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Theory of Generalized Compactness in Generalized Topological Spaces: Part II. Countable, Sequential and Local Properties

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Abstract: In a recent paper, a novel class of generalized compact sets (briefly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compact sets*) in generalized topological spaces (briefly, $\mathscr{T}_{\mathfrak{g}}$-*spaces*) has been studied. In this paper, the concept is further studied and, other derived concepts called countable, sequential, and local generalized compactness (*countable, sequential, local* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*) in $\mathscr{T}_{\mathfrak{g}}$-spaces are also studied relatively. The study reveals that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness and countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, sequential $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness and, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is a generalized topological property (briefly, $\mathscr{T}_{\mathfrak{g}}$-*property*). Diagrams establish the various relationships amongst these types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness presented here and in the literature, and a nice application supports the overall theory.

Keywords: Generalized topological space ($\mathscr{T}_{\mathfrak{g}}$-*space*), generalized compactness ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*), countable generalized compactness (*countable* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*), sequential generalized compactness (*sequential* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*), local generalized compactness (*local* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*).

1. Introduction

Since the study of such fundamental topological invariants as ordinary and generalized compactness in ordinary and generalized topological spaces (briefly, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-*compactness in* $\mathscr{T}$-*spaces* and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness in* $\mathscr{T}_{\mathfrak{g}}$-*spaces*), a variety of weaker and stronger forms of $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$compactness in $\mathscr{T}$-spaces and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces have been introduced and investigated [1–3, 5–8, 13–19].

Bacon [2] studied a class of $\mathscr{T}$-spaces in which closed countably $\mathfrak{T}$-compact subsets are always $\mathfrak{T}$-compact. Butcher and Joseph [3] gave theorems embracing known characterizations of many of the $\mathfrak{g}$-$\mathfrak{T}$-compactness properties. El-Monsef et al. [6] generalized and studied the notions

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of $\mathfrak{T}$-compactness, para $\mathfrak{T}$-compactness, and many weak forms of such types of $\mathfrak{T}$-compactness. Greever [7] studied the extent to which Hausdorff $\mathscr{T}$-spaces with various combinations of $\mathfrak{T}$-compactness can exist, just to name a few.

Having studied a novel class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets in $\mathscr{T}_{\mathfrak{g}}$-spaces recently [12], it is proposed in this paper to advance the study a step further by studying other properties and other derived concepts called countable, sequential, local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces relatively.

The paper is organized as follows: In Section 2, preliminary notions are described in Subsection 2.1 and the main results of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathscr{T}_{\mathfrak{g}}$-space are reported in Section 3. In Section 4, the establishment of the relationships among various types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are discussed in Subsection 4.1. To support the work, a nice application of the concept of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Subsection 4.2. Finally, Subsection 4.3 provides concluding remarks and future directions of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathscr{T}_{\mathfrak{g}}$-space.

2. Theory

2.1. Preliminaries

Standard references for notations and concepts are [9–12]. The mathematical structures $\mathfrak{T} \stackrel{\text{def}}{=} (\Omega, \mathscr{T})$ and $\mathfrak{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{g}})$, respectively, are $\mathscr{T}$, $\mathscr{T}_{\mathfrak{g}}$-spaces [9], on both of which no separation axioms are assumed unless otherwise mentioned [4, 10]. A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ endowed with a $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\mathrm{H}}$-axiom is called a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} \stackrel{\text{def}}{=} \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$ [9–11]. The sets I_n^0, I_n^* and I_∞^0, I_∞^*, respectively, are finite and infinite index sets [9]. Sets of the class $\mathscr{T}_{\mathfrak{g}}$ and of its complement class $\neg\mathscr{T}_{\mathfrak{g}}$, respectively, are called $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets [9]. The class $\mathfrak{g}$-ν-$\mathrm{S}\big[\mathfrak{T}_{\mathfrak{g},\Lambda}\big] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}$-$\nu$-$\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category $\nu \in I_3^0$ (briefly, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-sets*) [9, 12]. Accordingly, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets [9] are

$$\mathfrak{g}\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \quad = \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}\left[\mathfrak{T}_{\mathfrak{g}}\right]. \tag{1}$$

Definition 2.1 **($(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-Map *[9]*)** *A map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ from a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ into a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ is called a $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map.*

Definition 2.2 **($\mathfrak{g}$-ν-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-Map *[9]*)** *Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be $\mathscr{T}_{\mathfrak{g}}$-spaces, and let $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathscr{L}_{\mathfrak{g}}[\Sigma]$. Then, $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is called a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map if and only if, for every $(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$ there corresponds $(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$ such* *that:*

$$\left[\pi_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right) \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right] \vee \left[\pi_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right]. \tag{2}$$

A $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-map is of category* ν *if and only if it is in the class of* $\mathfrak{g}$-ν-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-maps:*

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} :\ &\left(\forall \mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}\right)\left(\exists \mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \\ &\left[\left(\pi_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right) \vee \left(\pi_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right)\right]\big\}. \end{aligned}\tag{3}$$

Definition 2.3 *The classes of* $\mathfrak{g}$-ν-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-open and* $\mathfrak{g}$-ν-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-closed maps, respectively, are:*

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M_O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] &\stackrel{\text{def}}{=} \left\{\pi_{\mathfrak{g}} : \left(\forall \mathscr{O}_{\mathfrak{g},\omega}\right)\left(\exists \mathscr{O}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right)\left[\pi_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right]\right\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{M_K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] &\stackrel{\text{def}}{=} \left\{\pi_{\mathfrak{g}} : \left(\forall \mathscr{K}_{\omega}\right)\left(\exists \mathscr{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right)\left[\pi_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right]\right\}. \end{aligned}\tag{4}$$

Accordingly, the class of all $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-maps [9] are

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] &= \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] \\ &= \bigcup_{(\nu,\mathrm{E}) \in I_3^0 \times \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{M_E}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup_{\mathrm{E} \in \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{M_E}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \end{aligned}\tag{5}$$

Definition 2.4 ($\mathfrak{g}$-ν-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$**-Continuous** *[9]***)** *Let* $\mathfrak{T}_{\mathfrak{g},\Omega} = \left(\Omega, \mathscr{T}_{\mathfrak{g},\Omega}\right)$ *and* $\mathfrak{T}_{\mathfrak{g},\Sigma} = \left(\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma}\right)$ *be* $\mathscr{T}_{\mathfrak{g}}$*-spaces, and let* $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$*. Then,* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is said to be* $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-continuous if and only if, for every* $\left(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}\right) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$ *there corresponds* $\left(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}\right) \in \mathscr{T}_{\mathfrak{g},\Omega} \times \neg\mathscr{T}_{\mathfrak{g},\Omega}$ *such that:*

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right) \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right] \vee \left[\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right]. \tag{6}$$

A $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-continuous map is of category* ν *if and only if it is in the class of* $\mathfrak{g}$-ν-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-continuous maps:*

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} :\ &\left(\forall \mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}\right)\left(\exists \mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \\ &\left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right) \vee \left(\pi_{\mathfrak{g}}^{-1}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}. \end{aligned}\tag{7}$$

Definition 2.5 ($\mathfrak{g}$-ν-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$**-Irresolute** *[9]***)** *Let* $\mathfrak{T}_{\mathfrak{g},\Omega} = \left(\Omega, \mathscr{T}_{\mathfrak{g},\Omega}\right)$ *and* $\mathfrak{T}_{\mathfrak{g},\Sigma} = \left(\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma}\right)$ *be* $\mathscr{T}_{\mathfrak{g}}$*-spaces, and let* $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathscr{L}_{\mathfrak{g}}[\Omega]$*. Then,* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is said to be* $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma}\right)$*-irresolute if and only if, for every* $\left(\mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}\right) \in \mathscr{T}_{\mathfrak{g},\Sigma} \times \neg\mathscr{T}_{\mathfrak{g},\Sigma}$ *there corresponds* $\left(\mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}\right) \in$

$\mathscr{T}_{\mathfrak{g},\Omega} \times \neg \mathscr{T}_{\mathfrak{g},\Omega}$ *such that:*

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right) \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right] \vee \left[\pi_{\mathfrak{g}}^{-1}\left(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right]. \tag{8}$$

A $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-irresolute map is of category* ν *if and only if it is in the class of* $\mathfrak{g}$-ν-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-irresolute maps:*

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}}:\ &\left(\forall \mathscr{O}_{\mathfrak{g},\sigma}, \mathscr{K}_{\mathfrak{g},\sigma}\right)\left(\exists \mathscr{O}_{\mathfrak{g},\omega}, \mathscr{K}_{\mathfrak{g},\omega}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \\ &\left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\sigma}\right)\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{O}_{\mathfrak{g},\omega}\right)\right) \vee \left(\pi_{\mathfrak{g}}^{-1}\left(\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\sigma}\right)\right) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathscr{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}.\end{aligned} \tag{9}$$

The classes of $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous and $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute maps, respectively, are:

$$\mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right], \quad \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \tag{10}$$

By a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open set and a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed set are meant a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ and a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ satisfying $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ and $\mathscr{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$, respectively. Likewise, by a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open set of category ν and a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed set of category ν are meant a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ and a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ satisfying $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})$ and $\mathscr{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})$, respectively; $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-sets of category ν will be called $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}$-sets [9].

Given the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, $\mathscr{R}_{\mathfrak{g}}$ is said to be *equivalent* to $\mathscr{S}_{\mathfrak{g}}$, written $\mathscr{R}_{\mathfrak{g}} \sim \mathscr{S}_{\mathfrak{g}}$, if and only if, there exists a $\mathfrak{T}_{\mathfrak{g}}$-map $\pi_{\mathfrak{g}}: \mathscr{R}_{\mathfrak{g}} \longrightarrow \mathscr{S}_{\mathfrak{g}}$ which is bijective. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is *finite* if and only if $\mathscr{S}_{\mathfrak{g}} = \emptyset$ or $\mathscr{S}_{\mathfrak{g}} \sim I_{\mu}^{*}$ for some $\mu \in I_{\infty}^{*}$; otherwise, the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}$ is said to be *infinite*. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is *denumerable* and satisfies the condition $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) = \aleph_0$ (*aleph-null*) if and only if $\mathscr{S}_{\mathfrak{g}} \sim I_{\infty}^{*}$. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}}$ is called *countable* if and only if it is *finite* or *denumerable* [9].

The symbol $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ denotes a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets sequence of category ν in $\mathfrak{T}_{\mathfrak{g}}$ [9, 11]. The sequences $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$, $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$, and $\langle \mathscr{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$, respectively, are simply said to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-covering*, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open covering*, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed covering* of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ whose cardinality is at most $\sigma \in I_{\infty}^{*}$ if and only if the corresponding relations $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\alpha}$, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\alpha}$ and $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{V}_{\mathfrak{g},\alpha}$ hold true [9, 11]. The map

$$\vartheta : \langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}} \longrightarrow \langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \tag{11}$$

is said to realise a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering $\langle \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$ from the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathscr{S}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ if and only if $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{S}_{\mathfrak{g},\vartheta(\alpha)}$ [9, 11]. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a

$\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compact if and only if, for every $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$,

$$\exists\big\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\big\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} : \quad \mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}, \tag{12}$$

where $\vartheta(\sigma) = \mathrm{card}\big(I_{\vartheta(\sigma)}^{*}\big) \leq \mathrm{card}(I_{\sigma}^{*}) = \sigma$ [9, 11]. The class of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-compact sets is:

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{A}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \Big\{\mathscr{S}_{\mathfrak{g}} : \big[\forall\big\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\big\rangle_{\alpha\in I_{\sigma}^{*}}\big]\big[\exists\big\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\big\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\big]$$

$$\Big(\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\Big)\Big\}. \tag{13}$$

A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-refinement [9, 11] of another $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathscr{R}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\beta\in I_{\mu}^{*}}$ of the same $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}}$ if and only if:

$$\big(\forall\alpha \in I_{\sigma}^{*}\big)\big(\exists\beta \in I_{\mu}^{*}\big)\big[\mathscr{S}_{\mathfrak{g},\alpha} \subseteq \mathscr{R}_{\mathfrak{g},\beta}\big]. \tag{14}$$

Definition 2.6 ($\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-Space [9, 11]) *A $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space denoted $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} \stackrel{\mathrm{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ if and only if each $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{T}_{\mathfrak{g}}$ has a finite $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering.*

By $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} \stackrel{\mathrm{def}}{=} \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{CA}]}\big)$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]} \stackrel{\mathrm{def}}{=} \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{SA}]}\big)$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{LA}]} \stackrel{\mathrm{def}}{=} \big(\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{LA}]}\big)$, respectively, are meant *countably*, *sequentially*, and *locally* $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-spaces; by a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{E}]}\big)$ is meant $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = \bigvee_{\nu\in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = \big(\Omega, \bigvee_{\nu\in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{E}]}\big) = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{E}]}\big)$, where $\mathrm{E} \in \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$.

Definition 2.7 (Finite Intersection Property [9, 11]) *A sequence $\langle \mathscr{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets is said to have the "finite intersection property" if and only if every finite subsequence of the type $\langle \mathscr{S}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}$ has a non-empty intersection:*

$$\forall\big\langle \mathscr{S}_{\mathfrak{g},\beta(\alpha)}\big\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \prec \big\langle \mathscr{S}_{\mathfrak{g},\alpha}\big\rangle_{\alpha\in I_{\sigma}^{*}} : \bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}} \mathscr{S}_{\mathfrak{g},\beta(\alpha)} \neq \emptyset. \tag{15}$$

Definition 2.8 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Accumulation Point [9, 11]) *A point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point" (or "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-limit point", "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-cluster point", "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived point") of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of $\mathfrak{T}_{\mathfrak{g}}$ if and only if every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$*

containing ξ *(whether* $\xi \in \mathscr{S}_{\mathfrak{g}}$ *or* $\xi \notin \mathscr{S}_{\mathfrak{g}}$*) contains at least a point* $\zeta \in \mathscr{S}_{\mathfrak{g}} \setminus \{\xi\}$*:*

$$\xi \in \mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \;\Rightarrow\; \mathscr{S}_{\mathfrak{g}} \cap \left(\mathscr{U}_{\mathfrak{g},\xi} \setminus \{\xi\}\right) \neq \emptyset. \tag{16}$$

The set $\mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ *of all* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-accumulation points is called the "*$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived set of* $\mathscr{S}_{\mathfrak{g}}$*".*

Definition 2.9 (Countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Compact *[9, 11]***)** *A* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is said to be "countably* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact" if and only if every infinite* $\mathfrak{T}_{\mathfrak{g}}$*-subset* $\mathscr{R}_{\mathfrak{g}} \subset \mathscr{S}_{\mathfrak{g}}$ *of* $\mathscr{S}_{\mathfrak{g}}$ *has at least one* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-accumulation point* $\xi \in \mathscr{S}_{\mathfrak{g}}$*.*

Definition 2.10 (Sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact *[9, 11]***)** *A* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is "sequentially* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact" if and only if every sequence* $\langle \xi_{\alpha} \in \mathscr{S}_{\mathfrak{g}} \rangle_{\alpha \in I_{\infty}^{*}}$ *in* $\mathscr{S}_{\mathfrak{g}}$ *contains a subsequence* $\langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}} \prec \langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$ *which converges to a point* $\xi \in \mathscr{S}_{\mathfrak{g}}$*.*

Definition 2.11 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Neighborhood *[9, 11]***)** *Let* $\xi \in \mathfrak{T}_{\mathfrak{g}}$ *be a point in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. A* $\mathfrak{T}_{\mathfrak{g}}$*-subset* $\mathscr{N}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ *of* $\mathfrak{T}_{\mathfrak{g}}$ *is a "*$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-neighborhood of* ξ*" if and only if* $\mathscr{N}_{\mathfrak{g}}$ *is a* $\mathfrak{T}_{\mathfrak{g}}$*-superset of a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open set* $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ *containing* ξ*:*

$$\left(\xi, \mathscr{N}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g},\xi}\right) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]: \quad \xi \in \mathscr{U}_{\mathfrak{g},\xi} \subseteq \mathscr{N}_{\mathfrak{g}}. \tag{17}$$

The class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-neighborhoods of* $\xi \in \mathfrak{T}_{\mathfrak{g}}$*, defined as*

$$\mathfrak{g}\text{-N}[\xi] \stackrel{\text{def}}{=} \left\{\mathscr{N}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \; \left(\exists \mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\right)\left[\xi \in \mathscr{U}_{\mathfrak{g},\xi} \subseteq \mathscr{N}_{\mathfrak{g}}\right]\right\}, \tag{18}$$

is called the "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-neighborhood system of* ξ*".*

Definition 2.12 (Locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Compact *[9, 11]***)** *A* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ *of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is said to be "locally* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact" if and only if, given any* $\left(\xi, \mathscr{N}_{\mathfrak{g},\xi}\right) \in \mathscr{S}_{\mathfrak{g}} \times \mathfrak{g}\text{-N}[\xi]$*, there is a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-neighborhood* $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-N}[\xi]$ *of* ξ *such that* $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \subset \mathscr{N}_{\mathfrak{g},\xi}$ *and* $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}\left(\hat{\mathscr{N}}_{\mathfrak{g},\xi}\right) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$*.*

By omitting the subscript $\mathfrak{g}$ in almost all symbols of the above definitions, we obtain very similar definitions but in a $\mathscr{T}_{\Lambda}$-space; see [9, 11, 12].

3. Main Results

The main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are presented in this section.

Lemma 3.1 *If* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *is a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set of a* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ *and suppose* $\xi \notin \mathscr{S}_{\mathfrak{g}}$*, then there exists* $\left(\mathscr{U}_{\mathfrak{g},\alpha}, \mathscr{U}_{\mathfrak{g},\beta}\right) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ *such that* $\left(\{\xi\}, \mathscr{S}_{\mathfrak{g}}\right) \subseteq \left(\mathscr{U}_{\mathfrak{g},\alpha}, \mathscr{U}_{\mathfrak{g},\beta}\right)$ *and* $\bigcap_{\mu=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\mu} = \emptyset$*.*

Proof Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ and suppose $\xi \notin \mathscr{S}_{\mathfrak{g}}$. Since $\xi \notin \mathscr{S}_{\mathfrak{g}}$, it results that $\zeta \in \mathscr{S}_{\mathfrak{g}}$ implies $\xi \notin \{\zeta\}$. But by hypothesis, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ and therefore, there exists $\left(\mathscr{U}_{\mathfrak{g},\zeta}, \hat{\mathscr{U}}_{\mathfrak{g},\zeta}\right) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\left(\xi, \zeta\right) \in \mathscr{U}_{\mathfrak{g},\zeta} \times \hat{\mathscr{U}}_{\mathfrak{g},\zeta}$ and $\mathscr{U}_{\mathfrak{g},\zeta} \cap \hat{\mathscr{U}}_{\mathfrak{g},\zeta} = \emptyset$. Hence, it follows that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\zeta \in \mathscr{S}_{\mathfrak{g}}} \hat{\mathscr{U}}_{\mathfrak{g},\zeta}$, meaning that $\langle \hat{\mathscr{U}}_{\mathfrak{g},\zeta} \rangle_{\zeta \in \mathscr{S}_{\mathfrak{g}}}$ is a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g}}$. But $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, there exists $\langle \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)} \rangle_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \prec \langle \hat{\mathscr{U}}_{\mathfrak{g},\zeta} \rangle_{\zeta \in \mathscr{S}_{\mathfrak{g}}}$ such that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)}$. Now let

$$\mathscr{U}_{\mathfrak{g},\alpha} = \bigcap_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\zeta(\mu)}, \quad \mathscr{U}_{\mathfrak{g},\beta} = \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)}.$$

It is evidently that, $\left(\mathscr{U}_{\mathfrak{g},\alpha}, \mathscr{U}_{\mathfrak{g},\beta}\right) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, since $\left(\mathscr{U}_{\mathfrak{g},\zeta(\mu)}, \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)}\right) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}$. Furthermore, $\left(\{\xi\}, \mathscr{S}_{\mathfrak{g}}\right) \subseteq \left(\mathscr{U}_{\mathfrak{g},\alpha}, \mathscr{U}_{\mathfrak{g},\beta}\right)$, since $\xi \in \mathscr{U}_{\mathfrak{g},\zeta(\mu)}$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}$. Lastly, let it be claimed that $\bigcap_{\mu=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\mu} = \emptyset$. Then, $\mathscr{U}_{\mathfrak{g},\zeta(\mu)} \cap \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)} = \emptyset$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}$ which, in turn, implies that $\mathscr{U}_{\mathfrak{g},\alpha} \cap \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)} = \emptyset$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}$. Hence,

$$\begin{aligned} \bigcap_{\mu=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\mu} = \mathscr{U}_{\mathfrak{g},\alpha} \cap \left(\bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)} \right) &= \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \left(\mathscr{U}_{\mathfrak{g},\alpha} \cap \hat{\mathscr{U}}_{\mathfrak{g},\zeta(\mu)}\right) \\ &= \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}}} \emptyset = \emptyset. \end{aligned}$$

This completes the proof of the lemma. □

Theorem 3.2 *Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$. If $\xi \notin \mathscr{S}_{\mathfrak{g}}$, then there exists a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\xi \in \mathscr{U}_{\mathfrak{g}} \subseteq \complement(\mathscr{S}_{\mathfrak{g}})$.*

Proof Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ and suppose $\xi \notin \mathscr{S}_{\mathfrak{g}}$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$, there exists then $\left(\mathscr{U}_{\mathfrak{g}}, \hat{\mathscr{U}}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\left(\{\xi\}, \mathscr{S}_{\mathfrak{g}}\right) \subseteq \left(\mathscr{U}_{\mathfrak{g}}, \hat{\mathscr{U}}_{\mathfrak{g}}\right)$ and $\mathscr{U}_{\mathfrak{g}} \cap \hat{\mathscr{U}}_{\mathfrak{g}} = \emptyset$. Hence, $\mathscr{U}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \emptyset$ and consequently, $\xi \in \mathscr{U}_{\mathfrak{g}} \subseteq \complement(\mathscr{S}_{\mathfrak{g}})$. This proves the theorem. □

Proposition 3.3 *Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$, then $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$.*

Proof Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})})$. It must be proved that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ which is equivalent to prove that $\complement(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. Let $\xi \in \complement(\mathscr{S}_{\mathfrak{g}})$; that is, $\xi \notin \mathscr{S}_{\mathfrak{g}}$. Since $\xi \notin \mathscr{S}_{\mathfrak{g}}$ there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\xi \in \mathscr{U}_{\mathfrak{g},\xi} \subseteq \complement(\mathscr{S}_{\mathfrak{g}})$. Consequently, $\complement(\mathscr{S}_{\mathfrak{g}}) = \bigcup_{\xi \in \complement(\mathscr{S}_{\mathfrak{g}})} \mathscr{U}_{\mathfrak{g},\xi}$. Therefore, $\complement(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, since $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\xi \in \complement(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. This proves the proposition. □

Lemma 3.4 *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-space whose $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is cofinite on Ω, then $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$.*

Proof Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space whose $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is cofinite on Ω and suppose $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open covering of Ω. Then, $\complement(\mathscr{U}_{\mathfrak{g},\alpha}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ for any chosen $\alpha \in I_{\sigma}^{*}$. Furthermore, since $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is cofinite on Ω, $\mathscr{U}_{\mathfrak{g},\alpha}$, it follows that, for every $\alpha \in I_{\sigma}^{*}$, $\complement(\mathscr{U}_{\mathfrak{g},\alpha})$ is a finite $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-closed set. Set $\complement(\mathscr{U}_{\mathfrak{g},\alpha}) = \left\{ \xi_{\beta(\alpha)} : (\alpha, \beta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*} \right\}$. Since $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open covering of Ω, for every $(\alpha, \beta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$, $\xi_{\beta(\alpha)} \in \complement(\mathscr{U}_{\mathfrak{g},\alpha})$ implies the existence of $\mathscr{U}_{\mathfrak{g},\gamma(\alpha)}$, where $\langle \mathscr{U}_{\mathfrak{g},\gamma(\alpha)} \rangle_{(\alpha,\gamma(\alpha)) \in I_{\sigma}^{*} \times I_{\gamma(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$, satisfying $\xi_{\beta(\alpha)} \in \mathscr{U}_{\mathfrak{g},\gamma(\alpha)}$. Hence, it follows that $\complement(\mathscr{U}_{\mathfrak{g},\alpha}) \subseteq \bigcup_{(\alpha,\gamma(\alpha)) \in I_{\sigma}^{*} \times I_{\gamma(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\gamma(\alpha)}$ and therefore,

$$\Omega = \mathscr{U}_{\mathfrak{g},\alpha} \cup \complement(\mathscr{U}_{\mathfrak{g},\alpha}) = \mathscr{U}_{\mathfrak{g},\alpha} \cup \Bigg(\bigcup_{(\alpha,\gamma(\alpha)) \in I_{\sigma}^{*} \times I_{\gamma(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\gamma(\alpha)} \Bigg).$$

Thus, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$. This completes the proof of the lemma. □

Theorem 3.5 *If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ is a pair of disjoint $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-compact sets of a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})})$, then there exists a pair $(\mathscr{U}_{\mathfrak{g},\alpha}, \mathscr{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ of disjoint $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-open sets such that $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{U}_{\mathfrak{g},\alpha}, \mathscr{U}_{\mathfrak{g},\beta})$.*

Proof Let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of disjoint $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}$-compact sets of a $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})})$ and suppose $\xi \in \mathscr{R}_{\mathfrak{g}}$. Then, since $\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \emptyset$, it results that $\xi \notin \mathscr{S}_{\mathfrak{g}}$. But by hypothesis, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ and consequently, there exists $(\mathscr{U}_{\mathfrak{g},\xi}, \hat{\mathscr{U}}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\{\xi\}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{U}_{\mathfrak{g},\xi}, \hat{\mathscr{U}}_{\mathfrak{g},\xi})$ and $\mathscr{U}_{\mathfrak{g},\xi} \cap \hat{\mathscr{U}}_{\mathfrak{g},\xi} = \emptyset$. Since $\xi \in \mathscr{U}_{\mathfrak{g},\xi}$, it follows that $\langle \mathscr{U}_{\mathfrak{g},\xi} \rangle_{\xi \in \mathscr{R}_{\mathfrak{g}}}$

is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering

$$\langle \mathscr{U}_{\mathfrak{g},\upsilon(\xi)} \rangle_{(\xi,\upsilon(\xi))\in\hat{\mathscr{R}}_{\mathfrak{g}}\times\mathscr{R}_{\mathfrak{g}}} \prec \langle \mathscr{U}_{\mathfrak{g},\xi} \rangle_{\xi\in\mathscr{R}_{\mathfrak{g}}},$$

where $\hat{\mathscr{R}}_{\mathfrak{g}} \subseteq \mathscr{R}_{\mathfrak{g}}$ is finite, can be selected so that $\mathscr{R}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathscr{R}}_{\mathfrak{g}}\times\mathscr{R}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\upsilon(\xi)}$. Furthermore, $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcap_{(\zeta,\vartheta(\zeta))\in\hat{\mathscr{S}}_{\mathfrak{g}}\times\mathscr{S}_{\mathfrak{g}}} \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\xi)}$, where $\hat{\mathscr{S}}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ is finite, since $\mathscr{S}_{\mathfrak{g}} \subseteq \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\xi)}$ for every $(\zeta,\vartheta(\zeta)) \in \hat{\mathscr{S}}_{\mathfrak{g}} \times \mathscr{S}_{\mathfrak{g}}$. Now let

$$\mathscr{U}_{\mathfrak{g},\alpha} = \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathscr{R}}_{\mathfrak{g}}\times\mathscr{R}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\upsilon(\xi)}, \quad \mathscr{U}_{\mathfrak{g},\beta} = \bigcap_{(\zeta,\vartheta(\zeta))\in\hat{\mathscr{S}}_{\mathfrak{g}}\times\mathscr{S}_{\mathfrak{g}}} \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\zeta)}.$$

Observe that $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta})$. Moreover, $(\mathscr{U}_{\mathfrak{g},\alpha},\mathscr{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, since $\mathscr{U}_{\mathfrak{g},\upsilon(\xi)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\xi,\upsilon(\xi)) \in \hat{\mathscr{R}}_{\mathfrak{g}} \times \mathscr{R}_{\mathfrak{g}}$ and $\hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\zeta)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\zeta,\vartheta(\zeta)) \in \hat{\mathscr{S}}_{\mathfrak{g}} \times \mathscr{S}_{\mathfrak{g}}$. The proof of the theorem is complete when the statement $\mathscr{U}_{\mathfrak{g},\alpha} \cap \mathscr{U}_{\mathfrak{g},\beta} = \emptyset$ is proved. First observe that, for every $(\xi,\zeta,\upsilon(\xi),\vartheta(\zeta)) \in \hat{\mathscr{R}}_{\mathfrak{g}} \times \hat{\mathscr{S}}_{\mathfrak{g}} \times \mathscr{R}_{\mathfrak{g}} \times \mathscr{S}_{\mathfrak{g}}$, the relation $\mathscr{U}_{\mathfrak{g},\upsilon(\xi)} \cap \hat{\mathscr{U}}_{\mathfrak{g},\vartheta(\zeta)} = \emptyset$ implies $\mathscr{U}_{\mathfrak{g},\upsilon(\xi)} \cap \mathscr{U}_{\mathfrak{g},\beta} = \emptyset$. Consequently,

$$\begin{aligned}\bigcap_{\mu=\alpha,\beta} \mathscr{U}_{\mathfrak{g},\mu} = \Bigg(\bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathscr{R}}_{\mathfrak{g}}\times\mathscr{R}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\upsilon(\xi)}\Bigg) \cap \mathscr{U}_{\mathfrak{g},\beta} &= \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathscr{R}}_{\mathfrak{g}}\times\mathscr{R}_{\mathfrak{g}}} \big(\mathscr{U}_{\mathfrak{g},\upsilon(\xi)} \cap \mathscr{U}_{\mathfrak{g},\beta}\big) \\ &= \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathscr{R}}_{\mathfrak{g}}\times\mathscr{R}_{\mathfrak{g}}} \emptyset = \emptyset.\end{aligned}$$

This proves the theorem. □

Theorem 3.6 *Let* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ *be* $\mathscr{T}_{\mathfrak{g}}$*-spaces. If* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map and* $\mathscr{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ *in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ *in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Proof Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathscr{T}_{\mathfrak{g},\Sigma})$ be given $\mathscr{T}_{\mathfrak{g}}$-spaces, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, $\mathscr{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$ and suppose $\langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})$ in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. Then,

$$\mathscr{S}_{\mathfrak{g},\omega} \subseteq \pi_{\mathfrak{g}}^{-1} \circ \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega}) \subseteq \pi_{\mathfrak{g}}^{-1}\Big(\bigcup_{\alpha\in I_{\sigma}^{*}} \mathscr{U}_{\mathfrak{g},\alpha}\Big) \subseteq \bigcup_{\alpha\in I_{\sigma}^{*}} \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha}).$$

Thus, $\langle \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha}) \rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g},\omega}$ in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, because $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ and for every $\alpha \in I_{\sigma}^{*}$, $\mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ implies $\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$. But, the relation $\mathscr{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ holds and consequently, there exists $\langle \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}) \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec$

$\langle\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha})\rangle_{\alpha\in I_{\sigma}^{*}}$ such that the relation $\mathscr{S}_{\mathfrak{g},\omega}\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)})$ holds. Accordingly,

$$\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})\subseteq\pi_{\mathfrak{g}}\circ\pi_{\mathfrak{g}}^{-1}\Bigg(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\Bigg)=\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}.$$

Thus, $\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})$ and hence, it follows that $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the theorem is complete. □

Theorem 3.7 *Let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute map, where $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=(\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ are $\mathscr{T}_{\mathfrak{g}}$-spaces. If $\mathscr{S}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\Omega}]$, then $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$.*

Proof Let $\mathscr{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute map, where $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=(\Sigma,\mathscr{T}_{\mathfrak{g},\Sigma})$ are $\mathscr{T}_{\mathfrak{g}}$-spaces. Suppose $\mathscr{S}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\Omega}]$, let $\langle\mathscr{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\Sigma}]\rangle_{\alpha\in I_{\sigma}^{*}}$ be any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})\subset\mathfrak{T}_{\mathfrak{g},\Sigma}$. Then, since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$, it follows, evidently, that the relation $\mathscr{S}_{\mathfrak{g},\omega}\bigcup_{\alpha\in I_{\sigma}^{*}}\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha})$ holds. On the other hand, since $\mathscr{S}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\Omega}]$, it results that, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\langle\mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ exists such that the relation $\mathscr{S}_{\mathfrak{g},\omega}\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)})$ holds. Consequently, it follows, then, that $\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$ and hence, $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$. The proof of the theorem is complete. □

Lemma 3.8 *Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space. If $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-K}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}]$, then $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-A}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}]$ in $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$.*

Proof Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=(\Omega,\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space and suppose $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-K}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}]$. Suppose $\langle\mathscr{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}]\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open covering of $\mathscr{S}_{\mathfrak{g}}$, then $\Omega=(\bigcup_{\alpha\in I_{\sigma}^{*}}\mathscr{U}_{\mathfrak{g},\alpha})\cup\complement(\mathscr{S}_{\mathfrak{g}})=\bigcup_{\alpha\in I_{\sigma}^{*}}(\mathscr{U}_{\mathfrak{g},\alpha}\cup\complement(\mathscr{S}_{\mathfrak{g}}))$, meaning that $\langle\mathscr{U}_{\mathfrak{g},\alpha}\cup\complement(\mathscr{S}_{\mathfrak{g}})\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open covering of $\mathscr{S}_{\mathfrak{g}}$ because, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-K}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}]$ implies $\complement(\mathscr{S}_{\mathfrak{g}})\in\mathfrak{g}\text{-O}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}]$. On the other hand, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$ is, by hypothesis, a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space. Thus, there exists $\langle\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\langle\mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ such that $\Omega=(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)})\cup\complement(\mathscr{S}_{\mathfrak{g}})$. But $\mathscr{S}_{\mathfrak{g}}\cap\complement(\mathscr{S}_{\mathfrak{g}})=\emptyset$ and hence, $\mathscr{S}_{\mathfrak{g}}\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$. This shows that any $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open covering $\langle\mathscr{U}_{\mathfrak{g},\alpha}\cup\complement(\mathscr{S}_{\mathfrak{g}})\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathscr{S}_{\mathfrak{g}}$

contains a finite $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ and hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ in $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$. The proof of the lemma is complete. □

Theorem 3.9 *Let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big)$ *be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space and let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})} = \big(\Sigma, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big)$ *be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space. If the* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-map* $\pi_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ *is a one-one* $\mathfrak{g}\text{-}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big)$*-continuous map, then* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \cong \pi_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big)$.

Proof Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space and let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})} = \big(\Sigma, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space, and suppose $\pi_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ is a one-one $\mathfrak{g}\text{-}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big)$-continuous map. Clearly, $\pi_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ is onto, and since it is, by hypothesis a one-one $\mathfrak{g}\text{-}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big)$-continuous map, it follows that $\pi_{\mathfrak{g}}^{-1} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$ exists. It must be shown that $\pi_{\mathfrak{g}}^{-1} \in \mathfrak{g}\text{-}\mathrm{C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}; \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big]$. Recall that $\pi_{\mathfrak{g}}^{-1} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$ is $\mathfrak{g}\text{-}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big)$-continuous if and only if, for every $\mathscr{K}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$, $\big(\pi_{\mathfrak{g}}^{-1}\big)^{-1}(\mathscr{K}_{\mathfrak{g},\omega}) = \pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(\mathrm{H})}\big]$ and $\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \subseteq \mathrm{im}\big(\pi_{\mathfrak{g}|\Sigma}\big)$. Clearly, $\mathscr{K}_{\mathfrak{g},\omega} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega})$, so $\mathscr{K}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$. But, $\mathscr{K}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ implies $\mathscr{K}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-}\mathrm{A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ in $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$. Furthermore, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}; \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big]$, it follows that $\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \in \mathfrak{g}\text{-}\mathrm{A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(\mathrm{H})}\big]$ and $\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \subseteq \mathrm{im}\big(\pi_{\mathfrak{g}|\Sigma}\big)$. But, $\pi_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\omega}) \in \mathfrak{g}\text{-}\mathrm{A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(\mathrm{H})}\big]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}\big]$. Accordingly, $\pi_{\mathfrak{g}}^{-1} \in \mathfrak{g}\text{-}\mathrm{C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}; \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big]$ and hence, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \cong \pi_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big)$. The proof of the theorem is complete. □

Proposition 3.10 *Let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$ *be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space and let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$ *be a* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space. If* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]} \supseteq \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},}^{(\mathrm{H})}$*, then* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},}^{(\mathrm{H})}$.

Proof Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$ be a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}\big)$, a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space, and suppose $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]} \supseteq \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},}^{(\mathrm{H})}$. Further, consider the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-map $\pi_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ defined by $\pi_{\mathfrak{g}}(\xi) = \xi$. Since $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]} \supseteq \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},}^{(\mathrm{H})}$, for every $\mathscr{O}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},}^{(\mathrm{H})}$, there exist $\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$ such that $\pi_{\mathfrak{g}}^{-1}\big(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}\big) = \mathscr{O}_{\mathfrak{g},\alpha} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\alpha})$. Consequently, $\pi_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \longrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ is a one-one and onto $\mathfrak{g}\text{-}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big)$-continuous map from a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$ to a $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ and therefore, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]} \cong \pi_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big)$. Hence, $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g},}^{(\mathrm{H})}$. The proof of the proposition is complete. □

Theorem 3.11 *If* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}[\mathfrak{T}_{\mathfrak{g}}]$ *is a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-compact set of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then it is* *also countably* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact in* $\mathfrak{T}_{\mathfrak{g}}$.

Proof Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and suppose $\mathscr{R}_{\mathfrak{g}} \subset \mathscr{S}_{\mathfrak{g}}$ be any infinite $\mathfrak{T}_{\mathfrak{g}}$-subset of $\mathscr{S}_{\mathfrak{g}}$. Equivalently proved, it must be shown that, the assumption that $\mathscr{R}_{\mathfrak{g}}$ has no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathscr{S}_{\mathfrak{g}}$ leads to a contradiction. Since $\mathscr{R}_{\mathfrak{g}} \subset \mathscr{S}_{\mathfrak{g}}$ is, by assumption, an infinite $\mathfrak{T}_{\mathfrak{g}}$-subset of $\mathscr{S}_{\mathfrak{g}}$ with no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathscr{S}_{\mathfrak{g}}$, it follows that, for every $\xi \in \mathscr{S}_{\mathfrak{g}}$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ which contains at most one point $\zeta \in \mathscr{R}_{\mathfrak{g}}$. It may be remarked, in passing, that $\langle \mathscr{U}_{\mathfrak{g},\xi} \rangle_{\xi \in \mathscr{S}_{\mathfrak{g}}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ for $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{\xi \in \mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\xi}$. Consequently, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\xi)} \rangle_{(\xi,\vartheta(\xi)) \in \mathscr{S}_{\mathfrak{g}} \times \hat{\mathscr{S}}_{\mathfrak{g}}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$, where $\hat{\mathscr{S}}_{\mathfrak{g}} \subset \mathscr{S}_{\mathfrak{g}}$, such $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi)) \in \mathscr{S}_{\mathfrak{g}} \times \hat{\mathscr{S}}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\vartheta(\xi)}$. But, for every $(\xi, \vartheta(\xi)) \in \mathscr{S}_{\mathfrak{g}} \times \hat{\mathscr{S}}_{\mathfrak{g}}$, $\mathscr{U}_{\mathfrak{g},\vartheta(\xi)}$ contains at most one point $\zeta \in \mathscr{R}_{\mathfrak{g}}$. Therefore, the infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathscr{R}_{\mathfrak{g}}$ of $\mathscr{S}_{\mathfrak{g}}$, satisfying $\mathscr{R}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi)) \in \mathscr{S}_{\mathfrak{g}} \times \hat{\mathscr{S}}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g},\vartheta(\xi)}$, can contain at most $\eta = \operatorname{card}(\hat{\mathscr{S}}_{\mathfrak{g}}) < \infty$ points. Accordingly, it follows that every infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathscr{R}_{\mathfrak{g}} \subset \mathscr{S}_{\mathfrak{g}}$ of $\mathscr{S}_{\mathfrak{g}}$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathscr{S}_{\mathfrak{g}}$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ is also countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$. This completes the proof of the theorem. □

Corollary 3.12 *Every* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *having the property that every countable* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open covering* $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ *of the* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$ *contains a finite* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open subcovering* $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ *of* $\mathfrak{T}_{\mathfrak{g}}$ *is a countably* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$.

Theorem 3.13 *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *are* $\mathscr{T}_{\mathfrak{g}}$*-spaces. If* $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *be a sequentially* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\operatorname{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is also a sequentially* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Proof Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ are $\mathscr{T}_{\mathfrak{g}}$-spaces, and suppose $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Omega}$. If $\langle \zeta_{\alpha} \in \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega}) \rangle_{\alpha \in I_{\infty}^{*}}$ be a sequence in $\operatorname{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$, then there exists a sequence $\langle \xi_{\alpha} \in \mathscr{S}_{\mathfrak{g}} \rangle_{\alpha \in I_{\infty}^{*}}$ in $\mathscr{S}_{\mathfrak{g}}$ such $\pi_{\mathfrak{g}}(\xi_{\alpha}) = \zeta_{\alpha}$ that for every $\alpha \in I_{\infty}^{*}$. But, by hypothesis, $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ is sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Therefore, there exists a subsequence $\langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}} \prec \langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}}$ which converges to a point $\xi \in \mathscr{S}_{\mathfrak{g}}$. On the other hand, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ and therefore, $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is sequentially $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous. Consequently, it results that $\langle \pi_{\mathfrak{g}}(\xi_{\vartheta(\alpha)}) \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}} = \langle \zeta_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}}$ converges to $\pi_{\mathfrak{g}}(\xi) \in \operatorname{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})$. Hence,

$\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. □

Proposition 3.14 *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *are* $\mathscr{T}_{\mathfrak{g}}$*-spaces. If* $\mathscr{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ *is a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathrm{A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ *is also* $\mathfrak{T}_{\mathfrak{g}}$*-compact in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Proof Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ are $\mathscr{T}_{\mathfrak{g}}$-spaces, and suppose $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \rangle_{\alpha \in I_{\eta}^{*}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g},\sigma} = \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$. Then, since the relation $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ holds, it results that $\langle \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha}) \rangle_{\alpha \in I_{\eta}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathscr{S}_{\mathfrak{g},\omega} = \pi_{\mathfrak{g}}^{-1}(\mathscr{S}_{\mathfrak{g},\sigma})$, because $\mathrm{O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$. Since $\mathscr{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$, a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}) \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\eta}^{*} \times I_{\vartheta(\eta)}^{*}} \prec \langle \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha}) \rangle_{\alpha \in I_{\eta}^{*}}$ exists, and such that, $\mathscr{S}_{\mathfrak{g},\omega} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\eta}^{*} \times I_{\vartheta(\eta)}^{*}} \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)})$. Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, it follows, consequently, that $\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega}) \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\eta}^{*} \times I_{\vartheta(\eta)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$. Therefore, $\langle \pi_{\mathfrak{g}}^{-1}(\mathscr{U}_{\mathfrak{g},\alpha}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \rangle_{\alpha \in I_{\eta}^{*}}$ is a finite $\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathscr{S}_{\mathfrak{g},\sigma} \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$. Hence, $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \in \mathrm{A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ is also $\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the proposition is complete. □

Theorem 3.15 *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *are* $\mathscr{T}_{\mathfrak{g}}$*-spaces. If* $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *is a countably* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is also a countably* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Proof Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ are $\mathscr{T}_{\mathfrak{g}}$-spaces, and suppose $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Omega}$. To prove that $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, let $\mathscr{S}_{\mathfrak{g},\sigma} \subseteq \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$ be an infinite $\mathfrak{T}_{\mathfrak{g}}$-subset of $\mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big)$. Then, a denumerable $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathscr{R}_{\mathfrak{g},\sigma} = \{\zeta_{\alpha} : \alpha \in I_{\infty}^{*}\} \subset \mathscr{S}_{\mathfrak{g},\sigma}$ exists. Since $\mathscr{R}_{\mathfrak{g},\sigma} \subset \mathscr{S}_{\mathfrak{g},\sigma} \subseteq \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) = \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$, there exists a denumerable $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathscr{R}_{\mathfrak{g},\omega} = \{\xi_{\alpha} : \alpha \in I_{\infty}^{*}\} \subset \mathscr{S}_{\mathfrak{g},\omega}$, with $\pi_{\mathfrak{g}}(\xi_{\alpha}) = \zeta_{\alpha}$ for every $\alpha \in I_{\infty}^{*}$. But, by hypothesis, $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ is countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Omega}$, so $\mathscr{R}_{\mathfrak{g},\omega}$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathscr{S}_{\mathfrak{g},\omega}$. Thus, $\xi \in \mathscr{R}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega}) \subseteq \mathscr{R}_{\mathfrak{g},\omega}$ and $\pi_{\mathfrak{g}}(\xi) \in \mathrm{im}\big(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}\big) = \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$; evidently, $\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ and therefore, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set $\mathscr{V}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ exists such that, $\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega}) = \mathscr{V}_{\mathfrak{g},\omega}$. But, by hypothesis, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$. Consequently, $\pi_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega})\big) \subseteq \pi_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega}) \cup \mathrm{der}_{\mathfrak{g}}\big(\pi_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega})\big) = \mathscr{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\sigma})$. But, $\xi \in \mathscr{R}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\omega})$ and therefore, $\pi_{\mathfrak{g}}(\xi) \in \mathscr{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\sigma})$. Now, $\pi_{\mathfrak{g}}(\xi) \in \mathscr{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\sigma})$, so let it be claimed that

$\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\mathscr{R}_{\mathfrak{g},\sigma}$. There are, then, two cases, namely, $\xi \notin \mathscr{R}_{\mathfrak{g},\omega}$ and $\xi \in \mathscr{R}_{\mathfrak{g},\omega}$.

I. *Case* $\xi \notin \mathscr{R}_{\mathfrak{g},\omega}$. If $\xi \notin \mathscr{R}_{\mathfrak{g},\omega}$, then $\pi_{\mathfrak{g}}(\xi) \notin (\mathscr{R}_{\mathfrak{g},\omega}) = \mathscr{R}_{\mathfrak{g},\sigma}$. But, $\pi_{\mathfrak{g}}(\xi) \in \mathscr{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g},\sigma})$ and consequently, $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\mathscr{R}_{\mathfrak{g},\sigma}$.

II. *Case* $\xi \in \mathscr{R}_{\mathfrak{g},\omega}$. If $\xi \in \mathscr{R}_{\mathfrak{g},\omega}$, choose a $\mu \in I^{*}_{\infty}$ such that $\xi = \xi_{\mu}$. Then, $\xi \notin \hat{\mathscr{R}}_{\mathfrak{g},\omega} = \{\xi_{\alpha} : \alpha \in I^{*}_{\infty} \setminus \{\mu\}\}$ and every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ containing ξ contains at least a point $\hat{\xi} \in \hat{\mathscr{R}}_{\mathfrak{g},\omega} = \{\xi_{\alpha} : \alpha \in I^{*}_{\infty} \setminus \{\mu\}\}$ and therefore, ξ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\hat{\mathscr{R}}_{\mathfrak{g},\omega}$. But, $\pi_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega}) = \{\zeta_{\alpha} : \alpha \in I^{*}_{\infty} \setminus \{\mu\}\}$ since, by hypothesis, $\pi_{\mathfrak{g}}(\xi_{\alpha}) = \zeta_{\alpha}$ for every $\alpha \in I^{*}_{\infty}$. Thus, $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\pi_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega})$ where $\pi_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega}) \subseteq \mathscr{R}_{\mathfrak{g},\sigma}$. Moreover, since $\pi_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega})) \subseteq \pi_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega}) \cup \mathrm{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\omega})) = \hat{\mathscr{R}}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{R}}_{\mathfrak{g},\sigma})$, it follows that, $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\hat{\mathscr{R}}_{\mathfrak{g},\sigma}$. Since $\mathscr{R}_{\mathfrak{g},\sigma} \subset \mathscr{S}_{\mathfrak{g},\sigma} \subseteq \mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) = \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$, $\pi_{\mathfrak{g}}(\xi)$ is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\mathscr{S}_{\mathfrak{g},\sigma}$ and $\pi_{\mathfrak{g}}(\xi) \in \mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) = \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$. Therefore, every infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathscr{S}_{\mathfrak{g},\sigma} \subseteq \mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})$ of $\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point in $\pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega})$ and hence, $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is also a countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the theorem is complete. □

Proposition 3.16 *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then every countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I^{*}_{\sigma}}$ of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set $\mathscr{S}_{\mathfrak{g}}$ is reducible to a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of the type $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I^{*}_{\sigma}}$ of $\mathscr{S}_{\mathfrak{g}}$.*

Proof Let it be assumed that $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact infinite set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Furthermore, assume that there exists a countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I^{*}_{\sigma}}$ of $\mathscr{S}_{\mathfrak{g}}$ with no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I^{*}_{\sigma}}$ of $\mathscr{S}_{\mathfrak{g}}$. Finally, introduce the sequence $\langle \xi_{\alpha} \in \mathscr{S}_{\mathfrak{g}} \rangle_{\alpha \in I^{*}_{\infty}}$ and define its elements in the following manner. Let $\vartheta(1) \in I^{*}_{\vartheta(\sigma)} \subset I^{*}_{\sigma}$ be the smallest integer in $I^{*}_{\vartheta(\sigma)}$ such that $\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(1)} \neq \emptyset$; choose $\xi_{1} \in \mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(1)}$. Let $\vartheta(2) \in I^{*}_{\vartheta(\sigma)} \subset I^{*}_{\sigma}$ be the least integer larger than $\vartheta(1)$ in $I^{*}_{\vartheta(\sigma)}$ such that $\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(2)} \neq \emptyset$; choose $\xi_{2} \in (\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(2)}) \setminus (\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(1)})$. Note that, such a point ξ_{2} always exists, for otherwise $\mathscr{U}_{\mathfrak{g},\vartheta(1)}$ covers $\mathscr{S}_{\mathfrak{g}}$. Continuing in this way, the properties of $\langle \xi_{\alpha} \rangle_{\alpha \in I^{*}_{\infty}}$, for every $\alpha \in I^{*}_{\infty} \setminus \{1\}$, are

$$\xi_{\alpha} \in \mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}, \quad \xi_{\alpha} \notin \textstyle\bigcup_{\nu \in I^{*}_{\alpha-1}} (\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(\nu)}), \quad \vartheta(\alpha) > \vartheta(\alpha - 1).$$

Let it be claimed that $\langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$ has no convergent subsequence $\langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}} \prec \langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$ in $\mathscr{S}_{\mathfrak{g}}$. Suppose $\xi \in \mathscr{S}_{\mathfrak{g}}$, then there exists a $\mu \in I_{\vartheta(\sigma)}^{*}$ such that $\xi \in \mathscr{U}_{\mathfrak{g},\vartheta(\mu)}$. Now, $\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(\mu)} \neq \emptyset$ since, $\xi \in \mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\vartheta(\mu)}$. Thus, there exists $\nu \in I_{\vartheta(\sigma)}^{*}$ such that, $\mathscr{U}_{\mathfrak{g},\vartheta(\nu)} = \mathscr{U}_{\mathfrak{g},\vartheta(\mu)}$. But, by the properties of the sequence $\langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$, $\alpha > \vartheta(\nu)$ implies $\xi_{\alpha} \notin \mathscr{U}_{\mathfrak{g},\vartheta(\mu)}$. Accordingly, since $\xi \in \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ no subsequence $\langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}} \prec \langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$ of $\langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$ converges to $\xi \in \mathscr{S}_{\mathfrak{g}}$. But, ξ was arbitrary and hence, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is not sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the proposition is complete. □

Theorem 3.17 *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then it is also locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$.*

Proof Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, for every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ such that $\mathscr{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$. It is clear that, for every $\xi \in \mathscr{S}_{\mathfrak{g}}$, there exists $\mathscr{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathscr{S}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g},\xi} = \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{U}_{\mathfrak{g},\xi}$ for some $(\alpha, \vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$. For every $(\alpha, \xi, \vartheta(\alpha), \upsilon(\alpha,\xi)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}} \times I_{\vartheta(\sigma)}^{*} \times I_{\upsilon(\sigma)}^{*}$, set $\mathscr{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} = \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{U}_{\mathfrak{g},\xi}$. Then, since $(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}, \mathscr{U}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\alpha, \xi, \vartheta(\alpha)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}} \times I_{\vartheta(\sigma)}^{*}$, there exists, for every $(\alpha, \xi, \vartheta(\alpha)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}} \times I_{\vartheta(\sigma)}^{*}$, a pair $(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}, \mathscr{O}_{\mathfrak{g},\xi}) \in \mathscr{T}_{\mathfrak{g}} \times \mathscr{T}_{\mathfrak{g}}$ of $\mathscr{T}_{\mathfrak{g}}$-open sets such that, $(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}, \mathscr{U}_{\mathfrak{g},\xi}) \subseteq (\mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}), \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi}))$. Consequently,

$$\mathscr{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} = \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{U}_{\mathfrak{g},\xi} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)}) \cap \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\xi}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{O}_{\mathfrak{g},\xi}) = \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\upsilon(\alpha,\xi)}),$$

where $\mathscr{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} = \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathscr{U}_{\mathfrak{g},\xi}$ for every $(\alpha, \xi, \vartheta(\alpha), \upsilon(\alpha,\xi)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}} \times I_{\vartheta(\sigma)}^{*} \times I_{\upsilon(\sigma)}^{*}$. Therefore, $\mathscr{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\alpha, \xi, \vartheta(\alpha), \upsilon(\alpha,\xi)) \in I_{\sigma}^{*} \times \mathscr{S}_{\mathfrak{g}} \times I_{\vartheta(\sigma)}^{*} \times I_{\upsilon(\sigma)}^{*}$. But, since $\xi \in \mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \subseteq \mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)})$ and $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \supset \mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, it results that,

$$\xi \in \mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \subseteq \mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}) \subset \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}.$$

Thus, given any $(\xi, \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}) \in \mathscr{S}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open neighborhood $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \in \mathfrak{g}\text{-N}[\xi]$ of ξ such that $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha,\xi)} \subset \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}$ and $\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)} \cup \mathrm{der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ implies that it is also locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. □

Corollary 3.18 *Every $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ having the property that every local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open* *covering* $\langle \mathscr{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ *of the* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$ *contains a finite* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open subcovering* $\langle \mathscr{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathscr{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ *of* $\mathfrak{T}_{\mathfrak{g}}$ *is a locally* $\mathfrak{g}$-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]})$.

Theorem 3.19 *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \longrightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ *are* $\mathscr{T}_{\mathfrak{g}}$*-spaces. If* $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *is a locally* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is also a locally* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

Proof Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ are $\mathscr{T}_{\mathfrak{g}}$-spaces, and suppose $\mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Since $\mathscr{S}_{\mathfrak{g},\omega}$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, for any given $(\xi, \mathscr{N}_{\mathfrak{g},\xi}) \in \mathscr{S}_{\mathfrak{g},\omega} \times \mathfrak{g}\text{-N}[\xi]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-N}[\xi]$ of ξ such that $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \subset \mathscr{N}_{\mathfrak{g},\xi}$ and $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$. Consequently, $\xi \in \hat{\mathscr{N}}_{\mathfrak{g},\xi} \subseteq \hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \subset \mathscr{N}_{\mathfrak{g},\xi}$ and thus, $\pi_{\mathfrak{g}}(\xi) \in \pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \subseteq \pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi})) \subset \pi_{\mathfrak{g}}(\mathscr{N}_{\mathfrak{g},\xi})$. But, $\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi})) \subseteq \pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \cup \pi_{\mathfrak{g}}(\mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}))$ because, by hypothesis, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$. Therefore,

$$\pi_{\mathfrak{g}}(\xi) \in \pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \subseteq \pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi})) \subseteq \pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \cup \mathrm{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi})) \subset \pi_{\mathfrak{g}}(\mathscr{N}_{\mathfrak{g},\xi}).$$

Since $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood in $\mathfrak{T}_{\mathfrak{g},\Omega}$ containing $\xi \in \mathscr{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$, $\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood in $\mathfrak{T}_{\mathfrak{g},\Sigma}$ containing $\pi_{\mathfrak{g}}(\xi) \in \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega}) \in \mathfrak{T}_{\mathfrak{g},\Sigma}$. Now $\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \cup \mathrm{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi})) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ by virtue of the statements $\hat{\mathscr{N}}_{\mathfrak{g},\xi} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ and $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$. In other words, for any given $(\pi_{\mathfrak{g}}(\xi), \pi_{\mathfrak{g}}(\mathscr{N}_{\mathfrak{g},\zeta})) = (\zeta, \mathscr{N}_{\mathfrak{g},\zeta}) \in \mathscr{S}_{\mathfrak{g},\sigma} \times \mathfrak{g}\text{-N}[\zeta]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood $\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) = \hat{\mathscr{N}}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-N}[\zeta]$ of $\pi_{\mathfrak{g}}(\xi) = \zeta$ such that $\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) = \hat{\mathscr{N}}_{\mathfrak{g},\zeta} \subseteq \mathscr{N}_{\mathfrak{g},\zeta} = \pi_{\mathfrak{g}}(\mathscr{N}_{\mathfrak{g},\xi})$ and $\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi}) \cup \mathrm{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\xi})) = \hat{\mathscr{N}}_{\mathfrak{g},\zeta} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathscr{N}}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$. Therefore, $\mathscr{S}_{\mathfrak{g},\sigma} \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. But, $\mathscr{S}_{\mathfrak{g},\sigma} = \pi_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\omega}) = \mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}})$. Hence, $\mathrm{im}(\pi_{\mathfrak{g}|\mathscr{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the theorem is complete.

□

4. Discussion

4.1. Categorical Classifications

Having adopted a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the dual purposes of the this section are firstly, to establish the various relationships amongst the elements of the sequences $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{E}]})\rangle_{\nu\in I_3^0}$ and $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}^{[\mathrm{E}]} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathscr{T}^{[\mathrm{E}]})\rangle_{\nu\in I_3^0}$ of $\mathfrak{g}$-ν-$\mathscr{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-spaces and $\mathfrak{g}$-ν-$\mathscr{T}^{[\mathrm{E}]}$-spaces, respectively, where $\mathrm{E} \in \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, and secondly, to illustrate them through diagrams.

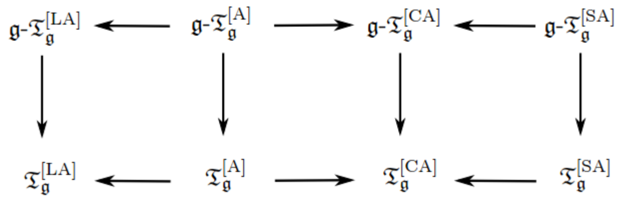

Figure 1: Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact spaces and $\mathfrak{T}_{\mathfrak{g}}$-compact spaces

It is plain that $\mathfrak{T}_{\mathfrak{g}}$-compactness implies both countable $\mathfrak{T}_{\mathfrak{g}}$-compactness and local countable $\mathfrak{T}_{\mathfrak{g}}$-compactness; sequential $\mathfrak{T}_{\mathfrak{g}}$-compactness implies countable $\mathfrak{T}_{\mathfrak{g}}$-compactness. Moreover, the following implications also hold: $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\mathrm{LA}} \longleftarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\mathrm{A}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\mathrm{CA}} \longleftarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\mathrm{A}}$, and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\mathrm{CA}} \longleftarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{\mathrm{SA}}$. Since the relation $\mathfrak{T}^{[\mathrm{E}]} \longleftarrow \mathfrak{g}\text{-}\mathfrak{T}^{[\mathrm{E}]}$ holds for every $\mathrm{E} \in \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, taking this last statement together with those preceding it into account, the diagram presented in Figure 1 follows, in which are illustrated the various relationships amongst the elements of $\langle \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \rangle_{\mathrm{E}\in\Lambda}$ and $\langle \mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \rangle_{\mathrm{E}\in\Lambda}$, where $\Lambda = \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$.

For each $\nu \in I_3^0$, these implications hold: $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{LA}]} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]}$. For each $\mathrm{E} \in \Lambda = \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, these implications also hold: $\mathfrak{g}\text{-}0\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \longleftarrow \mathfrak{g}\text{-}1\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$, $\mathfrak{g}\text{-}1\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \longleftarrow \mathfrak{g}\text{-}3\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$, and $\mathfrak{g}\text{-}2\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \longleftarrow \mathfrak{g}\text{-}3\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$. When all these implications are taken into consideration, the resulting compactness diagram so obtained is that presented in Figure 2. It is reasonably correct to call them $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-spaces of type E and of category ν, where $(\nu, \mathrm{E}) \in I_3^0 \times \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$. As in the papers of [7] and [17], among others, the manner we have positioned the arrows is solely to stress that, in general, none of the implications in Figures 1 and 2 is reversible.

In order to exemplify the notion of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-spaces of type E and of category ν, where $(\nu, \mathrm{E}) \in I_3^0 \times \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, a nice application is presented in the following section.

4.2. A Nice Application

Focusing on basic concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, we shall now present a nice application.

Let $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the $\mathfrak{g}$-topology on $\Omega = \mathbb{N}$ (set of positive integers) generated by $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets belonging to:

$$\mathscr{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{ \mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)} : \ (\forall \mu \in I_{\infty}^{*}) \big(\big[\mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \emptyset \big] \vee \big[\mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \{2\mu-1, 2\mu\} \big] \big) \right\};$$

$$\neg\mathscr{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{ \mathscr{K}_{\mathfrak{g},(2\mu-1,2\mu)} : \ (\forall \mu \in I_{\infty}^{*}) \big(\big[\mathscr{K}_{\mathfrak{g},(2\mu-1,2\mu)} = \mathbb{N} \big] \vee \big[\mathscr{K}_{\mathfrak{g},(2\mu-1,2\mu)} = \complement(\{2\mu-1, 2\mu\}) \big] \big) \right\},$$

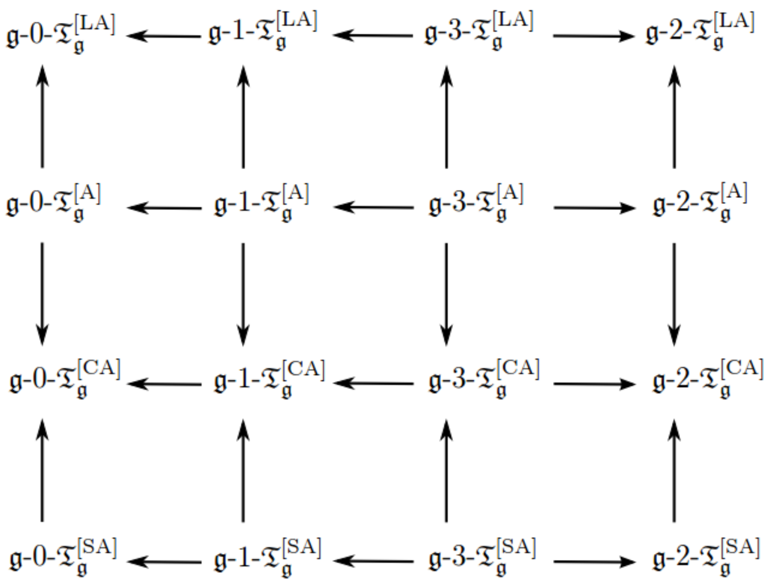

Figure 2: Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact spaces

respectively. As in the above case, it results that $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the relations $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big) \subseteq \{2\mu-1, 2\mu\} = \mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)}$ and, $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\mu\in I_{\sigma}^{*}} \mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big) = \bigcap_{\mu\in I_{\sigma}^{*}} \mathscr{T}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big)$ as well as $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\mu\in I_{\infty}^{*}} \mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big) = \bigcup_{\mu\in I_{\infty}^{*}} \mathscr{T}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big)$, since the two relations $\bigcap_{\mu\in I_{\sigma}^{*}} \mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \emptyset \in \mathscr{T}_{\mathfrak{g}}$ and $\bigcup_{\mu\in I_{\infty}^{*}} \mathscr{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \Omega \in \mathscr{T}_{\mathfrak{g}}$, respectively, hold. Therefore, $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega)$ is a $\mathscr{T}_{\mathfrak{g}}$-space and, moreover, since the relation $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega) = (\mathscr{T}, \Omega) = \mathfrak{T}$ holds, it is also a $\mathscr{T}$-space. Notice that $\big\langle \mathscr{O}_{\mathfrak{g},(2\alpha-1,2\alpha)} \big\rangle_{\alpha\in I_{\infty}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω, since $\mathscr{O}_{\mathfrak{g},(2\alpha-1,2\alpha)} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha \in I_{\infty}^{*}$ and furthermore, it is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω, since $\mathscr{O}_{\mathfrak{g},(2\alpha-1,2\alpha)} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},(2\alpha-1,2\alpha)}\big) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha \in I_{\infty}^{*}$. However, $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega)$, where $\Omega = \mathbb{N}$, is not a $\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space because $\big\langle \mathscr{O}_{\mathfrak{g},(2\alpha-1,2\alpha)} \big\rangle_{\alpha\in I_{\infty}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of Ω with no finite $\mathfrak{T}_{\mathfrak{g}}$-open subcovering.

As stated above, since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness, it follows, obviously, that it is also not a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space. On the other hand, $\mathfrak{T}_{\mathfrak{g}} = (\mathscr{T}_{\mathfrak{g}}, \Omega)$, where $\Omega = \mathbb{N}$, is also not a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact $\mathscr{T}_{\mathfrak{g}}$-space for the simple reason that sequence $\langle \xi_{\alpha} = \alpha \in \Omega \rangle_{\alpha\in I_{\infty}^{*}}$ in $\mathfrak{T}_{\mathfrak{g}}$ contains no subsequence of the type $\big\langle \xi_{\vartheta(\alpha)} \big\rangle_{(\alpha,\vartheta(\alpha)\in\Omega)\in I_{\infty}^{*}\times I_{\infty}^{*}} \prec \langle \xi_{\alpha} \rangle_{\alpha\in I_{\infty}^{*}}$ which converges to a point $\xi \in \Omega$. Hence, $\mathfrak{T}_{\mathfrak{g}}$ is not a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]}$-space which, then, implies that it is also not a $\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]}$-space.

Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a non-empty $\mathfrak{T}_{\mathfrak{g}}$-set in $\mathfrak{T}_{\mathfrak{g}}$. Then, it is no error to express it in the form $\mathscr{S}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}^{\mathrm{even}} \cup \mathscr{S}_{\mathfrak{g}}^{\mathrm{odd}}$, where $\mathscr{S}_{\mathfrak{g}}^{\mathrm{even}} = \big\{\mu : \ (\forall\alpha \in I_{\infty}^{*})\,[\mu = 2\alpha]\big\}$ and $\mathscr{S}_{\mathfrak{g}}^{\mathrm{odd}} = \big\{\mu : (\forall\alpha \in I_{\infty}^{*})\,[\mu = 2\alpha - 1]\big\}$. Since $\mathscr{S}_{\mathfrak{g}} \neq \emptyset$, consider an arbitrary point $\xi \in \mathscr{S}_{\mathfrak{g}}$. If $\xi \in \mathscr{S}_{\mathfrak{g}}^{\mathrm{even}}$

then, for every $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ containing ξ, $\mathscr{S}_{\mathfrak{g}}^{\text{even}} \cap \big(\mathscr{U}_{\mathfrak{g},\xi} \setminus \{\xi\}\big) = \emptyset$ and $\mathscr{S}_{\mathfrak{g}}^{\text{odd}} \cap \big(\mathscr{U}_{\mathfrak{g},\xi} \setminus \{\xi\}\big) \neq \emptyset$. But, if $\xi \in \mathscr{S}_{\mathfrak{g}}^{\text{odd}}$ then, for every $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{U}_{\mathfrak{g},\xi} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ containing ξ, $\mathscr{S}_{\mathfrak{g}}^{\text{even}} \cap \big(\mathscr{U}_{\mathfrak{g},\xi} \setminus \{\xi\}\big) \neq \emptyset$ and $\mathscr{S}_{\mathfrak{g}}^{\text{odd}} \cap \big(\mathscr{U}_{\mathfrak{g},\xi} \setminus \{\xi\}\big) = \emptyset$. In either case, it follows, then, that $\mathscr{S}_{\mathfrak{g}}$ have at least one $\mathfrak{T}_{\mathfrak{g}}$-accumulation point. Accordingly, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]}$-space. For every $\alpha \in I_{\infty}^{*}$, set $\mathscr{U}_{\mathfrak{g},2\alpha-1} = \{2\alpha-1\}$ and $\mathscr{U}_{\mathfrak{g},2\alpha} = \{2\alpha\}$. Accordingly, $\mathscr{U}_{\mathfrak{g},2\alpha-1}$, $\mathscr{U}_{\mathfrak{g},2\alpha} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ since $\mathscr{U}_{\mathfrak{g},2\alpha-1}$, $\mathscr{U}_{\mathfrak{g},2\alpha} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\mathscr{O}_{\mathfrak{g},(2\alpha-1,2\alpha)}\big) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha \in I_{\infty}^{*}$. Observe that, $\mathscr{S}_{\mathfrak{g}} \cap \big(\mathscr{U}_{\mathfrak{g},2\alpha-1} \setminus \{2\alpha-1\}\big) = \emptyset$ and $\mathscr{S}_{\mathfrak{g}} \cap \big(\mathscr{U}_{\mathfrak{g},2\alpha} \setminus \{2\alpha\}\big) = \emptyset$ for every $\alpha \in I_{\infty}^{*}$. This proves the existence of an infinite $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ with no $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-accumulation point and hence, $\mathfrak{T}_{\mathfrak{g}}$ is not a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]}$-space.

In relation to the above descriptions, further $\mathscr{T}_{\mathfrak{g}}$-properties amongst the $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-spaces $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{CA}]}\big)$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{SA}]}\big)$, and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{LA}]} = \big(\Omega, \mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{LA}]}\big)$ called, respectively, $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, *countably* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, *sequentially* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, and *locally* $\mathfrak{g}\text{-}\mathscr{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, can be discussed in a similar way by slight modifications of some $\mathscr{T}_{\mathfrak{g}}$-properties found in those cases.

4.3. Concluding Remarks

In a recent paper [11] the study of a novel class of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces was presented. In this paper, the concept is further studied and other derived concepts called countable, sequential, local $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathscr{T}_{\mathfrak{g}}$-spaces have also been studied relatively. It was shown that $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness implies local $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness and countable $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness, sequential $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness implies countable $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness is a generalized topological property (briefly, $\mathscr{T}_{\mathfrak{g}}$*-property*).

For future research, it would be interesting to develop the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness of mixed categories. More precisely, for some pair $(\nu,\mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\big\{\mathscr{U}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g},\nu} \cup \mathscr{U}_{\mathfrak{g},\mu} :\ (\mathscr{U}_{\mathfrak{g},\nu}, \mathscr{U}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Such a theory is what we thought would certainly be worth considering, and the discussion of this paper ends here.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Mohammad Irshad Khodabocus]: Thought and designed the research/problem, collected the data, contribution to completing the research and solving the problem, wrote the manuscript (%80).

Author [Noor-Ul-Hacq Sookia]: Contributed to research method or evaluation of data (%20).

Conflicts of Interest

The authors declare no conflict of interest.

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On Quasi-Conformally Flat Generalized Sasakian-Space Forms

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Abstract: In this paper, we classify quasi-conformally flat generalized Sasakian-space forms under the assumption that the characteristic vector field is Killing. Also, we classify quasi-conformally Weyl-symmetric generalized Sasakian-space forms.

Keywords: Generalized Sasakian-space forms, quasi-conformally flat, quasi-conformally Weyl-symmetric.

1. Introduction

In Riemannian geometry, many authors have studied curvature properties and to what extent they determined the manifold itself. Two important curvature properties are quasi-conformal flatness and Weyl-symmetry.

In [1], Alegre, Blair and Carriazo introduced and studied generalized Sasakian-space forms. These spaces are defined as follows: Given an almost contact metric manifold (M,ϕ,ξ,η,g), they say that M is a generalized Sasakian-space form if there exist three functions f_1, f_2 and f_3 on M such that

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X-g(X,Z)Y\} \\ &\quad +f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &\quad +f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1}$$

for any vector fields X,Y,Z on M, where R denotes the curvature tensor of M. In such a case, we will write $M(f_1,\ f_2,\ f_3)$.

Then, Kim studied conformally flat generalized Sasakian space forms [5].

In this paper, we study quasi-conformally flat generalized Sasakian-space forms and quasi-conformally Weyl-symmetric generalized Sasakian-space forms.

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2. Preliminaries

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η such that

$$\eta(\xi) = 1, \tag{2}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{3}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4}$$

for any vector fields X, Y on M [2]. Also,

$$\phi\xi = 0 \tag{5}$$

and

$$\eta \circ \phi = 0 \tag{6}$$

are deducible from these conditions. We define the fundamental 2-form Φ on M by $\Phi(X, Y) = g(X, \phi Y)$. An almost contact metric manifold M is said to be a contact metric manifold if $g(X, \phi Y) = d\eta(X, Y)$. If ξ is a Killing vector field, then the contact metric manifold is said to be a K-contact manifold. The almost contact metric structure of M is said to be normal if $[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi$, for any X, Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion tensor of ϕ. A normal contact metric manifold is called a Sasakian manifold. A normal almost contact metric manifold M with closed forms η and Φ is called a cosymplectic manifold. Cosymplectic manifolds are characterized by $\nabla_X \xi = 0$ and $(\nabla_X \phi)Y = 0$ for any vector fields X, Y on M. Given an almost contact metric manifold (M, ϕ, ξ, η, g), a ϕ-section of M at $p \in M$ is a plane section $\pi \subseteq T_p M$ spanned by a unit vector X_p orthogonal to ξ_p and ϕX_p. The ϕ-sectional curvature of π is defined by $g(R(X, \phi X)\phi X, X)$. A cosymplectic space-form, i.e., a cosymplectic manifold with constant ϕ-sectional curvature c, is a generalized Sasakian space-form with $f_1 = f_2 = f_3 = \frac{c}{4}$ [6]. It is known that the ϕ-sectional curvature of a generalized Sasakian-space form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$ [1].

For a $(2n+1)$-dimensional almost contact metric manifold (M, ϕ, ξ, η, g), $n \geq 1$, its Schouten tensor L is defined by

$$L = -\frac{1}{2n-1}Q + \frac{\tau}{4n(2n-1)}I, \tag{7}$$

where Q denotes the Ricci operator and τ is the scalar curvature of M. The Weyl conformal curvature tensor is given by

$$C(X,Y)Z = R(X,Y)Z - [g(LX,Z)Y - g(Y,Z)LX - g(LY,Z)X + g(X,Z)LY]. \tag{8}$$

In dimension > 3, that is $n > 1$, M is conformally flat if and only if $C = 0$, and in this case, L satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$ for any vector fields X, Y on M. In dimension 3, that is $n = 1$, $C = 0$ is automatically satisfied and M is conformally flat if and only if L satisfies $(\nabla_X L)Y - (\nabla_Y L)X = 0$ for any vector fields X, Y on M.

A symmetric tensor field T of type $(1,1)$ is a Codazzi tensor if it satisfies

$$(\nabla_X T)Y - (\nabla_Y T)X = 0.$$

For the later use, we give the following lemma which was proved Derdzinski.

Lemma 2.1 *[3, 4] Let T be a Codazzi tensor on a Riemannian manifold M. Then, we have the following:*

If T has more than one eigenvalue, then the eigenspaces for each eigenvalue v form an integrable subbundle V_v of constant multiplicity on open sets: If the multiplicity is greater than 1, then the integral submanifolds are umbilical submanifolds and each eigenfunction is constant along the integral submanifolds of its subbundle. Moreover, if v is constant on M, then the integral submanifolds of V_v are totally geodesic.

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space form. Then, the curvature tensor R of M is given by (1). From (1), we can easily see that

$$QX = \{2nf_1 + 3f_2 - f_3\}X - \{3f_2 + (2n-1)f_3\}\eta(X)\xi, \tag{9}$$

$$\tau = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{10}$$

Moreover, we can see that

$$LX = \{-\frac{1}{2}f_1 - \frac{3}{2(2n-1)}f_2\}X + \{\frac{3}{2n-1}f_2 + f_3\}\eta(X)\xi. \tag{11}$$

Therefore, the Weyl conformal curvature tensor C can be written as

$$\begin{aligned} C(X,Y)Z = {} & \frac{-3}{2n-1}f_2\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2g(X,\phi Y)\phi Z\} \\ & - \frac{3}{2n-1}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{12}$$

The notion of the quasi-conformal curvature tensor was defined by Yano and Sawaki [8]. According to them a quasi-conformal curvature tensor is defined by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z \\ &+b[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY] \\ &-\frac{\tau}{2n+1}[\frac{a}{2n}+2b][g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{13}$$

where a and b are constants, S is the Ricci tensor, Q is the Ricci operator and τ is the scalar curvature of the manifold M^{2n+1}. A Riemannian manifold (M^{2n+1},g), $(n>1)$, is called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C}=0$. If $a=1$ and $b=\frac{-1}{2n-1}$, then the quasi-conformal curvature tensor is reduced to the Weyl conformal curvature tensor.

A Riemannian manifold is said to be quasi-conformally Weyl-symmetric manifold if

$$R(X,Y)\cdot\tilde{C}=0,$$

where $\tilde{C}$ is the quasi-conformal curvature tensor.

On the other hand, from (1), we have

$$R(X,Y)\xi=(f_1-f_3)\{\eta(Y)X-\eta(X)Y\} \tag{14}$$

and

$$R(\xi,X)Y=(f_1-f_3)\{g(X,Y)\xi-\eta(Y)X\}. \tag{15}$$

3. Quasi-Conformally Flat Generalized Sasakian-Space Forms

Theorem 3.1 *Let $M(f_1,f_2,f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space form. Then, we have the following: (i) If $n>1$, then M is quasi-conformally flat if and only if $f_2=-\frac{(a+(2n-1)b)}{3(an+b)}f_3$, (ii) If M is quasi-conformally flat and ξ is a Killing vector field, then it is flat, or of constant curvature, or locally the product $N^1\times N^{2n}$, where N^1 is a 1-dimensional manifold and N^{2n} is a $2n$-dimensional almost Hermitian manifold of constant curvature. In any case, M is locally symmetric and has constant ϕ-sectional curvature.*

Proof Assume that $M(f_1,f_2,f_3)$ be a $(2n+1)$-dimensional generalized Sasakian−space form. Using (1), (9), (10) and equation $S(X,Y)=g(QX,Y)$ in (13), we obtain

$$\begin{aligned}\tilde{C}(X,Y)Z &= \frac{1}{2n+1}[(-3a+6b)f_2+(2a+2(2n-1)b)f_3]\{g(Y,Z)X-g(X,Z)Y\} \\ &+af_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &+[(a+(2n-1)b)f_3+3bf_2]\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X \\ &+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}.\end{aligned} \tag{16}$$

If $a = 1$ and $b = -\frac{1}{2n-1}$, then we obtain (13), that is, the quasi-conformal curvature tensor is reduced to the conformal curvature tensor.

Suppose that $M(f_1, f_2, f_3)$ is quasi-conformally flat and $n > 1$. Then, we have $\tilde{C} = 0$.

If we put $X = \phi Y$ in (16), then we find

$$\begin{aligned}
&\frac{1}{2n+1}[3(2b-a)f_2 + 2(a+(2n-1)b)f_3]\{g(Y,Z)\phi Y - g(\phi Y, Z)Y\} \\
&+af_2\{g(\phi Y, \phi Z)\phi Y - g(Y, \phi Z)\phi^2 Y + 2g(\phi Y, \phi Y)\phi Z\} \\
&+[(a+(2n-1)b)f_3 + 3bf_2]\{\eta(\phi Y)\eta(Z)Y - \eta(Y)\eta(Z)\phi Y \\
&+g(\phi Y, Z)\eta(Y)\xi - g(Y,Z)\eta(\phi Y)\xi\} = 0
\end{aligned} \tag{17}$$

or using (3) and (4) in (17), we obtain

$$\begin{aligned}
&\frac{1}{2n+1}[3(2b-a)f_2 + a(2n+1)f_2 \\
&+2(a+(2n-1)b)f_3]\{g(Y,Z)\phi Y - g(\phi Y, Z)Y\} \\
&+[af_2 + (a+(2n-1)b)f_3 + 3bf_2]\{-\eta(Y)\eta(Z)\phi Y - g(Y, \phi Z)\eta(Y)\xi\} \\
&+af_2\{2g(Y,Y)\phi Z - 2\eta(Y)\eta(Y)\phi Z\} = 0.
\end{aligned} \tag{18}$$

If we choose a unit vector U such that $g(U, \xi) = 0$ and put $Y = U$ in (18), then we have

$$\frac{1}{2n+1}[\{(2(n-1)a+6b)f_2+2(a+(2n-1)b)f_3\}\{g(U,Z)\phi U - g(\phi U, Z)U\} + 2(2n+1)af_2\phi Z] = 0. \tag{19}$$

Putting $Z = U$ in (19), we get

$$\{(2(n-1)a + 6b + 2(2n+1)a)f_2 + 2(a+(2n-1)b)f_3\}\phi U = 0.$$

Thus, we have

$$(2(n-1)a + 6b + 2(2n+1)a)f_2 + 2(a+(2n-1)b)f_3 = 0.$$

From this equation, we get

$$f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3. \tag{20}$$

Conversely, if $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3$, then from (16), we have $\tilde{C}(X,Y)Z = 0$ and hence, $M(f_1, f_2, f_3)$ is quasi-conformally flat. Therefore, when $n > 1$, $M(f_1, f_2, f_3)$ is conformally flat if and only if $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)}f_3$. Thus, the first part (i) of the Theorem 3.1 is proved.

For the proof of the second part (ii), we assume that $M(f_1, f_2, f_3)$ is quasi-conformally flat and ξ is Killing. Then, the Schouten tensor L of the manifold is a Codazzi tensor, that is,

$$(\nabla_X L)Y - (\nabla_Y L)X = 0 \tag{21}$$

for any vector fields X, Y on M. Also, if $n > 1$, then we have $f_2 = -\frac{(a+(2n-1)b)}{3(an+b)} f_3$ by the first part (i) and hence from (12), we obtain

$$
\begin{aligned}
LX &= [-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1} + b)f_3]X \\
&\quad +[\frac{(2n+1)(n-1)}{(2n-1)(na+b)}]af_3\eta(X)\xi. \qquad (22)
\end{aligned}
$$

Using (7), from (13), we get

$$
\begin{aligned}
\tilde{C}(X,Y)Z &= aR(X,Y)Z - (2n-1)b[g(LY,Z)X - g(LX,Z)Y \\
&\quad +g(Y,Z)LX - g(X,Z)LY] \qquad (23) \\
&\quad -\frac{\tau}{2n(2n+1)}(a+(2n-1)b)[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

If $n = 1$, then from (23), we get

$$
\begin{aligned}
\tilde{C}(X,Y)Z &= aR(X,Y)Z - b[g(LY,Z)X - g(LX,Z)Y \\
&\quad +g(Y,Z)LX - g(X,Z)LY] \qquad (24) \\
&\quad -\frac{\tau}{6}(a+b)[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

Since $M(f_1, f_2, f_3)$ is quasi-conformally flat, we can write $\tilde{C}(X,Y)Z = 0$, then we get

$$
\begin{aligned}
R(X,Y)Z &= \frac{b}{a}[g(LY,Z)X - g(LX,Z)Y \\
&\quad +g(Y,Z)LX - g(X,Z)LY] \qquad (25) \\
&\quad +\frac{\tau}{6}\frac{(a+b)}{a}[g(Y,Z)X - g(X,Z)Y]
\end{aligned}
$$

for any vector fields X, Y, Z. In the 3-dimensional manifold $M(f_1, f_2, f_3)$, the Schouten tensor is given by (11),

$$
LX = -\frac{1}{2}(f_1 + 3f_2)X + (3f_2 + f_3)\eta(X)\xi. \qquad (26)
$$

From (25) and (26), we obtain

$$
\begin{aligned}
R(X,Y)Z &= [f_1 + (\frac{a-2b}{a})f_2 - \frac{2}{3}(\frac{a+b}{a})f_3]\{g(Y,Z)X - g(X,Z)Y\} \\
&\quad +\frac{b}{a}(3f_2 + f_3)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \qquad (27) \\
&\quad +g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}.
\end{aligned}
$$

If we take

$$\begin{cases} f_1^* = f_1 + (\frac{a-2b}{a})f_2 - \frac{2}{3}(\frac{a+b}{a})f_3, \\ f_3^* = \frac{b}{a}(3f_2 + f_3), \end{cases} \tag{28}$$

then we can write

$$\begin{aligned} R(X,Y)Z &= f_1^*\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_3^*\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\}. \end{aligned}$$

Equation (26) gives

$$L\xi = (-\frac{1}{2}f_1 + \frac{3}{2}f_2 + f_3)\xi. \tag{29}$$

If X is a vector orthogonal to ξ, then we get

$$LX = -\frac{1}{2}(f_1 + 3f_2)X. \tag{30}$$

For $n > 1$, then from (22), we get

$$L\xi = -\frac{1}{2}[f_1 - \{\frac{1}{na+b}[(\frac{4n^2-2n-1}{2n-1})a + b]\}f_3]\xi. \tag{31}$$

If X is a vector orthogonal to ξ, then we have

$$LX = [-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1} + b)f_3]X. \tag{32}$$

Let $\xi, E_1, E_2, ..., E_{2n}$ be local orthonormal vector fields on $M(f_1, f_2, f_3)$. Then from (21), (22) and (32), we get

$$\begin{aligned} (\nabla_{E_i}L)E_j - (\nabla_{E_j}L)E_i &= -\frac{1}{2}(E_i f_1)E_j + \frac{1}{2}(E_j f_1)E_i \\ &+ \frac{1}{2(na+b)}(\frac{a}{2n-1} + b)[(E_i f_3)E_j - (E_j f_3)E_i] \\ &+ \frac{(2n+1)(n-1)}{(2n-1)(na+b)}af_3\eta(\nabla_{E_i}E_j - \nabla_{E_j}E_i)\xi = 0. \end{aligned} \tag{33}$$

Taking inner product with E_j in (33), we have

$$(E_j f_1) = \frac{1}{(na+b)}(\frac{a}{2n-1} + b)(E_j f_3). \tag{34}$$

Using (31), we obtain

$$(\nabla_{E_j} L)\xi + L \nabla_{E_j} \xi = -\frac{1}{2}\{f_1 - \frac{1}{(na+b)}[(\frac{4n^2-2n-1}{2n-1})a+b]f_3\} \nabla_{E_j} \xi$$
$$-\frac{1}{2}(E_j f_1)\xi + \frac{1}{2(na+b)}[(\frac{4n^2-2n-1}{2n-1})a+b](E_j f_3)\xi. \quad (35)$$

If we use (34) in (35), then we get

$$(\nabla_{E_j} L)\xi + L \nabla_{E_j} \xi = -\frac{1}{2}\{f_1 - \frac{1}{(na+b)}[(\frac{4n^2-2n-1}{2n-1})a+b]f_3\} \nabla_{E_j} \xi$$
$$+\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a(E_j f_3)\xi. \quad (36)$$

Since $\nabla_{E_j}\xi$ is orthogonal to ξ, using (32), we get

$$L(\nabla_{E_j}\xi) = [-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)f_3] \nabla_{E_j} \xi. \quad (37)$$

Thus from (36), we obtain

$$(\nabla_{E_j} L)\xi = [\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a]((E_j f_3)\xi + f_3 \nabla_{E_j} \xi). \quad (38)$$

Since ξ is Killing, then we get

$$(\nabla_\xi L)E_j + L(\nabla_\xi E_j) = [-\frac{1}{2}\xi(f_1) + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)\xi(f_3)]E_j$$
$$+[-\frac{1}{2}f_1 + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)f_3] \nabla_\xi E_j, \quad (39)$$

where

$$L(\nabla_\xi E_j) = -\frac{1}{2}f_1 \nabla_\xi E_j + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)f_3 \nabla_\xi E_j. \quad (40)$$

Thus from (36), we have

$$(\nabla_\xi L)E_j = [-\frac{1}{2}\xi(f_1) + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)\xi(f_3)]E_j. \quad (41)$$

Since $(\nabla_{E_j} L)\xi = (\nabla_\xi L)E_j$, from (38) and (41), we get

$$[\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a]((E_j f_3)\xi + f_3 \nabla_{E_j} \xi) \quad (42)$$
$$= [-\frac{1}{2}\xi(f_1) + \frac{1}{2(na+b)}(\frac{a}{2n-1}+b)\xi(f_3)]E_j.$$

Taking inner product with E_j in (42), we obtain

$$\xi(f_1) = \frac{1}{(na+b)}(\frac{a}{2n-1} + b)\xi(f_3). \tag{43}$$

Taking inner product with ξ, from (42), we get

$$[\frac{(2n+1)(n-1)}{(2n-1)(na+b)}a]((E_j f_3)\xi + f_3 \nabla_{E_j} \xi) = 0, \tag{44}$$

this gives $E_j f_3 = 0$ and $f_3 \nabla_{E_j} \xi = 0$ $(j = 1, 2, ..., 2n)$. Combining this with $\nabla_\xi \xi = 0$ gives

$$f_3(\nabla_X \xi) = 0 \tag{45}$$

for any vector field X. From (45), we get

$$(Y f_3)(\nabla_X \xi) + f_3 \nabla_Y \nabla_X \xi = 0.$$

This equation and (45) give

$$(X f_3) \nabla_Y \xi - (Y f_3) \nabla_X \xi + f_3[\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi] = 0.$$

Multiplying this equation with f_3 and using (45), we get

$$f_3^2 R(X, Y)\xi = 0.$$

This equation and (14) give

$$f_3^2(f_1 - f_3)[\eta(Y)X - \eta(X)Y] = 0$$

from which we obtain $f_3(f_1 - f_3) = 0$.

Consider the case $f_1 = 0$. In this case, we have $f_3 = 0$ on M and hence, $f_2 = 0$. Thus, M is flat.

Next consider the case $f_1 \neq 0$. Differentiating $f_3(f_1 - f_3) = 0$ with ξ gives $\{f_1 + [\frac{1}{(na+b)}(\frac{a}{2n-1} + b) - 2]f_3\}\xi(f_3) = 0$. If $f_3(p) = 0$ at a point $p \in M$, then $f_1(p)\xi(f_3)(p) = 0$, where since $f_1 \neq 0$, we get $\xi(f_3) = 0$ at p. If $f_3(p) \neq 0$, then $f_3 = f_1$ in an open neighborhood U of p. Thus, $\{\frac{a(1+n-2n^2)}{(na+b)(2n-1)} f_3\}\xi(f_3) = 0$. For $n > 1$, since $1 + n - 2n^2 \neq 0$, we get $\xi(f_3) = 0$ on U. Thus, we have $\xi(f_3) = 0$ on M. Since $E_j f_3 = 0$ $(j = 1, 2, ..., 2n)$, f_3 is constant on M. Hence, we have:

(a) If $f_3 = 0$, then M is of constant curvature f_1.

(b) If $f_3 \neq 0$, then we have $f_1 = f_3$ and $\nabla_X \xi = 0$ for any vector X on M. Hence, the Schouten tensor L has two distinct constant eigenvalues $\frac{1}{2}f_1$ with multiplicity 1 and $-\frac{1}{2}f_1$ with multiplicity $2n$. Therefore, we have the decomposition $\mathcal{D} \oplus [\xi]$, where $\mathcal{D}$ is the distribution defined by $\eta = 0$ and $[\xi]$ is the distribution spanned by the vector ξ. By Lemma 2.1, $\mathcal{D}$ is integrable. Hence, M is locally product of an integral submanifold N^1 of $[\xi]$ and an integral submanifold N^{2n} of $\mathcal{D}$. Since the eigenvalue is constant on M, N^{2n} is a totally geodesic submanifold of M by Lemma 2.1. If we denote the restriction of ϕ in $\mathcal{D}$ by J, then

$$J^2X = \phi^2X = -X + \eta(X)\xi = -X$$

for any $X \in \mathcal{D}$. Hence, J defines an almost complex structure on N^{2n}.

Also, $g'(JX, JY) = g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) = g'(X, Y)$ for any $X, Y \in \mathcal{D}$, where g' is the induced metric on N^{2n} from g. Hence, (N^{2n}, J, g') is an almost Hermitian manifold. Since N^{2n} is a totally geodesic hypersurface of M, the equation of Gauss is given by

$$R(X, Y)Z = R'(X, Y)Z$$

for any vector fields X, Y and Z tangent to N^{2n}, where R' is the curvature tensor of N^{2n}. Thus, we get

$$R'(X, Y)Z = f_1[g'(Y, Z)X - g'(X, Z)Y]$$

and hence, N^{2n} is a space of constant curvature f_1. In any case, from the above arguments, we can easily see that $M(f_1, f_2, f_3)$ is locally symmetric. Since f_1 and f_3 are constants, we can see that M is of constant ϕ-sectional curvature. This completes the proof of the Theorem 3.1. □

The above theorem was proved in another ways by Kim [5] and Sarkar and De [7].

Remark 3.2 *In the Theorem* 1, *the condition "ξ is Killing vector field" cannot be removed. For example, given (N, J, g) with constant curvature c, say, a 6-dimensional sphere with nearly Kaehler structure [6], the warped product $M = \mathbb{R} \times_f N$, where $f > 0$ is a nonconstant function on $\mathbb{R}$, can be endowed with an almost contact metric structure (ϕ, ξ, η, g_f).*

4. Quasi-Conformally Weyl-Symmetric Generalized Sasakian-Space Forms

Let us consider a quasi-conformally Weyl-symmetric generalized Sasakian-space form $M(f_1, f_2, f_3)$. Then, the condition

$$R(X, Y) \cdot \tilde{C} = 0$$

holds on $M(f_1, f_2, f_3)$ for every vector fields X, Y. Hence, we have

$$\begin{aligned}(R(X, Y) \cdot \tilde{C})(U, V)W &= R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W \\ &\quad -\tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W = 0. \end{aligned} \tag{46}$$

So, for $X=\xi$ in (46), we have

$$R(\xi,Y)\tilde{C}(U,V)W-\tilde{C}(R(\xi,Y)U,V)W$$
$$-\tilde{C}(U,R(\xi,Y)V)W-\tilde{C}(U,V)R(\xi,Y)W=0. \tag{47}$$

From (15), we get

$$(f_1-f_3)\{g(Y,\tilde{C}(U,V)W)\xi-\eta(\tilde{C}(U,V)W)Y-g(Y,U)\tilde{C}(\xi,V)W$$
$$+\eta(U)\tilde{C}(Y,V)W-g(Y,V)\tilde{C}(U,\xi)W+\eta(V)\tilde{C}(U,Y)W$$
$$-g(Y,W)\tilde{C}(U,V)\xi+\eta(W)\tilde{C}(U,V)Y\}=0. \tag{48}$$

Taking the inner product of (48) with ξ, we obtain

$$(f_1-f_3)\{g(Y,\tilde{C}(U,V)W)-\eta(\tilde{C}(U,V)W)\eta(Y)-g(Y,U)\eta(\tilde{C}(\xi,V)W)$$
$$+\eta(U)\eta(\tilde{C}(Y,V)W)-g(Y,V)\eta(\tilde{C}(U,\xi)W)+\eta(V)\eta(\tilde{C}(U,Y)W)$$
$$+\eta(W)\eta(\tilde{C}(U,V)Y)\}=0. \tag{49}$$

Putting $Y=U$ in (49), we have

$$(f_1-f_3)\{g(U,\tilde{C}(U,V)W)-\eta(\tilde{C}(U,V)W)\eta(U)-g(U,U)\eta(\tilde{C}(\xi,V)W)$$
$$+\eta(U)\eta(\tilde{C}(U,V)W)-g(U,V)\eta(\tilde{C}(U,\xi)W)+\eta(V)\eta(\tilde{C}(U,U)W)$$
$$+\eta(W)\eta(\tilde{C}(U,V)U\}=0. \tag{50}$$

From (16), we get

$$\eta(\tilde{C}(X,Y)Z)=(\frac{a+(2n-1)b}{2n+1})[-3f_2+(1-2n)f_3]\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}. \tag{51}$$

Putting $Z=\xi$, the equation (51) turns into the form

$$\eta(\tilde{C}(X,Y)\xi)=0. \tag{52}$$

Thus, using (52) in (50), we obtain

$$(f_1-f_3)\{g(U,\tilde{C}(U,V)W)-g(U,U)\eta(\tilde{C}(\xi,V)W)$$
$$-g(U,V)\eta(\tilde{C}(U,\xi)W)+\eta(W)\eta(\tilde{C}(U,V)U)\}=0. \tag{53}$$

Let $\{e_i\}$, $1\leq i\leq 2n+1$, $(e_{2n+1}=\xi)$ be an orthonormal basis of the tangent space at any point. Then, the sum for $U=e_i$, $1\leq i\leq 2n+1$, of the relation (53) give us

$$(f_1-f_3)\{g(e_i,\tilde{C}(e_i,V)W)-g(e_i,e_i)\eta(\tilde{C}(\xi,V)W)$$
$$-g(e_i,V)\eta(\tilde{C}(e_i,\xi)W)+\eta(W)\eta(\tilde{C}(e_i,V)e_i)\}=0. \tag{54}$$

On the other hand, from (51), we have

$$\eta(\tilde{C}(\xi,V)W) = (\frac{a+(2n-1)b}{2n+1})[-3f_2+(1-2n)f_3]\{g(W,V)-\eta(W)\eta(V)\}. \quad (55)$$

Using (55) in (54), we get

$$(f_1-f_3)\{g(e_i,\tilde{C}(e_i,V)W)+2n(\frac{a+(2n-1)b}{2n+1})[3f_2+(1-2n)f_3]g(W,V)\}=0. \quad (56)$$

Also, from (16), we have

$$\begin{aligned}\tilde{C}(e_i,V)W &= \frac{1}{2n+1}[(-3a+6b)f_2+(2a+2(2n-1)b)f_3][g(W,V)e_i-g(W,e_i)V]\\ &+af_2[g(e_i,\phi W)\phi V-g(V,\phi W)\phi e_i+2g(e_i,\phi V)\phi W] \quad (57)\\ &+[(a+(2n-1)b)f_3+3bf_2][\eta(e_i)\eta(W)V-\eta(V)\eta(W)e_i\\ &+g(e_i,W)\eta(V)\xi-g(V,W)\eta(e_i)\xi].\end{aligned}$$

Taking the inner product of (57) with e_i, we get

$$g(\tilde{C}(e_i,V)W,e_i) = (\frac{a+(2n-1)b}{2n+1})(3f_2+(2n-1)f_3)[g(W,V)-(2n+1)\eta(W)\eta(V)]. \quad (58)$$

If we use (58) in (56), we get

$$(f_1-f_3)(a+(2n-1)b)(3f_2+(2n-1)f_3)[g(W,V)-\eta(W)\eta(V)]=0. \quad (59)$$

If $f_1 \neq f_3$ and $a \neq (2n-1)b$, then $3f_2+(2n-1)f_3=0$, that is,

$$f_2 = -\frac{(2n-1)}{3}f_3. \quad (60)$$

Hence, using (60) in (10), we obtain

$$\tau = 2n(2n+1)(f_1-f_3) \quad (61)$$

and using (60) in (9), we get

$$QX = 2n(f_1-f_3)X. \quad (62)$$

So, we have the following result:

Theorem 4.1 *Let $M(f_1,f_2,f_3)$ be a generalized Sasakian-space form. Then, M^{2n+1} $(n>1)$ is quasi-conformally Weyl-symmetric if and only if either $f_1=f_3$ or $f_2=-\frac{(2n-1)}{3}f_3$ (when $f_1 \neq f_3$), where $a \neq (2n-1)b$.*

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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On Bi-f-Harmonic Legendre Curves in Sasakian Space Forms

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Abstract: In this study, we consider bi-f-harmonic Legendre curves in Sasakian space forms. We investigate necessary and sufficient conditions for a Legendre curve to be bi-f-harmonic in various cases.

Keywords: Bi-f-harmonic curves, Legendre curves, Sasakian space forms.

1. Introduction

Let (N,g) and $(\bar{N},\bar{g})$ be two Riemannian manifolds and $\psi:(N,g)\to(\bar{N},\bar{g})$ be a smooth map. Then, let give the following definitions.

Definition 1.1 *Harmonic maps between two Riemannian manifolds are critical points of the energy functional*

$$E(\psi)=\frac{1}{2}\int_N |d\psi|^2 dv_g$$

for smooth maps $\psi:(N,g)\to(\bar{N},\bar{g})$. *Namely,* ψ *is called as harmonic if*

$$\tau(\psi)=-d^* d\psi = trace\nabla d\psi = 0.$$

Here $\tau(\psi)$, which is the tension field of ψ, is the Euler-Lagrange equation of the energy functional $E(\psi)$, d is the exterior differentiation, d^* is the codifferentiation, ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{N}}$ of $\bar{N}$ and the pull-back connection $\nabla^{\bar{N}}$ [1, 3, 8].

Definition 1.2 ψ *is called as biharmonic if it is critical point, for all variations, of the bienergy functional*

$$E_2(\psi)=\frac{1}{2}\int_N |\tau(\psi)|^2 dv_g.$$

It means that ψ *is a biharmonic map if bitension field* $\tau_2(\psi)$ *equals to*

$$\tau_2(\psi)=trace(\nabla^{\psi}\nabla^{\psi}-\nabla^{\psi}_{\nabla})\tau(\psi)-trace(R^{\bar{N}}(d\psi,\tau(\psi))d\psi)=0, \quad (1)$$

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where $R^{\bar{N}}$ is the curvature tensor field of $\bar{N}$ [3, 12].

It is easy to see that any harmonic map is a biharmonic map. On the other hand, a biharmonic map is called as proper biharmonic if it is not harmonic. Now, let us remind the definition of a bi-f-harmonic map.

Definition 1.3 *ψ is called as bi-f-harmonic if it is critical point of the bi-f-energy functional*

$$E_{f,2}(\psi) = \frac{1}{2}\int_N |\tau_f(\psi)|^2 dv_g,$$

where $\tau_f(\psi) = f\tau(\psi) + d\psi(gradf)$ is the f-tension field. The Euler-Lagrange equation for the bi-f-harmonic map is given by

$$\tau_{f,2}(\psi) = trace\big(\nabla^{\psi} f(\nabla^{\psi}\tau_f(\psi)) - f\nabla^{\psi}_{\nabla^N}\tau_f(\psi) + fR^{\bar{N}}(\tau_f(\psi), d\psi)d\psi\big) = 0, \tag{2}$$

here $\tau_{f,2}(\psi)$ is the bi-f-tension field of the map ψ and f is a smooth positive function on the domain [12].

Note that overall throughout this paper, we will use SSF instead of Sasakian space form for the sake of simplicity.

The authors of [14] summarized the relationship between biharmonic and bi-f-harmonic maps; by extending bienergy functional to bi-f-energy functional defining a new type of harmonic map called as bi-f-harmonic map.

Bi-f-harmonic maps were introduced by Ouakkas et al. in 2010 [9] and Perktaş et al. obtained bi-f-harmonicity conditions of curves in Riemannian manifolds and derived bi-f-harmonic equations for curves in various spaces such as Euclidean and hyperbolic space in 2019 [12]. Biharmonic Legendre curves were handled in SSF by Fetcu in 2008 [4] and were introduced by Özgür and Güvenç in generalized SSF and S-space forms in 2014 [10, 11]. Subsequently, f-biharmonic Legendre curves were examined by Özgür and Güvenç in SSF in 2017 and were studied by Güvenç in S-space forms in 2019 [6, 7].

Inspired by these papers, in this study, we examined bi-f-harmonic Legendre curves in Sasakian space form. Firstly, in Section 2, we remind definition and properties of a Sasakian space form. Then, in Section 3, we give our main theorems and corollaries.

2. Sasakian Space Forms

Let (N, g) be a framed metric manifold with $dim(N) = (2n + s)$ and a framed metric structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$, where $\alpha \in \{1, ..., s\}$; φ is a $(1,1)$ tensor field defining a $\varphi-$structure of rank $2n$;

$\xi_1, ..., \xi_s$ are vector fields; $\eta^1, ..., \eta^s$ are $1-$forms and g is a Riemannian metric on N. For all $K, L \in TN$ and $\alpha, \beta \in \{1, ..., s\}$, following formulas are satisfied;

$$\varphi^2 K = -K + \sum_{\alpha=1}^{s} \eta^\alpha(K)\xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad \varphi(\xi_\alpha) = 0, \quad \eta^\alpha \circ \varphi = 0, \tag{3}$$

$$g(\varphi K, \varphi L) = g(K, L) - \sum_{\alpha=1}^{s} \eta^\alpha(K)\eta^\alpha(L), \tag{4}$$

$$d\eta^\alpha(K, L) = g(K, \varphi L) = -d\eta^\alpha(L, K), \quad \eta^\alpha(K) = g(K, \xi). \tag{5}$$

If Nijenhuis tensor of φ equals to $-2d\eta^\alpha \otimes \xi_\alpha$ for all $\alpha \in \{1, ..., s\}$, then $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is called $S-$structure and if $s = 1$, a framed metric structure becomes an almost contact metric structure; an $S-$structure becomes a Sasakian structure, then we have [2, 11, 13]:

$$(\nabla_K \varphi)L = \sum_{\alpha=1}^{s} (g(\varphi K, \varphi L)\xi_\alpha + \eta^\alpha(L)\varphi^2 K), \tag{6}$$

$$\nabla \xi_\alpha = -\varphi, \quad \alpha \in \{1, ..., s\}. \tag{7}$$

A plane section in T_pN is a φ-section if there exists a vector $K \in T_pN$ being orthogonal to $\xi_1, ..., \xi_s$ such that $K, \varphi K$ span the section. The sectional curvature of a φ-section is called φ-sectional curvature such that a S-manifold of constant φ-section curvature c is called as S-space form. Finally, if $s = 1$, a S-space form becomes a Sasakian space form [2, 6, 7]. For a SSF, from equations (6) and (7), it is easy to see that

$$(\nabla_K \varphi)L = g(K, L)\xi - \eta(L)K, \tag{8}$$

$$\nabla_K \xi = -\varphi K \tag{9}$$

and the curvature tensor R of a SSF is given by

$$\begin{aligned} R(K, L)M &= \frac{c+3}{4}\big(g(L, M)K - g(K, M)L\big) \\ &+ \frac{c-1}{4}\Big(g(K, \varphi M)\varphi L - g(L, \varphi M)\varphi K + 2g(K, \varphi L)\varphi M + \eta(K)\eta(M)L \\ &- \eta(L)\eta(M)K + g(K, M)\eta(L)\xi - g(L, M)\eta(K)\xi\Big) \end{aligned} \tag{10}$$

for all $K, L, M \in TN$ [2].

Here let's remind the definition of a Legendre curve in a SSF.

Definition 2.1 *A Legendre curve of a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$ *is a one dimensional integral submanifold of* N *and* $\beta : I \to (N^{2n+1}, \varphi, \xi, \eta, g)$ *is a Legendre curve if* $\eta(T) = 0$*, where* T *is the tangent vector field of* β *[6, 7].*

3. Bi-f-harmonic Legendre Curves in Sasakian Space Forms

Let $\beta : I \longrightarrow N$ be an arc-length parametrized curve in a m-dimensional Riemannian manifold (N, g) and $u_1, u_2, , u_r$ are vector fields along β such that

$$\begin{aligned} u_1 = \beta^{'} &= T, \\ \nabla_{u_1} u_1 &= k_1 u_2, \\ \nabla_{u_1} u_2 &= -k_1 u_1 + k_2 u_3, \\ &\vdots \\ \nabla_{u_1} u_r &= -k_{r-1} u_{r-1}. \end{aligned} \tag{11}$$

Then, β is called a Frenet curve of osculating order r, here $k_1, \ldots, k_{r-1}$ are positive functions on I and $1 \leq r \leq m$. With the help of Definition 1.3, β is called a bi-f-harmonic curve if and only if following condition is hold [12],

$$\begin{aligned} \tau_{f,2}(\beta) &= (f f^{''})^{'} u_1 + (3 f f^{''} + 2(f^{'})^2) \nabla_{u_1} u_1 + 4 f f^{'} \nabla^2_{u_1} u_1 + f^2 \nabla^3_{u_1} u_1 + f^2 R(\nabla_{u_1} u_1, u_1) u_1 \\ &= 0. \end{aligned} \tag{12}$$

Now, let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form and $\beta : I \to N$ be a Legendre curve. Then, with the help of equation (11) and derivative of $\eta(T) = \eta(u_1) = 0$, following equality

$$\eta(u_2) = 0 \tag{13}$$

is obtained [7]. By using equations (10), (11) and (13), we get the following equalities

$$\begin{aligned} \nabla_{u_1} u_1 &= k_1 u_2, \\ \nabla_{u_1} \nabla_{u_1} u_1 &= \nabla^2_{u_1} u_1 = -k_1^2 u_1 + k_1^{'} u_2 + k_1 k_2 u_3, \\ \nabla_{u_1} \nabla_{u_1} \nabla_{u_1} u_1 &= \nabla^3_{u_1} u_1 = -3 k_1 k_1^{'} u_1 + \big(- k_1^3 + k_1^{''} - k_1 k_2^2 \big) u_2 \\ &\quad + \big(2 k_1^{'} k_2 + k_1 k_2^{'} \big) u_3 + k_1 k_2 k_3 u_4, \\ R(\nabla_{u_1} u_1, u_1) u_1 &= k_1 (\frac{c+3}{4}) u_2 + 3 k_1 (\frac{c-1}{4}) g(u_2, \varphi u_1) \varphi u_1. \end{aligned}$$

Then, by substutiting these equalities into the bi-f-harmonicity condition, namely into the equation (12), we obtain bi-f-harmonicity condition of a Legendre curve in a Sasakian space form as follows,

$$\begin{aligned}
\tau_{f,2}(\beta) &= [(ff'')' - 4k_1^2ff' - 3k_1k_1'f^2]u_1 \\
&+ \big[(3ff'' + 2(f')^2)k_1 + 4ff'k_1' + \big(- k_1^3 + k_1'' - k_1k_2^2 + k_1(\frac{c+3}{4})\big)f^2\big]u_2 \\
&+ [4ff'k_1k_2 + f^2(2k_1'k_2 + k_1k_2')]u_3 \\
&+ [k_1k_2k_3f^2]u_4 \\
&+ 3f^2k_1(\frac{c-1}{4})g(u_2, \varphi u_1)\varphi u_1 \\
&= 0. \qquad (14)
\end{aligned}$$

It should be noted that if function f is a constant, then bi-f-harmonicity condition turns into a biharmonicity condition. For this reason, the function f will be considered different from a constant throughout the paper.

Now, we give interpretations of bi-f-harmonicity condition given in equation (14).

Remark 3.1 *[12] The property of a curve being bi-f-harmonic in a n-dimensional space $(n > 3)$ does not depend on all its curvatures, but only on k_1, k_2 and k_3.*

Let $k = min\{r, 4\}$. From equation (14), β is a bi-f-harmonic curve if and only if $\tau_{f,2}(\beta) = 0$, namely,

(i) $c = 1$ or $\varphi u_1 \perp u_2$ or $\varphi u_1 \in sp\{u_2, ..., u_k\}$,

(ii) $g(\tau_{f,2}(\beta), u_i) = 0$ for all $i = 1, ..., k$.

Thus, we can give the following main theorem.

Theorem 3.2 *Let β be a non-geodesic Legendre curve of osculating order r in a Sasakian space form $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $k = min\{r, 4\}$. Then, β is a bi-f-harmonic curve if and only if*

(i) $c = 1$ or $\varphi u_1 \perp u_2$ or $\varphi u_1 \in sp\{u_2, ..., u_k\}$,

(ii) the first k of the following differential equations are satisfied

$$\begin{cases}
(ff'')' - 4k_1^2ff' - 3k_1k_1'f^2 = 0, \\
k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})g(u_2, \varphi u_1)^2, \\
4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' + 3(\frac{c-1}{4})g(u_2, \varphi u_1)g(u_3, \varphi u_1) = 0, \\
k_2k_3 + 3(\frac{c-1}{4})g(u_2, \varphi u_1)g(u_4, \varphi u_1) = 0.
\end{cases} \qquad (15)$$

From here on, we investigate results of Theorem 3.2 in eight cases.

Case I: If $c = 1$, then equation (15) reduces to

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2k_3 = 0. \end{cases}$$

Hence, we have Theorem 3.3.

Theorem 3.3 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $c = 1$. Then, β is a bi-f-harmonic curve iff following differential equations are satisfied*

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2k_3 = 0. \end{cases} \tag{16}$$

Also, we get the following corollary from Theorem 3.2.

Corollary 3.4 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $c = 1$. Then, β is a bi-f-harmonic curve iff either*

(i) β is of osculating order $r = 2$ and f, k_1 satisfy the following differential equations

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1 \end{cases}$$

or

(ii) β is of osculating order $r = 3$ and f, k_1, k_2 satisfy the following differential equations

$$\begin{cases} (ff^{''})^{'} - 4k_1^2 ff^{'} - 3k_1k_1^{'}f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0. \end{cases}$$

Proof It is known that if k_2 equals to zero, then β is called as of osculating order 2. Here, if we substitute zero, for k_2 in equation (16), third and fourth equations are vanished, then we obtain the differential equations given in (i). On the other hand, if k_3 equals to zero, then β is called as of osculating order 3 and similarly, substutiting zero for k_3 in equation (16), fourth equation is vanished, so we obtain the differential equations given in (ii).

Case II: If $c = 1$ and $(f.f^{''})^{'} = 0$, then equation (15) reduces to

$$\begin{cases} 4k_1^2 f f^{'} + 3k_1 k_1^{'} f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f^{''}}{f} + 2(\frac{f^{'}}{f})^2 + 4\frac{k_1^{'}}{k_1}\frac{f^{'}}{f} + \frac{k_1^{''}}{k_1} + 1, \\ 4k_2\frac{f^{'}}{f} + 2k_2\frac{k_1^{'}}{k_1} + k_2^{'} = 0, \\ k_2 k_3 = 0. \end{cases} \tag{17}$$

Hence, we have Theorem 3.5.

Theorem 3.5 *Let β be a Legendre curve with non-constant geodesic curvature in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c = 1$, $(f.f^{''})^{'} = 0$ and $n \geq 2$. Then, β is a bi-f-harmonic curve iff either*

(i) β is of osculating order $r = 2$ with $f = c_1 k_1^{-\frac{3}{4}}$, where c_1 is a positive integration constant and k_1 satisfy the following second order non-linear ordinary differential equation

$$16k_1^4 - 16k_1^2 - 33(k_1^{'})^2 + 20k_1 k_1^{''} = 0$$

or

(ii) β is of osculating order $r = 3$ with $f = c_1 k_1^{-\frac{3}{4}}$, $k_2 = c_2 k_1$, where c_1, c_2 are positive integration constants and k_1 satisfy the following second order non-linear ordinary differential equation

$$16(1 + c_2^2)k_1^4 + 20k_1 k_1^{''} - 33(k_1^{'})^2 - 16k_1^2 = 0.$$

Proof By using the first equation of (17), we get

$$\frac{f^{'}}{f} = -\frac{3}{4}\frac{k_1^{'}}{k_1}, \quad \frac{f^{''}}{f} = \frac{21}{16}(\frac{k_1^{'}}{k_1})^2 - \frac{3}{4}\frac{k_1^{''}}{k_1}. \tag{18}$$

Thus from equation (18), we obtain $f = c_1 k_1^{-\frac{3}{4}}$, where c_1 is an integration constant. Then, we know that if $k_2 = 0$, β is called as of osculating order $r = 2$ and if $k_2 = 0$, third and fourth equations of (17) are vanished. Finally, by substituting equation (18) to the second equation of (17), we obtain a second order non-linear ordinary differential equation $16k_1^4 - 16k_1^2 - 33(k_1^{'})^2 + 20k_1 k_1^{''} = 0$.

On the other hand, we know that if $k_3 = 0$, β is called as of osculating order $r = 3$ and if $k_3 = 0$, fourth equation of (17) is vanished. Then, by substituting equation (18) to the third equation of (17), we obtain that $k_2 = c_2 k_1$ for a positive integration constant c_2. Finally, by using these results in the second equation of (17), we get second order non-linear ordinary differential equation $16(1+c_2^2)k_1^4 + 20k_1 k_1'' - 33(k_1')^2 - 16k_1^2 = 0$. So, the proof is complete. □

Case III: If $c \neq 1$ and $\varphi u_1 \perp u_2$, then equation (15) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0, \\ k_2 k_3 = 0. \end{cases}$$

Then, before giving Theorem 3.7, we need the following proposition.

Proposition 3.6 *[5] Let β be a Legendre curve of osculating order 3 in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$ and $\varphi u_1 \perp u_2$. Then, $\{u_1, u_2, u_3, \varphi u_1, \nabla_{u_1}\varphi u_1, \xi\}$ is linearly independent at any point of β. Consequently, $n \geq 3$.*

Now, we can give Theorem 3.7.

Theorem 3.7 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ and $\varphi u_1 \perp u_2$. Then, β is a bi-f-harmonic curve iff following differential equations are satisfied*

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0, \\ k_2 k_3 = 0. \end{cases}$$

Now, we can introduce the Corollary 3.8 of Theorem 3.7.

Corollary 3.8 *Let β be a non-geodesic Legendre curve in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ and $\varphi u_1 \perp u_2$. Then, β is a bi-f-harmonic curve iff either*

(i) β is of osculating order $r = 2$ and f, k_1 satisfy the following differential equations

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4} \end{cases}$$

or

(ii) β *is of osculating order* $r = 3$, $n \geq 3$ *and* f, k_1, k_2 *satisfy the following differential equations*

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0. \end{cases}$$

Proof The proof is similar to the proof of Corollary 3.4. □

Now, let investigate the Case IV.

Case IV: If $c \neq 1$, $\varphi u_1 \perp u_2$ and $(ff'')' = 0$, then equation (15) reduces to

$$\begin{cases} 4k_1^2 ff' + 3k_1k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4}, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0, \\ k_2k_3 = 0. \end{cases}$$

Now, with the help of Proposition 3.6, we can give the Theorem 3.9.

Theorem 3.9 *Let* β *be a Legendre curve with non-constant geodesic curvature in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ *and* $\varphi u_1 \perp u_2$. *Then,* β *is a bi-*f*-harmonic curve iff either*

(i) β *is of osculating order* $r = 2$ *with* $f = c_1 k_1^{-\frac{3}{4}}$, $\{u_1, u_2, \varphi u_1, \nabla_{u_1}\varphi u_1, \xi\}$ *is linearly independent,* $n \geq 2$ *and* k_1 *satisfy the following second order non-linear ordinary differential equation*

$$16k_1^4 - 4(c+3)k_1^2 - 33(k_1')^2 + 20k_1k_1'' = 0$$

or

(ii) β *is of osculating order* $r = 3$ *with* $f = c_1 k_1^{-\frac{3}{4}}$, $k_2 = c_2 k_1$, $\{u_1, u_2, u_3, \varphi u_1, \nabla_{u_1}\varphi u_1, \xi\}$ *is linearly independent,* $n \geq 3$ *and* k_1 *satisfy the following second order non-linear ordinary differential equation*

$$16(1 + c_2^2)k_1^4 + 20k_1k_1'' - 33(k_1')^2 - 4(c+3)k_1^2 = 0.$$

Proof It is proved as similar to the proof of Theorem 3.5. □

Case V: Let $c \neq 1$ and $\varphi u_1 \parallel u_2$.

In this case, since $\varphi u_1 \parallel u_2$, we can write $\varphi u_1 = \mp u_2$. Hence, $g(u_2, \varphi u_1) = \mp 1, g(u_3, \varphi u_1) = g(u_3, \mp u_2) = 0$ and similarly, $g(u_4, \varphi u_1) = g(u_4, \mp u_2) = 0$. Then, equation (15) reduces to

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + c, \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' = 0, \\ k_2 k_3 = 0. \end{cases} \tag{19}$$

Remark 3.10 *In [11], it is proved that in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$ *if* $c \neq 1$ *and* $\varphi u_1 \parallel u_2$*, then* $k_2 = 1$.

Hence, we give the Theorem 3.11.

Theorem 3.11 *Let* β *be a non-geodesic Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ *and* $\varphi u_1 \parallel u_2$*. Then,* β *is a bi-f-harmonic curve iff it is of osculating order* $r = 3$ *with* $f = c_1 k_1^{-\frac{1}{2}}$ *and* k_1 *satisfies the following differential equations*

$$\begin{cases} 18(k_1')^3 - 11k_1 k_1' k_1'' + 4k_1^2 k_1''' + 8k_1^4 k_1' = 0, \\ 4k_1^4 - 3(k_1')^2 + 2k_1 k_1'' - 4(c-1)k_1^2 = 0. \end{cases}$$

Proof First of all from Remark 3.10, we know that $k_2 = 1$ and by choosing β as a curve of osculating order $r = 3$, we get $k_3 = 0$. Then, when we substitute these informations into the equation (19), we get

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + c - 1, \\ 2\frac{f'}{f} + \frac{k_1'}{k_1} = 0. \end{cases} \tag{20}$$

Then, with help of third equation of (20), we obtain

$$\frac{f'}{f} = -\frac{1}{2}\frac{k_1'}{k_1}, \quad \frac{f''}{f} = \frac{3}{4}(\frac{k_1'}{k_1})^2 - \frac{1}{2}\frac{k_1''}{k_1}. \tag{21}$$

Finally, if equation (21) is substituted into the first and second equation of (20), then two equations are found for k_1 and the proof is completed. □

Case VI: If $c \neq 1$, $\varphi u_1 \parallel u_2$ and $(ff'')' = 0$, then by using Remark 3.10, equation (15) reduces to

$$\begin{cases} 4k_1^2 ff' + 3k_1 k_1' f^2 = 0, \\ k_1^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + c - 1, \\ 2\frac{f'}{f} + \frac{k_1'}{k_1} = 0. \end{cases} \tag{22}$$

In this case, if we take into consideration first and third equations of (22), then it is easy to see that f is a constant. Therefore, we obtain Theorem 3.12.

Theorem 3.12 *There is no bi-f-harmonic Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$*, where* $c \neq 1$, $\varphi u_1 \parallel u_2$ *and* $(ff'')' = 0$.

Considering that f is a constant, then we get Corollary 3.13.

Corollary 3.13 *Let* β *be a Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$*, where* $c \neq 1$, $\varphi u_1 \parallel u_2$ *and* $(ff'')' = 0$*. Then,* β *is a biharmonic curve if and only if it is a helix with* $k_1 = \sqrt{c-1}$ *and* $k_2 = 1$.

Case VII: Let $c \neq 1$ and $g(u_2, \varphi u_1)$ is not equal to $-1, 0$ or 1.

Now, let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a SSF and $\beta : I \longrightarrow N$ be a Legendre curve of osculating order r, where $4 \leq r \leq 2n+1$ and $n \geq 2$. We know that if β is bi-f-harmonic, then $\varphi u_1 \in sp\{u_2, u_3, u_4\}$. Here, let denote the angle between φu_1 and u_2 by $\phi(t)$, namely,

$$g(u_2, \varphi u_1) = cos\phi(t). \tag{23}$$

By differentiating $g(u_2, \varphi u_1)$ along β with the help of (8) and (11), the equality

$$-\phi'(t) sin\phi(t) = k_2 g(u_3, \varphi u_1) \tag{24}$$

is obtained. Also, we can write

$$\varphi u_1 = g(u_2, \varphi u_1)u_2 + g(u_3, \varphi u_1)u_3 + g(u_4, \varphi u_1)u_4. \tag{25}$$

For details, see [7]. By using these results, we obtain Theorem 3.14 and Theorem 3.15.

Theorem 3.14 *Let* β *be a non-geodesic Legendre curve in a SSF* $(N^{2n+1}, \varphi, \xi, \eta, g)$, $c \neq 1$ *and* $g(u_2, \varphi u_1)$ *is not equal to* $-1, 0$ *or* 1*. Then,* β *is a bi-*f*-harmonic curve iff following differential* *equations are satisfied*

$$\begin{cases} (ff'')' - 4k_1^2 ff' - 3k_1 k_1' f^2 = 0, \\ k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})cos^2\phi(t), \\ 4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' + 3(\frac{c-1}{4})g(u_3,\varphi u_1)cos\phi(t) = 0, \\ k_2 k_3 + 3(\frac{c-1}{4})g(u_4,\varphi u_1)cos\phi(t) = 0. \end{cases}$$

Proof It is easy to see that if equation (23) substituted into equation (15), then the proof is completed. □

Case VIII: If $c \neq 1$ and $g(u_2, \varphi u_1)$ is not equal to $-1, 0$ or 1 and $(ff'')' = 0$, then equation (15) reduces to

$$4k_1^2 ff' + 3k_1 k_1' f^2 = 0, \tag{26}$$

$$k_1^2 + k_2^2 = 3\frac{f''}{f} + 2(\frac{f'}{f})^2 + 4\frac{k_1'}{k_1}\frac{f'}{f} + \frac{k_1''}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})cos^2\phi(t), \tag{27}$$

$$4k_2\frac{f'}{f} + 2k_2\frac{k_1'}{k_1} + k_2' + 3(\frac{c-1}{4})g(u_3,\varphi u_1)cos\phi(t) = 0, \tag{28}$$

$$k_2 k_3 + 3(\frac{c-1}{4})g(u_4,\varphi u_1)cos\phi(t) = 0. \tag{29}$$

Now, let give the interpretation of Case VIII.

First of all, from equation (26), it is easy to see that $\frac{f'}{f} = -\frac{3}{4}\frac{k_1'}{k_1}$ and $\frac{f''}{f} = \frac{3}{4}\left(\frac{k_1'}{k_1}\right)^2 - \frac{1}{2}\frac{k_1''}{k_1}$. Then, by using these equalities in the equations (27) and (28), we get

$$k_1^2 + k_2^2 = \frac{33}{16}(\frac{k_1'}{k_1})^2 - \frac{5}{4}\frac{k_1''}{k_1} + \frac{c+3}{4} + 3(\frac{c-1}{4})cos^2\phi(t), \tag{30}$$

$$-k_2(\frac{k_1'}{k_1}) + k_2' + 3(\frac{c-1}{4})g(u_3,\varphi u_1)cos\phi(t) = 0, \tag{31}$$

respectively. Then, by multiplying equation (31) with $2k_2$ and using equation (24), we get

$$2k_2 k_2' - 2k_2^2\frac{k_1'}{k_1} + 3(\frac{c-1}{4})(-2\phi'(t)cos\phi(t)sin\phi(t)) = 0. \tag{32}$$

Let ϕ be a constant. Then, from (24), we get $g(u_3, \varphi u_1) = 0$ and also, from (25), we get $g(u_4, \varphi u_1) = \mp sin\phi$ since $\|\varphi u_1\| = 1$. Finally, from (32), we obtain $k_2 = c_2 k_1$, where c_2 is a positive integration constant. Then, by using these informations, equations (29) and (30) reduces to $c_2k_1k_3 = \mp\frac{3(c-1)sin(2\phi(t))}{8}$ and

$$33(k_1^{'})^2 - 20k_1k_1^{''} + k_1^2\left(4(c+3) + 3(c-1)cos^2\phi(t) - 16k_1^2 - 16c_2^2k_1^2\right) = 0.$$

Now, we can state the Theorem 3.15.

Theorem 3.15 *Let β be a Legendre curve with non-constant geodesic curvature of osculating order r in a SSF $(N^{2n+1}, \varphi, \xi, \eta, g)$, where $c \neq 1$, $g(u_2, \varphi u_1)$ is not equal to $-1, 0$ or 1, $(ff^{''})^{'} = 0$, $r \geq 4$, $n \geq 2$ and ϕ be a constant. Then, β is a bi-f-harmonic curve iff $f = c_1k_1^{-\frac{3}{4}}, k_2 = c_2k_1$ and k_1, k_3 satisfy following differential equations*

$$33(k_1^{'})^2 - 20k_1k_1^{''} + k_1^2\left(4(c+3) + 3(c-1)cos^2\phi(t) - 16k_1^2 - 16c_2^2k_1^2\right) = 0,$$

$$c_2k_1k_3 = \mp\frac{3(c-1)sin(2\phi(t))}{8},$$

where c_1 and c_2 are positive integration constants.

Declaration of Ethical Standards
The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest
The author declares no conflicts of interest.

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Remarks on Some Soliton Types with Certain Vector Fields

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Abstract: This paper mainly aims to investigate some soliton kinds with certain vector fields on Riemannian manifolds and gives some notable geometric results as regards such vector fields. Also, in this paper some special tensors that have an important place in Riemannian geometry are discussed and given some significant links between these tensors. Finally, an example that supports one of our results is given.

Keywords: Ricci soliton, Yamabe soliton, conformal quadratic killing tensor, $\mathcal{Z}$-curvature tensor.

1. Introduction

Over the past few years, the theory of geometric flows has become a significant tool to determine the most geometric properties of the related object of the manifolds in Riemannian geometry. Ricci flow, one of the most important geometric flows, was defined by Hamilton so that he can find a canonical metric on a smooth manifold in [12]. Another important geometric flow is the Yamabe flow that Hamilton defined as a tool in order to construct metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a smooth manifold [11]. Such flows are evolution equations for Riemannian metric. After these works, many mathematicians have studied such geometric flows and other evolution equations arising in differential geometry.

In 1988, Hamilton defined the concepts of Yamabe and Ricci solitons in Riemannian geometry [11]. The formation of singularities in the Yamabe and Ricci flows are determined by these concepts, which evolve only by diffeomorphisms and scaling. Also, the limit of the solutions of the Yamabe and Ricci flow are appeared by Yamabe and Ricci solitons, respectively. Whereas a special solution of the Yamabe flow is the Yamabe soliton, a special solution of the Ricci flow is the Ricci soliton. Ricci and Yamabe solitons are equals to each other in dimension $n = 2$. But $n > 2$, these solitons do not have such an equivalence.

Let M be a Riemannian manifold together with the Riemannian metric g and let S and r

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be the Ricci tensor and scalar curvature of M, respectively. For a vector field ξ being tangent to M, if the following

$$(£_\xi g)(W,F) + 2S(W,F) + 2\lambda g(W,F) = 0 \qquad (\lambda \in \mathbb{R}) \tag{1}$$

is satisfied, then this manifold is called a Ricci soliton. Similarly, if the following

$$(£_\xi g)(W,F) = 2(r - \lambda)g(W,F) \qquad (\lambda \in \mathbb{R}) \tag{2}$$

is satisfied, then this manifold is called a Yamabe soliton. Here, $£_\xi g$ stands for the Lie-derivative of g with respect to ξ and $W,F \in \Gamma(TM)$. A Yamabe (Ricci) soliton with vector field ξ is denoted by (M,g,ξ,λ). If ξ is Killing or zero in (1), then the Ricci soliton becomes trivial and in such a case, the metric becomes an Einstein metric. Hence, Ricci solitons can be considered as a generalization of Einstein manifolds. Similarly, if ξ is Killing or zero in (2), then Yamabe solitons reduce to manifolds of constant scalar curvature.

Yamabe (Ricci) solitons can be categorized as steady, expanding or shrinking depending on the values of λ: $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$. In addition, Ricci (Yamabe) solitons are called gradient if ξ is the gradient $D\mu$ of a smooth function μ called potential function. Recently, several varied generalizations of Ricci and Yamabe solitons have been investigated comprehensively by many authors within the framework of the many context. For example, in 2011, Pigola et al. defined almost Ricci solitons which are a more general case of Ricci solitons by setting λ in (1) as a function [19]. Similarly, Barbosa and Ribeiro introduced an another class of Yamabe solitons evolution equation (2) by taking constant λ with a variable function and then, they called it almost Yamabe soliton in [1]. For the recent works, we refer to ([6, 7, 16, 17, 21, 26]) and references therein.

On the other hand, as it is well-known that vector fields have been used for studying differential geometry of manifolds since they characterize most geometric properties of the related object. They are widely used in several fields of differential geometry and physics. Also, they play a significant role in the study of Riemannian geometry. Therefore, many papers on a Riemannian manifold endowed with geometric vector fields so that this manifold admits a Ricci soliton or Yamabe soliton have been discussed by many mathematicians. For further readings, we refer to studies ([2–5, 14, 18, 25, 27]).

Motivated by these circumstances, we deal with Ricci solitons and Yamabe solitons, which have recently received considerable attention of many geometers, on Riemannian manifolds endowed with certain vector fields such as affine conformal, projective and concircular. Also, we investigate some geometric properties of notions of $\mathcal{Z}$-curvature tensor and conformal quadratic Killing tensor, which proves to be rich in geometrical structures. The present study is structured as follows. In Section 2, we recall some necessary and notations formulas that will be needed. The Section 3 is devoted to conclusion in which we present our results that are obtained in this paper.

2. Preliminaries

In this section, some required notions which will be used for later are recalled.

Let (M,g) be a Riemannian manifold and A be a second order symmetric tensor. For any $W,F,Z \in \Gamma(TM)$, if this tensor satisfies the following

$$(\nabla_W A)(F,Z) + (\nabla_F A)(Z,W) + (\nabla_Z A)(W,F) = 0,$$

then it is called a quadratic Killing tensor (as a generalization of a Killing vector). Likewise, if this tensor satisfies the following

$$(\nabla_W A)(F,Z) + (\nabla_F A)(Z,W) + (\nabla_Z A)(W,F) = \tag{3}$$
$$k(W)A(F,Z) + k(F)A(Z,W) + k(Z)A(W,F),$$

then it is called a conformal quadratic Killing tensor (as a generalization of a conformal Killing vector). Here, ∇ is the Levi-Civita connection of M and k is a 1-form. For more details as regards these tensors, we refer to ([22, 23, 28]).

Let ξ be a vector field on a Riemannian manifold (M,g). The vector field ξ is named as affine conformal, projective and concircular, respectively, if the followings [5, 8, 20]

$$(\pounds_\xi \nabla)(W,F) = W(\rho)F + F(\rho)W - g(W,F)D\rho, \tag{4}$$

$$(\pounds_\xi \nabla)(W,F) = p(W)F + p(F)W \tag{5}$$

and

$$\nabla_W \xi = \mu W, \tag{6}$$

where p is a 1-form, $D\rho$ is the gradient of ρ and μ, ρ are some smooth functions on M. If variable ρ in (4) is constant, then ξ is named as an affine vector field. Also, if $p = 0$ in (5), then the vector field ξ is called affine. The vector field ξ is named as concurrent if it satisfies (6) together with $\mu = 1$.

On the other hand, the Hessian tensor H^μ of a smooth function μ on (M,g) is given by

$$H^\mu(W,F) = g(\nabla_W(D\mu),F),$$

where $D\mu$ is the gradient of μ on M [4]. Also, the Hessian tensor H^μ is symmetric in W and F.

Now, we need the following lemma for later use.

Lemma 2.1 *[4] Let μ be a function on a Riemannian manifold M. Then, the gradient $D\mu$ of μ is a concircular vector field if and only if the Hessian H^μ of μ satisfies*

$$H^\mu(W,F) = fg(W,F) \tag{7}$$

for W, F tangent to M, where f is the function on M. Moreover, in such case the function f satisfies equation $\nabla_W v = fW$ with $v = D\mu$.

Also, from Lemma 2.1 and taking $\mu = f$ in (7), it can be easily seen that the gradient $D\mu$ of μ satisfies

$$\nabla_W D\mu = \mu W$$

for any $W \in \Gamma(TM)$.

3. Main Results

In this section, our main results that we obtained in this work are given.

Proposition 3.1 *If a Riemannian manifold (M, g) admits a Yamabe soliton with vector field V, then V is an affine conformal vector field on M.*

Proof Since (M, g) is a Yamabe soliton, from (2), one has

$$(\pounds_V g)(F, Z) = 2(r - \lambda)g(F, Z) \tag{8}$$

for any $F, Z \in \Gamma(TM)$. Differentiating the equation (8) covariantly along any vector field W provides

$$(\nabla_W \pounds_V g)(F, Z) = 2W(r)g(F, Z). \tag{9}$$

On the other hand, it follows from the formula (see Yano [24, p.23]) that we have the equality

$$(\pounds_V \nabla_W g - \nabla_W \pounds_V g - \nabla_{[V,W]} g)(F, Z) = \tag{10}$$
$$-g((\pounds_V \nabla)(W, F), Z) - g((\pounds_V \nabla)(W, Z), F).$$

Since the Riemannian metric g is parallel with respect to ∇, namely $\nabla g = 0$, the equality (10) turns into

$$(\nabla_W \pounds_V g)(F, Z) = g((\pounds_V \nabla)(W, F), Z) + g((\pounds_V \nabla)(W, Z), F). \tag{11}$$

Also, in view of (9) and (11), we immediately have

$$2W(r)g(F, Z) = g((\pounds_V \nabla)(W, F), Z) + g((\pounds_V \nabla)(W, Z), F). \tag{12}$$

If we rearrange cyclically W, F and Z in (12), then we obtain

$$2F(r)g(Z, W) = g((\pounds_V \nabla)(F, Z), W) + g((\pounds_V \nabla)(F, W), Z) \tag{13}$$

and

$$2Z(r)g(W,F) = g((\pounds_V\nabla)(Z,W),F) + g((\pounds_V\nabla)(Z,F),W). \tag{14}$$

Due to being $\pounds_V\nabla$ symmetric tensor of type $(1,2)$, that is, $(\pounds_V\nabla)(W,F) = (\pounds_V\nabla)(F,W)$, adding (12) and (13), we get

$$\begin{aligned} 2W(r)g(F,Z) + 2F(r)g(Z,W) &= 2g((\pounds_V\nabla)(W,F),Z) \\ &\quad + g((\pounds_V\nabla)(Z,W),F) \\ &\quad + g((\pounds_V\nabla)(Z,F),W) \end{aligned} \tag{15}$$

which together with the equation (14) gives the following

$$g((\pounds_V\nabla)(W,F),Z) = W(r)g(F,Z) + F(r)g(Z,W) - Z(r)g(W,F).$$

Using the fact that $Z(r) = g(Dr,Z)$, the above last equation can be written as

$$\begin{aligned} g((\pounds_V\nabla)(W,F),Z) &= W(r)g(F,Z) + F(r)g(Z,W) \\ &\quad - g(Dr,Z)g(W,F). \end{aligned} \tag{16}$$

Removing Z from both sides in (16) gives

$$(\pounds_V\nabla)(W,F) = W(r)F + F(r)W - g(W,F)Dr \tag{17}$$

which by (4) means that the vector field V is affine conformal on M. Thus, we get the requested result. □

Lemma 3.2 *Let V be a a concircular vector field on a Riemannian manifold (M,g). Then, V is also an affine conformal vector field on M.*

Proof It follows from (6) that we have

$$\nabla_Z V = \mu Z \tag{18}$$

for vector field Z being tangent to M. It follows from (18) and the definition of the covariant derivative, we arrive at

$$\nabla_F\nabla_Z V = F(\mu)Z + \mu\nabla_F Z \tag{19}$$

for vector field F being tangent to M. If we interchange the roles of F and Z in (19), then one has

$$\nabla_Z\nabla_F V = Z(\mu)F + \mu\nabla_Z F. \tag{20}$$

Moreover, taking $[F, Z]$ instead of Z in (18), we write

$$\nabla_{[F,Z]}V = \mu\nabla_F Z - \mu\nabla_Z F. \tag{21}$$

By means of (19), (20) and (21), we find that

$$R(F, Z)V = F(\mu)Z - Z(\mu)F \tag{22}$$

which by taking inner product on both sides of (22) by W yields

$$g(R(F, Z)V, W) = F(\mu)g(Z, W) - Z(\mu)g(F, W). \tag{23}$$

This is equivalent to

$$g(R(V, W)F, Z) = F(\mu)g(Z, W) - Z(\mu)g(F, W) \tag{24}$$

from which it follows that

$$R(V, W)F = F(\mu)W - g(F, W)D\mu. \tag{25}$$

On the other hand, as is known from Yano [24, p.23], that the following identity holds

$$(\pounds_V\nabla)(W, F) = R(V, W)F - \nabla_{\nabla_W F}V + \nabla_W\nabla_F V. \tag{26}$$

Making use of (6), (25) and (26), therefore we obtain

$$(\pounds_V\nabla)(W, F) = W(\mu)F + F(\mu)W - g(W, F)D\mu$$

which is the desired result. □

Remark 3.3 *If the vector field V is concurrent, then V is also an affine vector field on M.*

Let A be a geometric/physical quantity on a Riemannian manifold M. If it satisfies

$$\pounds_\xi A = 2\Omega A, \tag{27}$$

then A inherits symmetry with respect to vector field ξ. Here, $\pounds$ stands for the Lie derivative and Ω is a function on the manifold [10]. For more details related to symmetry inheritance applications on manifolds, please see ([9, 10, 13, 28]).

The metric inheritance symmetry is one of the most basic and widely used example for which $A = g$ in (27), in this case ξ is the conformal Killing vector field such that

$$(\pounds_\xi g)(W, F) = 2\Omega g(W, F).$$

Also, if Ω is zero in above equation, then ξ is the Killing vector field.

If $A = R$ in (27), then the equation takes the form

$$\pounds_\xi R = 2\Omega R.$$

This particular symmetry is called curvature inheritance, where R is the Riemannian curvature tensor. If $\Omega = 0$, then it is called curvature collineation, which M admits a special symmetry.

In the following theorem, we discuss the role of such symmetry inheritance for the Ricci tensor field of a Riemannian manifold M.

Theorem 3.4 *Let μ be a potential function of an almost gradient Ricci soliton (M, g, V, λ), where (M, g) has symmetry inheritance and μ is the function satisfying equation (6). Then, the Ricci tensor field of M has symmetry inheritance with respect to V if the Hessian H^μ of μ satisfies*

$$H^\mu(W, F) = \mu g(W, F) \tag{28}$$

for any $W, F \in \Gamma(TM)$.

Proof Let us consider that the Hessian H^μ of μ satisfies equation (28). Then, by Lemma 2.1, the gradient $D\mu$ of μ is a concircular vector field. Since M is an almost gradient Ricci soliton and from Lemma 3.2, the concircular vector field V satisfies

$$(\pounds_V \nabla)(W, F) = W(\mu)F + F(\mu)W - g(W, F)D\mu \tag{29}$$

for vector fields W and F being tangent to M. Due to $V = D\mu$, the equation (29) transforms into

$$(\pounds_V \nabla)(W, F) = g(W, V)F + g(F, V)W - g(W, F)V. \tag{30}$$

Differentiating (30) covariantly along any vector field T and using the property of being V concircular, we obtain

$$(\nabla_T \pounds_V \nabla)(W, F) = \mu g(T, W)F + \mu g(T, F)W - \mu g(W, F)T. \tag{31}$$

Similarly, by a straight forward calculation, we also have

$$(\nabla_W \pounds_V \nabla)(T, F) = \mu g(W, T)F + \mu g(W, F)T - \mu g(T, F)W. \tag{32}$$

From [24], it is well known that

$$(\pounds_V R)(T, W)F = (\nabla_T \pounds_V \nabla)(W, F) - (\nabla_W \pounds_V \nabla)(T, F). \tag{33}$$

After inserting (31) and (32) in (33), we deduce that

$$(\pounds_V R)(T, W)F = 2\mu g(T, F)W - 2\mu g(W, F)T. \tag{34}$$

Furthermore, operating inner product with arbitrary vector field X in (34) yields

$$g((\pounds_V R)(T,W)F,X) = 2\mu g(T,F)g(W,X) - 2\mu g(W,F)g(T,X). \tag{35}$$

Setting $T = X = e_i$ in (35) and summing up over i $(i = 1,2,...,n)$, we derive that

$$(\pounds_V S)(W,F) = -2\mu(n-1)g(W,F). \tag{36}$$

Here, $\{e_i\}$ stands for the orthonormal basis of T_pM for all $p \in M$.

On the other hand, since M admits an almost gradient Ricci soliton with concircular vector field V, we find from (1) and (18) that

$$S(F,Z) = -(\lambda + \mu)g(F,Z) \tag{37}$$

for any $F, Z \in \Gamma(TM)$. Replacing F with V in (37) gives

$$S(V,Z) = -(\lambda + \mu)g(V,Z). \tag{38}$$

Owing to Lemma 3.2, putting $F = W = e_i$ in (23) and taking summation over i, we have

$$S(Z,V) = -(n-1)g(D\mu,Z). \tag{39}$$

Then, with the help of (38) and (39), one can see that $\lambda + \mu = n-1$. Using this in (37) provides

$$S(F,Z) = -(n-1)g(F,Z). \tag{40}$$

Keeping in mind (36) and from (40), we get

$$(\pounds_V S)(W,F) = 2\mu S(W,F) \tag{41}$$

which gives the conclusion. Hence, the proof is completed. □

The next result gives a necessary condition for the Ricci tensor field of M to be conformal quadratic Killing.

Theorem 3.5 *Let (M,g) be a Ricci soliton with vector field V, where V is either an affine conformal or a projective vector field. Then, the Ricci tensor field of M is conformal quadratic Killing.*

Proof Let us take into account that (M,g) is a Ricci soliton with affine conformal vector field V. Then, the equation (1) can be written as

$$(\pounds_V g)(F,Z) = -2S(F,Z) - 2\lambda g(F,Z) \tag{42}$$

for any $F, Z \in \Gamma(TM)$. Performing the properties of Lie derivative and Levi-Civita connection in (42), we infer that

$$(\nabla_W \pounds_V g)(F, Z) = -2(\nabla_W S)(F, Z) \tag{43}$$

for W being tangent to M. On the other hand, making use of the equation (4), we get

$$g((\pounds_V \nabla)(W, F), Z) + g((\pounds_V \nabla)(W, Z), Y) = 2g(W, D\rho)g(F, Z). \tag{44}$$

By virtue of the equations (11), (43) and (44), we have

$$(\nabla_W S)(F, Z) = -g(W, D\rho)g(F, Z). \tag{45}$$

If we stand for the the dual 1-form of $D\rho$ by $-\phi$, then equation (45) becomes

$$(\nabla_W S)(F, Z) = \phi(W)g(F, Z). \tag{46}$$

Also, if W, F and Z are cyclically displaced in (46), then we find that

$$(\nabla_F S)(Z, W) = \phi(F)g(Z, W) \tag{47}$$

and

$$(\nabla_Z S)(W, F) = \phi(Z)g(W, F). \tag{48}$$

By combining the equalities (46), (47) and (48), we obtain

$$(\nabla_W S)(F, Z) + (\nabla_F S)(Z, W) + (\nabla_Z S)(W, F) = \\ \phi(W)g(F, Z) + \phi(F)g(Z, W) + \phi(Z)g(W, F)$$

which by (3) means that the Ricci tensor field of M is a conformal quadratic Killing tensor.

When the above steps are done for the projective vector field V, it can be easily showed that the Ricci tensor field of M is conformal quadratic Killing. Hence, the proof is completed. □

In 2012, as a general concept of the Einstein gravitational tensor in General relativity, generalized $(0, 2)$ symmetric $\mathcal{Z}$ tensor was introduced by Mantica and Molina. According to them, such a tensor is defined by [15]

$$\mathcal{Z}(W, F) = S(W, F) + fg(W, F) \tag{49}$$

for f being a smooth function on M and S being the Ricci tensor field of M.

The next theorem presents an important relationship for curvature tensor $\mathcal{Z}$ and conformal quadratic Killing.

Theorem 3.6 *Let (M, g) be a Riemannian manifold admitting $\mathcal{Z}$-curvature tensor. Then, the tensor $\mathcal{Z}$ is conformal quadratic Killing if and only if the Ricci tensor field of M is conformal quadratic Killing.*

Proof Taking covariant derivative of (49) along T and using the fact that $T(f) = df(T)$, we get

$$(\nabla_T \mathcal{Z})(W, F) = (\nabla_T S)(W, F) + df(T)g(W, F) \tag{50}$$

for any $W, F, T \in \Gamma(TM)$. By a combinatorial combination, we find

$$(\nabla_W \mathcal{Z})(F, T) = (\nabla_W S)(F, T) + df(W)g(F, T) \tag{51}$$

and

$$(\nabla_F \mathcal{Z})(T, W) = (\nabla_F S)(T, W) + df(F)g(T, W). \tag{52}$$

By adding the equations (50), (51) and (52) provides

$$\begin{aligned}(\nabla_T \mathcal{Z})(W, F) + (\nabla_W \mathcal{Z})(F, T) + (\nabla_F \mathcal{Z})(T, W) = \\ (\nabla_T S)(W, U) + (\nabla_W S)(U, T) + (\nabla_U S)(T, W) \\ +df(T)g(W, F) + df(W)g(F, T) + df(F)g(T, W).\end{aligned} \tag{53}$$

Now, if $\mathcal{Z}$ is conformal quadratic Killing, then we write

$$\begin{aligned}(\nabla_T \mathcal{Z})(W, F) + (\nabla_W \mathcal{Z})(F, T) + (\nabla_F \mathcal{Z})(T, W) = \\ \alpha(T)g(W, F) + \alpha(W)g(F, T) + \alpha(F)g(T, W).\end{aligned} \tag{54}$$

Therefore, by combining (53) with (54) and using the linearity property of the 1-forms, we get

$$\begin{aligned}(\nabla_T S)(W, F) + (\nabla_W S)(F, T) + (\nabla_F S)(T, W) = (\alpha - df)(T)g(W, F) \\ +(\alpha - df)(W)g(F, T) + (\alpha - df)(F)g(T, W).\end{aligned} \tag{55}$$

If we take ϕ instead of the 1-form $\alpha - df$ in (55), then one has

$$\begin{aligned}(\nabla_T S)(W, F) + (\nabla_W S)(F, T) + (\nabla_F S)(T, W) = \\ +\phi(T)g(W, F) + \phi(W)g(F, T) + \phi(F)g(T, W).\end{aligned}$$

This, by (3), implies that the Ricci tensor field of M is a conformal quadratic Killing tensor.

Conversely, let the Ricci tensor field of M admitting $\mathcal{Z}$-curvature tensor be a conformal quadratic Killing tensor. Then, we have

$$\begin{aligned}(\nabla_T S)(W, F) + (\nabla_W S)(F, T) + (\nabla_F S)(T, W) = \\ k(T)g(W, F) + k(W)g(F, T) + k(F)g(T, W).\end{aligned} \tag{56}$$

After substituting (56) into (53) and using the linearity of the 1-forms, we obtain

$$(\nabla_T \mathcal{Z})(W,F) + (\nabla_W \mathcal{Z})(F,T) + (\nabla_F \mathcal{Z})(T,W) = \tag{57}$$
$$\varphi(T)g(W,F) + \varphi(W)g(F,T) + \varphi(F)g(T,W),$$

where we have used $\varphi = k + df$. Consequently, by (3) the tensor $\mathcal{Z}$ is conformal quadratic Killing. Therefore, we arrive at the desired result. □

The proof of following corollary easily follows from Theorem 3.6.

Corollary 3.7 *Let (M,g) be a Riemannian manifold admitting $\mathcal{Z}-$curvature tensor. Then, we have the followings:*

i) If tensor $\mathcal{Z}$ is quadratic Killing, then the Ricci tensor field of M is conformal quadratic Killing.

ii) If the Ricci tensor field of M is quadratic Killing, then tensor $\mathcal{Z}$ is conformal quadratic Killing.

Example 3.8 *[2] We set the three-dimensional manifold as*

$$M = \{(u,v,t) \in \mathbb{R}^3, t > 0\},$$

where (u,v,t) are the Cartesian coordinates in $\mathbb{R}^3$. Take

$$g := \frac{1}{t^2}\{du \otimes du + dv \otimes dv + dt \otimes dt\},$$
$$\eta := -\frac{1}{t}dt, \qquad V := -t\frac{\partial}{\partial t}.$$

Here, η denotes the dual 1-form of the vector field V. Let E_1, E_2 and E_3 be the vector fields in $\mathbb{R}^3$ such that these vector fields are linearly independent given by:

$$E_1 = t\frac{\partial}{\partial u}, \quad E_2 = t\frac{\partial}{\partial v} \quad \text{and} \quad E_3 = -t\frac{\partial}{\partial t}.$$

Then, we have

$$\eta(E_1) = 0, \qquad \eta(E_2) = 0, \qquad \eta(E_3) = 1,$$
$$[E_2, E_1] = 0, \qquad [E_3, E_2] = -E_2, \qquad [E_1, E_3] = E_1.$$

On the other hand, we find from Koszul's formula for the Riemannian metric g:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_3} E_1 = 0,$$
$$\nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_3} E_2 = 0,$$
$$\nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_3 = E_2, \quad \nabla_{E_3} E_3 = 0.$$

Also, with the help of the above equations we find

$$R(E_1,E_2)E_3=0,\qquad R(E_3,E_1)E_2=0,\qquad R(E_2,E_3)E_1=0,$$
$$R(E_1,E_2)E_2=-E_1,\qquad R(E_2,E_1)E_1=-E_2,\qquad R(E_1,E_3)E_3=-E_1,$$
$$R(E_3,E_1)E_1=-E_3,\qquad R(E_2,E_3)E_3=-E_2,\qquad R(E_3,E_2)E_2=-E_3.$$

Utilizing the expressions of the curvature tensors, we obtain

$$S(E_1,E_1)=S(E_2,E_2)=S(E_3,E_3)=-2 \qquad \textit{and} \qquad S(E_i,E_j)=0 \tag{58}$$

for all $i\neq j$ $(i,j=1,2,3)$. *As* $\{E_1,E_2,E_3\}$ *forms a basis of* M, *the followings can be written as*

$$W=a_1E_1+a_2E_2+a_3E_3,$$
$$Z=b_1E_1+b_2E_2+b_3E_3,$$
$$F=c_1E_1+c_2E_2+c_3E_3$$

for any vector field $W,Z,F\in\Gamma(TM)$, *where* $a_i,b_i,c_i\in\mathbb{R}^+$ *for* $i=1,2,3$. *Then, by a straight forward calculation, one has*

$$\nabla_F W=-a_1c_1E_3+a_3c_1E_1-a_2c_2E_3+a_3c_2E_2, \tag{59}$$
$$\nabla_F Z=-b_1c_1E_3+b_3c_1E_1-b_2c_2E_3+b_3c_2E_2. \tag{60}$$

From the above equations, we find that

$$S(W,Z) = -2(a_1b_1+a_2b_2+a_3b_3), \tag{61}$$
$$S(\nabla_F W,Z) = 2a_1c_1b_3-2a_3c_1b_1+2a_2c_2b_3-2a_3c_2b_2, \tag{62}$$
$$S(W,\nabla_F Z) = 2b_1c_1a_3-2b_3c_1a_1+2b_2c_2a_3-2b_3c_2a_2. \tag{63}$$

Therefore, we have

$$(\nabla_F S)(W,Z)=0. \tag{64}$$

Similarly, we obtain

$$(\nabla_W S)(Z,F),\qquad (\nabla_Z S)(F,W)=0. \tag{65}$$

In this case, from (64) and (65), S *is a quadratic Killing tensor.*

Furthermore, using (59)-(63) in (49) gives

$$\mathcal{Z}(W,Z) = (f-2)(a_1b_1+a_2b_2+a_3b_3), \tag{66}$$
$$\mathcal{Z}(\nabla_F W,Z) = (2-f)a_1c_1b_3+(f-2)a_3c_1b_1+(2-f)2a_2c_2b_3 +(f-2)a_3c_2b_2, \tag{67}$$
$$\mathcal{Z}(W,\nabla_F Z) = (2-f)b_1c_1a_3+(f-2)b_3c_1a_1+(2-f)b_2c_2a_3 +(f-2)b_3c_2a_2. \tag{68}$$

From (64)-(68) and owing to the fact that $g(X,Y) = a_1b_1 + a_2b_2 + a_3b_3$, *we get*

$$(\nabla_F \mathcal{Z})(W,Z) = df(F)g(W,Z). \tag{69}$$

Likewise,

$$(\nabla_W \mathcal{Z})(Z,F) = df(W)g(Z,F), \quad (\nabla_Z \mathcal{Z})(F,W) = df(Z)g(F,W). \tag{70}$$

Thus $\mathcal{Z}$ *is a conformal quadratic Killing tensor, which verifies Corollary 3.7* (ii).

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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A Note on Transitive Action of the Extended Modular Group on Rational Numbers

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Abstract: The extended modular group $\overline{\Gamma}$ is isomorphic to the amalgamated free product of two dihedral groups D_2 and D_3 with amalgamation $\mathbb{Z}_2$. This group acts on rational numbers transitively. In this study, we obtain elements in the extended modular group that are mappings between given two rationals. Also, we express these elements as a word in generators. We use interesting relations between continued fractions and the Farey graph.

Keywords: Extended modular group, continued fractions, Farey graph.

1. Introduction

The modular group $\Gamma = PSL(2,\mathbb{Z})$ is the projective special linear group of 2×2 matrices over the ring of integers with determinant one. This group is the quotient group $SL(2,\mathbb{Z})/\pm I$, hence, each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the same element with its negative $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. Modular group acts on the upper half plane $\mathbb{H}$ via linear fractional transformations $z \to \frac{az+b}{cz+d}$. These transformations are orientation preserving isometries of $\mathbb{H}$. Modular group is generated by two elements,

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

By taking $S = TU = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, the presentation of Γ is

$$\Gamma =< T,S : T^2 = S^3 = I >\cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

Let us denote the set $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc = -1 \right\}$. The corresponding transformations of elements in G are anti-automorphisms. Thus, the extended modular group can be

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defined as $\overline{\Gamma} = PSL(2,\mathbb{Z}) \cup G$. The extended modular group is isomorphic to free product of two dihedral groups of order 4 and 6, amalgamated with a cyclic group of order 2, i.e.,

$$\overline{\Gamma} =< T, S, R : T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I >\cong D_2 *_{\mathbb{Z}_2} D_3,$$

where $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a reflection map.

In this study, we focus on the action of the extended modular group $\overline{\Gamma}$ on rational numbers. Every rational number has a reduced fraction $\frac{p}{q} = \frac{-p}{-q}$, where $p, q \in \mathbb{Z}$ and $(p,q) = 1$. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. Consider the element $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$ and the corresponding Möbius transformation $V(z) = \frac{az+b}{cz+d}$. The image of $\frac{p}{q}$ is

$$V(\frac{p}{q}) = \frac{ap+bq}{cp+dq}.$$

Here $\frac{ap+bq}{cp+dq}$ is also a reduced fraction. Additionally, the Diophantine equation $px - qy = \pm 1$ is solvable since $(p,q) = 1$. Hence, it is possible to find an element $W = \begin{pmatrix} p & x \\ q & y \end{pmatrix} \in \overline{\Gamma}$ such that $W(\infty) = \frac{p}{q}$. As a result, the action of the extended modular group on rationals is transitive [14]. Our aim is to find an element $V \in \overline{\Gamma}$ for given two rationals $\frac{p}{q}, \frac{p'}{q'}$ such that $V(\frac{p}{q}) = \frac{p'}{q'}$. Also, we represent V as a word in generators.

2. Motivation and Background Materials

In this section, we give some information about continued fractions, the Farey sequence, the Farey graph and relations to the extended modular group. For more information see [1, 2, 6].

There are impressive relations between the modular group and continued fractions. Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = U^{r_0}.T.U^{r_1}.\cdots.U^{r_n}.T^i \in \Gamma$ where $r_j \in \mathbb{Z}$ and $i = 0,1$. The corresponding Möbius transformation of this element is

$$V(z) = U^{r_0}.T.U^{r_1}.\cdots.U^{r_n}.T(z) = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots r_{n-1} - \cfrac{1}{r_n - \frac{1}{z}}}}}. \tag{1}$$

In addition, the image of infinity is a continued fraction expansion of $\frac{a}{c}$. This expansion is the *Rosen continued fraction* defined in [11] for $\lambda = 1$, and it is called *integer continued fraction expansion.*

In this expansion, for $i \le n$, $C_i = \frac{p_i}{q_i} = [r_0; r_1, ..., r_i]$ is called ith convergent of the expansion. It can be seen by calculation $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. On the other hand, it is possible to make connections between integer continued fractions and the Farey sequence.

The Farey sequence of order n is a complete and ordered set of reduced rational numbers in the interval $[0,1]$ which have denominators less than or equal to n.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\},$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\},$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\},$$

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}.$$

It can be seen that if $\frac{a}{c}$ and $\frac{b}{d}$ appears one after another in some F_n, then $ad - bc = \pm 1$. We called such rationals Farey neighbours. All Farey neighbours of a rational x is denoted by $\mathcal{N}(x)$. The Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ defined as,

$$\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}.$$

All Farey neigbours of a rational number can be obtained by Farey sum. More clearly, if a rational $\frac{p}{q}$ first appears in F_n by Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ in F_{n-1}, i.e., $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d} = \frac{p}{q}$, then $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours of $\frac{p}{q}$. Here $\frac{a}{c}$ and $\frac{b}{d}$ are called Farey parents of $\frac{p}{q}$ and conversely, $\frac{p}{q}$ is called Farey child of $\frac{a}{c}$ and $\frac{b}{d}$. If $\frac{a_i}{c_i}$ is a Farey neighbour of $\frac{p}{q}$, then $\frac{a_i}{c_i} \oplus \frac{p}{q}$ is also a Farey neighbour of $\frac{p}{q}$.

Observe that every F_n includes F_{n-1} and new members are obtained by Farey sum of its neighbours. For instance $\frac{1}{2} \in F_2$ is the Farey sum of $\frac{0}{1}$ and $\frac{1}{1}$ in F_1. This rule is known as the mediant rule. It should be noted that if the denominator of Farey sum of two neighbours in F_{n-1} exceeds n, then this rational number will not appear in F_n since the definition of the Farey sequence. Definition of the Farey sequence can be extended to $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ by assuming $\infty = \frac{1}{0}$. Hence, for a given rational $\frac{a}{c}$, it is known that $\frac{a}{c}$ has finite integer continued fraction expansion. In addition, $\frac{b}{d}$ is the penultimate convergent of the integer continued fraction expansion of $\frac{a}{c}$. This yields $ad - bc = \pm 1$, in other words, $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours. As a result, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$.

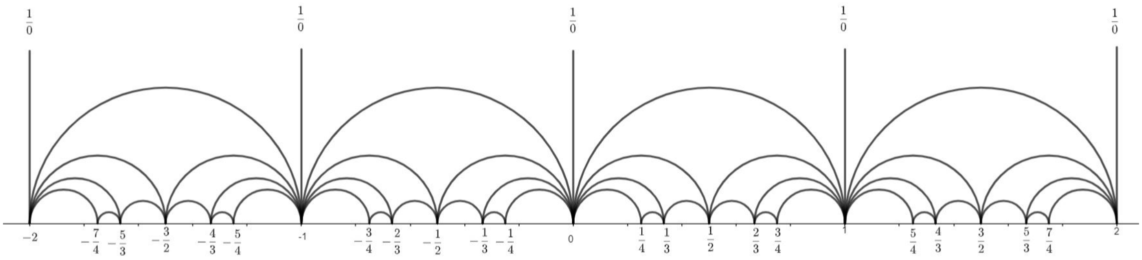

Figure 1: The Farey graph

The Farey graph is a graph with vertex set $\hat{\mathbb{Q}}$. Two reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent if and only if $ps - rq = \pm 1$, i.e., they are Farey neighbours. The edge between two vertices is drawn by a hyperbolic line in $\mathbb{H}$. The edges between $\frac{1}{0} = \infty$ and every integer a are vertical lines. To construct the graph, first join the vertices $\frac{1}{0}, \frac{0}{1}$ and $\frac{1}{1}$ and obtain a big triangle. By induction, if the endpoints of a long edge are $\frac{a}{c}$ and $\frac{b}{d}$, then the label of the third vertex of the triangle is $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$, see in Figure 1.

In recent years, many studies have contributed the continued fractions related to the action of some subgroups Möbius transformations. In [2], integer continued fraction expansions and geodesic expansions are studied from the perspective of graph theory. Short and Walker used Rosen continued fractions as paths in a class of graphs in hyperbolic geometry [13]. Same authors also studied connections between even integer continued fractions and the Farey graph [12]. Relations between cusp points and Fibonacci numbers are studied in [7] using Farey graph and continued fractions. Algebraic and combinatorial properties of continued fractions and modular group related with Farey graph are given in [10]. Besides that some relations between elliptic elements and circuits in graph for normalisers of subgroups of $PSL(2,\mathbb{R})$ are examined in [4, 5].

3. Main Results

Firstly, we obtain matrix representation of the elements in $\overline{\Gamma}$ which the corresponding transformation is a mapping between given two rationals. For a given reduced rational $x = \frac{p}{q}$ and a neighbour $y = \frac{r}{s} \in \mathcal{N}(x)$, we know $ps - rq = \pm 1$. Thus, we have $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \overline{\Gamma}$. In addition, if $y = \frac{r}{s}$ is on the left side of $x = \frac{p}{q}$ in the Farey graph that is $y < x$, then $ps - rq = 1$ and $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \Gamma$. In other words, the corresponding transformation $\frac{pz+r}{qz+s}$ is an automorphism. In this case, it is possible to construct an anti-automorphism element by taking $-r$ for r and $-s$ for s, i.e., $\begin{pmatrix} p & -r \\ q & -s \end{pmatrix}$. Similar observations can be done for the case $y > x$. For convenience throughout this paper, we need to define a location function $\mu_x : \mathcal{N}(x) \to \{-1, +1\}$ for neighbours of a rational:

$$\mu_x(y) = \begin{cases} 1 & , y < x \\ -1 & , y > x \end{cases}.$$

Now we are ready to obtain a mapping between two rationals.

Lemma 3.1 *Let $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q}$ and $\frac{r}{s} \in \mathcal{N}\left(\frac{p}{q}\right)$, $\frac{r'}{s'} \in \mathcal{N}\left(\frac{p'}{q'}\right)$. Then the corresponding transformation of the element*

$$V = \begin{pmatrix} p's - r'q & pr' - p'r \\ q's - qs' & ps' - q'r \end{pmatrix}$$

maps the rational $\frac{p}{q}$ to $\frac{p'}{q'}$. Moreover,

- *If $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = 1$, then the corresponding transformation of V is an automorphism.*
- *If $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = -1$, then the corresponding transformation of V is an anti-automorphism.*

Proof Let $\frac{p}{q}$ be a reduced rational and $\frac{r}{s}$ be a Farey neighbour of $\frac{p}{q}$. Then, we have from the definition of Farey neighbour $ps - rq = \pm 1$. Hence, $V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \overline{\Gamma}$. In addition, cusp point of this element is $\frac{p}{q}$. Similarly, we have the element $V_2 = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} \in \overline{\Gamma}$ with cusp point $\frac{p'}{q'}$. Finally, $V = V_2.V_1^{-1}$ is the element $V\left(\frac{p}{q}\right) = \frac{p'}{q'}$.

The equality $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = 1$ tells us both V_1 and V_2 are automorphism or anti-automorphism simultaneously. This yields V is an automorphism. The case $\mu_{\frac{p}{q}}\left(\frac{r}{s}\right).\mu_{\frac{p'}{q'}}\left(\frac{r'}{s'}\right) = -1$ can be interpreted similarly. □

Obtaining an element that maps $\frac{p}{q}$ to $\frac{p'}{q'}$ via Lemma 3.1 requires Farey neighbours one for each. Following corollary is an answer to what if $\frac{p}{q}$ and $\frac{p'}{q'}$ are adjacent.

Corollary 3.2 *Let the reduced rationals $\frac{p}{q}$ and $\frac{p'}{q'}$ be adjacent in the Farey graph. Then, the corresponding transformation of the element*

$$V = \begin{pmatrix} p'q' - pq & p^2 - p'^2 \\ q'^2 - q^2 & pq - p'q' \end{pmatrix}$$

maps $\frac{p}{q}$ to $\frac{p'}{q'}$. Furthermore, V is a reflection.

Proof Since $\frac{p}{q}$ and $\frac{p'}{q'}$ are adjacent in the Farey graph, we can take $V_1 = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ and $V_2 = \begin{pmatrix} p' & p \\ q' & q \end{pmatrix}$. Proof follows similar to the proof of Lemma 3.1. It can be seen easily that $\mu_{\frac{p}{q}}\left(\frac{p'}{q'}\right).\mu_{\frac{p'}{q'}}\left(\frac{p}{q}\right) = -1$. This equality proves that V is an anti-automorphism. It is obvious that V is a reflection since $Tr(V) = 0$. □

Considering the arguments mentioned before Lemma 3.1, we can map $\frac{p}{q}$ to $\frac{p'}{q'}$ via an elliptic element. It is enough to take $V_2 = \begin{pmatrix} p' & -p \\ q' & -q \end{pmatrix}$ in the proof of Corolary 3.2. Therefore, we omit the proof of the following corollary.

Corollary 3.3 *Let the reduced rationals $\frac{p}{q}$ and $\frac{p'}{q'}$ be adjacent in the Farey graph. Then, the corresponding transformation of the element*

$$V = \begin{pmatrix} p'q' + pq & -p^2 - p'^2 \\ q'^2 + q^2 & -pq - p'q' \end{pmatrix}$$

maps $\frac{p}{q}$ to $\frac{p'}{q'}$. Furthermore, V is an elliptic element of order 2 in Γ.

Now our aim is to obtain a generalization of Lemma 3.1. For doing this, we need more information about Farey neighbours. As we mentioned in the motivation section, the Farey sequence of level n is a complete and ordered set of reduced rationals which have denominators less than or equal to n. Every F_n includes F_{n-1}. New members obtained via mediant rule. More clearly, if $\frac{a}{c}$ and $\frac{b}{d}$ is contained in F_{n-1}, then the mediant of these two terms $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$ is contained in F_n on one condition that $c + d \leq n$. If a reduced rational $\frac{p}{q}$ first appears in F_n via Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ in F_{n-1}, then $\frac{a}{c}$ and $\frac{b}{d}$ is called Farey parents of $\frac{p}{q}$. After that all Farey neighbours of $\frac{p}{q}$ will be of the form,

$$\frac{a}{c} < \frac{a}{c} \oplus \frac{p}{q} = \frac{p+a}{q+c} < \frac{p+a}{q+c} \oplus \frac{p}{q} = \frac{2p+a}{2q+c} < \ldots < \frac{p}{q} < \ldots \frac{p}{q} \oplus \frac{b}{d} = \frac{p+b}{q+d} < \frac{b}{d}.$$

A basic result of this, is the following lemma.

Lemma 3.4 *Let $\frac{p}{q}$ be a reduced rational number and $\frac{r}{s}, \frac{r'}{s'} \in \mathcal{N}\left(\frac{p}{q}\right)$. Then, there exists an integer k such that*

$$\begin{pmatrix} r' \\ s' \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} . \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

Here we consider the element $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \overline{\Gamma}$ as an automorphism (or anti-automorphism). Using Lemma 3.4, we can construct another automorphism (anti-automorphism) element with cusp point $\frac{p}{q}$.

Lemma 3.5 *Let $V_1, V_2 \in \overline{\Gamma}$ with same cusp point. Then there exists an integer k such that*

$$V_1.U^k = V_2.$$

Here $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the parabolic generator of $\overline{\Gamma}$.

Proof Suppose the common cusp point is $\frac{p}{q}$. For neighbours $\frac{r}{s}, \frac{r'}{s'} \in \mathcal{N}\left(\frac{p}{q}\right)$, we can think V_1 and V_2 as

$$V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \text{ and } V_2 = \begin{pmatrix} p & r' \\ q & s' \end{pmatrix}.$$

From Lemma 3.4, we have an integer k such that

$$\frac{r'}{s'} = \frac{kp + r}{kq + s}.$$

Hence, we complete the proof by considering the generator $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$\begin{pmatrix} p & r' \\ q & s' \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k.$$

□

The above lemma tells us V_1 and $V_1.U^k$ have common cusp point for every integer k, and that is the key for the following theorem which is a generalization of Lemma 3.1.

Theorem 3.6 *Let $\frac{p}{q}, \frac{p'}{q'}$ be reduced fractions in $\mathbb{Q}$ and $\frac{r}{s} \in \mathcal{N}\left(\frac{p}{q}\right)$, $\frac{r'}{s'} \in \mathcal{N}\left(\frac{p'}{q'}\right)$. Then for every $k \in \mathbb{Z}$, the corresponding transformation of the element*

$$V = \begin{pmatrix} p's - r'q - kp'q & pr' - p'r + kpp' \\ q's - qs' - kqq' & ps' - q'r + kpq' \end{pmatrix}$$

maps the rational $\frac{p}{q}$ to $\frac{p'}{q'}$.

Proof We use a similar technique of the proof of Lemma 3.1. From Lemma 3.5, the elements

$$V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k_1} \text{ and } V_2 = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} . \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k_2}$$

have cusp points $\frac{p}{q}$ and $\frac{p'}{q'}$, respectively, for $k_1, k_2 \in \mathbb{Z}$. Then, $V_2.V_1^{-1}$ is the desired element V for $k_2 - k_1 = k \in \mathbb{Z}$. □

4. Words in Generators

In this section, we consider relations between Farey paths and integer continued fractions. Using these relations, we obtain extended modular group elements as words in terms of generators that correspond a transformation between given two rationals.

Theorem 4.1 *Let* $\frac{p}{q} = [r_0, r_1, ..., r_n]$ *and* $\frac{p'}{q'} = [s_0, s_1, ..., s_m]$ *be reduced rationals. Then, the automorphism in the extended modular group that maps* $\frac{p}{q}$ *to* $\frac{p'}{q'}$ *has the word form*

$$W(U,T,R) = U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.T.U^{k}.T.U^{-r_n}.T.U^{-r_{n-1}}.T.\cdots.T.U^{-r_0} \tag{2}$$

for every integer k. *In addition, the anti-automorphism has the word form*

$$W'(U,T,R) = U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.R.U^{k}.T.U^{-r_n}.T.U^{-r_{n-1}}.T.\cdots.T.U^{-r_0}. \tag{3}$$

Proof First we map $\frac{p}{q}$ to 0. Considering the equality (1),

$$U^{-r_n}.T.U^{-r_{n-1}}.T.\cdots.T.U^{-r_0}(\frac{p}{q}) = 0.$$

The two ordered elliptic generator T maps 0 to infinity. Then, the parabolic generator U fixes infinity. Finally, the cusp point of the element

$$U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.T$$

is $\frac{p'}{q'}$ which proves the result. The second part of the proof can be done by considering the element with cusp point $\frac{p'}{q'}$ as $U^{s_0}.T.U^{s_1}.T.\cdots.U^{s_m}.R$. □

Since the modular group is isomorphic to the free product of the cyclic groups of orders 2 and 3, every element can be expressed as a word in T and S. Considering $U = TS$, we get two blocks,

$$TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad and \quad TS^2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Hence, every element $V \in \overline{\Gamma}$ has a word form,

$$V = S^{i}.(TS)^{m_0}.(TS^2)^{n_0}.(TS)^{m_1}.(TS^2)^{n_1}.\cdots.(TS)^{m_k}.(TS^2)^{n_k}.T^{j}.R^{t},$$

where $i = 0, 1, 2$, $j, t = 0, 1$. The powers of the blocks are positive integers but m_0 and n_k may be zero. This form is called *block reduced form*. Every element in the extended modular group has block reduced form. For instance, $RTS^2RTSTS^2TS^2RTS$ can be expressed as $(TS)^2.(TS^2)^3R$. Trace classes of the modular group and the extended modular group were studied in [3, 8]. Corresponding transformations of these blocks are related to simple continued fraction expansions. Here we obtain block forms of the words given in Theorem 4.1.

Theorem 4.2 *The block reduced form of the elements given in (2) and (3) are*

$$\begin{aligned} W_{BRF} &= (TS)^{s_0-1}.(TS^2).(TS)^{s_1-2}.(TS^2).\cdots.(TS)^{s_m-2}.(TS^2).(TS)^{k-1}. \\ &\quad (TS^2)^{r_n-1}.(TS).(TS^2)^{r_{n-1}-2}.(TS).\cdots.(TS).(TS^2)^{r_1-2}.(TS).(TS^2)^{r_0-1}.T \end{aligned}$$

$$\begin{aligned} W'_{BRF} &= (TS)^{s_0-1}.(TS^2).(TS)^{s_1-2}.(TS^2).\cdots.(TS)^{s_{m-1}-2}.(TS^2).(TS)^{s_m-1}. \\ &\quad (TS^2)^{k}.(TS)^{r_n-1}.(TS^2).(TS)^{r_{n-1}-2}.(TS^2).\cdots.(TS)^{r_1-2}.(TS^2).(TS)^{r_0-1}.T.R, \end{aligned}$$

respectively.

Proof First we take $U = T.S$ in (2).

$$\begin{aligned} W_{BRF} &= (TS)^{s_0}.T.(TS)^{s_1}.T.\cdots.(TS)^{s_m}.T.(TS)^{k}.T.(TS)^{-r_n}.T.(TS)^{-r_{n-1}}.T.\cdots.T.(TS)^{-r_0} \\ &= (TS)^{s_0-1}.TS.T.TS.(TS)^{s_1-2}TS.T.\cdots.T.TS.(TS)^{s_m-2}.TS.T.TS.(TS)^{k-1}. \\ &\quad T.(S^2T)^{r_n}.T.(S^2T)^{r_{n-1}}.T.\cdots.T.(S^2T)^{r_0}. \end{aligned}$$

Since the elliptic generator T is of order 2 and S is of order 3, we have the block reduced form of the word as stated. The second part of the proof can be obtained similarly with relations

$$RS = S^2R,$$

$$TR = RT.$$

□

Before we sum up all our results, we make connections with Farey paths. A path in a graph consists of consecutive adjacent vertices. So, a Farey path $< v_1, v_2, ..., v_n >$ is a path such that $v_i = \frac{p_i}{q_i}$ for $i = 1, 2, ..., n$ are reduced rationals and since the consecutive v_i's are adjacent, we have $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. As Farey graph is connected, there always be a path between two rationals.

For a given reduced rational $\frac{p}{q} = [r_0; r_1, ..., r_n]$, the ith convergent of the integer continued fraction expansion of $\frac{p}{q}$ defined as $C_i = \frac{p_i}{q_i} = [r_0; r_1, ..., r_i]$ for $0 \le i \le n$, where $C_0 = \frac{p_0}{q_0} = \frac{r_0}{1}$ and $C_n = \frac{p_n}{q_n} = \frac{p}{q}$. Furthermore, we know that $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. Hence, every consecutive pair C_i and C_{i-1} are Farey neighbours. Also, this situation can be thought as $< \infty, C_0, C_1, ..., C_{n-1}, C_n >$ is a path from ∞ to $\frac{p}{q}$. Finally, every integer continued fraction expansion of a rational $\frac{p}{q}$ is related to a path from ∞ to $\frac{p}{q}$. Moreover, the shortest integer continued fraction of $\frac{p}{q}$ is related to a geodesic path from ∞ to $\frac{p}{q}$.

Now we give an example to explain all our results.

Example 4.3 *Let the given reduced rationals be* $\frac{7}{11}$ *and* $\frac{12}{7}$. *We find elements* $V \in \overline{\Gamma}$ *that the corresponding transformation* $V(z)$ *maps* $\frac{7}{11}$ *to* $\frac{12}{7}$, *i.e.,* $V(\frac{7}{11}) = \frac{12}{7}$. *We observe the following two paths from infinity to rationals* $\frac{7}{11}$ *and* $\frac{12}{7}$,

$$v =< \infty, 1, \frac{2}{3}, \frac{7}{11} >,$$

$$v' =< \infty, 2, \frac{9}{5}, \frac{7}{4}, \frac{12}{7} > .$$

The penultimate vertex $\frac{2}{3}$ *in path* v, *is the neighbour of* $\frac{7}{11}$ *such that* $\mu_{\frac{7}{11}}(\frac{2}{3}) = -1$. *Similarly, the neighbour of* $\frac{12}{7}$ *is* $\frac{7}{4}$, $\mu_{\frac{12}{7}}(\frac{7}{4}) = -1$. *By Lemma 3.1, we have the hyperbolic element,*

$$V = \begin{pmatrix} -41 & 25 \\ -23 & 14 \end{pmatrix} \in \Gamma.$$

The corresponding transformation is $V(z) = \frac{-41z+25}{-23z+14}$. *Hence, we obtain* $V(\frac{7}{11}) = \frac{12}{7}$. *For the neighbour* $\frac{2}{3}$, *taking* -2 *for* 2 *and* -3 *for* 3 *in Lemma 3.1, we have the element,*

$$V_1 = \begin{pmatrix} 113 & -73 \\ 65 & -42 \end{pmatrix}$$

which the corresponding transformation $V_1(z) = \frac{113\bar{z}-73}{65\bar{z}-42}$ *is a glide-reflection. To express* V *and* V_1 *as words in generators we need the integer continued fraction expansions of* $\frac{7}{11}$ *and* $\frac{12}{7}$. *The consecutive vertices in Farey path are the concecutive convergents of the integer continued fraction expansion. The convergents of* $\frac{7}{11}$ *are* $1, \frac{2}{3}$ *and* $\frac{7}{11}$. *For* $\frac{12}{7}$, *we have the convergents* $2, \frac{9}{5}, \frac{7}{4}$ *and* $\frac{12}{7}$. *Hence, one can calculate the integer continued fractions*

$$\frac{7}{11} = [1, 3, 4],$$

$$\frac{12}{7} = [2, 5, 1, 3].$$

From Theorem 4.1, we have the words

$$W = U^2.T.U^5.T.U.T.U^3.T.U^k.T.U^{-4}.T.U^{-3}.T.U^{-1},$$

$$W' = U^2.T.U^5.T.U.T.U^3.R.U^k.T.U^{-4}.T.U^{-3}.T.U^{-1}.$$

The elements V and V_1 have the word forms W and W' for $k = 0$, respectively. Finally, we express W and W' in block reduced forms by Theorem 4.2,

$$W_{BRF} = (TS).(TS^2).(TS)^3.(TS^2).(TS)^{-1}.(TS^2).(TS).(TS^2).(TS)^{k-1}.(TS^2)^3.(TS).(TS^2).(TS).T,$$

$$W'_{BRF} = (TS).(TS^2).(TS)^3.(TS^2).(TS)^{-1}.(TS^2).(TS)^2.(TS^2)^k.(TS)^3.(TS^2).(TS).(TS^2).T.R.$$

We substitute S^2T for the fifth term $(TS)^{-1}$ in each word,

$$W_{BRF} = (TS).(TS^2).(TS)^2.(TS^2)^2.(TS)^{k-1}.(TS^2)^3.(TS).(TS^2).(TS).T,$$

$$W'_{BRF} = (TS).(TS^2).(TS)^2.(TS^2).(TS).(TS^2)^k.(TS)^3.(TS^2).(TS).(TS^2).T.R.$$

5. Conclusion

For given two reduced rationals $\frac{p}{q}$ and $\frac{p'}{q'}$, we obtain elements $V \in \overline{\Gamma}$ such that $V(\frac{p}{q}) = \frac{p'}{q'}$. We use the relations between paths in the Farey graph and continued fractions to get V as a word in terms of generators. We also obtain the block reduced form that is a word contains only finite ordered elements. For future research one can consider the blocks,

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } h = TSR = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

defined in [9]. Powers of these matrices have only Fibonacci entries. Koruoğlu proved that every element can be written as a word in powers of f and h. This word is called *New Block Reduced Form* [9]. Obtaining new block reduced form of the words given in this study, makes relations to the Fibonacci sequence.

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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