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Obtaining the Parametric Equation of the Curve of the Sun's Apparent Movement by Using Quaternions

Deniz Güçler[1*], F. Nejat Ekmekci[1], Yusuf Yaylı[1] and Mustafa Helvacı[2]

[1]*Department of Mathematics, Faculty of Science, Ankara University, Ankara, Turkey*
[2]*Informatics Institute, Istanbul Technical University, Istanbul, Turkey*
[*]*Corresponding author*

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Abstract

This article aims to express the daily and yearly apparent movement of the Sun in the same curve by using quaternions as a rotation operator. To achieve this, the daily and yearly apparent movement of the Sun, the algebraical structure of quaternions, and how quaternions work as rotation operators have been examined. For each of the apparent movements of the Sun, a quaternion that will work as a rotation operator has been determined. Afterward, these two rotation operators have been applied to the vector that is found between point $(0,0,0)$ and the accepted starting point of the apparent movement of the Sun. As a result, a curve on a sphere is obtained. The importance of this study is to emphasize the use of quaternions in other areas of study and to provide the science of astronomy with a new outlook with regards to expressing the apparent movement of the Sun.

1. Introduction

Astronomy is considered the oldest science in the world. Humankind has always observed the stars in the sky and especially the Sun. At the end of these observations, it was noticed that the daily and yearly movement of the Sun followed a certain cycle. By observing the Sun's movement in the sky the formation of the night-day and the seasons was noted.

For thousands of years, mankind accepted that Earth was the center of the universe and believed that the Sun, like all other celestial bodies rotated around the Earth. However, Copernicus proved that this belief was not accurate because it was the Earth that rotated around the Sun [1]. After this discovery, the expression of "the Sun's movement" was replaced with the expression of "the Sun's apparent movement". Even though the daily and yearly apparent movement of the Sun occurs at the same time, in calculation these movements are considered separable. The two main reasons for why these movements are considered separable are: firstly, the dyad Earth-Sun is not alone in the solar system which means that the problem does not remain limited to the two-body problem. Secondly, the difference between the periods of the daily and yearly movement is too big.

Showing the daily and yearly apparent movement of the Sun in the same curve is important in helping understand these movements, especially for young astronomers. At the same time, there exist situations in which great precision is not required but where nonetheless finding these two movements in the same curve would be useful. In many areas, such as using solar panels, planning agricultural activities, and in determining prayer time, doing the calculation of this curve would bring many benefits.

In our time astronomy problems that have in their base periodical repetition of the movement find a solution by using spherical trigonometry and Kepler's Laws [2]. Solving this problem by using the rotation matrix is theoretically possible from the mathematical perspective, however, using this method is considerably difficult. Therefore, the question arises, is it possible to obtain a faster mathematical approach to calculate the apparent movement of the Sun that would take the place of the rotation matrices or the long calculations of Kepler's equations?

There are some studies done in this direction in the relevant literature. In 1996, M. Kummer proved that one can obtain the orbit's parameters by solving Kepler's equations with the Hamilton systems [3]. This study, on the other hand, has researched whether there can be easier and faster solutions done by using quaternions and the conclusion has been that quaternions can indeed be used in analyzing the apparent movement of the Sun.

Email addresses and ORCID numbers: **deniz.gucler@yahoo.com, 0000-0003-0376-0294 (D. Güçler), ekmekci@science.ankara.edu.tr, 0000-0003-1246-2395 (F. N. Ekmekci), yayli@science.ankara.edu.tr, 0000-0003-4398-3855 (Y. Yaylı), helvacim@itu.edu.tr, 0000-0002-4049-8072 (M. Helvacı)**

To understand and present the problem the author has benefited from the references [1]- [3] and [5] - [15]. The details about the quaternions can be viewed from the references [4] and [16] - [22]. The information needed for the other calculations is found in the references [23] - [24].

2. Notations and Preliminaries

2.1. Quaternion algebra

The quaternion, a hyper-complex number of rank 4, was invented by Hamilton. The most important rule of this invention is:

$$i^2 = j^2 = k^2 = ijk = -1$$

i, j and k are the components of the vector part of the quaternion.
Henceforth the quaternions will be denoted with the letters q, p or r. i, j and k will be used to represent the standard ortogonal base of $\mathbb{R}^3$. Accordingly:

$$i = (1,0,0)\,,\ j = (0,1,0)\,,\ k = (0,0,1)$$

The quaternion, from the Latin kuattur meaning four, can be thought of as a quadruplet of the real numbers. This makes it an element of $\mathbb{R}^4$. Accordingly, quaternion q can be expressed as below where q_0, q_1, q_2, q_3 are each a real number

$$q = (q_0, q_1, q_2, q_3)$$

or the quaternion q is accordingly:

$$\alpha = iq_1 + jq_2 + kq_3$$
$$q = q_0 + \alpha = q_0 + iq_1 + jq_2 + kq_3$$

where q_0 is the scalar part and α is the vector part. Throughout the article, q will be displayed with $q = q_0 + \alpha$.
Some algebraic properties of the quaternions are given as follows:

$$q + p = (q_0 + p_0) + i(q_1 + p_1) + j(q_2 + p_2) + k(q_3 + p_3)$$
$$aq = aq_0 + iaq_1 + jaq_2 + kaq_3 \quad , \quad a \in \mathbb{R}$$

Multiplication of quaternions is done according to the following rule

$$i^2 = j^2 = k^2 = ijk = -1 \ \text{ and } \ ij = k = -ij, jk = i = -kj, ki = j = -ij$$

for $p = p_0 + \alpha_p = p_0 + ip_1 + jp_2 + kp_3$ and $q = q_0 + \alpha_q = q_0 + iq_1 + jq_2 + kq_3$

$$\begin{aligned} p \times q &= (p_0 + ip_1 + jp_2 + kp_3) \times (q_0 + iq_1 + jq_2 + kp_3) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(iq_1 + jq_2 + kq_3) + q_0(p_0 + ip_1 + jp_2 + kp_3) \\ &+ i(p_2q_3 - p_3q_2) + j(p_3q_1 - p_1q_3) + k(p_1q_2 - p_2q_1) \\ &= p_0q_0 - \langle \alpha_p, \alpha_q \rangle + p_0\alpha_q + q_0\alpha_p + \alpha_p \wedge \alpha_q \end{aligned}$$

"$\langle\ ,\ \rangle$" represents the scalar product of vectors and "$\wedge$" represents the cross-produc of vectors.
Let q be a quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ then q's complex conjugent is:

$$q^* = q_0 - iq_1 - jq_2 - kq_3$$

Finally, we can state that the set of quaternions together with the addition and multiplication operation satisfies the properties of a field except that multiplication is not commutative. Before quaternions are expressed as a rotation operator the definition of pure quaternions will be given.

Definition 2.1. *The quaternion whose scalar part is zero is called a pure quaternion.*

According to the definition above, the set of pure quaternions is one-to-one correspondent with the $v \in \mathbb{R}^3$ vector set. It can be shown that for any $v \in \mathbb{R}^3$ and for whichever $q \in \mathbb{R}^4$, there can be found $w_1 = q \times v \times q^*$ vector $w_1 \in \mathbb{R}^3$ and $w_2 = q^* \times v \times q$ vector $w_2 \in \mathbb{R}^3$.
The unit quaternion $q = q_0 + \alpha$ satisfies the following equality $q_0^2 + |\alpha|^2 = 1$. It is known that for whichever φ angle $\cos^2\varphi + \sin^2\varphi = 1$. In this case, a φ angle which would make possible the equations below can be found:

$$\cos^2\varphi = q_0^2 \text{ and } \sin^2\varphi = |\alpha|^2.$$

If we select the φ angle in $-\pi < \varphi \leq \pi$, this angle will simultaneously have a singular value. In light of this data, the quaternion that will be used as a rotation operator is:

$$q = q_0 + \alpha = \cos\varphi + u\sin\varphi \ \text{ and } \ q^* = q_0 - \alpha = \cos\varphi - u\sin\varphi$$

where

$$u = \frac{\alpha}{|\alpha|} = \frac{\alpha}{\sin\varphi}.$$

Theorem 2.2. *For any $Q = Q_0 + \mathbf{Q} = \cos\varphi + u\sin\varphi$ unit quaternion (where Q_0 is the scalar part and $\mathbf{Q}$ is the vector part of the quaternion) and for any vector $v \in \mathbb{R}^3$ the action of the operator*

$$L_Q(v) = Q \times v \times Q^*$$

on v may be interpreted geometrically as a rotation of the vector v through an angle 2φ about $\mathbf{Q}$ as the axis of the rotation, [4].

In addition: the action of the operator $L_Q(v) = Q^* \times v \times Q$ on v may be interpreted geometrically as a rotation of the vector v through an angle 2φ in a negative direction about $\mathbf{Q}$ as the axis of the rotation.

Theorem 2.3. *Suppose that k and r are unit quaternions that define the quaternion rotation operators:*

$$L_k(u) = k \times u \times k^* \text{ and } L_r(v) = r \times v \times r^*.$$

Then the quaternion product $r \times k$ defines a quaternion operator L_{rk} which represents a sequence of operators, L_k followed by L_r. The axis and the angels of rotation are those represented by the quaternion product, $q = r \times k$ [4].

In this study, two methods will be used to solve the problem. The first method will benefit from the characteristic of quaternions used as rotation operators. The second method will use the rotation matrix, which is a product of the unit quaternion. This matrix is as below:
for $Q = q_0 + iq_1 + jq_2 + kq_3$ unit quaternion, the rotation matrix D_Q is shown below [4].

$$D_Q = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \tag{2.1}$$

and let $\beta = (\beta_1, \beta_2, \beta_3)$ be the vector that is obtained by the rotation of vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ then:

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \tag{2.2}$$

2.2. The Sun's daily and yearly apparent movement

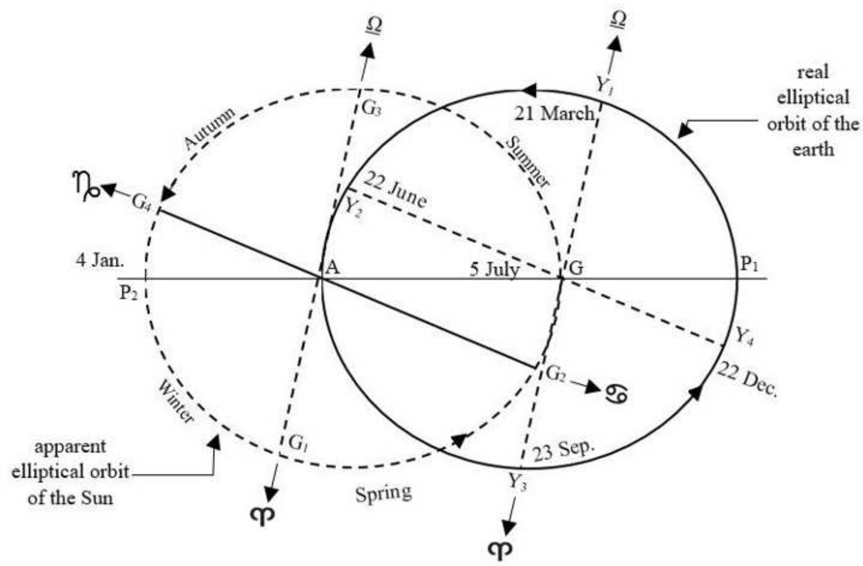
2.2.1. The Sun's daily apparent movement

The Earth rotates around its axis in a positive direction every day, so from the west to the east. Because the movement of the Earth cannot be felt, it is perceived instead that it's the other celestial bodies that rotate from the east to the west around the axis of the celestial sphere which in itself is the lengthening of the axis of the Earth. Among these celestial bodies, there is the Sun. So it can be said that the Sun in appearance moves every day in the negative direction in the celestial sphere. This movement occurs with a particular velocity in an orbit parallel to the celestial equator plane. The celestial equator plane is the lengthening of the Earth's equator plane [5].

2.2.2. The Sun's yearly apparent movement

The Earth orbits around the Sun in an elliptical orbit and a positive direction, in the elliptical plane throughout the year. However, in appearance, it is the Sun that orbits around the Earth in the same plane and a positive direction. The angle between the elliptical plane and the equatorial plane is $23^0 27'$. This plane forms a $23^0 27'$ angle with the plane of the celestial equator. If in the center of the celestial system instead of the Sun we placed the Earth and then drew the apparent elliptical orbit of the Sun,the orbit in Figure 2.1 would be obtained. To obtain this orbit the Earth will be imagined as fixed and the Sun as the body that rotates around it. Because the Earth's orbit is well-known the Earth will be fixed in what will be called point A henceforth which is found in its orbit. When the Earth is on day 21 March at the Y_1 point the Sun appears in the direction of Aries. If we transfer point Y_1 to point A and find point G_1 for which $AG_1 = Y_1G$ and AG_1 is parallel to Y_1G, it would mean that the Sun would appear at point G_1 at this date. In the same manner, if P_1G to AP_2, Y_2G to AG_2, Y_3G to AG_3, and Y_4G to AG_4 are transferred a new ellipse is formed which has at its center point A. This is the Sun's yearly apparent elliptical orbit. Every year the Sun moves in this elliptical orbit. Below are five important points that concern this orbit [5].

1. Both orbits are found in the same plane and this plane is the elliptical plane.
2. The Earth is found in one of the focal points of the apparent elliptic orbit.
3. These two ellipses are equal in shape and size.
4. The rotation period is the same in both and it is a one-star year long.
5. Both rotations are in the positive direction.



G: Sun, A: Earth, ♈: Aries, ♎: Libra, ♋: Cancer, ♑: Capricorn direction P_1 = Earth's perihelion, P_2 = Sun's perihelion

Figure 2.1: The Earth's orbit and the Sun's apparent orbit

3. Obtaining the Parametric Equation of the Curve of Both the Daily and Yearly Apparent Movement the Sun Makes in the Celestial Sphere by Using Quaternions

In this paper, it is assumed the apparent movement of the Sun occurs in ideal conditions. This means that the Earth will rotate around the Sun with a constant angular velocity (this velocity will be accepted as equal to the yearly average angular velocity of the Earth around the Sun) and it will be accepted that the orbit of rotation will be circular instead of elliptic. So, it will be accepted that the apparent movement of the Sun in the ecliptic plane will occur in a circular orbit with a constant angular velocity.

Firstly, it is necessary to define the problem in physical terms.

Let us accept that a celestial body completes a circular motion in plane E that intersects with plane XY in axis x and forms with it an ε angle. Let us also accept that this movement starts from point $P=(1,0,0)$ in a positive direction, and under force, F_1 completes a circular movement with a constant angular velocity w_1. Lastly, let us also accept that a force $F_2 = c\,F_1$, $c>2$ (there is a linear relationship between the scalar magnitude of the forces), forces the same celestial body to move parallel to plane XY in a positive direction with a constant angular velocity w_2. In this case, the celestial body whose vectors are linear independent is under the effect of two forces and is bound to both velocities. This body, however, will not move parallel to either plane XY or plane E instead it will move with the unified velocity in a different direction. How can we express the celestial body's interaction with the velocities w_1 and w_2?

Between the scalar magnitudes of w_1 and w_2 velocities, a linear relation is found. This linear relation will be the same as the linear relation between the scalar magnitudes of F_1 and F_2. In the same manner, the θ and φ angles these angular velocities trace in the same unit of time will also have the same linear relationship between their magnitudes. So $\varphi = c\theta$ because the forces are directly proportional to the angular velocities and the angular velocities are directly proportional to the angles they trace. To conclude, the curve that this celestial body traces on the sphere is a product of two rotations. One of the rotations will be in a positive direction around the axis of the plane E (let this axis be called N) and the other will be in a positive direction around axis Z.

Let plane E represent the elliptic plane while plane XY represents the plane of the celestial equator and angle ε represents the angle $\varepsilon = 23^0 27'$ which is the angle that is formed from the intersection of the celestial equatorial plane and the ecliptic plane (Figure 3.1). In this case, point $(0,0,0)$ represents the Earth. In addition, the positive direction of axis X will represent the Aries constellation. The direction of the vector $(0,-\cos\varepsilon,-\sin\varepsilon)$ will represent the Capricorn constellation. The direction of the vector $(0,\cos\varepsilon,\sin\varepsilon)$ will represent the Cancer constellation. The negative direction of axis X will represent the Libra constellation.

Now let us show the daily apparent movement of the Sun. This movement occurs in a negative direction parallel to the celestial equatorial plane. In this case, the second rotation movement in the negative direction of the celestial body that was presented in the problem above represents the movement of the daily apparent movement of the Sun.

Finally, above, it was stated that between the scalar magnitudes of w_1 (if we adapt w_1 to the velocity of the Sun this corresponds with the velocity of the Sun's movement in the elliptical plane) and w_2 (if we adapt w_2 to the velocity of the Sun this corresponds with the velocity of the movement the Sun makes parallel to the celestial equatorial plane) exists a linear relation. The same linear relation exists between the angles these velocities trace. In this case; because $w_2 = 365,25\,w_1$ (when the Sun rotates once around the ecliptic axis it rotates $365,25$ times parallel to the celestial equatorial plane) $\varphi = 365,25\theta$. So, $c = 365,25$.

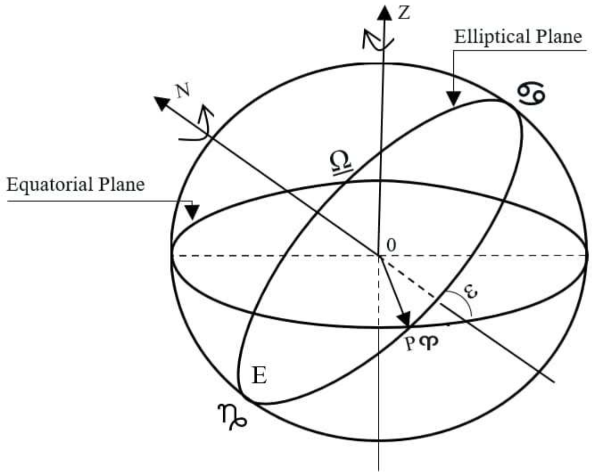


Figure 3.1: The system in which the apparent movement of the Sun occurs

Let Q_1 be the quaternion that will realize the movement in the positive direction around axis N. Let Q_2 be the quaternion that will realize the movement in the positive direction around axis Z. With the help of these two quaternions, the parametric equation of the curve of the daily and yearly apparent movement the Sun makes in the celestial sphere will be obtained. The starting point of the movement is $P=(1,0,0)$ which coincides with the Aries constellation. The vector OP that is found in the direction of the Earth-Aries constellation is $v=(1,0,0)$. First, let this vector be transferred to the quaternion space so:
$v_1=(1,0,0)$ vector $\rightarrow w_1=0+i+0j+0k=i$ corresponds to a pure quaternion. The first rotation movement will be realized around axis $u=-j\sin\varepsilon+k\cos\varepsilon$ with θ angle. The second rotation movement will be realized around axis k with a φ angle in a negative direction. In this case, the Q_1 and Q_2 quaternions that will operate as rotation operators are: For $a=\sin\varepsilon$ and $b=\cos\varepsilon$,

$$Q_1=\cos\left(\frac{\theta}{2}\right)-ja\sin\left(\frac{\theta}{2}\right)+kb\sin\left(\frac{\theta}{2}\right)$$

and

$$Q_2=\cos\left(\frac{\varphi}{2}\right)+k\sin\left(\frac{\varphi}{2}\right).$$

It is stated that the second rotation movement (daily movement) occurs around axis k in the negative direction. If the necessary adjustments are made, instead of $Q_2=\cos\left(\frac{\varphi}{2}\right)+k\sin\left(\frac{\varphi}{2}\right)$ for the second rotation, the complex conjugate of Q_2 will be used.

$$Q_2^*=\cos\left(\frac{\varphi}{2}\right)-k\sin\left(\frac{\varphi}{2}\right).$$

According to Theorem 2.3, for $L_{Q_1}(w_1)=Q_1\times w_1\times Q_1^*$, $L_{Q_2^*}(w_2)=Q_2^*\times w_2\times Q_2$, and $w_2=Q_1\times w_1\times Q_1^*$

$$L_{Q_2^*Q_1}(w_1)=(Q_2^*\times Q_1)\times w_1\times(Q_2^*\times Q_1)^*.$$

If $Q_2^*\times Q_1=Q$ and $w_1=i$ then

$$L_{Q_2^*Q_1}(w_1)=Q\times i\times Q^*.$$

So the calculations are as such:

$$Q=Q_2{}^*\times Q_1=\left(\cos\left(\frac{\varphi}{2}\right)-k\sin\left(\frac{\varphi}{2}\right)\right)\times\left(\cos\left(\frac{\theta}{2}\right)-ja\sin\left(\frac{\theta}{2}\right)+kb\sin\left(\frac{\theta}{2}\right)\right)$$

$$\begin{aligned}Q=&\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)-ia\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-ja\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\\&+k\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)\end{aligned}$$

$$L=Q\times i\times Q^*=L_0+iL_1+jL_2+kL_3$$

$$L_0=0$$

$$\begin{aligned}
L_1 &= \left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2+a^2\sin^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\\
&-a^2\cos^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)-\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^2\\
&=\cos^2\left(\frac{\varphi}{2}\right)\cos^2\left(\frac{\theta}{2}\right)+2b\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)+b^2\sin^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\\
&-a^2\sin^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)-b^2\cos^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\\
&+2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\\
&=\cos^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)-b^2\sin^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)\\
&-a^2\sin^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)+\left(2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right)\left(2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\\
&=\cos^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)-\sin^2\left(\frac{\theta}{2}\right)\left(a^2+b^2\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)\\
&+\left(2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right)\left(2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\\
&=\left(\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)+\left(2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right)
\end{aligned}$$

$$L_1=\cos\varphi\cos\theta+b\sin\varphi\sin\theta.$$

Likewise:

$$L_2=b\cos\varphi\sin\theta-\sin\varphi\cos\theta$$

$$L_3=a\sin\theta$$

then

$$\begin{aligned}
L_{Q_2{}^*Q_1}(w_1)&=Q\times i\times Q^*=i(\cos\varphi\cos\theta+b\sin\varphi\sin\theta)+j(b\cos\varphi\sin\theta-\sin\varphi\cos\theta)\\
&\quad+ka\sin\theta=w.
\end{aligned}$$

When vector w that was obtained in the quaternion space is transferred to vector v in the real space:

$$v=(x,y,z)=(\cos\varphi\cos\theta+b\sin\varphi\sin\theta\,,\ b\cos\varphi\sin\theta-\sin\varphi\cos\theta\,,\ a\sin\theta).$$

If $c>2$, $0\leq\theta\leq 2\pi$, $0\leq\varphi\leq n\pi$, n and c are constants and $\varphi=c\theta$ are kept in mind then:

$$\begin{aligned}
X&=\cos\theta\cos(c\theta)+b\sin\theta\sin(c\theta)\\
Y&=b\sin\theta\cos(c\theta)-\cos\theta\sin(c\theta)\\
Z&=a\sin\theta
\end{aligned}$$

$c=365,25$ and $0\leq\theta\leq 2\pi$, $a=\sin 23^0 27'$ and $b=\cos 23^0 27'$.

The quaternion that will be used for the first rotation movement, was defined before as:
$Q_1=\cos\left(\frac{\theta}{2}\right)-ja\sin\left(\frac{\theta}{2}\right)+kb\sin\left(\frac{\theta}{2}\right)$. From here, we have:

$$\begin{aligned}
q_{10}&=\cos\left(\frac{\theta}{2}\right)\\
q_{11}&=0\\
q_{12}&=-a\sin\left(\frac{\theta}{2}\right)\\
q_{13}&=b\sin\left(\frac{\theta}{2}\right)
\end{aligned}$$

According to (2.1) rotation matrix A which is produced by the unit quaternion above is:

$$A=\begin{bmatrix} 2\cos^2\left(\frac{\theta}{2}\right)-1 & -2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & -2a\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ 2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & 2\cos^2\left(\frac{\theta}{2}\right)-1+2\left(-a\sin\left(\frac{\theta}{2}\right)\right)^2 & -2ab\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ 2\cos\left(\frac{\theta}{2}\right)a\sin\left(\frac{\theta}{2}\right) & -2ab\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & 2\cos^2\left(\frac{\theta}{2}\right)-1+2\left(b\sin\left(\frac{\theta}{2}\right)\right)^2 \end{bmatrix}.$$

The quaternion that will be used for the second rotation movement, was defined before as:
$Q_2{}^*=\cos\left(\frac{\varphi}{2}\right)-k\sin\left(\frac{\varphi}{2}\right)$ From here:

$$\begin{aligned} q_{20}{}^* &= \cos\left(\frac{\varphi}{2}\right) \\ q_{21}{}^* &= 0 \\ q_{22}{}^* &= 0 \\ q_{23}{}^* &= -\sin\left(\frac{\varphi}{2}\right) \end{aligned}$$

According to (2.1) rotation matrix B which is produced by the unit quaternion above is:

$$B=\begin{bmatrix} 2\cos^2\left(\frac{\varphi}{2}\right)-1 & 2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) & 0 \\ -2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) & 2\cos^2\left(\frac{\varphi}{2}\right)-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let matrix be the resultant matrix of matrixes and then:

$$C=BA$$

When the necessary calculations are done:

$$C=\begin{bmatrix} 2q_0^2-1+2q_1^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 2q_0^2-1+2q_2^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 2q_0^2-1+2q_3^2 \end{bmatrix}$$

where

$$\begin{aligned} q_0 &= \cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ q_1 &= -a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ q_2 &= -a\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ q_3 &= b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right) \end{aligned} \tag{3.1}$$

As expected, the values in equation (3.1) are the same as the values of $Q=Q_2{}^*\times Q_1$.
According to (2.2), the vector $w=(w_1,w_2,w_3)$ obtained when rotation matrix C is applied in vector $\vec{v}=(1,0,0)$ is:

$$w=C\vec{v}$$

$$w(w_1,w_2,w_3)=\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}=\begin{bmatrix} 2q_0^2-1+2q_1^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 2q_0^2-1+2q_2^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 2q_0^2-1+2q_3^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$w_1=2q_0^2-1+2q_1^2=2\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2-1+2\left(-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2$$

$$w_1=\cos\varphi\cos\theta+b\sin\varphi\sin\theta$$

$$
\begin{aligned}
w_2 &= (2q_1q_2 + 2q_0q_3) \\
&= 2\left(-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(-a\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) \\
&+ 2\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right) + b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)
\end{aligned}
$$

$$w_2 = b\cos\varphi\sin\theta - \sin\varphi\cos\theta$$

$$
\begin{aligned}
w_3 &= 2\left(-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) \\
&- 2\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right) + bsin\left(\frac{\varphi}{2}\right)\ sin\left(\frac{\theta}{2}\right)\right)\left(-a\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) = 2a\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)
\end{aligned}
$$

$$w_3 = a\sin\theta$$

$$w = (w_1, w_2, w_3) = (\cos\varphi\cos\theta + b\sin\varphi\sin\theta\,,\ b\cos\varphi\sin\theta - \sin\varphi\cos\theta\,,\ a\sin\theta)\,.$$

If $c > 2$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq n\pi$, n and c constants and $\varphi = c\theta$, are kept in mind then:

$$
\begin{aligned}
w_1 &= X = \cos\theta\cos(c\theta) + b\sin\theta\sin(c\theta) \\
w_2 &= Y = b\sin\theta\cos(c\theta) - \cos\theta\sin(c\theta) \\
w_3 &= Z = a\sin\theta
\end{aligned}
\tag{3.2}
$$

$$c = 365,25 \text{ and } 0 \leq \theta \leq 2\pi,\ a = \sin 23^0 27' \text{ and } b = \cos 23^0 27'$$

If the graphic of the equation (3.2) we obtained above was drawn, the three-dimensional graphic shown in Figure 3.2 will be acquired. This curve covers the entirety of the sphere found between the planes $z = -\sin 23^0 27'$ and $z = \sin 23^0 27'$ because the constant c is $c = 365,25$. For this reason, to be able to comprehend the shape of the curve, $c = 12$ is chosen instead of $c = 365,25$ and this way the graphic shown in Figure 3.3 is obtained. As shown in Figure 3.3, the curve is a spherical spiral limited between the planes $z = -\sin 23^0 27'$ and $z = \sin 23^0 27'$. If in equation (3.2) $\varepsilon = 90^0$ then the parametric equation of the spherical spiral is procured.

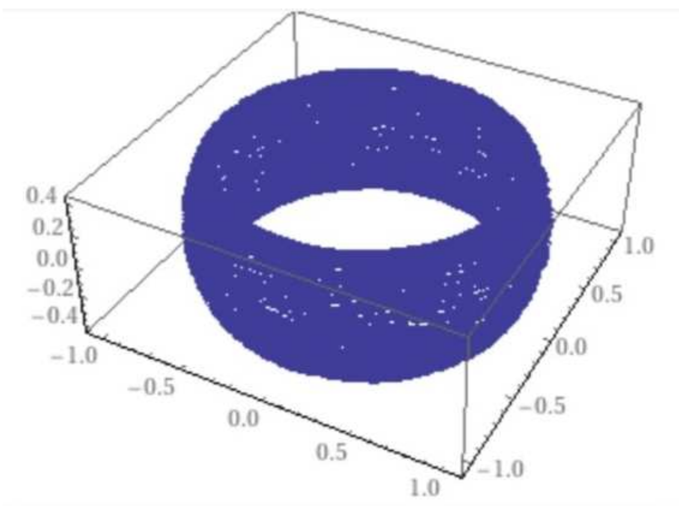


Figure 3.2: The curve of the apparent movement of the Sun for $c = 365,25$

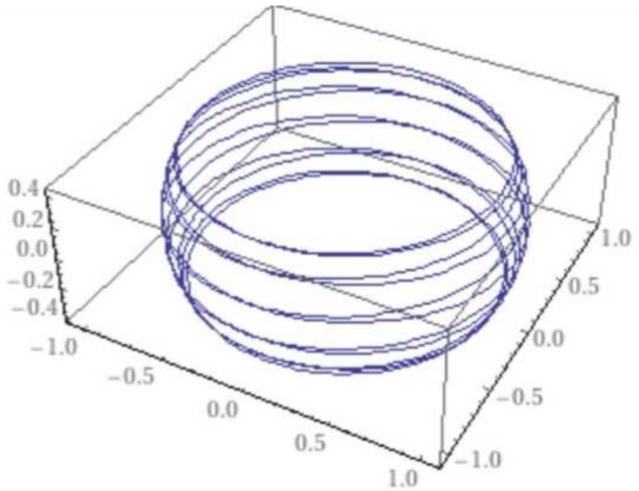


Figure 3.3: The curve of the apparent movement of the Sun for $c = 12$

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Nonexistence of Global Solutions for the Strongly Damped Wave Equation with Variable Coefficients

Erhan Pişkin[1*] and Ayşe Fidan[2]

[1] *Department of Mathematics, Dicle University, Diyarbakır, Turkey*
[2] *Department of Mathematics, Institute of Natural and Applied Sciences, Dicle University, Diyarbakır, Turkey*
[*] *Corresponding author*

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Abstract

In this work, we deal with the wave equation with variable coefficients. Under proper conditions on variable coefficients, we prove the nonexistence of global solutions.

1. Introduction

In this paper, we are concerned with the following problem:

$$\begin{cases} u_{tt}-\Delta u-\Delta u_t \ +\mu_1(t)\,|u_t|^{p-2}u_t=\mu_2(t)\,|u|^{q-2}u, & x\in\Omega,\ t>0, \\ u(x,t)=0, & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x)\,,\ \ u_t(x,0)=u_1(x)\,, & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n\in N)$, with a smooth boundary $\partial\Omega$, $p\geq 2$, $q>2$, $\mu_1(t)$ is a non-negative function of t and $\mu_2(t)$ is a positive functions of t. The quantity $|u_t|^{p-2}u_t$ is a damping term which assures global existence, and $|u|^{q-2}u$ is the source term which contributes to nonxistence of global solutions. $\mu_1(t)$ and $\mu_2(t)$ can be regarded as two control buttons which can dominate the polarity between damping term and source term.
In the absence of the strong damping term Δu_t, and $\mu_1(t)=\mu_2(t)\equiv 1$, then the problem (1.1) can be reduced to the following wave equation

$$u_{tt}-\Delta u+|u_t|^{p-2}u_t=|u|^{q-2}u.$$

Many authors established the existence, nonexistence and decay of solutions, see [1–6]. The interaction between nonlinear damping $(|u_t|^{p-2}u_t)$ and the source term $(|u|^{q-2}u)$ makes the problem more interesting. Levine [2,3] first studied the interaction between the linear damping $(p=2)$ and source term by using Concavity method. But this method can't be applied in the case of a nonlinear damping term. Georgiev and Todorova [1] extended Levine's result to the nonlinear case $(p>2)$. They showed that solutions with negative initial energy blow up in finite time. Later, Vitillaro in [6] extended these results to situations where the nonlinear damping and the solution has positive initial energy.
In [7], Yu investigated the equation with constant coefficients

$$u_{tt}-\Delta u-\Delta u_t \ +|u_t|^{p-2}u_t=|u|^{q-2}u. \tag{1.2}$$

He showed globality, boundedness, blow-up, convergence up to a subsequence towards the equilibria and exponential stability. Gerbi and Said-Houari [8] proved exponential decay of solutions (1.2) for $p=2$.

Email addresses and ORCID numbers: **episkin@dicle.edu.tr, 0000-0001-6587-4479 (E. Pişkin), afidanmat@gmail.com, 0000-0001-6988-8333 (A. Fidan)**

Zheng et al. [9] considered the Petrovsky equation

$$u_{tt}+\Delta^2 u+k_1(t)|u_t|^{m-2}u_t=k_2(t)|u|^{p-2}u$$

in a bounded domain. They proved the blow up of solutions.
In this paper, we established the nonexistence of solutions. To our best knowledge, the nonexistence of solutions of the wave equation with variable coefficients not yet studied.
This paper is organized as follows: In the next section, we present some lemmas, notations and local existence theorem. In section 3, the nonexistence of global solutions are given.

2. Preliminaries

In order to state the main results to problem (1.1) more clearly, we start to our work by introducing some notations and lemmas which will be used in this paper. Throughout this paper $\|u\|_p=\|u\|_{L^p(\Omega)}$ and $\|u\|_2=\|u\|$ denote the usual $L^p(\Omega)$ norm and $L^2(\Omega)$ norm, respectively. Also, $W_0^{m,2}(\Omega)=H_0^m(\Omega)$ is a Hilbert spaces (see [10, 11], for details).

Lemma 2.1. *[4]. Assume that*

$$\begin{cases} 2\le q<\infty, & n\le 2,\\ 2<q<\frac{2(n-1)}{n-2}, & n\ge 3.\end{cases}$$

Then, there exist a positive constant $C>1$, depending on Ω only, such that

$$\|u\|_q^s\le C\left(\|\nabla u\|^2+\|u\|_q^q\right) \tag{2.1}$$

for any $u\in H_0^1(\Omega)$ and $2\le s\le q$.

Lemma 2.2. *Assume that $p\ge 2$, $q>2$, $\mu_1(t)$ is a nonnegative function of t, $\mu_2(t)$ is a positive functions of t and $\mu_2'(t)\ge 0$. Let $u(t)$ be a solution of problem (1.1) then the energy functional $E(t)$ is non-increasing, namely $E'(t)\le 0$.*

Proof. Multiplying the equation (1.1) with u_t and integrating with respect to x over the domain Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2-\frac{\mu_2(t)}{q}\|u\|_q^q\right)=-\mu_1(t)\|u_t\|_p^p-\|\nabla u_t\|^2-\frac{\mu_2'(t)}{q}\|u\|_q^q. \tag{2.2}$$

By the equality (2.2), we get

$$E'(t)=-\mu_1(t)\|u_t\|_p^p-\|\nabla u_t\|^2-\frac{\mu_2'(t)}{q}\|u\|_q^q\le 0,$$

and $E(t)\le E(0)$, where

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2-\frac{\mu_2(t)}{q}\|u\|_q^q, \tag{2.3}$$

and

$$E(0)=\frac{1}{2}\|u_1\|^2+\frac{1}{2}\|\nabla u_0\|^2-\frac{\mu_2(0)}{q}\|u_0\|_q^q.$$

□

In order to obtain our main results, we set

$$H(t)=-E(t). \tag{2.4}$$

In the following remark, C denotes a generic constant that varies from line to line. Combining (2.1), (2.3) and (2.4), we obtain

Remark 2.3. *Assume that*

$$\begin{cases} 2\le q<\infty, & n\le 2,\\ 2<q<\frac{2(n-1)}{n-2}, & n\ge 3\end{cases}$$

and energy functional $E(t)<0$. Then, there exist a positive constant C, depending only on Ω, such that

$$\|u\|_q^s\le C\left(H(t)+\|u_t\|^2+\left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right) \tag{2.5}$$

for any $u\in H_0^1(\Omega)$ and $2\le s\le q$.

Next, we state the local existence theorem that can be established by combining arguments of [1, 12].

Theorem 2.4. *(Local existence). Suppose that*

$$\begin{cases} 2\le q<\infty, & n\le 2,\\ 2<q<\frac{2(n-1)}{n-2}, & n\ge 3.\end{cases}$$

Then, for any given $(u_0,u_1)\in\left(H_0^1(\Omega)\times L^2(\Omega)\right)$, the problem (1.1) has a local solution satisfying

$$u\in C\left([0,T]:H_0^1(\Omega),u_t\in C\left([0,T];L^2(\Omega)\right)\cap L^p(\Omega,[0,T])\right)$$

for some $T>0$.

3. Nonexistence of Global Solutions

In this section, we will consider the nonexistence of global solutions for the problem (1.1). By using the same techniques as in [9].

Theorem 3.1. *Let the assumptions of Lemma 2.2 hold. And assume that $\mu_1(t)$ is a nonnegative function of t, $\mu_2(t)$ is a positive functions of t, $\mu_2'(t) \geq 0$ and*

$$\lim_{t\to\infty} \mu_1(t)\,\mu_2(t)^{\alpha(p-1)}$$

exists, where

$$0 < \alpha \leq \min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}.$$

Then the solution of Eq. (1.1) blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)}$$

if $q > p$ and the initial energy function

$$E(0) < 0,$$

where

$$L(0) = [H(0)]^{1-\alpha} + \varepsilon \int_\Omega u_0 u_1 dx > 0.$$

Proof. From (2.2)-(2.4), we have

$$\frac{d}{dt}H(t) = \mu_1(t)\|u_t\|_p^p + \|\nabla u_t\|^2 + \frac{\mu_2'(t)}{q}\|u\|_q^q \geq 0 \tag{3.1}$$

for almost, every $t \in [0,T)$. Therefore

$$0 < H(0) \leq H(t) \leq \frac{\mu_2(t)}{q}\|u\|_q^q,\ t \in [0,T). \tag{3.2}$$

Define

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon}{2}\|\nabla u\|^2 \tag{3.3}$$

where $\varepsilon > 0$ is small to be chosen later, and

$$0 < \alpha \leq \min\left\{\frac{q-2}{2q}, \frac{q-p}{q(p-1)}\right\}. \tag{3.4}$$

Differentiating (3.3) with respect to t and combining the first equation of (1.1), we have

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega \left(u u_{tt} + u_t^2\right)dx + \varepsilon\int \nabla u \nabla u_t dx \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int \nabla u\nabla u_t dx \\ &\quad + \varepsilon\int_\Omega\left(u\Delta u + u\Delta u_t - \mu_1(t)|u_t|^{p-1}u + \mu_2(t)u^q + u_t^2\right)dx \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad + \varepsilon\mu_2(t)\|u\|_q^q - \varepsilon\mu_1(t)\int_\Omega |u_t|^{p-1}u dx. \end{aligned} \tag{3.5}$$

Due to the Hölder's and Young's inequalities, we have

$$\begin{aligned} \left|\mu_1(t)\int_\Omega |u_t|^{p-1}u dx\right| &\leq \mu_1(t)\int_\Omega |u_t|^{p-1}u dx \\ &\leq \left(\int_\Omega \mu_1(t)|u_t|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega \mu_1(t)|u|^p dx\right)^{\frac{1}{p}} \\ &\leq \frac{p-1}{p}\mu_1(t)\delta^{-\frac{p}{p-1}}\|u_t\|_p^p + \frac{\delta^p}{p}\mu_1(t)\|u\|_p^p, \end{aligned} \tag{3.6}$$

where δ is positive constant to be determined later. According to the conditions $\mu_1(t) \geq 0, \mu_2'(t) \geq 0$ and (3.1), we get

$$H'(t) \geq \mu_1(t)\|u_t\|_p^p. \tag{3.7}$$

Combining (2.3), (2.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned} L'(t) \geq\ & \left[(1-\alpha)H^{-\alpha}(t) - \frac{p-1}{p}\varepsilon\delta^{-\frac{p}{p-1}}\right]H'(t) \\ & +\varepsilon\left(qH(t) - \frac{\delta^p}{p}\mu_1(t)\|u_t\|_p^p\right) \\ & +\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2. \end{aligned} \tag{3.8}$$

Since the integral is taken over the variable x, it is reasonable to take δ depending on variable t. From (3.2), we obtain

$$0 < H^{-\alpha}(t) \leq H^{-\alpha}(0),$$

for every $t>0$. Hence $H^{-\alpha}(t)$ is a positive function and bounded. Thus, by taking $\delta^{-\frac{p}{p-1}} = mH^{-\alpha}(t)$, for large m to be specified later, and substituting in (3.8), we get

$$\begin{aligned} L'(t) \geq\ & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2 \\ & +\varepsilon\left[qH(t) - \frac{m^{1-p}}{p}\mu_1(t)H^{\alpha(p-1)}(t)\|u\|_p^p\right]. \end{aligned} \tag{3.9}$$

By using the (2.3), (2.4), (3.2) and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$ $(q>p)$, we arrive at $\|u\|_p^p \leq C\|u\|_q^p$ and

$$\begin{aligned} L'(t) \geq\ & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2 \\ & +\varepsilon\left[qH(t) - \frac{Cm^{1-p}}{p}\mu_1(t)\left(\frac{\mu_2(t)}{q}\right)^{\alpha(p-1)}\|u\|_q^{p+q\alpha(p-1)}\right]. \end{aligned} \tag{3.10}$$

From (3.4), we get $2 \leq s = p + q\alpha(p-1) \leq q$. Combining (2.3), (2.4), Remark 2.3 and (3.10), we obtain

$$\begin{aligned} L'(t) \geq\ & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{q}{2}+1\right)\|u_t\|^2 + \varepsilon\left(\frac{q}{2}-1\right)\|\nabla u\|^2 \\ & +\varepsilon\left[qH(t) - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\left(H(t)+\|u_t\|_2^2+\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right] \\ \geq\ & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) + \varepsilon\left(\frac{q+2}{2} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right)H(t) \\ & +\varepsilon\left[\frac{q+6}{4} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\right]\|u_t\|^2 \\ & +\varepsilon\left[\frac{q-2}{2q}\mu_2(t) - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\left(\frac{\mu_2(t)}{q}+1\right)\right]\|u\|_q^q, \end{aligned} \tag{3.11}$$

where $C_1 = \frac{C}{pq^{\alpha(p-1)}}$. Since $\lim_{t\to\infty}\mu_1(t)\mu_2(t)^{\alpha(p-1)}$ exists, $\mu_1(t)\mu_2(t)^{\alpha(p-1)}$ is bounded for every $t>0$. Then, we choose m large enough so that the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$ in (3.11) are strictly positive. Therefore, we arrive at

$$\begin{aligned} L'(t) \geq\ & \left[(1-\alpha) - \frac{p-1}{p}\varepsilon m\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \left(\frac{\mu_2(t)}{q}+1\right)\|u\|_q^q\right], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \beta =\ \min\Big\{ & \frac{q+2}{2} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t), \\ & \frac{q+6}{4} - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t), \\ & \frac{q-2}{2q}\mu_2(t) - C_1m^{1-p}\mu_2(t)^{\alpha(p-1)}\mu_1(t)\Big\} \end{aligned}$$

is the minimum of the coefficients of $H(t)$, $\|u_t\|^2$ and $\|u\|_q^q$. Once m is fixed, we can take ε small enough so that $1-\alpha-\frac{p-1}{p}\varepsilon m \geq 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0u_1dx > 0. \tag{3.13}$$

Then (3.12) becomes

$$L'(t) \geq \varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \left(\frac{\mu_2(t)}{q} + 1\right)\|u\|_q^q\right] \geq 0. \tag{3.14}$$

Then, we have

$$L(t) \geq L(0) > 0. \tag{3.15}$$

For the definition of $L(t)$ (see (3.3)) we have

$$\begin{aligned}\left|\int_\Omega uu_t dx\right| &\leq \|u\|\,\|u_t\| \\ &\leq C\|u\|_q\|u_t\|\end{aligned} \tag{3.16}$$

using Hölder's inequality and the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$ $(q > p)$. Thanks to Young's inequality, we have

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} &\leq C\|u\|_q^{\frac{1}{1-\alpha}}\|u_t\|^{\frac{1}{1-\alpha}} \\ &\leq C\left(\|u\|_q^{\frac{2}{1-2\alpha}} + \|u_t\|^2\right)\end{aligned} \tag{3.17}$$

from (3.4), we arrive at $\frac{2}{1-2\alpha} < q$.
Combining (3.17) and Remark 2.3, we get

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}} \leq C\left(H(t) + \|u_t\|_2^2 + \left(\frac{\mu_2(t)}{q} + 1\right)\|u\|_q^q\right). \tag{3.18}$$

Therefore, we obtain

$$\begin{aligned}L^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t dx\right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}}\left(H(t) + \left|\varepsilon\int_\Omega uu_t dx\right|^{\frac{1}{1-\alpha}}\right) \\ &\leq C\left(H(t) + \|u_t\|_2^2 + \left(\frac{\mu_2(t)}{q} + 1\right)\|u\|_q^q\right).\end{aligned} \tag{3.19}$$

Combining (3.14), (3.15) and (3.19), we have

$$L'(t) \geq \gamma L^{\frac{1}{1-\alpha}}(t) \tag{3.20}$$

where γ is a constant depending only on C, β and ε. Integrating (3.20), we arrive at

$$L^{\frac{1}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\gamma t}. \tag{3.21}$$

If

$$t \to \left[\frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)}\right]^-, \quad L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\gamma t \to 0.$$

Hence, $L(t)$ blows up in finite time T^* and

$$T^* \leq \frac{1-\alpha}{\alpha\gamma L^{\frac{\alpha}{1-\alpha}}(0)},$$

which complete the proof of the Theorem. □

4. Conclusion

In this paper, we obtained the nonexistence of global solutions for a strongly damped wave equation with variable coefficients. This improves and extends many results in the literature.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Connected Square Network Graphs

Burhan Selçuk[1]

[1]*Department of Computer Engineering, Faculty of Engineering, Karabük University, Karabük, Turkey*

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Abstract

In this study, connected square network graphs are introduced and two different definitions are given. Firstly, connected square network graphs are shown to be a Hamilton graph. Further, the labelling algorithm of this graph is obtained by using gray code. Finally, its topological properties are obtained, and conclusion are given.

1. Introduction

Nowadays, many types of interconnection network topologies have been extensively studied by researchers. The most popular of these are trees, cycles, grids, tori, meshes and hypercubes. A new network topology will be introduced in this study. This new network structure is obtained using squares and called the Connected Square Network Graph (CSNG). Two different definitions are given for the connected square network graph. Firstly, it is obtained by combining a finite number of squares in 2D space. Secondly, it is obtained recursively from square and compound cubes in the first way.

In the literature, hypercube, and its variants (Folded Hypercube, Crossed Cube and the Hierarchical Cubic Network) have been studied extensively in the interconnection network [1–9]. Karcı and Selçuk introduced new hypercube variants and investigated it's Hamilton-like features. These; Fractal Cubic Network Graph (FCNG) [10] uses the fractal structures and Connected Cubic Network Graph [11] uses hypercube. They investigated the topological properties of new hypercube variants.

Motivated by the [10] and [11], a new network structure will be defined in this study. The outline of this study is as follows. Section 2 informs basic information about graph theory and explains the definitions of CSNG. Section 3 investigates the analytical properties of CSNG and is obtained Hamiltonian properties of CSNG is obtained. Labelling algorithm for this graph is given in Section 4. In Section 5, topological features of CSNG are obtained and a projection for future work is presented.

2. Preliminaries

Rest of the study, $G=(V,E)$ is a graph where V is a vertex set and E is a edge set. (x,y) is an edge in E where $(x,y)\in G$. The degree of vertex $x\in V(G)$ is denoted by $\deg(x)$ and $d(x,y)$ is a shortest path from x to y in G.

"$||$" indicates the concatenation of two strings. The Hamming distance is $\sum_{i=0}^{n-1}(a_i\oplus b_i)$ since $\oplus$ is bitwise-XOR operation. $S(2)$ is denoted a square in $2D$ space. The $2D$ coordinate system is given below:

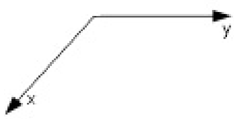

Figure 2.1: Two-dimensional coordinate space

Two different definitions be given to obtain these graphs, in this section.

***Email addresses and ORCID numbers*: bselcuk@karabuk.edu,tr 0000-0002-5141-5148 (B. Selçuk)**

Definition 2.1. ($CSNG$): Let $CSNG(0,0) = S(2)$ (Fig 2.(a)). $CSNG(k,m)$ can be defined in two steps.
Case I. Construction in one direction

(i) Suppose $\sum_{i=0}^{m} 2^i$ squares with common two nodes (an edge) are connected along the y-axis. This graph will be called a $CSNG(0,m)$. For example, the mesh structure given in Figure 2(b)-(c) are $CSNG(0,1)$ and $CSNG(0,2)$.
(ii) Suppose $\sum_{i=0}^{k}$ squares with common two nodes (an edge) are connected along the x-axis. This graph will be called a $CSNG(k,0)$.

Case II. Construction in two directions

(i) Suppose $\sum_{i=0}^{m} 2^i$ $CSNG(k,0)$s with common two nodes (an edge) are connected along the y-axis. This graph will be called a CSNG(k,m).
(ii) Suppose $\sum_{i=0}^{k} 2^i$ $CSNG(0,m)$s with common lower and upper surfaces (one surface) are connected along the x-axis. This graph will be called a $CSNG(k,m)$.

$CSNG(0,0)$ is represented by $S(2)$ in the Figure 2.2-(a). In Figure 2.2-(b) (Figure 2.2-(c)), $CSNG(0,1)$ ($CSNG(0,2)$) is obtained by combining 3 (7) squares with one side in common. $CSNG(1,2)$ is obtained by combining 3-$CSNG(0,2)$s which have top and bottom horizontal surfaces to be in common in Figure 3.1-(b).

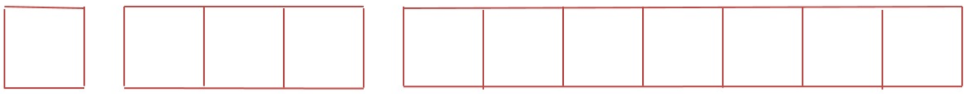

Figure 2.2: a. $CSNG(0,0)$, b. $CSNG(0,1)$, c. $CSNG(0,2)$, respectively

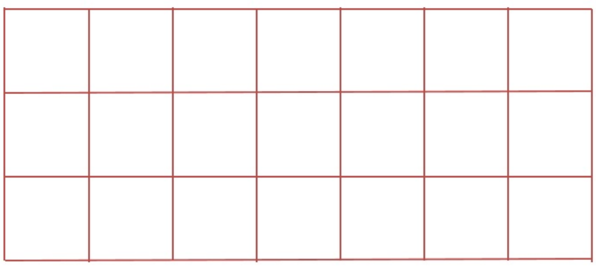

Figure 2.3: $CSNG(1,2)$

Definition 2.2. Two $CSNG(k,m-1)$s (or $CSNG(k-1,m)$s) can be merged to construct a new mesh of size doubling the size of $CSNG(k,m) = G(V,E), k \geq 0, m \geq 0$. There are two situations:

(i) If doubling dimension is x, then the nodes and edges in $0||CSNG(k-1,m)$ and $1||CSNG(k-1,m)$ are also included in $CSNG(k,m) = G(V_x,E_x)$. If $\forall v_i \in V,\ p = 0,\ldots,k+m-1,\ 2^p \leq Label(v_i) \leq 2^p+1,\ |k-m| \leq 1$, then $\forall (0||v_i, 1||v_i) \in E_x$.
(ii) If doubling dimension is y, then the nodes and edges in $0||CSNG(k,m-1)$ and $1||CSNG(k,m-1)$ are also included in $CSNG(k,m) = G(V_y,E_y)$. If $\forall v_i \in V, Label(v_i)$ is even, $Label(v_i) < 2^{k+m}$, $|k-m| \leq 1$, then $\forall (0||v_i, 1||v_i) \in E_y$.

$CSNG(0,1)$ and $CSNG(0,2)$ can be constructed using definition 2.2-(i) in Fig. 2.4-(a) and Fig. 3.1-(a), respectively. Similarly, $CSNG(1,0)$ and $CSNG(1,2)$ can be constructed using definition 2.2-(ii) in Fig. 2.4-(b) and Fig. 3.2-(a), respectively.

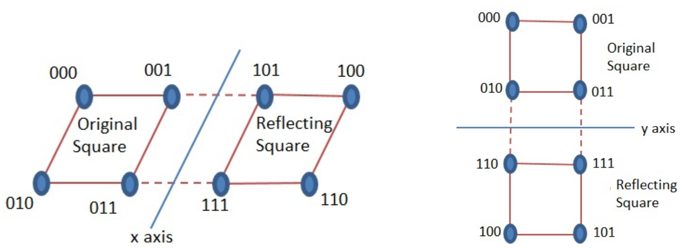

Figure 2.4: a. $CSNG(0,1)$, b. $CSNG(0,1)$, respectively

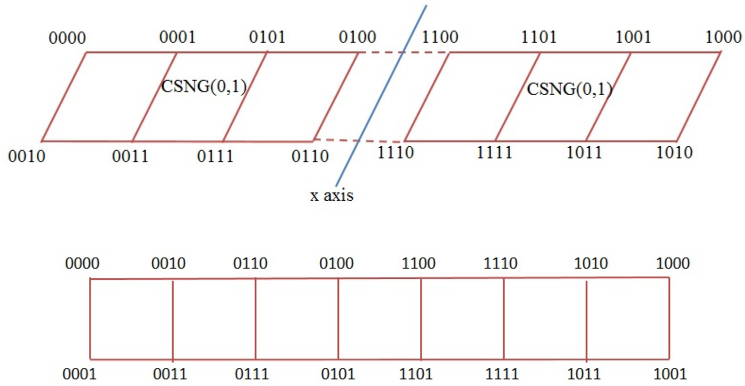

Figure 3.1: a. Construction of $CSNG(0,2)$ using Definiton 2.2, b. Labelling of $CSNG(0,2)$, respectively

3. Topological Features of Connected Square Network Graphs

3.1. Hamilton features of $CSNG(0,m)$ ($CSNG(k,0)$)

In this subsection, we analyzed Hamilton features of $CSNG(0,m)$ ($CSNG(k,0)$). Firstly, we give an example. $CSNG(0,2)$ is a Hamilton graph labelled with a 4-bit gray code in Fig. 3.1-(b).

Theorem 3.1. Suppose $\sum_{i=0}^{m} 2^i$-squares with common two nodes (an edge) are connected along the y-axis in definition 2.1-(a). This graph, $CSNG(0,m)$, has $3\times 2^{m+1}-2$ edges and 2^{m+2} nodes. Further, $CSNG(0,m)$ is a Hamilton graph and is labeled with a $m+2$-bit gray code.

Proof. The total node number of nodes of $CSNG(0,m)$ can be calculated by using definition 2.1-(a) and mathematical induction.

First Step: Let $m=2$. Suppose $\sum_{i=0}^{2} 2^i = 7$-squares with common two nodes (an edge) are connected along the y-axis. The total number node is along the y-axis $2\left(\sum_{i=0}^{2} 2^i+1\right) = 2^{2+2}$.

Hypothesis Step: Let $m=n-1$. Suppose $\sum_{i=0}^{n-1} 2^i$-squares with common two nodes (an edge) are connected along the y-axis. Assume that $CSNG(0,n-1)$ has 2^{n+1} nodes.

Final Step: Let $m=n$. Suppose $\sum_{i=0}^{n} 2^i$-squares with common two nodes (an edge) are connected along the y-axis. The following equation applies for the proof of final step:

$$\sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} 2^i + 2^n.$$

$CSNG(0,n)$ is obtained by adding 2^n $S(2)$ to the $CSNG(0,n-1)$ with 2 edges in common. Namely,

$$\left(\sum_{i=0}^{n} 2^i\right) S(2) = \left(\sum_{i=0}^{n-1} 2^i\right) S(2) + 2^n S(2),$$

$$CSNG(0,n) = CSNG(0,n-1) + 2^n S(2).$$

Hence, total node number of $CSNG(0,n)$ is $2^{n+1}+2^n\times 2 = 2^{n+2}$.

Secondly, the total number edge is along the x-axis $2\sum_{i=0}^{m} 2^i = 2\left(2^{m+1}-1\right) = 2.2^{m+1}-2$. The total number edge is along the y-axis $\sum_{i=0}^{m} 2^i + 1 = 2^{m+1}$. Total edge number of $CSNG(0,m)$ is $3.2^{m+1}-2$.

Finally, we showed that $CSNG(0,m)$ is a Hamilton graph. Mathematical induction will be used for proof.

First Step: Let $m=2$. $CSNG(0,2)$ is a Hamilton graph which is labelled with help of 4-bit Gray code seen in Fig. 3.1-(b).

Hypothesis Step: Let $m=n-1$. Suppose $CSNG(0,n-1)$ is a Hamilton graph which is labelled with help of $n+1$-bit Gray code and has 2^{n+1} nodes.

Final Step: Let $m=n$. The following equality is obtained

$$CSNG(0,n) = 0||CSNG(0,n-1) \cup 1||CSNG(0,n-1)$$

since $CSNG(0,n)$ has $2^{n+2} = 2.2^{n+1}$ nodes. Suppose x_i and x_j are two nodes in $CSNG(0,n-1)$ and $x_i \oplus x_j = 1$. The edges $(0||Label(x_i), 1||Label(x_i))$ and $(0||Label(x_j), 1||Label(x_j))$ are in $CSNG(0,n-1)$ and they are in Hamilton circuit in $CSNG(0,n)$. Namely, $CSNG(0,n)$ is a Hamilton graph which is labelled with help of $n+2$-bit Gray code and has 2^{n+2} nodes.

Similar results can be obtained in $CSNG(k,0)$.

3.2. Hamilton features of $CSNG(k,m)$

In this subsection, we analyzed Hamilton features of $CSNG(k,m)$. Firstly, we give an example. $CSNG(1,2)$ is a Hamilton graph labelled with a 5-bit gray code in Fig. 3.2-(b).

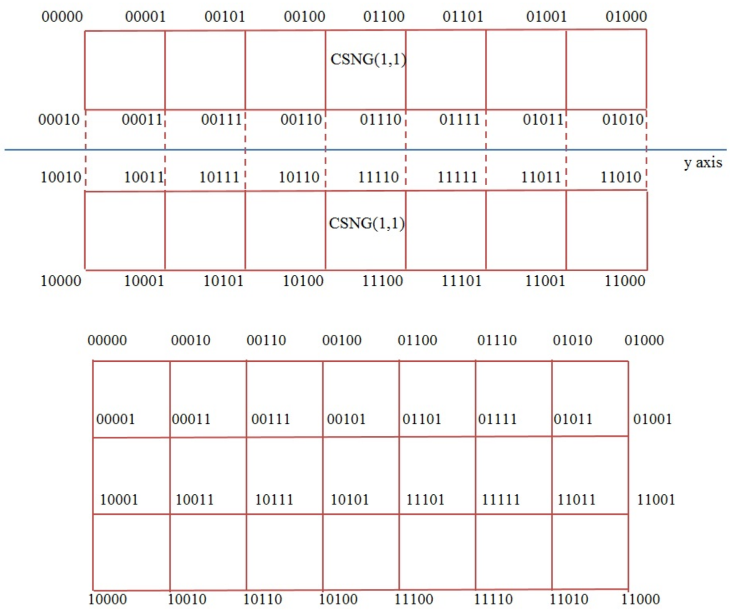

Figure 3.2: a. Construction of $CSNG(1,2)$ using Definiton 2.2, b. Labeling of $CSNG(1,2)$

Theorem 3.2. Suppose $\sum_{i=0}^{k}$-$CSNG(0,m)$s with common lower and upper surfaces (one surface) are connected along the x-axis. This graph will be called a $CSNG(k,m)$. $CSNG(k,m)$ has 2^{k+m+2} nodes and $2^{k+m+3}-2^{m+1}-2^{k+1}$ edges. Further, $CSNG(k,m)$ is a Hamilton graph and is labeled with a $k+m+2$-bit gray code.
Proof. Firstly, $CSNG(k,m)$ is consist of $\sum_{i=0}^{k}$-$CSNG(0,m)$s. Besides, $CSNG(0,m)$ is consist of $\sum_{j=0}^{m}$-$CSNG(0,0)$s where $CSNG(0,0)$ is a $S(2)$ square. Hence,

$$CSNG(k,m)=\sum_{i=0}^{k}CSNG(0,m)=\sum_{i=0}^{k}\sum_{j=0}^{m}CSNG(0,0)S(2).$$

Node numbers of $CSNG(k,m)$ is

$$\left(\sum_{i=0}^{k}2^{i}+1\right)\left(\sum_{j=0}^{m}2^{j}+1\right)=2^{k+1}2^{m+1}=2^{k+m+2}.$$

Because there are $\sum_{i=0}^{k}2^{i}+1$ nodes along the x-axis and $\sum_{j=0}^{m}2^{j}+1$ nodes along the y-axis.
Secondly, total number of edges along the x-axis is $\left(\sum_{i=0}^{m}2^{i}\right)\left(\sum_{j=0}^{k}2^{j}+1\right)$ and, total number of edges along the y-axis is $\left(\sum_{i=0}^{k}2^{i}\right)\left(\sum_{j=0}^{m}2^{j}+1\right)$.
Total number of edges of $CSNG(k,m)$ is

$$\left(\sum_{i=0}^{m}2^{i}\right)\left(\sum_{j=0}^{k}2^{j}+1\right)+\left(\sum_{i=0}^{k}2^{i}\right)\left(\sum_{j=0}^{m}2^{j}+1\right)=(2^{m+1}-1)2^{k+1}+(2^{k+1}-1)2^{m+1}=2^{k+m+3}-2^{m+1}-2^{k+1}.$$

A similar proof of Theorem 3.1 can be done to show that $CSNG(k,m)$ is a Hamilton graph.

4. Labelling Algorithm

In this section, an algorithm will be designed to label $CSNG(k,m)$ with the help of the reference [12].

Example 4.1. Let $k=1, m=2$ and $S=\{00\ 01\ 11\ 10\}$, $inv\_S=\{10\ 11\ 01\ 00\}$. Assume that $CSNG(0,0)=S$, $inv\_CSNG(0,0)=inv\_S$ where inv_CSNG is reverse sorting of $CSNG$. It can be calculation for $k+m=3$ iteration.
1. Iteration $(k=0,m=1)$*:*

$$\begin{aligned}CSNG(0,1)&=0||CSNG(0,0)\cup 1||inv\_CSNG(0,0)\\&=0||\{00\ 01\ 11\ 10\}\cup 1||\{10\ 11\ 01\ 00\}\\&=\{000\ 001\ 011\ 010\ 110\ 111\ 101\ 100\}\end{aligned}$$

and

$$\begin{aligned}inv\_CSNG(0,1)&=1||CSNG(0,0)\cup 0||inv\_CSNG(0,0)\\&=1||\{00011110\}\cup 0||\{10110100\}\\&=\{100\ 101\ 111\ 110\ 010\ 011\ 001\ 000\}\end{aligned}$$

2. Iteration $(k=0, m=2)$*:*

$$\begin{aligned} CSNG(0,2) &= 0||CSNG(0,1)\cup 1||inv\_CSNG(0,1) \\ &= 0||\{000\ 001\ 011\ 010\ 110\ 111\ 101\ 100\}\cup 1||\{100\ 101\ 111\ 110\ 010\ 011\ 001\ 000\} \\ &= \{0000\ 0001\ 0011\ 0010\ 0110\ 0111\ 0101\ 0100\ 1100\ 1101\ 1111\ 1110\ 1010\ 1011\ 1001\ 1000\} \end{aligned}$$

and

$$\begin{aligned} inv\_CSNG(0,2) &= 1||CSNG(0,1)\cup 0||inv\_CSNG(0,1) \\ &= 1||\{000\ 001\ 011\ 010\ 110\ 111\ 101\ 100\}\cup 0||\{100\ 101\ 111\ 110\ 010\ 011\ 001\ 000\} \\ &= \{1000\ 1001\ 1011\ 1010\ 1110\ 1111\ 1101\ 1100\ 0100\ 0101\ 0111\ 0110\ 0010\ 0011\ 0001\ 0000\} \end{aligned}$$

3. Iteration $(k=1, m=2)$*.*

$$\begin{aligned} CSNG(1,2) &= 0||CSNG(0,2)\cup 1||inv\_CSNG(0,2) \\ &= 0||\{0000\ 0001\ 0011\ 0010\ 0110\ 0111\ 0101\ 0100\ 1100\ 1101\ 1111\ 1110\ 1010\ 1011\ 1001\ 1000\}\cup \\ &\quad 1||\{1000\ 1001\ 1011\ 1010\ 1110\ 1111\ 1101\ 1100\ 0100\ 0101\ 0111\ 0110\ 0010\ 0011\ 0001\ 0000\} \end{aligned}$$

That is, labelling of nodes of $CSNG(1,2)$ is

00000 00001 00011 00010 00110 00111 00101 00100
01100 01101 01111 01110 01010 01011 01001 01000
11000 11001 11011 11010 11110 11111 11101 11100
10100 10101 10111 10110 10010 10011 10001 10000.

Remark 4.2. The Algorithm 1 finds the labeling of $CSNG(k,m)$ using recursive process. The running time of the Algorithm 1 is $O(p)$ where $p=\max(k,m)$. (Algorithm 1 in Appendix)

5. Comparison Results

Connected square network graphs are scalable. It has been shown that $CSNG(k,m)$ is an Hamiltonian graph and is not an Euler graph. $CSNG(0,m)$ (or $CSNG(k,0)$) has nodes with 2 and 3 are degree nodes and total node number of is 2^{m+2} in Table 1. $CSNG(k,m)$ has nodes with $2,3$ and 4 are degree nodes and total node number of is 2^{k+m+2} in Table 2. The edge-node relationship for $CSNG(0,m)$ and $CSNG(k,m)$ is given in Table 3 and Table 4, respectively. (Tables in Appendix)

Remark 5.1. (see [13]) Sum connectivity-index of $CSNG(0,m)$ is calculated as follows

$$\begin{aligned} \chi_\alpha(G) &= \sum_{(x,y)\in E}(\deg x+\deg y)^\alpha \\ &= 2.4^\alpha+4.5^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-6)6^\alpha \end{aligned}$$

and sum connectivity-index of $CSNG(k,m)$ is calculated as follows

$$\begin{aligned} \chi_\alpha(G) &= \sum_{(x,y)\in E}(\deg x+\deg y)^\alpha \\ &= 8.5^\alpha+(2^{m+2}+2^{k+2}-12)6^\alpha+(2^{m+2}+2^{k+2}-8)7^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-2^{m+3}-2^{k+3}+12)8^\alpha \end{aligned}$$

where $\alpha\in R$.
The general Randic index $R_\alpha(G)$ of $CSNG(0,m)$ is calculated as follows

$$\begin{aligned} R_\alpha(G) &= \sum_{(x,y)\in E}(\deg x \deg y)^\alpha \\ &= 2.4^\alpha+4.6^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-6)9^\alpha \end{aligned}$$

and, the general Randic index $R_\alpha(G)$ of $CSNG(k,m)$ is calculated as follows

$$\begin{aligned} R_\alpha(G) &= \sum_{(x,y)\in E}(\deg x \deg y)^\alpha \\ &= 8.6^\alpha+(2^{m+2}+2^{k+2}-12)9^\alpha+(2^{m+2}+2^{k+2}-8)12^\alpha+(2^{k+m+3}-2^{m+1}-2^{k+1}-2^{m+3}-2^{k+3}+12)16^\alpha \end{aligned}$$

where $\alpha=-1,-1/2,1/2,1$.

6. Conclusion

In this paper, connected square network graphs are introduced. Two different definitions are given to obtain connected square network graphs. The topological properties of these graphs have been investigated and it has been proven to be a Hamilton graph. These graphs can be thought of as a hypercube variant. A labeling algorithm is given that reinforces this idea.

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The author contributed to the writing of this paper. The author read and approved the final manuscript.

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7. Appendix

Algorithm 1: This algorithm calculate labelled of $CSNG(k,m)$.

Data: k, m, $S = \{00\ 01\ 11\ 10\}$, $inv\_S = \{10\ 11\ 01\ 00\}$
Result: labelled of $CSNG(k,m)$
begin
    $CSNG(0,0) = S$
    $inv\_CSNG(0,0) = inv\_S$
    for $j = 1$ *to* m **do**
        $CSNG(0,j) = 0||CSNG(0,j-1) \cup 1||inv\_CSNG(0,j-1)$
        $inv\_CSNG(0,j) = 1||CSNG(0,j-1) \cup 0||inv\_CSNG(0,j-1)$
    for $i = 1$ *to* k **do**
        $CSNG(i,j) = 0||CSNG(i-1,j) \cup 1||inv\_CSNG(i-1,j)$
        $inv\_CSNG(i,j) = 1||CSNG(i-1,j) \cup 0||inv\_CSNG(i-1,j)$
    return $CSNG(i,j)$

Table 1: The number of degree of nodes of $CSNG(0,m)$

$deg(2)$	$deg(3))$	Total Node
4	$2^{m+2}-4$	2^{m+2}

Table 2: The number of degree of nodes of $CSNG(k,m)$

$deg(2)$	$deg(3))$	$deg(4))$	Total Node
4	$2^{m+2}+2^{k+2}-8$	$2^{k+m+2}-2^{m+2}-2^{k+2}+4$	2^{k+m+2}

Table 3: The number of the edges of $CSNG(0,m)$

$(deg(2),deg(2))$	$(deg(2),deg(3))$	$(deg(3),deg(3))$	Total Edge
2	4	$2^{k+m+3}-2^{m+1}-2^{k+1}-6$	$2^{k+m+3}-2^{m+1}-2^{k+1}$

Table 4: The number of the edges of $CSNG(k,m)$

$(deg(2),deg(3))$	$(deg(3),deg(3))$	$(deg(3),deg(4))$	$(deg(4),deg(4))$	Total Edge
8	$2^{m+2}+2^{k+2}12$	$2^{m+2}+2^{k+2}-8$	$2^{k+m+3}-2^{m+1}-2^{k+1}-2^{m+3}-2^{k+3}+12$	$2^{k+m+3}-2^{m+1}-2^{k+1}$

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The New Iterative Approximating of Endpoints of Multivalued Nonexpansive Mappings in Banach Spaces

Makbule Kaplan Özekes[1]

[1] *Sinop University, Faculty of Education, Sinop, Turkey*

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Abstract

The purpose of this paper is to introduce a modified iteration process to approximate endpoints of multivalued nonexpansive mappings in Banach space. We prove weak and strong convergence theorems of proposed iterative scheme under some suitable assumptions in the framework of a uniformly convex Banach space.

1. Introduction and Preliminaries

In this study, we shall denote by $\mathbb{N}$ the set of natural numbers. Let $(E, \|.\|)$ be a Banach space and C be a nonempty convex subset of E. The distance from a $x \in E$ to a nonempty subset $C \subset E$ is defined by

$$dist\,(x,C) := \inf\{\|x-z\| : z \in C\}.$$

The radius of C relative to x is defined by

$$R(x,C) = \sup\{\|x-z\| : z \in C\}.$$

Definition 1.1. *A Banach space E is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there is a $\delta > 0$ such that for every $x, y \in E$*

$$\left.\begin{array}{c}\|x\| \leq 1\\ \|y\| \leq 1\\ \|x-y\| \geq \varepsilon\end{array}\right\} \Rightarrow \frac{\|x+y\|}{2} \leq 1-\delta.$$

We shall denote the family of nonempty compact subsets of C by $K(C)$. The Hausdorff metric H on $K(C)$ is defined as follows:

$$H(A,B) = \max\left\{\sup_{x\in A} dist\,(x,B),\ \sup_{y\in B} dist\,(y,A)\right\} \text{ for } A,B \in K(C).$$

A multivalued mapping $T : C \to K(C)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|, \quad \text{for each } \ x,y \in C.$$

A point $x \in K$ is a fixed point of a multivalued mapping $T : C \to K(C)$ if $x \in T(x)$. Moreover,if $T(x) = \{x\}$, then x is called an endpoint (or a stationary point) of T . We shall denote the set of all endpoints and the set of all fixed points of T by E_T (or $End(T)$) and F_T, respectively. It is clear that $End(T) \subseteq Fix(T)$. Endpoint for multivalued mappings is an important concept. Many researchers have studied the exsitence of an endpoint of a multivalued mapping. In 1980, Aubin and Siegel [1] proved that every multivalued dissipative mapping on a complete

Email address and ORCID number: **mkaplan@sinop.edu.tr, 0000-0002-7962-702X (M. Kaplan Özekes)**

metric space has always an endpoint. In 1986, Corley [2] showed that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of certain multivalued mapping. In 2018, Panyanak [3] showed that the modified Ishikawa iteration process converge to an endpoint of a multivalued nonexpansive mapping in Banach spaces. In 2020, Laokul [4] proved Browder's convergence theorem for multivalued mappings in Banach space without the endpoint condition by using the notion of diametrically regular mapping. Abdeljawad et al. [5] introduced the modified S- iteration process for finding endpoints of multivalued nonexpansive mappings in Banach spaces. Ullah et al. [6] proved the strong and Δ-convergence results of endpoints for multivalued generalized nonexpansive in Metric spaces.

Definition 1.2. *A Banach space $(E,\|.\|)$ is said to have Opial property [7] if for each sequence $\{x_n\}$ in E which weakly converges to $x \in E$ and $y \neq x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Definition 1.3. *[3] A mapping $T: C \to K(C)$ is said to satisfy condition (J) if there exists a nondecreasing function $h: [0,\infty) \to [0,\infty)$ with $h(0)=0$, $h(r)>0$ for $r \in (0,\infty)$ such that*

$$R(x,T(x)) \geq h(dist(x,End(T)) \text{ for all } x \in C.$$

Definition 1.4. *[3] The mapping $T: C \to K(C)$ is said to be semicompact if for any sequence $\{x_n\}$ in C such that*

$$\lim_{n\to\infty} R(x_n, T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in C$ such that $\lim_{k\to\infty} x_{n_k} = q$.

Definition 1.5. *A sequence $\{x_n\}$ in E is said to be Fejěr monotone with respect to C if*

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

for all $p \in C$ and $n \in \mathbb{N}$.

The purpose of this paper is to introduce a modified iteration process to approximate endpoints of multivalued nonexpansive mappings in Banach space.
Let C be a nonempty subset of a Banach space and $T: C \to K(C)$ be a nonexpansive multivalued mapping. Let $\alpha_n, \beta_n, \gamma_n \in [a,b] \subset (0,1)$ are real sequences. We introduce our iteration process as follows: $x_1 \in C$

$$z_n = (1-\gamma_n)x_n + \gamma_n v_n, \;\; n \in \mathbb{N}$$

where $v_n \in T(x_n)$ such that$\|x_n - v_n\| = R(x_n, T(x_n))$, and

$$y_n = (1-\beta_n)v_n + \beta_n w_n \tag{1.1}$$

where $w_n \in T(z_n)$ such that$\|z_n - w_n\| = R(z_n, T(z_n))$, and

$$x_{n+1} = (1-\alpha_n)v_n + \alpha_n u_n$$

where $u_n \in T(y_n)$ such that $\|y_n - u_n\| = R(y_n, T(y_n))$.
Following lemmas will be useful to prove our main results.

Lemma 1.6. *[3] For a multivalued mapping $T: C \to K(C)$, the following statements hold.*

(i) $x \in F(T) \Leftrightarrow dist(x,T(x)) = 0$.
(ii) $x \in End(T) \Leftrightarrow R(x,T(x)) = 0$.
(iii) If T is nonexpansive, the mapping $g: C \to \mathbb{R}$ defined by $g(x) := R(x,T(x))$ is continuous.

Lemma 1.7. *[8] A Banach space E is uniformly convex if and only if an arbitrary $k>0$, there exists a strictly increasing continuous function $\Psi: [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that*

$$\lim_{n\to\infty} \|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\Psi(\|x-y\|),$$

for all $x,y \in B_k(0) = \{x \in X : \|x\| \leq k\}$, and $\alpha \in [0,1]$.

Lemma 1.8. *[9] Let $\{\alpha_n\},\{\beta_n\}$ be two real sequences such that*

(i) $0 \leq \alpha_n, \beta_n < 1$,
(ii) $\beta_n \to 0$ as $n \to \infty$,
(iii) $\sum \alpha_n \beta_n = \infty$,
Let $\{\delta_n\}$ be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1-\beta_n)\delta_n < \infty$. Then $\{\delta_n\}$ has a subsequence which converges to zero.

Definition 1.9. *[10] Let $T: C \to CB(C)$ be a multivalued mapping. A sequence $\{x_n\}$ in C is called an approximate fixed point sequence (resp. an approximate endpoint sequence) for T if $\lim\limits_{n\to\infty} dist(x_n, T(x_n)) = 0$ (resp. $\lim\limits_{n\to\infty} R(x_n, T(x_n)) = 0$). The mapping T is said to have the approximate fixed point property (resp. the approximate endpoint property) if it has an approximate fixed point sequence (resp. an approximate endpoint sequence) in C.*

Let C be a nonempty subset of a metric space (X,d) and $\{x_n\}$ be a bounded sequence in X. The asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C,\{x_n\}) = inf\left\{\limsup_{n\to\infty} d(x_n,x) : x\in C\right\}.$$

The asymptotic center of $\{x_n\}$ relative to E is defined by

$$A(C,\{x_n\}) = \left\{x\in C : \limsup_{n\to\infty} d(x_n,x) = r(C,\{x_n\})\right\}.$$

Lemma 1.10. *[11] Let C be a nonempty closed convex subset of a uniformly convex Banach space and $T : C \to K(C)$ be a mutlivalued nonexpansive mapping. Then the following implication holds:*

$$\{x_n\}\subseteq C,\ x_n \rightharpoonup x,\ R(x_n, T(x_n)) \to 0 \Rightarrow x\in End(T).$$

Proposition 1.11. *[10] Let C be a nonempty subset of a metric space (X,d), $\{x_n\}$ be a sequence in E, and $T : C \to K(X)$ be a mapping. Then $R(x_n,T(x_n)) \to 0$ if and only if $dist(x_n,T(x_n)) \to 0$ and $diam(T(x_n)) \to 0$.*

Theorem 1.12. *[10] Let $(X,\|.\|)$ be a uniformly convex Banach space, C be a nonempty bounded closed convex subset of X, and $T : C \to K(C)$ be a nonexpansive mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

2. Main Results

We start with the following lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of an uniformly convex Banach space E and $T : C \to K(C)$ be a multivalued nonexpansive mapping with $E_T \neq \emptyset$. Let $\{x_n\}$ be a sequence as defined in (1.1). Then $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p\in E_T$.*

Proof. Let $p\in End(T)$. By (1.1), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)v_n+\alpha_n u_n - p\|\\ &\le (1-\alpha_n)\|v_n-p\|+\alpha_n\|u_n-p\|\\ &= (1-\alpha_n)dist(v_n,T(p))+\alpha_n dist(u_n,T(p))\\ &\le (1-\alpha_n)H(T(x_n),T(p))+\alpha_n H(T(y_n),T(p))\\ &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|. \end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\|y_n-p\| &= \|(1-\beta_n)v_n+\beta_n w_n - p\|\\ &\le (1-\beta_n)\|v_n-p\|+\beta_n\|w_n-p\|\\ &= (1-\beta_n)dist(v_n,T(p))+\beta_n dist(w_n,T(p))\\ &\le (1-\beta_n)H(T(x_n),T(p))+\beta_n H(T(z_n),T(p))\\ &\le (1-\beta_n)\|x_n-p\|+\beta_n\|z_n-p\| \end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\|z_n-p\| &= \|(1-\gamma_n)x_n+\gamma_n v_n - p\|\\ &\le (1-\gamma_n)\|x_n-p\|+\gamma_n\|v_n-p\|\\ &= (1-\gamma_n)\|x_n-p\|+\gamma_n dist(v_n,T(p))\\ &\le (1-\gamma_n)\|x_n-p\|+\gamma_n H(T(x_n),T(p))\\ &\le (1-\gamma_n)\|x_n-p\|+\gamma_n\|x_n-p\|\\ &= \|x_n-p\|. \end{aligned}\tag{2.3}$$

Using (2.3) and (2.2) ,we obtain

$$\|y_n-p\| \le (1-\beta_n)\|x_n-p\|+\beta_n\|x_n-p\| = \|x_n-p\|$$

which implies that

$$\|x_{n+1}-p\| \le (1-\alpha_n)\|x_n-p\|+\alpha_n\|x_n-p\| = \|x_n-p\|.$$

Thus$\{\|x_n-p\|\}$ is nonincreasing sequence and bounded, which implies that $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in E_T$. Also $\{x_n\}$ is bounded. □

Theorem 2.2. *Let E be a uniformly convex Banach space with Opial property, C be a nonempty closed convex subset of E and $T : C \to K(C)$ be a multivalued nonexpansive mapping with $E_T \neq \emptyset$. If $\{x_n\}$ is the sequence defined by (1.1) with $\alpha_n,\beta_n,\gamma_n \in [a,b] \subset (0,1)$ for all n in $\mathbb{N}$, then $\{x_n\}$ converges weakly to an element of E_T.*

Proof. Fix $p \in E_T$. Then, as in the proof of Lemma 2.1,$\{x_n\}$is bounded and so $\{y_n\}$,$\{z_n\}$ are bounded. Therefore, there exists $k > 0$ such that $x_n - p$, $y_n - p$, $z_n - p \in B_k(0)$ for all $n \geq 0$. Since E is a uniformly convex, by Lemma 1.7, there exists a strictly increasing continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\gamma_n)x_n + \gamma_n v_n - p\|^2 \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|v_n - p\|^2 - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n dist^2(v_n, T(p)) - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n H^2(T(x_n), T(p)) - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) \\
&\leq \|x_n - p\|^2 - \gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|).
\end{aligned} \tag{2.4}$$

By Lemma 1.7 and (2.4), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)v_n + \beta_n w_n - p\|^2 \\
&\leq (1-\beta_n)\|v_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n)dist^2(v_n, T(p)) + \beta_n dist^2(w_n, T(p)) - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n)H^2(T(x_n), T(p)) + \beta_n H^2(T(z_n), T(p)) - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\Psi(\|v_n - w_n\|) \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|)
\end{aligned} \tag{2.5}$$

from (2.4), (2.5) and by Lemma 1.7 , we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)v_n + \alpha_n u_n - p\|^2 \\
&\leq (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n)dist^2(v_n, T(p)) + \alpha_n dist^2(u_n, T(p)) - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n)H^2(T(x_n), T(p)) + \alpha_n H^2(T(y_n), T(p)) - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\Psi(\|v_n - u_n\|) \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|).
\end{aligned} \tag{2.6}$$

so,

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|).$$

This implies that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n(1-\gamma_n)\Psi(\|x_n - v_n\|) < \infty$$

By Lemma 1.8, we have $\lim_{n\to\infty}\Psi(\|x_n - v_n\|) = 0$. As Ψ is strictly increasing and continuous, we get $\lim_{n\to\infty}\|x_n - v_n\| = 0$. Hence

$$\lim_{n\to\infty} R(x_n, T(x_n)) = \lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{2.7}$$

We want to show that $\{x_n\}$ converges weakly to an element of E_T. For this, it must be shown that $\{x_n\}$ has unique weak subsequential limit in E_T. Therefore, we assume that there are subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. By (2.7), $\lim_{n_i\to\infty} R(x_{n_i}, T(x_{n_i})) = 0$. It follows from Lemma 1.10 that $u \in E_T$. Similarly, we can be shown that $v \in E_T$. Now, suppose $u \neq v$. By Lemma 2.1 and the Opial property, we get

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{n_i\to\infty}\|x_{n_i} - u\| \\
&< \lim_{n_i\to\infty}\|x_{n_i} - v\| \\
&= \lim_{n\to\infty}\|x - v\| \\
&= \lim_{n_j\to\infty}\|x_{n_j} - v\| \\
&< \lim_{n_j\to\infty}\|x_{n_j} - u\| \\
&= \lim_{n\to\infty}\|x_n - u\|
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to an element of E_T.

$\square$

Next, we prove strong convergence theorems in uniformly convex Banach spaces.

Theorem 2.3. *Let E, C and T be as in Theorem 2.2. Let $\{x_n\}$be the sequence defined by (1.1) with $\alpha_n, \beta_n, \gamma_n \in [a,b] \subset (0,1)$. If T is semi-compact, then $\{x_n\}$ converges strongly to an element of E_T.*

Proof. In view of (2.6), we have

$$\alpha_n \beta_n \gamma_n (1-\gamma_n) \Psi(\|x_n - v_n\|) < \infty .$$

By Lemma 1.8, there exists subsequence $\{v_{n_k}\}$ and $\{x_{n_k}\}$ of $\{v_n\}$ and $\{x_n\}$, respectively, such that $\lim_{k\to\infty} \Psi(\|x_{n_k} - v_{n_k}\|) = 0$. Since Ψ is strictly increasing and continuous, $\lim_{k\to\infty} \|x_{n_k} - v_{n_k}\| = 0$. So,

$$\lim_{k\to\infty} R(x_{n_k}, T(x_{n_k})) = \lim_{k\to\infty} \|x_{n_k} - v_{n_k}\| = 0. \tag{2.8}$$

Conversely, T is semicompact, we may assume, by passing through a subsequence, that $x_{n_k} \to q$ for some $q \in C$. We need show that $q \in E_T$ and $x_n \to q$. By Lemma 1.6 (iii), together with (2.8), we have

$$R(q, T(q)) = \lim_{k\to\infty} R(x_{n_k}, T(x_{n_k})) = 0. \tag{2.9}$$

It follows from Lemma 1.6 (ii) that $q \in E_T$. By Lemma 2.1 $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in E_T$ and hence q is the strong limit of $\{x_n\}$. □

Proposition 2.4. *[12] Let C be a nonempty closed subset of a Banach space and $\{x_n\}$ be a Fejer monotone sequence with respect to C. Then $\{x_n\}$ converges strongly to an element of C if and only if $\lim_{n\to\infty} dist(x_n, C) = 0$.*

Theorem 2.5. *Let E, C, T and $\{x_n\}$ be as in Theorem 2.2. If T satisfies condition (J), then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. Since T is a nonexpansive mapping, E_T is closed. As T satisfies condition (J), $\lim_{n\to\infty} dist(x_n, E_T) = 0$. Lemma 2.1 implies that $\{x_n\}$ is Fejer monotone with to respect E_T. By Proposition 2.4, $\{x_n\}$ converges strongly to an element of E_T. □

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Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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Generalized Woodall Numbers: An Investigation of Properties of Woodall and Cullen Numbers via Their Third Order Linear Recurrence Relations

Yüksel Soykan[1*] and Vedat İrge[1]

[1] *Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey*
[*] *Corresponding author*

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Abstract

In this paper, we investigate the generalized Woodall sequences and we deal with, in detail, four special cases, namely, modified Woodall, modified Cullen, Woodall and Cullen sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1. Introduction

The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS [22]). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [6] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.
The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, \ldots$$

(sequence A002064 in the OEIS).
Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,6,9,10,11,13,15,16,17,18] and references therein.
Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$R_n = 4R_{n-1} - 4R_{n-2} - 1,$$
$$C_n = 4C_{n-1} - 4C_{n-2} + 1.$$

Email addresses and ORCID numbers: **ysoykan@beun.edu.tr, 0000-0002-1895-211X (Y. Soykan), vedatirge_1986@hotmail.com, 0000-0003-2878-3505 (V. İrge)**

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \tag{1.1}$$
$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{1.2}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Woodall, Cullen numbers) via their third order linear recurrence relations (1.1) and (1.2). First, we recall some properties of generalized Tribonacci numbers.
The generalized (r,s,t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1.3}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers.
This sequence has been studied by many authors, see for example [3,4,5,7,8,14,19,20,21,24,25,27,28,29].
The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1,2,3,\ldots$ when $t \neq 0$. Therefore, recurrence (1.3) holds for all integer n.
As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.4}$$

whose roots are

$$\alpha = \alpha(r,s,t) = \frac{r}{3} + A + B,$$
$$\beta = \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B,$$
$$\gamma = \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B,$$

where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3},$$
$$\Delta = \Delta(r,s,t) = \frac{r^3t}{27} - \frac{r^2s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \ \omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that we have the following identities

$$\alpha + \beta + \gamma = r,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -s,$$
$$\alpha\beta\gamma = t.$$

In the case of two distinct roots, i.e., $\alpha = \beta \neq \gamma$, Binet's formula can be given as follows:

Theorem 1.1. *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$) Binet's formula of generalized Tribonacci numbers is*

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3\gamma^n$$

where

$$A_1 = \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha-\gamma)^2},$$
$$A_2 = \frac{W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha-\gamma)},$$
$$A_3 = \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha-\gamma)^2}.$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.2. *Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.5}$$

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{1.6}$$

For matrix formulation (1.6), see [12]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

Now, we present Simson's formula of generalized Tribonacci numbers.

Theorem 1.3 (Simson's Formula of Generalized Tribonacci Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{1.7}$$

Proof. For a proof, see Soykan [23]. □

Next, we consider two special cases of the generalized (r,s,t) sequence $\{W_n\}$ which we call them (r,s,t) and Lucas (r,s,t) sequences. (r,s,t) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r,s,t) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = rG_{n+2} + sG_{n+1} + tG_n, \quad G_0 = 0, G_1 = 1, G_2 = r, \tag{1.8}$$

$$H_{n+3} = rH_{n+2} + sH_{n+1} + tH_n, \quad H_0 = 3, H_1 = r, H_2 = 2s + r^2. \tag{1.9}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)},$$

$$H_{-n} = -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)}$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences (1.8)-(1.9) hold for all integers n.

In the case of two distinct roots, i.e., $\alpha = \beta \neq \gamma$, for all integers n, Binet's formula of (r,s,t) and Lucas (r,s,t) numbers (using initial conditions in (1.8)-(1.9)) can be expressed as follows:

Theorem 1.4. *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$) For all integers n, Binet's formula of (r,s,t) and Lucas (r,s,t) numbers are*

$$G_n = \left(\frac{-\gamma}{(\alpha-\gamma)^2} + \frac{1}{(\alpha-\gamma)}n \right) \times \alpha^n + \frac{\gamma}{(\alpha-\gamma)^2}\gamma^n,$$

$$H_n = 2\alpha^n + \gamma^n,$$

respectively.

Lemma 1.2 gives the following results as particular examples (generating functions of (r,s,t) and Lucas (r,s,t) numbers).

Corollary 1.5. *Generating functions of (r,s,t) and Lucas (r,s,t) numbers are*

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1-rx-sx^2-tx^3},$$

$$\sum_{n=0}^{\infty} H_n x^n = \frac{3-2rx-sx^2}{1-rx-sx^2-tx^3},$$

respectively.

The following theorem shows that the generalized Tribonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.6. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Proof. For the proof, see Soykan [26, Theorem 2.]. □

Now, we present a basic relation between $\{H_n\}$ and $\{W_n\}$ which can be used to write H_n in terms of W_n.

Lemma 1.7. *The following equality is true:*

$(W_2^3 + (t+rs)W_1^3 + t^2W_0^3 + (r^2-s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2+rt)W_0W_1^2 + 2stW_0^2W_1 + (rs-3t)W_0W_1W_2)H_n = (3W_2^2 + (r^2-s)W_1^2 + rtW_0^2 - 4rW_1W_2 - 2sW_0W_2 + (rs-3t)W_0W_1)W_{n+2} + (-2rW_2^2 + 3tW_1^2 - 2sW_1W_2 - 3tW_0W_2 + 3rsW_1^2 + 2stW_0^2 + 2r^2W_1W_2 + 2s^2W_0W_1 + rsW_0W_2 + 2rtW_0W_1)W_{n+1} + (-sW_2^2 + (s^2+rt)W_1^2 + 3t^2W_0^2 + (rs-3t)W_1W_2 + 2rtW_0W_2 + 4stW_0W_1)W_n.$

Proof. It is given in Soykan [25]. □
Using Theorem 1.6, we have the following corollary, see Soykan [26, Corollary 6].

Corollary 1.8. *For $n \in \mathbb{Z}$, we have*

(a)

$$G_{-n} = \frac{1}{t^{n+1}}((2rt - s^2)G_n^2 + tG_{2n} + sG_{n+2}G_n - (3t + rs)G_{n+1}G_n).$$

(b)

$$H_{-n} = \frac{1}{2t^n}(H_n^2 - H_{2n}).$$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 3$ in Theorem 1.6,

$$\begin{aligned} G_{-n} &= t^{-n}(G_{2n} - H_nG_n + \frac{1}{2}(H_n^2 - H_{2n})G_0) = t^{-n}(G_{2n} - H_nG_n), \\ H_{-n} &= t^{-n}(H_{2n} - H_nH_n + \frac{1}{2}(H_n^2 - H_{2n})H_0) = \frac{1}{2t^n}(H_n^2 - H_{2n}), \end{aligned}$$

respectively.

2. Generalized Woodall Sequence

In this paper, we consider the case $r = 5, s = -8, t = 4$. A generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.
The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (2.1) holds for all integer n.
Theorem 1.1 can be used to obtain Binet formula of generalized Woodall numbers. Binet formula of generalized Woodall numbers can be given as
(two distinct roots case: $\alpha = \beta \neq \gamma$)

$$W_n = (A_1 + A_2n) \times \alpha^n + A_3\gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2}, \\ A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)}, \\ A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}. \end{aligned}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= \beta = 2, \\ \gamma &= 1. \end{aligned}$$

So,

$$W_n = (A_1 + A_2n) \times 2^n + A_3$$

where

$$\begin{aligned} A_1 &= -W_2 + 4W_1 - 3W_0, \\ A_2 &= \frac{W_2 - 3W_1 + 2W_0}{2}, \\ A_3 &= W_2 - 4W_1 + 4W_0, \end{aligned}$$

i.e.,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{2.2}$$

The first few generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Woodall numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$
6	$196W_0 - 324W_1 + 129W_2$	$(247W_0 - 243W_1 + 60W_2)$
7	$516W_0 - 836W_1 + 321W_2$	$\frac{1}{256}(1004W_0 - 995W_1 + 247W_2)$
8	$1284W_0 - 2052W_1 + 769W_2$	$\frac{1}{256}(1013W_0 - 1008W_1 + 251W_2)$
9	$3076W_0 - 4868W_1 + 1793W_2$	$\frac{1}{1024}(4072W_0 - 4061W_1 + 1013W_2)$
10	$7172W_0 - 11268W_1 + 4097W_2$	$\frac{1}{1024}(4083W_0 - 4077W_1 + 1018W_2)$
11	$16388W_0 - 25604W_1 + 9217W_2$	$\frac{1}{4096}(16356W_0 - 16343W_1 + 4083W_2)$
12	$36868W_0 - 57348W_1 + 20481W_2$	$\frac{1}{4096}(16369W_0 - 16362W_1 + 4089W_2)$
13	$81924W_0 - 126980W_1 + 45057W_2$	$\frac{1}{16384}(65504W_0 - 65489W_1 + 16369W_2)$

Now, we define four special cases of the sequence $\{W_n\}$. Modified Woodall sequence $\{G_n\}_{n\geq 0}$, modified Cullen sequence $\{H_n\}_{n\geq 0}$, Woodall sequence $\{R_n\}$ and Cullen sequence $\{C_n\}$ are defined, respectively, by the third-order recurrence relations

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \quad G_0 = 0, G_1 = 1, G_2 = 5, \tag{2.3}$$
$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad H_0 = 3, H_1 = 5, H_2 = 9, \tag{2.4}$$
$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \tag{2.5}$$
$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{2.6}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)},$$
$$H_{-n} = 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)},$$
$$R_{-n} = 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)},$$
$$C_{-n} = 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (2.3)-(2.6) hold for all integer n.

Next, we present the first few values of the modified Woodall, modified Cullen, Woodall and Cullen numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
G_n	0	1	5	17	49	129	321	769	1793	4097	9217	20481	45057	98305
G_{-n}		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{11}{16}$	$\frac{13}{16}$	$\frac{57}{64}$	$\frac{15}{16}$	$\frac{247}{256}$	$\frac{251}{256}$	$\frac{1013}{1024}$	$\frac{509}{512}$	$\frac{4083}{4096}$	$\frac{4089}{4096}$
H_n	3	5	9	17	33	65	129	257	513	1025	2049	4097	8193	16385
H_{-n}		2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$	$\frac{2049}{2048}$	$\frac{4097}{4096}$
R_n	-1	1	7	23	63	159	383	895	2047	4607	10239	22527	49151	106495
R_{-n}		$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{11}{8}$	$-\frac{5}{4}$	$-\frac{37}{32}$	$-\frac{35}{32}$	$-\frac{135}{128}$	$-\frac{33}{32}$	$-\frac{521}{512}$	$-\frac{517}{512}$	$-\frac{2059}{2048}$	$-\frac{1027}{1024}$	$-\frac{8205}{8192}$
C_n	1	3	9	25	65	161	385	897	2049	4609	10241	22529	49153	106497
C_{-n}		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{27}{32}$	$\frac{29}{32}$	$\frac{121}{128}$	$\frac{31}{32}$	$\frac{503}{512}$	$\frac{507}{512}$	$\frac{2037}{2048}$	$\frac{1021}{1024}$	$\frac{8179}{8192}$

G_n, H_n, R_n and C_n are the sequences A000337, A000051 (and A048578), A003261 and A002064 in [22], respectively. Note that $\{H_n\}$ satisfies the following second order linear recurrence:

$$H_n = 3H_{n-1} - 2H_{n-2}, \;\; H_0 = 3, H_1 = 5$$

and satisfies the following first order non-linear recurrence:

$$H_n = 2H_{n-1} - 1, \;\; H_0 = 3.$$

For all integers n, modified Woodall, modified Cullen, Woodall and Cullen numbers (using initial conditions in (2.2)) can be expressed using Binet's formulas as

$$G_n = (n-1)2^n + 1$$
$$H_n = 2^{n+1} + 1$$
$$R_n = n \times 2^n - 1$$
$$C_n = n \times 2^n + 1$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 2.1. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Woodall sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

Proof. Take $r = 5, s = -8, t = 4$ in Lemma 1.2. □

The previous lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of modified Woodall, modified Cullen, Woodall and Cullen numbers are*

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1 - 5x + 8x^2 - 4x^3},$$
$$\sum_{n=0}^{\infty} H_n x^n = \frac{3 - 10x + 8x^2}{1 - 5x + 8x^2 - 4x^3},$$
$$\sum_{n=0}^{\infty} R_n x^n = \frac{-1 + 6x - 6x^2}{1 - 5x + 8x^2 - 4x^3},$$
$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - 2x + 2x^2}{1 - 5x + 8x^2 - 4x^3},$$

respectively.

3. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Woodall sequence $\{W_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized Woodall Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -2^{2n-4}(W_2 - 4W_1 + 4W_0)(W_2 - 3W_1 + 2W_0)^2.$$

Proof. Take $r = 5, s = -8, t = 4$ in Theorem 1.3. □

The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers n, Simson formula of modified Woodall, modified Cullen, Woodall and Cullen numbers are given as*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -2^{2n-2},$$
$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = 0,$$
$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_n \\ R_{n+1} & R_n & R_{n-1} \\ R_n & R_{n-1} & R_{n-2} \end{vmatrix} = 2^{2n-2},$$
$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -2^{2n-2},$$

respectively.

4. Some Identities

In this section, we obtain some identities of generalized Woodall, modified Woodall, modified Cullen, Woodall and Cullen numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{G_n\}$.

Lemma 4.1. *The following equalities are true:*

(a) $16W_n=(52W_0-47W_1+11W_2)G_{n+4}+(199W_1-216W_0-47W_2)G_{n+3}+4(57W_0-54W_1+13W_2)G_{n+2}$.
(b) $4W_n=(11W_0-9W_1+2W_2)G_{n+3}+(40W_1-47W_0-9W_2)G_{n+2}+(52W_0-47W_1+11W_2)G_{n+1}$.
(c) $4W_n=(8W_0-5W_1+W_2)G_{n+2}+(25W_1-36W_0-5W_2)G_{n+1}+4(11W_0-9W_1+2W_2)G_n$.
(d) $W_n=W_0G_{n+1}+(-5W_0+W_1)G_n+(8W_0-5W_1+W_2)G_{n-1}$.
(e) $W_n=W_1G_n+(-5W_1+W_2)G_{n-1}+4W_0G_{n-2}$.
(f) $4(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2G_n$
$=(8W_1^2-5W_1W_2-4W_0W_1+W_2^2)W_{n+4}+(-36W_1^2-5W_2^2+20W_0W_1-4W_0W_2+25W_1W_2)W_{n+3}+4(4W_0^2+16W_1^2+2W_2^2-16W_0W_1+5W_0W_2-11W_1W_2)W_{n+2}$.
(g) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2G_n$
$=(W_1^2-W_0W_2)W_{n+3}+(4W_0^2-8W_0W_1+5W_0W_2-W_1W_2)W_{n+2}+(8W_1^2+W_2^2-4W_0W_1-5W_1W_2)W_{n+1}$.
(h) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2G_n$
$=(4W_0^2+5W_1^2-8W_0W_1-W_1W_2)W_{n+2}+(W_2^2-4W_0W_1+8W_0W_2-5W_1W_2)W_{n+1}+4(W_1^2-W_0W_2)W_n$.
(i) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2G_n$
$=(20W_0^2+25W_1^2+W_2^2-44W_0W_1+8W_0W_2-10W_1W_2)W_{n+1}+4(-8W_0^2+16W_0W_1-W_2W_0-9W_1^2+2W_2W_1)W_n+4(4W_0^2+5W_1^2-8W_0W_1-W_1W_2)W_{n-1}$.
(j) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2G_n$
$=(68W_0^2+89W_1^2+5W_2^2-156W_0W_1+36W_0W_2-42W_1W_2)W_n+4(-36W_0^2+80W_0W_1-16W_0W_2-45W_1^2+19W_1W_2-2W_2^2)W_{n-1}+4(20W_0^2+25W_1^2+W_2^2-44W_0W_1+8W_0W_2-10W_1W_2)W_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n=a\times G_{n+4}+b\times G_{n+3}+c\times G_{n+2}$$

and solving the system of equations

$$\begin{aligned}W_0&=a\times G_4+b\times G_3+c\times G_2\\ W_1&=a\times G_5+b\times G_4+c\times G_3\\ W_2&=a\times G_6+b\times G_5+c\times G_4\end{aligned}$$

we find that $a=\frac{1}{16}(52W_0-47W_1+11W_2), b=-\frac{1}{16}(216W_0-199W_1+47W_2), c=\frac{1}{4}(57W_0-54W_1+13W_2)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma 4.1 can be proved by induction as well.
Next, we present a few basic relations between $\{W_n\}$ and $\{H_n\}$.

Lemma 4.2. *The following equalities are true:*

(a) $2(2W_0-3W_1+W_2)(4W_0-4W_1+W_2)H_n=(8W_0-10W_1+3W_2)W_{n+4}+(36W_1-28W_0-11W_2)W_{n+3}+2(12W_0-16W_1+5W_2)W_{n+2}$.
(b) $(2W_0-3W_1+W_2)(4W_0-4W_1+W_2)H_n=(6W_0-7W_1+2W_2)W_{n+3}+(24W_1-20W_0-7W_2)W_{n+2}+2(8W_0-10W_1+3W_2)W_{n+1}$.
(c) $(2W_0-3W_1+W_2)(4W_0-4W_1+W_2)H_n=(10W_0-11W_1+3W_2)W_{n+2}+2(18W_1-16W_0-5W_2)W_{n+1}+4(6W_0-7W_1+2W_2)W_n$.
(d) $(2W_0-3W_1+W_2)(4W_0-4W_1+W_2)H_n=(18W_0-19W_1+5W_2)W_{n+1}+4(15W_1-14W_0-4W_2)W_n+4(10W_0-11W_1+3W_2)W_{n-1}$.
(e) $(2W_0-3W_1+W_2)(4W_0-4W_1+W_2)H_n=(34W_0-35W_1+9W_2)W_n+4(27W_1-26W_0-7W_2)W_{n-1}+4(18W_0-19W_1+5W_2)W_{n-2}$.

Now, we give a few basic relations between $\{W_n\}$ and $\{R_n\}$.

Lemma 4.3. *The following equalities are true:*

(a) $8W_n=(42W_1-39W_0-11W_2)R_{n+4}+(151W_0-161W_1+42W_2)R_{n+3}+(151W_1-144W_0-39W_2)R_{n+2}$.
(b) $8W_n=(49W_1-44W_0-13W_2)R_{n+3}+(168W_0-185W_1+49W_2)R_{n+2}+4(42W_1-39W_0-11W_2)R_{n+1}$.
(c) $2W_n=(15W_1-13W_0-4W_2)R_{n+2}+(49W_0-56W_1+15W_2)R_{n+1}+(49W_1-44W_0-13W_2)R_n$.
(d) $2W_n=(19W_1-16W_0-5W_2)R_{n+1}+(60W_0-71W_1+19W_2)R_n+4(15W_1-13W_0-4W_2)R_{n-1}$.
(e) $W_n=(12W_1-10W_0-3W_2)R_n+2(19W_0-23W_1+6W_2)R_{n-1}+2(19W_1-16W_0-5W_2)R_{n-2}$.
(f) $2(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2R_n$
$=(-12W_0^2+36W_0W_1-13W_0W_2-26W_1^2+18W_1W_2-3W_2^2)W_{n+4}+(52W_0^2+108W_1^2+12W_2^2-152W_0W_1+53W_0W_2-73W_1W_2)W_{n+3}+(-48W_0^2+140W_0W_1-48W_0W_2-100W_1^2+67W_1W_2-11W_2^2)W_{n+2}$.
(g) $2(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2R_n$
$=(-8W_0^2+28W_0W_1-12W_0W_2-22W_1^2+17W_1W_2-3W_2^2)W_{n+3}+(48W_0^2+108W_1^2+13W_2^2-148W_0W_1+56W_0W_2-77W_1W_2)W_{n+2}+4(-12W_0^2+36W_0W_1-13W_0W_2-26W_1^2+18W_1W_2-3W_2^2)W_{n+1}$.
(h) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2R_n$
$=(4W_0^2-W_1^2-W_2^2-4W_0W_1-2W_0W_2+4W_1W_2)W_{n+2}+2(4W_0^2+18W_1^2+3W_2^2-20W_0W_1+11W_0W_2-16W_1W_2)W_{n+1}+2(-8W_0^2+28W_0W_1-12W_0W_2-22W_1^2+17W_1W_2-3W_2^2)W_n$.
(i) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2R_n$
$=(28W_0^2+31W_1^2+W_2^2-60W_0W_1+12W_0W_2-12W_1W_2)W_{n+1}+2(-24W_0^2+44W_0W_1-4W_0W_2-18W_1^2+W_1W_2+W_2^2)W_n+4(4W_0^2-W_1^2-W_2^2-4W_0W_1-2W_0W_2+4W_1W_2)W_{n-1}$.

(j) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2R_n$
$=(92W_0^2+119W_1^2+7W_2^2-212W_0W_1+52W_0W_2-58W_1W_2)W_n+4(-52W_0^2+116W_0W_1-26W_0W_2-63W_1^2+28W_1W_2-3W_2^2)W_{n-1}+4(28W_0^2+31W_1^2+W_2^2-60W_0W_1+12W_0W_2-12W_1W_2)W_{n-2}.$

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 4.4. *The following equalities are true:*

(a) $8W_n=(25W_0-22W_1+5W_2)C_{n+4}+(95W_1-105W_0-22W_2)C_{n+3}+(112W_0-105W_1+25W_2)C_{n+2}.$
(b) $8W_n=(20W_0-15W_1+3W_2)C_{n+3}+(71W_1-88W_0-15W_2)C_{n+2}+4(25W_0-22W_1+5W_2)C_{n+1}.$
(c) $2W_n=(3W_0-W_1)C_{n+2}+(8W_1-15W_0-W_2)C_{n+1}+(20W_0-15W_1+3W_2)C_n.$
(d) $2W_n=(3W_1-W_2)C_{n+1}+(3W_2-7W_1-4W_0)C_n+4(3W_0-W_1)C_{n-1}.$
(e) $W_n=(4W_1-2W_0-W_2)C_n+2(3W_0-7W_1+2W_2)C_{n-1}+2(3W_1-W_2)C_{n-2}.$
(f) $2(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(4W_0^2+10W_1^2+W_2^2-12W_0W_1+3W_0W_2-6W_1W_2)W_{n+4}+(-12W_0^2+40W_0W_1-11W_0W_2-36W_1^2+23W_1W_2-4W_2^2)W_{n+3}+(16W_0^2+44W_1^2+5W_2^2-52W_0W_1+16W_0W_2-29W_1W_2)W_{n+2}.$
(g) $2(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(8W_0^2+14W_1^2+W_2^2-20W_0W_1+4W_0W_2-7W_1W_2)W_{n+3}+(-16W_0^2+44W_0W_1-8W_0W_2-36W_1^2+19W_1W_2-3W_2^2)W_{n+2}+4(4W_0^2+10W_1^2+W_2^2-12W_0W_1+3W_0W_2-6W_1W_2)W_{n+1}.$
(h) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(12W_0^2+17W_1^2+W_2^2-28W_0W_1+6W_0W_2-8W_1W_2)W_{n+2}+2(-12W_0^2+28W_0W_1-5W_0W_2-18W_1^2+8W_1W_2-W_2^2)W_{n+1}+2(8W_0^2+14W_1^2+W_2^2-20W_0W_1+4W_0W_2-7W_1W_2)W_n.$
(i) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(36W_0^2+49W_1^2+3W_2^2-84W_0W_1+20W_0W_2-24W_1W_2)W_{n+1}+2(-40W_0^2+92W_0W_1-20W_0W_2-54W_1^2+25W_1W_2-3W_2^2)W_n+4(12W_0^2+17W_1^2+W_2^2-28W_0W_1+6W_0W_2-8W_1W_2)W_{n-1}.$
(j) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(100W_0^2+137W_1^2+9W_2^2-236W_0W_1+60W_0W_2-70W_1W_2)W_n+4(-60W_0^2+140W_0W_1-34W_0W_2-81W_1^2+40W_1W_2-5W_2^2)W_{n-1}+4(36W_0^2+49W_1^2+3W_2^2-84W_0W_1+20W_0W_2-24W_1W_2)W_{n-2}.$

Now, we give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 4.5. *The following equalities are true:*

$$\begin{aligned}
4H_n &= 5G_{n+4}-19G_{n+3}+18G_{n+2},\\
2H_n &= 3G_{n+3}-11G_{n+2}+10G_{n+1},\\
H_n &= 2G_{n+2}-7G_{n+1}+6G_n,\\
H_n &= 3G_{n+1}-10G_n+8G_{n-1},\\
H_n &= 5G_n-16G_{n-1}+12G_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{G_n\}$ and $\{R_n\}$.

Lemma 4.6. *The following equalities are true:*

$$\begin{aligned}
8G_n &= -13R_{n+4}+49R_{n+3}-44R_{n+2},\\
2G_n &= -4R_{n+3}+15R_{n+2}-13R_{n+1},\\
2G_n &= -5R_{n+2}+19R_{n+1}-16R_n,\\
G_n &= -3R_{n+1}+12R_n-10R_{n-1},\\
G_n &= -3R_n+14R_{n-1}-12R_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
8R_n &= -11G_{n+4}+43G_{n+3}-40G_{n+2},\\
2R_n &= -3G_{n+3}+12G_{n+2}-11G_{n+1},\\
2R_n &= -3G_{n+2}+13G_{n+1}-12G_n,\\
R_n &= -G_{n+1}+6G_n-6G_{n-1},\\
R_n &= G_n+2G_{n-1}-4G_{n-2}.
\end{aligned}$$

Now, we give a few basic relations between $\{G_n\}$ and $\{C_n\}$.

Lemma 4.7. *The following equalities are true:*

$$\begin{aligned}
8G_n &= 3C_{n+4}-15C_{n+3}+20C_{n+2},\\
2G_n &= -C_{n+2}+3C_{n+1},\\
G_n &= -C_{n+1}+4C_n-2C_{n-1},\\
G_n &= -C_n+6C_{n-1}-4C_{n-2},
\end{aligned}$$

and

$$
\begin{aligned}
8C_n &= 5G_{n+4} - 21G_{n+3} + 24G_{n+2},\\
2C_n &= G_{n+3} - 4G_{n+2} + 5G_{n+1},\\
2C_n &= G_{n+2} - 3G_{n+1} + 4G_n,\\
C_n &= G_{n+1} - 2G_n + 2G_{n-1},\\
C_n &= 3G_n - 6G_{n-1} + 4G_{n-2}.
\end{aligned}
$$

Next, we present a few basic relations between $\{H_n\}$ and $\{R_n\}$.

Lemma 4.8. *The following equalities are true:*

$$
\begin{aligned}
4H_n &= -3R_{n+4} + 13R_{n+3} - 14R_{n+2},\\
2H_n &= -R_{n+3} + 5R_{n+2} - 6R_{n+1},\\
H_n &= R_{n+1} - 2R_n,\\
H_n &= 3R_n - 8R_{n-1} + 4R_{n-2}.
\end{aligned}
$$

Now, we give a few basic relations between $\{H_n\}$ and $\{C_n\}$.

Lemma 4.9. *The following equalities are true:*

$$
\begin{aligned}
4H_n &= 5C_{n+4} - 19C_{n+3} + 18C_{n+2},\\
2H_n &= 3C_{n+3} - 11C_{n+2} + 10C_{n+1},\\
H_n &= 2C_{n+2} - 7C_{n+1} + 6C_n,\\
H_n &= 3C_{n+1} - 10C_n + 8C_{n-1},\\
H_n &= 5C_n - 16C_{n-1} + 12C_{n-2}.
\end{aligned}
$$

Next, we present a few basic relations between $\{R_n\}$ and $\{C_n\}$.

Lemma 4.10. *The following equalities are true:*

$$
\begin{aligned}
4R_n &= -6C_{n+4} + 23C_{n+3} - 21C_{n+2},\\
4R_n &= -7C_{n+3} + 27C_{n+2} - 24C_{n+1},\\
R_n &= -2C_{n+2} + 8C_{n+1} - 7C_n,\\
R_n &= -2C_{n+1} + 9C_n - 8C_{n-1},\\
R_n &= -C_n + 8C_{n-1} - 8C_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
4C_n &= -6R_{n+4} + 23R_{n+3} - 21R_{n+2},\\
4C_n &= -7R_{n+3} + 27R_{n+2} - 24R_{n+1},\\
C_n &= -2R_{n+2} + 8R_{n+1} - 7R_n,\\
C_n &= -2R_{n+1} + 9R_n - 8R_{n-1},\\
C_n &= -R_n + 8R_{n-1} - 8R_{n-2}.
\end{aligned}
$$

5. On the Recurrence Properties of Generalized Woodall Sequence

Taking $r = 5, s = -8, t = 4$ in Theorem 1.6, we obtain the following Proposition.

Proposition 5.1. *For $n \in \mathbb{Z}$, generalized Woodall numbers (the case $r = 5, s = -8, t = 4$) have the following identity:*

$$W_{-n} = 4^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0)$$

where

$$H_n = \frac{((10W_0 - 11W_1 + 3W_2)W_{n+2} - 2(16W_0 - 18W_1 + 5W_2)W_{n+1} + 4(6W_0 - 7W_1 + 2W_2)W_n)}{(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)} \tag{5.1}$$

Note that if we take $r = 5, s = -8, t = 4$ in Lemma 1.7 (or using Lemma 4.2 (c)) we get (5.1).
From the above Proposition 5.1 and Corollary 1.8, we have the following Corollary 5.2 which gives the connection between the special cases of generalized Woodall sequence at the positive index and the negative index: for modified Woodall, modified Cullen, Woodall and Cullen numbers: take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 5$, take $W_n = H_n$ with $H_0 = 3, H_1 = 5, H_2 = 9$, $W_n = R_n$ with $R_0 = -1, R_1 = 1, R_2 = 7$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3, C_2 = 9$, respectively. Note that in this case $H_n = H_n$.

Corollary 5.2. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Modified Woodall sequence:*

$$G_{-n} = 4^{-n}(-6G_n^2 + G_{2n} - 2G_{n+2}G_n + 7G_{n+1}G_n).$$

(b) *Modified Cullen sequence:*

$$H_{-n}=2^{-2n-1}\left(H_n^2-H_{2n}\right).$$

(c) *Woodall sequence:*

$$R_{-n}=2^{-2n-1}(-R_{n+1}^2+R_{2n+1}+2R_{n+1}R_n).$$

(d) *Cullen sequence:*

$$C_{-n}=2^{-2n-1}(4C_{n+2}^2+49C_{n+1}^2+24C_n^2-2C_{2n+2}+7C_{2n+1}-4C_{2n}-28C_{n+1}C_{n+2}+20C_nC_{n+2}-70C_nC_{n+1}).$$

6. Sum Formulas

The following Theorem 6.1 presents some formulas of of generalized Woodall numbers numbers with indices in arithmetic progression.

Theorem 6.1. *For all integers m and j, we have the following sum formula:*

$$\sum_{k=0}^{n}W_{mk+j}=\frac{1}{2\left(2^m-1\right)^2}(\Gamma_1+\Gamma_2+\Gamma_3)$$

where

$$\begin{aligned}\Gamma_1 &= ((j+mn-2)2^{mn+2m+j}-(j+m+mn-2)2^{mn+m+j}+(m-j+2)2^{m+j}+(j-2)2^j+2(n+1)(2^m-1)^2)W_2,\\ \Gamma_2 &= (-(3j+3mn-8)2^{mn+2m+j}+(3j+3m+3mn-8)2^{mn+m+j}+(3j-3m-8)2^{m+j}-(3j-8)2^j-8(n+1)(2^m-1)^2)W_1,\\ \Gamma_3 &= 2((j+mn-3)2^{mn+2m+j}-(j+m+mn-3)2^{mn+m+j}+(m-j+3)2^{m+j}+(j-3)2^j+4(n+1)(2^m-1)^2)W_0.\end{aligned}$$

Proof. Use the Binet's formula of generalized Woodall numbers, i.e.,

$$W_n=((-W_2+4W_1-3W_0)+\frac{W_2-3W_1+2W_0}{2}n)\times 2^n+(W_2-4W_1+4W_0).\ \square$$

The following Proposition 6.2 presents some formulas of generalized Woodall numbers numbers with positive subscripts.

Proposition 6.2. *For $n\geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n}W_k=((n-3)2^n+n+3)W_2-((3n-11)2^n+4n+11)W_1+((n-4)2^{n+1}+4n+9)W_0$.
(b) $\sum_{k=0}^{n}W_{2k}=\frac{1}{9}(((3n-4)2^{2n+2}+9n+16)W_2-12((3n-5)2^{2n}+3n+5)W_1+((6n-11)2^{2n+2}+36n+53)W_0)$.
(c) $\sum_{k=0}^{n}W_{2k+1}=\frac{1}{9}(((6n-5)2^{2n+2}+9n+20)W_2-3((6n-7)2^{2n+2}+12n+25)W_1+4((3n-4)2^{2n+2}+9n+16)W_0)$.

Proof. Take $m=1,j=0$; $m=2,j=0$ and $m=2,j=1$, respectively, in Theorem 6.1. $\square$
From Theorem 6.1, we have the following Corollary.

Corollary 6.3. *For all integers m and j, we have the following sum formulas:*

(a) $\sum_{k=0}^{n}G_{mk+j}=\frac{1}{(2^m-1)^2}((j+mn-1)2^{mn+2m+j}-(j+m+mn-1)2^{mn+m+j}+(n+1)2^{2m}-(n+1)2^{m+1}-(j-m-1)2^{m+j}$
$+(j-1)2^j+n+1)$.
(b) $\sum_{k=0}^{n}H_{mk+j}=\frac{1}{(2^m-1)}(2^{mn+m+j+1}+(n+1)2^m-2^{j+1}-n-1)$.
(c) $\sum_{k=0}^{n}R_{mk+j}=\frac{1}{(2^m-1)^2}((j+mn)2^{mn+2m+j}-(j+m+mn)2^{mn+m+j}-(n+1)2^{2m}+(n+1)2^{m+1}+(m-j)2^{m+j}+2^jj-n-1)$.
(d) $\sum_{k=0}^{n}C_{mk+j}=\frac{1}{(2^m-1)^2}((j+mn)2^{mn+2m+j}-(j+m+mn)2^{mn+m+j}+(n+1)2^{2m}-(n+1)2^{m+1}+(m-j)2^{m+j}+2^jj+n+1)$.

From the last Proposition 6.2 (or using Corollary 6.3), we have the following Corollary 6.4 which gives sum formulas of modified Woodall numbers (take $W_n=G_n$ with $G_0=0,G_1=1,G_2=5$).

Corollary 6.4. *For $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}G_k=(n-2)2^{n+1}+n+4$.
(b) $\sum_{k=0}^{n}G_{2k}=\frac{1}{9}((6n-5)2^{2n+2}+9n+20)$.
(c) $\sum_{k=0}^{n}G_{2k+1}=\frac{1}{9}((3n-1)2^{2n+4}+9n+25)$.

Taking $W_n=H_n$ with $H_0=3,H_1=5,H_2=9$ in the last Proposition 6.2 (or using Corollary 6.3), we have the following Corollary 6.5 which presents sum formulas of modified Cullen numbers.

Corollary 6.5. *For $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n}H_k=2^{n+2}+n-1$.
(b) $\sum_{k=0}^{n}H_{2k}=\frac{1}{3}(2^{2n+3}+3n+1)$.
(c) $\sum_{k=0}^{n}H_{2k+1}=\frac{1}{3}(2^{2n+4}+3n-1)$.

From the last Proposition 6.2 (or using Corollary 6.3), we have the following Corollary 6.6 which gives sum formulas of Woodall numbers (take $W_n=R_n$ with $R_0=-1,R_1=1,R_2=7$).

Corollary 6.6. *For $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} R_k = (n-1)(2^{n+1}-1)$.
(b) $\sum_{k=0}^{n} R_{2k} = \frac{1}{9}((3n-1)2^{2n+3}-9n-1)$.
(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3}-9n+1)$.

Taking $W_n = C_n$ with $C_0 = 1, C_1 = 3, C_2 = 9$ in the last Proposition 6.2, we have the following Corollary 6.7 which presents sum formulas of Cullen numbers.

Corollary 6.7. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} C_k = (n-1)2^{n+1}+n+3$.
(b) $\sum_{k=0}^{n} C_{2k} = \frac{1}{9}((3n-1)2^{2n+3}+9n+17)$.
(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3}+9n+19)$.

7. Matrices Related With Generalized Woodall numbers

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 4$. From (2.1) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and from (1.6) (or using (7.1) and induction) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = G$ in (7.1) we have

$$\begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -8W_n + 4W_{n-1} & 4W_n \\ W_n & -8W_{n-1} + 4W_{n-2} & 4W_{n-1} \\ W_{n-1} & -8W_{n-2} + 4W_{n-3} & 4W_{n-2} \end{pmatrix}$$

Theorem 7.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$
(b) $C_1 A^n = A^n C_1$
(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 5, s = -8, t = 4$ in Soykan [25, Theorem 5.1.]. □
Some properties of matrix A^n can be given as

$$A^n = 5A^{n-1} - 8A^{n-2} + 4A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 4^n$$

for all integer m and n.

Corollary 7.2. *For all integers n, we have the following formulas for the modified Woodall, Woodall and Cullen numbers.*

(a) *Modified Woodall Numbers.*

$$A^{n}=\begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n}=\begin{pmatrix} G_{n+1} & -8G_{n}+4G_{n-1} & 4G_{n} \\ G_{n} & -8G_{n-1}+4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2}+4G_{n-3} & 4G_{n-2} \end{pmatrix}$$
$$=\begin{pmatrix} 2^{n+1}n+1 & 4\times 2^{n}-6\times 2^{n}n-4 & 4\times 2^{n}n-4\times 2^{n}+4 \\ 2^{n}n-2^{n}+1 & 5\times 2^{n}-3\times 2^{n}n-4 & 2\times 2^{n}n-4\times 2^{n}+4 \\ \frac{1}{2}2^{n}n-2^{n}+1 & 2^{n+2}-\frac{3}{2}2^{n}n-4 & 2^{n}n-3\times 2^{n}+4 \end{pmatrix}.$$

(b) *Woodall Numbers.*

$$A^{n}=\frac{1}{2}\begin{pmatrix} -5R_{n+3}+19R_{n+2}-16R_{n+1} & 24R_{n+2}-92R_{n+1}+76R_{n} & 4(-5R_{n+2}+19R_{n+1}-16R_{n}) \\ -5R_{n+2}+19R_{n+1}-16R_{n} & 24R_{n+1}-92R_{n}+76R_{n-1} & 4(-5R_{n+1}+19R_{n}-16R_{n-1}) \\ -5R_{n+1}+19R_{n}-16R_{n-1} & 24R_{n}-92R_{n-1}+76R_{n-2} & 4(-5R_{n}+19R_{n-1}-16R_{n-2}) \end{pmatrix}.$$

(c) *Cullen Numbers.*

$$A^{n}=\begin{pmatrix} -C_{n+2}+4C_{n+1}-2C_{n} & 6C_{n+1}-26C_{n}+16C_{n-1} & 4(-C_{n+1}+4C_{n}-2C_{n-1}) \\ -C_{n+1}+4C_{n}-2C_{n-1} & 6C_{n}-26C_{n-1}+16C_{n-2} & 4(-C_{n}+4C_{n-1}-2C_{n-2}) \\ -C_{n}+4C_{n-1}-2C_{n-2} & 6C_{n-1}-26C_{n-2}+16C_{n-3} & 4(-C_{n-1}+4C_{n-2}-2C_{n-3}) \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 7.1 (a).
(b) Note that, from Lemma 4.6, we have

$$2G_{n}=-5R_{n+2}+19R_{n+1}-16R_{n}.$$

Using the last equation and (a), we get required result.
(c) Note that, from Lemma 4.7, we have

$$G_{n}=-C_{n+1}+4C_{n}-2C_{n-1}.$$

Using the last equation and (a), we get required result. □

Theorem 7.3. *For all integers m,n, we have*

$$\begin{aligned} W_{n+m} &= W_{n}G_{m+1}+W_{n-1}(-8G_{m}+4G_{m-1})+4W_{n-2}G_{m} \\ &= W_{n}G_{m+1}+(-8W_{n-1}+4W_{n-2})G_{m}+4W_{n-1}G_{m-1} \end{aligned} \tag{7.2}$$

Proof. Take $r=5, s=-8, t=4$ in Soykan [25, Theorem 5.2.]. □
By Lemma 4.1, we know that

$$\begin{aligned}(4W_{0}-4W_{1}+W_{2})(2W_{0}-3W_{1}+W_{2})^{2}G_{m} &= (4W_{0}^{2}+5W_{1}^{2}-8W_{0}W_{1}-W_{1}W_{2})W_{m+2} \\ &\quad +(W_{2}^{2}-4W_{0}W_{1}+8W_{0}W_{2}-5W_{1}W_{2})W_{m+1}+4(W_{1}^{2}-W_{0}W_{2})W_{m}.\end{aligned}$$

so (7.2) can be written in the following form
$(4W_{0}-4W_{1}+W_{2})(2W_{0}-3W_{1}+W_{2})^{2}W_{n+m}=W_{n}((4W_{0}^{2}+5W_{1}^{2}-8W_{0}W_{1}-W_{1}W_{2})W_{m+3}+(W_{2}^{2}-4W_{0}W_{1}+8W_{0}W_{2}-5W_{1}W_{2})W_{m+2}+4(W_{1}^{2}-W_{0}W_{2})W_{m+1})+(-8W_{n-1}+4W_{n-2})((4W_{0}^{2}+5W_{1}^{2}-8W_{0}W_{1}-W_{1}W_{2})W_{m+2}$
$+(W_{2}^{2}-4W_{0}W_{1}+8W_{0}W_{2}-5W_{1}W_{2})W_{m+1}+4(W_{1}^{2}-W_{0}W_{2})W_{m})+4W_{n-1}((4W_{0}^{2}+5W_{1}^{2}-8W_{0}W_{1}-W_{1}W_{2})W_{m+1}+(W_{2}^{2}-4W_{0}W_{1}+8W_{0}W_{2}-5W_{1}W_{2})W_{m}+4(W_{1}^{2}-W_{0}W_{2})W_{m-1})$.

Corollary 7.4. *For all integers m,n, we have*

$$\begin{aligned} G_{n+m} &= G_{n}G_{m+1}+G_{n-1}(-8G_{m}+4G_{m-1})+4G_{n-2}G_{m}, \\ H_{n+m} &= H_{n}G_{m+1}+H_{n-1}(-8G_{m}+4G_{m-1})+4H_{n-2}G_{m}, \\ R_{n+m} &= R_{n}G_{m+1}+R_{n-1}(-8G_{m}+4G_{m-1})+4R_{n-2}G_{m}, \\ C_{n+m} &= C_{n}G_{m+1}+C_{n-1}(-8G_{m}+4G_{m-1})+4C_{n-2}G_{m}, \end{aligned}$$

and

$$\begin{aligned} 2R_{m+n} &= -5R_{n}R_{m+3}+(19R_{n}+40R_{n-1}-20R_{n-2})R_{m+2} \\ &\quad +4(-4R_{n}-43R_{n-1}+19R_{n-2})R_{m+1}+4(51R_{n-1}-16R_{n-2})R_{m}-64R_{n-1}R_{m-1}, \\ 2C_{m+n} &= -C_{n}C_{m+3}+(3C_{n}+8C_{n-1}-4C_{n-2})C_{m+2} \\ &\quad +4(-7C_{n-1}+3C_{n-2})C_{m+1}+12C_{n-1}C_{m}. \end{aligned}$$

Taking $m=n$ in the last Corollary we obtain the following identities:

$$
\begin{aligned}
G_{2n} &= 4G_{n-1}^2 + (G_{n+1} - 8G_{n-1} + 4G_{n-2})G_n, \\
H_{2n} &= H_n G_{n+1} - 4\left(2H_{n-1} - H_{n-2}\right) G_n + 4H_{n-1}G_{n-1}, \\
R_{2n} &= R_n G_{n+1} - 4\left(2R_{n-1} - R_{n-2}\right) G_n + 4R_{n-1}G_{n-1}, \\
C_{2n} &= C_n G_{n+1} - 4\left(2C_{n-1} - C_{n-2}\right) G_n + 4C_{n-1}G_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
2R_{2n} &= -5R_nR_{n+3} + (19R_n + 40R_{n-1} - 20R_{n-2})R_{n+2} + 4(-4R_n - 43R_{n-1} + 19R_{n-2})R_{n+1} \\
&\quad + 4(51R_{n-1} - 16R_{n-2})R_n - 64R_{n-1}R_{n-1}, \\
2C_{2n} &= -C_nC_{n+3} + (3C_n + 8C_{n-1} - 4C_{n-2})C_{n+2} + 4(-7C_{n-1} + 3C_{n-2})C_{n+1} + 12C_{n-1}C_n.
\end{aligned}
$$

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