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Generalized Rayleigh-Quotient Formulas for the Real Parts, Imaginary Parts, and Moduli of Simple Eigenvalues of Compact Operators

Ludwig Kohaupt[1*]

Abstract
In an earlier paper, the author derived generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of the eigenvalues of *diagonalizable matrices*. More precisely, max-, min-max-, min-, and max-min-formulas were obtained. In this paper, we extend these results to the eigenvalues of linear *nonsymmetric compact operators* with *simple eigenvalues* in a *Hilbert space*. As an application, a new formula for the spectral radius is derived. An example arising from a boundary value problem in Mathematical Physics illustrates the general results, and numerical computations underpin the theoretical findings. In addition, the Euler column is treated from the area of Elastomechanics, which is complemented by references to other examples from this area.

Keywords: Generalized Rayleigh-Quotients, Hilbert space, Real parts, Imaginary parts, and moduli of eigenvalues, Simple eigenvalues of compact operators
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1. Introduction

In [16], the author derived generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of the eigenvalues of diagonalizable nonsymmetric matrices, that is, in the case of a finite-dimensional space. In this paper, we extend these results to the eigenvalues of nonsymmetric compact operators with simple eigenvalues in an infinite-dimensional Hilbert space. Some arguments in the proofs are similar to those in the finite-dimensional case, but others are very different from them.

The paper is structured as follows. In Section 2, as a basis for what follows, functions of an operator in a Banach space are discussed which is taken from [18]. Section 3 contains the expansion of a linear nonsymmetric compact operator and of a pertinent projection operator in a Hilbert space. In Sections 4 - 6, generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli are given respectively followed in Section 7 by generalized Rayleigh-quotient formulas for real eigenvalues. In Section 8, the general results are employed to obtain a new formula for the spectral radius. Section 9 presents new generalized numerical ranges, and in Section 10 an example from the area of a boundary value problem is given along with the results of numerical computations. In Section 11, it is discussed what consequences changes in the arrangement of the eigenvalues will have. Section 12 contains the conclusion and an outlook to future work. Finally, the references follow. Besides the cited references, the following non-cited ones are given: [1] - [3], [6], [8], [9], [12] - [15], [17], [19], [21], [22], [29], and [30] since the author thinks that they could be of interest to the reader in the context of the treated subject. We mention that the Remarks are not enumerated.

2. Functions of an Operator in a Banach Space

This section is of fundamental importance for what follows; it is taken from the corresponding section in [18]. The results are obtained in a Banach space of which a Hilbert space is a particular case.

Let $\{0\} \neq E$ be a Banach space over the field $\mathbb{F} = \mathbb{C}$. Whereas in [10, Chapter I] it is supposed that $dim E < \infty$, here we assume that $dim E = \infty$. As was shown in [26] based on findings of [24], the following results taken from [10, Chapter I] are valid not only for $dim E < \infty$, but also for $dim E = \infty$ if the space is complete.

Let $p(\zeta)$ be the polynomial

$$p(\zeta) = \alpha_0 + \alpha_1 \zeta + \cdots \alpha_n \zeta^n, \; \zeta \in \mathbb{C} \tag{2.1}$$

with $\alpha_j \in \mathbb{C}, \; j = 0, 1, \cdots, n$. Then the polynomial $p(T) \in B(E)$ is defined by

$$p(T) = \alpha_0 + \alpha_1 T + \cdots \alpha_n T^n, \; \zeta \in \mathbb{C}, \tag{2.2}$$

see [10, Chapter I, §3.3]. Making use of the resolvent

$$R(\zeta) := (T - \zeta)^{-1}, \; \zeta \in \mathbb{C}, \tag{2.3}$$

one can now define the function $\phi(T)$ of T for a more general class of functions $\phi(\zeta)$.

Before we do this, we mention that linear compact operators need not have eigenvalues. For example, Volterra integral operators have no eigenvalues. On the other hand, consider a symmetric linear compact operator. Then, such an operator has at least one eigenvalue, and all eigenvalues are real and simple. For these operators, there may exit only a finite number of eigenvalues. Further, there is at most a countable set of eigenvalues with the only possible accumulation point zero, and there exists a set of pertinent pairwise orthonormal eigenvectors. Further, it is known that the non-zero elements of the spectrum consist solely of eigenvalues and that, if there is a countable set of eigenvalues, the associated sequence tends to zero. For all this, see [27, Chapter 6].

Further, according to [7, Theorem 44.1, p.191], one has $\sigma(T) \setminus \{0\} = \sigma_P(T) \setminus \{0\}$ where $\sigma(T)$ is the spectrum of T and $\sigma_P(T)$ the point spectrum consisting of the eigenvalues of T.

Taking this into account, for our general linear compact operator $T \in B(E)$, we suppose that the spectrum $\sigma(T)$ of T has a countable set of non-zero eigenvalues λ_j and that the sequence of eigenvalues tends to zero.

Additionally, we suppose that $0 \notin \sigma(T)$ so that $N(T) = \{0\}$ since without this condition, we cannot obtain relation (2.11) resp. (2.14) below.

Now, suppose that $\phi(\zeta)$ is holomorphic in a domain D of the complex plane containing all the eigenvalues $\lambda_j \neq 0$ of T, and let $C \subset D$ be a simple closed smooth curve with positive direction enclosing all the eigenvalues λ_j in its interior. Then, $\phi(T)$ is defined by the *Dunford-Taylor integral*

$$\phi(T) = -\frac{1}{2\pi i}\int_C \phi(\zeta) R(\zeta)\, d\zeta = -\frac{1}{2\pi i}\int_C \phi(\zeta)(T-\zeta)^{-1}\, d\zeta. \tag{2.4}$$

This is an analogue of the Cauchy integral formula in the Theory of Functions, see [11, Part I, §15, p. 61]. More generally, the curve C may consist of several simple closed rectifiable Jordan curves C_k having a positive direction with interiors D'_k such that the union of the D'_k contains all the eigenvalues of T. We note that (2.4) does not depend on C as long as C satisfies these conditions. For the C_k, we can use the circles $C_k = \{z \in \mathbb{C} \,|\, |z - \lambda_k| = r_k\}$ with sufficiently small radii r_k.

It can be verified that, for the polynomial

$$\phi(\zeta) = p(\zeta) = \alpha_0 + \alpha_1 \zeta + \cdots \alpha_n \zeta^n, \; \zeta \in \mathbb{C} \tag{2.5}$$

with $\alpha_j \in \mathbb{C}, \; j = 0, 1, \cdots, n$, the Dunford-Taylor integral (2.4) is equal to (2.2).

For the special case

$$\phi(\zeta) = p(\zeta) = \zeta, \tag{2.6}$$

we obtain

$$T = -\frac{1}{2\pi i}\int_C T R(\zeta)\, d\zeta = T\left(-\frac{1}{2\pi i}\int_C R(\zeta)\, d\zeta\right) = \left(-\frac{1}{2\pi i}\int_C R(\zeta)\, d\zeta\right) T. \tag{2.7}$$

Now, we set

$$P := -\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta. \tag{2.8}$$

According to [10, Chapter I, §5, Section 3], P is a continuous projection operator onto the *algebraic eigenspace* $X = P(E) = R(P)$, where $R(P)$ means the range of P. Thus, from (2.7) and (2.8), one obtains

$$T = TP = PT = PTP. \tag{2.9}$$

Now, let the radii r_k be chosen such that

$$C_j \cap C_k = \emptyset,\ j \neq k,\ j,k = 1,2,3,\cdots. \tag{2.10}$$

Then,

$$P = -\frac{1}{2\pi i}\int_C R(\zeta)\,d\zeta = \sum_{j=1}^{\infty}\left(-\frac{1}{2\pi i}\int_{C_j} R(\zeta)\,d\zeta\right) = \sum_{j=1}^{\infty} P_j \tag{2.11}$$

with

$$P_j = -\frac{1}{2\pi i}\int_{C_j} R(\zeta)\,d\zeta,\ j = 1,2,3,\cdots. \tag{2.12}$$

At this point, we needed the assumption $0 \notin \sigma(T)$ since otherwise any circle C_0 about $\lambda_0 = 0$ would eventually intersect with the circles C_k for sufficiently large k so that we would not have (2.10) for $j,k \in (0,1,2,3,\cdots)$. Let J be the sequence

$$J := (1,2,3,\cdots). \tag{2.13}$$

Then, (2.11) can be written as

$$\boxed{P = \sum_{j=1}^{\infty} P_j = \sum_{j\in J} P_j.} \tag{2.14}$$

Because of (2.10), one has

$$P_jP_k = P_kP_j = P_j\delta_{jk},\ j,k \in J. \tag{2.15}$$

Herewith,

$$P_j(E) =: X_j \tag{2.16}$$

is the *algebraic eigenspace of T associated with the eigenvalue* λ_j.

From (2.9), (2.11), and (2.15), we obtain

$$T = PT = TP = PTP = \sum_{j\in J} P_jT = \sum_{j\in J} TP_j = \sum_{j\in J} P_jTP_j, \tag{2.17}$$

and so

$$\begin{aligned} R(T) = T(E) &= (PT)(E) = (TP)(E) = (PTP)(E) \\ &= \sum_{j\in J}(P_jT)(E) = \sum_{j\in J}(TP_j)(E) = \sum_{j\in J}(P_jTP_j)(E). \end{aligned} \tag{2.18}$$

3. Expansion of a Linear Compact Operator and of a Pertinent Projection Operator in Hilbert Space

Together with Section 2, this section forms a basis for what follows. The statements are taken over from [18], but most of the proofs are omitted.

(i) The Conditions (C1) - (C4)

We assume the following conditions:

(C1) $\{0\} \neq H$ is a Hilbert space over the field $\mathbb{F} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$

(C2) $0 \neq T \in B(H)$ is compact having countably many simple non-zero eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ with $\lim_{k\to\infty} \lambda_k = 0$ pertinent to the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$. Further, $0 \notin \sigma(T)$.

(C3) The eigenvectors of the adjoint T^* of T with the eigenvalues $\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \cdots$ are $\psi_1, \psi_2, \psi_3, \cdots$

(C4) $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j = 1, 2, 3 \cdots$

One has the following theorem.

Theorem 3.1. *(Biorthonormality relations for $\lambda_j \neq \lambda_k$, $j \neq k$)*

Let the conditions (C1) - (C4) be fulfilled. Then, with appropriate normalization, the eigenvectors $\chi_1, \chi_2, \chi_3, \cdots$ and $\psi_1, \psi_2, \psi_3, \cdots$ are biorthonormal, that is,

$$(\chi_j, \psi_k) = \delta_{jk},\ j, k \in J. \tag{3.1}$$

Proof. See [18, Theorem 3.1]. □

Furthermore, we obtain the following theorem.

Theorem 3.2. *(Expansion of Tu as well as of Pu in a series of eigenvectors) Let the conditions (C1) - (C4) be fulfilled. Then,*

$$Tu = \sum_{j\in J} \lambda_j (u, \psi_j)\chi_j,\ u \in H \tag{3.2}$$

as well as

$$Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H. \tag{3.3}$$

Proof. See [18, Theorem 3.2]. □

Remark. *From (3.2) we conclude that*

$$\overline{[\chi_1, \chi_2, \chi_3, \cdots]} = T(H) = R(T).$$

Further, from (3.3),

$$P : H \mapsto \overline{[\chi_1, \chi_2, \chi_3, \cdots]}.$$

□

Moreover, in [18, Theorem 3.3], we have proven the following theorem.

Theorem 3.3. *Let the conditions (C1) - (C4) be fulfilled. Then, we obtain*

$$u = Pu = \sum_{j\in J} (u, \psi_j)\chi_j,\ u \in H \tag{3.4}$$

and the projection operator

$$P_0 = I - P : H \mapsto N(T) = \{0\} \iff P_0 = 0. \tag{3.5}$$

For the next theorem, we define new subspaces of H. For every $j = 1, 2, \ldots$, let

$$N_{\chi,j} := \{u \in H \,|\, u = \sum_{k=1}^{j} \alpha_k \chi_k \text{ with } \alpha_k \in \mathbb{C},\, k = 1, 2, \ldots, j\} =: [\chi_1, \ldots, \chi_j], \tag{3.6}$$

$j = 1, 2, \ldots$ and

$$N_{\chi,j,\mathbb{R}} := \{u \in H \,|\, u = \sum_{k=1}^{j} \beta_k \chi_k \text{ with } \beta_k \in \mathbb{R},\, k = 1, 2, \ldots, j\} = [\chi_1, \ldots, \chi_j]_{\mathbb{R}},\ j = 1, 2, \ldots \tag{3.7}$$

$j = 1, 2, \ldots$ as well as

$$\begin{aligned} N_\chi &:= N_{\chi,\infty} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \alpha_k \chi_k \text{ exists in H with } \alpha_k \in \mathbb{C}, k = 1, 2, \ldots\} \\ &= \overline{[\chi_1, \chi_2, \ldots]} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} N_{\chi,\mathbb{R}} &:= N_{\chi,\infty,\mathbb{R}} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \beta_k \chi_k \text{ exists in H with } \beta_k \in \mathbb{R}, k = 1, 2, \ldots\} \\ &= \overline{[\chi_1, \chi_2, \ldots]}_{\mathbb{R}}. \end{aligned} \tag{3.9}$$

Likewise, we define

$$N_{\psi,j} := \{u \in H \,|\, u = \sum_{k=1}^{j} \alpha_k \psi_k \text{ with } \alpha_k \in \mathbb{C},\, k = 1, 2, \ldots, j\} =: [\psi_1, \ldots, \psi_j], \tag{3.10}$$

$j = 1, 2, \ldots$ and

$$N_{\psi,j,\mathbb{R}} := \{u \in H \,|\, u = \sum_{k=1}^{j} \beta_k \psi_k \text{ with } \beta_k \in \mathbb{R},\, k = 1, 2, \ldots, j\} = [\psi_1, \ldots, \psi_j]_{\mathbb{R}}, \tag{3.11}$$

$j = 1, 2, \ldots$ as well as

$$\begin{aligned} N_\psi &:= N_{\psi,\infty} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \alpha_k \psi_k \text{ exists in H with } \alpha_k \in \mathbb{C}, k = 1, 2, \ldots\} \\ &= \overline{[\psi_1, \psi_2, \ldots]} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} N_{\psi,\mathbb{R}} &:= N_{\psi,\infty,\mathbb{R}} := \{u \in H \,|\, u = \textstyle\sum_{k=1}^{\infty} \beta_k \psi_k \text{ exists in H with } \beta_k \in \mathbb{R}, k = 1, 2, \ldots\} \\ &= \overline{[\psi_1, \psi_2, \ldots]}_{\mathbb{R}}. \end{aligned} \tag{3.13}$$

After these preparations, we are able to prove the following theorem.

Theorem 3.4. *Let the conditions (C1) - (C4) be fulfilled. Then,*

$$(Tu, v) = \sum_{j \in J} \lambda_j (u, \psi_j)(\chi_j, v),\ u, v \in H \tag{3.14}$$

and

$$(u, v) = (Pu, v) = \sum_{j \in J} (u, \psi_j)(\chi_j, v),\ u, v \in H \tag{3.15}$$

where

$$(u, \psi_j), (\chi_j, v) \in \mathbb{R},\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}},\ j \in J \tag{3.16}$$

leading to

$$Re\,(Tu, v) = \sum_{j \in J} Re\,\lambda_j (u, \psi_j)(\chi_j, v),\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}},\ j \in J. \tag{3.17}$$

Proof. Let $u \in N_{\chi,\mathbb{R}}$ and $v \in N_{\psi,\mathbb{R}}$. Then,

$$u = \sum_{j \in J} (u, \psi_j) \chi_j \tag{3.18}$$

and

$$v = \sum_{k \in J} (v, \chi_k) \psi_k \tag{3.19}$$

implying

$$(Tu, v) = \sum_{j,k \in J} \lambda_j (u, \psi_j) \overline{(v, \chi_k)} (\chi_j, \psi_k) \tag{3.20}$$

so that with (3.1) relation (3.14) follows.

Further, let $u \in N_{\chi,\mathbb{R}}$. Then,

$$u = \sum_{j \in J} \alpha_j \chi_j$$

with elements $\alpha_j \in \mathbb{R},\ j \in J$ so that

$$(u, \psi_j) = \sum_{k \in J} \alpha_k (\chi_k, \psi_j) = \alpha_j \in \mathbb{R}.$$

Correspondingly, for $v \in N_{\psi,\mathbb{R}}$, one has $(\chi_j, v) \in \mathbb{R}$ so that (3.16) is proven. Relation (3.17) is a direct consequence of (3.14) and (3.16). The expression in (3.15) follows in a similar way as that in (3.14) by using (3.4). ☐

Next, we want to define vector spaces similar to those in [16, (16), (17)], namely

$$M_{\chi,1,\mathbb{R}} := N_{\chi,\mathbb{R}} = \overline{[\chi_1, \chi_2, \ldots]}_{\mathbb{R}}, \tag{3.21}$$

$$\begin{aligned} M_{\chi,j,\mathbb{R}} &:= \{u \in N_{\chi,\mathbb{R}} \,|\, (u, \psi_k) = 0,\, k = 1, 2, \ldots, j-1\} \\ &= [\psi_1, \ldots, \psi_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}},\ j = 2, 3, \ldots \end{aligned} \tag{3.22}$$

where $M_{\chi,j,\mathbb{R}}$ is called an *orthogonal complement in* $N_{\chi,\mathbb{R}}$ and

$$M_{\psi,1,\mathbb{R}} := N_{\psi,\mathbb{R}} = \overline{[\psi_1, \psi_2, \ldots]}_{\mathbb{R}}, \tag{3.23}$$

$$\begin{aligned} M_{\psi,j,\mathbb{R}} &:= \{u \in N_{\psi,\mathbb{R}} \,|\, (u, \chi_k) = 0,\, k = 1, 2, \ldots, j-1\} \\ &= [\chi_1, \ldots, \chi_{j-1}]^{\perp}_{N_{\psi,\mathbb{R}}},\ j = 2, 3, \ldots \end{aligned} \tag{3.24}$$

where $M_{\psi,j,\mathbb{R}}$ *is called an orthogonal complement in* $N_{\psi,\mathbb{R}}$. The next lemma characterizes these spaces.

Lemma 3.5. *Let the conditions (C1) - (C4) be fulfilled as well as* $\{\chi_1, \chi_2, \ldots\}$ *and* $\{\psi_1, \psi_2, \ldots\}$ *be sets of biorthogonal eigenvectors of* T *and* T^* *respectively, i.e., such that*

$$(\chi_i, \psi_j) = \delta_{ij},\ i, j = 1, 2, \ldots. \tag{3.25}$$

Then,

$$M_{\chi,j,\mathbb{R}} = \overline{[\chi_j, \chi_{j+1}, \ldots]}_{\mathbb{R}},\ j = 1, 2, \ldots \tag{3.26}$$

and

$$M_{\psi,j,\mathbb{R}} = \overline{[\psi_j, \psi_{j+1}, \ldots]}_{\mathbb{R}},\ j = 1, 2, \ldots \tag{3.27}$$

Proof. The proof is done for (3.26) and $j = 3$. The general case can be made by induction. The proof of (3.27) is similar. So, we have to prove

$$\begin{aligned} M_{\chi,3,\mathbb{R}} &:= \{u \in N_{\chi,\mathbb{R}} \,|\, (u,\psi_k) = 0,\, k = 1,2\} = [\psi_1,\psi_2]^{\perp}_{N_{\chi,\mathbb{R}}} \\ &= \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}} \end{aligned} \tag{3.28}$$

(i) $\underline{\overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}} \subset M_{\chi,3,\mathbb{R}}}$:

Let $u \in \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}}$. Then, $u = \sum_{k=3}^{\infty} \beta_j \chi_j$ with elements $\beta_j \in \mathbb{R},\ j = 3,4,\ldots$. Let $s \in \{1,2\}$. This entails, due to Theorem 3.1, $(u,\psi_s) = \sum_{j=3}^{\infty} \beta_j (\chi_j,\psi_s) = 0$ so that $u \in M_{\chi,3,\mathbb{R}}$. Therefore, (i) is proven.

(ii) $\underline{M_{\chi,3,\mathbb{R}} \subset \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}}}$:

Let $u \in M_{\chi,3,\mathbb{R}}$. This implies $u \in N_{\chi,\mathbb{R}}$ and $(u,\psi_j) = 0,\ j = 1,2$. Now, $u = \sum_{k=1}^{\infty} \beta_k \chi_k$ with $\beta_k = (u,\psi_k) \in \mathbb{R},\ k = 1,2,\ldots$ leading to $u = \sum_{k=3}^{\infty} \beta_k \chi_k$ since $(u,\psi_k) = 0,\ k = 1,2$ so that $u \in \overline{[\chi_3,\chi_4,\ldots]}_{\mathbb{R}}$. Therefore, (ii) is proven. □

Now, let $u \in N_{\chi,\mathbb{R}}$ with $u = \sum_{k=1}^{\infty} \alpha_k \chi_k$ and $\alpha_k \in \mathbb{R}$ as well as $v \in N_{\psi,\mathbb{R}}$ with $v = \sum_{k=1}^{\infty} \beta_k \psi_k$ and $\beta_k \in \mathbb{R}$. Then, due to Theorem 3.1,

$$(u,v) = \sum_{k=1}^{\infty} \alpha_k \beta_k. \tag{3.29}$$

In order to facilitate the manner of speaking, we say that the *scalar product* (u,v) *of* $u \in N_{\chi,\mathbb{R}}$ *and* $v \in N_{\psi,\mathbb{R}}$ *is strongly positive iff* $\alpha_k \beta_k \geq 0,\ k = 1,2,\ldots$ *and* $\sum_{k=1}^{\infty} \alpha_k \beta_k > 0$. For short, we write

$$(u,v) >> 0.$$

Remark. *One has* $\alpha_k = (u,\psi_k),\ u \in N_{\chi,\mathbb{R}}$ *and* $\beta_k = (\chi_k,v),\ v \in N_{\psi,\mathbb{R}}$ *for* $k = 1,2,\ldots$. *Therefore,* $(u,v) >> 0$ *means* $(u,\psi_k)(\chi_k,v) \geq 0,\ k = 1,2,\ldots$ *and* $(u,v) = \sum_{k=1}^{\infty} (u,\psi_k)(\chi_k,v) > 0$. □

Remark. *For* $(u,v) >> 0$*, one can admit linear combinations* $u = \sum_{k=1}^{\infty} \alpha_k \chi_k$ *and* $v = \sum_{k=1}^{\infty} \beta_k \psi_k$ *with* $\alpha_k, \beta_k \in \mathbb{C},\ k = 1,2,\ldots$ *such that* $\alpha_k \overline{\beta}_k = |\alpha_k \beta_k|,\ k = 1,2,\ldots$ *and* $\sum_{k=1}^{\infty} |\alpha_k \beta_k| > 0$. *For example, all elements* $\alpha_k, \beta_k \in \mathbb{C}$ *with* $\alpha_k = |\alpha_k| e^{i\varphi_k}$ *and* $\beta_k = |\beta_k| e^{i\varphi_k}$ *where* φ_k *is in* $0 \leq \varphi_k < 2\pi,\ k = 1,2,\ldots$ *are acceptable.* □

Remark. *At this point, we mention that, due to (2.14) and (2.17), it follows that we have the convergence*

$$P^{(n)} = \sum_{j=1}^{n} P_j \to P\ (n \to \infty)$$

and

$$T^{(n)} = \sum_{j=1}^{n} P_j T P_j \to T\ (n \to \infty)$$

in $B(H)$ *so that, e.g., the operators* T *and* P *defined in (3.2) and (3.3) are approximated by their partial sums not only pointwise, but even in the norm of* $B(H)$. □

4. Generalized Rayleigh-Quotient Formulas for the Real Parts of the Eigenvalues

In the sequel, we suppose that the non-zero eigenvalues are arranged according to

$$Re\lambda_1 \geq Re\lambda_2 \geq Re\lambda_3 \geq \ldots. \tag{4.1}$$

Such an arrangement is possible, for instance, if the real parts of all eigenvalues are positive. An arrangement that is always possible will be dealt with in Section 11.

One has the following generalized max-representation.

Theorem 4.1. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of* T *be arranged according to (4.1). Moreover, let the vector spaces* $M_{\chi,j,\mathbb{R}}$ *resp.* $M_{\psi,j,\mathbb{R}}$ *for* $j \in J$ *be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$Re\,\lambda_j = \max_{\substack{(u,v)>>0 \\ u \in M_{\chi,j,\mathbb{R}},\, v \in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)},\ j \in J. \tag{4.2}$$

The maximum is attained for $u = \chi_j,\ v = \psi_j$.

Proof. One uses (3.17) as starting point, i.e.,

$$Re\,(Tu,v) = \sum_{j\in J} Re\,\lambda_j (u,\psi_j)(\chi_j,v),\; u\in N_{\chi,\mathbb{R}},\, v\in N_{\psi,\mathbb{R}}$$

with

$$Re\,\lambda_j,\; (u,\psi_j),\; (\chi_j,v)\in\mathbb{R},\; u\in N_{\chi,\mathbb{R}},\; v\in N_{\psi,\mathbb{R}}.$$

Let $j\in J$ be arbitrarily chosen, but fixed as well as $u\in M_{\chi,j,\mathbb{R}}\subset N_{\chi,\mathbb{R}}$ and $v\in M_{\psi,j,\mathbb{R}}\subset N_{\psi,\mathbb{R}}$ also be arbitrarily chosen, but fixed with $(u,v)>>0$. Then,

$$\begin{aligned} Re\,(Tu,v) &= \sum_{k=j}^{\infty} Re\,\lambda_k (u,\psi_k)(\chi_k,v) \\ &\le \max_{k=j,j+1,\ldots} Re\,\lambda_k \sum_{k=j}^{\infty}(u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j \sum_{k=1}^{\infty}(u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j (u,v),\; u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}},\; (u,v)>>0, \end{aligned}$$

that is,

$$\frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_j,\; u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}},\; (u,v)>>0$$

and thus

$$\max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_j.$$

Now, $Re\,\lambda_j$ is attained for $u=\chi_j\in M_{\chi,j,\mathbb{R}}$ and $v=\psi_j\in M_{\psi,j,\mathbb{R}}$. Thus, because of $(\chi_j,\psi_j)>>0$,

$$Re\,\lambda_j = \frac{Re\,(T\chi_j,\psi_j)}{(\chi_j,\psi_j)} \le \max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_j$$

so that (4.2) is proven. □

For the next theorem, we need the following *denotation of codimension*. A subspace $M\subset H$ has codimension j for $j\in J$ denoted by $\mathbf{codim\,M=j}$ if there exist linearly independent vectors $v_1,\ldots,v_j\in H$ such that

$$M=[v_1,\ldots,v_j]^{\perp} := [v_1,\ldots,v_j]_H^{\perp} = \{u\in H\,|\,(u,v_k)=0,\; k=1,\ldots,j\}.$$

Further, we set

$$\operatorname{codim} M = 0$$

if $M=H$. Next, we prove a generalized min-max-representation.

Theorem 4.2. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (4.1).*

Then, for every $j\in J$ and every subspace $M_\chi\subset N_{\chi,\mathbb{R}}$ and $M_\psi\subset N_{\psi,\mathbb{R}}$ with $\operatorname{codim} M_\chi = \operatorname{codim} M_\psi = j-1$, the following inequalities are valid:

$$Re\,\lambda_j \le \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_1, \tag{4.3}$$

and the following min-max-representation formulas hold:

$$Re\,\lambda_j = \min_{\substack{\operatorname{codim} M_\chi=j-1\\ \operatorname{codim} M_\psi=j-1}} \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)},\; j\in J. \tag{4.4}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\; M_\psi = M_{\psi,j,\mathbb{R}}. \tag{4.5}$$

Proof. (4.3): For all subspaces $M_\chi \subset N_{\chi,\mathbb{R}}$, one has

$$\max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq \max_{\substack{(u,v)>>0\\ u\in N_{\chi,\mathbb{R}}, v\in N_{\psi,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} = Re\,\lambda_1. \tag{4.6}$$

In case $j=1$, it follows by definition of codim $M_\chi =$ codim $M_\psi = 0$ that $M_\chi = N_{\chi,\mathbb{R}}$ and $M_\psi = N_{\psi,\mathbb{R}}$ and thus the equal sign in (4.3); further, (4.4) reduces to (4.3) with the equal signs instead of the signs $\leq$. Now, let $j \geq 2$. Then, there exist linearly independent vectors $u_1,\ldots,u_{j-1}$ and $v_1,\ldots,v_{j-1}$ with

$$M_\chi = [u_1,\ldots,u_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}}, \qquad M_\psi = [v_1,\ldots,v_{j-1}]^{\perp}_{N_{\psi,\mathbb{R}}}. \tag{4.7}$$

Define

$$z_\chi = \sum_{i=1}^{j} \alpha_i \chi_i$$

and determine the coefficients $\alpha_1,\ldots,\alpha_j$ by the $j-1$ linear equations

$$(z_\chi, u_k) = \sum_{i=1}^{j} \alpha_i (\chi_i, u_k) = 0,\; k = 1,\ldots,j-1. \tag{4.8}$$

This system of $j-1$ linear equations and j unknowns has a nontrivial solution

$$z_\chi \neq 0,\; z_\chi \in M_\chi = [u_1,\ldots,u_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}}. \tag{4.9}$$

Now, define

$$z_\psi = \sum_{i=1}^{j} \alpha_i \psi_i \tag{4.10}$$

with the same coefficients α_i as in z_χ. Then,

$$z_\psi \neq 0. \tag{4.11}$$

Further,

$$(z_\chi, z_\psi) = \sum_{i=1}^{j} \alpha_i^2 > 0 \tag{4.12}$$

so that $(z_\chi, z_\psi) >> 0$. Moreover,

$$z_\psi \in [z_\psi]_{\mathbb{R}} \subset M_{\psi,z_\psi} \tag{4.13}$$

where M_{ψ,z_ψ} is any subspace of $N_{\psi,\mathbb{R}}$ with codimension $j-1$ containing the element z_ψ. From the above, it follows

$$Re\,(Tz_\chi, z_\psi) = \sum_{i,k=1}^{j} \alpha_i\, Re\,\lambda_i\, \alpha_k\,(\chi_i, \psi_k) = \sum_{i=1}^{j} Re\,\lambda_i\, \alpha_i^2. \tag{4.14}$$

Now, $\alpha_i \in \mathbb{R}$, $i = 1,\ldots,j$. Therefore,

$$Re\,(Tz_\chi, z_\psi) \geq (\min_{i=1,\ldots,j} Re\,\lambda_i) \sum_{i=1}^{j} \alpha_i^2 = Re\,\lambda_j\,(z_\chi, z_\psi) \tag{4.15}$$

leading to

$$\frac{Re\,(Tz_\chi, z_\psi)}{(z_\chi, z_\psi)} \geq Re\,\lambda_j. \tag{4.16}$$

Moreover, due to (4.1),

$$Re\,(Tu,v) \le (\max_{j=1,2,\ldots} Re\,\lambda_j)\sum_{j=1}^{\infty}(u,\psi_j)\,(\chi_j,v) = Re\,\lambda_1\,(u,v),$$

$(u,v) >> 0,\, u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}}$ so that

$$Re\,\lambda_1 \ge \frac{(Tu,v)}{(u,v)},\ (u,v) >> 0, u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}}. \tag{4.17}$$

This implies

$$Re\,\lambda_j \le \frac{Re\,(Tz_\chi,z_\psi)}{(z_\chi,z_\psi)} \le \max_{\substack{(u,v)>>0\\ u\in M_{\chi,z_\chi}, v\in M_{\psi,z_\psi}}} \frac{Re\,(Tu,v)}{(u,v)} \le \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \le Re\,\lambda_1. \tag{4.18}$$

Therefore, (4.3) is proven.

Proof of (4.4): From (4.3), we conclude

$$\min_{\substack{\text{codim}\, M_\chi=j-1\\ \text{codim}\, M_\psi=j-1}} \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \ge Re\,\lambda_j. \tag{4.19}$$

On the other hand, from Theorem 4.1,

$$Re\,\lambda_j = \max_{\substack{(u,v)>>0\\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \ge \min_{\substack{\text{codim}\, M_\chi=j-1\\ \text{codim}\, M_\psi=j-1}} \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \tag{4.20}$$

since

$$M_{\chi,j,\mathbb{R}} = \overline{[\chi_j,\chi_{j+1},\ldots]}_{\mathbb{R}} = [\psi_1,\ldots,\psi_{j-1}]^{\perp}_{N_{\chi,\mathbb{R}}} \tag{4.21}$$

and

$$M_{\psi,j,\mathbb{R}} = \overline{[\psi_j,\psi_{j+1},\ldots]}_{\mathbb{R}} = [\chi_1,\ldots,\chi_{j-1}]^{\perp}_{N_{\psi,\mathbb{R}}} \tag{4.22}$$

so that codim $M_{\chi,j,\mathbb{R}} = j-1$ and codim $M_{\psi,j,\mathbb{R}} = j-1$.

Relations (4.19) and (4.20) imply (4.4).

The last assertion follows from (4.21) and (4.22). □

The next theorem contains a generalized min-representation.

Theorem 4.3. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (4.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11). Then,*

$$Re\,\lambda_j = \min_{\substack{(u,v)>>0\\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)},\ j \in J. \tag{4.23}$$

The minimum is attained for $u = \chi_j$, $v = \psi_j$.

Proof. Due to (3.17),

$$Re\,(Tu,v) = \sum_{j\in J} Re\,\lambda_j (u,\psi_j)(\chi_j,v),\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}$$

with

$$Re\,\lambda_j,\ (u,\psi_j),\ (\chi_j,v) \in \mathbb{R},\ u \in N_{\chi,\mathbb{R}},\ v \in N_{\psi,\mathbb{R}}.$$

Let $j \in J$ be arbitrarily chosen, but fixed as well as $u \in N_{\chi,j,\mathbb{R}} \subset N_{\chi,\mathbb{R}}$ and $v \in N_{\psi,j,\mathbb{R}} \subset N_{\psi,\mathbb{R}}$ also be arbitrarily chosen, but fixed with $(u,v) >> 0$. Then, with (4.1),

$$\begin{aligned} Re\,(Tu,v) &= \sum_{k=1}^{j} Re\,\lambda_k (u,\psi_k)(\chi_k,v) \\ &\geq \min_{k=1,\ldots,j} Re\,\lambda_k \sum_{k=1}^{j} (u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j \sum_{k=1}^{j} (u,\psi_k)(\chi_k,v) \\ &= Re\,\lambda_j (u,v), \end{aligned}$$

that is,

$$\frac{Re\,(Tu,v)}{(u,v)} \geq Re\,\lambda_j,\; u \in N_{\chi,j,\mathbb{R}}, v \in N_{\psi,j,\mathbb{R}},\; (u,v) >> 0$$

and therefore,

$$\min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \geq Re\,\lambda_j.$$

Now, $Re\,\lambda_j$ is attained for $u = \chi_j \in N_{\chi,j,\mathbb{R}}$ and $v = \psi_j \in N_{\psi,j,\mathbb{R}}$, that is,

$$Re\,\lambda_j = \frac{Re\,(T\chi_j,\psi_j)}{(\chi_j,\psi_j)} \geq \min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \geq Re\,\lambda_j$$

so that (4.23) is proven. ☐

Next, we derive the following generalized max-min-representation of $Re\lambda_j$.

Theorem 4.4. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (4.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11).*

Then, for every $j \in J$ and every subspace $N_\chi \subset N_{\chi,\mathbb{R}}$ and $N_\psi \subset N_{\psi,\mathbb{R}}$ with $\dim N_\chi = \dim N_\psi = j$, the following inequalities are valid:

$$\min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq Re\,\lambda_j, \tag{4.24}$$

and the following max-min-representation formulas hold:

$$Re\,\lambda_j = \max_{\substack{dim N_\chi = j \\ dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)},\; j \in J. \tag{4.25}$$

The maximum is attained for

$$N_\chi = N_{\chi,j,\mathbb{R}},\; N_\psi = N_{\psi,j,\mathbb{R}}. \tag{4.26}$$

Proof. Let $j \in J$, and let $N_\chi \subset N_{\chi,\mathbb{R}}$ as well as $N_\psi \subset N_{\psi,\mathbb{R}}$ be subspaces with $dim N_\chi = dim N_\psi = j$. Then,

$$\min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq \min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \leq \frac{Re\,(T\chi_j,\psi_j)}{(\chi_j,\psi_j)} = Re\,\lambda_j \tag{4.27}$$

so that (4.24) follows. From (4.27), we conclude

$$\max_{\substack{dim N_\chi = j \\ dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq Re\,\lambda_j. \tag{4.28}$$

Further, (4.23) implies

$$Re\,\lambda_j = \min_{\substack{(u,v)>>0 \\ u\in N_{\chi,j,\mathbb{R}},v\in N_{\psi,j,\mathbb{R}}}} \frac{Re\,(Tu,v)}{(u,v)} \leq \max_{\substack{dim N_\chi = j \\ dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u\in N_\chi, v\in N_\psi}} \frac{Re\,(Tu,v)}{(u,v)} \leq Re\lambda_j. \tag{4.29}$$

From (4.28) and (4.29), we deduce (4.25) and that the maximum is attained for $N_\chi = N_{\chi,j,\mathbb{R}}$, $N_\psi = N_{\psi,j,\mathbb{R}}$. □

Changes in the Finite-Dimensional Case

In this case, the Hilbert space H over the field $\mathbb{F} = \mathbb{C}$ can be identified with $\mathbb{C}^n$ and the compact operator with an $n \times n$-matrix. Further, one has $J = (1,\ldots,n)$ instead of $J = (1,2,\ldots)$, and (4.1) is replaced by

$$Re\,\lambda_1 \geq \ldots \geq Re\,\lambda_n.$$

Moreover, the condition $\lim_{j\to\infty} \lambda_j = 0$ is omitted.

As a consequence, Theorems 4.1 - 4.4 deliver [16, Theorems 4 - 7] where the proofs of the theorems in this paper are essentially different from those in [16]. Beyond this, the proof of Theorem 4.2 is more detailed than the proof of [16, Theorem 5.]

5. Generalized Rayleigh-Quotient Formulas for the Imaginary Parts of the Eigenvalues

In this section, we want to state formulas for the representation of the imaginary parts of the eigenvalues of the compact operator $0 \neq T \in B(H)$ by Rayleigh quotients that generalize existing ones. We remind the reader that, in this paper beginning with Section 3, all eigenvalues are assumed to be simple. We obtain max-, min-max-, min-, and max-min-representations corresponding to those in Section 4.

Similarly to (4.1) we suppose that the eigenvalues of the compact operator T are arranged such that

$$Im\,\lambda_1 \geq Im\,\lambda_2 \geq Im\,\lambda_3 \geq \ldots. \tag{5.1}$$

First, we want to state a relation corresponding to that of (3.17).

Lemma 5.1. *Let the conditions (C1) - (C4) be fulfilled. Then, with the denotations of Theorem 3.1,*

$$Im\,(Tu,v) = \sum_{j\in J} Im\,\lambda_j (u,\psi_j)(\chi_j,v),\; u\in N_{\chi,\mathbb{R}},\; v\in N_{\psi,\mathbb{R}}. \tag{5.2}$$

Proof. Equation (5.2) follows directly from Theorem 3.4, Formulas (3.14) and (3.16). □

One has a series of theorems for the imaginary parts of the eigenvalues corresponding to those of Theorems 4.1 - 4.4 in Section 4. These Theorems 5.2 - 5.5 are stated without proofs since the only difference is that (5.1) and (5.2) are used instead of (4.1) and (3.17).

Theorem 5.2. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}}$ resp. $M_{\psi,j,\mathbb{R}}$ for $j\in J$ be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$Im\,\lambda_j = \max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}},v\in M_{\psi,j,\mathbb{R}}}} \frac{Im\,(Tu,v)}{(u,v)},\; j\in J. \tag{5.3}$$

The maximum is attained for $u=\chi_j$, $v=\psi_j$.

Theorem 5.3. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1).*

Then, for every $j \in J$ and every subspace $M_\chi \subset N_{\chi,\mathbb{R}}$ and $M_\psi \subset N_{\psi,\mathbb{R}}$ with $\operatorname{codim} M_\chi = \operatorname{codim} M_\psi = j-1$*, the following inequalities are valid:*

$$Im\,\lambda_j \le \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Im\,(Tu,v)}{(u,v)} \le Im\,\lambda_1, \tag{5.4}$$

and the following min-max-representation formulas hold:

$$Im\,\lambda_j = \min_{\substack{\operatorname{codim} M_\chi=j-1\\ \operatorname{codim} M_\psi=j-1}} \max_{\substack{(u,v)>>0\\ u\in M_\chi, v\in M_\psi}} \frac{Im\,(Tu,v)}{(u,v)},\ j\in J. \tag{5.5}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\ M_\psi = M_{\psi,j,\mathbb{R}}. \tag{5.6}$$

The next theorem contains a generalized min-representation of $Im\,\lambda_j$.

Theorem 5.4. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11). Then,*

$$Im\,\lambda_j = \min_{\substack{(u,v)>>0\\ u\in N_{\chi,j,\mathbb{R}}, v\in N_{\psi,j,\mathbb{R}}}} \frac{Im\,(Tu,v)}{(u,v)},\ j\in J. \tag{5.7}$$

The minimum is attained for $u = \chi_j$, $v = \psi_j$.

Next, we derive the following generalized max-min-representation of $Im\,\lambda_j$.

Theorem 5.5. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (5.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11).*

Then, for every $j \in J$ and every subspace $N_\chi \subset N_{\chi,\mathbb{R}}$ and $N_\psi \subset N_{\psi,\mathbb{R}}$ with $\dim N_\chi = \dim N_\psi = j$*, the following inequalities are valid:*

$$\min_{\substack{(u,v)>>0\\ u\in N_\chi, v\in N_\psi}} \frac{Im\,(Tu,v)}{(u,v)} \le Im\,\lambda_j, \tag{5.8}$$

and the following max-min-representation formulas hold:

$$Im\,\lambda_j = \max_{\substack{\dim N_\chi=j\\ \dim N_\psi=j}} \min_{\substack{(u,v)>>0\\ u\in N_\chi, v\in N_\psi}} \frac{Im\,(Tu,v)}{(u,v)},\ j\in J. \tag{5.9}$$

The maximum is attained for

$$N_\chi = N_{\chi,j,\mathbb{R}},\ N_\psi = N_{\psi,j,\mathbb{R}}. \tag{5.10}$$

Changes in the Finite-Dimensional Case

In this case, the Hilbert space H over the field $\mathbb{F} = \mathbb{C}$ can be identified with $\mathbb{C}^n$ and the compact operator with an $n\times n$-matrix. Further, one has $J = (1,\ldots,n)$ instead of $J = (1,2,\ldots)$, and (5.1) is replaced by

$$Im\,\lambda_1 \ge \ldots \ge Im\,\lambda_n.$$

Further, the condition $\lim_{j\to\infty} \lambda_j = 0$ is omitted.

As a consequence, Theorems 5.2 - 5.5 deliver [16, Theorems 9 - 12].

6. Generalized Rayleigh-Quotient Formulas for the Moduli of the Eigenvalues

Whereas in Sections 4 and 5 max-, min-max-, min-, and max-min-representations with generalized Rayleigh quotients could be obtained, it seems that, for the moduli of the eigenvalues, only a max-representation is possible.

We now deduce this max-representation. For this, we suppose that the eigenvalues $\lambda_1, \lambda_2, \ldots$ are arranged such that

$$|\lambda_1| \geq |\lambda_2| \geq \ldots . \tag{6.1}$$

Herewith, one has the following theorem.

Theorem 6.1. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (6.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}}$ resp. $M_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$|\lambda_j| = \max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)}, \; j \in J. \tag{6.2}$$

The maximum is attained for $u = \chi_j$, $v = \psi_j$.

Proof. One uses (3.14) and (3.16) as starting point, i.e.,

$$(Tu,v) = \sum_{j\in J} \lambda_j (u,\psi_j)(\chi_j,v), \; u \in N_{\chi,\mathbb{R}}, \, v \in N_{\psi,\mathbb{R}}$$

with

$$(u,\psi_j), \; (\chi_j,v) \in \mathbb{R}, \; u \in N_{\chi,\mathbb{R}}, \; v \in N_{\psi,\mathbb{R}}.$$

Let $j \in J$ be arbitrarily chosen, but fixed as well as $u \in M_{\chi,j,\mathbb{R}} \subset N_{\chi,\mathbb{R}}$ and $v \in M_{\psi,j,\mathbb{R}} \subset N_{\psi,\mathbb{R}}$ also be arbitrarily chosen, but fixed with $(u,v) >> 0$. Then,

$$\begin{aligned} |(Tu,v)| &= |\sum_{k=j}^{\infty} \lambda_k (u,\psi_k)(\chi_k,v)| \\ &\leq \max_{k=j,j+1,\ldots} |\lambda_k| \sum_{k=j}^{\infty} (u,\psi_k)(\chi_k,v) \\ &= |\lambda_j| \sum_{k=1}^{\infty} (u,\psi_k)(\chi_k,v) \\ &= |\lambda_j| (u,v), \; u \in M_{\chi,j,\mathbb{R}}, v \in M_{\psi,j,\mathbb{R}}, \; (u,v) >> 0, \end{aligned}$$

that is,

$$\frac{|(Tu,v)|}{(u,v)} \leq |\lambda_j|, \; u \in M_{\chi,j,\mathbb{R}}, v \in M_{\psi,j,\mathbb{R}}, \; (u,v) >> 0$$

and thus

$$\max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)} \leq |\lambda_j|.$$

Now, $|\lambda_j|$ is attained for $u = \chi_j \in M_{\chi,j,\mathbb{R}}$ and $v = \psi_j \in M_{\psi,j,\mathbb{R}}$. Thus, because of $(\chi_j,\psi_j) >> 0$,

$$|\lambda_j| = \frac{|(T\chi_j,\psi_j)|}{(\chi_j,\psi_j)} \leq \max_{\substack{(u,v)>>0 \\ u\in M_{\chi,j,\mathbb{R}}, v\in M_{\psi,j,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)} \leq |\lambda_j|$$

so that (6.2) is proven. □

7. Generalized Rayleigh-Quotient Formulas for Real Eigenvalues

When all eigenvalues of a compact operator T are real and simple, then

$$\sigma(T) \subset \mathbb{R}$$

and

$$Re\,\lambda_j = \lambda_j,\ j = 1,2,\ldots.$$

We mention that, in particular, $\lambda_j(T^*T) \in \mathbb{R}_0^+ := \{x \in \mathbb{R}\,|\,x \geq 0\} \subset \mathbb{R}$. For $\sigma(T) \subset \mathbb{R}$, from Section 4 one gets the following corollaries where correspondingly to (4.1), we suppose that the eigenvalues are arranged such that

$$\lambda_1 \geq \lambda_2 \geq \ldots. \tag{7.1}$$

The corollaries are obtained as follows.

Corollary 7.1. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1). Moreover, let the vector spaces $M_{\chi,j,\mathbb{R}}$ resp. $M_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.21), (3.22) resp. (3.23), (3.24) or (3.26) resp. (3.27). Then,*

$$\lambda_j = \max_{\substack{(u,v)>>0 \\ u \in M_{\chi,j,\mathbb{R}},\, v \in M_{\psi,j,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)},\ j \in J. \tag{7.2}$$

The maximum is attained for $u = \chi_j$, $v = \psi_j$.

Corollary 7.2. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1).*

Then, for every $j \in J$ and every subspace $M_\chi \subset N_{\chi,\mathbb{R}}$ and $M_\psi \subset N_{\psi,\mathbb{R}}$ with $\operatorname{codim} M_\chi = \operatorname{codim} M_\psi = j-1$, the following inequalities are valid:

$$\lambda_j \leq \max_{\substack{(u,v)>>0 \\ u \in M_\chi,\, v \in M_\psi}} \frac{(Tu,v)}{(u,v)} \leq \lambda_1, \tag{7.3}$$

and the following min-max-representation formulas hold:

$$\lambda_j = \min_{\substack{\operatorname{codim} M_\chi = j-1 \\ \operatorname{codim} M_\psi = j-1}} \max_{\substack{(u,v)>>0 \\ u \in M_\chi,\, v \in M_\psi}} \frac{(Tu,v)}{(u,v)},\ j \in J. \tag{7.4}$$

The minimum is attained for

$$M_\chi = M_{\chi,j,\mathbb{R}},\ M_\psi = M_{\psi,j,\mathbb{R}}. \tag{7.5}$$

The next corollary contains a generalized min-representation of λ_j.

Corollary 7.3. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11). Then,*

$$\lambda_j = \min_{\substack{(u,v)>>0 \\ u \in N_{\chi,j,\mathbb{R}},\, v \in N_{\psi,j,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)},\ j \in J. \tag{7.6}$$

The minimum is attained for $u = \chi_j$, $v = \psi_j$.

Corollary 7.4. *Let the conditions (C1) - (C4) be fulfilled. Further, let the eigenvalues of T be arranged according to (7.1). Moreover, let the vector spaces $N_{\chi,j,\mathbb{R}}$ resp. $N_{\psi,j,\mathbb{R}}$ for $j \in J$ be defined by (3.7) resp. (3.11).*

Then, for every $j \in J$ and every subspace $N_\chi \subset N_{\chi,\mathbb{R}}$ and $N_\psi \subset N_{\psi,\mathbb{R}}$ with $\dim N_\chi = \dim N_\psi = j$, the following inequalities are valid:

$$\min_{\substack{(u,v)>>0 \\ u \in N_\chi, v \in N_\psi}} \frac{(Tu,v)}{(u,v)} \leq \lambda_j, \tag{7.7}$$

and the following max-min-representation formulas hold:

$$\lambda_j = \max_{\substack{\dim N_\chi = j \\ \dim N_\psi = j}} \min_{\substack{(u,v)>>0 \\ u \in N_\chi, v \in N_\psi}} \frac{(Tu,v)}{(u,v)}, \; j \in J. \tag{7.8}$$

The maximum is attained for $N_\chi = N_{\chi,j,\mathbb{R}}$, $N_\psi = N_{\psi,j,\mathbb{R}}$.

Changes in the Finite-Dimensional Case

In this case, the Hilbert space H over the field $\mathbb{F} = \mathbb{C}$ can be identified with $\mathbb{C}^n$ and the compact operator with an $n \times n$-matrix. Further, one has $J = (1, \dots, n)$ instead of $J = (1, 2, \dots,$ and (7.1) is replaced by

$$\lambda_1 \geq \ldots \geq \lambda_n.$$

Further, the condition $\lim_{j\to\infty} \lambda_j = 0$ is omitted.

As a consequence, Corollaries 7.1 - 7.4 deliver [16, Corollaries 14 - 17].

8. Application to New Formula for Spectral Radius

In this section, an application of the obtained results is presented. More precisely, a new formula for the spectral radius $\rho(T)$ is derived. First, known formulas for this quantity are recapitulated.

Known formulas for the spectral radius $\rho(T)$

Let the conditions (C1) - (C4) be fulfilled. One formula is given by

$$\rho(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}, \tag{8.1}$$

see [10, Chapter I, p. 27], where the expression on the right-hand member of (8.1) is independent of the norm $\|\cdot\|$.

If $\mathbb{F} = \mathbb{C}$, another representation is

$$\rho(T) = \max_{j=1,2,\ldots} |\lambda_j|, \tag{8.2}$$

cf. [10, Chapter I, (5.10), p. 38].

New formula for the spectral radius $\rho(T)$

Let the conditions (C1) - (C4) be fulfilled, and let the eigenvalues of T be arranged according to (6.1).

Then, from Theorem 6.1, as Application, we deduce the new formula

$$\rho(T) = \max_{\substack{(u,v)>>0 \\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)}. \tag{8.3}$$

Proof. This follows from (6.2) as well as $M_{\chi,1,\mathbb{R}} = N_{\chi,\mathbb{R}}$, $M_{\psi,1,\mathbb{R}} = N_{\psi,\mathbb{R}}$ according to (3.21) and (3.23) as well as (3.9) and (3.13) since

$$\rho(T) = \max_{j=1,2,\ldots} |\lambda_j| = |\lambda_1| = \max_{\substack{(u,v)>>0 \\ u \in M_{\chi,1,\mathbb{R}}, v \in M_{\psi,1,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)} = \max_{\substack{(u,v)>>0 \\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}}} \frac{|(Tu,v)|}{(u,v)}. \tag{8.4}$$

□

9. New Generalized Numerical Ranges

In this section, a series of known numerical ranges is recapitulated, and new numerical ranges of a compact operator are defined. The new generalized numerical ranges are defined for compact operators with simple eigenvalues similarly as for diagonalizable matrices in [16].

Known numerical range of $T \in B(H)$ with respect to the Hilbert space H
Following [25, Section 5.4,(5)], the *numerical range* of $T \in B(H)$ is defined by

$$W_H(T) = \{z \in \mathbb{C} \,|\, z = \frac{(Tu,u)}{(u,u)},\ 0 \neq u \in H\} \tag{9.1}$$

which is a convex subset of $\mathbb{C}$. Applying this definition to T^*T instead to T, we obtain

$$W_H(T^*T) = \{x \in \mathbb{R}_0^+ \,|\, x = \frac{(T^*Tu,u)}{(u,u)} = \frac{(Tu,Tu)}{(u,u)} \geq 0,\ 0 \neq u \in H\} \tag{9.2}$$

which is a convex subset of $\mathbb{R}_0^+$. One has

$$W_H(T^*T) = [\inf_{j=1,2,\ldots} \lambda_j(T^*T), \sup_{j=1,2,\ldots} \lambda_j(T^*T)] = [\frac{1}{\|T^{-1}\|_2^2}, \|T\|_2^2] \tag{9.3}$$

where $1/\|T^{-1}\|_2^2$ has to be interpreted as zero if T^{-1} does not exist.

Generalized numerical range of $T \in B(H)$ with respect to the subspaces N_χ and N_ψ
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *generalized range* of T with respect to the subspaces N_χ and N_ψ by

$$W_{N_\chi,N_\psi,gen.}(T) = \{z \in \mathbb{C} \,|\, z = \frac{(Tu,v)}{(u,v)},\ (u,v) >> 0,\ u \in N_\chi, v \in N_\psi\} \tag{9.4}$$

Real part of the numerical range of $T \in B(H)$ with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *real part of the generalized numerical range* of T with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$ by

$$Re\,[W_{N_{\chi,\mathbb{R}},N_{\psi,\mathbb{R}},gen.}](T) = \{x \in \mathbb{R} \,|\, x = \frac{Re\,(Tu,v)}{(u,v)},\ (u,v) >> 0,\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}\}. \tag{9.5}$$

Imaginary part of the numerical range of $T \in B(H)$ with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *imaginary part of the generalized numerical range* of T with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$ by

$$Im\,[W_{N_{\chi,\mathbb{R}},N_{\psi,\mathbb{R}},gen.}](T) = \{y \in \mathbb{R} \,|\, y = \frac{Im\,(Tu,v)}{(u,v)},\ (u,v) >> 0,\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}\}. \tag{9.6}$$

Modulus of the generalized numerical range of $T \in B(H)$ with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$
Let the conditions (C1) - (C4) be fulfilled. Then, we define the *modulus of the generalized numerical range* of T with respect to the subspaces $N_{\chi,\mathbb{R}}$ and $N_{\psi,\mathbb{R}}$ by

$$|W_{N_{\chi,\mathbb{R}},N_{\psi,\mathbb{R}},gen.}(T)| = \{x \in \mathbb{R}_0^+ \,|\, x = \frac{|(Tu,v)|}{(u,v)},\ (u,v) >> 0,\ u \in N_{\chi,\mathbb{R}}, v \in N_{\psi,\mathbb{R}}\}. \tag{9.7}$$

10. Examples from the Area of Boundary Eigenvalue Problems

In this section, we check some of the formulas of Section 7 on an example of a nonsymmetric compact operator with nonnegative simple eigenvalues from the area of Mathematical Physics. More precisely, we check the validity of the following relation

$$\frac{(Tu,v)}{(u,v)} \in [\inf_{j=1,2,\ldots} \lambda_j(T), \sup_{j=1,2,\ldots} \lambda_j(T)]$$

for a series of vectors $u \in N_{\chi,\mathbb{R}}$, $v \in N_{\psi,\mathbb{R}}$ with $(u,v) >> 0$, which is a consequence of Theorems 7.2 and 7.4.

10.1 A Non-Selfadjoint BEVP with Ordinary Differential Operator of 2nd Order

(i) *The Differential Operators L and L_+ and Pertinent BEVPs*

As an example, we choose the non-selfadjoint Boundary Eigenvalue Problem (for short: BEVP) with ordinary differential operator of 2nd order in [18]. The differential operator L is given by

$$(Lu)(x) = -u''(x) + p_0\, u'(x) + q_0\, u(x),\ 0 \le x \le l \tag{10.1}$$

with the real constants p_0, q_0 where we restrict q_0 to $q_0 > 0$ and with the boundary conditions

$$u(0) = u(l) = 0. \tag{10.2}$$

The formally adjoint differential operator L_+ is given by

$$(L_+ v)(x) = -v''(x) - p_0\, v'(x) + q_0\, v(x),\ 0 \le x \le l \tag{10.3}$$

with the boundary conditions

$$v(0) = v(l) = 0. \tag{10.4}$$

The pertinent BEVPs read

$$\pi_{2,\mu} :\ Lu = \mu u,\ u \in H_D = D(L) \tag{10.5}$$

where

$$H_D = \{u \in C^2[0,l] \,|\, u(0) = u(l) = 0\} \tag{10.6}$$

and

$$\pi_{2,\overline{\mu},+} :\ L_+ v = \overline{\mu} v,\ v \in H_{D,+} = D(L_+). \tag{10.7}$$

where

$$H_{D,+} = H_D. \tag{10.8}$$

(ii) The Eigenvalues and Eigenfunctions

The eigenvalues of L and L_+ are given by

$$\mu = \overline{\mu} = \mu_j = \overline{\mu}_j = \frac{j^2\pi^2}{l^2} + D,\ j \in J \tag{10.9}$$

with the quantity

$$D = D(p_0, q_0) = (\frac{p_0}{2})^2 + q_0 \tag{10.10}$$

so that

$$\lambda_j = \frac{1}{\mu_j} = \frac{1}{\frac{j^2\pi^2}{l^2} + D},\ j \in J. \tag{10.11}$$

The biorthonormal eigenfunctions are found to be

$$\chi_j(x) = \sqrt{\frac{2}{l}}\, exp(\frac{p_0}{2}x)\, \sin j\pi\frac{x}{l},\ 0 \le x \le l,\ j \in J \tag{10.12}$$

and

$$\psi_j(x) = \sqrt{\frac{2}{l}}\, exp(-\frac{p_0}{2}x)\, \sin j\pi\frac{x}{l},\ 0 \le x \le l,\ j \in J \tag{10.13}$$

so that we have

$$(\chi_j, \psi_k) = \int_0^l \chi_j(x)\, \psi_k(x)\, dx = \frac{2}{l} \int_0^l \sin j\pi \frac{x}{l} \sin k\pi \frac{x}{l}\, dx = \delta_{jk},\ 0 \le x \le l,\ j,k \in J. \tag{10.14}$$

(iii) The Green's Function of $L_{p_0,q_0}\, u = 0,\ u(0) = u(l) = 0$
A set of fundamental solutions of $L_{p_0 \cdot q_0} = 0$, i.e., when $\mu = 0$, is given by

$$u_1(x) = e^{\frac{p_0}{2}x} \sinh \sqrt{D} x \tag{10.15}$$

$$u_2(x) = e^{\frac{p_0}{2}x} \cosh \sqrt{D} x \tag{10.16}$$

with

$$D = D(p_0, q_0) = (\frac{p_0}{2})^2 + q_0$$

in (10.10). Based on these fundamental solutions, the Green's functions pertinent to the BVPs $L_{p_0 \cdot q_0}\, u = 0,\ u(0) = u(l) = 0$ resp. $L_{+,p_0 \cdot q_0}\, v = 0,\ v(0) = v(l) = 0$ are given by

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{\sinh \sqrt{D}x\ \sinh \sqrt{D}\,(l-s)}{\sqrt{D} \sinh \sqrt{D}\, l} \exp\Big(\dfrac{p_0}{2}(x-s)\Big),\ 0 \le x \le s \le l, \\ G_2(x,s) = \dfrac{\sinh \sqrt{D}\,(l-x)\ \sinh \sqrt{D}s}{\sqrt{D} \sinh \sqrt{D}\, l} \exp\Big(\dfrac{p_0}{2}(x-s)\Big)\ 0 \le s \le x \le l, \end{cases} \tag{10.17}$$

resp.

$$G_+(x,s) = \begin{cases} G_{+,1}(x,s) = \dfrac{\sinh \sqrt{D}\,(l-x)\ \sinh \sqrt{D}s}{\sqrt{D} \sinh \sqrt{D}\, l} \exp\Big(\dfrac{p_0}{2}(s-x)\Big),\ 0 \le x \le s \le l, \\ G_{+,2}(x,s) = \dfrac{\sinh \sqrt{D}x\ \sinh \sqrt{D}\,(l-s)}{\sqrt{D} \sinh \sqrt{D}\, l} \exp\Big(\dfrac{p_0}{2}(s-x)\Big)\ 0 \le s \le x \le l, \end{cases} \tag{10.18}$$

so that, because of $D = D(p_0, q_0)$,

$$G(x,s) = G(x,s; p_0, q_0) \tag{10.19}$$

and

$$G_+(x,s) = G^T(x,s) = G(s,x) = G(s,x; -p_0, q_0) \tag{10.20}$$

in accordance with the fact that, for the pertinent operators, one has $G_+ = G^T$, see [18].
(iv) The Compact Operators T and $T_+ = T^* = T^T$
The inverse operators $T := G := L_+^{-1}$ and $T_+ := G_+ := L_+^{-1}$ are given by

$$(Tu)(x) = (Gu)(x) = (L^{-1}u)(x) = \int_0^l G(x,s; p_0, q_0)\, u(s)\, ds,\ u \in C([0,l], \mathbb{R}) \subset C[0,l] \tag{10.21}$$

where $C([0,l], \mathbb{R})$ is the set of real-valued continuous functions on $[0,l]$ endowed with the norm $\|\cdot\|_2$, and

$$(T_+u)(x) = (G_+u)(x) = (L_+^{-1}u)(x) = \int_0^l G^T(x,s; -p_0, q_0)\, u(s)\, ds,\ u \in C([0,l], \mathbb{R}) \tag{10.22}$$

with the eigenvalues

$$\lambda_j(T) = \lambda_j(G) = \lambda_j(T^T) = \lambda_j(G^T) = \frac{1}{\mu_j(L)} = \frac{1}{\dfrac{j^2\pi^2}{l^2} + D},\ j \in J, \tag{10.23}$$

and the same eigenfunctions χ_j in (10.12) resp. ψ_j in (10.13). From (10.23), we have

$$\lim_{j\to\infty} \lambda_j(T) = 0. \tag{10.24}$$

Further,

$$\inf_{j=1,2,\dots} \lambda_j(T) = 0, \qquad \sup_{j=1,2,\dots} \lambda_j(T) = \lambda_1(T) = \frac{1}{\frac{\pi^2}{l^2} + D} = \frac{1}{\frac{\pi^2}{l^2} + (\frac{p_0}{2})^2 + q_0}. \tag{10.25}$$

Now, due to [18, Theorem 3.3, (3.14)] and since $\chi_j(x) \in \mathbb{R},\ 0 \le x \le l$, one has

$$C([0,l],\mathbb{R}) \subset N_{\chi,\mathbb{R}} \subset L_2(0,l).$$

Therefore, from (7.1) and (7.3), we obtain

$$0 \le \frac{(Tu,v)}{(u,v)} \le \frac{1}{\frac{\pi^2}{l^2} + (\frac{p_0}{2})^2 + q_0},\ (u,v) >> 0,\ u,v \in C([0,l],\mathbb{R}). \tag{10.26}$$

(v) Special case $p_0 = q_0 = 0$
We mention that, in the particular case $p_0 = q_0 = 0$, we obtain

$$\mu_j = \frac{j^2\pi^2}{l^2},\ j \in J,$$

$$\lambda_j = \frac{l^2}{j^2\pi^2},\ j \in J,$$

$$\chi_j(x) = \psi_j(x) = \varphi_j(x) = \sqrt{\frac{2}{l}}\sin j\pi\frac{x}{l},\ 0 \le x \le l,\ j \in J,$$

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{x(l-s)}{l},\ 0 \le x \le s \le l, \\ G_2(x,s) = \dfrac{s(l-x)}{l},\ 0 \le s \le x \le l. \end{cases}$$

In this special case, we have

$$\frac{(Tu,v)}{(u,v)} \in [0;l^2/\pi^2],\ (u,v) >> 0,\ u,v \in C([0,l];\mathbb{R})). \tag{10.27}$$

10.2 Computations with Computer Algebra

In the particular case $p_0 = q_0 = 0$, using the symbolic-function feature of Matlab, one obtains the following Table 10.1.

i_u	i_v	u	v	$(Tu,v))$	(u,v)	$(Tu,v))/(u,v)$
1	1	1	1	$\frac{l^3}{12}$	l	$\frac{1}{12}\,l^2$
2	1	x	1	$\frac{l^4}{24}$	$\frac{l^2}{2}$	$\frac{1}{12}\,l^2$
1	2	1	x	$\frac{l^4}{24}$	$\frac{l^2}{2}$	$\frac{1}{12}\,l^2$
3	1	x^2	1	$\frac{l^5}{40}$	$\frac{l^3}{3}$	$\frac{3}{40}\,l^2$
2	2	x	x	$\frac{l^5}{45}$	$\frac{l^3}{3}$	$\frac{1}{15}\,l^2$
1	3	1	x^2	$\frac{l^5}{40}$	$\frac{l^3}{3}$	$\frac{3}{40}\,l^2$
4	1	x^3	1	$\frac{l^6}{60}$	$\frac{l^4}{4}$	$\frac{1}{15}\,l^2$
3	2	x^2	x	$\frac{l^6}{72}$	$\frac{l^4}{4}$	$\frac{1}{18}\,l^2$
2	3	x	x^2	$\frac{l^6}{72}$	$\frac{l^4}{4}$	$\frac{1}{18}\,l^2$
1	4	1	x^3	$\frac{l^6}{60}$	$\frac{l^4}{4}$	$\frac{1}{15}\,l^2$
5	1	x^4	1	$\frac{l^7}{84}$	$\frac{l^5}{5}$	$\frac{5}{84}\,l^2$
4	2	x^3	x	$\frac{l^7}{105}$	$\frac{l^5}{5}$	$\frac{1}{21}\,l^2$
3	3	x^2	x^2	$\frac{l^7}{112}$	$\frac{l^5}{5}$	$\frac{5}{112}\,l^2$
2	4	x	x^3	$\frac{l^7}{105}$	$\frac{l^5}{5}$	$\frac{1}{21}\,l^2$
1	5	1	x^4	$\frac{l^7}{84}$	$\frac{l^5}{5}$	$\frac{5}{84}\,l^2$
6	1	$1+x$	1	$\frac{(l+2)\,l^3}{24}$	$\frac{l\,(l+2)}{2}$	$\frac{1}{12}\,l^2$
1	6	1	$1+x$	$\frac{(l+2)\,l^3}{24}$	$\frac{l\,(l+2)}{2}$	$\frac{1}{12}\,l^2$
6	6	$1+x$	$1+x$	$\frac{(4l^2+15l+15)\,l^3}{180}$	$l+l^2\,(\frac{l}{3}+1)$	$f(l)=g(l)\,l^2$

Table 10.1: Computer-Algebra Results

with

$$f(l)=\frac{(4l^2+15l+15)\,l^3}{180\,(l+l^2\,(\frac{l}{3}+1))}=\frac{4l^5+15l^4+15l^3}{60l^3+180l^2+180l}=l^2\,\frac{4l^2+15l+15}{60l^2+180l+180}=g(l)\,l^2$$

where

$$g(l)=\frac{4l^2+15l+15}{60l^2+180l+180}.$$

The function $y=g(x)$ for $0\leq x\leq 10$ is illustrated in *Fig. 10.1*.

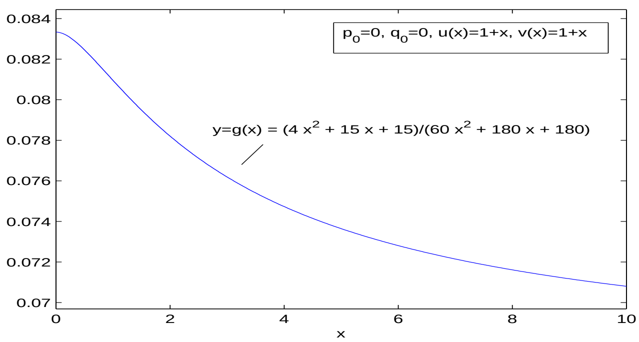

Fig. 10.1: Curve y=g(x) for $0\leq x\leq 10$

One has

$$\lim_{l\to 0} g(l)=\frac{15}{180}=\frac{1}{12}\in[0;\frac{1}{\pi^2}]$$

and

$$\lim_{l\to\infty} g(l) = \lim_{l\to\infty} \frac{4+\frac{15}{l}+\frac{15}{l^2}}{60+\frac{180}{l}+\frac{180}{l^2}} = \frac{4}{60} = \frac{1}{15} \in [0; \frac{1}{\pi^2}]$$

as well as

$$g'(x) = -\frac{x\,(x+2)}{20\,(x^2+3x+3)} < 0,\; x > 0$$

so that $y = g(x), x > 0$ is strictly monotonically decreasing. In *Fig. 10.2*, the curve $y = g(x)$ for $1 \le x \le 3$ is shown.

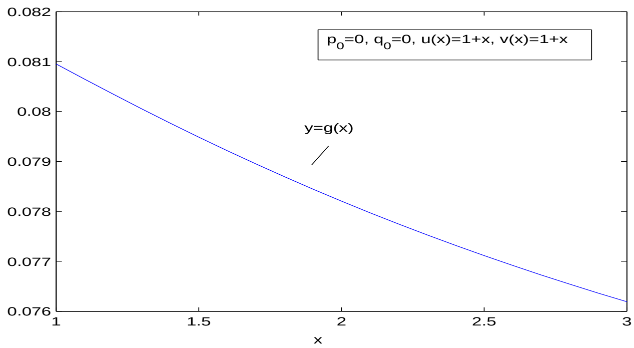

Fig. 10.2: Curve y=g(x) for $1 \le x \le 3$

At this point, we introduce the denotation of *reduced length*. Apparently,

$$\frac{(Tu,v)}{(u,v)} \in [0; \frac{1}{\pi^2}\, l^2]$$

for all values in *Table 10.1* which confirms (10.27) for $p_0 = q_0 = 0$, and for the largest eigenvalue of T, one has

$$\lambda_1(T) = \max_{\substack{(u,v)>>0\\ u\in N_{\chi,\mathbb{R}},\, v\in N_{\psi,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)} = \max_{\substack{(u,v)>>0\\ u,v\in C([0,l;\mathbb{R}])}} \frac{(Tu,v)}{(u,v)} = \frac{(T\chi_1,\psi_1)}{(\chi_1,\psi_1)} = \frac{l^2}{\pi^2}.$$

Correspondingly to this formula, for $u, v \in C([0,l],\mathbb{R})$ with $(u,v) >> 0$, we define the *reduced length* $l_{red,D=0}$ by

$$Q_{Ray} := \frac{(Tu,v)}{(u,v)} = \frac{l^2_{red,D=0}}{\pi^2}$$

implying

$$l^2_{red,D=0} = Q_{Ray}\,\pi^2.$$

For $u = \chi_1$, $v = \psi_1$, we get back

$$l^2_{red,D=0} = \frac{l^2}{\pi^2}\,\pi^2 = l^2$$

or

$$l_{red,D=0} = l,$$

as it must be. For $i_u = 5$, $i_v = 1$, i.e., for $u(x) = x^4,\; v(x) = 1$, *Table 10.1* delivers $Q_{Ray} = 5/84\, l^2$ and therefore

$$l_{red,D=0} = \pi\sqrt{\frac{5}{84}}\, l \doteq 0.766470\, l < l$$

and for $i_u = 6$, $i_v = 6$, i.e., for $u(x) = 1+x$, $v(x) = 1+x$, *Table 10.1* gives $Q_{Ray} \in [1/15\,l^2, 1/12\,l^2]$ so that

$$l_{red,D=0} \in \pi[\frac{1}{\sqrt{15}}\,l; \frac{1}{\sqrt{12}}\,l] \doteq [0.2581988\,l; 0.288675\,l] \subset [0,l]$$

The interpretation of $l_{red,D=0}$ is as follows. If the length l is replaced by $l_{red,D=0}$ for the index pair (i_u, i_v) resp. the pair of functions $u, v \in C([0,l], R)$ with $(u,v) >> 0$, then $\lambda_1(T) = \max\limits_{\substack{(u,v)>>0 \\ u,v \in C([0,l];R)}} \dfrac{(Tu,v)}{(u,v)}$ is attained for the pair of functions pertinent to the pair of indices (i_u, i_v) in *Table 10.1*.

10.3 Numerical Computations

If $p_0 \neq 0$ or $q_0 \neq 0$, then the results obtained by the Computer Algebra using the symbolic-function feature of Matlab get complicated. So, in this subsection, we use numerical integration methods to compute the Rayleigh quotients $(Tu,v)/(u,v)$. For the computation of

$$(Tu)(x) = \int_0^l G(x,s)\,u(s)\,ds = \int_0^x G_2(x,s)\,u(s)\,ds + \int_x^l G_1(x,s)\,u(s)\,ds,$$

we employ the Matlab routine *dblquad*, and for $(Tu,v) = \int_0^l (Tu)(x)\,v(x)\,dx$ as well as $(u,v) = \int_0^l u(x)\,v(x)\,dx$ the Matlab routine *quadl* that is based on the Simpson rule.

As to the *reduced length* $l_{red,D}$ for the general case when $D = (\frac{p_0}{2})^2 + q_0$ is not necessary equal to zero, we depart from the formula

$$\lambda_1(T) = \max_{\substack{(u,v)>>0 \\ u \in N_{\chi,\mathbb{R}},\, v \in N_{\psi,\mathbb{R}}}} \frac{(Tu,v)}{(u,v)} = \frac{1}{\frac{\pi^2}{l^2} + D} = \frac{l^2}{\pi^2 + D\,l^2}$$

since the maximum is attained for $u = \chi_1 \in C([0,l];\mathbb{R}) \subset N_{\chi,\mathbb{R}}$ and $v = \psi_1 \in C([0,l];\mathbb{R}) \subset N_{\chi,\mathbb{R}}$. In analogy to this formula, we define

$$Q_{Ray} = \frac{(Tu,v)}{(u,v)} := \frac{l^2_{red,D}}{\pi^2 + D\,l^2_{red,D}}$$

leading to

$$l^2_{red,D} = Q_{Ray}\,(\pi^2 + D\,l^2_{red,D}).$$

This implies

$$l^2_{red,D}\,(1 - D\,Q_{Ray}) = \pi^2\,Q_{Ray}$$

or

$$l^2_{red,D} = \pi^2\,\frac{Q_{Ray}}{1 - D\,Q_{Ray}}$$

leading to

$$l_{red,D} = \pi\,\frac{\sqrt{Q_{Ray}}}{\sqrt{1 - D\,Q_{Ray}}} = \pi\,\frac{\sqrt{\frac{(Tu,v)}{(u,v)}}}{\sqrt{1 - D\,\frac{(Tu,v)}{(u,v)}}}.$$

Special Case: $\boxed{u = \chi_1,\, v = \psi_1}$

In this case, we obtain

$$Q_{Ray} = \frac{(T\chi_1, \psi_1)}{(\chi_1, \psi_1)} = \lambda_1 = \frac{l^2}{\pi^2 + D\,l^2} = \frac{l^2_{red,D}}{\pi^2 + D\,l^2_{red,D}}$$

implying

$$l_{red,D} = l,$$

as it must be. In order to test the numerical computations, we begin with the special case $\boxed{p_0 = q_0 = 0}$. The pertinent computations for $y = (Tu,v)/(u,v)\,\frac{1}{l^2}$ with $u(x) = 1+x,\, v(x) = 1+x,\ 1 \le x \le 3$ deliver the same numerical values as for $y = g(l),\ 1 \le l \le 3$ in *Table 10.2*. This is illustrated in *Fig. 10.3*.

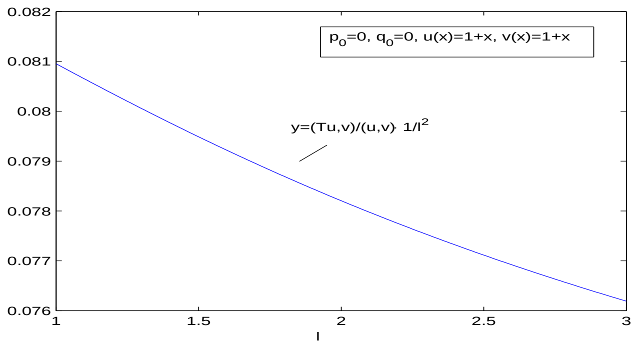

Fig. 10.3: Curve $y = ((Tu,v)/(u,v))/l^2$ for $0 \le l \le 3$

From this, one can expect that the numerical computations for the other pairs u, v of functions are reliable.

For $\boxed{p_0 = 0, q_0 = 1}$ and $\boxed{u(x) = 1,\ v(x) = 1}$, we have computed a series of variants for $l = 1.0(0.1)3.0$ given in *Table 10.2*.

k	l	$\frac{(Tu,v)}{(u,v))}$	$\frac{(Tu,v)}{(u,v))}\ \frac{1}{l^2}$	$\frac{1}{\pi^2+Dl^2}$	$l_{red,D}$	$\frac{l_{red,D}}{l}$
1	1.000000	0.083334	0.08333422	0.052995	0.947231	0.947231
2	1.100000	0.100834	0.08333422	0.052995	1.052045	0.956405
3	1.200000	0.120001	0.08333422	0.052995	1.160117	0.966764
4	1.300000	0.140834	0.08333387	0.052995	1.271937	0.978413
5	1.400000	0.163334	0.08333358	0.052995	1.388071	0.991480
6	1.500000	0.187501	0.08333358	0.052995	1.509175	1.006117
7	1.600000	0.213334	0.08333350	0.052995	1.636004	1.022503
8	1.700000	0.240834	0.08333350	0.052995	1.769458	1.040857
9	1.800000	0.270000	0.08333346	0.052995	1.910604	1.061447
10	1.900000	0.300834	0.08333346	0.052995	2.060739	1.084599
11	2.000000	0.333334	0.08333346	0.052995	2.221444	1.110722
12	2.100000	0.367501	0.08333346	0.052995	2.394687	1.140327
13	2.200000	0.403334	0.08333346	0.052995	2.582953	1.174070
14	2.300000	0.440834	0.08333346	0.052995	2.789440	1.212800
15	2.400000	0.480001	0.08333346	0.052995	3.018349	1.257645
16	2.500000	0.520834	0.08333341	0.052995	3.275340	1.310136
17	2.600000	0.563334	0.08333341	0.052995	3.568273	1.372413
18	2.700000	0.607501	0.08333341	0.052995	3.908442	1.447571
19	2.800000	0.653334	0.08333341	0.052995	4.312825	1.540295
20	2.900000	0.700834	0.08333341	0.052995	4.808408	1.658072
21	3.000000	0.750001	0.08333341	0.052995	5.441409	1.813803

Table 10.2: Computational Results for $p_0 = 0, q_0 = 1,\ u(x) = 1, v(x) = 1$

For $\boxed{p_0 = 0, q_0 = 1}$ and $\boxed{u(x) = 1+x,\ v(x) = 1+x}$, we have computed a series of variants for $l = 1.0(0.1)3.0$ given in

Table 10.3. This is illustrated in *Fig. 10.4*.

k	l	$\frac{(Tu,v)}{(u,v))}$	$\frac{(Tu,v)}{(u,v))}\ \frac{1}{l^2}$	$\frac{1}{\pi^2+Dl^2}$	$l_{red,D}$	$\frac{l_{red,D}}{l}$
1	1.000000	0.080953	0.08095276	0.052995	0.932388	0.932388
2	1.100000	0.097584	0.08064818	0.052995	1.033086	0.939169
3	1.200000	0.115702	0.08034841	0.052995	1.136372	0.946976
4	1.300000	0.135292	0.08005448	0.052995	1.242657	0.955890
5	1.400000	0.156344	0.07976723	0.052995	1.352407	0.966005
6	1.500000	0.178846	0.07948727	0.052995	1.466148	0.977432
7	1.600000	0.202790	0.07921501	0.052995	1.584482	0.990301
8	1.700000	0.228167	0.07895064	0.052995	1.708107	1.004769
9	1.800000	0.254969	0.07869424	0.052995	1.837836	1.021020
10	1.900000	0.283189	0.07844575	0.052995	1.974631	1.039279
11	2.000000	0.312821	0.07820516	0.052995	2.119642	1.059821
12	2.100000	0.343858	0.07797231	0.052995	2.274262	1.082982
13	2.200000	0.376296	0.07774703	0.052995	2.440198	1.109181
14	2.300000	0.410129	0.07752910	0.052995	2.619579	1.138948
15	2.400000	0.445354	0.07731832	0.052995	2.815102	1.172959
16	2.500000	0.481965	0.07711445	0.052995	3.030249	1.212100
17	2.600000	0.519961	0.07691725	0.052995	3.269615	1.257544
18	2.700000	0.559336	0.07672649	0.052995	3.539424	1.310898
19	2.800000	0.600089	0.07654194	0.052995	3.848361	1.374415
20	2.900000	0.642216	0.07636335	0.052995	4.209008	1.451382
21	3.000000	0.685714	0.07619049	0.052995	4.640441	1.546814

Table 10.3: Computational Results for $p_0 = 0$, $q_0 = 1$, $u(x) = 1+x$, $v(x) = 1+x$

m

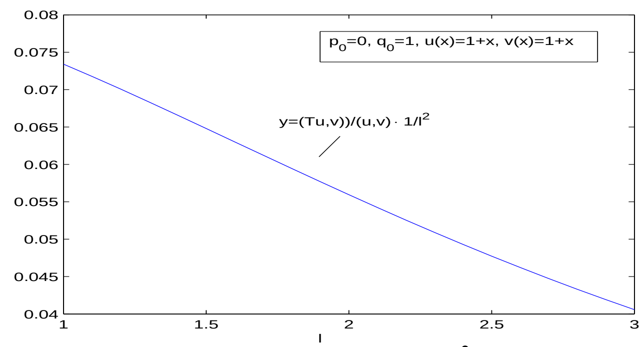

Fig. 10.4: Curve $y = ((Tu,v)/(u,v))/l^2$ for $0 \le l \le 3$

10.4 Computational Aspects

In this subsection, we say something about the used computer equipment, the computational times, and the Matlab numerical integration programs *quadl* and *dblquad*.

(i) As to the *computer equipment*, the following hardware was available: an Intel Core 2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software, for the computations, we used Matlab Version 7.11.

(ii) The *computation time* t of an operation was determined by the command sequence $t_1 = clock; operation; t = etime(clock, t_1)$. It is put out in seconds rounded to four decimal places. For the computation of the values in *Table 10.2*, the computation time was $t = 2.3400\,s$.

(iii) The double integrals $I_1 := (Tu,v)_1 := \int_0^l \int_x^l G_1(x,s)\,u(s)\,v(x)\,ds\,dx$ and $I_2 := (Tu,v)_2 := \int_0^l \int_0^x G_2(x,s)\,u(s)\,v(x)\,ds\,dx$ are computed by the Matlab commands

$$I_1 = dblquad(@(x,s)\,G1uv(x,s).*(x <= s),0,l,0,l,[],@quadl);$$

and

$$I_2 = dblquad(@(x,s)\,G2uv(x,s).*(s <= x),0,l,0,l,[],@quadl);$$

where

$$y = G1uv(x,s) = G1(x,s)*u(s)*v(x)$$

and

$$y = G2uv(x,s) = G2(x,s) * u(s) * v(x)$$

are defined in corresponding m-files. The quantity (Tu,v) is obtained as the sum of I_1 and I_2. The default absolut tolerance for *quadl* is $tol = 1.0e-6$.

The scalar product (u,v) is computed by the Matlab command

$$uv = quadl(@(t), fuv(t), 0, l);$$

where

$$y = fuv(t) = u(t). * v(t);$$

is defined in an associated m-file. Here, it is of interest to note that this command worked correctly for all function pairs u, v in *Table 10.2* except for the function pair $u(x) = 1$, $v(x) = 1$. It does neither work if one replaces $u(x) = 1$, $v(x) = 1$ by $u(x) = x^0$, $v(x) = x^0$, but it works correctly if one choose as replacements $u(x) = x+1-x$, $v(x) = x+1-x$. This is, of course, a shortcoming of the program and should be remedied by the company Mathworks.

10.5 Examples of Buckling Problems in Elastomechanics

In this subsection, we use some verbatim passages from [28].

(i) The Euler Column

As a simple example of a problem from Eleastomechanics, we choose the buckling of a slender elastic bar of length l with hinged ends, also called Euler column, see [28, Section 2.1, pp. 46-49] and [23, Section 7.2, pp. 218-226]. We assume that the bar with constant cross-section is compressed by a centrically applied force F. We further assume that the unloaded bar is exactly straight. When the critical force F_{crit} is applied, besides the undeformed shape, there exists a neighbouring shape with lateral deflection $w \neq 0$, see *Fig. 10.5*.

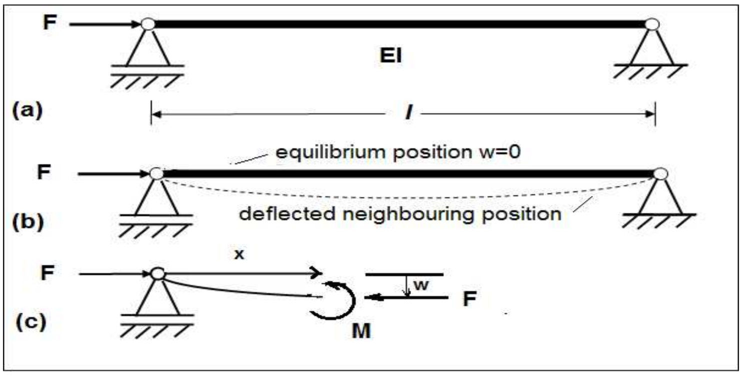

Fig. 10.5: Euler Buckling Column

In order to determine F_{crit}, it is necessary to set up the equilibrium conditions for the deflected shape, i.e., for the deformed bar. (Hereby, the change of the length can be neglected.) If one cuts the bar at the place x (*Fig.10.5 (c)*), then from the equilibrium of the bending moment about the left end taken counterclockwise for the deformed bar, one obtains

$$\overset{\curvearrowleft}{0}: \quad M - F\,w = 0. \tag{10.28}$$

Here, we have taken into account that, under horizontal force, there is no vertical bearing reaction. Substituting this in the law of elasticity $-EI\,w'' = M$ for the shearless bending bar leads to

$$-EI\,w'' = F\,w. \tag{10.29}$$

With the abbreviation

$$\mu = \frac{F}{EI}, \tag{10.30}$$

the *buckling equation* reads

$$-w'' = \mu\, w. \tag{10.31}$$

The boundary conditions for the hinges at the ends have the form

$$w(0) = w(l) = 0. \tag{10.32}$$

The BEVP consisting of (10.31) and (10.32) has the eigenvalues

$$\mu = \mu_j = j^2\, \frac{\pi^2}{l^2},\; j = 1,2,\ldots \tag{10.33}$$

and the eigenfunctions read

$$\chi_j(x) = \psi_j(x) = \varphi_j(x) = \sqrt{\frac{2}{l}}\,\sin j\,\pi\,\frac{x}{l},\; j = 1,2,\ldots. \tag{10.34}$$

As already found in Subsection 10.1, the Green's function $G(x,s) = G(x,s,p_0 \to 0, q_0 \to 0)$ turn into

$$G(x,s) = \begin{cases} G_1(x,s) = \dfrac{x\,(l-s)}{l},\; 0 \le x \le s \le l, \\[2ex] G_2(x,s) = \dfrac{s\,(l-x)}{l},\; 0 \le s \le x \le l, \end{cases}$$

so that the largest eigenvalue is

$$\lambda_1 = \frac{l^2}{\pi^2}.$$

In [23, p.223, Fig. 7/5], the critical forces for other boundary conditions such as clamped end - hinged end can be found.

(ii) Some References to Other Problems of Mathematical Physics and Engineering

Many examples for Eigenvalue Problems that can be treated by the methods of the paper may be found in the classical books [4],[5], [20], and [28].

In [4, Chapter I, pp. 5-39], one finds examples from the area of Engineering Mechanics. Further, there is a list of examples at the end of this book, cf. pages 406-456.

In [5, Chapter V, pp. 234-343], one finds vibratory and eigenvalue problems of Mathematical Physics.

The book [20, Chapter V, pp. 168-221] contains eigenvalue problems with many examples from Elastomechanics.

Books on the Theory of Elastic Stability such as [28] written primarily for engineers are full of examples from this field.

11. Changes for Other Arrangements of the Eigenvalues

(i) Changes for the Real Parts of the Eigenvalues

An arrangement of the eigenvalues as in (4.1) is possible, for instance, when all real parts are positive. However, such an arrangement is not possible if there are infinitely many eigenvalues with negative real parts and infinitely many eigenvalues with positive real parts.

In the general case that contains the last-mentioned one we proceed similarly as in [26, Section 15] for symmetric compact operators in Hilbert space: So, the sequence of eigenvalues and eigenvectors will be numbered such that eigenvalues with positive real parts have positive indices and eigenvalues with negative real parts have negative indices. Accordingly, there are sequences of numbers J_+, J_- whereby the finite resp. infinite sequence of eigenvalues can be arranged in the form

$$Re\,\lambda_{-1} \le Re\,\lambda_{-2} \le \ldots Re\,\lambda_{-k} \le \ldots < 0 \le \ldots \le Re\,\lambda_j \ldots \le Re\,\lambda_2 \le Re\,\lambda_1 \tag{11.1}$$

for $j \in J_+$, $k \in J_-$. For the index sequences J_+, J_-, it may happen that $J_+ = \emptyset$, $J_+ = (1,2,\ldots,m^+)$, or $J_+ = (1,2,\ldots)$ and $J_- = \emptyset$, $J_- = (1,2,\ldots,m^-)$, or $J_- = (1,2,\ldots)$, depending on whether no, finitely many, or infinitely many eigenvalues of T with positive real resp. negative real parts exist. Herewith, the formula (3.2) turns into

$$Tu = \sum_{j\in J_+} \lambda_j\,(u,\psi_j)\,\chi_j + \sum_{k\in J_-} \lambda_{-k}\,(u,\psi_{-k})\,\chi_{-k} \tag{11.2}$$

and the formula (3.3) into

$$Pu = \sum_{j \in J_+} (u, \psi_j)\, \chi_j + \sum_{k \in J_-} (u, \psi_{-k})\, \chi_{-k} \tag{11.3}$$

Further, the formulas (3.14), (3.15), (3.17) respectively become

$$(Tu, v) = \sum_{j \in J_+} \lambda_j (u, \psi_j)(\chi_j, v) + \sum_{k \in J_-} \lambda_{-k} (u, \psi_{-k})(\chi_{-k}, v), \tag{11.4}$$

$u, v \in H$,

$$(u, v) = (Pu, v) = \sum_{j \in J_+} (u, \psi_j)(\chi_j, v) + \sum_{k \in J_-} (u, \psi_{-k})(\chi_{-k}, v), \tag{11.5}$$

$u, v \in H$,

$$Re\,(Tu, v) = \sum_{j \in J_+} Re\, \lambda_j (u, \psi_j)(\chi_j, v) + \sum_{k \in J_-} Re\, \lambda_{-k} (u, \psi_{-k})(\chi_{-k}, v), \tag{11.6}$$

$u \in N_{\chi,\mathbb{R}}$, $v \in N_{\psi,\mathbb{R}}$.

At this point, we make the important remark that the eigenvalues of $-T$ are obtained by multiplying the eigenvalues of T by -1. Therefore, it is sufficient to characterize the positive real parts of the eigenvalues by extremal principles since the corresponding statements on the negative real parts of the eigenvalues are obtained by applying the formulas for the operator $-T$ resp. the pertinent expression $\frac{Re\,(-Tu,v)}{(u,v)}$.

It is left to the reader to show that the formulas in Theorems 4.1 - 4.4 remain valid for J_+ instead of J for the arrangement (11.1).

(ii) Changes for the Imaginary Parts of the Eigenvalues

As to the imaginary parts of the eigenvalues, considerations similar to those in (i) have to be taken into account.

(iii) Moduli of the Eigenvalues

It is not necessary to make any changes in the arrangement (6.1) for the moduli of the eigenvalues.

12. Conclusion and Outlook to Future Work

In this paper, it could be shown that generalized Rayleigh-quotient formulas for the real parts, imaginary parts, and moduli of simple eigenvalues of nonsymmetric compact operators can be derived that resemble corresponding results for diagonalizable matrices. Since the underlying Hilbert space is assumed to be infinite-dimensional, the proofs differ, in part, significantly from those in the finite-dimensional case of matrices. For instance, in the proof of Theorem 4.2, the denotation of codimension of a subspace of a Hilbert space was necessary that can be avoided in the finite-dimensional case, cf. [26, Section 15, pp.84-85].

In a subsequent paper, the results of this paper will be extended to defective, more precisely, to not necessarily simple eigenvalues of nonsymmetric compact operators.

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Competing interests

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Dynamics and Bifurcation of $x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}}$

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Abstract
The main goal of this paper is to study the bifurcation of a second order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\dots$$

with positive parameters α,β,A,B,C and non-negative initial conditions $\{x_{-k},x_{-k+1},\dots,x_0\}$. We study the dynamic behavior and the direction of the bifurcation of the period-two cycle. Numerical discussion with figures are given to support our results.

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1. Introduction

In this paper we studies the second order rational difference equation

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\dots, \tag{1.1}$$

with positive parameters α,β,A,B and C and non-negative initial conditions $\{x_{-k},x_{-k+1},\dots,x_0\}$. We focus on the dynamic behavior of the positive fixed points and the type of bifurcation exists where the change of stability occurs.

Equation (1.1) was studied by Lin-Xia Hu, Wan-Tong Li, Hong-Wu Xu in [4]. Boundedness, invariant intervals, semicycles and global stability of the positive fixed point was investigated. Also it was studied by Ladas in [5] and [1].

Recently, bifurcation and dynamics of higher order nonlinear difference equations have been studied in [8, 7, 6, 3].

Changing of variables convert the second-order rational difference equation with five positive parameters

$$x_{n+1}=\frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}},\quad n=0,1,2,\dots$$

into

$$y_{n+1}=\frac{p+qy_{n-1}}{1+y_n+ry_{n-1}},\quad n=0,1,2,\dots,$$

with three positive parameters p , q and r.
In this paper, regarding p as a parameter, we investigate the existence of Period-Doubling bifurcation and use the normal form theory for discrete dynamical system to determine the direction of bifurcation of period-two cycle. Then, we give numerical discussion with figures to support our results.

2. Dynamics of $y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}$

In this section we study the stability of the positive fixed points of

$$y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}. \tag{2.1}$$

Note that the discrete difference equation (2.1) has the unique positive fixed point

$$\bar{y} = \frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{2(1+r)}.$$

In order to convert equation (2.1) to a second dimensional system with three positive parameters p,q, and r, let $u_n = x_{n-1}$ and $w_n = x_n$. We have the following system

$$u_{n+1} = w_n,$$

$$w_{n+1} = \frac{p+qu_n}{1+w_n+ru_n}, n = 0,1,2,\dots. \tag{2.2}$$

System (2.2) has the unique positive fixed point $(u^*,w^*)^T = (\bar{y},\bar{y})^T$.

The Jacobian matrix associated with system (2.2) at the positive fixed point is

$$JF(u,w)\mid_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} & -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix}.$$

Note that

$$\det(JF(\bar{y},\bar{y})) = -\frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$$

and

$$tr(JF(\bar{y},\bar{y})) = -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$$

where det and tr denote the determinant and trace of the Jacobian matrix J, respectively.
We will use the following lemmas.

Lemma 2.1. *[2] Consider the map $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$ ba a C^1 map, where G is an open subset of $\mathbb{R}^2$, $\bar{x}$ is a fixed point of f, $A = Jf(\bar{x})$ and $\rho(A)$ is the spectral norm of A where $\rho(A) = max_i\{\mid \lambda_i \mid, \lambda_i$ are the eigenvalues of $A\}$. Then the following statement hold true:*

1. *If $\rho(A) < 1$, then $\bar{x}$ is asymptotically stable.*
2. *If $\rho(A) > 1$, then $\bar{x}$ is unstable.*
3. *If $\rho(A) = 1$, then $\bar{x}$ may or may not be stable.*

Lemma 2.2. *[2] Consider the map*

$$x \to f(x), \quad x \in \mathbb{R}^2,$$

with $\bar{x}$ as a fixed point of f and $A = Jf(\bar{x})$. Then $\rho(A) < 1$ if and only if

$$\mid trA \mid -1 < \det A < 1$$

where trA and $\det A$ denote trace and determinant of the matrix A respectively.

Theorem 2.3. *[9] The equilibrium point $\bar{y}$ of (2.1) is locally asymptotically stable if one of the following holds*

1. $q \leq 1$
2. $q > 1$ *and* $(r-1)(q-1)^2 + 4pr^2 > 0$.

Proof: We want to show that

$$\mid \frac{q - r\bar{y}}{1+\bar{y}+r\bar{y}} \mid < 1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} < 2.$$

That is equivalent to

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \mid \frac{q - r\bar{y}}{1+\bar{y}+r\bar{y}} \mid < 1 \ \ and \ \ \frac{\bar{y}}{1+\bar{y}+r\bar{y}} > -1.$$

The first inequality is equivalent to

$$\mid q - r\bar{y} \mid < 1 + r\bar{y}. \tag{2.3}$$

If $q - r\bar{y} < 0$, then(2.3) becomes $r\bar{y} - q < 1 + r\bar{y}$ and this is obvious .
If $q - r\bar{y} \geq 0$, then(2.3) becomes $q - r\bar{y} < 1 + r\bar{y}$,
or

$$q - 1 < 2r\bar{y}. \tag{2.4}$$

If $q \leq 1$, then (2.4) holds. If $q > 1$, then

$$r\bar{y} > r\frac{\sqrt{(q-1)^2 + 4p(1+r)}}{r+1} > r\sqrt{(q-1)^2 + 4p(1+r)}$$

and if $(r-1)(q-1)^2 + 4pr^2 > 0$, multiply both sides by $r+1$ we can get

$$(r^2-1)(q-1)^2 + 4pr^2(1+r) > 0.$$

Rearrange the terms of the previous inequality, we get

$$r^2((q-1)^2 + 4p(1+r)) > (q-1)^2.$$

Take the square of both sides, we obtain

$$r\sqrt{(q-1)^2 + 4p(1+r)} > (q-1).$$

Now, add $r(q-1)$ for both sides, we have

$$r(q - 1 + \sqrt{(q-1)^2 + 4p(1+r)} > (r+1)(q-1).$$

That is equivalent to

$$2r(r+1)\bar{y} > (r+1)(q-1),$$

or

$$2r\bar{y} > q - 1.$$

This shows in this case inequality (2.4) holds and hence

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \mid \frac{q - r\bar{y}}{1+\bar{y}+r\bar{y}} \mid < 1.$$

Note that the second inequality $1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} < 2$ is always true.
So in both cases the equilibrium point $\bar{y}$ is locally asymptotically stable.

3. Existence of Period-Doubling Bifurcation

In this section we will study the bifurcation of (2.1). we will use the following theorem.

Lemma 3.1. *[2] Consider the map*

$$x \to f(x,\alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R}. \tag{3.1}$$

Let $A = Jf(x^*,\alpha^*)$ where (x^*,α^*) is a fixed point of $f(x,\alpha)$. Then the following hold

1. *If $\det A = -trA - 1$, then the eigenvalues of A are $\lambda_1 = -\det A$ and $\lambda_2 = -1$.*
2. *If $\det A = trA - 1$, then $\lambda_1 = 1$ and $\lambda_2 = \det A$.*
3. *If $|\,trA\,| - 1 < \det A$ and $\det A = 1$, then A has complex eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ where $\theta = cos^{-1}(\frac{trA}{2})$.*

Corollary 3.2. *For the one-parameter of two-dimensional map*

$$x \to f(x,\alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R}, \tag{3.2}$$

with the fixed point (x^*,α^*) and $A = Jf(x^*,\alpha^*)$, then the following hold

1. *If $\det A = -trA - 1$, then the system (3.2) undergoes a period-doubling bifurcation.*
2. *If $\det A = trA - 1$, then then the system (3.2) undergoes a saddle-node bifurcation.*
3. *If $|\,trA\,| - 1 < \det A$ and $\det A = 1$, then the system (3.2) undergoes a Neimark-Sacker bifurcation.*

Using the previous corollary, system (2.2) can not undergoes a saddle-node or Neimark-Sacker bifurcation.

Theorem 3.3. *The fixed point $(\bar{y},\bar{y})^T$ of the system (2.2) undergoes a period-doubling (flip) bifurcation when $p = \frac{(1-r)(q-1)^2}{4r^2}$ if $q > 1$ and $r < 1$.*

Proof: Assume that $q > 1$ and $r < 1$. Corollary (3.2) implies that period-doubling bifurcation occurs if $\det(JF(\bar{y},\bar{y})^T) = -tr(JF(\bar{y},\bar{y})) - 1$.
That is equivalent to

$$-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - 1,$$

$$-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{\bar{y}-(1+\bar{y}+r\bar{y})}{1+\bar{y}+r\bar{y}} - 1,$$

or

$$-(q-r\bar{y}) = \bar{y}-(1+\bar{y}+r\bar{y}).$$

That is equivalent to

$$2r\bar{y} = q-1,$$

or

$$2\bar{y} = \frac{q-1}{r}.$$

Substitute the value of $\bar{y}$, we obtain

$$\frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{1+r} = \frac{q-1}{r},$$

or

$$q-1+\sqrt{(1-q)^2+4p(1+r)} = q-1+\frac{q-1}{r}.$$

Take the square of both sides, we get

$$(1-q)^2+4p(1+r)=(\frac{q-1}{r})^2,$$

multiply both sides by r^2

$$r^2[(1-q)^2+4p(1+r)]=(q-1)^2,$$

or

$$(r^2-1)(q-1)^2+4pr^2(1+r)=0.$$

Since $r>0$, $r+1\neq 0$, so we can divide into $1+r$. We obtain

$$(r-1)(q-1)^2+4pr^2=0,$$

$$p=\frac{(1-r)(q-1)^2}{4r^2}.$$

4. Direction of The Period-Doubling (Flip) Bifurcation

In this section we will use the normal form theory for discrete dynamical system to find the direction of the period-doubling bifurcation of system (2.2) which exists at $p=\frac{(1-r)(q-1)^2}{4r^2}$. Firstly, we shift the fixed point $(\bar{y},\bar{y})^T$ to the origin. Let

$$x_n=u_n-\bar{y},\;\; z_n=w_n-\bar{y}.$$

System (2.2) corresponds to

$$x_{n+1}=z_n,$$

$$z_{n+1}=\frac{p+q(x_n+\bar{y})}{1+(z_n+\bar{y})+r(x_n+\bar{y})}-\bar{y}, \tag{4.1}$$

or

$$Y_{n+1}=AY_n+G(Y_n), \tag{4.2}$$

where

$$A=\begin{pmatrix} 0 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \end{pmatrix}, Y_n=\begin{pmatrix} x_n \\ z_n \end{pmatrix},$$

and

$$G(Y)=\frac{1}{2}B(Y,Y)+\frac{1}{6}C(Y,Y,Y)+O(\parallel Y\parallel^3),$$

$$B(Y,Y)=\begin{pmatrix} 0 \\ B_2(Y,Y) \end{pmatrix} \text{ and } C(Y,Y,Y)=\begin{pmatrix} 0 \\ C_2(Y,Y,Y) \end{pmatrix},$$

where

$$B_2(\phi,\psi)=-\frac{2r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}\phi_1\psi_1+\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}[\phi_1\psi_2+\phi_2\psi_1]+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}\phi_2\psi_2,$$

and

$$\begin{aligned} C_2(\phi,\psi,\eta)=&6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\phi_1\psi_1\eta_1+\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}[\phi_1\psi_1\eta_2+\phi_2\psi_0\eta_1+\phi_1\psi_2\eta_1]\\ &+\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3}[\phi_1\psi_2\eta_2+\phi_2\psi_1\eta_2+\phi_1\psi_2\eta_2]-6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}\phi_2\psi_2\eta_2. \end{aligned}$$

Let q and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$, respectively. We have $Aq = -q$ and $A^T p^* = -p^*$, where

$$q \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ and } p^* \sim \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 \end{pmatrix}.$$

Normalize p^* and q,

$$< p^*, q >= \sum_{i=1}^{2} p_i^* q_i = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - 1.$$

Take

$$p = \xi * \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 \end{pmatrix}, \text{ where } \xi = \frac{1}{-1-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}} = -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}.$$

The critical eigenspace T^c corresponding to the eigenvalue λ is a one-dimensional map, and is spanned by the eigenvector q. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to the other eigenvalue than λ. Note that the matrix $A - \lambda I$ which is equivalent to the matrix $A + T$ has common invariant spaces with the matrix A, we conclude that $y \in T^{su}$ if and only if $< p, y >= 0$. Any vector $x \in \mathbb{R}^2$ can be decomposed as

$$x = uq + y,$$

where $uq \in T^c, y \in T^{su}$, and

$$u =< p, x >,$$

$$y = x- < p, x > q. \tag{4.3}$$

In the coordinates (u, y), the map (4.2) can be written as

$$\tilde{u} = \lambda u+ < p, F(uq+y) >,$$

$$\tilde{y} = Ay + F(uq+y)- < p, F(uq+y) > q. \tag{4.4}$$

Using Taylor expansions, (4.4) can be written as

$$\tilde{u} = \lambda u + \frac{1}{2}\sigma u^2 + u < b, y > + \frac{1}{6}\delta u^3 + \dots,$$

$$\tilde{y} = Ay + \frac{1}{2}au^2 + \dots, \tag{4.5}$$

where $u \in \mathbb{R}^1, y \in \mathbb{R}^2, \sigma, \delta \in \mathbb{R}^1, a, b \in \mathbb{R}^2$ and $< b, y >= \sum_{i=1}^{2} b_i y_i$ is the standard scaler product $< b, y >$ can be expressed as

$$< b, y >=< p, B(q, y) > .$$

The center manifold of (4.5) has the representation

$$y = V(u) = \frac{1}{2} w_2 u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^2$, so that $< p, w >= 0$. The vector w_2 satisfies

$$(A - I)w_2 + a = 0.$$

Note that the matrix $A-I$ is invertible in $\mathbb{R}^2$ because $\lambda=1$ is not an eigenvalue of A. Thus, we have

$$w_2=-(A-I)^{-1}a,$$

and the restriction of (4.5) to the center manifold takes the form

$$\tilde{u}=-u+\frac{1}{2}\sigma u^2+\frac{1}{6}(\delta-3<p,B(q,(A-I)^{-1}a)>)u^3+O(u^4),$$

where

$$\sigma=<p,B(q,q)>,\delta=<p,C(q,q,q)>,\text{ and } a=B(q,q)-<p,B(q,q)>q.$$

Using the identity $(A-I)^{-1}q=-\frac{1}{2}q$, the restricted map can be written as

$$\tilde{u}=-u+a(0)u^2+b(0)u^3+O(u^4), \tag{4.6}$$

where

$$a(0)=\frac{1}{2}<p,B(q,q)>,$$

and

$$b(0)=\frac{1}{6}<p,C(q,q,q)>-\frac{1}{4}(<p,B(q,q)>)^2-\frac{1}{2}<p,B(q,(A-I)^{-1}B(q,q))>.$$

$$B(q,q)=\begin{pmatrix}0\\ -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}}-2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}\end{pmatrix},$$

$$<p,B(q,q)>=-\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}[-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}}-2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}],$$

$$C(q,q,q)=\begin{pmatrix}0\\ 6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}-3\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}+3\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3}+6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}\end{pmatrix},$$

$$<p,C(q,q,q)>=-\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}[6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}-3\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}+3\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3}+6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}],$$

$$(A-I)^{-1}=\begin{pmatrix}-1 & 1\\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\end{pmatrix}^{-1}=\frac{1+\bar{y}+r\bar{y}}{2\bar{y}}\begin{pmatrix}-1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} & -1\\ -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -1\end{pmatrix},$$

$$(A-I)^{-1}B(q,q)=\frac{1+\bar{y}+r\bar{y}}{2\bar{y}}\begin{pmatrix}-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}}-2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}\\ -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}}-2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}\end{pmatrix},$$

$$B(q,(A-I)^{-1}B(q,q))=\frac{1+\bar{y}+r\bar{y}}{2\bar{y}}\begin{pmatrix}0\\ S\end{pmatrix}$$

where

$$S=[\frac{2r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}+\frac{2\bar{y}}{(1+\bar{y}+r\bar{y})^2}][-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}-2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}],$$

$$\begin{aligned}<p,B(q,(A-I)^{-1}B(q,q))>=&2\frac{r^2(q-r\bar{y})^2}{\bar{y}(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2}-2\frac{\bar{y}}{(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2}\\ &+2\frac{r(q-r\bar{y})(2r\bar{y}-q)}{\bar{y}(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2}+2\frac{2r\bar{y}-q}{(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2}.\end{aligned}$$

The map (4.6) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4),$$

where

$$c(0) = a^2(0) + b(0).$$

Thus, the critical normal form coefficient $c(0)$ allows us to predict the direction of bifurcation of period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{2} < p, B(q,(A-I)^{-1}B(q,q)) > .$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $p = \frac{(1-r)(1-q)^2}{4r^2}$.

5. Numerical Discussion

In this section we give numerical examples which support our results in the previous sections. Figure that we get using Matlab will be attached with example to illustrate the bifurcation.

Example 5.1. *Consider equation (2.1). In this example we fix the parameters q, r and consider p as bifurcation parameter. Take $q = 1.1$, $r = 0.09$ and $0 < p \leq 2$. Equation (2.1) becomes*

$$y_{n+1} = \frac{p + 1.1y_{n-1}}{1 + y_n + 0.09y_{n-1}}, n = 0, 1, 2, \ldots \tag{5.1}$$

The planer form corresponding to equation (5.1) is

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{p+1.1y_1(n)}{1+y_2(n)+0.09y_1(n)}. \end{pmatrix} \tag{5.2}$$

Positive equilibrium point of system (5.2) is $(\bar{y}, \bar{y})$, where $\bar{y} = \frac{0.1+\sqrt{0.01+4.36p}}{2.18}$. Theorem (3.3) determined the bifurcation point at $(r-1)(1-q)^2 + 4pr^2 = 0$. So, the fixed point undergoes a period-doubling bifurcation at $p = 0.2808642$.

$$q = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } p = \begin{pmatrix} 0.39539749 \\ -0.60460251 \end{pmatrix},$$

$$B(q,q) = \begin{pmatrix} 0 \\ 0.71303782 \end{pmatrix},$$

$$< p, B(q,q) >= -0.43110446,$$

$$C(q,q,q) = \begin{pmatrix} 0 \\ -0.4797597 \end{pmatrix},$$

$$< p, C(q,q,q) >= 0.2900639,$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ 0.65397924 & -1.34602076 \end{pmatrix},$$

$$B(q,(A-I)^{-1}B(q,q)) = \begin{pmatrix} 0 \\ 0.8212105 \end{pmatrix},$$

$$< p, B(q,(A-I)^{-1}B(q,q)) >= -0.49947486,$$

$$c(0) = 0.20139345 > 0.$$

So, this verify that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $p = 0.2808642$. See figure (5.1).

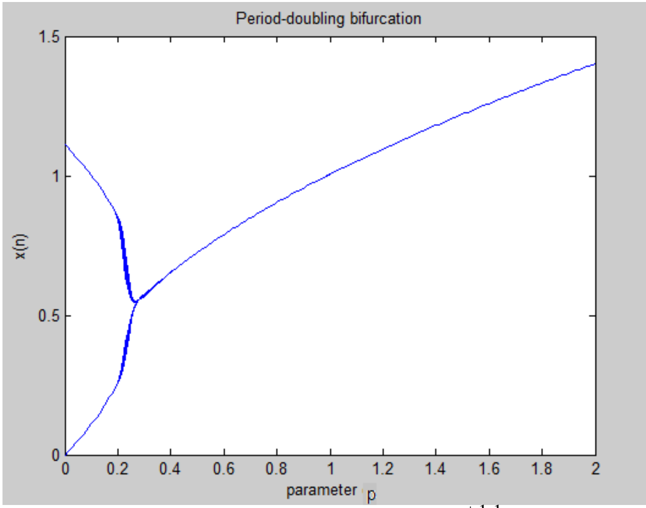

Figure 5.1. Period-doubling bifurcation of $y_{n+1} = \frac{p+1.1y_{n-1}}{1+y_i+0.09y_{i-1}}$, p is a parameter.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Automatic Continuity of Almost Derivations on Frechet Q-Algebras

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Abstract
In 1971 R. L. Carpenter proved that every derivation on a semisimple commutative Frechet algebra with identity is continuous. The concept of almost derivations on Frechet algebras is introduced in this article. Also, R. L. Carpenter result motivates us to ask an open question: Is every almost derivation on semisimple commutative Frechet algebras continuous? Moreover, a partial answer to this open question is derived in the sense that every almost derivation T on semisimple commutative Frechet Q-algebras Λ, with an additional condition on Λ, is continuous. Furthermore, an example is provided to illustrate our main result.

Keywords: Automatic continuity, Almost derivation, Frechet Q-algebras
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1. Introduction

We provide a brief outline of definitions and known outcomes in this section. For more details, one may refer to [2, 7]. All vector spaces are considered over the complex field, and we assume that all algebras are unital.

Definition 1.1. *A normed algebra Λ is an algebra with a norm $||.||$, which also satisfies $||p.q|| \leq ||p||.||q||$, $\forall\ p,q \in \Lambda$. A complete normed algebra is called a Banach algebra.*

Definition 1.2. *An algebra with a Hausdorff topology is called a topological algebra if all algebraic operations are jointly continuous.*

Definition 1.3. *[2] The Jacobson radical $rad(\Lambda)$ of an algebra Λ is the intersection of all maximal right(or left) ideals. An algebra is said to be semisimple if $rad(\Lambda) = \{0\}$.*

Definition 1.4. *[2] The spectrum $\sigma_\Lambda(p)$ of an element p of an algebra Λ is the set of all complex numbers λ such that $\lambda.1 - p$ is not invertible in Λ. The spectral radius $r_\Lambda(p)$ of an element $p \in \Lambda$ is defined by $r_\Lambda(p) = \sup\{|\lambda| : \lambda \in \sigma_\Lambda(p)\}$.*

If $(\Lambda, ||.||)$ is a Banach algebra, then $r_\Lambda(p) = \lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$. Also, for any algebra Λ, we have $rad(\Lambda) = \{p \in \Lambda : r_\Lambda(pq) = 0$, for every $q \in \Lambda\}$. See [2].

Definition 1.5. *[2] If $T : \Lambda \to \Gamma$ is a linear map from a Banach algebra Λ to a Banach algebra Γ, then the separating space of T is defined as the set*
$S(T) = \{q \in \Gamma :$ there exists $(p_n)_{n=1}^{\infty}$ in Λ such that $p_n \to 0$ and $Tp_n \to q\}$.

Also, $S(T)$ is a closed linear subspace of Γ and moreover, by the closed graph theorem, T is continuous if and only if $S(T) = \{0\}$. For a proof, see [2].

A complete metrizable topological algebra is called an F-algebra. A topological algebra Λ is said to be a LMC algebra if its topology is induced by a separating family of submultiplicative seminorms. A Frechet algebra is a LMC algebra which is also an F-algebra. A Q-algebra is a topological algebra in which the set of all invertible elements is open. A metrizable LMC algebra is written in the form $(\Lambda, (p_n)_{n=1}^{\infty})$, where $(p_n)_{n=1}^{\infty}$ is a separating sequence and each p_n is a submultiplicative seminorm (i.e. $p_n(x.y) \leq p_n(x).p_n(y), \forall x, y \in \Lambda$) satisfying $p_n(x) \leq p_{n+1}(x), \forall n, \forall x \in \Lambda$, in which the topology on Λ is induced by the seminorms $p_n, n = 1, 2, \ldots$. Also, a sequence (x_k) in the Frechet algebra $(\Lambda, (p_n))$ converges to $x \in \Lambda$ if and only if $p_n(x_k - x) \to 0$ for each $n \in N$, as $k \to \infty$. In a Frechet Q-algebra, spectral radius of every element is a finite number. Every Banach algebra is a Frechet Q-algebra.

Definition 1.6. *[6] Let Λ be an algebra. A linear map $T : \Lambda \to \Lambda$ is called derivation if $T(p.q) = p.T(q) + T(p).q, \forall p, q \in \Lambda$.*

Next, we introduce almost derivations on Frechet algebras.

Definition 1.7. *Let $(\Lambda, (p_n))$ be a Frechet algebra. A linear map $T : \Lambda \to \Lambda$ is called an almost derivation if there are $\varepsilon_n \geq 0$ such that $p_n(T(p.q) - p.T(q) - T(p).q) \leq \varepsilon_n p_n(p)\, p_n(q)$; $\forall n \in N, \forall p, q \in \Lambda$.*

Remark 1.8. *If $\varepsilon_n = 0$, for every n, then almost derivations on Λ turn out to be derivation on Λ, because (p_n) is a separating sequence of seminorms on Λ. Also, every derivation is an almost derivation, for every $\varepsilon_n \geq 0$.*

A conjecture of Kaplansky [6] can be stated in the following question form. Is every derivation on semisimple Banach algebra continuous? Kaplansky conjecture was proved by Johnson and Sinclair [5] in 1968. In 1971, R. L. Carpenter [1] proved that every derivation on a semisimple commutative Frechet algebra with identity is continuous. There are some recent articles [8, 9, 10, 11, 12] for automatic continuity of derivations in the theory of topological algebras.

Now, we write an open question for almost derivations on Frechet algebras.

Problem 1.9. *Let $T : (\Lambda, (p_n)) \to (\Lambda, (p_n))$ be an almost derivation on a semisimple commutative Frechet algebra $(\Lambda, (p_n))$. Is T continuous?*

Also, we derive a partial solution to this open Problem 1.9. More specifically, we prove that every almost derivation T on a semisimple commutative Frechet Q-algebra $(\Lambda, (p_n))$, with an additional condition on $(\Lambda, (p_n))$, is continuous.

2. Main Result

Definition 2.1. *[4] If $T : \Lambda \to \Gamma$ is a linear map from a Frechet algebra Λ to a Frechet algebra Γ, then the separating space of T is defined by*
$S(T) = \{q \in \Gamma$: there exists $(q_n)_{n=1}^{\infty}$ in Λ such that $q_n \to 0$ and $Tq_n \to q\}$.

Theorem 2.2. *Let $(\Lambda, (p_n))$ be a Frechet algebra. If $T : \Lambda \to \Lambda$ is an almost derivation, then the separating space $S(T)$ is a closed two sided ideal in $(\Lambda, (p_n))$.*

Proof. Obviously $S(T)$ is a closed linear subspace of $(\Lambda, (p_n))$.

Now, we prove that $S(T)$ is an ideal in Λ. Let $b \in S(T)$ and $c \in \Lambda$. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ in Λ such that $a_n \to 0$, and $T(a_n) \to b$. Let $w = T(c)$. Also we have $p_k(c.a_n) \leq p_k(c).p_k(a_n) \to 0, \forall k$, as $n \to \infty$. Since T is almost derivation, we have

$$\begin{aligned} p_k(T(c.a_n) - c.b) &\leq p_k(T(c.a_n) - c.T(a_n) - T(c).a_n) + p_k(c.T(a_n) + w.a_n - c.b) \\ &\leq p_k(T(c.a_n) - c.T(a_n) - T(c).a_n) + p_k(c.T(a_n) - c.b) + p_n(w.a_n) \\ &\leq \varepsilon_k p_k(c)\, p_k(a_n) + p_k(c)\, p_k(T(a_n) - b) + p_k(w.a_n). \end{aligned}$$

Since $p_k(T(a_n) - b) \to 0$, $p_k(a_n) \to 0$ and $p_k(w.a_n) \leq p_k(w).p_k(a_n) \to 0$, $\forall k$, as $n \to \infty$, we have $p_k(T(c.a_n) - c.b) \to 0$, and hence $T(c.a_n) \to c.b$, when $c.a_n \to 0$. Therefore, we conclude that $c.b \in S(T)$. Similarly $b.c \in S(T)$. Hence $S(T)$ is a two sided ideal in Λ. □

Theorem 2.3. *Let $(\Lambda, (p_n))$ be a Frechet Q-algebra such that Λ is semisimple, and r_Λ is continuous on Λ. If $T : \Lambda \to \Lambda$ is an almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a), \forall a \in \Lambda$, then T is continuous.*

Proof. Let $b \in S(T)$. Then there exists $(a_n)_{n=1}^{\infty}$ in Λ such that $a_n \to 0$ and $Ta_n \to b$. Since $r_\Lambda(Ta) \leq r_\Lambda(a)$ and $r_\Lambda(a_n) \to 0$, we have $r_\Lambda(Ta_n) \to 0$. Also, we have $r_\Lambda(Ta_n) \to r_\Lambda(b)$. So, we conclude that $r_\Lambda(b) = 0$. By Theorem 2.2, $S(T)$ is an ideal in Λ. For every $c \in \Lambda, b.c \in S(T)$. Therefore $r_\Lambda(b.c) = 0$. Also, $rad(\Lambda) = \{a_1 \in \Lambda : r_\Lambda(a_1.a_2) = 0, \forall a_2 \in \Lambda\}$, and hence $b \in rad(\Lambda)$. So, $S(T) \subseteq rad(\Lambda)$. Since Λ is semisimple, $S(T) = \{0\}$. Therefore T is continuous, by the closed graph theorem. □

Corollary 2.4. *Let $(\Lambda, (p_n))$ be a commutative Frechet Q-algebra such that Λ is semisimple. If $T : \Lambda \to \Lambda$ is an almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a), \forall a \in \Lambda$, then T is continuous.*

Proof. If Λ is a commutative Frechet Q-algebra, then the spectral radius function r_Λ is uniformly continuous. See, for example ([3], Theorem 6.18). □

This Corollary 2.4 is a partial solution to the Problem 1.9.

Corollary 2.5. *Let Λ be a commutative semisimple Banach algebra. If $T : \Lambda \to \Lambda$ is an almost derivation with $r_\Lambda(Ta) \leq r_\Lambda(a), \forall a \in \Lambda$, then T is continuous.*

Proof. If Λ is a commutative Banach algebra, then spectral radius function r_Λ is continuous on Λ. □

Example 2.6. *Let $(\Lambda, (p_n))$ be a semisimple commutative Frechet Q-algebra. A linear map $T : \Lambda \to \Lambda$ is defined by $T(a) = \beta a$, $\forall a \in \Lambda$ where $(\varepsilon_n =)\beta \in (0, \infty)$. Since*

$$p_n(T(p.q) - p.T(q) - T(p).q) = p_n(\beta p.q - p.\beta q - \beta p.q) = p_n(-\beta p.q) \leq |-\beta| p_n(p).p_n(q),$$

T is an almost derivation but not a derivation on $(\Lambda, (p_n))$. Since Λ is a Q-algebra, there exists $k \in N$ such that $r_\Lambda(a) = \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}}$, $\forall a \in \Lambda$. See, for example ([3], Theorem 6.18). So

$$r_\Lambda(Ta) = r_\Lambda(\beta a) = \lim_{n\to\infty}(p_k((\beta a)^n))^{\frac{1}{n}} = |\beta| \lim_{n\to\infty}(p_k(a^n))^{\frac{1}{n}} \leq r_\Lambda(a).$$

All hypotheses of Corollary 2.4 are satisfied, so T is continuous.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Bifurcations Along the Spiral Organization of the Periodicity in a Hopfield Neural Network

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Abstract
In this paper we report numerical results with respect to a certain Hopfield-type three-neurons network, where the activation function is of the type hyperbolic tangent function. Specifically, we investigate a place in a two-dimensional parameter-space of this system where typical periodic structures, the so-called shrimps, are embedded in a chaotic region. We show that these periodic structures are self-organized as a spiral that coil up toward a focal point, while undergo period-adding bifurcations. We also indicate the locations along this spiral in the parameter-space, where such bifurcations happen.

Keywords: Chaos, Lyapunov Exponents Spectrum, Parameter-Space, Period-Adding Bifurcation.
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1. Introduction

A Hopfield neural network [1] is an important mathematical model in artificial neurocomputing [2]. It is a continuous-time nonlinear dynamical system which is modeled by a set of n autonomous first-order nonlinear ordinary differential equations given by

$$C_i\dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^{n} w_{ij}v_j + I_i, \quad i = 1,2,\ldots,n, \tag{1.1}$$

where $v_j = f_j(x_j)$, x_i are real dynamical variables, C_i, R_i, and I_i are control parameters, and w_{ij} are the elements of an $n\times n$ matrix, namely the weight matrix or the connectivity matrix, which describes the strength of the connections between the n neurons that make up the network. The neuron activation function $f_j(x_j)$ is a bounded monotonic differentiable function usually represented by any smooth function.

A low-dimensional form of the mathematical model (1.1) is investigated here, namely the one concerned with a system composed of three neurons, and whose behavior depends on two control parameters, α and γ, and which is given by

$$\begin{aligned}
\dot{x}_1 &= -x_1 + 1.5f_1(x_1) + 2.9f_2(x_2) + \alpha f_3(x_3),\\
\dot{x}_2 &= -x_2 - 1.5f_1(x_1) + 1.18f_2(x_2),\\
\dot{x}_3 &= -x_3 + \gamma f_1(x_1) - 22f_2(x_2) + 0.47f_3(x_3).
\end{aligned} \tag{1.2}$$

To obtain system (1.2), we consider in system (1.1) the weight matrix equal to

$$\begin{pmatrix} 1.5 & 2.9 & \alpha \\ -1.5 & 1.18 & 0 \\ \gamma & -22 & 0.47 \end{pmatrix},$$

$n = 3$, $C_i = R_i = 1$ and $I_i = 0$, for $i = 1,2,3$. The neuron activation function considered in our study, which defines the nonlinearity in system (1.2), is a single hyperbolic tangent function given by $f_j(x_j) = \tanh(x_j)$, plotted in Fig. 1.1 in order to illustrate.

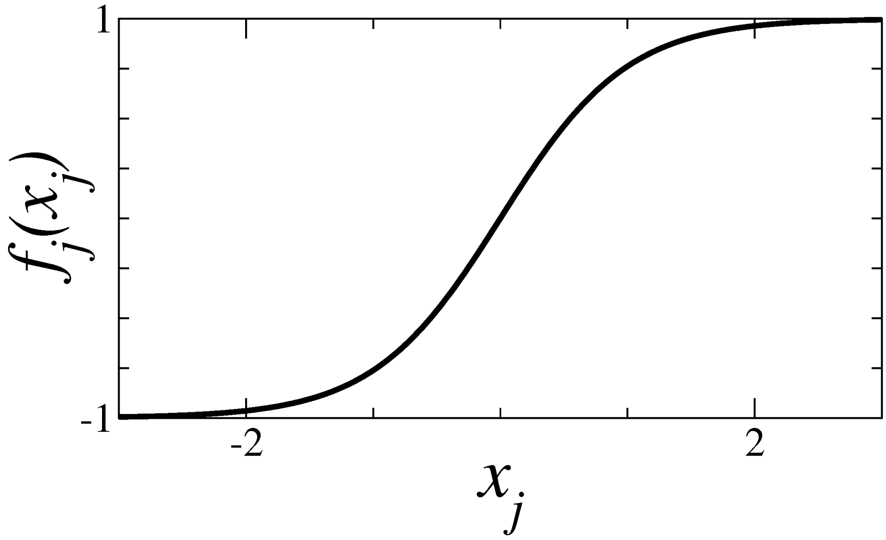

Figure 1.1. The hyperbolic tangent activation function that stands for the nonlinearity in system (1.2).

Huang and Huang [3] present various results concerning system (1.2), with the parameter α kept fixed at 0.8, involving mainly Lyapunov exponents, bifurcation diagrams, and attractors in the phase-space. Periodic and chaotic attractors were reported in [3], as a function of an other parameter (β), namely that which is the coefficient of the term $f_1(x_1)$ in the $\dot{x}_2$ equation in system (1.2). Therefore, the investigation carried out concerning system (1.2) and reported in [3] considered only a small quantity of points in a two-dimensional (α,β) parameter-space, more specifically those points located along the straight line α=0.8. A system similar to system (1.2), with a different weight matrix was presented by Chen and co-workers [4]. This time the dynamics of the system was investigated again varying only one parameter, by means of Lyapunov exponents spectra, phase-space portraits, and bifurcation diagrams. The authors present a numerical verification of the existence of a horseshoe in this system, for a certain parameter value. Lyapunov exponents spectrum, power spectrum, bifurcation diagrams, and topological horseshoe theory were used by Zheng and co-workers [5] to numerically investigate another three-neuron one-parameter chaotic Hopfield-type network with the hyperbolic tangent as the activation function. The existence of a horseshoe in the system was proved for a certain value of the variable parameter.

A two-dimensional parameter-space of system (1.2) was investigated by Mathias and Rech [6, 7]. However, the parameters and/or ranges of parameters considered there are different from those we consider here. Reference [6] also considers a piecewise-linear function as the activation function, by replacing the hyperbolic tangent function. The parameter-spaces obtained by using the two activation functions, obviously one at a time, are compared in Mathias and Rech [6], being the existence of organized periodic structures embedded in chaotic regions verified in both cases.

The main objective in this work is to investigate a particular region of the two-dimensional (α,γ) parameter-space of system (1.2), where we have detected a spiral periodic structure that coil up toward a focal point, while undergo period-adding bifurcations. More specifically, we are interested in making a numerical estimate of the location of the points along this spiral, where the involved bifurcations occur. The paper is organized as follows. In Sect. 2 we present and discuss a parameter-space diagram related with the model (1.2), which show different colored regions signifying different dynamical behaviors, namely chaotic and periodic. Finally, concluding remarks are given in Sect. 3.

2. Numerical Results

Figure 2.1(a) shows a global view of the (α,γ) parameter-space of system (1.2), namely for $0.4 \le \alpha \le 1.2$ and $-5 \le \gamma \le 15$,

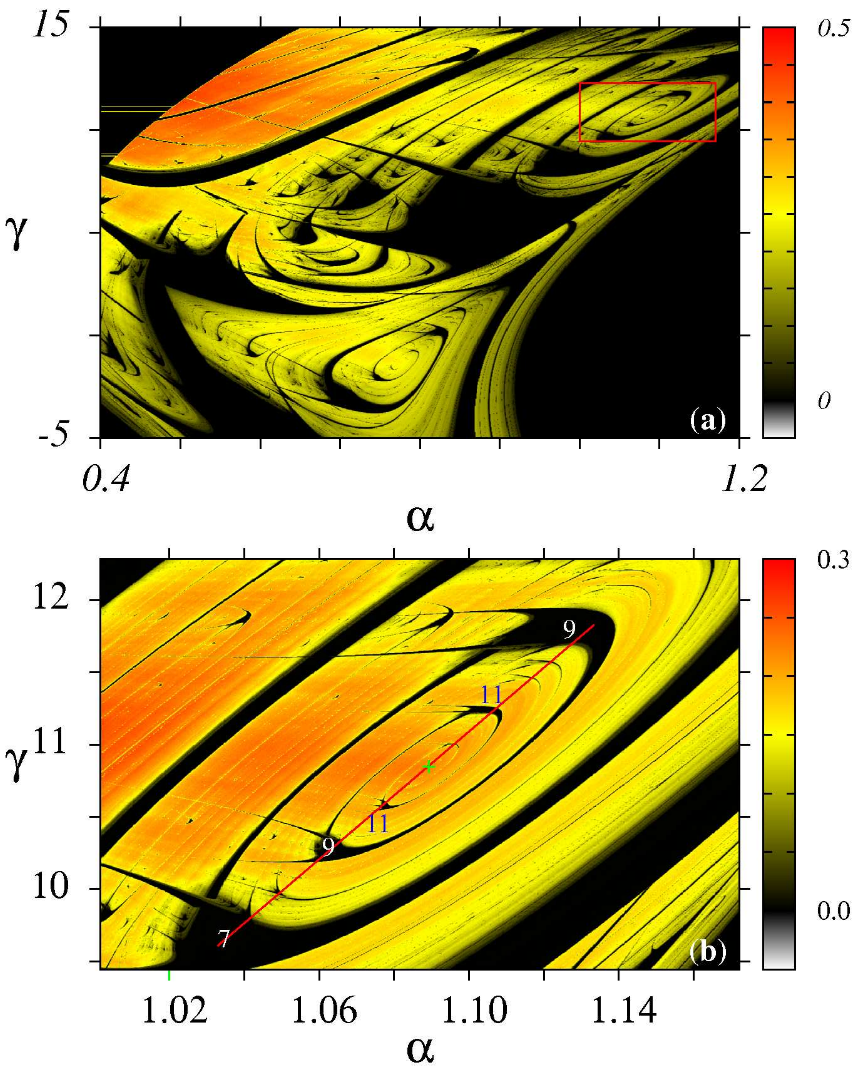

Figure 2.1. (a) Global view of the (α, γ) parameter-space of system (1.2), showing different dynamical behaviors. (b) Amplification of the boxed region in (a). Numbers refer to periods (see the text).

while in Fig. 2.1(b) one can see a particular region, that within the boxed region in Fig. 2.1(a) for which $1.00 \leq \alpha \leq 1.17$ and $9.44 \leq \gamma \leq 12.28$. Color in diagrams of Fig. 2.1 is related to the magnitude of the respective largest Lyapunov exponent (LLE). A positive LLE is indicated by a continuously changing yellow to red color, a negative LLE is indicated by a continuously changing white to black color, and the black color itself indicates a zero LLE, according to the scale shown in the column at right side in the diagram. Therefore, each point in Fig. 2.1 was painted according to the dynamical behavior presented, which was characterized by the related LLE. A chaotic region, for which the LLE is greater than zero, is painted in yellow-red, while a periodic region, for which the LLE is equal to zero, is painted in black.

Diagrams in Fig. 2.1 were obtained by computing the LLE on a grid of $10^3 \times 10^3$ (α, γ) parameters, always using an algorithm based in that present in Wolf and co-workers [8]. For each of the one million parameter sets (one million points in each diagram of Fig. 2.1), system (1.2) was integrated by using a fourth order Runge-Kutta algorithm, with a fixed time step size equal to 10^{-2}, being dropped the first 1×10^6 integration steps, regarded as a transient. For the computation of the average involved in the calculation of each one of the one million LLE, were considered the subsequent 1×10^6 integration steps.

Integrations were performed along lines of a constant parameter γ, starting always at the minor value of the parameter α. For instance, the diagram in Fig. 2.1(b) was constructed from an arbitrary initial condition for a fixed pair of parameters, in fact the two lowest $\alpha = 1.00$ and $\gamma = 9.44$. The variables (x_1, x_2, x_3) at the end of the integration for this pair of parameters were used to initialize the integration for the next pair, and so forth up to the highest value of both parameters, namely $\alpha = 1.17$ and $\gamma = 12.28$, be achieved. In other words, the procedure *following the attractor* along lines of fixed γ was used.

Both diagrams in Fig. 2.1 show dynamical behaviors of the Hopfield neural network (1.2), where we identify an intricate mixture of chaotic and periodic regions, represented respectively by yellow-red and black colors. They indicate how variations in the connection weight α, between third and first neurons, and γ, between first and third neurons, affect the dynamical behavior of the system (1.2). Figures 2.1(a) and 2.1(b) may be interpreted, each one of them, as presenting a chaotic region, in yellow-red, with several periodic regions in black, embedded in it. In other words, as the parameters α and γ are varied, we may observe regions on the (α, γ) parameter-space, where periodic structures appear embedded in a chaotic region.

Numbers identifying some periodic structures in black in Fig. 2.1(b) refer to the lower period (henceforth referred as period) of the respective structure, once bifurcations may occur when we move from the center to the periphery of each periodic structure. Period here is assumed as being the number of local maxima of the variable x_3, represented by X_3, in one complete trajectory on the phase-space attractor.

Some features, concerned with the above-mentioned periodic structures embedded in the chaotic region of the (α, γ) parameter-space in Figs. 2.1(a) and 2.1(b), deserve prominence and will be discussed in the continuation. For instance, it is remarkable the arrangement of periodic structures in the form of a spiral, that appears embedded in the chaotic set inside the boxed region in Fig. 2.1(a), and that appears amplified in Fig. 2.1(b). Note in Fig. 2.1(b) that this spiral structure coil up clockwise around a focal point marked with the plus sign in green, where the spiral itself initiates or terminates. By walking clockwise along this spiral in Fig. 2.1(b), moving along the *leg* joined to the period-7 structure at the lower portion of the parameter-space, we arrive at the period-9 structure at the upper portion. Continuing the moving, now from this period-9 structure, along the *leg* joined to it, we arrive at the period-9 structure at the lower portion. As can be concluded from inspection of Fig. 2.1(b), this behavior is recurrent, and the result is the $\ldots 7 \to 9 \to 9 \to 11 \to 11 \to 13 \to 13 \to \ldots$ periodic sequence, as we move along the *legs* of the periodic structures, closer and closer to focal point of the spiral. Therefore, this result may be interpreted as a period-adding sequence, whose increment in the period is equal to two, as the spiral is covered towards its focal point.

Numbers identifying the periods of some structures in Fig. 2.1(b) were determined with the assistance of the bifurcation diagram in Fig. 2.2, which was constructed by considering points along the red straight line $\gamma = 22.11\alpha - 12.50$ in Fig. 2.1(b),

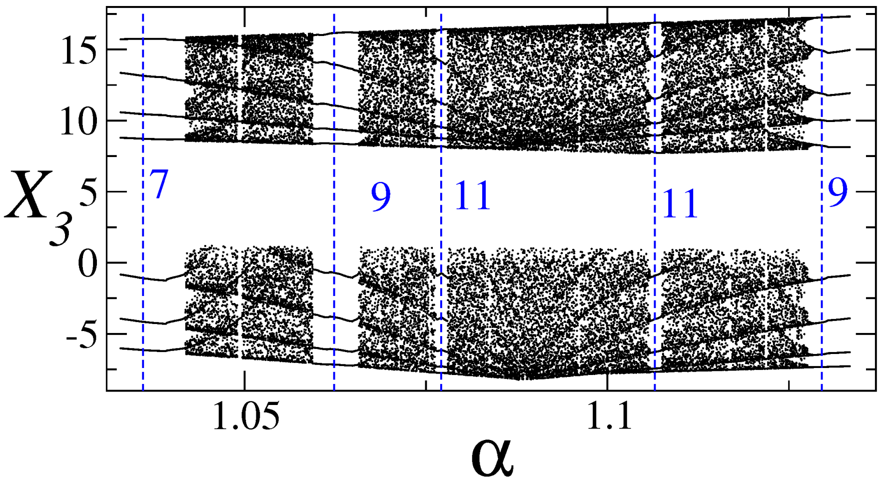

Figure 2.2. Bifurcation diagram for points along the red straight line $\gamma = 22.11\alpha - 12.50$ in Fig. 2.1(b), with $1.03293 \leq \alpha \leq 1.13337$. For each value of α was plotted the number of local maxima of the variable x_3, namely X_3, in one complete trajectory on the phase-space attractor.

for $1.03293 \leq \alpha \leq 1.13337$. Diagram in Fig. 2.2 considers the number of local maxima of the variable x_3 in one complete trajectory on the phase-space attractor, as a function of the parameter α. For each of the 10^3 values of the parameter α, were plotted 60 values of the local maximum X_3. As can be easily checked, each periodic window in Fig. 2.2 is related to a periodic structure of same number and position in Fig. 2.1(b).

Such spiral organization was observed before in different systems, modeled by different sets of nonlinear first-order ordinary differential equations, involving different mathematical functions. Some examples include electronic circuits [9, 10, 11], the Rössler model [12], a chemical oscillator [10], modified optical injection semiconductor lasers [13], a tumor growth model [14], an ecological model [15], the Lorenz-Stenflo system [16], and a thermal convection system [17], just to mention a few. The global mechanism behind the origin of the spiral organization of periodic structures in two dimensional parameter-spaces was reported simultaneously by Barrio and co-workers [18] and Vitolo and co-workers [19], having already been carried out experimental observation of such structures in electronic circuits [20].

Figure 2.3(a) shows the same particular region of the (α, γ) parameter-space of system (1.2) shown in Fig. 2.1(b), namely

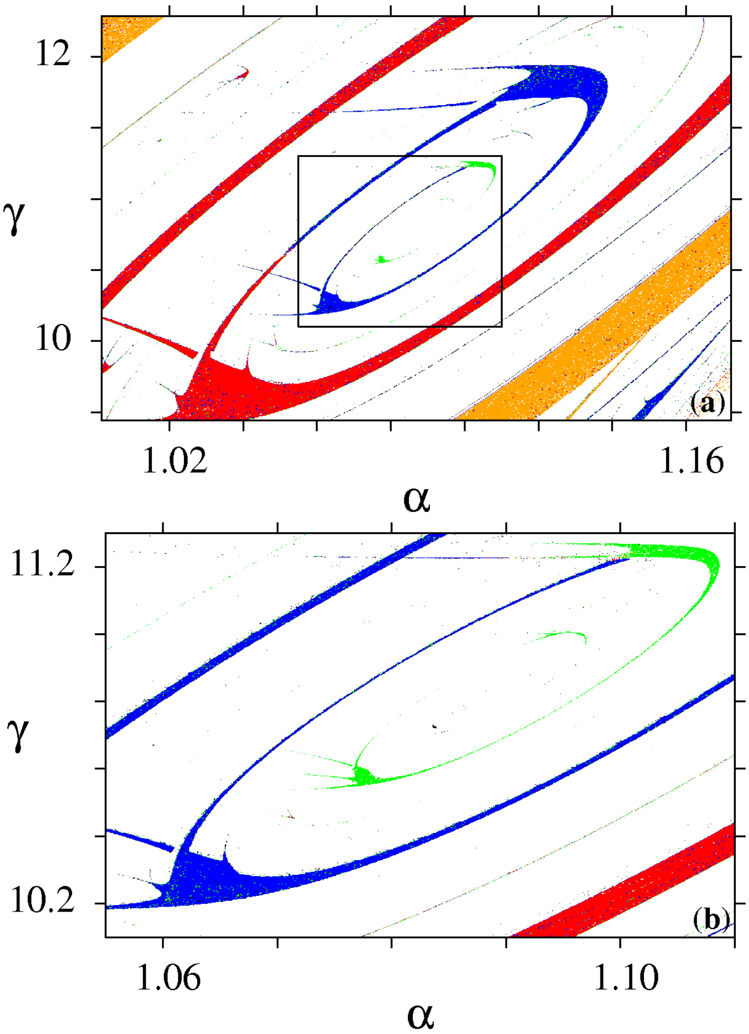

Figure 2.3. Both diagrams show periodic and chaotic behaviors in the (α, γ) parameter-space of system (1.2), with a hyperbolic tangent as the neuron activation function. Orange, red, blue, and green are associated respectively to periods 5, 7, 9, and 11.

for $1.00 \leq \alpha \leq 1.17$ and $9.44 \leq \gamma \leq 12.28$, while in Fig. 2.3(b) is shown an enlargement of the region inside the box in Fig. 2.3(a), for $1.055 \leq \alpha \leq 1.11$ and $10.1 \leq \gamma \leq 11.3$. Both plots are periodicity diagrams in the (α, γ) parameter-space, *i.e.*, the dynamical characterization of each point was made by using the period of the related trajectory in the phase-space, instead of the LLE. Therefore, each diagram in Fig. 2.3 provide us with more information than the one in Fig. 2.1(b), since the former discriminate different periodic regions and chaos, while the second only discriminate chaotic and periodic regions. As we said before, period is defined as the number of local maxima of the variable x_3 in one complete trajectory on the phase-space attractor. A period-k orbit is detected when $|(X_3)_k - (X_3)_0| < |(X_3)_0/10^3|$, $k = 1, 2, \ldots$.

Color in diagrams of Fig. 2.3 is related to the period of the respective structure. Period-5, period-7, period-9, and period-11 structures are painted respectively in orange, red, blue, and green. Structures with other period values are painted in white, as well as the chaotic regions. As before in obtaining diagrams in Fig. 2.1, system (1.2) was integrated by using a fourth order Runge-Kutta algorithm, with a fixed time step size equal to 10^{-2}, being dropped the first 1×10^6 integration steps, regarded as a transient. After this integration time, subsequent few integration steps were considered to determine the period for each pair of parameters (α, γ) in diagrams of Fig. 2.3.

In addition to discriminating different periods and chaos, diagrams in Fig. 2.3 provide us with additional information regarding the bifurcations that occur as the spiral is traversed in the direction of its focal point. As can be seen in Fig. 2.3(a), the bifurcation $7 \to 9$ occurs along the *leg* joining the period-7 and period-9 structures, where the color changes from red to blue. Figure 2.3(b) shows that the bifurcation $9 \to 11$ occurs along the *leg* joining the period-9 and period-11 structures, where the color changes from blue to green. Enlargement in an suitable region of Fig. 2.3(b) would show the location of the bifurcation $11 \to 13$, along the *leg* joining the period-11 and period-13 structures. Continuing with this procedure, *i.e.*, by producing enlargement in a suitable region of the previously obtained figure, which is not shown here, it would be possible to see the location of the bifurcation $13 \to 15$. Therefore, it is not difficult to conclude that the locations of the bifurcations $13 \to 15$, $15 \to 17$, $17 \to 19$, and so on, can be determined by considering different length scales in the (α, γ) parameter-space of system (1.2), *i.e.*, by considering diagrams resulting from enlargements of enlargements in Fig. 2.3(b).

3. Summary

In this work we have investigated a Hopfield-type three-neurons network, with a hyperbolic tangent function as the activation function. We have found a place in a two-dimensional parameter-space of this system, where typical periodic structures, the so-called shrimps, are embedded in a chaotic region, organized themselves in a spiral that coil up toward a focal point, while period-adding bifurcations occur. We have indicated the location along this spiral in the parameter-space, where these bifurcations happen.

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Author’s contributions

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Conformal Quasi-Hemi-Slant Riemannian Maps

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Abstract
In this paper, we state some geometric properties of conformal quasi-hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We give necessary and sufficient conditions for certain distributions to be integrable and get examples. For such distributions, we examine which conditions define totally geodesic foliations on base manifold. In addition, we apply notion of pluriharmonicity to get some relations between horizontally homothetic maps and conformal quasi-hemi-slant Riemannian maps.

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1. Introduction

The theory of Riemannian submersions between Riemannian manifolds was initially studied by O'Neill [17] and Gray [10]. Particularly, the concept of Riemannian submersions [7] and isometric immersions [6] were studied by Falcitelli and Chen. Then, Riemannian submersions were studied in various types as an anti-invariant, a semi-invariant, a slant, a hemi-slant, etc [13, 25]. Submersions between almost Hermitian manifolds expanded to almost Hermitian submersions [30]. Then, this concept was generalized to the notion of Riemannian map by Fischer [8]. Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions. Riemannian submersions have many application. Let $\Phi : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rank\Phi < \min\{\dim(M_1), \dim(M_2)\}$. Then the tangent bundle TM_1 of M_1 has the following decomposition:

$$TM_1 = ker\Phi_* \oplus (ker\Phi_*)^\perp.$$

Since $rank\Phi < \min\{\dim(M_1), \dim(M_2)\}$, always we have $(range\Phi_*)^\perp$. In this way, tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (range\Phi_*) \oplus (range\Phi_*)^\perp.$$

A smooth map $\Phi : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $\Phi^h_{*p_1} : (ker\Phi_{*p_1})^\perp \longrightarrow (range\Phi_*)$ is a linear isometry. Hence, a Riemannian map satisfies the equation

$$g_1(Z_1, Z_2) = g_2(\Phi_*(Z_1), \Phi_*(Z_2)) \tag{1.1}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $ker\Phi_* = \{0\}$ and $(range\Phi_*)^\perp = \{0\}$ [7]. An important application field of Riemannian maps is the eikonal equation. It acts as a bridge between geometric optics and physical optics. Also, Riemannian maps and their applications studied by Garcia-Rio and Kupeli in semi-Riemannian geometry [9]. Recently, some optimal inequalities for Riemannian maps from Riemannian manifolds onto space forms were established in [12].

Moreover, Şahin introduced any other types of Riemannian maps [20, 21, 22, 23], see also [18, 19]. In further studies, in particular Akyol, Şahin and Yanan searched this type submersions [1, 2, 3, 4, 31] and Riemannian maps [26, 27, 32, 35] under conformality case, see also [11]. All these studies have many applications as texture mapping, remeshing and simulation [14], computer graphics and medical imaging fields [28], brain mapping research [29]. For a comprehensive study in which these issues are introduced and their applications are given, see [25]. We say that $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rank\Phi_{*p} \leq \min\{m, n\}$ and Φ_{*p} maps the horizontal space $(ker(\Phi_{*p})^{\perp})$ conformally onto $range(\Phi_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(\Phi_{*p}(Z_1), \Phi_{*p}(Z_2)) = \lambda^2(p) g_M(Z_1, Z_2) \tag{1.2}$$

for $Z_1, Z_2 \in \Gamma((ker(\Phi_{*p})^{\perp})$. Also Φ is called conformal Riemannian if Φ is conformal Riemannian at each $p \in M$ [24].

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on M such that $J^2 = -\mathbb{I}$ where $\mathbb{I}$ denotes the identity transformation of TM and

$$g_M(E, F) = g_M(JE, JF), \forall E, F \in \Gamma(TM). \tag{1.3}$$

Let (M, g_M, J) is an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_E J)F = 0, \tag{1.4}$$

we say M is a Kaehler manifold [36].

Therefore, in section 2, we present necessary background concepts to be used in this paper. In section 3, we study conformal quasi-hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We introduce some properties as integrability conditions and totally geodesic foliation defining of distributions. In section 4, we use the concept of pluriharmonicity to introduce relations between horizontally homothetic maps and conformal quasi-hemi-slant Riemannian maps.

2. Preliminaries

In this section, we give several definitions and results to be used throughout the study for conformal quasi-hemi-slant Riemannian maps. Let $\Phi : (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. The second fundamental form of Φ is defined by

$$(\nabla\Phi_*)(E, F) = \overset{N}{\nabla}{}^{\Phi}_{E}\Phi_*(F) - \Phi_*(\overset{M}{\nabla}_E F) \tag{2.1}$$

for $E, F \in \Gamma(TM)$. The second fundamental form $\nabla\Phi_*$ is symmetric [15].

Then we define O'Neill's tensor fields $\mathscr{T}$ and $\mathscr{A}$ for Riemannian submersions as

$$\mathscr{A}_E F = h\overset{M}{\nabla}_{hE} vF + v\overset{M}{\nabla}_{hE} hF, \tag{2.2}$$

$$\mathscr{T}_E F = h\overset{M}{\nabla}_{vE} vF + v\overset{M}{\nabla}_{vE} hF \tag{2.3}$$

for $E, F \in \Gamma(TM)$ with the Levi-Civita connection $\overset{M}{\nabla}$ of g_M [17]. For any $E \in \Gamma(TM)$, $\mathscr{T}_E$ and $\mathscr{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. Also, $\mathscr{T}$ is vertical, $\mathscr{T}_E = \mathscr{T}_{vE}$, and $\mathscr{A}$ is horizontal, $\mathscr{A}_E = \mathscr{A}_{hE}$. Note that the tensor field $\mathscr{T}$ is symmetric on the vertical distribution [17]. Additionally, from (2.2) and (2.3) we have

$$\overset{M}{\nabla}_{\xi_1} \xi_2 = \mathscr{T}_{\xi_1} \xi_2 + \hat{\nabla}_{\xi_1} \xi_2, \tag{2.4}$$

$$\overset{M}{\nabla}_{\xi_1} Z_1 = h\overset{M}{\nabla}_{\xi_1} Z_1 + \mathscr{T}_{\xi_1} Z_1, \tag{2.5}$$

$$\overset{M}{\nabla}_{Z_1} \xi_1 = \mathscr{A}_{Z_1} \xi_1 + v\overset{M}{\nabla}_{Z_1} \xi_1, \tag{2.6}$$

$$\overset{M}{\nabla}_{Z_1} Z_2 = h\overset{M}{\nabla}_{Z_1} Z_2 + \mathscr{A}_{Z_1} Z_2 \tag{2.7}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$, where $\hat{\nabla}_{\xi_1}\xi_2 = v\overset{M}{\nabla}_{\xi_1}\xi_2$ [7].

If a vector field Z on M is related to a vector field Z' on N, we say Z is a projectable vector field. If Z is both a horizontal and a projectable vector field, we say Z is a basic vector field on M. From now on, when we mention a horizontal vector field, we always consider a basic vector field [5].

On the other hand, let $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a conformal Riemannian map between Riemannian manifolds. Then, we have

$$(\nabla\Phi_*)(Z_1, Z_2)\,|_{range\Phi_*} = Z_1(\ln\lambda)\Phi_*(Z_2) + Z_2(\ln\lambda)\Phi_*(Z_1) - g_M(Z_1, Z_2)\Phi_*(grad(\ln\lambda)), \tag{2.8}$$

where $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$. Hence from (2.8), we obtain $\overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2)$ as

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) &= \Phi_*(h\overset{M}{\nabla}_{Z_1}Z_2) + Z_1(\ln\lambda)\Phi_*(Z_2) + Z_2(\ln\lambda)\Phi_*(Z_1) \\ &- g_M(Z_1, Z_2)\Phi_*(grad(\ln\lambda)) + (\nabla\Phi_*)^\perp(Z_1, Z_2), \end{aligned} \tag{2.9}$$

where $(\nabla\Phi_*)^\perp(Z_1, Z_2)$ is the component of $(\nabla\Phi_*)(Z_1, Z_2)$ on $(range\Phi_*)^\perp$ for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ [26, 27].

Lastly, a map Φ from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if Φ satisfies the following equation

$$(\nabla\Phi_*)(E, F) + (\nabla\Phi_*)(JE, JF) = 0 \tag{2.10}$$

for $E, F \in \Gamma(TM)$ [16].

3. Conformal quasi-hemi-slant Riemannian map

We give definition of conformal quasi-hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. In the rest of this paper, we take (M, g_M, J) as a Kaehler manifold.

Definition 3.1. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal Riemannian map such that its vertical distribution $ker\Phi_*$ admits three orthogonal distributions D, D_θ and $D_\perp$ which are invariant, slant and anti-invariant distributions, respectively, i.e.*

$$ker\Phi_* = D \oplus D_\theta \oplus D_\perp. \tag{3.1}$$

Then, we say Φ is a conformal quasi-hemi-slant Riemannian map and the angle θ is called the quasi-hemi-slant angle of the map.

Here, we say that;

i) Φ is a conformal hemi-slant Riemannian map [33] if $D = \{0\}$.

ii) Φ is a conformal semi-invariant Riemannian map [27] if $D_\theta = \{0\}$.

iii) Φ is a conformal semi-slant Riemannian map [34] if $D_\perp = \{0\}$.

Therefore, conformal quasi-hemi-slant Riemannian maps are generalization of conformal hemi-slant Riemannian maps, conformal semi-invariant Riemannian maps and conformal semi-slant Riemannian maps. Hence, all these maps provide examples to conformal quasi-hemi-slant Riemannian maps.

We say that conformal quasi-hemi-slant Riemannian map Φ is a *proper conformal quasi-hemi-slant Riemannian map* if the invariant distribution $D \neq \{0\}$, the anti-invariant distribution $D_\perp \neq \{0\}$ and the slant angle $\theta \neq 0, \frac{\pi}{2}$. Now, we give an explicit example to proper condition.

Example 3.2. *Define a map by $\Phi : \mathbb{R}^8 \longrightarrow \mathbb{R}^5$ by*

$$\Phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \longrightarrow (x_1 + x_2, x_3 - x_5, \sqrt{2}x_4, b, c)$$

with $b, c \in \mathbb{R}$. We get the horizontal distribution

$$(ker\Phi_*)^\perp = \{Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Z_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, Z_3 = \sqrt{2}(\frac{\partial}{\partial x_4})\}$$

and the vertical distribution

$$ker\Phi_* = \{\xi_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \xi_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \xi_3 = \frac{\partial}{\partial x_6}, \xi_4 = \frac{\partial}{\partial x_7}, \xi_5 = \frac{\partial}{\partial x_8}\},$$

respectively. By the complex structure J of $\mathbb{R}^8$ such that $J = (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$, we have

$$J\xi_1 = Z_1,\ J\xi_2 = \frac{1}{\sqrt{2}}Z_3 + \xi_3,\ J\xi_3 = \frac{1}{2}Z_2 - \frac{1}{2}\xi_2,\ J\xi_4 = \xi_5,\ J\xi_5 = -\xi_4.$$

Hence, we obtain the distributions as $D = sp\{\xi_4, \xi_5\}$, $D_\theta = sp\{\xi_2, \xi_3\}$ and $D_\perp = sp\{\xi_1\}$. Therefore, Φ is a proper conformal quasi-hemi-slant Riemannian map with $\lambda = \sqrt{2}$ and the slant angle $\theta = \frac{\pi}{4}$.

Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then we have

$$TM = ker\Phi_* \oplus (ker\Phi_*)^\perp. \tag{3.2}$$

A vertical vector field ξ can be written as

$$\xi = \tilde{P}\xi + \tilde{Q}\xi + \tilde{R}\xi \tag{3.3}$$

where $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ are projections to D, D_θ and $D_\perp$, respectively. We get

$$J\xi = \phi\xi + \psi\xi \tag{3.4}$$

where $\phi\xi \in \Gamma(ker\Phi_*)$ and $\psi\xi \in \Gamma((ker\Phi_*)^\perp)$. We obtain $\psi\tilde{P}\xi = 0$, $\phi\tilde{R}\xi = 0$ and

$$J\xi = \phi\tilde{P}\xi + \phi\tilde{Q}\xi + \psi\tilde{Q}\xi + \psi\tilde{R}\xi \tag{3.5}$$

from (3.3) and (3.4). So, we can write

$$J(ker\Phi_*) = D \oplus \phi D_\theta \oplus \psi D_\theta \oplus J(D_\perp). \tag{3.6}$$

From (3.6), we have

$$(ker\Phi_*)^\perp = \psi D_\theta \oplus J(D_\perp) \oplus \mu \tag{3.7}$$

where μ is the orthogonal complement distributions of $\psi D_\theta \oplus J(D_\perp)$ in $(ker\Phi_*)^\perp$ and μ is the invariant with respect to J. At last, for $Z \in \Gamma((ker\Phi_*)^\perp)$, we have

$$JZ = BZ + CZ \tag{3.8}$$

where $BZ \in \Gamma(\psi D_\theta \oplus J(D_\perp))$ and $CZ \in \Gamma(\mu)$.

Here that easily we obtain from (3.1) - (3.7);

$$\phi D_\theta = D_\theta,\ \phi D_\perp = \{0\},\ B\psi D_\theta = D_\theta,\ B\psi D_\perp = D_\perp,\ \psi D = \{0\}, \tag{3.9}$$

$$\phi^2 + B\psi = -\mathbb{I},\ \ \psi\phi + C\psi = 0,\ \ \psi B + C^2 = \mathbb{I},\ \ \phi B + BC = 0. \tag{3.10}$$

Following theorem has same proof with hemi-slant submanifolds as hemi-slant Riemannian maps; see Theorem 3.6. of [23].

Theorem 3.3. *Let Φ be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is a conformal quasi-hemi-slant Riemannian map if and only if there exists a constant $\lambda \in [0,1]$ and a distribution $\mathscr{D}$ on $ker\Phi_*$ such that*

i) $\mathscr{D} = \{\xi \in \Gamma(ker\Phi_*) | \phi^2\xi = \lambda\xi\}$,

ii) for any $\xi \in \Gamma(ker\Phi_*)$ orthogonal to $\mathscr{D}$, we have $\phi\xi = 0$.

Further, we have $\lambda = -\cos^2\theta$ where θ is the slant angle of Φ.

Hence, we have followings from Theorem 3.3.

$$g_M(\phi V_1,\phi V_2)=\cos^2\theta g_M(V_1,V_2), \tag{3.11}$$

$$g_M(\psi V_1,\psi V_2)=\sin^2\theta g_M(V_1,V_2) \tag{3.12}$$

for any $V_1,V_2\in\Gamma(D_\theta)$.

Recall that always the vertical distribution $ker\Phi_*$ is integrable [25]. Now, we give integrability conditions for certain distributions on total manifolds.

Theorem 3.4. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the invariant distribution D is integrable if and only if*

$$\hat{\nabla}_{U_1}JU_2-\hat{\nabla}_{U_2}JU_1\in\Gamma(D\oplus D_\perp),$$

$$\phi\mathscr{T}_{U_1-U_2}\psi\xi\in\Gamma(D_\theta\oplus D_\perp)$$

for $U_1,U_2\in\Gamma(D)$ and $\xi\in\Gamma(D_\theta\oplus D_\perp)$.

Proof. Since $\mathscr{T}$ is skew-symmetric and from equations (1.4), (2.4), (3.4), we get

$$g_M(\overset{M}{\nabla}_{U_1}U_2,\xi)=g_M(\hat{\nabla}_{U_1}JU_2,\phi\xi)+g_M(\phi\mathscr{T}_{U_1}\psi\xi,U_2) \tag{3.13}$$

for $U_1,U_2\in\Gamma(D)$ and $\xi\in\Gamma(D_\theta\oplus D_\perp)$. Now, changing the roles of U_1 and U_2, we obtain

$$g_M([U_1,U_2],\xi)=g_M(\hat{\nabla}_{U_1}JU_2-\hat{\nabla}_{U_2}JU_1,\phi\xi)+g_M(\phi\mathscr{T}_{U_1}\psi\xi,U_2)-g_M(\phi\mathscr{T}_{U_2}\psi\xi,U_1). \tag{3.14}$$

The proof is complete from equation (3.14). □

Theorem 3.5. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the slant distribution D_θ is integrable if and only if*

$$\begin{aligned}\lambda^2 g_M(h\overset{M}{\nabla}_{V_1}\psi V_2-h\overset{M}{\nabla}_{V_2}\psi V_1,\psi\tilde{R}\xi) &= g_N((\nabla\Phi_*)(V_1-V_2,\tilde{P}\xi),\Phi_*(\psi\phi V_2))+g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))\\ &- g_N((\nabla\Phi_*)(V_2,\phi V_1),\Phi_*(\psi\tilde{R}\xi))+g_N((\nabla\Phi_*)(V_2,J\tilde{P}\xi),\Phi_*(\psi V_1))\\ &- g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2))\end{aligned}$$

for $V_1,V_2\in\Gamma(D_\theta)$ and $\xi\in\Gamma(D\oplus D_\perp)$.

Proof. Now, from equations (2.4), (2.5), (3.3) and (3.5), we have

$$\begin{aligned}g_M(\overset{M}{\nabla}_{V_1}V_2,\xi) &= g_M(\overset{M}{\nabla}_{V_1}\phi V_2,J\tilde{P}\xi)+g_M(\overset{M}{\nabla}_{V_1}\phi V_2,\psi\tilde{R}\xi)+g_M(\mathscr{T}_{V_1}\psi V_2+h\overset{M}{\nabla}_{V_1}\psi V_2,J\tilde{P}\xi+\psi\tilde{R}\xi)\\ &= -g_M(\overset{M}{\nabla}_{V_1}J\phi V_2,\tilde{P}\xi)+g_M(\mathscr{T}_{V_1}\phi V_2,\psi\tilde{R}\xi)+g_M(\mathscr{T}_{V_1}\psi V_2,J\tilde{P}\xi)+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi)\end{aligned}$$

for $V_1,V_2\in\Gamma(D_\theta)$ and $\xi\in\Gamma(D\oplus D_\perp)$. Since $\mathscr{T}$ is skew-symmetric and from (1.2), (2.1), Theorem 1., we have

$$\begin{aligned}g_M(\overset{M}{\nabla}_{V_1}V_2,\xi) &= -g_M(\overset{M}{\nabla}_{V_1}\phi^2V_2,\tilde{P}\xi)-g_M(\overset{M}{\nabla}_{V_1}\psi\phi V_2,\tilde{P}\xi)\\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))-g_M(\mathscr{T}_{V_1}J\tilde{P}\xi,\psi V_2)+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi)\\ &= \cos^2\theta g_M(\overset{M}{\nabla}_{V_1}V_2,\tilde{P}\xi)-g_M(\mathscr{T}_{V_1}\psi\phi V_2,\tilde{P}\xi)\\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2))+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}g_M(\overset{M}{\nabla}_{V_1}V_2,\xi)-\cos^2\theta g_M(\overset{M}{\nabla}_{V_1}V_2,\tilde{P}\xi) &= g_M(\mathscr{T}_{V_1}\tilde{P}\xi,\psi\phi V_2)-\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))\\ &+ \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2))+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi).\end{aligned} \tag{3.15}$$

In equation (3.15), if we change the roles of V_1 and V_2 we obtain

$$\begin{aligned} g_M([V_1,V_2],\xi)-\cos^2\theta g_M([V_1,V_2],\tilde{P}\xi) &= \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_2,\tilde{P}\xi),\Phi_*(\psi\phi V_2))-\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\tilde{P}\xi),\Phi_*(\psi\phi V_2)) \\ &+ g_M(h\overset{M}{\nabla}_{V_1}\psi V_2-h\overset{M}{\nabla}_{V_2}\psi V_1,\psi\tilde{R}\xi)+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2)) \\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_2,J\tilde{P}\xi),\Phi_*(\psi V_1))+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_2,\phi V_1),\Phi_*(\psi\tilde{R}\xi)) \\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi)). \end{aligned} \tag{3.16}$$

Therefore, the proof is clear from (3.16). □

Here, we know that integrability case of the anti-invariant distribution $D_\perp$ is same with Theorem 3.8. in [23]. We have next theorem for horizontal distribution.

Theorem 3.6. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the horizontal distribution $(ker\Phi_*)^\perp$ is integrable if and only if*

$$\begin{aligned} &\lambda^2 g_M(v\overset{M}{\nabla}_{Z_2}BZ_1-v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi) \\ &+\lambda^2\{CZ_2(\ln\lambda)g_M(Z_1,\psi\xi)-CZ_1(\ln\lambda)g_M(Z_2,\psi\xi)+\psi\xi(\ln\lambda)g_M(Z_2,CZ_1)-\psi\xi(\ln\lambda)g_M(Z_1,CZ_2)\} \\ &=g_N((\nabla\Phi_*)(Z_2,BZ_1)-(\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi))+g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2)) \\ &-g_N((\nabla\Phi_*)(Z_2,\phi\xi),\Phi_*(CZ_1))+g_N(\overset{N}{\nabla^\Phi}_{Z_1}\Phi_*(CZ_2)-\overset{N}{\nabla^\Phi}_{Z_2}\Phi_*(CZ_1),\Phi_*(\psi\xi)) \end{aligned}$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$ and $\xi\in\Gamma(ker\Phi_*)$.

Proof. To show the horizontal distribution $(ker\Phi_*)^\perp$ is integrable, we only search $0=g_M([Z_1,Z_2],\xi)$ for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$ and $\xi\in\Gamma(ker\Phi_*)$. Since $\mathscr{A}$ is skew-symmetric and from definitions (1.2), (1.4), equations (2.6), (3.4), (3.7) we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{Z_1}Z_2,\xi) &= g_M(\mathscr{A}_{Z_1}BZ_2,\psi\xi)+g_M(h\overset{M}{\nabla}_{Z_1}CZ_2,\psi\xi)+g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi)-g_M(\mathscr{A}_{Z_1}\phi\xi,CZ_2) \\ &= \frac{1}{\lambda^2}g_N(\Phi_*(\mathscr{A}_{Z_1}BZ_2),\Phi_*(\psi\xi))+\frac{1}{\lambda^2}g_N(\Phi_*(h\overset{M}{\nabla}_{Z_1}CZ_2),\Phi_*(\psi\xi)) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi)-\frac{1}{\lambda^2}g_N(\Phi_*(\mathscr{A}_{Z_1}\phi\xi),\Phi_*(CZ_2)) \end{aligned}$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$ and $\xi\in\Gamma(ker\Phi_*)$. Using (2.1), (2.8) and (2.9), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{Z_1}Z_2,\xi) &= -\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi))+\frac{1}{\lambda^2}g_N(\Phi_*(h\overset{M}{\nabla}_{Z_1}CZ_2),\Phi_*(\psi\xi)) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi)+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2)) \\ &= -\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi)) \\ &+ \frac{1}{\lambda^2}g_N(\overset{N}{\nabla^\Phi}_{Z_1}\Phi_*(CZ_2),\Phi_*(\psi\xi))-CZ_2(\ln\lambda)g_M(Z_1,\psi\xi) \\ &+ \psi\xi(\ln\lambda)g_M(Z_1,CZ_2)+g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi) \\ &+ \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2)). \end{aligned} \tag{3.17}$$

Similarly, if we change the roles of Z_1 and Z_2 in (3.17) we finally obtain,

$$\begin{aligned}
g_M([Z_1,Z_2],\xi) &= \frac{1}{\lambda^2}\{g_N((\nabla\Phi_*)(Z_2,BZ_1)-(\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi)) \\
&+ g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2))-g_N((\nabla\Phi_*)(Z_2,\phi\xi),\Phi_*(CZ_1)) \\
&+ g_N(\overset{N}{\nabla^\Phi}_{Z_1}\Phi_*(CZ_2)-\overset{N}{\nabla^\Phi}_{Z_2}\Phi_*(CZ_1),\Phi_*(\psi\xi))\} \\
&- CZ_2(\ln\lambda)g_M(Z_1,\psi\xi)+\psi\xi(\ln\lambda)g_M(Z_1,CZ_2) \\
&+ CZ_1(\ln\lambda)g_M(Z_2,\psi\xi)-\psi\xi(\ln\lambda)g_M(Z_2,CZ_1) \\
&+ g_M(v\overset{M}{\nabla}_{Z_1}BZ_2-v\overset{M}{\nabla}_{Z_2}BZ_1,\phi\xi).
\end{aligned} \tag{3.18}$$

Hence, we get the proof from (3.18). □

Note that if $(\nabla\Phi_*)(E,F)=0$ for all $E,F\in\Gamma(TM)$ the map Φ is said to be totally geodesic map [25]. Using this notion we have followings.

Theorem 3.7. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the invariant distribution D defines a totally geodesic foliation on M if and only if*

$$C\mathscr{T}_{U_1}\phi U_2+\psi\hat{\nabla}_{U_1}\phi U_2=0$$

for $U_1,U_2\in\Gamma(D)$.

Proof. Since D is an invariant distribution we have $\psi U_2=0$. From (2.1), (2.4), (3.4) and (3.8) we get

$$\begin{aligned}
(\nabla\Phi_*)(U_1,U_2) &= -\Phi_*(\overset{M}{\nabla}_{U_1}U_2) \\
&= \Phi_*(C\mathscr{T}_{U_1}\phi U_2+\psi\hat{\nabla}_{U_1}\phi U_2)
\end{aligned} \tag{3.19}$$

for $U_1,U_2\in\Gamma(D)$. The proof can be seen from (3.19). □

Theorem 3.8. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the slant distribution D_θ defines totally geodesic foliation on M if and only if*

$$\mathscr{T}_{V_1}B\psi V_2=0$$

for $V_1,V_2\in\Gamma(D_\theta)$.

Proof. From definition of second fundamental form, (3.4) and (3.8), we have

$$\begin{aligned}
(\nabla\Phi_*)(V_1,V_2) &= \Phi_*(J\overset{M}{\nabla}_{V_1}JV_2) \\
&= \Phi_*(\overset{M}{\nabla}_{V_1}J\phi V_2)+\Phi_*(\overset{M}{\nabla}_{V_1}J\psi V_2) \\
&= \Phi_*(\overset{M}{\nabla}_{V_1}\phi^2V_2+\overset{M}{\nabla}_{V_1}\psi\phi V_2)+\Phi_*(\overset{M}{\nabla}_{V_1}B\psi V_2+\overset{M}{\nabla}_{V_1}C\psi V_2)
\end{aligned}$$

for $V_1,V_2\in\Gamma(D_\theta)$. From (3.9), (3.10) and Theorem 3.3., we obtain

$$\begin{aligned}
&= \Phi_*(-\cos^2\theta\overset{M}{\nabla}_{V_1}V_2)+\Phi_*(h\overset{M}{\nabla}_{V_1}\psi\phi V_2)+\Phi_*(\mathscr{T}_{V_1}B\psi V_2+h\overset{M}{\nabla}_{V_1}C\psi V_2) \\
\cos^2\theta\Phi_*(\overset{M}{\nabla}_{V_1}V_2) &= \Phi_*(\mathscr{T}_{V_1}B\psi V_2).
\end{aligned} \tag{3.20}$$

The proof is clear from (3.20). □

In a similar way, we have easily the next theorems.

Theorem 3.9. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the anti-invariant distribution $D_\perp$ defines a totally geodesic foliation on M if and only if*

$$\psi\mathscr{T}_{W_1}\psi W_2+Ch\overset{M}{\nabla}_{W_1}\psi W_2=0$$

for $W_1,W_2\in\Gamma(D_\perp)$.

Proof. From (1.4), (2.5), (3.4) and (3.8), we obtain

$$\begin{aligned}(\nabla\Phi_*)(W_1,W_2) &= \Phi_*(J\overset{M}{\nabla}_{W_1}JW_2)\\ &= \Phi_*(J\mathscr{T}_{W_1}\psi W_2 + Jh\overset{M}{\nabla}_{W_1}\psi W_2)\\ &= \Phi_*(\psi\mathscr{T}_{W_1}\psi W_2 + Ch\overset{M}{\nabla}_{W_1}\psi W_2) \qquad (3.21)\end{aligned}$$

for $W_1,W_2\in\Gamma(D_\perp)$. We have the proof from (3.21). □

Theorem 3.10. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the vertical distribution $ker\Phi_*$ defines a totally geodesic foliation on M if and only if*

$$\psi\{\mathscr{T}_{\xi_1}\psi\xi_2+\hat{\nabla}_{\xi_1}\phi\xi_2\}+C\{h\overset{M}{\nabla}_{\xi_1}\psi\xi_2+\mathscr{T}_{\xi_1}\phi\xi_2\}=0$$

for $\xi_1,\xi_2\in\Gamma(ker\Phi_*)$.

Recall that if $h(grad(\ln\lambda))=0$, the map Φ is said to be horizontally homothetic map [5].

Theorem 3.11. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, any two conditions below imply the third condition;*

i) The horizontal distribution $(ker\Phi_*)^\perp$ defines a totally geodesic foliation on M,

ii) The map Φ is a horizontally homothetic map,

iii)

$$\overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_2)=\Phi_*(J[JZ_1,Z_2])+(\nabla\Phi_*)^\perp(CZ_1,CZ_2)+\Phi_*(\mathscr{A}_{CZ_2}BZ_1+\mathscr{A}_{CZ_1}BZ_2+\mathscr{T}_{BZ_1}BZ_2)$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$.

Proof. Firstly, from (2.1) and (2.9) we have

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{JZ_1}JZ_2) &= \overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_1)-(\nabla\Phi_*)(BZ_1,BZ_2)-(\nabla\Phi_*)(CZ_2,BZ_1)-(\nabla\Phi_*)(CZ_1,BZ_2)\\ &- (\nabla\Phi_*)^\perp(CZ_1,CZ_2)-CZ_1(\ln\lambda)\Phi_*(CZ_2)-CZ_2(\ln\lambda)\Phi_*(CZ_1)+g_M(CZ_1,CZ_2)\Phi_*(grad(\ln\lambda))\\ &= \overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_1)-(\nabla\Phi_*)^\perp(CZ_1,CZ_2)-\Phi_*(\mathscr{A}_{CZ_2}BZ_1+\mathscr{A}_{CZ_1}BZ_2+\mathscr{T}_{BZ_1}BZ_2)\\ &- CZ_1(\ln\lambda)\Phi_*(CZ_2)-CZ_2(\ln\lambda)\Phi_*(CZ_1)\\ &+ g_M(CZ_1,CZ_2)\Phi_*(grad(\ln\lambda)) \qquad (3.22)\end{aligned}$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$. On the other hand, we have

$$\overset{M}{\nabla}_{JZ_1}JZ_2=J[JZ_1,Z_2]+J\overset{M}{\nabla}_{Z_2}JZ_1. \qquad (3.23)$$

Putting equation (3.23) in (3.22), we obtain

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{Z_2}Z_1) &= \Phi_*(J[JZ_1,Z_2])-\overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_1)\\ &+ \Phi_*(\mathscr{A}_{CZ_2}BZ_1+\mathscr{A}_{CZ_1}BZ_2+\mathscr{T}_{BZ_1}BZ_2)\\ &+ (\nabla\Phi_*)^\perp(CZ_1,CZ_2)+CZ_1(\ln\lambda)\Phi_*(CZ_2)\\ &+ CZ_2(\ln\lambda)\Phi_*(CZ_1)-g_M(CZ_1,CZ_2)\Phi_*(grad(\ln\lambda)). \qquad (3.24)\end{aligned}$$

Now, suppose that (i) and (ii) are satisfied in (3.24). We have $\Phi_*(\overset{M}{\nabla}_{Z_2}Z_1)=0$ and

$$0 = CZ_1(\ln\lambda)\Phi_*(CZ_2)+CZ_2(\ln\lambda)\Phi_*(CZ_1)-g_M(CZ_1,CZ_2)\Phi_*(grad(\ln\lambda)) \qquad (3.25)$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$, respectively. So, we obtain (iii) clearly. If (ii) and (iii) are provided in (3.24), we have (3.25) and

$$\overset{N}{\nabla}{}^{\Phi}{}_{JZ_1}\Phi_*(CZ_2) \;=\; \Phi_*(J[JZ_1,Z_2]) + (\nabla\Phi_*)^\perp(CZ_1,CZ_2) + \Phi_*(\mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2 + \mathscr{T}_{BZ_1}BZ_2), \tag{3.26}$$

respectively. Easily, we obtain $\Phi_*(\overset{M}{\nabla}_{Z_2}Z_1) = 0$. Hence, we say that the horizontal distribution $(ker\Phi_*)^\perp$ defines totally geodesic foliation on M. At last, if (i) and (iii) are provided in (3.24), we obtain (3.25). For $CZ_1 \in \Gamma(\mu)$ in (3.25), we get

$$0 = \lambda^2 CZ_2(\ln\lambda) g_M(CZ_1, CZ_1). \tag{3.27}$$

Hence, we obtain $0 = CZ_2(\ln\lambda)$. It means λ is a constant on μ. Then, for $\xi \in \Gamma(ker\Phi_*)$ and $\psi\xi \in \Gamma(\psi D_\theta \oplus J(D_\perp))$ in (3.25) with $CZ_1 = CZ_2$, we get

$$0 = \lambda^2 \psi\xi(\ln\lambda) g_M(CZ_1, CZ_2). \tag{3.28}$$

Hence, we obtain $0 = \psi\xi(\ln\lambda)$. It means λ is a constant on $\psi D_\theta \oplus J(D_\perp)$. Hence, λ is a constant on horizontal distribution. We obtain (iii) from (3.27) and (3.28). The proof is complete. □

Here, we have conditions for the map Φ which defines a totally geodesic foliations on M.

Theorem 3.12. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the map Φ defines a totally geodesic foliations on M if and only if*

i) The map Φ is a horizontally homothetic map,

ii)

$$\begin{aligned}\overset{N}{\nabla}{}^{\Phi}{}_{E}\Phi_*(\bar F) - \overset{N}{\nabla}{}^{\Phi}{}_{\bar E}\Phi_*(\bar F) \;&=\; \Phi_*(\mathscr{T}_{\hat E}\hat F + h\overset{M}{\nabla}_{\hat E}\bar F + \mathscr{A}_{\bar E}\hat F)\\ &-\; (\nabla\Phi_*)^\perp(\bar E, \bar F)\end{aligned}$$

holds for $E, F \in \Gamma(TM)$ where $\bar E, \bar F$ and $\hat E, \hat F$ show horizontal and vertical parts of E, F, respectively.

Proof. Using definition of second fundamental form of a map, (2.4), (2.5) and (2.6) we have

$$\begin{aligned}(\nabla\Phi_*)(E,F) \;&=\; \overset{N}{\nabla}{}^{\Phi}{}_{E}\Phi_*(\bar F) - \Phi_*(\overset{M}{\nabla}_E F)\\ &=\; \overset{N}{\nabla}{}^{\Phi}{}_{E}\Phi_*(\bar F) - \Phi_*(\overset{M}{\nabla}_{\hat E}\hat F + \overset{M}{\nabla}_{\hat E}\bar F + \overset{M}{\nabla}_{\bar E}\hat F + \overset{M}{\nabla}_{\bar E}\bar F)\\ &=\; \overset{N}{\nabla}{}^{\Phi}{}_{E}\Phi_*(\bar F) - \Phi_*(\mathscr{T}_{\hat E}\hat F + h\overset{M}{\nabla}_{\hat E}\bar F + \mathscr{A}_{\bar E}\hat F) - \Phi_*(\overset{M}{\nabla}_{\bar E}\bar F)\end{aligned} \tag{3.29}$$

for $E, F \in \Gamma(TM)$. Here, from equation (2.9) in (3.29), we obtain

$$\begin{aligned}(\nabla\Phi_*)(E,F) \;&=\; \overset{N}{\nabla}{}^{\Phi}{}_{E}\Phi_*(\bar F) - \Phi_*(\mathscr{T}_{\hat E}\hat F + h\overset{M}{\nabla}_{\hat E}\bar F + \mathscr{A}_{\bar E}\hat F)\\ &-\; \overset{N}{\nabla}{}^{\Phi}{}_{\bar E}\Phi_*(\bar F) + (\nabla\Phi_*)^\perp(\bar E,\bar F) + \bar E(\ln\lambda)\Phi_*(\bar F)\\ &+\; \bar F(\ln\lambda)\Phi_*(\bar E) - g_M(\bar E,\bar F)\Phi_*(grad(\ln\lambda)).\end{aligned} \tag{3.30}$$

Because of Φ defines a totally geodesic foliations on M, we have (3.30). Suppose that Φ is a horizontally homothetic map, we have from (3.30)

$$0 = \bar E(\ln\lambda)\Phi_*(\bar F)\bar F(\ln\lambda)\Phi_*(\bar E) - g_M(\bar E,\bar F)\Phi_*(grad(\ln\lambda)). \tag{3.31}$$

We obtain from (3.31)

$$0 = \lambda^2 \bar F(\ln\lambda) g_M(\bar E, \bar E) \tag{3.32}$$

for $\bar E \in \Gamma((ker\Phi_*)^\perp)$. We have $0 = \bar F(\ln\lambda)$ from (3.32). It means λ is a constant on horizontal distribution. So, Φ is a horizontally homothetic map and (i) is satisfied. If (i) satisfies in (3.30), we obtain

$$0 = \overset{N}{\nabla}{}^{\Phi}{}_{E}\Phi_*(\bar F) - \Phi_*(\mathscr{T}_{\hat E}\hat F + h\overset{M}{\nabla}_{\hat E}\bar F + \mathscr{A}_{\bar E}\hat F) - \overset{N}{\nabla}{}^{\Phi}{}_{\bar E}\Phi_*(\bar F) + (\nabla\Phi_*)^\perp(\bar E,\bar F). \tag{3.33}$$

From (3.33), we obtain (ii). The proof is complete. □

4. Pluriharmonic conformal quasi-hemi-slant Riemannian map

In this section, we examine some geometric properties of certain distributions with respect to notion of pluriharmonicity, see equation (2.10) or [16]. We present D-pluriharmonicity in the following.

Theorem 4.1. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $D-$pluriharmonic map, then one of the below assertions imply the second assertion,*

i) D defines a totally geodesic foliation on M,

ii) $C\mathscr{T}_{JU_1}U_2 + \psi\hat{\nabla}_{JU_1}U_2 = 0$

for $U_1, U_2 \in \Gamma(D)$.

Proof. Initially, using definition of pluriharmonic map, we have

$$0 = (\nabla\Phi_*)(U_1,U_2) + (\nabla\Phi_*)(JU_1,JU_2) \tag{4.1}$$

for $U_1, U_2 \in \Gamma(D)$. By some calculations, we obtain from (4.1)

$$\begin{aligned} \Phi_*(\overset{M}{\nabla}_{U_1}U_2) &= -\Phi_*(J(\mathscr{T}_{JU_1}U_2 + \hat{\nabla}_{JU_1}U_2)) \\ \Phi_*(\overset{M}{\nabla}_{U_1}U_2) &= -\Phi_*(C\mathscr{T}_{JU_1}U_2 + \psi\hat{\nabla}_{JU_1}U_2). \end{aligned} \tag{4.2}$$

If (i) is satisfied in (4.2) we have $\Phi_*(\overset{M}{\nabla}_{U_1}U_2) = 0$. So, we obtain

$$C\mathscr{T}_{JU_1}U_2 + \psi\hat{\nabla}_{JU_1}U_2 = 0.$$

(ii) is provided. In a similar way, if (ii) is satisfied in (4.2), easily one can get (i). □

Theorem 4.2. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $D_\theta-$pluriharmonic map, then two of the below assertions imply the third assertion,*

i) D_θ defines a totally geodesic foliation on M,

ii) λ is a constant on ψD_θ and $(\nabla\Phi_*)^\perp(\psi V_1, \psi V_2) = 0$,

iii)

$$\cos^2\theta(C\mathscr{T}_{\phi V_1}V_2 + \psi\hat{\nabla}_{\phi V_1}V_2) = \psi\mathscr{T}_{\phi V_1}\psi\phi V_2 + Ch\overset{M}{\nabla}_{\phi V_1}\psi\phi V_2 - \mathscr{A}_{\psi V_2}\phi V_1 - \mathscr{A}_{\psi V_1}\phi V_2.$$

for $V_1, V_2 \in \Gamma(D_\theta)$.

Proof. If Φ is a $D_\theta-$pluriharmonic map, we have

$$0 = (\nabla\Phi_*)(V_1,V_2) + (\nabla\Phi_*)(JV_1,JV_2) \tag{4.3}$$

for $V_1, V_2 \in \Gamma(D_\theta)$. Using symmetry property of second fundamental form and from equations (2.4), (2.5), (2.6) and (2.9), we get

$$
\begin{aligned}
0 &= -\Phi_*(\overset{M}{\nabla}_{V_1}V_2)-\Phi_*(\overset{M}{\nabla}_{\phi V_1}\phi V_2)-\Phi_*(\overset{M}{\nabla}_{\psi V_2}\phi V_1)\\
&- \Phi_*(\overset{M}{\nabla}_{\psi V_1}\phi V_2)+(\nabla\Phi_*)^{\perp}(\psi V_1,\psi V_2)+\psi V_1(\ln\lambda)\Phi_*(\psi V_2)\\
&+ \psi V_2(\ln\lambda)\Phi_*(\psi V_1)-g_M(\psi V_1,\psi V_2)\Phi_*(grad(\ln\lambda))\\
\Phi_*(\overset{M}{\nabla}_{V_1}V_2) &= \Phi_*(J\overset{M}{\nabla}_{\phi V_1}J\phi V_2)-\Phi_*(\mathscr{A}_{\psi V_2}\phi V_1+\mathscr{A}_{\psi V_1}\phi V_2)\\
&+ (\nabla\Phi_*)^{\perp}(\psi V_1,\psi V_2)+\psi V_1(\ln\lambda)\Phi_*(\psi V_2)\\
&+ \psi V_2(\ln\lambda)\Phi_*(\psi V_1)-g_M(\psi V_1,\psi V_2)\Phi_*(grad(\ln\lambda))\\
\Phi_*(\overset{M}{\nabla}_{V_1}V_2) &= -\cos^2\theta\Phi_*(C\mathscr{T}_{\phi V_1}V_2+\psi\hat{\nabla}_{\phi V_1}V_2)\\
&+ \Phi_*(\psi\mathscr{T}_{\phi V_1}\psi\phi V_2+Ch\overset{M}{\nabla}_{\phi V_1}\psi\phi V_2)\\
&- \Phi_*(\mathscr{A}_{\psi V_2}\phi V_1+\mathscr{A}_{\psi V_1}\phi V_2)+(\nabla\Phi_*)^{\perp}(\psi V_1,\psi V_2)\\
&+ \psi V_1(\ln\lambda)\Phi_*(\psi V_2)+\psi V_2(\ln\lambda)\Phi_*(\psi V_1)\\
&- g_M(\psi V_1,\psi V_2)\Phi_*(grad(\ln\lambda)).
\end{aligned}
\tag{4.4}
$$

Now, if (i) and (ii) are satisfied in (4.4), we have $\Phi_*(\overset{M}{\nabla}_{V_1}V_2)=0$ and

$$0 = \psi V_1(\ln\lambda)\Phi_*(\psi V_2)+\psi V_2(\ln\lambda)\Phi_*(\psi V_1)-g_M(\psi V_1,\psi V_2)\Phi_*(grad(\ln\lambda)), \tag{4.5}$$

$$0 = (\nabla\Phi_*)^{\perp}(\psi V_1,\psi V_2), \tag{4.6}$$

respectively. Then, we get from (4.4)

$$
\begin{aligned}
0 &= -\cos^2\theta\Phi_*(C\mathscr{T}_{\phi V_1}V_2+\psi\hat{\nabla}_{\phi V_1}V_2)+\Phi_*(\psi\mathscr{T}_{\phi V_1}\psi\phi V_2+Ch\overset{M}{\nabla}_{\phi V_1}\psi\phi V_2)\\
&- \Phi_*(\mathscr{A}_{\psi V_2}\phi V_1+\mathscr{A}_{\psi V_1}\phi V_2).
\end{aligned}
\tag{4.7}
$$

So, (iii) is satisfied at (4.7). If (ii) and (iii) are satisfied in (4.4), we clearly have equations (4.5), (4.6) and (4.7) in (4.4). Therefore, we obtain (i). Lastly, suppose that (i) and (iii) are satisfied in (4.4). Then, we get (4.5) and (4.6). In (4.5), we obtain from (1.2)

$$
\begin{aligned}
0 &= \lambda^2\psi V_1(\ln\lambda)g_M(\psi V_2,\psi V_1)+\lambda^2\psi V_2(\ln\lambda)g_M(\psi V_1,\psi V_1)-\lambda^2 g_M(\psi V_1,\psi V_2)\psi V_1(\ln\lambda)\\
0 &= \lambda^2\psi V_2(\ln\lambda)g_M(\psi V_1,\psi V_1)
\end{aligned}
\tag{4.8}
$$

for $\psi V_1\in\Gamma(D_\theta)$. At (4.8), we have $0=\psi V_2(\ln\lambda)$. It means $0=\psi D_\theta(\ln\lambda)$ which implies that λ is a constant on ψD_θ. Hence, (ii) is satisfied. The proof is completed. □

Similarly, we have the following theorem.

Theorem 4.3. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $D_\perp-$pluriharmonic map, then one of the below assertions imply the second assertion,*

i) $D_\perp$ defines a totally geodesic foliation on M,

ii) λ is a constant on $JD_\perp$ and $(\nabla\Phi_*)^{\perp}(JW_1,JW_2)=0$

for $W_1,W_2\in\Gamma(D_\perp)$.

Now, we search properties of horizontal and vertical pluriharmonic maps, respectively.

Theorem 4.4. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $(ker\Phi_*)^{\perp}-$pluriharmonic map, then any two of the below assertions imply the third assertion,*

i) $(ker\Phi_*)^{\perp}$ defines a totally geodesic foliation on M,

ii) λ is a constant on μ,

iii) $\overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2)=\Phi_*(\mathscr{T}_{BZ_1}BZ_2+\mathscr{A}_{CZ_2}BZ_1+\mathscr{A}_{CZ_1}BZ_2)-(\nabla\Phi_*)^{\perp}(CZ_1,CZ_2)$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$.

Proof. From definition of pluriharmonic map, we have

$$0 = (\nabla\Phi_*)(Z_1, Z_2) + (\nabla\Phi_*)(JZ_1, JZ_2) \tag{4.9}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$. By some calculations from (2.8) and (2.9) in (4.9), we get

$$\begin{aligned} 0 &= \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) - \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2) + (\nabla\Phi_*)^{\perp}(CZ_1, CZ_2) \\ &- \Phi_*(\mathscr{T}_{BZ_1}BZ_2 + \mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2) + CZ_1(\ln\lambda)\Phi_*(CZ_2) \\ &+ CZ_2(\ln\lambda)\Phi_*(CZ_1) - g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{4.10}$$

If (i) and (ii) are satisfied in (4.10), we have

$$0 = \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2), \tag{4.11}$$

$$0 = CZ_1(\ln\lambda)\Phi_*(CZ_2) + CZ_2(\ln\lambda)\Phi_*(CZ_1) - g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda)). \tag{4.12}$$

So, we get (iii) from (4.10) such that

$$\overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) = \Phi_*(\mathscr{T}_{BZ_1}BZ_2 + \mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2) - (\nabla\Phi_*)^{\perp}(CZ_1, CZ_2). \tag{4.13}$$

If (ii) and (iii) are satisfied in (4.10), we have equations (4.12) and (4.13). Clearly, we obtain $0 = \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2)$ which implies that $(ker\Phi_*)^{\perp}$ defines a totally geodesic foliation on M. Lastly, if (i) and (iii) are satisfied in (4.10), we obtain (4.12). From (1.2) in (4.12) we get

$$\begin{aligned} 0 &= \lambda^2 CZ_1(\ln\lambda)g_M(CZ_2, CZ_1) + \lambda^2 CZ_2(\ln\lambda)g_M(CZ_1, CZ_1) - \lambda^2 g_M(CZ_1, CZ_2)CZ_1(\ln\lambda) \\ 0 &= \lambda^2 CZ_2(\ln\lambda)g_M(CZ_1, CZ_1) \end{aligned} \tag{4.14}$$

for $CZ_1, CZ_2 \in \Gamma(\mu)$. Here, we have $CZ_2(\ln\lambda) = 0$ which implies that λ is a constant on μ. (ii) is provided. The proof is completed. □

Theorem 4.5. *Let* $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal quasi-hemi-slant Riemannian map. If* Φ *is a* $ker\Phi_*-$*pluriharmonic map, then any two of the below assertions imply the third assertion,*

i) $ker\Phi_*$ *defines a totally geodesic foliation on M,*

ii) Φ *is a horizontally homothetic map and* $(\nabla\Phi_*)^{\perp}(\psi\xi_1, \psi\xi_2) = 0$*,*

iii) $\mathscr{T}_{\phi\xi_1}\phi\xi_2 + \mathscr{A}_{\psi\xi_2}\phi\xi_1 + \mathscr{A}_{\psi\xi_1}\phi\xi_2 = 0$

for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$.

Proof. From equations (2.5), (2.6), (2.9), (2.10) and (3.4), we get

$$\begin{aligned} 0 &= (\nabla\Phi_*)(\xi_1, \xi_2) + (\nabla\Phi_*)(J\xi_1, J\xi_2) \\ 0 &= -\Phi_*(\overset{M}{\nabla}_{\xi_1}\xi_2) + (\nabla\Phi_*)(\phi\xi_1, \phi\xi_2) + (\nabla\Phi_*)(\psi\xi_2, \phi\xi_1) \\ &+ (\nabla\Phi_*)(\psi\xi_1, \phi\xi_2) + (\nabla\Phi_*)(\psi\xi_1, \psi\xi_2) \\ \Phi_*(\overset{M}{\nabla}_{\xi_1}\xi_2) &= -\Phi_*(\mathscr{T}_{\phi\xi_1}\phi\xi_2 + \mathscr{A}_{\psi\xi_2}\phi\xi_1 + \mathscr{A}_{\psi\xi_1}\phi\xi_2) \\ &+ (\nabla\Phi_*)^{\perp}(\psi\xi_1, \psi\xi_2) + \psi\xi_1(\ln\lambda)\Phi(\psi\xi_2) \\ &+ \psi\xi_2(\ln\lambda)\Phi(\psi\xi_1) - g_M(\psi\xi_1, \psi\xi_2)\Phi_*(grad(\ln\lambda)) \end{aligned} \tag{4.15}$$

for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$. The proof of (i) and (iii) are clear to see. So, we only give proof for (ii). Suppose that (i) and (iii) are provided in (4.15). One can see easily that $(\nabla\Phi_*)^{\perp}(\psi\xi_1, \psi\xi_2) = 0$ and we get

$$0 = \psi\xi_1(\ln\lambda)\Phi(\psi\xi_2) + \psi\xi_2(\ln\lambda)\Phi(\psi\xi_1) - g_M(\psi\xi_1, \psi\xi_2)\Phi_*(grad(\ln\lambda)). \tag{4.16}$$

Since Φ is a conformal map, we obtain from (4.16)

$$0 = \lambda^2\psi\xi_2(\ln\lambda)g_M(\psi\xi_1, \psi\xi_1) \tag{4.17}$$

for $\psi\xi_1 \in \Gamma(\psi D_\theta \oplus JD_\perp)$. It means $0 = \psi\xi_2(\ln\lambda)$ which implies that λ is a constant on $\psi D_\theta \oplus JD_\perp$. On the other hand, we obtain from (4.16)

$$0 = -\lambda^2 CX(\ln\lambda) g_M(\psi\xi_1, \psi\xi_1) \tag{4.18}$$

for $CX \in \Gamma(\mu)$ with $\psi\xi_1 = \psi\xi_2$. It means $0 = CX(\ln\lambda)$ which implies that λ is a constant on μ. Hence, equations (4.17) and (4.18) give us that Φ is a horizontally homothetic map. (ii) is provided. The proof is completed. □

Lastly, we examine mixed pluriharmonicity on conformal quasi-hemi-slant Riemannian maps such that

$$0 = (\nabla\Phi_*)(Z, \xi) + (\nabla\Phi_*)(JZ, J\xi)$$

for $\xi \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$.

Theorem 4.6. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a mixed−pluriharmonic map, then any of the below assertions imply the second assertion,*

i- Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(CZ, \psi\xi) = 0$,

ii- $\mathscr{A}_Z\xi = \mathscr{T}_{BZ}\phi\xi + \mathscr{A}_{\psi\xi}BZ + \mathscr{A}_{CZ}\phi\xi$

for $\xi \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$.

Proof. From definition of mixed pluriharmonic map, we get

$$\begin{aligned} 0 &= -\Phi_*(\mathscr{A}_Z\xi) + \Phi_*(\mathscr{T}_{BZ}\phi\xi + \mathscr{A}_{\psi\xi}BZ + \mathscr{A}_{CZ}\phi\xi) \\ &+ (\nabla\Phi_*)^\perp(CZ, \psi\xi) + CZ(\ln\lambda)\Phi_*(\psi\xi) + \psi\xi(\ln\lambda)\Phi_*(CZ) \end{aligned} \tag{4.19}$$

for $\xi \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$. If (i) is satisfied in (4.19) we have $(\nabla\Phi_*)^\perp(CZ, \psi\xi) = 0$ and

$$0 = CZ(\ln\lambda)\Phi_*(\psi\xi) + \psi\xi(\ln\lambda)\Phi_*(CZ). \tag{4.20}$$

So, one can obtain (ii) easily. Now, if (ii) is satisfied in (4.19) we obtain easily $(\nabla\Phi_*)^\perp(CZ, \psi\xi) = 0$. Then, from (4.20) we obtain

$$0 = \lambda^2 \psi\xi(\ln\lambda) g_M(CZ, CZ) \tag{4.21}$$

for $CZ \in \Gamma(\mu)$. It means $0 = \psi\xi(\ln\lambda)$ which implies that λ is a constant on $\psi D_\theta \oplus JD_\perp$. On the other hand, from (4.20) we obtain

$$0 = \lambda^2 CZ(\ln\lambda) g_M(\psi\xi, \psi\xi) \tag{4.22}$$

for $\psi\xi \in \Gamma(\psi D_\theta \oplus JD_\perp)$. It means $0 = CZ(\ln\lambda)$ which implies that λ is a constant on μ. Hence, (4.21) and (4.22) give us that Φ is a horizontally homothetic map. (i) is provided. The proof is completed. □

5. Conclusion

In this paper, integrability conditions and conditions for defining a totally geodesic foliation by certain distributions were found. Then, by applying the notion of pluriharmonicity onto conformal quasi-hemi-slant Riemannian maps we obtained relations among pluriharmonicity, horizontally homotheticness and totally geodesicness.

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