be easily proved using elementary linear algebra knowledge for first three cases. We will only give proofs for the fourth case.

Lemma 2. *Let $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$ be a split-complex matrix with $\Delta = 0$. Then, the only eigenvalue is $\boldsymbol{\lambda} = \frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}$. So, we have different cases to determine the matrix $\mathbf{P}$, satisfying the equality $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ where,*

$$\mathbf{D} = \begin{bmatrix} \boldsymbol{\lambda} & 1/2 \\ 0 & \boldsymbol{\lambda} \end{bmatrix}$$

i): *If $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}, \mathbf{a}_{21}$ are not null*

$$\mathbf{P} = \begin{bmatrix} \mathbf{a}_{11} - \mathbf{a}_{22} & 1 \\ 2\mathbf{a_{21}} & 0 \end{bmatrix}.$$

ii): *If $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{21}$ are null, $\mathbf{a}_{12}$ is not null*

$$\begin{bmatrix} 2\mathbf{a_{12}} & 0 \\ \mathbf{a}_{22} - \mathbf{a}_{11} & 1 \end{bmatrix}.$$

iii): *If $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}$ are null, $\mathbf{a}_{21}$ is not null*

$$\mathbf{P} = \begin{bmatrix} \mathbf{a}_{11} - \mathbf{a}_{22} & 1 \\ 2\mathbf{a_{21}} & 0 \end{bmatrix}.$$

Theorem 1. *Let $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$ be a split-complex matrix with $\Delta = 0$. Then*

$$e^{\mathbf{A}} = e^{\boldsymbol{\lambda}}\mathbf{P}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}\mathbf{P}^{-1}$$

where $\mathbf{P}$ should be chosen according to the cases given in above Lemma.

Example 1. *Let's calculate exponential of the split-complex matrix*

$$\mathbf{A} = \begin{bmatrix} 3-j & 1+2j \\ 2-2j & 1+j \end{bmatrix}.$$

Since $\Delta = 0$, $\mathbf{a}_{21}$ is null and $\mathbf{a}_{12}$ is not null, we find

$$\mathbf{P} = \begin{bmatrix} 2\mathbf{a}_{12} & 0 \\ \mathbf{a}_{22}-\mathbf{a}_{11} & 1 \end{bmatrix} = \begin{bmatrix} 4j+2 & 0 \\ 2j-\mathbf{2} & 1 \end{bmatrix}.$$

Therefore, using the theorem and $(2j+1)^{-1} = \frac{-1}{3} + \frac{2}{3}j$, *we obtain*

$$e^{\mathbf{A}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} = e^2 \begin{bmatrix} 2-j & 2j+1 \\ 2-2j & j \end{bmatrix}.$$

In the case of $\Delta = 0$, if the numbers $\mathbf{a}_{11} - \mathbf{a}_{22}, \mathbf{a}_{12}, \mathbf{a}_{21}$ are null, then there will be a different case other than the three cases given in lemma. Before calculating exponential of a matrix for this case, let's give a lemma.

Lemma 3. *Let* $\mathbf{A} = [\mathbf{a}_{ij}]$ *be a* 2×2 *split-complex matrix with* $\Delta = 0$. *If the numbers* $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ *are null, then the matrix* $\mathbf{A}$ *in the form*

$$\begin{bmatrix} x+jy & -uy\,(j+\epsilon) \\ y\dfrac{j+\epsilon}{u} & x-2y\epsilon-jy \end{bmatrix}$$

where $u \neq 0$, $x, y \in \mathbb{R}$ *and* $\epsilon = \pm 1$.

Proof. If the numbers $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ are null, all of them are same type since $\Delta = 0$. They can be written as $\mathbf{a}_{11} - \mathbf{a}_{22} = \mu_1\,(1+\epsilon h)$, $\mathbf{a}_{12} = \mu_2\,(1+\epsilon h)$ and $\mathbf{a}_{21} = \mu_3\,(1+\epsilon h)$ for $\mu_i \in \mathbb{R}$. So, we have

$$\mathbf{a}_{11} - \mathbf{a}_{22} = \mu_1\,(1+\epsilon j) \Leftrightarrow \mu_1^2\,(1+\epsilon j)^2 = -4\mu_2\mu_3\,(1+\epsilon j)^2 \Leftrightarrow \mu_1^2 = -4\mu_2\mu_3.$$

Let the number be $\mathbf{a}_{11} = x + yj$, then $\mathbf{a}_{22} = x + yj - \mu_1\,(1+\epsilon j)$. So, $\mathbf{A}$ will be

$$\mathbf{A} = \begin{bmatrix} x+yj & \mu_2\,(1+\epsilon j) \\ \mu_3\,(1+\epsilon j) & x+yj-\mu_1\,(1+\epsilon j) \end{bmatrix}.$$

Suppose that $\overrightarrow{\mathbf{u}} = (u, 1)$, $0 \neq u \in \mathbb{R}$ is an eigenvector corresponding to the eigenvalue $\lambda = \frac{\mathbf{a}_{11} + \mathbf{a}_{22}}{2}$. So, from the equality $A\overrightarrow{\mathbf{u}} = \boldsymbol{\lambda}\overrightarrow{\mathbf{u}}$ and equality of split-complex numbers, we obtain

$$\mu_2 = -\epsilon u y, \quad \mu_3 = \frac{\mu_1}{u} - \epsilon\frac{y}{u}, \quad x - \epsilon y = \frac{\mathrm{tr}\mathbf{A}}{2}.$$

Moreover, from the equality $\mu_1^2 = -4\mu_2\mu_3$, we find

$$\mu_1^2 = 4\epsilon y\mu_1 - 4y^2 \Rightarrow \mu_1^2 - 4\epsilon y\mu_1 + 4y^2 = 0 \Rightarrow (-\mu_1 + 2y\epsilon)^2 = 0 \Rightarrow \mu_1 = 2y\epsilon.$$

Therefore, we obtain $\mu_1 = 2\epsilon y$, $\mu_2 = -\epsilon u y$, $\mu_3 = \frac{\epsilon y}{u}$ and

$$\mathbf{A} = \begin{bmatrix} x+jy & -uy\,(j+\epsilon) \\ y\dfrac{j+\epsilon}{u} & x-2y\epsilon-jy \end{bmatrix}.$$

Also, the only eigenvalue of this matrix is $x - \epsilon y$. □

Theorem 2. *If the entries* $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ *of the matrix* $\mathbf{A}$ *are null, then*

$$e^{\mathbf{A}} = e^{x-\epsilon y} \begin{bmatrix} \epsilon y + jy + 1 & -uy\,(j+\epsilon) \\ \dfrac{1}{u}y\,(j+\epsilon) & 1 - jy - \epsilon y \end{bmatrix}$$

where $\mathbf{a}_{11} = x + yj$ *and* $\overrightarrow{\mathbf{u}} = (u, 1)$ *is the only eigenvector of the matrix* A.

Proof. The matrix $\mathbf{A}$ can be written as $A = \mathbf{PDP}^{-1}$ where

$$\mathbf{P} = \begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} x - \epsilon y & \frac{y}{u}(j+\epsilon) \\ 0 & x - \epsilon y \end{bmatrix}$$

Therefore, we have

$$e^{\mathbf{D}} = e^{x-\epsilon y} \begin{bmatrix} 1 & \frac{y}{u}(j+\epsilon) \\ \mathbf{0} & 1 \end{bmatrix}$$

and

$$\begin{aligned} e^{\mathbf{A}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1} &= e^{x-\epsilon y} \begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{y}{u}(j+\epsilon) \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -u \end{bmatrix} \\ &= e^{x-\epsilon y} \begin{bmatrix} \epsilon y + jy + 1 & -uy(j+\epsilon) \\ \frac{1}{u}y(j+\epsilon) & 1 - jy - \epsilon y \end{bmatrix}. \end{aligned}$$

□

Example 2. *Let's calculate exponential of the split-complex matrix*

$$\mathbf{A} = \begin{bmatrix} 4j+3 & -8j-8 \\ 2j+2 & -4j-5 \end{bmatrix}.$$

Since $\Delta = 0$ *and* $\mathbf{a}_{11} - \mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ *are null, we use above theorem. From the equalities,* $\mathbf{a}_{11} = x + yj = 3 + 4j$, $\mathbf{a}_{12} = -uy(j+\epsilon) = -8j - 8$, *we get* $x = 3$, $y = 4$, $\epsilon = 1$ *and* $u = 2$, *thus we obtain*

$$e^{\mathbf{A}} = e^{-1} \begin{bmatrix} 5+4j & -8(j+1) \\ 2(j+1) & -4j-3 \end{bmatrix}.$$

Theorem 3. *Let* $\mathbf{A} = [\mathbf{a}_{ij}]$ *be a* 2×2 *split-complex matrix with* $\Delta = 0$. *Then,*

$$e^{\mathbf{A}} = e^{\boldsymbol{\lambda}} \left[(1 - \boldsymbol{\lambda}) I + \mathbf{A} \right]. \tag{2}$$

Proof. In the case $\Delta = 0$, we can write $\mathbf{A} = \mathbf{PDP}^{-1}$ where $\mathbf{P}$ and $\mathbf{D}$ can be chosen as Lemma 2 above. Therefore, according to Lemma 1,

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} \boldsymbol{\lambda} & 1/2 \\ 0 & \boldsymbol{\lambda} \end{bmatrix} \mathbf{P}^{-1} \Rightarrow e^{\mathbf{A}} = e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \mathbf{P}^{-1}$$

is written. Hence, we get

$$\begin{aligned} e^{\mathbf{A}} &= e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \mathbf{P}^{-1} \\ &= e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} 1 - \boldsymbol{\lambda} + \boldsymbol{\lambda} & 1/2 \\ 0 & 1 - \boldsymbol{\lambda} + \boldsymbol{\lambda} \end{bmatrix} \mathbf{P}^{-1} \\ &= e^{\boldsymbol{\lambda}} \mathbf{P} (1 - \boldsymbol{\lambda}) I \mathbf{P}^{-1} + e^{\boldsymbol{\lambda}} \mathbf{P} \begin{bmatrix} \boldsymbol{\lambda} & 1/2 \\ 0 & \boldsymbol{\lambda} \end{bmatrix} \mathbf{P}^{-1} \end{aligned}$$

$$= e^{\boldsymbol{\lambda}}\left(\left(1-\boldsymbol{\lambda}\right)I+\mathbf{A}\right).$$

On the other hand, if the entries $\mathbf{a}_{11}-\mathbf{a}_{22}$, $\mathbf{a}_{12}$, $\mathbf{a}_{21}$ of the matrix A are null, the only eigenvalue is $\lambda = x - y\epsilon$, and eigenvector is $\overrightarrow{\mathbf{u}} = (u,1)$. So, we can write as $\mathbf{A} = PDP^{-1}$ where

$$\mathbf{P}=\begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \qquad ve \qquad \mathbf{D}=\begin{bmatrix} \lambda & \mathbf{k} \\ 0 & \lambda \end{bmatrix}, \quad \mathbf{k}=\frac{y\left(j+\epsilon\right)}{u}.$$

So, we get again

$$e^{\mathbf{A}} = e^{\lambda}\mathbf{P}\begin{bmatrix} 1 & \mathbf{k} \\ 0 & 1 \end{bmatrix}\mathbf{P}^{-1} = e^{\lambda}\left(\mathbf{P}\left(1-\lambda\right)I\mathbf{P}^{-1}+\mathbf{A}\right) = e^{\boldsymbol{\lambda}}\left(\left(1-\boldsymbol{\lambda}\right)I+\mathbf{A}\right).$$

If we write the matrices, we have

$$\begin{aligned} e^{\mathbf{A}} &= e^{\boldsymbol{\lambda}}\left(\left(1-x+y\epsilon\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} x+jy & -uy\left(j+\epsilon\right) \\ y\dfrac{j+\epsilon}{u} & x-2y\epsilon-jy \end{bmatrix}\right) \\ &= e^{x-\epsilon y}\begin{bmatrix} \epsilon y+jy+1 & -uy\left(j+\epsilon\right) \\ \dfrac{1}{u}y\left(j+\epsilon\right) & 1-jy-\epsilon y \end{bmatrix}. \end{aligned}$$

□

Corollary 1. *Let* $\mathbf{A} = [\mathbf{a}_{ij}]$ *be a* 2×2 *split-complex matrix with* $\Delta = 0$. *Then,*

$$e^{\mathbf{A}} = e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\begin{bmatrix} 1+\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & 1-\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} \end{bmatrix}. \tag{3}$$

Proof. According to Theorem 3,

$$\begin{aligned} e^{\mathbf{A}} &= e^{\boldsymbol{\lambda}}\left[\left(1-\boldsymbol{\lambda}\right)\mathbf{I}+\mathbf{A}\right] \\ &= e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\left(\begin{bmatrix} 1-\boldsymbol{\lambda} & 0 \\ 0 & 1-\boldsymbol{\lambda} \end{bmatrix}+\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix}\right) \\ &= e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\left(\begin{bmatrix} 1-\dfrac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2} & 0 \\ 0 & 1-\dfrac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2} \end{bmatrix}+\begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix}\right) \\ &= e^{\frac{\mathbf{a}_{11}+\mathbf{a}_{22}}{2}}\begin{bmatrix} 1+\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & 1-\dfrac{\mathbf{a}_{11}-\mathbf{a}_{22}}{2} \end{bmatrix} \end{aligned}$$

is obtained. □

3.2. **Case 2 : $\Delta \neq 0$ and $\Delta \notin \mathbb{P}_0$.** Now, we will deal with how to find the exponent of a split-complex matrix, when discriminant (Δ) is not a null number. In this case, we have two primary eigenvalues $\boldsymbol{\lambda}_{1,2} = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} \pm \sqrt{\Delta}\right)$, since $\Delta \neq 0$. For the split-complex matrix $\mathbf{A} = [\mathbf{a}_{ij}]_{2\times 2}$, we can write as $\mathbf{A} = \mathbf{PDP}^{-1}$ where

$$\mathbf{P} = \begin{bmatrix} \mathbf{a}_{11} - \mathbf{a}_{22} - \sqrt{\Delta} & 2\mathbf{a}_{21} \\ 2\mathbf{a}_{12} & \mathbf{a}_{22} - \mathbf{a}_{11} + \sqrt{\Delta} \end{bmatrix} \text{ and } \mathbf{D} = \frac{1}{2}\begin{bmatrix} \mathrm{tr}\mathbf{A} - \sqrt{\Delta} & 0 \\ 0 & \mathrm{tr}\mathbf{A} + \sqrt{\Delta} \end{bmatrix}.$$

Notice that $\det \mathbf{P} = 2\sqrt{\Delta}\left(\mathbf{a}_{11} - \mathbf{a}_{22} - \sqrt{\Delta}\right)$, so the matrix $\mathbf{P}$ does not have an inverse, if $\mathbf{a}_{11} - \mathbf{a}_{22} - \sqrt{\Delta} \in \mathbb{P}_0$.

For example, for the split-complex matrix

$$\mathbf{A} = \begin{bmatrix} j+1 & 3+2j \\ 1+2j & j+3 \end{bmatrix},$$

we have

$$\mathbf{P} = \begin{bmatrix} -4j-6 & 4j+2 \\ 4j+6 & 4j+6 \end{bmatrix}$$

and this matrix has no inverse.

Theorem 4. *Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of any matrix $\mathbf{A} \in \mathbb{M}_{2\times 2}(\mathbb{P})$. If $\Delta \neq 0$ and $\Delta \notin \mathbb{P}_0$, then, $\boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1$ is not null. So, exponential of $\mathbf{A}$ is*

$$e^{\mathbf{A}} = \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1}\mathbf{A}. \tag{4}$$

Proof. Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of the matrix $\mathbf{A} \in \mathbb{M}_{2\times 2}(\mathbb{P})$. So we can write

$$\mathbf{A} = \mathbf{P}\begin{bmatrix} \boldsymbol{\lambda}_1 & 0 \\ 0 & \boldsymbol{\lambda}_2 \end{bmatrix}\mathbf{P}^{-1}.$$

Similarly, the matrix $\mathbf{P}$ here is a matrix of eigenvectors whose columns correspond to the eigenvalues $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$. Therefore, we can write

$$e^{\mathbf{A}} = \mathbf{P}\begin{bmatrix} e^{\boldsymbol{\lambda}_1} & 0 \\ 0 & e^{\boldsymbol{\lambda}_2} \end{bmatrix}\mathbf{P}^{-1}.$$

Hence, we get

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{P}\begin{bmatrix} e^{\boldsymbol{\lambda}_1} & 0 \\ 0 & e^{\boldsymbol{\lambda}_2} \end{bmatrix}\mathbf{P}^{-1} \\ &= \mathbf{P}\begin{bmatrix} \dfrac{\boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} + \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2} & 0 \\ 0 & \dfrac{\boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} + \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2} \end{bmatrix}\mathbf{P}^{-1} \end{aligned}$$

$$= \mathbf{P} \begin{bmatrix} \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} & 0 \\ 0 & \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \end{bmatrix} \mathbf{P}^{-1}$$

$$+ \mathbf{P} \begin{bmatrix} \frac{\boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} & 0 \\ 0 & \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_2} - \boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \end{bmatrix} \mathbf{P}^{-1}$$

$$= \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \mathbf{A}.$$

□

Theorem 5. *Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of any matrix $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$. If $\Delta \neq 0$ and $\Delta \notin \mathbb{P}_0$, then, $\boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1$ is not null. So, exponential of $\mathbf{A}$ is*

$$e^{\mathbf{A}} = \frac{\mathbf{m}}{\Delta} \begin{bmatrix} \sqrt{\Delta}\cosh\sqrt{\Delta} + (\mathbf{a}_{11} - \mathbf{a}_{22})\sinh\sqrt{\Delta} & 2\mathbf{a}_{12}\sinh\sqrt{\Delta} \\ 2\mathbf{a}_{21}\sinh\sqrt{\Delta} & \sqrt{\Delta}\cosh\sqrt{\Delta} - (\mathbf{a}_{11} - \mathbf{a}_{22})\sinh\sqrt{\Delta} \end{bmatrix} \tag{5}$$

where $\mathbf{m} = e^{(\mathrm{tr}\mathbf{A})/2}$ and $\Delta = (\mathrm{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ be the eigenvalues of the matrix $\mathbf{A} \in \mathbb{M}_{2\times 2}(\mathbb{P})$. We know that

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sin x = \frac{e^x - e^{-x}}{2}.$$

On the other hand, we have $\lambda_1 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} - \sqrt{\Delta}\right)$, $\lambda_2 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)$ and $\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1 = \Delta$. Therefore, using the Theorem 4, and the equalities

$$e^{\boldsymbol{\lambda}_2} = e^{\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)/2} = e^{(\mathrm{tr}\mathbf{A})/2} e^{\sqrt{\Delta}} = \mathbf{mk},$$
$$e^{\boldsymbol{\lambda}_1} = \mathbf{mk}^{-1},$$

where $\mathbf{k} = e^{\sqrt{\Delta}}$, we find

$$\begin{aligned} e^{\mathbf{A}} &= \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \mathbf{A} \\ &= \frac{\boldsymbol{\lambda}_2 \mathbf{mk}^{-1} - \boldsymbol{\lambda}_1 \mathbf{mk}}{\Delta} I + \frac{\mathbf{mk} - \mathbf{mk}^{-1}}{\Delta} \mathbf{A} \\ &= \frac{\mathbf{m}}{\Delta}\left(\left(\boldsymbol{\lambda}_2 \mathbf{k}^{-1} - \boldsymbol{\lambda}_1 \mathbf{k}\right) I + \left(\mathbf{k} - \mathbf{k}^{-1}\right)\mathbf{A}\right) \\ &= \frac{\mathbf{m}}{\Delta} \begin{bmatrix} \boldsymbol{\lambda}_2 \mathbf{k}^{-1} - \boldsymbol{\lambda}_1 \mathbf{k} + \mathbf{a}_{11}\left(\mathbf{k} - \mathbf{k}^{-1}\right) & \mathbf{a}_{12}\left(\mathbf{k} - \mathbf{k}^{-1}\right) \\ \mathbf{a}_{21}\left(\mathbf{k} - \mathbf{k}^{-1}\right) & \boldsymbol{\lambda}_2 \mathbf{k}^{-1} - \boldsymbol{\lambda}_1 \mathbf{k} + \mathbf{a}_{22}\left(\mathbf{k} - \mathbf{k}^{-1}\right) \end{bmatrix} \end{aligned}$$

$$= \frac{\mathbf{m}}{\Delta}\begin{bmatrix} \boldsymbol{\lambda}_2\mathbf{k}^{-1} - \boldsymbol{\lambda}_1\mathbf{k} + \mathbf{2a}_{11}\sinh\sqrt{\Delta} & 2\mathbf{a}_{12}\sinh\sqrt{\Delta} \\ 2\mathbf{a}_{21}\sinh\sqrt{\Delta} & \boldsymbol{\lambda}_2\mathbf{k}^{-1} - \boldsymbol{\lambda}_1\mathbf{k} + \mathbf{2a}_{22}\sinh\sqrt{\Delta} \end{bmatrix}.$$

If we write $\lambda_1 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} - \sqrt{\Delta}\right)$ and $\lambda_2 = \frac{1}{2}\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)$, we have

$$\begin{aligned} \boldsymbol{\lambda}_2\mathbf{k}^{-1} - \boldsymbol{\lambda}_1\mathbf{k} &= \frac{1}{2}\left(\mathrm{tr}\mathbf{A} + \sqrt{\Delta}\right)\mathbf{k}^{-1} - \frac{1}{2}\left(\mathrm{tr}\mathbf{A} - \sqrt{\Delta}\right)\mathbf{k} \\ &= (\mathrm{tr}\mathbf{A})\,\frac{\mathbf{k}^{-1} - \mathbf{k}}{2} + \sqrt{\Delta}\frac{\mathbf{k}^{-1} + \mathbf{k}}{2} \\ &= \sqrt{\Delta}\cosh\sqrt{\Delta} - (\mathbf{a}_{11} + \mathbf{a}_{22})\sinh\sqrt{\Delta}. \end{aligned}$$

Thus, we obtain 5. □

Notice that if Δ is a null number, this formula does not work.

3.3. **Case 3 : $\Delta \neq 0$ and $\Delta \in \mathbb{P}_0$.**

Theorem 6. *Let $\mathbf{A} = [\mathbf{a}_{ij}] \in \mathbb{M}_{2\times 2}(\mathbb{P})$ be a split-complex matrix with $\Delta \in \mathbb{P}_0$ and $\Delta \neq 0$. If eigenvectors of $\mathbf{A}$ are not a real vector, $\mathbf{A}$ cannot be diagonalized.*

Proof. If $\Delta \in \mathbb{P}_0$, then we can write that $\Delta = x(1 + \varepsilon h)$, $x \in \mathbb{R}$. Therefore we have

$$\sqrt{\Delta} = \frac{\sqrt{2x}}{2}(1 + \varepsilon j).$$

So, eigenvalues of $\mathbf{A}$ will be

$$\lambda_{1,2} = \frac{2\mathrm{tr}\mathbf{A} \pm \sqrt{2x}(1 + \varepsilon j)}{4}.$$

Also, the eigenvector matrix $\mathbf{P}$ can be found as

$$\mathbf{P} = \begin{bmatrix} 2(\mathbf{a}_{11} - \mathbf{a}_{22}) - \sqrt{2x}(1 + \varepsilon j) & 2(\mathbf{a}_{11} - \mathbf{a}_{22}) + \sqrt{2x}(1 + \varepsilon j) \\ 4\mathbf{a}_{21} & 4\mathbf{a}_{21} \end{bmatrix}.$$

Determinant of this matrix is

$$\det\mathbf{P} = -8\sqrt{2x}a_{21}(1 + j\varepsilon) \in \mathbb{P}_0.$$

So, $\mathbf{P}^{-1}$ is not defined. □

Example 3. *For the matrix,*

$$\mathbf{A} = \begin{bmatrix} 1 + 2j & 1 - j \\ 2 + 2j & j \end{bmatrix}, \tag{6}$$

we have $\Delta = 2j + 2 \in \mathbb{P}_0$. Although the primary eigenvalues of this matrix are different from each other, it cannot be diagonalized. We have to use another method to find the exponential of this matrix.

Remark 1. *Using the fact that $\boldsymbol{\lambda}(1+j)=0 \Leftrightarrow \boldsymbol{\lambda}=0$, we can write the eigenvectors of some matrices as real vectors. Some of this case,* $\mathbf{A}$ *can be diagonalized. For example, the matrix*

$$\begin{bmatrix} 3 & -j-1 \\ 2j+2 & -3j \end{bmatrix}$$

can be written as,

$$\begin{bmatrix} 3 & -j-1 \\ 2j+2 & -3j \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2-j & 0 \\ 0 & 1-2j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}.$$

Notice that $a_{11}-a_{22}$, a_{11}, a_{21} and Δ are null split-complex numbers in the form $x(1+j)$, $x \in \mathbb{R}$ and we do not need to use the identity $j^2=1$. Here, Δ is also a square of a split-complex number.

Remark 2. *The exponentials of the $n \times n$ split-complex matrix $\mathbf{A}=X+Yj$ can be computed by converting them to $2n \times 2n$ real matrices with the isomorphism*

$$\mathcal{P}(\mathbf{A}) = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix},$$

and using the help of Jordan form and the useful property $e^{\mathcal{P}(\mathbf{X})} = \mathcal{P}\left(e^{\mathbf{X}}\right)$. The matrix $\mathcal{P}(\mathbf{A})$ is called the real matrix representation of the split-complex matrix $\mathbf{A}$. *It is known that any $n \times n$ complex matrix A can be written as sum of a diagonalizable matrix B and nilpotent matrix N_0 where the matrices B and N commute. Remember that if N is a nilpotent matrix, then N^k is zero matrix for $k \in \mathbb{Z}^+$. The Jordan matrix decomposition of a square matrix A is $A=PJP^{-1}$ where J is a Jordan matrix [29]. It means that a square complex matrix A is similar to a block diagonal matrix J. In this case, we can write as*

$$J = D + N$$

where D is the diagonal and N is strictly triangular and thus nilpotent matrix. Then, we have

$$A = P(D+N)P^{-1} = PDP^{-1} + PNP^{-1}.$$

Therefore, any $n \times n$ complex matrix A can be written as the diagonalizable matrix $B=PDP^{-1}$ and nilpotent matrix $N_0=PNP^{-1}$, since

$$(PNP^{-1})^k = PN(P^{-1}P)N(P^{-1}N \cdots NP)NP^{-1} = PN^kP^{-1} = 0.$$

Also, the matrices $B=PDP^{-1}$ and $N_0=PNP^{-1}$ commute. This property allows us to simplify the calculation of a matrix exponential.

$$e^A = e^{PJP^{-1}} = e^{P(D+N)P^{-1}} = Pe^{D+N}P^{-1} = Pe^De^NP^{-1}.$$

Example 4. *Let's find the exponential of the matrix 6*

$$\mathbf{A} = \begin{bmatrix} 1+2j & 1-j \\ 2+2j & j \end{bmatrix}.$$

We know that it is not diagonalized and $\Delta = 2j + 2 \in \mathbb{P}_0$. *Therefore, we convert it to the* 4×4 *real matrix*

$$\mathcal{P}(\mathbf{A}) = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 0 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

This matrix also cannot be diagonalized. But, we can write in Jordan form as

$$\mathcal{P}(\mathbf{A}) = P^{-1} J P = P^{-1}(D + N) P$$

where $N^2 = 0$. *Therefore, we obtain*

$$\begin{aligned} e^{\mathcal{P}(\mathbf{A})} &= P e^{N} e^{D} P^{-1} \\ &= \begin{bmatrix} 0 & 1 & 1/2 & -1 \\ 1 & 2 & -1/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 1 & 2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} e^{1} & 0 & 0 & 0 \\ 0 & e^{3} & 0 & 0 \\ 0 & 0 & e^{-1} & 0 \\ 0 & 0 & e^{-1} & e^{-1} \end{bmatrix} \begin{bmatrix} -1 & 1/2 & -1 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-1} + e^{3} & 2e^{-1} & e^{3} - e^{-1} & -2e^{-1} \\ 2e^{3} - 2e & e^{-1} + e & 2e^{3} - 2e & e - e^{-1} \\ e^{3} - e^{-1} & -2e^{-1} & e^{-1} + e^{3} & 2e^{-1} \\ 2e^{3} - 2e & e - e^{-1} & 2e^{3} - 2e & e^{-1} + e \end{bmatrix} \end{aligned}$$

since $e^{N} = I + N$. *As a result, according to equality* $e^{\mathcal{P}(\mathbf{A})} = \mathcal{P}\left(e^{\mathbf{A}}\right)$, *we find*

$$e^{\mathbf{A}} = \frac{1}{2} \begin{bmatrix} e^{-1} + e^{3} - j\left(e^{-1} - e^{3}\right) & 2e^{-1} - 2je^{-1} \\ 2e^{3} - 2e - j\left(2e - 2e^{3}\right) & e^{-1} + e - j\left(e^{-1} - e\right) \end{bmatrix}.$$

Conclusion Let $\mathbf{A} = [\mathbf{a}_{ij}]$ be a 2×2 split-complex matrix, we can compute exponential of $\mathbf{A}$ using the formulas :

- If $\triangle = 0$, then $e^{\mathbf{A}} = e^{\boldsymbol{\lambda}}\left[(1 - \boldsymbol{\lambda}) I + \mathbf{A}\right]$, where $\boldsymbol{\lambda}$ is only eigenvalue of $\mathbf{A}$.
- If $\triangle \neq 0$ and $\triangle$ is not a null split-complex number, then
$$e^{\mathbf{A}} = \frac{\boldsymbol{\lambda}_2 e^{\boldsymbol{\lambda}_1} - \boldsymbol{\lambda}_1 e^{\boldsymbol{\lambda}_2}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} I + \frac{e^{\boldsymbol{\lambda}_2} - e^{\boldsymbol{\lambda}_1}}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1} \mathbf{A}.$$
- If $\triangle$ is a null split-complex number, we do not give a direct computation formula without converting it to a real matrix.

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References

[1] Abłamowicz, R., Matrix exponential via Clifford algebras, *Journal of Nonlinear Mathematical Physics*, 5(3) (1998), 294-313. doi: 10.2991/jnmp.1998.5.3.5

[2] Baker, A., Matrix Groups: An Introduction to Lie Group Theory, Springer Science & Business Media, 2012. doi: 10.1007/978-1-4471-0183-3

[3] Bernstein, D. S., Orthogonal matrices and the matrix exponential, *SIAM Review*, 32(4) (1990), 673. doi: 10.1137/1032130

[4] Bernstein, D. S., So, W., Some explicit formulas for the matrix exponential, *IEEE Transactions on Automatic Control*, 38(8) (1993), 1228-1232. doi: 10.1109/9.233156

[5] Borota, N. A., Flores, E., Osler, T. J., Spacetime numbers the easy way, *Mathematics and Computer Education*, 34(2) (2000), 159.

[6] Catoni, F., Boccaletti, D., Cannata, R., Catoni, V., Zampetti, P., Geometry of Minkowski Space-time, Springer Science & Business Media, 2011. doi: 10.1007/978-3-642-17977-8

[7] Erdoğdu, M., Özdemir, M., On exponential of split quaternionic matrices, *Applied Mathematics and Computation*, 315 (2017), 468-476. doi: 10.1016/j.amc.2017.08.007

[8] Erdoğdu, M., Özdemir, M., Matrices over hyperbolic split quaternions, *Filomat*, 30(4) (2016), 913-920. doi: 10.2298/FIL1604913E

[9] Ersoy, S., Akyigit, M., One-parameter homothetic motion in the hyperbolic plane and Euler-Savary formula, *Advances in Applied Clifford Algebras*, 21(2) (2011), 297-313. doi: 10.1007/s00006-010-0255-3

[10] Fjelstad, P., Extending special relativity via the perplex numbers, *American Journal of Physics*, 54(5) (1986), 416-422. doi: 10.1119/1.14605

[11] Amorim, R. G. G. D., Santos, W. C. D., Carvalho, L. B., Massa, I. R., A physical approach of perplex numbers, *Revista Brasileira de Ensino de Física*, 40(3) (2018). doi: 10.1590/1806-9126-RBEF-2017-0356

[12] Gürses, N., Şentürk, G. Y., Yüce, S., A study on dual-generalized complex and hyperbolic-generalized complex numbers, *Gazi University Journal of Science*, 34(1) (2021), 180-194.

[13] Harkin, A. A., Harkin, J. B., Geometry of generalized complex numbers, *Mathematics Magazine*, 77(2) (2004), 118-129. doi: 10.1080/0025570X.2004.11953236

[14] Kisil, V. V., Geometry of Mobius Transformations: Elliptic, Parabolic and Hyperbolic Actions of SL2(R). World Scientific, 2012.

[15] Laksov, D., Diagonalization of matrices over rings, *Journal of Algebra*, 376 (2013), 123-138. doi: 10.1016/j.jalgebra.2012.10.029

[16] Leonard, I. E., The matrix exponential, *SIAM review*, 38(3) (1996), 507-512. doi: 10.1137/S0036144595286488

[17] Machen, C., The exponential of a quaternionic matrix, *Rose-Hulman Undergraduate Mathematics Journal*, 12(2) (2011), 3.

[18] McDonald Bernard, R., Lynn McDonald, et al., Linear algebra over commutative rings, volume 87, *Courier Corporation,* 1984.

[19] Poodiack, R. D., LeClair, K. J., Fundamental theorems of algebra for the perplexes, *The College Mathematics Journal*, 40(5) (2009), 322-336. doi: 10.4169/074683409X475643

[20] Özyurt, G., Alagöz, Y., On hyperbolic split quaternions and hyperbolic split quaternion matrices, *Advances in Applied Clifford Algebras*, 28(5) (2018), 1-11. doi: 10.1007/s00006-018-0907-2

[21] Richter, R. B., Wardlaw, W. P., Diagonalization over commutative rings, *The American Mathematical Monthly*, 97(3) (1990), 223-227. doi: 10.1080/00029890.1990.11995580

[22] Rooney, J., On the three types of complex number and planar transformations, *Environment and Planning B: Planning and Design*, 5(1) (1978), 89-99. doi: 10.1068/b050089

[23] Sobczyk, G., The hyperbolic number plane, *The College Mathematics Journal*, 26(4) (1995), 268-280. doi: 10.1080/07468342.1995.11973712

[24] Sobczyk, G., Complex and Hyperbolic Numbers, In New Foundations in Mathematics (pp. 23-42). Birkhäuser, Boston. doi: 10.1007/978-0-8176-8385-6_2

[25] Sporn, H., Pythagorean triples, complex numbers, and perplex numbers, *The College Mathematics Journal*, 48(2) (2017), 115-122. doi: 10.4169/college.math.j.48.2.115

[26] Tapp, T., Matrix groups for undergraduates, *student math*, 2005.

[27] Ulrych, S., Representations of Clifford algebras with hyperbolic numbers, *Advances in Applied Clifford Algebras*, 18(1) (2008), 93-114. doi: 10.1007/s00006-007-0057-4

[28] Yaglom, I. M., A Simple Non-Euclidean Geometry and Its Physical Basis: An Elementary Account of Galilean Geometry and the Galilean Pinciple of Relativity, Springer Science & Business Media, 2012. doi: 10.1007/978-1-4612-6135-3

[29] Zhang, F., Matrix Theory: Basic Results and Techniques, Springer Science & Business Media, 2011.

http://communications.science.ankara.edu.tr

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A NOVEL ANALYSIS OF INTEGRAL INEQUALITIES IN THE FRAME OF FRACTIONAL CALCULUS

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Abstract. In this paper, we define and explore the new family of exponentially convex functions which are called exponentially s–convex functions. We attain the amazing examples and algebraic properties of this newly introduced function. In addition, we find a novel version of Hermite-Hadamard type inequality in the support of this newly introduced concept via the frame of classical and fractional calculus (non-conformable and Riemann-Liouville integrals operator). Furthermore, we investigate refinement of Hermite-Hadamard type inequality by using exponentially s–convex functions via fractional integral operator. Finally, we elaborate some Ostrowski type inequalities in the frame of fractional calculus. These new results yield us some generalizations of the prior results.

1. Introduction

Convex functions are significant in the hypothesis of numerical inequalities, some notable outcomes are immediate ramifications of these functions. The ideas of different sorts of new convex functions are developed from the basic definition of a convex function. The generalizations, extensions and refinements of these functions are proved to be very beneficial in mathematical analysis, financial mathematics, mathematical statistics, optimization theory, etc. For the attention of readers, see the reference [1–4].

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In literature, the celebrated Hermite-Hadamard double Inequality [5] for convex function on an interval of the real line, discovered by C. Hermite and J. Hadamard individually, has been the hot topic for extensive research, which is stated as,

Let a function $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$ is a convex function on $\mathbb{A}$ in $\mathbb{R}$ and $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$, then

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi(x)dx \leq \frac{\varphi(\delta_1)+\varphi(\delta_2)}{2}. \qquad (1)$$

Let φ is a concave function, then this inequality 1 is reversed. The inequality 1 provides upper and lower bounds for the integral means of the convex function φ. The inequality 1 has different kinds of forms with correspondence to different kinds of convexities such as s–convex function, h–convex function, p–convex function, log–convex function, exponential convex function, exponential type convex function, MT–convex function, tgs–convex function, n–polynomial convex function, preinvex functions etc.
Recently, few researchers have been studying on the properties and applications of exponential type convexity, for more information, we refer interested readers to go through [6–9].
In the literature of inequalities the Hadamard inequality and Ostrowski type inequality appear in different forms for various convex functions. In [10] and [11] for the first time, Hermite-Hadamard inequality and Ostrowski inequality was studied for Riemann-Liouville fractional integrals respectively and after it, researchers started to get many versions of these for different kinds of fractional integral operators and functions.
In the literature, Ostroswki Inequality [12] is defined as follows:
Let $\varphi : I \subset [0,\infty) \longrightarrow R$ be a differentiable mapping on I°, the interior of the interval I, such that $\varphi' \in \mathcal{L}[\delta_1, \delta_2]$, where $\delta_1, \delta_2 \in I$ with $\delta_1 < \delta_2$. If $|\varphi'(z)| \leq \mathbb{M}$, for all $x \in [\delta_1, \delta_2]$, then the following inequality holds:

$$\left|\varphi(z) - \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi(x)\,dx\right| \leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}\left[\frac{(z-\delta_1)^2+(\delta_2-z)^2}{2}\right], \qquad (2)$$

which gives an upper bound for approximations of the integral average by $\varphi(z)$ at point $z \in [\delta_1, \delta_2]$
For some recent generalizations about this inequality please see [13], [14] and the references therein. In [15], the authors have established some Ostrowski type inequalities for s-convex function in the second sense. In the recent past many generalizations for Ostrowski type inequality have been performed via different directions like on coordinates, on quantum calculus, on different fractional integral operators like Riemann -Liouville, Katugampola, Caputo, Caputo Fabrizio, ψ- generalized fractional operator, etc.

2. Preliminaries

In this section, we recall some known concepts.

Definition 1. *[16] Let $\varphi : \mathbb{A} \to \mathbb{R}$ be a real valued function. A function φ is said to be convex, if*

$$\varphi\left(\lambda\delta_1 + (1-\lambda)\,\delta_2\right) \leq \lambda\varphi\left(\delta_1\right) + (1-\lambda)\,\varphi\left(\delta_2\right), \tag{3}$$

holds for all $\delta_1, \delta_2 \in \mathbb{A}$ and $\lambda \in [0,1]$.

Definition 2. *[17] Let $s \in (0,1]$, then $\varphi : [0,+\infty) \to \mathbb{R}$ is known as s–convex in the 2^{nd} sense, if*

$$\varphi\left(\lambda\delta_1 + (1-\lambda)\,\delta_2\right) \leq \lambda^s\varphi\left(\delta_1\right) + (1-\lambda)^s\,\varphi\left(\delta_2\right), \tag{4}$$

holds $\forall\ \delta_1, \delta_2 \in [0,+\infty)$ and $\lambda \in [0,1]$.

Dragomir et al. investigated and explored a novel version of Hadamard's inequality in the mode of s–convex functions in the 2^{nd} sense in the published article [18]. In the last few decades or so, fractional calculus can be seen gaining a lot of attention as the most researched subject of mathematics. Its importance is prominent from the fact that many real-life problems are well interpreted and modeled using the theory of fractional calculus. It is also seen that various branches of engineering and applied science have been using the tools and techniques of fractional calculus. It is mainly due to the two mathematicians, L'Hospital and Leibnitz that fractional calculus is so popular nowadays. After this many mathematicians developed different new types of fractional operators and worked upon them to generalize inequalities like Hermite-Hadamard, Ostrowski, Opial, Jensen, Hermite-Hadamard-mercer, Oslen type, etc. The authors examined and celebrated conformable and non-conformable derivative in the published articles [19] and [20]. Both fractional integral operators have a lot of meaningful and useful applications, see the references [21–29].

Definition 3. *Let $\varphi : \mathbb{A} \subseteq [0,\infty) \to \mathbb{R}$ be a real valued function, then the non–conformable derivative of φ is defined by*

$$_{N_3^\alpha}\varphi(x) = \lim_{\epsilon\to 0}\frac{\varphi(\lambda + \epsilon\lambda^\alpha) - \varphi(\lambda)}{\epsilon},$$

where $\alpha \in (0,1)$ and $\lambda \in \mathbb{A}$.

If $\exists\ _{N_3^\alpha}\varphi(\lambda)$ and is finite, then φ is a α–differentiable at λ.
If φ at λ is a differentiable, then

$$_{N_3^\alpha}\varphi(\lambda) = \lambda^\alpha\varphi^{'}(\lambda).$$

Definition 4. *[30] For each $\varphi \in L[\delta_1, \delta_2]$ and $0 < \delta_1 < \delta_2$, we define*

$$_{N_3}J_u^\alpha\varphi(x) = \int_u^x \lambda^{-\alpha}\varphi(\lambda)d\lambda,$$

for every $x, u \in [\delta_1, \delta_2]$ and $\alpha \in \mathbb{R}$.

Definition 5. *[30] For each function $\varphi \in L[\delta_1, \delta_2]$ and $\delta_1 < \delta_2$, we define the fractional integrals*

$$ {}_{N_3}J^{\alpha}_{\delta_1^+}\varphi(x) = \int_{\delta_1}^{x} (x-\lambda)^{-\alpha}\varphi(\lambda)d\lambda, $$

$$ {}_{N_3}J^{\alpha}_{\gamma_2^-}\varphi(x) = \int_{x}^{\delta_2} (\lambda-x)^{-\alpha}\varphi(\lambda)d\lambda, $$

for every $x \in [\gamma_1, \gamma_2]$ and $\alpha \in \mathbb{R}$.

Remark 1. *In the above definitions, if we put $\alpha = 0$ then we get the classical integrals which is represented by ${}_{N_3}J^{\alpha}_{\delta_1^+}\varphi(x) = {}_{N_3}J^{\alpha}_{\delta_2^-}\varphi(x) = \int_{\delta_1}^{\delta_2}\varphi(\lambda)d\lambda$.*

It is remarkable that M.Z. Sarikaya et al. (see in [10]) proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *[10] Suppose $\varphi : \mathbb{A} = [\delta_1, \delta_2] \to \mathbb{R}$ is a positive mapping with $\delta_2 > \delta_1$ and $\varphi \in L[\delta_1, \delta_2]$. If φ is a convex function on $[\delta_1, \delta_2]$, then the following inequalities for fractional integrals holds:*

$$ \varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^{\alpha}}\left\{J^{\alpha}_{\delta_1^+}\varphi(\delta_2) + J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right\} \leq \frac{\varphi(\delta_1) + \varphi(\delta_2)}{2}, $$

with $\alpha > 0$.

Definition 6. *[10] Let $\varphi \in L[\delta_1, \delta_2]$. Then Riemann–Liouville fractional integrals of order $\alpha > 0$ with $\delta_1 \geq 0$ are defined as follows:*

$$ J^{\alpha}_{\delta_1^+}\varphi(z) = \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{z}(z-\lambda)^{\alpha-1}\varphi(\lambda)\,d\lambda, \quad z > \delta_1 $$

and

$$ J^{\alpha}_{\delta_2^-}\varphi(z) = \frac{1}{\Gamma(\alpha)}\int_{z}^{\delta_2}(\lambda-z)^{\alpha-1}\varphi(\lambda)\,d\lambda, \quad z < \delta_2. \tag{5} $$

where $\Gamma(r)$ is the Gamma function defined by

$$ \Gamma(r) = \int_0^{\infty} e^{-y}y^{r-1}dy. $$

Since $a(a > 0)$ will stand for the parameter of the incomplete gamma function (see [31]:8.2.1)

$$ \gamma(a, r) = \int_0^{r} e^{-y}y^{a-1}dy. $$

For further details one may, see [32, 33, 35].

We compose the paper in the following manner, In section 3, we will give the idea of exponentially s–convex functions, examples, and its properties. In section 4, we will give the generalizations of (H-H)–type inequality in the support of the newly introduced idea. In section 5, we will investigate the new version of Hermite–Hadamard type inequality and its refinements for exponentially s–convex function via a fractional integral operator. In section 6, we will also obtain some Ostrowski type inequalities for the exponentially s–convex function φ for fractional integral inequalities. In section 7, a brief conclusion will be given as well.

3. Exponentially s–Convex Function and Its Properties

The main aim of this section is to define the new family of convex functions, which are called exponentially s–convex functions. In the manner of this newly introduced concept, we obtain some examples and algebraic properties.

Definition 7. *Let $s \in [\ln 2.4, 1]$. Then $\varphi : \mathbb{A} \subset \mathbb{R} \to \mathbb{R}$ is known to be exponentially s–convex function, if*

$$\varphi\left(\lambda\delta_1 + (1-\lambda)\delta_2\right) \leq \left(e^{s\lambda} - 1\right)\varphi\left(\delta_1\right) + \left(e^{s(1-\lambda)} - 1\right)\varphi\left(\delta_2\right), \tag{6}$$

holds $\forall \delta_1, \delta_2 \in \mathbb{R}$ and $\lambda \in [0,1]$.

Remark 2. *In above Definition 7, if $s = 1$, then we get exponential type convexity given by İşcan in [6].*

Remark 3. *The range of the exponentially s–convex functions for some fixed $s \in [\ln 2.4, 1]$ is $[0, +\infty)$.*

Lemma 1. *The following inequalities $(e^{s\lambda} - 1) \geq \lambda^s$ and $(e^{s(1-\lambda)} - 1) \geq (1-\lambda)^s$ are holds, if for all $\lambda \in [0,1]$ and for some fixed $s \in [\ln 2.4, 1]$*

Proof. The proof is evident. □

Proposition 1. *Every nonnegative s–convex function is exponentially s–convex function for $s \in [\ln 2.4, 1]$.*

Proof. By using Lemma 1, for $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned}\varphi\left(\lambda\delta_1 + (1-\lambda)\delta_2\right) &\leq \lambda^s\varphi\left(\delta_1\right) + (1-\lambda)^s\varphi\left(\delta_2\right)\\ &\leq \left(e^{s\lambda} - 1\right)\varphi\left(\delta_1\right) + \left(e^{(1-\lambda)s} - 1\right)\varphi\left(\delta_2\right).\end{aligned}$$

□

Proposition 2. *Every exponentially s–convex function for $s \in [\ln 2.4, 1]$ is an h–convex function with $h(\lambda) = \left(e^{\lambda} - 1\right)$*

Proof.

$$\begin{aligned}\varphi\left(\lambda\delta_1 + (1-\lambda)\delta_2\right) &\leq \left(e^{\lambda} - 1\right)\varphi\left(\delta_1\right) + \left(e^{1-\lambda} - 1\right)\varphi\left(\delta_2\right)\\ &\leq h(\lambda)\varphi(\delta_1) + h(1-\lambda)\varphi(\wp_2).\end{aligned}$$

□

Example 1. *Dragomir have investigated that in the published article [18], the non-negative function* $\varphi(x) = x^{ls}$, $x > 0$ *is* s*–convex function for the all mention conditions* $s \in (0,1)$, *where* $1 \leq l \leq \frac{1}{s}$, . *Then according to Proposition 2, it is also exponentially* s*–convex function for some fixed* $s \in [\ln 2.4, 1)$.

Example 2. $\varphi(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$ *for* $m > 1, q \geq 1$ *is a non-negative* s*-convex function. Then according to Proposition 2, it is also exponentially* s*–convex function for some fixed* $s \in [\ln 2.4, 1)$.

Theorem 2. *Let* $\varphi : [0,\delta] \to \mathbb{A}$ *be* s*–convex function for* $s \in [\ln 2.4, 1]$ *and* $\phi : \mathbb{A} \to \mathbb{R}$ *is non-decreasing and exponentially convex function. Then the function* $\phi \circ \varphi : [0,\delta] \to \mathbb{R}$ *is exponentially* s*–convex function.*

Proof. For all $\delta_1, \delta_2 \in [0,\delta]$ and $\lambda \in [0,1]$, and for $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned}(\phi \circ \varphi)(\lambda\delta_1 + (1-\lambda)\delta_2) &= \phi(\varphi(\lambda\delta_1 + (1-\lambda)\delta_2)) \leq \phi(\lambda^s \varphi(\delta_1) + (1-\lambda)^s \varphi(\delta_2)) \\ &\leq \left(e^{s\lambda} - 1\right)(\phi \circ \varphi)(\delta_1) + \left(e^{(1-\lambda)s} - 1\right)(\phi \circ \varphi)(\delta_2).\end{aligned}$$

□

Remark 4. *If we choose* $s = 1$ *in above Theorem (2), then we get Theorem (2.2) in [6].*

4. New Generalizations of (H–H) Type Inequality Using Exponentially s–Convex Functions

The aim of this section is to find the new generalization of Hermite–Hadamard type inequality for the exponentially s–convex function for φ in the frame of simple calculus and also we attain the novel version of Hermite–Hadamard type inequality in the manner of newly introduced idea in the frame of fractional calculus by the non-conformable integral operator.

Theorem 3. *Suppose* $s \in [\ln 2.4, 1]$, $\alpha \in (0,1]$, $\delta_2 > \delta_1$ *and* $\varphi : \mathbb{A} = [\delta_1, \delta_2] \to \mathbb{R}$ *is exponentially* s*–convex function such that* $\varphi \in L[\delta_1, \delta_2]$. *Then one has*

$$\begin{aligned}&\frac{1}{2\left(e^{\frac{s}{2}} - 1\right)} \varphi\left(\frac{\delta_1 + \delta_2}{2}\right) \leq \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \varphi(u)\ du \\ &\leq \left(\frac{e^s - s - 1}{s}\right) \left[\varphi(\delta_1) + \varphi(\delta_2)\right].\end{aligned} \tag{7}$$

Proof. Let $z_1, z_2 \in \mathbb{A}$. Then it follows from the exponentially s–convex function for φ on $\mathbb{A}$ that

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leq \left(e^{\frac{s}{2}} - 1\right)[\varphi(z_1) + \varphi(z_2)] \tag{8}$$

Suppose

$$z_1 = \lambda\delta_2 + (1-\lambda)\delta_1 \quad \text{and} \quad z_2 = \lambda\delta_1 + (1-\lambda)\delta_2.$$

Then (8) leads to

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)\left[\varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right)\right]. \tag{9}$$

Now integrating on both sides in the last inequality with respect to λ from 0 to 1, we obtain

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)\left[\int_0^1 \varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right)d\lambda+\int_0^1 \varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right)\, d\lambda\right].$$

$$\frac{1}{2\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi\left(u\right)\, du,$$

which gives the proof of first part of inequality of (7). Next, we show the second part of inequality of (7). Let $\lambda\in[0,1]$. Then from the fact that φ is exponentially s–convex function, we obtain

$$\varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right) \leq \left(e^{s\lambda}-1\right)\varphi\left(\delta_2\right)+\left(e^{s(1-\lambda)}-1\right)\varphi\left(\delta_1\right) \tag{10}$$

and

$$\varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right) \leq \left(e^{s\lambda}-1\right)\varphi\left(\delta_1\right)+\left(e^{s(1-\lambda)}-1\right)\varphi\left(\delta_2\right). \tag{11}$$

By adding the above inequalities, we obtain

$$\begin{aligned}&\varphi\left(\lambda\delta_2+(1-\lambda)\,\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\,\delta_2\right)\\ &\leq \left[\varphi\left(\delta_1\right)+\varphi\left(\delta_2\right)\right]\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}.\end{aligned} \tag{12}$$

Now integrating on both sides by above equation with respect to λ from 0 to 1, then making the change of variable, we obtain

$$\begin{aligned}&2\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\varphi\left(u\right)\, du.\\ &\leq \left[\varphi\left(\delta_1\right)+\varphi\left(\delta_2\right)\right]\int_0^1\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda,\end{aligned}$$

which leads to the conclusion that

$$\leq 2\left(\frac{e^s-s-1}{s}\right)\left[\varphi\left(\delta_1\right)+\varphi\left(\delta_2\right)\right].$$

The proof is completed. □

Remark 5. *If we choose $s=1$, then Theorem 3 becomes to* [*Theorem 3.1, [6]*].

Theorem 4. *Let $\varphi:\mathbb{A}=[\delta_1,\delta_2]\to\mathbb{R}$ be a positive function with $0\leq\delta_1\leq\delta_2$ and φ be a integrable function on closed interval set δ_1 and δ_2. If φ is exponentially*

s–convex function, then the following inequalities for fractional integral namely non-conformable integral operator with $\alpha < 0$ and $s \in [\ln 2.4, 1]$ holds:

$$\frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \le \frac{1-\alpha}{(\delta_2-\delta_1)^{1-\alpha}}\left[\,_{N_3}J^{\alpha}_{a^+}\varphi(x) +_{N_3} J^{\alpha}_{b^-}\varphi(x)\right] \tag{13}$$
$$\le (1-\alpha)[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1 \lambda^{-\alpha}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda.$$

Proof. Let $\sigma_1, \sigma_2 \in \mathbb{A}$. Then it follows from the exponentially s–convex function for φ on $\mathbb{A}$ that

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right) \le \left(e^{\frac{s}{2}}-1\right)\left[\varphi(\sigma_1)+\varphi(\sigma_2)\right]. \tag{14}$$

Suppose

$$\sigma_1 = \lambda\delta_2 + (1-\lambda)\,\delta_1 \quad \text{and} \quad \sigma_2 = \lambda\delta_1 + (1-\lambda)\,\delta_2.$$

Then (14) leads to

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \le \left(e^{\frac{s}{2}}-1\right)\left[\varphi(\lambda\delta_2+(1-\lambda)\,\delta_1)+\varphi(\lambda\delta_1+(1-\lambda)\,\delta_2)\right]. \tag{15}$$

Now integrating on both sides in the last inequality with respect to λ from 0 to 1 and multiply both sides by $\lambda^{-\alpha}$, we obtain

$$\frac{1}{1-\alpha}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \le \left(e^{\frac{s}{2}}-1\right)\left[\int_0^1 \lambda^{-\alpha}\varphi(\lambda\delta_2+(1-\lambda)\,\delta_1)\,d\lambda + \int_0^1 \lambda^{-\alpha}\varphi(\lambda\delta_1+(1-\lambda)\,\delta_2)\,d\lambda\right],$$
$$\frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \le \frac{1-\alpha}{(\delta_2-\delta_1)^{1-\alpha}}\left[N_3J^{\alpha}_{a^+}\varphi(x) + N_3J^{\alpha}_{b^-}\varphi(x)\right],$$

which gives the proof of first part of inequality of (13). Next, we show the second part of inequality of (13). Let $\lambda \in [0,1]$. Then from the fact that φ is exponentially s–convex function, we obtain

$$\varphi(\lambda\delta_2+(1-\lambda)\,\delta_1) \le \left(e^{s\lambda}-1\right)\varphi(\delta_2)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_1) \tag{16}$$

and

$$\varphi(\lambda\delta_1+(1-\lambda)\,\delta_2) \le \left(e^{s\lambda}-1\right)\varphi(\delta_1)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_2). \tag{17}$$

By adding the above inequalities, we obtain

$$\varphi(\lambda\delta_2+(1-\lambda)\,\delta_1)+\varphi(\lambda\delta_1+(1-\lambda)\,\delta_2) \tag{18}$$
$$\le [\varphi(\delta_1)+\varphi(\delta_2)]\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}.$$

Now integrating on both sides by above equation with respect to λ from 0 to 1 and multiply $\lambda^{-\alpha}$ both sides, then making the change of variable, we obtain

$$\frac{1}{(\delta_2-\delta_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{a^+}\varphi(x)+{}_{N_3}J^{\alpha}_{b^-}\varphi(x)\right]$$
$$\leq [\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\lambda^{-\alpha}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda,$$

which leads to the conclusion that

$$\frac{1-\alpha}{(\delta_2-\delta_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{a^+}\varphi(x)+{}_{N_3}J^{\alpha}_{b^-}\varphi(x)\right]$$
$$\leq (1-\alpha)[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\lambda^{-\alpha}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda.$$

The proof is completed. □

Remark 6. (i) *If we choose $s=1$ and $\alpha=0$ then Theorem 3 becomes to* $\Big[$*Theorem 3.1, [6]*$\Big]$.
(ii) *If we choose $\alpha=0$ then Theorem 4, then we attain the Theorem 3.*

5. Hermite–Hadamard Type Inequality and Its Refinements For Exponentially s–Convex Function Via Fractional Integral Operator

The main key of this section is to obtain the new sort of Hermite–Hadamard inequality in the manner of new introduced concept in the frame of fractional calculus namely Riemann-Liouville integral operator. Also we attain the refinement of this inequality.

Theorem 5. *Let $\varphi:\mathbb{A}=[\delta_1,\delta_2]\to\mathbb{R}$ be a positive function with $0\leq\delta_1\leq\delta_2$ and φ be a integrable function on closed interval set δ_1 and δ_2. If φ is exponentially s–convex function, then the following inequalities for fractional integral namely Riemann-Liouville with $\alpha>0$ and $s\in[\ln 2.4,1]$ holds:*

$$\frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right] \tag{19}$$
$$\leq\alpha[\varphi(\delta_1)+\varphi(\delta_2)]\int_0^1\lambda^{\alpha-1}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda.$$

Proof. Let $\sigma_1,\sigma_2\in\mathbb{A}$. Then it follows from the exponentially s–convex function for φ on $\mathbb{A}$ that

$$\varphi\left(\frac{\sigma_1+\sigma_2}{2}\right)\leq\left(e^{\frac{s}{2}}-1\right)[\varphi(\sigma_1)+\varphi(\sigma_2)] \tag{20}$$

$$\textit{Suppose}\quad \sigma_1=\lambda\delta_2+(1-\lambda)\,\delta_1\qquad \textit{and}\quad \sigma_2=\lambda\delta_1+(1-\lambda)\,\delta_2.$$

Then (20) leads to

$$\varphi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \left(e^{\frac{s}{2}}-1\right)\left[\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right)\right]. \tag{21}$$

Now integrating on both sides in the last inequality with respect to λ from 0 to 1 and multiply both sides by $\lambda^{\alpha-1}$, we obtain

$$\begin{aligned}
\frac{1}{\alpha}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \left(e^{\frac{s}{2}}-1\right)\left[\int_0^1 \lambda^{\alpha-1}\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right)d\lambda\right.\\
&\quad\left.+\int_0^1 \lambda^{\alpha-1}\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right)\,d\lambda\right],\\
\frac{1}{\alpha\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{\Gamma(\alpha)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right],\\
\frac{1}{\left(e^{\frac{s}{2}}-1\right)}\varphi\left(\frac{\delta_1+\delta_2}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right],
\end{aligned}$$

which gives the proof of first part of inequality of (19).
Next, we show the second part of inequality of (19). Let $\lambda \in [0,1]$. Then from the fact that φ is exponentially s–convex function, we obtain

$$\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_2)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_1) \tag{22}$$

and

$$\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right) \leq \left(e^{s\lambda}-1\right)\varphi(\delta_1)+\left(e^{s(1-\lambda)}-1\right)\varphi(\delta_2). \tag{23}$$

By adding the above inequalities, we obtain

$$\begin{aligned}
&\varphi\left(\lambda\delta_2+(1-\lambda)\delta_1\right)+\varphi\left(\lambda\delta_1+(1-\lambda)\delta_2\right) \qquad (24)\\
&\leq \left[\varphi(\delta_1)+\varphi(\delta_2)\right]\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}.
\end{aligned}$$

Now integrating on both sides by above equation with respect to λ from 0 to 1 and multiply $\lambda^{\alpha-1}$ both sides, then making the change of variable, we obtain

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right]\\
&\leq \left[\varphi(\delta_1)+\varphi(\delta_2)\right]\int_0^1 \lambda^{\alpha-1}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda,
\end{aligned}$$

which leads to the conclusion that

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{(\delta_2-\delta_1)}\left[J^{\alpha}_{\delta_1^+}\varphi(\delta_2)+J^{\alpha}_{\delta_2^-}\varphi(\delta_1)\right]\\
&\leq \alpha\left[\varphi(\delta_1)+\varphi(\delta_2)\right]\int_0^1 \lambda^{\alpha-1}\left\{\left(e^{s\lambda}-1\right)+\left(e^{s(1-\lambda)}-1\right)\right\}d\lambda.
\end{aligned}$$

The proof is completed. □

Remark 7. (i) *If we choose* $s=1$ *and* $\alpha=1$ *then Theorem 3 becomes to* $\Big[$*Theorem 3.1, [6]*$\Big]$.
(ii) *If we choose* $\alpha=1$ *then Theorem 5, then we attain the Theorem 3.*

Next we find the refinement of Hermite–Hadamard type inequality using exponentially s–convex function via fractional integral operator. In order to obtain the following result, we need the following lemma.

Lemma 2. *[34] Let* $\varphi:\mathbb{A}\longrightarrow\mathbb{R}$ *be a differentiable mapping on* $\mathbb{A}^\circ$, *where* $\delta_1,\delta_2\in\mathbb{A}^\circ$ *with* $0\le\delta_1\le\delta_2$. *If* $\varphi'\in\mathcal{L}[\delta_1,\delta_2]$, *then the following equality for fractional integral holds*

$$\frac{\varphi(\delta_1)+\varphi(\delta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2)+I^\alpha_{\delta_2^-}\varphi(\delta_1)\right]$$
$$=\frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1\left[(1-\lambda)^\alpha-\lambda^\alpha\right]\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)d\lambda\right\}.$$

Theorem 6. *Let* $\varphi:\mathbb{A}=[\delta_1,\delta_2]\subset[0,\infty)\longrightarrow\mathbb{R}$ *be a differentiable mapping on* (δ_1,δ_2) *with* $\delta_1<\delta_2$ *such that* $\varphi'\in L[\delta_1,\delta_2]$. *If* $|\varphi'|^q$ *is an exponentially s-convex function on* $[\delta_1,\delta_2]$*for some fixed* $s\in[ln2.4,1]$ *and* $q\ge1$. *Then the following fractional inequality holds true.*

$$\left|\tfrac{\varphi(\delta_1)+\varphi(\delta_2)}{2}-\tfrac{\Gamma(\alpha+1)}{2(\delta_2-\delta1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2)+I^\alpha_{\delta_2^-}\varphi(\delta1)\right]\right|$$
$$\le\tfrac{(\delta_2-\delta_1)}{2}\left[\tfrac{2}{\alpha+1}\left(1-\tfrac{1}{2^\alpha}\right)\right]^{\frac{q-1}{q}}$$
$$\times\Bigg\{|\varphi'(\delta_1)|^q\tfrac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}-|\varphi'(\delta_1)|^q\tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(1+\alpha)e^s]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_2)|^q\tfrac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}-|\varphi'(\delta_2)|^q\tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(3+\alpha)e^s]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_1)|^q\tfrac{1}{4}\left(\tfrac{2(-2+2^{-\alpha})}{1+\alpha}+\tfrac{(4-2^{-\alpha})e^\alpha}{2+\alpha}\right)-|\varphi'(\delta_1)|^q\tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(3+\alpha)e^s]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_2)|^q\tfrac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}-|\varphi'(\delta_2)|^q\tfrac{2^{(-2-\alpha)}[-2(2+\alpha)+(1+\alpha)e^s]}{(1+\alpha)(2+\alpha)}\Bigg\}^{\frac{1}{q}}.$$

Proof. Suppose that $q=1$. From lemma (2) and using properties of modulus, we have

$$\left|\frac{\varphi(\delta_1)+\varphi(\delta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2)+I^\alpha_{\delta_2^-}\varphi(\delta_1)\right]\right|$$
$$=\frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1|(1-\lambda)^\alpha-\lambda^\alpha|\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|d\lambda\right\}$$
$$\le\frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1|(1-\lambda)^\alpha-\lambda^\alpha|\left[(e^{s\lambda}-1)|\varphi'(\delta_1)|+(e^{s(1-\lambda)}-1)|\varphi'(\delta_2)|\right]d\lambda\right\}$$
$$\le\frac{(\delta_2-\delta_1)}{2}\left\{\int_0^{1/2}\left[(1-\lambda)^\alpha-\lambda^\alpha\right]\left[(e^{s\lambda}-1)|\varphi'(\delta_1)|+(e^{s(1-\lambda)}-1)|\varphi'(\delta_2)|\right]d\lambda\right.$$

$$+\int_{1/2}^{1}\left[\lambda^{\alpha}-(1-\lambda)^{\alpha}\right]\left[(e^{s\lambda}-1)|\varphi'(\delta_1)|+(e^{s(1-\lambda)}-1)|\varphi'(\delta_2)|\right]d\lambda\Bigg\}$$

$$=\frac{(\delta_2-\delta_1)}{2}\Bigg\{|\varphi'(\delta_1)|\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s\lambda}-1)d\lambda-|\varphi'(\delta_1)|\int_0^{1/2}\lambda^{\alpha}(e^{s\lambda}-1)d\lambda$$

$$+|\varphi'(\delta_2)|\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s(1-\lambda)}-1)d\lambda-|\varphi'(\delta_2)|\int_0^{1/2}\lambda^{\alpha}(e^{s(1-\lambda)}-1)d\lambda$$

$$+|\varphi'(\delta_1)|\int_{1/2}^{1}\lambda^{\alpha}(e^{s\lambda}-1)d\lambda-|\varphi'(\delta_1)|\int_{1/2}^{1}(1-\lambda)^{\alpha}(e^{s\lambda}-1)d\lambda$$

$$+|\varphi'(\delta_2)|\int_{1/2}^{1}\lambda^{\alpha}(e^{s(1-\lambda)}-1)d\lambda-|\varphi'(\delta_2)|\int_{1/2}^{1}(1-\lambda)^{\alpha}(e^{s(1-\lambda)}-1)d\lambda\Bigg\}$$

$$=\frac{(\delta_2-\delta_1)}{2}\Bigg\{|\varphi'(\delta_1)|\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$-|\varphi'(\delta_1)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$+|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$-|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$+|\varphi'(\delta_1)|\frac{1}{4}\left(\frac{2(-2+2^{-\alpha})}{1+\alpha}+\frac{(4-2^{-\alpha})e^{\alpha}}{2+\alpha}\right)$$

$$-|\varphi'(\delta_1)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$+|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$-|\varphi'(\delta_2)|\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)},$$

where,

$$\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s(1-\lambda)}-1)\,d\lambda=\frac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$\int_0^{1/2}\lambda^{\alpha}(e^{s(1-\lambda)}-1)\,d\lambda=\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$

$$\int_0^{1/2}(1-\lambda)^{\alpha}(e^{s\lambda}-1)\,d\lambda=\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$\int_0^{1/2} \lambda^\alpha (e^{s\lambda}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$\int_{1/2}^{1} (1-\lambda)^\alpha (e^{s\lambda}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$\int_{1/2}^{1} \lambda^\alpha (e^{s\lambda}-1)\,d\lambda = \frac{1}{4}\left(\frac{2(-2+2^{-\alpha})}{1+\alpha}+\frac{(4-2^{-\alpha})e^\alpha}{2+\alpha}\right)$$
$$\int_{1/2}^{1} (1-\lambda)^\alpha (e^{s(1-\lambda)}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$\int_{1/2}^{1} \lambda^\alpha (e^{s(1-\lambda)}-1)\,d\lambda = \frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}.$$

This completes the proof of this case. Suppose now that $q>1$, since $|\varphi|^q$ is an exponential s-convex function, we have

$$|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|^q \le (e^{s\lambda}-1)|\varphi(\delta_1)|^q + (e^{s(1-\lambda)}-1)|\varphi(\delta_2)|^q$$

Now using Hölders Inequality for $\frac{1}{p}+\frac{1}{q}=1$

$$\left|\frac{\varphi(\delta_1)+\varphi(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^\alpha}\left[I^\alpha_{\delta_1^+}\varphi(\delta_2)+I^\alpha_{\delta_2^-}\varphi(\delta_1)\right]\right|$$
$$\le \frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|d\lambda\right\}$$
$$= \frac{(\delta_2-\delta_1)}{2}\left\{\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|^{1-1/q}\,|(1-\lambda)^\alpha-\lambda^\alpha|^{1/q}\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|d\lambda\right\}$$
$$\le \frac{(\delta_2-\delta_1)}{2}\left\{\left(\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\,d\lambda\right)^{\frac{q-1}{q}}\left(\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\,|\varphi'(\lambda\delta_1+(1-\lambda)\delta_2)|^q d\lambda\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(\delta_2-\delta_1)}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]^{\frac{q-1}{q}}\left(\int_0^1 |(1-\lambda)^\alpha-\lambda^\alpha|\left[(e^{s\lambda}-1)|\varphi(\delta_1)|^q\right.\right.$$
$$\left.\left.+(e^{s(1-\lambda)}-1)|\varphi(\delta_2)|^q\right]d\lambda\right)^{\frac{1}{q}} = \frac{(\delta_2-\delta_1)}{2}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]^{\frac{q-1}{q}}$$
$$\times\left\{|\varphi'(\delta_1)|^q\frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)}\right.$$
$$-|\varphi'(\delta_1)|^q\frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}$$
$$+|\varphi'(\delta_2)|^q\frac{2^{(-2-\alpha)}\left[4+2\alpha-e^s-\alpha e^s+2^{(2+\alpha)}(-2-\alpha+(1+\alpha)e^s)\right]}{(1+\alpha)(2+\alpha)}$$

$$
\begin{aligned}
&-|\varphi'(\delta_2)|^q \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)} \\
&+|\varphi'(\delta_1)|^q \frac{1}{4}\left(\frac{2(-2+2^{-\alpha})}{1+\alpha}+\frac{(4-2^{-\alpha})e^{\alpha}}{2+\alpha}\right) \\
&-|\varphi'(\delta_1)|^q \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(3+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)} \\
&+|\varphi'(\delta_2)|^q \frac{2^{(-2-\alpha)}\left[4+2\alpha-3e^s-\alpha e^s-2^{(2+\alpha)}(2+\alpha-e^s)\right]}{(1+\alpha)(2+\alpha)} \\
&-|\varphi'(\delta_2)|^q \frac{2^{(-2-\alpha)}\left[-2(2+\alpha)+(1+\alpha)e^s\right]}{(1+\alpha)(2+\alpha)}\Bigg\}^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof of the Theorem. □

6. Ostrowski Type Inequalities for Exponentially s–Convexity via Fractional Integral

In this section, we established new Ostrowski type inequalities for exponentially s–convexity via Riemann-Liouville fractional integral. A useful and interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals. In order to prove our results, we need the following identity.(see in [11, 35]).

Lemma 3. *Suppose a mapping $\varphi : \mathbb{A} \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{A}^o$, where $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$. If $\varphi' \in L[\delta_1, \delta_2]$, for all $z \in [\delta_1, \delta_2]$ and $\alpha > 0$, then the following equality holds:*

$$
\begin{aligned}
&\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\left\{J^{\alpha}_{z-}\ \varphi(\delta_1)\ +\ J^{\alpha}_{z+}\ \varphi(\delta_2)\right\} \\
&= \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ \varphi'(\lambda z+(1-\lambda)\delta_1)\ d\lambda \\
&- \frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ \varphi'(\lambda z+(1-\lambda)\delta_2)\ d\lambda, \qquad (25)
\end{aligned}
$$

where Γ is the Euler gamma function.

Theorem 7. *Suppose a mapping $\varphi : \mathbb{A} \subseteq R \to R$ is differentiable on $\mathbb{A}^o$, where $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$. If $|\varphi'|$ is exponentially s–convex on $[\delta_1, \delta_2]$ for some $s \in [\ln 2.4, 1]$, $\varphi' \in L[\delta_1, \delta_2]$ and $|\varphi'(z)| \leq \mathbb{M}$, for all $z \in [\delta_1, \delta_2]$, $\alpha > 0$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\left\{J^{\alpha}_{z-}\ \varphi(\delta_1)\ +\ J^{\alpha}_{z+}\ \varphi(\delta_2)\right\}\right| \\
&\leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}
\end{aligned}
$$

$$\times\Bigg[(z-\delta_1)^{\alpha+1}\Bigg\{\Bigg(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\Bigg)$$
$$-\Bigg(\frac{(\gamma(\alpha+1,s)-\Gamma(\alpha+1))e^{s}}{s^{\alpha+1}}+\frac{1}{\alpha+1}\Bigg)\Bigg\}$$
$$+(\delta_2-z)^{\alpha+1}\Bigg\{\Bigg(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\Bigg)$$
$$-\Bigg(\frac{(\gamma(\alpha+1,s)-\Gamma(\alpha+1))e^{s}}{s^{\alpha+1}}+\frac{1}{\alpha+1}\Bigg)\Bigg\}\Bigg]. \tag{26}$$

Proof. From Lemma 3 and since $|\varphi'|$ is exponentially s–convexity and $|\varphi'(z)|\leq\mathbb{M}$, we have

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J_{z^-}^{\alpha}\ \varphi(\delta_1)\ +\ J_{z^+}^{\alpha}\ \varphi(\delta_2)\}\right|$$
$$\leq\ \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\left|\varphi'(\lambda z+(1-\lambda)\delta_1)\right|d\lambda\ \ \frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\left|\varphi'(\lambda z+(1-\lambda)\delta_2)\right|d\lambda.$$

$$\leq\ \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\Big\{\left(e^{s\lambda}-1\right)|\varphi'(z)|+\Big(e^{s(1-\lambda)}-1\Big)|\varphi'(\delta_1)|\Big\}d\lambda$$
$$+\ \frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\Big\{\left(e^{s\lambda}-1\right)|\varphi'(z)|+\Big(e^{s(1-\lambda)}-1\Big)|\varphi'(\delta_1)|\Big\}d\lambda$$
$$\leq\ \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\Big\{|\varphi'(z)|\int_0^1\lambda^{\alpha}\left(e^{s\lambda}-1\right)d\lambda+|\varphi'(\delta_1)|\int_0^1\lambda^{\alpha}\Big(e^{s(1-\lambda)}-1\Big)d\lambda\Big\}$$
$$+\ \frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\Big\{|\varphi'(z)|\int_0^1\lambda^{\alpha}\left(e^{s\lambda}-1\right)d\lambda+|\varphi'(\delta_2)|\int_0^1\lambda^{\alpha}\Big(e^{s(1-\lambda)}-1\Big)d\lambda\Big\}$$
$$\leq\ \frac{\mathbb{M}}{(\delta_2-\delta_1)}\times(z-\delta_1)^{\alpha+1}\Bigg\{\Bigg(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\Bigg)$$
$$-\Bigg(\frac{(\gamma(\alpha+1,s)-\Gamma(\alpha+1))e^{s}}{s^{\alpha+1}}+\frac{1}{\alpha+1}\Bigg)\Bigg\}$$
$$+\frac{\mathbb{M}}{(\delta_2-\delta_1)}\times(\delta_2-z)^{\alpha+1}\Bigg\{\Bigg(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\Bigg)$$
$$-\Bigg(\frac{(\gamma(\alpha+1,s)-\Gamma(\alpha+1))e^{s}}{s^{\alpha+1}}+\frac{1}{\alpha+1}\Bigg)\Bigg\}.$$

After simplification, we get(26). The proof is completed. □

Corollary 1. *Under the similar consideration in Theorem 7, by choosing* $s=1$, *we obtain*

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J_{z^-}^{\alpha}\ \varphi(\delta_1)\ +\ J_{z^+}^{\alpha}\ \varphi(\delta_2)\}\right|$$

$$\leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}$$
$$\times\Bigg[(z-\delta_1)^{\alpha+1}\Bigg\{\Bigg(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\Bigg)$$
$$-\Bigg((\gamma(\alpha+1,1)-\Gamma(\alpha+1))e+\frac{1}{\alpha+1}\Bigg)\Bigg\}$$
$$+(\delta_2-z)^{\alpha+1}\Bigg\{\Bigg(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\Bigg)$$
$$-\Bigg((\gamma(\alpha+1,1)-\Gamma(\alpha+1))e+\frac{1}{\alpha+1}\Bigg)\Bigg\}\Bigg].$$

Theorem 8. *Suppose a mapping $\varphi:\mathbb{A}\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable on $\mathbb{A}^o$, where $\delta_1,\delta_2\in\mathbb{A}$ with $\delta_1<\delta_2$. If $|\varphi'|^q$ is exponentially s–convex on $[\delta_1,\delta_2]$ for some $s\in[\ln 2.4,1]$, $q>1$, $q^{-1}=1-p^{-1}$, $\varphi'\in L[\delta_1,\delta_2]$ and $|\varphi'(z)|\leq\mathbb{M}$, for all $z\in[\delta_1,\delta_2]$, with $\alpha>0$, then the following inequality holds:*

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J^{\alpha}_{z^-}\ \varphi(\delta_1)\ +\ J^{\alpha}_{z^+}\ \varphi(\delta_2)\}\right|$$
$$\leq\frac{2^{\frac{1}{q}}\mathbb{M}}{(\delta_2-\delta_1)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}$$
$$\times\left[(z-\delta_1)^{\alpha+1}\ \left\{\left(\frac{e^s-s-1}{s}\right)\right\}^{\frac{1}{q}}\ +\ (\delta_2-z)^{\alpha+1}\ \left\{\left(\frac{e^s-s-1}{s}\right)\right\}^{\frac{1}{q}}\right].\quad(27)$$

Proof. From Lemma 3 and famous Hölder's inequality, we have

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J^{\alpha}_{z^-}\ \varphi(\delta_1)\ +\ J^{\alpha}_{z^+}\ \varphi(\delta_2)\}\right|$$
$$\leq\frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\ |\varphi'(\lambda z+(1-\lambda)\delta_1)|\ d\lambda$$
$$+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1\lambda^{\alpha}\ |\varphi'(\lambda z+(1-\lambda)\delta_2)|\ d\lambda$$
$$\leq\frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1\lambda^{\alpha p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1|\varphi'(\lambda z+(1-\lambda)\delta_1)|^q d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1\lambda^{\alpha p}d\lambda\right)^{\frac{1}{p}}\left(\int_0^1|\varphi'(\lambda z+(1-\lambda)\delta_2)|^q d\lambda\right)^{\frac{1}{q}}.\quad(28)$$

Since $|\varphi'|^q$ is exponentially s–convexity and $|\varphi'(z)|\leq\mathbb{M}$, we obtain

$$\int_0^1|\varphi'(\lambda z+(1-\lambda)\delta_1)|^q d\lambda=\int_0^1\left\{\left(e^{s\lambda}-1\right)|\varphi'(z)|^q+\left(e^{s(1-\lambda)}-1\right)|\varphi'(\delta_1)|^q\right\}d\lambda$$

$$\leq \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right) + \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right)$$
$$\leq 2\mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right) \tag{29}$$

and

$$\int_0^1 |\varphi'(\lambda z + (1-\lambda)\delta_2)|^q \, d\lambda = \int_0^1 \left\{ \left(e^{s\lambda} - 1\right) |\varphi'(z)|^q + \left(e^{s(1-\lambda)} - 1\right) |\varphi'(\delta_2)|^q \right\} d\lambda$$
$$\leq \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right) + \mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right)$$
$$\leq 2\mathbb{M}^q \left(\frac{e^s - s - 1}{s}\right). \tag{30}$$

By connecting (29) and (30) with (28), we get (27). The proof is completed. □

Corollary 2. *Under the similar consideration in Theorem 8, by choosing $s = 1$, we obtain*

$$\left| \left(\frac{(z-\delta_1)^\alpha + (\delta_2 - z)^\alpha}{\delta_2 - \delta_1}\right) \varphi(z) - \frac{\Gamma(\alpha+1)}{\delta_2 - \delta_1} \{ J^\alpha_{z^-} \varphi(\delta_1) + J^\alpha_{z^+} \varphi(\delta_2) \} \right|$$
$$\leq \frac{2^{\frac{1}{q}} \mathbb{M}}{(\delta_2 - \delta_1)} \left(\frac{1}{\alpha p + 1}\right)^{\frac{1}{p}} \left[(z-\delta_1)^{\alpha+1} \left\{ (e-2) \right\}^{\frac{1}{q}} + (\delta_2 - z)^{\alpha+1} \left\{ (e-2) \right\}^{\frac{1}{q}} \right].$$

Theorem 9. *Suppose a mapping $\varphi : \mathbb{A} \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{A}^o$, where $\delta_1, \delta_2 \in \mathbb{A}$ with $\delta_1 < \delta_2$. If $|\varphi'|^q$ is exponentially s–convex on $[\delta_1, \delta_2]$ for some $s \in [\ln 2.4, 1]$, $q \geq 1$, $q^{-1} = 1 - p^{-1}$, $\varphi' \in L[\delta_1, \delta_2]$ and $|\varphi'(z)| \leq \mathbb{M}$, for all $z \in [\delta_1, \delta_2]$, with $\alpha > 0$, then the following inequality holds:*

$$\left| \left(\frac{(z-\delta_1)^\alpha + (\delta_2 - z)^\alpha}{\delta_2 - \delta_1}\right) \varphi(z) - \frac{\Gamma(\alpha+1)}{\delta_2 - \delta_1} \{ J^\alpha_{z^-} \varphi(\delta_1) + J^\alpha_{z^+} \varphi(\delta_2) \} \right|$$
$$\leq \frac{\mathbb{M}}{(\delta_2 - \delta_1)} \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times \left[(z-\delta_1)^{\alpha+1} \left\{ \left(\frac{\gamma(\alpha+1, -s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1} \right) \right.\right.$$
$$\left. - \left((\gamma(\alpha+1, s) - \Gamma(\alpha+1)) s^{-\alpha-1} e^s - \frac{1}{\alpha+1} \right) \right\}^{\frac{1}{q}}$$
$$+ (\delta_2 - z)^{\alpha+1} \left\{ \left(\frac{\gamma(\alpha+1, -s) - \Gamma(\alpha+1)}{(-s)^\alpha s} - \frac{1}{\alpha+1} \right) \right.$$
$$\left.\left. - \left((\gamma(\alpha+1, s) - \Gamma(\alpha+1)) s^{-\alpha-1} e^s - \frac{1}{\alpha+1} \right) \right\}^{\frac{1}{q}} \right]. \tag{31}$$

Proof. From Lemma 3 and power mean inequality, we have

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J^{\alpha}_{z^-}\ \varphi(\delta_1)\ +\ J^{\alpha}_{z^+}\ \varphi(\delta_2)\}\right|$$
$$\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ |\varphi'(\lambda z+(1-\lambda)\delta_1)|\ d\lambda$$
$$+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\int_0^1 \lambda^{\alpha}\ |\varphi'(\lambda z+(1-\lambda)\delta_2)|\ d\lambda$$
$$\leq \frac{(z-\delta_1)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1 \lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda^{\alpha}|\varphi'(\lambda z+(1-\lambda)\delta_1)|^q d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\delta_2-z)^{\alpha+1}}{\delta_2-\delta_1}\left(\int_0^1 \lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1 \lambda^{\alpha}|\varphi'(\lambda z+(1-\lambda)\delta_2)|^q d\lambda\right)^{\frac{1}{q}} \tag{32}$$

Since $|\varphi'|^q$ is exponentially s–convexity and $|\varphi'(z)|\leq \mathbb{M}$, we obtain

$$\int_0^1 \lambda^{\alpha}\ |\varphi'(\lambda z+(1-\lambda)\delta_1)|^q\ d\lambda$$
$$=\ \int_0^1 \lambda^{\alpha}\ \left\{\left(e^{s\lambda}-1\right)|\varphi'(z)|^q+\left(e^{s(1-\lambda)}-1\right)|\varphi'(\delta_1)|^q\right\}d\lambda$$
$$\leq\ \mathbb{M}^q\left\{\left(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\right)\right.$$
$$\left.-\left((\gamma(\alpha+1,s)-\Gamma(\alpha+1))\,s^{-\alpha-1}e^s-\frac{1}{\alpha+1}\right)\right\} \tag{33}$$

and

$$\int_0^1 \lambda^{\alpha}\ |\varphi'(\lambda z+(1-\lambda)\delta_2)|^q\ d\lambda$$
$$=\ \int_0^1 \lambda^{\alpha}\ \left(e^{s\lambda}-1\right)|\varphi'(z)|^q+\left(e^{s(1-\lambda)}-1\right)|\varphi'(\delta_2)|^q\left.\right\}d\lambda$$
$$\leq\ \mathbb{M}^q\left\{\left(\frac{\gamma(\alpha+1,-s)-\Gamma(\alpha+1)}{(-s)^{\alpha}s}-\frac{1}{\alpha+1}\right)\right.$$
$$\left.-\left((\gamma(\alpha+1,s)-\Gamma(\alpha+1))\,s^{-\alpha-1}e^s-\frac{1}{\alpha+1}\right)\right\}. \tag{34}$$

By connecting (33) and (34) with (32), we get (31). The proof is completed. □

Corollary 3. *Under the similar consideration in Theorem 9, by choosing $s=1$, we obtain*

$$\left|\left(\frac{(z-\delta_1)^{\alpha}+(\delta_2-z)^{\alpha}}{\delta_2-\delta_1}\right)\varphi(z)-\frac{\Gamma(\alpha+1)}{\delta_2-\delta_1}\{J^{\alpha}_{z^-}\ \varphi(\delta_1)\ +\ J^{\alpha}_{z^+}\ \varphi(\delta_2)\}\right|$$

$$\leq \frac{\mathbb{M}}{(\delta_2-\delta_1)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[(z-\delta_1)^{\alpha+1}\ \left\{\left(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\right)\right.\right.$$
$$\left.-\left((\gamma(\alpha+1,1)-\Gamma(\alpha+1))\,e-\frac{1}{\alpha+1}\right)\right\}^{\frac{1}{q}}$$
$$+\ (\delta_2-z)^{\alpha+1}\ \left\{\left(\frac{\gamma(\alpha+1,-1)-\Gamma(\alpha+1)}{(-1)^{\alpha}}-\frac{1}{\alpha+1}\right)\right.$$
$$\left.\left.-\left((\gamma(\alpha+1,1)-\Gamma(\alpha+1))\,e-\frac{1}{\alpha+1}\right)\right\}^{\frac{1}{q}}\right].$$

7. Conclusion

In this article, the authors showed the new class of exponentially s–convex functions, derive several new versions of the Hermite-Hadamard inequality using the class of exponentially s–convex functions in the frame of classical and fractional calculus. We have obtained some refinement of Hermite–Hadamard inequality. Finally, we have attained some Ostrowski type inequalities for exponentially s–convexity via fractional integral. We hope the consequences and techniques of this article will energize and inspire the researchers to explore a more interesting sequel in this area.

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References

[1] Xi, B. Y., Qi, F., Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means, *J. Funct. Spaces. Appl.,* Article ID 980438 (2012), 1–14. https://doi.org/10.1155/2012/980438

[2] Özcan, S., İşcan, İ., Some new Hermite Hadamard type integral inequalities for the s–convex functions and theirs applications, *J. Inequal. Appl.*, 201 (2019), 1–14.

[3] Butt, S. I., Budak, H., Tariq, M., Nadeem, M., Integral inequalities for n–polynomial s–type preinvex functions with applications, *Math. Methods Appl. Sci.*, (2021). https://doi.org/10.1002/mma.7465

[4] Hudzik, H., Maligranda, L., Some remarks on s-convex functions, *Aequationes Math*, 48 (1994), 100-111.

[5] Hadamard, J., Étude sur les propriétés des fonctions entiéres en particulier d'une fonction considérée par Riemann, *J. Math. Pures. Appl.*, 58 (1893), 171–215.
[6] Kadakal, M., İşcan, İ., Exponential type convexity and some related inequalities, *J. Inequal. Appl.*, 82 (2020), 1–9.
[7] Butt, S. I., Tariq, M., Aslam, A., Ahmad, H., Nofel, T. A., Hermite–Hadamard type inequalities via generalized harmonic exponential convexity, *J. Funct. Spaces*, (2021), 1–12. https://doi.org/10.1155/2021/5533491
[8] Butt, S. I., Kashuri, A., Tariq, M., Nasir, J., Aslam, A., Geo, W., Hermite–Hadamard–type inequalities via n–polynomial exponential–type convexity and their applications, *Adv. Differ. Equ.*, 508 (2020).
[9] Butt, S. I., Kashuri, A., Tariq, M., Nasir, J., Aslam, A., Geo, W., n–polynomial exponential–type p–convex function with some related inequalities and their applications, *Heliyon*, (2020). DOI: 10.1016/j.heliyon.2020.e05420
[10] Sarikaya, M. Z., Set, E., Yaldiz, H., Basak, N., Hermite Hadamard inequalities for fractional integrals and related fractional inequalities, *Mathematical and Computer Modelling*, 57(9-10) (2003), 2403-2407.
[11] Set, E., New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, *Computers and Mathematics with Applications*, 63(7) (2012), 1147-1154.
[12] Ostrowski, A., Über die Absolutabweichung einer differentiebaren funktion von ihren integralmittelwert, *Comment. Math. Helv.*, 10 (1938), 226–227.
[13] Mohsen, B. B., Awan, M. U., Javed, M. Z., Noor, M. A., Noor, K. I., Some new Ostrowski-type inequalities involving σ-fractional integrals, *J. Math.*, (2021). https://doi.org/10.1155/2021/8850923
[14] Akhtar, N., Awan, M. U., Javed, M. Z., Rassias, M. T., Mihai, M. V., Noor, M. A., Noor, K. I., Ostrowski type inequalities involving harmonically convex functions and applications, *Symmetry*, 13(2) (2021).
[15] Alomari, M., Darus, M., Dragomir, S. S., Cerone, P., Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Applied Mathematics Letters*, 23(9) (2010), 1071-1076.
[16] Niculescu, C. P., Persson, L. E., Convex Functions and Their Applications, Springer, New York, 2006.
[17] Set, E., Özdemir, M. E., Sarikaya, M. Z., New inequalities of Ostrowski's type for s-convex functions in the second sense with applications, arXiv preprint arXiv:1005.0702, (2010).
[18] Dragomir, S. S., Fitzpatrik, S., The Hadamard inequality for $s-$convex functions in the second sense, *Demonstratio Math.*, 32(4) (1999), 687-696.
[19] Khalil, R., Horani, M. A., Yousaf, A., Sababheh, M., A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264 (2014), 65–70.
[20] Nápoles valdés, J. E., Guzman, P. M., Lugo, L. M., Some new results on non–confromable fractional calculus, *Adv. Dyn. Syst. Appl.*, 13 (2018), 167–175.
[21] Gómez–Aguilar, J. F., Novel analytic solutions of the fractional drude model, *Optik*, 168 (2018), 728–740.
[22] Gómez–Aguilar, J. F., Analytic and numerical solutions of a nonlinear alcoholism model via variable–order fractional differential equations, *Phys. A Stat. Mech. Appl.*, 494 (2018), 52–75.
[23] Ghanim, F., Al-Janaby, H. F., Bazighifan, O., Some new extensions on fractional differential and integral properties for Mittag-Leffler confluent hypergeometric function, *Fractal Fract.*, 5 (2021), 143.
[24] El-Deeb, A. A.-M., Bazighifan, O., Awrejcewicz, J. A., Variety of dynamic Steffensen-type inequalities on a general time scale, *Symmetry,* 13 (2021), 1738.
[25] Elayaraja, R., Ganesan, V., Bazighifan, O., Cesarano, C., Oscillation and asymptotic properties of differential equations of third-order, *Axioms*, 10 (2021), 192.

[26] Tariq, M., Sahoo, S. K., Nasir, J., Awan, S. K., Some Ostrowski type integral inequalities using Hypergeometric Functions, *J. Frac. Calc. Nonlinear Sys.*, 2 (2021), 24–41.

[27] Tariq, M., Nasir, J., Sahoo, S. K., Mallah, A. A., A note on some Ostrowski type inequalities via generalized exponentially convex function, *J. Math. Anal. Model.*, 2 (2021), 1–15. https://doi.org/10.48185/jmam.v2i2.216

[28] Tariq, M., New Hermite–Hadamard type inequalities via p–harmonic exponential type convexity and applications, *U. J. Math. Appl.*, 4 (2021), 59–69.

[29] Tariq, M., New Hermite–Hadamard type and some related inequalities via s-type p-convex function, *IJSER.*, 11(12) (2020), 498-508.

[30] Valdes, J. E. N., Rodriguez, J. M., Sigarreta, J. M., New Hermite–Hadamard type inequalities involving non-conformable integral operators, *Symmetry*, 11 (2019), 1–11.

[31] Olver, F. W. J., Lozier, D. W., Boisvert, R. F., Clark, F. C., The NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010.

[32] Miller, K., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, Inc., New York, 1993.

[33] Podlubni, I., Fractional Differential Equations, Academic Press, San Diego, 1999.

[34] Set, E., Sarikaya, M. Z., Özdemir, M. E., Yildirim, H., The Hermite–Hadamard inequality for some convex functions via fractional integral and related results, *JAMSI.*, (2011).

[35] Noor, A. M., Noor, K. I., Awan, M. U., Fractional Ostrowski inequalities for s-Godunova-Levin functions, *I. J. Anal. Appl.*, 5(2) (2014), 167–173.

http://communications.science.ankara.edu.tr

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ON THE SPECTRUM OF THE UPPER TRIANGULAR DOUBLE BAND MATRIX $U(a_0, a_1, a_2; b_0, b_1, b_2)$ OVER THE SEQUENCE SPACE c

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Abstract. The upper triangular double band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ is defined on a Banach sequence space by

$$U(a_0, a_1, a_2; b_0, b_1, b_2)(x_n) = (a_n x_n + b_n x_{n+1})_{n=0}^{\infty}$$

where $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod3)$. The class of the operator

$$U(a_0, a_1, a_2; b_0, b_1, b_2)$$

includes, in particular, the operator $U(r, s)$ when $a_k = r$ and $b_k = s$ for all $k \in \mathbb{N}$, with $r, s \in \mathbb{R}$ and $s \neq 0$. Also, it includes the upper difference operator; $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}$. In this paper, we completely determine the spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum, and the compression spectrum of the operator $U(a_0, a_1, a_2; b_0, b_1, b_2)$ over the sequence space c.

1. Introduction

Spectral theory is an important branch of mathematics. It also has many applications in physics. It is used, for example, to determine atomic energy levels in quantum mechanics. The resolvent set, which is the complement of the spectrum set of band matrices, can be used in such problems.

In this paper, we will calculate spectral decomposition of $U(a_0, a_1, a_2; b_0, b_1, b_2)$ matrix. $U(a_0, a_1, a_2; b_0, b_1, b_2)$ matrix is studied in c_0 sequence space by Durna and Kılıç [9] therefore some result is omit becuse it is similar with [9].

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$A : X \to Y$ be a bounded linear operator where X and Y are two Banach spaces. We will show the image set of A with set $R(A) = \{y \in Y : y = Ax,\ x \in X\}$. $B(X)$ is defined as in $B : X \to X$ all bounded, linear operators.

$A : D(A) \to X$ is a linear operator including $D(A) \subset X$, where $D(A)$ show the domain of A and X is a complex normed space. Let $A_\lambda := \lambda I - A$ for $A \in B(X)$ and $\lambda \in \mathbb{C}$ where I show the identity operator. A_λ^{-1} is defined as the resolvent operator of A.

The resolvent set of A consist from the set of complex numbers λ of A such that A_λ^{-1} exists, is continuous and, is defined on a set which is dense in X, signified by $\rho(A, X)$. The complement of $\rho(A, X)$ i.e. $\sigma(A, X) = \mathbb{C} \backslash \rho(A, X)$ is the spectrum of A.

Spectrum $\sigma(A, X)$ is the union of three sets which are disjoint, as follows: If A_λ^{-1} does not exist $\lambda \in \mathbb{C}$ belongs to the point spectrum. If A_λ^{-1} is defined on a dense subspace of X and is unbounded then $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(A, X)$ of A. If A_λ^{-1} exists, but its domain of definition is not dense in X then A_λ^{-1} may be bounded or unbounded. In this case $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(A, X)$.

$$\sigma(A, X) = \sigma_p(A, X) \cup \sigma_c(A, X) \cup \sigma_r(A, X) \tag{1}$$

is obtained by from above definitons and these sets are two by two discrete between them.

The all, bounded, convergent, null and bounded variation sequences are denoted by w, ℓ_∞, c, c_0 and bv, respectively. Moreover the spaces of all $p-$absolutely summable sequences and $p-$bounded variation sequences are denoted by ℓ_p, bv_p, respectively.

We notice that the dual space of c is norm isomorphic to the Banach space

$$\ell_1 = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} |x_k| < \infty \right\}.$$

Many Authors studied the spectrum and fine spectrum of linear operators on some sequence spaces. Some of the operators studied on the spectrum are as follows: The q-Cesáro matrices with $0 < q < 1$ on c_0 was studied by Yıldırım [19] in 2020, the difference operator over the sequence space bv_p by Akhmedov and Başar [1] in 2007 and forward difference operator on the Hahn space by Yeşilkayagil and Kirişci [16] in 2016.

2. Fine Spectrum

The upper triangular double band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ is defined on a Banach sequence space by

$$U(a_0, a_1, a_2; b_0, b_1, b_2)(x_n) = (a_n x_n + b_n x_{n+1})_{n=0}^{\infty}$$

where $a_x = a_y, b_x = b_y$ for $x \equiv y(mod3)$. The class of the operator $U(a_0, a_1, a_2; b_0, b_1, b_2)$ includes, in particular, the operator $U(r, s)$ when $a_k = r$ and $b_k = s$ for all $k \in \mathbb{N}$, with $r, s \in \mathbb{R}$ and $s \neq 0$. Also, it includes the upper difference operator; $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}$. These operators have been studied in [14] and [11], respectively. $U(a_0, a_1, a_2; b_0, b_1, b_2)$ is an infinite matrix of form

$$U(a_0,a_1,a_2;b_0,b_1,b_2)=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & b_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & b_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & b_1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (b_0, b_1, b_2 \neq 0). \tag{2}$$

In this work, we will calculate spectral decomposition of above matrix.

Lemma 1 ([3], p.6)**.** *The matrix $B = (b_{nk})$ gives rise to a bounded linear operator $T \in (c; c)$ from c to itself if and only if*
(i) the rows of B are in ℓ_1 and their ℓ_1 norm are bounded,
(ii) the colums of B are in c,
(iii) the squence of row sums of B is in c.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1. *$U(a_0, a_1, a_2; b_0, b_1, b_2) : c \to c$ is a bounded linear operator and the norm is $\|U(a_0, a_1, a_2; b_0, b_1, b_2)\| = \max\{|a_0| + |b_0|, |a_1| + |b_1|, |a_2| + |b_2|\}$.*

Notation 1. *Throughout this study we will demonstrate as*

$$M = \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\},$$

∂M is the boundary of the set M and $\overset{\circ}{M}$ is interior of the set M.

Theorem 1. *$\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \overset{\circ}{M}$.*

Proof. Proof is similar to proof of [9, Theorem 1]. □

Lemma 2 ([3], p.267)**.** *Let $T : c \to c$ be a bounded linear operator. If $T^* : \ell_1 \to \ell_1$, $T^*g = g \circ T$, $g \in c^* \cong \ell_1$, then T and T^* have matrix representations $B = (b_{nk})$ and B^* respectively. In here*

$$B^* = \begin{pmatrix} \bar{\chi} & v_0 - \bar{\chi} & v_1 - \bar{\chi} & v_2 - \bar{\chi} & \cdots \\ u_0 & b_{00} - u_0 & b_{10} - u_0 & b_{20} - u_0 & \cdots \\ u_1 & b_{01} - u_1 & b_{11} - u_1 & b_{21} - u_1 & \cdots \\ u_2 & b_{02} - u_2 & b_{12} - u_2 & b_{22} - u_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$u_n = \lim_{m\to\infty} b_{m,n} \quad , \quad v_n = \sum_{m=0}^{\infty} b_{n,m}$$

and

$$\bar{\chi} = \lim_{n\to\infty} v_n.$$

In this section, we will take $a_n + b_n = a_{n+1} + b_{n+1} = s$, herein $a_x = a_y$, $b_x = b_y$, $x \equiv y \,(\mathrm{mod}\, 3)$.

From Lemma 2 the adjoint of $U(a_0, a_1, a_2; b_0, b_1, b_2) : c \to c$ is the matrix

$$U(a_0, a_1, a_2; b_0, b_1, b_2)^* = \begin{pmatrix} s & 0 \\ 0 & U^t \end{pmatrix}$$

and $U(a_0, a_1, a_2; b_0, b_1, b_2) \in B(\ell_1)$.

Lemma 3 (Goldberg [13, p.59])**.** *T has a dense range $\Leftrightarrow$ T^* is 1-1.*

Lemma 4 (Goldberg [13, p.60])**.** *T has a bounded inverse $\Leftrightarrow$ T^* is onto.*

Theorem 2. $\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \{s\}$.

Proof. Let η be an eigenvalue of the operator $U(a_0, a_1, a_2; b_0, b_1, b_2)^*$. Then there exists $u \neq \theta = (0, 0, 0, ...)$ in ℓ_1 such that $U(a_0, a_1, a_2; b_0, b_1, b_2)^* u = \eta u$.
Then, we obtain

$$su_0 = \eta u_0 \tag{3}$$
$$a_0 u_1 = \eta u_1 \tag{4}$$
$$b_0 u_1 + a_1 u_2 = \eta u_2 \tag{5}$$
$$b_1 u_2 + a_2 u_3 = \eta u_3 \tag{6}$$
$$b_2 u_3 + a_0 u_4 = \eta u_4 \tag{7}$$
$$\vdots$$

Then we have if $\eta = s$, then from (3) $u_0 \in \mathbb{C}$, from (4) and etc. $u_1 = u_2 = u_3 = \cdots = u_n = \cdots = 0$. If $\eta \neq s$, then from (3) $u_0 = 0$, from (4) $\eta = a_0$. Therefore from (7) $u_3 = 0$, from (6) $u_2 = 0$, from (5) $u_1 = 0$ and etc. So $u_0 = u_1 = u_2 = \cdots = u_n = \cdots = 0$. Hereby, $\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c^* \cong \ell_1) = \{s\}$. □

Theorem 3. $\sigma_r(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \{s\}$.

Proof. Owing to $\sigma_r(A, c) = \sigma_p(A^*, c^* \cong \ell_1) \backslash \sigma_p(A, c)$, required result is given by Theorems 1 and 2 □

Lemma 5.

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{3n+t} a_k b_{nk} \right) = \sum_{k=1}^{\infty} a_{3k+t} \left(\sum_{n=k}^{\infty} b_{n,3k+t} \right) \ , \ t = 0, 1, 2$$

herein (a_k) and (b_{nk}) are real numbers.

Proof.

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{k=0}^{3n+t} a_k b_{nk}\right)\\
= &\sum_{k=0}^{3+t} a_k b_{1k} + \sum_{k=0}^{6+t} a_k b_{2k} + \sum_{k=0}^{9+t} a_k b_{3k} + \cdots + \sum_{k=0}^{3n+t} a_k b_{nk} + \cdots\\
= &a_0 b_{10} + a_1 b_{11} + a_2 b_{12} + a_3 b_{13} + a_4 b_{14} + a_5 b_{15}\\
&+ a_0 b_{20} + a_1 b_{21} + a_2 b_{22} + a_3 b_{23} + a_4 b_{24} + a_5 b_{25} + a_6 b_{26} + a_7 b_{27} + a_8 b_{28}\\
&+ a_0 b_{30} + a_1 b_{31} + a_2 b_{32} + a_3 b_{33} + a_4 b_{34} + a_5 b_{35} + a_6 b_{36} + a_7 b_{37} + a_8 b_{38}\\
&+ a_9 b_{39} + a_{10} b_{3,10} + a_{11} b_{3,11}\\
&+ \ldots\\
&+ a_0 b_{n0} + a_1 b_{n1} + \cdots + a_{3n+2} b_{n,3n+2}\\
&+ \cdots\\
= &a_0 \sum_{n=1}^{\infty} b_{n0} + a_1 \sum_{n=1}^{\infty} b_{n1} + a_2 \sum_{n=1}^{\infty} b_{n2} + a_{3+t} \sum_{n=1}^{\infty} b_{n,3+t} + a_{6+t} \sum_{n=2}^{\infty} b_{n,6+t}\\
&+ \cdots + a_{3k+t} \sum_{n=k}^{\infty} b_{n,3k+t}\\
= &\sum_{k=0}^{\infty} a_{3k+t}\left(\sum_{n=k}^{\infty} b_{n,3k+t}\right)
\end{aligned}$$

□

Theorem 4.

$$\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \partial M \setminus \{s\} \quad \textit{and} \quad \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M.$$

Proof. Let $v = (v_n) \in \ell_1$ be such that $(U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I)^* u = v$ for some $u = (u_n)$. Then we get following system of linear equations:

$$\begin{gathered}
(s-\lambda)u_0 = v_0\\
(a_0-\lambda)u_1 = v_1\\
b_0 u_1 + (a_1-\lambda)u_2 = v_2\\
\vdots
\end{gathered}$$

$$\left.\begin{array}{r}
(s-\lambda)u_0 = v_0\\
(a_0-\lambda)u_1 = v_1\\
b_2 u_{3n} + (a_0-\lambda)u_{3n+1} = v_{3n+1}\\
b_0 u_{3n+1} + (a_1-\lambda)u_{3n+2} = v_{3n+2}\\
b_1 u_{3n+2} + (a_2-\lambda)u_{3n+3} = v_{3n+3}\\
\vdots
\end{array}\right., n \geq 0$$

Solving above equations, we have

$$\begin{aligned} u_0 &= \frac{1}{s-\lambda} v_0 \\ u_{3n+t} &= \frac{1}{a_{t+2}-\lambda} \left[\sum_{k=1}^{3n+t} (-1)^{3n+t-k} v_k \prod_{v=0}^{3n+t-k-1} \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right], \quad t=0,1,2;\ n=1,2,\ldots \end{aligned}$$

Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod3)$ and we accept that $\prod_{v=0}^{-1} \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} = 1$. Therefore we get

$$\begin{aligned} \sum_{n=0}^{\infty} |u_n| &= |u_0| + |u_1| + |u_2| + |u_3| + \cdots \\ &= |u_0| + |u_1| + |u_2| + \sum_{n=1}^{\infty} |u_{3n+t}| \\ &= |u_0| + |u_1| + |u_2| \\ &\quad + \sum_{n=1}^{\infty} \left| \frac{1}{a_{t+2}-\lambda} \left[\sum_{k=1}^{3n+t} (-1)^{3n+t-k} v_k \prod_{\nu=0}^{3n+t-k-1} \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right] \right| \\ &\le \left| \frac{1}{s-\lambda} v_0 \right| + \left| \frac{1}{a_0-\lambda} v_1 \right| + \left| \frac{1}{a_1-\lambda} v_2 - \frac{b_0}{(a_0-\lambda)(a_1-\lambda)} v_1 \right| \\ &\quad + \frac{1}{|a_{t+2}-\lambda|} \sum_{n=1}^{\infty} \left[\sum_{k=1}^{3n+t} |v_k| \prod_{\nu=0}^{3n+t-k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right| \right] \end{aligned}$$

Thus the inequality is gotten;

$$\left| \sum_{n=0}^{\infty} u_n \right| \le G + \max_{m=0}^{2} \frac{1}{|a_m-\lambda|} \sum_{n=1}^{\infty} \left[\sum_{k=1}^{3n+t} |v_k| \prod_{\nu=0}^{3n+t-k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right| \right] \tag{8}$$

where

$$G = \left| \frac{1}{s-\lambda} v_0 \right| + \left| \frac{1}{a_0-\lambda} v_1 \right| + \left| \frac{1}{a_1-\lambda} v_2 - \frac{b_0}{(a_0-\lambda)(a_1-\lambda)} v_1 \right|$$

Now, we consider the sum $\sum_{n=1}^{\infty} \left[\sum_{k=1}^{3n+t} |v_k| \prod_{\nu=0}^{3n+t-k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right| \right]$. In Lemma 5 if we take $a_k = |v_k|$ and $b_{nk} = \prod_{\nu=0}^{3n+t-k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right|$ then we have

$$\sum_{n=1}^{\infty} \left[\sum_{k=1}^{3n+t} |v_k| \prod_{\nu=0}^{3n+t-k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu}-\lambda} \right| \right]$$

$$= \sum_{k=1}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} \prod_{\nu=0}^{3n+t-(3k+t)-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| \right]$$
$$= \sum_{k=1}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} \prod_{\nu=0}^{3n-3k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| \right].$$

Also since $\prod_{\nu=0}^{3n-3k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| = |d|^{n-k}, t = 0, 1, 2$ setting $d = \frac{b_2 b_1 b_0}{(a_2 - \lambda)(a_1 - \lambda)(a_0 - \lambda)}$ while $|d| < 1$, the last equation turns into the sum

$$\begin{aligned} \sum_{k=0}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} \prod_{\nu=0}^{3n-3k-1} \left| \frac{b_{3n+t+1-\nu}}{a_{3n+t+1-\nu} - \lambda} \right| \right] &= \sum_{k=0}^{\infty} |v_{3k+t}| \left[\sum_{n=k}^{\infty} |d|^{n-k} \right] \\ &= \sum_{k=0}^{\infty} |v_{3k+t}| \left(\frac{1}{1-|d|} \right) \\ &\quad \frac{1}{1-|d|} \|v\|_{\ell_1}. \end{aligned}$$

Then since $|d| < 1$ we get

$$\left| \sum_{n=0}^{\infty} u_n \right| \leq G + \max_{m=0}^{2} \frac{1}{|a_m - \lambda|} \frac{1}{1-|d|} \|v\|_{\ell_1}.$$

So, we have $v = (v_n) \in \ell_1$, $u = (u_n) \in \ell_1$ if $|d| = \left| \frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} \right| < 1$. Consequently, if for $\lambda \in \mathbb{C}$, $|a_2 - \lambda| |a_1 - \lambda| |a_0 - \lambda| > |b_2| |b_1| |b_0|$, then $(u_n) \in \ell_1$. Thus, the operator $(U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I)^*$ is onto if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| > |b_0| |b_1| |b_2|$. Then by Lemma 4, $U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I$ has a bounded inverse if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| > |b_0| |b_1| |b_2|$. Therefore,

$$\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}.$$

Owing to $\sigma(A, c)$ is the disjoint union of $\sigma_p(A, c)$, $\sigma_r(A, c)$ and $\sigma_c(A, c)$, thence

$$\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \subseteq \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\}.$$

By Theorem 1, we get

$$\begin{aligned} \{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\} &= \sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \\ &\subset \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c). \end{aligned}$$

Since, $\sigma(A, c)$ is closed

$$\begin{aligned} \overline{\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\}} &\subset \overline{\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)} \\ &= \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c), \end{aligned}$$

and hence $\{\lambda \in \mathbb{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| \leq |b_0| |b_1| |b_2|\} \subset \sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c)$. Therefore, $\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M$ and so $\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = M \backslash \left(\mathring{M} \cup \{s\}\right) = \partial M \backslash \{s\}$. □

3. Subdivision of the Spectrum

Subdivision of the spectrum; consists of three subsets of the spectrum that need not be discrete as follows:

The sequence $(x_n) \in X$ that satisfy the conditions of $\|x_n\| = 1$ and $\|Ax_n\| \to 0$ as $n \to \infty$ is called a Weyl sequence for A.

The set

$$\sigma_{ap}(A, X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - A\} \tag{9}$$

show the approximate point spectrum of A. The set

$$\sigma_\delta(A, X) := \{\lambda \in \sigma(A, X) : \lambda I - A \text{ is not surjective}\} \tag{10}$$

show defect spectrum of A. Finally, the set

$$\sigma_{co}(A, X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - A)} \neq X\} \tag{11}$$

show compression spectrum in the literature.

The below Proposition is extremely important for obtaining the subdivision of the spectrum of $U(a_0, a_1, a_2; b_0, b_1, b_2)$ in c.

Proposition 1 ([2], Proposition 1.3)**.** *The spectrum and subspectrum of an operator $A \in B(X)$ and its adjoint $A^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(A^*, X^*) = \sigma(A, X)$, *(b)* $\sigma_c(A^*, X^*) \subseteq \sigma_{ap}(A, X)$,
(c) $\sigma_{ap}(A^*, X^*) = \sigma_\delta(A, X)$, *(d)* $\sigma_\delta(A^*, X^*) = \sigma_{ap}(A, X)$,
(e) $\sigma_p(A^*, X^*) = \sigma_{co}(A, X)$, *(f)* $\sigma_{co}(A^*, X^*) \supseteq \sigma_p(A, X)$,
(g) $\sigma(A, X) = \sigma_{ap}(A, X) \cup \sigma_p(A^*, X^*) = \sigma_p(A, X) \cup \sigma_{ap}(A^*, X^*)$.

Goldberg's Classification of Spectrum

If $A \in B(X)$, then there are three cases for $R(A)$:

(I) $R(A) = X$, (II) $\overline{R(A)} = X$, but $R(A) \neq X$, (III) $\overline{R(A)} \neq X$
and three cases for A^{-1}:

(1) A^{-1} exists and bounded, (2) A^{-1} exists but boundless, (3) A^{-1} does not exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 (see [13]).

$\sigma(A, X)$ can be divided into subdivisions $I_2\sigma(A, X) = \emptyset$, $I_3\sigma(A, X)$, $II_2\sigma(A, X)$, $II_3\sigma(A, X)$, $III_1\sigma(A, X)$, $III_2\sigma(A, X)$, $III_3\sigma(A, X)$. For example, if $T = \lambda I - A$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(A, X)$.

By the definitions given above and introduction, we can write following Table 1.

TABLE 1. Subdivisions of the spectrum of a linear operator

		1	2	3
		A_λ^{-1} exists and is bounded	A_λ^{-1} exists and is unbounded	A_λ^{-1} does not exists
I	$R(\lambda I - A) = X$	$\lambda \in \rho(A, X)$	–	$\lambda \in \sigma_p(A, X)$ $\lambda \in \sigma_{ap}(A, X)$
II	$\overline{R(\lambda I - A)} = X$	$\lambda \in \rho(A, X)$	$\lambda \in \sigma_c(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$	$\lambda \in \sigma_p(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$
III	$\overline{R(\lambda I - A)} \neq X$	$\lambda \in \sigma_r(A, X)$ $\lambda \in \sigma_\delta(A, X)$ $\lambda \in \sigma_{co}(A, X)$	$\lambda \in \sigma_r(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$ $\lambda \in \sigma_{co}(A, X)$	$\lambda \in \sigma_p(A, X)$ $\lambda \in \sigma_{ap}(A, X)$ $\lambda \in \sigma_\delta(A, X)$ $\lambda \in \sigma_{co}(A, X)$

The articles mentioned in the Section 2, are related to the discretization of the spectrum defined by Goldberg. However, subdivision of the spectrum was examined on certain sequence space in [4], [6], [7]. Moreover, the spectrum and fine spectrum was calculated in [5], [8], [10] , [12], [15], [17], [18].

Theorem 5. *If* $|\lambda - a_0| \, |\lambda - a_1| \, |\lambda - a_2| < |b_0| \, |b_1| \, |b_2|$, *then*

$$\lambda \in I_3\sigma(U\,(a_0, a_1, a_2; b_0, b_1, b_2)\,, c).$$

Proof. Proof is similar to proof of [9, Theorem 5]. □

Corollary 2. $III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$, $III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \{s\}$.

Proof. If $|\lambda - a_0| \, |\lambda - a_1| \, |\lambda - a_2| > |b_0| \, |b_1| \, |b_2|$ then the operator $U\,(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I$ has a bounded inverse from proof of Theorem 3 and $\lambda = s$ does not satisfy the inequality $|\lambda - a_0| \, |\lambda - a_1| \, |\lambda - a_2| > |b_0| \, |b_1| \, |b_2|$. Owing to

$$\begin{aligned} \sigma_r(U(a_0, a_1, a_2; b_0, b_1, b_2), c) &= III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \\ &\cup III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) \end{aligned}$$

from Table 1, we obtain $III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$,
$III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \{s\}$. □

Corollary 3. $II_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = III_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c) = \emptyset$.

Proof. Since

$$\begin{aligned}\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2),c) &= I_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c)\\ &\cup II_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c)\\ &\cup III_3\sigma\left(U(a_0,a_1,a_2;b_0,b_1,b_2),c\right)\end{aligned}$$

in Table 1, $\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = I_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c)$ from Theorem 1 and Theorem 5. Thus
$II_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = III_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = \emptyset$. □

Theorem 6. *(a)* $\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = \partial M$,
(b) $\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = M$,
(c) $\sigma_{co}(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = \{s\}$.

Proof. (a) From Table 1, we obtain

$$\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = \sigma\left(U(a_0,a_1,a_2;b_0,b_1,b_2),c\right)\backslash I_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c)\,.$$

So using Theorem 4 and 5 with $a_n+b_n = a_{n+1}+b_{n+1} = S$, the required result is gotten.

(b) From Table 1, we obtain

$$\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = \sigma((U(a_0,a_1,a_2;b_0,b_1,b_2),c)\backslash III_1\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c).$$

And so $\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),c) = M$ from Corollary 2.

(c) By Proposition 1 (e), we obtain

$$\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,c^*) = \sigma_{co}(U(a_0,a_1,a_2;b_0,b_1,b_2),c).$$

Using Theorem 2 with $a_n+b_n = a_{n+1}+b_{n+1}$, the required result is gotten. □

Corollary 4. *(a)* $\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,c^*\cong \ell_1) = \partial M$,
(b) $\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,c^*\cong \ell_1) = M$.

Proof. By Proposition 1 (c) and (d), we obtain

$$\sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,c^*\cong \ell_1) = \sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2),c)$$

and

$$\sigma_\delta(U(a_0,a_1,a_2;b_0,b_1,b_2)^*,c^*\cong \ell_1) = \sigma_{ap}(U(a_0,a_1,a_2;b_0,b_1,b_2),c).$$

from Theorem 6 (a) and (b) with $a_n+b_n = a_{n+1}+b_{n+1} = S$, the required results are gotten. □

4. Results

We can generalize our operator

$$U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1})=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & b_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $b_0,b_1,\ldots,b_{n-1}\neq 0$.

In parallel with our study, the following results are valid for the n-entry upper triangular double band matrix above.

Theorem 7. *The following results are valid, where* $T=\left\{\lambda\in\mathbb{C}:\prod_{k=0}^{n-1}\left|\frac{\lambda-a_k}{b_k}\right|\leq 1\right\}$, $\mathring{T}$ *be the interior of the set* T *and* ∂T *be the boundary of the set* T *and for* $a_n+b_n=a_{n+1}+b_{n+1}=t$

(1) $\sigma_p(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=\mathring{T}$,
(2) $\sigma_p(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1})^*,c^*\cong\ell_1)=\{t\}$,
(3) $\sigma_r(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=\{t\}$,
(4) $\sigma_c(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=\partial T\setminus\{t\}$,
(5) $\sigma(U(a_0,a_1,\ldots,a_{n-1};b_0,b_1,\ldots,b_{n-1}),c)=T$.

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Declaration of Competing Interests The authors declare that they have no competing interest.

References

[1] Akhmedov, A. M., Başar, F., The fine spectra of the difference operator over the sequence space bv_p, $(1\leq p<\infty)$, *Acta. Math. Sin. (Engl Ser)*, 23(10) (2007), 1757–1768. doi.org/10.1007/s10114-005-0777-0
[2] Appell, J., De Pascale, E., Vignoli, A., Nonlinear Spectral Theory, Walter de Gruyter, Berlin, New York, 2004.
[3] Wilansky, A., Summabilitiy Through Functional Analysis, Amsterdam, North Holland, 1984.
[4] Başar, F., Durna, N., Yildirim, M., Subdivisions of the spectra for generalized difference operator over certain sequence spaces, *Thai J. Math.*, 9(1) (2011), 285-295.
[5] Das, R., On the spectrum and fine spectrum of the upper triangular matrix $U(r_1,r_2;s_1,s_2)$ over the sequence space c_0, *Afr. Math.*, 28 (2017), 841-849. doi.org/10.1007/s13370-017-0486-8

[6] Durna, N., Yildirim, M., Subdivision of the spectra for factorable matrices on c_0, *GU J. Sci.*, 24(1) (2011), 45-49.
[7] Durna, N., Subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c, *ADYU Sci.*, 6(1) (2016), 31-43.
[8] Durna, N., Yildirim, M., Kılıç, R., Partition of the spectra for the generalized difference operator $B(r, s)$ on the sequence space cs, *Cumhuriyet Sci. J.*, 39(1) (2018), 7-15.
[9] Durna, N., Kılıç, R., Spectra and fine spectra for the upper triangular band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ over the sequence space c_0, *Proyecciones J. Math.*, 38(1) (2019), 145-162.
[10] Durna, N., Subdivision of spectra for some lower triangular double-band matrices as operators on c_0, *Ukr. Mat. Zh.*, 70(7) (2018), 913-922.
[11] Dündar, E., Başar, F., On the fine spectrum of the upper triangular double band matrix Δ^+ on sequence space c_0, *Math. Commun.*, 18 (2013), 337–348.
[12] El-Shabrawy, S. R., Abu-Janah, S. H., Spectra of the generalized difference operator on the sequence spaces and bv_0 and h, *Linear and Multilinear Algebra*, 66(1) (2017), 1691-1708. doi.org/10.1080/03081087.2017.1369492
[13] Goldberg, S., Unbounded Linear Operators, McGraw Hill, New York, 1966.
[14] Karakaya, V., Altun, M., Fine spectra of upper triangular double-band matrices, *J. Comput. Appl. Math.*, 234 (2010), 1387–1394. doi.org/10.1016/j.cam.2010.02.014
[15] Tripathy, B. C., Das, R., Fine spectrum of the upper triangular matrix $U(M, 0, 0, s)$ over the squence spaces c_0 and c, *Proyecciones J. Math.*, 37(1) (2018), 85-101.
[16] Yeşilkayagil, M., Kirişci, M., On the fine spectrum of the forward difference operator on the Hahn space, *Gen. Math. Notes*, 33(2) (2016), 1-16.
[17] Yildirim, M., Durna, N., The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on ℓ_p, $(1 < p < \infty)$, *J. Inequal. Appl.*, 2017(193) (2017), 1-13. DOI 10.1186/s13660-017-1464-2
[18] Yildirim, M., Mursaleen, M., Doğan, Ç., The spectrum and fine spectrum of generalized Rhaly-Cesàro matrices on c_0 and c, *Operators and Matrices*, 12(4) (2018), 955-975. doi:10.7153/oam-2018-12-58
[19] Yildirim, M., The spectrum and fine spectrum of q-Cesaro matrices with $0 < q < 1$ on c_0, *Numer. Func. Anal. Optim.*, 41(3) (2020), 361-377. doi.org/10.1080/01630563.2019.1633666

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A FRACTIONAL ORDER MODEL OF HEPATITIS B TRANSMISSION UNDER THE EFFECT OF VACCINATION

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Abstract. In this paper we present a fractional order mathematical model to explain the spread of Hepatitis B Virus (HBV) in a non-constant population. The model we propose includes both vertical and horizontal transmission of the infection and also vaccination at birth and vaccination of the susceptible class. We also use a frequency dependent transmission rate in the model. We give results on existence of equilibrium points of the model and analyze the stability of the disease-free equilibrium. Finally, numerical simulations of the model are presented.

1. Introduction

Hepatitis B is a serious liver infection caused by Hepatitis B Virus (HBV). According to World Health Organization (WHO), an estimated 296 million people are living with HBV infection and in 2019 almost 820000 people died due to HBV related liver diseases [33]. However, immunization of newborns and susceptible individuals is a very effective strategy to control the transmission of the disease [34].

There are basically two different transmission types for HBV. When blood, semen or another body fluid from a person infected with HBV enters to the body of a non-infected person, horizontal transmission occurs. The virus can also be vertically transmitted [33]. Vertical transmission is the transmission of the virus from an infected mother to the baby at birth. Most of the infected individuals recover from the disease and gain immunity, however some develop chronic HBV infection. Chronic HBV infection can lead some life-threatening diseases like cirrhosis and liver cancer. The incubation period for HBV is an average of 120 days [21]. Once an individual is diagnosed with HBV, the infection is considered as acute infection for 6 months but if the infection lasts more than 6 months, it is considered as

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chronic infection [21]. For adults, recovery rate from acute infection with immunity is 95% while for infants and children, this rate is dramatically low [7].

In early 1980's a general dynamical model considering immunization, composed of partial differential equations is proposed and the idea of suitability of the given model for HBV transmission was put forward for the first time [1]. In 1994, the first differential equation model specifically for transmission of HBV infection including vaccination is proposed [16]. Since then, many researchers studied on models with vaccination (See [12]).

Medley et al. introduced a compartment epidemic model for HBV infection with immunization of children born to carrier mothers and newborn babies [22]. In this study population in the absence of disease is assumed to be constant. Zou et al. modified the model given in [22] considering the lifelong immunity gained after recovery and waning vaccine-induced immunity in a non-constant population by assuming only horizontal transmission of the disease [35]. In both of these models, transmission rate is assumed to be density dependent and also transmission occurs only through carriers ans acute infectious individual. However, HBV may transmit during its incubation period [33].

In recent years many mathematicians studied on fractional order epidemic models ([3], [17], [18]). Ullah et al. introduced a fractional order epidemic model for HBV transmission with density dependent transmission rate using Caputo-Fabrizio derivative in which only the immunization of children born to carrier mothers is considered [29]. Farman et al. analyzed an epidemic model for HBV infection consists of differential equations in Caputo sense. In this model, the population is assumed to be constant in the absence of the disease [6]. In this study we propose a more general model using fractional differential equations of Caputo sense considering newborn vaccination and also vaccination of susceptible individuals regardless of age. The reason for using fractional differential equations is to reflect the memory effect in the spread of the disease to the mathematical model ([4], [23]). In this model we consider both horizontal and vertical transmission of the disease. HBV infection is a long term infection so, ignoring the demographic structure of the population is not realistic. In the model we propose, we also consider the demographic properties of the population. Transmission rates used in epidemic models can be classified in two major forms: density dependent transmission rate and frequency dependent transmission rate [9]. Density dependent transmission rate is commonly assumed for smaller populations and specifically in modeling airborne transmitted diseases, nevertheless frequency dependent transmission rate is commonly assumed for large, heterogeneous populations and in modeling vector-borne or sexually transmitted diseases ([9], [8]). In all of the above mentioned models, transmission rates are assumed to be density dependent, however we assume frequency dependent transmission rate. We first introduce the model then analyze the equilibrium points of the model and finally we give the numerical simulations for the constructed model.

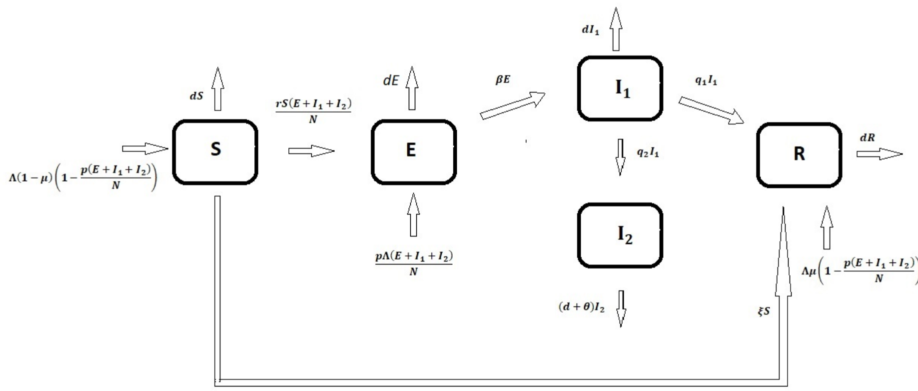

FIGURE 1. The schematic diagram of the proposed model.

2. MODEL DERIVATION

Definitions of fractional order integral and fractional order derivative in Caputo sense [19] are presented in the Appendix. Due to its nature, Caputo fractional derivative is widely used in mathematical modeling of real life problems. We also use Caputo derivative in our model. The main reason for using fractional derivative rather than the integer order derivative is the memory effect that is considered in the fractional order differential equations. Like most of the biological dynamics, dynamics of the transmission of epidemic diseases have a short memory effect [23].

The schematic diagram of the proposed epidemic model to explain the spread of HBV infection is given in Figure 1. The total population, $N(t)$, is partitioned into five classes namely susceptible, exposed, acute infectious, chronic infectious and recovered classes denoted by $S(t), E(t), I_1(t), I_2(t)$ and $R(t)$, respectively. The individuals in susceptible class are healthy individuals who are candidates for contracting the disease. The individuals in $E(t)$ class are infected individuals for whom the virus is in its incubation period. In the model there are two more infectious classes. After the symptoms are seen in an infectious individual, he/she is assumed to pass to the acute infectious class. If an acute infectious person cannot recover from the disease in a specific time interval which depends on the structure of the disease, he/she is assumed to be chronic infectious. Acute and chronic infectious compartments are denoted with $I_1(t)$ and $I_2(t)$, respectively. d is the natural death rate of the population and θ is the death rate related to the fatal diseases caused by the infection. Particularly for HBV infection, secondary fatal liver related diseases arise for the chronic infectious individuals that enhances the death rate. We assume that vaccination rate at birth is μ and the rate of vaccination of susceptible class is ξ. Also vaccinated individuals gain immunity and pass to the recovered class. Since, HBV is a virus that can be vertically transmitted which means an infected

TABLE 1. Variables and parameters used in the model.

$S(t)$:	Number of susceptible individuals at time t
$E(t)$:	Number of exposed individuals at time t
$I_1(t)$:	Number of acute infectious individuals at time t
$I_2(t)$:	Number of chronic infectious individuals at time t
$R(t)$:	Number of recovered individuals with immunity at time t
μ :	Immunization rate by vaccination at birth
Λ :	Number of recruits per unit time
d :	Natural death rate
p :	Probability of having an exposed baby for exposed and infectious classes
r :	Transmission coefficient (both exposed and infectious individuals can transmit the disease)
ξ :	Immunization rate of susceptible class
q_1:	Recovery rate from acute HBV infection
q_2:	Rate of developing chronic disease after acute Hepatitis B infection.
θ :	Disease related death rate
β :	The rate at which exposed individuals pass to acute infectious class

individual (we only consider the vertical transmission from mother) may transmit the disease to its babies before birth, the parameter p is defined as the probability of having an exposed baby for the infected individuals. All of the parameters used in the model are explained in Table 1 .

These assumptions lead to the following system of differential equations with $0 < \alpha < 1$,

$$\begin{aligned}
&D^\alpha S = (1-\mu)\Lambda\left(1-\tfrac{p(E+I_1+I_2)}{N}\right) - S\left(\tfrac{r(E+I_1+I_2)}{N} + d + \xi\right),\\
&D^\alpha E = (rS + p\Lambda)\tfrac{(E+I_1+I_2)}{N} - (\beta+d)E,\\
&D^\alpha I_1 = \beta E - (q_1+q_2+d)I_1,\\
&D^\alpha I_2 = q_2 I_1 - (\theta+d)I_2,\\
&D^\alpha R = q_1 I_1 + \mu\Lambda\left(1-\tfrac{p(E+I_1+I_2)}{N}\right) + \xi S - dR
\end{aligned} \tag{1}$$

and the initial conditions

$$S(0) = S_0,\ E(0) = E_0,\ I_1(0) = I_{10},\ I_2(0) = I_{20},\ R(0) = R_0, \tag{2}$$

where $N(t) = S(t) + E(t) + I_1(t) + I_2(t) + R(t)$ and $(S, E, I_1, I_2, R) \in R_+^5$. Using system (1), we obtain

$$D^\alpha N(t) = \Lambda - dN - \theta I_2. \tag{3}$$

Theorem 1. *The initial value problem (1)-(2) has a unique solution and the solution remains in* R_+^5.

Proof. The existence and uniqueness of the solution of (1)-(2) in $(0, \infty)$ can be shown by using [13]. We now show the positive invariance of the domain R_+^5.

Since,

$$
\begin{aligned}
D^{\alpha}S \quad | \quad {}_{S=0} &= \Lambda(1-\mu)\left(1-\frac{p\left(E+I_1+I_2\right)}{N}\right) \geq 0, \\
D^{\alpha}E \quad | \quad {}_{E=0} &= \frac{rS(I_1+I_2)}{S+I_1+I_2+R} + \frac{p\Lambda(I_1+I_2)}{S+I_1+I_2+R} \geq 0, \\
D^{\alpha}I_1 \quad | \quad {}_{I_1=0} &= \beta E \geq 0, \\
D^{\alpha}I_2 \quad | \quad {}_{I_2=0} &= q_2 I_1 \geq 0, \\
D^{\alpha}R \quad | \quad {}_{R=0} &= q_1 I_1 + \mu\Lambda\left(1-\frac{p\left(E+I_1+I_2\right)}{S+E+I_1+I_2}\right) + \xi S \geq 0,
\end{aligned}
$$

on every hyperplane bounding the nonnegative orthant, the vector field points into R_{+}^{5}.

□

It is clear that $N\left(t\right)$ also remains nonnegative.
Let $\Omega = \left\{\left(S\left(t\right), E\left(t\right), I_1\left(t\right), I_2\left(t\right), R\left(t\right)\right) \in R_{+}^{5} : 1 \leq N\left(t\right) \leq \Lambda/d\right\}$.

Lemma 1. *The set Ω is positively invariant with respect to system (1).*

Proof. (3) implies that

$$
\begin{aligned}
D^{\alpha}N(t) &\leq -dN\left(t\right)+\Lambda, \\
0 &< \alpha < 1.
\end{aligned}
$$

So,

$$
N\left(t\right) \leq \left(N_0 - \frac{\Lambda}{d}\right) E_{\alpha}\left(-dt^{\alpha}\right) + \frac{\Lambda}{d}.
$$

Consequently, $N\left(t\right) \leq \frac{\Lambda}{d}$, if $N_0 \leq \frac{\Lambda}{d}$.

□

For the sake of simplicity in calculations, we use the system

$$
\begin{array}{l}
D^{\alpha}S = \left(1-\mu\right)\Lambda\left(1-\frac{p(E+I_1+I_2)}{N}\right) - S\left(\frac{r(E+I_1+I_2)}{N}+d+\xi\right), \\
D^{\alpha}E = \left(rS+p\Lambda\right)\frac{(E+I_1+I_2)}{N} - \left(\beta+d\right)E, \\
D^{\alpha}I_1 = \beta E - (q_1+q_2+d)I_1, \\
D^{\alpha}I_2 = q_2 I_1 - \left(\theta+d\right)I_2, \\
D^{\alpha}N(t) = \Lambda - dN - \theta I_2
\end{array} \tag{4}
$$

that can be obtained by (1) and (3) with the initial conditions

$$
S\left(0\right)=S_0,\ E\left(0\right)=E_0,\ I_1\left(0\right)=I_{10},\ I_2\left(0\right)=I_{20},\ N\left(0\right)=N_0.
$$

3. Equilibrium Points and Stability

System (4) has a disease free equilibrium (DFE) at $H_0 = \left(\frac{\Lambda(1-\mu)}{d+\xi}, 0, 0, 0, \frac{\Lambda}{d}\right)$. The positive equilibrium appears at $H_1 = (S^*, E^*, I_1^*, I_2^*, N^*)$ where

$$\begin{aligned} S^* &= \left(\frac{N^* A_0 A_1}{A_0 A_1 + A_0 + 1} - p\Lambda\right)\frac{1}{r}, \\ E^* &= A_0 A_1 I_2^*, \\ I_1^* &= A_0 I_2^* \\ N^* &= \frac{\Lambda - \theta I_2^*}{d} \end{aligned}$$

and

$$\begin{aligned} A_0 &= \frac{d+\theta}{q_2}, \\ A_1 &= \frac{(d+q_1+q_2)}{\beta}, \end{aligned}$$

if the condition

$$\frac{A_0 A_1 (d+\xi)}{r(1-\mu)\, d\, (A_0 A_1 + A_0 + 1)} > 1, \qquad \mu < 1$$

holds true.

Basic reproduction number, denoted by R_0^*, for an infection is the number of secondary infections caused by one infected individual introduced to a totally susceptible population. Therefore, it is assumed to be a treshold value for the infection to persist. Jacobian method is commonly used to determine the value of R_0^* in epidemic models. However, it is not easy to overcome the algebraic work needed to apply Jacobian method to models with multiple infectious compartments. Next generation matrix (NGM) method is an alternative method to find the value of R_0^*. The details of the NGM method can be found in [2], [30], [31] and [32]. We first give the outline of the NGM method and apply it to the model given by (4).

Consider the system given with

$$\frac{dX}{dt} = G(X).$$

Let $X = (x_1, x_2, ..., x_n)^T$ be the number of individuals in each compartment of the epidemic model and let the first m compartments ($m < n$) are composed of infected individuals. Consider the equations represented in the form

$$\frac{dx_i}{dt} = \mathcal{F}_i(X) - \mathcal{V}_i(X), \qquad i = 1, 2, ..., m \tag{5}$$

where $\mathcal{F}_i(X)$ is the rate of appearance of new infections in compartment i and $\mathcal{V}_i(X)$ is the rate of transitions between the infected compartments. Here $\mathcal{F}_i$ and

$\mathcal{V}_i$ are assumed to be in $\mathcal{C}^2$. FV^{-1} is called the next generation matrix where

$$F = \left[\frac{\partial \mathcal{F}_i\left(X^*\right)}{\partial x_j}\right] \text{ and } V = \left[\frac{\partial \mathcal{V}_i\left(X^*\right)}{\partial x_j}\right], \qquad 1 \leq i, j \leq m$$

and the spectral radius of the NGM is the basic reproduction number.

Theorem 2. *([32]) If X_0 is a disease free equilibrium of the system $\frac{dx_i}{dt} = \mathcal{F}_i\left(X\right) - \mathcal{V}_i\left(X\right)$ then X_0 is locally asymptotically stable if $R_0^* = \rho\left(FV^{-1}\right) < 1$, but unstable if $R_0^* > 1$.*

Remark 1. *Consider an epidemic model given by the integer order system*

$$\frac{dX}{dt} = G\left(X\right) \tag{6}$$

and its fractional order counterpart

$$\frac{d^\alpha X}{dt^\alpha} = G\left(X\right). \tag{7}$$

Systems (6) and (7) have the same equilibrium points. Let X^* be the disease free equilibrium point for both models. If X^* is stable for (6), then it is also stable for (7). But the converse is not always true. Therefore, Theorem 1 gives only a sufficient condition for the stability of X^* for (7).

We now consider the system consisting of three infected compartments of the model (4),

$$\begin{array}{l} D^\alpha E = \left(rS + p\Lambda\right) \frac{\left(E + I_1 + I_2\right)}{N} - \left(\beta + d\right) E \\ D^\alpha I_1 = \beta E - \left(q_1 + q_2 + d\right) I_1 \\ D^\alpha I_2 = q_2 I_1 - \left(\theta + d\right) I_2 \end{array} \tag{8}$$

and split the system in the form (5).

Let $X = (S, E, I_1, I_2, N)$ and define

$$\begin{aligned} \mathcal{F}_1\left(X\right) &= \frac{\left(r + p\Lambda\right) S \left(E + I_1 + I_2\right)}{N}, \\ \mathcal{F}_2\left(X\right) &= 0, \\ \mathcal{F}_3\left(X\right) &= 0, \\ \mathcal{V}_1\left(X\right) &= \left(\beta + d\right) E, \\ \mathcal{V}_2\left(X\right) &= -\beta E + \left(d + q_1 + q_2\right) I_1, \\ \mathcal{V}_3\left(X\right) &= \left(d + \theta\right) I_2 - q_1 I_1. \end{aligned}$$

So,

$$F \mid_{H_0} = \begin{bmatrix} \frac{rd(1-\mu)}{d+\xi} + pd & \frac{rd(1-\mu)}{d+\xi} + pd & \frac{rd(1-\mu)}{d+\xi} + pd \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V\mid_{H_0}=\begin{bmatrix}(\beta+d) & 0 & 0\\ -\beta & d+q_1+q_2 & 0\\ 0 & -q_2 & d+\theta\end{bmatrix}$$

and

$$R_0^*=\rho\left(FV^{-1}\right)=\frac{dp\,(d+\xi)+d\,(1-\mu)\,r}{(\beta+d)\,(d+\xi)}.$$

Theorem 3. *DFE of system (4) is locally asymptotically stable, if $R_0^*<1$ and unstable if $R_0^*>1$.*

Proof. The first part of the theorem is a direct consequence of NGM method and Remark 1. In order to prove the unstability condition, we apply the Jacobian method. The characteristic equations of system (4) for the DFE is

$$(-d-\xi-\lambda)\,(-d-\lambda)\,P_3(\lambda)=0$$

where $P_3(\lambda)=-\lambda^3+a_2\lambda^2+a_1\lambda+a_0$ with

$$\begin{aligned}a_2 &= (d+\beta)\,(R_0^*-1)-(A_1\beta+A_0q_2)\\ a_1 &= (A_1\beta+A_0q_2)\,(d+\beta)\,(R_0^*-1)-A_0A_1\beta q_2+(d+\beta)\,R_0^*\beta\\ a_0 &= (d+\beta)\,\beta q_2\,(A_0A_1\,(R_0^*-1)+R_0^*\,(A_0+1)).\end{aligned}$$

If $R_0^*>1$ then $a_0>0$. Applying Descartes' rule of signs, we see that P_3 has at least one positive root, that is DFE is unstable. □

Theorem 4. *DFE of the system (4) is globally asymptotically stable in Ω if $R_0^*<1$ and the following condition holds:*

$$\frac{(q_1+q_2+d)\,[(\beta+d)\,(\theta+d)+q_2\,(r+p)]}{(r+p)\,(q_1+q_2+d+\beta)\,(\theta+d+R_0^*q_2)}\geq 1. \tag{9}$$

Proof. Consider the Lyapunov function

$$L=A_1E+A_2I_1+A_3I_2,$$

where

$$\begin{aligned}A_1 &= (q_1+q_2+d)\,(\theta+d)\,R_0^*,\\ A_2 &= (r+p)\,(\theta+d+R_0^*q_2),\\ A_3 &= (r+p)\,(q_1+q_2+d)\,R_0^*.\end{aligned} \tag{10}$$

Using system (4), we have

$$\begin{aligned}D^\alpha L \leq\ & E\,[A_1\,(r+p)-A_1\,(\beta+d)+A_2\beta]\\ &+I_1\,[A_1\,(r+p)-A_2\,(q_1+q_2+d)+A_3q_2]\\ &+I_2\,[A_1\,(r+p)-A_3\,(\theta+d)].\end{aligned}$$

Substituting $A_1,\ A_2$ and A_3 as given in (10), we obtain

$$\begin{aligned}D^\alpha L \leq\ & (E+I_1)\,(R_0^*-1)\,(q_1+q_2+d)\,(r+p)\,(\theta+d)\\ &+E[(\beta+q_1+q_2+d)\,(r+p)\,(\theta+d+R_0^*q_2)\end{aligned}$$

$$
\begin{aligned}
&\quad - (q_1 + q_2 + d)\, R_0^* \left((\theta + d)(\beta + d) + q_2 (r + p)\right)] \\
&\leq \quad 0,
\end{aligned}
$$

if $R_0^* < 1$ and (9) holds. Consequently, using LaSalle's invariance principle, we conclude that DFE is globally stable in Ω. □

4. Numerical Solutions of the Model Using Data of Turkey

Hepatitis B virus infection is a serious public health issue in Turkey as well as the rest of the world. There are two phases of the infection namely acute and chronic. Once a person is diagnosed with chronic HBV infection he/she may develop new HBV related fatal diseases like cirrhosis and liver carcinoma. A traditional SIR model is performed for explaining HBV transmission in Turkey and transmission coefficient for this model is estimated for two different values for birth and natural death rate of the population [10]. We also simulate our model using data of Turkey.

According to the data provided by Turkish Statistical Institution (TUIK), the average number of people born in Turkey every year is 1303000 and the average death rate in Turkey is 0.00521 between the years 2010 and 2020 [28].

The most effective method to control the spread of HBV is the immunization of the individuals in the population. In our model there are two parameters related to the immunization. The first one is μ that represents the efficient immunization rate of the newborns. In Turkey since 1997, every baby born in hospitals is being vaccinated after birth. The immunity is gained after three doses of vaccine with 95% [20]. In Turkey nearly 94% of the births take place in hospitals and the newborns receive the first dose after birth but only 75% percent of them take three doses of vaccine ([24]). For the parameter that represents the efficient vaccination rate at birth, μ, we use the estimated value 0.66975 that is the product of 0.95, 0.94 and 0.75.

Vertical transmission of HBV is important for the models explaining the dynamics of the spread of HBV because the rate of developing chronic Hepatitis B is $70\% - 90\%$ for the babies who are born infected ([25]). The rate of having an exposed baby for the infected mothers is known to be almost 90% and according to TUIK ([28]), the rate of giving birth for the population is 1.7% in Turkey. So, we set the parameter $p = 0.0153$. β is assumed to be the the rate at which the exposed individuals pass to the acute infectious compartment, that is closely related with the incubation period of the virus. The incubation period of HBV is known to be $60 - 180$ days and for the simulations we assume it to be 120 days and set $\beta = 360/120 = 3$.

The average recovery rates from acute infection for adults, children and babies are 95%, 50% and 10%, respectively. Using the demographic data of Turkey we use the weighted average for the recovery rate for Hepatitis B as 3.5008 considering the average recovery duration 90 days. We also use the value 0.2496 for the parameter q_2, that is the rate of developing chronic HBV infection for the acute infectious compartment. This value is calculated by $q_2 = 2(1 - q_1/4)$.

Table 2. Initial values for system (4).

$N(0)$	$74724.269(\times 10^3)$
Hepatitis B prevalence ([26])	4.57%
Number of HBV infectious people	$3414.899(\times 10^3)$
$I_1(0)$ ([5])	$406.372(\times 10^3)$
$I_2(0)$	$3008.526(\times 10^3)$
$E(0)$ (assumed)	$100(\times 10^3)$
$R(0)$ ([27])	$23837.041(\times 10^3)$
$S(0)$	$47372.33(\times 10^3)$

Chronic HBV infection causes liver related fatal diseases and chronic HBV related deaths are due to liver cancer with 55% and cirrhosis and other liver diseases with 45% ([15]). Disease related death rate, θ, is estimated to be 2.2×10^{-5} ([14]). We also assume that the vaccination rate for the susceptible compartment is 0.0001.

Transmission coefficient of the disease does not only dependent on the type of the virus but also depends on the social structure of the population we work on. So, we simulate the model for different values of transmission coefficient, r ($r = 0.8, 1$).

We start the simulation from 2011, when the prevalence of HBV in Turkey was 4.57% ([26]). According to the Annual Epidemiological Report (2010-2014) on Hepatitis B of European Center for Disease Prevention and Control (ECDC), 11.9% of reported Hepatitis B cases are acute ([5]). Also the rate of anti-HBs positivity which is a marker for gained immunity rate for Turkey is 31.9% ([27]). We use the values given in Table 2 to determine the initial values of system (4).

Basic reproduction numbers are calculated as 0.0862849 and 0.107849 for $r = 0.8$ and $r = 1$, respectively. The solutions of the proposed model using the above mentioned parameters for different values of α are represented in the figures (2)-(5).

5. Conclusion

Epidemic diseases and their health and economic consequences are one of the major problems in the world. The first step to control the spread of a disease is to understand its dynamics. Mathematical models are very convenient tools to understand how a disease spread. Although statistical analysis of the data about the spread of a disease gives a foresight about the future, it generally ignores the dynamical feature of the process. Also, collecting appropriate data needs a long time and also it is too expensive. In this paper we propose an epidemic model to explain the spread of Hepatitis B. Hepatitis B epidemic is a long term epidemic unlike the seasonal diseases. So, the population in the model is assumed to be non-constant. Also, due to the nature of HBV both vertical and horizontal transmissions are considered in the model. We also use a fractional order system to reflect the memory

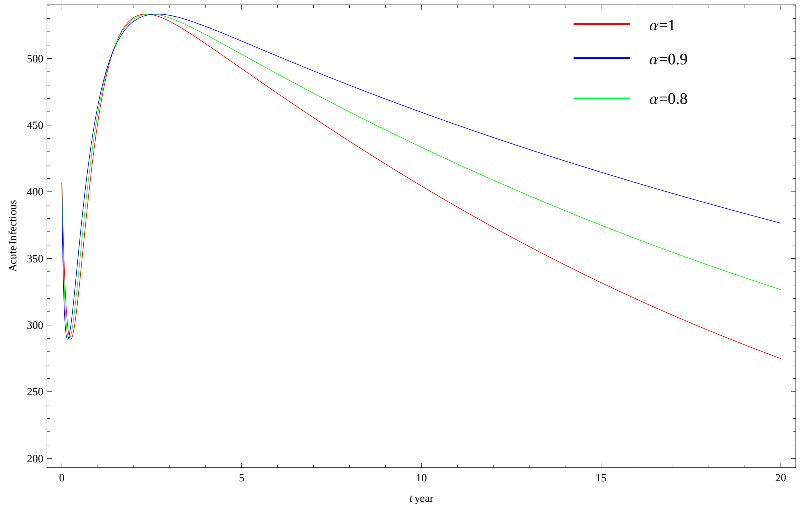

FIGURE 2. Acute infectious compartment for $r = 0.8$.

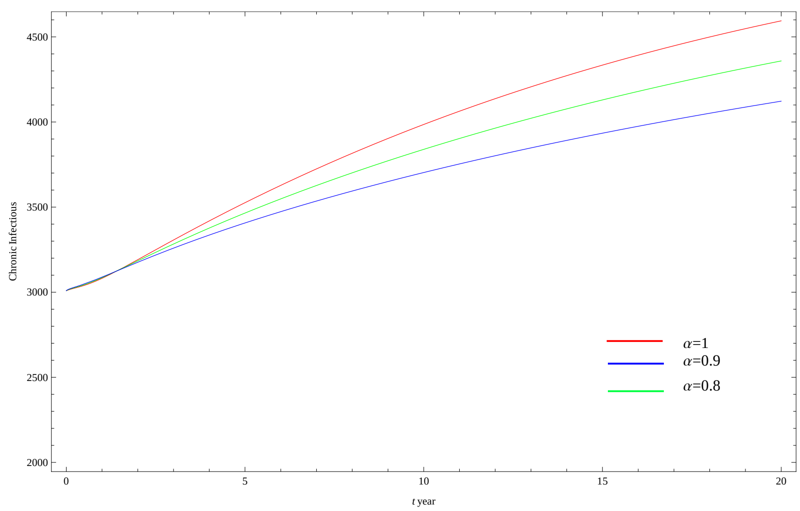

FIGURE 3. Chronic infectious compartment for $r = 0.8$.

effect of the epidemic. After determining the equilibrium points of the model, we give local stability analysis of the disease free equilibrium. We also give numerical simulations for the model. The parameters used in the simulations are obtained using previously published research and the numerical solutions are plotted for two different values of the transmission coefficient. The solutions are presented for

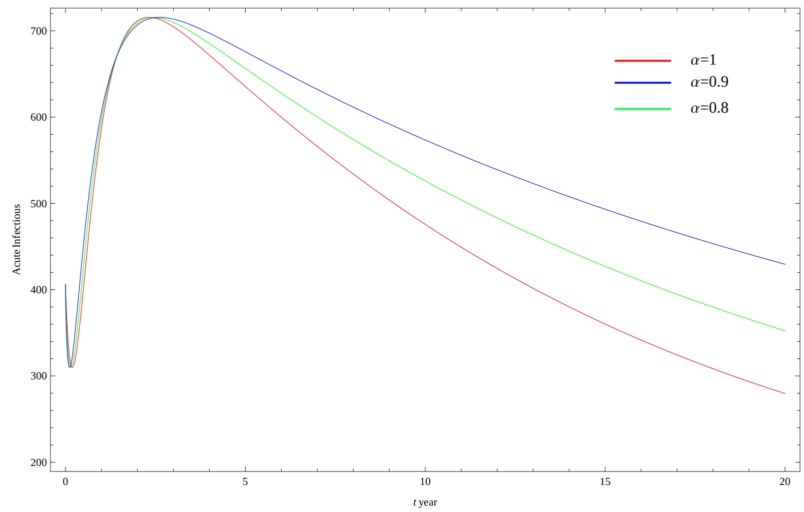

FIGURE 4. Acute infectious compartment for $r = 1$.

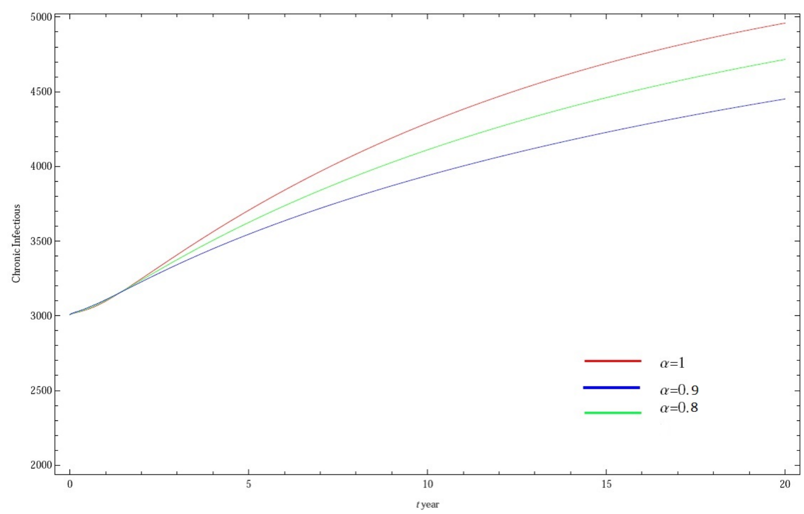

FIGURE 5. Chronic infectious compartment for $r = 1$.

$\alpha = 1, 0.9, 0.8$. Data for the incidence of the disease is easily reachable for Hepatitis B. But for the simulations of the proposed model, we need the prevalence data and the only comprehensive data for the prevalence of HBV infection in Turkey is given in 2011 ([26]). This model may give a foresight for the future of HBV infection in Turkey under the mentioned scenario.

Appendix

Definition 1. *[19] Riemann-Liouville fractional order integral of order* $\alpha > 0$ *for a function* $f : R^+ \to R$ *is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

and Caputo fractional order derivative of order $\alpha \in (n-1, n)$ *of* $f(t)$ *is defined by*

$$D^\alpha f(t) = I^{n-\alpha} D^\alpha f(t)$$

where $n = \lfloor \alpha \rfloor - 1$ *and* $D = d/dt$*. Here and elsewhere* Γ *denotes the Gamma function.*

Declaration of Competing Interests The authors declare that they have no competing interests.

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References

[1] Anderson, R. M., May, R. M., Vaccination and herd immunity, *Nature*, 318 (1985), 323-329.

[2] Diekmann, O., Heesterbeek, J. A. P., Metz, J. A. J., On the definition and computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, 28 (1990), 365-382.

[3] Ding, Y., Ye, H., A fractional-order differential equation model of HIV infection of $CD4^+$T-Cells, *Mathematical and Computer Modeling*, 50 (2009), 386-392.

[4] El-Saka, H. A. A., The fractional-order SIS epidemic model with variable population size, *Journal of Egyptian Mathematical Society*, 22(1) (2014), 50-54.

[5] European Center for Disease Prevention and Control, *Hepatitis B - Annual Epidemiological Report* (2016). https://www.ecdc.europa.eu/en/publications-data/hepatitis-b-annual-epidemiological-report-2016-2014-data / Accessed 21.09.2020.

[6] Farman, M., Ahmad, A., Umer, S. A., Hafeez, A., A mathematical analysis and modeling of hepatitis B model with non-integer time fractional derivative, *Communications in Mathematics and Applications,* 10(3) (2019), 571-584.

[7] Fattorich, G., Bortolotti, F., Donato, F., Natural history of chronic hepatitis B: special emphasis on disease progression and prognostic factors, *J. Hepatol.,* 48(2) (2008), 335-352. doi: 10.1016/j.jhep.2007.11.011

[8] Ferrari, M. J., Perkins, S. E., Pomeroy, L. W., Bjornstad, O. N., Pathogens, social networks, and the paradox of transmission scaling, *Interdisciplinary Perspectives on Infectious Diseases,* Article ID 267049 (2011). doi:10.1155/2011/267049.

[9] Geard, N., Glass, K., McCaw, J. M., McBryde, E. S., Korb, K. B., Keeling, M. J., McVernon, J., The effects of demographic change on disease transmission and vaccine impact in a household structured population, *J. Epidemics,* 13 (2015), 56-64.

[10] Golgeli, M., A Mathematical model of hepatitis B transmission in Turkey, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 68(2) (2019), 1586-1595.

[11] Khan, T., Zaman, G., Chohan, M. I., The transmission dynamic and optimal control of acute and chronic hepatitis B, *Journal of Biological Dynamics*, 11(1) (2017), 172-189.

[12] Liang, P., Zu, J., Zhiang, G., A Literature review of mathematical models of hepatitis B virus transmission applied to immunization strategies from 1994 to 2015, *J. Epidemiol.*, 28(5) (2018), 221-229.

[13] Lin, W., Global existence theory and chaos control of fractional differential equations, *JMAA,* 332 (2007), 709–726.

[14] Marcelin, P., Pequignot, F., Delarocque-Astagneau, E., Zarski, J., Ganne, N., Hillon, P., Antona, D., Bovet, M., Mechain, M., Asselah, T., Desenclos, J., Jougla, E., Mortality related to chronic hepatitis B and chronic hepatitis C in France: Evidence for the role of HIV coinfection and alcohol consumption, *Journal of Hepatology*, 48 (2008), 200-207.

[15] Mårdh, O., Quinten, C., Amato-Gauci, A. J., Duffell, E., Mortality from liver diseases attributable to hepatitis B and C in the EU/EEA – descriptive analysis and estimation of 2015 baseline, *Infectious Diseases,* (2020). doi:10.1080/23744235.2020.1766104

[16] Mc Lean, A. R., Blumberg, B. S., Modeling the impact of mass vaccination against hepatitis B. I. model formulation and parameter estimation, *Proc.Biol.Sci.,* 256 (1994), 7-15.

[17] Mouaouine, A., Boukhouima, A., Hattaf, K., Yousuf, N., A fractional order SIR epidemic model with nonlinear incidence rate, *Advances in Difference Equations*, 160 (2018).

[18] Özalp, N., Demirci, E., A Fractional order SEIR model with vertical transmission, *Mathematical and Computer Modelling*, 54(1-2) (2011), 1-6.

[19] Podlubny, I., Fractional Differential Equations, Volume 198: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, (1st ed.), Academic Press, 1998.

[20] Republic of Turkey Ministry of Health, Vaccine Portal, asi.saglik.gov.tr/liste/4-hepatit-b-hastaligi-nedir.html / Accessed 16.09.2020.

[21] Kodani, M., Schillie, S. F., Hepatitis B, manual for the surveillance of vaccine-preventable diseases, Edited by: Roush, S. W., Baldy, L. M., Hall, M. A. K., Centers for Disease Control and Prevention, Atlanta GA, March 13 2020.

[22] Medley, G. F., Lindop, N. A., Edmunds, W. J., Nokes, D. J., Hepatitis-B virus endemicity: heterogeneity, catastrophic dynamics and control, *Nature Medicine*, 7 (2001), 619-624. https://doi.org/10.1038/87953

[23] Saeedian, M., Khalighi, M., Azimi-Tafreshi, N., Jafari, G. R., Ausloos, M., Memory effects on epidemic evolutions: The susceptible-infected-recovered epidemic model, *Physical Review E*, 95(022409) (2017).

[24] Tosun, S., Hepatit B Aşılaması ve Ülkemizde Hepatit Aşılama Sonuçları, In Tabak, F., Balık, İ. (Eds.), *Viral Hepatit 2013* (pp.413-39), Viral Hepatitle Savaşım Derneği Yayını, İstanbul Medikal Yayıncılık, İstanbul, 2013.

[25] Tosun, S., Pregnancy and hepatitis B virus infection, *Medditerr. J. Infect. Microb. Animicrob.,* 5(4) (2016).

[26] Toy, M., Onder, F. O., Wörmann, T., Bozdayi, A. M., Schalm, S. W., Borsboom, G. J., van Rosmalen, J., Richardus, J. H., Yurdaydin, C., Age and region specific hepatitis B prevalence in Turkey estimated using generalized linear mixed models: a systematic review, *BMC Infectious Diseases,* 11(337) (2011). doi: 10.1186/1471-2334-11-337

[27] Tozun, N., Ozdogan, O., Cakaloglu, Y., Idilman, R., Karasu, Z., Akarca, U., Kaymakoglu, S., Ergonul, O., Seroprevalence of hepatitis B and C virus infections and risk factors in Turkey: a fieldwork TURHEP study, *Clin. Microbiol. Infect.,* 21(11) (2015), 1020-1026. doi: 10.1016/j.cmi.2015.06.028

[28] Turkish Statistical Institution, TÜİK. http://www.tuik.gov.tr/PreIstatistikTablo.do?istab_id =1636/ Accessed 23.09.2020.

[29] Ullah, S., Khan, M. A., Farooq, M., A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative, *The European Physical Journal Plus,* 133(237) (2018). doi: 10.1140/epjp/i2018-12072-4.

[30] Van den Driessche, P., Watmough, J., Reproduction numbers and sub-treshold equilibria for compartmental models of disease transmission, *Mathematical Biosciences,* 180(1-2) (2002), 29-48.

[31] Van den Driessche, P., Watmough, J., Further Notes on the Basic Reproduction Number, Mathematical Epidemiology (pp.159-178), Springer, Berlin, Heidelberg, 2008.

[32] Van den Driessche, P., Reproduction numbers of infectious disease models, *Infectious Disease Modelling*, 2(3) (2017), 288-303.

[33] World Health Organization, https://www.who.int/news-room/fact-sheets/detail/hepatitis-b/ Accessed 28 May 2022.

[34] World Health Organization Western Pacific Region, hepatitis B control through immunization : a reference guide, WHO, ISBN: 9789290616696, 2014. https://www.who.int/news-room/fact-sheets/detail/hepatitis-b/ Accessed 28 May 2022.

[35] Zou, L., Zhang, W., Ruan, S., Modeling the transmission dynamics and control of hepatitis B virus in China, *Journal of Theoretical Biology*, 262(2) (2010), 330-338.

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ORLICZ-LACUNARY CONVERGENT TRIPLE SEQUENCES AND IDEAL CONVERGENCE

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Abstract. In the present paper we introduce and study Orlicz lacunary convergent triple sequences over n-normed spaces. We make an effort to present the notion of g_3-ideal convergence in triple sequence spaces. We examine some topological and algebraic features of new formed sequence spaces. Some inclusion relations are obtained in this paper. Finally, we investigate ideal convergence in these spaces.

1. Introduction and Preliminaries

Several authors involving Duran [16], King [32], Lorentz [38], Moricz and Rhoades [46], Schafer [57] have worked the space of almost convergent sequences. The notion of strong almost convergence was considered by Maddox [39]. In [40], Maddox defined a generalization of strong almost convergence. Related articles with the topic almost convergence and strong almost convergence can be seen in [3,8–15,53].

In 1922, Banach defined normed linear spaces as a set of axioms. Since then, mathematicians keep on trying to find a proper generalization of this concept. The first notable attempt was by Vulich [60]. He introduced K-normed space in 1937. In another process of generalization, Siegfried Gähler [20] introduced 2-metric in 1963. As a continuation of his research, Gähler [21] proposed a mathematical structure, called 2-normed space, as a generalization of normed linear space which has been subsequently worked by many researchers [22–26, 33, 35, 41, 45].

In order to extend convergence of sequences, the notion of statistical convergence was given by Fast [17] for the real sequences. Afterward, it was further researched

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from sequence point of view and connected with the summability theory (see [5, 7, 18, 31, 42–44, 47]) and has been generalized to the thought of 2-normed space by Gürdal and Pehlivan [27]. Recently, Alotaibi and Alroqi [1] extended this concept in paranormed space. The concept of paranorm is a generalization of absolute value (see [48]). We can refer to [4, 34] which are connected with this topic. The studies of double and triple sequences have seen rapid growth. The initial work on double sequences was establised by Bromwich [6]. The concept of regular convergence for double sequences was introduced by Hardy [29]. Quite recently, Zeltser [61] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Recently, Mursaleen and Edely [50] and Şahiner et al. [58] considered the idea of statistical convergence for multiple sequences. Fridy and Orhan [19] defined the concept of lacunary statistical convergence. The double lacunary statistical convergence was worked by Patterson and Savaş [55]. For details about definition of sequence spaces, Orlicz sequence spaces and paranormed spaces one can see [37, 51, 54, 56].

Since sequence convergence plays a very significant role in the essential theory of mathematics, there are many convergence notions in summability theory, in classical measure theory, in approximation theory, and in probability theory, and the relationships between them are examined. The concerned reader may consult Gürdal et al. [26], and Hazarika et al. [30], the monographs [2] and [49] for the background on the sequence spaces and related topics. Inspired by this, in this chapter, a further investigation into the mathematical features of triple sequences will be made. Section 2 recalls some known definitions and theorems in summability theory. In Section 3, we study the concept of Musielak-Orlicz lacunary almost and strongly almost convergent triple sequences over n-normed spaces and introduce the notion of g_3-ideal convergence in a paranormed triple sequence spaces, where the base space is an n-normed space. In addition we investigate some topological and algebraic features of newly formed sequence spaces. In addition to these definitions, natural inclusion theorems shall also be presented.

Now we remind the n-normed space which was determined in [23] and some definitions on n-normed space (see [28]).

Definition 1. *Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$ (Here we allow d to be infinite). A real-valued function $\|.,...,.\|$ on X^n satisfying the subsequent four features*

(i) $\|x_1, x_2, ..., x_n\| = 0$ *iff* $x_1, x_2, ..., x_n$ *are linearly dependent;*

(ii) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation;*

(iii) $\|x_1, x_2, ..., x_{n-1}, \alpha x_n\| = |\alpha| \ \|x_1, x_2, ..., x_{n-1}, x_n\|$, *for any* $\alpha \in \mathbb{R}$*;*

(iv) $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$,

is called an n-norm on X and the pair $(X, \|.,...,.\|)$ is called an n-normed space.

It is well-known fact from the following corollary that n-normed spaces are actually normed spaces.

Corollary 1. *([23]) Every n-normed space is an $(n-r)$-normed space for all $r = 1, ..., n-1$. Especially, every n-normed space is a normed space.*

Example 1. *A standard example of an n-normed space is $X = \mathbb{R}^n$ equipped with the n-norm is*

$\|x_1, x_2, ..., x_{n-1}, x_n\| :=$ *the volume of the n-dimensional parallelepiped spanned by $x_1, x_2, ..., x_{n-1}, x_n$ in X.*

Observe that in any n-normed space $(X, \|., ..., .\|)$ we acquire

$$\|x_1, x_2, ..., x_{n-1}, x_n\| \geq 0$$

and

$$\|x_1, x_2, ..., x_{n-1}, x_n\| = \|x_1, x_2, ..., x_{n-1}, x_n + \alpha_1 x_1 + ... + \alpha_{n-1} x_{n-1}\|$$

for all $x_1, x_2, ..., x_n \in X$ and $\alpha_1, ..., \alpha_{n-1} \in \mathbb{R}$.

Definition 2. *A sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is said to convergent to an $l \in X$ if*

$$\lim_{k \to \infty} \|x_k - l, y_1, y_2, ..., y_{n-1}\| = 0$$

for every $y_1, y_2, ..., y_{n-1} \in X$.

Definition 3. *A sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is called a Cauchy sequence if*

$$\lim_{k,l \to \infty} \|x_k - x_l, y_1, y_2, ..., y_{n-1}\| = 0$$

for every $y_1, y_2, ..., y_{n-1} \in X$.

By the convergence of a triple sequence we mean the convergence in the Pringsheim sense, i.e. a triple sequence $x = (x_{ijk})$ has Pringsheim limit ξ (indicated by $P - \lim x = \xi$) provided that given $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $|x_{ijk} - \xi| < \varepsilon$ wherever $i, j, k > n$, (see [58]). We shall denote more briefly as $P-$convergent. The triple sequence $x = (x_{ijk})$ is bounded if there is $K > 0$ such that $|x_{ijk}| < K$ for all i, j and k.

Definition 4. *A subset K of $\mathbb{N}^3$ is called to have natural density $\delta_3(K)$ if*

$$\delta_3(K) = P - \lim_{n,k,l \to \infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars signify the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{pqr})$ is named to be statistically convergent to ξ in Pringsheim's sense provided that for every $\varepsilon > 0$,

$$\delta_3\left(\left\{(n, k, l) \in \mathbb{N}^3 : p \leq n, q \leq k, r \leq l, \ \ |x_{pqr} - \xi| \geq \varepsilon\right\}\right) = 0.$$

Statistical convergence was further generalized in the paper [36] utilizing the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is named a non-trivial ideal. A non-trivial ideal $\mathcal{I}$ is named admissible if $\mathcal{I}$ includes all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will signify an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be any ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$ named the filter connected with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Definition 5. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)$ be a real sequence. We say that the sequence x is $\mathcal{I}$-convergent to $\xi \in \mathbb{R}$ provided that for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\} \in \mathcal{I}$.*

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergent, see the references in [52].

Definition 6. *([59]) Let $\mathcal{I}_3$ be an admissible ideal on $\mathbb{N}^3$, then a triple sequence (x_{jkl}) is named to be $\mathcal{I}_3$-convergent to ξ in Pringsheim's sense if for every $\varepsilon > 0$,*

$$\left\{(j,k,l) \in \mathbb{N}^3 : |x_{jkl} - \xi| \geq \varepsilon\right\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3\text{-}\lim x_{jkl} = \xi$.

Remark 1. *(i) Let $\mathcal{I}_3(f)$ be the family of all finite subsets of $\mathbb{N}^3$. Then $\mathcal{I}_3(f)$ convergence coincides with the convergence of triple sequences in [58].*

(ii) Let $\mathcal{I}_3(\delta) = \left\{A \subset \mathbb{N}^3 : \delta_3(A) = 0\right\}$. Then $\mathcal{I}_3(\delta)$ convergence coincides with the statistical convergence in [58].

2. Main Results

Following the above definitions and results, we aim in this section to introduce some new notions of Orlicz lacunary convergent triple sequences and g_3-ideal convergence over n-normed spaces. In addition to these definition, also some topological and algebraic properties of newly formed sequence spaces have been established.

A triple sequence $x = (x_{ijk})$ of real numbers is called to be almost convergent to a limit ξ if

$$\lim_{p,q,r\to\infty} \sup_{\alpha,\beta,\gamma\geq 0} \left| \frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} (x_{ijk} - \xi) \right| = 0. \tag{1}$$

In this case, ξ is called the f_3-limit of x and the space of all almost convergent triple sequences is denoted by f_3,

$$f_3 = \left\{ x = (x_{ijk}) : \lim_{p,q,r\to\infty} |h_{pqr\alpha\beta\gamma}(x) - \xi| = 0, \text{ uniformly in } \alpha, \beta, \gamma \right\},$$

where

$$h_{pqr\alpha\beta\gamma}(x) = \frac{1}{(p+1)(q+1)(r+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} x_{i+\alpha, j+\beta, k+\gamma}.$$

The set of all strongly almost convergent triple sequences is denoted by $[f_3]$.

Let $\mathcal{M} = (M_{mno})$ be a Musielak-Orlicz function, $u = (u_{mno})$ be a triple sequence of positive real numbers and $p = (p_{mno})$ be a bounded sequence of positive real numbers. We indicate the space of all sequences defined over $(X, \|., ..., .\|)$ by $w(n-X)$. Now we itendify the following sequence spaces for some ρ and for every nonzero $y_1, y_2, ..., y_{n-1} \in X$:

$$[M, u, F, p, \|., ..., .\|] = \{x \in w(n-X) :$$
$$\lim_{p,q,r\to\infty} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno}\left(\left\|\frac{h_{pqr\alpha\beta\gamma}(x-\xi)}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right]^{p_{mno}} = 0,$$
$$\text{uniformly in } \alpha, \beta, \gamma \geq 1\}$$

and

$$[M, u, [F], p, \|., ..., .\|] = \{x \in w(n-X) :$$
$$\lim_{p,q,r\to\infty} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno}\left(h_{pqr\alpha\beta\gamma}\left(\left\|\frac{x-\xi}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right)\right]^{p_{mno}} = 0,$$
$$\text{uniformly in } \alpha, \beta, \gamma \geq 1\},$$

where $h_{pqr\alpha\beta\gamma}(x)$ is defined as in (1). We write $[\mathcal{M}, u, [F], p, \|., ..., .\|] - \lim x = \xi$. Also we have

$$[\mathcal{M}, u, [F], p, \|., ..., .\|] \subset [\mathcal{M}, u, F, p, \|., ..., .\|] \subset [M, u, \ell^{\infty}, p, \|., ..., .\|]$$

holds from the inequality:

$$\left\|\frac{h_{pqr\alpha\beta\gamma}(x-\xi)}{\rho}, y_1, ..., y_{n-1}\right\| = \left\|\frac{\frac{1}{(p+1)(q+1)(r+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} (x_{i+\alpha, j+\beta, k+\gamma} - \xi)}{\rho}, y_1, ..., y_{n-1}\right\|$$
$$\leq \frac{1}{(p+1)(q+1)(r+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} \left\|\frac{x_{i+\alpha, j+\beta, k+\gamma} - \xi}{\rho}, y_1, ..., y_{n-1}\right\| = h_{pqr\alpha\beta\gamma}\left(\left\|\frac{x-\xi}{\rho}, y_1, ..., y_{n-1}\right\|\right).$$

Furthermore, the triple sequence $\theta_3 = \theta_{v,\eta,\tau} = \{(m_v, n_\eta, o_\tau)\}$ is called triple lacunary sequence if there exist three increasing sequences of integers such that

$$m_0 = 0, \; h_v = m_v - m_{v-1} \to \infty \text{ as } v \to \infty,$$

$$n_0 = 0, \; h_\eta = n_\eta - n_{\eta-1} \to \infty \text{ as } \eta \to \infty,$$

and

$$o_0 = 0, \; h_\tau = o_\tau - o_{\tau-1} \to \infty \text{ as } \tau \to \infty.$$

Let $m_{v,\eta,\tau} = m_v n_\eta o_\tau$, $h_{v,\eta,\tau} = h_v h_\eta h_\tau$ and $I_{v,\eta,\tau}$ is determined as follows:

$$I_{v,\eta,\tau} = \{(m, n, o) : m_{v-1} < m \leq m_v, n_{\eta-1} < n \leq n_\eta \text{ and } o_{\tau-1} < o \leq o_\tau\},$$

$$s_v = \frac{m_v}{m_{v-1}}, s_\eta = \frac{n_\eta}{n_{\eta-1}},\ s_r = \frac{o_\tau}{o_{\tau-1}} \text{ and } s_{v,\eta,\tau} = s_v s_\eta s_\tau.$$

Let $D \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The number

$$\delta_3^{\theta_3}(D) = \lim_{v,\eta,\tau} \frac{1}{h_{v,\eta,\tau}} \left|\{(m,n,o) \in I_{v,\eta,\tau} : (m,n,o) \in D\}\right|$$

is named to be the θ_3-density of D, provided the limit exists.

The spaces of lacunary almost and strongly almost convergent triple sequences in n-normed spaces are identified as follows:

$$[M,u,F_\theta,p,\|.,...,.\|] = \{x \in (x_{ijk}) \in w(n-X) : \lim_{v,\eta,\tau\to\infty} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno}M_{mno}\left\|\frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} \left(\frac{x_{i+\alpha,j+\beta,k+\gamma}-\xi}{\rho}, y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} = 0, \text{ uniformly in } \alpha,\beta,\gamma \geq 1\}$$

and

$$[M,u,[F_\theta],p,\|.,...,.\|] = \{x \in (x_{ijk}) \in w(n-X) : \lim_{v,\eta,\tau\to\infty} \frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno}M_{mno}\left\|\left(\frac{x_{i+\alpha,j+\beta,k+\gamma}-\xi}{\rho}, y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} = 0, \text{ uniformly in } \alpha,\beta,\gamma \geq 1\},$$

where

$$F_\theta = \left\{ \begin{array}{c} x = (x_{ijk}) : \lim_{v,\eta,\tau\to\infty} \left\|\frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} (x_{i+\alpha,j+\beta,k+\gamma}-\xi), y_1,...,y_{n-1}\right\|, \\ \text{uniformly in } \alpha,\beta,\gamma \end{array} \right\}$$

and

$$[F_\theta] = \left\{ \begin{array}{c} x = (x_{ijk}) : \lim_{v,\eta,\tau\to\infty} \frac{1}{h_{v,\eta,\tau}} \sum_{i,j,k\in I_{v\eta\tau}} \|(x_{i+\alpha,j+\beta,k+\gamma}-\xi), y_1,...,y_{n-1}\|, \\ \text{uniformly in } \alpha,\beta,\gamma \end{array} \right\}.$$

Lemma 1. *For a given $\varepsilon > 0$ and let $x = (x_{ijk})$ be a strongly almost convergent triple sequence. Then there exist p_0, q_0, r_0, α_0, β_0 and γ_0 such that*

$$\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho}, y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}} < \varepsilon$$

for all $p_{mno} \geq 1$, $p \geq p_0$, $q \geq q_0$, $r \geq r_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$, for every nonzero $y_1,...,y_{n-1} \in X$. Then $x \in [\mathcal{M},u,[F],p,\|.,...,.\|]$.

Proof. Given $\varepsilon > 0$. Take p_0', q_0', r_0', α_0, β_0, γ_0 such that

$$\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho}, y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}} < \frac{\varepsilon}{2} \tag{2}$$

for all $p \geq p_0'$, $q \geq q_0'$, $r \geq r_0'$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$. Now we have to prove only that there are p_0'', q_0'', r_0'' such that for $p > p_0''$, $q > q_0''$, $r > r_0''$, $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$.

$$\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} < \varepsilon. \tag{3}$$

By selecting $p_0 = \max(p_0', p_0'')$, $q_0 = \max(q_0', q_0'')$ and $r_0 = \max(r_0', r_0'')$, (3) will holds for $p \geq p_0$, $q \geq q_0$, $r \geq r_0$ and $\forall \alpha, \beta, \gamma$. Take α_0, β_0, γ_0 be fixed,

$$\sum_{i=0}^{\alpha_0-1} \sum_{j=0}^{\beta_0-1} \sum_{k=0}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_1. \tag{4}$$

$$\sum_{i=\alpha}^{\alpha_0-1} \sum_{j=\beta}^{\beta_0-1} \sum_{k=\gamma_0}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_2. \tag{5}$$

$$\sum_{i=\alpha}^{\alpha_0-1} \sum_{j=\beta_0}^{\beta+q-1} \sum_{k=\gamma}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_3. \tag{6}$$

$$\sum_{i=\alpha}^{\alpha_0-1} \sum_{j=\beta_0}^{\beta+q-1} \sum_{k=\gamma_0}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_4. \tag{7}$$

$$\sum_{i=\alpha_0}^{\alpha+p-1} \sum_{j=\beta}^{\beta_0-1} \sum_{k=\gamma}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_5. \tag{8}$$

$$\sum_{i=\alpha_0}^{\alpha+p-1} \sum_{j=\beta}^{\beta_0-1} \sum_{k=\gamma_0}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_6. \tag{9}$$

$$\sum_{i=\alpha_0}^{\alpha+p-1} \sum_{j=\beta_0}^{\beta+q-1} \sum_{k=\gamma}^{\gamma_0-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k} - \xi}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} = K_7. \tag{10}$$

Now taking $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$ and $p > \alpha_0$, $q > \beta_0$, $r > \gamma_0$, we have from (4-10)

$$\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$= \frac{1}{pqr}\left(\sum_{i=\alpha}^{\alpha_0-1}+\sum_{i=\alpha_0}^{\alpha+p-1}\right)\left(\sum_{j=\beta}^{\beta_0-1}+\sum_{j=\beta_0}^{\beta+q-1}\right)\left(\sum_{k=\gamma}^{\gamma_0-1}+\sum_{k=\gamma_0}^{\gamma+r-1}\right)\times$$

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}+$$

$$+\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha_0+p-1}\sum_{j=\beta_0}^{\beta_0+q-1}\sum_{k=\gamma_0}^{\gamma_0+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}+\frac{\varepsilon}{2} \text{ from (2).}$$

Taking p, q, r sufficiently large, we can obtain

$$\frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}+\frac{\varepsilon}{2}<\varepsilon.$$

gives (3). This concludes the proof. □

Theorem 1. *Let* $p_{mno} \geq 1 \; \forall \; m, n, o$ *and for every* θ_3 *we have* $[\mathcal{M}, u, [F_\theta], p, \|.,...,.\|] = [\mathcal{M}, u, [F], p, \|.,...,.\|]$.

Proof. Let $\{x_{ijk}\} \in [\mathcal{M}, u, [F_\theta], p, \|.,...,.\|]$. Then for given $\varepsilon > 0$, there exist p_0, q_0, r_0 and ξ such that

$$\frac{1}{h_{\upsilon,\eta,\tau}}\sum_{i=\alpha}^{\alpha+h_\upsilon-1}\sum_{j=\beta}^{\beta+h_\eta-1}\sum_{k=\gamma}^{\gamma+h_\tau-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-L}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}<\varepsilon$$

for $\upsilon \geq \upsilon_0$, $\eta \geq \eta_0$, $\tau \geq \tau_0$ and $\alpha = U_{\upsilon-1}+1+a$, $\beta = V_{\eta-1}+1+a$, $\gamma = Z_{\tau-1}+1+a$, $a \geq 0$. Let $p \geq h_\upsilon$, $q \geq h_\eta$, $r \geq h_\tau$ write $p = ch_\upsilon + \theta$, $q = bh_\eta + \theta$, $r = dh_\tau + \theta$

where b, c, d are integers. Since b, c, $d \geq 1$. Now

$$\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \frac{1}{pqr}\sum_{i=\alpha}^{\alpha+(c+1)h_\upsilon-1}\sum_{j=\beta}^{\beta+(b+1)h_\eta-1}\sum_{j=\gamma}^{\gamma+(d+1)h_\tau-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$= \frac{1}{pqr}\sum_{u'=0}^{c}\sum_{i=\alpha+u'h_\upsilon}^{\alpha+\left(u'+1\right)h_\upsilon-1}\sum_{u'=0}^{b}\sum_{j=\beta+u'h_\eta}^{\beta+\left(u'+1\right)h_\eta-1}\sum_{u'=0}^{d}\sum_{j=\gamma+\upsilon'h_\tau}^{\gamma+\left(u'+1\right)h_\tau-1}\times$$

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}$$

$$\leq \frac{(d+1)(c+1)(b+1)}{pqr}h_\upsilon h_\eta h_\tau\varepsilon \leq \frac{4cbdh_\upsilon h_\eta h_\tau\varepsilon}{pqr}\ \ (d,c,b\geq 1).$$

For $\frac{h_\upsilon h_\eta h_\tau}{pqr} \leq 1$, since $\frac{cbdh_\upsilon h_\eta h_\tau}{pqr} \leq 1$

$$\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}} \leq 4\varepsilon.$$

Using Lemma 1, we get $[\mathcal{M},u,[F_\theta],p,\|.,...,.\|] \subseteq [\mathcal{M},u,[F],p,\|.,...,.\|]$. □

Lemma 2. *Assume for a given $\varepsilon > 0$ there exist p_0, q_0, r_0 and α_0, β_0, γ_0 such that*

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} < \varepsilon$$

for all $p \geq p_0$, $q \geq q_0$, $r \geq r_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$, for every nonzero $y_1,...,y_{n-1} \in X$ and for some $\rho > 0$. Then $x \in [\mathcal{M},F,u,p,\|.,...,.\|]$.

Proof. Assume $\varepsilon > 0$. Take p_0', q_0', r_0', α_0, β_0, γ_0 such that

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} < \frac{\varepsilon}{2} \tag{11}$$

for all $p \geq p_0'$, $q \geq q_0'$, $r \geq r_0'$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$. By Lemma 1, it is enought to denote that there exist p_0'', q_0'', r_0'' such that for $p \geq p_0''$, $q \geq q_0''$, $r \geq r_0''$, $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} < \varepsilon$$

Since α_0, β_0, γ_0 are fixed, let $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$ and $p > \alpha_0$, $q > \beta_0$, $r > \gamma_0$. According to (4-10), we obtain

$$
\begin{aligned}
&\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
\leq\ &\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta}^{\beta_0-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta}^{\beta_0-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha_0-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta}^{\beta_0-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma}^{\gamma_0-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \tfrac{K_1}{pqr}+\tfrac{K_2}{pqr}+\tfrac{K_3}{pqr}+\tfrac{K_4}{pqr}+\tfrac{K_5}{pqr}+\tfrac{K_6}{pqr}+\tfrac{K_7}{pqr}+\\
&+\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\tfrac{1}{pqr}\sum_{i=\alpha_0}^{\alpha+p-1}\sum_{j=\beta_0}^{\beta+q-1}\sum_{k=\gamma_0}^{\gamma+r-1}\left(\tfrac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}
\end{aligned}
\tag{12}
$$

Let $p-\alpha_0 > p_0'$, $q-\beta_0 > q_0'$, $r-\gamma_0 > r_0'$. Then, for $0 \leq \alpha \leq \alpha_0$, $0 \leq \beta \leq \beta_0$, $0 \leq \gamma \leq \gamma_0$, we get $p+\alpha-\alpha_0 \geq p_0'$, $q+\beta-\beta_0 \geq q_0'$, $r+\gamma-\gamma_0 \geq r_0'$. From (11), we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{(p+\alpha+\alpha_0)(q+\beta+\beta_0)(r+\gamma+\gamma_0)}\sum_{i=\alpha_0}^{\alpha_0+p+\alpha-\alpha_0}\sum_{j=\beta_0}^{\beta_0+q+\beta-\beta_0}\sum_{k=\gamma_0}^{\gamma_0+r+\gamma-\gamma_0}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\frac{\varepsilon}{2}. \tag{13}$$

From (12) and (13), we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}$$

$$\leq \frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}$$

$$+\frac{(p+\alpha-\alpha_0)(q+\beta-\beta_0)(r+\gamma-\gamma_0)}{pqr}\frac{\varepsilon}{2}$$

$$\leq \frac{K_1}{pqr}+\frac{K_2}{pqr}+\frac{K_3}{pqr}+\frac{K_4}{pqr}+\frac{K_5}{pqr}+\frac{K_6}{pqr}+\frac{K_7}{pqr}+\frac{\varepsilon}{2}<\varepsilon,$$

for sufficiently large p,q,r. □

Theorem 2. *(i) For every* θ,
$[\mathcal{M},u,F_\theta,p,\|.,...,.\|]\cap[\mathcal{M},u,l^\infty,p,\|.,...,.\|]=[\mathcal{M},u,F,p,\|.,...,.\|]$.
(ii) For every θ, $[\mathcal{M},u,F_\theta,p,\|.,...,.\|]\nsubseteq[\mathcal{M},u,l^\infty,p,\|.,...,.\|]$.

Proof. (i) Let $\{x_{ijk}\}\in[\mathcal{M},u,F_\theta,p,\|.,...,.\|]\cap[\mathcal{M},u,l^\infty,p,\|.,...,.\|]$ $\forall\varepsilon>0$ there exist υ_0, η_0 and τ_0 such that

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{h_{\upsilon,\eta,\tau}}\sum_{i=\alpha}^{\alpha+h_\upsilon-1}\sum_{j=\beta}^{\beta+h_\eta-1}\sum_{k=\gamma}^{\gamma+h_\tau-1}\left(\frac{x_{i,j,k}-L}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}<\frac{\varepsilon}{2} \tag{14}$$

for $\upsilon,\eta,\tau \geq \upsilon_0,\eta_0,\tau_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$, $\alpha = U_{\upsilon-1}+1+a$, $\beta = V_{\eta-1}+1+a$, $\gamma = Z_{\tau-1}+1+a$, $a \geq 0$. Let integers $p \geq h_\upsilon$, $q \geq h_\eta$, $r \geq h_\tau$, $b,c,d \geq 1$. Then

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}M_{mno}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}$$

$$\leq \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{\mu=0}^{c-1}\sum_{i=\alpha+\mu h_\upsilon}^{\alpha+(\mu+1)h_\upsilon-1}\sum_{\psi=0}^{b-1}\sum_{j=\beta+\psi h_\eta}^{\beta+(\psi+1)h_\eta-1}\sum_{\varphi=0}^{d-1}\sum_{k=\gamma+\varphi h_\tau}^{\gamma+(\varphi+1)h_\tau-1}\right.\right.$$

$$\left.\left.\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}+\frac{1}{pqr}$$

$$= \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\sum_{i=\alpha+ch_\upsilon}^{\alpha+p-1}\sum_{j=\beta+bh_\eta}^{\beta+q-1}\sum_{j=\gamma+dh_\tau}^{\gamma+r-1}\left\|\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}. \tag{15}$$

Since $\{x_{ijk}\} \in [\mathcal{M},u,l^\infty,p,\|.,...,.\|]$ for all i,j,k, we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} < K,$$

From (14) and (15), we have

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}}$$

$$\leq \quad \frac{1}{pqr}dcb.h_\upsilon h_\eta h_\tau\frac{\varepsilon}{2}+\frac{Kh_{\upsilon,\eta,\tau}}{pqr}$$

for $\frac{h_\upsilon h_\eta h_\tau}{pqr} \leq 1$, since $\frac{dcbh_\upsilon h_\eta h_\tau}{pqr} \leq 1$ and $\frac{Kh_{\upsilon,\eta,\tau}}{pqr}$ can be made less than $\frac{\varepsilon}{2}$, taking p, q, r sufficiently large so

$$\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left\|\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left(\frac{x_{i,j,k}-\xi}{\rho},y_1,...,y_{n-1}\right)\right\|\right]^{p_{mno}} < \varepsilon$$

for $\upsilon,\eta,\tau \geq \upsilon_0,\eta_0,\tau_0$, $\alpha \geq \alpha_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$.

Therefore, $[\mathcal{M},u,F_\theta,p,\|.,...,.\|] \cap [\mathcal{M},u,l^\infty,p,\|.,...,.\|] \subseteq [\mathcal{M},u,F,p,\|.,...,.\|]$.

(ii) Let $\{x_{ijk}\} = (-1)^{ijk}(ijk)^\psi$ where ψ is constant with $0<\psi<1$. Then

$$\sum_{i=\alpha}^{\alpha+h_\upsilon-1}\sum_{j=\beta}^{\beta+h_\eta-1}\sum_{j=\gamma}^{\gamma+h_\tau-1}x_{ijk}, \quad (\alpha,\ \beta,\ \gamma \geq 0)$$

Let $X=\mathbb{R}^n$. It is straight forward to verify that $\{x_{ijk}\} \in [\mathcal{M},u,F_\theta,p,\|.,...,.\|]$ with $\xi=0$. But $\{x_{ijk}\}$ is not bounded. $\square$

Theorem 3. *The sequence space* $[\mathcal{M}, u, [F], p, \|., ..., .\|]$ *is a linear topological space total paranormed by*

$$g_3(x) = \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1 \\ 0\neq y_1,\ldots,y_{n-1}\in X}} \left(\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} \right)$$

$$= \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1 \\ 0\neq y_1,\ldots,y_{n-1}\in X}} \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} u_{mno} M_{mno} \left[\left(h_{pqr\alpha\beta\gamma} \left(\left\| \frac{x_{i,j,k}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right) \right]^{p_{mno}}.$$

Proof. Obviously $g_3(x) = 0 \Leftrightarrow x = 0$, $g_3(x) = g_3(-x)$ and g_3 is subadditive. Let $(x^{(m)})$ in $[\mathcal{M}, u, [F], p, \|., ..., .\|]$ such that $g_3(x^{(m)} - x) \to 0$ as $m \to \infty$ and (v_{mno}) be any sequence of scalars such that $v_{mno} \to v$ as $m, n, o \to \infty$. Since

$$g_3\left(x^{(m)}\right) \leq g_3(x) + g_3\left(x^{(m)} - x\right)$$

holds by subadditivity of $g_3, g_3\left(x^{(m)}\right)$ is bounded. Thus, we acquire

$$\begin{aligned}
& g_3\left(v_{mno}x^{(m)} - \nu x\right) \\
= & \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1 \\ 0\neq y_1,\ldots,y_{n-1}\in X}} \Big(\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \\
& \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{v_{mno}x_{ijk}^{(m)} - v x_{ijk}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} \Big) \\
\leq & |v_{mno} - v| \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1 \\ 0\neq y_1,\ldots,y_{n-1}\in X}} \Big(\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \\
& \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k}^{(m)}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} \Big) \\
& + |v| \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1 \\ 0\neq y_1,\ldots,y_{n-1}\in X}} \Big(\frac{1}{pqr} \sum_{i=\alpha}^{\alpha+p-1} \sum_{j=\beta}^{\beta+q-1} \sum_{k=\gamma}^{\gamma+r-1} \\
& \sum_{m,n,o=1,1,1}^{\infty,\infty,\infty} \left[u_{mno} M_{mno} \left(\left\| \frac{x_{i,j,k}^{(m)} - x_{ijk}}{\rho}, y_1, ..., y_{n-1} \right\| \right) \right]^{p_{mno}} \Big) \\
= & |v_{mno} - v| \, g_3\left(x^{(m)}\right) + |v| \, g_3\left(x^{(m)} - x\right) \to 0
\end{aligned}$$

as $m, n, o \to \infty$. This concludes the proof. □

Definition 7. *A triple sequence $x=(x_{ijk})$ is named to be strongly p-Cesàro $\mathcal{I}$-summable $(0<p<\infty)$ to a limit ξ in $([\mathcal{M},[F],u,,p,\|.,...,.\|],g_3)$ if*

$$\left\{(i,j,k)\in\mathbb{N}^3:\frac{1}{mno}\sum_{i,j,k=1,1,1}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p\geq\varepsilon\right\}\in\mathcal{I}_3$$

for every $\varepsilon>0$ and we write it as $x_{ijk}\to\xi\left[C,g_3\left(\mathcal{I}\right)\right]_p$.

Definition 8. *A triple sequence $x=(x_{ijk})$ is said to be g_3-ideal convergent to a number L in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$ if for each $\varepsilon>0$*

$$\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-\xi e\right)\geq\varepsilon\right\}\in\mathcal{I}_3,$$

where

$$\begin{aligned}&g_3\left(x_{ijk}-\xi e\right)\\=&\sup_{\substack{p,q,r\geq1,\ \alpha,\beta,\gamma\geq1\\0\neq y_1,...,y_{n-1}\in X}}\left(\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\right.\\&\left.\sum_{m,n,o=1,1,1}^{\infty,\infty,\infty}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{ijk}-\xi e}{\rho},y_1,...,y_{n-1}\right\|\right)\right]^{p_{mno}}\right.\end{aligned}$$

By $\mathcal{I}^3_{([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)}$ we denote set of all $g_3(\mathcal{I})$-convergent sequences in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$.

Definition 9. *A triple sequence $x=(x_{ijk})$ is said to be g_3-ideal Cauchy in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$ if for every $\varepsilon>0$ there exist three numbers $P=P(\varepsilon)$, $Q=Q(\varepsilon)$, $R=R(\varepsilon)$ such that*

$$\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-x_{PQR}\right)\geq\varepsilon\right\}\in\mathcal{I}_3.$$

Theorem 4. *If a triple sequence is ideal convergent in $([\mathcal{M},u,[F],p,\|.,...,.\|],g_3)$, then its limit is unique.*

Proof. For given $\varepsilon>0$ we define sets as:

$$K_1\left(\varepsilon\right)=\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-\xi_1\right)\geq\frac{\varepsilon}{2}\right\}$$

and

$$K_2\left(\varepsilon\right)=\left\{(i,j,k)\in\mathbb{N}^3:g_3\left(x_{ijk}-\xi_2\right)\geq\frac{\varepsilon}{2}\right\}.$$

and suppose $g_3(\mathcal{I})-\lim x=\xi_1$ and $g_3(\mathcal{I})-\lim x=\xi_2$. Since $g_3(\mathcal{I})-\lim x=\xi_1$, we have $K_1(\varepsilon)\in\mathcal{I}_3$. Similarly, $g_3(\mathcal{I})-\lim x=\xi_2$ we have $K_2(\varepsilon)\in\mathcal{I}_3$. Now let $K(\varepsilon)=K_1(\varepsilon)\cup K_2(\varepsilon)$. Then $K(\varepsilon)\in\mathcal{I}_3$ and therefore, $K^c(\varepsilon)$ is a non-empty set and $K^c(\varepsilon)\in\mathcal{F}\left(\mathcal{I}_3\right)$. If $(i,j,k)\in(\mathbb{N}\times\mathbb{N}\times\mathbb{N})\setminus K\left(\varepsilon\right)$, then we have

$$g_3\left(\xi_1-\xi_2\right)\leq g_3\left(x_{ijk}-\xi_1\right)+g_3\left(x_{ijk}-\xi_2\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $g_3(\xi_1 - \xi_2) = 0$ and hence $\xi_1 = \xi_2$. □

Theorem 5. *Let $x = (x_{ijk}) \in ([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$ be ideal convergent to ξ iff there exists a set*

$$K = \{(i_p, j_q, k_r) \in \mathbb{N}^3 : i_1 < i_2 < ... < i_p < ...,\ j_1 < j_2 < ... < j_q < ...,\ k_1 < k_2 < ... < k_r < ...\}$$

with $K \in \mathcal{F}(\mathcal{I}_3)$ such that $g_3(x_{i_p j_q k_r} - \xi) \to 0$ as $i_p, j_q, k_r \to \infty$.

Proof. Let $g_3(\mathcal{I}) - \lim x = \xi$. Now write for $v = 1, 2, ...$

$$K_s(\varepsilon) = \left\{(p, q, r) \in \mathbb{N}^3 : g_3\left(x_{i_p j_q k_r} - \xi\right) \le 1 + \frac{1}{v}\right\}$$

and

$$M_v(\varepsilon) = \left\{(p, q, r) \in \mathbb{N}^3 : g_3\left(x_{i_p j_q k_r} - \xi\right) > \frac{1}{v}\right\}.$$

Then $K_s \in \mathcal{I}_3$. Also

$$M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ... \tag{16}$$

and

$$M_v \in \mathcal{F}(\mathcal{I}_3),\ v = 1, 2, ... \tag{17}$$

As we know $\{x_{i_p j_q k_r}\}$ is g_3-convergent to ξ. Assume $\{x_{ijk}\}$ is not g_3-convergent to ξ. Therefore, there is $\varepsilon > 0$ such that $g_3(x_{i_p j_q k_r} - \xi) \le \varepsilon$ for infinitely many terms.

Let

$$M_\varepsilon = \left\{(p, q, r) \in \mathbb{N}^3 : g_3(x_{i_p j_q k_r} - \xi) > \varepsilon\right\}$$

and $\varepsilon > \frac{1}{v}$, $(v = 1, 2, ...)$. Then $M_\varepsilon \in \mathcal{I}_3$ and by (16) $M_v \subset M_\varepsilon$. Hence, $M_v \in \mathcal{I}_3$ which contradicts (17) and we get that $\{x_{ijk}\}$ is g_3-convergent to ξ.

Conversely, suppose that there exists a subset

$$K = \left\{(i_p, j_q, k_r) \in \mathbb{N}^3 : i_1 < ... < i_p < ...,\ j_1 < ... < j_q < ...,\ k_1 < ... < k_r < ...\right\}$$

with $K \in \mathcal{F}(\mathcal{I}_3)$ such that $g_3\text{-}\lim_{p,q,r\to\infty} x_{i_p j_q k_r} = \xi$ then there exists an $N(\varepsilon)$ such that

$$g_3(x_{ijk} - \xi) < \varepsilon \text{ for } i, j, k > N.$$

Let

$$K_\varepsilon = \left\{(i, j, k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) \ge \varepsilon\right\}$$

and

$$K' = \{(i_{N+1}, j_{N+1}, k_{N+1}), (i_{N+2}, j_{N+2}, k_{N+2}), ...\}.$$

Then $K' \in \mathcal{F}(\mathcal{I}_3)$ and $K_\varepsilon \subseteq (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \diagdown K'$ which implise that $K_\varepsilon \in \mathcal{I}_3$. Hence $g_3(\mathcal{I}) - \lim x = \xi$. □

Theorem 6. *Let $g_3(\mathcal{I}) - \lim x = \xi_1$ and $g_3(\mathcal{I}) - \lim y = \xi_2$. Then*
(i) $g_3(\mathcal{I}) - \lim(x \pm y) = \xi_1 \pm \xi_2$
(ii) $g_3(\mathcal{I}) - \lim(\alpha x) = \alpha\xi_1$, $\alpha \in \mathbb{R}$.

Proof. It is easy to prove, so we omit it. □

Theorem 7. *If* $0 < p < \infty$ *and* $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]_p$, *then* (x_{ijk}) *is* g_3*-ideal convergent to* ξ *in* $([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$.

Proof. Let $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]$. Then we have

$$\begin{aligned}\frac{1}{mno}\sum_{i,j,k=1,1,1}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p &\geq \frac{1}{mno}\sum_{\substack{i,j,k=1\\ (g_3(x_{ijk}-\xi e))\geq\varepsilon}}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p\\ &\geq \frac{\varepsilon^p}{mno}\left|\left\{(i,j,k)\in\mathbb{N}^3 : g_3\left(x_{ijk}-\xi e\right)\geq\varepsilon\right\}\right|\end{aligned}$$

and

$$\frac{1}{\varepsilon^p mno}\sum_{i,j,k=1,1,1}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p \geq \frac{1}{mno}\left|\left\{(i,j,k)\in\mathbb{N}^3 : g_3\left(x_{ijk}-\xi e\right)\geq\varepsilon\right\}\right|$$

That is

$$\left\{(i,j,k)\in\mathbb{N}^3 : g_3\left(x_{ijk}-\xi e\right)\geq\varepsilon\right\}\in\mathcal{I}_3.$$

Hence (x_{ijk}) is g_3-ideal convergent to ξ in $([\mathcal{M}, u, [F], p, \|., ..., .\|], g_3)$. □

Theorem 8. *If* $x = (x_{ijk})$ *is* $g_3(\mathcal{I})$*-convergent to* ξ *in* $([\mathcal{M}, [F], u, p, \|., ..., .\|], g_3)$ *then* $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]_p$.

Proof. Suppose that $x = (x_{ijk})$ is g_3-ideal convergent to ξ in $([\mathcal{M}, [F], u, p, \|., ..., .\|], g_3)$. Then for $\varepsilon > 0$, we have $K_\varepsilon \in \mathcal{I}_3$, where $K_\varepsilon = \left\{(i,j,k)\in\mathbb{N}^3 : g_3\left(x_{ijk}-\xi e\right)\geq\varepsilon\right\}$. Since $x = (x_{ijk}) \in [\mathcal{M}, u, l^\infty, p, \|., ..., .\|]$, then there exists $K > 0$ such that

$$\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi e}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right]^{p_{mno}} \leq K$$

for all i, j, k. Thus,

$$g_3\left(x_{ijk}-\xi e\right) = \sup_{\substack{p,q,r\geq 1,\ \alpha,\beta,\gamma\geq 1\\ 0\neq y_1,...,y_{n-1}\in X}}\left(\frac{1}{pqr}\sum_{i=\alpha}^{\alpha+p-1}\sum_{j=\beta}^{\beta+q-1}\sum_{k=\gamma}^{\gamma+r-1}\left[u_{mno}M_{mno}\left(\left\|\frac{x_{i,j,k}-\xi e}{\rho}, y_1, ..., y_{n-1}\right\|\right)\right]^{p_{mno}}\right) \leq K.$$

Hence

$$\begin{aligned}\frac{1}{mno}\sum_{i,j,k=1,1,1}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p &= \frac{1}{mno}\sum_{\substack{i,j,k=1,1,1\\ i,j,k\notin K_\varepsilon}}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p\\ &+\frac{1}{mno}\sum_{\substack{i,j,k=1,1,1\\ i,j,k\in K_\varepsilon}}^{m,n,o}\left(g_3\left(x_{ijk}-\xi e\right)\right)^p \leq \varepsilon^p + \frac{K^p}{mno}\left|K_\varepsilon\right|.\end{aligned}$$

Then the set K_ε on the right hand side of above inequality belongs to $\mathcal{I}_3$. Therefore, $x_{ijk} \to \xi\left[C, g_3(\mathcal{I})\right]_p$. Hence the proof is concluded. □

Let X and Y be two triple sequence spaces. We use the notation $X_{reg} \subset Y_{reg}$ to mean if the triple sequence x converges to a limit ξ in X then the sequence x converges to the same limit in Y.

Theorem 9. $\left(\mathcal{I}^3_{([\mathcal{M},u,[F],p,\|.,\dots,.\|],g_3)}\right)_{reg} = \left([C, g_3(\mathcal{I})]_p\right)_{reg}.$

Proof. By combining Theorem (7) and Theorem (8) we have the proof. □

Theorem 10. *Let a complete paranormed space be* $([\mathcal{M}, u, [F], p, \|., \dots, .\|], g_3)$. *Then a sequence in* $[\mathcal{M}, u, [F], p, \|., \dots, .\|]$ *is* g_3*-ideal convergent iff it is* g_3*-ideal Cauchy.*

Proof. Let $g_3(\mathcal{I}_3) - \lim x = \xi$. Then, we get $X(\varepsilon) \in \mathcal{I}_3$, where

$$X(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) \geq \frac{\varepsilon}{2}\right\}.$$

This implies

$$X^c(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}_3).$$

Let $m, n, o \in X^c(\varepsilon)$. Then $g_3(x_{mno} - \xi) < \frac{\varepsilon}{2}$. Now let

$$Y(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) \geq \varepsilon\right\}.$$

We need to demonstrate that $Y(\varepsilon) \subset X(\varepsilon)$. Let $(i,j,k) \in Y(\varepsilon)$. Then $g_3(x_{mno} - x_{ijk}) \geq \varepsilon$ and therefore $g_3(x_{ijk} - \xi) \geq \varepsilon$, that is $(i,j,k) \in X(\varepsilon)$. Otherwise, if $g_3(x_{ijk} - \xi) < \varepsilon$ then

$$\varepsilon \leq g_3(x_{ijk} - x_{mno}) \leq g_3(x_{ijk} - \xi) + g_3(x_{mno} - \xi) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which is not possible. Thus, $Y(\varepsilon) \subset X(\varepsilon)$ and therefore, $x = (x_{ijk})$ is g_3-ideal convergent sequences.

Conversely, let $x = (x_{ijk})$ is g_3-ideal Cauchy but not g_3-ideal convergent sequences. Then there exist $(t', w', v') \in \mathbb{N}^3$ such that

$$D(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - x_{t'w'v'}) \geq \varepsilon\right\} \in \mathcal{I}_3$$

and $G(\varepsilon) \in \mathcal{I}_3$, where

$$G(\varepsilon) = \left\{(i,j,k) \in \mathbb{N}^3 : g_3(x_{ijk} - \xi) < \frac{\varepsilon}{2}\right\},$$

that is, $G^c(\varepsilon) \in \mathcal{F}(\mathcal{I}_3)$, since $g_3(x_{ijk} - x_{mno}) \leq 2g_3(x_{ijk} - \xi) < \varepsilon$. If $g_3(x_{ijk} - \xi) < \frac{\varepsilon}{2}$ then $D^c(\varepsilon) \in \mathcal{I}_3$, that is, $D(\varepsilon) \in \mathcal{F}(\mathcal{I}_3)$ which leads to a contradiction. Hence, the result is obtained. □

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References

[1] Alotaibi, A., Alroqi, A. M., Statistical convergence in a paranormed space, *J. Ineq. Appl.*, 2012(1) (2012), 1-6. https://doi.org/10.1186/1029-242X-2012-39

[2] Başar, F., Summability Theory and Its Applications, Bentham Science Publishers, İstanbul, 2012. https://doi.org/10.2174/97816080545231120101

[3] Başarır, M., On some new sequence spaces, *Riv. Math. Univ. Parma.*, 51 (1992), 339-347.

[4] Başarır, M., Konca, Ş., Kara, E. E., Some generalized difference statistically convergent sequence spaces in 2-normed space, *J. Ineq. Appl.*, 2013(177) (2013), 1-12. https://doi.org/10.1186/1029-242X-2013-177

[5] Belen, C., Mohiuddine, S. A., Generalized weighted statistical convergence and application, *Appl. Math. Comput.*, 219 (2013), 9821-9826. https://doi.org/10.1016/j.amc.2013.03.115

[6] Bromwich, T. J., An Introduction to the Theory of Infinite Series, Macmillan and Co. Ltd., New York, 1965.

[7] Connor, J. S., The statistical and strong p-Cesaro convergence of sequences, *Analysis*, 8 (1988), 47-63. https://doi.org/10.1524/anly.1988.8.12.47

[8] Das, G., Mishra, S. K., Banach limits and lacunary strong almost convergence, *J. Orissa Math. Soc.*, 2(2) (1983), 61-70.

[9] Das, G., Patel, B. K., Lacunary distribution of sequences, *Indian J. Pure Appl. Math.*, 20(1) (1989), 64-74.

[10] Das, G., Sahoo, S. K., On some sequence spaces, *J. Math. Anal. Appl.*, 164 (1992), 381-398.

[11] Das, B., Tripathy, B. C., Debnath, P., Bhattacharya, B., Statistical convergence of complex uncertain triple sequence, *Comm. Statist. Theory Methods*, in press. https://doi.org/10.1080/03610926.2020.1871016

[12] Das, B., Tripathy, B. C., Debnath, P., Bhattacharya, B., Almost convergence of complex uncertain double sequences, *Filomat*, 35(1) (2021), 61–78. https://doi.org/10.2298/FIL2101061D

[13] Das, B., Tripathy, B. C., Debnath, P., Nath, J., Bhattacharya, B., Almost convergence of complex uncertain triple sequences, *Proc. Nat. Acad. Sci. India Sect. A*, 91(2) (2021), 245-256. https://doi.org/10.1007/s40010-020-00721-w

[14] Das, B., Tripathy, B. C., Debnath, P., Bhattacharya, B., Characterization of statistical convergence of complex uncertain double sequence, *Anal. Math. Phys.*, 10(4) (2020), 1-20. https://doi.org/10.1007/s13324-020-00419-7

[15] Das, B., Tripathy, B. C., Debnath, P., Bhattacharya, B., Study of matrix transformation of uniformly almost surely convergent complex uncertain sequences, *Filomat*, 34(14) (2021), 4907-4922. https://doi.org/10.2298/FIL2014907D

[16] Duran, J. P., Infinite matrices and almost convergence, *Math. Zeit.*, 128 (1972), 75-83.

[17] Fast, H., Sur la convergence statistique, *Colloq. Math.*, 2 (1951), 241-244.

[18] Fridy, J. A., On statistical convergence, *Analysis*, 5 (1985), 301-313. https://doi.org/10.1524/anly.1985.5.4.301

[19] Fridy, J. A., Orhan, C., Lacunary statistical convergence, *Pacific J. Math.*, 160 (1993), 43-51.

[20] Gähler, S., 2-metrische Räume und ihre topologische Struktur, *Math. Nachr.*, 26 (1963), 115-148. https://doi.org/10.1002/mana.19630260109

[21] Gähler, S., Linear 2-normietre rume, *Math. Nachr.*, 28 (1965), 1-43. https://doi.org/10.1002/mana.19640280102

[22] Gunawan, H., On n-inner product, n-norms and the Cauchy-Schwartz inequality, *Sci. Math. Jpn.*, 5 (2001), 47-54.

[23] Gunawan, H., Mashadi, M., On n-normed spaces, *Int. J. Math. Sci.,* 27(10) (2001), 631-639. https://doi.org/10.1155/s0161171201010675

[24] Gürdal, M., Şahiner, A., Statistical approximation with a sequence of 2-Banach spaces, *Math. Comput. Modelling*, 55(3-4) (2012), 471-479. https://doi.org/10.1016/j.mcm.2011.08.026

[25] Gürdal, M., Şahiner, A., Açık, I., Approximation Theory in 2-Banach spaces, *Nonlinear Anal.*, 71(5-6) (2009), 1654-1661.

[26] Gürdal, M., Sarı, N., Savaş, E., A-statistically localized sequences in n-normed spaces, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 69(2) (2020), 1484-1497. https://doi.org/10.31801/cfsuasmas.704446

[27] Gürdal, M., Pehlivan, S., The statistical convergence in 2-normed spaces, *Southeast Asian Bull. Math.*, 33(2) (2009), 257-264.

[28] Gürdal, M., Pehlivan, S., The statistical convergence in 2-banach spaces, *Thai. J. Math.,* 2(1) (2004), 107-113.

[29] Hardy, G. H., On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, 19 (1917) 86-95. https://doi.org/10.1112/plms/s2-1.1.124

[30] Hazarika, B., Alotaibi, A., Mohiudine, S. A., Statistical convergence in measure for double sequences of fuzzy-valued functions, *Soft Comput.*, 24(9) (2020), 6613-6622. https://doi.org/10.1007/s00500-020-04805-y

[31] Kadak, U., Mohiuddine, S. A., Generalized statistically almost convergence based on the difference operator which includes the (p, q)-Gamma function and related approximation theorems, *Results Math.*, 73(9) (2018), 1-31. https://doi.org/10.1007/s00025-018-0789-6

[32] King, J. P., Almost summable sequences, *Proc. Amer. Math. Soc.*, 17 (1966), 1219-1225. https://doi.org/10.1090/S0002-9939-1966-0201872-6

[33] Konca, S., Başarir, M., Almost convergent sequences in 2-normed space and g-statistical convergence, *J. Math. Anal.*, 4 (2013), 32-39.

[34] Konca, Ş., Başarır, M., Generalized difference sequence spaces associated with a multiplier sequence on a real n-normed space, *J. Ineq. Appl.*, 2013(335) (2013), 1-18. https://doi.org/10.1186/1029-242X-2013-335

[35] Konca, Ş., Idris, M., Gunawan, H., A new 2-inner product on the space of p-summable sequences, *J. Egyptian Math. Soc.,* 24 (2016), 244-249. https://doi.org/10.1016/j.joems.2015.07.001

[36] Kostyrko, P., Macaj, M., Šalát, T., $\mathcal{I}$-convergence, *Real Anal. Exchange*, 26(2) (2000), 669-686.

[37] Lindenstrauss, J., Tzafriri, L., On Orlicz sequence spaces, *Israel J. Math.*, 10 (1971), 379-390. https://doi.org/10.1007/BF02771656

[38] Lorentz, G. G., A contribution to the theory of divergent sequences, *Acta Math.*, 80 (1948), 167-190. https://doi.org/10.1007/BF02393648

[39] Maddox, I. J., A new type of convergence, *Math. Proc. Camb. Phil. Soc.*, 83 (1978), 61-64.

[40] Maddox, I. J., On strong almost convergence, *Math. Proc. Phil. Soc.*, 85 (1979), 345-350. https://doi.org/10.1017/S0305004100055766

[41] Misiak, A., n-inner product spaces, *Math. Nachr.*, 140 (1989), 299-319. https://doi.org/10.1002/mana.19891400121

[42] Mohiuddine, S. A., Alamri, B. A. S., Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math.*, 113(3) (2019), 1955-1973. https://doi.org/10.1007/s13398-018-0591-z

[43] Mohiuddine, S. A., Asiri, A., Hazarika, B., Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, *Int. J. Gen. Syst.*, 48(5) (2019), 492-506. https://doi.org/10.1080/03081079.2019.1608985

[44] Mohiuddine, S. A., Hazarika, B., Alghamdi, M. A., Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems, *Filomat*, 33(14) (2019), 4549-4560. https://doi.org/10.2298/FIL1914549M

[45] Mohiuddine, S. A., Şevli, H., Cancan, M., Statistical convergence in fuzzy 2-normed space, *J. Comput. Anal. Appl.*, 12(4) (2010), 787-798. https://doi.org/10.2298/FIL1204673M

[46] Moricz, F., Rhoades, B. E., Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Cambridge Philos. Soc.*, 104 (1988), 283-294.

[47] Mursaleen, M., Karakaya, V., Erturk, M., Gursoy, F., Weighted statistical convergence and its application to Korovkin type approximation theorem, *Appl. Math. Comput.*, 218 (2012), 9132-9137. https://doi.org/10.1016/j.amc.2012.02.068

[48] Mursaleen, M., Elements of Metric Spaces, Anamaya Publ., New Delhi, ISBN 81-88342-42-4, 2005.

[49] Mursaleen, M., Başar, F., Sequence Spaces: Topics in Modern Summability Theory, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton London New York, ISBN 9780367819170, 2020.

[50] Mursaleen, M., Edely, O. H., Statistical convergence of double sequences, *J. Math. Anal. Appl.*, 288 (2003), 223-231. https://doi.org/10.1016/j.jmaa.2003.08.004

[51] Mursaleen, M., Generalized spaces of difference sequences, *J. Math. Anal. Appl.*, 203 (1996), 738-745. https://doi.org/10.1006/jmaa.1996.0409

[52] Nabiev, A., Pehlivan, S., Gürdal, M., On $\mathcal{I}$-Cauchy sequences, *Taiwanese J. Math.*, 11(2) (2007), 569-576.

[53] Nath, J., Tripathy, B. C., Das, B., Bhattacharya, B., On strongly almost λ-convergence and statistically almost λ-convergence in the environment of uncertainty, *Int. J. Gen. Syst.*, in press. https://doi.org/10.1080/03081079.2021.1998032

[54] Parasher, S. D., Choudhary, B., Sequence spaces defined by Orlicz function, *Indian J. Pure Appl. Math.*, 25 (1994), 419-428.

[55] Patterson, R. F., Savaş, E., Lacunary statistical convergence of double sequences, *Math. Commun.*, 10 (2005), 55-61. https://doi.org/10.1186/1029-242X-2014-480

[56] Raj, K., Sharma, S. K., Applications of double lacunary sequences to n-norm, *Acta Univ. Sapientiae Mathematica*, 7 (2015), 67-88. https://doi.org/10.1515/ausm-2015-0005

[57] Schaefer, P., Infinite matrices and invariant means, *Proc. Amer. Math. Soc.*, 36 (1972), 104-110.

[58] Şahiner, A., Gürdal, M., Düden, F. K., Triple sequences and their statistical convergence, *Selçuk J. Appl. Math.*, 8(2) (2007), 49-55.

[59] Şahiner, A., Tripathy, B. C., Some $\mathcal{I}$-related properties of triple sequences, *Selçuk J. Appl. Math.*, 9(2) (2008), 9-18.

[60] Vulich, B., On a generalized notion of convergence in a Banach space, *Ann. Math.*, 38(1) (1937), 156-174. https://doi.org/10.2307/1968517

[61] Zeltser, M., Investigation of Double Sequence Spaces by Soft and Hard Analytical Methods, Diss. Math. Univ. Tartu, 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

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POISSON AND NEGATIVE BINOMIAL REGRESSION MODELS FOR ZERO-INFLATED DATA: AN EXPERIMENTAL STUDY

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Abstract. Count data regression has been widely used in various disciplines, particularly health area. Classical models like Poisson and negative binomial regression may not provide reasonable performance in the presence of excessive zeros and overdispersion problems. Zero-inflated and Hurdle variants of these models can be a remedy for dealing with these problems. As well as zero-inflated and Hurdle models, alternatives based on some biased estimators like ridge and Liu may improve the performance against to multicollinearity problem except excessive zeros and overdispersion. In this study, ten different regression models including classical Poisson and negative binomial regression with their variants based on zero-inflated, Hurdle, ridge and Liu approaches have been compared by using a health data. Some criteria including Akaike information criterion, log-likelihood value, mean squared error and mean absolute error have been used to investigate the performance of models. The results show that the zero-inflated negative binomial regression model provides the best fit for the data. The final model estimations have been obtained via this model and interpreted in detail. Finally, the experimental results suggested that models except the classical models should be considered as powerful alternatives for modelling count and give better insights to the researchers in applying statistics on working similar data structures.

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1. Introduction

From past to present, due to it's interpretability and simplicity abilities, regression analysis have been widely used in various fields. Particularly, it has been more attractive in machine learning field as a result of processing increasing data in this information era. The aim of a classical regression is to establish a mathematical model between a response variable and a group of explanatory variables to get some insights and some inferences. The structure of response variable is critical to determine an appropriate regression model. Although a continuous response variables is the most common one in regression models, there has been great interest to develop models on discrete variables. Count data regression is an effective way for this purpose. When the response variable is the count of some occurrences of an event and non-negative valued, count data regression models can produce convincing results. Classical regression models are based on some assumptions like normality, linearity and homoskedasticity. These assumptions are adversely affected in the existence of count response variable. In a such model's results may be to biased, unstable and poor on generalization.

In order to more suitable models for count data, some alternative models based on poisson and negative binomial distributions have been proposed. Poisson regression (PR) and negative binomial regression (NBR) models have been used with great interest in many areas like medicine, biostatistics, biology, finance, demography, astronomy, business and management, earth sciences, communication and insurance [1,2]. The underlying remarks of attention to count data regression can be given as follows: (1) interpretability, (2) easy-implementation, (3)good performance and (4) wide application area. However, it is not rare to face excessive zero values meaning no-occurrence in the variable of interest. In such situation, classical PRR and NBR models may not be sufficiently enough for count data modelling. For example, the number of cigarettes smoked daily or the number of weekly deaths due to the cancer in a hospital may be zero. The excess zeros cause a problem calles overdispersion which has the effect on increasing the sample variance [3]. Due to the overdispersion,the conditional variance could be bigger than conditional mean unlike the assumption based on the equality of them. That's why, the effect of overdispersion over classical poisson regression is more severe than negative binomial regression. Even NBR can be used to a certain extent in the existence of excessive zeros, zero inflated poisson regression (ZPR), zero inflated negative binomial regression (ZNBR), poisson Hurdle regression (PHR) and negative binomial Hurdle regression (NBHR) models have been proposed to deal with overdispersion problem. On the other side, a phenomenon call multicollinearity corresponding high correlations between two or more explanatory variables is another common problem in real word applications. In count data regression models, multicollinearity affected the significance of each variable and the stability performances. As a solution to multicollinearity in classical linear regression models, ridge regression estimator and Liu estimator were proposed by Hoerl and Kennard [4] and Liu [5], respectively. The superiorities of ridge estimator have been extended to count data regression via poisson ridge regression (PRR) and negative binomial ridge regression (NBRR) models which were proposed by Månsson and Shukur [6] and Månsson [7], respectively. Similarly, poisson Liu regression (PLR) and negative binomial Liu regression (NBLR) models based on Liu estimator were developed by Månsson and Kibria [8] and Månsson [9], respectively.

Both classical poisson and negative binomial regression models and the corresponding zero-inflated versions have been extensively used in real word applications. Wang and Famoye [10] were used the generalized poisson regression (GPR) and classical poisson and negative binomial regression models to investigate household fertility decisions. Famoye and Singh [11] proposed zero-inflated generalized poisson regression (ZGPR) model to make predictions on domestic violence data set and found this model adequate over its competitors like PR, GPR, ZPR and ZNBR. Bandyopadhyay et al. [12] used Hurdle and zero-inflated regression models on drug addiction data set. Similarly, Buu et al. [13] developed a new variable selection method for ZPR on a data set related with alcohol addiction. Mouatassim and Ezzahid [14] applied PR and ZPR models to make estimation on private health insurance data set. Xie et al. [15] made a comparison between zero-inflated models and NBR model by using a smoking data set and found that NBR model could provide convincing estimations for predicting zero-excessive data. Liyanage et al. [16] used PR model to estimate the prevalence of end-stage kidney disease and worldwide use of renal replacement therapy (RRT) and made some projections about the needs to 2030. Martinez et al. [17] compared PR and NBR models performance on a data set related with severe chronic obstructive pulmonary disease. Oliveira et al. [18] presented a comparative study on the usage of ZPR and ZNBR models for radiation-induced chromosome aberration data and developed a score test for ZPR model. Tang et al. [19] studied on zero-and-one-inflated poisson models, proposed a sampling approach with a simulation study and evaluated the performance via two real data sets. Chai et al. [20] used ZNBR model to estimate ship sinking accident mortalities. This study investigated the effective factors on medical visit and the performance of regression models on it's estimation. Ten different regression models including PR, NBR, ZPR, ZNBR, PHR, NBHR, PRR, PLR, NBRR and NBLR have been considered in evaluation. The data set is obtained from Deb and Trivedi's study [21].

The rest of this study is organized as follows. The background methodology is reviewed in Section 2. The appropriate selection methods of ridge and Liu biasing parameters are presented in Section 3. In Section 4, some information about data set are described. Performance evaluation is given in Section 5. Some conclusion and discussions are summarized in Section 6.

2. Backround Methodology

In this section, we review the aforementioned models briefly. Firstly, we give models related with conventional poisson regression including PR, PRR, PLR, ZPR and PHR. Secondly, models based on negative binomial regression including NBR, NBRR, NBLR, ZNBR and NBHR are presented.

2.1. Models based on Poisson Regression. Poisson regression models have been widely used and popular in various fields due to its advantages like the interpretability and suitability on inference count data. The conventional PR models are based on poisson distribution which is given as follows:

$$f\left(y_i|x_i\right)=\frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!},\ y=0,1,2,...$$

where μ is the mean parameter defined by $E\left[y_i|x_i\right]=\mu_i=\exp\left(x_i^{'}\beta\right)$, x_i corresponds the ith row of $X_{n\times(p+1)}$ data matrix, $\beta_{(p+1)\times 1}$ is the coefficients vector and y is the response variable. Poisson log likelihood function is defined as follows:

$$L\left(\mu,y\right)=\sum_{i=1}^{n}\left\{y_i\ln\left(\mu_i\right)-\mu_i-\ln\left(y_i!\right)\right\}$$

Then, poisson log likelihood function can be expressed by considering $\mu_i=\exp\left(x_i^{'}\beta\right)$ equation as follows:

$$L\left(\beta,y\right)=\sum_{i=1}^{n}\left\{y_i\left(x_i^{'}\beta\right)-\exp\left(x_i^{'}\beta\right)-\ln\left(y_i!\right)\right\}$$

The first derivative or gradient vector of $L\left(.\right)$ likelihood function can be obtained as

$$\frac{\partial\left(L\left(\beta,y\right)\right)}{\partial\beta}=\sum_{i=1}^{n}\left(y_i-\exp\left(x_i^{'}\beta\right)\right)x_i^{'}$$

An iterative algorithm like iteratively reweighted least squares [22] can be used to get the solutions of above equation. Based on IRLS algorithm the maximum likelihood estimations of β can be given as follows:

$$\hat{\beta}_{PR}=\left(X^{'}\hat{W}_{IRLS-PR}X\right)^{-1}X^{'}\hat{W}_{IRLS-PR}\hat{z}$$

where
$\hat{W}_{IRLS-PR}=\text{diag}\left[\hat{\mu}_i\left(\hat{\beta}_{PR}\right)\right]$ and $z_i=\log\left(\hat{\mu}_i\left(\hat{\beta}_{PR}\right)\right)+\left(y_i-\hat{\mu}_i\left(\hat{\beta}_{PR}\right)\right)/\hat{\mu}_i\left(\hat{\beta}_{PR}\right)$.
The steps of IRLS are repeated until a converge criterion is met. Each of $\hat{W}$ and z parameters is updated to maximize likelihood function. The threshold for converge is usually determined as 10^{-6} [23]. The matrix $\left(X^{'}\hat{W}_{IRLS-PR}X\right)$ is adversely affected due to multicollinearity. In other words, when the explanatory variables

are highly correlated, the deviations of coefficients will be unstable and the estimation variance increases. Similar to classical linear regression, ridge regression estimator can be integrated to poisson regression to deal with this issue. Månsson and Shukur [6] proposed PRR method which considers ridge regression estimator in poisson regression estimation process to deal with multicollinearity and defined as follows:

$$\hat{\beta}_{PRR} = \left(X^{'}\hat{W}_{IRLS-PR}X + kI\right)^{-1} X^{'}\hat{W}_{IRLS-PR}X\hat{\beta}_{PR},\ k \geq 0$$

As an alternative to ridge regression, Liu estimator which was proposed by Liu [5] can compete with ridge estimator for dealing with multicollinearity. One superiority of Liu estimator over ridge is to have a linear form of biasing parameters unlike non-linear form in ridge estimator. That's why, PLR was proposed by Månsson et al. [8] as follows:

$$\hat{\beta}_{PLR} = \left(X^{'}\hat{W}_{IRLS-PR}X + I\right)^{-1} \left(X^{'}\hat{W}_{IRLS-PR}X + dI\right)\hat{\beta}_{PR},\ 0 < d < 1$$

As well as multicollinearity, overdispersion is a critical problem on the performance of models based on poisson regression. The main reason of this problem is to observe excessive zeros in data set and this situation is not rare in practical real applications. The positive count values can be estimated more accurate via poisson distribution but zero counts cause zero-inflation on the model. To deal with overdispersion, zero-inflated and Hurdle regression models have been proposed. These models account excessive zeros separately in estimation process. The modelling stage consist two parts, one for estimation of positive counts and the other one for estimation of zero count values. Generally, while logistic or probit model is used to estimate zero values in a binary logic, classical poisson distribution is used for positive count values. Although zero inflated and Hurdle models have similar estimation process, there are some differences between them. Firstly, Hurdle models consider zero values independently from count values and creates a different process (i.e. different distribution) generating zeros. So, only one process can produce zero count values. At the end of estimation process, the likelihood function is calculated by using the mixture of different distributions. In zero-inflated models, the observation is possible in each of two processes. Unlike Hurdle models, the process used in estimation count values can produce zero values. Secondly, the distribution used in second part of estimation is the truncated version of conventional distribution. For example, the truncated poisson distribution is considered in Hurdle regression and non-zero count estimations are guaranteed in this way. ZPR model was proposed by Lambert [24]. The assumption underlying ZPR model is expressed as follows:

$$y_i \sim \left\{ \begin{matrix} 0, with\ probability\ 1-\alpha_i \\ Poisson\left(\mu_i\right),\ with\ probability\ \alpha_i \end{matrix} \right\}$$

The unconditional probability distribution of ZPR model can be given as

$$P_{ZPR}(y_i) = \left\{ \begin{matrix} \alpha_i + (1-\alpha_i)e^{-\mu_i},\ y_i = 0 \\ (1-\alpha_i)\frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!},\ y_i \geq 1;\ 0 \leq \alpha_i \leq 1 \end{matrix} \right\}$$

Similar to ZPR models, the distribution of ZHR model is defined as follows:

$$P_{ZHR}(y_i) = \left\{ \begin{matrix} (1-\alpha_i),\ y_i = 0 \\ \alpha_i \frac{f(j;\mu_i)}{1-f(0;\mu_i)},\ y_i = j;\ j = 1, 2, ... \end{matrix} \right\}$$

where $f(j;\mu_i)$ is a truncated poisson distribution.

2.2. **Models based on Negative Binomial Regression.** Negative binomial regression model is a powerful tool which is highly effective on a wide range of count data applications. Various types of NBR model have been described. The most well-known and used type is termed as NB2 model referring quadratic form of variance function. Unlike to conventional PR model, the conventional NBR model can deal with over-dispersion problem. NBR model accounts over-dispersion by adding an additional term into probability function and this function is defined as follows:

$$P_{NBR}(y_i) = \left\{ \frac{\Gamma(y_i + 1/\theta)}{\Gamma(y_i+1)\Gamma(1/\theta)} \left(\frac{1}{1+\theta\mu_i} \right)^{1/\theta} \left(1 - \frac{1}{1+\theta\mu_i} \right)^{y_i} \right\}$$

where θ is the overdispersion parameter. It is clear that θ is becomes zero, NBR model will be equivalent to PR model. The log likelihood function of NBR model can be expressed as

$$L(\mu, y, \theta) = \sum_{i=1}^{n} y_i \ln\left(\frac{\theta\mu_i}{1+\theta\mu_i}\right) - \frac{1}{\theta}\ln(1+\theta\mu_i) + \ln\Gamma\left(y_i + \frac{1}{\theta}\right) - \ln\Gamma(y_i+1) - \ln\Gamma\left(\frac{1}{\theta}\right)$$

This likelihood function can be re-expressed in terms of coefficients vector as follows:

$$L(\beta_j, y, \theta) = \sum_{i=1}^{n} y_i \ln\left(\frac{\theta\exp\left(x_i'\beta\right)}{1+\theta\exp\left(x_i'\beta\right)}\right) - \frac{1}{\theta}\ln\left(1+\theta\exp\left(x_i'\beta\right)\right) + \ln\Gamma\left(y_i + \frac{1}{\theta}\right) - \ln\Gamma(y_i+1) - \ln\Gamma\left(\frac{1}{\theta}\right)$$

Similar to approach in PR model, IRLS algorithm can be used to obtain the solution of this likelihood function and the solution is described as

$$\hat{\beta}_{NBR} = \left(X'\hat{W}_{IRLS-NBR}X\right)^{-1} X'\hat{W}_{IRLS-NBR}\hat{z}$$

where $\hat{W}_{IRLS-NBR}$ and $\hat{z}$ are obtained via IRLS algorithm. In order to improve the conventional NBR model against to multicollinearity problem, NBRR and NBLR model have been propose by Månsson [7] and Månsson [9], respectively. NBRR model is defined as follows:

$$\hat{\beta}_{NBRR} = \left(X'\hat{W}_{IRLS-NBR}X + kI\right)^{-1} X'\hat{W}_{IRLS-NBR}X\hat{\beta}_{NBR},\ k \geq 0$$

NBLR model is also given as

$$\hat{\beta}_{NBLR} = \left(X'\hat{W}_{IRLS-NBR}X + I\right)^{-1}\left(X'\hat{W}_{IRLS-NBR}X + dI\right)\hat{\beta}_{NBR},\ 0 < d < 1$$

Although NBR model is actually effective on dealing with overdispersion problem, some alternatives have been also proposed to improve NBR model's performance. By carrying similar process like in poisson models, zero-inflated negative binomial regression and negative binomial Hurdle regression models are defined as follows:

$$P_{ZNBR}(y_i) = \left\{\begin{matrix} \alpha_i + (1-\alpha_i)\left(\frac{1}{1+\theta\mu_i}\right)^{1/\theta},\ y_i = 0 \\ (1-\alpha_i)\frac{\Gamma\left(y_i+\frac{1}{\theta}\right)}{\Gamma(y_i+1)\Gamma\left(\frac{1}{\theta}\right)}\left(\frac{1}{1+\theta\mu_i}\right)^{1/\theta}\left(\frac{\theta\mu_i}{1+\theta\mu_i}\right)^{y_i},\ y_i \geq 1 \end{matrix}\right\}$$

$$P_{NBHR}(y_i) = \left\{\begin{matrix} \alpha_i,\ y_i = 0 \\ (1-\alpha_i)\frac{\Gamma\left(y_i+\frac{1}{\theta}\right)^{\alpha_i}}{\left[1-\left(\frac{1}{1+\theta\mu_i}\right)^{1/\theta}\right]\Gamma(y_i+1)\Gamma\left(\frac{1}{\theta}\right)}\left(\frac{1}{1+\theta\mu_i}\right)^{1/\theta}\left(\frac{\theta\mu_i}{1+\theta\mu_i}\right)^{y_i},\ y_i \geq 1 \end{matrix}\right\}$$

where α corresponds the probability of zero counts.

3. Selection Ridge and Liu Parameters

The selection of ridge and Liu biasing parameters significantly affect on the performance of PRR, NBRR, PLR and NBLR models. Although there have been extensive literature on the determination of these parameters, there is no consensus among these studies. Various different methods have been proposed for each model. For the models based on ridge estimators, the most well-known method is defined by Hoerl and Kennard as follows:

$$k = \left(\frac{\hat{\sigma}^2}{\hat{\alpha}^2_{\max}}\right)$$

where $\hat{\alpha} = \gamma\hat{\beta}_{ML}$ and γ corresponds the eigenvector of $X'\hat{W}X$. $\hat{W}$ matrix is obtained via each model, separately. Some alternative estimators have been proposed in order to avoiding underestimating of optimal k value via above method. The following estimator has been suggested by Månsson [6] and Månsson [7] for PRR and NBRR models and considered in this study:

$$k_1 = \max\left(\frac{1}{m_j}\right)$$

where $m_j = \sqrt{\hat{\sigma}^2/\hat{\alpha}^2_j}$.

Similarly, Månsson et al. [8] and Månsson [9] suggested the following selection method on the selection of Liu biasing parameters for PLR and NBLR models:

$$d_1 = \max\left(0, \min\left(\frac{\hat{\alpha}_j^2 - 1}{\left(1/\hat{\lambda}_j\right) + \hat{\alpha}_j^2}\right)\right)$$

where $\hat{\lambda}_j$ is the jth eigenvalue of $X'\hat{W}X$.

4. Performance Evaluation

In order to investigate the performance of regression models, a real data set has been used. The data set has been obtained from Deb and Trivedi's study [21] and is available in [25]. This data set includes 4406 individuals and 12 attributes. The dependent variable is the number of pysician office visits. The number of days of hospital stays, the health status, age, gender, marital status, education, family income, private insurance status, the number of chronic diseases, job status and disability status are considered as the explanatory variables in this study. There is no missing values in data set. The existence of outliers has been examined by using the range of ± 3 standard deviation of eah variable. To investigate the existence of outliers, we considered both z-score and individually ± 3 standard deviation approach with some graphical methods including box-plot. Mahalanobis distance was used to investigate the multivariate outliers but there have been no critical outliers in terms of this measurement potentially affecting the performance. At the end of univariate outlier deletion process, the remain 4182 individuals are used in experiments.

The data set has been splitted as train and testing data sets with the ratios %70 and %30, respectively. Regression models have been fitted via train data set and tested via testing data set. The distribution of frequencies for each split is given in Fig. 1. All the experimental study has been carried out via R software [26] and related packages including MASS [28], lmtest [29], pscl [30] and countreg [30].

4.1. **Training Regression Models.**

In the training phase, six models including PR, ZPR, PHR, NBR, ZNBR and NBHR have been fitted on the training data set. Akaike information criterion (AIC) and log likelihood values have been calculated to determine the best fit. The obtained results has been given in Table 1. According to results in Table 1, NBR, ZNBR and NBHR models which provide the minimum AIC and LL values, have been found as the best models on the training data set. Although the performances of these models are similar to each other, the best one among them is the NBHR model. The worst model for our training data set is the

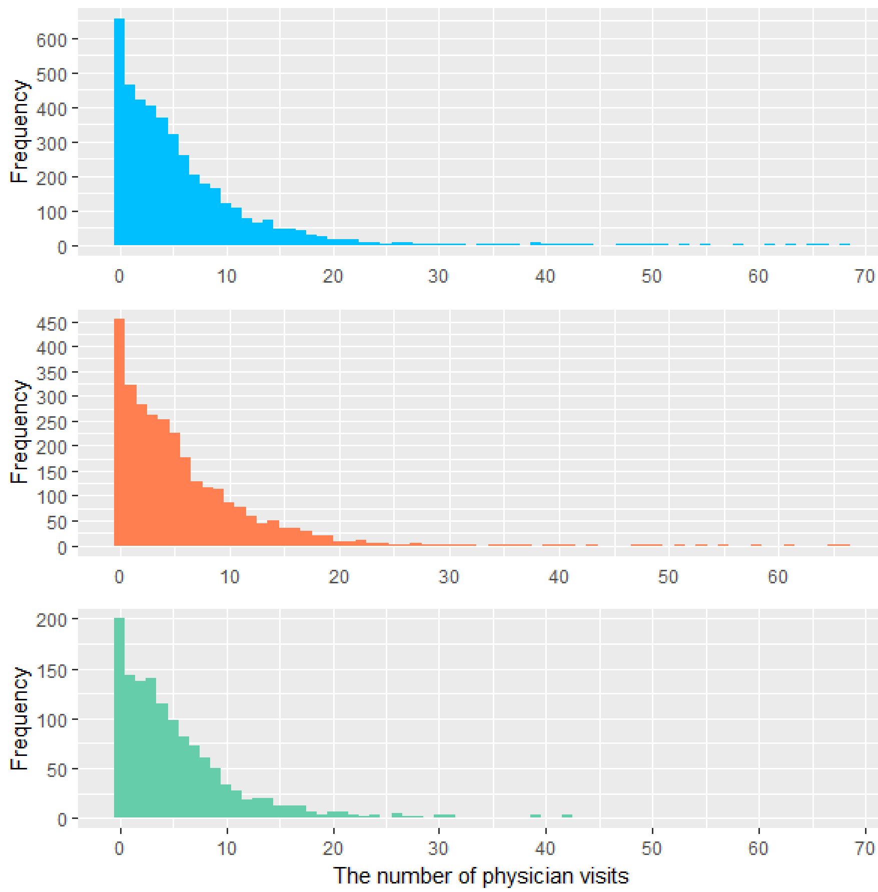

FIGURE 1. The distribution of frequencies in each data split

classical PR model.

Table 1. AIC and LL values of regression models for training data set

Model	AIC	LL
PR	23239.538	-11606.769
NBR	16091.514	-8031.757
ZPR	20968.239	-10458.119
ZNBR	16003.561	-7974.781
PHR	20968.497	-10458.249
NBHR	15993.192	-7969.596

A log likelihood ratio test has been carried out in nested models to decide whether an overdispersion parameters in negative binomial regression model is necessary. Models based on negative binomial regression and poisson regression have been examined among themselves, separately. The results are given in Table 2. Based on Table 2, it can be said that all comparisons of nested models have been as significant. This result means that models based on negative binomial regression can be seen as more suitable at the point of dealing with overdispersion. Besides, Vuong test [27] is applied training results for non-nested models to determine the model's performance of dealing with excessive zeros. A positive value of test statistic in Vuong test means that the first model is more reasonable than the second one. The test statistics and significance values are given in Table 2. According to Vuong test's results, PHR and ZPR are more preferable as statistically than PR. Similarly, ZNBR and NBHR are found as more suitable than classical NBR model. Only the differences between ZNBR&NBHR and ZPR&PHR are not statistically significant.

Table 2. LRT and Vuong test results based on pair comparisons on training data set

LRT	NBR-PR		ZNBR-ZPR		NBHR-PHR	
Value	7150	($p<0.001$)	4966.7	($p<0.001$)	4977.3	($p<0.001$)
Vuong test	PR-PHR	ZNBR-NBHR	PR-ZPR	NBR-ZNBR	ZPR-PHR	NBR-NBHR
z statistic	-14.183	-1.037	-14.186	-5.008	0.745	-5.352
p	($p<0.001$)	($p=0.1499$)	($p<0.001$)	($p<0.001$)	($p=0.2508$)	($p<0.001$)

As well as LRT and Vuong tests, rootogram graphs have been obtained to examine the model fits as visually. In rootogram graphs, the closeness of the bars to the x axis, is proportional to the goodness of fit. These graphs are given in Figure 2. It can be said that models based on negative binomial regression are seen more better than the models related with poisson regression. Among these models, ZNBR and NBHR are seen more suitable than PR. Although the basic NBR model can be a convincing alternative in the existence of excessive zeros and overdispersion, it's zero inflated and Hurdle variants is more powerful for an effective estimation.

4.2. **Testing Regression Models.** In the training phase, all regression models have been fitted to the training data and some performance measurements like AIC and LL have been given to compare six models except the ones based on ridge and Liu estimators. All outweights vectors have been obtained for all models. By using these outweights, regression models including PRR, PLR, NBRR and NBLR have been tested on the unseen (i.e. testing) data set. The mean squared error (MSE), it's square root (RMSE) and the mean absolute error (MAE) are used as performance criteria. Unstandardized residuals have been considered in the calculation of these measures. The testing results are given in Table 3. Based on Table 3, it is shown that ZNBR, PHR and ZPR models provide better results than their competitors. These models give the smaller MSE, RMSE and MAE values. Although the performances of ZNBR, PHR and ZPR models are similar each other, the best fit is obtained with PHR model. Models based on ridge and Liu estimators are slightly poorer than the rest of models. Overall, the NBLR model is found as

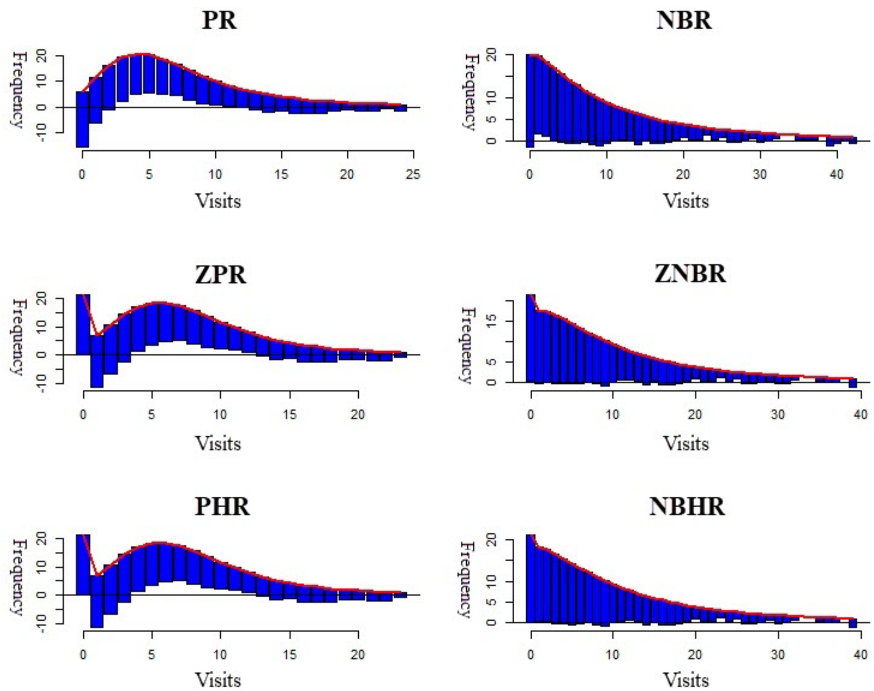

FIGURE 2. The rootogram graphs of each regression model

the worst model for our data set.

Table 3. The comparison of regression models on the testing data set

Model	MSE	RMSE	MAE
PR	39.8572	6.31325	3.9699
NBR	40.7568	6.38410	3.9847
ZPR	39.6489	6.29673	3.9449
ZNBR	39.8083	6.30938	3.9318
PHR	39.6447	6.29640	3.9450
NBHR	40.0699	6.33007	3.9500
PRR	40.8191	6.38898	3.9842
PLR	40.8198	6.38903	3.9844
NBRR	41.0495	6.40698	3.9919
NBLR	41.0499	6.4070	3.9920

4.3. **Determination of the Best Model and Interpretation.** When the results in training and testing phases are examinated together with the rootogram graphs, ZNBR model can be seen as the best model for the data set in this study. In the training phase, NBR, ZNBR and NBHR models have similar performances according to the AIC and LL values. NBR, ZNBR and NBHR models show the best fit as visually in rootogram graphs. Finally, ZPR, ZNBR and PHR models are found better than their competitors based on testing performance. Besides, LRT tests is found significant between ZPR and ZNBR models. By virtue of these comments, it can be said that the most suitable model for the data set is ZNBR. The final fit and coefficients have been obtained using ZNBR model. All results of final modelling are given in Table 4. The number of days of hospital stays, the health status, education, age, gender, marital status, family income, private insurance status, the number of chronic diseases, job status and disability status

Table 4. The fitting results of ZNBR model

	Count Part			Zero Part		
Variable	β	exp (β)	Se	β	exp (β)	Se
Intercept	1.593*	4.920	0.21580	4.574*	96.9350	1.45082
#Days of hospital stays	0.280*	1.324	0.02995	-0.533	0.587	0.34171
Health status (perfect)	-0.276*	0.759	0.06445	0.406	1.500	0.28061
Health status (bad)	0.264*	1.303	0.04949	-0.505	0.603	0.58746
#Chronic diseases	0.137*	1.147	0.01308	-1.127*	0.324	0.16908
Disability status (yes)	0.117*	1.124	0.04180	0.039	1.040	0.34310
Age (/10 years)	-0.057*	0.945	0.02730	-0.600*	0.549	0.18999
Gender (male)	-0.026	0.974	0.03552	1.044*	2.840	0.24176
Marital status (married)	-0.111*	0.895	0.03699	-0.747*	0.474	0.25823
Education	0.020*	1.020	0.00471	-0.097*	0.908	0.03041
Family income (x10000$)	0.007	1.007	0.00882	-0.022	0.978	0.06752
Job status (yes)	-0.022	0.978	0.05314	-0.334	0.716	0.37298
Private insurance (yes)	0.137*	1.147	0.04292	-0.172*	0.310	0.22918

Coefficients for each part of zero-inflated negative binomial regression have been given separately. Count part corresponds the modelling results on the individuals who are visited by the pyhsician more than zero. Similarly, zero part results belong the the people who are never visited by the pyhsician. As mentioned before, zero-inflated models present a mixture of different distributions who are able to model count and zero dependent values in a separate process.

According to the count part results in Table 4, the significant variables are #days of hospital stays, health status, #chronic diseases, disability status, age, marital status, education and private insurance ($p < 0.05$). The physicians tend to visit about %24 more to people whose perception of health is perfect than the ones having normal perception. The people having some disabilities in their normal life are more likely to be visited (approximately %15). Similarly, private insurance status is effective on the number of visits. The existence of private insurance provides more visits (%14.7) than normal status. Married people are approximately %10 less likely to be visited than single people. On the other hand, gender, family income and job status are not statistically significant on the number of physician visits ($p < 0.05$).

When observed the results in zero part, it can be seen that #chronic diseases, age, gender, marital status, education and private insurance are statistically significant on the modelling the zero counts ($p < 0.05$). The results show that people with chronic diseases are less likely (approximately %67) not to be visited by the physicians. This results is reasonable because of the risk of those people. The physicians tend not to visit men 2.84 times less likely than women. The comments is supportive with count part when marital status, education and private insurance variable are examined. The more likely situation in count part corresponds the less likely one in zero part. While an occurrance for a particulary variable in count part can increase the possibility of visits, it can decrease for the zero-part or vice versa. Compared with count part, there is more insignificant variables in zero part. #days of hospital stays, health status, disability status, family income and job status are not the effective factors ($p < 0.05$).

5. Conclusions and Discussions

In this study, we have used poisson and negative binomial regression models and their variants based on zero-inflated, Hurdle, ridge and Liu approaches to model a medical data. Zero-inflated, Hurdle and biased (i.e. ridge and Liu) models have been considered in order to get better solution in the presence of excessive zeros and overdispersion. Because of the prevalence of these problems in real word application, classical regression models can be a remedy. As a result of comprehensive experimental study, it can be said that zero-inflated and Hurdle models may provide better results than their competitors. The researchers facing excessive zeros and overdispersion problems may consider these models as powerfull alternatives to get useful insight about their application. Moreover, poisson and negative binomial regression models based on ridge and Liu estimators should be taken into consideration in the presence of multicollinearity in addition to excessive zeros and overdispersion problems.

Although the mentioned models can provide useful insights for modeling count data, it should be noted that the performance comparison is valid for data sets having similar underlying structure. In the future works, we will carried out a comprehensive simulation study to achieve better generalization performance. Different types of data sets belonging various domains, larger dimensions and more detailed parameter selection process can be seen as the limitations of this study.

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References

[1] Dobson, A. J., Barnett, A. G., An Introduction to Generalized Linear Models, Chapman and Hall/CRC, 2008. https://doi.org/10.1201/9781315182780

[2] Chatterjee, S., Hadi, A. S., Regression Analysis by Example, John Wiley & Sons, 2015. https://doi.org/10.1002/0470055464

[3] Cameron, A. C., Trivedi, P. K., Regression Analysis of Count Data, 6th edn., Cambridge University Press, New York, 2007. http://dx.doi.org/10.1017/CBO9780511814365

[4] Hoerl, E., Kennard, R. W., Ridge regression: biased estimation for nonorthogonal problems, *Technometrics,* 12 (1970), 55–67. https://doi.org/10.1080/00401706.1970.10488634

[5] Liu, K., A new class of biased estimate in linear regression, *Commun. Stat. Theory Methods,* 22 (1993), 393-402. https://doi.org/10.1080/03610929308831027

[6] Månsson, K., Shukur, G., A Poisson ridge regression estimator. *Econ. Model.,* 28 (2011), 1475-1481. https://doi.org/10.1016/j.econmod.2011.02.030

[7] Månsson, K., On ridge estimators for the negative binomial regression model, *Econ. Model.,* 29 (2012), 178-184. https://doi.org/10.1016/j.econmod.2011.09.009

[8] Månsson, K., Kibria, B. M. G., Sjolander, P., Shukur, G., Improved Liu estimators for the poisson regression model. *Int. J. Stat. Probab.,* 1 (2012), 2-6. https://doi.org/10.5539/ijsp.v1n1p2

[9] Månsson, K., Developing a Liu estimator for the negative binomial regression model: method and application. *J. Stat. Comput. Simul.,* 83 (2013), 1773-1780. https://doi.org/10.1080/00949655.2012.673127

[10] Wang, W., Famoye, F., Modeling household fertility decisions with generalized Poisson regression, *Journal of Population Economics*, 10(3) (1997), 273-283. https://doi.org/10.1007/s001480050043

[11] Famoye, F., Singh, K. P., Zero-inflated generalized Poisson regression model with an application to domestic violence data, *Journal of Data Science,* 4(1) (2006), 117-130. https://doi.org/10.6339/JDS.2006.04(1).257

[12] Bandyopadhyay, D., DeSantis, S. M., Korte, J. E., Brady, K. T., Some considerations for excess zeroes in substance abuse research, *The American Journal of Drug and Alcohol Abuse*, 37(5) (2011), 376-382. https://doi.org/10.3109/00952990.2011.568080

[13] Buu, A., Johnson, N. J., Li, R., Tan, X., New variable selection methods for zero-inflated count data with applications to the substance abuse field, *Statistics in Medicine*, 30(18) (2011), 2326-2340. https://doi.org/10.1002/sim.4268

[14] Mouatassim, Y., Ezzahid, E. H., Poisson regression and zero-inflated Poisson regression: application to private health insurance data, *European Actuarial Journal,* 2(2) (2012), 187-204. https://doi.org/10.1007/s13385-012-0056-2

[15] Xie, H., Tao, J., McHugo, G. J., Drake, R. E., Comparing statistical methods for analyzing skewed longitudinal count data with many zeros: An example of smoking cessation, *Journal of Substance Abuse Treatment,* 45(1) (2013), 99-108. https://doi.org/10.1016/j.jsat.2013.01.005

[16] Liyanage, T., Ninomiya, T., Jha, V., Neal, B., Patrice, H. M., Okpechi, I., Zhao, M-H., Lv, J., Garg, A. X., Knight, J., Rodgers, A., Gallagher, M., Kotwal, S., Cass, A., Perkovic, V., Worldwide access to treatment for end-stage kidney disease: a systematic review, *The Lancet,* 385(9981) (2015), 1975-1982. https://doi.org/10.1016/s0140-6736(14)61601-9

[17] Martinez, F. J., Calverley, P. M., Goehring, U. M., Brose, M., Fabbri, L. M., Rabe, K. F., Effect of roflumilast on exacerbations in patients with severe chronic obstructive pulmonary

disease uncontrolled by combination therapy (REACT): a multicentre randomised controlled trial, *The Lancet,* 385(9971) (2015), 857-866. https://doi.org/10.1183/13993003.00158-2017

[18] Oliveira, M., Einbeck, J., Higueras, M., Ainsbury, E., Puig, P., Rothkamm, K., Zero-inflated regression models for radiation-induced chromosome aberration data: A comparative study, *Biometrical Journal,* 58(2) (2016), 259-279. https://doi.org/10.1002/bimj.201400233

[19] Tang, Y., Liu, W., Xu, A., Statistical inference for zero-and-one-inflated poisson models, *Statistical Theory and Related Fields,* 1(2) (2017), 216-226. https://doi.org/10.1002/bimj.201400233

[20] Chai, T., Xiong, D., Weng, J., A zero-inflated negative binomial regression model to evaluate ship sinking accident mortalities, *Transportation Research Record*, 2672(11) (2018), 65–72. https://doi.org/10.1177

[21] Deb, P., Trivedi, P. K., Demand for medical care by the elderly: a finite mixture approach, *Journal of Applied Econometrics*, 12(3) (1997), 313-336. http://www.jstor.org/stable/2285252?origin=JSTOR-pdf

[22] Garthwaite, P. H., Jolliffe, I. T., Jones, B., Statistical Inference, Oxford University Press, Oxford, 2002. https://doi.org/10.1017/S0025557200173425

[23] Hilbe, J. M., Negative Binomial Regression, Cambridge University Press, Cambridge, 2011. https://doi.org/10.1017/CBO9780511973420

[24] Lambert, D., Zero-inflated Poisson regression with an application to defects in manufacturing, *Technometrics,* 34 (1992), 1-14. https://doi.org/10.2307/1269547

[25] https://www.jstatsoft.org/article/view/v016i09

[26] Core Team, R., R: A language and environment for statistical computing, Vienna: Austria: R foundation for Statistical Computing, (2016). http://www.R-project.org/

[27] Vuong, Q. H., Likelihood ratio tests for model selection and nonnested hypotheses, *Econometrica,* 57(2) (1989), 30-33. https://doi.org/10.2307/1912557

[28] Venables, W. N., Ripley, B. D., Modern Applied Statistics with S, Fourth Edition, Springer, New York, 2002. https://www.stats.ox.ac.uk/pub/MASS4/.

[29] Zeileis, A., Hothorn, T., Diagnostic checking in regression relationships, *R News*, 2(3) (2002) 7-10. https://CRAN.R-project.org/doc/Rnews/.

[30] Zeileis, A., Kleiber C, Jackman, S., Regression models for count data in R, *Journal of Statistical Software*, 27(8) (2008), 1-25. http://www.jstatsoft.org/v27/i08/.

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