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A NEW SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL INCLUSION PROBLEMS IN SEMI-INNER PRODUCT SPACES

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Abstract. In this work we reflect a new system of generalized nonlinear variational inclusion problems in 2-uniformly smooth Banach spaces. By using resolvent operator technique, we offer an iterative algorithm for figuring out the approximate solution of the said system. The motive of this paper is to review the convergence analysis of a system of generalized nonlinear variational inclusion problems in 2-uniformly smooth Banach spaces. The proposition used in this paper can be considered as an extension of propositions for examining the existence of solution for various classes of variational inclusions considered and studied by many authors in 2-uniformly smooth Banach spaces.

1. Introduction

In recent past, variational inequalities have been elongated in dissimilar directions and sections of studies, using peculiar and ingenious techniques. One of such conception is variational inclusions. Numerous problems that exist in engineering, optimization and control situations can be designed by free boundary problems which conveys to variational inequality and variational inclusion problems. For details, please refer [1-5, 8-14, 18, 20-23, 25, 26].

2. Resolvent Operator and Formulation of Problem

Let X be a real 2-uniformly smooth Banach space equipped with norm $\|.\|$ and a semi-inner product $[.,.]$. Let $C(X)$ be the family of all nonempty compact subsets of X and 2^X be the power set of X.

We need the following definitions and results from the literature.

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Definition 1. *Let X be a vector space over the field F of real or complex numbers. A functional $[.,.] : X \times X \to F$ is called a semi-inner product if it satisfies the following:*

(i) $[x+y, z] = [x, z] + [y, z], \ \forall x, y, z \in X$;
(ii) $[\lambda x, y] = \lambda[x, y], \ \forall \lambda \in F$ *and* $x, y \in X$;
(iii) $[x, x] > 0$, *for* $x \neq 0$;
(iv) $|[x, y]|^2 \leq [x, x][y, y]$.

The pair $(X, [.,.])$ is called a semi-inner product space.

We observe that $||x|| = [x, x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [7] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to find a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x, y] = 0$ if and only if y is orthogonal to x, that is if and only if $||y|| \leq ||y + \lambda x||, \ \forall$ scalars λ.
(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y], \ \forall x \in X$.
(iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $\text{Re}[y, x + \lambda y] \to \text{Re}[y, x]$ as $\lambda \to 0$.

The sequence space $l^p, \ p > 1$ and the function space $L^p, \ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 1. [19] *The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[x, y] = \frac{1}{||y||_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \ x, y \in l^p.$$

Example 2. [7, 19] *The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$[f, g] = \frac{1}{||g||_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn(g(x)) d\mu, \ f, g \in L^p.$$

Definition 2. [19, 24] *Let X be a real Banach space. Then:*

(i) *The modulus of smoothness of X is defined as*

$$\rho_X(t) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| = t, \ t > 0 \right\}.$$

(ii) *X is said to be uniformly smooth if* $\lim\limits_{t \to 0} \dfrac{\rho_X(t)}{t} = 0$.

(iii) *X is said to be p-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c\ t^p,\ p > 1$. Clearly, X is 2-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c\ t^2$.*

Lemma 1. [19, 24] *Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) *X is 2-uniformly smooth.*
(ii) *There is a constant $c > 0$ such that for every $x, y \in X$, the following inequality holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, f_x\rangle + c||y||^2,$$

where $f_x \in J(x)$ and $J(x) = \{x^\star \in X^\star : \langle x, x^\star\rangle = ||x||^2 \text{ and } ||x^\star|| = ||x||\}$ is the normalized duality mapping.

Remark 1. [19] *Every normed linear space is a semi-inner product space (see[15]). In fact by Hahn Banach theorem, for each $x \in X$, there exists atleast one functional $f_x \in X^\star$ such that $\langle x, f_x\rangle = ||x||^2$. Given any such mapping f from X into $X^\star$, we can verify that $[y, x] = \langle y, f_x\rangle$ defines a semi-inner product. Hence we can write (ii) of above Lemma as*

$$||x + y||^2 \leq ||x||^2 + 2[y, x] + c||y||^2,\ \forall x, y \in X.$$

The constant c is chosen with best possible minimum value. We call c, as the constant of smoothness of X.

Example 3. *The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,*

$$||x + y||^2 \leq ||x||^2 + 2[y, x] + (p - 1)||y||^2.$$

Here the constant of smoothness is $p - 1$.

Definition 3. [16, 19] *Let X be a real 2-uniformly smooth Banach space. A mapping $S : X \to X$ is said to be:*

(i) *monotone, if $[Sx - Sy, x - y] \geq 0,\ \forall x, y \in X$,*
(ii) *strictly monotone, if $[Sx - Sy, x - y] > 0,\ \forall x, y \in X$, and equality holds if and only if $x = y$,*
(iii) *r-strongly monotone if there exists a positive constant $r > 0$ such that*

$$[Sx - Sy, x - y] \geq r||x - y||^2,\ \forall x, y \in X,$$

(iv) *δ-Lipschitz continuous, if there exists a constant $\delta > 0$ such that*

$$\|S(x) - S(y)\| \leq \delta\|x - y\|,\ \forall x, y \in X,$$

(v) *η-monotone, if $[Sx - Sy, \eta(x, y)] \geq 0,\ \forall x, y \in X$,*
(vi) *strictly η-monotone, if $[Sx - Sy, \eta(x, y)] > 0,\ \forall x, y \in X$, and equality holds if and only if $x = y$,*
(vii) *r-strongly η-monotone if there exists a positive constant $r > 0$ such that*

$$[Sx - Sy, \eta(x, y)] \geq r||x - y||^2,\ \forall x, y \in X,$$

(viii) *ξ-cocoercive if there exists a constant $\xi > 0$ such that*

$$[Sx - Sy, x - y] \geq \xi ||Sx - Sy||^2, \ \forall x, y \in X,$$

(ix) *relaxed (ξ, δ)-cocoercive if there exist two constants $\xi, \delta > 0$ such that*

$$[Sx - Sy, x - y] \geq -\xi ||Sx - Sy||^2 + \delta \|x - y\|^2, \ \forall x, y \in X.$$

For $\xi = 0$ S is δ-strongly monotone.

This class of mappings is more general than the class of strongly monotone mappings.

Definition 4. *Let X be a 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be single-valued mappings and $M : X \times X \to 2^X$ be multi-valued mapping. Then*

(i) *η is said to be accretive, if*

$$\Big[\eta(x, y), x - y\Big] \geq 0, \ \forall x, y \in X.$$

(ii) *η is said to be strictly accretive, if*

$$\Big[\eta(x, y), x - y\Big] > 0, \ \forall x, y \in X.$$

and equality holds only when $x = y$.

(iii) *η is said to be r-strongly-accretive if there exists a constant $r > 0$ such that*

$$\Big[\eta(x, y), x - y\Big] \geq r \|x - y\|^2, \ \forall x, y \in X.$$

(iv) *η is said to be m-Lipschitz continuous, if there exists a constant $m > 0$ such that*

$$\|\eta(x, y)\| \leq m \|x - y\|, \ \forall x, y \in X,$$

(v) *M is said to be η-accretive in the first argument if*

$$\Big[u - v, \eta(x, y)\Big] \geq 0, \ \forall x, y \in X, \ \forall u \in M(x, t), v \in M(y, t), \ \text{for each fixed } t \in X,$$

(vi) *μ-strongly η-accretive if there exists a positive constant $\mu > 0$ such that*

$$[u - v, \eta(x, y)] \geq \mu \|x - y\|^2, \ \forall x, y \in X, u \in M(x, t), v \in M(y, t).$$

Definition 5. *Let X be a 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be single-valued mappings, $M : X \times X \to 2^X$ be a multi-valued mapping, then M is said to be $m - \eta -$accretive mapping if for each fixed $t \in X, M(., t)$ is η-accretive in the first argument and $(I + \rho M(., t))X = X$, $\forall \rho > 0$.*

Theorem 1. *Let X be a 2-uniformly smooth Banach space. Let $\eta : X \times X \to X$ be q-strongly accretive mapping. Let $M : X \times X \to 2^X$ be $m - \eta -$accretive mapping. If the following inequality : $\Big[u - v, \eta(x, y)\Big] \geq 0$, holds $\forall (y, v) \in$ Graph $(M(., t))$, then $(x, u) \in$ Graph$(M(., t))$, where Graph $(M(., t)) := \{(x, u) \in X \times X : u \in M(x, t)\}$.*

Theorem 2. *Let $\eta : X \times X \to X$ be q-strongly accretive mapping. Let $M : X \times X \to 2^X$ be $m-\eta-$accretive mapping. Then the mapping $(I+\rho M(.,t))^{-1}$ is single-valued, $\forall\ \rho > 0$.*

Definition 6. *Let $\eta : X \times X \to X$ be single-valued mapping. Let $M : X \times X \to 2^X$ be $m-\eta-$accretive mapping. Then for each fixed $t \in X$, the resolvent operator $R_{\rho,\eta}^{M(.,t)} : X \to X$ is defined by*

$$R_{\rho,\eta}^{M(.,t)}(x) = (I+\rho M(.,t))^{-1}(x),\ \forall x \in X.$$

Theorem 3. *Let $\eta : X \times X \to X$ be p-Lipschitz continuous and q-strongly accretive mapping. Let $M : X \times X \to 2^X$ be $m-\eta-$accretive mapping. Then for each fixed $t \in X$ the resolvent operator of M, $R_{\rho,\eta}^{M(.,t)}(x) = (I+\rho M(.,t))^{-1}(x)$ is $\frac{p}{q}-$Lipschitz continuous, that is,*

$$\left\| R_{\rho,\eta}^{M(.,t)}(x) - R_{\rho,\eta}^{M(.,t)}(y) \right\| \le L\|x-y\|,\ \forall x, y, t \in X.$$

where $L = \dfrac{p}{q}$.

Definition 7. *The Hausdorff metric $D(\cdot,\cdot)$ on $CB(X)$, is defined by*

$$D(A,B) = \max\left\{ \sup_{u\in A} \inf_{v\in B} d(u,v),\ \sup_{v\in B} \inf_{u\in A} d(u,v) \right\},\ A, B \in CB(X),$$

where $d(\cdot,\cdot)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 8. [6] *A set-valued mapping $T : X \to CB(X)$ is said to be γ-D-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that*

$$D(T(x), T(y)) \le \gamma \|x-y\|,\ \forall x, y \in X.$$

Theorem 4. [17] *Let $T : X \to CB(X)$ be a set-valued mapping on X and (X,d) be a complete metric space. Then:*

(i) *For any given $\nu > 0$ and for any given $u, v \in X$ and $x \in T(u)$, there exists $y \in T(v)$ such that*

$$d(x,y)\ \le\ (1+\nu) D(T(u), T(v));$$

(ii) *If $T : X \to C(X)$, then (i) holds for $\nu = 0$, (where $C(X)$ denotes the family of all nonempty compact subsets of X).*

Lemma 2. *Let $\{b^n\}$ be a sequence of nonnegative real numbers such that*

$$b^{n+1} \le (1-a^n) b^n + c^n + h^n,\ \forall n \ge n_0,$$

where n_0 is a nonnegative integer, $\{a^n\}$ is a sequence in $(0,1)$ with $\sum\limits_{n=1}^{\infty} a^n = \infty$, $c^n = o(a^n)$ and $\sum\limits_{n=0}^{\infty} h^n < \infty$. Then $\lim\limits_{n\to\infty} b^n = 0$.

Definition 9. *A mapping $S : X\times X\times X \to X$ is said to be relaxed $(\xi, \delta)-$cocoercive if there exist constants $\xi, \delta > 0$ such that*

$$\Big[S(x, y, z) - S(x_1, y_1, z_1), x - x_1\Big] \geq -\xi\Big\|S(x, y, z) - S(x_1, y_1, z_1)\Big\|^2 + \delta\Big\|x - x_1\Big\|^2,$$

$$\forall x, x_1, y, y_1, z, z_1 \in X. \tag{1}$$

Definition 10. *A mapping $S : X\times X\times X \to X$ is said to be β-Lipschitz continuous in the first variable if there exist constant $\beta > 0$ such that*

$$\Big\|S(x, y, z) - S(x_1, y_1, z_1)\Big\| \leq \beta\Big\|x - x_1\Big\|,\ \forall x, x_1, y, y_1, z, z_1 \in X. \tag{2}$$

Now, we formulate our main problem.
For each $i = 1, 2, 3$, let $N_i : X\times X\times X \to X,\ f_i : X \to X, \eta_i : X\times X \to X$ be single-valued mappings. Let $A_i, B_i, F_i : X \to C(X)$ be set-valued mappings. Suppose that $M_i : X \times X \to 2^X$ is $m_i - \eta_i-$ accretive mapping. Then we consider the following system of generalized nonlinear variational inclusion problems (in short, SGNVIP): Find $(x_1, x_2, x_3) \in X \times X \times X, u_i \in A_i(x_i), v_i \in B_i(x_i), w_i \in F_i(x_i)$ such that

$$\left.\begin{aligned}
&0 \in f_1(x_1) - f_1(x_2) + \rho_1\{N_1(u_2, u_3, u_1) + M_1(f_1(x_1), x_1)\}\\
&0 \in f_2(x_2) - f_2(x_3) + \rho_2\{N_2(v_3, v_1, v_2) + M_2(f_2(x_2), x_2)\}\\
&0 \in f_3(x_3) - f_3(x_1) + \rho_3\{N_3(w_1, w_2, w_3) + M_3(f_3(x_3), x_3)\},\ \forall\rho_i > 0.
\end{aligned}\right\} \tag{3}$$

Special Cases:

I. If in problem (3), $f_1(x_1) = G(x), f_1(x_2) = H(x)$, such that $G, H : X \to X, f_2 = f_3 \equiv 0, N_1 = N_2 = N_3 \equiv 0, \rho_1 = \rho_2 = \rho_3 = 1$, then problem (3) reduces to the following problem: Find $x \in X$ such that

$$0 \in G(x) - H(x) + M(G(x), x). \tag{4}$$

This type of problem has been considered and studied by Sahu *et al.*[19].

3. Iterative Algorithm

First, we give the following technical lemma:

Lemma 3. *Let X be a real 2-uniformly smooth Banach space. Let for each $i \in \{1, 2, 3\}$ N_i, f_i, η_i be single-valued mappings. Let $A_i, B_i, F_i : X \to C(X)$ be set-valued mappings, $M_i : X \times X \to 2^X$ be $m_i - \eta_i-$accretive mappings. Then (x_i, u_i, v_i, w_i) where $x_i \in X, u_i \in A_i(x_i), v_i \in B_i(x_i), w_i \in F_i(x_i)$ is a solution of*

(3) if and only if (x_i, u_i, v_i, w_i) satisfies

$$\left.\begin{aligned} f_1(x_1) &= R^{M_1(.,x_1)}_{\rho_1,\eta_1}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ f_2(x_2) &= R^{M_2(.,x_2)}_{\rho_2,\eta_2}\Big\{f_2(x_3) - \rho_2 N_2(v_3,v_1,v_2)\Big\} \\ f_3(x_3) &= R^{M_3(.,x_3)}_{\rho_3,\eta_3}\Big\{f_3(x_1) - \rho_3 N_3(w_1,w_2,w_3)\Big\} \end{aligned}\right\} \tag{5}$$

where $R^{M_i(.,x_i)}_{\rho_i,\eta_i} = \Big(I + \rho_i M_i(.,x_i)\Big)^{-1}$ *are the resolvent operators.*

Proof. Let (x_i, u_i, v_i, w_i) is a solution of (3), then we have

$$\begin{aligned} &f_1(x_1) = R^{M_1(.,x_1)}_{\rho_1,\eta_1}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ &\iff f_1(x_1) = \Big(I + \rho_1 M_1(.,x_1)\Big)^{-1}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ &\iff f_1(x_1) + \rho_1 M_1(f_1(x_1),x_1) = \Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\ &\iff 0 \in f_1(x_1) - f_1(x_2) + \rho_1\{N_1(u_2,u_3,u_1) + M_1(f_1(x_1),x_1)\}. \end{aligned}$$

Proceeding likewise by using (5), we have

$$\begin{aligned} &f_2(x_2) = R^{M_2(.,x_2)}_{\rho_2,\eta_2}\Big\{f_2(x_3) - \rho_2 N_2(v_3,v_1,v_2)\Big\} \\ &\iff 0 \in f_2(x_2) - f_2(x_3) + \rho_2\{N_2(v_3,v_1,v_2) + M_2(f_2(x_2),x_2)\} \end{aligned}$$

and

$$\begin{aligned} &f_3(x_3) = R^{M_3(.,x_3)}_{\rho_3,\eta_3}\Big\{f_3(x_1) - \rho_3 N_3(w_1,w_2,w_3)\Big\} \\ &\iff 0 \in f_3(x_3) - f_3(x_1) + \rho_3\{N_3(w_1,w_2,w_3) + M_3(f_3(x_3),x_3)\}. \end{aligned}$$

□

Lemma 3 allows us to suggest the following iterative algorithm for finding the approximate solution of (3).

Iterative Algorithm 1. *For each $i = \{1,2,3\}$ given $\{x_i^0, u_i^0, v_i^0, w_i^0\}$ where $x_i^0 \in X_i, u_i^0 \in A_i(x_i^0), v_i^0 \in B_i(x_i^0), w_i^0 \in F_i(x_i^0)$ compute the sequences $\{x_i^n, u_i^n, v_i^n, w_i^n\}$ defined by the iterative schemes*

$$f_3(x_3^n) = R^{M_3(.,x_3^n)}_{\rho_3,\eta_3}\Big\{f_3(x_1^n) - \rho_3 N_3(w_1^n,w_2^n,w_3^n)\Big\}$$

$$f_2(x_2^n) = R^{M_2(.,x_2^n)}_{\rho_2,\eta_2}\Big\{f_2(x_3^n) - \rho_2 N_2(v_3^n,v_1^n,v_2^n)\Big\}$$

$$x_1^{n+1} = (1-\alpha^n)x_1^n + \alpha^n\Big(x_1^n - f_1(x_1^n) + R^{M_1(.,x_1^n)}_{\rho_1,\eta_1}\Big\{f_1(x_2^n) - \rho_1 N_1(u_2^n,u_3^n,u_1^n)\Big\}\Big)$$

where α^n is a sequence of real numbers such that $\sum_{n=0}^{\infty} \alpha^n = \infty,\ \forall n \geq 0.$

4. Existence of Solution and Convergence Analysis

Theorem 5. *For each $i \in \{1,2,3\}$, let X be a real 2-uniformly smooth Banach space with k as constant of smoothness. Let $N_i : X \times X \times X \to X$ be a relaxed (ξ_i, δ_i)-cocoercive and ν_i-Lipschitz continuous in the first argument. Let f_i be a relaxed (r_i, s_i)-cocoercive and β_i-Lipschitz continuous in the first argument. Let $A_i, B_i, F_i : X_i \to C(X_i)$ be set-valued mappings such that A_i is $L_{A_i} - D-$Lipschitz continuous, B_i is $L_{B_i} - D-$Lipschitz continuous and F_i is $L_{F_i} - D-$Lipschitz continuous. In addition, if there are constants $t_i > 0$ such that*

$$\left\| R_{\rho_i,\eta_i}^{M_i(.,x_i^n)}(z_i) - R_{\rho_i,\eta_i}^{M_i(.,x_i)}(z_i) \right\|_i \leq t_i \|x_i^n - x_i\|_i, \ \forall z_i \in X_i \tag{6}$$

and

$$1 - (t_2 + \Phi_5) > 0, \ 1 - (t_3 + \Phi_6) > 0$$

such that

$$0 < \Bigg(\Phi_4 + \Phi_4 \frac{L_1 L_2 L_3 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)}$$

$$+ \frac{L_1 L_2 L_3 \Phi_1 \Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)} + t_1 \Bigg) < 1, \tag{7}$$

where

$$\Phi_1 = \sqrt{1 + 2\rho_1(\xi_1 \nu_1^2 L_{A_2}^2 - \delta_1) + k\rho_1^2 \nu_1^2 L_{A_2}^2}; \ \Phi_2 = \sqrt{1 + 2\rho_2(\xi_2 \nu_2^2 L_{B_3}^2 - \delta_2) + k\rho_2^2 \nu_2^2 L_{B_3}^2}.$$

$$\Phi_3 = \sqrt{1 + 2\rho_3(\xi_3 \nu_3^2 L_{F_1}^2 - \delta_3) + k\rho_3^2 \nu_3^2 L_{F_1}^2}; \ \Phi_4 = \sqrt{1 + 2(r_1 \beta_1^2 - s_1) + k\beta_1^2}.$$

$$\Phi_5 = \sqrt{1 + 2(r_2 \beta_2^2 - s_2) + k\beta_2^2}; \ \Phi_6 = \sqrt{1 + 2(r_3 \beta_3^2 - s_3) + k\beta_3^2}.$$

Then the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{w_i^n\}$ generated by above iterative algorithm 1 converges strongly to (x_i, u_i, v_i, w_i) where (x_i, u_i, v_i, w_i) is a solution of above problem (3).

Proof. From Lemma 3, Iterative Algorithm 1, (6) and by using Theorem 3, it follows that

$$\left\| x_1^{n+1} - x_1 \right\|$$

$$
\begin{aligned}
&= \Big\|(1-\alpha^n)x_1^n + \alpha^n\Big(x_1^n - f_1(x_1^n) + R_{\rho_1,\eta_1}^{M_1(.,x_1^n)}\Big\{f_1(x_2^n) - \rho_1 N_1(u_2^n,u_3^n,u_1^n)\Big\}\Big) \\
&\quad -\Big[(1-\alpha^n)x_1 + \alpha^n\Big(x_1 - f_1(x_1) + R_{\rho_1,\eta_1}^{M_1(.,x_1)}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\}\Big)\Big]\Big\| \\
&\leq (1-\alpha^n)\Big\|x_1^n - x_1\Big\| + \alpha^n\Big\|(x_1^n - x_1) - (f_1(x_1^n) - f_1(x_1))\Big\| \\
&\quad +\alpha^n\Big\|R_{\rho_1,\eta_1}^{M_1(.,x_1^n)}\Big\{f_1(x_2^n) - \rho_1 N_1(u_2^n,u_3^n,u_1^n)\Big\} \\
&\quad -R_{\rho_1,\eta_1}^{M_1(.,x_1^n)}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\
&\quad +R_{\rho_1,\eta_1}^{M_1(.,x_1^n)}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\} \\
&\quad -R_{\rho_1,\eta_1}^{M_1(.,x_1)}\Big\{f_1(x_2) - \rho_1 N_1(u_2,u_3,u_1)\Big\}\Big\| \\
&\leq (1-\alpha^n)\Big\|x_1^n - x_1\Big\| + \alpha^n\Big\|(x_1^n - x_1) - (f_1(x_1^n) - f_1(x_1))\Big\| \\
&\quad +\alpha^n L_1\Big\|f_1(x_2^n) - f_1(x_2) - \rho_1\Big(N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1)\Big)\Big\| \\
&\quad +\alpha^n t_1\|x_1^n - x_1\| \\
&\leq (1-\alpha^n)\Big\|x_1^n - x_1\Big\| + \alpha^n\Big\|(x_1^n - x_1) - (f_1(x_1^n) - f_1(x_1))\Big\| \\
&\quad +\alpha^n L_1\Big\|(x_2^n - x_2) - (f_1(x_2^n) - f_1(x_2)\Big\| \\
&\quad +\alpha^n L_1\Big\|(x_2^n - x_2) - \rho_1\Big(N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1)\Big)\Big\| \\
&\quad +\alpha^n t_1\|x_1^n - x_1\|. \qquad (8)
\end{aligned}
$$

Since N_1 is relaxed (ξ_1, δ_1)-cocoercive and ν_1-Lipschitz continuous in the first argument, therefore by using Remark 1 , it follows that

$$
\Big\|(x_2^n - x_2) - \rho_1\Big(N_1(u_2^n,u_3^n,u_1^n) - N_1(u_2,u_3,u_1)\Big)\Big\|^2
$$

$$
\begin{aligned}
&= \Big\|x_2^n - x_2\Big\|^2 - 2\rho_1\Big[N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1), x_2^n - x_2\Big] \\
&\quad + k\rho_1^2\Big\|N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1)\Big\|^2 \\
&\leq \Big\|x_2^n - x_2\Big\|^2 - 2\rho_1\Big\{-\xi_1\Big\|N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1)\Big\|^2 + \delta_1\Big\|x_2^n - x_2\Big\|^2\Big\} \\
&\quad + k\rho_1^2\nu_1^2\Big\|u_2^n - u_2\Big\|^2 \\
&\leq \Big\|x_2^n - x_2\Big\|^2 + 2\rho_1\xi_1\nu_1^2\Big\|u_2^n - u_2\Big\|^2 \\
&\quad - 2\rho_1\delta_1\Big\|x_2^n - x_2\Big\|^2 + k\rho_1^2\nu_1^2\Big\|u_2^n - u_2\Big\|^2 \\
&\leq \Big\|x_2^n - x_2\Big\|^2 + 2\rho_1\xi_1\nu_1^2\Big(D(A_2(x_2^n), A_2(x_2))\Big)^2 \\
&\quad - 2\rho_1\delta_1\Big\|x_2^n - x_2\Big\|^2 + k\rho_1^2\nu_1^2\Big(D(A_2(x_2^n), A_2(x_2))\Big)^2 \\
&\leq \Big\|x_2^n - x_2\Big\|^2 + 2\rho_1\xi_1\nu_1^2 L_{A_2}^2\Big\|x_2^n - x_2\Big\|^2 \\
&\quad - 2\rho_1\delta_1\Big\|x_2^n - x_2\Big\|^2 + k\rho_1^2\nu_1^2 L_{A_2}^2\Big\|x_2^n - x_2\Big\|^2 \\
&\leq \Big(1 + 2\rho_1(\xi_1\nu_1^2 L_{A_2}^2 - \delta_1) + k\rho_1^2\nu_1^2 L_{A_2}^2\Big)\Big\|x_2^n - x_2\Big\|^2
\end{aligned}
$$

$$
\Longrightarrow \Big\|(x_2^n - x_2) - \rho_1\Big(N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1)\Big)\Big\| \leq \Phi_1\Big\|x_2^n - x_2\Big\| \tag{9}
$$

where

$$
\Phi_1 = \sqrt{1 + 2\rho_1(\xi_1\nu_1^2 L_{A_2}^2 - \delta_1) + k\rho_1^2\nu_1^2 L_{A_2}^2}.
$$

Also

$$
\begin{aligned}
\Big\|x_2^n - x_2\Big\| &= \Big\|(x_2^n - x_2) - (f_2(x_2^n) - f_2(x_2)) + (f_2(x_2^n) - f_2(x_2))\Big\| \\
&\leq \Big\|(x_2^n - x_2) - (f_2(x_2^n) - f_2(x_2))\Big\| \\
&\quad + \Big\|f_2(x_2^n) - f_2(x_2)\Big\|.
\end{aligned} \tag{10}
$$

Now,

$$
\begin{aligned}
&\Big\|f_2(x_2^n)-f_2(x_2)\Big\| \\
=\ &\Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3^n)-\rho_2N_2(v_3^n,v_1^n,v_2^n)\Big\}-R_{\rho_2,\eta_2}^{M_2(.,x_2)}\Big\{f_2(x_3)-\rho_2N_2(v_3,v_1,v_2)\Big\}\Big\| \\
=\ &\Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3^n)-\rho_2N_2(v_3^n,v_1^n,v_2^n)\Big\} \\
&-R_{\rho_2,\eta_2}^{M_2(.,x_2)}\Big\{f_2(x_3)-\rho_2N_2(v_3,v_1,v_2)\Big\}\Big\| \\
\leq\ &\Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3^n)-\rho_2N_2(v_3^n,v_1^n,v_2^n)\Big\} \\
&-R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3)-\rho_2N_2(v_3,v_1,v_2)\Big\}\Big\| \\
&+\Big\|R_{\rho_2,\eta_2}^{M_2(.,x_2^n)}\Big\{f_2(x_3)-\rho_2N_2(v_3,v_1,v_2)\Big\} \\
&-R_{\rho_2,\eta_2}^{M_2(.,x_2)}\Big\{f_2(x_3)-\rho_2N_2(v_3,v_1,v_2)\Big\}\Big\| \\
\leq\ &L_2\Big\|f_2(x_3^n)-f_2(x_3)-\rho_2\Big(N_2(v_3^n,v_1^n,v_2^n)-N_2(v_3,v_1,v_2)\Big)\Big\| \\
&+t_2\Big\|x_2^n-x_2\Big\| \\
\leq\ &L_2\Big\|(x_3^n-x_3)-\Big(f_2(x_3^n)-f_2(x_3)\Big)\Big\| \\
&+L_2\Big\|(x_3^n-x_3)-\rho_2\Big(N_2(v_3^n,v_1^n,v_2^n)-N_2(v_3,v_1,v_2)\Big)\Big\| \\
&+t_2\Big\|x_2^n-x_2\Big\|. \qquad (11)
\end{aligned}
$$

Since N_2 is relaxed (ξ_2,δ_2)-cocoercive and ν_2-Lipschitz continuous in the first argument, therefore by using Remark 1, we have

$$\Big\|(x_3^n-x_3)-\rho_2\Big(N_2(v_3^n,v_1^n,v_2^n)-N_2(v_3,v_1,v_2)\Big)\Big\|^2$$

$$
\begin{aligned}
&= \Big\|x_3^n - x_3\Big\|^2 - 2\rho_2\Big[N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2), x_3^n - x_3\Big] \\
&\quad + k\rho_2^2\Big\|N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 - 2\rho_2\Big\{ - \xi_2\Big\|N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big\|^2 + \delta_2\Big\|x_3^n - x_3\Big\|^2\Big\} \\
&\quad + k\rho_2^2\nu_2^2\Big\|v_3^n - v_3\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 - 2\rho_2\Big\{ - \xi_2\nu_2^2\Big\|v_3^n - v_3\Big\|^2 + \delta_2\Big\|x_3^n - x_3\Big\|^2\Big\} \\
&\quad + k\rho_2^2\nu_2^2\Big\|v_3^n - v_3\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 + 2\rho_2\xi_2\nu_2^2\Big(D(B_3(x_3^n), B_3(x_3))\Big)^2 \\
&\quad - 2\rho_2\delta_2\Big\|x_3^n - x_3\Big\|^2 + k\rho_2^2\nu_2^2\Big(D(B_3(x_3^n), B_3(x_3))\Big)^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 + 2\rho_2\xi_2\nu_2^2L_{B_3}^2\Big\|x_3^n - x_3\Big\|^2 \\
&\quad - 2\rho_2\delta_2\Big\|x_3^n - x_3\Big\|^2 + k\rho_2^2\nu_2^2L_{B_3}^2\Big\|x_3^n - x_3\Big\|^2 \\
&\leq \Big(1 + 2\rho_2(\xi_2\nu_2^2L_{B_3}^2 - \delta_2) + k\rho_2^2\nu_2^2L_{B_3}^2\Big)\Big\|x_3^n - x_3\Big\|^2
\end{aligned}
$$

$$
\Longrightarrow \Big\|(x_3^n - x_3) - \rho_2\Big(N_2(v_3^n, v_1^n, v_2^n) - N_2(v_3, v_1, v_2)\Big)\Big\| \leq \Phi_2\Big\|x_3^n - x_3\Big\| \tag{12}
$$

where

$$
\Phi_2 = \sqrt{1 + 2\rho_2(\xi_2\nu_2^2L_{B_3}^2 - \delta_2) + k\rho_2^2\nu_2^2L_{B_3}^2}.
$$

Since f_2 is relaxed (r_2, s_2)-cocoercive and β_2-Lipschitz continuous, therefore by using Remark 1, we have

$$
\Big\|x_3^n - x_3 - (f_2(x_3^n) - f_2(x_3))\Big\|^2
$$

$$\begin{aligned}
&= \Big\|x_3^n - x_3\Big\|^2 - 2\Big[f_2(x_3^n) - f_2(x_3), x_3^n - x_3\Big] + k\Big\|f_2(x_3^n) - f_2(x_3)\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 - 2\Big\{-r_2\Big\|f_2(x_3^n) - f_2(x_3)\Big\|^2 + s_2\Big\|x_3^n - x_3\Big\|^2\Big\} + k\beta_2^2\Big\|x_3^n - x_3\Big\|^2 \\
&\leq \Big\|x_3^n - x_3\Big\|^2 + 2r_2\beta_2^2\Big\|x_3^n - x_3\Big\|^2 - 2s_2\Big\|x_3^n - x_3\Big\|^2 + k\beta_2^2\Big\|x_3^n - x_3\Big\|^2 \\
&\leq \Big(1 + 2(r_2\beta_2^2 - s_2) + k\beta_2^2\Big)\Big\|x_3^n - x_3\Big\|^2
\end{aligned}$$

$$\Longrightarrow \Big\|x_3^n - x_3 - (f_2(x_3^n) - f_2(x_3))\Big\| \leq \Phi_5\Big\|x_3^n - x_3\Big\| \tag{13}$$

where

$$\Phi_5 = \sqrt{1 + 2(r_2\beta_2^2 - s_2) + k\beta_2^2}.$$

Similarly

$$\Big\|x_2^n - x_2 - (f_2(x_2^n) - f_2(x_2))\Big\| \leq \Phi_5\Big\|x_2^n - x_2\Big\|. \tag{14}$$

Substituting (12), (13) in (11), we have

$$\Big\|f_2(x_2^n) - f_2(x_2)\Big\| \leq L_2(\Phi_2 + \Phi_5)\Big\|x_3^n - x_3\Big\| + t_2\Big\|x_2^n - x_2\Big\|. \tag{15}$$

Combining (10), (14) and (15), we have

$$\begin{aligned}
\Big\|x_2^n - x_2\Big\| &\leq \Phi_5\Big\|x_2^n - x_2\Big\| + L_2(\Phi_2 + \Phi_5)\Big\|x_3^n - x_3\Big\| + t_2\Big\|x_2^n - x_2\Big\| \\
&\leq (\Phi_5 + t_2)\Big\|x_2^n - x_2\Big\| + L_2(\Phi_2 + \Phi_5)\Big\|x_3^n - x_3\Big\|. \qquad (16)
\end{aligned}$$

Again, we have

$$\begin{aligned}
\Big\|x_3^n - x_3\Big\| &= \Big\|(x_3^n - x_3) - (f_3(x_3^n) - f_3(x_3)) + (f_3(x_3^n) - f_3(x_3))\Big\| \\
&\leq \Big\|(x_3^n - x_3) - (f_3(x_3^n) - f_3(x_3))\Big\| + \Big\|(f_3(x_3^n) - f_3(x_3))\Big\|. \qquad (17)
\end{aligned}$$

Now,

$$\begin{aligned}
&\Big\|f_3(x_3^n) - f_3(x_3)\Big\| \\
&= \Big\|R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1^n) - \rho_3 N_3(w_1^n, w_2^n, w_3^n)\Big\} - R_{\rho_3,\eta_3}^{M_3(.,x_3)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\}\Big\| \\
&\leq \Big\|R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1^n) - \rho_3 N_3(w_1^n, w_2^n, w_3^n)\Big\} - R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\}\Big\| \\
&\quad + \Big\|R_{\rho_3,\eta_3}^{M_3(.,x_3^n)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\} - R_{\rho_3,\eta_3}^{M_3(.,x_3)}\Big\{f_3(x_1) - \rho_3 N_3(w_1, w_2, w_3)\Big\}\Big\| \\
&\leq L_3\Big\|f_3(x_1^n) - f_3(x_1) - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\| + t_3\Big\|x_3^n - x_3\Big\| \\
&\leq L_3\Big\|x_1^n - x_1 - \Big(f_3(x_1^n) - f_3(x_1)\Big)\Big\|
\end{aligned}$$

$$+L_3\Big\|x_1^n - x_1 - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\| + t_3\Big\|x_3^n - x_3\Big\|. \qquad (18)$$

Since N_3 is relaxed (ξ_3, δ_3)-cocoercive and ν_3-Lipschitz continuous in the first argument, therefore by using Remark 1, we have

$$\begin{aligned}
&\Big\|(x_1^n - x_1) - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\|^2 \\
= \;& \Big\|x_1^n - x_1\Big\|^2 - 2\rho_3\Big[N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3), x_1^n - x_1\Big] \\
& + k\rho_3^2\Big\|N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big\|^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 - 2\rho_3\Big\{ - \xi_3\Big\|N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big\|^2 + \delta_3\Big\|x_1^n - x_1\Big\|^2\Big\} \\
& + k\rho_3^2\nu_3^2\Big\|w_1^n - w_1\Big\|^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 + 2\rho_3\xi_3\nu_3^2\Big\|w_1^n - w_1\Big\|^2 \\
& - 2\rho_3\delta_3\Big\|x_1^n - x_1\Big\|^2 + k\rho_3^2\nu_3^2\Big\|w_1^n - w_1\Big\|^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 + 2\rho_3\xi_3\nu_3^2\Big(D(F_1(x_1^n), F_1(x_1))\Big)^2 \\
& - 2\rho_3\delta_3\Big\|x_1^n - x_1\Big\|^2 + k\rho_3^2\nu_3^2\Big(D(F_1(x_1^n), F_1(x_1))\Big)^2 \\
\leq \;& \Big\|x_1^n - x_1\Big\|^2 + 2\rho_3\xi_3\nu_3^2 L_{F_1}^2\Big\|x_1^n - x_1\Big\|^2 \\
& - 2\rho_3\delta_3\Big\|x_1^n - x_1\Big\|^2 + k\rho_3^2\nu_3^2 L_{F_1}^2\Big\|x_1^n - x_1\Big\|^2 \\
\leq \;& \Big(1 + 2\rho_3(\xi_3\nu_3^2 L_{F_1}^2 - \delta_3) + k\rho_3^2\nu_3^2 L_{F_1}^2\Big)\Big\|x_1^n - x_1\Big\|^2
\end{aligned}$$

$$\Longrightarrow \Big\|(x_1^n - x_1) - \rho_3\Big(N_3(w_1^n, w_2^n, w_3^n) - N_3(w_1, w_2, w_3)\Big)\Big\| \leq \Phi_3\Big\|x_1^n - x_1\Big\| \qquad (19)$$

where

$$\Phi_3 = \sqrt{1 + 2\rho_3(\xi_3\nu_3^2 L_{F_1}^2 - \delta_3) + k\rho_3^2\nu_3^2 L_{F_1}^2}.$$

Since f_3 is relaxed (r_3, s_3)-cocoercive and β_3-Lipschitz continuous, therefore by using Remark 1, it follows that

$$\Big\|x_1^n - x_1 - (f_3(x_1^n) - f_3(x_1))\Big\|^2$$

$$
\begin{aligned}
&= \Big\|x_1^n - x_1\Big\|^2 - 2\Big[f_3(x_1^n) - f_3(x_1), x_1^n - x_1\Big] \\
&\quad + k\Big\|f_3(x_1^n) - f_3(x_1)\Big\|^2 \\
&\leq \Big\|x_1^n - x_1\Big\|^2 - 2\Big\{-r_3\Big\|f_3(x_1^n) - f_3(x_1)\Big\|^2 + s_3\Big\|x_1^n - x_1\Big\|^2\Big\} \\
&\quad + k\beta_3^2\Big\|x_1^n - x_1\Big\|^2 \\
&\leq \Big\|x_1^n - x_1\Big\|^2 + 2r_3\beta_3^2\Big\|x_1^n - x_1\Big\|^2 \\
&\quad - 2s_3\Big\|x_1^n - x_1\Big\|^2 + k\beta_3^2\Big\|x_1^n - x_1\Big\|^2 \\
&\leq \Big(1 + 2(r_3\beta_3^2 - s_3) + k\beta_3^2\Big)\Big\|x_1^n - x_1\Big\|^2 \\
&\quad \Longrightarrow \Big\|x_1^n - x_1 - (f_3(x_1^n) - f_3(x_1))\Big\| \leq \Phi_6\Big\|x_1^n - x_1\Big\|, \qquad (20)
\end{aligned}
$$

where

$$\Phi_6 = \sqrt{1 + 2(r_3\beta_3^2 - s_3) + k\beta_3^2}.$$

Similarly

$$\Big\|x_3^n - x_3 - (f_3(x_3^n) - f_3(x_3))\Big\| \leq \Phi_6\Big\|x_3^n - x_3\Big\|. \qquad (21)$$

Substituting (19), (20) in (18), we have

$$\Big\|f_3(x_3^n) - f_3(x_3)\Big\| \leq L_3(\Phi_3 + \Phi_6)\Big\|x_1^n - x_1\Big\| + t_3\Big\|x_3^n - x_3\Big\|. \qquad (22)$$

Combining (17), (21) and (22), we have

$$
\begin{aligned}
&\Big\|x_3^n - x_3\Big\| \\
&\leq \Phi_6\Big\|x_3^n - x_3\Big\| + L_3\Big(\Phi_3 + \Phi_6\Big)\Big\|x_1^n - x_1\Big\| + t_3\Big\|x_3^n - x_3\Big\| \\
&\leq \Big(\Phi_6 + t_3\Big)\Big\|x_3^n - x_3\Big\| + L_3\Big(\Phi_3 + \Phi_6\Big)\Big\|x_1^n - x_1\Big\| \\
&\Longrightarrow \Big(1 - (t_3 + \Phi_6)\Big)\Big\|x_3^n - x_3\Big\| \leq L_3\Big(\Phi_3 + \Phi_6\Big)\Big\|x_1^n - x_1\Big\| \\
&\quad \Longrightarrow \Big\|x_3^n - x_3\Big\| \leq \frac{L_3\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_3 + \Phi_6)\Big)}\ \Big\|x_1^n - x_1\Big\|. \qquad (23)
\end{aligned}
$$

Substituting (23) in (16), we have

$$\begin{aligned}
&\Big\|x_2^n - x_2\Big\| \\
&\leq (t_2 + \Phi_5)\Big\|x_2^n - x_2\Big\| + \frac{L_2L_3\Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_3 + \Phi_6)\Big)} \ \Big\|x_1^n - x_1\Big\| \\
\Longrightarrow &\Big(1 - (t_2 + \Phi_5)\Big)\Big\|x_2^n - x_2\Big\| \leq \frac{L_2L_3\Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_3 + \Phi_6)\Big)} \ \Big\|x_1^n - x_1\Big\| \\
\Longrightarrow &\Big\|x_2^n - x_2\Big\| \leq \frac{L_2L_3\Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)} \ \Big\|x_1^n - x_1\Big\|. \qquad (24)
\end{aligned}$$

Substituting (24) in (9),

$$\begin{aligned}
&\Big\|(x_2^n - x_2) - \rho_1\Big(N_1(u_2^n, u_3^n, u_1^n) - N_1(u_2, u_3, u_1)\Big)\Big\| \\
&\leq \frac{L_2L_3\Phi_1\Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)} \ \Big\|x_1^n - x_1\Big\|. \qquad (25)
\end{aligned}$$

Since f_1 is relaxed (r_1, s_1)-cocoercive and β_1-Lipschitz continuous, therefore following the same procedure as in (13), (20), we have

$$\Big\|x_1^n - x_1 - (f_1(x_1^n) - f_1(x_1))\Big\| \leq \Phi_4\Big\|x_1^n - x_1\Big\|, \qquad (26)$$

and similarly, we have

$$\Big\|x_2^n - x_2 - (f_1(x_2^n) - f_1(x_2))\Big\| \leq \Phi_4\Big\|x_2^n - x_2\Big\|, \qquad (27)$$

where

$$\Phi_4 = \sqrt{1 + 2(r_1\beta_1^2 - s_1) + k\beta_1^2}.$$

Combining (24) and (27)

$$\begin{aligned}
&\Big\|x_2^n - x_2 - (f_1(x_2^n) - f_1(x_2))\Big\| \\
&\leq \Phi_4\frac{L_2L_3\Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)} \ \Big\|x_1^n - x_1\Big\|. \qquad (28)
\end{aligned}$$

Substituting (25), (26), (28) in (8), it follows that

$$\begin{aligned}
&\Big\|x_1^{n+1} - x_1^n\Big\| \\
&\leq \Bigg\{(1 - \alpha^n) + \alpha^n\Phi_4 + \alpha^n\Phi_4\frac{L_1L_2L_3\Big(\Phi_2 + \Phi_5\Big)\Big(\Phi_3 + \Phi_6\Big)}{\Big(1 - (t_2 + \Phi_5)\Big)\Big(1 - (t_3 + \Phi_6)\Big)}
\end{aligned}$$

$$+\alpha^n \frac{L_1L_2L_3\Phi_1\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)} + \alpha^n t_1\Bigg\} \Big\|x_1^n - x_1\Big\|$$

$$\leq \Bigg\{1-\alpha^n\Bigg(1-\Phi_4-\Phi_4\frac{L_1L_2L_3\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)}$$

$$-\frac{L_1L_2L_3\Phi_1\Big(\Phi_2+\Phi_5\Big)\Big(\Phi_3+\Phi_6\Big)}{\Big(1-(t_2+\Phi_5)\Big)\Big(1-(t_3+\Phi_6)\Big)} - t_1\Bigg)\Bigg\} \Big\|x_1^n - x_1\Big\|$$

$$\leq (1-\alpha^n(1-\hbar)) \Big\|x_1^n - x_1\Big\|. \tag{29}$$

where $\hbar < 1$ by assumption (7). Therefore by using Lemma 2, $\{x_i^n\}$ converges strongly to a solution of (3). This completes the proof. □

5. Conclusion

A new system of generalized nonlinear variational inclusion problems has been introduced in semi-inner product spaces. Using resolvent operator technique, an iterative algorithm has been constructed to solve the proposed system and the convergence analysis of the iterative algorithm has been investigated. The obtained results generalizes many known classes of variational inequalities and variational inclusions in the literature. The results presented can be used for approximation solvability of some different classes of problems in the literature.

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References

[1] Basem, S. M., Iterative approximation of a solution of a multi-valued variational-like inclusion involving δ-strongly maximal $P-\eta-$ monotone mapping in real Hilbert space, *IJMSEA*, 10(2) (2016), 197–206.

[2] Bhat, M.I., Shafi, S., Malik, M. A., H-mixed accretive mapping and proximal point method for solving a system of generalized set-valued variational inclusions, *Numerical Functional Analysis and Optimization*, (2021). https://doi.org/10.1080/01630563.2021.1933527

[3] Cai, G., Bu, S., Convergence analysis for variational inequality problems and fixed point problems in 2-uniformly smooth and uniformly convex Banach spaces, *Math. Comput. Model.*, 55 (2012), 538–546.

[4] Ceng, L. C., Shang, M., Strong convergence theorems for variational inequalities and common fixed-point problems using relaxed mann implicit iteration methods, *Mathematics,* 7(5) (2019), 1–16. doi: 10.3390/math7050424

[5] Ceng, L. C., Postolache, M., Yao, Y., Iterative algorithms for a system of variational inclusions in Banach spaces, *Symmetry,* 11 (2019), 1–12. doi: 10.3390/sym11060811

[6] Chang, S. S., Kim, J. K., Kim, K. H., On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces, *J. Math. Anal. Appl.,* 268 (2002), 89–108.

[7] Giles, J. R., Classes of semi-inner product spaces, *Trans. Am. Math. Soc.,* 129 (1967), 436–446.

[8] Gong, X., Wang, W., A new convergence theorem in a reflexive Banach space, *J. Nonlinear Sci. Appl.,* 9 (2016), 1891–1901.

[9] Gupta, S., Singh, M., Generalized monotone mapping and resolvent equation with an application, *J. Math. Comput. Sci.*, 11(2) (2021), 1767–1783.

[10] Gupta, S., Husain, S., Mishra, V. N., Variational inclusion governed by $\alpha\beta - H((.,.),(.,.))$-mixed accretive mapping, *Filomat*, 31(20) (2017), 6529–6542.

[11] Jeong, J. U., Convergence of parallel iterative algorithms for a system of nonlinear variational inequalities in Banach spaces, *J. Appl. Math. and Informatics,* 34 (2016), 61–73.

[12] Kim, J. K., Bhat, M. I., Shafi, S., Convergence and stability of a perturbed mann iterative algorithm with errors for a system of generalized variational-like inclusion problems in q-uniformly smooth Banach spaces, *Communications in Mathematics and Applications*, 12(1) (2021), 29-50. doi: 10.26713/cma.v12i1.1401

[13] Kim, J. K., Bhat, M. I., Shafi, S., Convergence and stability of iterative algorithm of system of generalized implicit variational-like inclusion problems using $(\theta,\varphi,\gamma)-$ relaxed cocoercivity, *Nonlinear Functional Analysis and Applications*, 26(4) (2021), 749–780. https://doi.org/10.22771/nfaa.2021.26.04.07

[14] Liu, Y., Kong, H., Strong convergence theorems for relatively nonexpansive mappings and Lipschitz-continuous monotone mappings in Banach spaces, *Indian J. Pure Appl. Math.,* 50(4) (2019), 1049–1065.

[15] Lumer, G., Semi inner product spaces, *Trans. Am. Math. Soc.,* 100 (1961), 29–43.

[16] Luo, X. P., Huang, N. J., (H,ϕ)-η-monotone operators in Banach spaces with an application to variational inclusions, *Appl. Math. Comput.,* 216 (2010), 1131–1139.

[17] Nadler, S. B., Multivalued contraction mappings, *Pacific J. Math.,* 30 (1969), 475–488.

[18] Sahu, N. K., Chadli, O., Mohapatra, R. N., Variational inequalities in semi-inner product spaces, *Computational Mathematics and Variational Analysis, Springer Optimization and Its Applications*, 159 (2020), 421–439. doi: 10.1007/978-3-030-44625-3-23

[19] Sahu, N. K., Mohapatra, R. N., Nahak, C., Nanda, S., Approximation solvability of a class of A-monotone implicit variational inclusion problems in semi-inner product spaces, *Appl. Math. Comput.,* 236 (2014), 109-117.

[20] Shafi, S., System of generalized nonlinear variational-like inclusion problems in 2-uniformly smooth Banach spaces, *International Journal of Nonlinear Analysis and Applications*, 13(1) (2022), 267-287. http://dx.doi.org/10.22075/ijnaa.2020.20694.2197

[21] Shafi, S., Mishra, L. N., Existence of solution and convergence of resolvent iterative algorithms for a system of nonlinear variational inclusion problem, *Electronic Journal of Mathematical Analysis and Applications*, 9(2) (2021), 48–60. http://math-frac.org/Journals/EJMAA/

[22] Shafi, S., Mishra, V. N., $(A(.,.),\eta)$-monotone mappings and a system of generalized set-valued variational-like inclusion problem, *Jnanabha*, 51(1) (2021), 245–253.

[23] Sow, T. M. M., Diop, C., Gueye, M. M., General iterative algorithm for solving system of variational inequality problems in real Banach spaces, *Results in Nonlinear Analysis,* 3(1) (2020), 1–11.

[24] Xu, H. K., Inequalities in Banach spaces with applications, *Nonlinear Anal.,* 16(12) (1991), 1127–1138.

[25] Xu, H. K., Muglia, L., On solving variational inequalities defined on fixed point sets of multivalued mappings in Banach spaces, *J. Fixed Point Theory Appl.,* 22(79) (2020). doi.org/10.1007/s11784-020-00817-1

[26] Xu, Y., Guan, J., Tang, Y., Su, Y., Multivariate systems of nonexpansive operator equations and iterative algorithms for solving them in uniformly convex and uniformly smooth Banach spaces with applications, *Journal of Inequalities and Applications,* 2018(37) (2018). doi.org/10.1186/s13660-018-1629-7

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ON EIGENFUNCTIONS OF HILL'S EQUATION WITH SYMMETRIC DOUBLE WELL POTENTIAL

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Abstract. Throughout this paper the asymptotic approximations for eigenfunctions of eigenvalue problems associated with Hill's equation satisfying periodic and semi-periodic boundary conditions are derived when the potential is symmetric double well. These approximations are used to determine the Green's functions of the related problems. Then, the obtained results are adapted to the Whittaker-Hill equation which has the symmetric double well potential and is widely investigated in the literature.

1. Introduction

Consider the Hill's equation

$$y'' + [\lambda - q(x)]y = 0, \quad x \in [0, a] \tag{1}$$

under the periodic boundary conditions $y(0) = y(a)$, $y'(0) = y'(a)$, or the semi-periodic boundary conditions $y(0) = -y(a)$, $y'(0) = -y'(a)$. Here, λ is a real parameter and the potential $q(x)$ is a real-valued, absolutely continuous and periodic function with period a such that

$$\int_0^a q(t)dt = 0.$$

The equation (1) is fundamental for the quantum mechanical treatment of atomic and molecular phenomena. This kind of equation was first used by Hill [21] in modelling of the moon motion. It also appears in the theory of particle orbits in linear accelerators and alternating gradient synchrotrons, because the field structures are periodic [10, 25, 32].

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The derivation of asymptotic formulae for eigenvalues and eigenfunctions of Hill's equation, when restrictive conditions were imposed on q, is of interest in its own right and has a long history. Exact solutions of differential equations are unfortunately rare in applied mathematics and physics. Asymptotical interpretation of the differential equations plays an important role in understanding the behaviour of such differential equations [5,12,27,30,31]. Motivation for studying eigenvalue and eigenfunction asymptotics has come from several different types of problems including instability intervals and gaps of eigenvalues [3,4,11,15,22,26], the derivation and properties of the Green's function [7–9,13,14,23], inverse spectral theory and theory on reconstructing the potential function from knowledge of spectral data [16,19], and the general theory of periodic potentials [2,6,18,24,28].

The main purpose of this paper is to determine asymptotic formulae for the eigenfunctions of the Hill's equation with $q(x)$ being of a symmetric double well potential under the periodic and semi-periodic boundary conditions . We call q a double well potential, if there are points $x_1 < x_2 < x_3$ in $[0, a]$ such that q is monotone decreasing on $[0, x_1]$ and $[x_2, x_3]$ and is monotone increasing elsewhere. In this work, it is assumed in particular that the potential function q is a continuous function on $[0, a]$ which is symmetric on $[0, a]$ as well as on $[0, \frac{a}{2}]$ and non-increasing on $[0, \frac{a}{4}]$, that is, $q(x) = q(a-x) = q(\frac{a}{2} - x)$, mathematically.

Denote by λ_n and μ_n $(n = 0, 1, 2, ...)$ the periodic and semi-periodic eigenvalues of (1), respectively. These eigenvalues are interlaced in the following way:

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < ... \to \infty.$$

Başkaya [4] obtained the asymptotic approximations of the periodic and semi-periodic eigenvalues of (1) having symmetric double well potential such that, as $n \to \infty$

$$\begin{aligned} \begin{matrix}\lambda_{2n+1}\\ \lambda_{2n+2}\end{matrix} &= \frac{4(n+1)^2\pi^2}{a^2} \mp \frac{1}{(n+1)\pi}\Big|\int_0^{a/4} q'(t)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\Big| \\ &- \frac{a}{16(n+1)^2\pi^2}[aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)dt \\ &- 8\int_0^{a/4} tq(t)q'(t)dt] + o(n^{-2}) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \begin{matrix}\mu_{2n}\\ \mu_{2n+1}\end{matrix} &= \frac{(2n+1)^2\pi^2}{a^2} - \frac{a}{4(2n+1)^2\pi^2}[aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)dt \\ &- 8\int_0^{a/4} tq(t)q'(t)dt] + o(n^{-2}). \end{aligned} \tag{3}$$

In Section 2, the eigenfunctions of (1) corresponding to the eigenvalues, λ_n and μ_n, given by (2) and (3) are investigated. By using the estimates on the eigenfunctions, the Green's function asymptotics related to the Hill's equation are derived in Section 3. Here, the method developed by Fulton [20] is followed. In Section 4, the obtained results for the eigenfunctions and Green's functions are adapted to the Whittaker-Hill equation

$$\frac{d^2\psi}{dz^2} + [\lambda + 2kcos(2z) + 2lcos(4z)]\psi = 0$$

where λ, k, l are real. This equation arises after separating the wave equation using paraboloidal coordinates [1] and is equivalent to a time-independent Schrödinger equation,

$$-\alpha\frac{d^2\psi}{d\theta^2} + V(\theta)\psi = \varepsilon\psi,$$

that describes the internal rotational (torsinal) problem of a given molecular system around a dihedral angle $\theta = 2z$. $\varepsilon = \alpha\lambda/4$ is the energy eigenvalue of the eigenfunction $\psi = \psi(\theta)$ and $V(\theta) = V_1\cos(\theta) + V_2\cos(2\theta)$ is a period 2π function representing a symmetric periodic double well potential with $V_1 = -\alpha k/2$ and $V_2 = -\alpha l/2$ (see [29]).

The following results obtained in [18] will be used to determine the eigenfunctions.

Let $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ be the linearly independent solutions of (1) with the initial conditions

$$\phi_1(0,\lambda) = 1, \quad \phi_1'(0,\lambda) = 0, \quad \phi_2(0,\lambda) = 0, \quad \phi_2'(0,\lambda) = 1. \tag{4}$$

Theorem 1. *[18, §4.3] Assume that $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are the solutions of (1) satisfying (4). Let $q(x)$ be an absolutely continuous function. Then, as $\lambda \to \infty$,*

$$\phi_1(x,\lambda) = \cos(x\sqrt{\lambda}) + \frac{1}{2}\lambda^{-\frac{1}{2}}Q(x)\sin(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-1}\left\{q(x) - q(0) - \frac{1}{2}Q^2(x)\right\} \times \cos(x\sqrt{\lambda}) + o(\lambda^{-1}),$$

$$\phi_2(x,\lambda) = \lambda^{-\frac{1}{2}}\sin(x\sqrt{\lambda}) - \frac{1}{2}\lambda^{-1}Q(x)\cos(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-\frac{3}{2}}\left\{q(x) + q(0) - \frac{1}{2}Q^2(x)\right\}\sin(x\sqrt{\lambda}) + o(\lambda^{-\frac{3}{2}})$$

where

$$Q(x) = \int_0^x q(t)dt. \tag{5}$$

2. Asymptotics of Eigenfunctions

In this section we obtain the asymptotic approximations for eigenfunctions of (1) satisfying the periodic and semi-periodic boundary conditions.

Before, we prove the following lemma for $q(x)$ being of a symmetric double well potential.

Lemma 1. *If $q(x)$ is a symmetric double well potential on $[0,a]$, then*

$$\int_0^x q(t)dt = xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt. \qquad (6)$$

Proof. Using integration by parts and $q(x) = q(a-x) = q(\frac{a}{2} - x)$, it is obtained that

$$\begin{aligned}
\int_0^x q(t)dt &= tq(t)\Big|_{t=0}^{x} - \int_0^x tq'(t)dt \\
&= xq(x) - \left[\int_0^{a/2} tq'(t)dt + \int_{a/2}^{x} tq'(t)dt\right] \\
&= xq(x) - \left[-\int_0^{a/2} tq'(a-t)dt + \int_{a/2}^{x} tq'(t)dt\right] \\
&= xq(x) - \int_a^{a/2} (a-t)q'(t)dt - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) + a[q(t)]\Big|_{t=a/2}^{a} - \int_{a/2}^{a} tq'(t)dt - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) + a\left[q(a) - q\left(\frac{a}{2}\right)\right] - \int_{a/2}^{a} tq'(t)dt - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) - \int_{a/2}^{a} tq'(t)dt - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) - \left[\int_0^{a/4} tq'(t)dt + \int_{a/4}^{a/2} tq'(t)dt\right] - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) - \left[-\int_0^{a/4} tq'\left(\frac{a}{2} - t\right)dt + \int_{a/4}^{a/2} tq'(t)dt\right] - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) - \left[\int_{a/2}^{a/4} \left(\frac{a}{2} - t\right)q'(t)dt + \int_{a/4}^{a/2} tq'(t)dt\right] - \int_{a/2}^{x} tq'(t)dt \\
&= xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt.
\end{aligned}$$

□

Theorem 2. *Let $q(x)$ be a symmetric double well potential on $[0, a]$. Then as $\lambda \to \infty$, for the solutions of (1) with the initial conditions (4), we have*

$$\phi_1(x,\lambda) = \cos(x\sqrt{\lambda}) + \frac{1}{2}\lambda^{-\frac{1}{2}}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt$$
$$- \int_{a/4}^{x} tq'(t)dt\} \sin(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-1}\{q(x) - q(0) - \frac{1}{2}[xq(x) + \frac{a}{2}[q\left(\frac{a}{2}\right)$$
$$- q\left(\frac{a}{4}\right)] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt]^2\} \cos(x\sqrt{\lambda}) + o(\lambda^{-1}), \qquad (7)$$

$$\phi_2(x,\lambda) = \lambda^{-\frac{1}{2}} \sin(x\sqrt{\lambda}) - \frac{1}{2}\lambda^{-1}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt$$
$$- \int_{a/4}^{x} tq'(t)dt\} \cos(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-\frac{3}{2}}\{q(x) + q(0) - \frac{1}{2}[xq(x) + \frac{a}{2}[q\left(\frac{a}{2}\right)$$
$$- q\left(\frac{a}{4}\right)] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt]^2\} \sin(x\sqrt{\lambda}) + o(\lambda^{-\frac{3}{2}}). \qquad (8)$$

Proof. If we use Theorem 1 and substitute (6) in (5), the proof is done. □

Theorem 3. *The eigenfunctions of the periodic problem having symmetric double well potential satisfy, as $n \to \infty$*

$$\phi_1(x,n) = \cos\frac{2(n+1)\pi x}{a} + \frac{a}{4(n+1)\pi}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right]$$
$$- \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt\} \sin\frac{2(n+1)\pi x}{a} + \frac{a^2}{16(n+1)^2\pi^2}$$
$$\times \{q(x) - q(0) - \frac{1}{2}[xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt$$
$$- \int_{a/4}^{x} tq'(t)dt]^2\} \cos\frac{2(n+1)\pi x}{a} + o(n^{-2}),$$

$$\phi_2(x,n) = \frac{a}{2(n+1)\pi} \sin\frac{2(n+1)\pi x}{a} - \frac{a^2}{8(n+1)^2\pi^2}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right]$$
$$- \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt\} \cos\frac{2(n+1)\pi x}{a} + \frac{a^3}{32(n+1)^3\pi^3}$$
$$\times \{q(x) + q(0) - \frac{1}{2}[xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt$$
$$- \int_{a/4}^{x} tq'(t)dt]^2\} \sin\frac{2(n+1)\pi x}{a} + o(n^{-3}).$$

Theorem 4. *The eigenfunctions of the semi-periodic problem having symmetric double well potential satisfy, as $n \to \infty$*

$$\phi_1(x,n) = \cos\frac{(2n+1)\pi x}{a} + \frac{a}{2(2n+1)\pi}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt\}\sin\frac{(2n+1)\pi x}{a} + \frac{a^2}{4(2n+1)^2\pi^2} \times \{q(x) - q(0) - \frac{1}{2}[xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt]^2\}\cos\frac{(2n+1)\pi x}{a} + o(n^{-2}),$$

$$\phi_2(x,n) = \frac{a}{(2n+1)\pi}\sin\frac{(2n+1)\pi x}{a} - \frac{a^2}{2(2n+1)^2\pi^2}\{xq(x) + \frac{a}{2}[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt\}\cos\frac{(2n+1)\pi x}{a} + \frac{a^3}{4(2n+1)^3\pi^3}\{q(x) + q(0) - \frac{1}{2}[xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt]^2\}\sin\frac{(2n+1)\pi x}{a} + o(n^{-3}).$$

To prove Theorem 3 and Theorem 4, the related eigenvalues given by (2) and (3) are substituted in Theorem 2.

We also have asymptotic formulae for the derivatives of $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$. We will use them in calculation of the Green's functions.

Lemma 2. *Consider the equation (1) having symmetric double well potential. As $\lambda \to \infty$, for the derivatives of its solutions, $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ which satisfy (4), we have*

$$\phi_1'(x,\lambda) = -\lambda^{\frac{1}{2}}\sin(x\sqrt{\lambda}) + \frac{1}{2}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt\}\cos(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-\frac{1}{2}}\{q(x) + q(0) + \frac{1}{2}[xq(x) + \frac{a}{2}[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt]^2\}\sin(x\sqrt{\lambda}) + o(\lambda^{-\frac{1}{2}}), \tag{9}$$

$$\phi_2'(x,\lambda) = \cos(x\sqrt{\lambda}) + \frac{1}{2}\lambda^{-\frac{1}{2}}\{xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt$$

$$-\int_{a/4}^{x} tq'(t)dt\}\sin(x\sqrt{\lambda})-\frac{1}{4}\lambda^{-1}\{q(x)-q(0)+\frac{1}{2}[xq(x)+\frac{a}{2}[q\left(\frac{a}{2}\right)$$
$$-q\left(\frac{a}{4}\right)]-\int_{a/4}^{a/2} tq'(t)dt-\int_{a/4}^{x} tq'(t)dt]^2\}\cos(x\sqrt{\lambda})+o(\lambda^{-1}). \qquad (10)$$

Proof. Here, the proof of (9) will be shown. The proof of (10) is similar to that.

If $q(x)$ is a piecewise continuous function, then, as $\lambda\to\infty$,

$$\phi_1(x,\lambda)=\cos(x\sqrt{\lambda})+\lambda^{-\frac{1}{2}}\int_0^x \sin\{(x-t)\sqrt{\lambda}\}q(t)\cos(t\sqrt{\lambda})dt$$
$$+\lambda^{-1}\int_0^x \sin\{(x-t)\sqrt{\lambda}\}q(t)dt\int_0^t \sin\{(t-u)\sqrt{\lambda}\}q(u)\cos(u\sqrt{\lambda})du$$
$$+O(\lambda^{-\frac{3}{2}}) \qquad (11)$$

(see [18, §4.3]). The usual variation of constants formula [17, §2.5] gives

$$\phi_1(x,\lambda)=\cos(x\sqrt{\lambda})+\lambda^{-\frac{1}{2}}\int_0^x \sin\{(x-t)\sqrt{\lambda}\}q(t)\phi_1(t,\lambda)dt.$$

If we arrange this formula, one can write

$$\phi_1(x,\lambda)=\cos(x\sqrt{\lambda})+\lambda^{-\frac{1}{2}}\{\sin(x\sqrt{\lambda})\int_0^x \cos(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt$$
$$-\cos(x\sqrt{\lambda})\int_0^x \sin(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt\}. \qquad (12)$$

It is obtained by differentiating (12) with respect to x and substituting $\phi_1(t,\lambda)$ from (11) in the integral that

$$\phi_1'(x,\lambda)=-\lambda^{\frac{1}{2}}\sin(x\sqrt{\lambda})+\lambda^{-\frac{1}{2}}\{\lambda^{\frac{1}{2}}\cos(x\sqrt{\lambda})\int_0^x \cos(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt$$
$$+\lambda^{\frac{1}{2}}\ sin(x\sqrt{\lambda})\int_0^x \sin(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt\}$$
$$=-\lambda^{\frac{1}{2}}\sin(x\sqrt{\lambda})+\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)\phi_1(t,\lambda)dt$$
$$=-\lambda^{\frac{1}{2}}\sin(x\sqrt{\lambda})+\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)\cos(t\sqrt{\lambda})dt$$
$$+\lambda^{-\frac{1}{2}}\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)dt\int_0^t \sin\{(t-u)\sqrt{\lambda}\}q(u)\cos(u\sqrt{\lambda})du$$
$$+O(\lambda^{-1}). \qquad (13)$$

If differentiability conditions are imposed on $q(x)$, (13) can be made more precise. Assume that $q(x)$ is absolutely continuous. This implies that $q'(x)$ exists almost everywhere and is integrable. Under these conditions, let consider the second term on the right of (13). We have

$$\begin{aligned}
&\textstyle\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)\cos(t\sqrt{\lambda})dt\\
&\qquad= \frac{1}{2}\int_0^x \left[\cos(x\sqrt{\lambda})+\cos\{(x-2t)\sqrt{\lambda}\}\right]q(t)dt\\
&\qquad= \frac{1}{2}Q(x)\cos(x\sqrt{\lambda})+\frac{1}{2}\int_0^x \cos\{(x-2t)\sqrt{\lambda}\}q(t)dt\\
&\qquad= \frac{1}{2}Q(x)\cos(x\sqrt{\lambda})+\frac{1}{2}[-\frac{1}{2}\lambda^{-\frac{1}{2}}q(t)sin\{(x-2t)\sqrt{\lambda}\}\Big|_{t=0}^{x}\\
&\qquad\quad+\frac{1}{2}\lambda^{-\frac{1}{2}}\int_0^x q'(t)\sin\{(x-2t)\sqrt{\lambda}\}dt]\\
&\qquad= \frac{1}{2}Q(x)\cos(x\sqrt{\lambda})+\frac{1}{4}\lambda^{-\frac{1}{2}}\left[q(x)+q(0)\right]\sin(x\sqrt{\lambda})\\
&\qquad\quad+\frac{1}{4}\lambda^{-\frac{1}{2}}\int_0^x q'(t)\sin\{(x-2t)\sqrt{\lambda}\}dt.
\end{aligned}$$

The right-hand integral on the last equality is $o(1)$ as $\lambda\to\infty$ by the Riemann-Lebesgue Lemma. So,

$$\begin{aligned}
\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)\cos(t\sqrt{\lambda})dt &= \frac{1}{2}Q(x)\cos(x\sqrt{\lambda})+\frac{1}{4}\lambda^{-\frac{1}{2}}\left[q(x)+q(0)\right]\\
&\quad\times\sin(x\sqrt{\lambda})+o(\lambda^{-\frac{1}{2}}).
\end{aligned}\tag{14}$$

Also, from [18, §4.3]

$$\begin{aligned}
\int_0^x \sin\{(x-t)\sqrt{\lambda}\}q(t)\cos(t\sqrt{\lambda})dt &= \frac{1}{2}Q(x)\sin(x\sqrt{\lambda})+\frac{1}{4}\lambda^{-\frac{1}{2}}\left[q(x)-q(0)\right]\\
&\quad\times\cos(x\sqrt{\lambda})+o(\lambda^{-\frac{1}{2}}).
\end{aligned}\tag{15}$$

For the third term on the right of (13), together with (15) we find

$$\begin{aligned}
&\textstyle\lambda^{-\frac{1}{2}}\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)dt\int_0^t \sin\{(t-u)\sqrt{\lambda}\}q(u)\cos(u\sqrt{\lambda})du\\
&\qquad= \frac{1}{2}\lambda^{-\frac{1}{2}}\int_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)Q(t)\sin(t\sqrt{\lambda})dt+O(\lambda^{-1})\\
&\qquad= \frac{1}{4}\lambda^{-\frac{1}{2}}\int_0^x \left[\sin(x\sqrt{\lambda})-\sin\{(x-2t)\sqrt{\lambda}\}\right]q(t)Q(t)dt+O(\lambda^{-1})\\
&\qquad= \frac{1}{4}\lambda^{-\frac{1}{2}}\sin(x\sqrt{\lambda})\left[\frac{Q^2(t)}{2}\right]\Big|_{t=0}^{x}+o(\lambda^{-\frac{1}{2}})\\
&\qquad= \frac{1}{8}\lambda^{-\frac{1}{2}}Q^2(x)\sin(x\sqrt{\lambda})+o(\lambda^{-\frac{1}{2}}),
\end{aligned}\tag{16}$$

again by using the Riemann-Lebesgue Lemma. From (14) and (16), it is obtained that

$$\phi_1'(x,\lambda) = -\lambda^{\frac{1}{2}}\sin(x\sqrt{\lambda}) + \frac{1}{2}Q(x)\cos(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-\frac{1}{2}}\left\{q(x)+q(0)+\frac{1}{2}Q^2(x)\right\} \times \sin(x\sqrt{\lambda}) + o(\lambda^{-\frac{1}{2}}). \tag{17}$$

Using (6) in (5) and substituting this in (17) prove (9).

□

3. Asymptotics of Green's Functions

In this section, we aim to improve asymptotic formulae for Green's functions of the periodic and semi-periodic problems with symmetric double well potential. The Green's function $G(x,\zeta,\lambda)$ is given by

$$G(x,\zeta,\lambda) = \begin{cases} \frac{\phi_1(\zeta,\lambda)\phi_2(x,\lambda)}{w(\lambda)}, & 0 \le \zeta \le x \le a \\ \frac{\phi_1(x,\lambda)\phi_2(\zeta,\lambda)}{w(\lambda)}, & 0 \le x \le \zeta \le a \end{cases} \tag{18}$$

(see [20]). Here, $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are linearly independent solutions of (1) satisfying (4). And, we define $w(\lambda)$ as follows

$$w(\lambda) := \phi_1(x,\lambda)\phi_2'(x,\lambda) - \phi_1'(x,\lambda)\phi_2(x,\lambda). \tag{19}$$

It is known as the Wronskian function of $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$.

Theorem 5. *Suppose that the equation (1) has the symmetric double well potential and its independent solutions, $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ satisfy the initial conditions (4). Then, the Green's function of the problem is, as $\lambda \to \infty$*

$$\begin{aligned} G(x,\zeta,\lambda) &= \lambda^{-\frac{1}{2}}\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda}) - \frac{1}{2}\lambda^{-1}[D(x)\cos(\zeta\sqrt{\lambda})\cos(x\sqrt{\lambda}) \\ &\quad - D(\zeta)\sin(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})] + \frac{1}{4}\lambda^{-\frac{3}{2}}\{[q(\zeta)+q(x) - \frac{1}{2}(D^2(\zeta) \\ &\quad + D^2(x))]\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda}) - D(\zeta)D(x)\sin(\zeta\sqrt{\lambda})\cos(x\sqrt{\lambda})\} \\ &\quad + o(\lambda^{-\frac{3}{2}}), \qquad 0 \le \zeta \le x \le a \end{aligned}$$

where

$$D(x) := xq(x) + \frac{a}{2}\left[q\left(\frac{a}{2}\right) - q\left(\frac{a}{4}\right)\right] - \int_{a/4}^{a/2} tq'(t)dt - \int_{a/4}^{x} tq'(t)dt. \tag{20}$$

Similar result holds for $0 \le x \le \zeta \le a$ changing the role of ζ and x.

Proof. We begin to the proof by evaluating the Wronskian function $w(\lambda)$. For this reason, we substitute (7), (8), (9) and (10) into (19). Hence,

$$w(\lambda) = 1 - \frac{1}{4}\lambda^{-1}\left[q(x) - q(0) + \frac{1}{2}D^2(x)\right]\cos^2(x\sqrt{\lambda}) + \frac{1}{4}\lambda^{-1}$$

$$
\begin{aligned}
&\times \left[q(x)+q(0)-\frac{1}{2}D^2(x)\right]\sin^2(x\sqrt{\lambda})\\
&+\frac{1}{4}\lambda^{-1}D^2(x)+\frac{1}{4}\lambda^{-1}\left[q(x)-q(0)-\frac{1}{2}D^2(x)\right]\cos^2(x\sqrt{\lambda})\\
&-\frac{1}{4}\lambda^{-1}\left[q(x)+q(0)+\frac{1}{2}D^2(x)\right]\sin^2(x\sqrt{\lambda})+o(\lambda^{-1})\\
&=1-\frac{1}{4}\lambda^{-1}D^2(x)+\frac{1}{4}\lambda^{-1}D^2(x)+o(\lambda^{-1})\\
&=1+o(\lambda^{-1}).
\end{aligned}
$$

From that, we can write

$$
\frac{1}{w(\lambda)}=\frac{1}{1+o(\lambda^{-1})}=1+o(\lambda^{-1}). \tag{21}
$$

Finally, using (7), (8), (21) in (18) we find

$$
\begin{aligned}
\frac{\phi_1(\zeta,\lambda)\phi_2(x,\lambda)}{w(\lambda)} &= \{\cos(\zeta\sqrt{\lambda})+\frac{1}{2}\lambda^{-\frac{1}{2}}D(\zeta)\sin(\zeta\sqrt{\lambda})+\frac{1}{4}\lambda^{-1}[q(\zeta)-q(0)\\
&\quad-\frac{1}{2}D^2(\zeta)]\cos(\zeta\sqrt{\lambda})+o(\lambda^{-1})\}\\
&\quad\times\{\lambda^{-\frac{1}{2}}\sin(x\sqrt{\lambda})-\frac{1}{2}\lambda^{-1}D(x)\cos(x\sqrt{\lambda})+\frac{1}{4}\lambda^{-\frac{3}{2}}\\
&\quad\times\left[q(x)+q(0)-\frac{1}{2}D^2(x)\right]\sin(x\sqrt{\lambda})+o(\lambda^{-\frac{3}{2}})\}\{1+o(\lambda^{-1})\}\\
&=\{\lambda^{-\frac{1}{2}}\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})-\frac{1}{2}\lambda^{-1}D(x)\cos(\zeta\sqrt{\lambda})\cos(x\sqrt{\lambda})\\
&\quad+\frac{1}{4}\lambda^{-\frac{3}{2}}\left[q(x)+q(0)-\frac{1}{2}D^2(x)\right]\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})\\
&\quad+\frac{1}{2}\lambda^{-1}D(\zeta)\sin(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})-\frac{1}{4}\lambda^{-\frac{3}{2}}D(\zeta)D(x)\\
&\quad\times\sin(\zeta\sqrt{\lambda})\cos(x\sqrt{\lambda})+\frac{1}{4}\lambda^{-\frac{3}{2}}\left[q(\zeta)-q(0)-\frac{1}{2}D^2(\zeta)\right]\\
&\quad\times\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})+o(\lambda^{-\frac{3}{2}})\}\times\{1+o(\lambda^{-1})\}\\
&=\lambda^{-\frac{1}{2}}\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})-\frac{1}{2}\lambda^{-1}[D(x)\cos(\zeta\sqrt{\lambda})\cos(x\sqrt{\lambda})\\
&\quad-D(\zeta)\sin(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})]+\frac{1}{4}\lambda^{-\frac{3}{2}}\\
&\quad\times\{\left[q(\zeta)+q(x)-\frac{1}{2}\left(D^2(\zeta)+D^2(x)\right)\right]\cos(\zeta\sqrt{\lambda})\sin(x\sqrt{\lambda})\\
&\quad-D(\zeta)D(x)\sin(\zeta\sqrt{\lambda})\cos(x\sqrt{\lambda})\}+o(\lambda^{-\frac{3}{2}}).
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 6. *Green's function of the periodic problem with symmetric double well potential satisfies, as $n \to \infty$*

$$G(x,\zeta,n) = \frac{a}{2(n+1)\pi}\cos\frac{2(n+1)\pi\zeta}{a}\sin\frac{2(n+1)\pi x}{a} - \frac{a^2}{8(n+1)^2\pi^2}$$
$$\times [D(x)\cos\frac{2(n+1)\pi\zeta}{a}\cos\frac{2(n+1)\pi x}{a}$$
$$- D(\zeta)\sin\frac{2(n+1)\pi\zeta}{a}\sin\frac{2(n+1)\pi x}{a}]$$
$$+ \frac{a^3}{32(n+1)^3\pi^3}\{\left[q(\zeta)+q(x)-\frac{1}{2}\left(D^2(\zeta)+D^2(x)\right)\right]$$
$$\times \cos\frac{2(n+1)\pi\zeta}{a}\sin\frac{2(n+1)\pi x}{a} - D(\zeta)D(x)\sin\frac{2(n+1)\pi\zeta}{a}$$
$$\times \cos\frac{2(n+1)\pi x}{a}\} + o(n^{-3})$$

for $0 \le \zeta \le x \le a$. Similar result holds for $0 \le x \le \zeta \le a$ changing the role of ζ and x.

Theorem 7. *Green's function of the semi-periodic problem with symmetric double well potential satisfies, as $n \to \infty$*

$$G(x,\zeta,n) = \frac{a}{(2n+1)\pi}\cos\frac{(2n+1)\pi\zeta}{a}\sin\frac{(2n+1)\pi x}{a} - \frac{a^2}{2(2n+1)^2\pi^2}$$
$$\times [D(x)\cos\frac{(2n+1)\pi\zeta}{a}\cos\frac{(2n+1)\pi x}{a}$$
$$- D(\zeta)\sin\frac{(2n+1)\pi\zeta}{a}\sin\frac{(2n+1)\pi x}{a}] + \frac{a^3}{4(2n+1)^3\pi^3}$$
$$\times \{\left[q(\zeta)+q(x)-\frac{1}{2}\left(D^2(\zeta)+D^2(x)\right)\right]\cos\frac{(2n+1)\pi\zeta}{a}$$
$$\times \sin\frac{(2n+1)\pi x}{a} - D(\zeta)D(x)\sin\frac{(2n+1)\pi\zeta}{a}\cos\frac{(2n+1)\pi x}{a}\}$$
$$+ o(n^{-3})$$

for $0 \le \zeta \le x \le a$. Similar result holds for $0 \le x \le \zeta \le a$ changing the role of ζ and x.

To prove Theorem 6 and Theorem 7, the related eigenvalues given by (2) and (3) are used together with Theorem 5.

4. The Whittaker-Hill Equation

Consider the Whittaker-Hill equation

$$y'' + [\lambda + 2k\cos(2x) + 2\ell\cos(4x)]y = 0, \quad x \in [0, 2\pi], \quad \lambda, k, \ell \in \mathbb{R} \tag{22}$$

under the periodic boundary conditions $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$, or the semi-periodic boundary conditions $y(0) = -y(2\pi)$, $y'(0) = -y'(2\pi)$. Here, our goal is to seek the eigenfunction and Green's function asymptotics of the described problem. This problem is a special case of (1) when $q(x) = 2k\cos(2x) + 2\ell\cos(4x)$ and $a = 2\pi$. Also, note that q is a continuous function on $[0, 2\pi]$ which is symmetric on $[0, 2\pi]$ as well as on $[0, \pi]$ and non-increasing on $[0, \frac{\pi}{2}]$, i. e., $q(x) = q(2\pi - x) = q(\pi - x)$. So, we say that q is a symmetric double well potential (see Figure 1). Last of all, we can apply the obtained results in Sections 2 and 3 to this problem.

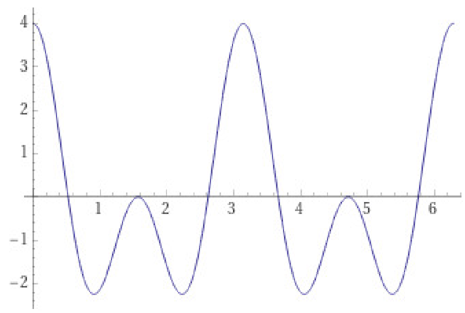

Figure 1. Graph of q when $k = \ell = 1$

Following two theorems give the results about the eigenfunctions.

Theorem 8. *The eigenfunctions of the Whittaker-Hill equation satisfying the periodic boundary conditions are, as $n \to \infty$*

$$\begin{aligned}\phi_1(x,n) &= \cos((n+1)x) + \frac{1}{2(n+1)}\left[ksin(2x) + \frac{\ell}{2}\sin(4x)\right]\sin((n+1)x)\\ &\quad + \frac{1}{4(n+1)^2}\{2k\cos(2x) + 2\ell\cos(4x) - 2(k+\ell) - \frac{1}{2}[k\sin(2x)\\ &\quad + \frac{\ell}{2}\sin(4x)]^2\}\cos((n+1)x) + o(n^{-2}),\end{aligned}$$

$$\begin{aligned}\phi_2(x,n) &= \frac{1}{n+1}\sin((n+1)x) - \frac{1}{2(n+1)^2}\left[ksin(2x) + \frac{\ell}{2}\sin(4x)\right]\cos((n+1)x)\\ &\quad + \frac{1}{4(n+1)^3}\{2k\cos(2x) + 2\ell\cos(4x) + 2(k+\ell) - \frac{1}{2}[k\sin(2x)\\ &\quad + \frac{\ell}{2}\sin(4x)]^2\}\sin((n+1)x) + o(n^{-3}).\end{aligned}$$

Theorem 9. *The eigenfunctions of the Whittaker-Hill equation satisfying the semi-periodic boundary conditions are, as* $n \to \infty$

$$\phi_1(x,n) = \cos\frac{(2n+1)x}{2} + \frac{1}{2n+1}\left[k\sin(2x) + \frac{\ell}{2}\sin(4x)\right]\sin\frac{(2n+1)x}{2} + \frac{1}{(2n+1)^2}\{2k\cos(2x) + 2\ell\cos(4x) - 2(k+\ell) - \frac{1}{2}[k\sin(2x) + \frac{\ell}{2}\sin(4x)]^2\}\cos\frac{(2n+1)x}{2} + o(n^{-2}),$$

$$\phi_2(x,n) = \frac{2}{2n+1}\sin\frac{(2n+1)x}{2} - \frac{2}{(2n+1)^2}\left[k\sin(2x) + \frac{\ell}{2}\sin(4x)\right]\cos\frac{(2n+1)x}{2} + \frac{2}{(2n+1)^3}\{2k\cos(2x) + 2\ell\cos(4x) + 2(k+\ell) - \frac{1}{2}[k\sin(2x) + \frac{\ell}{2}\sin(4x)]^2\}\sin\frac{(2n+1)x}{2} + o(n^{-3}).$$

To prove Theorem 8 and Theorem 9, we take $q(x) = 2k\cos(2x) + 2\ell\cos(4x)$ and $a = 2\pi$ in Theorem 3 and Theorem 4, respectively.

Following two theorems give the results about Green's functions.

Theorem 10. *Green's function of the Whittaker-Hill equation under periodic boundary conditions is, as* $n \to \infty$

$$G(x,\zeta,n) = \frac{1}{(n+1)}\cos((n+1)\zeta)\sin((n+1)x) - \frac{1}{2(n+1)^2}\{[k\sin(2x) + \frac{\ell}{2}\sin(4x)] \times \cos((n+1)\zeta)\cos((n+1)x) - [k\sin(2\zeta) + \frac{\ell}{2}\sin(4\zeta)]\sin((n+1)\zeta) \times \sin((n+1)x)\} + \frac{1}{4(n+1)^3}\{[2k[\cos(2\zeta) + \cos(2x)] + 2\ell[\cos(4\zeta) + \cos(4x)] - \frac{1}{2}[(k\sin(2\zeta) + \frac{\ell}{2}\sin(4\zeta))^2 + (k\sin(2x) + \frac{\ell}{2}\sin(4x))^2]] \times \cos((n+1)\zeta)\sin((n+1)x) - [k\sin(2\zeta) + \frac{\ell}{2}\sin(4\zeta)][k\sin(2x) + \frac{\ell}{2}\sin(4x)]\sin((n+1)\zeta)\cos((n+1)x)\} + o(n^{-3})$$

for $0 \le \zeta \le x \le 2\pi$. *Similar result holds for* $0 \le x \le \zeta \le 2\pi$ *changing the role of* ζ *and* x.

Theorem 11. *Green's function of the Whittaker-Hill equation under semi-periodic boundary conditions is, as* $n \to \infty$

$$G(x,\zeta,n) = \frac{2}{2n+1}\cos\frac{(2n+1)\zeta}{2}\sin\frac{(2n+1)x}{2} - \frac{2}{(2n+1)^2}\{[ksin(2x)$$

$$+\frac{\ell}{2}\sin(4x)]\cos\frac{(2n+1)\zeta}{2}\cos\frac{(2n+1)x}{2}-[k\sin(2\zeta)+\frac{\ell}{2}\sin(4\zeta)]$$
$$\times\sin\frac{(2n+1)\zeta}{2}\sin\frac{(2n+1)x}{2}\}+\frac{2}{(2n+1)^3}\{[2k(\cos(2\zeta)+\cos(2x))$$
$$+2\ell(\cos(4\zeta)+\cos(4x))-\frac{1}{2}[(k\sin(2\zeta)+\frac{\ell}{2}\sin(4\zeta))^2+(k\sin(2x)$$
$$+\frac{\ell}{2}\sin(4x))^2]]\cos\frac{(2n+1)\zeta}{2}\sin\frac{(2n+1)x}{2}-[k\sin(2\zeta)+\frac{\ell}{2}\sin(4\zeta)]$$
$$\times[k\sin(2x)+\frac{\ell}{2}\sin(4x)]\sin\frac{(2n+1)\zeta}{2}\cos\frac{(2n+1)x}{2}\}+o(n^{-3})$$

for $0\le\zeta\le x\le 2\pi$*. Similar result holds for* $0\le x\le\zeta\le 2\pi$ *changing the role of* ζ *and* x*.*

To prove Theorem 10 and Theorem 11, we first calculate (20) for $q(x)=2k\cos(2x)+2\ell\cos(4x)$ and $a=2\pi$. We find

$$\begin{aligned}D(x)&=2kx\cos 2x+2\ell x\cos 4x+4\pi k+4k\int_{\pi/2}^{\pi}t\sin 2tdt+8\ell\int_{\pi/2}^{\pi}t\sin 4tdt\\&\quad+4k\int_{\pi/2}^{x}t\sin 2tdt+8\ell\int_{\pi/2}^{x}t\sin 4tdt\\&=k\sin 2x+\frac{\ell}{2}\sin 4x.\end{aligned}$$

Then, we substitute the obtained result of $D(x)$ in Theorem 6 and Theorem 7, respectively. The proof is done.

Declaration of Competing Interests The author declares that they have no competing interests.

References

[1] Arscott, F. M., The Whittaker-Hill equation and the wave equation in paraboloidal coordinates, *Proc. R. Soc. Edinb. A: Math.*, 67(4) (1967), 265–276. https://doi.org/10.1017/S008045410000813X

[2] Bartuccelli, M., Gentile, G., Wright, J. A., On a class of Hill's equations having explicit solutions, *Appl. Math. Lett.*, 26(10) (2013), 1026–1030. https://doi.org/10.1016/j.aml.2013.05.005

[3] Başkaya, E., On the gaps of Neumann eigenvalues for Hill's equation with symmetric double well potential, *Tbil. Math. J.*, 8 (2021), 139-145.

[4] Başkaya, E., Periodic and semi-periodic eigenvalues of Hill's equation with symmetric double well potential, *TWMS J. of Apl. Eng. Math.*, 10(2) (2020), 346–352.

[5] Bender, C. M., Orszag, S. A., Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.

[6] Bognar, G., Periodic and antiperiodic eigenvalues for half-linear version of Hill's equation, *International Journal of Mathematical Models and Methods in Applied Sciences*, 2(1) (2007), 33–37.

[7] Cabada, A., Cid, J. A., On comparison principles for the periodic Hill's equation, *J. London Math. Soc.*, 86(1) (2012), 272–290. https://doi.org/10.1112/jlms/jds001
[8] Cabada, A., Cid, J. A., Lopez-Somoza, L., Green's functions and spectral theory for the Hill's equation, *Appl. Math. Comput.*, 286 (2016), 88–105. https://doi.org/10.1016/j.amc.2016.03.039
[9] Cabada, A., Cid, J. A., Lopez-Somoza, L., Maximum Principles for the Hill's Equation, Academic Press, 2018.
[10] Casperson, L. W., Solvable Hill equation, *Phys. Rev. A.*, 30 (1984), 2749– 2751. https://doi.org/10.1103/PhysRevA.30.2749
[11] Coşkun, H., Başkaya, E., Kabataş, A., Instability intervals for Hill's equation with symmetric single well potential, *Ukr. Math. J.*, 71(6) (2019), 977–983. DOI: 10.1007/s11253-019-01692-x
[12] Coşkun, H., Kabataş, A., Asymptotic approximations of eigenfunctions for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential, *Math. Scand.*, 113 (2013), 143-160. https://doi.org/10.7146/math.scand.a-15486
[13] Coşkun, H., Kabataş, A., Green's function of regular Sturm-Liouville problem having eigenparameter in one boundary condition, *Turk. J. Math. Comput. Sci.*, 4 (2016), 1-9.
[14] Coşkun, H., Kabataş, A., Başkaya, E., On Green's function for boundary value problem with eigenvalue dependent quadratic boundary condition, *Bound. Value Probl.*, (2017). http://dx.doi.org/10.1186/s13661-017-0802-0
[15] Coşkun, H., On the spectrum of a second order periodic differential equation, *Rocky Mountain J. Math.*, 33 (2003), 1261–1277. DOI: 10.1216/rmjm/1181075461
[16] Coşkun, H., Some inverse results for Hill's equation, *J. Math. Anal. Appl.*, 276(2) (2002), 833–844. https://doi.org/10.1016/S0022-247X(02)00454-7
[17] Eastham, M. S. P., Theory of Ordinary Differential Equations, Van Nostrand Reinhold, 1970.
[18] Eastham, M. S. P., The Spectral Theory of Periodic Differential Equations, Scottish Academic Press, Edinburgh and London, 1973.
[19] Flaschka, H., On the inverse problem for Hill's operator, *Arch. Rational. Mech. Anal.*, 59 (1975), 293–304.
[20] Fulton, C. T., Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. R. Soc. Edinb. A: Math.*, 77(3-4) (1977), 293–308. https://doi.org/10.1017/S030821050002521X
[21] Hill, G. W., On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and the moon, *Acta Math.*, 8 (1886), 1–36. DOI: 10.1007/BF02417081
[22] Huang, M. J., The first instability interval for Hill equations with symmetric single well potentials, *Proc. Amer. Math. Soc.*, 125(3) (1997), 775–778.
[23] Kabataş, A., Eigenfunction and Green's function asymptotics for Hill's equation with symmetric single well potential, *Ukrains'kyi Matematychnyi Zhurnal*, 74(2) (2022), 191-203. https://doi.org/10.37863/umzh.v74i2.6246
[24] Magnus, W., Winkler, S., Hill's Equation, Dover Publications, New York, 2004.
[25] Mathews, J. , Walker, R. L., Mathematical Methods of Physics, W. A. Benjamin, New York, 1965.
[26] McKean, H. P., Van Moerbeke, P., The spectrum of Hill's equation, *Invent. Math.*, 30 (1975), 217–274.
[27] Moriguchi, H., An improvement of the WKBJ Method in the presence of turning points and the asymptotic solutions of a class of Hill equations, *J. Phys. Soc. Jpn.*, 14(12) (1959), 1771-1796. https://doi.org/10.1143/JPSJ.14.1771
[28] Pokutnyi, A. A., Periodic solutions of the Hill equation, *J. Math. Sci.*, 197(1) (2014), 114–121. http://dx.doi.org/10.1007/s10958-014-1707-4
[29] Roncaratti, L. F., Aquilanti, V., Whittaker-Hill equation, Ince polynomials, molecular torsional modes, *Int. J. Quantum Chem.*, 110(3) (2010), 716-730. http://dx.doi.org/10.1002/qua.22255

[30] Say, F., First-order general differential equation for multi-level asymptotics at higher levels and recurrence relationship of singulants, *Comptes Rendus. Mathématique*, 359(10) (2021), 1267-1278. https://doi.org/10.5802/crmath.264

[31] Say, F., Late-order terms of second order ODEs in terms of pre-factors, *Hacet. J. Math. Stat.*, 50(2) (2021), 342-350. https://doi.org/10.15672/hujms.657267

[32] Slater, J. C., The design of linear accelerators, *Rev. Mod. Phys.*, 20(3) (1948), 473–518. https://doi.org/10.1103/RevModPhys.20.473

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TAIL DEPENDENCE ESTIMATION BASED ON SMOOTH ESTIMATION OF DIAGONAL SECTION

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Abstract. This paper is mainly developed around the diagonal section which is strongly related to tail dependence coefficients as defined in Nelsen [19]. Hence, we propose a flexible method for estimating tail dependence coefficients based on the new smooth estimation of the diagonal section based on the Bernstein polynomial approximation. To assess the performance of the new estimators we conduct the Monte-Carlo simulation study. As a result of the simulation study, both estimators perform satisfactory performance. Also, the estimation methods are illustrated by real data examples.

1. Introduction

Let X and Y be the random variable having the joint distribution function H and the marginals F and G, respectively. The copula C is the function that links the multivariate joint distribution function to its marginal distributions due to the following relationship proposed by Sklar [24]:

$$H(x,y) = C(F(x), G(y)).$$

Copula C is unique if and only if marginals F and G are continuous. Also, it satisfies the following properties

(1) $C(0,u) = C(u,0) = 0$ for all $u \in [0,1]$
(2) $C(1,u) = C(u,1) = u$ for all $u \in [0,1]$
(3) for all $u,\ u',\ v,\ v' \in [0,1]$ with $u < u'$ and $v < v'$

$$V_C([u,u'] \times [v,v']) = C(u,u') - C(u,v') - C(u',v) + C(u,v) \geq 0$$

where $V_C([u,u'] \times [v,v'])$ is the $C - volume$ of the rectangle $[u,u'] \times [v,v']$.

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The contribution of this study is two-fold: first, we are proposing a smooth estimation of diagonal section of the copula. Second, we estimate the tail dependence coefficients using the smooth estimation of the diagonal section.

The diagonal section of copulas is an important aspect in the field of dependence modelling. Especially, the diagonal section provides some pieces of information about the tail dependence behaviour of the bivariate random variables (Joe [18]). Thus, estimation of the diagonal section is a crucial part of the estimation of tail dependence coefficients between bivariate random variables. For this reason, we propose a non-parametric smooth estimation of the diagonal section based on the Bernstein polynomial approximation. The proposed estimation method is a continuous approximation of the classical estimation which has jump discontinuous. In a copula framework, estimation procedures based on the polynomial approximation is not new: see, e.g., Susam and Hudaverdi [21]- [22], Dimitrova et al. [6], Durante and Okhrin [8], Ambrard and Girard [1].

The second aim of this paper is tail dependence estimation based on the plug-in method. Because there is a direct relationship between the diagonal section and tail dependence coefficients as defined in Nelsen [19], the tail dependence estimation method is mainly developed around the smooth estimation of the diagonal section. The use of the Bernstein estimator of the diagonal section reduced the complexity of the tail dependence estimation coefficients. Moreover, the proposed estimation method of the tail dependence coefficient is flexible according to its polynomial degree, hence the error of the estimation may be reduced by increasing the degree of the Bernstein polynomial. There are some papers which introduces the tail dependence estimation in the literature e.g., Susam and Erdogan [23], Ferreira [13], Schmidt and Stadtmuller [20], Ferreira and Ferreira [14], Frahm et al. [16], Caillault and Guégan [4], Goegebeur and Guillou [17]. The plug-in estimation of tail dependence based on Bernstein polynomial approximation is not a new idea. Susam and Erdogan [23] proposed a tail dependence estimation using the plug-in principle. Their tail dependence estimation is mainly developed around the smooth estimation of the Kendall distribution function of Archimedean copula family. The main difference of this article from the Susam and Erdogan [23] is that our proposed tail dependence estimator is applicable to all copula families such as Elliptical, Extreme value, etc.

The paper is organized as follows. In section 2, we propose the smooth estimation of the diagonal section using Bernstein polynomial approximation and investigate its properties. Also, we conduct a simulation study to measure its performance. In section 3, we deal with the estimation of tail dependence coefficients using the smooth estimation of the diagonal section. Moreover, we investigate its performance. As an illustration, we apply the proposed tail dependence estimation method to the Danube data set. Finally, the conclusion is given in the last section.

2. Estimation of Diagonal Section

In this section, firstly, we review basic definitions and properties about diagonal section of copulas, which can be found, for instance, in Durante et al. [9] and Durante et al. [10]. Then, we investigated the smooth estimation of diagonal section of copulas based on the Bernstein polynomial approaximation.

$\delta_C : [0,1] \to [0,1]$, called diagonal section of copula, is the function defined by $\delta(t) = C(t,t)$. Let us consider that X and Y are uniformly distributed on the unit interval. Moreover, suppose that $W = \max(X,Y)$ is distributed according to the cumulative distribution function (cdf) H. The behaviour of the random variable W is determined by the diagonal section of the copula $C_{X,Y}$, such that $\delta_C(t) = H_W(t)$ (Durante et al. [10]). Diagonal section of the copula has the following properties:

(D1) $\delta_C(0) = 0$ and $\delta_C(1) = 1$;
(D2) $\delta_C(t) \le t$ for all $t \in [0,1]$;
(D3) $\delta_C(t)$ is non-decreasing function;
(D4) δ_C is $2 - \textit{Lipschitz}$, such that $\left|\delta_C(t_2) - \delta_C(t_1)\right| \le 2\left|t_2 - t_1\right|$ for all $t_2, t_1 \in [0,1]$.

Let $\{(X_1,Y_1),\dots,(X_n,Y_n)\}$ be a random sample of (X,Y) from cdf $H(x,y)$. The inference is then based on the pseudo-samples defined as

$$U_i = \frac{R(X_i)}{n},\ V_i = \frac{R(Y_i)}{n},\ i = 1,\dots,n;$$

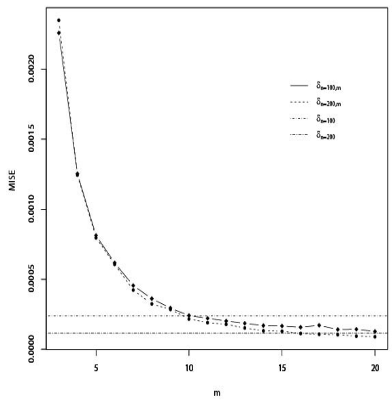
where $R(.)$ is the rank of random variable. Hence, the pair of random variables (U,V) yield an approximate sample from the copula $C(u,v)$. The non-parametric estimation of diagonal section relies on the pseudo-observations $w_i = \max\left(u_i, v_i\right), i = 1,\dots,n$ which have the distribution function $C(w,w)$. It is natural to non-parametric estimate the diagonal section given by

$$\delta_n(t) = \frac{1}{n}\sum_{i=0}^{n} \mathbf{I}\left(w_i \le t\right),\ t \in [0,1]; \tag{1}$$

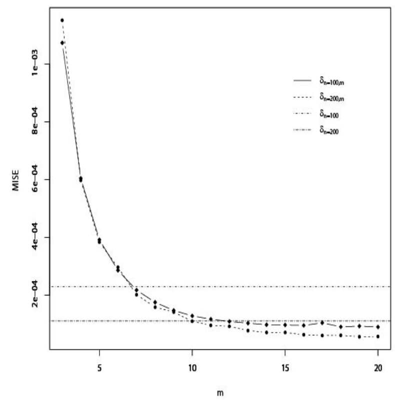
which by the Glivenko–Cantelli lemma converges to the true cdf. Erderly [12] investigated properties of empirical diagol section δ_n. An empirical diagonal section can be written by following:

$$\delta_n(t) = C_n(t,t),\ t \in [0,1]$$

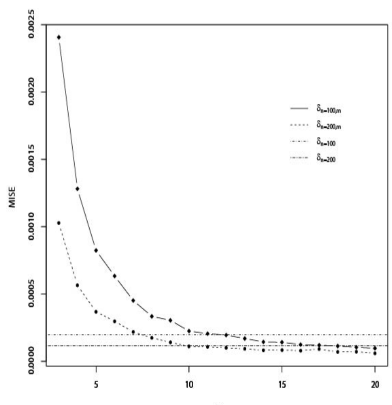
where C_n is the empirical copula defined by Deheuvels [5]. Hence, the properties of δ_n may be imvestigated using the properties of empirical copula and empirical



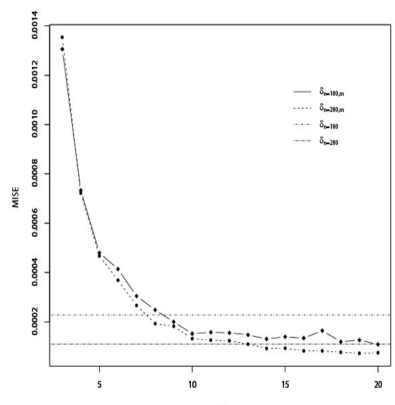
(A) Gumbel Copula with $\tau = 0.25$



(B) Gumbel Copula with $\tau = 0.50$



(C) Clayton Copula with $\tau = 0.25$

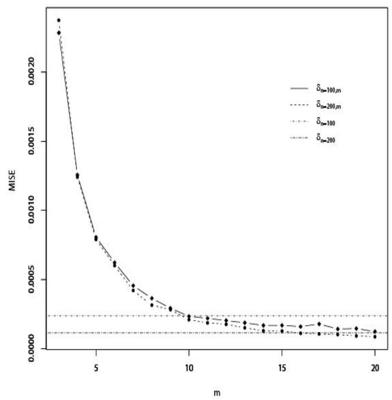


(D) Clayton Copula with $\tau = 0.50$

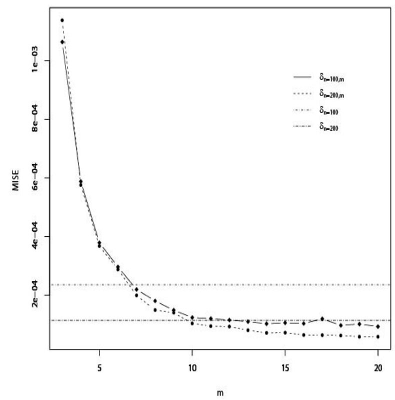
FIGURE 1. MISE values for some Archimdean copulas with $\tau = 0.25, 0.5$

cdf. It is clear that $\delta_n(0) = 0$, $\delta_n(1) = 1$ and $\delta_n(t)$ is non-decreasing function. Moreover, by the Fréchet–Hoeffding bounds for empirical copula:

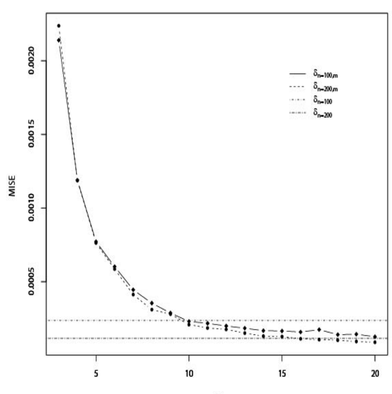
$$\max(2t - 1, 0) \leq \delta_n(t) \leq t,\ t \in [0, 1],$$



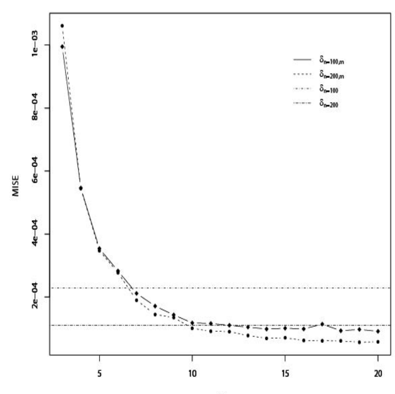
(A) Normal Copula with $\tau = 0.25$



(B) Normal Copula with $\tau = 0.50$



(C) Student-t Copula with $\tau = 0.25$



(D) Student-t Copula with $\tau = 0.50$

FIGURE 2. MISE values for some Elliptical copulas with $\tau = 0.25, 0.5$

hence property D3 is also satisfied. Erderly [12] also proved the propoerty given by:

$$\delta_n(\frac{i+1}{n}) - \delta_n(\frac{i}{n}) \in \{0, \frac{1}{n}, \frac{2}{n}\}, \ i = 1, \ldots, n. \tag{2}$$

Because of the δ_n has jump discontinuities, estimating a continuous distribution function may not be a good choice. Hence, in this paper, we propose a smooth estimation of δ_n using Bernstein polynomial approximation. The Bernstein estimator of order $(m > 0)$ of the diagonal section δ is defined as,

$$\delta_{m,n}(t) = \sum_{k=0}^{m} \delta_n(\frac{k}{m}) P_{k,m}(t),\ t \in [0,1]$$

where $P_{k,m}(t) = \binom{m}{k} t^k (1-t)^{m-k}$ is the binomial probability. The following theorem defined in Feller [15] helps us to prove the consistency of the Bernstein empirical diagonal section.

Theorem 1. *If $f(t)$ is a bounded and continuous function on the interval $[0,1]$, then as $m \to \infty$*

$$f_m^*(t) = \sum_{k=0}^{m} f(\frac{k}{m}) P_{k,m}(t) \to f(t)$$

The following theorem states that the Bernstein empirical diagonal section is a consistent estimator of $\delta(t)$.

Theorem 2. *Let δ be a continuous diagonal section on the interval $[0,1]$. If $m, n \to \infty$, then $\sup_{t\in[0,1]} |\delta_{n,m}(t) - S(t)| \to 0$ a.s.*

Proof. Recall Theorem 1 of f_m^* for any f.

$$\sup_{t\in[0,1]} |\delta_{n,m}(t) - \delta(t)| \leq \sup_{t\in[0,1]} |\delta_{n,m}(t) - \delta_m^*(t)| + \sup_{t\in[0,1]} |\delta_m^*(t) - \delta(t)|.$$

As

$$\delta_{n,m}(t) - \delta^*(t) = \sum_{k=0}^{m} \big(\delta_n(t) - \delta(t)\big) P_{k,m}(t)$$

we have

$$\sup_{t\in[0,1]} |\delta_{n,m}(t) - \delta_m^*(t)| \leq \max_{0\leq k\leq m} |\delta_n(\frac{k}{m}) - \delta(\frac{k}{m})| \leq \sup_{t\in[0,1]} |\delta_n(t) - \delta(t)|$$

Then, $\sup_{t\in[0,1]} |\delta_n(t) - \delta(t)| \to 0$ a.s as $n \to \infty$. See also, Babu et al. [2]. □

The next proposition investigates the properties of the Bernstein empirical diagonal section:

Proposition 1. *The Bernstein empirical diagonal section with order $m > 0$ has the following properties:*

(P1) $\delta_{n,m}(0) = 0$ *and* $\delta_{n,m}(1) = 1$*;*
(P2) $\delta_{n,m}(t) \leq t$ *for all* $t \in [0,1]$*;*
(P3) $\delta_{n,m}(t)$ *is non-decreasing function;*
(P4) $\delta_{n,m}$ *is* $2-$*Lipschitz.*

Proof. From the endpoint property of Berntein ploynomial, $\delta_{m,n}(1) = \delta_n(1) = 1$ and $\delta_{m,n}(0) = \delta_n(0) = 0$. See Duncan [7]. We know that $\delta_n(t) \leq t$, hence we can write $\delta_n(\frac{i}{m}) = \frac{i}{m} - r_i,\ i = 1, \ldots, m$ then

$$\begin{aligned}
\delta_{m,n}(t) &= \sum_{i=0}^{m} \delta_n(\frac{i}{m}) \binom{m}{k} t^k (1-t)^{m-k} \\
&= \sum_{k=0}^{m} (\frac{k}{m} - r_k) \binom{m}{k} t^k (1-t)^{m-k} \\
&= \sum_{k=0}^{m} (\frac{k}{m}) \binom{m}{i} t^k (1-t)^{m-k} - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} \\
&= t \sum_{k=1}^{m} \binom{m-1}{k-1} t^{k-1} (1-t)^{m-k} - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} \\
&= t \sum_{l=0}^{m-1} t^l (1-t)^{m-l-1} \binom{m-1}{l} - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} \\
&= t - \sum_{k=0}^{m} r_k \binom{m}{k} t^k (1-t)^{m-k} < t.
\end{aligned}$$

Thus $\delta_{m,n}(t) \leq t$ is satisfied for all $t \in [0,1]$. The first derivative of the $\delta_{m,n}$ can be obtained as

$$\delta'_{m,n}(t) = m \sum_{k=0}^{m-1} \left(\delta_n(\frac{k+1}{m}) - \delta_n(\frac{k}{m})\right) t^k (1-t)^{m-k-1} \binom{m-1}{k}$$

, see Duncan [7]. Becasue δ_n is non-decreasing function such that

$$\delta_n(\frac{k+1}{m}) - \delta_n(\frac{k}{m}) \geq 0,\ k = 0, \ldots, m-1;$$

then $\delta'_{m,n}(t) \geq 0,\ t \in [0,1]$. We note that a function $f : [a,b] \to \Re$ is said to be a Lipschitz if there is a constant L such that

$$\left|f(x_2) - f(x_1)\right| \leq L\left|x_2 - x_1\right|,\ \forall\, x_2, x_1 \in [a,b],$$

where Lipschitz constant of f equals to $\sup_{x \in [0,1]} \left|f'(t)\right|$. Brown et al. [3] showed that Bernstein polynomial approximation defined as

$$B(t) = \sum_{k=0}^{m} f(\frac{k}{m}) P_{k,m}(t),\ t \in [0,1]$$

is $L-$*Lipschitz* function. Hence, the Lipschitz constant L eqauls to $L = \sup_{t \in [0,1]} \left|B'(t)\right|$. We know that $\delta_{m,n}(t)$ is non-decreasing function and $\delta'_{m,n}(1)$ equals to $m\Big(\delta_n(\frac{m}{m}) -$

$\delta_n(\frac{m-1}{m})\Big) \in \{0,1,2\}$. Hence, the Lipschitz constant of $\delta_{m,n}(t)$ can be calculated as

$$L = \sup_{t\in[0,1]} \left|\delta'_{m,n}(t)\right| = 2.$$

□

To measure the performance of the proposed estimator, we conduct Monte-Carlo simulation study. Gumbel, Clayton (Archimedean) and Normal, Student-t (Elliptical) copulas that have parameters coressponding to Kendall's tau as $\tau = 0.25$, 0.50 are used to generate the data. Specifically, 10.000 Monte-Carlo samples of size $n = 100, 200$ are generated from each copula, and the performance of the Bernstein empirical diagonal section with order $m = 3, \ldots, 30$ are measured by means of the Mean Integrated Squared error (MISE) defined as

$$MISE(\delta) = E\Big(\int_0^1 \big(\delta_{m,n}(t) - \delta(t)\big)^2 dt\Big).$$

Simulation results are shown in Figures 1 and 2 for the Archimedean and Elliptical copulas, respectively. From these figures, it is clear that MISE scores of the Bernstein empirical diagonal section gets closure to the true cdf when both order m and sample size n are increased for all copula classes. Moreover, the Bernstein empirical diagonal section $\delta_{m,n}$ outperforms to classical one δ_n for all possible situations.

3. Tail Dependence Estimation

In this section, firstly, we will be introducing the tail dependence concept. Then, we investigate the plug-in estimators for the upper and lower tail dependence coefficients based on the smooth estimation of the diagonal section discussed in Section 2.

An crucial part of the dependence between the variables in the upper-right quadrant and in the lower-left quadrant of $\mathbf{I^2}$. In general, most dependence measures associate the entire distribution of two or more random variables. However, the dependence between the upper part of the distribution may be different than the mid-range and/or lower part of the distribution (Embrechts et al. [11]). Let X and Y be continuous random variables with margins F and G, respectively. Nelsen [19] shows that the tail dependence coefficients depend on the derivative of diagonal section are given by following:

$$\lambda_U = 2 - \lim_{t\to 1^-} \frac{1 - C(t,t)}{1-t} = 2 - \delta'_C(1^-), \;\; \lambda_L = \lim_{t\to 0^+} \frac{C(t,t)}{t} = \delta'_C(0^+). \qquad (3)$$

In general, the tail dependence between variables may strongly depend on the choice of model or estimation technique (Frahm et al. [16]). For this reason, to estimate the tail dependence coefficients we prefer to use smooth estimation of diagonal section of copula which outperforms the classical estimator as shown in section 2. The estimation of the tail dependence coeffcients investigated in next propsition.

Proposition 2. *Let $\delta_{m,n}(.)$ be the estimator of diagonal section based on Bernstein polynomial approximation and $\delta_n(.)$ be empirical diagonal section. The estimation of the lower tail and the upper tail dependence for copulas are obtained by*

$$\hat{\lambda}_L = m\Big(\delta_n(\frac{1}{m})\Big)$$
$$\hat{\lambda}_U = 2 - m\Big((1 - \delta_n(\frac{m-1}{m})\Big)$$

The proof of the Proposition 2 can be easley done using the properties of Bernstein polynomials. It is obvious that there are clear link beetwen the tail dependence estimations $\hat{\lambda}_L$, $\hat{\lambda}_U$ and the Bernstein polynomial degree m.

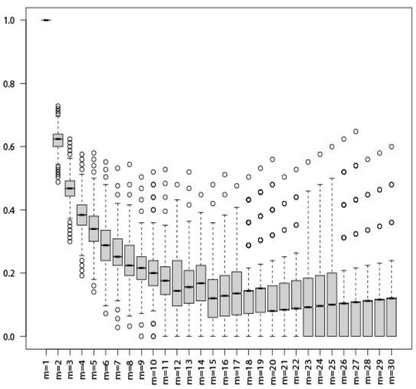
Table 1. The values of the tail dependence and dependence parameter for Gumbel, Clayton, Normal and Student-t copula for different level of dependence

Copula	τ	θ	λ_U	λ_L
Gumbel	0.25	1.3333	0.3182	0
	0.50	2	0.5857	0
Clayton	0.25	0.6666	0	0.3535
	0.50	2	0	0.7071
Normal	0.25	0.3826	0	0
	0.50	0.7071	0	0
Student	0.25	0.3826	0.1953	0.1953
	0.50	0.7071	0.3968	0.3968

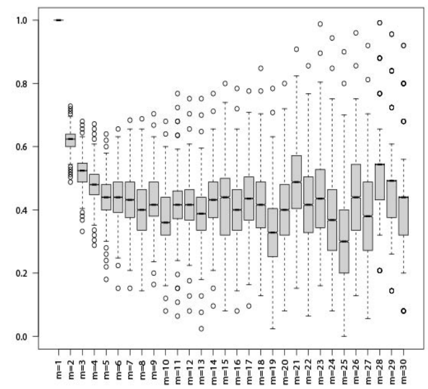
To asses the performance of the tail dependence estimation, we simulate $K = 10.000$ times bivariate random of sample size $n = 250,\ 750$, respectively, from Gumbel, Clayton, Normal and Student-t copulas with Kendall's tau $\tau = 0.25,\ 0.50$. The value of the upper tail dependence (λ_U), lower tail dependence (λ_L) and the dependence parameter (θ) for Gumbel, Clayton, Normal and Student-t copulas with $\tau = 0.25,\ 0.50$ are given in Table 1.

The boxplots of the results of the tail dependence estimations obtained after K Monte-Carlo samples of size $n = 250,\ 750$ from Gumbel, Clayton, Normal and Student-t copulas for varying Kendall's tau values $\tau = 0.25,\ 0.50$ are displayed in Figs. 3-6.

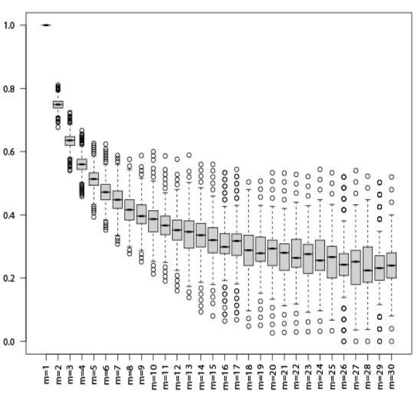
The following results can be obtained:



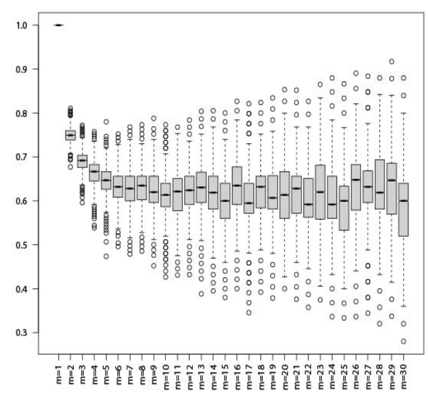
(A) λ_L estimation for $\tau =$ 0.25 and $n = 250$



(B) λ_U estimation for $\tau =$ 0.25 and $n = 250$



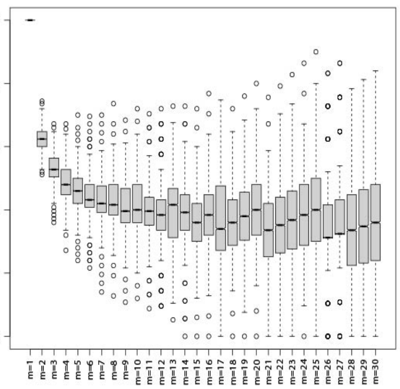
(C) λ_L estimation for $\tau =$ 0.5 and $n = 750$



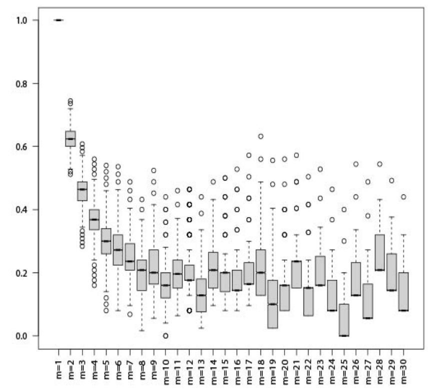
(D) λ_U estimation for $\tau =$ 0.5 and $n = 750$

FIGURE 3. Box-plots of the estimation of the tail dependence coefficients of Gumbel copula

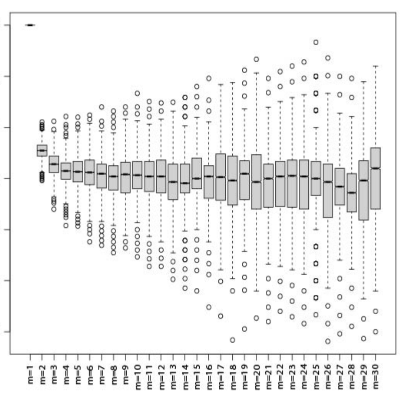
(1) For copulas studied in this paper, the upper tail dependence and lower tail dependence estimation converge to its true value defined in Table 1, regardless of Kendall's tau and sample size.



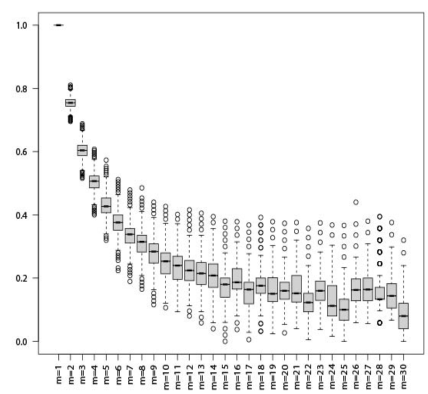
(A) λ_L estimation for $\tau = 0.25$ and $n = 250$



(B) λ_U estimation for $\tau = 0.25$ and $n = 250$



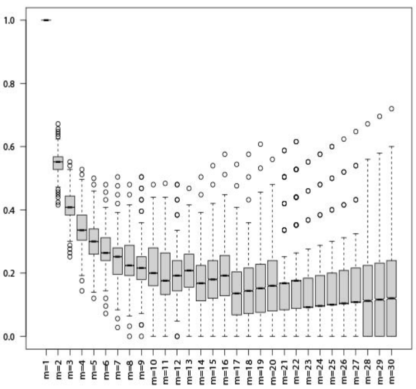
(C) λ_L estimation for $\tau = 0.5$ and $n = 750$



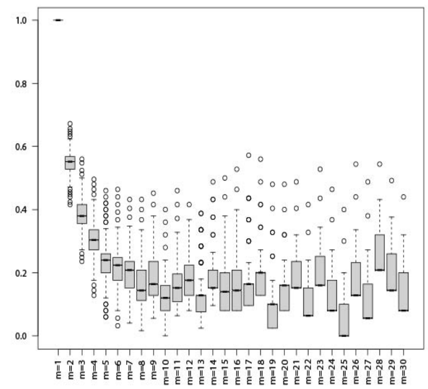
(D) λ_U estimation for $\tau = 0.5$ and $n = 750$

FIGURE 4. Box-plots of the estimation of the tail dependence coefficients of Clayton copula

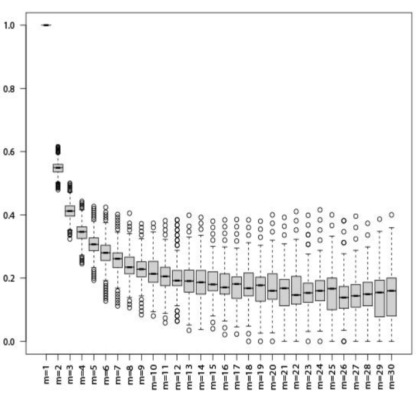
(2) It is obvious that variance of the upper tail dependence and lower tail dependence estimation increases when the Bernstein polynomial degree increases in all situations.



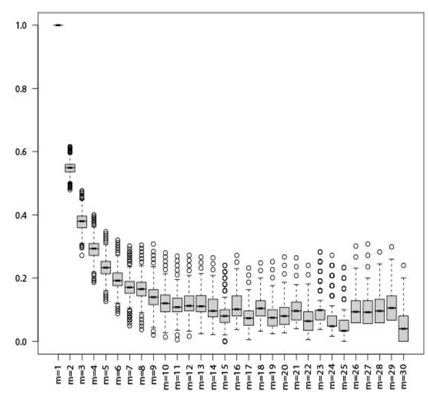
(A) λ_L estimation for $\tau =$ 0.25 and $n = 250$



(B) λ_U estimation for $\tau =$ 0.25 and $n = 250$



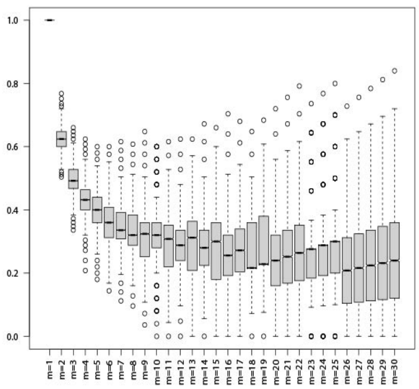
(C) λ_L estimation for $\tau =$ 0.5 and $n = 750$



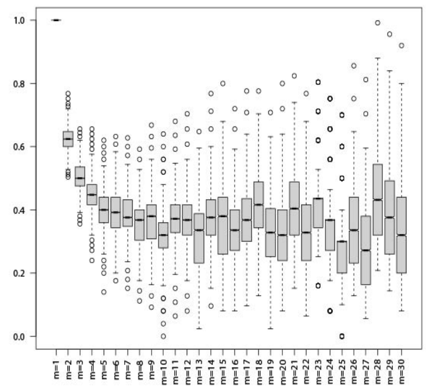
(D) λ_U estimation for $\tau =$ 0.5 and $n = 750$

FIGURE 5. Box-plots of the estimation of the tail dependence coefficients of Normal copula

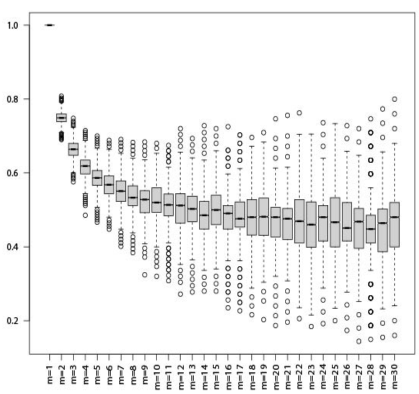
(3) For Gumbel copula with sample size $n = 750$ and Kendall's tau $\tau = 0.50$, to estimate the λ_L approaches its true value, the polynomial degree of estimation should be chosen higher than 30.



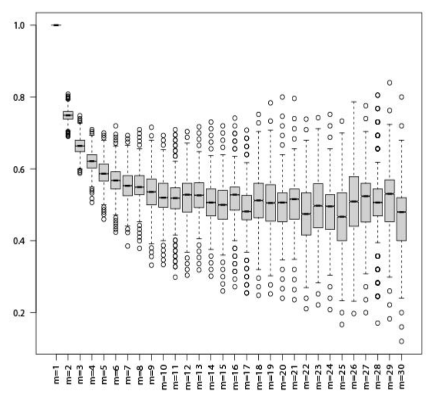
(A) λ_L estimation for $\tau = 0.25$ and $n = 250$



(B) λ_U estimation for $\tau = 0.25$ and $n = 250$



(C) λ_L estimation for $\tau = 0.5$ and $n = 750$



(D) λ_U estimation estimation for $\tau = 0.5$ and $n = 750$

FIGURE 6. Box-plots of the estimation of the tail dependence coefficients of Student-t copula

4. Case Study

In this section, in order to demonstrate the estimation methods of diagonal section and tail dependence coefficients, we use the Danube data set which is available in the R package copula. According to this package, the Danube data set contains ranks of base flow observations from the Global River Discharge project of the Oak Ridge National Laboratory Distributed Active Archive Centre (ORNL DAAC), a NASA data centre. The measurements are the monthly average flow rate for two stations situated at Scharding (Austria) on the Inn River and Nagymaros (Hungary) on the Danube.

The scatter plot of the pseudo-observations of the Danube data set is displayed in Figure 7. In this figure, symmetrical dependence structures are observed. From this figure, it seems that the Danube data set has a heavy right tail dependence structure and mild left tail dependence structure. Figure 8 represents the estimation of upper tail dependence and lower tail dependence coefficient for polynomial degree $m = 1, \ldots, 30$. As it is expected estimation of upper tail dependece is greater than the lower tail dependence estimation for all poynomial degrees. From figure 8, $\hat{\lambda}_U$ approaximates to 0.5 and $\hat{\lambda}_L$ approaximates to 0.20.

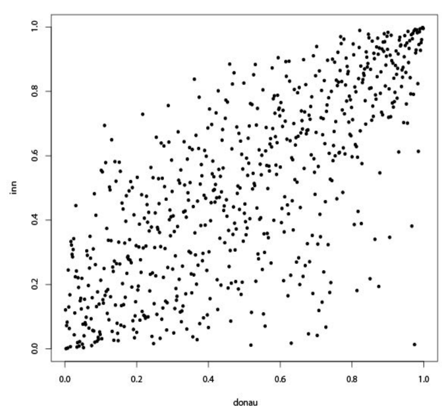
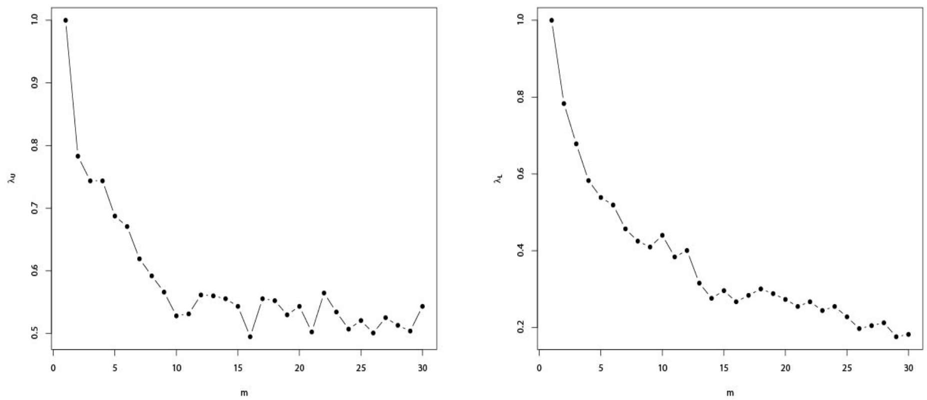


Figure 7. Scatter plot of Danube data set

5. Conclusion

In this paper, we have presented a smooth estimation of the diagonal section based on the Bernstein polynomial approximation. The new estimator is flexible according to its polynomial degree; the error of the estimation may be decreased



(A) Estimation of λ_U (B) Estimation of λ_L

FIGURE 8. Estimation of λ_U and λ_L for Danube data set

when the polynomial degree increases. Moreover, Bernstein diagonal section outperforms the empirical diagonal section for the higher polynomial degrees. Also, considering the strong relationship between the diagonal section and the tail dependence coefficient, we propose the tail dependence coefficients estimation method via Bernstein diagonal section. According to the simulation results and real data example, the tail dependence coefficients estimation method has a satisfactory performance.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have apepeared to influence the work reported in this paper.

References

[1] Amblard, C., Girard, S., Estimation procedures for a semiparametric family of bivariate copulas, *Journal of Computational and Graphical Statistics*, 14(2) (2005), 363-377. DOI: 10.1198/106186005X48722

[2] Babu, G. J., Canty, A. J., Chaubey, Y. P., Application of Bernstein polynomials for smooth estimation of a distribution and density function, *Journal of Statistical Planning and Inference*, 105 (2002), 377-392. DOI:10.1016/S0378-3758(01)00265-8

[3] Brown, B. M., Elliot, D., Paget, D. F., Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function, *Journal of Approximation Theory*, 49 (1987), 196-199. DOI:10.1016/0021-9045(87)90087-6

[4] Caillault, C., Guégan, D., Empirical estimation of tail dependence using copulas: application to Asian markets, *Quantitative Finance*, 5 (2007), 489-501. DOI: 10.1080/14697680500147853

[5] Deheuvels, P., La fonction de dépendance empirique et ses propriétés un test non paramétrique d indépendance, *Acad. Roy. Belg. Bull. Cl. Sci.*, 65(5) (1979), 274-292.

[6] Dimitrova, D., Kaishev, K., Penev, I., GeD spline estimation of multivariate Archimedean copulas, *Computational Statistics and Data Analysis*, 52 (2008), 3570-3582. DOI: 10.1016/j.csda.2007.11.010

[7] Duncan, M., Applied Geometry for Computer Graphics and CAD, Springer Verlag, London, 2005.

[8] Durante, F., Okhrin, O., Estimation procedures for exchangeable Marshall copulas with hydrological application, *Stoch. Environ. Res. Risk Assess*, 29 (2015), 205-226. DOI:10.1007/s00477-014-0866-7

[9] Durante, F., Fernández-Sánchez, J., Pappadá, R., Copulas, diagonals, and tail dependence, *Fuzzy Sets and Systems*, 264 (2015), 22-41. DOI: 10.1016/j.fss.2014.03.014

[10] Durante, F., Kolesárová, A., Mesiar, R., Sempi, C., Copulas with given diagonal sections: novel constructions and applications, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 15(4) (2007), 397-410. DOI: 10.1142/S0218488507004753

[11] Embrechts, P., Mcneil, A., Straumann, D., Correlation and dependence in risk management: properties and pitfalls. In: Risk management: value at risk and beyond, *Cambridge University Press*, Cambridge, (2002), 176-223.

[12] Erdely, A., Exact distribution under independence of the diagonal section of the empirical copula, *Kybernetika*, 44(6) (2008), 826-845.

[13] Ferreira, M., Nonparametric estimation of the tail-dependence coefficient, *REVSTAT-Statistical Journal*, 11(1) (2013), 1-16.

[14] Ferreira, H., Ferreira, M., Tail dependence between order statistics, *Journal of Multivariate Analysis*, 105(1) (2012), 176-192. DOI:10.1016/j.jmva.2011.09.001

[15] Feller, W., An Introduction to Probability Theory and its Applications, Wiley, New York, 1965.

[16] Frahm, G., Junker, M., Schmidt, R., Estimating the tail-dependence coefficient: properties and pitfalls, *Insur. Math. Econ.*, 37(1) (2005), 80-100. DOI:10.1016/j.insmatheco.2005.05.008

[17] Goegebeur, Y., Guillou, A., Asymptotically unbiased estimation of the coefficient of tail dependence, *Scandinavian Journal of Statistics*, 40(1) (2012), 174-189. DOI:10.2307/23357259

[18] Joe, H., Multivariate Models and Dependence Concepts, Chapman Hall, London, 1997.

[19] Nelsen, R., An Introduction to Copulas, Springer Series in Statistics, Springer, New York, 2006.

[20] Schmidt, R., Stadtmuller, U., Non-parametric estimation of tail dependence, *Scand. J. Stat.*, 33(2) (2006), 307-335. DOI: 10.1111/j.1467-9469.2005.00483.x

[21] Susam, S. O., Hudaverdi, B. U., Testing independence for Archimedean copula based on Bernstein estimate of Kendall distribution function, *Journal of Statistical Computation and Simulation*, 88(13) (2018), 2589-2599. DOI:10.1080/00949655.2018.1478978

[22] Susam, S. O., Hudaverdi, B. U., A goodness-of-fit test based on Bézier curve estimation of Kendall distribution, *Journal of Statistical Computation and Simulation*, 90(13) (2020), 1194-1215. DOI:10.1080/00949655.2020.1720680

[23] Susam, S. O., Erdogan, M., Plug-in estimation of dependence characteristics of Archimedean copula via Bézier curve, *RevStat-Statistical Journal*, (In Press).

[24] Sklar, A., Fonctions de repartition a n dimensions et leurs marges, *Publications de lInstitut de Statistique de lUniversite de Paris*, 8 (1959), 229-231.

http://communications.science.ankara.edu.tr

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ON THE MAXIMUM MODULUS OF A COMPLEX POLYNOMIAL

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Abstract. In this paper we impose distinct restrictions on the moduli of the zeros of $p(z) = \sum_{v=0}^{n} a_v z^v$ and investigate the dependence of $\|p(Rz) - p(\sigma z)\|$, $R > \sigma \geq 1$ on M_α and $M_{\alpha+\pi}$, where $M_\alpha = \max_{1\leq k\leq n} |p(e^{i(\alpha+2k\pi)/n})|$ and on certain coefficients of $p(z)$. This paper comprises several results, which in particular yields some classical polynomial inequalities as special cases. Moreover, the problem of estimating $p\left(1 - \frac{w}{n}\right)$, $0 < w \leq n$ given $p(1) = 0$ is considered.

1. Introduction

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n over $\mathbb{C}$. Then it is well known that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

The result in (1) is sharp and equality holds when $p(z) = \lambda z^n$, where $\lambda \in \mathbb{C}$.

The inequality (1), known as Bernstein's inequality, was proved by Bernstein [4] in 1926, however it was also proved earlier by Riesz [14]. By the maximum modulus principle, $\max_{|z|\leq 1} |p(z)| = \max_{|z|=1} |p(z)|$ and so if we consider $\|p\| = \max_{|z|=1} |p(z)|$, then inequality (1) can be written as

$$\|p'\| \leq n \|p\|. \tag{2}$$

For $R \geq 1$, the inequality pertaining to the estimate of $\|p\|$ on a large circle

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$|z| = R$ given below is well known [11, Problem 269] or [15].

$$\max_{|z|=R} |p(z)| \leq R^n \|p\|, \tag{3}$$

equality holds in (3) when $p(z) = \lambda z^n$, $\lambda \in \mathbb{C}$.

Marden [9], Milovanović et al. [10] and Rahman and Schmeisser [12] have presented an exceptional introduction to this topic. Frappier, Rahman and Ruscheweyh [6] were able to refine (1) under the same hypothesis, by replacing the estimate of the maximum modulus of $|p(z)|$ on a unit circle $|z| = 1$ with the estimate of the maximum modulus of $|p(z)|$ taken over $(2n)^{th}$ roots of unity. The maximum modulus of $|p(z)|$ taken over $(2n)^{th}$ roots of unity may be less than the maximum modulus of $|p(z)|$ on unit circle $|z| = 1$ which is shown by a simple example $p(z) = z^n + ia, a > 0$. In fact they proved that

$$\|p'\| \leq n \max_{1\leq k\leq 2n} |p(e^{ik\pi/n})|. \tag{4}$$

As an improvement of (4) A.Aziz [2] showed that the maximum modulus of $|p(z)|$ taken over $(2n)^{th}$ roots of unity in (4) can be replaced by maximum modulus of $|p(z)|$ taken over n^{th} roots of the equation $w^n = e^{i\alpha}$. In fact he proved that, for a polynomial $p(z)$ of degree n and for every $\alpha \in \mathbb{R}$,

$$\|p'\| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{5}$$

where

$$M_\alpha = \max_{1\leq k\leq n} |p(e^{i(\alpha+2k\pi)/n})| \tag{6}$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha + \pi$. The result is sharp and equality in (5) holds for the polynomial $p(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1$.

As an application of inequality (5) A.Aziz [2] was able to establish the following refinement of (3).
For a polynomial $p(z)$ of degree n, and for every α and $R > 1$

$$\|p(Rz) - p(z)\| \leq \frac{R^n - 1}{2}[M_\alpha + M_{\alpha+\pi}], \tag{7}$$

where M_α is defined by (6) and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha + \pi$. The result is the best possible and equality in (7) holds for $p(z) = z^n + re^{i\alpha}, -1 \leq r \leq 1$.

In the same paper A.Aziz [2] also proved that if $p(z)$ is a polynomial of degree n such that $p(1) = 0$, then for $0 < w \leq n$

$$\left|p\left(1 - \frac{w}{n}\right)\right| \leq \frac{1}{2}\left[1 - \left(1 - \frac{w}{n}\right)^n\right]\{M_0 + M_\pi\}, \tag{8}$$

where M_α is defined by (6). The result is the best possible and equality in (8) holds for $p(z) = z^n - 1$.

The study of mathematical objects associated with Bernstein type inequalities has been very active over the years, many papers are published each year in a variety of journals and different approaches are being employed for different purposes. In the present article we have come up with the similar type of inequalities, their refined and improved forms. If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then one would expect, the further developments of the upper bound estimate in (1). In fact, P. Erdös conjectured and later P.D. Lax [8] proved that if $p(z) \neq 0$ in $|z| < 1$, then

$$\|p'\| \leq \frac{n}{2}\|p\|. \tag{9}$$

The result is best possible and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$. In this connection A. Aziz [2], improved the inequality (5) by showing that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every given real α

$$\|p'\| \leq \frac{n}{2}(M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2}, \tag{10}$$

where M_α is defined by (6) for all real α. The result is the best possible and equality in (10) holds for $p(z) = z^n + e^{i\alpha}$. Furthermore, A. Aziz [2] also established that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every given real α and $R > 1$

$$\|p(Rz) - p(z)\| \leq \frac{R^n - 1}{2}[M_\alpha^2 + M_{\alpha+\pi}^2]^{1/2}, \tag{11}$$

where M_α is defined by (6). The result is the best possible and equality in (11) holds for $p(z) = z^n + e^{i\alpha}$. By estimating the minimum modulus of $|p(z)|$ on the unit circle inequality (11) was refined and generalized by Ahmad [1]. In fact proved the following result.

If $p(z)$ is a polynomial of degree n having all its zero in $|z| \geq 1$ and $m = \min_{|z|=1} |p(z)|$, then for all real λ and $R > r \geq 1$

$$\|p(Rz) - p(rz)\| \leq \frac{R^n - r^n}{2}[M_\lambda^2 + M_{\lambda+\pi}^2 - 2m^2]^{1/2}, \tag{12}$$

where M_λ is defined by (6). Just replace argument α of z simply by λ, unless otherwise stated. In the same paper Ahmad [1] also proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z| \geq k \geq 1$ and $m = \min_{|z|=1} |p(z)|$, then for all real λ and $R > r \geq 1$

$$\|p(Rz) - p(rz)\| \leq \frac{R^n - r^n}{\sqrt{2(1+k^2)}}[M_\lambda^2 + M_{\lambda+\pi}^2 - 2m^2]^{\frac{1}{2}}, \tag{13}$$

where M_λ is defined by (6).

While establishing the inequality analogous to (11) for the class of polynomials having all zeros in $|z| \leq k, k \leq 1$, M. H. Gulzar [7] proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z| \leq k \leq 1$, then for all real λ and $R > 1$

$$\|p(Rz) - p(z)\| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2n})}}[M_\lambda^2 + M_{\lambda+\pi}^2]^{\frac{1}{2}}, \tag{14}$$

where M_λ is defined by (6) and $M_{\lambda+\pi}$ is obtained by replacing λ by $\lambda+\pi$ in M_λ. While seeking the generalization of (14). Formerly, in the same paper Ahmad [1] proved that if $p(z)$ is a polynomial of degree n having all its zero in $|z| \le k \le 1$, then for all real λ and $R > r \ge 1$

$$\|p(Rz) - p(rz)\| \le \frac{R^n - r^n}{\sqrt{2(1+k^{2n})}}[M_\lambda^2 + M_{\lambda+\pi}^2]^{\frac{1}{2}}. \tag{15}$$

We conclude this section by stating the following result for the case when $p(z)$ has no zero in $|z| < k, k \le 1$.
If $p(z)$ is a polynomial of degree n and $p(z)$ has no zero in $|z| < k, k \le 1$, then for every real α and $R > 1$

$$\|p(Rz) - p(z)\| \le \frac{R^n - 1}{\sqrt{2(1+k^{2n})}}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}, \tag{16}$$

provided $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$. The result is best possible and equality in (16) holds for $p(z) = z^n + k^n$. This result is ascribed to Rather and Shah [13].

2. Lemmas

Lemma 1. *If $p(z)$ is a polynomial of degree n having all its zeros $|z| \le k \le 1$, then for all real λ*

$$|p'(z)| \le \frac{n}{2^{\frac{1}{2}}(1+k^{2n})^{\frac{1}{2}}}[M_\lambda^2 + M_{\lambda+\pi}^2]^{\frac{1}{2}}.$$

This lemma is a special case of the result due to M.H.Gulzar [7].

Lemma 2. *If $P(z)$ is a polynomial of degree n, then for $R \ge 1$*

$$\max_{|z|=R}|p(z)| \le R^n\|p\| - 2\frac{(R^n-1)}{n+2}|a_0| - |a_1|\left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right], \ \textit{for } n > 2 \tag{17}$$

and

$$\max_{|z|=R}|p(z)| \le R^2\|p\| - \frac{(R-1)}{2}[(R+1)|a_0| + (R-1)|a_1|], \ \textit{for } n = 2. \tag{18}$$

The above lemma is ascribed to Dewan et.al [5].

Lemma 3. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \ge k \ge 1$, then for $|z| = 1$*

$$k|p'(z)| \le |np(z) - zp'(z)| - nm,$$

where $m = \min_{|z|=k}|p(z)|$.

Lemma 3 is a special case of a result due to A. Aziz and N. A. Rather [3].

Lemma 4. *If $p(z)$ is a polynomial of degree n, then for $|z| = 1$ and for every real λ*

$$|p'(z)|^2 + |np(z) - zp'(z)|^2 \leq \frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].$$

The above lemma is due to A.Aziz [2].

Lemma 5. *If $p(z)$ is a polynomial of degree n which has no zeros in $|z| < k$, $k \geq 1$ and $m = \min_{|z|=k}|p(z)|$ then for every real α*

$$\|p'\| \leq \frac{n}{\sqrt{2(1+k^2)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}},$$

where M_α is defined by (6).

Lemma 6. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$ and $m = \min_{|z|=k}|p(z)|$, then for $|z| = 1$*

$$k^n\|p'\| + nm \leq \|q'\|,$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

Lemmas 5 and 6 are due to Rather and Shah [13].

3. Main Results

In this paper we first prove the generalization of inequality (7) which is ascribed to A.Aziz [2]. More precisely we prove the following result.

Theorem 1. *If $p(z)$ is a polynomial of degree n, then for every real α and $R > \sigma \geq 1$*

$$\begin{aligned}\|p(Rz) - p(\sigma z)\| \leq \frac{R^n - \sigma^n}{2}[M_\alpha + M_{\alpha+\pi}] - \frac{2|a_1|}{n+1}\left(\frac{R^n - \sigma^n}{n} - (R-\sigma)\right) \\ - 2|a_2|\left[\frac{(R^n - \sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2} - \sigma^{n-2}) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right], \\ \text{for } n > 3\end{aligned} \tag{19}$$

and

$$\begin{aligned}\|P(Rz) - P(\sigma z)\| \leq \frac{R^3 - \sigma^3}{2}[M_\alpha + M_{\alpha+\pi}] - |a_1|\left(\frac{R^3 - \sigma^3 - 3(R-\sigma)}{6}\right) \\ - |a_2|\left[\frac{(R-1)^3 - (\sigma-1)^3}{3}\right], \text{ for } n = 3,\end{aligned} \tag{20}$$

where M_α is defined by (6) and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha+\pi$. The result is the best possible and equality in (19) and (20) holds for $p(z) = z^n + re^{i\alpha}$, $-1 \leq r \leq 1$.

Proof. Let $n > 3$. Since $p(z)$ is a polynomial of degree $n > 3$, therefore $p'(z)$ is of degree $n \geq 3$, applying inequality (17) of Lemma 2 we obtain for all $v \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(ve^{i\theta})| \leq v^{n-1}\|p'\| - 2\frac{(v^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{v^{n-1}-1}{n-1} - \frac{v^{n-3}-1}{n-3}\right].$$

Using inequality (5) we get,

$$|p'(ve^{i\theta})| \leq \frac{nv^{n-1}}{2}(M_\alpha + M_{\alpha+\pi}) - 2\frac{(v^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{v^{n-1}-1}{n-1} - \frac{v^{n-3}-1}{n-3}\right].$$

For each θ, $0 \leq \theta < 2\pi$ and $R > \sigma \geq 1$, it follows that

$$\begin{aligned}
|p(Re^{i\theta}) - p(\sigma e^{i\theta})| &= \left|\int_\sigma^R e^{i\theta}p'(ve^{i\theta})dv\right| \\
&\leq \int_\sigma^R |p'(ve^{i\theta})|dv \\
&\leq \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)}{2}\int_\sigma^R v^{n-1}dv - \frac{2|a_1|}{n+1}\int_\sigma^R (v^{n-1}-1)dv \\
&- 2|a_2|\int_\sigma^R \left(\frac{v^{n-1}-1}{n-1} - \frac{v^{n-3}-1}{n-3}\right)dv \\
&= \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)}{2}\frac{(R^n - \sigma^n)}{n} - \frac{2|a_1|}{n+1}\left(\frac{R^n - \sigma^n}{n} - (R-\sigma)\right) \\
&- 2|a_1|\left[\frac{(R^n - \sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2} - \sigma^{n-2}) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right],
\end{aligned}$$

equivalently

$$\begin{aligned}
\|p(Rz) - p(\sigma z)\| \leq \frac{R^n - \sigma^n}{2}[M_\alpha + M_{\alpha+\pi}] - \frac{2|a_1|}{n+1}\left(\frac{R^n - \sigma^n}{n} - (R-\sigma)\right) \\
- 2|a_2|\left[\frac{(R^n - \sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2} - \sigma^{n-2}) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right].
\end{aligned}$$

This is the desired result for $n > 3$. Furthermore the case for $n = 3$ follows on the same lines but instead of using inequality (17) of Lemma 2 we use inequality (18) of the same Lemma. $\square$

Theorem 2. *If $p(z)$ is a polynomial of degree n such that $p(1) = 0$, then for $0 < w \leq n$ and $\alpha = 0$*

$$\left|p\left(1-\frac{w}{n}\right)\right| \leq \frac{1}{2}\left[1-\left(1-\frac{w}{n}\right)^n\right]\{M_0+M_\pi\} - \frac{2|a_{n-1}|}{n+1}\left(\frac{1-(1-w/n)^n}{n} - \frac{w}{n}(1-w/n)^{n-1}\right) - 2|a_{n-2}|\chi(w,n), \text{ for } n>3 \tag{21}$$

and

$$\left|p\left(1-\frac{w}{n}\right)\right| \leq \frac{1}{2}\left[1-\left(1-\frac{w}{n}\right)^3\right]\{M_0+M_\pi\} - \frac{|a_{n-1}|}{6}\left(1-\left(1-\frac{w}{3}\right)^3 - w\left(1-\frac{w}{3}\right)^2\right) - \frac{|a_{n-2}|}{3}\left(\frac{w}{3}\right)^3, \text{ for } n=3, \tag{22}$$

where

$$\chi(w,n) = \left[\frac{1-(1-w/n)^n - w(1-w/n)^{n-1}}{n(n-1)} - \frac{(1-w/n)^2-(1-w/n)^n-(w-2w/n)(1-w/n)^{n-1}}{(n-2)(n-3)}\right]$$

and M_0 is defined by (6). The result is the best possible and equality in (21) holds for $p(z) = z^n - 1$.

Proof. **Case I, n > 3**: If $t(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then $|t(z)| = |p(z)|$ for $|z| = 1$ and by the hypothesis we have $t(1) = \overline{p(1)} = 0$. On using inequality (19) of Theorem 1 to the polynomial $t(z)$ for $\alpha = 0$ and $\sigma = 1$, we get for $R > 1$

$$|t(R)| \leq \frac{R^n-1}{2}[M_0+M_\pi] - \frac{2|a_{n-1}|}{n+1}\left(\frac{R^n-\sigma^n}{n} - (R-\sigma)\right) - 2|a_{n-2}|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)} - \frac{(R^{n-2}-\sigma^{n-2})-(n-2)(R-\sigma)}{(n-2)(n-3)}\right].$$

This gives for $R > 1$

$$|t(1/R)| \leq \frac{1}{2}(1-R^{-n})[M_0+M_\pi] - \frac{2|a_{n-1}|}{n+1}\left(\frac{1-R^{-n}}{n} - (R^{1-n}-R^{-n})\right) - 2|a_{n-2}|\left[\frac{(1-R^{-n})-n(R^{1-n}-R^{-n})}{n(n-1)} - \frac{(R^{-2}-R^{-n})-(n-2)(R^{1-n}-R^{-n})}{(n-2)(n-3)}\right].$$

Since $0 < w \leq n$, so that $(1-w/n)^{-1} > 1$ and therefore, in particular, replace R by $(1-w/n)^{-1} > 1$ and after simplification we have,

$$\begin{aligned}\left|p\left(1-\frac{w}{n}\right)\right| \leq \frac{1}{2}\left[1-\left(1-\frac{w}{n}\right)^n\right]\{M_0+M_\pi\} \\ -\frac{2|a_{n-1}|}{n+1}\left(\frac{1-(1-w/n)^n}{n}-\frac{w}{n}(1-w/n)^{n-1}\right) \\ -2|a_{n-2}|\chi(w,n),\end{aligned}$$

where

$$\begin{aligned}\chi(w,n) = \Bigg[&\frac{1-(1-w/n)^n-w(1-w/n)^{n-1}}{n(n-1)} \\ &-\frac{(1-w/n)^2-(1-w/n)^n-(w-2w/n)(1-w/n)^{n-1}}{(n-2)(n-3)}\Bigg].\end{aligned}$$

Case II, n = 3: This can be established identically as above by using inequality (20) of Theorem 1. □

Now we present the refinement of inequality (12). Here we are able to prove

Theorem 3. *If $p(z)$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \geq 1$ and $m = \min_{|z|=1}|p(z)|$, then for all real α and $R > \sigma \geq 1$*

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq \frac{R^n-\sigma^n}{2}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n}-(R-\sigma)\right) \\ -2|a_2|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)}-\frac{(R^{n-2}-\sigma^{n-2})-(n-2)(R-\sigma)}{(n-2)(n-3)}\right], \\ \text{if } n>3\end{aligned} \tag{23}$$

and

$$\begin{aligned}\|p(Rz)-p(\sigma z)\| \leq \frac{R^3-\sigma^3}{2}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - |a_1|\left(\frac{(R^3-\sigma^3)-3(R-\sigma)}{6}\right) \\ -|a_2|\left[\frac{(R-1)^2-(\sigma-1)^3}{3}\right], \text{ if } n=3,\end{aligned} \tag{24}$$

Proof. Since $p(z)$ has all its zeros in $|z| \geq 1$ and $m = \min_{|z|=1}|p(z)|$, therefore by Lemma 3 with $k=1$, we have for $|z|=1$

$$(|p'(z)|+mn)^2 \leq |np(z)-zp'(z)|^2.$$

. This in conjunction with Lemma 4 gives

$$\begin{aligned}|p'(z)|^2 + (|p'(z)| + mn)^2 &\leq |p'(z)|^2 + |np(z) - zp'(z)|^2 \\ &\leq \frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].\end{aligned}$$

Since we have $(|p'(z)| + mn)^2 = |p'(z)|^2 + (mn)^2 + 2mn|p'(z)|$.
This gives

$$(|p'(z)| + mn)^2 \geq |p'(z)|^2 + (mn)^2.$$

Therefore, we have

$$\|p'\| \leq \frac{n}{2}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}. \tag{25}$$

Applying inequality (17) of Lemma 2 with $R = s \geq 1$ to the polynomial $p'(z)$ which is of degree $n-1$, we obtain for $n > 3$

$$|p'(se^{i\theta})| \leq s^{n-1}\|p'\| - \frac{2(s^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

With the help of inequality (25), we obtain for $n > 3$

$$|p'(se^{i\theta})| \leq \frac{ns^{n-1}}{2}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}} - \frac{2(s^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

Now for each $0 \leq \theta < 2\pi$ and $R > \sigma \geq 1$, we have

$$\begin{aligned}|p(Re^{i\theta}) - p(\sigma e^{i\theta})| &= \left|\int_\sigma^R e^{i\theta}p'(se^{i\theta})ds\right| \\ &\leq \int_\sigma^R |p'(se^{i\theta})|ds \\ &\leq \frac{n}{2}[M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2]^{\frac{1}{2}}\int_\sigma^R s^{n-1}ds - \frac{2|a_1|}{n+1}\int_\sigma^R (s^{n-1}-1)ds \\ &- 2|a_2|\int_\sigma^R \left(\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right)ds \\ &= \frac{R^n - \sigma^n}{2}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n - \sigma^n}{n} - (R-\sigma)\right) \\ &- 2|a_2|\left[\frac{(R^n - \sigma^n) - n(R-\sigma)}{n(n-1)} - \frac{(R^n - \sigma^n) - (n-2)(R-\sigma)}{(n-2)(n-3)}\right],\end{aligned}$$

which implies

$$\|p(Rz)-p(\sigma z)\| \leq \frac{R^n-\sigma^n}{2}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n}-(R-\sigma)\right) - 2|a_2|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)} - \frac{(R^n-\sigma^n)-(n-2)(R-\sigma)}{(n-2)(n-3)}\right].$$

This proves the result in case $n>3$. For the case $n=3$, the result follows from similar lines but instead of using inequality (17) of Lemma 2, we use inequality (18) of the same Lemma and this proves the theorem completely. □

As a refinement of inequality (13), we prove the following result.

Theorem 4. *If $p(z)$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \geq k \geq 1$ and $m = \min_{|z|=k}|p(z)|$, then for all real α and $R > \sigma \geq 1$*

$$\|p(Rz)-p(\sigma z)\| \leq \frac{R^n-\sigma^n}{\sqrt{2(1+k^2)}}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\sigma^n}{n}-(R-\sigma)\right) - 2|a_2|\left[\frac{(R^n-\sigma^n)-n(R-\sigma)}{n(n-1)} - \frac{(R^n-\sigma^n)-(n-2)(R-\sigma)}{(n-2)(n-3)}\right], \quad \textit{if } n>3 \tag{26}$$

and

$$\|p(Rz)-p(\sigma z)\| \leq \frac{R^3-\sigma^3}{\sqrt{2(1+k^2)}}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}} - |a_1|\left(\frac{(R^3-\sigma^3)-3(R-\sigma)}{6}\right) - |a_2|\left[\frac{(R-1)^2-(\sigma-1)^3}{3}\right], \quad \textit{if } n=3, \tag{27}$$

where M_α is defined by (6).

Proof. The proof of this theorem follows easily on using arguments similar to that used in the proof of Theorem 3 but instead of using inequality (25) we use Lemma 5. We omit the details. □

Next we establish the upper bound estimate for $\|p(Rz)-p(\xi z)\|$ and thereby prove the following improvement of inequality (15).

Theorem 5. *Let $p(z)$ be a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for all real α and $R > \xi \geq 1$*

$$\begin{aligned}\|p(Rz) - p(\xi z)\| \leq \frac{R^n - \xi^n}{\sqrt{2(1+k^{2n})}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n - \xi^n}{n} - (R - \xi)\right) \\ - 2|a_2|\left[\frac{(R^n - \xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)}\right], \\ \text{for } n > 3 \end{aligned} \tag{28}$$

and

$$\begin{aligned}\|p(Rz) - p(\xi z)\| \leq \frac{R^3 - \xi^3}{\sqrt{2(1+k^6)}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - |a_1|\left(\frac{(R^3 - \xi^3) - 3(R-\xi)}{6}\right) \\ - |a_2|\left[\frac{(R-1)^3 - (\xi-1)^3}{3}\right], \text{ for } n = 3. \end{aligned} \tag{29}$$

Proof. Let $n > 3$. Since $p(z)$ is a polynomial of degree $n > 3$, it follows that $p'(z)$ is a polynomial of degree $n \geq 3$. Hence applying inequality (17) of Lemma 2 to the polynomial $p'(z)$ with $k = s \geq 1$, we have for $n > 3$

$$|p'(se^{i\theta})| \leq s^{n-1}\|p'\| - 2\frac{(s^n - 1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1} - 1}{n-1} - \frac{s^{n-3} - 1}{n-3}\right]$$

This gives with the help of Lemma 1,

$$|p'(se^{i\theta})| \leq s^{n-1}\left[\frac{n}{2^{\frac{1}{2}}(1+k^{2n})^{\frac{1}{2}}}[M_\alpha^2 + M_{\alpha+\pi}^2]^{\frac{1}{2}}\right] - 2\frac{(s^n-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right].$$

Hence for each θ, $0 \leq \theta < 2\pi$ and $R > \xi \geq 1$

$$\begin{aligned}
|p(Re^{i\theta}) - p(\xi e^{i\theta})| &= \left| \int_{\xi}^{R} e^{i\theta} p'(se^{i\theta}) ds \right| \\
&\leq \int_{\xi}^{R} |p'(se^{i\theta})| ds \\
&\leq \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^{2n})^{\frac{1}{2}}} \int_{\xi}^{R} s^{n-1} ds - \frac{2|a_1|}{n+1} \int_{\xi}^{R} (s^{n-1} - 1) ds \\
&- 2|a_2| \int_{\xi}^{R} \left(\frac{s^{n-1} - 1}{n-1} - \frac{s^{n-3} - 1}{n-3} \right) ds \\
&= \frac{n(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^{2n})^{\frac{1}{2}}} \frac{(R^n - \xi^n)}{n} - \frac{2|a_1|}{n+1} \left(\frac{R^n - \xi^n}{n} - (R - \xi) \right) \\
&- 2|a_1| \left[\frac{(R^n - \xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)} \right].
\end{aligned}$$

This implies,

$$\begin{aligned}
\|p(Rz) - p(\xi z)\| \leq \frac{R^n - \xi^n}{\sqrt{2}(1+k^{2n})^{\frac{1}{2}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1} \left(\frac{R^n - \xi^n}{n} - (R-\xi) \right) \\
- 2|a_2| \left[\frac{(R^n - \xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2} - \xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)} \right].
\end{aligned}$$

This is the desired result for the case $n > 3$. For $n = 3$, using inequality (18) of Lemma 2 with $k = s \geq 1$ to the polynomial $p'(z)$ we obtain

$$|p'(se^{i\theta})| \leq s^2 \|p'\| - \frac{(s-1)}{2}[(s+1)|a_1| + (s-1)|a_2|].$$

As before, again this gives with the help of Lemma 1 that

$$|p'(se^{i\theta})| \leq s^2 \frac{3}{\sqrt{2}(1+k^6)^{\frac{1}{2}}} (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - \frac{(s-1)}{2}[(s+1)|a_1| + (s-1)|a_2|].$$

Now for each θ, $0 \leq \theta < 2\pi$ and $R > \xi \geq 1$

$$\begin{aligned}
|p(Re^{i\theta}) - p(\xi e^{i\theta})| &\leq \int_{\xi}^{R} |p'(se^{i\theta})| ds \\
&\leq \int_{\xi}^{R} \left[\frac{3(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^6)^{\frac{1}{2}}} s^2 - \frac{s^2-1}{2}|a_1| - (s-1)^2|a_2| \right] ds \\
&= \frac{3(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}}{\sqrt{2}(1+k^6)^{\frac{1}{2}}} \frac{R^3-\xi^3}{3} - \frac{1}{2}\left[\frac{R^3-\xi^3}{3} - (R-\xi)\right]|a_1| \\
&\quad - \left[\frac{(R-1)^3 - (\xi-1)^3}{3}\right]|a_2|,
\end{aligned}$$

i.e,

$$\begin{aligned}
\|p(Rz) - p(\xi z)\| \leq \frac{R^3-\xi^3}{2^{\frac{1}{2}}(1+k^6)^{\frac{1}{2}}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}} - |a_1|\left(\frac{(R^3-\xi^3) - 3(R-\xi)}{6}\right) \\
- |a_2|\left[\frac{(R-1)^3-(\xi-1)^3}{3}\right].
\end{aligned}$$

This proves the theorem for the case $n = 3$. □

Finally we present the refinement and generalization for the upper bound of inequality (16). More precisely we prove the following result.

Theorem 6. *Let $p(z)$ be a polynomial of degree $n \geq 3$ which has no zeros in $|z| < k$, $k \leq 1$ and $m = \min_{|z|=k}|p(z)|$ then for all real α and $R > \xi \geq 1$*

$$\begin{aligned}
\|p(Rz) - p(\xi z)\| \leq \frac{R^n-\xi^n}{\sqrt{2(1+k^{2n})}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\xi^n}{n} - (R-\xi)\right) \\
- 2|a_2|\left[\frac{(R^n-\xi^n) - n(R-\xi)}{n(n-1)} - \frac{(R^{n-2}-\xi^{n-2}) - (n-2)(R-\xi)}{(n-2)(n-3)}\right], \\
\textit{if } n > 3
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
\|p(Rz) - p(\xi z)\| \leq \frac{R^3-\xi^3}{\sqrt{2(1+k^6)}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}} - |a_1|\left(\frac{(R^3-\xi^3) - 3(R-\xi)}{6}\right) \\
- |a_2|\left[\frac{(R-1)^3-(\xi-1)^3}{3}\right], \ \textit{if } n = 3,
\end{aligned} \tag{31}$$

provided $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on $|z| = 1$, where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$. The result is best possible and equality in (30) holds for $p(z) = z^n + k^n$.

Proof. Since $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, therefore,

$$|q'(z)| = |np(z) - zp'(z)| \text{ for } |z| = 1.$$

By hypothesis $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on $|z| = 1$. If we consider

$$\max_{|z|=1} |p'(z)| = |p(e^{i\alpha})|,\ 0 \le \alpha < 2\pi$$

then it is clear that,

$$\max_{|z|=1} |q'(z)| = |q(e^{i\alpha})|,\ 0 \le \alpha < 2\pi.$$

Since $p(z)$ does not vanish in $|z| < k, k \le 1$ and $m = \min\limits_{|z|=k} |p(z)|$. Therefore by Lemma 6 and by using above maximum values of $|p'(z)|$ and $|q'(z)|$, we get

$$(k^n|p'(e^{i\alpha})| + nm)^2 \le |q'(e^{i\alpha})|^2.$$

This gives with the help of Lemma 4

$$|p'(e^{i\alpha})|^2 + (k^n|p'(e^{i\alpha})| + nm)^2 \le |p'(e^{i\alpha})|^2 + |q'(e^{i\alpha})|^2$$
$$\frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].$$

Since

$$(k^n|p'(e^{i\alpha})| + nm)^2 \ge k^{2n}|p'(e^{i\alpha})|^2 + n^2m^2.$$

Consequently,

$$|p'(e^{i\alpha})|^2 + k^{2n}|p'(e^{i\alpha})|^2 + n^2m^2 \le \frac{n^2}{2}[M_\alpha^2 + M_{\alpha+\pi}^2].$$

Equivalently,

$$|p'(e^{i\alpha})|^2 \le \frac{n^2}{2(1+k^2)}[M_\lambda^2 + M_{\lambda+\pi}^2 - 2m^2]$$

and therefore, we have

$$\|p'\| \le \frac{n}{\sqrt{2(1+k^{2n})}}[M_\lambda^2 + M_{\lambda+\pi}^2 - 2m^2]^{\frac{1}{2}}. \tag{32}$$

Since $p(z)$ is a polynomial of degree $n > 3$, it follows that $p'(z)$ is a polynomial of degree $n \ge 3$. Hence applying inequality (17) of Lemma 2 to the polynomial $p'(z)$ with $k = s \ge 1$, we have for $n > 3$

$$|p'(se^{i\theta})| \le s^{n-1}\|p'\| - 2\frac{(s^{n-1}-1)}{n+1}|a_1| - 2|a_2|\left[\frac{s^{n-1}-1}{n-1} - \frac{s^{n-3}-1}{n-3}\right],$$

This in conjunction with (32) gives,

$$|p'(se^{i\theta})| \leq s^{n-1}\left[\frac{n}{\sqrt{2(1+k^{2n})}}[M_\alpha^2+M_{\alpha+\pi}^2-2m^2]^{\frac{1}{2}}\right] - 2\frac{(s^{n-1}-1)}{n+1}|a_1|$$
$$-2|a_2|\left[\frac{s^{n-1}-1}{n-1}-\frac{s^{n-3}-1}{n-3}\right].$$

Hence for each θ, $0 \leq \theta < 2\pi$ and $R > \xi \geq 1$

$$|p(Re^{i\theta})-p(\xi e^{i\theta})| = \left|\int_\xi^R e^{i\theta}p'(se^{i\theta})ds\right|$$
$$\leq \int_\xi^R |p'(se^{i\theta})|ds$$
$$\leq \frac{n(M_\alpha^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}}{\sqrt{2(1+k^{2n})}}\int_\xi^R s^{n-1}ds - \frac{2|a_1|}{n+1}\int_\xi^R (s^{n-1}-1)ds$$
$$-2|a_2|\int_\xi^R\left(\frac{s^{n-1}-1}{n-1}-\frac{s^{n-3}-1}{n-3}\right)ds$$
$$= \frac{n(M_\alpha^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}}{\sqrt{2(1+k^{2n})}}\frac{(R^n-\xi^n)}{n} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\xi^n}{n}-(R-\xi)\right)$$
$$-2|a_1|\left[\frac{(R^n-\xi^n)-n(R-\xi)}{n(n-1)}-\frac{(R^{n-2}-\xi^{n-2})-(n-2)(R-\xi)}{(n-2)(n-3)}\right].$$

This implies,

$$\|p(Rz)-p(\xi z)\| \leq \frac{R^n-\xi^n}{\sqrt{2(1+k^{2n})}}(M_\alpha^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}} - \frac{2|a_1|}{n+1}\left(\frac{R^n-\xi^n}{n}-(R-\xi)\right)$$
$$-2|a_2|\left[\frac{(R^n-\xi^n)-n(R-\xi)}{n(n-1)}-\frac{(R^{n-2}-\xi^{n-2})-(n-2)(R-\xi)}{(n-2)(n-3)}\right].$$

This proves inequality (30). For the proof of inequality (31), we use inequality (18) of Lemma 2 rather than inequality (17) of the same Lemma. □

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References

[1] Ahmad, I., Some generalizations of Bernstein type inequalities, *Int. J. Modern Math. Sci.,* 4(3) (2012), 133-138.

[2] Aziz, A., A refinement of an inequality of S. Bernstein, *J. Math. Anal. Appl.,* 144 (1989), 226-235. https://doi.org/10.1016/0022-247X(89)90370-3

[3] Aziz, A., Rather, N. A., New L^q inequalities for polynomials, *Mathematical Inequalities and Applications,* 2 (1998), 177-191.

[4] Bernstein, S. N., ÉMales Et La Meilleure Approximation Des Fonctions Analytiques D'une Variable RÉElle, Gauthier-Villars, Paris, 1926.

[5] Dewan, K. K., Kaur, J., Mir, A., Inequalities for the derivative of a polynomial, *J. Math. Anal. Appl.,* 269 (2002), 489-499. https://doi.org/10.1016/S0022-247X(02)00030-6

[6] Frappier, C., Rahman, Q. I., Ruscheweyh, St., New inequalities for polynomials, *Trans. Amer. Math. Soc.,* 288 (1985), 69-99. https://doi.org/10.2307/2000427

[7] Gulzar, M. H., Inequalities for a polynomial and its derivative, *International Journal of Mathematical Archive,* 3(2) (2012), 528-533.

[8] Lax, P. D., Proof of a conjecture of P. Erdös on the derivatives of a polynomial, *Bull. Amer. Math. Soc.*, 50 (1944), 509-513.

[9] Marden, M., Geometry of Polynomials, Second Edition, Mathematical Surveys, Amer. Math. Soc. Providence R.I. 1996.

[10] Milovanović, G. V., Mitrinović, D. S., Rassias, Th.M., Topics in Polynomials Extremal Problems, Inequalities, Zeros, World Scientific, Singapore, 1994.

[11] Pólya, G., Szegö, G., Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925.

[12] Rahman, Q. I., Schmeisser, G., Analytic Theory of Polynomials, Oxford University Press, New York, 2002.

[13] Rather, N. A., Shah, M. A., On the derivative of a polynomial, *Applied Mathematics*, 3 (2012), 746-749. http://dx.doi.org/10.4236/am.2012.37110

[14] Riesz, M., Eine trigonometrische interpolation formel und einige Ungleichung für Polynome, *Jahresber. Dtsch. Math.Verein*, 23 (1914), 354-368.

[15] Riesz, M., Über einen Sat'z des Herrn Serge Bernstein, *Acta Math.*, 40 (1916), 337-347.

http://communications.science.ankara.edu.tr

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$\alpha-$SASAKIAN, $\beta-$KENMOTSU AND TRANS-SASAKIAN STRUCTURES ON THE TANGENT BUNDLE

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Abstract. This paper consists of two main sections. In the first part, we give some general information about the almost contact manifold, $\alpha-$Sasakian, $\beta-$Kenmotsu and Trans-Sasakian Structures on the manifolds. In the second part, these structures were expressed on the tangent bundle with the help of lifts and the most general forms were tried to be obtained.

1. Introduction

1.1. Lifts of Vector Fields.

Definition 1. *Let M^n be an $n-$dimensional differentiable manifold of class C^∞ and let $T_p(M^n)$ be the tangent space of M^n at a point p of M^n. Then the set [12]*

$$T(M^n) = \bigcup_{p\in M^n} T_p(M^n) \tag{1}$$

is called the tangent bundle over the manifold M^n.

For any point $\tilde{p}$ of $T(M^n)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M^n) \to M^n$, Thus $\pi(\tilde{p}) = p$, where $\pi : T(M^n) \to M^n$ defines the bundle projection of $T(M^n)$ over M^n. The set $\pi^{-1}(p)$ is called the fibre over $p \in M^n$ and M^n the base space.

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1.1.1. *Vertical Lifts.* If f is a function in M^n, we write f^v for the function in $T(M^n)$ obtained by forming the composition of $\pi : T(M^n) \to M^n$ and $f : M^n \to R$, so that

$$f^v = fo\pi. \tag{2}$$

Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^v(\tilde{p}) = f^v(\sigma, \theta) = fo\pi(\tilde{p}) = f(p) = f(\sigma). \tag{3}$$

Thus the value of $f^v(\tilde{p})$ is constant along each fibre $T_p(M^n)$ and equal to the value $f(p)$. We call f^v the vertical lift of f [12].

Let $\sigma \in \Im_0^1(T(M^n))$ be such that $\sigma f^v = 0$ for all $f \in \Im_0^0(M^n)$. Then we say that σ is a vertical vector field. Let $\begin{bmatrix} \sigma^h \\ \sigma^{\bar{h}} \end{bmatrix}$ be components of σ with respect to the induced coordinates. Then σ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{bmatrix} \sigma^h \\ \sigma^{\bar{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^{\bar{h}} \end{bmatrix}. \tag{4}$$

Suppose that $\sigma \in \Im_0^1(M^n)$, so that is a vector field in M^n. We define a vector field σ^v in $T(M^n)$ by

$$\sigma^v(\iota\ \zeta) = (\zeta\sigma)^v \tag{5}$$

ζ being an arbitrary $1-$form in M^n. We cal σ^v the vertical lift of σ [12].

Let $\zeta \in \Im_1^0(T(M^n))$ be such that $\zeta(\sigma)^v = 0$ for all $\sigma \in \Im_0^1(M^n)$. Then we say that ζ is a vertical $1-$form in $T(M^n)$. We define the vertical lift ζ^v of the $1-$form ζ by

$$\zeta^v = (\zeta_i)^v (dx^i)^v \tag{6}$$

in each open set $\pi^{-1}(U)$, where $(U; x^h)$ is coordinate neighbourhood in M^n and ζ is given by $\zeta = \zeta_i dx^i$ in U. The vertical lift ζ^v with local expression $\zeta = \zeta_i dx^i$ has components of the form

$$\zeta^v : (\zeta^i, 0) \tag{7}$$

with respect to the induced coordinates in $T(M^n)$.

Vertical lift has the following formulas [10,12]:

$$\begin{aligned} (f\sigma)^v &= f^v\sigma^v,\ I^v\sigma^v = 0,\ \eta^v(\sigma^v) = 0, \\ (f\eta)^v &= f^v\eta^v,\ [\sigma^v, \theta^v] = 0,\ \varphi^v\sigma^v = 0, \\ \sigma^v f^v &= 0,\ \sigma^v f^v = 0 \end{aligned} \tag{8}$$

hold good, where $f \in \Im_0^0(M_n^n)$, $\sigma, \theta \in \Im_0^1(M_n^n)$, $\eta \in \Im_1^0(M_n^n)$, $\varphi \in \Im_1^1(M_n^n)$, $I = id_{M_n^n}$.

1.1.2. *Complete Lifts.* If f is a function in M^n, we write f^c for the function in $T(M^n)$ defined by

$$f^c = \iota(df) \tag{9}$$

and call f^c the comple lift of f. The complete lift f^c has the local expression

$$f^c = y^i \partial_i f = \partial f \tag{10}$$

with respect to the induced coordinates in $T(M^n)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $\sigma \in \Im_0^1(M^n)$. We define a vector field σ^c in $T(M^n)$ by

$$\sigma^c f^c = (\sigma f)^c, \tag{11}$$

f being an arbitrary function in M^n and call σ^c the complete lift of σ in $T(M^n)$ [3,12]. The complete lift σ^c with components x^h in M^n has components

$$\sigma^c = \begin{pmatrix} \sigma^h \\ \partial \sigma^h \end{pmatrix} \tag{12}$$

with respect to the induced coordinates in $T(M^n)$.

Suppose that $\zeta \in \Im_1^0(M^n)$, then a 1−form ζ^c in $T(M^n)$ defined by

$$\zeta^c(\sigma^c) = (\zeta\sigma)^c \tag{13}$$

σ being an arbitrary vector field in M^n. We call ζ^c the complete lift of ζ. The complete lift ζ^c of ζ with components ζ_i in M^n has components of the form

$$\zeta^c : (\partial \zeta_{i,} \zeta_i) \tag{14}$$

according to the induced coordinates in $T(M^n)$ [3].

$$\begin{array}{rcl} \sigma^c f^v &=& (\sigma f)^v,\ \eta^v(\sigma^c) = (\eta(\sigma))^v, \\ (f\sigma)^c &=& f^c \sigma^v + f^v \sigma^c = (\sigma f)^c, \\ \sigma^v f^c &=& (\sigma f)^v,\ \varphi^v \sigma^c = (\varphi\sigma)^v, \\ \varphi^c \sigma^v &=& (\varphi\sigma)^v,\ (\varphi\sigma)^c = \varphi^c \sigma^c, \\ \eta^v(\sigma^c) &=& (\eta(\sigma))^c,\ \eta^c(\sigma^v) = (\eta(\sigma))^v, \\ [\sigma^v, \theta^c] &=& [\sigma, \theta]^v,\ I^c = I,\ I^v \sigma^c = \sigma^v,\ [\sigma^c, \theta^c] = [\sigma, \theta]^c. \end{array} \tag{15}$$

1.2. **Almost Contact Manifolds.** An almost contact manifold is an odd-dimensional C^∞ manifold whose structural group can be reduced to $U(x) \times 1$. This is equivalent to the existence of a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1−form η satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. From these conditions one can deduce that $\phi\xi = 0$ and $\eta \circ \phi = 0$. A Riemannian metric g is compatible with these structure tensors if

$$g(\phi\sigma, \phi\theta) = g(\sigma, \theta) - \eta(\sigma)\eta(\theta) \tag{16}$$

and we refer to an almost contact metric structure (ϕ, ξ, η, g). Note also that $\eta(\sigma) = g(\sigma, \xi)$.

Let M^n be an almost contact manifold and define an almost complex structure J on $M^n \times R$ by

$$J(\sigma, f\frac{d}{dt}) = (\phi\sigma - f\xi, \eta(\sigma)\frac{d}{dt}). \tag{17}$$

A Sasakian manifold is a normal contact metric manifold. It is well known that the Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_\sigma \phi)\theta = g(\sigma,\theta)\xi - \eta(\theta)\sigma, \tag{18}$$

again see e.g. [1].

2. $\alpha-$Sasakian and $\beta-$Kenmotsu structures on the tangent bundle

A $\alpha-$Sasakian structure [6] which may be defined by the requirement

$$(\nabla_\sigma \phi)\theta = \alpha(g(\sigma,\theta)\xi - \eta(\theta)\sigma), \tag{19}$$

where α is a non-zero constant. Setting $\theta = \xi$ in this formula, one readily obtains

$$\nabla_\sigma \xi = -\alpha\phi\sigma \tag{20}$$

Theorem 1. *Let a vector field ξ, ϕ be a tensor field of type $(1,1)$, $1-$form η satisfying $\phi^2 = -I + \eta \otimes \xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. A $\alpha-$Sasakian structure on tangent bundle defined by*

$$(\nabla^c_{\sigma^c}\phi^c)\theta^c = \alpha((g(\sigma,\theta))^V\xi^C + (g(\sigma,\theta))^C\xi^V - (\eta(\theta))^C\sigma^V - (\eta(\theta))^V\sigma^C),$$

where g is a Riemannian metric, α is a non-zero constant. In addition, if we put $\theta = \xi$, we get

$$\nabla^C_{\sigma^C}\xi^C = -\alpha\phi^C\sigma^C.$$

Proof. From (19) , we get the $\alpha-$Sasakian structure on the bundle

$$\begin{aligned}(\nabla^c_{\sigma^c}\phi^c)\theta^c &= \nabla^C_{\sigma^C}\phi^c\theta^C - \phi^C\,\nabla^C_{\sigma^C}\,\theta^C\\ &= \alpha((g(\sigma,\theta))^V\xi^C + (g(\sigma,\theta))^C\xi^V - (\eta(\theta))^C\sigma^V - (\eta(\theta))^V\sigma^C).\end{aligned}$$

If we put $\theta = \xi$, we get

$$\begin{aligned}(\nabla^C_{\sigma^C}\phi^C)\xi^C &= \nabla^C_{\sigma^C}\phi^C\xi^C - \phi^C\,\nabla^C_{\sigma^C}\,\xi^C\\ &= -\phi^C\,\nabla^C_{\sigma^C}\,\xi^C\\ &= \alpha(\eta(\sigma)^V\xi^C + (\eta(\sigma))^C\xi^V - (\eta(\xi))^C\sigma^V - (\eta(\xi)^V\sigma^C)\\ &= \alpha((\eta(\sigma))^V\xi^C + (\eta(\sigma))^C\xi^V - \sigma^C)\\ -\phi^C\,\nabla^C_{\sigma^C}\,\xi^C &= \alpha(\phi^C)^2\sigma^C\\ \nabla^C_{\sigma^C}\xi^C &= -\alpha\phi^C\sigma^C\end{aligned}$$

□

In particular the almost contact metric structure in this case satisfies

$$(\nabla_\sigma \phi)\theta = g(\phi\sigma, \theta)\xi - \eta(\theta)\phi\sigma \tag{21}$$

and an almost contact metric manifold satisfying this condition is called a Kenmotsu manifold [6,7]. Again one has the more general notion of a $\beta-$Kenmotsu structure [6] which may be defined by

$$(\nabla_\sigma \phi)\theta = \beta(g(\phi\sigma, \theta)\xi - \eta(\theta)\phi\sigma), \tag{22}$$

where β is a non-zero constant. From the condition one may readily deduce that

$$\nabla_\sigma \xi = \beta(\sigma - \eta(\sigma)\xi). \tag{23}$$

Theorem 2. *Let ϕ be a tensor field of type $(1,1)$, a vector field ξ, $1-$form η satisfying $\phi^2 = -I + \eta \otimes \xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. A $\beta-$Kenmotsu structure on tangent bundle defined by*

$$((\nabla_\sigma \phi)\theta)^C = \beta((g(\phi\sigma, \theta))^V \xi^C + (g(\phi\sigma, \theta))^C \xi^V - (\eta(\theta))^C (\phi\sigma)^V - (\eta(\theta))^V (\phi\sigma)^C),$$

where g is a Riemannian metric, β is a non-zero constant. In addition, if we put $\theta = \xi$, we get

$$\nabla^C_{\sigma^C} \xi^C = \beta(\sigma^C - ((\eta(\sigma))\xi)^C). \tag{24}$$

Proof. From (22) , we get the $\beta-$Kenmotsu structure on the bundle

$$\begin{aligned} ((\nabla_\sigma \phi)\theta)^C &= \nabla^C_{\sigma^C} \phi^C \theta^C - \phi^C \nabla^C_{\sigma^C} \theta^C \\ &= \beta((g(\phi\sigma, \theta))^V \xi^C + (g(\phi\sigma, \theta))^C \xi^V - (\eta(\theta))^C (\phi\sigma)^V - (\eta(\theta))^V (\phi\sigma)^C) \end{aligned}$$

If we put $\theta = \xi$, we get

$$\begin{aligned} -\phi^C \nabla^C_{\sigma^C} \xi^C &= \beta((\eta(\phi\sigma))^V \xi^C + (\eta(\phi\sigma))^C \xi^V - (\eta(\xi))^C (\phi\sigma)^V - (\eta(\xi))^V (\phi\sigma)^C) \\ -\phi^C \nabla^C_{\sigma^C} \xi^C &= \beta(-(\phi\sigma)^C + (\eta(\phi\sigma\ ))^V \xi^C + (\eta(\phi\sigma))^C \xi^V) \\ \phi^2 \nabla^C_{\sigma^C} \xi^C &= \beta((\phi\sigma)^C - (\eta(\phi\sigma))^V \xi^C - (\eta(\phi\sigma))^C \xi^V) \\ \phi \nabla^C_{\sigma^C} \xi^C &= \beta(\phi^C \sigma^C - \eta^V (\phi^C \sigma^C)\xi^C - \eta^C (\phi^C \sigma^C)\xi^V) \\ \nabla^C_{\sigma^C} \xi^C &= \beta(\sigma^C - (\eta^V \sigma^C)\xi^C - (\eta^C \sigma^C)\xi^V) \\ \nabla^C_{\sigma^C} \xi^C &= \beta(\sigma^C - (\eta\sigma)^V \xi^C - (\eta\sigma)^C \xi^V) \\ \nabla^C_{\sigma^C} \xi^C &= \beta(\sigma^C - ((\eta(\sigma))\xi)^C) \end{aligned}$$

□

3. Trans-Sasakian manifolds on the tangent bundle

An almost contact metric structure (ϕ, ξ, η, g) on M^n is trans-Sasakian [9] if $(M^n \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M^n \times R$ defined by (17) and G is the product metric on $M^n \times R$. This expressed by the condition

$$(\nabla_\sigma \phi)\theta = \alpha(g(\sigma, \theta)\xi - \eta(\theta)\sigma) + \beta(g(\phi\sigma, \theta)\xi - \eta(\theta)\phi\sigma) \tag{25}$$

for functions α and β on M^n, and we shall say that the trans-Sasakian structure is of type (α,β); in particular, it is normal and it generalizes both $\alpha-Sasakian$ and $\beta-Kenmotsu$ structures. From the formula one obtain

$$\bigtriangledown_\sigma \xi = -\alpha\phi\sigma + \beta(\sigma - \eta(\sigma)\xi), \tag{26}$$

$$(\bigtriangledown_\sigma\eta)(\theta) = -\alpha g(\phi\sigma,\theta) + \beta(g(\sigma,\theta) - \eta(\sigma)\eta(\theta)), \tag{27}$$

$$(\bigtriangledown_\sigma\phi)(\theta,Z) = \alpha(g(\sigma,Z)\eta(\theta) - g(\sigma,\theta)\eta(Z)) - \beta(g(\sigma,\phi Z)\eta(\theta) - g(\sigma,\phi\theta)\eta(Z)), \tag{28}$$

where ϕ is the fundamental $2-$form of the structure, given by $\phi(\sigma,\theta) = g(\sigma,\phi\theta)$.

Theorem 3. *Let ϕ be a tensor field of type $(1,1)$, a vector field ξ, $1-$form η satisfying $\phi^2 = -I + \eta\otimes\xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta\circ\phi = 0$. A trans-Sasakian structure on tangent bundle defined by*

$$(\bigtriangledown^C_{\sigma^C}\phi^C)\theta^C = \alpha((g(\sigma,\theta))^V\xi^C + (g(\sigma,\theta))^C\xi^V - (\eta(\theta))^C\sigma^V - (\eta(\theta))^V\sigma^C)$$
$$+\beta((g(\phi\sigma,\theta))^V\xi^C + (g(\phi\sigma,\theta))^C\xi^V - (\eta(\theta))^C(\phi\sigma)^V - (\eta(\theta))^V(\phi\sigma)^C),$$

where g is a Riemannian metric, α,β are non-zero constants. In addition, if we put $\theta = \xi$, we get

$$\bigtriangledown^C_{\sigma^C}\xi^C = -\alpha\phi^C\sigma^C + \beta(\sigma^C - ((\eta(\sigma))\xi)^C)$$

Proof. From (25), we get the trans-Sasakian structure on the bundle

$$(\bigtriangledown^C_{\sigma^C}\phi^C)\theta^C = \bigtriangledown^C_{\sigma^C}\phi^C\theta^C - \phi^C\bigtriangledown^C_{\sigma^C}\theta^C$$
$$= \alpha((g(\sigma,\theta))^V\xi^C + (g(\sigma,\theta))^C\xi^V - (\eta(\theta))^C\sigma^V - (\eta(\theta))^V\sigma^C)$$
$$+\beta((g(\phi\sigma,\theta))^V\xi^C + (g(\phi\sigma,\theta))^C\xi^V - (\eta(\theta))^C(\phi\sigma)^V - (\eta(\theta))^V(\phi\sigma)^C).$$

If we put $\theta = \xi$ and using the formulas of (8),(15), similarly we get

$$\bigtriangledown^C_{\sigma^C}\xi^C = -\alpha\phi^C\sigma^C + \beta(\sigma^C - ((\eta(\sigma))\xi)^C).$$

□

Theorem 4. *Let a vector field ξ, ϕ be a tensor field of type $(1,1)$, $1-$form η satisfying $\phi^2 = -I + \eta\otimes\xi$ i.e. $\eta(\xi) = 1$, $\phi\xi = 0$ and $\eta\circ\phi = 0$. The term $(\bigtriangledown^C_{\sigma^C}\eta^C)\theta^C$ in a trans-Sasakian structure on tangent bundle defined by*

$$(\bigtriangledown^C_{\sigma^C}\eta^C)\theta^C = -\alpha g^C(\phi^C\sigma^C,\theta^C) + \beta g^C(\phi^C\sigma^C,\phi^C\theta^C),$$

where g is a Riemannian metric, α,β is a non-zero constant.

Proof. From (27) , we get

$$\begin{aligned}(\bigtriangledown^C_{\sigma^C}\eta^C)\theta^C &= \bigtriangledown^C_{\sigma^C}\eta^C\theta^C - \eta^C\bigtriangledown^C_{\sigma^C}\theta^C\\ &= \bigtriangledown^C_{\sigma^C}(g(\theta,\xi))^C - (g(\bigtriangledown_\sigma\theta,\xi))^C\\ &= \bigtriangledown^C_{\sigma^C}g^C)(\theta^C,\xi^C) + g^C(\bigtriangledown^C_{\sigma^C}\theta^C,\xi^C) + g^C(\theta^C,\bigtriangledown^C_{\sigma^C}\xi^C)\\ &\quad -g^C(\bigtriangledown^C_{\sigma^C}\theta^C,\xi^C)\end{aligned}$$

$$
\begin{aligned}
&= g^C(\theta^C, \bigtriangledown^C_{\sigma^C}\xi^C) = g^C(\theta^C, -\alpha\phi^C\sigma^C + \beta(\sigma^C((\eta(\sigma))\xi)^C) \\
&= -\alpha g^C(\theta^C, \phi^C\sigma^C) + \beta g^C(\theta^C, \sigma^C - ((\eta(\sigma))\xi)^C) \\
&= -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta g^C(\theta^C, \sigma^C - (\eta(\sigma))^V\xi^C - (\eta(\sigma))^C\xi^V) \\
&= -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta(g^C(\theta^C, \sigma^C) - (\eta(\sigma))^V g^C(\theta^C, \xi^C) \\
&\quad -(\eta(\sigma))^C g^C(\theta^C, \xi^V)) \\
&= -\alpha g^C(\phi^C\sigma^C, \theta^C) + \beta g^C(\phi^C\sigma^C, \phi^C\theta^C),
\end{aligned}
$$

where $g^C(\theta^C, \xi^V) = (\eta(\theta))^V$ and $g^C(\phi^C\sigma^C, \phi^C\theta^C) = g^C(\sigma^C, \theta^C) - (\eta(\sigma))^C(\eta(\theta))^V - (\eta(\sigma))^V(\eta(\theta))^C$. □

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References

[1] Blair, D. E., Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics 509, Springer, 1976.

[2] Capursi, M., Some remarks on the product of two almost contact manifolds, *An. Sti. Univ. Al. I. Cuza* XXX, (1984), 75-79.

[3] Das Lovejoy, S., Fiberings on almost r-contact manifolds*, Publicationes Mathematicae,* 43 (1993), 161-167. DOI: 10.5486/PMD.1993.1258

[4] Gray, A., Hervella, L. M., The sixteen classes of almost Hermitianmanifolds and their linear invariant, *Ann. Math. Pura Appl.,* 4(123) (1980), 35-58. https://doi.org/10.1007/BF01796539

[5] Ianus, S., Smaranda, D., Some remarkable structures on the product of an almost contact metric manifold with the real line, *Papers from the National Colloquium on Geometry and Topology, Univ. Timisoara,* (1977), 107-110.

[6] Janssesns, D., Vanhecke, L., Almost contact structures and curvature tensors, *Kodai Math. J.,* 4 (1981), 1-27.

[7] Kenmotsu, K., A class of almost contact Riemannian manifolds, *Tôhoku Math. J.,* 24 (1972), 93-103.

[8] Morimoto, A., On normal almost contact structures, *J. Math. Soc. Japan,* 15 (1963), 420-436.

[9] Oubina, J.A., New classes of almost contact metric structures, *Publicationes Mathematicae Debrecen,* 32 (1985), 187-193.

[10] Omran, T., Sharffuddin, A., Husain, S.I., Lift of structures on manifolds, *Publications de 1'Instıtut Mathematıqe, Nouvelle serie*, 360(50) (1984), 93–97.

[11] Tanno, S., The automorphism groups of almost contact Riemannian manifolds, *Tôhoku Math. J.,* 21 (1969), 21-38.

[12] Yano, K., Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973.

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SOFT SEMI-TOPOLOGICAL POLYGROUPS

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Abstract. By removing the condition that the inverse function is continuous in soft topological polygroups, we will have less constraint to obtain the results. We offer different definitions for soft topological polygroups and eliminate the inverse function continuity condition to have more freedom of action.

1. Introduction

To answer the types of uncertainties that abound in various sciences, we insert soft sets into mathematical structures. Specifically, we equip topological polygroups with soft sets. This is a process that began in 1934 by Marty [16] with the introduction of hypergroups and continued with the introduction of soft sets by Molodtsov in 1999 [17]. Since then, many efforts have been made to deepen the discussion, some of which we can mention below.

A good description of the Groupoides, demi-hypergroupes et hypergroupes is given by M. Koskas in [14], also useful information about the Soft subsets and soft product operations is provided by F. Feng, Y.M. Li in [8]. There is a beautiful writing about the topological spaces from the S. Nazmul, SK. Samanta under the name Neighbourhood properties of soft topological spaces in [20], also about Soft set theory by P. K. Maji, R. Biswas and A. R. Roy in [15], Soft topological groups and rings by T. Shah and S. Shaheen in [27], On soft topological hypergroups by G. Oguz in [24], On soft topological spaces by M. Shabir and M. Naz in [26]. Only a genius like T. Hida can write such a beautiful story about the Soft topological group in [11], also G. Oguz with article Soft topological hyperstructure in [25] and M. Shabir, M. Naz With their own handwriting about the On soft topological spaces in [26]. If you want to read interesting articles about the topological polygroups,

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you can read Heidari's article about the Topological polygroups in [9], also about the Idealistic soft topological hyperrings by G. Oguz in [23] and A new view on topological polygroups by G. Oguz in [22], Soft sets and soft groups by H. Aktas and N. Cagman in [1], Prolegomena of Hypergroup Theory by P. Corsini in [5].

2. Preliminaries

2.1. **Soft Sets.** Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and A be a non-empty subset of E. A pair $(\mathbb{F}, A)$ is called a soft set over U, where $\mathbb{F}$ is a mapping given by $\mathbb{F}: A \to P(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $a \in A, \mathbb{F}(a)$ may be considered as the set of approximate elements of the soft set$(\mathbb{F}, A)$. Clearly a soft set is not a set. For two soft sets $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ over a common universe U, we say that $(\mathbb{F}, A)$ is a soft subset of $(\mathbb{G}, B)(i.e., (\mathbb{F}, A)\widehat{\subset}(\mathbb{G}, B))$ if $A \subseteq B$ and $\mathbb{F}(a) \subseteq \mathbb{G}(a)$ for all $a \in A$. $(\mathbb{F}, A)$ is said to be a soft super set of $(\mathbb{G}, B)$, if $(\mathbb{G}, B)$ is a soft subset of $(\mathbb{F}, A)$ and it is denoted by $(\mathbb{F}, A)\widehat{\supset}(\mathbb{G}, B)$. Two soft sets $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ over a common universe U are said to be soft equal if $(\mathbb{F}, A)$ is a soft subset of $(\mathbb{G}, B)$ and $(\mathbb{G}, B)$ is a soft subset of $(\mathbb{F}, A)$. A soft set $(\mathbb{F}, A)$ over U is said to be a NULL soft set, denoted by $\widehat{\varnothing}$, if $\mathbb{F}(a) = \varnothing$ (null set) for all $a \in A$. A soft set $(\mathbb{F}, A)$ over U is said to be ABSOLUTE soft set, denoted by $\widehat{A}$, if $\mathbb{F}(a) = U$ for all $a \in A$. $(\mathbb{F}, A)$ AND $(\mathbb{G}, B)$ denoted by $(\mathbb{F}, A)\widehat{\wedge}(\mathbb{G}, B)$ is defined by $(\mathbb{F}, A)\widehat{\wedge}(\mathbb{G}, B) = (\mathbb{H}, A \times B)$, where $\mathbb{H}((a, b)) = \mathbb{F}(a) \cap \mathbb{G}(b)$ for all $(a, b) \in A \times B$. $(\mathbb{F}, A)$ OR $(\mathbb{G}, B)$ denoted by $(\mathbb{F}, A)\widehat{\vee}(\mathbb{G}, B)$ is defined by $(\mathbb{F}, A)\widehat{\vee}(\mathbb{G}, B) = (O, A \times B)$ where, $O((a, b)) = \mathbb{F}(a) \cup \mathbb{G}(b)$ for all $(a, b) \in A \times B$. Union of two soft sets $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ over the common universe U denoted by $(\mathbb{F}, A)\widehat{\cup}(\mathbb{G}, B)$ is defined by $(\mathbb{H}, C)$, where $C = A \cup B$ and for all $a \in C$,

$$\mathbb{H}(a) = \begin{cases} \mathbb{F}(a) & \text{if } a \in A - B \\ \mathbb{G}(a) & \text{if } a \in B - A \\ \mathbb{F}(a) \cup \mathbb{G}(a) & \text{if } a \in A \cap B. \end{cases}$$

Bi-intersection of two soft sets $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ over the common universe U is the soft set $(\mathbb{H}, C)$ is defined by $(\mathbb{F}, A)\widehat{\cap}(\mathbb{G}, B) = (\mathbb{H}, C)$, where $C = A \cap B$ and $\mathbb{H}(a) = \mathbb{F}(a) \cap \mathbb{G}(a)$ for all $a \in C$. Extended intersection of two soft sets $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ over the common universe U denoted by $(\mathbb{F}, A) \cap_E (\mathbb{G}, B)$ and is defined by $(\mathbb{H}, C)$, where $C = A \cup B$ and for all $a \in C$,

$$\mathbb{H}(a) = \begin{cases} \mathbb{F}(a) & \text{if } a \in A - B \\ \mathbb{G}(a) & \text{if } a \in B - A \\ \mathbb{F}(a) \cap \mathbb{G}(a) & \text{if } a \in A \cap B. \end{cases}$$

Let $(\mathbb{F}, A)$ be a soft set. The set $Supp(\mathbb{F}, A) = \{a \in A : \mathbb{F}(a) \neq \varnothing\}$ is called the support of the soft set $(\mathbb{F}, A)$. A soft set is said to be non-null if its support is not equal to the empty set. If A is equal to E we write $\mathbb{F}$ instead of $(\mathbb{F}, A)$. Let $\theta : U \longmapsto U'$ be a function and $\mathbb{F}(resp.\mathbb{F}')$ be a soft set over $U(resp.U')$ with a parameter set E. Then $\theta(\mathbb{F})(resp.\theta^{-1}(\mathbb{F}'))$ is the soft set on $U'(resp.U)$ is defined by $(\theta(\mathbb{F}))(e) = \theta(\mathbb{F}(e))(resp.(\theta^{-1}(\mathbb{F}'))(e) = \theta^{-1}(\mathbb{F}'(e)))$. We will use the symbol $\mathbb{F}^{\widehat{c}}$ to denote soft complement of $\mathbb{F}$ and is defined by $\mathbb{F}^{\widehat{c}}(e) = U \setminus \mathbb{F}(e)(e \in E)$. Let $\mathbb{F}$ be a soft set over U and x be an element of U we call x is a soft element of $\mathbb{F}$, if $x \in \mathbb{F}(e)$ for all parameters $e \in E$ and denoted by $x \widehat{\in} \mathbb{F}$. We recall the above definitions from [11,27].

2.2. **Polygroups.** Let H be a non-empty set. A mapping $\circ : H \times H \longmapsto P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the family of non-empty subsets of H. The couple $(H, \circ)$ is called a hypergroupoid. In the above definition,if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a quasihypergroup if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup [5].

Let $(H, \circ)$ be a semihypergroup and A be a non-empty subset of H. We say that A is a complete part of H if for any non-zero natural number n and for all $a_1, \ldots, a_n$ of H, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \varnothing \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

The complete parts were introduced for the first time by Koskas [14]. Let $(G, \circ)$ and $(H, *)$ be two hypergroups. A map $f : G \longmapsto H$, is called a homomorphism if for all x, y of G, we have $f(x \circ y) \subseteq f(x) * f(y)$; a good homomorphism if for all x, y of G, we have $f(x \circ y) = f(x) * f(y)$;f is an isomorphism if it is a good homomorphism ,and its inverse f^{-1} is a homomorphism, too.

Definition 1. *A special sub class of hypergroups is the class of polygroups.A polygroup is a system $P = < P, \circ, e, -1 >$,where $\circ : P \times P \longmapsto P^*(P)$, $e \in P$, -1 is a unitary operation on P and the following axioms hold for all $x, y, z \in P$:*

(1) $(x \circ y) \circ z = x \circ (y \circ z)$;
(2) $e \circ x = x \circ e = x$;
(3) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x, e^{-1} = e, (x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. A non-empty subset K of a polygroup P is a subpolygroup of P if and only if $a, b \in K$ implies $a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$.

The subpolygroup N of P is normal in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for all $a \in P$.

Theorem 1. *Let N be a normal subpolygroup of P then:*

(1) *$Na = aN$ for all $a \in P$;*
(2) *$(aN)(bN) = abN$ for all $a, b \in P$;*
(3) *$aN = bN$ for all $b \in aN$.*

EXAMPLE 1. *Let P be $\{1, 2\}$ and hyperoperation $\divideontimes$ be as follow:*

$\divideontimes$	1	2
1	1	2
2	2	$\{1, 2\}$

With the above multiplication table, P is a polygroup [7].

Let P is polygroup and $(\mathbb{F}, A)$ be a soft set on P. Then $(\mathbb{F}, A)$ is called a (normal)soft polygroup on P if $\mathbb{F}(x)$ be a (normal)subpolygroup of P for all $x \in Supp(\mathbb{F}, A)$.

EXAMPLE 2. *Let P be $\{e, a, b\}$ and multiplication table be:*

$\circ$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a\}$

Subpolygroups of P are $\varnothing, P, \{e\}, \{e, a\}$. Let A be equal with P and define soft set $\mathbb{F}$ as follow:

$$\mathbb{F}(x) = \begin{cases} \{e\} & \text{if } x = e \\ \{e, a\} & \text{if } x = a \\ \{e, a, b\} & \text{if } x = b \end{cases}$$

Therefore $(\mathbb{F}, A)$ is a soft polygroup. We recall the above definitions and theorems from [7].

2.3. **Topological Hyperstructure.** Suppose that T is a topology on G, where G is a group, then (G, T) is called a topological group over G if φ and $^{-1}$ are continuous, where φ and $^{-1}$ are as follow:

(1) The mapping $\varphi : G \times G \longmapsto G$ is defined by $\varphi(g, h) = gh$ and $G \times G$ is endowed with the product topology.
(2) The mapping $^{-1} : G \longmapsto G$ is defined by $^{-1}(g) = g^{-1}$ [10].

If the condition (2) of previous definition is not met, then the (G, T) is called semi-topological group over G.

Let $(\mathbb{F}, A)$ be a soft set over G. Then the $(\mathbb{F}, A, T)$ is called soft topological group over G if the following conditions hold:

(1) $\mathbb{F}(a)$ be a subgroup of G for all $a \in A$.
(2) The mapping $\varphi : (x, y) \longmapsto xy$ of the topological space $\mathbb{F}(a) \times \mathbb{F}(a)$ onto $\mathbb{F}(a)$ be continuous for all $a \in A$.
(3) The mapping $^{-1} : \mathbb{F}(a) \longmapsto \mathbb{F}(a)$ is defined by $^{-1}(g) = g^{-1}$ be continuous for all $a \in A$.

If the condition (3) of previous definition is not met, then the $(\mathbb{F}, A, T)$ is called soft semi-topological group over G.

In [9] is proved that condition continuity φ is equivalent to following statement;

If $U \subseteq G$ is open, and $gh \in U$, then there exist open sets V_g and V_h with the property that $g \in V_g$,$h \in V_h$, and $V_g V_h = \{v_1 v_2 | v_1 \in V_g, v_2 \in V_h\} \subseteq U$.

Also, condition continuity $^{-1}$ is equivalent to following statement; If U subset of G is open, then $U^{-1} = \{g^{-1} | g \in U\}$ be open.

Let (H, T) be a topological space. The following theorem give us a topology on $P^*(H)$ that is induced by T.

Theorem 2. *Let (H, T) be a topological space. Then the family β consisting of all sets $S_V = \{U \in P^*(H) \mid U \subseteq V\}, V \in T$ is a base for a topology on $P^*(H)$. This topology is denoted by T^* [12].*

Let (H, T) be a topological space, where $(H, \circ)$ be a hypergroup. Then the triple $(H, \circ, T)$ is called a topological hypergroup if the following functions are continuous:

(1) The mapping $\varphi : (x, y) \longmapsto x \circ y$, from $H \times H$ onto $P^*(H)$;
(2) The mapping $\psi : (x, y) \longmapsto x/y$, from $H \times H$ onto $P^*(H)$, where $x/y = \{z \in H | x \in z \circ y\}$.

If the condition (2) of previous definition is not met, then $(H, \circ, T)$ is called a semi-topological hypergroup.

Let (P, T) be a topological space, where $(P, \circ, e, ^{-1})$ be a polygroup. Then the (P, T) is called a topological polygroup (in short TP) if the following axioms hold:

(1) The mapping $\circ : P \times P \longmapsto P^*(P)$ be continuous, where $\circ(x, y) = x \circ y$;
(2) The mapping $^{-1} : P \longmapsto P$ be continuous, where $^{-1}(x) = -x$.

We can combine items (1),(2) and present the following case:

The mapping $\varphi : P \times P \longmapsto P^*(P)$ be continuous, where $\varphi(x, y) = x \circ y^{-1}$.

The following theorem help us to determine the continuity of hyperoperation. We us to use the following theorem for the continuity test.

Theorem 3. *The hyperoperation $\circ : P \times P \longmapsto P^*(P)$ is continuous, where P is a polygroup $\Longleftrightarrow \forall a, b \in P$ and $C \in T$ with the property that $a \circ b \subseteq C$ then there exist $A, B \in T$ with the property that $a \in A$ and $b \in B$ and $A \circ B \subseteq C$ [9].*

EXAMPLE 3. *[18] Let P be $\{e, a, b, c\}$ and multiplication table be:*

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Hyperoperation $\circ : P \times P \longmapsto P^*(P)$ is continuous with topologies:

T_{dis},
T_{ndis},
$T_1 = \{\varnothing, P, \{e, b\}\}$,
$T_2 = \{\varnothing, P, \{e\}, \{b\}\}$,
since $x^{-1} = x$ for all $x \in P$, inverse operation is identity and identity function is continuous with every topology, it follows that P with topologies T_1, T_2 is topological polygroup.

Hyperoperation $\circ : P \times P \longmapsto P^*(P)$ with below topologies is not continuous.

$T_3 = \{\varnothing, P, \{e\}\}$,
$T_4 = \{\varnothing, P, \{a\}\}$,
$T_5 = \{\varnothing, P, \{b\}\}$,
$T_6 = \{\varnothing, P, \{c\}\}$,
$T_7 = \{\varnothing, P, \{e, a\}\}$,
$T_8 = \{\varnothing, P, \{e, c\}\}$,
$T_9 = \{\varnothing, P, \{a, b\}\}$,
$T_{10} = \{\varnothing, P, \{a, c\}\}$,
$T_{11} = \{\varnothing, P, \{b, c\}\}$,
$T_{12} = \{\varnothing, P, \{e, a, b\}\}$,
$T_{13} = \{\varnothing, P, \{e, a, c\}\}$,
$T_{14} = \{\varnothing, P, \{e, b, c\}\}$,
$T_{15} = \{\varnothing, P, \{a, b, c\}\}$,
$T_{16} = \{\varnothing, P, \{e\}, \{a\}\}$.

If the condition (2) of previous defintion is not met, then $(P, \circ, e, ^{-1}, T)$ is called a semi-topological polygroup.

3. Soft Semi-Topological Polygroups

The first definition we provide for soft semi-topological polygroups is as follows, and the examples and results that follow from this definition will be given below.

Definition 2. *Let T be a topology on a polygroup P. Let $(\mathbb{F}, A)$ be a soft set over P. Then the system $(\mathbb{F}, A, T)$ said to be soft semi-topological polygroup over P if the following axioms hold:*

(a) *$\mathbb{F}(a)$ is a subpolygroup of P for all $a \in A$.*
(b) *The mapping $(x, y) \longmapsto x \circ y$ of the topological space $\mathbb{F}(a) \times \mathbb{F}(a)$ onto $P^*(\mathbb{F}(a))$ is continuous for all $a \in A$.*

Topology T on P induces topologies on $\mathbb{F}(a)$, $\mathbb{F}(a) \times \mathbb{F}(a)$ and by Theorem 2 on $P^*(\mathbb{F}(a))$.

If A be $\{e, a_1, a_2, ...\}$, B be $\{e, b_1, b_2, ...\}$, and the table for $*$ in $A[B]$ be the following form:

	e	a_1	a_2	...	b_1	b_2	...
e	e	a_1	a_2	...	b_1	b_2	...
a_1	a_1	a_1a_1	a_1a_2	...	b_1	b_2	...
a_2	a_2	a_2a_1	a_2a_2	...	b_1	b_2	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_1	b_1	b_1	b_1	...	$b_1 * b_1$	$b_1 * b_2$	...
b_2	b_2	b_2	b_2	...	$b_2 * b_1$	$b_2 * b_2$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then several special cases of the algebra $A[B]$ are useful [6,7]. Before describing them we need to assign names to the 2-elements polygroups. Let 2 denotes the group $\mathbb{Z}_2$ and let 3 denotes the polygroup $\mathbb{S}_3//\langle(12)\rangle \cong \mathbb{Z}_3/T$, where T is the special conjugation with blocks $\{0\}, \{1, 2\}$. The multiplication table for 3 is

	0	1
0	0	2
1	1	$\{0, 1\}$

The system $3[M]$ is the result of adding a new identity to the polygroup $[M]$. The system $2[M]$ is almost as good. For example, suppose that R is the system with table

	0	1	2
0	0	1	2
1	1	$\{0, 2\}$	$\{1, 2\}$
2	2	$\{1, 2\}$	$\{0, 1\}$

Example 4. *With the above description, polygroup* $2[R]$ *will be as follows:*

$\circ$	0	a	1	2
0	0	a	1	2
a	a	0	1	2
1	1	1	$\{0, a, 2\}$	$\{1, 2\}$
2	2	2	$\{1, 2\}$	$\{0, a, 1\}$

Hyperoperation $\circ : 2[R] \times 2[R] \longmapsto P^*(2[R])$ *is not continuous with the following topologies:*

$T_1 = \{\varnothing, 2[R], \{0\}\}$,
$T_2 = \{\varnothing, 2[R], \{a\}\}$,
$T_3 = \{\varnothing, 2[R], \{1\}\}$,
$T_4 = \{\varnothing, 2[R], \{2\}\}$,
$T_5 = \{\varnothing, 2[R], \{0, 1\}\}$,
$T_6 = \{\varnothing, 2[R], \{0, 2\}\}$,

$T_7 = \{\varnothing, 2[R], \{a, 1\}\}$,
$T_8 = \{\varnothing, 2[R], \{a, 2\}\}$,
$T_9 = \{\varnothing, 2[R], \{1, 2\}\}$,
$T_{10} = \{\varnothing, 2[R], \{0, a, 1\}\}$,
$T_{11} = \{\varnothing, 2[R], \{0, a, 2\}\}$,
$T_{12} = \{\varnothing, 2[R], \{a, 1, 2\}\}$,
$T_{13} = \{\varnothing, 2[R], \{0, 1, 2\}\}$.
But $\circ : 2[R] \times 2[R] \longmapsto P^*(2[R])$ *is continuous with*

$$T_{14} = \{\varnothing, 2[R], \{0, a\}\}, T_{15} = \{\varnothing, 2[R], \{0\}, \{a\}\}$$

This means that $(2[R], T_{dis})$, $(2[R], T_{ndis})$, $(2[R], T_{14})$ *and* $(2[R], T_{15})$ *are semi-topological polygroups. Subpolygroups of* $2[R]$ *are* $\varnothing, 2[R], \{0\}, \{0, a\}$. *Let* A *be a arbitrary set and* $a_1, a_2, a_3 \in A$ *and define a soft set* $\mathbb{F}$ *by*

$$\mathbb{F}(x) = \begin{cases} \{0\} & \textit{if } x = a_1 \\ \{0, a\} & \textit{if } x = a_2 \\ 2[R] & \textit{if } x = a_3 \\ \varnothing & \textit{otherwise.} \end{cases}$$

In conclusion $(\mathbb{F}, A, T_{14})$ *and* $(\mathbb{F}, A, T_{15})$ *are soft semi-topological polygroups [18].*

EXAMPLE 5. *Polygroup* $3[R]$ *will be as follows:*

$\circ$	0	a	1	2
0	0	a	1	2
a	a	$\{0, a\}$	1	2
1	1	1	$\{0, a, 2\}$	$\{1, 2\}$
2	2	2	$\{1, 2\}$	$\{0, a, 1\}$

Hyperoperation $\circ : 3[R] \times 3[R] \longmapsto P^*(3[R])$ *is not continuous with the following topologies:*
$T_1 = \{\varnothing, 3[R], \{a\}\}$,
$T_2 = \{\varnothing, 3[R], \{1\}\}$,
$T_3 = \{\varnothing, 3[R], \{2\}\}$,
$T_4 = \{\varnothing, 3[R], \{0, 1\}\}$,
$T_5 = \{\varnothing, 3[R], \{0, 2\}\}$,
$T_6 = \{\varnothing, 3[R], \{a, 1\}\}$,
$T_7 = \{\varnothing, 3[R], \{a, 2\}\}$,
$T_8 = \{\varnothing, 3[R], \{1, 2\}\}$,
$T_9 = \{\varnothing, 3[R], \{0, a, 1\}\}$,
$T_{10} = \{\varnothing, 3[R], \{0, a, 2\}\}$,
$T_{11} = \{\varnothing, 3[R], \{a, 1, 2\}\}$.
Nevertheless hyperoperation $\circ : 3[R] \times 3[R] \longmapsto P^*(3[R])$ *is continuous with*
$T_{12} = \{\varnothing, 3[R], \{0\}\}$,
$T_{13} = \{\varnothing, 3[R], \{0, a\}\}$,
$T_{14} = \{\varnothing, 3[R], \{0\}, \{a\}\}$.

Therefore, $(3[R], (T_i)_{i=12,13,14})$ *are semi-topological polygroups. Subpolygroups of* $3[R]$ *are* $\varnothing, 3[R], \{0\}, \{0, a\}$. *Let* A *be* $3[R]$ *and define a soft set* $\mathbb{F}$ *by*

$$\mathbb{F}(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ \{0, a\} & \text{if } x = a \\ 3[R] & \text{if } x = 1 \\ \varnothing & \text{if } x = 2. \end{cases}$$

Then, $(\mathbb{F}, A, (T_i)_{i=12,13,14}))$ *is a soft semi-topological polygroup. Now, let* A *be arbitrary set and* $a_1, a_2 \in A$ *and define a soft set* $\mathbb{F}$ *by*

$$\mathbb{F}(x) = \begin{cases} \varnothing & \text{if } x = a_1 \\ \{0, a\} & \text{if } x = a_2 \\ \{0\} & \text{otherwise.} \end{cases}$$

In this case $(\mathbb{F}, A, (T_i)_{i=3,4,5,8,9,10})$ *are soft semi-topological polygroups.*

Theorem 4. *[22] Let* $(\mathbb{F}, A)$ *be a soft polygroup over* P *and* (P, T) *be a semi-topological polygroup. then* $(\mathbb{F}, A, T)$ *is a soft semi-topological polygroup over* P.

Theorem 5. *[22] Let* $(\mathbb{F}, A, T)$ *and* $(\mathbb{G}, B, T)$ *be soft semi-topological polygroups over* P. *Then* $(\mathbb{F}, A, T)\widehat{\cap}(\mathbb{G}, B, T)$ *and* $(\mathbb{F}, A, T)\cap_E(\mathbb{G}, B, T)$ *are soft semi-topological polygroup over* P.

Theorem 6. *[22] If* $(\mathbb{F}_i, A_i, T)$ *be a nonempty family of soft semi-topological polygroups, then* $\widehat{\cap}_{i\in I}(\mathbb{F}_i, A_i, T)$ *is a soft semi-topological polygroup over* P.

Theorem 7. *[22] Let* $(\mathbb{F}, A, T)$ *and* $(\mathbb{G}, B, T)$ *be soft semi-topological polygroups over* P. *Then* $(\mathbb{F}, A, T)\widehat{\wedge}(\mathbb{G}, B, T)$ *and* $(\mathbb{F}, A, T)\widehat{\cup}(\mathbb{G}, B, T)$ *are soft semi-topological polygroup.*

Theorem 8. *[22] Let* $(\mathbb{F}_i, A_i, T)$ *be a nonempty family of soft semi-topological polygroups over* P. *Then* $\widehat{\wedge}_{i\in I}(\mathbb{F}_i, A_i, T)$ *and* $\widehat{\cup}_{i\in I}(\mathbb{F}_i, A_i, T)$ *are soft semi-topological polygroup.*

Definition 3. *Let* $(\mathbb{F}, A, T)$ *be a soft semi-topological polygroup over* P *. Then* $(\mathbb{G}, B, T)$ *is called a soft semi-topological subpolygroup (resp. normal subpolygroup) of* $(\mathbb{F}, A, T)$ *if the following items hold:*

(a) *B subset of A and $\mathbb{G}(b)$ is a subpolygroup (resp. normal subpolygroup) of $\mathbb{F}(b)$ for every $b \in supp(\mathbb{G}, B)$.*
(b) *the mapping $(x, y) \longmapsto x \circ y$ of the topological space $\mathbb{G}(b) \times \mathbb{G}(b)$ onto $P^*(\mathbb{G}(b))$ is continuous for every $b \in supp(\mathbb{G}, B)$.*

Theorem 9. *Let* $(\mathbb{F}, A, T)$ *be a soft semi-topological polygroup over P, and* $(\mathbb{G}_i, B_i, T)_{i\in I}$ *be a non-empty family of (normal) soft semi-topological subpolygroups of* $(\mathbb{F}, A, T)$. *Then*

(1) *If $\cap_{i\in I} B_i \neq \varnothing$, then $\widehat{\cap}_{i\in I}(\mathbb{G}_i, B_i, T)$ is a (normal) soft subpolygroup of $(\mathbb{F}, A, T)$.*

(2) *If* $B_i \cap B_j = \varnothing$ *for all* $i, j \in I$ *and* $i \neq j$*, then* $(\cap_E)_{i\in I}(\mathbb{G}_i, B_i, T)$ *is a (normal) soft subpolygroup of* $(\mathbb{F}, A, T)$*.*
(3) *If* $B_i \cap B_j = \varnothing$ *for all* $i, j \in I$ *and* $i \neq j$*, then* $\widehat{\cup}_{i\in I}(\mathbb{G}_i, B_i, T)$ *is a (normal) soft subpolygroup of* $(\mathbb{F}, A, T)$*.*
(4) *The* $\widehat{\wedge}_{i\in I}(\mathbb{G}_i, B_i, T)$ *is a (normal) soft subpolygroup of the soft polygroup* $\widehat{\wedge}_{i\in I}(\mathbb{F}, A, T)$*.*

Proof.
(1) Suppose that $C = \cap_{i\in I}(B_i)$ and $\mathbb{H}(c) = \cap_{i\in I}(\mathbb{G}_i(c))$ Furthermore $C \subseteq A$ and $\mathbb{H}(c)$ is a (normal) soft subpolygroup of A and the mapping in Definition 3 (b) is continuous on $\mathbb{H}(c)$.
(2) Give $C = \cup_{i\in I}(B_i), \mathbb{H}(c) = \mathbb{G}_i(c)$ where $c \in B_i$ and $\mathbb{H}(c)$ is a (normal) soft subpolygroup of $F(c)$ and the mapping in Definition 3 (b) is continuous on $\mathbb{H}(c)$.
(3) Take $C = \cup_{i\in I} B_i, \mathbb{H}(c) = \mathbb{G}_i(c)$, where $c \in B_i$ thus $B_i \subseteq A$ notably $\cup_{i\in I}(B_i) \subseteq A$ in conclusion $\mathbb{H}(c) = \mathbb{G}_i(c)$ is a (normal) soft subpolygroup of $\mathbb{F}(c)$ and the mapping in Definition 3 (b) is continuous on $\mathbb{H}(c)$.
(4) Select $C = \times_{i\in I}(B_i), \mathbb{H}((c_i)_{i\in I}) = \cap_{i\in I}\mathbb{G}_i((c_i)_{i\in I})$ and $\mathbb{G}_i(c_i)$ is a (normal) soft subpolygroup of $\times_{i\in I}\mathbb{F}(c_i)$ in conclusion the mapping in Definition 3 (b) is continuous on $\mathbb{H}((c_i)_{i\in I})$. □

Definition 4. *Let* $(\mathbb{F}, A, T)$ *and* $(\mathbb{G}, B, \xi)$ *be the soft semi-topological polygroups over* P_1 *and* P_2*, where* T *and* ξ *are topologies are defined over* P_1 *and* P_2 *respectively. Let* $f : P_1 \longmapsto P_2$ *and* $g : A \longmapsto B$ *be two mappings. Then the pair* (f, g) *is called a soft semi-topological polygroup homomorphism if the following condition true:*

(a) f *be strong epimorphism and* g *be surjection.*
(b) $f(\mathbb{F}(a)) = \mathbb{G}(g(a))$*.*
(c) $f_a : (\mathbb{F}(a), T_{\mathbb{F}(a)}) \longmapsto (\mathbb{G}(g(a)), \xi_{\mathbb{G}(g(a))})$ *is continuous.*

Then $(\mathbb{F}, A, T)$ is said to be soft semi-topologically homomorphic to $(\mathbb{G}, B, \xi)$ and denoted by$(\mathbb{F}, A, T) \sim (\mathbb{G}, B, \xi)$. If f is a polygroup isomorphism, g is bijective and f_a is continuous as well as open, then the pair (f, g) is called a soft semi-topological polygroup isomorphism. In this case $(\mathbb{F}, A, T)$ is soft topologically isomorphic to $(\mathbb{G}, B, \xi)$, which is denoted by $(\mathbb{F}, A, T) \simeq (\mathbb{G}, B, \xi)$.

Theorem 10. *If* $(\mathbb{F}, A, T) \sim (\mathbb{G}, B, \xi)$ *and* $(\mathbb{F}, A, T)$ *is a normal soft polygroup over* P*, then* $(\mathbb{G}, B, \xi)$ *is a normal soft polygroup over* Q*, where* $(\mathbb{F}, A, T)$ *and* $(\mathbb{G}, B, \xi)$ *be soft semi-topological polygroups over* P *and* Q*.*

Proof. Let (f, g) be a soft semi-topological homomorphism from $(\mathbb{F}, A)$ to $(\mathbb{G}, B)$. For all $x \in supp(\mathbb{F}, A)$, $\mathbb{F}(x)$ is a normal subpolygroup of P; then $f(\mathbb{F}(x))$ is a normal subpolygroup of Q. For all $y \in supp(\mathbb{G}, B)$, there exists $x \in supp(\mathbb{F}, A)$ with the property that $g(x) = y$. In conclusion $\mathbb{G}(y) = \mathbb{G}(g(x)) = f(\mathbb{F}(x))$ is a normal subpolygroup of Q. Thus $(\mathbb{G}, B)$ is a normal soft polygroup on Q.

□

Theorem 11. *Let N be a normal subpolygroup of P, and $(\mathbb{F}, A, T)$ be a soft semi-topological polygroup over P. Then $(\mathbb{F}, A, T) \sim (\mathbb{G}, A, T)$, where $\mathbb{G}(x) = \mathbb{F}(x)/N$ for all $x \in A$, and $N \subseteq \mathbb{F}(x)$ for all $x \in supp(\mathbb{F}, A)$.*

Proof. Firstly $supp(\mathbb{G}, A) = supp(\mathbb{F}, A)$ and we know that P/N is a factor polygroup. Since for every $x \in supp(\mathbb{F}, A), \mathbb{F}(x)$ is a subpolygroup of P and $N \subseteq \mathbb{F}(x)$, it follows that $\mathbb{F}(x)/N$ is also a factor polygroup, which is a subpolygroup of P/N. Thus $(\mathbb{G}, A)$ is a soft polygroup over P/N. Therefore $f : P \longmapsto P/N, f(a) = aN$. Clearly, f is a strong epimorphism. In other words $g : A \longmapsto A, g(x) = x$. Then g is a surjective mapping. For all $x \in supp(\mathbb{F}, A), f(\mathbb{F}(x)) = \mathbb{F}(x)/N = \mathbb{G}(x) = \mathbb{G}(g(x))$. For all $x \in A - supp(\mathbb{F}, A)$, notably $f(\mathbb{F}(x)) = \varnothing = \mathbb{G}(g(x))$. Therefore, (f, g) is a soft semi-topological homomorphism, and $(\mathbb{F}, A, T) \sim (\mathbb{G}, B, \xi)$.

□

Definition 5. *Closure of $(\mathbb{F}, A, T)$ denoted by $(\overline{\mathbb{F}}, A, T)$ and is defined by $\overline{\mathbb{F}}(a) = \overline{\mathbb{F}(a)}$ where $\overline{\mathbb{F}(a)}$ is the closure of $\mathbb{F}(a)$ in topology on P.*

Theorem 12. *[9] Let P be a semi-topological polygroup with the property that every open subset of P is a complete part. Then:*

(1) *If K is a subhypergroup of P, then as well as $\overline{K}$.*
(2) *If K is a subpolygroup of P, then as well as $\overline{K}$.*

Theorem 13. *Let $(\mathbb{F}, A, T)$ be a soft semi-topological polygroup over a semi-topological polygroup (P, T) and every open subset of P is a complete part Then:*

(1) *$(\overline{\mathbb{F}}, A, T)$ is also a soft semi-topological polygroup over (P, T).*
(2) *$(\mathbb{F}, A, T) \widehat{\subset} (\overline{\mathbb{F}}, A, T)$.*

Proof. (1) By Theorem 12 $\overline{\mathbb{F}(a)}$ is subpolygroup P and since (P, T) is a semi-topological polygroup, it follows that condition (b) of Definition 2 holds on $\overline{\mathbb{F}(a)}$.
(2) It is clear.

□

Definition 6. *Let $(\mathbb{F}, A), (\mathbb{G}, B)$ be soft sets over polygroup $< P, e, \circ, -1 >$ define $(\mathbb{F}, A) \widehat{\circ} (\mathbb{G}, B) = (H, C)$ where $C = A \cup B$ for all $a \in C$, and*

$$H(a) = \begin{cases} \mathbb{F}(a) & \text{if } a \in A - B \\ \mathbb{G}(a) & \text{if } a \in B - A \\ \mathbb{F}(a) \circ \mathbb{G}(a) & \text{if } a \in A \cap B \end{cases}$$

Theorem 14. *[9] Let A and B be subsets of polygroup P with the property that every open subset of P is a complete part. Then:*

(1) $\overline{A} \circ \overline{B} \subseteq \overline{A \circ B}$.
(2) $(\overline{A})^{-1} = \overline{(A^{-1})}$.

Theorem 15. *[9] In every topological space (X, T) if $A, B \subseteq X$ we have:*

(1) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.
(2) $\overline{A} \cap \overline{B} = \overline{A \cap B}$.

Theorem 16. *Let $(\mathbb{F}, A, T)$, $(\mathbb{F}, B, T)$ be soft semi-topological polygroups over a semi-topological polygroup (P, T) and every open subset of P is a complete part Then:*

(1) $(\overline{\mathbb{F}}, A, T)\widehat{\cup}(\overline{\mathbb{G}}, B, T) = \overline{(\mathbb{F}, A, T)\widehat{\cup}(\mathbb{G}, B, T)}$.
(2) $(\overline{\mathbb{F}}, A, T)\widehat{\cap}(\overline{\mathbb{G}}, B, T) = \overline{(\mathbb{F}, A, T)\widehat{\cap}(\mathbb{G}, B, T)}$.
(3) $(\overline{\mathbb{F}}, A, T)\widehat{\wedge}(\overline{\mathbb{G}}, B, T) = \overline{(\mathbb{F}, A, T)\widehat{\wedge}(\mathbb{G}, B, T)}$.
(4) $(\overline{\mathbb{F}}, A, T)\widehat{\circ}(\overline{\mathbb{G}}, B, T)\widehat{\subseteq}\overline{(\mathbb{F}, A, T)\widehat{\circ}(\mathbb{G}, B, T)}$.
(5) $(\overline{F}, A, T) \cap_E (\overline{\mathbb{G}}, B, T) = \overline{(\mathbb{F}, A, T) \cap_E (\mathbb{G}, B, T)}$.

Proof. (1) Let a be element of $A - B$. then $(\overline{\mathbb{F}}, A, T)\widehat{\cup}(\overline{\mathbb{G}}, B, T)(a) = (\overline{\mathbb{F}}, A, T)(a) = \overline{\mathbb{F}(a)}$ In conclusion, $\overline{(\mathbb{F}, A, T)\widehat{\cup}(\mathbb{G}, B, T)}(a) = \overline{\mathbb{F}}(a) = \overline{\mathbb{F}(a)}$.

Let a be element of $B - A$. Then $(\overline{\mathbb{F}}, A, T)\widehat{\cup}(\overline{\mathbb{G}}, B, T)(a) = (\overline{\mathbb{G}}, B, T)(a) = \overline{\mathbb{G}(a)}$ In conclusion, $\overline{(\mathbb{F}, A, T)\widehat{\cup}(\mathbb{G}, B, T)}(a) = \overline{G}(a) = \overline{\mathbb{G}(a)}$.

Let a be element of $A \cap B$. Then $(\overline{\mathbb{F}}, A, T)\widehat{\cup}(\overline{\mathbb{G}}, B, T)(a) = \overline{\mathbb{F}(a)} \cup \overline{\mathbb{G}(a)}$ In conclusion, $\overline{(\mathbb{F}, A, T)\widehat{\cup}(\mathbb{G}, B, T)}(a) = \overline{\mathbb{F}(a) \cup \mathbb{G}(a)}$. By Theorem 15 proof is complete.

(4) Let a be element of $A - B$. Then $(\overline{\mathbb{F}}, A, T)\widehat{\circ}(\overline{\mathbb{G}}, B, T)(a) = (\overline{\mathbb{F}}, A, T)(a) = \overline{\mathbb{F}(a)}$ In conclusion, $\overline{(\mathbb{F}, A, T)\widehat{\circ}(\mathbb{G}, B, T)}(a) = \overline{\mathbb{F}}(a) = \overline{\mathbb{F}(a)}$.

Let a be element of $B - A$. Then $(\overline{\mathbb{F}}, A, T)\widehat{\circ}(\overline{\mathbb{G}}, B, T)(a) = (\overline{\mathbb{G}}, B, T)(a) = \overline{\mathbb{G}(a)}$ In conclusion, $\overline{(\mathbb{F}, A, T)\widehat{\circ}(\mathbb{G}, B, T)}(a) = \overline{\mathbb{G}}(a) = \overline{\mathbb{G}(a)}$.

Let a be element of $A \cap B$. Then $(\overline{\mathbb{F}}, A, T)\widehat{\circ}(\overline{\mathbb{G}}, B, T)(a) = \overline{\mathbb{F}(a)} \circ \overline{\mathbb{G}(a)}$ In conclusion, $\overline{(\mathbb{F}, A, T)\widehat{\circ}(\mathbb{G}, B, T)}(a) = \overline{\mathbb{F}(a) \circ \mathbb{G}(a)}$. By Theorem 14 proof is complete.

Other items are similar (1) or (4). □

The second definition of soft semi-topological polygroups is as follows, and this definition is based on soft topologies and soft continuity. The results of this definition follow. To distinguish the latter Definition from the previous one, we use distinct symbols.

A family θ of soft sets over U is called a soft topology on U if the following axioms hold:

(1) $\widehat{\varnothing}$ and $\widehat{U}$ are in θ,
(2) θ is closed under finite soft intersection,
(3) θ is closed under (arbitrary) soft union.

We will use the symbol (U, θ, E) to denote a soft topological space and soft set $\mathbb{F}$ is called a soft close set if $\mathbb{F}^{\widehat{c}}$ is soft open set, where each member of θ said to be a soft open set [4, 26].

EXAMPLE 6. *Let U be $\mathbb{Z}_2$ and θ be $\{\widehat{\varnothing}, \{e_2\} \times \mathbb{Z}_2, \widehat{\mathbb{Z}_2}\}$, where $E = \{e_1, e_2\}$ and $\{e_2\} \times \mathbb{Z}_2$ be soft set $\mathbb{F} : E \longmapsto P(\mathbb{Z}_2)$ with the property that $\mathbb{F}(e_1) = \varnothing; \mathbb{F}(e_2) = \mathbb{Z}_2$. Then $(\mathbb{Z}_2, \theta, E)$ is soft topological space.*

EXAMPLE 7. *Let P be $\{e, a, b, c\}$ and hyperoparation $\circ$ be as follow:*

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

polygroup P with topologies $\theta_1 = \{\widehat{\varnothing}, \{e_1\} \times P, \widehat{P}\}$, $\theta_2 = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$ are soft topological spaces.

Closure of $\mathbb{F}$ denoted by $\widehat{Cl}(\mathbb{F})$ and define soft intersection of all soft closed supersets of $\mathbb{F}$, where $\mathbb{F}$ be soft set over U.

A soft set $\mathbb{F}$ said to be a soft neighborhood of x if there exists a soft open set $\mathbb{G}$ with the property that $x \widehat{\in} \mathbb{G} \widehat{\subseteq} \mathbb{F}$, where x be an element of the universe U. The soft neighborhood system of x we will consider the collection of all soft neighborhoods of x.

Let V be a subset of the universe U. A soft set $\mathbb{F}$ said to be a soft neighborhood of V if there exists a soft open set $\mathbb{G}$ with the property that $V \widehat{\subseteq} \mathbb{G} \widehat{\subseteq} \mathbb{F}$. (i.e $\forall e \in E$: $V \subseteq \mathbb{G}(e) \subseteq \mathbb{F}(e)$).

The collection of all soft neighborhoods of V said to be the soft neighborhood system of V.

Definition 7. *Let P_1, P_2 be polygroups and $(P_1, \theta_1, E), (P_2, \theta_2, E)$ be soft topological spaces. The function $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ said to be a soft continuous function if for all $x \in P_1$ and for all soft neighborhood $\mathbb{F}_{\varphi(x)}$ of $\varphi(x)$, there exists a soft neighborhood $\mathbb{F}_x$ of x with the property that $\varphi(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_{\varphi(x)}$.*

Theorem 17. *The function $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ is soft continuous function if and only if for every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed.*

Proof. This is easily seen to be an equivalence relation. □

Theorem 18. *Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be function in this case, for every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed if and only if for all soft set $\mathbb{F}$, we have $\varphi(\widehat{Cl}(\mathbb{F})) \widehat{\subseteq} \widehat{Cl}(\varphi(\mathbb{F}))$.*

Proof. (i) $\Longleftarrow$ Let $\mathbb{F}'$ be soft closed set . Then we have $\varphi(\varphi^{-1}(\mathbb{F}')) \widehat{\subseteq} \mathbb{F}'$. The soft closeness of $\mathbb{F}'$, together with the assumption (for all soft set $\mathbb{F}$, we have $\varphi(\widehat{Cl}(\mathbb{F})) \widehat{\subseteq} \widehat{Cl}(\varphi(\mathbb{F}))$), proves that

$$\varphi(\widehat{Cl}(\varphi^{-1}(\mathbb{F}')))\widehat{\subseteq}\widehat{Cl}(\varphi(\varphi^{-1}(\mathbb{F}')))\widehat{\subseteq}\mathbb{F}'$$

Therefore, it holds that $\widehat{Cl}(\varphi^{-1}(\mathbb{F}'))\widehat{\subseteq}\varphi^{-1}(\mathbb{F}')\widehat{\subseteq}\widehat{Cl}(\varphi^{-1}(\mathbb{F}'))$, which shows that $\varphi^{-1}(\mathbb{F}')$ is soft closed.

(ii) $\Longrightarrow$ We have $\mathbb{F}\widehat{\subseteq}\varphi^{-1}(\widehat{Cl}(\varphi(\mathbb{F})))$ for any soft set $\mathbb{F}$. Since (for every soft closed set $\mathbb{F}'$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft closed), we have $\widehat{Cl}(\mathbb{F})\widehat{\subseteq}\varphi^{-1}(\widehat{Cl}(\varphi(\mathbb{F})))$. Thus, we have

$$\varphi(\widehat{Cl}(\mathbb{F}))\widehat{\subseteq}\varphi(\varphi^{-1}(\widehat{Cl}(\varphi(\mathbb{F}))))\widehat{=}\widehat{Cl}(\varphi(\mathbb{F}))$$

□

Theorem 19. *Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a function. If for all soft open set $\mathbb{F}' \in \theta_2$, the inverse image $\varphi^{-1}(\mathbb{F}')$ is also soft open set then φ is a soft continuous function.*

Proof. For all $x \in P_1$ and a soft open neighborhood $\mathbb{F}'$ of $\varphi(x), \varphi^{-1}(\mathbb{F}')$ is a soft open set having x as a soft element. Since $\varphi(\varphi^{-1}(\mathbb{F}'))\widehat{\subseteq}\mathbb{F}'$, give $F = \varphi^{-1}(\mathbb{F}')$ in this case $\varphi(\mathbb{F})\widehat{\subseteq}\mathbb{F}'$. □

EXAMPLE 8. *We prove that the opposite Theorem 19 is not true.*

Let P_1 be $< \{u\}, \theta_1, \{e_1, e_2\} >$ and P_2 be $< \{u\}, \theta_2, \{e_1, e_2\} >$, where

$$\theta_1 = \{\widehat{\varnothing}, \{(e_1, u), (e_2, u)\}\}$$
$$\theta_2 = \{\widehat{\varnothing}, \{(e_2, u)\}, \{(e_1, u), (e_2, u)\}\}$$

In soft topologies, $\{e_1, e_2\} \times \{u\}$ is the soft neighborhood of the point u. Thus $id : P_1 \longmapsto P_2$ satisfies in second part Theorem 19. However, $id^{-1}(\{(e_2, u)\})$ is not soft open in P_1, showing that the inverse images of soft open sets are, in general, not soft open. Show that, not only $id : P_1 \longmapsto P_2$ but also $id^{-1} : P_2 \longmapsto P_1$ satisfy in second part Theorem 19.

Definition 8. *A bijection $\varphi : P_1 \longmapsto P_2$ said to be a soft homeomorphism between (P_1, θ_1, E) and (P_2, θ_2, E) if φ and φ^{-1} are soft continuous.*

Theorem 20. *Let $\varphi : (P_1, \theta_1, E) \longmapsto (P_2, \theta_2, E)$ be a soft continuous function and for all soft open set $\mathbb{F}_2 \in \theta_2$, there exists a soft open set $\mathbb{F}_1 \in \theta_1$ with the property that for all $x \in P_1$; $x\widehat{\in}\mathbb{F}_1$ if and only if $x\widehat{\in}\varphi^{-1}(\mathbb{F}_2)$.*

Proof. For every $x \in P_1$ with $\varphi(x)\widehat{\in}\mathbb{F}_2$, choose a soft open $\mathbb{F}_x \in \theta_1$ with the property that $x\widehat{\in}\mathbb{F}_x$ and $\varphi(\mathbb{F}_x)\widehat{\subseteq}\mathbb{F}_2$. Then define $\mathbb{F}_1 = \widehat{\bigcup}\{\mathbb{F}_x | x \in P_1, \varphi(x)\widehat{\in}\mathbb{F}_2\}$ is the desired soft open set. □

Definition 9. *Let $(P, \circ, e,^{-1})$ be a polygroup and θ be a soft topology on P with a parameter set E. then (P, θ, E) is a soft semi-Topological polygroup if the following item true:*

For each soft neighborhood $\mathbb{F}$ *of* $p \circ q$*, where* $(p,q) \in P \times P$ *there exist soft neighborhoods* $\mathbb{F}_p$ *and* $\mathbb{F}_q$ *of* p *and* q *with the property that* $\mathbb{F}_p \circ \mathbb{F}_q \widehat{\subseteq} \mathbb{F}$.

Every soft semi-topological group is soft semi-Topological polygroup.

EXAMPLE 9. *Let* E *be* $\{e_1, e_2\}$ *and* θ *be* $\{\widehat{\varnothing}, \{(e_1, \overline{1})\}, \widehat{\mathbb{Z}_2}\}$*. Conclusion* $(\mathbb{Z}_2, \theta, E)$ *is a soft semi-*T*opological polygroup.*

EXAMPLE 10. *Let* P *be* $\{e, a, b, c\}$ *and hyperoparation* $\circ$ *be as follow:*

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e,a\}$	c	$\{b,c\}$
b	b	c	e	a
c	c	$\{b,c\}$	a	$\{e,a\}$

And E *be* $\{e_1, e_2, e_3\}$*. Then the polygroup* P *with each of the following topologies*

$$\theta_1 = \{\widehat{\varnothing}, \{e_1\} \times P, \widehat{P}\}$$
$$\theta_2 = \{\widehat{\varnothing}, \{e_2\} \times P, \widehat{P}\}$$
$$\theta_3 = \{\widehat{\varnothing}, \{e_3\} \times \{a,b\}, \widehat{P}\}$$
$$\theta_3 = \{\widehat{\varnothing}, \{e_3\} \times \{a,b\}, \{e_1\} \times \{e,b\}, \widehat{P}\}$$
$$\theta_4 = \{\widehat{\varnothing}, \{e_3\} \times \{a,b\}, \{e_1\} \times \{e,b\}, \{e_2\} \times \{e,b,c\}, \widehat{P}\}$$

is a soft semi-T*opological polygroup.*

The family of soft sets Θ, is said to be a soft indiscrete (soft discrete) topology on P if $\Theta = \{\widehat{\varnothing}, \widehat{P}\}(\Theta = SS(P))$, in this case (P, Θ) is called a soft indiscrete space (soft discrete space) over P, where $SS(P)$ is the set of all soft sets over P [26].

EXAMPLE 11. *Every polygroup with soft discrete or indiscrete topology is a soft semi-*T*opological polygroup.*

If we want to merge the previous two Definitions of soft semi-topological polygroups into one Definition, it will be as follows. We will show with an example how the generalized Definition refers to the first Definition and under what conditions the second Definition.

Definition 10. *Let* (P, θ, A) *be a soft topology on* P *and* $(\mathbb{F}, E)$ *be a soft set over* P*, where* $A \neq E$ *are sets of parameters. Then* $(\mathbb{F}, \theta, A, E, \circ)$ *is called a generalized soft semi-topological polygroup over* P *if the following axioms satisfies:*

(1) $\mathbb{F}(e)$ *is a subpolygroup of* P *for all* $e \in E$.
(2) *For all* $e \in E$ *and every soft open neighborhoods* $\mathbb{F}_{p \circ q}$ *of* $p \circ q$ *subset of* $\mathbb{F}(e)$*, there exist an soft open neighborhood* $\mathbb{F}_p$ *of* p *and an soft open neighborhood* $\mathbb{F}_q$ *of* q*, such that* $\mathbb{F}_p \circ \mathbb{F}_q \widehat{\subseteq} \mathbb{F}_{p \circ q}$*, with the restricted soft topology* θ *to* $\mathbb{F}(e)$ *which is denoted by* $\theta\mid_{\mathbb{F}(e)}$.

The following example proves that the two Definitions soft semi-topological polygroup are a special case of Definition 10.

Example 12. *Let (F, E) be $\widehat{P}$ in this case $(\mathbb{F}, \theta, A, E, \circ)$ is a soft semi-Topological polygroup via Definition 9 and if A be a single member set then $(\mathbb{F}, \theta, A, E, \circ)$ is a soft semi-topological polygroup via Definition 2. It should be noted that in case that set A contains a parameter, the soft topology becomes a normal topology.*

Example 13. *Let $P = (\mathbb{Z}_4, +)$, $\theta = \{\widehat{\varnothing}, \widehat{\mathbb{Z}_4}, \{(a_1, \{\widehat{0}, \widehat{2}\}), (a_2, \varnothing)\}, \{(a_1, \{\widehat{1}, \widehat{3}\}), (a_2, \mathbb{Z}_4)\}\}$, where $A = \{a_1, a_2\}$ and $E = \{e_1, e_2\}$, $(E, F) = \{(e_1, \{\widehat{0}, \widehat{2}\}), (e_2, \mathbb{Z}_4)\}$. In this case we have $\theta\mid_{\mathbb{F}(e_1)} = \{\widehat{\varnothing}, \{\widehat{0}, \widehat{2}\}, \{(a_1, \{\widehat{0}, \widehat{2}\}), (a_2, \varnothing)\}, \{(a_1, \varnothing), (a_2, \{\widehat{0}, \widehat{2}\})\}\}$, and $\theta\mid_{\mathbb{F}(e_2)} = \theta$.*

With above condition $(\mathbb{F}, \theta, A, E, +)$ is a generalized soft semi-topological polygroup over P.

Definition 11. *Let $(\mathbb{F}, \theta, A, E, \circ)$ be a generalized soft semi-topological polygroup over P and $\mathbb{G}$ be a soft subset of $\mathbb{F}$. Then $(\mathbb{G}, \theta, A, E, \circ)$ sub-gstp(sub-generalized soft semi-topological polygroup) of $(\mathbb{F}, \theta, A, E, \circ)$ if $(\mathbb{G}, \theta, A, E, \circ)$ also is a generalized soft semi-topological polygroup over P.*

Example 14. *Let $(\mathbb{F}, \theta, A, E, +)$ be in Example 13, in conclusion $(\mathbb{F}, \theta, A, E, +)$ $(\mathbb{F}, \theta, \{a_1\}, E, +)$, $(\mathbb{Z}_4, \theta, A, E, +)$ are sub-gstp of $(\mathbb{F}, \theta, A, E, +)$.*

Definition 12. *Let $(P, \circ, e_n, ^{-1})$ and $(Q, \star, e'_n, ^{-1})$ be polygruops if $P^* \subseteq P$, $Q \subseteq Q$ with the property that $(\widehat{P^*}, \theta, A, E, \circ)$,$(\widehat{Q}, \theta, A, E, \star)$ are generalized soft semi-topological polygroup over P^* and Q then $F = (f_1, f_2)$ said to be a morphism if the following conditions are true:*

(i) *$f_1 : (P, \theta, A) \longmapsto (Q, \theta, A)$ is soft continuous.*
(ii) *$f_2 : (P, \circ) \longmapsto (Q, \star)$ is a polygroup homomorphism.*

Theorem 21. *The image of a generalized soft semi-topological polygroup under a morphism, is also a generalized soft semi-topological polygroup.*

Proof. Let $(P, \circ, e_n, ^{-1})$, $(Q, \star, e'_n, ^{-1})$ be polygruops, $P^* \subseteq P$, $Q \subseteq Q$ with the property that $(\widehat{P^*}, \theta, A, E, \circ)$,$(\widehat{Q}, \theta, A, E, \star)$ are generalized soft semi-topological polygroup over P^* and Q and $F = (f_1, f_2)$be a morphism. since for every $e \in E$, $f_2(F(e))$ is subpolygroup of Q as f_2 is a polygroup homomorphism, it follows that $F((\widehat{P^*}, \theta, A, E, \circ))$ is a generalized soft semi-topological polygroup. Furthemore the composition of two continuous functions is continuous, this proves the second and third conditions. □

Definition 13. *Let $(\mathbb{F}, \theta, A, E, \circ)$ be a generalized soft semi-topological polygroup over P. The $(\mathbb{F}, \theta, A, E, \circ)$ is called $T_i$generalized soft semi-topological polygroup if (P, θ, A) is a soft $T_i$space.*

Theorem 22. *[11]Let $(\mathbb{F}, \theta, A, E, \circ)$ be a generalized soft semi-topological polygroup over P. the following items are equivalents:*

(i) $(\mathbb{F}, \theta, A, E, \circ)$ T_0*generalized soft semi-topological polygroup.*
(ii) $(\mathbb{F}, \theta, A, E, \circ)$ T_1*generalized soft semi-topological polygroup.*
(iii) $(\mathbb{F}, \theta, A, E, \circ)$ T_2*generalized soft semi-topological polygroup.*

Let P,Q,R are polygroups and hyperoperation of polygroups is "$\circ$" and $SS(P)$ is all soft sets are defined on the set of parameters E. Note that in a polygroup, the combination of two members will be a set.

Definition 14. *[13] Consider $\mathbb{F}_A \in SS(P)$, $\mathbb{G}_B \in SS(Q)$ and $\psi : P \longmapsto Q$, $\varphi : A \longmapsto B$ be two mappings. The (φ, ψ) is a soft mapping from $\mathbb{F}_A$ to $\mathbb{G}_B$ denoted by $(\varphi, \psi) : \mathbb{F}_A \longmapsto \mathbb{G}_B$ if and only if*

$$\psi(\mathbb{F}_A(a)) = \mathbb{G}_B(\varphi(a)), \forall a \in A.$$

We consider that all soft sets are defined on the set of parameters E and all soft mappings are defined with respect to the identity on E. Note that if $(id_E, f) : \mathbb{F} \longmapsto \mathbb{G}$ is a soft mapping we write f instead of (id_E, f).

Definition 15. *The cartesian product of $\mathbb{F}_A$ and $\mathbb{G}_B$ is shown with soft set $(\mathbb{F}_A \widehat{\times} \mathbb{G}_B) \in SS(P \times Q)$, such that $(\mathbb{F}_A \widehat{\times} \mathbb{G}_B)(a, b) = \mathbb{F}_A(a) \times \mathbb{G}_B(b), \forall (a, b) \in A \times B$, where $\mathbb{F}_A \in SS(P)$ and $\mathbb{G}_B \in SS(Q)$ [3].*

Throughout this section, we will deal with soft topological spaces defined over a soft set $\mathbb{F} \in SS(P)$. Thus, we will recall the following Definition for soft topology [26].

Definition 16. *Consider $\mathbb{F} \in SS(P)$ and Θ be a family of soft subsets of $\mathbb{F}$ and*

(i) $\widehat{\varnothing}, \mathbb{F} \in \Theta$;
(ii) Θ *is closed under finite intersection;*
(iii) Θ *is closed under arbitrary union.*

We say that Θ is a soft topology on $\mathbb{F}$ and $(\mathbb{F}, \Theta)$ is called the soft topological space (in short STS) and $V \in SS(P)$ is called a soft open set if $V \in \Theta$ [4].

Example 15. *Assume that $E = \mathbb{R}^+$ (the set of all positive real numbers), where $\mathbb{R}$ be the set of all real numbers. Let $\varepsilon \in E$ and $\mathbb{F}_\varepsilon \in SS(\mathbb{R})$ such that $\mathbb{F}_\varepsilon(e) = (e-\varepsilon, e+\varepsilon)$, for all $e \in E$. Consider $\Theta = \{\mathbb{F}_\varepsilon \mid \varepsilon \in E\}$. Then $(\mathbb{R}, \Theta)$ is a soft semi-topological space [2].*

Definition 17. *Assume that (P, Θ) and (Q, Λ) are soft topological spaces and f be mapping $f : P \longmapsto Q$ then*

(1) *If f satisfies in the condition $\mathbb{F} \in \Theta \Longrightarrow f(\mathbb{F}) \in \Lambda$, then f is said to be soft open;*
(2) *f is said to be soft continuous, if and only if for any $x \in P$ and any soft open neighborhoods $\mathbb{F}_{f(x)}$ of $f(x)$, there exist an soft open neighborhood $\mathbb{F}_x$ of x such that $f(x) \widehat{\in} f(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_{f(x)}$;*

(3) *If f is bijective and f , f^{-1} are soft continuous, then f is said to be soft homeomorphism;*
(4) *Assume that $\mathbb{F} \in SS(P)$ and $\mathbb{G} \in SS(Q)$, then the mapping $f : \mathbb{F} \longmapsto \mathbb{G}$ is said to be soft continuous, if and only if for any $x \widehat{\in} \mathbb{F}$ and any soft open neighborhoods $\mathbb{F}_{f(x)}$ of $f(x)$, there exist an soft open neighborhood $\mathbb{F}_x$ of x such that$f(x) \widehat{\in} f(\mathbb{F}_x) \widehat{\subseteq} \mathbb{F}_{f(x)}$ [11].*

In the above Definition, $f(x)$ may be a set. In particular, when f is hyperoperation of polygroup.

Definition 18. *Assume that (P, Θ) and (Q, Λ) be soft topological spaces. We can make soft product topological space $(P \times Q, \Theta \widehat{\times} \Lambda)$, where the collection of all unions of soft sets in $\{\mathbb{F} \widehat{\times} \mathbb{G} \mid \mathbb{F} \in \Theta, \mathbb{G} \in \Lambda\}$ is a soft topology on $P \times Q$ and it is said to be soft product topology on $P \times Q$ and denoted by $(\Theta \widehat{\times} \Lambda)$ [19].*

Theorem 23. *Assume that (P, Θ) and (Q, Λ) is soft topological spaces. Then $proj_p : (P \times Q, \Theta \widehat{\times} \Lambda) \longmapsto (P, \Theta)$ and $proj_q : (P \times Q, \Theta \widehat{\times} \Lambda) \longmapsto (Q, \Lambda)$ are soft continuous and soft open too the smallest soft topology on $P \times Q$ for which $proj_p, proj_q$ be soft continuous is $\Theta \widehat{\times} \Lambda$ [19].*

Theorem 24. *The mapping $f : (R, \phi) \longmapsto (P \times Q, \Theta \widehat{\times} \Lambda)$ is soft continuous, if and only if the mappings $(proj_q \circ f)$ and $(proj_p \circ f)$ are soft continuous, where $(P, \Theta), (Q, \Lambda)$ and (R, ϕ) are soft topological spaces [19].*

Theorem 25. *Assume that $f : P \longmapsto Q$ and $g : Q \longmapsto R$ be soft continuous. Then the mapping $g \circ f$ is soft continuous, where (P, Θ) , (Q, Λ) and (R, ϕ) be soft topological spaces [19].*

Definition 19. *The set β is a base for a soft topological space (P, Θ) if we can make every soft open set in Θ as a union of elements of β [26].*

Definition 20. *Suppose that Q is subset of P and (P, Θ) is a soft topological space. Then the set $\Theta_{\widehat{Q}} = \{\widehat{Q} \widehat{\cap} \mathbb{F} \mid \mathbb{F} \in \Theta\}$ is said to be the soft relative topology on Q, and $(Q, \Theta_{\widehat{Q}})$ is a soft subspace of (P, Θ) [26].*

Theorem 26. *Assume that (P, Θ) is a soft topological space and $\mathbb{F} \in SS(P)$. Then the collection $\Theta_{\mathbb{F}} = \{\mathbb{F} \widehat{\cap} \mathbb{G} \mid \mathbb{G} \in \Theta\}$ is a soft topology over $\mathbb{F}$.*

Proof. The first, Θ is closed under the finite intersection and arbitrary union for all soft sets over P that is indeed $\Theta_{\mathbb{F}}$ is closed under the finite intersection and arbitrary union since the elements of $\Theta_{\mathbb{F}}$ are soft sets over P.

The second, since $\Theta_{\mathbb{F}} = \{\mathbb{F} \widehat{\cap} \mathbb{G} \mid \mathbb{G} \in \Theta\}$ and $\mathbb{F} \widehat{\cap} \mathbb{G} \widehat{\subseteq} \mathbb{F}$, it follows that element soft $\Theta_{\mathbb{F}}$ are soft subsets of $\mathbb{F}$. Moreover, since (P, Θ) be a soft topological space over P, then $\widehat{P}, \widehat{\varnothing} \in \Theta$. So, $\mathbb{F} = \mathbb{F} \widehat{\cap} \widehat{P} \in \Theta_{\mathbb{F}}$ and $\widehat{\varnothing} = \mathbb{F} \widehat{\cap} \widehat{\varnothing} \in \Theta_{\mathbb{F}}$. □

$(\mathbb{F}, \Theta_{\mathbb{F}})$ is referred to as a soft subspace of (P, Θ), where $\Theta_{\mathbb{F}}$ is said to be the soft relative topology on $\mathbb{F}$.

Theorem 27. *The union of two STS is not necessary a STS. However, the intersection of two STS is a STS [21].*

Definition 21. *Assume that Θ is a soft topology on P and $\mathbb{F} \in SS(P)$ is a soft polygroup, then the soft topological space $(\mathbb{F}, \Theta)$ is said to be soft semi-topological soft polygroup over P (in short SSTSP) if the soft mappings $f : (a, b) \longmapsto a \circ b$ from $(\mathbb{F}\widehat{\times}\mathbb{F}, \Theta\widehat{\times}\Theta)$ to $(\mathbb{F}, \Theta_{\mathbb{F}})$ is soft continuous.*

Definition 22. *The sum of $\mathbb{F}$ and $\mathbb{G}$ is the soft set $\mathbb{F}\widehat{\circ}\mathbb{G} \in SS(P)$, such that $(\mathbb{F}\widehat{\circ}\mathbb{G})(e) = \mathbb{F}(e) \circ \mathbb{G}(e)$, for all $e \in E$, where that $\mathbb{F}, \mathbb{G} \in SS(P)$ are soft polygroups.*

The following theorem presents an equivalent definition for SSTSP.

Theorem 28. *Suppose that $\mathbb{F}$ is a soft polygroup over P where Θ is a soft topology on P. Then $(\mathbb{F}, \Theta)$ is an SSTSP over P if and only if the following condition be true:*

For all $a, b\widehat{\in}\mathbb{F}$ and every soft open neighborhoods $\mathbb{F}_{a\circ b}$ of $a \circ b$, there exist an soft open neighborhood $\mathbb{F}_a$ of a and an soft open neighborhood $\mathbb{F}_b$ of b, such that $\mathbb{F}_a\widehat{\circ}\mathbb{F}_b\widehat{\subseteq}\mathbb{F}_{a\circ b}$.

Proof. $[\Rightarrow]$ The first assume that $(\mathbb{F}, \Theta)$ is an SSTSP. Then $f : (a, b) \longmapsto a \circ b$ from $(\mathbb{F}\widehat{\times}\mathbb{F}, \Theta\widehat{\times}\Theta)$ to $(\mathbb{F}, \Theta_{\mathbb{F}})$, is soft continuous. Suppose that $a, b\widehat{\in}\mathbb{F}$, and $\mathbb{F}_{a\circ b}$ of an arbitrary soft open neighborhood of $f(a, b) = a \circ b$. Then by soft-continuity in Definition 17, for every $(a, b) \in \mathbb{F}\widehat{\times}\mathbb{F}$ and every soft open neighborhoods $\mathbb{F}_{f(a,b)}$ of $f(a, b)$, there is an soft open neighborhood $\mathbb{F}_{(a,b)}$ of (a, b) such that $a \circ b\widehat{\in}f(\mathbb{F}_{(a,b)})\widehat{\subseteq}\mathbb{F}_{f(a,b)}$.

Now $\mathbb{F}_{(a,b)}$ is a soft open set in $\Theta\widehat{\times}\Theta$, which means there exist $\{\mathbb{F}_{a_i}, \mathbb{F}_{b_i} \in \Theta, i \in I\}$ such that $\mathbb{F}_{(a,b)} = \bigcup_{i\in I} \mathbb{F}_{a_i}\widehat{\times}\mathbb{F}_{b_i}$. That shows there exist $i \in I$ such that $a\widehat{\in}\mathbb{F}_{a_i}$ and $b\widehat{\in}\mathbb{F}_{b_i}$. So, $\mathbb{F}_{a_i}\widehat{\times}\mathbb{F}_{b_i} \in \Theta\widehat{\times}\Theta$ and $\mathbb{F}_{a_i}\widehat{\times}\mathbb{F}_{b_i}\widehat{\subseteq}\mathbb{F}_{(a,b)}$ and

$$\mathbb{F}_{a_i}\widehat{\circ}\mathbb{F}_{b_i} = f(\mathbb{F}_{a_i}\widehat{\times}\mathbb{F}_{b_i})\widehat{\subseteq}f(\mathbb{F}_{(a,b)})\widehat{\subseteq}\mathbb{F}_{f(a,b)}.$$

$[\Leftarrow]$ For all $a, b\widehat{\in}\mathbb{F}$ and every soft open neighborhoods $\mathbb{F}_{a\circ b}$ of $a \circ b$, there exist an soft open neighborhood $\mathbb{F}_a$ of a and an soft open neighborhood $\mathbb{F}_b$ of b, such that $\mathbb{F}_a\widehat{\circ}\mathbb{F}_b\widehat{\subseteq}\mathbb{F}_{a\circ b}$.

However, $\mathbb{F}_a\widehat{\circ}\mathbb{F}_b = f(\mathbb{F}_a\widehat{\times}\mathbb{F}_b)$, since $a\widehat{\in}\mathbb{F}_a$ and $b\widehat{\in}\mathbb{F}_b$ and they are soft open neighborhoods in Θ, then $\mathbb{F}_a\widehat{\times}\mathbb{F}_b$ is an soft open neighborhood in $\Theta\widehat{\times}\Theta$ contains (a, b). Therefore, by Definition of soft continuity 17, the mapping f is soft continuous. □

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References

[1] Aktas, H., Cagman, N., Soft sets and soft groups, *Information Sciences*, 77(13) (2007), 2726-2735. https://doi.org/10.1016/j.ins.2006.12.008

[2] Aygunoglu, A., Aygun, H., Some notes on soft topological spaces, *Neural Comput. & Applic.*, 21(1) (2012), 113-119. https://doi.org/10.1007/s00521-011-0722-3

[3] Babitha, K.V., Sunil, J.J., Soft set relations and functions, *Comput. Math. Appl.*, 60(7) (2010), 1840-1849. https://doi.org/10.1016/j.camwa.2010.07.014

[4] Cagman, N., Karatas, S., Enginoglu, S., Soft topology, *Comput. Math. Appl.*, 62(1) (2011), 351-358. https://doi.org/10.1016/j.camwa.2011.05.016

[5] Corsini, P., Prolegomena of Hypergroup Theory, Aviani Editore, Tricesimo, Italy, 1993.

[6] Comer, S.D., Extension of polygroups by polygroups and their representations using color schemes, Universal algebra and lattice theory (Puebla, 1982), 91–103, *Lecture Notes in Math.*, 1004, Springer, Berlin, 1983. https://doi.org/10.1007/BFb0063431

[7] Davvaz, B., Polygroup Theory and Related Systems, World Scientific Publishing Co. Pte.Ltd., Hackensack, NJ, 2013. https://doi.org/10.1142/8593

[8] Feng, F., Li, Y.M., Soft subsets and soft product operations, *Information Sciences*, 232 (2013), 44-57. https://doi.org/10.1016/j.ins.2013.01.001

[9] Heidari, D., Davvaz, B., Modarres, S.M.S., Topological polygroups, *Bull. Malays. Math. Sci. Soc.*, 39 (2016), 707-721. https://doi.org/10.1007/s40840-015-0136-y

[10] Hewitt, E., Ross, K.A., Abstract Harmonic Analysis, vol. 1, Springer Verlag, Berlin, 1963.

[11] Hida, T., Soft topological group, *Annals Fuzzy Mathematics and Informatics*, 8(6) (2014), 1001-1025.

[12] Hoskova-Mayerova, S., Topological hypergroupoids, *Comput. Math. Appl.*, 64(9) (2012), 2845-2849. https://doi.org/10.1016/j.camwa.2012.04.017

[13] Kharal, A., Ahmad, B., Mappings on soft classes, *New Math Nat. Comput.*, 7(3) (2011), 471-481. https://doi.org/10.1142/S1793005711002025

[14] Koskas, M., Groupoides, demi-hypergroupes et hypergroupes, *J. Math. Pures Appl.*, 49(9) (1970), 155-192.

[15] Maji, P.K., Biswas, R., Roy, A.R., Soft set theory, *Comput. Math. Appl.* 45(4-5) (2003), 555-562. https://doi.org/10.1016/S0898-1221(03)00016-6

[16] Marty, F., *Sur une généralization de la notion de groupe*, 8^{iem}, Congress Math. Scandinaves, Stockholm, (1934) 45-49.

[17] Molodtsov, D., Soft set Theory-First Results, *Comput. Math. Appl.*, 37 (1999), 19-31. https://doi.org/10.1016/S0898-1221(99)00056-5

[18] Mousarezaei, R., Davvaz, B., On soft topological polygroups and their examples, *International Journal of Fuzzy Logic and Intelligent Systems*, 21(1) (2021), 29-37. http://doi.org/10.5391/IJFIS.2021.21.1.29

[19] Nazmul, S., Samanta, S.K., Group soft topology, J. Fuzzy Math., 22(2) (2014), 435-450.

[20] Nazmul, S., Samanta, S.K., Neighbourhood properties of soft topological spaces, *Annals Fuzzy Mathematics and Informatics*, 6(1) (2013), 1-15.

[21] Nazmul, S., Samanta, S.K., Soft topological soft groups, *Mathematical Sciences*, 6 (2012) 1-10. https://doi.org/10.1186/2251-7456-6-66

[22] Oguz, G., A new view on topological polygroups, *Turkish Journal of Science*, 5(2) (2020), 110-117.

[23] Oguz, G., Idealistic soft topological hyperrings, *Scientific Inquiry and Review*, 4(3) (2020), 61-70.

[24] Oguz, G., On soft topological hypergroups, *Journal of Hyperstructures*, 9(2) (2021), 81-95.

[25] Oguz, G., Davvaz, B., Soft topological hyperstructure, *Journal of Intelligent and Fuzzy Systems*, 40(5) (2021), 8755-8764. https://doi.org/10.3233/JIFS-200242

[26] Shabir, M., Naz, M., On soft topological spaces, *Comput. Math. Appl.*, 61(7) (2011), 1786-1799. https://doi.org/10.1016/j.camwa.2011.02.006

[27] Shah, T., Shaheen, S., Soft topological groups and rings, *Annals Fuzzy Mathematics and Informatics*, 7(5) (2014), 725-743.

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A DIFFERENT APPROACH TO BOUNDEDNESS OF THE $B-$MAXIMAL OPERATORS ON THE VARIABLE LEBESGUE SPACES

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Abstract. By using the $L_{p(\cdot)}-$boundedness of a maximal operator defined on homogeneous space, it has been shown that the $B-$maximal operator is bounded. In the present paper, we aim to bring a different approach to the boundedness of the $B-$maximal operator generated by generalized translation operator under a continuity assumption on $p(\cdot)$. It is noteworthy to mention that our assumption is weaker than uniform Hölder continuity.

1. Introduction

Nowadays, there is a big attention on the singular integral operator and maximal operators which are defined on variable Lebesgue spaces. The problem that such operators are bounded under which conditions is well-studied and it is the main topic of harmonic analysis. $L_{p(\cdot)}-$boundedness of the Hardy-Littlewood maximal operator and singular integral operators have been investigated in [1–5].

This study is dealing with the boundedness of maximal operator generated by the Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad 1 \leq k \leq n,$$

which has big importance in harmonic analysis. In [8], Guliyev has obtained the $L_{p,\gamma}-$boundedness of the $B-$maximal operator. Moreover, in [6, 12], it has been shown that the $B-$maximal operator is $L_{p(\cdot),\gamma}-$bounded by using the $L_{p(\cdot)}-$ boundedness of a maximal operator whose domain is a homogeneous space.

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In this study, we obtain that the $B-$maximal operator is bounded on the variable Lebesgue spaces. Here, there are some difficulties while studying the theory of variable Lebesgue spaces. One of them, the generalized translation operator is in general not continuous on the spaces $L_{p(\cdot),\gamma}$. Particularly, if $p(\cdot)$ is not constant, then the generalized translation operator T^y is not continuous on the variable Lebesgue spaces. But, it is still possible to overcome these difficulties by taking some regularity conditions on this exponent function. In [7], it has been obtained that the generalized translation operator on the spaces $L_{p(\cdot),\gamma}$ is bounded. The construction of the article is as follows: The first section is devoted to introduction. In the second section, we recall some basic concepts, notations and some known results which we need throughout the paper. In the third section, we present that the $B-$maximal operator on the spaces $L_{p(\cdot),\gamma}$ is bounded under suitable assumptions by a different approach.

2. Preliminaries

Now, we pause to collect some basic concepts, notations and known results which are beneficial for us.

Let $x = (x', x'')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_k > 0, 1 \le k \le n\}$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, and $S_+ = \{x \in \mathbb{R}^n_{k,+} : |x| = 1\}$. Denote by $B_+(x, r)$ the open ball of radius r centered at x, namely,
$B_+(x, r) = \{y \in \mathbb{R}^n_{k,+} : |x - y| < r\}$. Let $B_+(0, r) \subset \mathbb{R}^n_{k,+}$ be a measurable set, then

$$|B_+(0,r)|_\gamma = \int_{B_+(0,r)} (x')^\gamma dx = \omega(n,k,\gamma) r^{n+|\gamma|},$$

where $\omega(n,k,\gamma) = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$.

We will now introduce the spaces $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and recall the basic properties of it. Let $\mathcal{P}(\mathbb{R}^n_{k,+})$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty]$. The elements of $\mathcal{P}(\mathbb{R}^n_{k,+})$ are called variable exponent functions and also let

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n_{k,+}} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} p(x).$$

Given $p(\cdot)$, the conjugate exponent function is as follows:

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}^n_{k,+}.$$

The analog of log-Hölder continuity for variable Lebesgue spaces related to the Laplace-Bessel differential operator is defined by the following.

Definition 1. *Given a function $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$, $p(\cdot)$ is called log-Hölder continuous on $\mathbb{R}^n_{k,+}$, if there exist constants $C_0, C_\infty > 0$ and p_∞ such that for all $|x - y| \le \frac{1}{2}$, and $x, y \in \mathbb{R}^n_{k,+}$,*

$$|p(x) - p(y)| \le \frac{C_0}{-\log|x-y|}, \tag{1}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}, \tag{2}$$

where $p_\infty = \lim_{x\to\infty} p(x) > 1$. If (1) and (2) hold for $p(\cdot)$, then it is denoted by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$, and $p(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n_{k,+})$, respectively.

Lemma 1. *[7] Let $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty)$ be continuous. The followings are equivalent:*

(i) *$p(\cdot)$ is uniformly continuous with $|p(x) - p(y)| \le \dfrac{C_0}{\ln|x-y|^{-1}}$ for all $0 < |x - y| \le \frac{1}{2}$.*

(ii) *$|B_+|_\gamma^{p_- - p_+} \le C_1$ holds for all open balls B_+.*

The space $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is known as the set of measurable functions f such that for a variable exponent $p(\cdot) : \mathbb{R}^n_{k,+} \to [1, \infty]$,

$$\|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \inf\left\{\lambda > 0 :\ \rho_{p(\cdot),\gamma}(f/\lambda) \le 1\right\} < \infty,$$

where

$$\rho_{p(\cdot),\gamma} := \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)} (x')^\gamma dx.$$

Note that the variable Lebesgue space $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is a Banach space for $1 < p_- \le p(x) \le p_+ < \infty$.

The definition of the generalized translation operator is as follows:

$$T^y f(x) := C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''\right] d\gamma(\alpha),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma(\frac{\gamma_i+1}{2})[\Gamma(\frac{\gamma_i}{2})]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos\alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $1 \le k \le n$, and $d\gamma(\alpha) = \prod_{i=1}^k \sin^{\gamma_i - 1}\alpha_i\, d\alpha_i$ [13, 14]. Notice that the generalized translation operator is related to the Laplace-Bessel differential operator.

The definition of the $B-$convolution operator is as follows:

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma dy.$$

Given a function $f \in L^{\text{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, then the maximal operator associated with the Laplace-Bessel differential operator ($B-$maximal operator) (see [8]) is as follows:

$$M_\gamma f(x) = \sup_{r>0} |B_+(0,r)|^{-1}_\gamma \int_{B_+(0,r)} T^y |f(x)| (y')^\gamma dy.$$

Let $B_+ \in \mathbb{R}^n_{k,+}$ be an arbitrary ball and $f \in L^{\text{loc}}_{1,\gamma}(\mathbb{R}^n_{k,+})$, then define

$$M_{\gamma,B_+} f := |B_+(0,r)|^{-1}_\gamma \int_{B_+} T^y |f(x)| (y')^\gamma dy.$$

By taking supremum over all balls centered at x, one can easily observe that

$$M_\gamma f := \sup_{B_+(x)} M_{\gamma,B_+(x)} f.$$

As mentioned earlier, the variable Lebesgue spaces $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ have some undesired properties about the generalized translation operator. In order to overcome this problem, it is necessary to give some smoothness conditions on $p(\cdot)$. The following theorem states the necessary condition for the boundedness of generalized translation operator.

Theorem 1. *[7] Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$ with $1 < p_- \le p_+ < \infty$. Then for all $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+}) \cap \mathcal{S}'_+(\mathbb{R}^n_{k,+})$ with* supp $F_B f \subset \{\xi \in \mathbb{R}^n_{k,+} : |\xi| \le 2^{v+1}\}$, $v \in \mathbb{N}_0$,

$$\|T^y f(x)\|_{p(\cdot),\gamma} \le c \exp\left((2 + 2^{vn}|y|) c_{\log}(p)\right) \|f\|_{p(\cdot),\gamma},$$

holds, where $c > 0$ is independent of v.

3. Main Results

This section is devoted to our main results. First of all we obtain some lemmas which we need to prove that the $B-$maximal operator is bounded on variable Lebesgue spaces.

Lemma 2. *Let $p(\cdot) \in \mathbb{R}^n_{k,+}$ be as in Lemma 1. Then there exists a positive constant $C(p,\gamma) > 0$ such that*

$$(M_\gamma f(x))^{\frac{p(x)}{p_-}} \le C(p,\gamma) \left(M_\gamma (|f|^{\frac{p(\cdot)}{p_-}})(x) + 1 \right), \quad \textit{for all} \quad x \in \mathbb{R}^n_{k,+},$$

holds for all $\|f\|_{p(\cdot),\gamma} \le 1$.

Proof. Define $q(\cdot) := \dfrac{p(\cdot)}{p_-}$, then $q(\cdot)$ is also as in Lemma 1. Let $\|f\|_{p(\cdot),\gamma} \le 1$, then $\rho_{p(\cdot),\gamma}(f) \le 1$. By Theorem 1, for $r \ge \dfrac{1}{2}$, we get

$$(M_\gamma f)^{q(x)} = \left(|B_+|^{-1}_\gamma \int_{B_+} T^y |f(x)| (y')^\gamma dy \right)^{q(x)}$$

$$\leq \left(|B_+|_\gamma^{-1}\int_{B_+}\left(\frac{1}{p(y)}T^y|f(x)|^{p(y)}(y')^\gamma+\frac{1}{p'(y)}(y')^\gamma\right)dy\right)^{q(x)}$$

$$\leq \left(|B_+|_\gamma^{-1}\int_{B_+}\frac{1}{p(y)}T^y|f(x)|^{p(y)}(y')^\gamma dy+|B_+|_\gamma^{-1}\int_{B_+}\frac{1}{p'(y)}(y')^\gamma dy\right)^{q(x)}$$

$$\leq \left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{p(y)}(y')^\gamma dy+|B_+|_\gamma^{-1}\int_{B_+}(y')^\gamma dy\right)^{q(x)}$$

$$\leq \left(|B_+|_\gamma^{-1}\int_{B_+}|f(x)|^{p(y)}(y')^\gamma dy+|B_+|_\gamma^{-1}\int_{B_+}(y')^\gamma dy\right)^{q(x)}$$

$$\leq \left(|B_+|_\gamma^{-1}\int_{B_+}(|f(x)|^{p(y)}+1)(y')^\gamma dy\right)^{q(x)}$$

$$\leq \left(|B_+|_\gamma^{-1}\rho_{p(\cdot),\gamma}(f)+1\right)^{q(x)}$$

$$\leq \left(|B_+(0,\frac{1}{2})|_\gamma^{-1}+1\right)^{q_+}.$$

If $0<r<\dfrac{1}{2}$, then $|B_+|_\gamma\leq(2r)^{n+|\gamma|}<1$, and

$$(M_\gamma f)^{q(x)}=\left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|(y')^\gamma dy\right)^{q(x)}$$

$$\leq\left[\left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{q_-}(y')^\gamma dy\right)^{\frac{1}{q_-}}\left(|B_+|_\gamma^{-1}\int_{B_+}(y')^\gamma dy\right)^{\frac{1}{q'_-}}\right]^{q(x)}$$

$$\leq\left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{q_-}(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}$$

$$\leq\left(|B_+|_\gamma^{-1}\int_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}$$

$$\leq\left(|B_+|_\gamma^{-1}\int_{B_+}(T^y|f(x)|^{q(y)}+1)(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}$$

$$\leq |B_+|_\gamma^{-\frac{q(x)}{q_-}}3^{q_+}\left(\frac{1}{3}\int_{B_+}(T^y|f(x)|^{q(y)}+1)(y')^\gamma dy\right)^{\frac{q(x)}{q_-}}.$$

Since,

$$\frac{1}{3}\int_{B_+}(T^y|f(x)|^{q(y)}+1)(y')^\gamma dy \le \frac{1}{3}\int_{B_+}(T^y|f(x)|^{p(y)}+2)(y')^\gamma dy$$
$$\le \frac{1}{3}\int_{B_+}T^y|f(x)|^{p(y)}(y')^\gamma dy+\frac{2}{3}|B_+|_\gamma<1,$$

and from Lemma 1, we obtain

$$(M_\gamma f)^{q(x)} \le |B_+|_\gamma^{-\frac{q(x)}{q_-}}3^{q_+}\left(\frac{1}{3}\int_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy+\frac{2}{3}|B_+|_\gamma\right)$$
$$\le |B_+|_\gamma^{-\frac{q(x)}{q_-}}|B_+|_\gamma 3^{q_+-1}\left(\int_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy+2\right)$$
$$\le |B_+|_\gamma^{\frac{q_--q_+}{q_-}}3^{q_+-1}\left(\oint_{B_+}T^y|f(x)|^{q(y)}(y')^\gamma dy+2\right)$$
$$\le C_0\,3^{q_+-1}(M_\gamma(|f|^{q(y)})+2).$$

If one takes supremum over all balls B_+, then the proof is completed. □

Lemma 3. *Let $p(\cdot)\in\mathbb{R}^n_{k,+}$ be as in Lemma 1 and be constant outside some ball $B_+(0,r)$. Then there exist a constant $C(p,\gamma)>0$, and $h\in L_{1,\infty,\gamma}(\mathbb{R}^n_{k,+})\cap L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ such that*

$$(M_\gamma f(x))^{\frac{p(x)}{p_-}}\le C(p,\gamma)\,M_\gamma\left(|f|^{\frac{p(\cdot)}{p_-}}\right)(x)+h(x)\quad \textit{for a.a. } x\in\mathbb{R}^n_{k,+},$$

holds for all $\|f\|_{p(\cdot),\gamma}\le 1$.

Proof. Define $q(\cdot):=\dfrac{p(\cdot)}{p_-}$, and $q_\infty:=\dfrac{p_\infty}{p_-}$, then $q(\cdot)$ satisfies the equivalent conditions of Lemma 1. Let $\|f\|_{p(\cdot),\gamma}\le 1$, then $\rho_{p(\cdot),\gamma}(f)\le 1$. Split $f=f_0+f_1$ such that $f_0:=\chi_{B_+}f$, and $f_1:=\chi_{\mathbb{R}^n_{k,+}\backslash B_+}f$. Thus, for all $x\in B_+(0,2r)$,

$$(M_\gamma f(x))^{q(x)}\le C(q,\gamma)\,(M_\gamma(|f|^{q(\cdot)})+1). \tag{3}$$

Now let $x\in\mathbb{R}^n_{k,+}\backslash B_+(0,2r)$. Then $|x|-r\ge\frac{1}{2}|x|$, and $|B_+(x,|x|-r)|_\gamma\ge C\,|x|^{n+|\gamma|}$. Since supp$f_0\subset B_+(x,r)$, and from Theorem 1, we get

$$(M_\gamma f_0(x))^{q(x)}\le\left(\sup_{|x|-r<r}|B_+(x,r)|_\gamma^{-1}\int_{B_+(x,r)}T^y|f_0(x)|(y')^\gamma dy\right)^{q(x)}$$
$$\le\left(|B_+(x,|x|-r)|_\gamma^{-1}\int_{B_+(x,r)}T^y|f(x)|(y')^\gamma dy\right)^{q(x)}$$

$$
\begin{aligned}
&\leq \left(C\,|x|^{-n-|\gamma|} \int_{B_+(x,r)} T^y |f(x)| (y')^\gamma dy \right)^{q(x)} \\
&\leq \left(C\,|x|^{-n-|\gamma|} \int_{B_+(x,r)} |f(x)| (y')^\gamma dy \right)^{q(x)} \\
&\leq \left(C\,|x|^{-n-|\gamma|} \int_{B_+(x,r)} (|f(x)|^{p(y)} + 1)(y')^\gamma dy \right)^{q(x)} \\
&\leq \left(C\,|x|^{-n-|\gamma|} \rho_{p(\cdot),\gamma}(f) \right)^{q(x)} \\
&\leq C(q,\gamma)|x|^{-n-|\gamma|}. \qquad (4)
\end{aligned}
$$

Moreover, for $x \in \mathbb{R}^n_{k,+} \backslash B_+(0,2r)$,

$$
\begin{aligned}
(M_\gamma f_1(x))^{q(x)} &= \left(\fint_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} |T^y f_1(x)| (y')^\gamma dy \right)^{q(x)} \\
&\leq \left(\fint_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} |T^y f_1(x)| (y')^\gamma dy \right)^{q_\infty} \\
&\leq \fint_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} T^y |f_1(x)|^{q_\infty} (y')^\gamma dy \\
&\leq \fint_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} T^y |f_1(x)|^{q(x)} (y')^\gamma dy \\
&\leq M_\gamma(|f|^{q(x)})(x). \qquad (5)
\end{aligned}
$$

By (3), (4) and (5), we obtain

$$
\begin{aligned}
(M_\gamma f(x))^{q(x)} &\leq \chi_{B_+(0,2r)} (M_\gamma f(x))^{q(x)} + \chi_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} (M_\gamma f_0(x) + M_\gamma f_1(x))^{q(x)} \\
&\leq \chi_{B_+(0,2r)} (M_\gamma f(x))^{q(x)} + C(q,\gamma)\chi_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} \left((M_\gamma f_0(x))^{q(x)} + (M_\gamma f_1(x))^{q(x)} \right) \\
&\leq C(q,\gamma)\, M_\gamma(|f|^{q(\cdot)})(x) + \chi_{B_+(0,2r)} C(q,\gamma) \\
&+ \left(\sup_{x \in \mathbb{R}^n_{k,+} \backslash B_+(0,2r)} \fint_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} (y')^\gamma dy \right)^{q(x)} \\
&\leq C(q,\gamma)\, M_\gamma(|f|^{q(\cdot)})(x) + \underbrace{\chi_{B_+(0,2r)} C(q,\gamma) + \chi_{\mathbb{R}^n_{k,+} \backslash B_+(0,2r)} C(q,\gamma)\,|x|^{-n-|\gamma|}}_{=:h},
\end{aligned}
$$

for all $x \in \mathbb{R}^n_{k,+}$. The fact that $h \in L_{1,\infty,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ proves the lemma. □

Now we can present our main theorem.

Theorem 2. *Let $p(\cdot)$ be as in Lemma 3 with $p_- > 1$. Then M_γ is bounded on $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, i.e.*

$$\|M_\gamma f\|_{p(\cdot),\gamma} \le C(p,\gamma)\,\|f\|_{p(\cdot),\gamma}.$$

Proof. Since $M_\gamma(\lambda f) = \|\lambda\| M_\gamma f$, we have $\|M_\gamma f\|_{p(\cdot),\gamma} \le C$, for all $\|f\|_{p(\cdot),\gamma} \le 1$. Since $p_+ < \infty$, it is sufficient to illustrate $\rho_{p(\cdot),\gamma}(M_\gamma f) \le C$ for all $\|f\|_{p(\cdot),\gamma} \le 1$. Let $f \in L_{p(\cdot),\gamma}$ with $\|f\|_{p(\cdot),\gamma} \le 1$. Then $\rho_{p(\cdot),\gamma}(M_\gamma f) \le 1$. Moreover, let $q(\cdot) := p(\cdot)/p_-$. By Lemma 3, there exists $h \in L_{1,\infty,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ such that $(M_\gamma f)^{q(\cdot)} \le C(p,\gamma)\, M_\gamma(|f|^{q(\cdot)}) + h$. Thus,

$$\begin{aligned}
\rho_{p(\cdot),\gamma}(M_\gamma f) &= \int_{\mathbb{R}^n_{k,+}} |M_\gamma f|^{p(x)} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\sup_{B_+} \int_{B_+} T^y |f(x)| (y')^\gamma dy \right)^{p(x)} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\sup_{B_+} \int_{B_+} T^y |f(x)| (y')^\gamma dy \right)^{q(x)p_-} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\left(\sup_{B_+} \int_{B_+} T^y |f(x)| (y')^\gamma dy \right)^{q(x)} \right)^{p_-} (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(|M_\gamma f|^{q(x)} \right)^{p_-} (x')^\gamma dx \\
&= \left\| (M_\gamma f)^{q(x)} \right\|_{p_-,\gamma}^{p_-} \\
&\le \left(C(p,\gamma) \left\| M_\gamma(|f|^{q(x)}) \right\|_{p_-,\gamma} + \|h\|_{p_-,\gamma} \right)^{p_-}
\end{aligned}$$

holds and since $p_- > 1$, one can see that the $B-$maximal operator $M_\gamma f$ is continuous on $L_{p_-,\gamma}(\mathbb{R}^n_{k,+})$. Therefore, we obtain that

$$\begin{aligned}
\rho_{p(\cdot),\gamma}(M_\gamma f) &\le \left(C(p,\gamma)\, \|M_\gamma(|f|^{q(x)})\|_{p_-,\gamma} + \|h\|_{p_-,\gamma} \right)^{p_-} \\
&= \left(C(p,\gamma)\, \rho_{p(\cdot),\gamma}(f)^{\frac{1}{p_-}} + \|h\|_{p_-,\gamma} \right)^{p_-} \le C(p,\gamma),
\end{aligned}$$

and this completes the proof. □

4. Concluding Remarks

The Hardy-Littlewood maximal operators, singular integral operators, rough integral operator, its commutators and their boundedness on the various function spaces are crucial topics of Harmonic Analysis. In this study, we have shown that the $B-$maximal operator on the variable Lebesgue spaces is bounded under suitable assumptions by a different approach. The boundedness of this operator plays a significant role in order to obtain the boundedness of the singular integral operator, fractional integral operator and its commutators. The fractional versions of these operators have recently become an active area of research (see [9–11, 15, 16]). As a future direction of this study, one might extend to the case that the Laplace-Bessel differential operators with coefficient such as $a(x)$ that could be continuous or Vanishing Mean Oscillation functions.

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References

[1] Adamowicz, T., Harjulehto, P., Hästö, P., Maximal operator in variable exponent Lebesgue spaces on unbounded quasimetric measure spaces, *Math. Scand.*, 116(1) (2015), 5–22. https://doi.org/10.7146/math.scand.a-20448

[2] Cruz-Uribe, D., Fiorenza, A., Neugebauer, C., The maximal function on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.*, 28 (2003), 223–238.

[3] Cruz-Uribe, D., Fiorenza, A., Martell, J., Pérez, C., The boundedness of classical operators on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.*, 31 (2006), 239–264.

[4] Diening, L., Maximal function on Orlicz-Musielak spaces and generalized Lebesgue space, *Bull. Sci. Math.*, 129(8) (2005), 657–700. https://doi.org/10.1016/j.bulsci.2003.10.003

[5] Diening, L., Maximal function on generalized Lebesgue spaces $L^{p(x)}$, *Mathematical Ineq. & App.*, 7(2) (2004), 245–253. dx.doi.org/10.7153/mia-07-27

[6] Ekincioglu, I., Guliyev, V. S., Kaya, E., B_n-maximal operator and B_n-singular integral operators on variable exponent Lebesgue spaces, *Mathematica Slovaca*, 70(4) (2020), 893–902. https://doi.org/10.1515/ms-2017-0401

[7] Ekincioglu, I., Shishkina, E. L., Kaya, E., On the boundedness of the generalized translation operator on variable exponent Lebesgue spaces, *Acta Applicandae Mathematicae*, 173 (2021), 1–14. https://doi.org/10.1007/s10440-021-00411-8

[8] Guliev, V. S., On maximal function and fractional integral, associated with the Bessel differential operator, *Math. Inequal. Appl.*, 6(2) (2003), 317–330. dx.doi.org/10.7153/mia-06-30

[9] Gürbüz, F., Değişken üstlü Lebesgue uzaylarında kaba çekirdekli kesirli maksimal ve integral operatörlerinin genelleştirilmiş komütatörleri için bazı tahminler için Mühendislik, Matematik ve Doğa Bilimlerinde Yenilikçi Yaklaşımlar, Ekin Yayınevi, Türkiye, 2019, 1–11.

[10] Gürbüz, F., Ding, S., Han, H., Long, P., Characterizations of Rough Fractional-Type Integral Operators on Variable Exponent Vanishing Morrey-Type Spaces, *in Topics in Contemporary Mathematical Analysis and Applications*, CRC Press, 2020. 10.1201/9781003081197-4

[11] Gürbüz, F., Ding, S., Han, H., Long, P., Norm inequalities on variable exponent vanishing Morrey type spaces for the rough singular type integral operators, *Int. J. Nonlinear Sci. Numer. Simul.*, 22(6) (2021), 721–739. https://doi.org/10.1515/ijnsns-2019-0180

[12] Kaya, E., A note on maximal operators related to Laplace-Bessel differential operators on variable exponent Lebesgue spaces, *Open Mathematics*, 19(1) (2021), 306–315. https://doi.org/10.1515/math-2021-0041

[13] Klyuchantsev, M. I., On singular integrals generated by the generalized shift operator I, *Sibirsk. Math. Zh.*, 11(4) (1970), 810–821. http://mi.mathnet.ru/eng/smj/v11/i4/p810

[14] Levitan, B. M., Bessel function expansions in series and Fourier integrals, *Uspekhi Mat. Nauk*, 6:2(42) (1951), 102–143. http://mi.mathnet.ru/eng/umn/v6/i2/p102

[15] Sarfraz, N., Jarad, F., Estimates for a rough fractional integral operator and its commutators on $p-$adic central Morrey spaces, *Fractal Fract.*, 6(2) (2022), 117. https://doi.org/10.3390/fractalfract6020117

[16] Sarfraz, N., Aslam, M., Jarad, F., Boundedness for commutators of rough $p-$adic Hardy operator on $p-$adic central Morrey spaces, *J. Funct. Spaces*, 2021 (2021), 1–5. https://doi.org/10.1155/2021/4886197

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EIGENVALUE PROBLEMS FOR A CLASS OF STURM-LIOUVILLE OPERATORS ON TWO DIFFERENT TIME SCALES

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ABSTRACT. In this study, we consider a boundary value problem generated by the Sturm-Liouville equation with a frozen argument and with non-separated boundary conditions on a time scale. Firstly, we present some solutions and the characteristic function of the problem on an arbitrary bounded time scale. Secondly, we prove some properties of eigenvalues and obtain a formulation for the eigenvalues-number on a finite time scale. Finally, we give an asymptotic formula for eigenvalues of the problem on another special time scale: $\mathbb{T} = [\alpha, \delta_1] \cup [\delta_2, \beta]$.

1. Introduction

A Sturm-Liouville equation with a frozen argument has the form

$$-y''(t) + q(t)y(a) = \lambda y(t),$$

where $q(t)$ is the potential function, a is the frozen argument and λ is the complex spectral parameter. The spectral analysis of boundary value problems generated with this equation is studied in several publications [3], [15], [16], [26], [33] and references therein. This kind problems are related strongly to non-local boundary value problems and appear in various applications [4], [12], [31] and [38].

A Sturm-Liouville equation with a frozen argument on a time scale $\mathbb{T}$ can be given as

$$-y^{\Delta\Delta}(t) + q(t)y(a) = \lambda y^{\sigma}(t), \ t \in \mathbb{T}^{\kappa^2} \tag{1}$$

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where $y^{\Delta\Delta}$ and σ denote the second order Δ-derivative of y and forward jump operator on $\mathbb{T}$, respectively, $q(t)$ is a real-valued continuous function, $a \in \mathbb{T}^{\kappa} := \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$, $y^{\sigma}(t) = y(\sigma(t))$ and $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$.

Spectral properties the classical Sturm-Liouville problem on time scales were given in various publications (see e.g. [1], [2], [5]- [9], [11], [17]- [25], [27]- [30], [34]- [37], [39] and references therein). However, there is no any publication about the Sturm-Liouville equation with a frozen argument on an arbitrary time scale.

In the present paper, we consider a boundary value problem which is generated by equation (1) and the following boundary conditions

$$U(y) := a_{11} y(\alpha) + a_{12} y^{\Delta}(\alpha) + a_{21} y(\beta) + a_{22} y^{\Delta}(\beta) \quad (2)$$

$$V(y) := b_{11} y(\alpha) + b_{12} y^{\Delta}(\alpha) + b_{21} y(\beta) + b_{22} y^{\Delta}(\beta) \quad (3)$$

where $\alpha = \inf \mathbb{T}$, $\beta = \rho(\sup \mathbb{T})$, $\alpha \neq \beta$ and a_{ij}, $b_{ij} \in \mathbb{R}$ for $i, j = 1, 2$. We aim to give some properties of some solutions and eigenvalues of (1)-(3) for two different cases of $\mathbb{T}$

For the basic notation and terminology of time scales theory, we recommend to see [10], [13], [14] and [32].

2. Preliminaries

Let $S(t, \lambda)$ and $C(t, \lambda)$ be the solutions of (1) under the initial conditions

$$S(a, \lambda) = 0,\ S^{\Delta}(a, \lambda) = 1, \quad (4)$$

$$C(a, \lambda) = 1,\ C^{\Delta}(a, \lambda) = 0, \quad (5)$$

respectively. Clearly, $S(t, \lambda)$ and $C(t, \lambda)$ satisfy

$$\begin{aligned} S^{\Delta\Delta}(t, \lambda) + \lambda S^{\sigma}(t, \lambda) &= 0 \\ C^{\Delta\Delta}(t, \lambda) + \lambda C^{\sigma}(t, \lambda) &= q(t), \end{aligned}$$

respectively and so these functions and their Δ-derivatives are entire on λ for each fixed t (see [34]).

Lemma 1. *Let $\varphi(t, \lambda)$ be the solution of (1) under the initial conditions $\varphi(a, \lambda) = \delta_1$, $\varphi^{\Delta}(a, \lambda) = \delta_2$ for given numbers δ_1, δ_2. Then $\varphi(t, \lambda) = \delta_1 C(t, \lambda) + \delta_2 S(t, \lambda)$ is valid on $\mathbb{T}$.*

Proof. It is clear that the function $y(t, \lambda) = \delta_1 C(t, \lambda) + \delta_2 S(t, \lambda)$ is the solution of the initial value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + \lambda y^{\sigma}(t) &= q(t)\delta_1 \\ y(a, \lambda) &= \delta_1 \\ y^{\Delta}(a, \lambda) &= \delta_2. \end{aligned}$$

We obtain by taking into account uniqueness of the solution of an initial value problem that $y(t, \lambda) = \varphi(t, \lambda)$. □

Consider the function

$$\Delta(\lambda) : \det \begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix}. \tag{6}$$

It is obvious $\Delta(\lambda)$ is also entire.

Theorem 1. *The zeros of the function $\Delta(\lambda)$ coincide with the eigenvalues of the problem (1)-(3).*

Proof. Let λ_0 be an eigenvalue and $y(t,\lambda_0) = \delta_1 C(t,\lambda_0) + \delta_2 S(t,\lambda_0)$ is the corresponding eigenfunction, then $y(t,\lambda_0)$ satisfies (2) and (3). Therefore,

$$\begin{aligned} &\delta_1 U(C(t,\lambda_0)) + \delta_2 U(S(t,\lambda_0)) = 0, \\ &\delta_1 V(C(t,\lambda_0)) + \delta_2 V(S(t,\lambda_0)) = 0. \end{aligned}$$

It is obvious that $y(t,\lambda_0) \neq 0$ iff the coefficients-determinant of the above system vanishes, i.e., $\Delta(\lambda_0) = 0$. □

Since $\Delta(\lambda)$ is an entire function, eigenvalues of the problem (1)-(3) are discrete.

3. Eigenvalues of (1)-(3) on a Finite Time Scale

Let $\mathbb{T}$ be a finite time scale such that there are m (or r) many elements which are larger (or smaller) than a in $\mathbb{T}$. Assume $m \geq 1$, $r \geq 0$ and $r + m \geq 2$. It is clear that the number of elements of $\mathbb{T}$ is $n = m + r + 1$. We can write $\mathbb{T}$ as follows

$$\mathbb{T} = \left\{\rho^r(a), \rho^{r-1}(a), ..., \rho^2(a), \rho(a), a, \sigma(a), \sigma^2(a), ..., \sigma^{m-1}(a), \sigma^m(a)\right\},$$

where $\sigma^j = \sigma^{j-1} \circ \sigma$, $\rho^j = \rho^{j-1} \circ \rho$ for $j \geq 2$, $\rho^r(a) = \alpha$ and $\sigma^{m-1}(\alpha) = \beta$.

Lemma 2. *i) If $r \geq 3$ and $m \geq 2$, the following equalities hold for all λ*

$$S(\alpha,\lambda) = (-1)^r \mu^{\rho}(a) \left[\mu^{\rho^2}(a)\mu^{\rho^3}(a)...\mu^{\rho^r}(a)\right]^2 \lambda^{r-1} + O\left(\lambda^{r-2}\right)$$

$$S^{\sigma}(\alpha,\lambda) = (-1)^{r-1} \mu^{\rho}(a) \left[\mu^{\rho^2}(a)\mu^{\rho^3}(a)...\mu^{\rho^{r-1}}(a)\right]^2 \lambda^{r-2} + O\left(\lambda^{r-3}\right)$$

$$S(\beta,\lambda) = S^{\sigma^{m-1}}(a,\lambda) = (-1)^m \left[\mu(a)\mu^{\sigma}(a)...\mu^{\sigma^{m-3}}(a)\right]^2 \lambda^{m-2}\mu^{\sigma^{m-2}}(a) + O\left(\lambda^{m-3}\right)$$

$$S^{\sigma}(\beta,\lambda) = S^{\sigma^m}(a,\lambda) = (-1)^{m+1} \left[\mu(a)\mu^{\sigma}(a)...\mu^{\sigma^{m-2}}(a)\right]^2 \lambda^{m-1}\mu^{\sigma^{m-1}}(a) + O\left(\lambda^{m-2}\right)$$

$$C(\alpha,\lambda) = (-1)^r \left[\mu^{\rho}(a)\mu^{\rho^2}(a)...\mu^{\rho^r}(a)\right]^2 \lambda^r + O\left(\lambda^{r-1}\right)$$

$$C^{\sigma}(\alpha,\lambda) = (-1)^{r-1} \left[\mu^{\rho}(a)\mu^{\rho^2}(a)...\mu^{\rho^{r-1}}(a)\right]^2 \lambda^{r-1} + O\left(\lambda^{r-2}\right)$$

$$C(\beta,\lambda) = C^{\sigma^{m-1}}(a,\lambda) = (-1)^m \mu(a) \left[\mu^{\sigma}(a)\mu^{\sigma^2}(a)...\mu^{\sigma^{m-3}}(a)\right]^2 \mu^{\sigma^{m-2}}(a)\lambda^{m-2} + O\left(\lambda^{m-3}\right)$$

$$C^{\sigma}(\beta,\lambda) = C^{\sigma^m}(a,\lambda) = (-1)^{m+1} \mu(a) \left[\mu^{\sigma}(a)\mu^{\sigma^2}(a)...\mu^{\sigma^{m-2}}(a)\right]^2 \mu^{\sigma^{m-1}}(a)\lambda^{m-1} + O\left(\lambda^{m-2}\right),$$

where $O(\lambda^l)$ *denotes a polynomial whose degree is* l.

ii) If $r \in \{0,1,2\}$ *or* $m \in \{0,1\}$, *degrees of all above functions are vanish.*

Proof. It is clear from $f^\sigma(t) = f(t)+\mu(t)f^\Delta(t)$ that $S^\sigma(a,\lambda) = \mu(a)$ and $C^\sigma(a,\lambda) = 1$. On the other hand, since $S(t,\lambda)$ and $C(t,\lambda)$ satisfy (1) then the following equalities hold for each $t \in \mathbb{T}^\kappa$ and for all λ.

$$\begin{aligned} S^{\sigma^2}(t,\lambda) &= \left(1+\frac{\mu(t)}{\mu^\sigma(t)} - \lambda\mu(t)\mu^\sigma(t)\right)S^\sigma(t,\lambda) \\ &\quad -\frac{\mu^\sigma(t)}{\mu(t)}S(t,\lambda) \end{aligned} \tag{7}$$

$$\begin{aligned} C^{\sigma^2}(t,\lambda) &= \left(-\mu(t)\mu^\sigma(t)\lambda + 1 + \frac{\mu(t)}{\mu^\sigma(t)}\right)C^\sigma(t,\lambda) \\ &\quad -\frac{\mu^\sigma(t)}{\mu(t)}C(t,\lambda) + \mu(t)\mu^\sigma(t)q(t) \end{aligned} \tag{8}$$

It can be calculated from (7) and (8) that

$$\begin{aligned} S^{\sigma^j}(a,\lambda) &= (-1)^{j+1}\left(\mu(a)\mu^\sigma(a)...\mu^{\sigma^{j-2}}(a)\right)^2\mu^{\sigma^{j-1}}(a)\lambda^{j-1} \\ &\quad +O\left(\lambda^{j-2}\right) \end{aligned} \tag{9}$$

$$\begin{aligned} S^{\rho^j}(a,\lambda) &= (-1)^j\mu^\rho(a)\left(\mu^{\rho^2}(a)\mu^{\rho^3}(a)...\mu^{\rho^j}(a)\right)^2\lambda^{j-1} \\ &\quad +O\left(\lambda^{j-2}\right) \end{aligned} \tag{10}$$

$$\begin{aligned} C^{\sigma^k}(a,\lambda) &= (-1)^{k+1}\mu(a)\left(\mu^\sigma(a)\mu^{\sigma^2}(a)...\mu^{\sigma^{k-2}}(a)\right)^2\mu^{\sigma^{k-1}}(a)\lambda^{k-1} \\ &\quad +O\left(\lambda^{k-2}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} C^{\rho^k}(a,\lambda) &= (-1)^k\left(\mu^\rho(a)\mu^{\rho^2}(a)...\mu^{\rho^k}(a)\right)^2\lambda^k \\ &\quad +O\left(\lambda^{k-1}\right) \end{aligned} \tag{12}$$

for $j = 2,3,...m$ and $k = 2,3,...,r$. Using (9)-(12) and taking into account $\alpha = \rho^r(a)$ and $\beta = \sigma^{m-1}(\alpha)$ we have our desired relations. □

Corollary 1. $\deg C(\alpha,\lambda)S^\sigma(\beta,\lambda) = \begin{cases} r+m-1, & r>0 \text{ and } m>1 \\ 1, & \text{the other cases} \end{cases}$.

Lemma 3. *The following equlaties hold for all* $\lambda \in \mathbb{C}$.

$$\begin{aligned} S^\sigma(\alpha,\lambda)C(\alpha,\lambda) - S(\alpha,\lambda)C^\sigma(\alpha,\lambda) &= A\lambda^\delta + O\left(\lambda^{\delta-1}\right) \\ S^\sigma(\beta,\lambda)C(\beta,\lambda) - S(\beta,\lambda)C^\sigma(\beta,\lambda) &= B\lambda^\gamma + O\left(\lambda^{\gamma-1}\right) \end{aligned}$$

where $A=(-1)^{r}\mu(\alpha)\mu^{\rho}(a)\left[\mu^{\rho^{2}}(a)...\mu^{\rho^{r-1}}(a)\right]^{2}\mu^{\rho^{r}}(a)q(\alpha)$,

$B=(-1)^{m-1}\mu(\beta)\left[\mu(a)\mu^{\sigma}(a)...\mu^{\sigma^{m-2}}(a)\right]^{2}q(\rho(\beta))$,

$\delta=\begin{cases} r-2, & r\geq 3\\ 0, & r<3\end{cases}$ *and* $\gamma=\begin{cases} m-2, & m\geq 3\\ 0, & m<3.\end{cases}$

Proof. Consider the function

$$\varphi(t,\lambda):=\frac{1}{\mu(t)}\left[S^{\sigma}(t,\lambda)C(t,\lambda)-S(t,\lambda)C^{\sigma}(t,\lambda)\right] \tag{13}$$

It is clear that

$$\varphi(t,\lambda):=\left[S^{\Delta}(t,\lambda)C(t,\lambda)-S(t,\lambda)C^{\Delta}(t,\lambda)\right]=W\left[C(t,\lambda),S(t,\lambda)\right]$$

and it is the solution of initial value problem

$$\begin{aligned}\varphi^{\Delta}(t) &= -q(t)S^{\sigma}(t,\lambda)\\ \varphi(a) &= 1\end{aligned}$$

Therefore, we can obtain the following relations

$$\varphi^{\sigma}(t,\lambda) = \varphi(t,\lambda)-\mu(t)q(t)S^{\sigma}(t,\lambda), \tag{14}$$

$$\varphi^{\rho}(t,\lambda) = \varphi(t,\lambda)+\mu^{\rho}(t)q(\rho(t))S(t,\lambda). \tag{15}$$

By using (9), (10), (14) and (15), the proof is completed. □

Corollary 2. *i)* $\deg\left(S^{\sigma}(\alpha,\lambda)C(\alpha,\lambda)-S(\alpha,\lambda)C^{\sigma}(\alpha,\lambda)\right)<\deg C(\alpha,\lambda)S^{\sigma}(\beta,\lambda)$,

ii) $\deg\left(S^{\sigma}(\beta,\lambda)C(\beta,\lambda)-S(\beta,\lambda)C^{\sigma}(\beta,\lambda)\right)<\deg C(\alpha,\lambda)S^{\sigma}(\beta,\lambda)$.

The next theorem gives the number of eigenvalues of the problem (1)-(3) on $\mathbb{T}$. Recall $n=m+r+1$ denotes the number of elements of $\mathbb{T}$ and put
$A=\begin{pmatrix} a_{11}\mu(\alpha)-a_{12} & b_{11}\mu(\alpha)-b_{12}\\ a_{22} & b_{22}\end{pmatrix}$.

Theorem 2. *If* $\det A\neq 0$*, the problem (1)-(3) has exactly* $n-2$ *many eigenvalues with multiplications, otherwise the eigenvalues-number of (1)-(3) is least than* $n-2$*.*

Proof. Since $\mathbb{T}$ is finite, $\Delta(\lambda)$ is a polinomial and its degree gives the number eigenvalues of the problem. It can be calculated from (6)-(14) that

$$\begin{aligned}\Delta(\lambda) \quad = \quad & \frac{1}{\mu(\alpha)\,\mu(\beta)} \det \begin{pmatrix} a_{11}\mu(\alpha) - a_{12} & b_{11}\mu(\alpha) - b_{12} \\ a_{22} & b_{22} \end{pmatrix} C(\alpha,\lambda)S^{\sigma}(\beta,\lambda) \\ & + \frac{1}{\mu(\alpha)} \det \begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{pmatrix} \left(S^{\sigma}(\alpha,\lambda)\, C(\alpha,\lambda) - S(\alpha,\lambda)\, C^{\sigma}(\alpha,\lambda)\right) \\ & + \frac{1}{\mu(\beta)} \det \begin{pmatrix} a_{21} & a_{22} \\ b_{21} & b_{22} \end{pmatrix} \left(S^{\sigma}(\beta,\lambda)\, C(\beta,\lambda) - S(\beta,\lambda)\, C^{\sigma}(\beta,\lambda)\right) \\ & + O(\lambda^{n+m-2}).\end{aligned}$$

According to Corollary 1 and Corollary 2, if $\det A \neq 0$,
$\deg \Delta(\lambda) = \deg C(\alpha,\lambda)S^{\sigma}(\beta,\lambda) = m + r - 1 = n - 2.$ □

Corollary 3. *i) The eigenvalues-number of (1)-(3) depends only on the elements-number of $\mathbb{T}$ and the coefficients of the boundary conditions (2) and (3). On the other hand, it does not depend on $q(t)$ and a (neither value nor location of a on $\mathbb{T}$). ii) If $\det A \neq 0$, the eigenvalues-number of (1)-(3) and the elements-number of $\mathbb{T}$ determine uniquely each other.*

Remark 1. *As is known, all eigenvalues of the classical Sturm-Liouville problem with separated boundary conditions on time scales are real and algebraicly simple [2]. However, the Sturm-Liouville problem with the frozen argument may have non-real or non-simple eigenvalues even if it is equipped with separated boundary conditions.*

We end this section with two example problems that have non-real or non-simple eigenvalues.

Example 1. *Consider the following problem on* $\mathbb{T} = \{0, 1, 2, 3, 4, 5\}$.

$$L_1 : \begin{cases} -y^{\Delta\Delta}(t) + q_1(t)y(3) = \lambda y^{\sigma}(t), \;\; t \in \{0, 1, 2, 3\} \\ y^{\Delta}(0) = 0 \\ y^{\Delta}(4) + y(4) = 0, \end{cases}$$

where $q_1(t) = \begin{cases} 0 & t = 0 \\ 1 & t = 1 \\ 0 & t = 2 \\ 2 & t = 3 \end{cases}$. *Eigenvalues of* L_1 *are* $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$, $\lambda_3 = \frac{3}{2} + \frac{1}{2}\sqrt{5}$, $\lambda_4 = \frac{3}{2} - \frac{1}{2}\sqrt{5}$.

Example 2. *Consider the following problem on* $\mathbb{T} = \{0, 1, 2, 3, 4, 5\}$.

$$L_2 : \begin{cases} -y^{\Delta\Delta}(t) + q_2(t)y(3) = \lambda y^{\sigma}(t), \;\; t \in \{0, 1, 2, 3\} \\ y^{\Delta}(0) + 2y(0) = 0 \\ y^{\Delta}(4) + y(4) = 0, \end{cases}$$

where $q_2(t)=\begin{cases} -1 & t=0 \\ 2 & t=1 \\ 0 & t=2 \\ 1 & t=3 \end{cases}$ *. Eigenvalues of* L_2 *are* $\lambda_1=\lambda_2=\lambda_3=2,\ \lambda_4=3.$

4. Eigenvalues of (1)-(3) on the Time Scale $\mathbb{T}=[\alpha,\delta_1]\cup[\delta_2,\beta]$

In this section, we investigate eigenvalues of the problem (1)-(3) on another special time scale: $\mathbb{T}=[\alpha,\delta_1]\cup[\delta_2,\beta]$, where $\alpha<a<\delta_1<\delta_2<\beta$. We assume that $a\in(\alpha,\delta_1)$. The similar results can be obtained in the case when $a\in(\delta_2,\beta)$.

The following relations are valid on $[\alpha,\delta_1]$ (see [15]).

$$\begin{aligned} S(t,\lambda) &= \frac{\sin\sqrt{\lambda}\,(t-a)}{\sqrt{\lambda}} \\ C(t,\lambda) &= \cos\sqrt{\lambda}\,(t-a)+\int_a^t \frac{\sin\sqrt{\lambda}\,(t-\xi)}{\sqrt{\lambda}}q(\xi)d\xi \end{aligned}$$

The following asymptotic relations for the solutions $S(t,\lambda)$ and $C(t,\lambda)$ can be proved by using a method similar to that in [35].

$$S(t,\lambda)=\begin{cases} \dfrac{\sin\sqrt{\lambda}\,(t-a)}{\sqrt{\lambda}}, & t\in[\alpha,\delta_1], \\ \delta^2\sqrt{\lambda}\cos\sqrt{\lambda}\,(\delta_1-a)\sin\sqrt{\lambda}(\delta_2-t)+O\left(\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{16}$$

$$S^{\Delta}(t,\lambda)=\begin{cases} \cos\sqrt{\lambda}\,(t-a), & t\in[\alpha,\delta_1), \\ -\delta^2\lambda\cos\sqrt{\lambda}\,(\delta_1-a)\cos\sqrt{\lambda}(\delta_2-t)+O\left(\sqrt{\lambda}\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{17}$$

$$C(t,\lambda)=\begin{cases} \cos\sqrt{\lambda}\,(t-a)+O\left(\dfrac{1}{\sqrt{\lambda}}\exp|\tau|\,|t-a|\right), & t\in[\alpha,\delta_1], \\ -\delta^2\lambda\sin\sqrt{\lambda}\,(\delta_1-a)\sin\sqrt{\lambda}(\delta_2-t)+O\left(\sqrt{\lambda}\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{18}$$

$$C^{\Delta}(t,\lambda)=\begin{cases} -\sqrt{\lambda}\sin\sqrt{\lambda}\,(t-a)+O\left(\exp|\tau|\,|t-a|\right), & t\in[\alpha,\delta_1), \\ \delta^2\lambda^{3/2}\sin\sqrt{\lambda}\,(\delta_1-a)\cos\sqrt{\lambda}(\delta_2-t)+O\left(\lambda\exp|\tau|\,(t-a-\delta)\right), & t\in[\delta_2,\beta], \end{cases} \tag{19}$$

where $\delta=\delta_2-\delta_1$, $\tau=\mathrm{Im}\sqrt{\lambda}$ and O denotes Landau's symbol.

Lemma 4. *The following equlaties hold for all* $\lambda\in\mathbb{C}$ *and* $t\in\mathbb{T}$.

$$C^{\Delta}(t,\lambda)S\,(t,\lambda)-C(t,\lambda)S^{\Delta}\,(t,\lambda)=O\left(\sqrt{\lambda}\exp|\tau|\,(\beta-\alpha-\delta)\right)$$

Proof. It is clear the function

$$\varphi(t,\lambda) := C^{\Delta}(t,\lambda)S(t,\lambda) - C(t,\lambda)S^{\Delta}(t,\lambda)$$

satisfies initial value problem

$$\begin{aligned} \varphi^{\Delta}(t) &= q(t)\,S^{\sigma}(t,\lambda),\ t\in[\alpha,\delta_1] \\ \varphi(a) &= 1 \end{aligned}$$

and

$$\begin{aligned} \varphi^{\Delta}(t) &= q(t)\,S^{\sigma}(t,\lambda),\ t\in[\delta_2,\beta] \\ \varphi(\delta_2) &= \varphi(\delta_1) + \delta q(\delta_1)S(\delta_2,\lambda). \end{aligned}$$

Hence, we get proof by using (16). □

Theorem 3. *i) The problem (1)-(3) on* $\mathbb{T} = [\alpha,\delta_1]\cup[\delta_2,\beta]$ *has countable many eigenvalues such as* $\{\lambda_n\}_{n\geq 0}$.
ii) The numbers $\{\lambda_n\}_{n\geq 0}$ *are real for sufficiently large* n.
iii) If $a_{22}b_{12} - a_{12}b_{22} \neq 0$ *and* $\beta-\delta_2 = \delta_1-\alpha$*, the following asymptotic formula holds for* $n\to\infty$.

$$\sqrt{\lambda_n} = \frac{(n-1)\pi}{2(\beta-\delta_2)} + O\left(\frac{1}{n}\right) \tag{20}$$

Proof. The proof of (i) is obvious, since $\Delta(\lambda)$ is entire on λ.

By calculating directly, we get

$$\begin{aligned} \Delta(\lambda) &= \det\begin{pmatrix} U(C) & V(C) \\ U(S) & V(S) \end{pmatrix} \\ &= (a_{22}b_{12}-a_{12}b_{22})\left[C^{\Delta}(\beta,\lambda)S^{\Delta}(\alpha,\lambda) - C^{\Delta}(\alpha,\lambda)S^{\Delta}(\beta,\lambda)\right] + \\ &\quad + (a_{22}b_{21}-a_{21}b_{22})\left[C^{\Delta}(\beta,\lambda)S(\beta,\lambda) - C(\beta,\lambda)S^{\Delta}(\beta,\lambda)\right] + \\ &\quad + (a_{12}b_{11}-a_{11}b_{12})\left[C^{\Delta}(\alpha,\lambda)S(\alpha,\lambda) - C(\alpha,\lambda)S^{\Delta}(\alpha,\lambda)\right] \\ &\quad + O\left(\lambda\exp|\tau|(\beta-\alpha-\delta)\right). \end{aligned}$$

It follows from (16)-(19) and Lemma 4 that

$$\begin{aligned} \Delta(\lambda) &= (a_{22}b_{12}-a_{12}b_{22})\delta^2\lambda^{3/2}\sin\sqrt{\lambda}(\delta_1-\alpha)\cos\sqrt{\lambda}(\beta-\delta_2) \\ &\quad + O\left(\lambda\exp|\tau|(\beta-\alpha-\delta)\right) \end{aligned}$$

is valid for $|\lambda|\to\infty$. Thus, we obtain the proof of (ii).

Since $a_{22}b_{12}-a_{12}b_{22}\neq 0$ and $\beta-\delta_2=\delta_1-\alpha$, the numbers $\{\lambda_n\}_{n\geq 0}$ are roots of

$$\lambda^2\frac{\sin 2\sqrt{\lambda}(\beta-\delta_2)}{\sqrt{\lambda}} + O\left(\lambda\exp 2|\tau|(\beta-\delta_2)\right) = 0. \tag{21}$$

Now, we consider the region

$$G_n := \{\lambda\in\mathbb{C} : \lambda=\rho^2, |\rho| < \frac{n\pi}{2(\beta-\delta_2)} + \varepsilon\}$$

where ε is sufficiently small number. There exist some positive constants C_ε such that, $\left|\lambda^2 \frac{\sin 2\sqrt{\lambda}(\beta-\delta_2)}{\sqrt{\lambda}}\right| \geq C_\varepsilon |\lambda|^{3/2} \exp 2|\tau|(\beta-\delta_2)$ for sufficiently large $\lambda \in \partial G_n$. Therefore, by applying Rouche's theorem to (21) on G_n, we can show that (20) holds for sufficiently large n. □

Remark 2. *Since $\mu(\alpha) = 0$ in the considered time scale, the term $a_{22}b_{12} - a_{12}b_{22}$ is not another than $detA$ in section 3.*

5. Conclusion

In this paper, we give some spectral properties of a boundary value problem generated by the Sturm-Liouville equation with a frozen argument and with non-separated boundary conditions on time scales. We focus on two different time scales: a finite set and a union of two discrete closed intervals. On the finite set, we obtain a formulation for some solutions, characteristic function and the eigenvalues-number of the problem. On the other time scale, we give some properties and an asymptotic formula for eigenvalues.

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References

[1] Adalar, İ., Ozkan, A. S., An interior inverse Sturm–Liouville problem on a time scale, *Analysis and Mathematical Physics*, 10(4) (2020), 1-10. https://doi.org/10.1007/s13324-020-00402-2

[2] Agarwal, R. P., Bohner, M., Wong, P. J. Y., Sturm-Liouville eigenvalue problems on time scales, *Appl. Math. Comput.* 99 (1999), 153–166. https://doi.org/10.1016/S0096-3003(98)00004-6

[3] Albeverio S., Hryniv, R. O., Nizhink, L. P., Inverse spectral problems for non-local Sturm-Liouville operators, (1975), 2007-523-535. https://doi.org/10.1088/0266-5611/23/2/005

[4] Albeverio, S., Nizhnik, L., Schrödinger operators with nonlocal point interactions, *J. Math. Anal. Appl.*, 332(2) (2007). https://doi.org/10.1016/j.jmaa.2006.10.070

[5] Allahverdiev, B. P., Tuna, H., Conformable fractional Sturm–Liouville problems on time scales, *Mathematical Methods in the Applied Sciences,* (2021). https://doi.org/10.1002/mma.7925

[6] Allahverdiev, B. P., Tuna, H., Dissipative Dirac operator with general boundary conditions on time scales, *Ukrainian Mathematical Journal,* 72(5) (2020). https://doi.org/10.37863/umzh.v72i5.546

[7] Allahverdiev, B. P., Tuna, H., Investigation of the spectrum of singular Sturm–Liouville operators on unbounded time scales, *São Paulo Journal of Mathematical Sciences,* 14(1) (2020), 327-340. https://doi.org/10.1007/s40863-019-00137-4

[8] Amster, P., De Nápoli, P., Pinasco, J. P., Eigenvalue distribution of second-order dynamic equations on time scales considered as fractals, *J. Math. Anal. Appl.,* 343 (2008), 573–584. https://doi.org/10.1016/j.jmaa.2008.01.070

[9] Amster, P., De Nápoli, P., Pinasco, J. P., Detailed asymptotic of eigenvalues on time scales, *J. Differ. Equ. Appl.,* 15 pp. (2009), 225–231. https://doi.org/10.1080/10236190802040976

[10] Atkinson, F., Discrete and Continuous Boundary Problems, *Academic Press, New York,* 1964. https://doi.org/10.1002/zamm.19660460520

[11] Barilla D., Bohner, B., Heidarkhani, S., Moradi, S., Existence results for dynamic Sturm–Liouville boundary value problems via variational methods, *Applied Mathematics and Computation,* 409, 125614 (2021). https://doi.org/10.1016/j.amc.2020.125614

[12] Berezin, F. A., Faddeev, L. D., Remarks on Schrödinger equation, *Sov. Math.—Dokl.,* 137 (1961), 1011–4.

[13] Bohner, M., Peterson, A., Dynamic Equations on Time Scales, *Birkhäuser, Boston, MA,* 2001.

[14] Bohner, M., Peterson, A., Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003. https://doi.org/10.1007/978-1-4612-0201-1

[15] Bondarenko, N. P., Buterin, S. A., Vasiliev, S.V., An inverse problem for Sturm -Liouville operators with frozen argument, *Journal of Mathematical Analysis and Applications,* 472(1) (2019), 1028-1041. https://doi.org/10.1016/j.jmaa.2018.11.062

[16] Buterin, S., Kuznetsova, M., On the inverse problem for Sturm–Liouville-type operators with frozen argument, rational case, *Comp. Appl. Math.,* 39(5) (2020). https://doi.org/10.1007/s40314-019-0972-8

[17] Davidson, F. A., Rynne, B. P., Global bifurcation on time scales, *J. Math. Anal. Appl.,* 267 (2002), 345–360. https://doi.org/10.1006/jmaa.2001.7780

[18] Davidson, F. A., Rynne, B. P., Self-adjoint boundary value problems on time scales, *Electron. J. Differ. Equ.,* 175 (2007), 1–10.

[19] Davidson, F. A., Rynne, B. P., Eigenfunction expansions in L^2 spaces for boundary value problems on time-scales, *J. Math. Anal. Appl.,* 335 (2007), 1038–1051. https://doi.org/10.1016/j.jmaa.2007.01.100

[20] Erbe, L., Hilger, S., Sturmian theory on measure chains, *Differ. Equ. Dyn. Syst.,* 1 (1993), 223–244.

[21] Erbe, L., Peterson, A., Eigenvalue conditions and positive solutions, *J. Differ. Equ. Appl.,* 6 (2000), 165–191. https://doi.org/10.1080/10236190008808220

[22] Guseinov, G. S., Eigenfunction expansions for a Sturm-Liouville problem on time scales, *Int. J. Differ. Equ.,* 2 (2007), 93–104. https://doi.org/10.37622/000000

[23] Guseinov, G. S., An expansion theorem for a Sturm-Liouville operator on semi-unbounded time scales, *Adv. Dyn. Syst. Appl.,* 3 (2008), 147–160. https://doi.org/10.37622/000000

[24] Heidarkhani, S., Bohner, B., Caristi, G., Ayazi F., A critical point approach for a second-order dynamic Sturm–Liouville boundary value problem with p -Laplacian, *Applied Mathematics and Computation,* 409 (2021), 125521. https://doi.org/10.1016/j.amc.2020.125521

[25] Heidarkhani, S., Moradi, S., Caristi G., Existence results for a dynamic Sturm–Liouville boundary value problem on time scales, *Optimization Letters,* 15 (2021), 2497–2514. https://doi.org/10.1007/s11590-020-01646-4

[26] Hu, Y. T., Bondarenko, N. P., Yang, C. F., Traces and inverse nodal problem for Sturm–Liouville operators with frozen argument, *Applied Mathematics Letters,* 102 (2020), 106096. https://doi.org/10.1016/j.aml.2019.106096

[27] Hilscher, R. S., Zemanek, P., Weyl-Titchmarsh theory for time scale symplectic systems on half line, *Abstr. Appl. Anal.,* 738520, (2011), 41 pp. https://doi.org/10.1155/2011/738520

[28] Huseynov, A., Limit point and limit circle cases for dynamic equations on time scales, *Hacet. J. Math. Stat.,* 39 (2010), 379–392.

[29] Huseynov, A., Bairamov, E., On expansions in eigenfunctions for second order dynamic equations on time scales, *Nonlinear Dyn. Syst. Theo.,* 9 (2009), 7–88.

[30] Kong, Q., Sturm-Liouville problems on time scales with separated boundary conditions, *Results Math.,* 52 (2008), 111–121. https://doi.org/10.1007/s00025-007-0277-x

[31] Krall, A. M., The development of general differential and general differential-boundary systems, *Rocky Mount. J. Math.,* 5 (1975), 493–542.

[32] Lakshmikantham, V., Sivasundaram, S., Kaymakcalan, B., Dynamic Systems on Measure Chains, Kluwer Academic Publishers, Dordrecht, 1996. https://doi.org/10.1007/978-1-4757-2449-3

[33] Nizhink, L. P., Inverse eigenvalue problems for non-local Sturm Liouville problems, *Methods Funct. Anal. Topology,* 15(1) (2009), 41-47.

[34] Ozkan, A. S., Sturm-Liouville operator with parameter-dependent boundary conditions on time scales, *Electron. J. Differential Equations,* 212 (2017), 1-10.

[35] Ozkan, A. S., Adalar, I., Half-inverse Sturm-Liouville problem on a time scale, *Inverse Probl.* 36 (2020), 025015. https://doi.org/10.1088/1361-6420/ab2a21

[36] Rynne, B. P., L2 spaces and boundary value problems on time-scales, *J. Math. Anal. Appl.,* 328 (2007), 1217–1236. https://doi.org/10.1016/j.jmaa.2006.06.008

[37] Sun, S., Bohner, M., Chen, S., Weyl-Titchmarsh theory for Hamiltonian dynamic systems, *Abstr. Appl. Anal. Art.*, 514760 (2010), 18 pp . https://doi.org/10.1155/2010/514760

[38] Wentzell A. D., On boundary conditions for multidimensional diffusion processes, *Theory Probab.,* 4 (1959), 164–77. (Engl. Transl.) https://doi.org/10.1137/1104014

[39] Yurko, V. A., Inverse problems for Sturm-Liouville differential operators on closed sets, *Tamkang Journal of Mathematics,* 50(3) (2019), 199-206. https://doi.org/10.5556/j.tkjm.50.2019.3343

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MULTIVALENT HARMONIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATION

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Abstract. In the present investigation we study a subclass of multivalent harmonic functions involving multiplier transformation. An equivalent convolution class condition and a sufficient coefficient condition for this class is acquired. We also show that this coefficient condition is necessary for functions belonging to its subclass. As an application of coefficient condition, a necessary and sufficient hypergeometric inequality is also given. Further, results on bounds, inclusion relation, extreme points, a convolution property and a result based on the integral operator are obtained.

1. Introduction

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real-valued harmonic in $\mathbb{D}$. In any simply-connected domain $\mathbb{D} \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, where h is called the analytic part and g is called the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [6]). Let H denote a class of harmonic functions $f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

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It should be worthy to note that the family H reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is if $g = 0$.

The concept of multivalent harmonic complex valued functions by using argument principle, was given by Duren et al. [8]. Using this concept, Ahuja and Jahagiri [1], [2] introduced a class $H(m)$ of m-valent harmonic and orientation preserving functions $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are m-valent functions of the form

$$h(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n,\ g(z) = \sum_{n=m}^{\infty} b_n z^n, |b_m| < 1, m \in \mathbb{N} = \{1, 2, 3....\} \quad (1)$$

which are analytic in $\Delta = \{z : |z| < 1\}$. For $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ complex parameters $\alpha_i\,(i = 1, 2, ..., p)$ and $\beta_i(\neq -n, n \in \mathbb{N})$ $(i = 1, 2, .., q)$, the generalized hypergeometric function ${}_pF_q(\alpha_1, ...\alpha_p, \beta_1, ...\beta_q; z) = {}_pF_q\left((\alpha_i)\,;(\beta_i)\,;z\right)$ is defined by

$${}_pF_q\left((\alpha_i)\,;(\beta_i)\,;z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots \left(\beta_q\right)_n n!}\ z^n \quad (p \le q + 1; z \in \Delta) \quad (2)$$

where $(\lambda)_n$ represents the Pochhammer symbol defined, in terms of Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{c} 1, \quad n = 0, \lambda \neq 0 \\ \lambda(\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1)\,, n \in \mathbb{N} \end{array} \right\}$$

The convolution of two analytic functions $\phi(z) = \sum_{n=o}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ defined on Δ is an analytic function given by

$$\phi(z) * \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) * \phi(z)\,.$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, p)$ and $\beta_j \in \mathbb{C} \backslash \{0, -1, 2, \ldots\}$ $(j = 1, 2, \ldots, q)$, Dziok and Srivastava [9] introduced the following operator for an analytic function $h(z)$ of the form (1) is given by

$$\begin{aligned} H_m^{p,q}[\alpha_1]\,h(z) &= z^m\ {}_pF_q(\alpha_1, ...\alpha_p; \beta_1, ...\beta_q; z) * h(z) && (3) \\ &= z^m + \sum_{n=m+1}^{\infty} \theta_n([\alpha_1]\,; p, q) a_n z^n && (4) \end{aligned}$$

where

$$\theta_n([\alpha_1];p,q)=\frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{1}{(n-m)!},n\geq m. \tag{5}$$

Several results on harmonic functions by involving generalised hypergeometric functions and involving certain linear operator have recently been studied in [3–5, 11–16, 18, 19, 22]. Motivated with the operator defined by Srivastava et al. in [20], we define a mutiplier operator $\pounds^{t,[\alpha_1]}_{\lambda,p,q}$ for an analytic function $h(z)$ of the form (1) as follows:

$\pounds^{0,[\alpha_1]}_{\lambda,p,q}h(z)=h(z)$

$\pounds^{1,[\alpha_1]}_{\lambda,p,q}h(z)=\pounds^{\alpha_1}_{\lambda,p,q}h(z)=(1-\lambda)H^{p,q}_m[\alpha_1]h(z)+\frac{\lambda z}{mz'}\left(H^{p,q}_m[\alpha_1]h(z)\right)',(\lambda\geq 0)$

$\pounds^{2,\alpha_1}_{\lambda,p,q}h(z)=\pounds^{\alpha_1}_{\lambda,p,q}\left(\pounds^{1,[\alpha_1]}_{\lambda,p,q}h(z)\right)$

and in general for $t\in\mathbb{N}$,

$\pounds^{t,\alpha_1}_{\lambda,p,q}h(z)=\pounds^{\alpha_1}_{\lambda,p,q}\left(\pounds^{t-1,\alpha_1}_{\lambda,p,q}h(z)\right).$

The series expression is given by

$$\pounds^{t,\alpha_1}_{\lambda,p,q}h(z)=z^m+\sum_{n=m+1}^{\infty}\theta^t_n(\alpha_1;\lambda;p;q)a_nz^n, \tag{6}$$

where

$$\theta^t_n([\alpha_1];\lambda;p;q)=\left(\frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{[m+\lambda(n-m)]}{m(n-m)!}\right)^t,(n\in\mathbb{N},n\geq m,t\in\mathbb{N}_0). \tag{7}$$

Similarly for the analytic function $g(z)$ given in (1),

$$\pounds^{t,\gamma_1}_{\lambda,r,s}g(z)=\sum_{n=m}^{\infty}\phi^t_n([\gamma_1];\lambda;r;s)b_nz^n \tag{8}$$

where

$$\phi^t_n([\gamma_1];\lambda;r;s)=\left(\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{[m+\lambda(n-m)]}{m(n-m+1)!}\right)^t,(n\in\mathbb{N},n\geq m,t\in\mathbb{N}_0). \tag{9}$$

We note that when $t=1$ and $\lambda=0$ the linear operator $\pounds^{t,\alpha_1}_{\lambda,p,q}$ would reduce to the operator $H^{p,q}_m[\alpha_1]$ which includes (as its special cases) various other linear operators introduced and studied by Hohlov et al. [7], Owa [17] and Ruscheweyh [21].

Now, for $f = h+\overline{g} \in H(m)$ (where $h(z)$ and $g(z)$ are of the form (1)), in terms of the operators defined in (6) and (8) we defined a linear operator $\pounds^{t,p,q}_{\lambda,r,s}([\alpha_1];[\gamma_1]) := \mathcal{I} : H(m) \to H(m)$ by

$$\mathcal{I}f(z){=}\pounds^{t,\alpha_1}_{\lambda,p,q}h(z) + \overline{\pounds^{t,\gamma_1}_{\lambda,r,s}g(z)}. \tag{10}$$

For the purpose of this paper, on applying the linear operator $\mathcal{I}f(z)$, motivated with the class defined in [10] we define a class $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ of functions $f \in H(m)$ if it satisfy the condition

$$\Re\left\{(1-\lambda)\frac{\mathcal{I}f(z)}{z^m} + \lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'} + \lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}\right\} > \frac{\beta}{m} \tag{11}$$

where $\lambda \geq 0, 0 \leq k \leq 1$, $0 \leq \beta < m$ and $z = re^{i\theta}(r<1,\theta\in\mathbb{R})$, $z' = \frac{\partial}{\partial\theta}(z), z'' = \frac{\partial^2}{\partial\theta^2}(z), (\mathcal{I}f(z))' = \frac{\partial}{\partial\theta}(\mathcal{I}f(z))$ and $(\mathcal{I}f(z))'' = \frac{\partial^2}{\partial\theta^2}(\mathcal{I}f(z))$.

Based on some particular values of λ and k, we denote following classes:

(1) for $\lambda = 0$, $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;0,k\right) = A^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$
(2) for $\lambda = 1$, $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;1,k\right) = B^t_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$
(3) for $k = 0$, $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,0\right) = C^t_{m,}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$
(4) for $k = 1$, $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,1\right) = D^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$
(5) for $\lambda = 1$ and $k = 0$ $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;1,0\right) = E^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta\right)$
(6) for $\lambda = 1$ and $k = 1$ $R^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;1,1\right) = F^t_m\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta\right)$

Let $\widetilde{H}(m)$ be a subclass of $H(m)$ whose members $f = h+\overline{g}$ are such that, h and g are of the form

$$h(z) = z^m - \sum_{n=m+1}^{\infty}|a_n|z^n,\ \ g(z) = \sum_{n=m}^{\infty}|b_n|z^n, |b_m| < 1. \tag{12}$$

We further denote $\widetilde{R}^t_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right) = R^t_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right) \cap \widetilde{H}(m)$.

In this paper, an equivalent convolution class condition is derived and a coefficient inequality is obtained for the functions $f = h+\overline{g} \in H(m)$ to be in the class $R^t_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$. It is also proved that this inequality is necessary for $f = h+\overline{g}$ to be in $\widetilde{R}^t_{m,k}\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$ class. As an application of coefficient inequality a necessary and sufficient hypergeometric inequality is also given. Further, based on the coefficient inequality, results on bounds,inclusion relations, extreme points, convolution and convex combination and on an integral operator are obtained.

Throughout in this paper, we consider that the parameters involved in the operator $\pounds^{t,p,q}_{\lambda,r,s}[m,[\alpha_1];[\gamma_1]]$ such as $\alpha_i\,(i=1,2,...,p)$, $\gamma_i\,(i=1,2,...,r)$, $\beta_i\,(i=1,2,..,q)$,

δ_i $(i=1,2,..,s)$, are positive real and $\theta_n^t(\alpha_1;\lambda;p;q)$, $\phi_n^t(\gamma_1;\lambda;r;s)$ given by (7), (9) are bounded with $\theta_n^t(\alpha_1;\lambda;p;q)\geq\frac{n}{m}$, $\phi_n^t(\gamma_1;\lambda;r;s)\geq\frac{n}{m}$ $(n\geq m)$.

2. Coefficient Inequality

Theorem 1. *Let $\lambda\geq 0, 0\leq k\leq 1$, $0<\beta\leq m, m\in\mathbb{N}$. If the function $f=h+\overline{g}\in H(m)$ (where h and g are of the form (1)) satisfies*

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t(\alpha_1;\lambda;p;q)\,|a_n|+$$
$$\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t(\gamma_1;\lambda;r;s)\,|b_n|\leq 1, \tag{13}$$

then f is sense-preserving, harmonic multivalent in Δ and $f\in R_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$.

Proof. Under the given parametric constraints, we have

$$\frac{n}{m}\leq\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n \text{ and } \frac{n}{m}\leq\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n, n\geq m. \tag{14}$$

Thus, for $f=h+\overline{g}\in H(m)$, where h and g are of the form (1), we get

$$\begin{aligned}
|h'(z)| &\geq m\,|z|^{m-1}-\sum_{n=m+1}^{\infty}n\,|a_n|\,|z|^{n-1}\geq m\,|z|^{m-1}\left[1-\sum_{n=m+1}^{\infty}\frac{n}{m}\,|a_n|\right]\\
&\geq m\,|z|^{m-1}\left[1-\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n\,|a_n|\right]\\
&\geq m\,|z|^{m-1}\left[\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n\,|b_n|\right]>\sum_{n=m}^{\infty}n\,|b_n|\,|z|^{n-1}\\
&\geq |g'(z)|
\end{aligned}$$

which proves that $f(z)$ is sense preserving in Δ. Now to show that $f\in R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$, we need to show (11), that is

$$\Re\left\{(1-\lambda)\frac{\mathcal{I}f(z)}{z^m}+\lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'}+\lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}\right\}>\frac{\beta}{m}, z\in\Delta. \tag{15}$$

Suppose

$$A(z)=(1-\lambda)\frac{\mathcal{I}f(z)}{z^m}+\lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'}+\lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}.$$

It is suffices to show that

$$\left|\frac{A(z)-1}{A(z)-\frac{2\beta}{m}+1}\right|<1.$$

Series expansion of $A(z)$ is given by

$$A(z) = 1+\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left\{1+\lambda(\frac{n}{m}-1)(\frac{kn}{m}+1\right\}a_nz^{n-m}+ \sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left\{1+\lambda(\frac{n}{m}+1)(\frac{kn}{m}-1\right\}b_n\overline{z}^nz^{-m}$$

and we have

$$\left|A(z)-\frac{2\beta}{m}+1\right|-|A(z)-1|$$

$$= \Big|2(1-\tfrac{\beta}{m})+\textstyle\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left\{1+\lambda(\tfrac{n}{m}-1)(\tfrac{kn}{m}+1\right\}a_nz^{n-m} +\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left\{1+\lambda(\tfrac{n}{m}+1)(\tfrac{kn}{m}-1)\right\}b_n\overline{z}^nz^{-m}\Big|$$
$$- \Big|\textstyle\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left\{1+\lambda(\tfrac{n}{m}-1)(\tfrac{kn}{m}+1\right\}a_nz^{n-m} +\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left\{1+\lambda(\tfrac{n}{m}+1)(\tfrac{kn}{m}-1\right\}b_n\overline{z}^nz^{-m}\Big|$$

$$\geq \frac{1}{m}\Bigg[2(m-\beta)-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|m+\lambda(n-m)(\frac{kn}{m}+1)\right||a_n|\left|z^{n-m}\right| -\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|m+\lambda(n+m)(\frac{kn}{m}-1)\right||b_n||\overline{z}^n|\left|z^{-m}\right|- \sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|m+\lambda(n-m)(\frac{kn}{m}+1)\right||a_n|\left|z^{n-m}\right| -\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|m+\lambda(n+m)(\frac{kn}{m}-1)\right||b_n||\overline{z}^n|\left|z^{-m}\right|\Bigg]$$

$$= \frac{1}{m}\Bigg[2(m-\beta)-2\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|m+\lambda(n-m)(\frac{kn}{m}+1\right||a_n|\left|z^{n-m}\right| -2\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|m+\lambda(n+m)(\frac{kn}{m}-1)\right||b_n||\overline{z}^n|\left|z^{-m}\right|\Bigg]$$
$$\geq 0$$

by (13) when $z=r\to 1$ and this proves Theorem 1. □

In our next result we show that the above sufficient coefficient condition is also necessary for functions in the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$.

Theorem 2. *Let $\lambda\geq 0, 0\leq k\leq 1,\ 0<\beta\leq m, m\in\mathbb{N}$.and let the function $f=h+\overline{g}\in\widetilde{H}(m)$ be such that h and g are given by (12). Then $f\in$*

$\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ *if and only if* (13) *holds. The inequality* (13) *is sharp for the function given by*

$$f(z) = z^m - \sum_{n=m+1}^{\infty} \frac{m(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1];\lambda;p;q)} |x_n| z^n \quad (16)$$
$$+ \sum_{n=m}^{\infty} \frac{m(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1];\lambda;r;s)} |y_n| \overline{z^n},$$

$\sum_{n=m+1}^{\infty}|x_n| + \sum_{n=m}^{\infty}|y_n| = 1.$

Proof. The if part, follows from Theorem 1. To prove the "only if part" let $f = h + \overline{g} \in \widetilde{H}(m)$ be such that h and g are given by (12) and $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then for $z = re^{i\theta}$ in Δ we obtain

$$\Re\left\{(1-\lambda)\frac{\mathcal{I}f(z)}{z^m} + \lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'} + \lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}\right\} > \frac{\beta}{m}$$

$$= \Re\left\{(1-\lambda)\frac{\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}h(z) + \overline{\mathcal{L}_{\lambda,r,s}^{t,\gamma_1}g(z)}}{z^m} + \lambda(1-k)\frac{z\left(\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}h(z)\right)' - z\left(\overline{\mathcal{L}_{\lambda,r,s}^{t,\gamma_1}g(z)}\right)'}{mz^m}\right\}$$
$$+ \Re\left\{\lambda k\frac{z^2\left(\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}h(z)\right)'' + z\left(\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}h(z)\right)' + z^2\left(\overline{\mathcal{L}_{\lambda,r,s}^{t,\gamma_1}g(z)}\right)'' + z\left(\overline{\mathcal{L}_{\lambda,r,s}^{t,\gamma_1}g(z)}\right)'}{m^2z^m}\right\}$$

$$\geq 1 - \sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|1+\lambda(\frac{n}{m}-1)(\frac{kn}{m}+1\right||a_n|\left|z^{n-m}\right| -$$
$$\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|1+\lambda(\frac{n}{m}+1)(\frac{kn}{m}-1)\right| b_n|\overline{z}^n|\left|z^{-m}\right|$$
$$> \frac{\beta}{m}.$$

The above inequality must hold for all $z \in \Delta$. in particular $z = r \to 1$ yields the required condition (13). Sharpness of the result can easily be verified for the function given by (16). □

Corollary 1. $f \in \widetilde{A}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{m}{(m-\beta)}\theta_n^t([\alpha_1];p;q)|a_n| +$$

$$\sum_{n=m}^{\infty} \frac{m}{(m-\beta)} \phi_n^t([\gamma_1];r;s)\,|b_n| \le 1$$

holds.

Corollary 2. $f \in \widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1];p;q)\,|a_n| +$$

$$\sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1];r;s)\,|b_n| \le 1$$

holds.

Corollary 3. $f \in \widetilde{C}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)m\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1];\lambda;p;q)\,|a_n| +$$

$$\sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)m\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1];\lambda;r;s)\,|b_n| \le 1$$

holds.

Corollary 4. $f \in \widetilde{D}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n^2-m^2)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1];\lambda;p;q)\,|a_n| +$$

$$\sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n^2-m^2)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1];\lambda;r;s)\,|b_n| \le 1$$

holds.

Corollary 5. $f \in \widetilde{E}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)m\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1];p;q)\,|a_n| +$$

$$\sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)m\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1];r;s)\,|b_n| \le 1$$

holds.

Corollary 6. $f \in \widetilde{F}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+(n^2-m^2)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1];p;q)\,|a_n| +$$

$$\sum_{n=m}^{\infty}\frac{\left|m^2+(n^2-m^2)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;r;s)\,|b_n|\leq 1$$

holds.

On applying coefficient inequality (13), we get a sufficient condition in the form of hypergeometric inequality for certain function $f=h+\overline{g}\in H(m)$ to be in $R_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ class and it is proved that this inequality is necessary for certain $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$.

Corollary 7. *Let $\lambda\geq 0, 0\leq k\leq 1$, $0<\beta\leq m, m\in\mathbb{N}$, and let the function $f=h+\overline{g}\in H(m)$ where h and g are of the form (1) be such that*

$$|a_n|\leq\frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|},n\geq m+1 \tag{17}$$

and (18)

$$|b_n|\leq\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|},n\geq m. \tag{19}$$

If (in case $p=q+1$) $\sum_{i=1}^{q}\beta_i-\sum_{i=1}^{p}\alpha_i>0$ and (in case $r=s+1$) $\sum_{i=1}^{s}\delta_i-\sum_{i=1}^{r}\gamma_i>0$,
the hypergeometric inequality

$$\left[\left[{}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right)-1\right]\frac{(m+\lambda(n-m))}{m}\right]^t+ \tag{20}$$

$$\left[{}_rF_s\left((\gamma_i)\,;(\delta_i)\,;1\right)\frac{(m+\lambda(n-m))}{m}\right]^t\leq 1$$

holds, then $f\in R_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$. Further, if

$$\begin{aligned} f(z) &= z^m-\sum_{n=m+1}^{\infty}\frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}\;z^n \\ &\quad+\sum_{n=m}^{\infty}\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}\overline{z^n} \\ &\in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right), \end{aligned} \tag{21}$$

then (20) holds.

Proof. To prove the result, we need to show by Theorem 1 the inequality:

$$S_1\;:=\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n|$$

$$+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n|$$
$$\leq\ 1.$$

By (17) and (19), we get by (20),

$$\begin{aligned}S_1 &\leq \sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1]\,;\lambda;p;q)+\sum_{n=m}^{\infty}\phi_n^t([\gamma_1]\,;\lambda;r;s)\\ &= \left[\left[{}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right)-1\right]\frac{(m+\lambda(n-m))}{m}\right]^t\\ &+\left[{}_rF_s\left((\gamma_i)\,;(\delta_i)\,;1\right)\frac{(m+\lambda(n-m))}{m}\right]^t\leq 1\end{aligned}$$

where, under the given conditions

$$\begin{aligned}\sum_{n=m+1}^{\infty}\theta_n^T([\alpha_1]\,;\lambda;p;q) &= \left[\left(\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_i)_n}\frac{1}{n!}-1\right)\frac{(m+\lambda(n-m))}{m}\right]^t\\ &= \left[\left[{}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right)-1\right]\frac{(m+\lambda(n-m))}{m}\right]^t.\end{aligned}$$

Similarly

$$\begin{aligned}\sum_{n=m}^{\infty}\phi_n^T([\gamma_1]\,;\lambda;r;s) &= \left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_n}{\prod_{i=1}^{s}(\delta_i)_n}\frac{1}{n!}\frac{(m+\lambda(n-m))}{m}\right]^t\\ &= \left[{}_rF_s\left((\gamma_i)\,;(\delta_i)\,;1\right)\frac{(m+\lambda(n-m))}{m}\right]^t.\end{aligned}$$

Further, (20) holds by Theorem 2, if $f(z)$ of the form (21) belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$. This proves the result. □

In particular if we take $\lambda=0, t=1$ we get the following hypergeometric inequality which is sufficient for certain function $f=h+\overline{g}\in H(m)$ to be in $R_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ class, and this inequality is necessary for certain $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$.

Corollary 8. *Let $\lambda\geq 0, 0\leq k\leq 1$, $0<\beta\leq m, m\in\mathbb{N}$, and let the function $f=h+\overline{g}\in H(m)$ where h and g are of the form (1) be such that*

$$|a_n|\ \leq\ \frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}, n\geq m+1 \qquad (22)$$

$$|b_n| \quad \leq \quad \frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}, n \geq m. \tag{23}$$

If (in case $p = q+1$) $\sum_{i=1}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i > 0$ and (in case $r = s+1$) $\sum_{i=1}^{s} \delta_i - \sum_{i=1}^{r} \gamma_i > 0$, the hypergeometric inequality

$${}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right) +_r F_s\left((\gamma_i)\,;(\delta_i)\,;1\right) \leq 2 \tag{24}$$

holds, then $f \in R^t_{m,k}\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda\right)$. Further, if

$$\begin{aligned} f(z) \quad &= \quad z^m - \sum_{n=m+1}^{\infty} \frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}\; z^n \\ &\quad + \sum_{n=m}^{\infty} \frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}\overline{z^n} \\ &\in \quad \widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right), \end{aligned} \tag{25}$$

then (20) holds.

3. Inclusion Relation

The inclusion relations between the classes $\widetilde{B}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$ and $\widetilde{A}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$ for different values of λ In this section inclusion relation between the classes and for different values of $\widetilde{B}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$ $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$

Theorem 3. *for $n \in \{1, 2, 3..\}$ and $0 \leq \beta < m$, we have*

(1) $\widetilde{B}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right) \subset \widetilde{A}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$
(2) $\widetilde{B}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right) \subset \widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right), 0 \leq \lambda \leq 1$
(3) $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right) \subset \widetilde{B}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right), \lambda \geq 1.$

Proof. (i) Let $f(z) \in \widetilde{B}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$. in view of corollaries 1 and 2, we have

$$\begin{aligned} &\sum_{n=m+1}^{\infty} \frac{m}{(m-\beta)}\theta^t_n([\alpha_1]\,;p;q)\,|a_n| + \sum_{n=m}^{\infty} \frac{m}{(m-\beta)}\phi^t_n([\gamma_1]\,;r;s)\,|b_n| \\ &\leq \sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta^t_n([\alpha_1]\,;p;q)\,|a_n| + \\ &\sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi^t_n([\gamma_1]\,;r;s)\,|b_n| \leq 1 \end{aligned}$$

(ii)Let $f(z) \in \widetilde{B}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$.For $0 \le \lambda \le 1$,we can write

$$\begin{aligned}
& \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; \lambda; p; q)|a_n| + \\
& \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; \lambda; r; s)|b_n| \\
\le & \sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; p; q)|a_n| + \\
& \sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; r; s)|b_n| \\
\le & \; 1
\end{aligned}$$

by corollary 2 and (ii) follows from Theorem 2.

(iii) By the Theorem 2, if $\lambda \ge 1$,we have

$$\begin{aligned}
& \sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; p; q)|a_n| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; r; s)|b_n| \\
\le & \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; \lambda; p; q)|a_n| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; \lambda; r; s)|b_n| \\
\le & \; 1.
\end{aligned}$$

Therefore the result follows from corollary 2. □

4. Bounds

Our next theorems provide the bounds for the function in the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ which are followed by a covering result for this class.

Theorem 4. *Let $\lambda \ge 0, 0 \le k \le 1$, $0 < \beta \le m, m \in \mathbb{N}$. if $f = h + \overline{g} \in \widetilde{H}(m)$, where h and g are of the form (12) belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then for $|z| = r < 1$,*

$$|\mathcal{I}f(z)| \le (1+|b_m|)\, r^m + \frac{mr^{m+1}}{m+1}\left(1 - \frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}|b_m|\right), \tag{26}$$

and

$$|\mathcal{I}f(z)| \geq (1-|b_m|)\, r^m - \frac{m}{m+1}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right) r^{m+1}. \tag{27}$$

The result is sharp.

Proof. Let $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then on using (13), related to (12), by (10), we get for $|z| = r < 1$,

$$\begin{aligned}
&|\mathcal{I}f(z)| \\
\leq\ & (1+|b_m|)\, r^m + \sum_{n=m+1}^{\infty}\left(\theta^t([\alpha_1]; \lambda; p; q)\,|a_n| + \phi_n^t([\gamma_1]; \lambda; r; s)\,|b_n|\right) r^n \\
\leq\ & (1+|b_m|)\, r^m + r^{m+1}\sum_{n=m+1}^{\infty}\left(\theta_n^t([\alpha_1]; \lambda; p; q)\,|a_n| + \phi_n^t([\gamma_1]; \lambda; r; s)\,|b_n|\right) \\
\leq\ & (1+|b_m|)\, r^m + \frac{mr^{m+1}}{m+1}\left(\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]; \lambda; p; q)\,|a_n| + \right. \\
& \left. \sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]; \lambda; r; s)\,|b_n|\right) \\
\leq\ & (1+|b_m|)\, r^m + \frac{mr^{m+1}}{m+1}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right)
\end{aligned}$$

which proves the result (26). The result (27) can similarly be obtained. The bounds (26) and (27) are sharp for the function given by

$$f(z) = z^m + |b_m|\,\overline{z^m} + \frac{m}{(m+1)\,\phi_{m+1}^t([\gamma_1]; \lambda; r; s)}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right)\overline{z^{m+1}}$$

for $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m,\ |b_m| < \frac{1-\frac{\beta}{m}}{1+2\lambda(k-1)}$. □

Corollary 9. *Let $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m, m \in \mathbb{N}$. If $f = h + \overline{g} \in \widetilde{H}(m)$ with h and g are of the form (12) belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then*

$$\left\{\omega : |\omega| < 1 - \frac{m}{m+1} + \left(\frac{m\,(1+2\lambda(k-1))}{(m+1)\left(1-\frac{\beta}{m}\right)} - 1\right)|b_m|\right\} \subset f(\Delta).$$

Theorem 5. *Let $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m, m \in \mathbb{N}$ and let*

$$\delta_{m+1}^t([\alpha_1]; [\gamma_1]; \lambda; p; q; r; s) \leq \min\left(\theta_n^t([\alpha_1]; \lambda; p; q), \phi_n^t([\gamma_1]; \lambda; r; s)\right), n \geq m+1$$

If $f = h + \overline{g} \in \widetilde{H}(m)$, *where* h *and* g *are of the form* (12), *belongs to the class* $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, *then for* $|z| = r < 1$,

$$|f(z)| \leq (1 + |b_m|)\, r^m + \frac{m}{(m+1)\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} \left(1 - \frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}} |b_m|\right) r^{m+1}, \tag{28}$$

and

$$|f(z)| \geq (1 - |b_m|)\, r^m - \frac{m}{(m+1)\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} \left(1 - \frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}} |b_m|\right) r^{m+1}. \tag{29}$$

The result is sharp.

Proof. Let $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then on using (13), from (12), we get for $|z| = r < 1$,

$$\begin{aligned}
|f(z)| &\leq (1 + |b_m|)\, r^m + \sum_{n=m+1}^{\infty} (|a_n| + |b_n|)\, r^n \\
&\leq (1 + |b_m|)\, r^m + r^{m+1} \sum_{n=m+1}^{\infty} (|a_n| + |b_n|) \\
&\leq (1 + |b_m|)\, r^m + \frac{r^{m+1}}{\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} \\
&\times \sum_{n=m+1}^{\infty} \left(\theta_n^t([\alpha_1];\lambda;p;q)\, |a_n| + \phi_n^t([\gamma_1];\lambda;r;s)\, |b_n|\right) \\
&\leq (1 + |b_m|)\, r^m + \frac{m r^{m+1}}{(m+1)\, \delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} \\
&\quad \left(\sum_{n=m+1}^{\infty} \frac{\left|m^2 + \lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1];\lambda;p;q)\, |a_n| + \right. \\
&\quad \left. \sum_{n=m}^{\infty} \frac{\left|m^2 + \lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1];\lambda;r;s)\, |b_n|\right) \\
&\leq (1 + |b_m|)\, r^m \\
&+ \frac{m r^{m+1}}{(m+1)\, \delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} \left(1 - \frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}} |b_m|\right) r^{m+1},
\end{aligned}$$

which proves (28). The result (29) can similarly be obtained. The bounds (28) and (29) are sharp for the function given by

$$f(z) = z^m + |b_m|\, \overline{z^m} + \frac{m r^{m+1}}{(m+1)\, \delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} \left(1 - \frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}} |b_m|\right) \overline{z^{m+1}}$$

for $|b_m| < \frac{1-\frac{\beta}{m}}{1+2\lambda(k-1)}$. □

Corollary 10. *Let* $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m, m \in \mathbb{N}$ *and let* $\delta^t_{m+1}([\alpha_1]\,;[\gamma_1]\,;\lambda;p;q;r;s) \leq \min\left(\theta^t_n([\alpha_1]\,;\lambda;p;q), \phi^t_n([\gamma_1]\,;\lambda;r;s)\right),\ n \geq m+1.$ *If* $f = h + \overline{g} \in \widetilde{H}(m)$ *with* h *and* g *are of the form* (12) *belongs to the class* $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$*, then*

$$\left\{\omega : |\omega| < 1 - \frac{m}{(m+1)\delta^t_{m+1}([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} + \left(\frac{m(1+2\lambda(k-1))}{(m+1)\left(1-\frac{\beta}{m}\right)\delta^t_{m+1}([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} - 1\right)|b_m|\right\} \subset f(\Delta).$$

5. Extreme Points

In this section, we determine the extreme points for the class $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$.

Theorem 6. *let* $f = h + \overline{g} \in \widetilde{H}(m)$ *and*

$$\begin{aligned} h_m(z) &= z^m, \\ h_n(z) &= z^m - \frac{m(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta^t_n([\alpha_1]\,;\lambda;p;q)}z^n \quad (n \geq m+1), \\ g_n(z) &= z^m + \frac{m(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi^t_n([\gamma_1]\,;\lambda;r;s)}\overline{z^n} \quad (n \geq m), \end{aligned}$$

then the function $f \in \widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ *if and only if it can be expressed as* $f(z) = \sum_{n=m}^{\infty}(x_n h_n(z) + y_n g_n(z))$ *where* $x_n \geq 0, y_n \geq 0$ *and* $\sum_{n=m}^{\infty}(x_n + y_n) = 1$*. In particular, the extreme points of* $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. Suppose that

$$f(z) = \sum_{n=m}^{\infty}(x_n h_n(z) + y_n g_n(z))$$

Then,

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty}(x_n + y_n)z^m - \sum_{n=m+1}^{\infty}\frac{m(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta^t_n([\alpha_1]\,;\lambda;p;q)}x_n z^n \\ &\quad + \sum_{n=m}^{\infty}\frac{m(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi^t_n([\gamma_1]\,;\lambda;r;s)}y_n\overline{z^n} \\ &= z^m - \sum_{n=m+1}^{\infty}\frac{m(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta^t_n([\alpha_1]\,;\lambda;p;q)}x_n z^n \end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=m}^{\infty}\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}y_n\overline{z^n}\\
\in\quad &\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)
\end{aligned}
$$

by Theorem 2, since,

$$
\begin{aligned}
&\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\\
&\times\left(\frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1]\,;\lambda;p;q)}x_n\right)\\
&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\\
&\times\left(\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}y_n\right)\\
=\quad &\sum_{n=m+1}^{\infty}x_n+\sum_{n=m}^{\infty}y_n=1-x_m\le 1.
\end{aligned}
$$

Conversely, let $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ and let

$$|a_n|=\frac{m\,(m-\beta)\,x_n}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1]\,;\lambda;p;q)}$$

and

$$|b_n|=\frac{m\,(m-\beta)\,y_n}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}$$

and

$$x_m=1-\sum_{n=m+1}^{\infty}x_n-\sum_{n=m}^{\infty}y_n,$$

then, we get

$$
\begin{aligned}
f\,(z)\quad &=\quad z^m-\sum_{n=m+1}^{\infty}|a_n|\,z^n+\sum_{n=m}^{\infty}|b_n|\,\overline{z}^n\\
&=\quad h_m(z)-\sum_{n=m+1}^{\infty}\frac{m\,(m-\beta)\,x_n}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1]\,;\lambda;p;q)}x_nz^n\\
&\quad+\sum_{n=m}^{\infty}\frac{m\,(m-\beta)\,y_n}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}y_n\overline{z^n}\\
&=\quad h_m(z)+\sum_{n=m+1}^{\infty}\left(h_n(z)-h_m(z)\right)x_n+\sum_{n=m}^{\infty}\left(g_n(z)-h_m(z)\right)y_n
\end{aligned}
$$

$$= h_m(z)\left(1-\sum_{n=m+1}^{\infty}x_n-\sum_{n=m}^{\infty}y_n\right)+\sum_{n=m+1}^{\infty}h_n(z)x_n+\sum_{n=m}^{\infty}g_n(z)y_n$$
$$= \sum_{n=m}^{\infty}\left(x_nh_n(z)+y_ng_n(z)\right).$$

This proves the Theorem 6. □

6. Convolution and Convex Combinations

In this section, we show that the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ is invariant under convolution and convex combinations of its members.

Let the function $f=h+\overline{g}\in\widetilde{H}(m)$ where h and g are of the form (12) and

$$F(z)=z^m-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|A_n\right|z^n+\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|B_n\right|\overline{z^n}\in\widetilde{H}(m). \tag{30}$$

The convolution between the functions of the class $\widetilde{H}(m)$ is defined by
$(f*F)(z)=f(z)*F(z)=z^m-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|a_nA_n\right|z^n+\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|b_nB_n\right|\overline{z^n}$

Theorem 7. *Let* $\lambda\geq 0, 0\leq k\leq 1, 0<\beta\leq m, m\in\mathbb{N}$, *if* $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ *and* $F\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$, *then* $f*F\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$.

Proof. Let $f=h+\overline{g}\in\widetilde{H}(m)$,where h and g are of the form (12) and $F\in\widetilde{H}(m)$ of the form (30) be in $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ class. Then by theorem (2), we have

$$\begin{aligned}&\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\left(m-\beta\right)}\theta_n^t([\alpha_1];\lambda;p;q)\left|A_n\right|\\&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\left(m-\beta\right)}\phi_n^t([\gamma_1];\lambda;r;s)\left|B_n\right|\\\leq\ & 1\end{aligned}$$

which in view of (14), yields

$$\begin{aligned}\left|A_n\right| &\leq \frac{m\left(m-\beta\right)}{\left|m^2+\lambda(n-m)(kn+m)\right|\theta_n^t([\alpha_1];\lambda;p;q)}\leq\frac{m}{n}\leq 1, n\geq m+1\\ \left|B_n\right| &\leq \frac{m\left(m-\beta\right)}{\left|m^2+\lambda(n+m)(kn-m)\right|\phi_n^t([\gamma_1];\lambda;r;s)}\leq\frac{m}{n}\leq 1, n\geq m.\end{aligned}$$

Hence, by Theorem 2,

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\left(m-\beta\right)}\theta_n^t([\alpha_1];\lambda;p;q)\left|a_nA_n\right|$$

$$
\begin{aligned}
& + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n B_n| \\
\leq & \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n| \\
\leq & \ 1
\end{aligned}
$$

which proves that $f * F \in \widetilde{R}_m^t \left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$. □

We prove next that the class $\widetilde{R}_m^t \left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$ is closed under convex combination.

Theorem 8. *Let* $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m, m \in \mathbb{N}$, *the class* $\widetilde{R}_m^t \left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$ *is closed under convex combination.*

Proof. Let $f_j \in \widetilde{R}_m^t \left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$, $j \in \mathbb{N}$ be of the form

$$
f_j\,(z) = z^m - \sum_{n=m+1}^{\infty} |A_{j,n}|\, z^n + \sum_{n=m}^{\infty} |B_{j,n}|\, \overline{z^n}, j \in \mathbb{N}.
$$

Then by Theorem 2, we have for $j \in \mathbb{N}$,

$$
\begin{aligned}
& \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^T([\alpha_1]\,;\lambda;p;q)\,|A_{j,n}| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^T([\gamma_1]\,;\lambda;r;s)\,|B_{j,n}| \\
\leq & \ 1.
\end{aligned} \tag{31}
$$

For some $0 \leq t_j \leq 1$, let $\sum_{j=1}^{\infty} t_j = 1$, the convex combination of $f_j\,(z)$ may be written as

$$
\sum_{j=1}^{\infty} t_j \ f_j\,(z) = z^m - \sum_{n=m+1}^{\infty} \sum_{j=1}^{\infty} t_j\, |A_{j,n}|\, z^n + \sum_{n=m}^{\infty} \sum_{j=1}^{\infty} t_j\, |B_{j,n}|\, \overline{z^n}
$$

Now by (31),

$$
\begin{aligned}
& \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1]\,;\lambda;p;q) \sum_{j=1}^{\infty} t_j\, |A_{j,n}| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1]\,;\lambda;r;s) \sum_{j=1}^{\infty} t_j\, |B_{j,n}|
\end{aligned}
$$

$$= \sum_{j=1}^{\infty} t_j \left[\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1]\,;\lambda;p;q)\,|A_{j,n}| + \right.$$

$$\left. \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1]\,;\lambda;r;s)\,|B_{j,n}| \right] \le \sum_{j=1}^{\infty} t_j = 1$$

and so again by Theorem 2, we get $\sum_{j=1}^{\infty} t_j\ f_j\,(z) \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$. This proves the result. □

7. Integral Operator

In this section, we study a closure property of the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$ involving the generalized Bernardi Libera-Livingston Integral operator $L_{m,c}$ which is given for $f = h + \overline{g} \in \widetilde{H}\,(m)$ by

$$L_{m,c}(f) = \frac{c+m}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+m}{z^c}\int_0^z t^{c-1}g(t)dt}, \quad c > -m, z \in \Delta. \tag{32}$$

Theorem 9. *Let* $\lambda \ge 0, 0 \le k \le 1, 0 < \beta \le m, m \in \mathbb{N}$, *if* $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$, *then* $L_{m,c}(f) \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$.

Proof. Let $f = h + \overline{g} \in \widetilde{H}\,(m)$,where h and g are of the form (12), belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right)$. Then, it follows from (32) that

$$\begin{aligned} L_{m,c}(f) &= z^m - \sum_{n=m+1}^{\infty}\left(\frac{c+m}{c+n}\right)|a_n|\,z^n + \sum_{n=m}^{\infty}\left(\frac{c+m}{c+n}\right)|b_n|\,\overline{z^n} \\ &\in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,, [\gamma_1]_{r,s}\,; \beta; \lambda, k\right) \end{aligned}$$

by (13), since,

$$\begin{aligned} &\sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\left(\frac{c+m}{c+n}\right)\theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n| + \\ &\sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\left(\frac{c+m}{c+n}\right)\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n| \\ \le\ &\sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n| \\ &+ \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n| \\ \le\ &1. \end{aligned}$$

This proves the result. □

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References

[1] Ahuja, O. P., Jahangiri, J. M., Multivalent harmonic starlike functions, *Ann. Univ. Mariac Curie-Sklodowska Section A*, 55(1) (2001), 1–13.
[2] Ahuja, O. P., Jahangiri, J. M., Errata to Multivalent harmonic starlike function, *Ann. Univ. Mariac Curie-Sklodowska Section A* 56(1) (2002), 105.
[3] Ahuja, O. P., Aghalary, R., Joshi, S. B., Harmonic univalent functions associated with k-uniformly starlike functions, *Math. Sci. Res. J.,* 9(1) (2005), 9–17.
[4] Ahuja, O. P., Güney, H. Ö., Sakar, F. M., Certain classes of harmonic multivalent functions based on Hadamard product, *J. Inequal. Appl.,* 2010 (2009), Art. ID 759251, 12pp. https://doi.org/10.1155/2009/759251
[5] Güney, H. Ö., Ahuja, O. P., Inequalities involving multipliers for multivalent harmonic functions, *J. Inequal. Pure Appl. Math.,* 7(5) Art. 190 (2006), 1–9.
[6] Clunie, J., Sheil-Small, T., Harmonic univalent functions, *Ann. Acad. Aci. Fenn. Ser. A Math.,* 9 (1984), 3–25.
[7] Carlson, B. C., Shaffer, D. B., Starlike and prestarlike hypergeometric functions, *SIAM, J. Math. Anal.*, 15 (1984), 737–745.
[8] Duren, P., Hengartner, W., Laugesen, R. S., The argument principle for harmonic functions, *Amer. Math. Monthly,* 103 (1996), 411–415.
[9] Dziok, J., Srivastava, H. M., Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.,* 103 (1999), 1–13. https://doi.org/10.1016/S0096-3003(98)10042-5
[10] Ebadian, A., Tehranchi, A., On certain classes of harmonic p- valent functions by applying the Ruscheweyh derivatives, *Filomat,* 23(1) (2009), 91–101.
[11] Murugusundaramoorthy, G., Raina, R. K., On a subclass of harmonic functions associated with Wright's generalized hypergeometric functions, *Hacettepe J. Math. Stat.,* 38(2) (2009), 129–136.
[12] Murugusundaramoorthy, G., Vijaya, K., Starlike harmonic functions in parabolic region associated with a convolution structure, *Acta Univ. Sapientiae Math.*, 2(2) (2010), 168–183.
[13] Murugusundaramoorthy, G., Vijaya, K., Raina, R. K., A subclass of harmonic univalent functions with varying arguments defined by Dziok-Srivastava operator, *Archivun Mathematicum (BRNO) Tomus*, 45 (2009), 37-46.
[14] Murugusundaramoorthy, G., Harmonic starlike functions of complex order involving hypergeometric functions, *Math. Vesnik,* 64(4) (2012), 316–325.
[15] Murugusundaramoorthy, G., Uma, K., Harmonic univalent functions associated with generalized hypergeometric functions, *Bull. Math. Anal. Appl.,* 2(2) (2010), 69–76.
[16] Omar, R., Halim, S. A., Multivalent Harmonic Functions defined by Dziok-Srivastava operator, *Bull. Malays. Math. Sci. Soc.,* 35(3)(2) (2012), 601–610.
[17] Owa, S., On the distortion theorems - I, *Kyungpook. Math. J.*, 18 (1978) 53–59.

[18] Porwal, S., Some properties of a subclass of harmonic univalent functions defined by the multiplier transformations, *Indian J. Pure Appl. Math.,* 46(3) (2015), 309–335. https://doi.org/10.1007/s13226-015-0132-9

[19] Porwal, S., On a New Subclass of Harmonic Univalent Functions Defined by Multiplier Transformation, *Mathematica Moravica,* 19(2) (2015), 75–87. http://dx.doi.org/10.5937/MatMor1502075P

[20] Srivastava, H. M., Li, Shu-Hai, Tang, H., Certain classes of k-uniformly close-to-convex functions and other related functions defined by using the Dziok-Srivastava operator, *Bull. Math. Anal. Appl.*, 1(3) (2009), 49–63.

[21] Ruscheweyh, S., New criteria for univalent functions, *Proc. Amer. Math. Soc.,* 49 (1975), 109–115. https://doi.org/10.1090/S0002-9939-1975-0367176-1

[22] Atshan, W. G., Kulkarni, S. R., Raina, R. K., A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator, *Math. Vesnik*, 60 (2008), 207–213.

http://communications.science.ankara.edu.tr

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POWER SERIES METHODS AND STATISTICAL LIMIT SUPERIOR

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Abstract. Given a real bounded sequence $x = (x_j)$ and an infinite matrix $A = (a_{nj})$ the Knopp core theorem is equivalent to study the inequality $\limsup Ax \leq \limsup x$. Recently Fridy and Orhan [6] have considered some variants of this inequality by replacing $\limsup x$ with statistical limit superior $st - \limsup x$. In the present paper we examine similar type of inequalities by employing a power series method P, a non-matrix sequence-to-function transformation, in place of $A = (a_{nj})$.

1. Introduction

In order to investigate the effect of matrix transformations upon the derived set of a sequence $x = (x_j)$, Knopp [10] introduced the idea of the core of x and proved the well-known Core Theorem. This is equivalent to study the inequality $\limsup Ax \leq \limsup x$ for the finite matrix and bounded sequences $x = (x_j)$ where $Ax := \sum_{j=0}^{\infty} a_{nj}x_j$ ([12, 15]). Based on the recently introduced concept of a statistical cluster point [6], a definition is given for the statistical core by Fridy and Orhan [7]. They have also determined a class of regular matrices for which the inequality $\limsup Ax \leq st - \limsup x$ holds for real bounded sequences.

In the present paper, we consider similar type of inequalities by replacing the sequence to sequence transformation with a power series method which is a sequence to function transformation.

Recall that the core of the sequence $x = (x_j)$ is the closed convex hull of the set of limit points of the sequence $x = (x_j)$.

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Let (p_j) be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{j=0}^{\infty} p_j t^j$$

has radius of convergence R with $0 < R \leq \infty$.

Let

$$C_P := \left\{ f : (-R, R) \to \mathbb{R} \mid \lim_{0<t\to R^-} \frac{f(t)}{p(t)} \ \textit{exists} \right\}$$

and

$$C_{P_p} := \left\{ x = (x_j) : p_x(t) := \sum_{j=0}^{\infty} p_j t^j x_j \text{ has radius of convergence } \geq R \text{ and } p_x \in C_p \right\}$$

The functional $P - \lim : C_{P_p} \to \mathbb{R}$ defined by

$$P - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and the sequences $x = (x_j)$ is said to be $P-$ *convergent*. The method P is regular if and only if $\lim_{0<t\to R^-} \frac{p_j t^j}{p(t)} = 0$ for every j (see, e.g. [2]). We note that the Abel method is a particular case of a power series method ([17]).

From now on we assume that $t \in (0, R)$ and $0 < R \leq \infty$.

In the subsequent sections we give some inequalities by relating $\limsup_{t\to R^-} \frac{p_x(t)}{p(t)}$ to $\limsup x$ and st-$\limsup x$. These inequalities are motivated by those of Maddox [2], Orhan [15], and, Fridy and Orhan [7].

2. An Inequaility Related to Limit Superior

Let $Q_x(t) := \frac{p_x(t)}{p(t)}$. In this section for real bounded sequences $x = (x_j)$, we consider the inequality

$$\limsup_{t\to R^-} Q_x(t) \leq \limsup_j x_j$$

which may be interpreted as saying that

$$\mathcal{K}{-}core\{Q_x(t)\} \subseteq \mathcal{K}{-}core\{x\}$$

where $\mathcal{K}{-}core\{x\}$ denotes the usual Knopp core of x (see,e.g., [8, p.55]). Let ℓ^∞ denote the space of all real bounded sequences and let $L(x) := \limsup_n x_n$ and $l(x) := \liminf_n x_n$. Now we have the following

Theorem 1. *For every* $x = (x_j) \in \ell^\infty$ *we have*

$$\limsup_{t \to R^-} Q_x(t) \le \limsup_j x_j \tag{1}$$

if and only if P *is regular.*

Proof. Necessity. Let $x \in c$. Then by (1), we immediately get

$$-\limsup(-x) \le -\limsup_{t \to R^-} Q_{(-x)}(t)$$

Combining this with (1), one can have

$$\liminf x \le \liminf Q_x(t) \le \limsup Q_x(t) \le \limsup x.$$

Since $x \in c$,

$$\lim x = \lim_{t \to R^-} Q_x(t)$$

is obtained, i.e., P is regular.

Conversely, assume that P is regular. Let $x \in \ell^\infty$ and $\varepsilon > 0$. Then choose an index m so that $x_j < L(x) + \varepsilon$ whenever $j \ge m$. Hence we have

$$\begin{aligned} \sum_{j=0}^{\infty} p_j t^j x_j &= \sum_{j<m} p_j t^j x_j + \sum_{j \ge m} p_j t^j x_j \\ &\le \|x\| \sum_{j<m} p_j t^j + (L(x) + \varepsilon) \sum_{j=0}^{\infty} p_j t^j. \end{aligned}$$

Multiplying both sides by $\dfrac{1}{p(t)}$ we get

$$\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j \le \frac{\|x\|}{p(t)} \sum_{j<m} p_j t^j + (L(x) + \varepsilon)$$

Taking limit superior as $t \to R^-$ and using the regularity of P one can observe that

$$\limsup_{t \to R^-} Q_x(t) \le L(x) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we conclude that (1) holds, which proves the theorem. □

3. An Inequality Concerning Statistical Limit Superior

In this section, replacing limit superior by statistical limit superior of a real bounded sequence we prove an inequality.

Following the concepts of statistical convergence and statistical cluster points of a sequence $x = (x_j)$, Fridy and Orhan [7] have introduced the definition of statistical limit superior and inferior.

We first recall some terminology and notation. If $K \subseteq \mathbb{N}_0$ and $K_n := \{k \leq n : k \in K\}$ then $|K_n|$ denotes the cardinality of K_n. If the limit $\delta(K) := \lim_n \frac{1}{n+1} |K_n|$ exists, then we say that K has a natural (asymptotic) density. A sequence $x = (x_j)$ is statistically convergent to L, denoted $st - \lim x = L$, if for every $\varepsilon > 0$, $\delta(\{j : |x_j - L| \geq \varepsilon\}) = 0$, (see, e.g., [3, 5, 14, 16]).

The number γ is called a statistical cluster point of $x = (x_j)$ if for every $\varepsilon > 0$ the set $\{j : |x_j - \gamma| < \varepsilon\}$ does not have density zero ([6]).

Note that throughout the paper the statement $\delta(K) \neq 0$ means that either $\delta(K) > 0$ or K does not have natural density.

Following [7] we recall the following definitions and results. For a real number sequence $x = (x_j)$ let B_x denote the set:

$$B_x := \{b \in \mathbb{R} : \delta\{j : x_j > b\} \neq 0\};$$

similarly

$$A_x := \{a \in \mathbb{R} : \delta\{j : x_j < a\} \neq 0\}.$$

Then the statistical limit superior of x is given by

$$st - \limsup x := \begin{cases} \sup B_x & , if\ B_x \neq \varnothing \\ -\infty & , if\ B_x = \varnothing. \end{cases}$$

Also, the statistical limit inferior of x is given by

$$st - \liminf x := \begin{cases} \inf A_x & , if\ A_x \neq \varnothing \\ \infty & , if\ A_x = \varnothing. \end{cases}$$

If $\beta := st - \limsup x$ is finite, then for every $\varepsilon > 0$, $\delta\{j : x_j > \beta - \varepsilon\} \neq 0$ and $\delta\{j : x_j > \beta + \varepsilon\} = 0$. We also have that $st - \limsup x \leq \limsup x$.

Recall that, by $W_q\,(q > 0)$, we denote the space of all $x = (x_j)$ such that for some L,

$$\frac{1}{n+1} \sum_{j=0}^{n} |x_j - L|^q \to 0 \quad , \ (n \to \infty)$$

If $x \in W_q$ then we say that x is strongly Cesàro convergent with index q. When $q = 1$ this space is denoted by W and it is called the space of strong Cesàro convergent sequences ([13]). It is well-known that strong Cesàro convergence and statistical convergence are equivalent on bounded sequences ([1, 3, 9]).

In order to prove an inequality relating $Q_x(t)$ to $st - \limsup x$ we need the following result which is an analog of Theorem 1 of Maddox [13] (see also [4, 11]).

Note that $P-$density of $E \subseteq \mathbb{N}$ is defined by

$$\delta_P(E) := \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j \in E} p_j t^j$$

whenever the limit exists (see, [18]).

Theorem 2. *The power series method P transforms bounded strongly convergent sequences, leaving the strong limit invariant, into the space of convergent sequences if and only if P is regular and for any subset $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ implies that*

$$\delta_P(E) = 0. \tag{2}$$

Proof. Sufficiency. Let $x \in \ell^\infty$ and strongly convergent to L. In order to prove the sufficiency it is enough to show that

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j - L| = 0. \tag{3}$$

Let $\varepsilon > 0$ and let $E_\varepsilon := \{j \in \mathbb{N} : |x_j - L| \geq \varepsilon\}$.

Since $x = (x_j)$ bounded and strongly convergent to L, it is statistically convergent to L (see [3,9]). Hence $\delta(E_\varepsilon) = 0$. This implies, by the hypothesis that, $\delta_P(E_\varepsilon) = 0$. From

$$\begin{aligned} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j - L| &= \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j |x_j - L| + \frac{1}{p(t)} \sum_{j \in E_\varepsilon^c} p_j t^j |x_j - L| \\ &\leq \sup_j |x_j - L| \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j + \varepsilon, \end{aligned}$$

we have

$$\begin{aligned} \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j - L| &\leq \|x - Le\|_\infty \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j + \varepsilon \\ &\leq \varepsilon \end{aligned}$$

because

$$\delta_P(E_\varepsilon)) := \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j \in E_\varepsilon} p_j t^j = 0.$$

We obtain that (3) is true.
Necessity. Note that any convergent sequence is statistically convergent to the same value. Since statistical convergence and strong Cesàro convergence are equivalent on the space of bounded sequences, we observe that P is regular. Assume now that there is a subset $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ such that (2) fails. This implies that E is an infinite set.

So we may write $E = \{k_j : j \in \mathbb{N}\} = \{k_1, k_2, ...\}$. Since the continuous method is regular the corresponding matrix method is also regular. Hence by the Schur theorem there exists a bounded sequences $z = \left(z_{k_1}, z_{k_2}, ...z_{k_j}, ...\right)$ which is not summable by the regular matrix method. Now define a bounded sequence, $x = (x_k)$ as follows: $x_k = z_k$ if $k = k_j$ $(j = 0, 1, 2, ...)$ and $x_k = 0$ otherwise. Since $\delta(E) = 0$, it follows from the fact that

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}|x_k-0| &= \frac{1}{n+1}\sum_{k=0}^{n}|x_k| \\ &\leq \sup_k |x_k| \frac{1}{n+1}\sum_{k=0}^{n}\chi_E(k)\to 0,\ (n\to\infty)\end{aligned}$$

i.e., the sequence $x=(x_k)$ is a bounded statistically convergent sequence which is not summable by the regular discrete method. So it is not summable by the continuous method either. This contradicts the hypothesis. □

In the rest of the paper we use the following notation:

$$\alpha(x) := st-\liminf x \text{ and } \beta(x) := st-\limsup x$$

Theorem 3. *For every $x=(x_k)\in\ell^\infty$ we have*

$$\limsup_{t\to R^-} Q_x(t) \leq st-\limsup x \tag{4}$$

if and only if P is regular and that (2) *holds.*

Proof. Let $x\in\ell^\infty$. Suppose that (4) holds. Since $\beta(x)\leq\limsup x$ it follows from (4) and Theorem 1 that P is regular. On the other hand (4) implies that

$$-\beta(-x)\leq\liminf_{t\to R^-}Q_x(t)\leq\limsup_{t\to R^-}Q_x(t)\leq\beta(x). \tag{5}$$

If $x=(x_k)$ is a bounded statistically convergent sequence, (5) implies that

$$P-\lim x = st-\lim x.$$

Hence by Theorem 2, we observe that (2) holds.

Conversely, assume P is regular and (2) holds. Let x be bounded. Then $\beta(x)$ is finite. Given $\varepsilon>0$ let $E:=\{k\in\mathbb{N}: x_j>\beta(x)+\varepsilon\}$. Hence $\delta(E)=0$ and if $k\notin E$ then $x_j\leq\beta(x)+\varepsilon$.

For a fixed positive integer m we write

$$\begin{aligned}Q_x(t) &= \frac{1}{p(t)}\sum_{j<m}p_jt^jx_j+\frac{1}{p(t)}\sum_{j\geq m}p_jt^jx_j \\ &\leq \|x\|\frac{1}{p(t)}\sum_{j<m}p_jt^j+\frac{1}{p(t)}\sum_{\substack{j\geq m\\ j\notin E}}p_jt^jx_j+\frac{1}{p(t)}\sum_{\substack{j\geq m\\ j\in E}}p_jt^jx_j \\ &\leq \|x\|\frac{1}{p(t)}\sum_{j<m}p_jt^j+(\beta(x)+\varepsilon)\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j+\|x\|\frac{1}{p(t)}\sum_{j\in E}p_jt^j\end{aligned}$$

Taking the limit superior as $t\to R^-$ and using the regularity of P we get that

$$\limsup_{t\to R^-}Q_x(t)\leq(\beta(x)+\varepsilon)+\|x\|\,\delta_P(E).$$

Recall that $\delta_P(E) = 0$ by (2). Since ε is arbitrary we conclude that (4) holds. This proves the theorem. $\square$

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References

[1] Belen, C., Yildirim, M., Sümbül, C., On statistical and strong convergence with respect to a modulus function and a power series method, *Filomat,* 34(12) (2020), 3981-3993. https://doi.org/10.2298/FIL2012981B
[2] Boos, J., Classical and Modern Methods in Summability, Oxford University Press, 2000.
[3] Connor, J., The statistical and strong $p-$Cesàro convergence of sequences, *Analysis*, 8 (1988), 47-63. https://doi.org/10.1524/anly.1988.8.12.47
[4] Demirci, K., Khan, M. K., Orhan, C., Strong and A-statistical comparisons for sequences, *J. Math. Anal. Appl.,* 278 (2003), 27-33. https://doi.org/10.1016/S0022-247X(02)00456-0
[5] Fridy, J. A., On statistical convergence, *Analysis*, 5 (1985), 301-313. https://doi.org/10.1524/anly.1985.5.4.301
[6] Fridy, J. A., Statistical limit points, *Proc. Amer. Math. Soc.,* 118 (1993), 1187-1192.
[7] Fridy, J. A., Orhan, C., Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.,* 125 (1997), 3625-3631. Doi: S 0002-9939(97)04000-8.
[8] Hardy, G. H., Divergent Series, Oxford Univ. Press, London, 1949.
[9] Khan, M. K., Orhan, C., Matrix characterization of A-statistical convergence, *J. Math. Anal. Appl.,* 335 (2007), 406-417. https://doi.org/10.1016/j.jmaa.2007.01.084
[10] Knopp, K., Zur Theorie der Limitierungsverfahren (Erste Mittilung), *Math. Zeit.,* 31 (1930), 97-127.
[11] Kolk, E., Matrix summability of statistically convergent sequences, *Analysis,* 1993.
[12] Maddox, I. J., Some analogues of Knopp's core theorem, *Inter. J. Math. and Math. Sci.,* 2 (1979), 605-614. https://doi.org/10.1155/S0161171279000454
[13] Maddox, I. J., Steinhaus type theorems for summability matrices, *Proc. Amer. Math. Soc.,* 45 (1974), 209-213.
[14] Miller, H. I., A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.,* 347 (1995), 1881-1819.
[15] Orhan, C., Sublinear functionals and Knopp's core theorem, *Internat. J. Math. and Math. Sci.,* 2 (1979), 605-614. https://doi.org/10.1155/S0161171290000680
[16] Salat, T., On statistically convergent sequences of real numbers, *Math Slovaca,* 30(2) (1980), 139-140.
[17] Ünver, M., Abel summability in topological spaces, *Monatshefte für Mathematik,* 178(4) (2015), 633-643. https://doi.org/10.1007/s00605-014-0717-0
[18] Ünver, M., Orhan, C., Statistical convergence with respect to power series methods and applications to approximation theory, *Numerical and Functional Analysis and Optimization*, 40(5) (2019), 533-547. https://doi.org/10.1080/01630563.2018.1561467

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SOME HARDY-TYPE INTEGRAL INEQUALITIES WITH SHARP CONSTANT INVOLVING MONOTONE FUNCTIONS

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Abstract. In this work, we present some Hardy-type integral inequalities for $0 < p < 1$ via a second parameter $q > 0$ with sharp constant. These inequalities are new generalizations to the inequalities given bellow.

1. Introduction

It is well-known that for L^p spaces with $0 < p < 1$, the Hardy inequality is not satisfied for arbitrary non-negative functions, but is satisfied for non-negative monotone functions. Moreover the sharp constant was found in the Hardy type-inequality for non-negative monotone functions (see [4] for more details). Namely the following statement was proved there.

Theorem 1. *Let $0 < p < 1$:*

- *If $-\frac{1}{p} < \alpha < 1-\frac{1}{p}$, then for all functions f non-negative and non-increasing on $(0,+\infty)$*

$$\|x^{\alpha}(Hf)(x)\|_{L^p(0,+\infty)} \leq \left(1-\frac{1}{p}-\alpha\right)^{-\frac{1}{p}} \|x^{\alpha}f(x)\|_{L^p(0,+\infty)}. \tag{1}$$

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- *If* $\alpha < -\frac{1}{p}$*, then for all functions* f *non-negative and non-decreasing on* $(0, +\infty)$

$$\|x^{\alpha}(Hf)(x)\|_{L^p(0,+\infty)} \leq (p\,\beta(p, -\alpha p))^{\frac{1}{p}} \|x^{\alpha} f(x)\|_{L^p(0,+\infty)}. \quad (2)$$

- *If* $\alpha > 1 - \frac{1}{p}$*, then for all functions* f *non-negative and non-increasing on* $(0, +\infty)$

$$\left\|x^{\alpha}(\widetilde{H}f)(x)\right\|_{L^p(0,+\infty)} \leq (p\,\beta(p, \alpha p + 1 - p))^{\frac{1}{p}} \|x^{\alpha} f(x)\|_{L^p(0,+\infty)}. \quad (3)$$

Here

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt, \quad (\widetilde{H}f)(x) = \frac{1}{x}\int_x^{\infty} f(t)dt.$$

$\beta(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt$ *is the Euler -Beta function.*
The constants in the inequalities (1), (2), (3) are sharp.

In 2012 W.T. Sulaiman [5] extended Hardy's integral inequality as follows.

Theorem 2. *If* $f \geq 0,\ g > 0,\ x^{-1}g(x)$ *is non-decreasing* $p > 1,\ 0 < a < 1$ *and* $F(x) = \int_0^x f(t)dt$*, then*

$$\int_0^{\infty} \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{a(p-1)(1-a)^{p-1}} \int_0^{\infty} \left(\frac{xf(x)}{g(x)}\right)^p dx, \quad (4)$$

in particular if $a = \frac{1}{p}$*,* $g(x) = x$*, we obtain Hardy's inequality.*

Moreover he proved the reverse inequality.

Theorem 3. *If* $f \geq 0,\ g > 0,\ x^{-1}g(x)$ *is non-increasing* $0 < p < 1,\ a > 0$ *and* $F(x) = \int_0^x f(t)dt$*, then*

$$\int_0^{\infty} \left(\frac{F(x)}{g(x)}\right)^p dx \geq \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^{\infty} \left(\frac{xf(x)}{g(x)}\right)^p dx. \quad (5)$$

The following Lemmas were established in [4].

Lemma 1. *Let* $0 < p < 1,\ -\infty < a < b \leq +\infty$ *and* f *a non- negative non-increasing function on* (a, b)*, then*

$$\left(\int_a^b f(x)dx\right)^p \leq p\int_a^b f^p(x)(x-a)^{p-1}dx. \quad (6)$$

Lemma 2. *Let* $0 < p < 1,\ -\infty \leq a < b < +\infty$ *and* f *a non- negative non-decreasing function on* (a, b)*, then*

$$\left(\int_a^b f(x)dx\right)^p \leq p\int_a^b f^p(x)(b-x)^{p-1}dx. \quad (7)$$

The factor p is the best possible in inequalities (6) and (7).

About the Hardy inequality, its history and some related results one can consult [1], [2], [3], [6] and [7].

The aim of this work is includes two objectives, first the power weight function x^α in Theorem 1 is replaced by $g(x)$, where $x^{-\alpha}g(x)$ is non-decreasing or non-increasing function and we give a new some Hardy-type integral inequalities with sharp constant. The second objective is to present some generalizations for the weighted Hardy operator with $0 < p < 1$. Moreover we introduce a second parameter $q > 0$ for these generalizations.

2. Main Results

In this section, we present our results. We assume that f and g are non-negative Lebesgue measurable functions on $(0, +\infty)$.

Theorem 4. *Let $0 < p < 1$, $q > 0$, $g > 0$ and the function $x^\alpha g(x)$ is non-decreasing for $-\frac{1}{q} < \alpha < \frac{p-1}{q}$, then for all non-negative non-increasing function f we have*

$$\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx \leq \frac{p}{p-\alpha\, q-1}\int_0^\infty \frac{f^p(x)}{g^q(x)}dx. \tag{8}$$

The constant in (8) is sharp.

Proof.
Since f is non-increasing, then by Lemma 1 we get

$$\begin{aligned}
\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx &= \int_0^\infty x^{-p}g^{-q}(x)\left(\int_0^x f(t)dt\right)^p dx \\
&\leq p\int_0^\infty x^{-p}g^{-q}(x)\left(\int_0^x f^p(t)t^{p-1}dt\right)dx \\
&= p\int_0^\infty t^{p-1}f^p(t)\left(\int_t^{+\infty} x^{-p}g^{-q}(x)dx\right)dt \\
&\leq p\int_0^\infty t^{p-1}f^p(t)\left(\frac{t^{-\alpha}}{g(t)}\right)^q\left(\int_t^{+\infty} x^{-p+\alpha q}dx\right)dt \\
&= \frac{p}{p-\alpha q-1}\int_0^\infty t^{p-1}f^p(t)\frac{t^{-\alpha q}}{g^q(t)}t^{-p+\alpha q+1}dt \\
&= \frac{p}{p-\alpha q-1}\int_0^\infty \frac{f^p(t)}{g^q(t)}dt.
\end{aligned}$$

To proof that $\dfrac{p}{p-\alpha q-1}$ is the best possible, we put $g(x)=x^{-\alpha}$ and

$$f(x)=\begin{cases} 1 & \text{if } x\in(0,\xi),\\ \\ 0 & \text{if } x\in(\xi,+\infty).\end{cases}$$

Let RHS and LHS respectively be the right hand side and the left hand side of the inequality (8), then

$$\begin{aligned} RHS &= \int_0^\infty x^{\alpha q-p}\left(\int_0^x f(t)dt\right)^p dx \\ &= \frac{\xi^{\alpha q+1}}{\alpha q+1}, \end{aligned}$$

and

$$\begin{aligned} LHS &= \frac{p}{p-\alpha\, q-1}\int_0^\xi x^{\alpha q}dx \\ &= \frac{p}{p-\alpha\, q-1}\frac{\xi^{\alpha q+1}}{\alpha q+1}. \end{aligned}$$

Using $q=p$ in the Theorem 4, we get the following Corollary.

Corollary 1. *Let $0<p<1$, $g>0$ and the function $x^\alpha g(x)$ is non-decreasing for $-\frac{1}{p}<\alpha<\frac{p-1}{p}$, then for all non-negative non-increasing function f we have*

$$\left\|\frac{(Hf)(x)}{g(x)}\right\|_{L^p(0,+\infty)} \le \left(1-\alpha-\frac{1}{p}\right)^{-\frac{1}{p}}\left\|\frac{f(x)}{g(x)}\right\|_{L^p(0,+\infty)}. \tag{9}$$

The constant $\left(1-\alpha-\frac{1}{p}\right)^{-\frac{1}{p}}$ is sharp.

Remark 1. *If we take $g(x)=x^{-\alpha}$ in the inequality (9), we obtain the inequality (1).*

Theorem 5. *Let $0<p<1$, $q>0$, $g>0$ and the function $x^\alpha g(x)$ is non-decreasing for $\alpha<-\frac{1}{q}$, then for all non-negative non-decreasing function f we have*

$$\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx \le p\,\beta(p,-\alpha\, q)\int_0^\infty \frac{f^p(x)}{g^q(x)}dx, \tag{10}$$

where β is the Euler-Beta function. The constant in (10) is sharp.

Proof.
By using the Lemma 2, we get

$$\begin{aligned}
\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx &= \int_0^\infty x^{-p}g^{-q}(x)\left(\int_0^x f(t)dt\right)^p dx \\
&\leq p\int_0^\infty x^{-p}g^{-q}(x)\left(\int_0^x f^p(t)(x-t)^{p-1}dt\right)dx \\
&= p\int_0^\infty f^p(t)\left(\int_t^{+\infty} x^{-p}g^{-q}(x)(x-t)^{p-1}dx\right)dt \\
&\leq p\int_0^\infty f^p(t)\left(\frac{t^{-\alpha}}{g(t)}\right)^q\left(\int_t^{+\infty} x^{\alpha q-p}(x-t)^{p-1}dx\right)dt.
\end{aligned}$$

Using the change of variable $z=\frac{t}{x}$, then

$$\begin{aligned}
\int_t^{+\infty} x^{\alpha q-p}(x-t)^{p-1}dx &= \int_0^1\left(\frac{t}{z}\right)^{\alpha q-p}\left(\frac{t}{z}-t\right)^{p-1}\frac{t}{z^2}dz \\
&= t^{\alpha q}\int_0^1 z^{-\alpha q-1}(1-z)^{p-1}dz \\
&= t^{\alpha q}\beta(p,\,-\alpha q),
\end{aligned}$$

therefore

$$\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx \quad \leq p\beta(p,\,-\alpha q)\int_0^\infty\left(\frac{f^p(t)}{g^q(t)}\right)dt.$$

To proof that $p\,\beta(p,\,-\alpha\,q)$ is the best possible, we put $g(x)=x^{-\alpha}$ and

$$f(x)=\begin{cases} 0 & \text{if } x\in(0,\xi), \\ 1 & \text{if } x\in(\xi,+\infty). \end{cases}$$

Let RHS and LHS respectively be the right side and the left side of the inequality (10), then

$$\begin{aligned}
RHS &= \int_\xi^\infty x^{\alpha q-p}\left(\int_\xi^x f(t)dt\right)^p dx \\
&= \int_\xi^\infty x^{\alpha q-p}\,(x-\xi)^p\,dx,
\end{aligned}$$

let $\mu = \frac{\xi}{x}$, then we get

$$\begin{aligned} RHS &= \xi^{\alpha q+1} \int_0^1 \mu^{-\alpha q-2}(1-\mu)^p d\mu \\ &= \xi^{\alpha q+1} \beta(p+1,\, -\alpha q-1) \\ &= \tfrac{p}{|\alpha q+1|} \xi^{\alpha q+1} \beta(p,\, -\alpha q). \end{aligned}$$

On another side

$$\begin{aligned} LHS &= p\,\beta(p,\, -\alpha\, q) \int_\xi^{+\infty} x^{\alpha q} dx \\ &= p\,\beta(p,\, -\alpha\, q) \tfrac{1}{|\alpha q+1|} \xi^{\alpha q+1}. \end{aligned}$$

If we set $q = p$ in the Theorem 5, we get the following Corollary.

Corollary 2. *Let $0 < p < 1$, $g > 0$ and the function $x^\alpha g(x)$ is non-decreasing for $\alpha < -\frac{1}{q}$, then for all non-negative non-decreasing function f we have*

$$\left\| \frac{(Hf)(x)}{g(x)} \right\|_{L^p(0,+\infty)} \le (p\,\beta(p,\, -\alpha\, p))^{\frac{1}{p}} \left\| \frac{f(x)}{g(x)} \right\|_{L^p(0,+\infty)}. \tag{11}$$

The constant $(p\,\beta(p,\, -\alpha\, p))^{\frac{1}{p}}$ is sharp.

Remark 2. *If we take $g(x) = x^{-\alpha}$ in the inequality (11), we obtain the inequality (2).*

Theorem 6. *Let $0 < p < 1$, $q > 0$, $g > 0$ and the function $x^\alpha g(x)$ is non-increasing for $\alpha > \frac{p-1}{q}$, then for all non-negative non-increasing function f we have*

$$\int_0^\infty \frac{(\widetilde{Hf})^p(x)}{g^q(x)} dx \le p\,\beta(p,\, \alpha\, q+1-p) \int_0^\infty \frac{f^p(x)}{g^q(x)} dx, \tag{12}$$

the constant in (12) is sharp.

Proof.
By applying the Lemma 1, we obtain

$$\begin{aligned}\int_0^\infty \frac{(\widetilde{Hf})^p(x)}{g^q(x)}dx &= \int_0^\infty x^{-p}g^{-q}(x)\left(\int_x^\infty f(t)dt\right)^p dx\\ &\le p\int_0^\infty x^{-p}g^{-q}(x)\left(\int_x^\infty f^p(t)(t-x)^{p-1}dt\right)dx\\ &= p\int_0^\infty f^p(t)\left(\int_0^t x^{-p}g^{-q}(x)(t-x)^{p-1}dx\right)dt\\ &\le p\int_0^\infty f^p(t)\left(\frac{t^{-\alpha}}{g(t)}\right)^q\left(\int_0^t x^{\alpha q-p}(t-x)^{p-1}dx\right)dt.\end{aligned}$$

Using the change of variable $\nu=\frac{t-x}{t}$, then

$$\begin{aligned}\int_0^t x^{\alpha q-p}(t-x)^{p-1}dx &= \int_0^1 [(1-\nu)t]^{\alpha q-p}(\nu t)^{p-1}\,td\nu\\ &= t^{\alpha q}\int_0^1 \nu^{p-1}(1-\nu)^{\alpha q-p}d\nu\\ &= t^{\alpha q}\beta(p,\,\alpha q-p+1),\end{aligned}$$

thus

$$\int_0^\infty \frac{(\widetilde{Hf})^p(x)}{g^q(x)}dx \le p\beta(p,\,\alpha q-p+1)\int_0^\infty\left(\frac{f^p(t)}{g^q(t)}\right)dt.$$

The proof that $p\,\beta(p,\,\alpha q-p+1)$ is sharp, is similar to that of Theorem 5 with the function f defined as follows

$$f(x)=\begin{cases}1 & \text{if } x\in(0,\xi),\\ 0 & \text{if } x\in(\xi,+\infty).\end{cases}$$

If we put $q=p$ in the Theorem 6, we have the following Corollary.

Corollary 3. *Let $0<p<1$, $g>0$ and the function $x^\alpha g(x)$ is non-increasing for $\alpha<-\frac{1}{q}$, then for all non-negative non-increasing function f we have*

$$\left\|\frac{(\widetilde{Hf})(x)}{g(x)}\right\|_{L^p(0,+\infty)} \le (p\,\beta(p,\,\alpha p+1-p))^{\frac{1}{p}}\left\|\frac{f(x)}{g(x)}\right\|_{L^p(0,+\infty)}. \tag{13}$$

The constant $(p\,\beta(p,\,\alpha p+1-p))^{\frac{1}{p}}$ is sharp.

Remark 3. *If we take $g(x)=x^{-\alpha}$ in the inequality (13), we obtain the inequality (3).*

In the second part of this work, we consider Theorems 2 and 3 for weighted Lebesgue space. Let $0 < p < \infty$, the weighted Lebesgue space $L^p_w(0,\infty)$ is the space of all Lebesgue measurable functions f such that

$$\|f\|_{L^p_w(0,\infty)} = \left(\int_0^\infty |f(t)|^p \, w(t)dt\right)^{\frac{1}{p}} < \infty, \tag{14}$$

where w is the weight function (Lebesgue measurable and positive on $(0,\infty)$).

Theorem 7. *Let $f \geq 0$, $g > 0$, $0 < p < 1$, $0 < \alpha < 1$. If the function $\frac{w(x)}{g^p(x)}$ is non-increasing, then*

$$\left\|\frac{(Hf)(x)}{g(x)}\right\|_{L^p_w(0,\infty)} \leq C_1 \left\|\frac{f(x)}{g(x)}\right\|_{L^p_w(0,\infty)}, \tag{15}$$

where the constant $C_1 = \frac{1}{1-\alpha}$ is sharp.

Proof.
By using Holder's inequality, we have

$$\begin{aligned}
\left\|\tfrac{(Hf)(x)}{g(x)}\right\|^p_{L^p_w(0,\infty)} &= \int_0^\infty \frac{(Hf)^p(x)}{g^p(x)} w(x)dx \\
&= \int_0^\infty \frac{g^{-p}(x)}{x^p}\left(\int_0^x f(t)t^{\alpha(1-\frac{1}{p})}t^{\alpha(\frac{1}{p}-1)}dt\right)^p w(x)dx \\
&\leq \int_0^\infty \frac{g^{-p}(x)}{x^p} w(x)\left(\int_0^x t^{\alpha(p-1)} f^p(t)dt\right)\left(\int_0^x t^{-\alpha}dt\right)^p dx \\
&= \left(\frac{1}{1-\alpha}\right)^{p-1}\int_0^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)\left(\int_0^x t^{\alpha(p-1)} f^p(t)dt\right) dx \\
&= \left(\frac{1}{1-\alpha}\right)^{p-1}\int_0^\infty t^{\alpha(p-1)} f^p(t)\left(\int_t^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)dx\right) dt \\
&= \left(\frac{1}{1-\alpha}\right)^{p-1}\int_0^\infty \frac{f^p(t)}{g^p(t)} w(t)K(t)dt,
\end{aligned}$$

where

$$K(t) = \left[\frac{t^{\alpha(p-1)}g^p(t)}{w(t)}\left(\int_t^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)dx\right)\right].$$

Now we proof that $K(t)$ is finite for all $t > 0$. From the assumption $\frac{w(x)}{g^p(x)}$ is non-increasing, we deduce that

$$\int_t^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)dx \leq \frac{w(t)}{g^p(t)} \int_t^\infty x^{\alpha(p-1)-1}dx$$

$$= \frac{w(t)}{g^p(t)} \frac{t^{\alpha(p-1)}}{\alpha(1-p)},$$

hence

$$\text{for all } t > 0, \ K(t) < \infty.$$

Thus

$$\left\| \frac{(Hf)(x)}{g(x)} \right\|^p_{L^p_w(0,\infty)} \leq \frac{\sup\limits_{t>0} K(t)}{(1-\alpha)^{p-1}} \left\| \frac{f(x)}{g(x)} \right\|_{L^p_w(0,\infty)}$$

$$= C^p \left\| \frac{f(x)}{g(x)} \right\|^p_{L^p(0,+\infty)}.$$

To proof that $C_1 = \left(\frac{1}{1-\alpha}\right)$ is the best possible, taking $f(x) = x^{-\alpha}$, this gives us $(Hf)(x) = \frac{1}{1-\alpha} x^{-\alpha}$ and

$$\left\| \frac{(Hf)(x)}{g(x)} \right\|^p_{L^p_w(0,\infty)} = \frac{1}{(1-\alpha)^p} \int_0^\infty \left(\frac{1}{x^\alpha g(x)} \right)^p w(x)dx,$$

$$\left\| \frac{f(x)}{g(x)} \right\|^p_{L^p_w(0,\infty)} = \int_0^\infty \left(\frac{1}{x^\alpha g(x)} \right)^p w(x)dx.$$

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References

[1] Benaissa, B., Budak, H., On Hardy-type integral inequalities with negative parameter, *Turkish Journal of Inequalities*, 5(2) (2021), 42-47.

[2] Benaissa, B., Sarikaya, M. Z., Senouci, A., On some new Hardy-type inequalities, *J. Math. Meth. Appl. Sci.*, 43 (2020), 8488-8495. https/doi.org/10.1002/mma.6503

[3] Yang, B., On a new Hardy type integral inequalities, *Int. Math. Forum.*, 67(2) (2007), 3317–3322.

[4] Burenkov, V. I., On the best constant in Hardy's inequality with $0 < p < 1$ for monotone functions, *Proc. Steklov Inst. Math.,* 194 (1993), 59-63.
[5] Sulaiman, W. T., Some Hardy type integral inequalities, *Appl. Math. Lett,* 25 (2012), 520-525. https://doi.org/10.1016/j.aml.2011.09.050
[6] Kufner, A., Maligranda, L., Person, L. E., The Hardy Inequality: About Its History and Some Related Results, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[7] Banyat, S., More on some Hardy type integral inequalities, *J. Math. Inequal.*, 8(3) (2014), 497–501.

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COMBINATORIAL RESULTS OF COLLAPSE FOR ORDER-PRESERVING AND ORDER-DECREASING TRANSFORMATIONS

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Abstract. The full transformation semigroup $\mathcal{T}_n$ is defined to consist of all functions from $X_n = \{1, \ldots, n\}$ to itself, under the operation of composition. In [9], for any α in $\mathcal{T}_n$, Howie defined and denoted collapse by $c(\alpha) = \bigcup_{t \in im(\alpha)} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}$. Let $\mathcal{O}_n$ be the semigroup of all order-preserving transformations and $\mathcal{C}_n$ be the semigroup of all order-preserving and decreasing transformations on X_n under its natural order, respectively. Let $E(\mathcal{O}_n)$ be the set of all idempotent elements of $\mathcal{O}_n$, $E(\mathcal{C}_n)$ and $N(\mathcal{C}_n)$ be the sets of all idempotent and nilpotent elements of $\mathcal{C}_n$, respectively. Let U be one of $\{\mathcal{C}_n, N(\mathcal{C}_n), E(\mathcal{C}_n), \mathcal{O}_n, E(\mathcal{O}_n)\}$. For $\alpha \in U$, we consider the set $im^c(\alpha) = \{t \in im(\alpha) : |t\alpha^{-1}| \geq 2\}$. For positive integers $2 \leq k \leq r \leq n$, we define

$$\begin{aligned} \mathcal{U}(k) &= \{\alpha \in \mathcal{U} : t \in im^c(\alpha) \text{ and } |t\alpha^{-1}| = k\}, \\ \mathcal{U}(k,r) &= \{\alpha \in \mathcal{U}(k) : |\bigcup_{t \in im^c(\alpha)} t\alpha^{-1}| = r\}. \end{aligned}$$

The main objective of this paper is to determine $|\mathcal{U}(k,r)|$, and so $|\mathcal{U}(k)|$ for some values r and k.

1. Introduction

For any non-empty finite set X, the set $\mathcal{T}_X$ of all transformations of X (i.e., all maps X to itself) is a semigroup under composition, and is called the *full transformation semigroup* on X. For any $n \in \mathbb{N}$, if $X = X_n = \{1, \ldots, n\}$, then $\mathcal{T}_X$ is denoted by $\mathcal{T}_n$. A transformation $\alpha \in \mathcal{T}_n$ is called *order-preserving*, if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$ and *decreasing* (*increasing*), if $x\alpha \leq x$ ($x\alpha \geq x$) for all $x \in X_n$. The subsemigroup of all order-preserving transformations in $\mathcal{T}_n$ is denoted by $\mathcal{O}_n$ and the order-decreasing transformations in $\mathcal{T}_n$ is denoted by $\mathcal{D}_n$.

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The subsemigroup of all order-preserving and decreasing (increasing) transformations in $\mathcal{T}_n$ is denoted by $\mathcal{C}_n$ ($\mathcal{C}_n^+$) i.e., $\mathcal{C}_n = \mathcal{O}_n \cap \mathcal{D}_n$. Umar proved that $\mathcal{D}_n$ and $\mathcal{D}_n^+$ are isomorphic in [15, Corollary 2.7.]. Furthermore, Higgins proved that $\mathcal{C}_n$ and $\mathcal{C}_n^+$ are isomorphic semigroups in [8, Remarks, p. 290]. For any transformation α in $\mathcal{T}_n$, the *collapse*, the *fix*, the *image* and the *kernel* are denoted and definded, respectively, by

$$\begin{aligned} c(\alpha) &= \bigcup_{t\in im(\alpha)} \{t\alpha^{-1} : |t\alpha^{-1}| \geq 2\}, \quad (\alpha) = \{x \in X_n : x\alpha = x\}, \\ im(\alpha) &= \{x\alpha : x \in X_n\}, \text{ and } \ker(\alpha) = \{(x,y) : x\alpha = y\alpha \text{ for all } x, y \in X_n\}. \end{aligned}$$

Given transformation α in $\mathcal{T}_n$ is called *collapsible*, if there exists $t \in im(\alpha)$ such that $|t\alpha^{-1}| \geq 2$.

An element e of a semigroup S is called *idempotent* if $e^2 = e$ and the set of all idempotents in S is denoted by $E(S)$. An element a of a finite semigroup S with a zero, denoted by 0, is called *nilpotent* if $a^m = 0$ for some positive integer m, and furthermore, if $a^{m-1} \neq 0$, then a is called an *m-nilpotent* element of S. Note that zero element is an 1-nilpotent element. The set of all nilpotent elements of S is denoted by $N(S)$. It was proven a finite semigroup S with zero is nilpotent when exactly the unique idempotent of S is the zero element (see, [6, Proposition 8.1.2]). The reader is referred to [5] and [11] for additional details in semigroup theory.

Recall that *Fibonacci sequence* $\{f_n\}$ is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$, where $f_1 = f_2 = 1$ (see [10]). As proved in [13, Theorem 2.1], $|\mathcal{C}_n| = |\mathcal{C}_n^+| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th *Catalan number* for $n \geq 1$ (see, [7]). That is why $\mathcal{C}_n$ is also called the *Catalan monoid*. In [13, Proposition 2.3] and [8, Theorem 3.19], it has been shown that $|N(C_n)| = |N(C_n^+)| = C_{n-1}$ and $|E(\mathcal{C}_n)| = |E(\mathcal{C}_n^+)| = 2^{n-1}$. Also, from [10, Theorem 2.1 and Theorem 2.3], we have that $|\mathcal{O}_n| = \binom{2n-1}{n-1}$ and $|E(\mathcal{O}_n)| = f_{2n}$.

As indicated in [5] if $\alpha \in \mathcal{C}_n$, we use

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix} \tag{1}$$

to notifty that $im(\alpha) = \{a_1 = 1 < a_2 < \ldots < a_r\}$ and $a_i\alpha^{-1} = A_i$ for each $1 \leq i \leq r$. Furthermore, $A_1, A_2, \ldots, A_r$ which are pairwise disjoint subsets of X_n are called *blocks* of α. It is clear that such an α is an idempotent if and only if $a_i \in A_i$ for all i. As defined in [4] a set $K \subseteq X_n$ is called *convex* if K is in the form $[i, i+t] = \{i, i+1, \ldots, i+t-1, i+t\}$. A partition $P = \{A_1, \ldots, A_r\}$ of X_n for $1 \leq r \leq n$ is called an *ordered partition*, and written $P = (A_1 < \cdots < A_r)$ if $x < y$ for all $x \in A_i$ and $y \in A_{i+1}$ $(1 \leq i \leq r-1)$. For a given $\alpha \in \mathcal{C}_n$ let $im(\alpha) = \{a_1 = 1 < a_2 < \ldots < a_r\}$ and $A_i = a_i\alpha^{-1}$ for every $1 \leq i \leq r$. Then, the set of kernel clasess of α, $X_n/\ker(\alpha) = \{A_1, \ldots, A_r\}$, is an ordered convex partition of X_n. Since $N(\mathcal{C}_n)$ is a nilpotent subsemigroup of $\mathcal{C}_n$, if $\alpha \in N(\mathcal{C}_n)$, then $1\alpha = 2\alpha = 1$, and that $|1\alpha^{-1}| \geq 2$.

Several authors studied certain problems concerning combinatorial aspects of semigroup theory during the years. The vast majority of papers have been written in this area such as [3, 9, 12, 13, 15, 16]. The rank (minimal size of a generating set) and idempotent rank (minimal size of an idempotent generating set) of several transformations semigroups have been studied in [9], [12] and [16] by using the combinatorial methods. A mapping $\alpha : dom(\alpha) \subseteq X_n \to im(\alpha) \subseteq X_n$ is called a *partial transformation*, and the set of all partial transformations is a semigroup under composition and denoted by $\mathcal{P}_n$. In the articles [1] and [14] the numbers $|\mathcal{T}_n(k,r)|$ and $|\mathcal{P}_n(k,r)|$ were calculated for $r = k = 2, 3$. Since then, $\mathcal{T}_n(k,r)$ were determined for $r = k$ for $2 \leq k \leq n$ in [2]. In the present paper, we calculate $|\mathcal{C}_n(k,k)|$, $|\mathcal{C}_n(k,2k)|$, $|\mathcal{C}_n(2,n)|$, $|N(\mathcal{C}_n)(k,k)|$, $|N(\mathcal{C}_n)(k,2k)|$, $|N(\mathcal{C}_n)(2,n)|$, $|E(\mathcal{C}_n)(k,r)|$, $|\mathcal{O}_n(k,k)|$ and $|E(\mathcal{O}_n)(k,k)|$. These invariants could be interesting and useful in the study of structure of semigroups.

2. Collapsible elements in $\mathcal{C}_n$

Let $\mathcal{U}(k,r) = \mathcal{C}_n(k,r)$ for positive integers $2 \leq k \leq r \leq n$. Then, it is obvious that $|\mathcal{C}_n(k,r)| = 0$ if k does not divide r, and further $|\mathcal{C}_n(n,n)| = 1$. Note that 1_n which denotes identity element of $\mathcal{C}_n$ and $\mathcal{O}_n$ is the only non-collapsible element of $\mathcal{C}_n$ and $\mathcal{O}_n$ then, the number of collapsible elements in $\mathcal{C}_n$ and $\mathcal{O}_n$ are $C_n - 1$ and $\binom{2n-1}{n-1} - 1$, respectively. The proof of the next combinatorial result is easy and is omitted.

Lemma 1. *For positive integers k and n where $1 \leq k \leq n$,*

$$\sum_{i=1}^{n-k+1} \binom{n-i}{n-k-i+1} = \binom{n}{k}.$$

□

Theorem 1. *For positive integers k and n where $2 \leq k \leq n$,*

$$|\mathcal{C}_n(k,k)| = \binom{n}{k}.$$

Proof. For a given $\alpha \in \mathcal{C}_n(k,k)$ it is clear that there exists $i \in im(\alpha)$ such that $|i\alpha^{-1}| = k$ and $\min(i\alpha^{-1}) = i$. So,

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & [i, k+i-1] & \{k+i\} & \cdots & \{n\} \\ 1 & 2 & \cdots & i-1 & i & (k+i)\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \leq i \leq n-k+1$. As can be seen the above form, we choose elements of $im(\alpha)$ from the set $[i+1, n]$ for the set $[k+i, n]$. There are $\binom{n-(i+1)+1}{n-(k+i)+1} = \binom{n-i}{n-k-i+1}$ ways to do that. This yields, there are $\binom{n-i}{n-k-i+1}$ elements in $\mathcal{C}_n(k,k)$ for a fixed i. Since $1 \leq i \leq n-k+1$, it follows directly from Lemma 1 that

$$|\mathcal{C}_n(k,k)| = \sum_{i=1}^{n-k+1} \binom{n-i}{n-k-i+1} = \binom{n}{k}.$$

□

Our next result computes $|\mathcal{C}_n(k,2k)|$.

Proposition 1. *For positive integers k and n where $2 \le k \le n$,*

$$|\mathcal{C}_n(k,2k)| = \sum_{i=1}^{n-2k+1} \sum_{j=i+k}^{n-k+1} \sum_{l=j-k+1}^{j} \binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}.$$

Proof. Given $\alpha \in \mathcal{C}_n(k,2k)$, let $A_i = [i, k+i-1]$ and $A_j = [j, k+j-1]$ be any two blocks of α each of which contain k elements. So,

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & A_i & \{k+i\} & \cdots & A_j & \cdots & \{n\} \\ 1 & 2 & \cdots & i-1 & i & (k+i)\alpha & \cdots & j\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \le i \le n-2k+1$ and $i+k \le j \le n-k+1$. Let $j\alpha = l$ where $j-k+1 \le l \le j$. As can be seen above form, we choose elements of $im(\alpha)$ from the set $[i+1, l-1]$ for the set $[k+i, j-1]$ and from the set $[l+1, n]$ for the set $[k+j, n]$. However, this can be done $\binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}$ ways. This yields, there are $\binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}$ elements in $\mathcal{C}_n(k,2k)$ for fixed i, j and l. Since $1 \le i \le n-2k+1$, $i+k \le j \le n-k+1$ and $j-k+1 \le l \le j$, it follows quickly that

$$|\mathcal{C}_n(k,2k)| = \sum_{i=1}^{n-2k+1} \sum_{j=i+k}^{n-k+1} \sum_{l=j-k+1}^{j} \binom{l-i-1}{j-k-i}\binom{n-l}{n-k-j+1}.$$

□

Theorem 2. *For positive even integer $n \ge 2$,*

$$|\mathcal{C}_n(2,n)| = \frac{2}{(n+2)}\binom{n}{\frac{n}{2}}.$$

Proof. For any $\alpha \in \mathcal{C}_n(2,n)$, it is clear that n must be even, and so $|\mathcal{C}_n(n,2)| = 0$ if 2 does not divide n. Then, the result will clearly follow if we establish a bijection between $\mathcal{C}_n(2,n)$ and $\mathcal{C}_{\frac{n}{2}}$. Define a map $\theta : \mathcal{C}_n(2,n) \to \mathcal{C}_{\frac{n}{2}}$ by $(\alpha)\theta = \alpha'$ where

$$\begin{cases} (2i-1)\alpha = i\alpha' + i - 1, & i = 1, 2, \ldots, \frac{n}{2}; \\ (2i)\alpha = i\alpha' + i - 1, & i = 1, 2, \ldots, \frac{n}{2}, \end{cases}$$

that is,

$$\begin{cases} j\alpha = (\frac{j+1}{2})\alpha' + \frac{j-1}{2}, & j = 1, 3, \ldots, n-1; \\ j\alpha = \frac{j}{2}\alpha' + \frac{j-2}{2}, & j = 2, 4, \ldots, n. \end{cases}$$

This yields, θ is a well-defined bijection. Since $|\mathcal{C}_{\frac{n}{2}}| = C_{\frac{n}{2}}$, the proof is completed.

□

Example 1. *The function $\theta : \mathcal{C}_6(2,6) \to \mathcal{C}_{\frac{6}{2}}$ defined as in above is a bijection. Certainly,*

$$\mathcal{C}_6(2,6) = \left\{ \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 5 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} \\ 1 & 3 & 5 \end{pmatrix} \right\} \text{ and}$$

$$\mathcal{C}_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\},$$

as wanted. □

Let $\mathcal{U}(k,r) = N(\mathcal{C}_n)(k,r)$ for positive integers $2 \le k \le r \le n$. Clearly, $|N(\mathcal{C}_n)(k,r)| = 0$ if k does not divide r, and also $|N(\mathcal{C}_n)(n,n)| = 1$ and $|N(\mathcal{C}_n(n-1,n-1)| = n-2$. Note that $\alpha \in N(\mathcal{C}_n)$, $1\alpha = 2\alpha = 1$ and $i\alpha \le i-1$ for all $3 \le i \le n$, and so the number of collapsible emenets in $N(\mathcal{C}_n)$ is $|N(\mathcal{C}_n)| = C_{n-1}$.

Lemma 2. *For positive integers k and n where $2 \le k \le n$,*

$$|N(\mathcal{C}_n)(k,k)| = \binom{n-2}{n-k}.$$

Proof. Given $\alpha \in N(\mathcal{C}_n)(k,k)$, since $1\alpha = 2\alpha = 1$ and $|1\alpha^{-1}| = k$, we have

$$\alpha = \begin{pmatrix} [1,k] & \{k+1\} & \{k+2\} & \cdots & \{n\} \\ 1 & (k+1)\alpha & (k+2)\alpha & \cdots & n\alpha \end{pmatrix}.$$

As can be seen above form, we choose elements of $im(\alpha)$ from the set $[2,n]$ for the set $[k+1, n-1]$. However, there are

$$|N(\mathcal{C}_n)(k,k)| = \binom{n-2}{n-k}$$

ways to do that, as required. □

Proposition 2. *For positive integers k and n where $2 \le k \le n$,*

$$|N(\mathcal{C}_n)(k,2k)| = \sum_{j=k+1}^{n-k+1} \sum_{l=2}^{j} \binom{l-2}{j-k-1} \binom{n-l}{n-k-j+1}.$$

Proof. Given $\alpha \in N(\mathcal{C}_n)(k,2k)$, let $A_1 = [1,k]$ and $A_j = [j, k+j-1]$ be any two blocks of α which contain k elements. This yields,

$$\alpha = \begin{pmatrix} A_1 & \{k+1\} & \cdots & A_j & \{n\} \\ 1 & (k+1)\alpha & \cdots & j\alpha & n\alpha \end{pmatrix},$$

where $k+1 \le j \le n-k+1$. Let $j\alpha = l$ where $2 \le l \le j$. As can be seen above form, we choose element of $im(\alpha)$ from the set $[2, l-1]$ for the set $[k+1, j-1]$ and from the set $[l+1, n]$ for the set $[k+j, n]$. However, this can be done $\binom{l-2}{j-k-1}\binom{n-l}{n-k-j-1}$ ways. This yields, there are $\binom{l-2}{j-k-1}\binom{n-l}{n-k-j-1}$ elements in $N(\mathcal{C}_n)(k,2k)$ for fixed j and l. Since $k+1 \le j \le n-k+1$ and $2 \le l \le j$, it follows quickly that

$$|N(\mathcal{C}_n)(k,2k)| = \sum_{j=k+1}^{n-k+1} \sum_{l=2}^{j} \binom{l-2}{j-k-1}\binom{n-l}{n-k-j+1}.$$

□

Theorem 3. *For positive even integer $n \ge 2$,*

$$|N(\mathcal{C}_n)(2,n)| = \frac{2}{n}\binom{n-2}{\frac{n-2}{2}}.$$

Proof. Let α be any element of $N(\mathcal{C}_n)(n,2)$. Then, it is clear that n must be even, and so $|N(\mathcal{C}_n)(2,n)| = 0$ if 2 does not divide n. If we construct a bijection between $N(\mathcal{C}_{\frac{n}{2}})$ and $|N(\mathcal{C}_n)(2,n)|$, then this completes the proof. Define a map $\theta : N(\mathcal{C}_n)(2,n) \to N(\mathcal{C}_{\frac{n}{2}})$ by $(\alpha)\theta = \alpha'$ where

$$\begin{cases} (2i-1)\alpha = i\alpha' + i - 1, & i = 1,2,\ldots,\frac{n}{2}; \\ (2i)\alpha = i\alpha' + i - 1, & i = 1,2,\ldots,\frac{n}{2}, \end{cases}$$

that is,

$$\begin{cases} j\alpha = (\frac{j+1}{2})\alpha' + \frac{j-1}{2}, & j = 1,3,\ldots,n-1; \\ j\alpha = \frac{j}{2}\alpha' + \frac{j-2}{2}, & j = 2,4,\ldots,n. \end{cases}$$

Now it is easy to check that θ is a well-defined bijection. Since $|N(\mathcal{C}_{\frac{n}{2}})| = C_{\frac{n}{2}-1}$, the proof is complete. □

Example 2. *The function $\theta : N(\mathcal{C}_8)(2,8) \to N(\mathcal{C}_{\frac{8}{2}})$ defined as in above is a bijection. Indeed,* $= N(\mathcal{C}_8)(2,8) =$

$$\left\{ \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 3 & 5 \end{pmatrix}, \right.$$
$$\begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 3 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 4 & 5 \end{pmatrix},$$
$$\left. \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} \\ 1 & 2 & 4 & 6 \end{pmatrix} \right\} \text{ and}$$

$$N(\mathcal{C}_4) = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix} \right\},$$

as required. □

Let $\mathcal{U}(k,r) = E(\mathcal{C}_n)(k,r)$ for positive integers $2 \leq k \leq r \leq n$. Clearly, $|E(\mathcal{C}_n)(k,r)| = 0$ if k does not divide r, and also $|E(\mathcal{C}_n)(n,n)| = 1$. Note that the number of collapsible elements in $E(\mathcal{C}_n)$ is $2^{n-1} - 1$.

Theorem 4. *For positive integers k, r and n where $2 \leq k \leq r \leq n$ and $r = kt$,*

$$|E(\mathcal{C}_n)(k,r)| = \binom{n+t-r}{t}.$$

Proof. If $\alpha \in E(\mathcal{C}_n)(k,r)$ and $r = kt$, then $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_{n+t-r} \\ 1 & a_2 & \cdots & a_{n+t-r} \end{pmatrix}$, where $a_i \in A_i$ for all $1 \leq i \leq n+t-r$. Since $r = kt$, ordered partition of α contains $n+t-r$ blocks such that t blocks contain k elements and $n - kt$ blocks contain one element. Without loss of generality assume that each of the sets $A_1, A_2, \ldots, A_t$ contains k elements and each of the sets $A_{t+1}, A_{t+2}, \ldots, A_{n+t-r}$ contains one element. Since α is an idempotent, it is clear that α is the only element in $E(\mathcal{C}_n)(k,r)$ with this ordered partition. Hence, all elements of $E(\mathcal{C}_n)(k,r)$ are entirely determined by choosing t blocks which contain k elements. Since we choose t blocks $\binom{n+t-r}{t}$ ways, this completes the proof. □

The next result is clear from the definition of $\mathcal{U}(k)$ and $\mathcal{U}(k,r)$:

$$|\mathcal{U}(k)| = \sum_{i=1}^{t} |\mathcal{U}(k,ik)|,$$

where $t = \frac{n}{k}$.

Example 3. *We obtain $|E(\mathcal{C}_6)(2,4)| = \binom{6+2-4}{2} = 6$ by Theorem 4. Since $n = 6, r = 4, k = 2, t = 2$, each element in $E(\mathcal{C}_6)(2,4)$ have $6+2-4$ blocks such that 2 blocks contain 2 elements and 2 blocks are singletons. Indeed, $E(\mathcal{C}_6)(2,4) =$*

$$\left\{ \begin{pmatrix} \{1,2\} & \{3,4\} & \{5\} & \{6\} \\ 1 & 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4,5\} & \{6\} \\ 1 & 3 & 4 & 6 \end{pmatrix}, \begin{pmatrix} \{1,2\} & \{3\} & \{4\} & \{5,6\} \\ 1 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2,3\} & \{4,5\} & \{6\} \\ 1 & 2 & 4 & 6 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2,3\} & \{4\} & \{5,6\} \\ 1 & 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} \{1\} & \{2\} & \{3,4\} & \{5,6\} \\ 1 & 2 & 3 & 5 \end{pmatrix} \right\}.$$

Furthermore, $|E(\mathcal{C}_6)(2)| = \sum_{i=1}^{3} |E(\mathcal{C}_6)(2, i2)| = |E(\mathcal{C}_6)(2, 2)| + |E(\mathcal{C}_6)(2, 4)| + |E(\mathcal{C}_6)(2, 6)| = \binom{6+1-2}{1} + \binom{6+2-4}{2} + \binom{6+3-6}{3} = 12.$ □

3. Collapsible elements in $\mathcal{O}_n$

Let $U(k, r) = \mathcal{O}_n(k, r)$ for positive integers $2 \le k \le r \le n$. Then, it is clear that $|(\mathcal{O}_n)(r, k)| = 0$ if k does not divide r, and also $|(\mathcal{O}_n)(n, n)| = n$. By convention, we take $\binom{0}{0} = 1$ in the following theorem.

Theorem 5. *For positive integers k and n where $2 \le k \le n$,*

$$|\mathcal{O}_n(k, k)| = \sum_{i=1}^{n-k+1} \sum_{j=i}^{k+i-1} \binom{j-1}{i-1} \binom{n-j}{n-k-i+1}.$$

Proof. For any $\alpha \in \mathcal{O}_n(k, k)$, let

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & [i, k+i-1] & \{k+i\} & \cdots & \{n\} \\ 1\alpha & 2\alpha & \cdots & (i-1)\alpha & i\alpha & (k+i)\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \le i \le n-k+1$. As can be seen above form, the set of all value of $i\alpha$ is the set $[i, k+i-1]$ and for all distinct $m, r \in X_n \setminus [i, k+i-1]$, it is clear that $m\alpha \ne r\alpha$. Let $i\alpha = j$ where $i \le j \le k+i-1$. Then, we choose elements of $im(\alpha)$ for the left and right sides of $i\alpha = j$. For the left side, we choose elements from the set $[1, j-1]$ for the set $[1, i-1]$. There are $\binom{j-1}{i-1}$ ways to do that. For the right side, we choose the elements from the set $[j+1, n]$ for the set $[k+i, n]$. There are $\binom{n-j}{n-k-i+1}$ ways to do that. This yields, there are $\binom{j-1}{i-1}\binom{n-j}{n-k-i+1}$ elements in $\mathcal{O}_n(k, k)$ for fixed i and j. Since $1 \le i \le n-k+1$ and $i \le j \le k+i-1$, it follows that

$$|\mathcal{O}_n(k, k)| = \sum_{i=1}^{n-k+1} \sum_{j=i}^{k+i-1} \binom{j-1}{i-1} \binom{n-j}{n-k-i+1}.$$

□

Let $\mathcal{U}(k, r) = E(\mathcal{O}_n)(k, r)$ for positive integers $2 \le k \le r \le n$. Clearly, $|E(\mathcal{O}_n)(k, r)| = 0$ if k does not divide r. Notice that the number of collapsible elements in $E(\mathcal{O}_n)$ is $f_{2n} - 1$.

Lemma 3. *For positive integers k and n where $2 \le k \le n$,*

$$|E(\mathcal{O}_n)(k, k)| = k(n-k+1).$$

Proof. For any $\alpha \in \mathcal{O}_n(k, k)$, let

$$\alpha = \begin{pmatrix} \{1\} & \{2\} & \cdots & \{i-1\} & [i, k+i-1] & \{k+i\} & \cdots & \{n\} \\ 1\alpha & 2\alpha & \cdots & (i-1)\alpha & i\alpha & (k+i)\alpha & \cdots & n\alpha \end{pmatrix},$$

where $1 \le i \le n-k+1$. As can be seen above form, the set of all value of $i\alpha$ is the set $[i, k+i-1]$. Moreover, since α is an idempotent, $m\alpha = m$ for all $m \in X_n \setminus [i, k+i-1]$. Let $i\alpha = j$ where $i \le j \le k+i-1$. Then, it is easy to see that α is the only element in $E(\mathcal{O}_n)(k,k)$ for fixed i and j. Since $i \leq j \leq k+i-1$, there are k elements in $E(\mathcal{O}_n)(k,k)$ for fixed i. Since $1 \leq i \leq n-k+1$, it follows that

$$|E(\mathcal{O}_n)(k,k)| = k(n-k+1).$$

□

Declaration of Competing Interests The author has no competing interests to declare.

References

[1] Adenji, A.O., Science, C., Makanjuola, S.O., On some combinatorial results of collapse and properties of height in full transformation semigroups, *Afr. J. Comp. ICT.,* 1(2) (2008), 61-63.

[2] Adenji, A.O., Science, C., Makanjuola, S.O., A combinatorial property of the full transformation semigroup, *Afr. J. Comp. ICT.,* 2(1) (2009), 15-19.

[3] Ayık, G., Ayık, H., Koç, M., Combinatorial results for order-preserving and order-decreasing transformations, *Turk. J. Math.,* 35(4) (2011), 1-9. https://doi.org/10.3906/mat-1010-432

[4] Catarino P. M., Higgins, P.M., The monoid of orientation-preserving mappings on a chain, *Semigroup Forum,* 58(2) (1999), 190-206. https://doi.org/10.1007/s002339900014

[5] Clifford, A.H., Preston, G.B., The algebraic theory of semigroups. Vol.II., American Math. Soc., Providence, 1967.

[6] Ganyushkin, O., Mazorchuk, V., Classical Finite Transformation Semigroups, Springer-Verlag, London, 2009.

[7] Grimaldi R.P., Discrete and combinatorial mathematics, Pearson Education Inc., USA, 2003.

[8] Higgins, P.M., Combinatorial results for semigroups of order-preserving mappings, *Math. Proc. Cambridge Phil. Soc.,* 113(2) (1993), 281-296. https://doi.org/10.1017/S0305004100075964

[9] Howie, J. M., The subsemigroup generated by the idempotents of a full transformation semigroup, J. *London Math. Soc.,* 41(1) (1966), 707-716. https://doi.org/10.1112/jlms/s1-41.1.707

[10] Howie, J. M., Products of idempotents in certain semigroups of transformations, *Proc. Edinb. Math. Soc.,* 17(3) (1970/71), 223-236. https://doi.org/10.1017/S0013091500026936

[11] Howie, J.M., Fundamentals of Semigroup Theory, Oxford University Press, New York, 1995.

[12] Korkmaz, E., Ayık, H., Ranks of nilpotent subsemigroups of order-preserving and decreasing transformation semigroups, *Turk. J. Math.,* 45(4) (2021), 1626-1634. https://doi.org/10.3906/mat-2008-19

[13] Laradji, A., Umar, A., On certain finite semigroups of order-decreasing transformations I, *Semigroup Forum,* 69(2) (2004), 184-200. https://doi.org/10.1007/s00233-004-0101-9

[14] Mbah, M. A., Ndubisi, R. U., Achaku, D. T., On some combinatorial results of collapse in partial transformation semigroups, *Canadian Journal of Pure and Applied Sciences,* 1, (2020) 257-261. https://doi.org/10.15864/jmscm.1301

[15] Umar, A., Semigroups of order-decreasing transformations: The isomorphism theorem, *Semigroup Forum,* 53(1) (1996), 220–224. https://doi.org/10.1007/BF02574137

[16] Yağcı, M., Korkmaz, E., On nilpotent subsemigroups of the order-preserving and decreasing transformation semigroups, *Semigroup Forum*, 101(2) (2020), 486-289. https://doi.org/10.1007/s00233-020-10098-2

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COEFFICIENTS OF RANDIĆ AND SOMBOR CHARACTERISTIC POLYNOMIALS OF SOME GRAPH TYPES

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Abstract. Let G be a graph. The energy of G is defined as the summation of absolute values of the eigenvalues of the adjacency matrix of G. It is possible to study several types of graph energy originating from defining various adjacency matrices defined by correspondingly different types of graph invariants. The first step is computing the characteristic polynomial of the defined adjacency matrix of G for obtaining the corresponding energy of G. In this paper, formulae for the coefficients of the characteristic polynomials of both the Randić and the Sombor adjacency matrices of path graph P_n, cycle graph C_n are presented. Moreover, we obtain the five coefficients of the characteristic polynomials of both Randić and Sombor adjacency matrices of a special type of $3-$regular graph R_n.

1. Introduction

Let $G = (V, E)$ be a simple graph with the number of n vertices and m edges. If two vertices v_i and v_j are connected with an edge e, then they are called adjacent vertices and they are expressed as $e = v_iv_j$ or $e = v_jv_i$. If a vertex v is a terminal point of edge e, then they are called incident. Degree of a vertex v_i is the number of edges that are incident to the vertex v_i and it is denoted by $d(v_i)$. A graph does not contain any cycle is called acyclic. If there is a way between all vertices of a graph, then it is called connected. Connected acyclic graph is called tree. Path graph is a tree that is in the form of straight line with degrees of two vertices are one and degrees of other vertices are two and it is denoted by P_n. Cycle graph is a graph that contains only one cycle through all vertices and degrees of all vertices are two. It is denoted by C_n. If degrees of all vertices of G are k, then it is called $k-$regular graph.

Let $A = [a_{ij}]_{n\times n}$ be a matrix. If v_i and v_j are adjacent vertices then a_{ij} and a_{ji} are

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1 or else 0, see [1]. A is called adjacency matrix of G. Analogous with linear algebra, $\det(\lambda\cdot I-A)$ is called the characteristic polynomial of G and we denoted it by $P_G(\lambda)$. Roots of $P_G(\lambda)$ are called eigenvalues of G and the energy of G is defined as the summation of absolute values of the eigenvalues of G, see [6]. Furthermore, there are many topological invariants used in several researches. In [16], Randić index is a molecular descriptor defined by Milan Randić and denoted by $\sum_{v_iv_j\in E}\frac{1}{\sqrt{d(v_i)d(v_j)}}$. In [9], another important molecular descriptor recently introduced by Ivan Gutman with the name Sombor index is $\sum_{v_iv_j\in E}\sqrt{(d(v_i))^2+(d(v_j))^2}$. In addition to topological invariants, several adjacency matrix forms have been defined until today, for more details see [13]. With the help of various adjacency matrices defined by correspondingly different types of graph invariants, it is possible to study different types of graph energy such as laplacian energy, distance energy, Randić energy and Sombor energy, see for details [15]. Two of the well-known them are Randić and Sombor matrices that are related to the corresponding topological indices. Researchers have studied these notions from various aspects so far. Some studies on the subjects Randić and Sombor adjacency matrices and energies can be seen in [2, 4, 5, 8, 10–12, 14, 17]. The first step to obtaining the desired energy type of a graph G is to calculate the characteristic polynomials of the corresponding adjacency matrices. In this paper, we obtain formulae for each coefficient of both Randić and Sombor characteristic polynomials of path graph P_n and cycle graph C_n by using a well-known equation. Also, we present formulae for some coefficients of Randić and Sombor characteristic polynomials of a special type of $3-$regular graph.

2. Coefficients of Randić and Sombor Characteristic Polynomials of Path, Cycle and a Special Type of 3-Regular Graphs

Let $G=(V,E)$ be a simple graph with n vertices and m edges. The Randić matrix of G was mentioned in the substantial book [3] and the Sombor matrix was defined in [10]. We denote the Randić and Sombor adjacency matrices of G by $R(G)$ and $S(G)$, respectively. $R(G)=[r_{ij}]_{n\times n}$ and $S(G)=[s_{ij}]_{n\times n}$ are formed by using the adjacency of vertices as the following:

$$r_{ij}=\begin{cases}\frac{1}{\sqrt{d(v_i)d(v_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent}\\ 0, & \text{otherwise.}\end{cases}$$

$$s_{ij}=\begin{cases}\sqrt{(d(v_i))^2+(d(v_j))^2}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent}\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $R(G)$ and $S(G)$ are symmetric matrices with dimension $n\times n$. Let us denote the ordinary characteristic polynomial of G as follows:

$$P_G(\lambda)=\lambda^n+c_1\lambda^{n-1}+c_2\lambda^{n-2}+\cdots+c_{n-1}\lambda+c_n.$$

Let us denote the number of components in an elementary subgraph G' which are single edges and cycles as $\rho_0(G')$ and $\rho_1(G')$, respectively.

In [18], the formula for the coefficients of the ordinary characteristic polynomial are given by

$$c_k = \sum (-1)^{\rho_0(G')+\rho_1(G')} 2^{\rho_1(G')}, \tag{1}$$

where the summation is taken over all elementary subgraphs G' with k vertices for $1 \leq k \leq n$. At the present time, the formula is called Sachs theorem, for details and history of the theorem see [1,3,7].

Let ψ_{ij} denote the nonzero value in the entry ij of the adjacency matrix of a vertex-degree-based topological index of a regular graph G. As a natural result of the Sachs theorem, it is clear that the formula for each coefficient c'_k of the characteristic polynomial of the adjacency matrix of this vertex-degree-based topological index is obtained by

$$c'_k = (\psi_{ij})^k \sum (-1)^{\rho_0(G')+\rho_1(G')} 2^{\rho_1(G')},$$

where the summation is taken over all elementary subgraphs G' with k vertices for $1 \leq k \leq n$.

In this paper, we aim to obtain all coefficients of the Randić and Sombor characteristic polynomials of path graph P_n and regular graph C_n by using the numbers of elementary subgraphs. Similarly, we also aim to obtain some coefficients of the same characteristic polynomials of a special type of $3-$regular graph we call R_n. We begin with the Randić characteristic polynomial of P_n. Let us note that the Randić characteristic polynomial of P_2 is equal to $\lambda^2 - 1$. Moreover, let us denote the set of non-negative integer numbers and the set of positive integer numbers by $\mathbb{Z}^*$ and $\mathbb{Z}^+$, respectively.

Theorem 1. *Let $P_n = (V, E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c$ be the Randić characteristic polynomial of P_n, where $c_k \in \mathbb{R}, 1 \leq k \leq n-1$. The formulae for the coefficients c_ks of the Randić characteristic polynomial of P_n, where $n \geq 3$, are as follows:*

$$c_2 = (-1)^{\frac{k}{2}}(\tfrac{n-3}{4} + 1),$$

$$c_k = 0, \text{ where } k \in 2\mathbb{Z}^* + 1,$$

$$c_k = (-1)^{\frac{k}{2}} \Big[\tbinom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1} \cdot (\tfrac{1}{\sqrt{2}})^2 \cdot (\tfrac{1}{2})^{2(\frac{k}{2}-1)} + \tbinom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2} \cdot (\tfrac{1}{4}) \cdot (\tfrac{1}{2})^{2(\frac{k}{2}-2)} \cdot (\tfrac{1}{\sqrt{2}})^2$$

$$+ \sum_{j=\frac{k}{2}-1}^{n-2-\frac{k}{2}} \tbinom{j}{\frac{k}{2}-1} \cdot (\tfrac{1}{2})^k + \sum_{j=\frac{k}{2}-2}^{n-3-\frac{k}{2}} \tbinom{j}{\frac{k}{2}-2} \cdot (\tfrac{1}{4}) \cdot (\tfrac{1}{2})^{k-2} \Big], \text{ where } k \geq 4, k \in 2\mathbb{Z}^+.$$

Proof. First of all, it is clear that $c_2 = (-1)^{\frac{k}{2}}(\frac{n-3}{4} + 1)$ for all $n \geq 3$. By the Eqn. 1, we know that c_k consists of the contributions of several elementary subgraphs of

G with k vertices. Also, since P_n does not have any cycle we take into account only edges that do not have any common vertex. At this point, we will apply a method that involves an edge removing and continue calculation of remaining part. Let us consider a path graph P_n with n vertices whose vertices are labelled by $1, 2, \cdots, n$. For calculation of c_k, if we remove the edge v_1v_2, then remaining part with $k-2$ vertices consists of number of

$$\binom{\frac{k}{2}-2}{\frac{k}{2}-2} + \binom{\frac{k}{2}-1}{\frac{k}{2}-2} + \cdots + \binom{n-\frac{k}{2}-3}{\frac{k}{2}-2} + \binom{n-\frac{k}{2}-2}{\frac{k}{2}-2} = \binom{n-1-\frac{k}{2}}{\frac{k}{2}-1}$$

combinations. Moreover, if we remove any edge v_iv_{i+1} which is not terminal edges of P_n, then remaining part consists one of the numbers

$$\binom{\frac{k}{2}-1}{\frac{k}{2}-1}, \binom{\frac{k}{2}}{\frac{k}{2}-1}, \cdots, \binom{n-2-\frac{k}{2}}{\frac{k}{2}-1}.$$

Hence, contributions of elementary subgraphs that are in the form of $v_1v_2, \cdots, v_iv_j$ can be $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{2})^2$ or $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{\sqrt{2}})^2$. Hereby, the contribution of the type subgraphs that contribute to c_k in the $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{2})^2$ form is obtained as $\left[\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1} - \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\right] \cdot (\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^{2(\frac{k}{2}-1)}$. Moreover, the contribution of the other type subgraphs that contribute to c_k in the $(\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{\sqrt{2}})^2$ form is obtained as $\binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2} \cdot (\frac{1}{\sqrt{2}})^4 \cdot (\frac{1}{2})^{2(\frac{k}{2}-2)}$. Thus, the first part of the formula is obtained as $\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1} \cdot (\frac{1}{\sqrt{2}})^2 \cdot (\frac{1}{2})^{2(\frac{k}{2}-1)} + \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2} \cdot (\frac{1}{4}) \cdot (\frac{1}{2})^{2(\frac{k}{2}-2)} \cdot (\frac{1}{\sqrt{2}})^2$ by arranging the contribution statements above.

Furthermore, contributions of elementary subgraphs that are in the form of $v_av_b, \cdots, v_iv_j$ can be $(\frac{1}{2})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{2})^2$ or $(\frac{1}{2})^2 \cdot (\frac{1}{2})^2 \cdots (\frac{1}{2})^2 \cdot (\frac{1}{\sqrt{2}})^2$, where $a \neq 1, b \neq 2$ or $a \neq 2, b \neq 1$. Similar to the previous part of the proof, two contribution equations of c_k are obtained as $\sum_{j=\frac{k}{2}-1}^{n-2-\frac{k}{2}} \binom{j}{\frac{k}{2}-1} \cdot (\frac{1}{2})^k$ and $\sum_{j=\frac{k}{2}-2}^{n-3-\frac{k}{2}} \binom{j}{\frac{k}{2}-2} \cdot (\frac{1}{4}) \cdot (\frac{1}{2})^{k-2}$, respectively. As a result, since there is no other elementary subgraph contribution type, the proof is completed by summing all the above subgraph contributions. □

In the next corollary, we continue with the Sombor characteristic polynomial of P_n. Firstly, it is clear that the Sombor characteristic polynomial of P_2 is equal to $\lambda^2 - 2$.

Corollary 1. *Let $P_n = (V, E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c$ be the Sombor characteristic polynomial of P_n, where $c_k \in \mathbb{Z}, 1 \leq k \leq n-1$. The formulae for the coefficients c_ks of the Sombor characteristic polynomial of P_n, where $n \geq 3$, are as follows:*

$$c_2 = (-1)^{\frac{k}{2}}(8(n-3)+10),$$

$$c_k = 0,\ \ where\ k \in 2\mathbb{Z}^* + 1,$$

$$c_k = (-1)^{\frac{k}{2}}\Big[\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-1)} - 3\binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-2)}$$

$$+\sum_{j=\frac{k}{2}-1}^{n-2-\frac{k}{2}}\binom{j}{\frac{k}{2}-1}\cdot(\sqrt{8})^k - 3\sum_{j=\frac{k}{2}-2}^{n-3-\frac{k}{2}}\binom{j}{\frac{k}{2}-2}\cdot(\sqrt{8})^{(k-2)}\Big],\ \ where\ k \geq 4, k \in 2\mathbb{Z}^+.$$

Proof. Proof is the same with the proof of Thm. 1. Only difference originated from the difference between the Randić and Sombor adjacency matrices of P_n. □

Theorem 2. *Let $P_n = (V, E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Randić characteristic polynomial of P_n, where $c_k \in \mathbb{R}$. The formula for the coefficient c_n, where $n \geq 3$, of the Randić characteristic polynomial of P_n is as follows:*

$$c_k = 0,\ \ where\ k \in 2\mathbb{Z}^* + 1,$$

$$c_k = (-1)^{\frac{k}{2}}\Big[\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\tfrac{1}{\sqrt{2}})^2\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-1)} + \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\tfrac{1}{4})\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-2)}\cdot(\tfrac{1}{\sqrt{2}})^2\Big],\ \ otherwise.$$

Proof. First of all, it clear that $c_k = 0$, where $k \in 2\mathbb{Z}^* + 1$. Similarly to Thm. 1, let us consider a path graph P_n with n vertices whose vertices are labelled by $1, 2, \cdots, n$. We keep in view elementary subgraphs with n vertices that consist of disjoint edges since $n = k$. At this point, we have only one choice and it is $v_1v_2, v_3v_4, \cdots, v_{n-1}v_n$. Thus, by the proof of Thm. 1, we know that the contribution of this subgraph to c_k is equal to $\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\frac{1}{\sqrt{2}})^2\cdot(\frac{1}{2})^{2(\frac{k}{2}-1)} + \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\frac{1}{4})\cdot(\frac{1}{2})^{2(\frac{k}{2}-2)}\cdot(\frac{1}{\sqrt{2}})^2$. Finally, by using Eqn. 1, we have the result as follow:

$$c_k \;=\; (-1)^{\frac{k}{2}}\Big[\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\tfrac{1}{\sqrt{2}})^2\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-1)} + \binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\tfrac{1}{4})\cdot(\tfrac{1}{2})^{2(\frac{k}{2}-2)}\cdot(\tfrac{1}{\sqrt{2}})^2\Big].$$

□

Corollary 2. *Let $P_n = (V, E)$ be a path graph with n vertices and $n-1$ edges. Let $P_{P_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomial of P_n, where $c_k \in \mathbb{Z}$. The formula for the coefficient c_n, where $n \geq 3$, of the Sombor characteristic polynomial of P_n is as follows:*

$$c_k = 0,\ \ where\ k \in 2\mathbb{Z}^* + 1,$$

$$c_k = (-1)^{\frac{k}{2}}\Big[\binom{(n-1)-\frac{k}{2}}{\frac{k}{2}-1}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-1)} - 3\binom{(n-2)-\frac{k}{2}}{\frac{k}{2}-2}\cdot(\sqrt{5})^2\cdot(\sqrt{8})^{2(\frac{k}{2}-2)}\Big],\ \ otherwise.$$

Proof. Proof is the same with the proof of Thm. 2. Only difference originate from the definitions of Randić and Sombor adjacency matrices of P_n. □

For the next theorem, we denote the number of elementary subgraphs with k vertices by $N(c_k)$.

Theorem 3. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n$. The formulae for the coefficients c_k $(k = 2t, t \in \mathbb{Z}^+)$ of the Sombor characteristic polynomial of C_n are as follows:*

$$\begin{aligned}
c_2 &= -8n,\\
c_4 &= (8)^2\left(\binom{n-2}{2} + \binom{n-3}{1}\right),\\
c_6 &= -(8)^3\left(\binom{n-3}{3} + \binom{n-4}{2}\right),\\
c_8 &= (8)^4\left(\binom{n-4}{4} + \binom{n-5}{3}\right),\\
c_{10} &= -(8)^5\left(\binom{n-5}{5} + \binom{n-6}{4}\right),\\
&\vdots\\
c_k &= (-1)^{\frac{k}{2}} \cdot (8)^{\frac{k}{2}}\left(\binom{n-\frac{k}{2}}{\frac{k}{2}} + \binom{n-(\frac{k}{2}+1)}{\frac{k}{2}-1}\right),
\end{aligned}$$

in the case of $n = k$, then $c_n = c_k - 2 \cdot 8^{\frac{n}{2}}$, where c_k is as given above.

Proof. We know that c_k consist of the contributions of different elementary subgraphs of G with k vertices by Eqn. 1. For the coefficients c_k $(k = 2t, t \in \mathbb{Z}^+)$ of the Sombor characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n-1$, we take into account only elementary subgraphs that consist of disjoint edges without any elementary subgraph that does not involve any cycle. Similarly to proof of Thm. 1, we apply edge removing method so that we get the number of elementary subgraphs for forming $c_4, c_6, c_8, c_{10}, \cdots, c_k$, where $c_k \in \mathbb{R}$, $1 \leq k \leq n-1$, by using combinations as follows:

$$\begin{aligned}
N(c_4) &= \sum_{i=1}^{n-3}\binom{i}{1} + \binom{n-3}{1},\\
N(c_6) &= \sum_{i=2}^{n-4}\binom{i}{2} + \binom{n-4}{2},\\
N(c_8) &= \sum_{i=3}^{n-5}\binom{i}{3} + \binom{n-5}{3},
\end{aligned}$$

$$N(c_{10}) = \sum_{i=4}^{n-6} \binom{i}{4} + \binom{n-6}{4},$$

$$\vdots$$

$$N(c_k) = \sum_{i=\frac{k}{2}-1}^{n-(\frac{k}{2}+1)} \binom{i}{\frac{k}{2}-1} + \binom{n-(\frac{k}{2}-1)}{\frac{k}{2}-1}.$$

As a result, we get the desired result by using combination properties and Eqn. 1. In addition, if $n = k$, then there exists one possibility of elementary subgraph that consists of the cycle C_n itself. Therefore, in this case result is obtained as $c_n = c_k - 2 \cdot 8^{\frac{n}{2}}$, where c_k is as given above. □

In a cycle graph C_n, it is trivial that if k is odd, then $c_k = 0$ whenever $0 \leq k \leq n-1$. In the next corollary, the last part of the previous theorem is presented with a more explicit statement.

Corollary 3. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n$. The formula for the coefficient c_n of the Sombor characteristic polynomial of C_n is as follows:*

$$c_n = \begin{cases} -2^{\frac{3n+2}{2}}, & n = 2t+1, \textit{where } t \in \mathbb{Z}^+ \\ -2^{\frac{3n+4}{2}}, & n = 2t, \textit{where } t \in \{3, 5, 7, \cdots\} \\ 0, & n = 4t, \textit{where } t \in \mathbb{Z}^+. \end{cases}$$

Proof. Let us consider a cycle graph C_n. There are three possible cases of elementary subgraph of C_n with n vertices. The first case is $n = 2t + 1$, where $t \in \mathbb{Z}^+$. For this case, we have just an elementary subgraph that consists of C_n itself and contribution of this subgraph is equal to $-2 \cdot (2\sqrt{2})^n$ by using Eqn. 1.

Second case is $n = 2t$, where $t \in \{3, 5, 7, \cdots\}$. At this point, there are 2 types of elementary subgraphs with n vertices. These elementary subgraphs can consist either just a cycle C_n or $\frac{n}{2}$ disjoint edges. Therefore, contribution of these subgraphs is equal to $-2 \cdot 8^{\frac{n}{2}} - 2 \cdot 8^{\frac{n}{2}}$ that is $-2^{\frac{3n+4}{2}}$. Third case is $n = 4t$, where $t \in \mathbb{Z}^+$. Similarly to second case, there are two possible elementary subgraphs of C_n with n vertices. These consist of either just a cycle C_n or $\frac{n}{2}$ disjoint edges. At this point, since $\frac{n}{2}$ is even number contribution of these subgraphs is equal to $2 \cdot 8^{\frac{n}{2}} - 2 \cdot 8^{\frac{n}{2}}$ that is 0 by Eqn. 1. □

Corollary 4. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Randić characteristic* *polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n$. The formulae for the coefficients c_k $(k = 2t, t \in \mathbb{Z}^+)$ of the Randić characteristic polynomial of C_n are as follows:*

$$\begin{aligned}
c_2 &= -\frac{n}{4}, \\
c_4 &= (\frac{1}{4})^2 \left(\binom{n-2}{2} + \binom{n-3}{1} \right), \\
c_6 &= -(\frac{1}{4})^3 \left(\binom{n-3}{3} + \binom{n-4}{2} \right), \\
c_8 &= (\frac{1}{4})^4 \left(\binom{n-4}{4} + \binom{n-5}{3} \right), \\
c_{10} &= -(\frac{1}{4})^5 \left(\binom{n-5}{5} + \binom{n-6}{4} \right), \\
&\vdots \\
c_k &= (-1)^{\frac{k}{2}} \cdot (\frac{1}{4})^{\frac{k}{2}} \left(\binom{n-\frac{k}{2}}{\frac{k}{2}} + \binom{n-(\frac{k}{2}+1)}{\frac{k}{2}-1} \right),
\end{aligned}$$

in the case of $n = k$, then $c_n = c_k - 2 \cdot (\frac{1}{4})^{\frac{n}{2}}$, where c_k is as given above.

Proof. Proof can be followed by using Theorem 3. □

In the previous theorem, it is clear that if k is odd, then $c_k = 0$ as long as $0 \leq k \leq n-1$ for each cycle graph C_n. The case $n = k$ is presented in the next result.

Corollary 5. *Let $C_n = (V, E)$ be a cycle graph with $n \geq 3$ vertices and n edges. Let $P_{C_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Randić characteristic polynomials of C_n, where $c_k \in \mathbb{R}$ and $1 \leq k \leq n$. The formula for the coefficient c_n of the Randić characteristic polynomial of C_n is as follows:*

$$c_n = \begin{cases} -2^{1-n}, & n = 2t+1, \text{where } t \in \mathbb{Z}^+ \\ -2^{2-n}, & n = 2t, \text{where } t \in \{3, 5, 7, \cdots\} \\ 0, & n = 4t, \text{where } t \in \mathbb{Z}^+. \end{cases}$$

Proof. Proof can be followed by using Corollary 3. □

Let us define a special regular graph that consists of n $(n \geq 4, n = 2t, t \in \mathbb{Z}^+)$ vertices, $\frac{3n}{2}$ edges and degrees of all vertices are 3. Also vertices intersect each others in a point. We denote it by R_n. Let us demonstrate the structures of graphs R_6 and R_8 in Figure 1.

Theorem 4. *Let $R_n = (V, E)$ be a $3-$regular graph with n vertices and $\frac{3n}{2}$ edges as shown in Fig. 1. Let $P_{R_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Randić characteristic polynomial of R_n, where $c_k \in \mathbb{R}$. The formulae for some*

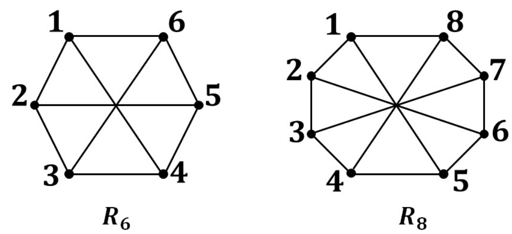

FIGURE 1. Graphs R_6 and R_8

coefficients of the Randić characteristic polynomial of R_n are as follows:

$$c_2 = -\tfrac{n}{6},$$

$$c_3 = 0, \text{ if } n=4, \text{ then } c_3 = -8 \cdot (\tfrac{1}{3})^3,$$

$$c_5 = 0, \text{ if } n=8, \text{ then } c_5 = -16 \cdot (\tfrac{1}{3})^5.$$

$$c_4 = \begin{cases} -3 \cdot (\frac{1}{3})^4, & n=4 \\ 0, & n=6 \\ -(\frac{1}{3})^4 n + (\frac{1}{3})^4 \left(\sum_{j=1}^{n-3} j + (n-3) + n\frac{n-4}{2} + \binom{\frac{n}{2}}{2} \right), & \text{otherwise.} \end{cases}$$

$$c_6 = \begin{cases} 0, & n=4 \\ 0, & n=6 \\ -(\frac{1}{3})^6 \left(\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3) \right. & \\ \left. +\frac{n}{2}(\frac{n}{2}-2) \right) + 2 \cdot (\frac{1}{3})^6 \left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\,\frac{n-4}{2}) \right) - 2 \cdot (\frac{1}{3})^6 \left(n + \frac{n}{2} \right), & n=10 \\ -(\frac{1}{3})^6 \left(\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3) \right. & \\ \left. +\frac{n}{2}(\frac{n}{2}-2) \right) + 2 \cdot (\frac{1}{3})^6 \left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\,\frac{n-4}{2}) \right) - (\frac{1}{3})^6 n, & \text{otherwise.} \end{cases}$$

Proof. It is clear that c_1 of $P_{R_n}(\lambda)$ is 0.

First of all, let us consider c_2. We know that the number of possible elementary subgraphs with 2 vertices is equal to the number of edges of R_n. Hence, since R_n is 3−regular, contribution of these elementary subgraphs to $c_2 = -(\frac{1}{3})^2 \frac{3n}{2} = -\frac{n}{6}$ by Eqn. 1.

Secondly, it is clear that 3−cycles just exist in R_n when n is equal to 4. Thus, by Eqn. 1 if $n=4$, then $c_3 = -(\frac{1}{3})^3 \cdot 2 \cdot 4$, otherwise $c_3 = 0$.

Thirdly, there exists 4 options for elementary subgraphs with 4 vertices. They can consist of 4−cycles that are in the form of cross labeling such as (1436) in R_6 in Fig. 1 and the number of possible elementary subgraphs in this form is $\frac{n}{2}$. The rest 3 options can be two disjoint edges that one belongs to C_n and other one is a diagonal edge, two disjoint edges that belong to C_n and lastly two disjoint edges that are diagonal edges, respectively. The number of possible elementary subgraphs in the form of second option is $n\frac{n-4}{2}$ because when we select an edge that belongs to C_n, we have $\binom{\frac{n-4}{2}}{1}$ possibility for an other diagonal edge. Since R_n has n vertices there are $n\frac{n-4}{2}$ elementary subgraphs in the second form. For the third option, the number of possible elementary subgraphs that are in the form of

$$\begin{aligned}&\{v_1v_2, v_3v_4\}, \{v_1v_2, v_4v_5\}, \cdots, \{v_1v_2, v_{n-1}v_n\},\\&\{v_2v_3, v_4v_5\}, \{v_2v_3, v_5v_6\}, \cdots, \{v_2v_3, v_{n-1}v_n\}, \{v_2v_3, v_nv_1\},\\&\vdots\\&\{v_{n-3}v_{n-2}, v_{n-1}v_n\}, \{v_{n-3}v_{n-2}, v_nv_1\}\end{aligned}$$

is equal to $1 + 2 + 3 + \cdots + (n-3) + (n-3)$. Also, it is clear that the number of possible elementary subgraphs of the last option is $\binom{\frac{n}{2}}{2}$. As a result, by using Eqn. 1 we get $c_4 = -2 \cdot (\frac{1}{3})^4 \frac{n}{2} + (\frac{1}{3})^4 \left(\sum_{j=1}^{n-3} j + (n-3) + n\frac{n-4}{2} + \binom{\frac{n}{2}}{2}\right)$. However, additionally when n is equal to 4, for the first option we have one more possible elementary subgraph that is C_4 itself so we get the result as $-3 \cdot (\frac{1}{3})^4$ by adding $-2 \cdot (\frac{1}{3})^4$. Moreover, when n is equal to 6, for the first option, we have six more possible elementary subgraphs that are C_4 itself so we get result as 0 by adding $-12 \cdot (\frac{1}{3})^4$.

Fourthly, there exists just one option for an elementary subgraph with 5 vertices that is a $5-$cycle C_5 itself and it can be possible only for R_n, where $n = 8$. Therefore, c_5 is obtained as $-16 \cdot (\frac{1}{3})^5$ by Eqn. 1.

Lastly, let us consider possible elementary subgraphs with 6 vertices, where $n \neq 6, 10$. One of the possible elementary subgraph types consisting of three edges that are in C_n are in the form

$$\begin{aligned}&\{v_1v_2, v_3v_4, v_5v_6\}, \{v_1v_2, v_3v_4, v_6v_7\}, \cdots, \{v_1v_2, v_{n-3}v_{n-2}, v_{n-1}v_n\},\\&\{v_2v_3, v_4v_5, v_6v_7\}, \{v_2v_3, v_4v_5, v_7v_8\}, \cdots, \{v_2v_3, v_{n-3}v_{n-2}, v_nv_1\}, \{v_2v_3, v_{n-2}v_{n-1}, v_nv_1\},\\&\vdots\\&\{v_{n-4}v_{n-3}, v_{n-2}v_{n-1}, v_nv_1\}.\end{aligned}$$

Possible number of these types is equal to $\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2}$. An another type can consist of three diagonal edges whose possible number is $\binom{\frac{n}{2}}{3}$. Another type can consist of one edge that is in C_n and other two edges are diagonal edges. As explained before possible number of these elementary subgraphs is $n\binom{\frac{n-4}{2}}{2}$. For another type of elementary subgraphs that consist of two edges in C_n and one in diagonal edges, we get the possible number $n(\frac{n}{2} - 3)(\frac{n}{2} - 4) + n(\frac{n}{2} - 3) + \frac{n}{2}(\frac{n}{2} - 2)$ by using processes as mentioned above. The number of possible elementary subgraphs that consist of cross labeling C_4 and an edge in C_n is $n(\frac{n-4}{2} - 1)$. Also, the number of possible elementary subgraphs that consist of cross labeling

C_4 and a diagonal edge is $(\frac{n}{2}\,\frac{n-4}{2})$. Moreover, the number of possible elementary subgraphs that consist of C_6 is $\frac{n}{2}$. Consequently, we get the formula by using Eqn. 1, where $n \neq 6, 10$. After all, additively when n is equal to 6, there is no possible elementary subgraph in the form of one edge that is in C_n and other two edges are diagonal edges. Therefore, for the $n = 6$ distinctively, we have $\sum_{j=2}^{2} \binom{j}{2} + \binom{2}{2} + \binom{\frac{6}{2}}{3} + 6(\frac{6}{2}-3)(\frac{6}{2}-4) + 6(\frac{6}{2}-3) + \frac{6}{2}(\frac{6}{2}-2)$ times possible elementary subgraphs that consist of disjoint edges of R_n and we have $(6 \cdot 0 + 3 \cdot 1)$ times possible elementary subgraphs that consist of one cross labeling C_4 and edge in R_6. Also, we have 6 possible elementary subgraphs that consist of C_6 and we have 6 possible elementary subgraphs consisting of an edge and a C_4 that is not cross labeling. As a consequence, privately for $n = 6$, we have the result $-6\cdot(\frac{1}{3})^6 + 6\cdot(\frac{1}{3})^6 - 12\cdot(\frac{1}{3})^6 + 12\cdot(\frac{1}{3})^6 = 0$ by using Eqn. 1. Finally, additively, if $n = 10$, there are n times more possible elementary subgraphs that consist of a C_6 so we have the formula by adding $-2 \cdot (\frac{1}{3})^6 n$ to the first formula. Thus, we complete the proof. □

Corollary 6. *Let $R_n = (V, E)$ be a 3−regular graph with n vertices and $\frac{3n}{2}$ edges as shown in Fig. 1. Let $P_{R_n}(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n$ be the Sombor characteristic polynomial of R_n, where $c_k \in \mathbb{R}$. The formulae for some coefficients of the Sombor characteristic polynomial of R_n are as follows:*

$$\begin{aligned} c_2 &= -27n, \\ c_3 &= 0, \ \textit{if } n = 4, \ \textit{then } c_3 = -8 \cdot (\sqrt{18})^3, \\ c_5 &= 0, \ \textit{if } n = 8, \ \textit{then } c_5 = -16 \cdot (\sqrt{18})^5, \end{aligned}$$

Also, we get the equations as follows:

$$c_4 = \begin{cases} -3 \cdot (\sqrt{18})^4, & n = 4 \\ 0, & n = 6 \\ -(\sqrt{18})^4 n + (\sqrt{18})^4 \left(\sum_{j=1}^{n-3} j + (n-3) + n\frac{n-4}{2} + \binom{\frac{n}{2}}{2} \right), & \textit{otherwise} \end{cases}$$

$$c_6 = \begin{cases} 0, & n = 4 \\ 0, & n = 6 \\ -(\sqrt{18})^6 \left(\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3) \right. & \\ \left. + \frac{n}{2}(\frac{n}{2}-2) \right) + 2 \cdot (\sqrt{18})^6 \left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\,\frac{n-4}{2}) \right) - 2 \cdot (\sqrt{18})^6 \left(n + \frac{n}{2} \right), & n = 10 \\ -(\sqrt{18})^6 \left(\sum_{j=2}^{n-4} \binom{j}{2} + \binom{n-4}{2} + \binom{\frac{n}{2}}{3} + n\binom{\frac{n-4}{2}}{2} + n(\frac{n}{2}-3)(\frac{n}{2}-4) + n(\frac{n}{2}-3) \right. & \\ \left. + \frac{n}{2}(\frac{n}{2}-2) \right) + 2 \cdot (\sqrt{18})^6 \left(n(\frac{n-4}{2}-1) + (\frac{n}{2}\,\frac{n-4}{2}) \right) - (\sqrt{18})^6 n, & \textit{otherwise} \end{cases}$$

Proof. The proof can be completed by simply replacing $\frac{1}{3}$ with $\sqrt{18}$ in the proof of the previous theorem. □

3. Conclusion

The Randić and the Sombor characteristic polynomials of P_n and C_n were obtained. Additionally, the formulae of five coefficients of the Randić and Sombor characteristic polynomials of R_n were presented. The Randić and the Sombor energies of P_n and C_n can be studied by using these presented results. Furthermore, various characteristic polynomials of some similar adjacency matrices defined according to some vertex-degree-based topological invariants can be obtained by using the number of elementary subgraphs that we presented in the theorems and corollaries. Especially, this study can also be extended to the multiplicative Sombor index associated with the Sombor index.

Declaration of Competing Interests The author has no competing interest to declare.

References

[1] Bapat, R. B., Graphs and Matrices, Springer, London, 2010. http://dx.doi.org/10.1007/978-1-84882-981-7
[2] Bozkurt, S. B., Güngör, A. D., Gutman, I., Çevik, A. S., Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.*, 64(1) (2010), 239–250.
[3] Cvetković, D. M., Doob, M., Sachs, H., Spectra of Graphs – Theory and Application, Academic Press, New York, 1980.
[4] Das, K. C., Sorgun, S., Xu, K., On the Randić energy of graphs, *MATCH Commun. Math. Comput. Chem.*, 72(1) (2014), 227–238.
[5] Ghanbari, N., On the Sombor characteristic polynomial and Sombor energy of a graph, arXiv: 2108.08552, 2021. https://doi.org/10.48550/arXiv.2108.08552
[6] Gutman, I., The energy of a graph, *Ber. Math-Statist. Sekt. Forschungsz. Graz*, 103 (1978), 1–22.
[7] Gutman, I., Impact of the Sachs theorem on theoretical chemistry: A participant's testimony, *MATCH Commun. Math. Comput. Chem.*, 48 (2003), 17–34.
[8] Gutman, I., Furtula, B., Bozkurt, S. B., On Randić energy, *Linear Algebra Appl.*, 442 (2014), 50–57. http://dx.doi.org/10.1016/j.laa.2013.06.010
[9] Gutman, I., Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.*, 86(1) (2021), 11–16.
[10] Gutman, I., Spectrum and energy of the Sombor matrix, *Vojno tehn. glas.*, 69(3) (2021), 551–561. http://dx.doi.org/10.5937/vojtehg69-31995
[11] Gutman, I., Redžepović, I., Rada, J., Relating energy and Sombor energy, *Contrib. Math.*, 4 (2021), 41–44. DOI: 10.47443/cm.2021.0054
[12] Gutman, I., Redžepović, I., Sombor energy and Huckel rule, *Discrete Math. Lett.*, 9 (2022), 67–71. DOI: 10.47443/dml.2021.s211
[13] Janežič, D., Miličević, A., Nikolić, S., Trinajstić, N., Graph Theoretical Matrices in Chemistry, CRC Press, Boca Raton, 2015. http://dx.doi.org/10.1201/b18389
[14] Jayanna, G. K., Gutman, I., On characteristic polynomial and energy of Sombor matrix, *Open J. Discret. Appl. Math.*, 4 (2021), 29–35. http://dx.doi.org/10.30538/psrp-odam2021.0062
[15] Li, X., Shi, Y., Gutman, I., Graph Energy, Springer, New York, 2012. http://dx.doi.org/10.1007/978-1-4614-4220-2
[16] Randić, M., On characterization of molecular branching, *J. Am. Chem. Soc.*, 97(23) (1975), 6609–6615. http://dx.doi.org/10.1021/ja00856a001

[17] Redžepović, I., Gutman, I., Comparing energy and Sombor energy–An empirical study, *MATCH Commun. Math. Comput. Chem.*, 88(1) (2022), 133–140.

[18] Sachs, H., Über selbstkomplementäre graphen, *Publ. Math. Debrecen*, 9 (1962), 270-288.

http://communications.science.ankara.edu.tr

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ON DIFFERENCE OF BIVARIATE LINEAR POSITIVE OPERATORS

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Abstract. In the present paper we give quantitative type theorems for the differences of different bivariate positive linear operators by using weighted modulus of continuity. Similar estimates are obtained via K-functional and for Chebyshev functionals. Moreover, an example involving Szász and Szász-Kantorovich operators is given.

1. Introduction

Studies in the theory of approximations have been going on for many years. During these times, the most well-known operator Bernstein operators, the best-known theorem for convergence was the Korovkin Theorem. Then, Szasz, Baskakov, Kantorovich operators are defined and their convergence properties are examined. Many researchers have defined various modification forms of these operators and examined their convergence properties and their applications are given. In recent years, some studies have been carried out to obtain general information between the convergence speeds of the operators by taking the difference of any two operators.

In the recent past, there is a growing interest in studying the difference of linear positive operators in approximation theory (see [1], [2], [3] and [6])

In 2006, Gonska et al., using Taylor's expansion with Peano remainder, gave a Theorem showing that the difference of two operators A and B can be limited by the concave majorant $\widetilde{\omega}$, where ω_k is the k-th order modulus of smoothness [11].

In 2016, A. M. Acu and I. Raşa obtained some inequalities using Taylor's formula and obtained some estimations by applying these inequalities on the differences of Linear Positive operators [1].

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In 2019, A. Aral et al. obtained some estimates for the difference of two general linear positive operators on unbounded interval [5].

In 2021, A. M. Acu et al. gave some theorems for the difference of linear positive operators of two variables defined on a simplex [4].

In this study, we will give some theorems given by A. Aral et al. [5] for univariate operators for bivariate operators.

This paper deals with the difference of certain bivariate operators defined on unbounded intervals. The differences are estimated in terms of weighted moduli of smoothness for the operators constructed with the same fundamental functions and different functionals in front of them.

2. Auxiliary Results

If we can calculate that the difference between the A and B operators is very small, we can learn the properties of the other by looking at the properties of one.

It is well-known that classical modulus of continuity is a very useful tool in order to determine the rate of convergence of the corresponding sequence of linear positive operators defined bounded interval, in case of unbounded intervals, It would be more appropriate to use a defined modulus of continuity in weighted function spaces. This allows to enlarge the continuous function space to weighted function space in approximation problems. For this purpose, we consider the modulus of continuity defined in suitable polynomial weighted space, defined for univariate case in [10] by Gadjieva and Doğru and for bivariate case in [12] by İspir and Atakut.

Let $\mathcal{D} := [0,\infty) \times [0,\infty)$ and $\rho(x,y) := 1 + x^2 + y^2$, $(x,y) \in \mathcal{D}$. Throughout the paper; $C(\mathcal{D})$ will denote the space of real-valued continuous functions on $\mathcal{D}$ and $C_B(\mathcal{D})$ will denote the space of all $f \in C(\mathcal{D})$ that are bounded on $\mathcal{D}$. Let $B_\rho(\mathcal{D})$ denote the space of functions f satisfying the inequality

$$|f(x,y)| \leq m_f \rho(x,y), \quad (x,y) \in \mathcal{D},$$

where m_f is a positive constant which depend on the function f. $B_\rho(\mathcal{D})$ is a linear normed space with the norm

$$\|f\|_\rho = \sup_{(x,y)\in\mathcal{D}} \frac{|f(x,y)|}{\rho(x,y)}. \tag{1}$$

Let $C_\rho(\mathcal{D})$ denote the subspace of all continuous functions belonging to $B_\rho(\mathcal{D})$. Also, let $C_\rho^*(\mathcal{D})$ denote the subspace of all functions $f \in C_\rho(\mathcal{D})$ for which there exists a constant k_f such that

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{|f(x,y)|}{\rho(x,y)} = k_f < \infty.$$

In the case of $k_f = 0$, we will write $C_\rho^0(\mathcal{D})$.

We use the weighted modulus of continuity, considered in [10] and [12], denoted by $\Omega_\rho(f,\cdot,\cdot)$ and given by

$$\Omega_\rho(f,\delta_1,\delta_2)=\sup_{(x,y)\in\mathcal{D},\ |h_1|<\delta_1,\ |h_2|<\delta_2}\frac{f(x+h_1,y+h_2)-f(x,y)}{(1+x^2+y^2)(1+h_1^2+h_2^2)};\ f\in C_\rho(\mathcal{D}). \tag{2}$$

The weighted modulus of continuity Ω_ρ satisfies the following properties for $f\in C_\rho^*(\mathcal{D})$:

i**:** $\Omega_\rho(f,\delta_1,\delta_2)\to 0$ as $\delta_1\to 0$ and $\delta_2\to 0$ for $\delta_1,\delta_2>0$.

ii**:** For any positive real numbers λ_1, λ_2, δ_1 and δ_2 the following relation

$$\Omega_\rho(f,\lambda_1\delta_1,\lambda_2\delta_2)\le 4(1+\lambda_1)(1+\lambda_2)\Omega_\rho(f,\delta_1,\delta_2) \tag{3}$$

holds.

In the sequel, we will use the notation that $e_{i,j}(x,y):=x^iy^j$, $i,j\in\mathbb{N}$, $(x,y)\in\mathcal{D}$, $\mathbf{1}$ denotes the constant function

$$\mathbf{1}:\mathcal{D}\to\mathbb{R},\ \mathbf{1}(x,y)=1,\ (x,y)\in\mathcal{D}, \tag{4}$$

and $\mathbb{D}$ denotes a linear subspace of $C(\mathcal{D})$, which contains $C_\rho(\mathcal{D})$. We also consider the positive linear functional $F:\mathbb{D}\to\mathbb{R}$ such that $F(\mathbf{1})=1$. Denoting

$$\theta_1^F:=F(e_{1,0})\ ,\ \theta_2^F:=F(e_{0,1}) \tag{5}$$

and

$$\mu_{i,j}^F:=F\left(\left(e_{1,0}-\theta_1^F\mathbf{1}\right)^i\left(e_{0,1}-\theta_2^F\mathbf{1}\right)^j\right),\quad i,j\in\mathbb{N}, \tag{6}$$

then one has

$$\begin{aligned}\mu_{1,0}^F &= 0,\ \mu_{2,0}^F=F(e_{1,0})^2-\left(\theta_1^F\right)^2\ge 0, \\ \mu_{0,1}^F &= 0,\ \mu_{0,2}^F=F(e_{0,1})^2-\left(\theta_2^F\right)^2\ge 0.\end{aligned} \tag{7}$$

Lemma 1. *For $(x,y)\in\mathcal{D}$, $f\in C_\rho^*(\mathcal{D})$ and $0<\delta_1,\delta_2\le 1$, we have*

$$|f(t,s)-f(x,y)|\le 256\rho(x,y)\left(1+\frac{(t-x)^4}{\delta_1^4}\right)\left(1+\frac{(s-y)^4}{\delta_2^4}\right)\Omega_\rho(f,\delta_1,\delta_2).$$

Proof. Using the inequality [5] with $\lambda_1=\frac{|t-x|}{\delta_1}$ ve $\lambda_2=\frac{|s-y|}{\delta_2}$, from (2) and (3), we have

$$\begin{aligned}|f(t,s)-f(x,y)| &\le 4\rho(x,y)\Omega_\rho(f,\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\\ &\quad\times\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)\end{aligned}$$

$$\le\begin{cases}16\rho(x,y)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega_\rho(f,\delta_1,\delta_2); & |t-x|\le\delta_1,|s-y|\le\delta_2\\ 16\rho(x,y)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega_\rho(f,\delta_1,\delta_2)\frac{(t-x)^4}{\delta_1^4}\frac{(s-y)^4}{\delta_2^4}; & |t-x|>\delta_1,|s-y|>\delta_2\end{cases}.$$

Therefore

$$|f(t,s)-f(x,y)| \leq 16\rho(x,y)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\left(1+\tfrac{(t-x)^4}{\delta_1^4}\right)\left(1+\tfrac{(t-y)^4}{\delta_2^4}\right)\Omega_\rho(f,\delta_1,\delta_2).$$

Choosing $0<\delta_1\leq 1,\ 0<\delta_2\leq 1$ for $f\in C_\rho^*(\mathcal{D})\,,\ (x,y)\in\mathcal{D}$, we get

$$|f(t,s)-f(x,y)| \leq 256\rho(x,y)\left(1+\frac{(t-x)^4}{\delta_1^4}\right)\left(1+\frac{(s-y)^4}{\delta_2^4}\right)\Omega_\rho(f,\delta_1,\delta_2)\,.$$

□

Now, we present the following estimate for the difference $\left|F(f)-f\left(\theta_1^F,\theta_2^F\right)\right|$.

Lemma 2. *Let f and all of its partial derivatives of order ≤ 2 belong to the space $C_\rho(\mathcal{D})$ and $0<\delta_1\leq 1,\ 0<\delta_2\leq 1$. Then we have*

$$\left|F(f)-f\left(\theta_1^F,\theta_2^F\right)\right| \leq M_f\rho\left(\theta_1^F,\theta_2^F\right)\left[\mu_{2,0}^F+\mu_{0,2}^F\right],$$

where

$$M_f := \max\left\{\|f_{xx}\|_\rho\,,\|f_{xy}\|_\rho\,,\|f_{yy}\|_\rho\right\}.$$

Proof. For $f\in C_\rho(\mathcal{D})$, $(t,s)\in\mathcal{D}$, using the Taylor formula we have

$$\begin{aligned}
&f(t,s)-f\left(\theta_1^F,\theta_2^F\right)\\
=\ & f_x\left(\theta_1^F,\theta_2^F\right)\left(t-\theta_1^F\right)+f_y\left(\theta_1^F,\theta_2^F\right)\left(s-\theta_2^F\right)+\frac{1}{2}\left\{f_{xx}(c_1,c_2)\left(t-\theta_1^F\right)^2\right.\\
&\left.+2f_{xy}(c_1,c_2)\left(t-\theta_1^F\right)\left(s-\theta_2^F\right)+f_{yy}(c_1,c_2)\left(s-\theta_2^F\right)^2\right\},
\end{aligned}$$

where (c_1,c_2) is a point on the line connecting $\left(\theta_1^F,\theta_2^F\right)$ and (t,s). Taking into account of the fact that $F(\mathbf{1})=1$ and (5), one has

$$\begin{aligned}
&F(f)-f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\\
=\ & f_x\left(\theta_1^F,\theta_2^F\right)\left(F(e_{1,0})-\theta_1^F F(\mathbf{1})\right)-f_y\left(\theta_1^F,\theta_2^F\right)\left(F(e_{0,1})-\theta_2^F F(\mathbf{1})\right)\\
&+\frac{1}{2}\left\{f_{xx}(c_1,c_2)\mu_{2,0}^F+2f_{xy}(c_1,c_2)\mu_{1,1}^F+f_{yy}(c_1,c_2)\mu_{0,2}^F\right\}. \qquad (8)
\end{aligned}$$

Using the facts

$$|f_{xx}(c_1,c_2)| \leq M_f\left(1+\left(\theta_1^F\right)^2+\left(\theta_2^F\right)^2\right),$$

$$|f_{xy}(c_1,c_2)| \leq M_f\left(1+\left(\theta_1^F\right)^2+\left(\theta_2^F\right)^2\right),$$

and

$$|f_{yy}(c_1,c_2)| \leq M_f\left(1+\left(\theta_1^F\right)^2+\left(\theta_2^F\right)^2\right),$$

and since

$$2\mu_{1,1}^F \leq \mu_{2,0}^F + \mu_{0,2}^F,$$

from (8) we get

$$\begin{aligned}\left|F(f) - f\left(\theta_1^F, \theta_2^F\right)\right| &\leq \frac{1}{2} M_f \left(1 + \left(\theta_1^F\right)^2 + \left(\theta_2^F\right)^2\right) \left\{\mu_{2,0}^F + 2\mu_{1,1}^F + \mu_{0,2}^F\right\} \\ &\leq M_f \left(1 + \left(\theta_1^F\right)^2 + \left(\theta_2^F\right)^2\right) \left[\mu_{2,0}^F + \mu_{0,2}^F\right].\end{aligned}$$

□

3. Difference of Bivariate Positive Linear Operators

In this section, we will give estimates for the difference of bivariate positive linear operators, on unbounded set $\mathcal{D}$, in terms of weighted modulus of continuity. Let $\mathbb{K}$ be a set of non-negative integers and consider a family of functions $p_{k,l} : \mathcal{D} \to \mathcal{D}$, $k, l \in \mathbb{K}$. We consider discrete operators given by

$$U(f; x, y) = \sum_{k,l \in \mathbb{K}} F_{k,l}(f) \, p_{k,l}(x, y) \ , \ V(f; x, y) = \sum_{k,l \in \mathbb{K}} G_{k,l}(f) \, p_{k,l}(x, y),$$

where $\sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) = 1$, $F_{k,l}, G_{k,l} : \mathbb{D} \to \mathbb{R}$ are positive linear functionals such that $F_{k,l}(\mathbf{1}) = 1$, $G_{k,l}(\mathbf{1}) = 1$. U and V are positive linear operators such that $U, V : \mathbb{D} \to B_\rho(\mathcal{D})$.

Theorem 1. *Let $f \in C_\rho^*(\mathcal{D})$ with all of its partial derivatives of order ≤ 2 belong to the space $C_\rho(\mathcal{D})$. Then we have*

$$|(U - V)(f; x, y)| \leq \delta_1 + \delta_2 + 2^8 \Omega_\rho(f, \delta_3, \delta_4) \left(1 + \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) \, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right)\right),$$

where

$$\delta_1 := M_f \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) \, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) \left[\mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}\right],$$

$$\delta_2 := M_f \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) \, \rho\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right) \left[\mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}\right],$$

$$\delta_3^4 = \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) \, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) \left(\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right)^4,$$

and

$$\delta_4^4 = \sum_{k,l \in \mathbb{K}} p_{k,l}(x, y) \, \rho\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) \left(\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right)^4.$$

Proof. We can write

$$
\begin{aligned}
\left|(U-V)(f;x,y)\right| &= \Bigg|\sum_{k,l\in\mathbb{K}}\Big\{F_{k,l}(f)-G_{k,l}(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)+f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\\
&\quad -f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)+f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\Big\}p_{k,l}(x,y)\Bigg|\\
&\le \sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\Big\{\left|F_{k,l}(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right|\\
&\quad +\left|G_{k,l}(f)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\\
&\quad +\left|f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\Big\}.
\end{aligned}
$$

Using Lemma 2, (5), (6) and (7), we get

$$
\left|F(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right|\le M_f\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left\{\mu_{2,0}^{F_{k,l}}+2\mu_{1,1}^{F_{k,l}}+\mu_{0,2}^{F_{k,l}}\right\}
$$

and

$$
\begin{aligned}
&\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left|F_{k,l}(f)-f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right|\\
\le\ & M_f\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left[\mu_{2,0}^{F_{k,l}}+\mu_{0,2}^{F_{k,l}}\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left|G_{k,l}(f)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\\
\le\ & M_f\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\left[\mu_{2,0}^{G_{k,l}}+\mu_{0,2}^{G_{k,l}}\right].
\end{aligned}
$$

Using Lemma 1, we get

$$
\begin{aligned}
&\left|f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right|\\
\le\ & 2^8\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\Omega_\rho(f,\delta_3,\delta_4)\\
&\times\left(1+\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\right)\left(1+\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right)\\
\le\ & 2^8\Omega_\rho(f,\delta_3,\delta_4)\left\{\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)+\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\right.
\end{aligned}
$$

$$+\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}$$
$$\left.+\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right\}$$

and we can write

$$\begin{aligned}
&\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left|f\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)-f\left(\theta_1^{G_{k,l}},\theta_2^{G_{k,l}}\right)\right| \\
\leq\ & 2^8\Omega_\rho(f,\delta_3,\delta_4)\left\{\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\right. \\
&+\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4} \\
&+\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4} \\
+\ &\left.\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right\} \\
=\ & 2^8\Omega_\rho(f,\delta_3,\delta_4)\left\{A_{0,0}+A_{1,0}+A_{0,1}+A_{1,1}\right\},
\end{aligned}$$

where

$$\begin{aligned}
A_{i,j} &= q_{k,l}(x,y)\left[\frac{\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4}{\delta_3^4}\right]^i\left[\frac{\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4}{\delta_4^4}\right]^j ;0\leq i,j\leq 1 \\
q_{k,l}(x,y) &= \sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right).
\end{aligned}$$

Choosing

$$\delta_3^4=\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4$$

and

$$\delta_4^4=\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4,$$

we reach to the desired result. □

4. Estimate via K-functional

In this section, we give an estimate for the difference of bivariate positive linear operators; in terms of K-functional. For this aim, we firstly recall the definition of K-functional. Let $C_B^2(\mathcal{D}) = \left\{ f \in C_B(\mathcal{D}) ; f^{(p,q)} \in C_B(\mathcal{D}), 1 \le p, q \le 2 \right\}$ where $f^{(p,q)}$ is (p,q) th-order partial derivative with respect to x, y of f, equipped with the norm

$$\|f\|_{C_B^2(\mathcal{D})} = \|f\|_{C_B(\mathcal{D})} + \sum_{i=1}^{2} \left\| \frac{\partial^i f}{\partial x^i} \right\|_{C_B(\mathcal{D})} + \sum_{i=1}^{2} \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C_B(\mathcal{D})}.$$

The Peetre K-functional of the function $f \in C_B(\mathcal{D})$ is given by

$$K(f;\delta) = \inf_{g \in C_B^2(\mathcal{D})} \left\{ \|f-g\|_{C_B(\mathcal{D})} + \delta \|g\|_{C_B^2(\mathcal{D})}, \ \delta > 0 \right\}.$$

It is known that there is a connection between the second order modulus of smoothness and Peetre's K-functional for all $\delta > 0$ as follows (see [9, p.192] or [7]):

$$K(f;\delta) \le C_0 \left\{ \omega_2\left(f; \sqrt{\delta}\right) + \min(1,\delta) \|f\|_{C_B(\mathcal{D})} \right\}.$$

Here, the constant C_0 is independent of δ and f, and 2nd order modulus of smoothness of f is a function $\omega_2 : C_B(\mathcal{D}) \times (0,\infty) \to [0,\infty)$ given by

$$\omega_2(f,\delta) = \sup_{0<\|h\|\le\delta} \sup_{x\in\mathcal{D}} \Delta_h^2 f(x), \quad \delta > 0,$$

where $\|.\|$ is the Euclidean norm in $\mathbb{R}^2$ and $\Delta_h^2 f$ is the 2nd order difference on $\mathcal{D}$ given by

$$\Delta_h^2 f(x) = \sum_{k=0}^{2} (-1)^{2-k} \binom{2}{k} f(x+kh), \ x \in \mathcal{D}, \ h \in \mathcal{D}.$$

Now, assume that $C_B^2(\mathcal{D}) \subset \mathbb{D}$, where, as it is mentioned in page 3, $\mathbb{D}$ is the linear subspace of $C(\mathcal{D})$ containing $C_\rho(\mathcal{D})$.

Lemma 3. *Let $f \in \mathbb{D} \cap C_B(\mathcal{D})$. Then*

$$\left| F(f) - f\left(\theta_1^F, \theta_2^F\right) \right| \le 2K\left(f; \frac{1}{4}\left[\mu_{2,0}^F + \mu_{0,2}^F\right]\right).$$

Proof. Let $g(x,y) \in C_B^2(\mathcal{D})$ and $(t,s) \in \mathcal{D}$. Using Taylor's expansion [8], we have

$$\begin{aligned} g(t,s) - g(x,y) &= \frac{\partial g(x,y)}{\partial x}(t-x) + \frac{\partial g(x,y)}{\partial y}(s-y) \\ &\quad + \int_x^t (t-u) \frac{\partial^2 g(u,y)}{\partial u^2} du + \int_y^s (s-v) \frac{\partial^2 g(x,v)}{\partial v^2} dv. \end{aligned}$$

Application of the functional F on both sides of the last formula gives

$$\left| F(g) - g\left(\theta_1^F, \theta_2^F\right) F(\mathbf{1}) \right|$$

$$
\begin{aligned}
&\le \left|g_x\left(\theta_1^F,\theta_2^F\right)\left(F\left(e_{1,0}\right)-\theta_1^F F\left(\mathbf{1}\right)\right)\right| + \left|g_y\left(\theta_1^F,\theta_2^F\right)\left(F\left(e_{0,1}\right)-\theta_2^F F\left(\mathbf{1}\right)\right)\right| \\
&\quad F\left(\left|\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}\right| du; x,y\right) + F\left(\left|\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv\right|; x,y\right) \\
&\le \frac{1}{2}\left\{\|g_{xx}\|_{C_B(\mathcal{D})}\left(F\left(e_{1,0}\right)-\theta_1^F F\left(\mathbf{1}\right)\right)^2 + \|g_{yy}\|_{C_B(\mathcal{D})}\left(F\left(e_{0,1}\right)-\theta_2^F F\left(\mathbf{1}\right)\right)^2\right\}.
\end{aligned}
$$

Taking into account of $F(\mathbf{1})=1$, (4), (5) and (6), we get

$$
\left|F(g)-g\left(\theta_1^F,\theta_2^F\right)\right| \le \frac{1}{2}\left\{\|g_{xx}\|_{C_B(\mathcal{D})}\mu_{2,0}^F + \|g_{yy}\|_{C_B(\mathcal{D})}\mu_{0,2}^F\right\}.
$$

Now, let $f\in\mathbb{D}\cap C_B(\mathcal{D})$ and $(t,s)\in\mathcal{D}$, then we have

$$
\begin{aligned}
&\left|F(f;x,y)-f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
=& \left|F(f-g+g;x,y)-f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})+g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
=& \left|F(f-g;x,y)+F(g;x,y)-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right. \\
-& \left. f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})+g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
\le& \left|F(f-g;x,y)\right| + \left|F(g;x,y)-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
+& \left|f\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})-g\left(\theta_1^F,\theta_2^F\right)F(\mathbf{1})\right| \\
\le& 2\|f-g\|_{C_B(\mathcal{D})}+\frac{1}{2}\left\{\|g_{xx}\|_{C_B(\mathcal{D})}\mu_{2,0}^F+\|g_{yy}\|_{C_B(\mathcal{D})}\mu_{0,2}^F\right\} \\
\le& 2\|f-g\|_{C_B(\mathcal{D})}+\frac{1}{2}\|g\|_{C_B^2(\mathcal{D})}\left[\mu_{2,0}^F+\mu_{0,2}^F\right].
\end{aligned}
$$

Therefore, taking the infimum on the right hand side over all $g\in C_B^2(D)$

$$
\begin{aligned}
\left|F(f;x,y)-f\left(\theta_1^F,\theta_2^F\right)\right| &\le \inf_{g\in C_B^2(D)}\left\{2\|f-g\|_{C_B(\mathcal{D})}+\frac{1}{2}\|g\|_{C_B^2(\mathcal{D})}\left[\mu_{2,0}^F+\mu_{0,2}^F\right]\right\} \\
&= 2K\left(f;\frac{1}{4}\left[\mu_{2,0}^F+\mu_{0,2}^F\right]\right).
\end{aligned}
$$

□

Now, the following theorem can be given.

Theorem 2. *Let $f\in\mathbb{D}\cap C_B(\mathcal{D})$ with all of its first order partial derivatives belong to $C_B(\mathcal{D})$. Then*

$$
|(U-V)(f;x,y)| \le 4K\left(f,\frac{1}{8}\eta(x,y)\right)+M_f'\mu(x,y),
$$

where $M_f' := \max\left\{\|f_x\|_{C_B(\mathcal{D})}, \|f_x\|_{C_B(\mathcal{D})}\right\}$,

$$\eta(x,y) := \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left(\lambda_{F_{k,l}} + \lambda_{G_{k,l}}\right),$$

with $\lambda_{F_{k,l}} := \mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}$, $\lambda_{G_{k,l}} := \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}$ *and*

$$\mu(x,y) = \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right| + \left|\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right|\right\}.$$

Proof. By the hypothesis, f is differentiable on the line connecting the points $\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right)$ and $\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right)$. From the mean value theorem for function of two variables (see, e.g., [7]), there is a point (c_1, c_2) on this line such that

$$f\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right) - f\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right) = f_x(c_1,c_2)\left(\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right) + f_y(c_1,c_2)\left(\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right)$$

holds. For $f \in \mathbb{D} \cap C_B(\mathcal{D})$, using Lemma 3, and the above formula, we have

$$\begin{aligned}
&|(U-V)(f;x,y)| \\
\leq\ & \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left|F_{k,l}(f) - G_{k,l}(f)\right| \\
\leq\ & \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|F_{k,l}(f) - f\left(\theta_1^{F_{k,l}}, \theta_2^{F_{k,l}}\right)\right| + \left|G_{k,l}(f) - f\left(\theta_1^{G_{k,l}}, \theta_2^{G_{k,l}}\right)\right|\right. \\
& \left.\left|f_x(c_1,c_2)\left(\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right) + f_y(c_1,c_2)\left(\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right)\right|\right\} \\
\leq\ & 2\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{K\left(f;\frac{1}{4}\left[\mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}\right]\right) + K\left(f;\frac{1}{4}\left[\mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}\right]\right)\right\} \\
& + \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\|f_x\|_{C_B(\mathcal{D})}\left|\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right| + \|f_y\|_{C_B(\mathcal{D})}\left|\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right|\right\} \\
=\ & 2\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{K\left(f;\frac{1}{4}\lambda_{F_{k,l}}\right) + K\left(f;\frac{1}{4}\lambda_{G_{k,l}}\right)\right\} \\
& + K_f\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}} - \theta_1^{G_{k,l}}\right| + \left|\theta_2^{F_{k,l}} - \theta_2^{G_{k,l}}\right|\right\},
\end{aligned}$$

where we denote

$\lambda_{F_{k,l}} := \mu_{2,0}^{F_{k,l}} + \mu_{0,2}^{F_{k,l}}$, $\lambda_{G_{k,l}} := \mu_{2,0}^{G_{k,l}} + \mu_{0,2}^{G_{k,l}}$ and $M_f' := \max\left\{\|f_x\|_{C_B(\mathcal{D})}, \|f_x\|_{C_B(\mathcal{D})}\right\}$

From the definition of K-functional, for a fixed $g \in C_B^2(\mathcal{D})$, we can write

$$|(U-V)(f;x,y)| \quad\leq\quad 4\|f-g\|_{C(\mathcal{D})}\sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)$$

$$
\begin{aligned}
&+\frac{1}{2}\|g\|_{C^2(\mathcal{D})}\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left(\lambda_{F_{k,l}}+\lambda_{G_{k,l}}\right)\\
&+M_f'\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right|+\left|\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right|\right\}\\
=\ &4K\left(f,\frac{1}{8}\eta(x,y)\right)+M_f'\mu(x,y),
\end{aligned}
$$

where

$$\eta(x,y):=\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left(\lambda_{F_{k,l}}+\lambda_{G_{k,l}}\right)$$

and

$$\mu(x,y)=\sum_{k,l\in\mathbb{K}}p_{k,l}(x,y)\left\{\left|\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right|+\left|\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right|\right\}.$$

□

Note that using (9), from the above theorem we obtain

$$|(U-V)(f;x,y)|\leq C_0\left\{\omega_2\left(f;\sqrt{\frac{1}{8}\eta(x,y)}\right)+\min(1,\lambda)\|f\|_{C_B(\mathcal{D})}\right\}+M_f'\mu(x,y).$$

5. Difference for Chebishev Functionals

For $f,g\in C_\rho$, we take the bivariate positive linear operators U and V defined at the beginning of this section. Assuming that $f,g,\ fg\in C_\rho\ (\mathcal{D})$, we consider the Chebishev functional of U given by $T^U(f,g):=U(fg)-U(f)U(g)$ (similarly for V) (see [5] and references therein). In this part, we give an upper estimate related to the difference $\left|T^U(f,g)-T^V(f,g)\right|$.

Theorem 3. *Let the functions f,g and fg belong to $C_\rho^*(\mathcal{D})$ and all of their partial derivatives of order ≤ 2 belong to $C_\rho(\mathcal{D})$. If*

$$
\begin{aligned}
\theta_1^{F_{k,l}}=\theta_1^{G_{k,l}}=\theta_1,\ \theta_2^{F_{k,l}}=\theta_2^{G_{k,l}}=\theta_2\ ,\\
U\left(1+(e_{1,0})^2+(e_{0,1})^2;x,y\right)\leq M\rho(x,y)
\end{aligned}
$$

and

$$V\left(1+(e_{1,0})^2+(e_{0,1})^2;x,y\right)\leq M\ \rho(x,y),$$

then we have

$$
\begin{aligned}
&\left|T^U(f,g;x,y)-T^V(f,g;x,y)\right|\\
\leq\ &(\delta_1+\delta_2)\left[1+M\rho(x,y)\left(\|f\|_\rho+\|g\|_\rho\right)\right]+2^8\left[1+q_{k,l}(x,y)\right]\\
&\times\left\{\Omega_\rho(fg,\delta_3,\delta_4)+M\rho(x,y)\left(\|f\|_\rho\Omega_\rho(g,\delta_3,\delta_4)+\|g\|_\rho\Omega_\rho(f,\delta_3,\delta_4)\right)\right\},
\end{aligned}
$$

where δ_1 *and* δ_2 *are the same as in Theorem 1 and*

$$q_{k,l}(x,y) = \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right).$$

Proof. From the definition of Chebyshev functionals, we can write

$$\begin{aligned}
& T^U(f,g;x,y) - T^V(f,g;x,y) \\
= \; & U(fg;x,y) - U(f;x,y)\,U(g;x,y) - V(fg;x,y) + V(f;x,y)\,V(g;x,y) \\
= \; & U(fg;x,y) - U(f;x,y)\,U(g;x,y) - U(f;x,y)\,V(g;x,y) + U(f;x,y)\,V(g;x,y) \\
& -V(fg;x,y) + V(f;x,y)\,V(g;x,y) \\
= \; & U(fg;x,y) - V(fg;x,y) - U(f;x,y)\left[U(g;x,y) - V(g;x,y)\right] \\
& -V(g;x,y)\left[U(f;x,y) - V(f;x,y)\right].
\end{aligned}$$

By taking absolute value of both sides we obtain

$$\begin{aligned}
& \left|T^U(f,g;x,y) - T^V(f,g;x,y)\right| \\
\leq \; & \left|U(fg;x,y) - V(fg;x,y)\right| + \left|U(f;x,y)\right|\left|U(g;x,y) - V(g;x,y)\right| \\
& + \left|V(g;x,y)\right|\left|U(f;x,y) - V(f;x,y)\right|.
\end{aligned}$$

From Theorem 1 we have

$$\begin{aligned}
& \left|U(fg;x,y) - V(fg;x,y)\right| \\
\leq \; & \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\left|F_{k,l}(fg;x,y) - G_{k,l}(fg;x,y)\right| \\
\leq \; & \delta_1 + \delta_2 + 2^8\Omega_\rho(fg,\delta_3,\delta_4)\left(1 + q_{k,l}(x,y)\right)
\end{aligned}$$

and

$$\begin{aligned}
& \left|U(f;x,y)\right|\left|U(g;x,y) - V(g;x,y)\right| \\
\leq \; & M\rho(x,y)\,\|f\|_\rho\left[\delta_1 + \delta_2 + 2^8\Omega_\rho(g,\delta_3,\delta_4)\left(1 + q_{k,l}(x,y)\right)\right]
\end{aligned}$$

$$\begin{aligned}
& \left|V(g;x,y)\right|\left[U(f;x,y) - V(f;x,y)\right] \\
\leq \; & M\rho(x,y)\,\|g\|_\rho\left[\delta_1 + \delta_2 + 2^8\Omega_\rho(f,\delta_3,\delta_4)\left(1 + q_{k,l}(x,y)\right)\right].
\end{aligned}$$

If necessary arrangements are made, the proof is completed. □

6. Application

If we take the well-known bivariate Szász operator as the operator U and the bivariate Szász-Kantorovich as the operator V given, respectively, by

$$U_{n,m}(f;x,y) = \sum_{k,l=0}^{\infty} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!} f\left(\frac{k}{n},\frac{l}{m}\right)$$

and

$$V_{n,m}\left(f;x,y\right)=\sum_{k,l=0}^{\infty}e^{-nx-my}\frac{\left(nx\right)^{k}}{k!}\frac{\left(my\right)^{l}}{l!}nm\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_{\frac{l}{m}}^{\frac{l+1}{m}}f\left(t,s\right)dsdt.$$

Theorem 4. *Let $f\in C_{\rho}^{*}\left(\mathcal{D}\right)$ with all of its partial derivatives of order ≤ 2 belong to the space $C_{\rho}\left(\mathcal{D}\right)$. Then we have*

$$\left|\left(U-V\right)\left(f;x,y\right)\right|\leq\delta_{2}+2^{8}\Omega_{\rho}\left(f,\delta_{3},\delta_{4}\right)\psi\left(x,y\right),$$

where

$$\delta_{2}\left(x,y\right)=\left\{1+\frac{\left(1+8nx+4nx^{2}\right)}{4n^{2}}+\frac{\left(1+8my+4my^{2}\right)}{4m^{2}}\right\}\left\{\frac{1}{3n^{2}}+\frac{1}{3m^{2}}\right\},$$

$$\delta_{3}^{4}\left(x,y\right)=\frac{1}{16n^{2}}+\frac{nx+4nx^{2}}{16n^{4}}+\frac{my+4my^{2}}{16n^{2}m^{2}},$$

$$\delta_{4}^{4}\left(x,y\right)=\frac{1}{16m^{2}}+\frac{nx+4nx^{2}}{16n^{2}m^{2}}+\frac{my+4my^{2}}{16m^{4}}$$

and

$$\psi\left(x,y\right)=2+x^{2}+y^{2}+\frac{x}{n}+\frac{y}{m}.$$

Proof. We use Theorem. By making simple calculations for the operators U and V given above, we have

$$F_{k,l}\left(f\right)=f\left(\frac{k}{n},\frac{l}{m}\right),$$

$$\theta_{1}^{F}=F_{k,l}\left(e_{1,0}\right)=\frac{k}{n},\qquad\theta_{2}^{F}=\frac{l}{m},$$

$$G_{k,l}\left(f\right)=nm\int_{\frac{k}{n}}^{\frac{k+1}{n}}\int_{\frac{l}{m}}^{\frac{l+1}{m}}f\left(t,s\right)dsdt,$$

$$\theta_{1}^{G}=G_{k,l}\left(e_{1,0}\right)=\frac{1}{2n}\left(2k+1\right),\qquad\theta_{2}^{F}=\frac{1}{2m}\left(2l+1\right),$$

$$\mu_{2,0}^{F}=F_{k,l}\left(\left(e_{1,0}-\frac{k}{n}\right)^{2}\right)=0,\ \ \mu_{0,2}^{F}=F_{k,l}\left(\left(e_{0,1}-\frac{l}{m}\right)^{2}\right)=0,$$

$$\mu_{2,0}^{G}=G_{k,l}\left(\left(e_{1,0}-\frac{k}{n}\right)^{2}\right)=\frac{1}{3n^{2}},\ \ \mu_{0,2}^{G}=G_{k,l}\left(\left(e_{0,1}-\frac{l}{m}\right)^{2}\right)=\frac{1}{3m^{2}}.$$

Therefore, we get

$$\begin{aligned}\delta_{1}\left(x,y\right)&=0,\\ \delta_{2}\left(x,y\right)&=\sum_{k,l}^{\infty}e^{-nx-my}\frac{\left(nx\right)^{k}}{k!}\frac{\left(my\right)^{l}}{l!}\left\{\left(1+\frac{\left(2k+1\right)^{2}}{4n^{2}}+\frac{\left(2l+1\right)^{2}}{4m^{2}}\right)\left\{\frac{1}{3n^{2}}+\frac{1}{4mn}+\frac{1}{3m^{2}}\right\}\right\}\\ &=\left\{1+\frac{\left(1+8nx+4nx^{2}\right)}{4n^{2}}+\frac{\left(1+8my+4my^{2}\right)}{4m^{2}}\right\}\left\{\frac{1}{3n^{2}}+\frac{1}{4mn}+\frac{1}{3m^{2}}\right\},\end{aligned}$$

$$
\begin{aligned}
\delta_3^4 &= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_1^{F_{k,l}}-\theta_1^{G_{k,l}}\right)^4\\
&= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\rho\left(\frac{k}{n},\frac{l}{m}\right)\left(\frac{k}{n}-\frac{1}{2n}(2k+1)\right)^4\\
&= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left(1+\frac{k^2}{n^2}+\frac{l^2}{m^2}\right)\left(\frac{1}{2n}\right)^4\\
&= \frac{1}{16n^2}+\frac{nx+4nx^2}{16n^4}+\frac{my+4my^2}{16n^2m^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_4^4 &= \sum_{k,l\in\mathbb{K}} p_{k,l}(x,y)\,\rho\left(\theta_1^{F_{k,l}},\theta_2^{F_{k,l}}\right)\left(\theta_2^{F_{k,l}}-\theta_2^{G_{k,l}}\right)^4\\
&= \sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left(1+\frac{k^2}{n^2}+\frac{l^2}{m^2}\right)\left(\frac{1}{2m}\right)^4\\
&= \frac{1}{16m^2}+\frac{nx+4nx^2}{16n^2m^2}+\frac{my+4my^2}{16m^4}.
\end{aligned}
$$

$$
\begin{aligned}
\psi(x,y) &= 1+\sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\rho\left(\frac{k}{n},\frac{l}{m}\right)\\
&= 1+\sum_{k,l\in\mathbb{K}} e^{-nx-my}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}\left(1+\frac{k^2}{n^2}+\frac{l^2}{m^2}\right)\\
&= 2+x^2+y^2+\frac{x}{n}+\frac{y}{m}.
\end{aligned}
$$

This completes the proof. □

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References

[1] Acu, A. M., Raşa, I., New estimates for the differences ofpositive linear operators, *Numer. Algor.,* 73(3) (2016), 775–789. https://doi.org/10.1007/s11075-016-0117-8

[2] Acu, A. M., Hodiş, S., Raşa, I., A survey on estimates for the differences of positive linear operators, *Constructive Mathematical Analysis*, (1)2 (2018), 113–127. https://doi.org/10.33205/cma.478408

[3] Acu, A. M., Raşa, I., Estimates for the differences of positive linear operators and their derivatives, *Numer. Algor.*, 85(1) (2020), 191–208. https://doi.org/10.1007/s11075-019-00809-4

[4] Acu, A. M., Başcanbaz-Tunca, G., Raşa, I., Differences of positive linear operators on simplices, *Journal of Function Spaces*, 2021 (2021), Article ID 5531577, 11 pages. https://doi.org/10.1155/2021/ 5531577

[5] Aral, A., Inoan, D., Raşa, I., On differences of linear positive operators, *Analysis and Mathematical Physics*, (9)3 (2019), 1227–1239. https://doi.org/10.1007/s13324-018-0227-7

[6] Aral, A., Erbay, H., A note on the difference of positive operators and numerical aspects, *Mediterr. J. Math.*, 17(45) (2020), 20 pp. https://doi.org/10.1007/s00009-020-1489-5

[7] Bartle, R. G., The Elements of Real Analysis*,* John Wiley & Sons Ltd, 1964.

[8] Begen, S., Ince Ilarslan, H. G., Degree of approximation for bivariate Százs-Kantorovich type based on Brenke type polynomials, *Honam Math. J.*, (2020), 42(2), 251–268. https://doi.org/10.5831/HMJ.2020.42.2.251

[9] Butzer, P. L., Berens, H., Semi-Groups of Operators and Approximation, Springer, New York, 1967.

[10] Gadjieva, E., Doğru, O., Szász Tipinde Operatörler Dizisinin Sürekli Fonksiyonlara Yaklaşımı, *II. Uluslararası Kızılırmak Fen Bilimleri Kongresi*, May 20-22, 1998.

[11] Gonska, H., Piţul, P., Raşa, I., On differences of positive linear operators, *Carpathian J. Math.*, 22(1-2) (2006), 65-78. https://www.jstor.org/stable/43998909

[12] İspir, N., Atakut, Ç., Approximation by modified Szász-Mirakjan operators on weighted spaces, *Proc. Indian Acad. Sci. Math. Sci.,* 112(4) (2002), 571–578. https://doi.org/10.1007/BF02829690

http://communications.science.ankara.edu.tr

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ON INEQUALITIES OF SIMPSON'S TYPE FOR CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. Fractional calculus and applications have application areas in many different fields such as physics, chemistry, and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. In this study, we prove an identity involving generalized fractional integrals by using differentiable functions. By utilizing this identity, we obtain several Simpson's type inequalities for the functions whose derivatives in absolute value are convex. Finally, we present some new results as the special cases of our main results.

1. Introduction

Simpson's rules are well-known ways for the numerical integration and numerical estimation of definite integrals. This method is known as developed by Thomas Simpson's (1710–1761). However, Johannes Kepler used the same approximation about 100 years ago, so that this method is also known as Kepler's rule. Simpson's rule includes the three-point Newton-Cotes quadrature rule, so estimation based on three steps quadratic kernel is sometimes called as Newton type results.

(1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\kappa_1}^{\kappa_2} \vartheta(\chi)\mathrm{d}\chi \approx \frac{\kappa_2-\kappa_1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right].$$

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(2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\kappa_1}^{\kappa_2} \vartheta(\chi)\mathrm{d}\chi \approx \frac{\kappa_2-\kappa_1}{8}\left[\vartheta(\kappa_1)+3\vartheta\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\vartheta\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\vartheta(\kappa_2)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimation known as Simpson's inequality:

Theorem 1. *Suppose that* $\vartheta : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is a four times continuously differentiable mapping on* (κ_1, κ_2) *and* $\left\|\vartheta^{(4)}\right\|_\infty = \sup_{\chi\in(\kappa_1,\kappa_2)} \left|\vartheta^{(4)}(\chi)\right| < \infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\vartheta(\kappa_1)+\vartheta(\kappa_2)}{2}+2\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\vartheta(\chi)d\chi\right|$$
$$\le \frac{1}{2880}\left\|\vartheta^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

In recent years, many authors have focused on Simpson's type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson's and Newton's type results for convex mappings, because convexity theory is an effective and powerful method for solving a large number of problems which arise within different branches of pure and applied mathematics. For example, Dragomir et al. [16] presented new Simpson's type results and their applications to quadrature formulas in numerical integration. What is more, some inequalities of Simpson's type for s-convex functions are deduced by Alomari et al. in [6]. Afterwards, Sarikaya et al. observed the variants of Simpson's type inequalities based on convexity in [42]. In [34] and [35], the authors provided some Newton's type inequalities for harmonic convex and p-harmonic convex functions. Additionally, new Newton's type inequalities for functions whose local fractional derivatives are generalized convex are given by Iftikhar et al. in [25]. For more recent developments, one can consult [2–5, 7, 11–15, 17, 18, 23, 36, 47].

2. Generalized Fractional Integrals

Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamic models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches. This subject has been studied by many scientists in terms of its widespread use [20, 21, 27, 30, 31, 37, 40, 44]. One of the most important applications of the fractional Integrals is the Hermite-Hadamard integral inequality (see, [1, 22, 26, 38, 39, 41]).

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [41].

Let's define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{\kappa_1+}I_\varphi \vartheta(\chi) = \int_{\kappa_1}^{\chi} \frac{\varphi(\chi-\tau)}{\chi-\tau} \vartheta(\tau) d\tau, \quad \chi > \kappa_1, \tag{1}$$

$$_{\kappa_2-}I_\varphi \vartheta(\chi) = \int_{\chi}^{\kappa_2} \frac{\varphi(\tau-\chi)}{\tau-\chi} \vartheta(\tau) d\tau, \quad \chi < \kappa_2. \tag{2}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (1) and (2) are mentioned below.

i) If we take $\varphi(\tau) = \tau$, the operator (1) and (2) reduce to the Riemann integral as follows:

$$I_{\kappa_1+}\vartheta(\chi) = \int_{\kappa_1}^{\chi} \vartheta(\tau) d\tau, \quad \chi > \kappa_1,$$

$$I_{\kappa_2-}\vartheta(\chi) = \int_{\chi}^{\kappa_2} \vartheta(\tau) d\tau, \quad \chi < \kappa_2.$$

ii) Let us consider $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$, $\alpha > 0$. Then, the operator (1) and (2) reduce to the Riemann-Liouville fractional integral as follows:

$$J_{\kappa_1+}^{\alpha}\vartheta(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\chi} (\chi-\tau)^{\alpha-1} \vartheta(\tau) d\tau, \quad \chi > \kappa_1,$$

$$J_{\kappa_2-}^{\alpha}\vartheta(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{\kappa_2} (\tau-\chi)^{\alpha-1} \vartheta(\tau) d\tau, \quad \chi < \kappa_2.$$

iii) For $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$, $\alpha, k > 0$, the operator (1) and (2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$J_{\kappa_1+,k}^{\alpha}\vartheta(\chi) = \frac{1}{k\Gamma_k(\alpha)} \int_{\kappa_1}^{\chi} (\chi-\tau)^{\frac{\alpha}{k}-1} \vartheta(\tau) d\tau, \quad \chi > \kappa_1,$$

$$J_{\kappa_2-,k}^{\alpha}\vartheta(\chi)=\frac{1}{k\Gamma_k(\alpha)}\int_{\chi}^{\kappa_2}(\tau-\chi)^{\frac{\alpha}{k}-1}\vartheta(\tau)d\tau,\quad \chi<\kappa_2.$$

Here,

$$\Gamma_k(\alpha)=\int_0^{\infty}\tau^{\alpha-1}e^{-\frac{\tau^k}{k}}d\tau,\quad \mathcal{R}(\alpha)>0$$

and

$$\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right),\quad \mathcal{R}(\alpha)>0;k>0$$

are given by Mubeen and Habibullah in [33].

In the literature, there are several papers on inequalities for generalized fractional integrals. For more information and unexplained subjects, we refer the reader to [8–10, 19, 24, 28, 29, 32, 46, 48] and the references therein.

3. Simpson's Type Inequalities for Generalized Fractional Integrals

Throughout this study for brevity, we define

$$\eta_1(\chi,\tau)=\int_0^{\tau}\frac{\varphi\left(\left(\kappa_2-\chi\right)u\right)}{u}du,\quad \nu_1(\chi,\tau)=\int_0^{\tau}\frac{\varphi\left(\left(\chi-\kappa_1\right)u\right)}{u}du.$$

Particularly, if we choose $\chi=\frac{\kappa_1+\kappa_2}{2}$, then we have

$$\eta_1\left(\frac{\kappa_1+\kappa_2}{2},\tau\right)=\nu_1\left(\frac{\kappa_1+\kappa_2}{2},\tau\right)=\Upsilon_1(\tau)=\int_0^{\tau}\frac{\varphi\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)u\right)}{u}du.$$

Lemma 1. *Let $\vartheta:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an absolutely continuous mapping (κ_1,κ_2) such that $\vartheta'\in L_1([\kappa_1,\kappa_2])$. Then, the following equality holds:*

$$\begin{aligned}&\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\\ =\;&\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\int_0^1\left(\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right)\vartheta'\left(\tau\chi+(1-\tau)\kappa_2\right)d\tau\\ &-\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\int_0^1\left(\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right)\vartheta'\left((1-\tau)\kappa_1+\tau\chi\right)d\tau.\end{aligned}$$

Proof. By using integration by parts, we have

$$\begin{aligned}H_1&=\int_0^1\left(\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right)\vartheta'\left(\tau\chi+(1-\tau)\kappa_2\right)d\tau \qquad (3)\\ &=\frac{1}{\kappa_2-\chi}\eta_1(\chi,1)\left[2\vartheta(\chi)+\vartheta(\kappa_2)\right]\end{aligned}$$

$$+\frac{3}{\chi-\kappa_2}\int_0^1 \vartheta\left(\tau\chi+(1-\tau)\kappa_2\right)\frac{\varphi\left((\kappa_2-\chi)\tau\right)}{\tau}d\tau$$
$$=\frac{1}{\kappa_2-\chi}\eta_1(\chi,1)\left[2\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{3}{\kappa_2-\chi}\int_\chi^{\kappa_2}\frac{\vartheta(u)\varphi(\kappa_2-u)}{\kappa_2-u}du$$
$$=\frac{\eta_1(\chi,1)}{\kappa_2-\chi}\left[2\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{3}{\kappa_2-\chi}\ {}_{\chi+}I_\varphi\vartheta(\kappa_2).$$

Similarly, we obtain

$$H_2=\int_0^1\left(\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right)\vartheta'\left(\tau\chi+(1-\tau)\kappa_1\right)d\tau \tag{4}$$
$$=\frac{\nu_1(\chi,1)}{\chi-\kappa_1}\left[-2\vartheta(\chi)-\vartheta(\kappa_1)\right]+\frac{3}{\chi-\kappa_1}\ {}_{\chi-}I_\varphi\vartheta(\kappa_1).$$

From (3) and (4), we get

$$\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}H_1-\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}H_2$$
$$=\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_\varphi\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_\varphi\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right].$$

This ends the proof of Lemma 1. □

Corollary 1. *Under assumptions of Lemma 1 with* $\chi=\frac{\kappa_1+\kappa_2}{2}$, *we obtain the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]$$
$$-\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\vartheta(\kappa_2)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\vartheta(\kappa_1)\right]$$
$$=\frac{\kappa_2-\kappa_1}{12\eta_1(\chi,1)}\int_0^1\left(\Upsilon_1(1)-3\Upsilon_1(\tau)\right)$$
$$\times\left[\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)-\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)\right]d\tau.$$

Corollary 2. *In Lemma 1, if we choose $\varphi(\tau)=\tau$ for all $\tau\in[\kappa_1,\kappa_2]$, then we obtain the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_{\chi}^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]$$
$$=\frac{\kappa_2-\chi}{6}\int_0^1(1-3\tau)\,\vartheta'(\tau\chi+(1-\tau)\kappa_2)\,d\tau$$
$$-\frac{\chi-\kappa_1}{6}\int_0^1(1-3\tau)\,\vartheta'((1-\tau)\kappa_1+\tau\chi)\,d\tau.$$

Corollary 3. *In Lemma 1, let us consider $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)},\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$. Then, we get the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J^\alpha_{\chi+}\vartheta(\kappa_2)}{(\kappa_2-\chi)^\alpha}+\frac{J^\alpha_{\chi-}\vartheta(\kappa_1)}{(\chi-\kappa_1)^\alpha}\right]$$
$$=\frac{\kappa_2-\chi}{6}\int_0^1(1-3\tau^\alpha)\,\vartheta'(\tau\chi+(1-\tau)\kappa_2)\,d\tau$$
$$-\frac{\chi-\kappa_1}{6}\int_0^1(1-3\tau^\alpha)\,\vartheta'((1-\tau)\kappa_1+\tau\chi)\,d\tau.$$

Corollary 4. *In Lemma 1, if we assign $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},k,\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we have the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J^\alpha_{\chi+,k}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J^\alpha_{\chi-,k}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]$$
$$=\frac{\kappa_2-\chi}{6}\int_0^1\left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'(\tau\chi+(1-\tau)\kappa_2)\,d\tau$$
$$-\frac{\chi-\kappa_1}{6}\int_0^1\left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'((1-\tau)\kappa_1+\tau\chi)\,d\tau.$$

Remark 1. *If we set $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollaries 2, 3 and 4, then we obtain the following identities*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\vartheta(\tau)\,d\tau$$

$$= \frac{\kappa_2-\kappa_1}{12}\left[\int_0^1 (1-3\tau)\,\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)d\tau - \int_0^1 (1-3\tau)\,\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)d\tau\right],$$

$$\begin{aligned}&\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\\ &= \frac{\kappa_2-\kappa_1}{12}\left[\int_0^1 (1-3\tau^{\alpha})\,\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)d\tau - \int_0^1 (1-3\tau^{\alpha})\,\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)d\tau\right],\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right] - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\\ &= \frac{\kappa_2-\kappa_1}{12}\left[\int_0^1 \left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)d\tau - \int_0^1 \left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)d\tau\right],\end{aligned}$$

respectively.

Theorem 2. *Assume that the assumptions of Lemma 1 hold. Assume also that the mapping $|\vartheta'|$ is convex on $[\kappa_1,\kappa_2]$. Then, we have the following inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right|$$

$$\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left[\Xi_1\left|\vartheta'(\chi)\right|+\Xi_2\left|\vartheta'(\kappa_2)\right|\right]+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left[\Xi_3\left|\vartheta'(\kappa_1)\right|+\Xi_4\left|\vartheta'(\chi)\right|\right],$$

where

$$\begin{cases} \Xi_1 = \int\limits_0^1 \tau \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| d\tau, \\ \Xi_2 = \int\limits_0^1 (1-\tau) \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| d\tau, \\ \Xi_3 = \int\limits_0^1 (1-\tau) \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| d\tau, \\ \Xi_4 = \int\limits_0^1 \tau \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| d\tau. \end{cases} \tag{5}$$

Proof. By taking modulus in Lemma 1, we obtain

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1) + 4\vartheta(\chi) + \vartheta(\kappa_2)\right] - \frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)} + \frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right| \tag{6}$$

$$\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\int_0^1 \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| \left|\vartheta'(\tau\chi + (1-\tau)\kappa_2)\right| d\tau$$

$$+ \frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\int_0^1 \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| \left|\vartheta'((1-\tau)\kappa_1 + \tau\chi)\right| d\tau.$$

With the help of the convexity of $|\vartheta'|$, we get

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1) + 4\vartheta(\chi) + \vartheta(\kappa_2)\right] - \frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)} + \frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right|$$

$$\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\int_0^1 \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| \left[\tau\left|\vartheta'(\chi)\right| + (1-\tau)\left|\vartheta'(\kappa_2)\right|\right] d\tau$$

$$+ \frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\int_0^1 \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| \left[(1-\tau)\left|\vartheta'(\kappa_1)\right| + \tau\left|\vartheta'(\chi)\right|\right] d\tau$$

$$= \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left[\Xi_1\left|\vartheta'(\chi)\right| + \Xi_2\left|\vartheta'(\kappa_2)\right|\right] + \frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left[\Xi_3\left|\vartheta'(\kappa_1)\right| + \Xi_4\left|\vartheta'(\chi)\right|\right].$$

This completes the proof of Theorem 2. □

Corollary 5. *Under assumptions of Theorem 2 with* $\chi = \frac{\kappa_1+\kappa_2}{2}$, *we have the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1) + 4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right) + \vartheta(\kappa_2)\right]\right.$$

$$\left. -\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\vartheta(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\vartheta(\kappa_1)\right]\right|$$

$$\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left[2\Xi_5\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|+\Xi_6\left[\left|\vartheta'\left(\kappa_2\right)\right|+\left|\vartheta'\left(\kappa_1\right)\right|\right]\right]$$
$$\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\Xi_5+\Xi_6\right)\left[\left|\vartheta'\left(\kappa_2\right)\right|+\left|\vartheta'\left(\kappa_1\right)\right|\right].$$

Here,

$$\Xi_5=\int_0^1 \tau\left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right|d\tau \text{ and } \Xi_6=\int_0^1 (1-\tau)\left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right|d\tau. \tag{7}$$

Corollary 6. *In Theorem 2, let us note that $\varphi(\tau)=\tau$ for all $\tau\in[\kappa_1,\kappa_2]$. Then, we obtain the inequality*

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_{\chi}^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]\right|$$
$$\leq \frac{\kappa_2-\chi}{6}\left[\frac{29}{54}\left|\vartheta'\left(\chi\right)\right|+\frac{8}{27}\left|\vartheta'\left(\kappa_2\right)\right|\right]+\frac{\chi-\kappa_1}{6}\left[\frac{8}{27}\left|\vartheta'\left(\kappa_1\right)\right|+\frac{29}{54}\left|\vartheta'\left(\chi\right)\right|\right].$$

Corollary 7. *In Theorem 2, if we select $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}, \alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we get the inequality*

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J_{\chi+}^{\alpha}\vartheta\left(\kappa_2\right)}{\left(\kappa_2-\chi\right)^{\alpha}}+\frac{J_{\chi-}^{\alpha}\vartheta\left(\kappa_1\right)}{\left(\chi-\kappa_1\right)^{\alpha}}\right]\right|$$
$$\leq \frac{\kappa_2-\chi}{6}\left[\Theta_1(\alpha)\left|\vartheta'\left(\chi\right)\right|+\Theta_2(\alpha)\left|\vartheta'\left(\kappa_2\right)\right|\right]$$
$$+\frac{\chi-\kappa_1}{6}\left[\Theta_2(\alpha)\left|\vartheta'\left(\kappa_1\right)\right|+\Theta_1(\alpha)\left|\vartheta'\left(\chi\right)\right|\right],$$

where

$$\Theta_1(\alpha)=\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{4-\alpha}{2\left(\alpha+2\right)}, \tag{8}$$
$$\Theta_2(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{4-3\alpha-\alpha^2}{2\left(\alpha+1\right)\left(\alpha+2\right)}.$$

Corollary 8. *In Theorem 2, consider $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, k,\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we have the following inequality*

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J_{\chi+,k}^{\alpha}\vartheta\left(\kappa_2\right)}{\left(\kappa_2-\chi\right)^{\frac{\alpha}{k}}}+\frac{J_{\chi-,k}^{\alpha}\vartheta\left(\kappa_1\right)}{\left(\chi-\kappa_1\right)^{\frac{\alpha}{k}}}\right]\right|$$
$$\leq \frac{\kappa_2-\chi}{6}\left[\Psi_1(\alpha,k)\left|\vartheta'\left(\chi\right)\right|+\Psi_2(\alpha,k)\left|\vartheta'\left(\kappa_2\right)\right|\right]$$
$$+\frac{\chi-\kappa_1}{6}\left[\Psi_2(\alpha,k)(\alpha)\left|\vartheta'\left(\kappa_1\right)\right|+\Psi_1(\alpha,k)\left|\vartheta'\left(\chi\right)\right|\right].$$

Here,

$$\Psi_1(\alpha,k)=\frac{\alpha}{\alpha+2k}\left(\frac{1}{3}\right)^{\frac{2k}{\alpha}}+\frac{4k-\alpha}{2(\alpha+2k)},\tag{9}$$
$$\Psi_2(\alpha,k)=\frac{2\alpha}{\alpha+k}\left(\frac{1}{3}\right)^{\frac{k}{\alpha}}-\frac{\alpha}{\alpha+2k}\left(\frac{1}{3}\right)^{\frac{2k}{\alpha}}+\frac{4k^2-3\alpha k-\alpha^2}{2(\alpha+k)(\alpha+2k)}.$$

Remark 2. *If we set $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 6, then Corollary 6 reduces to [43, Corollary 1].*

Remark 3. *Assume $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 7. Then, we obtain the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.\\&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{12}\left[\Theta_2(\alpha)\left(\left|\vartheta'(\kappa_2)\right|+\left|\vartheta'(\kappa_1)\right|\right)+2\Theta_1(\alpha)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right],\end{aligned}$$

which is given by Hai and Wang in [23].

Remark 4. *Assume $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 8. Then, we obtain the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.\\&\quad\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{12}\left[\Psi_2(\alpha,k)\left(\left|\vartheta'(\kappa_2)\right|+\left|\vartheta'(\kappa_1)\right|\right)+2\Psi_1(\alpha,k)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right].\end{aligned}$$

Theorem 3. *Suppose that the assumptions of Lemma 1 hold. Suppose also that the mapping $\left|\vartheta'\right|^q$, $q>1$, is convex on $[\kappa_1,\kappa_2]$. Then, we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right|\\&\leq\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\\&\quad+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder inequality (6), we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|\\
&\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\vartheta'\left(\tau\chi+(1-\tau)\kappa_2\right)\right|^q d\tau\right)^{\frac{1}{q}}\\
&\quad+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\vartheta'\left((1-\tau)\kappa_1+\tau\chi\right)\right|^q d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

By using convexity of $\left|\vartheta'\right|^q$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|\\
&\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\left(\tau\left|\vartheta'(\chi)\right|^q+(1-\tau)\left|\vartheta'(\kappa_2)\right|^q\right)d\tau\right)^{\frac{1}{q}}\\
&\quad+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\left((1-\tau)\left|\vartheta'\left(\kappa_1\right)\right|^q+\tau\left|\vartheta'\left(\chi\right)\right|^q\right)d\tau\right)^{\frac{1}{q}}\\
&=\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\\
&\quad+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof of Theorem 3. □

Corollary 9. *Under assumptions of Theorem 3 with* $\chi = \frac{\kappa_1+\kappa_2}{2}$, *we have the following inequalities*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.\\
&\left.\quad-\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\vartheta(\kappa_2)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\vartheta(\kappa_1)\right]\right|\\
&\le\frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1\left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right|^p d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right]\\
&\le\frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1\left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right|^p d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{\left|\vartheta'(\kappa_1)\right|^q+3\left|\vartheta'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\vartheta'(\kappa_1)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 10. *In Theorem 3, let us consider* $\varphi(\tau)=\tau$ *for all* $\tau\in[\kappa_1,\kappa_2]$. *Then, we obtain the inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_\chi^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]\right|\\
&\le\frac{1}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\\
&\quad\times\left[(\kappa_2-\chi)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}+(\chi-\kappa_1)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 11. *In Theorem 3, if we take* $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}, \alpha>0$ *for all* $\tau\in[\kappa_1,\kappa_2]$, *then we get the inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J_{\chi+}^{\alpha}\vartheta(\kappa_2)}{(\kappa_2-\chi)^\alpha}+\frac{J_{\chi-}^{\alpha}\vartheta(\kappa_1)}{(\chi-\kappa_1)^\alpha}\right]\right|\\
&\le\frac{1}{6}\left(\int_0^1\left|1-3\tau^\alpha\right|^p d\tau\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left[(\kappa_2-\chi)\left(\frac{|\vartheta'(\chi)|^q+|\vartheta'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}+(\chi-\kappa_1)\left(\frac{|\vartheta'(\chi)|^q+|\vartheta'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Corollary 12. *In Theorem 3, let us note that* $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, k,\alpha>0$ *for all* $\tau\in[\kappa_1,\kappa_2]$. *Then, we have the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J^{\alpha}_{\chi+,k}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J^{\alpha}_{\chi-,k}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]\right|$$

$$\leq\frac{1}{6}\left(\int_0^1\left|1-3\tau^{\frac{\alpha}{k}}\right|^p d\tau\right)^{\frac{1}{p}}$$

$$\times\left[(\kappa_2-\chi)\left(\frac{|\vartheta'(\chi)|^q+|\vartheta'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}+(\chi-\kappa_1)\left(\frac{|\vartheta'(\chi)|^q+|\vartheta'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Remark 5. *If we assign* $\chi=\frac{\kappa_1+\kappa_2}{2}$ *in Corollary 10, then Corollary 10 reduces to [43, Corollary 3].*

Remark 6. *Consider* $\chi=\frac{\kappa_1+\kappa_2}{2}$ *in Corollaries 11 and 12. Then, we obtain the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{12}\left(\int_0^1|1-3\tau^{\alpha}|^p d\tau\right)^{\frac{1}{p}}\left[\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+|\vartheta'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+|\vartheta'(\kappa_1)|^q}{2}\right)^{\frac{1}{q}}\right]$$

and

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{12}\left(\int_0^1\left|1-3\tau^{\frac{\alpha}{k}}\right|^p d\tau\right)^{\frac{1}{p}}\left[\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+|\vartheta'(\kappa_2)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\Bigg],$$

respectively.

Theorem 4. *Suppose that the assumptions of Lemma 1 hold. If the mapping* $\left|\vartheta'\right|^q$, $q\geq 1$, *is convex on* $[\kappa_1,\kappa_2]$, *then we have the following inequality*

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|$$

$$\leq\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|d\tau\right)^{1-\frac{1}{q}}\left(\Xi_1\left|\vartheta'\left(\chi\right)\right|^q+\Xi_2\left|\vartheta'\left(\kappa_2\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1\left(\chi,1\right)-3\nu_1\left(\chi,\tau\right)\right|d\tau\right)^{1-\frac{1}{q}}\left(\Xi_3\left|\vartheta'\left(\kappa_1\right)\right|^q+\Xi_4\left|\vartheta'\left(\chi\right)\right|^q\right)^{\frac{1}{q}},$$

where Ξ_i, $i=1,2,3,4$ *are defined as in equality (5).*

Proof. By applying power mean inequality (6), we get

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|$$

$$\leq\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|d\tau\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|\left|\vartheta'\left(\tau\chi+(1-\tau)\kappa_2\right)\right|^q d\tau\right)^{\frac{1}{q}}$$

$$+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|d\tau\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|\left|\vartheta'\left((1-\tau)\kappa_1+\tau\chi\right)\right|^q d\tau\right)^{\frac{1}{q}}.$$

Since $\left|\vartheta'\right|^q$ is convex, we obtain

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|$$

$$\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1 |\eta_1(\chi,1)-3\eta_1(\chi,\tau)|\,d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[\tau\,|\eta_1(\chi,1)-3\eta_1(\chi,\tau)|\,|\vartheta'(\chi)|^q\right.\right.$$
$$\left.\left.+(1-\tau)\,|\eta_1(\chi,1)-3\eta_1(\chi,\tau)|\,|\vartheta'(\kappa_2)|^q\right]d\tau\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left(\int_0^1 |\nu_1(\chi,1)-3\nu_1(\chi,\tau)|\,d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[(1-\tau)\,|\nu_1(\chi,1)-3\nu_1(\chi,\tau)|\,|\vartheta'(\kappa_1)|^q\right.\right.$$
$$\left.\left.+\tau\,|\nu_1(\chi,1)-3\nu_1(\chi,\tau)|\,|\vartheta'(\chi)|^q\,d\tau\right]\right)^{\frac{1}{q}}$$
$$=\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1 |\eta_1(\chi,1)-3\eta_1(\chi,\tau)|\,d\tau\right)^{1-\frac{1}{q}}\left(\Xi_1\,|\vartheta'(\chi)|^q+\Xi_2\,|\vartheta'(\kappa_2)|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left(\int_0^1 |\nu_1(\chi,1)-3\nu_1(\chi,\tau)|\,d\tau\right)^{1-\frac{1}{q}}\left(\Xi_3\,|\vartheta'(\kappa_1)|^q+\Xi_4\,|\vartheta'(\chi)|^q\right)^{\frac{1}{q}}.$$

This completes the proof of Theorem 4. □

Corollary 13. *Under assumptions of Theorem 4 with* $\chi=\frac{\kappa_1+\kappa_2}{2}$, *we have the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\vartheta(\kappa_2)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\vartheta(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1|\Upsilon_1(1)-3\Upsilon_1(\tau)|\,d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\Xi_5\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\Xi_6\,|\vartheta'(\kappa_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Xi_6\,|\vartheta'(\kappa_1)|^q+\Xi_5\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1 |\Upsilon_1(1)-3\Upsilon_1(\tau)|\,d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{\Xi_5\left|\vartheta'(\kappa_1)\right|^q+(\Xi_5+2\Xi_6)\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\Xi_5+2\Xi_6)\left|\vartheta'(\kappa_1)\right|^q+\Xi_5\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Here, Ξ_5 and Ξ_6 are defined as in equality (7).

Corollary 14. *In Theorem 4, if we choose $\varphi(\tau)=\tau$ for all $\tau\in[\kappa_1,\kappa_2]$, then we obtain the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_\chi^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]\right|$$
$$\leq\frac{\kappa_2-\chi}{6}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\frac{29}{54}\left|\vartheta'(\chi)\right|^q+\frac{8}{27}\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\frac{8}{27}\left|\vartheta'(\kappa_1)\right|^q+\frac{29}{54}\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}}.$$

Corollary 15. *In Theorem 4, let us note that $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)},\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$. Then, we have the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J^\alpha_{\chi+}\vartheta(\kappa_2)}{(\kappa_2-\chi)^\alpha}+\frac{J^\alpha_{\chi-}\vartheta(\kappa_1)}{(\chi-\kappa_1)^\alpha}\right]\right|$$
$$\leq\frac{\kappa_2-\chi}{6}\left(\Theta_3(\alpha)\right)^{1-\frac{1}{q}}\left(\Theta_1(\alpha)\left|\vartheta'(\chi)\right|^q+\Theta_2(\alpha)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6}\left(\Theta_3(\alpha)\right)^{1-\frac{1}{q}}\left(\Theta_2(\alpha)\left|\vartheta'(\kappa_1)\right|^q+\Theta_1(\alpha)\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}},$$

where $\Theta_i(\alpha)$, $i=1,2$ are defined as in equality (8) and

$$\Theta_3(\alpha)=2\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\left[1-\frac{1}{\alpha+1}\right]+\frac{3}{\alpha+1}-1.$$

Corollary 16. *In Theorem 4, if we set $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},k,\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we get the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J^\alpha_{\chi+,k}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J^\alpha_{\chi-,k}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]\right|$$

$$\leq \frac{\kappa_2-\chi}{6}\left(\Psi_3(\alpha,k)\right)^{1-\frac{1}{q}}\left(\Psi_1(\alpha,k)\left|\vartheta'(\chi)\right|^q+\Psi_2(\alpha,k)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6}\left(\Psi_3(\alpha,k)\right)^{1-\frac{1}{q}}\left(\Psi_2(\alpha,k)\left|\vartheta'(\kappa_1)\right|^q+\Psi_1(\alpha,k)\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}},$$

where $\Psi_i(\alpha,k)$, $i=1,2$ are defined as in equality (9) and

$$\Psi_3(\alpha,k)=2\left(\frac{1}{3}\right)^{\frac{k}{\alpha}}\left[1-\frac{k}{\alpha+k}\right]+\frac{3k}{(\alpha+k)}-1.$$

Remark 7. *Considering $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 14, then Corollary 14 reduces to [43, Theorem 10 (for $s=1$)].*

Remark 8. *If we take $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollaries 15 and 16, then we obtain the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{12}\left(\Theta_3(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Theta_1(\alpha)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\Theta_2(\alpha)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Theta_2(\alpha)\left|\vartheta'(\kappa_1)\right|^q+\Theta_1(\alpha)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

and

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{12}\left(\Psi_3(\alpha,k)\right)^{1-\frac{1}{q}}\left[\left(\Psi_1(\alpha,k)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\Psi_2(\alpha,k)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Psi_2(\alpha,k)\left|\vartheta'(\kappa_1)\right|^q+\Psi_1(\alpha,k)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right],$$

respectively.

4. Conclusion

In this paper, we used the concepts of fractional calculus and proved some new inequalities of Simpson's type inequalities for differentiable convex mappings. Moreover, we discussed the special cases of the main results and several new inequalities of Simpson's type for differentiable convex functions via the ordinary integral are obtained. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for co-ordinated convex functions in their future research.

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References

[1] Agarwal, P., Vivas-Cortez, M., Rangel-Oliveros, Y., Ali, M. A., New Ostrowski type inequalities for generalized s-convex functions with applications to some special means of real numbers and to midpoint formula, *AIMS Mathematics*, 7(1) (2022), 1429–1444. doi: 10.3934/math.2022084

[2] Ali, M. A., Budak, H., Zhang, Z., Yildirim, H., Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus, *Math. Methods Appl. Sci.*, 44(6) (2021), 4515–4540 https://doi.org/10.1002/mma.7048

[3] Ali, M. A., Budak, H., Abbas, M., Chu, Y.-M., Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^{κ_2}*-derivatives*, *Adv. Difference Equ.*, 2021(7) (2021). https://doi.org/10.1186/s13662-020-03163-1

[4] Ali, M. A., Chu, Y. M., Budak, H., Akkurt, A., Yildirim, H., New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions, *Adv. Difference Equ.*, 2021(64) (2021). https://doi.org/10.1186/s13662-021-03226-x

[5] Ali, M. A., Abbas, M., Budak, H., Agarwal, P., Murtaza, G., Chu, Y.-M., Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables, *Adv. Difference Equ.*, 2021(25) (2021). https://doi.org/10.1186/s13662-020-03195-7

[6] Alomari, M., Darus, M., Dragomir, S. S., New inequalities of Simpson's type for s-convex functions with applications, *RGMIA Res. Rep. Coll.*, 12(4) (2009).

[7] Budak, H., Erden, S., Ali, M. A., Simpson and Newton type inequalities for convex functions via newly defined quantum integrals, *Math. Methods Appl. Sci.*, 44(1) (2021), 378–390 https://doi.org/10.1002/mma.6742.

[8] Budak, H., Kara, H., Kapucu, R., New midpoint type inequalities for generalized fractional integral, *Comput. Methods Differ. Equ.*, 10(1) (2022), 93–108. DOI:10.22034/cmde.2020.40684.1772

[9] Budak, H., Pehlivan E., Kösem, P., On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals, *Sahand Communications in Mathematical Analysis*, 18(1) (2021), 73–88.

[10] Budak, H., Hezenci, F., Kara, H., On parameterized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integrals, *Math. Methods Appl. Sci.*, 44(17) (2021), 12522–12536. DOI: 10.1002/mma.7558

[11] Butt, S. I., Akdemir, A. O., Agarwal, P., Baleanu, D., Non-conformable integral inequalities of Chebyshev-Pólya-Szego type, *J. Math. Inequal.*, 15(4) (2021), 1391–1400. dx.doi.org/10.7153/jmi-2021-15-94

[12] Butt, S. I., Agarwal, P., Yousaf, S. Guirao, J. L., Generalized fractal Jensen and Jensen-Mercer inequalities for harmonic convex function with applications, *J. Inequal. Appl.*, 2022(1) (2022), 1–18. https://doi.org/10.1186/s13660-021-02735-3
[13] Butt, S. I., Akdemir, A. O., Nadeem, M., Raza, M. A., Gruss type inequalities via generalized fractional operators *Math. Methods Appl. Sci.*, 44(17) (2021), 12559–12574. https://doi.org/10.1002/mma.7563
[14] Butt, S. I., Yousaf, S., Akdemir, A. O., Dokuyucu, M. A., New Hadamard-type integral inequalities via a general form of fractional integral operators, *Chaos Solitons Fractals*, 148 (2021), 111025. https://doi.org/10.1016/j.chaos.2021.111025
[15] Butt, S. I., Nadeem, M., Tariq, M., Aslam, A., New integral type inequalities via Raina-convex functions and its applications *Commun. Fac. Sci. Univ. Ank. S´er. A1 Math. Stat.*, 70(2) (2021), 1011-1035. https://doi.org/10.31801/cfsuasmas.848853
[16] Dragomir, S. S., Agarwal, R. P., Cerone, P., On Simpson's inequality and applications, *J. Inequal. Appl.*, 5 (2000), 533–579.
[17] Du, T., Li, Y., Yang, Z., A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions, *Appl. Math. Comput.*, 293 (2017), 358–369. https://doi.org/10.1016/j.amc.2016.08.045
[18] Erden, S., Iftikhar, S., Delavar, R. M., Kumam, P., Thounthong P., Kumam, W., On generalizations of some inequalities for convex functions via quantum integrals, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM*, 114(3) (2020), 1–15. Doi: 10.1007/s13398-020-00841-3.
[19] Ertuğral, F., Sarikaya, M. Z., Simpson type integral inequalities for generalized fractional integral, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM*, 113(4) (2019), 3115–3124. https://doi.org/10.1007/s13398-019-00680-x
[20] Farid, G., Rehman, A., Zahra, M., On Hadamard inequalities for k-fractional integrals, *Nonlinear Funct. Anal. Appl.*, 21(3) (2016), 463–478.
[21] Gorenflo, R., Mainardi, F., Fractional Calculus: Integral and Differential Equations of Fractional Order, Wien: Springer-Verlag, 1997, 223–276.
[22] Hadamard, J., Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par, *Riemann, J. Math. Pures. et Appl.*, 58 (1893), 171–215.
[23] Hai, X., Wang S. H., Simpson type inequalities for convex function based on the generalized fractional integrals, *Turkish J. Math.*, 5(1) (2021), 1–15.
[24] Han, J., Mohammed, P. O., Zeng, H., Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function, *Open Math.*, 18(1) (2020), 794–806. https://doi.org/10.1515/math-2020-0038
[25] Iftikhar, S., Komam, P., Erden, S., Newton's type integral inequalities via local fractional integrals, *Fractals*, 28(3) (2020), 2050037, 13 pages. Doi: 10.1142/S0218348X20500371.
[26] Jain, S., Goyal, R., Agarwal, P., Guirao, J. L., Some inequalities of extended hypergeometric functions, *Mathematics*, 9(21) (2021), 2702. https://doi.org/10.3390/math9212702
[27] Katugampola, U. N., A new fractional derivative with classical properties, (2014) *e-print arXiv:1410.6535*.
[28] Kashuri, A., Ali, M. A., Abbas M., Budak, H., New inequalities for generalized m-convex functions via generalized fractional integral operators and their applications, *International Journal of Nonlinear Analysis and Applications*, 10(2) (2019), 275-299. doi: 10.22075/ijnaa.2019.18455.2014
[29] Kashuri, A., Liko, R., On Fejér type inequalities for convex mappings utilizing generalized fractional integrals, *Appl. Appl. Math.*, 15(1) (2020), 240–255.
[30] Khalil, R., Alomari, M., Yousef, A., Sababheh, M., A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264 (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002
[31] Miller, S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, New York, Wiley, 1993.

[32] Mohammed, P. O., Sarikaya, M. Z., On generalized fractional integral inequalities for twice differentiable convex functions, *J. Comput. Appl. Math.*, 372 (2020), 112740. https://doi.org/10.1016/j.cam.2020.112740

[33] Mubeen, S., Habibullah, G. M., k-Fractional integrals and application, *Int. J. Contemp. Math. Sciences*, 7(2) (2012), 89–94.

[34] Noor, M. A., Noor, K. I., Iftikhar, S., Some Newton's type inequalities for harmonic convex functions, *J. Adv. Math. Stud.*, 9(1) (2016), 07–16.

[35] Noor, M. A., Noor K. I., Iftikhar, S., Newton inequalities for p-harmonic convex functions, *Honam Math. J.*, 40(2) (2018), 239–250. https://doi.org/10.5831/HMJ.2018.40.2.239

[36] Park, J., On Simpson-like type integral inequalities for differentiable preinvex functions, *Appl. Math. Sci.*, 7(121) (2013), 6009–6021. http://dx.doi.org/10.12988/ams.2013.39498

[37] Podlubni, I., Fractional Differential Equations, San Diego, CA: Academic Press, 1999.

[38] Sarikaya, M. Z., Set, E., Yaldiz, H., Basak, N., Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modelling*, 57(9–10) (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

[39] Sarikaya, M. Z., Akkurt, A., Budak, H., Yildirim, M. E., Yildirim, H., Hermite-Hadamard's inequalities for conformable fractional integrals, *Konuralp Journal of Mathematics*, 8(2) (2020), 376-383.

[40] Sarikaya, M. Z., Ogunmez, H., On new inequalities via Riemann–Liouville fractional integration, *Abs Appl Anal.* 2012. Article ID 428983, 10 pages. doi:10.1155/2012/428983.

[41] Sarikaya, M. Z., Ertugral, F., On the generalized Hermite-Hadamard inequalities, *An. Univ. Craiova Ser. Mat. Inform.*, 47(1) (2020), 193–213.

[42] Sarikaya, M. Z., Set, E., Özdemir, M. E., On new inequalities of Simpson's type for convex functions, *RGMIA Res. Rep. Coll.*, 13(2) (2010), Article2.

[43] Sarikaya, M. Z., Set, E., Özdemir, M. E., On new inequalities of Simpson's type for s-convex functions, *Comput. Math. Appl.*, 60(8) (2020), 2191–2199. https://doi.org/10.1016/j.camwa.2010.07.033

[44] Set, E., Butt, S. I., Akdemir, A. O., Karaoglan, A., Abdeljawad, T., New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators, *Chaos Solitons Fractals*, 143 (2021), 110554. https://doi.org/10.1016/j.chaos.2020.110554

[45] Turkay, M. E., Sarikaya, M. Z., Budak, H., Yildirim, H., Some Hermite-Hadamard type inequalities for co-ordinated convex functions via generalized fractional integrals, Submitted, *ResearchGate Article: https://www.researchgate.net/publication/321803898*.

[46] Qi, F., Mohammed, P. O., Yao, J.-C., Yao, Y.-H., Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions, *J. Inequal. Appl.* 2019, 135 (2019). https://doi.org/10.1186/s13660-019-2079-6

[47] Vivas-Cortez, M., Ali, M. A., Kashuri, A., Sial, I. B., Zhang, Z., Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus, *Symmetry*, 12(9) (2020) 1476; https://doi.org/10.3390/sym12091476.

[48] Zhao, D., Ali, M. A., Kashuri, A., Budak, H., Sarikaya, M. Z., Hermite–Hadamard-type inequalities for the interval-valued approximately $h-$convex functions via generalized fractional integrals, *J. Inequal. Appl.*, 2020(222) (2020), 1–38. https://doi.org/10.1186/s13660-020-02488-5

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ASSOCIATED CURVES FROM A DIFFERENT POINT OF VIEW IN E^3

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Abstract. In this paper, tangent, principal normal and binormal wise associated curves are defined such that each of these vectors of any given curve lies on the osculating, normal and rectifying plane of its partner, respectively. For each associated curve, a new moving frame and the corresponding curvatures are formulated in terms of Frenet frame vectors. In addition to this, the possible solutions for distance functions between the curve and its associated mate are discussed. In particular, it is seen that the involute curves belong to the family of tangent associated curves in general and the Bertrand and the Mannheim curves belong to the principal normal associated curves. Finally, as an application, we present some examples and map a given curve together with its partner and its corresponding moving frame.

1. Introduction

In differential geometry, curves are named as associated if there exist a mathematical relation among them. Some of those known as involute-evolute curves, Bertrand curves, Mannheim curves and more recently the successor curves are the ones on which the researchers most referred ([1–5]). For such curves, the association is based upon the Frenet elements of the curves. There have been other studies using different frames such as Darboux and Bishop to associate curves, as well ([6–11]). From a distinct point of view, Choi and Kim (2012), introduced new associated curves of a given Frenet curve as the integral curves of vector fields [12]. Şahiner, on the other hand, established direction curves of "tangent" and "principal normal" indicatrix of any curve and provided some methods to portray helices and slant helices by using these curves in his studies, [13] and [14], respectively. In

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this study, we introduce another Frenet frame based associated curves such that the tangent, the principal normal and the binormal vectors of a given any curve lies on the osculating, normal and rectifying plane of its partner, respectively. For each associated curves a new moving frame is established and the distances between the curve and its offset are given. In particular, it is seen that the involute curves belong to the family of tangent associated curves. In addition, some traces of the Bertrand and Mannheim curves are found while examining principal normal and binormal associated curves. We also provided a few examples to illustrate the intuitive idea of this paper.

Since we refer the Frenet frame, the formulae and the curvatures of a regular curve, α through out the paper, we remind the definitions of these once again as:

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \quad N(s) = B(s) \times T(s), \quad B(s) = \frac{\alpha'(s) \times \alpha''(s)}{\|\alpha'(s) \times \alpha''(s)\|}, \tag{1}$$

$$\kappa(s) = \frac{\|\alpha'(s) \times \alpha''(s)\|}{\|\alpha'(s)\|^3}, \quad \tau(s) = \frac{\langle \alpha'(s) \times \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \times \alpha''(s)\|^2}, \tag{2}$$

$$\vec{T}'(s) = \nu\kappa(s)\vec{N}(s),\ \vec{N}'(s) = -\nu\kappa(s)\vec{T}(s) + \nu\tau(s)\vec{B}(s),\ \vec{B}'(s) = -\nu\tau(s)\vec{N}(s), \tag{3}$$

where $\nu = \|\alpha'(s)\|$ and, $\vec{T}$, $\vec{N}$, $\vec{B}$, κ and τ are called the tangent vector, the principal normal vector, the binormal vector, the curvature and the torsion of the curve, respectively.

2. Tangent Associated Curves

In this section we will define tangent associated curves such that the tangent vector of a given curve lies on the osculating, normal and rectifying plane of its mate. Let $\alpha(s) : I \subset \Re \to \Re^3$ be a unit speed curve and denote α^* as its associated mate. Assuming that $\{T^*, N^*, B^*\}$ is the Frenet frame of α^* we write the unit vectors lying on osculating, normal and rectifying plane of α^* as following:

$$O^* = \frac{aT^* + bN^*}{\sqrt{a^2 + b^2}}, \tag{4}$$

$$P^* = \frac{cN^* + dB^*}{\sqrt{c^2 + d^2}}, \tag{5}$$

$$R^* = \frac{eT^* + fB^*}{\sqrt{e^2 + f^2}}, \tag{6}$$

respectively, where $a, b, c, d, e, f \in \Re^+$ are some arbitrary positive real numbers. Note that, the representation of the arc length assumed parameter " s " of the main curve α was omitted throughout the paper for simplicity, unless otherwise stated.

Definition 1. *Let $\alpha(s) : I \subset \Re \to \Re^3$ be a unit speed curve and α^* be any regular curve. If the tangent vector, T of α is linearly dependent with the vector, O^*, then we name the curve α^* as $T - O^*$ associated curve of α.*

The following figure (Fig. 1) is given to illustrate the main idea for this and the next definitions.

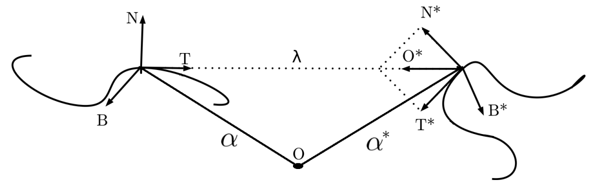


FIGURE 1. The curve α (left) and its $T-O^*$ associated mate α^* (right)

Theorem 1. *If α^* is $T-O^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α,α^*) pair is given by the following,*

$$\begin{aligned} T^* &= \frac{a}{\sqrt{a^2+b^2}}T+\frac{b}{\sqrt{a^2+b^2}}N \\ N^* &= \frac{b}{\sqrt{a^2+b^2}}T-\frac{a}{\sqrt{a^2+b^2}}N \\ B^* &= -B. \end{aligned}$$

Proof. Since α and α^* are defined as $T-O^*$ associated curves, we may write

$$\alpha^*(s)=\alpha(s)+\lambda(s)T(s). \tag{7}$$

By differentiating the relation (7), taking its norm and using the Frenet formulae given in (3), we have:

$$T^*=\frac{(1+\lambda')T+\lambda\kappa N}{\sqrt{(1+\lambda')^2+(\lambda\kappa)^2}}. \tag{8}$$

Now taking the second derivative of the equation (7) and referring again to (3) we write

$$\alpha^{*\prime\prime}=(\lambda''-\lambda\kappa^2)T+\left((1+\lambda')\kappa+(\lambda\kappa)'\right)N+\lambda\kappa\tau B.$$

The cross production of $\alpha^{*\prime}$ and $\alpha^{*\prime\prime}$ leads us the following form,

$$\alpha^{*\prime}\times\alpha^{*\prime\prime}=\left(\lambda^2\kappa^2\tau\right)T-\left((\lambda'+1)\lambda\kappa\tau\right)N+\left((\lambda'+1)\left(\left(\lambda'+1\right)\kappa+(\lambda\kappa)'\right)-\lambda\kappa(\lambda''-\lambda\kappa^2)\right)B. \tag{9}$$

By calling upon (1), we simply calculate N^* and B^* as

$$N^*=-\frac{\lambda\kappa\left(\left(\lambda'+1\right)\left(\left(\lambda'+1\right)\kappa+\lambda'\kappa+\lambda\kappa'\right)-\lambda\kappa\left(\lambda''-\lambda\kappa^2\right)\right)T}{\parallel\alpha^{*\prime}\parallel\parallel\alpha^{*\prime}\times\ \alpha^{*\prime\prime}\parallel}$$
$$+\frac{\left(\lambda'+1\right)\Bigg(\left(\lambda'+1\right)\left(\left(\lambda'+1\right)\kappa+\lambda'\kappa+\lambda\kappa'\right)-\lambda\kappa\left(\lambda''-\lambda\kappa^2\right)\Bigg)N}{\parallel\alpha^{*\prime}\parallel\parallel\alpha^{*\prime}\times\ \alpha^{*\prime\prime}\parallel}$$

$$+\frac{\lambda\kappa\tau\left(\lambda^2\kappa^2+\left(\lambda'+1\right)^2\right)B}{\parallel \alpha^{*\prime} \parallel \parallel \alpha^{*\prime}\times \alpha^{*\prime\prime} \parallel},$$

$$B^* = \frac{\lambda^2\kappa^2\tau T-\lambda\kappa\tau(\lambda'+1)N+\left((\lambda'+1)\left(\left(\lambda'+1\right)\kappa+(\lambda\kappa)'\right)-\lambda\kappa\left(\lambda''-\lambda\kappa^2\right)\right)B}{\parallel \alpha^{*\prime}\times \alpha^{*\prime\prime} \parallel}. \tag{10}$$

Note that we will refer these relations from (8) to (10) in the next two theorems. We call these as the raw relations.
Now, as we defined the curve α^* to be the $T-O^*$ associated curve of α, we deduce that $< T, T^* >=< O^*, T^* >$. By using this deduction and referring both the relation (4) and (8) we write

$$\frac{(1+\lambda')}{\sqrt{(1+\lambda')^2+(\lambda\kappa)^2}}=\frac{a}{\sqrt{a^2+b^2}}.$$

Simple elementary operations on this relation result the following linear ordinary differential equation (ODE) with $b\neq 0$ as

$$1+\lambda'=\frac{a}{b}\lambda\kappa. \tag{11}$$

When substituted the given ODE, (11) into (8) we complete the first part of the proof for T^*.
Similarly, another deductions can be drawn as

$$< T, N^* >=< O^*, N^* >, \text{ and } < T, B^* >=< O^*, B^* >= 0,$$

and using these we write

$$-\frac{\lambda\kappa\left[\left(\lambda'+1\right)\left(\left(\lambda'+1\right)\kappa+\lambda'\kappa+\lambda\kappa'\right)-\lambda\kappa\left(\lambda''-\lambda\kappa^2\right)\right]}{\parallel \alpha^{*\prime} \parallel \parallel \alpha^{*\prime}\times \alpha^{*\prime\prime} \parallel}=\frac{b}{\sqrt{a^2+b^2}}, \tag{12}$$

$$\lambda^2\kappa^2\tau=0, \tag{13}$$

respectively. Now when substituted the relations (11), (12) and (13) into both (8) and (10) we complete the proof. □

Corollary 1. *From (11) and (13) $\kappa,\lambda\neq 0$ that results $\tau=0$. Therefore it can be easily said that the curve α is a planar curve or equivalently there is no a space curve having a T associated partner such that its tangent lies on the osculating plane of its mate.*

Theorem 2. *If α^* is the $T-O^*$ associated curve of α then the curvature, κ^* and the torsion, τ^* of α^* are given as follows.*

$$\kappa^* = \frac{b}{\lambda\sqrt{a^2+b^2}},$$
$$\tau^* = 0.$$

Proof. By using the equations in (2) and the relation (11) with the fact that $\tau = 0$ the proof is completed. □

Theorem 3. *If α^* is the $T-O^*$ associated curve of α, then the distance between the corresponding points of α and α^* in E^3 is given as follows:*

$$d(\alpha,\alpha^*) = \left| e^{\int \frac{a}{b}\kappa}\left[-\int e^{-\int \frac{a}{b}\kappa} + c_1 \right] \right|, \tag{14}$$

where c_1 is an integral constant.

Proof. We rewrite (11) as

$$\lambda' - \frac{a}{b}\kappa\lambda = -1. \tag{15}$$

By taking μ as an integrating factor and multiplying the both hand sides of the latter equation by that we get

$$\mu\lambda' - \mu\frac{a}{b}\kappa\lambda = -\mu. \tag{16}$$

From the product rule of the composite form we write

$$(\mu\lambda)' = \mu\lambda' + \mu'\lambda \tag{17}$$

and equate the terms of (17) with those in the left hand side of the (16) we find

$$\mu' = -\mu\frac{a}{b}\kappa.$$

The solution for the integrating factor μ is given with

$$\int \frac{\mu'}{\mu} = -\int \frac{a}{b}\kappa \quad \Rightarrow \quad \mu = e^{-\int \frac{a}{b}\kappa + c}$$

On the other hand, the use of integrating factor let us to write following relation

$$[\mu\lambda]' = -\mu.$$

Integrating both hand sides of this equation

$$\mu\lambda + c_o = -\int \mu$$

and leaving λ all alone we get

$$\lambda = \frac{-\int \mu - c_o}{\mu}.$$

By substituting μ in place, we finally get

$$\lambda = e^{\int \frac{a}{b}\kappa}\Big[-\int e^{-\int \frac{a}{b}\kappa} + c_1\Big].$$

□

Definition 2. *Let $\alpha(s) : I \subset \Re \to \Re^3$ be a unit speed curve and α^* be any regular curve. If the tangent vector, T of α is linearly dependent with the vector, P^*, then we name the curve α^* as $T - P^*$ associated curve of α.*

Theorem 4. *If α^* is $T - P^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α, α^*) pair is given by the following,*

$$\begin{aligned} T^* &= N, \\ N^* &= \frac{-c}{\sqrt{c^2+d^2}}T + \frac{d}{\sqrt{c^2+d^2}}B, \\ B^* &= \frac{d}{\sqrt{c^2+d^2}}T + \frac{c}{\sqrt{c^2+d^2}}B. \end{aligned}$$

Proof. Since we defined the curve α^* to be as $T-P^*$ associated curve of α we could deduce that $< T, N^* > = < P^*, N^* >$. Using this, together with the relations (5) and (10) results the following:

$$-\frac{\lambda\kappa\left[\left(\lambda'+1\right)\left(\left(\lambda'+1\right)\kappa + \lambda'\kappa + \lambda\kappa'\right) - \lambda\kappa\left(\lambda'' - \lambda\kappa^2\right)\right]}{\parallel \alpha^{*\prime} \parallel \parallel \alpha^{*\prime} \times \ \alpha^{*\prime\prime} \parallel} = \frac{c}{\sqrt{c^2+d^2}}. \tag{18}$$

By the same manner, it can be derived that $< T, B^* > = < P^*, B^* >$ which results

$$\frac{\lambda^2\kappa^2\tau}{\parallel \alpha^{*\prime} \times \ \alpha^{*\prime\prime} \parallel} = \frac{d}{\sqrt{c^2+d^2}}. \tag{19}$$

Another deduction that $< T, T^* > = < P^*, T^* > = 0$ provides $1 + \lambda' = 0$ and so

$$\lambda = -s + c \tag{20}$$

where c is the integral constant. Utilizing these three relations, (18), (19) and (20) results what is stated in the theorem.
Note that, by substituting (20) first in both (18) and (19), we find the following relations

$$\frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{-c}{\sqrt{c^2+d^2}} \text{ and } \frac{\tau}{\sqrt{\kappa^2+\tau^2}} = \frac{d}{\sqrt{c^2+d^2}}, \tag{21}$$

respectively which points out that $c = -\kappa$ and $d = \tau$ and since by definition $\kappa \geq 0$, $c \leq 0$. □

Corollary 2. *It can be easily seen that if α^* is $T - P^*$ associated curve of α, then α^* is the involute of α.*

Theorem 5. *If α^* is the $T-P^*$ associated curve of α, then the curvature, κ^* and the torsion, τ^* of α^* are given as follows,*

$$\begin{aligned}\kappa^* &= \frac{\tau\sqrt{c^2+d^2}}{d\lambda\kappa} = \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa},\\ \tau^* &= \frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}.\end{aligned} \tag{22}$$

Proof. The proof can be easily done by using (20) and (21). □

Theorem 6. *If α^* is the $T-P^*$ associated curve of α, then the distance between the corresponding points of α and α^* in E^3 is given as follows:*

$$d(\alpha^*,\alpha) = \big|-s+c\big|. \tag{23}$$

Proof. The proof is trivial. □

Definition 3. *Let $\alpha(s): I\subset\Re\to\Re^3$ be a unit speed curve and α^* be any regular curve. If the tangent vector, T of α is linearly dependent with the vector, R^*, then we name the curve α^* as $T-R^*$ associated curve of α.*

Theorem 7. *If α^* is $T-R^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α,α^*) pair is given by the following,*

$$\begin{aligned}T^* &= \frac{e}{\sqrt{e^2+f^2}}T + \frac{f}{\sqrt{e^2+f^2}}N,\\ N^* &= B,\\ B^* &= \frac{f}{\sqrt{e^2+f^2}}T - \frac{e}{\sqrt{e^2+f^2}}N.\end{aligned}$$

Proof. Since we defined the curve α^* to be as $T-R^*$ associated curve of α we could deduce that $<T,T^*>=<R^*,T^*>$. By using this deduction and referring both the relation (6) and (8) we write

$$\frac{(1+\lambda')}{\sqrt{(1+\lambda')^2+(\lambda\kappa)^2}} = \frac{e}{\sqrt{e^2+f^2}},$$

and with some simple elementary operations on this relation we come up with the following linear ordinary differential equation (ODE), with $f\neq 0$.

$$1+\lambda' = \frac{e}{f}\lambda\kappa. \tag{24}$$

When substituted the given ODE into (8) we complete the first part of the proof for T^*.
Similarly, another deduction can be drawn as $<T,B^*>=<R^*,B^*>$ which results

$$\frac{\lambda^2\kappa^2\tau}{\|\alpha^{*\prime}\times\alpha^{*\prime\prime}\|} = \frac{f}{\sqrt{e^2+f^2}}, \text{ and so } \|\alpha^{*\prime}\times\alpha^{*\prime\prime}\| = \lambda^2\kappa^2\tau\frac{\sqrt{e^2+f^2}}{f}. \tag{25}$$

Now when substituted the relations (24) and (25) into (10) we complete the proof for B^*.
A final inference on the idea of $T-R^*$ association can be drawn as $<T,N^*>=<R^*,N^*>=0$. This puts the following equation forward

$$-\lambda\kappa\left[\left(\lambda'+1\right)\left(\left(\lambda'+1\right)\kappa+\lambda'\kappa+\lambda\kappa'\right)-\lambda\kappa\left(\lambda''-\lambda\kappa^2\right)\right]=0. \tag{26}$$

By substituting (24), (25) and (26) in (10) the proof is completed for N^* and all. ☐

Theorem 8. *If α^* is the $T-R^*$ associated curve of α, then the curvature, κ^* and the torsion τ^* of α^* are given as follows.*

$$\kappa^* = \frac{\tau f^2}{\lambda\kappa(e^2+f^2)},$$
$$\tau^* = \frac{f\left(\kappa^2\tau e^3+\kappa^2\tau e f^2+\tau^3 e f^2+\kappa\tau' e^2 f+\kappa\tau' f^3-\kappa'\tau e^2 f-\kappa'\tau f^3\right)}{\lambda\kappa\left(\tau^2 e^2 f^2+\tau^2 f^4+\kappa^2 e^4+2\,\kappa^2 e^2 f^2+\kappa^2 f^4\right)}.$$

Proof. By the equations given in (2) and substituting (24) and (25) into these, we may easily derive κ^*. On the other hand the third derivative of (7) is

$$\begin{aligned}\alpha^{*'''} &= \left(-3\,\lambda'\kappa^2-3\,\lambda\kappa\kappa'-\kappa^2+\lambda'''\right)T+\left(-\kappa^3\lambda-\lambda\kappa\tau^2+3\,\lambda''\kappa+3\,\lambda'\kappa'+\lambda\kappa''+\kappa'\right)N\\ &\quad+\left(3\,\lambda'\kappa\tau+\lambda\kappa\tau'+2\,\lambda\kappa'\tau+\kappa\tau\right)B.\end{aligned}$$

From (2) and using (24), τ^* can be computed as in the given above form. ☐

Theorem 9. *If α^* is the $T-R^*$ associated curve of α, then the distance between the corresponding points of α and α^* in E^3 is given as follows:*

$$d(\alpha^*,\alpha)=\left|e^{\int\frac{e}{f}\kappa}\left[-\int e^{-\int\frac{e}{f}\kappa}+c_2\right]\right| \tag{27}$$

Proof. The proof is the same as the proof of Theorem (3). ☐

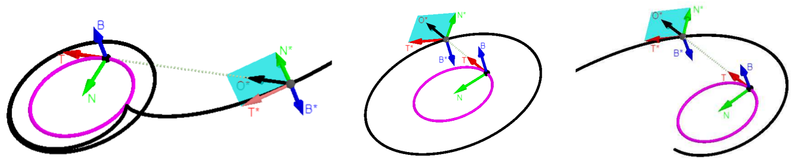
3. Examples

In this section, we provide an example for the tangent associated curves by considering each of the three different cases.

(1) Let α be chosen a unit speed circle as a planar curve given with a parameterization
$\alpha(s)=(cos(s),sin(s),0)$. Since α is chosen to be a circle $\kappa=1$. By taking $a=b=1$, the general solution for the given ODE in (11) is

$$\lambda(s)=1+e^s c_0$$

where c_0 is the integral constant.



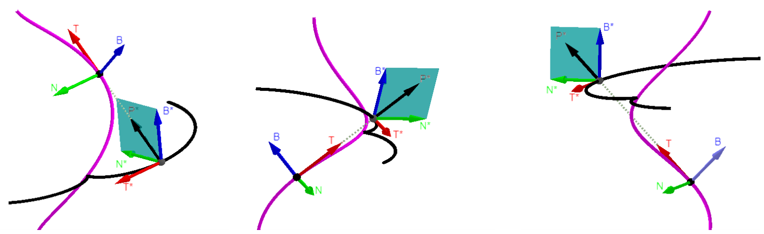
(a) $c_0 = -1$ (b) $c_0 = 0$ (c) $c_0 = 1$

FIGURE 2. The main curve α (pink) and its $T - O^*$ associated mate α^* (black)

(2) Let α be chosen a unit speed helix given with a parameterization $\alpha(s) = \frac{1}{\sqrt{2}}(cos(s), sin(s), s)$. Since $\kappa = \tau = \frac{1}{\sqrt{2}}$, the vector P^* should be formed by the values of c and d such that $-c = d = \frac{1}{\sqrt{2}}$. From theorem (6) we have

$$\lambda(s) = -s + c_0$$

where c_0 is the integral constant.



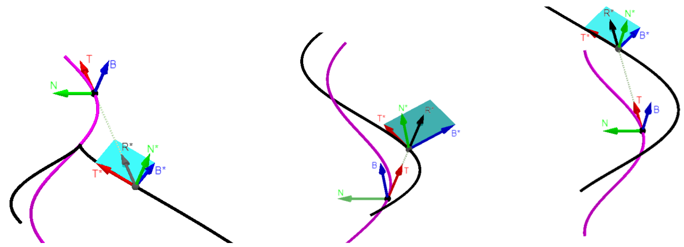
(a) $c_0 = -1$ (b) $c_0 = 0$ (c) $c_0 = 1$

FIGURE 3. The main curve α (pink) and its $T - P^*$ associated mate α^* (black)

(3) By referring the same curve given in (ii) we know that $\kappa = \frac{1}{\sqrt{2}}$. The general solution for the ODE in (11) for $e = f = 1$ is this time

$$\lambda(s) = \sqrt{2} + e^{\frac{\sqrt{2}}{2}s} c_0$$

where c_0 is the integral constant.
One of the animated versions for the figures can be found at the link below and for all figures see the author's profile.
https://www.geogebra.org/m/vnbzaghp



(a) $c_0 = -1$ (b) $c_0 = 0$ (c) $c_0 = 1$

FIGURE 4. The main curve α (pink) and its $T - R^*$ associated mate α^* (black)

4. Principal Normal Associated Curves

In this section, we define principal normal associated curves such that the principal normal vector of a given curve lies on the osculating, normal and rectifying plane of its mate.

Definition 4. *Let $\alpha(s) : I \subset \Re \to \Re^3$ be a unit speed curve and α^* be any regular curve. If the principal normal, N of α is linearly dependent with the vector, O^*, then we name the curve α^* as $N - O^*$ associated curve of α.*

Theorem 10. *If α^* is $N - O^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α, α^*) pair is given by the following,*

$$T^* = \frac{1}{\sqrt{a^2+b^2}}\left(\frac{(-\lambda\kappa+1)\,b}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}T + aN + \frac{\lambda\tau b}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}B\right),$$
$$N^* = \frac{b}{\sqrt{a^2+b^2}}\left(\frac{-\mathbf{M}}{\mathbf{M}\,(\lambda\kappa-1)-\mathbf{K}\lambda\tau}T + N + \frac{\mathbf{K}}{\mathbf{M}\,(\lambda\kappa-1)-\mathbf{K}\lambda\tau}B\right),$$
$$B^* = \frac{b(\mathbf{K}T+\mathbf{M}B)}{a(\mathbf{M}\,(\lambda\kappa-1)-\mathbf{K}\lambda\tau)},$$

where the coefficients $\mathbf{K}$ and $\mathbf{M}$ are

$$\mathbf{K} = \lambda'\left(\lambda\tau'+2\,\lambda'\tau\right) - \lambda\tau\left((-\lambda\kappa+1)\,\kappa - \lambda\tau^2+\lambda''\right),$$
$$\mathbf{M} = (-\lambda\kappa+1)\left((-\lambda\kappa+1)\,\kappa-\lambda\tau^2+\lambda''\right) - \lambda'\left(-\lambda\kappa'-2\,\lambda'\kappa\right).$$

Proof. Since α and α^* are defined as $N - O^*$ associated curves, we write

$$\alpha^* = \alpha + \lambda N. \tag{28}$$

By differentiating the relation (28), using the Frenet formulae given in (3) and taking the norm, we have:

$$T^* = \frac{(-\lambda\kappa+1)T + \lambda' N + \lambda\tau B}{\sqrt{(-\lambda\kappa+1)^2+(\lambda')^2+(\lambda\tau)^2}}. \tag{29}$$

Next taking the second derivative of the equation (28) and referring again to (3) result the following relation.

$$\alpha^{*\prime\prime} = \left(-\lambda\kappa' - 2\,\lambda'\kappa\right)T + \left(\left(-\lambda\kappa + 1\right)\kappa - \lambda\tau^2 + \lambda''\right)N + \left(\lambda\tau' + 2\,\lambda'\tau\right)B.$$

The cross production of $\alpha^{*\prime}$ and $\alpha^{*\prime\prime}$ leads us the following form,

$$\alpha^{*\prime} \times \ \alpha^{*\prime\prime} = \mathbf{K}T + \mathbf{L}N + \mathbf{M}B,$$

where $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ are assigned to be as

$$\begin{aligned}
\mathbf{K} &= \lambda'\left(\lambda\tau' + 2\,\lambda'\tau\right) - \lambda\tau\left(\left(-\lambda\kappa + 1\right)\kappa - \lambda\tau^2 + \lambda''\right),\\
\mathbf{L} &= \left(-\lambda\kappa + 1\right)\left(\lambda\tau' + 2\,\lambda'\tau\right) + \lambda\tau\left(-\lambda\kappa' - 2\,\lambda'\kappa\right),\\
\mathbf{M} &= \left(-\lambda\kappa + 1\right)\left(\left(-\lambda\kappa + 1\right)\kappa - \lambda\tau^2 + \lambda''\right) - \lambda'\left(-\lambda\kappa' - 2\,\lambda'\kappa\right),
\end{aligned} \tag{30}$$

for the sake of simplicity. Note that the norm, $\parallel \alpha^{*\prime} \times \ \alpha^{*\prime\prime} \parallel = \sqrt{\mathbf{K}^2 + \mathbf{L}^2 + \mathbf{M}^2}$. By referring again the definitions given by (1), we simply calculate N^*, and B^* as

$$N^* = \frac{(\mathbf{L}\lambda\tau - \mathbf{M}\lambda')T + (\mathbf{M}\left(\lambda\kappa - 1\right) - \mathbf{K}\lambda\tau)N + (\mathbf{K}\lambda' - \mathbf{L}\left(-\lambda\kappa + 1\right))B}{\sqrt{(-\lambda\kappa + 1)^2 + (\lambda')^2 + (\lambda\tau)^2}\sqrt{\mathbf{K}^2 + \mathbf{L}^2 + \mathbf{M}^2}}, \tag{31}$$

$$B^* = \frac{\mathbf{K}T + \mathbf{L}N + \mathbf{M}B}{\sqrt{\mathbf{K}^2 + \mathbf{L}^2 + \mathbf{M}^2}}.$$

The intuitive idea is as same as before. Since we defined α^* to be as the $N - O^*$ associated curve of α we can write that $< N, T^* > = < O^*, T^* >$. By using this together with the relations (4) and (29) we write

$$\frac{\lambda'}{\sqrt{(-\lambda\kappa + 1)^2 + (\lambda')^2 + (\lambda\tau)^2}} = \frac{a}{\sqrt{a^2 + b^2}}. \tag{32}$$

Similarly, we can write $< N, N^* > = < O^*, N^* >$ which results the following

$$\frac{\mathbf{M}\left(\lambda\kappa - 1\right) - \mathbf{K}\lambda\tau}{\sqrt{(-\lambda\kappa + 1)^2 + (\lambda')^2 + (\lambda\tau)^2}\sqrt{\mathbf{K}^2 + \mathbf{L}^2 + \mathbf{M}^2}} = \frac{b}{\sqrt{a^2 + b^2}}. \tag{33}$$

and by the same idea that $< N, B^* > = < O^*, B^* > = 0$, we get

$$\frac{\mathbf{L}}{\sqrt{\mathbf{K}^2 + \mathbf{L}^2 + \mathbf{M}^2}} = 0. \tag{34}$$

When substituted the given three relations (32), (33) and (34) into (29) and (31), we complete the proof. □

Note that none of the differential equations given above is solvable analytically. However we might solve them under some assumptions.

Corollary 3. *If the curve α is chosen to be a curve with constant curvatures like helix, then by referring the relation (32), we can derive*

$$\lambda'\left(\lambda'' - \frac{a^2}{b^2}\left((\lambda\kappa - 1)\kappa + \lambda\tau^2\right)\right) = 0,$$

which is solvable analytically in two folds. First, $\lambda' = 0$ *corresponding to that* λ *is a constant. If this is the case, then from the relation (32),* $a = 0$*, and if* $a = 0$ *then* $O^* = N^*$*. This is clearly the definition of the Bertrand curve, since* α^* *becomes* $N - N^*$ *associated curve of* α*.*
When considered the second factor of the latter relation we come up with a non homogeneous linear second order differential equation with constant coefficients. For this case, there we have a complex solution that is as

$$\lambda = \sin\left(\frac{ai\sqrt{\kappa^2+\tau^2}}{b}\right)c_1 + \cos\left(\frac{ai\sqrt{\kappa^2+\tau^2}}{b}\right)c_2 + \frac{\kappa}{\kappa^2+\tau^2}, \qquad i^2 = -1,$$

and since $sin(ix) = isinh(x)$ *and* $cos(ix) = cosh(x)$*, we can rewrite the solution as:*

$$\lambda = i\sinh\left(\frac{a\sqrt{\kappa^2+\tau^2}}{b}\right)c_1 + \cosh\left(\frac{a\sqrt{\kappa^2+\tau^2}}{b}\right)c_2 + \frac{\kappa}{\kappa^2+\tau^2},$$

where c_1 *and* c_2 *are integration constants.*
Now, by recalling the relations (30) and (34) under the assumption that α *is a helix like curve with constant curvatures, then we have*

$$\lambda'\tau(-2\lambda\kappa + 1) = 0.$$

This results that λ *is a constant of the form,* $\lambda = \dfrac{1}{2\kappa}$*.*

Theorem 11. *If* α^* *is the* $N - O^*$ *associated curve of* α*, then the curvature,* κ^* *and the torsion,* τ^* *of* α^* *are given as follows,*

$$\kappa^* = \frac{a^4(\mathbf{M}\,(\lambda\kappa - 1) - \mathbf{K}\lambda\tau)}{b(a^2+b^2)(\lambda')^4},$$

$$\tau^* = \frac{b\lambda'}{a(\mathbf{M}\,(\lambda\kappa - 1) - \mathbf{K}\lambda\tau)}\begin{pmatrix} \mathbf{K}\left(\lambda\kappa^3 + \lambda\kappa\tau^2 - 3\lambda'\kappa' - \lambda\kappa'' - 3\lambda''\kappa - \kappa^2\right) \\ +\mathbf{M}\left(\kappa\tau - \lambda\kappa^2\tau - \lambda\tau^3 + 3\lambda'\tau' + \lambda\tau'' + 3\lambda''\tau\right)\end{pmatrix}.$$

Proof. By taking the third derivative of (28) and using Frenet formulae, we have

$$\begin{aligned} \alpha^{*'''} &= \left(\lambda\kappa^3 + \lambda\kappa\tau^2 - 3\lambda'\kappa' - \lambda\kappa'' - 3\lambda''\kappa - \kappa^2\right)T \\ &\quad + \left(\lambda''' - 3\lambda'(\kappa^2+\tau^2) - 3\lambda(\kappa\kappa' + \tau\tau') + \kappa'\right)N \\ &\quad + \left(\kappa\tau - \lambda\kappa^2\tau - \lambda\tau^3 + 3\lambda'\tau' + \lambda\tau'' + 3\lambda''\tau\right)B. \end{aligned} \tag{35}$$

Now by recalling the relations (32), (33) and (34) to substitute these into the equations given in (2), we complete the proof. □

Definition 5. *Let* $\alpha(s) : I \subset \Re \to \Re^3$ *be a unit speed curve and* α^* *be any regular curve. If the principal normal,* N *of* α *is linearly dependent with the vector,* P^**, then we name the curve* α^* *as* $N - P^*$ *associated curve of* α*.*

Theorem 12. *If α^* is $N-P^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α,α^*) pair is given by the following,*

$$T^* = \frac{-\lambda\kappa+1}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}T + \frac{\lambda\tau}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}B,$$

$$N^* = \frac{1}{\sqrt{c^2+d^2}}\left(\frac{d\lambda\tau}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}T + cN + \frac{-d(-\lambda\kappa+1)}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}B\right),$$

$$B^* = \frac{d}{\sqrt{c^2+d^2}}\left(-\frac{\tau\left(\lambda\tau^2+\kappa^2\lambda-\kappa\right)}{\lambda\tau\kappa'-\kappa\tau'\lambda+\tau'}T + N + \frac{(\lambda\kappa-1)\left(\lambda\tau^2+(\kappa)^2\lambda-\kappa\right)}{\lambda\left(-\lambda\tau\kappa'+\kappa\tau'\lambda-\tau'\right)}B\right).$$

Proof. Now, since again we defined α^* to be as the $N-P^*$ associated curve of α we can write three of our associative relations as usual which are

- $< N, N^* > = < P^*, N^* >$,
- $< N, B^* > = < P^*, B^* >$,
- $< N, T^* > = < P^*, T^* > = 0$.

These relations this time result the following three equations

$$\bullet \quad \frac{\mathbf{M}(\lambda\kappa-1)-\mathbf{K}\lambda\tau}{\sqrt{(-\lambda\kappa+1)^2+(\lambda')^2+(\lambda\tau)^2}\sqrt{\mathbf{K}^2+\mathbf{L}^2+\mathbf{M}^2}} = \frac{c}{\sqrt{c^2+d^2}},$$

$$\bullet \quad \frac{\mathbf{L}}{\sqrt{\mathbf{K}^2+\mathbf{L}^2+\mathbf{M}^2}} = \frac{d}{\sqrt{c^2+d^2}}, \tag{36}$$

$$\bullet \quad \frac{\lambda'}{\sqrt{(-\lambda\kappa+1)^2+(\lambda')^2+(\lambda\tau)^2}} = 0.$$

When substituted the latter relations in (29) and (31) we complete the proof. □

Corollary 4. *Note that the third relation in (36) results that λ is constant. What we know from literature is that a Bertrand curve has a constant distance as well as the Mannheim curves (see [1], [4] and [2]). For Bertrand curves we also know that curves share the principal normal vectors as common, on the other hand for Mannheim curves, they share the property of the parallelization of principal normal and binormal vectors. By our result, we see that if the principal normal vector of any given curve coincides the unit vector spanned by principal normal and binormal vectors of its mate, then the distance of two curves is constant, in general.*

Theorem 13. *If α^* is the $N-P^*$ associated curve of α, then the curvature, κ^* and the torsion, τ^* of α^* are given as follows.*

$$\kappa^* = \frac{\mathbf{l}\,c^3\sqrt{c^2+d^2}}{d^4\left(\mathbf{m}(\lambda\kappa-1)-\mathbf{k}\lambda\tau\right)^3},$$

$$\tau^* = \frac{\mathbf{l}^2(c^2+d^2)}{d^2}\begin{pmatrix} \mathbf{k}\left(\kappa^3\lambda+\kappa\tau^2\lambda-\lambda\kappa''-\kappa^2\right)+\mathbf{l}\left(-3\lambda\tau\tau'-3\lambda\kappa'\kappa+\kappa'\right) \\ +\mathbf{m}\left(-\kappa^2\tau\lambda-\tau^3\lambda+\lambda\tau''+\kappa\tau\right)\end{pmatrix}$$

where $\mathbf{k}$, $\mathbf{l}$, *and* $\mathbf{m}$ *are the coefficients of which* $\mathbf{K}$, $\mathbf{L}$, *and* $\mathbf{M}$ *reformed with* $\lambda' = 0$, *respectively.*

Proof. By referring the relations (36) together with (35), the proof is trivial. □

Definition 6. *Let* $\alpha(s) : I \subset \Re \to \Re^3$ *be a unit speed curve and* α^* *be any regular curve. If the principal normal,* N *of* α *is linearly dependent with the vector,* R^*, *then we name the curve* α^* *as* $N - R^*$ *associated curve of* α.

Theorem 14. *If* α^* *is* $N - R^*$ *associated curve of* α, *then the relationship of the corresponding Frenet frames of* (α, α^*) *pair is given by the following,*

$$T^* = \frac{1}{\sqrt{e^2+f^2}}\left(\frac{(-\lambda\kappa+1)\,f}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}T + eN + \frac{\lambda\tau f}{\sqrt{(-\lambda\kappa+1)^2+(\lambda\tau)^2}}B\right),$$

$$\begin{aligned} N^* = \frac{ef}{\sqrt{e^2+f^2}}\Bigg(&\left(\frac{\lambda\tau}{\lambda'} - \frac{(-\lambda\kappa+1)\left((-\lambda\kappa+1)\,\kappa - \lambda\tau^2 + \lambda''\right) - \lambda'\left(-\lambda\kappa' - 2\,\lambda'\kappa\right)}{(\lambda\kappa-1)\left(\lambda\tau' + 2\,\lambda'\tau\right) + \lambda\tau\left(-\lambda\kappa' - 2\,\lambda'\kappa\right)}\right)T \\ &+ \left(\frac{\lambda'\left(\lambda\tau' + 2\,\lambda'\tau\right) - \lambda\tau\left((-\lambda\kappa+1)\,\kappa - \lambda\tau^2 + \lambda''\right)}{(\lambda\kappa-1)\left(\lambda\tau' + 2\,\lambda'\tau\right) + \lambda\tau\left(-\lambda\kappa' - 2\,\lambda'\kappa\right)} + \frac{\lambda\kappa - 1}{\lambda'}\right)B\Bigg), \end{aligned}$$

$$\begin{aligned} B^* = \frac{f}{\sqrt{e^2+f^2}}\Bigg(&\left(\frac{\lambda'\left(\lambda\tau' + 2\,\lambda'\tau\right) - \lambda\tau\left((-\lambda\kappa+1)\,\kappa - \lambda\tau^2 + \lambda''\right)}{(\lambda\kappa-1)\left(\lambda\tau' + 2\,\lambda'\tau\right) + \lambda\tau\left(-\lambda\kappa' - 2\,\lambda'\kappa\right)}\right)T + N \\ &+ \left(\frac{(-\lambda\kappa+1)\left((-\lambda\kappa+1)\,\kappa - \lambda\tau^2 + \lambda''\right) - \lambda'\left(-\lambda\kappa' - 2\,\lambda'\kappa\right)}{(\lambda\kappa-1)\left(\lambda\tau' + 2\,\lambda'\tau\right) + \lambda\tau\left(-\lambda\kappa' - 2\,\lambda'\kappa\right)}\right)B\Bigg) \end{aligned}$$

Proof. Now, since again we defined α^* to be as the $N - R^*$ associated curve of α we can write three of our associative relations as usual which are

- $< N, T^* > = < R^*, T^* >$,
- $< N, B^* > = < R^*, B^* >$,
- $< N, N^* > = < R^*, N^* > = 0$.

By using these we get

$$\begin{aligned} &\bullet \quad \frac{\lambda'}{\sqrt{(-\lambda\kappa+1)^2+(\lambda')^2+(\lambda\tau)^2}} = \frac{e}{\sqrt{e^2+f^2}}, \\ &\bullet \quad \frac{\mathbf{L}}{\sqrt{\mathbf{K}^2+\mathbf{L}^2+\mathbf{M}^2}} = \frac{f}{\sqrt{e^2+f^2}}, \\ &\bullet \quad \frac{\mathbf{M}\,(\lambda\kappa-1) - \mathbf{K}\lambda\tau}{\sqrt{(-\lambda\kappa+1)^2+(\lambda')^2+(\lambda\tau)^2}\sqrt{\mathbf{K}^2+\mathbf{L}^2+\mathbf{M}^2}} = 0. \end{aligned} \tag{37}$$

When substituted the above expressions into (29) and (31) the proof is complete. □

Corollary 5. *The only analytically solvable equation in (37) is the first one with the same assumption that* α *is helix like curve with constant curvatures. The possible solutions to that has already been discussed in Corollary (3).*

Theorem 15. *If α^* is the $N - R^*$ associated curve of α, then the curvature, κ^* and the torsion, τ^* of α^* are given as follows.*

$$\kappa^* = \frac{e^3\mathbf{L}}{f(e^2+f^2)(\lambda')^3},$$
$$\tau^* = \frac{\mathbf{L}^2(e^2+f^2)}{f^2}\begin{pmatrix} \mathbf{K}\left(\lambda\kappa^3+\lambda\kappa\tau^2-\lambda\kappa''-\kappa^2-3\,\lambda''\kappa-3\,\lambda'\kappa'\right) \\ +\mathbf{L}\left(-3\,\lambda\kappa\kappa'-3\,\lambda\tau\tau'-3\,\kappa^2\lambda'-3\,\lambda'\tau^2+\lambda'''+\kappa'\right) \\ +\mathbf{M}\left(-\lambda\kappa^2\tau-\lambda\tau^3+\lambda\tau''+\kappa\tau+3\,\lambda''\tau+3\,\lambda'\tau'\right)\end{pmatrix}$$

Proof. By recalling both the third derivative (35) and the relations (37) to substitute into curvatures in (2), we complete the proof. □

5. Binormal Associated Curves

In this section, we define binormal associated curves such that the binormal vector of a given curve lies on the osculating, normal and rectifying plane of its mate.

Definition 7. *Let $\alpha(s): I \subset \Re \to \Re^3$ be a unit speed curve and α^* be any regular curve. If the binormal, B of α is linearly dependent with the vector, O^*, then we name the curve α^* as $B - O^*$ associated curve of α.*

Theorem 16. *If α^* is $B - O^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α, α^*) pair is given by the following,*

$$T^* = \frac{1}{\sqrt{a^2+b^2}}\left(\frac{b}{\sqrt{1+\lambda^2\tau^2}}T - \frac{\lambda\tau b}{\sqrt{1+\lambda^2\tau^2}}N + aB\right),$$
$$N^* = \frac{b}{\sqrt{a^2+b^2}}\left(-\frac{\lambda'\mathbf{Y}}{a(\lambda\tau\mathbf{X}+\mathbf{Y})}T + \frac{\lambda\mathbf{X}}{\lambda\tau\mathbf{X}+\mathbf{Y}}N + B\right),$$
$$B^* = -\frac{b\lambda'}{a(\lambda\tau\mathbf{X}+\mathbf{Y})}\left(\mathbf{X}T+\mathbf{Y}N\right),$$

where the coefficients $\mathbf{X}$ and $\mathbf{Y}$ are

$$\mathbf{X} = -\lambda\tau\left(-\lambda\tau^2+\lambda''\right) - \lambda'\left(-\lambda\tau'-2\,\lambda'\tau+\kappa\right),$$
$$\mathbf{Y} = \lambda\tau^2 - \lambda'' + \lambda'\lambda\tau\kappa.$$

Proof. Since α and α^* are defined as $B - O^*$ associated curves, we write

$$\alpha^* = \alpha + \lambda B. \tag{38}$$

By differentiating the relation (38), using the Frenet formulae given in (3) and taking the norm, we have:

$$T^* = \frac{T - \lambda\tau N + \lambda' B}{\sqrt{1+\lambda^2\tau^2+(\lambda')^2}} \tag{39}$$

Next taking the second derivative of the equation (38) and referring again to (3) result the following relation.

$$\alpha^{*\prime\prime} = (\lambda\tau\kappa)T + (-\lambda\tau' - 2\,\lambda'\tau + \kappa)N + (-\lambda\tau^2 + \lambda'')B.$$

The cross production of $\alpha^{*\prime}$ and $\alpha^{*\prime\prime}$ leads us the following form,

$$\alpha^{*\prime} \times \ \alpha^{*\prime\prime} = \mathbf{X}T + \mathbf{Y}N + \mathbf{Z}B$$

where $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are assigned to be as

$$\begin{aligned} \mathbf{X} &= -\lambda\tau\left(-\lambda\tau^2 + \lambda''\right) - \lambda'\left(-\lambda\tau' - 2\,\lambda'\tau + \kappa\right), \\ \mathbf{Y} &= \lambda\tau^2 - \lambda'' + \lambda'\lambda\tau\kappa, \\ \mathbf{Z} &= -\lambda\tau' - 2\,\lambda'\tau + \kappa + \lambda^2\tau^2\kappa, \end{aligned} \tag{40}$$

for the sake of simplicity. Note that the norm, $\| \alpha^{*\prime} \times \ \alpha^{*\prime\prime} \| = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}$. By referring again the definitions given by (1), we simply calculate N^* and B^* as

$$\begin{aligned} N^* &= \frac{(\mathbf{Y}\lambda' + \mathbf{Z}\lambda\tau)T + (-\mathbf{X}\lambda + \mathbf{Z})N + (-\mathbf{X}\lambda\tau - \mathbf{Y})B}{\sqrt{1 + \lambda^2\tau^2 + (\lambda')^2}\sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}}, \\ B^* &= \frac{\mathbf{X}T + \mathbf{Y}N + \mathbf{Z}B}{\sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}}. \end{aligned} \tag{41}$$

The intuitive idea is as same as before. Since we defined α^* to be as the $B - O^*$ associated curve of α we can write that

- $< B, T^* > = < O^*, T^* >$,
- $< B, N^* > = < O^*, N^* >$,
- $< B, B^* > = < O^*, B^* > = 0$.

By using these together with the relations (4) and (39) we write

$$\begin{aligned} &\bullet \quad \frac{\lambda'}{\sqrt{1 + \lambda^2\tau^2 + (\lambda')^2}} = \frac{a}{\sqrt{a^2 + b^2}}, \\ &\bullet \quad \frac{-\mathbf{X}\lambda\tau - \mathbf{Y}}{\sqrt{1 + \lambda^2\tau^2 + (\lambda')^2}\sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}} = \frac{b}{\sqrt{a^2 + b^2}}, \\ &\bullet \quad \frac{\mathbf{Z}}{\sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}} = 0. \end{aligned} \tag{42}$$

Substituting these relations into (39) and (41), we complete the proof. □

Corollary 6. *If τ is taken to be constant, then from the first relation given in (42) we can derive the following:*

$$\lambda'(\lambda'' - \lambda\frac{a^2}{b^2}\tau^2) = 0.$$

This relation holds either $\lambda' = 0$, correspondingly that λ is constant or

$$\lambda = c_1 e^{\frac{a\tau}{b}} + c_2 e^{-\frac{a\tau}{b}},$$

as a result of the solution of second order differential equation, where c_1 and c_2 are the integration constants. If λ is taken to be constant then by the first relation of (42) $a = 0$, resulting that $O^* = N^*$. We remind that this is the definition of Mannheim curves.
On the other hand, when considered the third equation in (42) and recall (40), we have the following

$$\mathbf{Z} = -\lambda\tau' - 2\,\lambda'\tau + \kappa + \lambda^2\tau^2\kappa = 0.$$

Rearranging this equation by dividing each term with (-2τ) results

$$\lambda' + \lambda^2\left(\frac{-\tau\kappa}{2}\right) + \lambda\left(\frac{\tau'}{2\tau}\right) - \frac{\kappa}{2\tau} = 0, \tag{43}$$

which is clearly a Riccati type of differential equation. If $\lambda = \lambda_1$ is a particular solution for (43) then we have a general solution by substituting $\lambda = \lambda_1 + \frac{1}{\mu}$, that converts the Riccati equation into the following first order linear differential equation:

$$\mu' - \left(2\lambda_1\left(\frac{-\tau\kappa}{2}\right) + \left(\frac{\tau'}{2\tau}\right)\right)\mu = \left(\frac{-\tau\kappa}{2}\right) \tag{44}$$

where μ is an arbitrary function of the parameter, s. The solution for this (44) can be done by following the steps given in the proof of Theorem (3).

Theorem 17. *If α^* is the $B - O^*$ associated curve of α, then the curvature, κ^* and the torsion, τ^* of α^* are given as follows:*

$$\kappa^* = -\frac{a^4(\mathbf{X}\lambda\tau + \mathbf{Y})}{(\lambda')^2 b(a^2+b^2)\sqrt{a^2+b^2}},$$

$$\tau^* = \frac{b^2(\lambda')^2}{a^2(\mathbf{X}\lambda\tau + \mathbf{Y})^2}\Bigg(\mathbf{X}\left(\lambda\tau\kappa' + 3\,\lambda'\tau\kappa + 2\,\lambda\tau'\kappa - \kappa^2\right) + \mathbf{Y}\left(\lambda\tau^3 + \lambda\tau\kappa^2 - \lambda\tau'' - 3\,\lambda'\tau' - 3\,\lambda''\tau + \kappa'\right)\Bigg).$$

Proof. By taking the third derivative of (38) and using Frenet formulas, we have

$$\alpha^{*\prime\prime\prime} = (\lambda\tau\kappa' + 2\,\lambda\tau'\kappa + 3\,\lambda'\tau\kappa - \kappa^2)T + (\lambda\tau\kappa^2 + \lambda\tau^3 - \lambda\tau'' - 3\,\lambda''\tau - 3\,\lambda'\tau' + \kappa')N + (\lambda''' - 3\,\lambda\tau\tau' - 3\,\lambda'\tau^2 + \kappa\tau)B. \tag{45}$$

Now, using the relations given in (42) together with (45), to substitute into the definitions (2) lets us to complete the proof. □

Definition 8. *Let $\alpha(s) : I \subset \Re \to \Re^3$ be a unit speed curve and α^* be any regular curve. If the binormal, B of α is linearly dependent with the vector, P^*, then we name the curve α^* as $B - P^*$ associated curve of α.*

Theorem 18. *If α^* is $B-P^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α,α^*) pair is given by the following,*

$$T^* = \frac{1}{\sqrt{1+\lambda^2\tau^2}}T - \frac{\lambda\,\tau}{\sqrt{1+\lambda^2\tau^2}}N,$$
$$N^* = \frac{1}{\sqrt{c^2+d^2}}\left(\frac{d\lambda\tau}{\sqrt{1+\lambda^2\tau^2}}T - \frac{d(\lambda^3\tau^3-(-\lambda\tau'+\kappa+\lambda^2\tau^2))}{(-\lambda\tau'+\kappa+\lambda^2\tau^2)\sqrt{1+\lambda^2\tau^2}}N + cB\right),$$
$$B^* = \frac{1}{\sqrt{c^2+d^2}}\left(-\frac{c\lambda\tau}{\sqrt{1+\lambda^2\tau^2}}T - \frac{c}{\sqrt{1+\lambda^2\tau^2}}N + dB\right).$$

Proof. Now, since again we defined α^* to be as the $B-P^*$ associated curve of α we can write three of our associative relations as usual which are

- $< B, N^* > = < P^*, N^* >$,
- $< B, B^* > = < P^*, B^* >$,
- $< B, T^* > = < P^*, T^* > = 0$.

These relations this time result the following three equations

$$\bullet \quad \frac{-\mathbf{X}\lambda\tau - \mathbf{Y}}{\sqrt{1+\lambda^2\tau^2+(\lambda')^2}\sqrt{\mathbf{X}^2+\mathbf{Y}^2+\mathbf{Z}^2}} = \frac{c}{\sqrt{c^2+d^2}},$$
$$\bullet \quad \frac{\mathbf{Z}}{\sqrt{\mathbf{X}^2+\mathbf{Y}^2+\mathbf{Z}^2}} = \frac{d}{\sqrt{c^2+d^2}}, \tag{46}$$
$$\bullet \quad \frac{\lambda'}{\sqrt{1+\lambda^2\tau^2+(\lambda')^2}} = 0.$$

When substituted these relations, (46) in (39) and (41), we complete the proof. □

Corollary 7. *When taken into account the third relation of (46) we conclude that if the binormal vector of a given curve is linearly dependent with the unit vector lying on the normal plane of its mate, then the distance between these curves is constant.*

Theorem 19. *If α^* is the $B-P^*$ associated curve of α, then the curvature, κ^* and the torsion, τ^* of α^* are given as follows.*

$$\kappa^* = -\frac{d^2\sqrt{c^2+d^2}\left(\lambda\tau^2(1+\lambda^2\tau^2)\right)^3}{c^3(-\lambda\tau'+\kappa+\lambda^2\tau^2)^2},$$
$$\tau* = \frac{\begin{array}{c} d^2(\lambda^2\tau^3\left(\lambda\,\tau\kappa'+2\,\tau'\kappa\lambda-\kappa^2\right)+\lambda\tau^2\left(\lambda\,\tau\kappa^2+\lambda\,\tau^3-\tau''\lambda+\kappa'\right) \\ +(-\lambda\tau'+\kappa+\lambda^2\tau^2)\left(-3\,\tau'\tau\lambda+\kappa\tau\right)) \end{array}}{(c^2+d^2)(-\lambda\tau'+\kappa+\lambda^2\tau^2)^2}$$

Proof. By substituting the relations given in (37) and the third derivative (45) into the definitions given in (2), we complete the proof. □

Definition 9. *Let $\alpha(s) : I \subset \Re \to \Re^3$ be a unit speed curve and α^* be any regular curve. If the binormal, B of α is linearly dependent with the vector, R^*, then we name the curve α^* as $B - R^*$ associated curve of α.*

Theorem 20. *If α^* is $B - R^*$ associated curve of α, then the relationship of the corresponding Frenet frames of (α, α^*) pair is given by the following,*

$$
\begin{aligned}
T^* &= \frac{1}{\sqrt{e^2+f^2}}\left(\frac{f}{\sqrt{1+\lambda^2\tau^2}}T - \frac{\lambda\tau f}{\sqrt{1+\lambda^2\tau^2}}N + eB\right),\\
N^* &= \frac{ef}{\sqrt{e^2+f^2}}\left(\frac{\mathbf{Y}\lambda' + \mathbf{Z}\lambda\tau}{\mathbf{Z}\lambda'}T + \frac{-\mathbf{X}\lambda + \mathbf{Z}}{\mathbf{Z}\lambda'}\right),\\
B^* &= \frac{f}{\sqrt{e^2+f^2}}\left(\frac{\mathbf{X}}{\mathbf{Z}}T + \frac{\mathbf{Y}}{\mathbf{Z}}N + B\right).
\end{aligned}
$$

Proof. Now, since again we defined α^* to be as the $B - R^*$ associated curve of α we can write three of our associative relations as usual which are

- $< B, T^* >=< R^*, T^* >$,
- $< B, B^* >=< R^*, B^* >$,
- $< B, N^* >=< R^*, N^* >= 0$.

These relations this time result the following three equations

$$
\begin{aligned}
&\bullet \quad \frac{\lambda'}{\sqrt{1+\lambda^2\tau^2+(\lambda')^2}} = \frac{e}{\sqrt{e^2+f^2}},\\
&\bullet \quad \frac{\mathbf{Z}}{\sqrt{\mathbf{X}^2+\mathbf{Y}^2+\mathbf{Z}^2}} = \frac{f}{\sqrt{e^2+f^2}},\\
&\bullet \quad \frac{-\mathbf{X}\lambda\tau - \mathbf{Y}}{\sqrt{1+\lambda^2\tau^2+(\lambda')^2}\sqrt{\mathbf{X}^2+\mathbf{Y}^2+\mathbf{Z}^2}} = 0.
\end{aligned} \tag{47}
$$

For the last time when substituted (47) into (39) and (41), the proof is complete. □

Corollary 8. *The only analytically solvable equation is the first one of (47) with the same assumption given in Corollary (6). The possible solutions can be get by following the same steps as well.*

Theorem 21. *If α^* is the $B - R^*$ associated curve of α, then the curvature, κ^* and the torsion, τ^* of α^* are given as follows.*

$$
\begin{aligned}
\kappa^* &= \frac{\mathbf{Z}e^3}{f(e^2+f^2)(\lambda')^3},\\
\tau^* &= \frac{f^2}{\mathbf{Z}^2(e^2+f^2)}\left(\begin{array}{c}\mathbf{X}\left(\lambda\tau\kappa' + 2\,\lambda\tau'\kappa + 3\,\lambda'\tau\kappa - \kappa^2\right)\\ +\mathbf{Y}\left(\lambda\tau\kappa^2 + \lambda\tau^3 - \lambda\tau'' - 3\,\lambda''\tau - 3\,\lambda'\tau' + \kappa'\right)\\ +\mathbf{Z}\left(-3\,\lambda\tau\tau' - 3\,\left(\lambda'\right)\tau^2 + \kappa\tau + \lambda'''\right)\end{array}\right).
\end{aligned}
$$

Proof. Recall the relations (45) and (47) and substitute these in (2), the proof is complete. □

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References

[1] Bertrand, J., Mémoire sur la théorie des courbes à double courbure, *Journal de Mathématiques Pures et Appliquées* 15 (1850), 332–350.

[2] Mannheim, A., De l'emploi de la courbe représentative de la surface des normales principales d'une courbe gauche pour la démonstration de propriétés relatives à cette courbure, *C.R. Comptes Rendus des Séances de l'Académie des Sciences*, 86 (1878), 1254-1256.

[3] O'Neill, B., *Elementary differential geometry*, Academic Press Inc., New York, 1966.

[4] Liu, H., Wang, F., Mannheim partner curves in 3-space, *Journal of Geometry* 88(1-2) (2008), 120-126. https://doi.org/10.1007/s00022-007-1949-0

[5] Menninger, T., Characterization of the slant helix as successor curve of the general helix, *International Electronic Journal of Geometry*, 7(2) (2014), 84-91. https://doi.org/10.36890/iejg.593986

[6] Kazaz, M., Uğurlu, H. H., Önder, M., Oral, S., Bertrand partner D- curves in the Euclidean 3-space E^3, *Afyon Kocatepe University Journal of Science and Engineering*, 16(1) (2016), 76-83. https://doi.org/10.5578/fmbd.25270

[7] Kaya, O., Önder, M., New partner curves in the Euclidean 3-space, *International Journal of Geometry* 6(2) (2017), 41-50.

[8] Kaya, O., Önder, M., C-partner curves and their applications, *Differential Geometry-Dynamical Systems* 19 (2017), 64-74.

[9] Körpınar, T., Sarı aydın, M. T., Turhan, E., Associated curves according to Bishop frame in Euclidean 3 space, *Advanced Modeling and Optimization*, 15(3) (2013), 713-717.

[10] Masal, M., Azak, A. Z., Mannheim B-curves in the Euclidean 3-space E^3, *Kuwait Journal of Science*, 44(1) (2017), 36-41.

[11] Yılmaz, B., Has, A., Alternative partner curves in the Euclidean 3-space, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 69(1) (2020), 900-909. https://doi.org/10.31801/cfsuasmas.538177

[12] Choi, J. H., Kim, Y. H., Associated curves of a Frenet curve and their applications, *Applied Mathematics and Computation*, 218(18) (2012), 9116-9124. https://doi.org/10.1016/j.amc.2012.02.064

[13] Şahiner, B., Direction curves of principal normal indicatrix of a curve, *Journal of Technical Sciences*, 8(2) (2018), 46-54.

[14] Şahiner, B., Direction curves of tangent indicatrix of a curve, *Applied Mathematics and Computation*, 343 (2019), 273-284. https://doi.org/10.1016/j.amc.2018.09.021

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ON THE SOLUTIONS OF THE q-ANALOGUE OF THE TELEGRAPH DIFFERENTIAL EQUATION

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Abstract. In this work, q-analogue of the telegraph differential equation is investigated. The approximation solution of q-analogue of the telegraph differential equation is founded by using the Laplace transform collocation method (LTCM). Then, the exact solution is compared with the approximation solution for q-analogue of the telegraph differential equation. The results showed that the method is useful and effective for q-analogue of the telegraph differential equation.

1. Introduction

Quantum calculus (q-calculus) was initiated at the beginning of the 18th century by Euler [1]. The q-calculus is often called calculus without limits. It allows the substitution of the classical derivative with the q-derivative operator to deal with sets of non-differentiable functions. The q-calculus has an unexpected role in several mathematical areas such as fractal geometry, quantum theory, hypergeometric functions, orthogonal polynomials, the calculus of variation and theory of relativity. The works [2], [3] can be cited for some results related to the history of quantum calculus, its basic concepts and q-differential equations. In [4], [5], a q-analogue of Sturm-Liouville problems are investigated.

Partial differential equations are ubiquitous in mathematically-oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, general relativity, and quantum mechanics. In [6], an expansion theorem was proved for the analytic function in several variables which satisfies a system of q-partial differential equations by using the theory of functions of several variables and q-calculus. In [7], using the theory of functions of

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several complex variables, it was proved that if an analytic function in several variables satisfies a system of q-partial differential equations then, it can be expanded in terms of the product of the Rogers-Szegő polynomials. In [8], identities and evaluate integrals by expanding functions in terms of products of the q-hypergeometric polynomials was proved by homogeneous q-partial difference equations.

In [9], with the use of Laplace transform technique, a new form of trial function from the original equation is obtained. The unknown coefficients in the trial functions are determined using collocation method. In [10], using the Laplace transform collocation method (LTCM) and Daftar-Gejii-Jafaris method (DGJM), the fractional order time-varying linear dynamical system was investigated.

In this paper, the following the telegraph differential equation defined by q-difference operator which we call the q-analogue of the telegraph differential equation is studied

$$\begin{cases} D^2_{q,\eta}\varphi(\eta,\xi) + D_{q,\eta}\varphi(\eta,\xi) + \varphi(\eta,\xi) = D^2_{q,\xi}\varphi(\eta,\xi) + f(\eta,\xi), \\ 0 < \eta < L \quad 0 < \xi < L \quad 0 < q \le 1, \\ \varphi(0,\xi) = h(\xi), \quad D_{q,\eta}\varphi(0,\xi) = g(\xi) \\ \varphi(\eta,0) = \varphi(\eta,L) = 0, \end{cases} \tag{1}$$

where h, g and f are known continuous functions and the function φ is unknown function. $D_{q,\eta}\varphi(\eta,\xi) = \frac{\partial_q\varphi(\eta,\xi)}{\partial_q\eta}$, $D_{q,\xi}\varphi(\eta,\xi) = \frac{\partial_q\varphi(\eta,\xi)}{\partial_q\xi}$ are q-difference of $\varphi(\eta,\xi)$ respect to η and ξ, respectively. If $\alpha = 1$, and $q = 1$ then the equation (1) is called telegraph partial differential equation.

LTCM method is used for numerical solution of the problem (1). Using the Laplace transform method, the exact solution of the problem (1) and a new form of trial function from the basic equation are obtained.

2. Preliminaries

We first recall some basic definition in q-calculus.

Let parameter q be a positive real number and n a non-negative integer. $[n]_q$ denotes a q integer, defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Let $q > 0$ be given. We define a q-factorial, $[n]_q!$ of $k \in \mathbb{N}$, as

$$[n]_q! = \begin{cases} [1]_q[2]_q \dots [n]_q, & n = 1, 2, \dots \\ 1, & n = 0. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[n-r]_q![r]_q!}.$$

The q-shifted factorials (q-Pochhammer symbol) are defined for $a \in \mathbb{C}$ by

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{j=0}^{\infty}(1-aq^j).$$

The q-exponential function is given by

$$E_q(-z) = ((1-q)z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{[n]_q!} z^n.$$

For $t, x, y \in \mathbb{R}$ and $n \in \mathbb{Z} \geq 0$, the q-binomial formula is given by

$$(x+y)_q^n = \prod_{j=0}^{n-1}(x+q^j y) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j} q^{\frac{n(n-1)}{2}} y^j.$$

Let q be a positive number with $0 < q < 1$. Let f be a real or complex valued function on A (A is q-geometric set (see [4])). The q-difference operator D_q (the Jackson q-derivative) is defined as

$$D_q f(x) =: \frac{\partial_q f(x)}{\partial_q x} = \frac{f(x)-f(qx)}{x(1-q)}, \quad x \neq 0.$$

Let f and g are defined on a q-geometric set A such that the q-derivatives of f and g exist for all $x \in A$. Then, there is a non-symmetric formula for the q-differentiation of a product

$$D_q[f(x)g(x)] = f(qx)D_q g(x) + g(x)D_q f(x).$$

The q-integral usually associated with the name of Jackson is defined in the interval $(0,x)$, as

$$\int_0^x f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(xq^n)xq^n,$$

$$\int_0^x D_q f(t)d_qt = f(x) - f(0).$$

The q-integration for a function f over $[0,\infty)$ is defined as the following by Hahn (see [11])

$$\int_0^\infty f(t)d_qt = \sum_{n=-\infty}^{\infty}(1-q)q^n f(q^n).$$

The q-analogue of the Laplace transformed is defined by

$$F_q(s) = \pounds_q(f(t)) = \int_0^{\infty} E_q(-qst)f(t)d_qt \quad (s > 0). \tag{2}$$

From (2), we obtain

$$\pounds_q(\alpha f(t) + \beta g(t)) = \alpha \pounds_q(f(t)) + \beta \pounds_q(g(t)),$$

where α, β are constants. The q-analogue of the Gamma function is defined as the following in [13]

$$\Gamma_q(t) = \int_0^{\frac{1}{(1-q)}} x^{t-1} E_q(-qx)d_qx, \quad (t > 0) \tag{3}$$

From (2) and (3), we get

$$\pounds_q(1) = \frac{1}{s} \quad (s > 0), \quad \pounds_q(t) = \frac{1}{s^2} \quad (s > 0), ..., \pounds_q(t^n) = \frac{\Gamma_q(n+1)}{s^{n+1}} = \frac{[n]_q!}{s^{n+1}}.$$

3. LTCM for q-Analogue of the Telegraph Differential Equation

We shall obtain numerical solution of q-analogue of the telegraph differential equation using the method LTCM. Taking the Laplace transform of the problem (1), we get

$$D_{q,\eta}\varphi(0,\xi) - s\varphi(0,\xi) + s^2\varphi_q(s,\xi)$$

$$= -\pounds_q\{D_{q,\eta}\varphi(\eta,\xi)\} - \pounds_q\{\varphi(\eta,\xi)\} + \pounds_q\{D^2_{q,\xi}\varphi(\eta,\xi)\} + \pounds_q\{f(\eta,\xi)\} \tag{4}$$

After simple algebraic simplification and using initial condition of the problem (1), we have

$$\begin{aligned}\varphi_q(s,\xi) = \frac{1}{s^2}\,[D_{q,\eta}\varphi(0,\xi) + s\varphi(0,\xi) - \pounds_q\{D_{q,\eta}\varphi(\eta,\xi)\} - \pounds_q\{\varphi(\eta,\xi)\} \\ + \pounds_q\{D^2_{q,\xi}\varphi(\eta,\xi)\} + \pounds_q\{f(\eta,\xi)\}]\end{aligned} \tag{5}$$

The function $\varphi_q(\eta,\xi)$ and its derivative function in the equation (5) are replaced with a trial function of the form

$$\varphi_q = \varphi_q^0 + \sum_{i=1}^{n} c_i\varphi_q^i, \tag{6}$$

then we will obtain the following equation

$$\begin{aligned}\varphi_q(s,\xi)=\frac{1}{s^2}\Bigg[&D_{q,\eta}\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(0,\xi)\right)+s\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(0,\xi)\right)\\&-\pounds_q\left\{D_{q,\eta}\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(\eta,\xi)\right)\right\}-\pounds_q\left\{\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(\eta,\xi)\right\}\\&+\pounds_q\left\{D_{q,\xi}^2\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(\eta,\xi)\right)\right\}+\pounds_q\{f(\eta,\xi)\}\Bigg],\end{aligned}\tag{7}$$

where c_i are constants to be stated which satisfy the given conditions in the problem (1). Taking the inverse q-Laplace transform of the equation (7), we obtain

$$\begin{aligned}\varphi_q^{new}(\eta,\xi)=\pounds_q^{-1}\Bigg[\frac{1}{s^2}\Bigg[&D_{q,\eta}\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(0,\xi)\right)+s\left(\varphi_q^0(0,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(0,\xi)\right)\\&-\pounds_q\left\{D_{q,\eta}\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(\eta,\xi)\right)\right\}-\pounds_q\left\{\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(\eta,\xi)\right\}\\&+\pounds_q\left\{D_{q,\xi}^2\left(\varphi_q^0(\eta,\xi)+\sum_{i=1}^{n}c_i\varphi_q^i(\eta,\xi)\right)\right\}+\pounds_q\{f(\eta,\xi)\}\Bigg]\Bigg].\end{aligned}\tag{8}$$

Substituting the equality (8) into the problem (1), we get new collocating at points $\xi=\xi_k$ as following

$$D_{q,\eta}^2\varphi_q^{new}(\eta,\xi_k)+\varphi_q^{new}(\eta,\xi_k)+D_{q,\eta}\varphi_q^{new}(\eta,\xi_k)-D_{q,\xi}^2\varphi_q^{new}(\eta,\xi_k)=f(\eta,\xi_k)\tag{9}$$

where $\xi_k=\frac{L-0}{n+1}$, $k=1,2,\ldots,n$.

Now, we shall define the residual function by the following formula

$$R_n(\eta,\xi)=L[\varphi_q^{new}(\eta,\xi)]-f(\eta,\xi).\tag{10}$$

Here $\varphi_q^{new}(\eta,\xi)$ demonstrates the approximate solution, $\varphi(\eta,\xi)$ demonstrates the exact solution and

$$L[\varphi_q^{new}(\eta,\xi)]=D_{q,\eta}^2\varphi_q^{new}(\eta,\xi)+D_{q,\eta}\varphi_q^{new}(\eta,\xi)+\varphi_q^{new}(\eta,\xi)-D_{q,\xi}^2\varphi_q^{new}(\eta,\xi).\tag{11}$$

From the equality (11), we write

$$D_{q,\eta}^2\varphi_q^{new}(\eta,\xi)+D_{q,\eta}\varphi_q^{new}(\eta,\xi)+\varphi_q^{new}(\eta,\xi)-D_{q,\xi}^2\varphi_q^{new}(\eta,\xi)=f(\eta,\xi)+R_n(\eta,\xi),\tag{12}$$

Now since L is a linear operator, we obtain for the error function
$e_n=\varphi_q^{new}(\eta,\xi)-\varphi(\eta,\xi)$

$$D_{q,\eta}^2e_n(\eta,\xi)+D_{q,\eta}e_n(\eta,\xi)+e_n(\eta,\xi)-D_{q,\xi}^2e_n(\eta,\xi)=R_n(\eta,\xi).\tag{13}$$

From the conditions in the problem (1), we get

$$e_n(0,\xi) = D_{q,\eta}e_n(0,\xi) = D^2_{q,\eta}e_n(0,\xi) = 0, \tag{14}$$

$$e_n(\eta,0) = e_n(\eta,L) = 0. \tag{15}$$

By solving (13) subject to the homogeneous conditions (14) and (15), we obtain the error function $e_n(\eta,\xi)$. This allows us to calculate $\varphi(\eta,\xi)=u_n(\eta,\xi) + e_n(\eta,\xi)$ even for problems without known exact solutions.

4. Numerical Applications

In this section, we shall present one test example for implementation of the LTCM. In the following example, the numerical solution calculated by the this method will be compared with the exact solution.

Example 1. *Consider the following initial-boundary value problem for q-analogue of the telegraph differential equation*

$$\begin{cases} D^2_{q,\eta}\varphi(\eta,\xi) + D_{q,\eta}\varphi(\eta,\xi) + \varphi(\eta,\xi) = D^2_{q,\xi}\varphi(\eta,\xi) + [3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 \\ \qquad\qquad + \eta^3\xi^3 - [3]_q!\xi\eta^3 \\ 0<\eta<L \quad 0<\xi<L \quad 0<q\leq 1, \\ \varphi(0,\xi) = h(\xi), \quad D_{q,\eta}\varphi(0,\xi) = g(\xi) \\ \qquad \varphi(\eta,0) = 0 \quad \varphi(\eta,1) = \eta^3 \end{cases} \tag{16}$$

First, we shall calculate the example problem (16) by LTCM.

We assume that the trial function is the following form:

$$\varphi(\eta,\xi) = c_1\xi^2(\xi-1)\eta^3 + c_2\xi(\xi-1)^2\eta^3. \tag{17}$$

Taking the Laplace transform of the equation (16) and using the formula (7), we obtain

$$\begin{aligned} & -D_{q,\eta}\varphi_q(0,\xi) - s\varphi_q(0,\xi) + s^2\varphi_q(s,\xi) \\ = \; & -\pounds_q\left\{D_{q,\eta}\varphi(\eta,\xi)\right\} - \pounds_q\left\{\varphi(\eta,\xi)\right\} + \pounds\left\{D^2_{q,\xi}\varphi(\eta,\xi)\right\} \\ & +\pounds\left\{[3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 + \eta^3\xi^3 - [3]_q!\xi\eta^3\right\}. \end{aligned} \tag{18}$$

Using the initial condition of the problem (16), the formula (18) is obtained as:

$$\begin{aligned} \varphi_q(s,\xi) = \; & \frac{1}{s^2}\left[-\pounds_q\{D_{q,\eta}\varphi(\eta,\xi)\} - \pounds_q\{\varphi(\eta,\xi)\}\right. \\ & \left. +\pounds_q\{D^2_{q,\xi}\varphi(\eta,\xi)\} + \pounds_q\{[3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 + \eta^3\xi^3 - [3]_q!\xi\eta^3\}\right] \end{aligned} \tag{19}$$

Using the formulas (17) and (19), we obtain

$$\begin{aligned} \varphi_q(s,\xi) \; = \; & \frac{1}{s^2}\pounds_q\left\{\left(-[3]_q\xi^2(\xi-1)\eta^2 - \xi^2(\xi-1)\eta^3 + [3]_q!\xi\eta^3 - [2]_q!\eta^3\right)c_1\right. \\ & + \left(-[3]_q\xi(\xi-1)^2\eta^2 - \xi(\xi-1)^2\eta^3 + ([3]_q!\xi - [4]_q\eta^3\right)c_2 \\ & \left. +\pounds_q\left\{[3]_q!\eta\xi^3 + [3]_q\eta^2\xi^3 + \eta^3\xi^3 - [3]_q!\xi\eta^3\right\}\right\}. \end{aligned} \tag{20}$$

From the formula (20), we can get

$$\begin{aligned}\varphi_q(s,\xi) &= \left(-[3]_q!\xi^2(\xi-1)\frac{1}{s^5}-[3]_q!\xi^2(\xi-1)\frac{1}{s^6}+[3]_q!([3]_q!\xi-[2]_q!)\frac{1}{s^6}\right)c_1\\ &+\left(-[3]_q!\xi(\xi-1)^2\frac{1}{s^5}-[3]_q!\xi(\xi-1)^2\frac{1}{s^6}+[3]_q!([3]_q!\xi-[4]_q)\frac{1}{s^6}\right)c_2\\ &+\left(\frac{[3]_q!}{s^4}+\frac{[3]_q!}{s^5}+\frac{[3]_q!}{s^6}\right)\xi^3-\frac{[3]_q!^2}{s^6}\xi \qquad (21)\end{aligned}$$

Taking the inverse Laplace transform of (21), we get the following new trial solution:

$$\begin{aligned}\varphi_q^{new}(\eta,\xi) &= \left[\left(-\frac{\eta^4}{[4]_q}-\frac{\eta^5}{[4]_q[5]_q}\right)(c_1+c_2)+\eta^3+\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}\right]\xi^3\\ &+\left[\left(\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}\right)(c_1+2c_2)\right]\xi^2\\ &+\left[\frac{[3]_q!}{[4]_q[5]_q}\eta^5c_1-\left(\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}-\frac{[3]_q!}{[4]_q[5]_q}\eta^5\right)c_2-\frac{[3]_q!}{[4]_q[5]_q}\eta^5\right]\xi\\ &-\frac{\eta^5}{[5]_q}(c_1+c_2) \qquad (22)\end{aligned}$$

Substituting (22) into (16), we have the following residual formula:

$$\begin{aligned}R(\eta,\xi,c_1,c_2) &= D^2_{q,\eta}\varphi_q^{new}(\eta,\xi)+D_{q,\eta}\varphi_q^{new}(\eta,\xi)+\varphi_q^{new}(\eta,\xi)-D^2_{q,\xi}\varphi_q^{new}(\eta,\xi)\\ &-\left([3]_q!\eta+[3]_q\eta^2+\eta^3\right)\xi^3+[3]_q!\eta^3\xi \qquad (23)\end{aligned}$$

Taking the derivatives of the equation (22) as to ξ and η, and writing in the formula (23), we obtain

$$\begin{aligned}R(\eta,\xi,c_1,c_2) &= (A\xi^3-A\xi^2+D\xi-B-C)c_1\\ &+ (A\xi^3-2A\xi^2+(A+D)\xi-B-2C)c_2-A-D\\ &= 0, \qquad (24)\end{aligned}$$

where,

$$A=-\frac{\eta^5}{[4]_q[5]_q}-2\frac{\eta^4}{[4]_q}-2\eta^3-[3]_q\eta^2,$$

$$B=[4]_q\eta^3+\eta^4+\frac{\eta^5}{[5]_q},$$

$$C=[2]_q\left(\frac{\eta^4}{[4]_q}+\frac{\eta^5}{[4]_q[5]_q}\right),$$

$$D=\frac{2[3]_q!}{[4]_q[5]_q}\eta^5+\frac{2[3]_q!}{[4]_q}\eta^4+[3]_q!\eta^3.$$

From (24), we have

$$c_1 = \frac{A}{A\xi^3 - A\xi^2 + D\xi - B - C}$$

$$c_2 = \frac{D}{A\xi^3 - 2A\xi^2 + (A+D)\xi - B - 2C}.$$

Errors calculate by the following formula

$$Error = |exact \quad solution - approximate \quad solution|,$$

$$\epsilon = max|\varphi_{exact} - \varphi_{app}|,$$

where $\varphi_{exact} = \eta^3\xi^3$ is exact solution and $\varphi_{app} = c_1\xi^2(\xi-1)\eta^3 + c_2\xi(\xi-1)^2\eta^3$ is numerical solution that is obtained by using LTCM for the problem (16). As shows

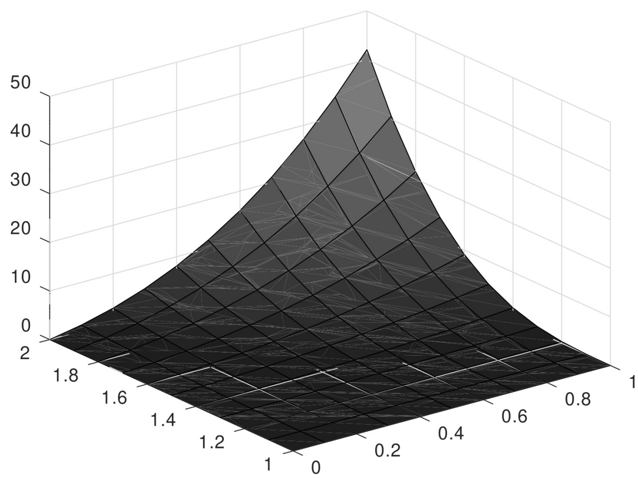

FIGURE 1. Gives the approximation solution of the example (16) for $1 \le \xi \le 2$, $0 \le \eta \le 1$ and $q = 0.01$.

from the figure of Figure 1 the difference better exact solution and approximation solutions is not clearly obvious. Therefore we present the numerical regents and error analysis in the following Table 1.

5. CONCLUSION

In this work, we adopted a combination of Laplace transform collocation method to develop numerical methods for the q-difference operator for the telegraph differential equation. Numerical example was considered to demonstrate the accuracy and efficiency of this method. The exact solution is compared with the approximate solution. Obtained results are given in the numerical error analysis Table 1 The simulations are showed for the exact and approximation solution.

$\xi = \eta$	α	*Exact Solution*	*LTCM method*	*Error Analysis*
0.99	0.01	0.941480149401000	0.769146969442938	0.172333179958063
0.5	0.01	0.015625000000000	0.032637283173177	0.017012283173177
0.5	0.5	0.015625000000000	0.037117402318906	0.021492402318906
0.5	0.99	0.015625000000000	0.027823738101408	0.012198738101408
0.1	0.01	1.0001×10^{-6}	2.3185×10^{-5}	2.2185×10^{-5}
0.1	0.5	1.0001×10^{-6}	1.6396×10^{-5}	1.5396×10^{-5}
0.1	0.99	1.0001×10^{-6}	1.1374×10^{-5}	1.0374×10^{-5}
0.01	0.01	1.000×10^{-12}	2.8934×10^{-10}	2.883410×10^{-10}
0.01	0.5	1.000×10^{-12}	1.7618×10^{-10}	1.7518×10^{-10}
0.01	0.99	1.000×10^{-12}	1.1629×10^{-10}	1.1529×10^{-10}

Table 1. Table error analysis of Example 1.

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References

[1] Euler, L., Introduction in Analysin Infinitorum, vol. 1, Lausanne, Switzerland, Bousquet, 1748.

[2] Ernst, T., The History of q-Calculus and a New Method, U.U.D.M. Report 2000, 16, Uppsala, Department of Mathematics, Uppsala University, 2000.

[3] Kac, V., Cheung, P., Quantum Calculus, Universitext, Springer, New York, 2002.

[4] Annaby, M.H., Mansour, Z.S., q-Fractional Calculus and Equations, Lecture Notes in Mathematics, vol. 2056, Springer, Heidelberg, 2012. https://doi.org/10.1007/978-3-642- 30898-7

[5] Karahan, D., Mamedov, Kh.R., Sampling theory associated with q-Sturm-Liouville operator with discontinuity conditions, *Journal of Contemporary Applied Mathematics*, 10(2) (2020), 1-9.

[6] Liu, Z.G., On a System of q-Partial Differential Equations with Applications to q-Series, In: *Andrews G., Garvan F., Analytic Number Theory, Modular Forms and q-Hypergeometric Series, ALLADI60 2016,* Springer Proceedings in Mathematics and Statistics, Vol. 221, Springer, 2017.

[7] Liu, Z.G., On the q-Partial Differential Equations and q-Series. In: *The Legacy of Srinivasa Ramanujan*, 213-250, Ramanujan Mathematical Society Lecture Notes Series, Vol. 20, Ramanujan Mathematical Society, Mysore, 2013.

[8] Cao, J., Homogeneous q-partial difference equations and some applications, *Advances in Applied Mathematics*, 84 (2017), 47-72. https://doi.org/10.1016/j.aam.2016.11.001

[9] Adewumi, A.O., Akindeinde, S.O., Aderogba, A.A., Ogundare, B.S., Laplace transform collocation method for solving hyperbolic telegraph equation, *International Journal of Engineering Mathematics*, (2017). https://doi.org/10.1155/2017/3504962

[10] Modanli, M., Laplace transform collocation and Daftar-Gejii-Jafaris method for fractional order time varying linear dynamical systems, *Physica Scripta*, 96(9) (2021), 094003. https://doi.org/10.1088/1402-4896/ac00e7

[11] Hahn, W., Beitrage zur Theorie der Heineschen Reihen (German), *Math. Nachr.,* 2 (1949), 340-379. https://doi.org/10.1002/mana.19490020604

[12] Daiz, R., Ternel C., q, k-generalized gamma and beta functions, *J. Nonlinear Math. Phys.,* 12(1) (2005), 118-131.

[13] De Sole, A., Kac, V.G., On integral representations of q-gamma and q-beta functions, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni*, 16(1) (2005), 11-29.

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TIMELIKE LOXODROMES ON LORENTZIAN HELICOIDAL SURFACES IN MINKOWSKI n–SPACE

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Abstract. In this paper, we examine timelike loxodromes on three kinds of Lorentzian helicoidal surfaces in Minkowski n–space. First, we obtain the first order ordinary differential equations which determine timelike loxodromes on the Lorentzian helicoidal surfaces in $\mathbb{E}_1^n$ according to the causal characters of their meridian curves. Then, by finding general solutions, we get the explicit parametrizations of such timelike loxodromes. In particular, we investigate the timelike loxodromes on the three kinds of Lorentzian right helicoidal surfaces in $\mathbb{E}_1^n$. Finally, we give an example to visualize the results.

1. Introduction

Loxodromes, which are also known as rhumb lines, are curves that make constant angles with the meridians on the Earth's surface. Geodesics which minimize the distance between two points on Earth's surface, are different from than loxodromes on Earth's surface, [26]. Only the equator and the meridians are both constant course angle and length minimizing. Since loxodromes give an efficient routing from one position to another by means of a constant course angle, they are still primarily used in navigation. For details, we refer to [1,2,25,27]. Since the Earth's surface can be thought as a Riemannian sphere, the notion of loxodromes can be

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broaden to an arbitrary surface of revolution, where meridians are copies of the profile curve.

In early of 20th century, C. A. Noble [21] studied the loxodrome on the surface of revolution in $\mathbb{E}^3$ and he also showed that the loxodrome on sphereoid projects stereographically into the same spiral as the loxodrome on the sphere which is tangent to the sphereoid along equator. Then, S. Kos et al. [19] and M. Petrović [23] got the differential equations related to the loxodromes on a sphere and a sphereoid and determined the length of such loxodromes, respectively.

Later, the topic of loxodromes has been studied on the rotational surfaces in Minkowski space which is important in general relativity. In 3–dimensional Minkowski space, there are three types of rotational surfaces with respect to the casual characters of rotation axes and the concept of angle to define loxodromes is not similar to Riemannian case. Therefore, the results in the Minkowski space are richer than the Euclidean space. The authors determined the parametrizations of spacelike and timelike loxodromes on rotational surfaces in $\mathbb{E}^3_1$ which have either spacelike meridians or timelike meridians in [3] and [4], respectively. For 4–dimensional Minkowski space, there are three types of rotation with 2–dimensional axes such as elliptic, hyperbolic and parabolic rotation leaving a Riemannian plane, a Lorentzian plane or a degenerate plane pointwise fixed, respectively. Then, M. Babaarslan and M. Gümüş found the explicit parametrizations of loxodromes on such rotational surfaces of $\mathbb{E}^4_1$ in [10].

Helicoidal surfaces are the natural generalizations of rotational surfaces and they play important roles in nature, science and engineering, see [17,18,22]. Thus, this generalization leads the studies to the loxodromes on helicoidal surfaces in [5–9]. Recently, M. Babaarslan and N. Sönmez constructed the three kinds of helicoidal surfaces in $\mathbb{E}^4_1$ by using rotation with 2–dimensional axes and translation in $\mathbb{E}^4_1$ and they also obtained the general form of spacelike and timelike loxodromes on such helicoidal surfaces in [11].

With the motivation from geometry, M. Babaarslan, B. B. Demirci, and R. Genç extended the notion of the helicoidal surfaces in $\mathbb{E}^4_1$ to higher dimensional Minkowski space and they made characterization of spacelike loxodromes on these helicoidal surfaces of $\mathbb{E}^n_1$ in [12]. In this context, this paper is a sequel of the article given by [12].

In this paper, we study timelike loxodromes on three types of Lorentzian helicoidal surfaces in Minkowski n–space $\mathbb{E}^n_1$. We find the equations of timelike loxodromes on such helicodial surfaces which have either spacelike meridians or timelike meridians and then we get the explicit parametrizations of these loxodromes by finding the general solution of the equations. As particular cases, we consider timelike loxodromes on each Lorentzian right helicoidal surfaces in $\mathbb{E}^n_1$. Finally, we give an illustrative example.

2. Preliminaries

Let $\mathbb{E}_s^n$ denote the pseudo–Euclidean space of dimension n and index s, i.e., $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) \mid x_1, x_2, ..., x_n \in \mathbb{R}\}$ equipped with the metric

$$ds^2 = \sum_{i=1}^{n-s} dx_i^2 - \sum_{j=n-s+1}^{n} dx_j^2. \tag{1}$$

For $s = 1$, $\mathbb{E}_1^n$ is known as the Minkowski space which is inspired by general relativity.

A vector v in $\mathbb{E}_1^n$ is called spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, and lightlike (or null) if $\langle v, v\rangle = 0$ and $v \neq 0$. The length of a vector v in $\mathbb{E}_1^n$ is given by $||v|| = \sqrt{|\langle v, v\rangle|}$ and v is said to be an unit vector if $||v|| = 1$.

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}_1^n$ be a smooth regular curve in $\mathbb{E}_1^n$, where I is an open interval. Then, the causal character of α is spacelike, timelike or lightlike if $\dot{\alpha}$ is spacelike, timelike or lightlike, respectively, where $\dot{\alpha} = d\alpha/dt$.

Let M be a pseudo–Riemannian surface in $\mathbb{E}_1^n$ given by a local parametrization $\mathbf{x}(u, v)$. Then, the coefficients of the first fundamental form of M are

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle, \tag{2}$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ denote the partial derivatives of $\mathbf{x}$ with respect u and v, respectively. Thus, the induced metric g of M in $\mathbb{E}_1^n$ is given by

$$g = Edu^2 + 2Fdudv + Gdv^2. \tag{3}$$

Also, a pseudo–Riemannian surface M in $\mathbb{E}_1^n$ is called a spacelike surface or a timelike surface if and only if $EG - F^2 > 0$ or $EG - F^2 < 0$, respectively. For the case $EG - F^2 = 0$, a pseudo–Riemannian surface M is called a lightlike surface. Throughout this work, we will assume that the surface is nondegenerate.

The length of the curve α on the pseudo–Riemannian surface M between two points u_0 and u_1 in $\mathbb{E}_1^n$ is given by

$$L = \int_{u_0}^{u_1} \sqrt{\left| E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2 \right|} du. \tag{4}$$

For later use, we give the following definition of Lorentzian angle in $\mathbb{E}_1^n$ by using [24].

Definition 1. *Let x and y be vectors in $\mathbb{E}_1^n$. Then, we have the following statements:*

i. *for a spacelike vector x and a timelike vector y, there is a unique nonnegative real number θ such that*

$$\langle x, y\rangle = \pm||x||||y|| \sinh\theta. \tag{5}$$

The number θ is called Lorentzian timelike angle between x and y.

ii. *for timelike vectors x and y, there is a unique nonnegative real number θ such that*

$$\langle x, y\rangle = ||x||||y|| \cosh\theta. \tag{6}$$

The number θ is called Lorentzian timelike angle between x and y. Note that $\theta = 0$ if and only if x and y are positive scalar multiples of each other.

By using [12], the definition of the helicoidal surfaces in $\mathbb{E}_1^n$ can be given as follows.

Let $\beta : I \subset \mathbb{R} \longrightarrow \Pi \subset \mathbb{E}_1^n$ be a smooth curve in a hyperplane $\Pi \subset \mathbb{E}_1^n$, P be a $(n-2)$-plane in the hyperplane $\Pi \subset \mathbb{E}_1^n$ and ℓ be a line parallel to P. A helicoidal surface in $\mathbb{E}_1^n$ is defined as a rotation of the curve β around P with a translation along the line ℓ. Here, the speed of translation is proportional to the speed of this rotation. Thus, there are three types of helicoidal surfaces in $\mathbb{E}_1^n$ as follows:

2.1. **Helicoidal surface of type I.** Let $\{e_1, e_2, ..., e_n\}$ be a standard orthonormal basis for $\mathbb{E}_1^n$. Then, we choose a Lorentzian $(n-2)$–subspace $\mathbf{P}_1$ generated by $\{e_3, e_4, ..., e_n\}$, Π_1 a hyperplane generated by $\{e_1, e_3, ..., e_n\}$ and a line ℓ_1 generated by e_n. Assume that $\beta_1 : I \longrightarrow \Pi_1 \subset \mathbb{E}_1^n$, $\beta_1(u) = (x_1(u), 0, x_3(u), ..., x_n(u))$, is a smooth regular curve lying in Π_1 defined on an open interval $I \subset \mathbb{R}$ and u is arc length parameter, that is, $x_1'^2(u) + x_3'^2(u) + ... - x_n'^2(u) = \varepsilon$ with $\varepsilon = \pm 1$. For $0 \le v < 2\pi$ and a positive constant c, we consider the surface M_1

$$H_1(u, v) = (x_1(u)\cos v, x_1(u)\sin v, x_3(u), ..., x_{n-1}(u), x_n(u) + cv) \tag{7}$$

which is the parametrization of the helicoidal surface obtained the rotation of the curve β_1 that leaves the Lorentzian subspace $\mathbf{P}_1$ pointwise fixed followed by the translation along ℓ_1. The surface M_1 in $\mathbb{E}_1^n$ is called a helicoidal surface of type I. Also, the surface M_1 is called a right helicoidal surface of type I in $\mathbb{E}_1^n$ if x_n is a constant function.

2.2. **Helicoidal surface of type II.** Let $\{e_1, e_2, ..., e_n\}$ be a standard orthonormal basis for $\mathbb{E}_1^n$. Then, we choose a Riemannian $(n-2)$–subspace $\mathbf{P}_2$ generated by $\{e_1, e_2, ..., e_{n-2}\}$, Π_2 a hyperplane generated by $\{e_1, ..., e_{n-2}, e_n\}$ and a line ℓ_2 generated by e_1. Assume that $\beta_2 : I \longrightarrow \Pi_2 \subset \mathbb{E}_1^n, \beta_2(u) = (x_1(u), ..., x_{n-2}(u), 0, x_n(u))$, is a smooth regular curve lying in Π_2 defined on an open interval $I \subset \mathbb{R}$ and u is an arc length parameter, that is, $x_1'^2(u) + x_2'^2(u) + ... - x_n'^2(u) = \varepsilon$ for $\varepsilon = \pm 1$. For $v \in \mathbb{R}$ and a positive constant c, we consider the surface M_2

$$H_2(u, v) = (x_1(u) + cv, x_2(u), ..., x_{n-2}(u), x_n(u)\sinh v, x_n(u)\cosh v) \tag{8}$$

which is the parametrization of the helicoidal surface obtained the rotation of the curve β_2 which leaves Riemannian subspace $\mathbf{P}_2$ pointwise fixed followed by the translation along ℓ_2. The surface M_2 in $\mathbb{E}_1^n$ is called a helicoidal surface of type II. Also, the surface M_2 is called a right helicoidal surface of type II in $\mathbb{E}_1^n$ if x_1 is a constant function.

2.3. **Helicoidal surface of type III.** Let define a pseudo–orthonormal basis $\{e_1, e_2, ..., \xi_{n-1}, \xi_n\}$ for $\mathbb{E}_1^n$ using a standard orthonormal basis $\{e_1, e_2, ..., e_{n-1}, e_n\}$ for $\mathbb{E}_1^n$ such that

$$\xi_{n-1} = \frac{1}{\sqrt{2}}(e_n - e_{n-1}) \text{ and } \xi_n = \frac{1}{\sqrt{2}}(e_n + e_{n-1}), \tag{9}$$

where $\langle \xi_{n-1}, \xi_{n-1} \rangle = \langle \xi_n, \xi_n \rangle = 0$ and $\langle \xi_{n-1}, \xi_n \rangle = -1$. Then, we choose a degenerate $(n-2)$–subspace $\mathbf{P}_3$ generated by $\{e_1, e_3, ..., \xi_{n-1}\}$, Π_3 a hyperplane generated by $\{e_1, e_3, ..., e_{n-2}, \xi_{n-1}, \xi_n\}$ and a line ℓ_3 generated by ξ_{n-1}. Assume that $\beta_3 : I \longrightarrow \Pi_3 \subset \mathbb{E}_1^n$, $\beta_3(u) = x_1(u)e_1 + x_3(u)e_3 + ... + x_{n-1}(u)\xi_{n-1} + x_n(u)\xi_n$, is a smooth curve lying in Π_3 defined on an open interval $I \subset \mathbb{R}$ and u is an arc length parameter, that is, $x_1'^2(u) + x_3'^2(u) + ... - 2x_{n-1}'(u)x_n'(u) = \varepsilon$ for $\varepsilon = \pm 1$. Then, we consider the surface M_3

$$\begin{aligned} H_3(u, v) =& x_1(u)e_1 + \sqrt{2}vx_n(u)e_2 + x_3(u)e_3 + ... + x_{n-2}(u)e_{n-2} \\ &+ (x_{n-1}(u) + v^2x_n(u) + cv)\xi_{n-1} + x_n(u)\xi_n \end{aligned} \tag{10}$$

which is the parametrization of the helicoidal surface obtained a rotation of the curve β_3 which leaves the degenerate subspace $\mathbf{P}_3$ pointwise fixed followed by the translation along ℓ_3. The surface M_3 in $\mathbb{E}_1^n$ is called the helicoidal surface of type III. If x_n is a constant function, then the helicoidal surface M_3 is called a right helicoidal surface of type III in $\mathbb{E}_1^n$.

Remark 1. *It can be easily seen that the helicoidal surfaces M_1–M_3 in $\mathbb{E}_1^n$ defined by (7), (8) and (10) reduce to the rotational surfaces in $\mathbb{E}_1^n$ for $c = 0$.*

3. Timelike Loxodrome on Timelike Helicoidal Surface of Type I in $\mathbb{E}_1^n$

In this section, we determine the parametrization of timelike loxodrome on the timelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (7).
Consider the timelike helicoidal surface of type I, M_1, in $\mathbb{E}_1^n$ given by (7). From a simple calculation, the induced metric g_1 on M_1 is defined by

$$g_1 = \varepsilon du^2 - 2cx_n'(u)dudv + (x_1^2(u) - c^2)dv^2. \tag{11}$$

Since M_1 is a timelike surface in $\mathbb{E}_1^n$, we have $\varepsilon x_1^2(u) - c^2(\varepsilon + x_n'^2(u)) < 0$. Assume that $\alpha_1(t) = H_1(u(t), v(t))$ is a timelike loxodrome on M_1 in $\mathbb{E}_1^n$, that is, $\alpha_1(t)$ intersects the meridian $m_1(u) = H_1(u, v_0)$ for a constant v_0 with a constant angle ϕ_0 at the point $p \in M_1$. Then, we have

$$\langle \dot{\alpha_1}(t), (m_1)_u \rangle = \varepsilon \frac{du}{dt} - cx_n'(u)\frac{dv}{dt}, \tag{12}$$

$$\varepsilon \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2 < 0. \tag{13}$$

In this context, there are two following cases occur with respect to the causal character of the meridian curve $m_1(u)$.

Case i. M_1 has a spacelike meridian curve $m_1(u)$, that is, $\varepsilon = 1$. Using the equations (12) and (13) in (5), we get

$$\sinh\phi_0 = \pm\frac{\frac{du}{dt} - cx_n'(u)\frac{dv}{dt}}{\sqrt{-\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2}}. \tag{14}$$

Case ii. M_1 has a timelike meridian curve $m_1(u)$, that is, $\varepsilon = -1$. Using the equations (12) and (13) in (6), we obtain

$$\cosh\phi_0 = -\frac{\frac{du}{dt} + cx_n'(u)\frac{dv}{dt}}{\sqrt{\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - (x_1^2(u) - c^2)\left(\frac{dv}{dt}\right)^2}}. \tag{15}$$

After a simple calculation in equations (14) and (15), we get the following lemma.

Lemma 1. *Let M_1 be a timelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (7). Then, $\alpha_1(t) = H_1(u(t), v(t))$ is a timelike loxodrome with $\dot{u} \neq 0$ if and only if one of the following differential equations is satisfied:*

(i.) *for having a spacelike meridian,*

$$(\sinh^2\phi_0(x_1^2(u) - c^2) + c^2x_n'^2(u))\dot{v}^2 - 2c\cosh^2\phi_0 x_n'(u)\dot{u}\dot{v} + \cosh^2\phi_0\dot{u}^2 = 0, \tag{16}$$

(ii.) *for having a timelike meridian,*

$$(\cosh^2\phi_0(x_1^2(u) - c^2) + c^2x_n'^2(u))\dot{v}^2 - 2c\sinh^2\phi_0 x_n'(u)\dot{u}\dot{v} - \sinh^2\phi_0\dot{u}^2 = 0, \tag{17}$$

where ϕ_0 is a nonnegative constant.

Theorem 1. *A timelike loxodrome on a timelike helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (7) is parametrized by $\alpha_1(u) = H_1(u, v(u))$, where $v(u)$ is given by one of the following functions:*

(i.) $v(u) = \pm\dfrac{1}{2\sinh\phi_0}\displaystyle\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2 - x_1^2(\xi)}}$,

(ii.) $v(u) = \pm\dfrac{1}{2\cosh\phi_0}\displaystyle\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2 - x_1^2(\xi)}}$,

(iii.) *for* $\sinh^2\phi_0(x_1^2(\xi) - c^2) + c^2x_n'^2(\xi) \neq 0$,

$$v(u) = \int_{u_0}^{u}\frac{2c\cosh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(c^2(x_n'^2(\xi) + 1) - x_1^2(\xi))}}{2\sinh^2\phi_0(x_1^2(\xi) - c^2) + 2c^2x_n'^2(\xi)}d\xi,$$

(iv.) *for* $\cosh^2\phi_0(x_1^2(\xi) - c^2) + c^2x_n'^2(\xi) \neq 0$,

$$v(u) = \int_{u_0}^{u}\frac{2c\sinh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(c^2(x_n'^2(\xi) - 1) + x_1^2(\xi))}}{2\cosh^2\phi_0(x_1^2(\xi) - c^2) + 2c^2x_n'^2(\xi)}d\xi,$$

where ϕ_0 is a nonnegative constant and $c > 0$ is a constant.

Proof. Assume that M_1 is a timelike helicoidal surface in $\mathbb{E}_1^n$ defined by (7) and $\alpha_1(t) = H_1(u(t), v(t))$ is a timelike loxodrome on M_1 in $\mathbb{E}_1^n$. From Lemma 1, we have the equations (16) and (17).

For a spacelike meridian, the equation (16) implies

$$(\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2-2c\cosh^2\phi_0 x_n'(u)\frac{dv}{du}+\cosh^2\phi_0=0. \quad (18)$$

If $\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)=0$, then the equation (18) becomes

$$2c\cosh^2\phi_0 x_n'(u)\frac{dv}{du}-\cosh^2\phi_0=0 \quad (19)$$

whose the solution is $v(u)=\frac{1}{2c}\int_{u_0}^{u}\frac{d\xi}{x_n'(\xi)}$. On the other side, $\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)=0$ implies $x_n'(u)=\pm\frac{\sinh\phi_0}{c}\sqrt{c^2-x_1^2(u)}$ for $\phi_0\neq 0$. Thus, we get the desired equation in (i). Also, we note that $c^2-x_1^2(u)>0$ due the fact that M_1 is a timelike surface in $\mathbb{E}_1^n$.

If $\sinh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)\neq 0$, it can be easily obtained that the solution $v(u)$ of the differential equation (18) is given by the integral in (iii).

Similarly, for a timelike meridian, the equation (17) implies

$$(\cosh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u))\left(\frac{dv}{du}\right)^2-2c\sinh^2\phi_0 x_n'(u)\frac{dv}{du}-\sinh^2\phi_0=0. \quad (20)$$

If $\cosh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)=0$, the equation (20) reduces to the following equation

$$2c\sinh^2\phi_0 x_n'(u)\frac{dv}{du}+\sinh^2\phi_0=0 \quad (21)$$

whose the solution is $v(u)=-\frac{1}{2c}\int_{u_0}^{u}\frac{d\xi}{x_n'(\xi)}$ for a nonzero constant ϕ_0. Since $x_n'(u)=\pm\frac{\cosh\phi_0}{c}\sqrt{c^2-x_1^2(u)}$, we get the desired equation in (ii). Also, we note that $c^2-x_1^2(u)>0$ due the fact that M_1 is a timelike surface in $\mathbb{E}_1^n$. If $\cosh^2\phi_0(x_1^2(u)-c^2)+c^2x_n'^2(u)\neq 0$, the solution $v(u)$ of the differential equation (20) is given by the integral in (iv). Thus, we get the parametrization of the loxodrome with respect to u parameter such that $\alpha_1(u)=H_1(u,v(u))$, where $v(u)$ is defined by one of the integrals in (i)-(iv). □

Now, we consider a timelike right helicoidal surface of type I in $\mathbb{E}_1^n$, denoted by M_1^R, that is,

$$H_1^R(u,v)=(x_1(u)\cos v, x_1(u)\sin v, x_3(u),\ldots,x_{n-1}(u), x_{n_0}+cv), \quad (22)$$

where $c\neq 0$ and x_{n_0} are constants. Then, from the equation in (iii) of Theorem 1, we give the following corollary.

Corollary 1. *A timelike loxodrome on a timelike right helicoidal surface of type I in $\mathbb{E}_1^n$ defined by (22) is parametrized by $\alpha_1^R(u)=H_1^R(u,v(u))$ where $v(u)$ is given by*

$$v(u)=\pm\coth\phi_0\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2-x_1^2(\xi)}} \quad (23)$$

for constant $\phi_0>0$.

Using the equation (4) and Corollary 1, we give the following statement:

Corollary 2. *The length of a timelike loxodrome on a timelike right helicoidal surface of type I in* $\mathbb{E}_1^n$ *defined by* (22) *between two points* u_0 *and* u_1 *is given by*

$$L = \left| \frac{u_1 - u_0}{\sinh \phi_0} \right|,$$

for constant $\phi_0 > 0$.

4. Timelike Loxodrome on Timelike Helicoidal Surface of Type II in $\mathbb{E}_1^n$

In this section, we determine the parametrization of timelike loxodrome on the timelike helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (8).

Consider the timelike helicoidal surface of type II, M_2, in $\mathbb{E}_1^n$ given by (8). From a simple calculation, the induced metric g_2 on M_2 is defined by

$$g_2 = \varepsilon du^2 + 2cx_1'(u)dudv + (c^2 + x_n^2(u))dv^2. \tag{24}$$

Since M_2 is a timelike surface in $\mathbb{E}_1^n$, we have $c^2(\varepsilon - x_1'^2(u)) + \varepsilon x_n^2(u) < 0$. Assume that $\alpha_2(t) = H_2(u(t), v(t))$ is a timelike loxodrome on M_2 in $\mathbb{E}_1^n$, that is, $\alpha_2(t)$ intersects the meridian $m_2(u) = H_2(u, v_0)$ for a constant v_0 with a constant angle ϕ_0 at the point $p \in M_2$. Then, we have

$$\langle \dot{\alpha}_2(t), (m_2)_u \rangle = \varepsilon \frac{du}{dt} + cx_1'(u)\frac{dv}{dt}, \tag{25}$$

$$\varepsilon \left(\frac{du}{dt}\right)^2 + 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} + (c^2 + x_n^2(u))\left(\frac{dv}{dt}\right)^2 < 0. \tag{26}$$

In this context, there are two following cases occur with respect to the causal character of the meridian curve $m_2(u)$.

Case i. M_2 has a spacelike meridian curve $m_2(u)$, that is, $\varepsilon = 1$. Using the equations (25) and (26) in (5), we get

$$\sinh \phi_0 = \pm \frac{\frac{du}{dt} + cx_1'(u)\frac{dv}{dt}}{\sqrt{-\left(\frac{du}{dt}\right)^2 - 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} - (c^2 + x_n^2(u))\left(\frac{dv}{dt}\right)^2}}. \tag{27}$$

Case ii. M_2 has a timelike meridian curve $m_2(u)$, that is, $\varepsilon = -1$. Using the equations (25) and (26) in (6), we obtain

$$\cosh \phi_0 = \frac{-\frac{du}{dt} + cx_1'(u)\frac{dv}{dt}}{\sqrt{\left(\frac{du}{dt}\right)^2 - 2cx_1'(u)\frac{du}{dt}\frac{dv}{dt} - (c^2 + x_n^2(u))\left(\frac{dv}{dt}\right)^2}}. \tag{28}$$

After a simple calculation in equations (27) and (28), we get the following lemma.

Lemma 2. *Let* M_2 *be a timelike helicoidal surface of type II in* $\mathbb{E}_1^n$ *defined by* (8). *Then,* $\alpha_2(t) = H_2(u(t), v(t))$ *is a timelike loxodrome with* $\dot{u} \neq 0$ *if and only if one of the following differential equations is satisfied:*

(i.) *for having a spacelike meridian,*

$$(\sinh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\dot{v}^2+2c\cosh^2\phi_0x_1'(u)\dot{u}\dot{v}+\cosh^2\phi_0\dot{u}^2=0, \tag{29}$$

(ii.) *for having a timelike meridian,*

$$(\cosh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\dot{v}^2+2c\sinh^2\phi_0x_1'(u)\dot{u}\dot{v}-\sinh^2\phi_0\dot{u}^2=0, \tag{30}$$

where ϕ_0 is a nonnegative constant.

Theorem 2. *A timelike loxodrome on a timelike helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (8) is parametrized by $\alpha_2(u)=H_2(u,v(u))$, where $v(u)$ is given by one of the following functions:*

$$\text{(i.)}\ \ v(u)=\int_{u_0}^{u}\frac{-2c\cosh^2\phi_0x_1'(\xi)\pm\sqrt{\sinh^2(2\phi_0)(c^2(x_1'^2(\xi)-1)-x_n^2(\xi))}}{2\sinh^2\phi_0(x_n^2(\xi)+c^2)+2c^2x_1'^2(\xi)}d\xi,$$

$$\text{(ii.)}\ \ v(u)=\int_{u_0}^{u}\frac{-2c\sinh^2\phi_0x_1'(\xi)\pm\sqrt{\sinh^2(2\phi_0)(x_n^2(\xi)+c^2(x_1'^2(\xi)+1))}}{2\cosh^2\phi_0(x_n^2(\xi)+c^2)+2c^2x_1'^2(\xi)}d\xi,$$

where ϕ_0 is a nonnegative constant.

Proof. Assume that M_2 is a timelike helicoidal surface in $\mathbb{E}_1^n$ defined by (8) and $\alpha_2(t)=H_2(u(t),v(t))$ is a timelike loxodrome on M_2 in $\mathbb{E}_1^n$. From Lemma 2, we have the equations (29) and (30).

For a spacelike meridian, the equation (29) implies

$$(\sinh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\left(\frac{dv}{du}\right)^2+2c\cosh^2\phi_0x_1'(u)\frac{dv}{du}+\cosh^2\phi_0=0. \tag{31}$$

Since $\sinh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u)\neq 0$ for all $u\in I\subset\mathbb{R}$, it can be easily obtained that the solution $v(u)$ of the differential equation (31) is given by the integral in (i).

Similarly, for a timelike meridian, the equation (30) implies

$$(\cosh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u))\left(\frac{dv}{du}\right)^2+2c\sinh^2\phi_0x_1'(u)\frac{dv}{du}-\sinh^2\phi_0=0. \tag{32}$$

Due to $\cosh^2\phi_0(x_n^2(u)+c^2)+c^2x_1'^2(u)\neq 0$, the solution $v(u)$ of the differential equation (32) is given by the integral in (ii). Thus, we get a parametrization of the loxodrome with respect to u parameter such that $\alpha_2(u)=H_2(u,v(u))$, where $v(u)$ is defined by one of the integrals in (i) and (ii). □

Now, we consider a timelike right helicoidal surface of type II in $\mathbb{E}_1^n$ denoted by M_2^R, that is,

$$H_2^R(u,v)=(x_{1_0}+cv,x_2(u),\ldots,x_{n-2}(u),x_n(u)\sinh v,x_n(u)\cosh v), \tag{33}$$

where $c\neq 0$ and x_{1_0} are constants. Since M_2^R is a timelike surface in $\mathbb{E}_1^n$, we have $\varepsilon(c^2+x_n^2(u))<0$. This inequality can be only satisfied when $\varepsilon=-1$. Thus, the meridian curve of M_2^R must be timelike. Then, from the equations in (ii) of Theorem 2, we give the following corollary.

Corollary 3. *A timelike loxodrome on a timelike right helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (33) is parametrized by $\alpha_2^R(u) = H_2^R(u, v(u))$, where $v(u)$ is given by*

$$v(u) = \pm \tanh \phi_0 \int_{u_0}^{u} \frac{d\xi}{\sqrt{x_n^2(\xi) + c^2}} \tag{34}$$

and $c, \phi_0 > 0$ are constants.

Using the equation (4) and Corollary 3, we give the following statement:

Corollary 4. *The length of a timelike loxodrome on a timelike right helicoidal surface of type II in $\mathbb{E}_1^n$ defined by (33) between two points u_0 and u_1 is given by*

$$L = \left| \frac{u_1 - u_0}{\cosh \phi_0} \right|, \tag{35}$$

where ϕ_0 is a nonnegative constant.

5. Timelike Loxodrome on Timelike Helicoidal Surface of Type III in $\mathbb{E}_1^n$

In this section, we determine the parametrization of timelike loxodrome on the timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by (10).

Consider the timelike helicoidal surface of type III, M_3, in $\mathbb{E}_1^n$ given by (10). The induced metric g_3 on M_3 is defined by

$$g_3 = \varepsilon du^2 - 2cx_n'(u)dudv + 2x_n^2(u)dv^2. \tag{36}$$

Since M_3 is a timelike surface in $\mathbb{E}_1^n$, we have $2\varepsilon x_n^2(u) - c^2 x_n'^2(u) < 0$. Assume that $\alpha_3(t) = H_3(u(t), v(t))$ is a timelike loxodrome on M_3 in $\mathbb{E}_1^n$, that is, $\alpha_3(t)$ intersects the meridian $m_3(u) = H_3(u, v_0)$ for a constant v_0 with a constant angle ϕ_0 at the point $p \in M_3$. Then, we have

$$\langle \dot\alpha_3(t), (m_3)_u \rangle = \varepsilon \frac{du}{dt} - cx_n'(u)\frac{dv}{dt}, \tag{37}$$

$$\varepsilon \left(\frac{du}{dt}\right)^2 - 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} + 2x_n^2(u)\left(\frac{dv}{dt}\right)^2 < 0. \tag{38}$$

In this context, there are two following cases occur with respect to the causal character of the meridian curve $m_3(u)$.
Case i. M_3 has a spacelike meridian curve $m_3(u)$, that is, $\varepsilon = 1$. Using the equations (37) and (38) in (5), we get

$$\sinh \phi_0 = \pm \frac{\frac{du}{dt} - cx_n'(u)\frac{dv}{dt}}{\sqrt{-\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - 2x_n^2(u)\left(\frac{dv}{dt}\right)^2}}. \tag{39}$$

Case ii. M_3 has a timelike meridian curve $m_3(u)$, that is, $\varepsilon = -1$. Using the equations (37) and (38) in (6), we obtain

$$\cosh\phi_0 = -\frac{\frac{du}{dt} + cx_n'(u)\frac{dv}{dt}}{\sqrt{\left(\frac{du}{dt}\right)^2 + 2cx_n'(u)\frac{du}{dt}\frac{dv}{dt} - 2x_n^2(u)\left(\frac{dv}{dt}\right)^2}}. \tag{40}$$

After a simple calculation in the equations (39) and (40), we get the following lemma.

Lemma 3. *Let M_3 be a timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by (10). Then, $\alpha_3(t) = H_3(u(t), v(t))$ is a timelike loxodrome with $\dot{u} \neq 0$ if and only if one of the following differential equations is satisfied:*

(i.) *for having a spacelike meridian,*

$$(2\sinh^2\phi_0 x_n^2(u) + c^2 x_n'^2(u))\dot{v}^2 - 2c\cosh^2\phi_0 x_n'(u)\dot{u}\dot{v} + \cosh^2\phi_0\dot{u}^2 = 0, \tag{41}$$

(ii.) *for having a timelike meridian,*

$$(2\cosh^2\phi_0 x_n^2(u) + c^2 x_n'^2(u))\dot{v}^2 - 2c\sinh^2\phi_0 x_n'(u)\dot{u}\dot{v} - \sinh^2\phi_0\dot{u}^2 = 0, \tag{42}$$

where ϕ_0 is a nonnegative constant.

Theorem 3. *A timelike loxodrome on a timelike helicoidal surface of type III in $\mathbb{E}_1^n$ defined by (10) is parametrized by $\alpha_3(u) = H_3(u, v(u))$, where $v(u)$ is given by one of the following functions:*

(i.) $$v(u) = \int_{u_0}^{u} \frac{2c\cosh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(c^2 x_n'^2(\xi) - 2x_n^2(\xi))}}{4\sinh^2\phi_0 x_n^2(\xi) + 2c^2 x_n'^2(\xi)} d\xi,$$

(ii.) $$v(u) = \int_{u_0}^{u} \frac{2c\sinh^2\phi_0 x_n'(\xi) \pm \sqrt{\sinh^2(2\phi_0)(2x_n^2(\xi) + c^2 x_n'^2(\xi))}}{4\cosh^2\phi_0 x_n^2(\xi) + 2c^2 x_n'^2(\xi)} d\xi,$$

where ϕ_0 is a nonnegative constant.

Proof. Assume that M_3 is a timelike helicoidal surface in $\mathbb{E}_1^n$ defined by (10) and $\alpha_3(t) = H_3(u(t), v(t))$ is a timelike loxodrome on M_3 in $\mathbb{E}_1^n$. From Lemma 3, we have the equations (41) and (42).

For a spacelike meridian, the equation (41) implies

$$(2\sinh^2\phi_0 x_n^2(u) + c^2 x_n'^2(u))\left(\frac{dv}{du}\right)^2 - 2c\cosh^2\phi_0 x_n'(u)\frac{dv}{du} + \cosh^2\phi_0 = 0. \tag{43}$$

Since $2\sinh^2\phi_0 x_n^2(u) + c^2 x_n'^2(u) \neq 0$ for all $u \in I \subset \mathbb{R}$, it can be easily obtained that the solution $v(u)$ of the differential equation (43) is given by the integral in (i).

Similarly, for a timelike meridian, the equation (42) implies

$$(2\cosh^2\phi_0 x_n^2(u) + c^2 x_n'^2(u))\left(\frac{dv}{du}\right)^2 - 2c\sinh^2\phi_0 x_n'(u)\frac{dv}{du} - \sinh^2\phi_0 = 0. \tag{44}$$

Due to $2\cosh^2\phi_0 x_n^2(u) + c^2 x_n'^2(u) \neq 0$, the solution $v(u)$ of the differential equation (44) is given by the integral in (ii). Thus, we get the parametrization of the loxodrome with respect to u parameter such that $\alpha_3(u) = H_3(u, v(u))$, where $v(u)$ is defined by one of the integrals in (i) and (ii). □

Note that the timelike right helicoidal surface of type III with the timelike meridian does not exist.

6. Visualization

In this section, we give an example to visualize our main results.

Example 1. *We consider the following spacelike profile curve:*

$$\beta_1(u) = (x_1(u), 0, x_3(u), ..., x_n(u)).$$

Then, we have the following parametrization of timelike helicoidal surface M_1*:*

$$H_1(u, v) = (x_1(u)\cos v, x_1(u)\sin v, x_3(u), ..., x_{n-1}(u), x_n(u) + cv).$$

By using (i) of Theorem 1, we have $v(u) = \pm\frac{1}{2\sinh\phi_0}\int_{u_0}^{u}\frac{d\xi}{\sqrt{c^2 - x_1^2(\xi)}}$. *If we choose* $x_1(\xi) = ck\sin\xi$ *for* $0 < k < 1$, *then* $v(u) = \pm\frac{1}{2c\sinh\phi_0}\int_{u_0}^{u}\frac{d\xi}{\sqrt{1-k^2\sin^2\xi}} = \pm\frac{1}{2c\sinh\phi_0}F(u,k)$, *where* $F(u,k)$ *is an elliptic integral of first kind (see [13]). Then, the parametrization of timelike loxodrome on timelike helicoidal surface* M_1 *in Minkowski* n*–space is given by*

$$\alpha_1(u) = (x_1(u)\cos v(u), x_1(u)\sin v(u), x_3(u), ..., x_{n-1}(u), x_n(u) + cv(u)),$$

where $v(u) = \pm\frac{1}{2c\sinh\phi_0}F(u,k)$ *for* $0 < k < 1$.

7. Conclusion

Loxodromes on various surfaces and hypersurfaces in different ambient spaces have been studied and many significant results have been obtained, see [3, 14–16, 20, 21, 28]. In this paper, we investigate the timelike loxodromes on Lorentzian helicoidal surfaces in Minkowski n–space which were constructed in [12], called type I, type II and type III. For this reason, we get the first order ordinary differential equations which determine the parametrizations of the timelike loxodromes on such helicoidal surfaces. Solving these equations, we obtain the explicit parametrizations of the such loxodromes parametrized by the parameter of the profile curves of the helicoidal surfaces. It is known that a particular case of helicoidal surfaces is right helicoidal surfaces. We observe that the Lorentzian right helicoidal surfaces appear only for the Lorentzian helicoidal surfaces of type I having spacelike meridians and the Lorentzian helicoidal surfaces of type II having timelike meridians. Hence, we look the parametrizations for timelike loxodromes on which the Lorentzian right helicoidal of $\mathbb{E}_1^n$ exist. Moreover, we find the lengths of such loxodromes which just depend on the points and the angle between the loxodromes and the meridians of the surfaces. Finally, we give a theoretical example to give the concept of the loxodromes. In [11], the graphical examples of the loxodromes can be found for the 4–dimensional Minkowski space. Hence, our results in this paper and [12] can be used as finding the parametrizations of spacelike and timelike loxodromes on the nondegenerate helicoidal surfaces in the Minkowski space with the higher dimension than four.

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References

[1] Alexander, J., Loxodromes: a rhumb way to go, *Math. Mag.*, 77(5) (2004), 349–356. https://doi.org/10.1080/0025570X.2004.11953279

[2] Aldea, N., Kopacz, P., Generalized loxodromes with application to time-optimal navigation in arbitrary wind, *J. Franklin Inst.*, 358(1) (2021), 776–799. https://doi.org/10.1016/j.jfranklin.2020.11.009

[3] Babaarslan, M., Yaylı, Y., Space–like loxodromes on rotational surfaces in Minkowski 3–space, *J. Math. Anal. Appl.*, 409(1) (2014), 288–298. https://doi.org/10.1016/j.jmaa.2013.06.035

[4] Babaarslan, M., Munteanu, M. I., Time–like loxodromes on rotational surfaces in Minkowski 3–space, *An. Ştiint. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.*, 61(2) (2015), 471–484.

[5] Babaarslan, M., Yaylı, Y., Differential equation of the loxodrome on a helicoidal surface, *J. Navig.*, 68(5) (2015), 962–970. https://doi.org/10.1017/S0373463315000181

[6] Babaarslan, M., Kayacık, M., Time–like loxodromes on helicoidal surfaces in Minkowski 3–space, *Filomat*, 31(14) (2017), 4405–4414. https://doi.org/10.2298/fil1714405b

[7] Babaarslan, M., Loxodromes on helicoidal surfaces and tubes with variable radius in $\mathbb{E}^4$, *Commun. Fac. Sci. Univ. Ank. Ser. A1, Math. Stat.*, 68(2) (2019), 1950–1958. https://doi.org/10.31801/cfsuasmas.455372

[8] Babaarslan, M., Kayacık, M. Differential equations of the space–like loxodromes on the helicoidal surfaces in Minkowski 3–space, *Differ. Equ. Dyn. Syst.*, 28(2) (2020), 495–512. https://doi.org/10.1007/s12591-016-0343-5

[9] Babaarslan, M., A note on loxodromes on helicoidal surfaces in Euclidean n–space, *Appl. Math. E-Notes*, 20 (2020), 458–461.

[10] Babaarslan, M., Gümüş, M., On the parametrizations of loxodromes on time–like rotational surfaces in Minkowski space–time, *Asian-Eur. J. Math.*, 14(5) (2021), 2150080. https://doi.org/10.1142/S1793557121500807

[11] Babaarslan, M., Sönmez, N., Loxodromes on non–degenerate helicoidal surfaces in Minkowksi space–time, *Indian J. Pure Appl. Math.*, 52(4) (2021), 1212–1228. https://doi.org/10.1007/s13226-021-00030-x

[12] Babaarslan, M., Demirci, B.B, Genç, R., Spacelike loxodromes on helicoidal surfaces in Lorentzian n–space, *Differ. Geom. Dyn. Syst.*, 24 (2022), 18-33.

[13] Byrd, P.F., Friedman, M.D., Handbook of Elliptic Integrals for Engineers and Physicists, Springer-Verlag Berlin Heidelberg, 1954.

[14] Blackwood, J., Dukehart, A., Javaheri, M., Loxodromes on hypersurfaces of revolution, *Involve*, 10(3) (2016), 465–472. https://doi.org/10.2140/involve.2017.10.465

[15] Carlton–Wippern, K. C., On loxodromic navigation, *J. Navig.*, 45(2) (1992), 292–297. https://doi.org/10.1017/s0373463300010791

[16] Caddeo, R., Onnis, I. I, Piu, P., Loxodromes on invariant surfaces in three–manifolds, *Mediterr. J. Math.*, 17(1) (2020), 1–24. https://doi.org/10.1007/s00009-019-1439-2

[17] Hitt, L. R., Roussos, I. M., Computer graphics of helicoidal surfaces with constant mean curvature, *An. Acad. Bras. Ci*, 63(3) (1991), 211–228.

[18] Jensen, B., Electronic states on the helicoidal surface, *Phys. Rev. A*, 80(2) (2009), 022101. https://doi.org/10.1103/PhysRevA.80.022101

[19] Kos, S., Filjar, R., Hess, M., Differential equation of the loxodrome on a rotational surface, *Proceedings of the 2009 International Technical Meeting of the Institute of Navigation*, (2009), 958–960.

[20] Manfio, F., Tojeiro R., Van der Veken J., Geometry of submanifolds with respect to ambient vector fields, *Ann. Mat. Pura Appl.*, 199(6) (2020), 2197–2225. https://doi.org/10.1007/s10231-020-00964-9

[21] Noble, C. A., Note on loxodromes, *Amer. M. S. Bull.*, 12(2) (1905), 116–119. https://doi.org/10.1090/S0002-9904-1905-01296-9

[22] Pollard, D. D., Fletcher, R. C., Fundamentals of Structural Geology, Cambridge University Press, 2005.

[23] Petrović M, Differential equation of a loxodrome on the sphereoid, *Int. J. Mar. Sci. Technol. "Our Sea"*, 54(3–4) (2007), 87–89. https://hrcak.srce.hr/16504

[24] Ratcliffe, J.G., Foundations of Hyperbolic Manifolds, Springer Graduate Texts in Mathematics, 149, Second Edition, 2006.

[25] Tseng, W.K, Earle, M. A., Guo, J.L., Direct and inverse solutions with geodetic latitude in terms of longitude for rhumb line sailing, *J. Navig.*, 65(3) (2012), 549–559. https://doi.org/10.1017/S0373463312000148

[26] Tseng, W.K, Chang, W.J., Analogues between 2D linear equations and great circle sailing, *J. Navig.*, 67(1) (2014), 101–112. https://doi.org/10.1017/S0373463313000532

[27] Weintrit, A., Kopacz, P., A novel approach to loxodrome (rhumb line), orthodrome (great circle) and geodesic line in ECDIS and navigation in general, *Int. J. Mar. Navig. Saf. Sea Transp.*, 5(4) (2011), 507–517.

[28] Yoon, D.W., Loxodromes and geodesics on rotational surfaces in a simply isotropic space, *J. Geom.*, 108(2) (2017), 429–435. https://doi.org/10.1007/s00022-016-0349-8

http://communications.science.ankara.edu.tr

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ROLE OF IDEALS ON σ-TOPOLOGICAL SPACES

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Abstract. In this writeup, we have discussed the role of ideals on σ-topological spaces. Using this idea, we have also studied and discussed two operators $()^{*\sigma}$ and ψ_σ. We have extended this concept to a new generalized set and investigated some basic properties of these concepts using $()^{*\sigma}$ and ψ_σ operators.

1. Introduction

In topological space, the idea of ideal was known by Kuratowski [7] and Vaidyanathswamy [13]. After that, in the ideal topological space, local function was introduced and studied by Vaidyanathswamy. Njåstad [12] has introduced compatability of the topology with the help of an ideal. In [5,6] Janković and Hamlett introduced further the characteristics of ideal topological spaces and ψ-operator was introduced by them in 1990. A new type of topology from original ideal topological space was also introduced. In this new topological space, a Kuratowski-closure operator was defined using the local function. Also from ψ-operator, they proved that interior operator can be deduced in the new topological space. In 2007, using ψ-operator Modak and Bandhyopadhyay in [8] introduced generalized open sets. The idea of ideal m-space was introduced by Al-Omari and Noiri in [1,2] and they also investigated two operators identical with ψ-operator and local function in 2012. Their

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extensive works related to this topic can be found in [3, 4].

The idea of σ-topological space have been introduced and studied here. In this paper, ideal σ-topological space has been introduced and two set operators σ-local and ψ_σ and their properties have been studied. Finally σ-codense ideal, σ-compatible ideal and $\psi_\sigma - C$ set using ψ_σ operator have been introduced. Further investigation of various properties of that knowledge have been studied.

2. Preliminaries

Related to this paper, we have discussed some definitions, examples and results in this article.

Definition 1. *A family γ of subsets of a set T is called σ-topology if the following conditions are satisfied:*
(i) $\emptyset, T \in \gamma$.
(ii) γ is closed under countable union.
(iii) γ is closed under finite intersection.
The couple (T, γ) is said to be a σ-topological space. The member of γ is called σ-open set in (T, γ) and the complement of σ-open set is called σ-closed set.

Note 1. *Every topology on a non-empty set T is a σ-topology but every σ-topology on T may not be a topology. For an example, let $T = \mathbb{R}$, set of all real numbers and $\gamma = \{\emptyset, \mathbb{R}\} \cup \{S \subset \mathbb{R} : S \text{ is countable}\}$. Then γ is σ-topology on T. But $\bigcup_{p \in \mathbb{R}\setminus\mathbb{Q}} \{p\} \notin \gamma$, i.e, γ is not closed under arbitrary union. Hence γ is not a topology on $T = \mathbb{R}$.*

Definition 2. *A non-empty family J of subsets of T is called an ideal on T, if*
(i) $M \in J$ and $N \subset M$ implies $N \in J$ (heredity).
(ii) $M \in J$ and $N \in J$ imply $M \cup N \in J$ (finite additivity).

Definition 3. *Let (T, γ) be a σ-topological space and $M \subset T$. The σ-interior and σ-closure of M in (T, γ) are defined as respectively:*

$$\cup\{V : V \subset M \text{ and } V \in \gamma\} \text{ and } \cap\{C : M \subset C \text{ and } T \setminus C \in \gamma\}$$

The σ-interior and σ-closure of M in (T, γ) are denoted as $Int^\sigma(M)$ and $Cl^\sigma(M)$ respectively.

Theorem 1. *Let (T, γ) be a σ-topological space and M, N be two subsets of T, then*
(i) $p \in Cl^\sigma(M)$ if and only if for any σ-open set V containing p, $V \cap M \neq \emptyset$.
(ii) If $M \subset N$ then $Cl^\sigma(M) \subset Cl^\sigma(N)$.

Proof. (i) Let $p \in Cl^\sigma(M)$. If possible let there exists a σ-open set V containing p such that $V \cap M = \emptyset$. This implies $M \subset T \setminus V$. Since $T \setminus V$ is σ-closed in T

containing M, so $Cl^{\sigma}(M) \subset T \setminus V$. This implies $Cl^{\sigma}(M) \cap V = \emptyset$, which contradicts the fact that $p \in Cl^{\sigma}(M) \cap V$. Thus if $p \in Cl^{\sigma}(M)$, then for any σ-open set V containing p, $V \cap M \neq \emptyset$.
Conversely, let for any σ-open set V containing p, $V \cap M \neq \emptyset$. If possible let $p \notin Cl^{\sigma}(M)$. Then $p \in T \setminus Cl^{\sigma}(M) = V$(say). This implies $V \cap Cl^{\sigma}(M) = \emptyset$ and hence $V \cap M = \emptyset$, as $M \subset Cl^{\sigma}(M)$, which contradicts our assumption. Hence $p \in Cl^{\sigma}(M)$.
(ii) Let $p \in Cl^{\sigma}(M)$. Then for any σ-open set V containing p, $V \cap M \neq \emptyset$. This implies $V \cap N \neq \emptyset$, since $M \subset N$. Thus $p \in Cl^{\sigma}(N)$. Hence $Cl^{\sigma}(M) \subset Cl^{\sigma}(N)$. □

Theorem 2. *Let (T, γ) be a σ-topological space and $M \subset T$, then $Int^{\sigma}(M) = T \setminus Cl^{\sigma}(T \setminus M)$.*

Proof. $Cl^{\sigma}(T \setminus M) = Cl^{\sigma}(M^c) = \cap\{F : M^c \subset F, F^c \in \gamma\}$ where $M^c = T \setminus M$ and $F^c = T \setminus F$. This implies $\{Cl^{\sigma}(T \setminus M)\}^c = \cup\{F^c : M \supset F^c, F^c \in \gamma\}$. Thus $T \setminus Cl^{\sigma}(T \setminus M) = Int^{\sigma}(M)$. Hence the result. □

Definition 4. *Let (T, γ) be a σ-topological space and $M \subset T$. Then M is called a σ-neighbourhood of $p \in T$, if there exists $V \in \gamma$ such that $p \in V \subset M$.*

Definition 5. *Let (T, γ) be a σ-topological space and J be an ideal on T. Then the triplicate (T, γ, J) is called an ideal σ-topological space.*

Definition 6. *Let (T, γ, J) be an ideal σ-topological space. Then*

$M^*(J, \gamma) = \{p \in T : M \cap V \notin J$ *for every* $V \in \gamma(p)\}$, *where* $\gamma(p) = \{V \in \gamma : p \in V\}$

is said to be the σ-local function of M with respect to J and γ.
When there is no confusion, we will write M^J or simply $M^{*\sigma}$ or $M^*(J, \gamma)$ and call it the "σ-local function of M".

Example 1. *Let $T = \{p, q, r\}$, $\gamma = \{\emptyset, T, \{p\}, \{p, q\}, \{p, r\}\}$ and $J = \{\emptyset, \{p\}\}$. Take $M = \{p, q\}$. Then $M^{*\sigma} = \{t \in T : M \cap V \notin J$ for every $V \in \gamma(t)\} = \{q\}$.*

Theorem 3. *Let (T, γ) be a σ-topological space with I and J ideals on T and let M and N be subsets of T. Then*
(i) $\emptyset^{*\sigma} = \emptyset$.
(ii) $(M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$.
(iii) If $M \subset N$ *then* $M^{*\sigma} \subset N^{*\sigma}$.
(iv) If $I_1 \in I$ *then* $I_1^{*\sigma} = \emptyset$.
(v) $I \subset J$ *implies* $M^{*\sigma}(J) \subset M^{*\sigma}(I)$.
(vi) $M^{*\sigma} \cup N^{*\sigma} = (M \cup N)^{*\sigma}$.
(vii) $(\bigcup_i M_i)^{*\sigma} = \bigcup_i (M_i^{*\sigma})$.
(viii) $(M \cap N)^{*\sigma} \subset M^{*\sigma} \cap N^{*\sigma}$.
(ix) $M^{*\sigma} \setminus N^{*\sigma} = (M \setminus N)^{*\sigma} \setminus N^{*\sigma}$.
(x) For any $O \in \gamma, O \cap (O \cap M)^{*\sigma} \subset O \cap M^{*\sigma}$.

(xi) For any $I_1 \in I, (M \cup I_1)^{*\sigma} = M^{*\sigma} = (M \setminus I_1)^{*\sigma}$.
(xii) $M^{*\sigma}(I \cap J) = M^{*\sigma}(I) \cup N^{*\sigma}(J)$.
(xiii) $\gamma \cap I = \{\emptyset\}$ *if and only if* $T = T^{*\sigma}$.
(xiv) $M^{*\sigma} \subset Cl^{\sigma}(M)$.

Proof. (i) Here $\emptyset^{*\sigma}$ =$\{p \in T : \emptyset \cap V \notin I$ for every $V \in \gamma(\text{p})\}$. But $\emptyset \cap V = \emptyset \in I$ for every $V \in \gamma(p)$. Thus $\emptyset^{*\sigma}$ contains no element of T. Therefore $\emptyset^{*\sigma} = \emptyset$.
(ii) Let $p \in (M^{*\sigma})^{*\sigma}$. Then for every $V \in \gamma(p), V \cap M^{*\sigma} \notin I$ and hence $V \cap M^{*\sigma} \neq \emptyset$. Let $y \in V \cap M^{*\sigma}$. Then $V \in \gamma(y)$ and $y \in M^{*\sigma}$. This implies $V \cap M \notin I$ and hence $p \in M^{*\sigma}$. Therefore $(M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$.
(iii) Let $p \in M^{*\sigma}$. Then for every $V \in \gamma(p), V \cap M \notin I$. Since $M \subset N$, therefore $V \cap M \subset V \cap N$. Since $V \cap M \notin I$, so $V \cap N \notin I$. This implies $p \in N^{*\sigma}$ and so $M^{*\sigma} \subset N^{*\sigma}$.
(iv) Since $I_1 \in I$. Then for every $V \in \gamma$, $V \cap I_1 \subset I_1 \in I$ and by heredity, $V \cap I_1 \in I$. So $I_1^{*\sigma}$ =$\{p \in T : I_1 \cap V \notin I$ for every $V \in \gamma(\text{p})\}$ implies $I_1^{*\sigma} = \emptyset$.
(v) Let $p \in M^{*\sigma}(J)$. Then for every $V \in \gamma(p)$, $M \cap V \notin J$ implies $M \cap V \notin I$ (since $I \subset J$). So $p \in M^{*\sigma}(I)$. Hence $M^{*\sigma}(J) \subset M^{*\sigma}(I)$.
(vi) We know $M \subset M \cup N$ and $N \subset M \cup N$. This implies $M^{*\sigma} \subset (M \cup N)^{*\sigma}$ and $N^{*\sigma} \subset (M \cup N)^{*\sigma}$ (by Theorem 3 (iii)). So $M^{*\sigma} \cup N^{*\sigma} \subset (M \cup N)^{*\sigma}$. For reverse inclusion, let $p \notin (M^{*\sigma} \cup N^{*\sigma})$. Then $p \notin M^{*\sigma}$ and $p \notin N^{*\sigma}$. So there exist V, $O \in \gamma(p)$ such that $V \cap M \in I$ and $O \cap N \in I$. This implies $(V \cap M) \cup (O \cap N) \in I$ since I is additive.
Now

$$\begin{aligned}(V \cap M) \cup (O \cap N) &= [(V \cap M) \cup O] \cap [(V \cap M) \cup N] \\ &= (V \cup O) \cap (M \cup O) \cap (V \cup N) \cap (M \cup N) \\ &\supset (V \cap O) \cap (M \cup N)\end{aligned}$$

This implies $(V \cap O) \cap (M \cup N) \in I$ (since I is hereditary). Since $V \cap O \in \gamma(p)$, $p \notin (M \cup N)^{*\sigma}$. Contrapositively $p \in (M \cup N)^{*\sigma}$ implies $p \in M^{*\sigma} \cup N^{*\sigma}$. Thus $(M \cup N)^{*\sigma} \subset M^{*\sigma} \cup N^{*\sigma}$. Hence we get $M^{*\sigma} \cup N^{*\sigma} = (M \cup N)^{*\sigma}$.
(vii) Proof is obvious and hence omitted.
(viii) We know $M \cap N \subset M$ and $M \cap N \subset N$. This implies $(M \cap N)^{*\sigma} \subset M^{*\sigma}$ and $(M \cap N)^{*\sigma} \subset M^{*\sigma}$ (by Theorem 3 (iii)). So $(M \cap N)^{*\sigma} \subset M^{*\sigma} \cap N^{*\sigma}$.

Independent Proof: If possible let $(M \cap N)^{*\sigma}$ not be a subset of $M^{*\sigma} \cap N^{*\sigma}$. Then there exists $p \in (M \cap N)^{*\sigma}$ but $p \notin M^{*\sigma} \cap N^{*\sigma}$. Now $p \in (M \cap N)^{*\sigma}$ implies $V \cap (M \cap N) \notin I$ for every $V \in \gamma(p)$, i.e., $(V \cap M) \cap (V \cap N) \notin I$ for every $V \in \gamma(p)$. This implies $V \cap M \notin I$ and $V \cap N \notin I$ for every $V \in \gamma(p)$. So $p \in M^{*\sigma}$ and $p \in N^{*\sigma}$ which implies $p \in M^{*\sigma} \cap N^{*\sigma}$ which contradicts the fact that $p \notin M^{*\sigma} \cap N^{*\sigma}$. Hence $(M \cap N)^{*\sigma} \subset M^{*\sigma} \cap N^{*\sigma}$.

(ix) We know $M = (M \setminus N) \cup (M \cap N)$. This implies

$$\begin{aligned} M^{*\sigma} &= [(M \setminus N) \cup (M \cap N)]^{*\sigma} \\ &= (M \setminus N)^{*\sigma} \cup (M \cap N)^{*\sigma} \ (by\ Theorem\ 3\ (vii)) \\ &\subset (M \setminus N)^{*\sigma} \cup N^{*\sigma} \ (by\ Theorem\ 3\ (iii)) \end{aligned}$$

This implies $M^{*\sigma} \setminus N^{*\sigma} \subset (M \setminus N)^{*\sigma} \setminus N^{*\sigma}$.

Again $M \setminus N \subset M$. Then $(M \setminus N)^{*\sigma} \subset M^{*\sigma}$ and hence $(M \setminus N)^{*\sigma} \setminus N^{*\sigma} \subset M^{*\sigma} \setminus N^{*\sigma}$. Thus we obtain $M^{*\sigma} \setminus N^{*\sigma} = (M \setminus N)^{*\sigma} \setminus N^{*\sigma}$

(x) We have $O \cap M \subset M$. This implies $(O \cap M)^{*\sigma} \subset M^{*\sigma}$ (by Theorem 3 (iv)). So $O \cap (O \cap M)^{*\sigma} \subset O \cap M^{*\sigma}$.

(xi) We have $M \subset (M \cup I_1)$. This implies $M^{*\sigma} \subset (M \cup I_1)^{*\sigma}$. Let $p \in (M \cup I_1)^{*\sigma}$. Then for every $V \in \gamma(p)$, $V \cap (M \cup I_1) \notin I$. This implies $V \cap M \notin I$. If not, suppose that $V \cap M \in I$. Since $V \cap I_1 \subset I_1 \in I$, by heredity $V \cap I_1 \in I$ and hence by finite additivity $(V \cap M) \cup (V \cap I_1) \in I$. This implies $V \cap (M \cup I_1) \in I$, a contradiction. Consequently $p \in M^{*\sigma}$. Therefore $(M \cup I_1)^{*\sigma} \subset M^{*\sigma}$. So $(M \cup I_1)^{*\sigma} = M^{*\sigma}$.

Also $M \setminus I_1 \subset M$ implies $(M \setminus I_1)^{*\sigma} \subset M^{*\sigma}$. For the converse, let $p \in M^{*\sigma}$, we claim that $p \in (M \setminus I_1)^{*\sigma}$. If not, there exists $V \in \gamma(p)$ such that $V \cap (M \setminus I_1) \in I$. This implies $I_1 \cup (V \cap (M \setminus I_1)) \in I$, since $I_1 \in I$ (by finite additivity). Thus $I_1 \cup (V \cap M) \in I$. So $V \cap M \in I$, a contradiction to the fact that $p \in M^{*\sigma}$. Hence $M^{*\sigma} \subset (M \setminus I_1)^{*\sigma}$. So $M^{*\sigma} = (M \setminus I_1)^{*\sigma}$. Consequently $(M \cup I_1)^{*\sigma} = M^{*\sigma} = (M \setminus I_1)^{*\sigma}$.

(xii) We have $I \cap J \subset I$ and $I \cap J \subset J$. This implies $M^{*\sigma}(I \cap J) \supset M^{*\sigma}(I)$ and $M^{*\sigma}(I \cap J) \supset M^{*\sigma}(J)$ (by Theorem 3 (v)). So $M^{*\sigma}(I \cap J) \supset M^{*\sigma}(I) \cup M^{*\sigma}(J)$.

For reverse, let $p \in M^{*\sigma}(I \cap J)$. Then for every $V \in \gamma(p), V \cap M \notin I \cap J$. Thus $V \cap M \notin I$ or $V \cap M \notin J$. This implies $p \in M^{*\sigma}(I)$ or $p \in M^{*\sigma}(J)$. These imply $p \in M^{*\sigma}(I) \cup M^{*\sigma}(J)$ and hence $M^{*\sigma}(I) \cup M^{*\sigma}(J) \supset M^{*\sigma}(I \cap J)$. So $M^{*\sigma}(I \cap J) = M^{*\sigma}(I) \cup M^{*\sigma}(J)$.

(xiii) From definition $T^{*\sigma} \subset T$.

For reverse inclusion let $p \in T$. If possible let $p \notin T^{*\sigma}$. Then there exists $V \in \gamma(p)$ such that $V \cap T \in I$. This implies $V \in I$, a contradiction. Hence $T \subset T^{*\sigma}$. Thus $T = T^{*\sigma}$.

Conversely, suppose that $T = T^{*\sigma}$ holds. If possible let $V \in \gamma \cap I$ and $p \in V$. Then $V \cap T \subset V \in \gamma \cap I$. This implies $V \cap T \in \gamma \cap I$ and hence $V \cap T \in I$. Thus $p \notin T^{*\sigma}$, a contradiction.

(xiv) Let $p \in M^{*\sigma}$. Then for every $V \in \gamma(p)$, $V \cap M \notin I$. This implies $V \cap M \neq \emptyset$, for all $p \in M^{*\sigma}$. Thus $p \in Cl^{\sigma}(M)$. Hence $M^{*\sigma} \subset Cl^{\sigma}(M)$. □

Result 1. *Let (T, γ) be a σ-topological space with J an ideal on T and $M \subset T$. Then $V \in \gamma$, $V \cap M \in J$ implies $V \cap M^{*\sigma} = \emptyset$.*

Proof. If possible let $V \cap M^{*\sigma} \neq \emptyset$ and let $p \in V \cap M^{*\sigma}$. This implies $p \in V$ and for all $N_p \in \gamma(p)$ such that $N_p \cap M \notin J$. Since $p \in V \in \gamma$ then $V \cap M \notin J$, which is a contradiction. Hence the result. $\square$

Result 2. *Let (T, γ) be a σ-topological space with J an ideal on T. Then $(M \cup M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$ for all $M \in \wp(T)$.*

Proof. Let $p \notin M^{*\sigma}$. Then there exists $V_p \in \gamma(p)$ such that $V_p \cap M \in J$. This implies $V_p \cap M^{*\sigma} = \emptyset$. This implies $V_p \cap (M \cup M^{*\sigma}) = (V_p \cap M) \cup (V_p \cap M^{*\sigma}) = V_p \cap M \in J$. Thus $p \notin (M \cup M^{*\sigma})^{*\sigma}$. Hence $(M \cup M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$. $\square$

Theorem 4. *Let (T, γ) be a σ-topological space with J an ideal on T. Then the operator $Cl^{*\sigma} : \wp(T) \to \wp(T)$ defined by $Cl^{*\sigma}(M) = M \cup M^{*\sigma}$ for all $M \in \wp(T)$, is a Kuratowski closure operator and it generates a σ-topology $\gamma^*(J) = \{M \in \wp(T) : Cl^{*\sigma}(M^c) = M^c\}$ which is finer than γ.*

Proof. (i) Since $\emptyset^{*\sigma} = \emptyset$, then $Cl^{*\sigma}(\emptyset) = \emptyset \cup \emptyset^{*\sigma} = \emptyset \cup \emptyset = \emptyset$.
(ii) $Cl^{*\sigma}(M) = M \cup M^{*\sigma}$. This implies $M \subset Cl^{*\sigma}(M)$.
(iii) $Cl^{*\sigma}(M \cup N) = (M \cup N) \cup (M \cup N)^{*\sigma} = (M \cup N) \cup (M^{*\sigma} \cup N^{*\sigma}) = (M \cup M^{*\sigma}) \cup (N \cup N^{*\sigma}) = Cl^{*\sigma}(M) \cup Cl^{*\sigma}(N)$.
(iv) Let $M, N \subset T$ with $M \subset N$. Then $Cl^{*\sigma}(M) = M \cup M^{*\sigma} \subset N \cup N^{*\sigma} = Cl^{*\sigma}(N)$. This implies $Cl^{*\sigma}(M) \subset Cl^{*\sigma}(N)$. We have $M \subset Cl^{*\sigma}(M)$. This implies $Cl^{*\sigma}(M) \subset Cl^{*\sigma}(Cl^{*\sigma}(M))$. But $Cl^{*\sigma}(Cl^{*\sigma}(M)) = Cl^{*\sigma}(M \cup M^{*\sigma}) = (M \cup M^{*\sigma}) \cup (M \cup M^{*\sigma})^{*\sigma} \subset (M \cup M^{*\sigma}) \cup M^{*\sigma} = M \cup M^{*\sigma} = Cl^{*\sigma}(M)$. Hence $Cl^{*\sigma}(Cl^{*\sigma}(M)) = Cl^{*\sigma}(M)$. Consequently $Cl^{*\sigma}(M)$ is a Kuratowski closure operator.
Now we have to show that $\gamma^*(J) = \{M \in \wp(T) : Cl^{*\sigma}(M^c) = M^c\}$ is a σ-topology on T.
Since $Cl^{*\sigma}(\emptyset) = \emptyset$, then $\emptyset^c \in \gamma^*(J)$. This implies $T \in \gamma^*(J)$. Also since $T \subset Cl^{*\sigma}(T) \subset T$, then $Cl^{*\sigma}(T) = T$. This implies $T^c \in \gamma^*(J)$. Hence $\emptyset \in \gamma^*(J)$
Let $M_1, M_2, ..., M_n, ... \in \gamma^*(J)$. Then $Cl^{*\sigma}(M_i^c) = M_i^c$ for all $i \in \mathbb{N}$. Now $\bigcap\limits_{i \in \mathbb{N}} M_i^c \subset M_i^c$ for all $i \in \mathbb{N}$. This implies $Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c) \subset Cl^{*\sigma}(M_i^c) = M_i^c$ for all $i \in \mathbb{N}$. This implies $Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c) \subset (\bigcap\limits_{i \in \mathbb{N}} M_i^c) \subset Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c)$. This implies $Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c) = (\bigcap\limits_{i \in \mathbb{N}} {M_i}^c)$. Thus $Cl^{*\sigma}(\bigcup\limits_{i \in \mathbb{N}} M_i)^c = (\bigcup\limits_{i \in \mathbb{N}} M_i)^c$. Hence $\bigcup\limits_{i \in \mathbb{N}} M_i \in \gamma^*(J)$. Therefore $\gamma^*(J)$ is closed under countable union.
Again let $M_j \in \gamma^*(J), j = 1, 2, 3, ...n$. Then $Cl^{*\sigma}(M_j^c) = M_j^c$ for all $j = 1, 2, 3, ...n$. Therefore $Cl^{*\sigma}\{(\bigcap\limits_{j=1}^{n} M_j)^c\} = Cl^{*\sigma}(\bigcup\limits_{j=1}^{n} M_j^c) = \bigcup\limits_{j=1}^{n} Cl^{*\sigma}(M_j^c) = \bigcup\limits_{j=1}^{n} (M_j^c) = (\bigcap\limits_{j=1}^{n} M_j)^c$. This implies $\bigcap\limits_{j=1}^{n} M_j \in \gamma^*(J)$. Therefore $\gamma^*(J)$ is closed under finite intersection. Thus $\gamma^*(J)$ is a σ-topology on T.

Next from Theorem 3 (xiv), we have $M^{*\sigma} \subset Cl^{\sigma}(M)$ implies $M \cup M^{*\sigma} \subset M \cup Cl^{\sigma}(M) = Cl^{\sigma}(M)$ implies $Cl^{*\sigma}(M) \subset Cl^{\sigma}(M)$. Hence $\gamma \subset \gamma^*(J)$. □

The member of $\gamma^*(J)$ is called $\sigma^*(J)$-open set or simply σ^*-open set and the complement of $\sigma^*(J)$-open set is called $\sigma^*(J)$-closed set or simply σ^*-closed set.

Result 3. *Let (T, γ) be a σ-topological space. If $J = \{\emptyset\}$, then $\gamma = \gamma^*(J)$.*

Proof. If $p \in Cl^{\sigma}(M)$, then (by Theorem 1 (i)), $V_p \cap M \neq \emptyset$, for all $V_p \in \gamma(p)$. This implies $V_p \cap M \notin \{\emptyset\} = J$, for all $V_p \in \gamma(p)$ implies $p \in M^{*\sigma}$ implies $p \in M \cup M^{*\sigma} = Cl^{*\sigma}(M)$. Since p is an arbitrary member of $Cl^{\sigma}(M)$, then $Cl^{\sigma}(M) \subset Cl^{*\sigma}(M)$. Also by Theorem 3 (xiv), $M^{*\sigma} \subset Cl^{\sigma}(M)$. This implies $M \cup M^{*\sigma} \subset M \cup Cl^{\sigma}(M)$ implies $Cl^{*\sigma}(M) \subset Cl^{\sigma}(M)$. Hence $Cl^{*\sigma}(M) = Cl^{\sigma}(M)$, for all $M \in \wp(T)$. Consequently $\gamma^*(J) = \gamma$ implies $\gamma = \gamma^*(\{\emptyset\})$. □

Theorem 5. *Let (T, γ) be a σ-topological space and J_1, J_2 be two ideals on T. If $J_1 \subset J_2$, then $\gamma^*(J_1) \subset \gamma^*(J_2)$.*

Proof. Let $O \in \gamma^*(J_1)$. Then $Cl^{*\sigma}_{J_1}(O^c) = O^c \Rightarrow O^c \cup O^{c^{*\sigma}}(J_1) = O^c$. This implies $O^{c^{*\sigma}}(J_1) \subset O^c$ implies $O^{c^{*\sigma}}(J_2) \subset O^{c^{*\sigma}}(J_1) \subset O^c$ (by Theorem 3 (v)). This implies $O^{c^{*\sigma}}(J_2) \cup O^c = O^c$ implies $Cl^{*\sigma}_{J_2}(O^c) = O^c$ implies $O \in \gamma^*(J_2)$. Since $O \in \gamma^*(J_1)$ is arbitrary, then $\gamma^*(J_1) \subset \gamma^*(J_2)$. □

Theorem 6. *Let (T, γ) be a σ-topological space with J an ideal on T. Then*
(i) $I \in J$ implies $I^c \in \gamma^*(J)$.
(ii) $M^{*\sigma} = Cl^{*\sigma}(M^{*\sigma})$, for all $M \in \wp(T)$.

Proof. : (i) We have for all $I \in J$, $(M \cup I)^{*\sigma} = M^{*\sigma}$. If we take $M = \emptyset$, then $I^{*\sigma} = \emptyset^{*\sigma} = \emptyset$, for all $I \in J$. Hence $Cl^{*\sigma}(I) = I \cup I^{*\sigma} = I \cup \emptyset = I$. Therefore $I^c \in \sigma^*(J)$. This implies I is $\gamma^*(J)$-closed, for all $I \in J$.
(ii) We have $(M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$. This implies $M^{*\sigma} = M^{*\sigma} \cup (M^{*\sigma})^{*\sigma} = Cl^{*\sigma}(M^{*\sigma})$. Hence $M^{*\sigma}$ is a $\sigma^*(J)$-closed. □

Theorem 7. *Let (T, γ) be a σ-topological space and $M \subset T$. Then M is σ^*-closed if and only if $M^{*\sigma} \subset M$.*

Proof. If M is σ^*-closed, then $M = Cl^{*\sigma}(M) = M \cup M^{*\sigma}$. This implies $M^{*\sigma} \subset M$. Conversely let $M^{*\sigma} \subset M$. This implies $M = M \cup M^{*\sigma} = Cl^{*\sigma}(M)$. Hence M is σ^*-closed. □

Theorem 8. *Let (T, γ_1) and (T, γ_2) be two σ-topological spaces and J be an ideal on T. Then $\gamma_1 \subset \gamma_2$ implies $M^{*\sigma}(J, \gamma_2) \subset M^{*\sigma}(J, \gamma_1)$.*

Proof. Let $p \in M^{*\sigma}(J, \gamma_2)$. This implies $V_p \cap M \notin J$ for all $V_p \in \gamma_2(p)$ implies $V_p \cap M \notin J$ for all $V_p \in \gamma_1(p)$. This implies $p \in M^{*\sigma}(J, \gamma_1)$. Since p is an arbitrary element of $M^{*\sigma}(J, \gamma_2)$, then $M^{*\sigma}(J, \gamma_2) \subset M^{*\sigma}(J, \gamma_1)$. □

Theorem 9. *Let (T,γ) be a σ-topological space and J be an ideal on T. Then the collection $\beta(J,\gamma)=\{M\setminus I : M\in\gamma, I\in J\}$ is a basis for the σ-topology $\gamma^*(J)$.*

Proof. Let $M\in\gamma^*(J)$ and $p\in M$. Then M^c is σ^*-closed, i.e, $Cl^{*\sigma}(M^c)=M^c$ and hence $M^c\cup(M^c)^{*\sigma}=M^c$ implies $(M^c)^{*\sigma}\subset M^c$. This implies $p\notin(M^c)^{*\sigma}$ and there exists $V_p\in\gamma(p)$ such that $V_p\cap M^c\in J$. Take $K=V_p\cap M^c$, then $p\notin K$ and $K\in J$. Thus $p\in V_p\setminus K=V_p\cap K^c=V_p\cap(V_p\cap M^c)^c=V_p\cap(V_p^c\cup M)=(V_p\cap V_p^c)\cup(V_p\cup M)=V_p\cap M\subset M$. Hence $p\in V_p\setminus K\subset M$, where $V_p\setminus K\in\beta(J,\gamma)$. Thus $\beta(J,\gamma)$ is an open base of $\gamma^*(J)$. □

The example given below proves that $M^{*\sigma}\cap N^{*\sigma}=(M\cap N)^{*\sigma}$ is not satisfied in general.

Example 2. *Let T=$\{p,q,r,s\}$, γ=$\{\emptyset,T,\{p\},\{s\},\{p,s\},\{q,s\},\{r,s\},\{p,r,s\},\{p,q,s\},\{q,r,s\}\}$, $J=\{\emptyset,\{p\}\}$. Then σ-open sets containing p are: T, $\{p\}$, $\{p,s\}$, $\{p,r,s\}$, $\{p,q,s\}$; σ-open sets containing q are: T, $\{q,s\}$, $\{p,q,s\}$, $\{q,r,s\}$; σ-open sets containing r are: T, $\{r,s\}$, $\{p,r,s\}$, $\{q,r,s\}$; σ-open sets containing s are: T, $\{s\}$, $\{p,s\}$, $\{q,s\}$, $\{r,s\}$, $\{p,q,s\}$, $\{p,r,s\}$, $\{q,r,s\}$. Take M=$\{p,q\}$ and N=$\{p,s\}$. Then $M^{*\sigma}=\{q\}$ and $N^{*\sigma}=\{q,r,s\}$ and hence $M^{*\sigma}\cap N^{*\sigma}=\{q\}$. Now $(M\cap N)^{*\sigma}=\{p\}^{*\sigma}=\emptyset$ and so $M^{*\sigma}\cap N^{*\sigma}\neq(M\cap N)^{*\sigma}$.*

3. ψ_σ-Operator

In this part, we have introduced another set operator ψ_σ in (T,γ,J). This operator is as like similar of ψ-operator [5,10], in ideal topological space. Definition of ψ_σ-operator is given below:

Definition 7. *Let (T,γ,J) be an ideal σ-topological space. An operator $\psi_\sigma:\wp(T)\to\gamma$ is defined as follows:*

for every $M\in\wp(T)$, $\psi_\sigma(M)$=$\{p\in T$: there exists a $V\in\gamma(p)$ such that $V\setminus M\in J\}$.

Observe that $(T\setminus M)^{*\sigma}=\{p\in T : V\cap(T\setminus M)\notin J \ for\ every\ V\in\gamma(p)\}$.
This implies

$$\begin{aligned} T\setminus(T\setminus M)^{*\sigma} &= T\setminus\{p\in T : V\cap(T\setminus M)\notin J\ for\ every\ V\in\gamma(p)\}\\ &=\{p\in T:\exists V\in\gamma(p)\ such\ that\ V\cap(T\setminus M)\in J\}\\ &=\{p\in T:\exists V\in\gamma(p)\ such\ that\ V\setminus M\in J\}\\ &=\psi_\sigma(M)\end{aligned}$$

Hence $\psi_\sigma(M)=T\setminus(T\setminus M)^{*\sigma}$.

Here we have to find out the value of $\psi_\sigma(M)$ of a set in σ-topological space.

Example 3. *Let T=$\{p,q,r,\}$, γ=$\{\emptyset,T,\{r\},\{p,r\},\{q,r\}\}$, $J=\{\emptyset,\{r\}\}$. Then for M=$\{p,q\}$, $\psi_\sigma(M)=T\setminus(T\setminus M)^{*\sigma}=T\setminus\{r\}^{*\sigma}=T\setminus\emptyset=T$.*

The characteristics of the operator ψ_σ has been studied in the following results:

Theorem 10. *Let (T,γ,J) be an ideal σ-topological space. Then the following properties hold:*
(i) If $M \subset N$, then $\psi_\sigma(M) \subset \psi_\sigma(N)$.
(ii) If $M, N \in \wp(T)$, then $\psi_\sigma(M) \cup \psi_\sigma(N) \subset \psi_\sigma(M \cup N)$.
(iii) If $M, N \in \wp(T)$, then $\psi_\sigma(M) \cap \psi_\sigma(N) = \psi_\sigma(M \cap N)$.
(iv) If $M \subset T$, then $\psi_\sigma(M) = \psi_\sigma(\psi_\sigma(M))$ if and only if $(T \setminus M)^{*\sigma} \subset ((T \setminus M)^{*\sigma})^{*\sigma}$.
(v) If $M \in J$, then $\psi_\sigma(M) = T \setminus T^{*\sigma}$.
(vi) If $M \subset T, J_1 \in J$, then $\psi_\sigma(M \setminus J_1) = \psi_\sigma(M)$.
(vii) If $M \subset T, J_1 \in J$, then $\psi_\sigma(M \cup J_1) = \psi_\sigma(M)$.
(viii) If $V \in \gamma$, then $V \subset \psi_\sigma(V)$.
(ix) If $(M \setminus N) \cup (N \setminus M) \in J$, then $\psi_\sigma(M) = \psi_\sigma(N)$.
(x) $Int^{\sigma^*}(M) = M \cap \psi_\sigma(M)$.

Proof. (i) $M \subset N$ implies $(T \setminus M) \supset (T \setminus N)$. This implies $(T \setminus M)^{*\sigma} \supset (T \setminus N)^{*\sigma}$ (by Theorem 3 (iii)). This implies $T \setminus (T \setminus M)^{*\sigma} \subset T \setminus (T \setminus N)^{*\sigma}$. Hence $\psi_\sigma(M) \subset \psi_\sigma(N)$.
(ii) We know $M \subset M \cup N$ and $N \subset M \cup N$. This implies $\psi_\sigma(M) \subset \psi_\sigma(M \cup N)$ and $\psi_\sigma(N) \subset \psi_\sigma(M \cup N)$ (by Theorem 10 (i)). Hence $\psi_\sigma(M) \cup \psi_\sigma(N) \subset \psi_\sigma(M \cup N)$.
(iii) Since $M \cap N \subset M$ and $M \cap N \subset N$. This implies $\psi_\sigma(M \cap N) \subset \psi_\sigma(M)$ and $\psi_\sigma(M \cap N) \subset \psi_\sigma(N)$ (by Theorem 10 (i)). Hence $\psi_\sigma(M \cap N) \subset \psi_\sigma(M) \cap \psi_\sigma(N)$.

For reverse inclusion let $p \in \psi_\sigma(M) \cap \psi_\sigma(N)$. Then $p \in \psi_\sigma(M)$ and $p \in \psi_\sigma(N)$. Then there exist $V, O \in \gamma(p)$ such that $V \setminus M \in J$ and $O \setminus N \in J$. This implies $(V \setminus M) \cup (O \setminus N) \in J$, since J is finite additive. Now

$$\begin{aligned}(V \setminus M) \cup (O \setminus N) &= [(V \cap M^c) \cup O] \cap [(V \cap M^c) \cup N^c] \\ &= (V \cup O) \cap (M^c \cup O) \cap (V \cup N^c) \cap (M^c \cup N^c) \\ &\supset (V \cap O) \cap (M^c \cup N^c) \\ &= (V \cap O) \setminus (M \cap N)\end{aligned}$$

This implies $(V \cap O) \setminus (M \cap N) \in J$, since J is heredity. Since $V \cap O \in \gamma(p)$ then $p \in \psi_\sigma(M \cap N)$. Thus $\psi_\sigma(M) \cap \psi_\sigma(N) \subset \psi_\sigma(M \cap N)$. Hence we obtain $\psi_\sigma(M) \cap \psi_\sigma(N) = \psi_\sigma(M \cap N)$.
(iv) Let $\psi_\sigma(M) = \psi_\sigma(\psi_\sigma(M))$. Then $T \setminus (T \setminus M)^{*\sigma} = T \setminus [T \setminus \psi_\sigma(M)]^{*\sigma} = T \setminus [T \setminus \{T \setminus (T \setminus \psi_\sigma(M))\}]^{*\sigma}$. This implies $(T \setminus M)^{*\sigma} = ((T \setminus M)^{*\sigma})^{*\sigma}$.

Conversely, suppose that $(T \setminus M)^{*\sigma} = ((T \setminus M)^{*\sigma})^{*\sigma}$ holds. Then $T \setminus (T \setminus M)^{*\sigma} = T \setminus ((T \setminus M)^{*\sigma})^{*\sigma} = T \setminus [T \setminus \{T \setminus (T \setminus \psi_\sigma(M))\}]^{*\sigma}$. This implies $\psi_\sigma(M) = T \setminus (T \setminus \psi_\sigma(M))^{*\sigma} = \psi_\sigma(\psi_\sigma(M))$.

(v) We have $\psi_\sigma(M) = T \setminus (T \setminus M)^{*\sigma} = T \setminus T^{*\sigma}$ (by Theorem 3 (xi)).

(vi) We have $T \setminus [T \setminus (M \setminus J_1)]^{*\sigma} = T \setminus [T \setminus (M \cap J_1^c)]^{*\sigma} = T \setminus [T \cap (M^c \cup J_1)]^{*\sigma} = T \setminus [(T \cap M^c) \cup (T \cap J_1)]^{*\sigma} = T \setminus [(T \setminus M) \cup J_1]^{*\sigma} = T \setminus (T \setminus M)^{*\sigma}$ (by Theorem 3 (xi)). So $\psi_\sigma(M \setminus J_1) = \psi_\sigma(M)$.

(vii) We have $T\setminus[T\setminus(M\cup J_1)]^{*\sigma} = T\setminus[T\cap(M^c\cap J_1^c)]^{*\sigma} = T\setminus[(T\setminus M)\setminus J_1]^{*\sigma} = T\setminus(T\setminus M)^{*\sigma}$ (by Theorem 3 (xi)). So $\psi_\sigma(M\cup J_1) = \psi_\sigma(M)$.

(viii) Let $p\in V$. Then $p\notin T\setminus V$ and hence $V\cap(T\setminus V)=\emptyset\in J$. Thus $p\notin(T\setminus V)^{*\sigma}$. This implies $p\in T\setminus(T\setminus V)^{*\sigma}$ and hence $p\in\psi_\sigma(V)$. So $V\subset\psi_\sigma(V)$.

(ix) Let $J_1 = M\setminus N$ and $J_2 = N\setminus M$. Since $J_1\cup J_2\in J$, then by heredity $J_1, J_2\in J$. Also $N = (M\setminus J_1)\cup J_2$. This implies $\psi_\sigma(N) = \psi_\sigma((M\setminus J_1)\cup J_2)$. So $\psi_\sigma(N) = \psi_\sigma((M\setminus J_1)$ and hence $\psi_\sigma(N) = \psi_\sigma(M)$, (by Theorem 10 (vi) and (vii)).

(x) Let $p\in M\cap\psi_\sigma(M)$. Then $p\in M$ and $p\in\psi_\sigma(M)$. Thus $p\in M$ and there exists a $V_p\in\gamma(p)$ such that $V_p\setminus M\in J$ implies $V_p\setminus(V_p\setminus M)$ is a σ^*-open neighborhood of p and hence $p\in Int^{\sigma^*}(M)$. Hence $M\cap\psi_\sigma(M)\subset Int^{\sigma^*}(M)$. Again, if $p\in Int^{\sigma^*}(M)$, then there exists a σ^*-open neighborhood $V_p\setminus I$ of p where $V_p\in\gamma$ and $I\in J$ such that $p\in V_p\setminus I\subset M$ which implies $V_p\setminus M\subset I$ and $V_p\setminus M\in J$. Hence $p\in M\cap\psi_\sigma(M)$. Hence $Int^{\sigma^*}(M) = M\cap\psi_\sigma(M)$. □

Note 2. *We have $V\subset\psi_\sigma(V)$, for every $V\in\gamma$. But there exists a set M which is not σ-open set but satisfies $M\subset\psi_\sigma(M)$.*

Example 4. *Let T=$\{p,q,r,\}$, γ=$\{\emptyset,T,\{r\},\{p,r\},\{q,r\}\}$, $J=\{\emptyset,\{r\}\}$. Then for M=$\{p,q\}$, $\psi_\sigma(M) = T\setminus(T\setminus M)^{*\sigma} = T\setminus\{r\}^{*\sigma} = T\setminus\emptyset = T$. Thus $M\subset\psi_\sigma(M)$ but M is not a σ-open set.*

The example given below shows that $\psi_\sigma(M)\cup\psi_\sigma(N) = \psi_\sigma(M\cup N)$ does not hold in general.

Example 5. *In Example 2 we consider $M=\{r,s\}$ and $N=\{q,r\}$. Then $\psi_\sigma(M) = T\setminus\{p,q\}^{*\sigma} = T\setminus\{q\} = \{p,r,s\}$ and $\psi_\sigma(N) = T\setminus\{p,s\}^{*\sigma} = T\setminus\{q,r,s\} = \{p\}$. Therefore $\psi_\sigma(M)\cup\psi_\sigma(N) = \{p,r,s\}$ and $\psi_\sigma(M\cup N) = T\setminus\{p\}^{*\sigma} = T\setminus\emptyset = T$. Hence $\psi_\sigma(M)\cup\psi_\sigma(N)\neq\psi_\sigma(M\cup N)$.*

Definition 8. *Let γ be a σ-topological space on a non empty set T. If an ideal J satisfies the property $\gamma\cap J=\{\emptyset\}$ then the ideal J is called σ-codense ideal.*

Theorem 11. *Let (T,γ,J) be an ideal σ-topological space. Then the properties given below are equivalent.*
(i) $\gamma\cap J=\{\emptyset\}$.
(ii) $\psi_\sigma(\emptyset)=\emptyset$.
(iii) If $J_1\in J$ then $\psi_\sigma(J_1)=\emptyset$.

Proof. $(i)\Rightarrow(ii)$: Let $\gamma\cap J=\{\emptyset\}$. Then $T=T^{*\sigma}$. Then $\psi_\sigma(\emptyset) = T\setminus(T\setminus\emptyset)^{*\sigma} = T\setminus T^{*\sigma} = \emptyset$.

$(ii)\Rightarrow(iii)$: Let $\psi_\sigma(\emptyset)=\emptyset$ holds. Then $\psi_\sigma(J_1) = T\setminus(T\setminus J_1)^{*\sigma} = T\setminus T^{*\sigma}$ (by Theorem 3 (xi)) $= T\setminus(T\setminus\emptyset)^{*\sigma} = \psi_\sigma(\emptyset)=\emptyset$.

$(iii)\Rightarrow(i)$: Let $J_1\in J$ be such that $\psi_\sigma(J_1)=\emptyset$. Now $\psi_\sigma(J_1)=\emptyset$ implies $T\setminus(T\setminus J_1)^{*\sigma}=\emptyset$. This implies $T\setminus T^{*\sigma}=\emptyset$, since $J_1\in J$ (by Theorem 3 (xi)). Thus $T=T^{*\sigma}$. Hence $\gamma\cap J=\{\emptyset\}$. □

4. σ-Compatible Ideal

In this section, we have studied a particular type of ideal and its several features. This ideal is as like similar of μ-compatible ideal [9] on supra topological space. This particular type of ideal is:

Definition 9. *Let (T,γ,J) be an ideal σ-topological space. We say the σ-structure is σ-compatible with the ideal J denoted $\gamma \sim J$, if the condition holds: for every $M \subset T$, if for all $p \in M$, there exists $V \in \gamma(p)$ such that $V \cap M \in J$, then $M \in J$.*

Theorem 12. *Let (T,γ,J) be an ideal σ-topological space. Then $\gamma \sim J$ if and only if $\psi_\sigma(M) \setminus M \in J$ for every $M \subset T$.*

Proof. Suppose $\gamma \sim J$. Observe that $p \in \psi_\sigma(M) \setminus M$ if and only if $p \notin M$ and there exists $V_p \in \gamma(p)$ such that $V_p \setminus M \in J$. Now for each $p \in \psi_\sigma(M) \setminus M$ and $V_p \in \gamma(p)$, $V_p \cap (\psi_\sigma(M) \setminus M) \in J$ (by heredity) and hence $(\psi_\sigma(M) \setminus M) \in J$, since $\gamma \sim J$.
Conversely, suppose the given condition holds and $M \subset T$ and assume that for each $p \in M$, there exists $V_p \in \gamma(p)$ such that $V_p \cap M \in J$. Observe that $\psi_\sigma(T \setminus M) \setminus (T \setminus M) = M \setminus M^{*\sigma}$=$\{p \in T$: there exists $V_p \in \gamma(p)$ such that $p \in V_p \cap M \in J\}$. Thus we have $M \subset \psi_\sigma(T \setminus M) \setminus (T \setminus M) \in J$ and hence $M \in J$, by heredity of J. □

Example 6. *Let T=$\{p,q\}$, γ=$\{\emptyset,T,\{p\},\{q\}\}$, $J = \{\emptyset, \{p\}\}$. Then $\emptyset^{*\sigma} = \emptyset$, $\{p\}^{*\sigma} = \emptyset$, $\{q\}^{*\sigma} = \{q\}$ and $\{T\}^{*\sigma} = \{q\}$. Then $\psi_\sigma(\emptyset) = T \setminus T^{*\sigma} = \{p,q\} \setminus \{q\} = \{p\}$, $\psi_\sigma(\{p\}) = T \setminus (T \setminus \{p\})^{*\sigma} = T \setminus \{q\}^{*\sigma} = T \setminus \{q\} = \{p\}$, $\psi_\sigma(\{q\}) = T \setminus (T \setminus \{q\})^{*\sigma} = T \setminus \{p\}^{*\sigma} = T \setminus \emptyset = T$, $\psi_\sigma(T) = T \setminus \emptyset^{*\sigma} = T \setminus \emptyset = T$. Then we see that $\psi_\sigma(\emptyset) \setminus \emptyset = \{p\} \in J$, $\psi_\sigma(\{q\}) \setminus \{q\} = T \setminus \{q\} = \{p\} \in J$, $\psi_\sigma(\{p\}) \setminus \{p\} = \{p\} \setminus \{p\} = \emptyset \in J$ and $\psi_\sigma(T) \setminus T = T \setminus T = \emptyset \in J$. So $\gamma \sim J$.*

Corollary 1. *Let (T,γ,J) be an ideal σ-topological space with $\gamma \sim J$. Then $\psi_\sigma(\psi_\sigma(M)) = \psi_\sigma(M)$ for every $M \subset T$.*

Proof. We know $\psi_\sigma(M) \subset \psi_\sigma(\psi_\sigma(M))$. Also since $\gamma \sim J$, then for every $M \subset T$, $\psi_\sigma(M) \setminus M \in J$. This implies $\psi_\sigma(M) \setminus M = J_1$ for some $J_1 \in J$. This implies $\psi_\sigma(M) \subset M \cup J_1$. Then $\psi_\sigma(\psi_\sigma(M)) \subset \psi_\sigma(M \cup J_1) = \psi_\sigma(M)$. Thus $\psi_\sigma(\psi_\sigma(M)) = \psi_\sigma(M)$. □

Example 7. *Consider T=$\{p,q\}$, γ=$\{\emptyset,T,\{p\},\{q\}\}$ and $J = \{\emptyset, \{p\}\}$. Then by Example 6, $\gamma \sim J$ and $\psi_\sigma(\psi_\sigma(\phi)) = \psi_\sigma(\emptyset)$, $\psi_\sigma(\psi_\sigma(\{p\})) = \psi_\sigma(\{p\})$, $\psi_\sigma(\psi_\sigma(\{q\})) = \psi_\sigma(T) = T = \psi_\sigma(\{q\})$ and $\psi_\sigma(\psi_\sigma(T)) = \psi_\sigma(T)$*

Newcomb in [11] has defined $M = N$ $(mod\ J)$, if $(M \setminus N) \cup (N \setminus M) \in J$. Further, he studied several characteristics of $M = N$ $(mod\ J)$. Here we shall observe that if $M = N$ $(mod\ J)$ then $\psi_\sigma(M) = \psi_\sigma(N)$.
Now we define Baire set in (T,γ,J).

Definition 10. *Let (T, γ, J) be an ideal σ-topological space. A subset M of T is called a Baire set with respect to γ and J denoted by $M \in \boldsymbol{B}_r(T, \gamma, J)$, if there exists a σ-open set $V \in \gamma$ such that $M = V$ $(mod\ J)$.*

Theorem 13. *Let (T, γ, J) be an ideal σ-topological space with $\gamma \sim J$. If $V \cup O \in \gamma$ and $\psi_\sigma(V) = \psi_\sigma(O)$, then $V = O$ $(mod\ J)$.*

Proof. $V \in \gamma$ implies $V \subset \psi_\sigma(V)$ and hence $V \setminus O \subset \psi_\sigma(V) \setminus O = \psi_\sigma(O) \setminus O \in J$. By heredity of J, $V \setminus O \in J$. Similarly, $O \setminus V \in J$. Then $(V \setminus O) \cup (O \setminus V) \in J$, by finite additivity of J. So $V = O$ $(mod\ J)$. □

Clearly, $M = N$ $(mod\ J)$ is an equivalence relation. In this favour, following theorem is observable:

Theorem 14. *Let (T, γ, J) be an ideal σ-topological space with $\gamma \sim J$. If $M, N \in \boldsymbol{B}_r(T, \gamma, J)$ and $\psi_\sigma(M) = \psi_\sigma(N)$. Then $M = N$ $(mod\ J)$.*

Proof. Let $V, O \in \gamma$ such that $M = V$ $(mod\ J)$ and $N = O$ $(mod\ J)$. Now $\psi_\sigma(M) = \psi_\sigma(N)$ and $\psi_\sigma(N) = \psi_\sigma(O)$ (by Theorem 10 (ix)). Since $\psi_\sigma(M) = \psi_\sigma(V)$ implies that $\psi_\sigma(V) = \psi_\sigma(O)$, hence $V = O$ $(mod\ J)$ (by Theorem 13). Hence $M = N$ $(mod\ J)$, by transitivity. □

Theorem 15. *Let (T, γ, J) be an ideal σ-topological space.*
(i) If $N \in \boldsymbol{B}_r(T, \gamma, J) \setminus J$, then there exists $M \in \gamma \setminus \{\emptyset\}$ such that $N = M$ $(mod\ J)$.
(ii) Let $\gamma \cap J = \{\emptyset\}$, then $N \in \boldsymbol{B}_r(T, \gamma, J) \setminus J$ if and only if there exists $M \in \gamma \setminus \{\emptyset\}$ such that $N = M$ $(mod\ J)$.

Proof. (i) Let $N \in \mathbf{B}_r(T, \gamma, J) \setminus J$, then $N \in \mathbf{B}_r(T, \gamma, J)$. Now if there does not exist $M \in \gamma \setminus \{\emptyset\}$ such that $N = M$ $(mod\ J)$, we have $N = \emptyset$ $(mod\ J)$. This implies $N \in J$, which is a contradiction.
(ii) Here we shall prove converse part only. Let $M \in \gamma \setminus \{\emptyset\}$ such that $N = M$ $(mod\ J)$. Then $M = (N \setminus J_2) \cup J_1$, where $J_2 = N \setminus M$, $J_1 = M \setminus N$ both belong to J. If $N \in J$, then $M \in J$, by heredity and additivity, which contradicts the fact $\gamma \cap J = \{\emptyset\}$. □

5. $\psi_\sigma - C$ SETS

Modak and Bandyopadhyay in [8] have introduced a generalized set with the help of ψ-operator in ideal topological space. In this part, we have studied a set with the help of ψ_σ-operator in (T, γ, J) space. Further, we have studied the properties of this type of sets.

Definition 11. *Let (T, γ, J) be an ideal σ-topological space. A subset M of T is called a $\psi_\sigma - C$ sets, if $M \subset Cl^\sigma(\psi_\sigma(M))$.*
The family of all ψ_σ-C sets in (T, γ, J) is denoted by $\psi_\sigma(T, \gamma)$.

Theorem 16. *Let (T, γ, J) be an ideal σ-topological space. If $M \in \gamma$ then $M \in \psi_\sigma(T, \gamma)$.*

Proof. By Theorem 10 (viii), it follows that $\gamma \subset \psi_\sigma(T, \gamma)$. □

Now by the given example we are to show that the reverse inclusion is not true:

Example 8. *From Example 4 we get $M \in \psi_\sigma(T, \gamma)$ but $M \notin \gamma$.*

By the following example, we are to show that any σ-closed in (T, γ, J) may not be a $\psi_\sigma - C$ set.

In the following example, by $C^\sigma(\gamma)$ we denote the family of all σ-closed sets in (T, γ).

Example 9. *We consider Example 2. Here $M = \{q\} \in C^\sigma(\gamma)$. Then $\psi_\sigma(M) = T \setminus (T \setminus M)^{*\sigma} = T \setminus \{p, r, s\}^{*\sigma} = T \setminus \{q, r, s\} = \{p\}$. Hence $Cl^\sigma(\psi_\sigma(M)) = \cap\{C : \psi_\sigma(M) \subset C, T \setminus C \in \gamma\} = \{p\}$. Therefore $M \in C^\sigma(\gamma)$ but $M \notin \psi_\sigma(T, \gamma)$.*

Theorem 17. *Let $\{M_\alpha : \alpha \in \Delta\}$ be a family of non-empty $\psi_\sigma - C$ sets in an ideal σ-topological space (T, γ, J), then $\bigcup_{\alpha \in \Delta} \in \psi_\sigma(T, \gamma)$.*

Proof. For each $\alpha \in \Delta$, $M_\alpha \subset Cl^\sigma(\psi_\sigma(M_\alpha)) \subset Cl^\sigma(\psi_\sigma(\bigcup_{\alpha \in \Delta} M_\alpha))$. This implies that $\bigcup_{\alpha \in \Delta} M_\alpha \subset Cl^\sigma(\psi_\sigma(\bigcup_{\alpha \in \Delta} M_\alpha))$. Thus $\bigcup_{\alpha \in \Delta} M_\alpha \in \psi_\sigma(T, \gamma)$. □

6. Conclusion

In this writeup, we have introduced a new topology called σ-topology and defined ideals on that spaces. Using this idea, we have discussed relationship of various operators namely $()^{*\sigma}$ operator, ψ_σ-operator. The result of this writeup can be extended to σ-connected sets, σ-compact sets, σ-paracompact sets. The separation axioms can also be introduced in this space. The other properties of ψ_σ-sets can be found and one can introduce some operators on this type of sets to the development of mathematical knowledge.

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References

1. Al-Omari, A., Noiri, T., On ψ_*-operator in ideal m-spaces, *Bol. Soc. Paran. Mat.*, (3s.), 30(1) (2012), 53-66. https://doi.org/10.5269/bspm.v30i1.12787
2. Al-Omari, A., Noiri, T., Corrigendum to on ψ_*-operator in ideal m-spaces, *Bol. Soc. Paran. Mat.,* (3s.), 30(2) (2012), 119-119. https://doi.org/10.5269/bspm.v30i2.14925
3. Al-Omari, A., Noiri, T., On operators in ideal minimal spaces, *Mathematica,* 58(81)(1-2) (2016), 3-13.
4. Al-Saadi, H., Al-Omari, A., Some operators in ideal topological spaces, *Missouri J. Math. Sci.,* 30(1) (2018), 59-71. https://doi.org/ 10.35834/mjms/1534384955
5. Hamlett, T.R., Janković, D., Ideals in topological spaces and set operator ψ, *Bull. U.M.I.,* 7(4-B) (1990), 863-874.
6. Janković, D., Hamlett, T.R., New topologies from old via ideals, *Amer. Math. Monthly,* 97 (1990), 295-300. https://doi.org/10.1080/00029890.1990.11995593
7. Kuratowski, K., Topology, Vol. 1, Academic Press, New York, 1966.
8. Modak, S., Bandhyopadhyay, C., A note on ψ-operator, *Bull. Malyas. Math. Sci. Soc.,* 30(1) (2007), 43-48.
9. Modak, S., Mistry, S., Ideal on supra topological space, *Int. Journal of Math. Analysis,* 6(1) (2012), 1-10.
10. Natkaniec, T., On I-continuity and I-semicontinuity points, *Math. Slovaca,* 36(3) (1986), 297-312.
11. Newcomb, R.L., Topologies Which are Compact Codulo an Ideal, Ph.D. Dissertation, Univ. of Cal. at Santa Barbara, 1967.
12. Njåstad, O., On some classes of nearly open sets, *Pacific J. Math.,* 15(3) (1965), 961-970. https://doi.org/10.2140/pjm.1965.15.961
13. Vaidyanathaswamy, R., Set Topology, Chelsea Publishing Co., New York, 1960.

http://communications.science.ankara.edu.tr

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THE MINKOWSKI TYPE INEQUALITIES FOR WEIGHTED FRACTIONAL OPERATORS

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Abstract. In this article, inequalities of reverse Minkowski type involving weighted fractional operators are investigated. In addition, new fractional integral inequalities related to Minkowski type are also established.

1. Introduction

Fractional analysis has drawn attention highly because of its aplications in different areas. Researchers focus on developing different fractional operators in the development of fractional analysis. These different fractional operators are also used in integral inequalities. Hence fractional analysis plays an important role in the development of inequality theory. One of the most useful fractional integral operator is Riemann-liouville fractional integral operator. Scientist who suggested that Riemann-Liouville fractional operator can be used in fractional analysis is Joseph Liouville ([20]). Then, several researchers studied these operators with different inequalities and thus introduced the notion of fractional conformable integrals. In [1], Abdeljawad presented the properties of the conformable fractional operators. Also, in [18], Khan et al. investigated fractional conformable derivatives operators. Similarly, several mathematicians have been interested in and studied conformable fractional operators ([17], [29]). In [15], Katugampola defined a new fractional derivative operator. Also, Katugampola developed a new approach to generalized fractional derivatives. Based on these operators, new theorems were proved by the researchers.

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In [1]- [32], some researchers used new fractional derivative or integral operators such as Riemann-Liouville, Caputo, Hadamard and Katugampola types. Many new results are obtained for functions in $L_p[a,b]$ which is defined as follows:

Definition 1. *For $p \in [1,\infty)$, if the function $\wp$ holds the following inequality*

$$\left(\int_a^b |\wp(\tau)|^p \, d\tau\right)^{\frac{1}{p}} < \infty,$$

then it is said to be in $L_p[a,b]$.

In the mathematical literature the Minkowski's inequality, which is very well known in the literature, has been stated as follows (see [12]):

Theorem 1. *$\int_a^b \wp^p(\tau)d\tau$ and $\int_a^b \hbar^p(\tau)d\tau$ are positive finite reals for $p \geq 1$. Then the inequality*

$$\left(\int_a^b (\wp(\tau)+\hbar(\tau))^p d\tau\right)^{\frac{1}{p}} \leq \left(\int_a^b \wp^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \hbar^p(\tau)d\tau\right)^{\frac{1}{p}}$$

holds.

The reverse Minkowski inequality for classical Riemann integrals is obtained by L. Bougoffa in [5] which is given as the following:

Theorem 2. *Let $\wp, \hbar \in L_p[a,b]$ be two positive functions, with $1 \leq p < \infty$, $0 < \int_a^b \wp^p(\tau)d\tau < \infty$ and $0 < \int_a^b \hbar^p(\tau)d\tau < \infty$. If $0 \leq n \leq \frac{\wp(\tau)}{\hbar(\tau)} \leq N$ for $n, N \in \mathbb{R}^+$ and every $\tau \in [a,b]$, then the inequality*

$$\left(\int_a^b \wp^p(\tau)d\tau\right)^{\frac{1}{p}} + \left(\int_a^b \hbar^p(\tau)d\tau\right)^{\frac{1}{p}} \leq c\left(\int_a^b (\wp(\tau)+\hbar(\tau))^p d\tau\right)^{\frac{1}{p}}$$

holds where $c = \frac{N(n+1)+(N+1)}{(n+1)(N+1)}$.

The following theorem is called "Young's inequality" (see [22]):

Theorem 3. *Let $[0,k]$ where $k>0$ be an interval and h be a function which is increasing and continuous on $[0,k]$. If $b \in [0,\hbar(k)]$, $a \in [0,k]$, $\hbar(0)=0$ and $\hbar^{-1}$ stands for the inverse function of h, then*

$$\int_0^a \hbar(\tau)d\tau + \int_0^b \hbar^{-1}(\tau)d\tau \geq ab. \tag{1}$$

Example 1. *The function $\hbar : (0,c) \to \mathbb{R}$, $\hbar(\tau) = \tau^{r-1}$ satisfies the conditions mentioned in Theorem 3 for $r>1$. Applying $\hbar$ to (1) we have*

$$\frac{1}{r}a^r + \frac{1}{s}b^s \geq ab, \qquad a,b \geq 0,\ r \geq 1 \text{ and } \frac{1}{r}+\frac{1}{s}=1.$$

In other terms, this inequality puts forward the relation between arithmetic mean and geometric mean.

Definition 2. *([4]) Let $\hbar \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}\hbar$ and $J^{\alpha}_{b^-}\hbar$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}\hbar(\tau) = \frac{1}{\Gamma(\alpha)}\int_a^{\tau} (\tau-\theta)^{\alpha-1}\hbar(\theta)d\theta, \quad \tau > a$$

and

$$J^{\alpha}_{b^-}\hbar(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\tau}^{b} (\theta-\tau)^{\alpha-1}\hbar(\theta)d\theta, \quad \tau < b$$

respectively where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$. By choosing $\alpha = 0$ in above definitions, we get the function $\hbar$ itself. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 3. *([27])Let (a,b) be an infinite or finite interval on positive real axis and let $\hbar$ is defined on (a,b) with $\hbar \in L_p(a,b)$. Then for $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, definitions of the left-sided and right-sided Hadamard fractional integrals of order α of a real function $\hbar$ are given as*

$$H^{\alpha}_{a^+}\hbar(\tau) = \frac{1}{\Gamma(\alpha)}\int_a^{\tau} \left(\log\frac{\tau}{\theta}\right)^{\alpha-1}\frac{\hbar(\theta)}{\theta}d\theta, \quad a < \tau < b$$

and

$$H^{\alpha}_{b^-}\hbar(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\tau}^{b} \left(\log\frac{\theta}{\tau}\right)^{\alpha-1}\frac{\hbar(\theta)}{\theta}d\theta, \quad a < \tau < b,$$

respectively.

Definition 4. *([16]) Let $[a,b]$ be a finite interval and $\hbar \in X^p_c(a,b)$ be a real function. Then for $\alpha \in \mathbb{C}$, $\rho > 0$, $Re(\alpha) > 0$, the definitions of left-sided and right sided Katugampola fractional integrals of order α of $\hbar$ are given as*

$${}^{\rho}I^{\alpha}_{a^+}\hbar(\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\tau} \frac{\theta^{\rho-1}}{(\tau^{\rho}-\theta^{\rho})^{1-\alpha}}\hbar(\theta)d\theta, \quad \tau > a$$

and

$${}^{\rho}I^{\alpha}_{b^-}\hbar(\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau}^{b} \frac{\theta^{\rho-1}}{(\theta^{\rho}-\tau^{\rho})^{1-\alpha}}\hbar(\theta)d\theta, \quad \tau < b,$$

respectively.

As the use of fractional integral operators increased, it became necessary to obtain more general versions of the new results obtaibed. Thus, weighted integral operators began to be presented. While new results are obtained with these operators, general versions of the results in the literature can also be obtained. One of the most effective weighted integral operator presented recently is given in the following:

Definition 5. *([24]) Let $\phi(\tau)$ be a monotonic, positive and increasing function on the finite interval $[a,b]$ and continuously differentiable on (a,b) with $\phi(0)=0$, $0\in[a,b]$. Then for $w(\tau)\neq 0$ and $w^{-1}(\tau)=\frac{1}{w(\tau)}$, the definitions of the weighted fractional integrals of a function (the left-side and right-side respectively) $\hbar$ with respect to ϕ on $[a,b]$ are given as*

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar\right)(\tau) = \frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(\theta)\left[\phi(\tau)-\phi(\theta)\right]^{\ell-1}\hbar(\theta)w(\theta)d\theta, \tag{2}$$

$$\left({}_{w}\Im_{b-}^{\ell:\phi}\hbar\right)(\tau) = \frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_\tau^b \phi'(\theta)\left[\phi(\theta)-\phi(\tau)\right]^{\ell-1}\hbar(\theta)w(\theta)d\theta, \quad \ell>0. \tag{3}$$

The fractional integral operator given above is studied on in this study because it can give very efficient results in terms of application. Since by choosing $\phi(\tau)=\tau$ and $w(\theta)=1$, the weighted fractional integral operators ((2) and (3)) reduce to the classical Riemann–Liouville fractional integral operators and by choosing other special cases, many forms of fractional integral operators can be obtained.

Obtaining some new general forms of the Minkowski type inequalities using weighted fractional operators is the main aim of this study.

2. Reverse Minkowski Inequalities for Weighted Fractional Operators

Theorem 4. *Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$, $\ell>0$, $p\geq 1$. If $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ holds for $n,N\in\mathbb{R}^+$ and $t\in[a,\tau]$, then*

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)+\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau)\leq c_1\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau), \tag{4}$$

with $c_1=\frac{N(n+1)+(N+1)}{(n+1)(N+1)}$.

Proof. Under the given condition $\frac{\wp(t)}{\hbar(t)}\leq N$, $t\in[a,\tau]$, it can be written as

$$\wp(t)\leq N(\wp(t)+\hbar(t))-N\wp(t)$$

which implies that

$$(N+1)^p\wp^p(t)\leq N^p(\wp(t)+\hbar(t))^p. \tag{5}$$

Multiplying both sides of (5) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating with respect to t from a to τ, we have

$$\begin{aligned}&\frac{(N+1)^p w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp^p(t)w(t)dt\\ \leq\ &\frac{N^p w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}(\wp+\hbar)^p(t)w(t)dt.\end{aligned}$$

Consequently, we can write

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \leq \frac{N}{N+1}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau). \tag{6}$$

On the other hand, as $n\hbar(t) \leq \wp(t)$, it follows

$$\left(1+\frac{1}{n}\right)^p \hbar^p(t) \leq \left(\frac{1}{n}\right)^p (\wp(t)+\hbar(t))^p. \tag{7}$$

Next, multiplying both sides of (7) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1} w(t)$ and then integrating with respect to t from a to τ, we obtain

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \frac{1}{n+1}\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau). \tag{8}$$

From (6) and (8), the required result follows. □

Remark 1. *Applying Theorem 4 for $\phi(\tau)=\tau$ and $w(\theta)=1$, we obtain Theorem 2.1 in [9].*

Remark 2. *In Theorem 4, if we choose $\phi(\tau)=\tau$, $w(\theta)=1$ and $\ell=1$, we have the reverse Minkowski inequality in [5].*

Inequality (4) is a version of reverse Minkowski inequality obtained with weighted fractional operators.

Theorem 5. *Let $\wp,\hbar \in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau > a > 0$, $\ell > 0$, $p \geq 1$. If $0 \leq n \leq \frac{\wp(t)}{\hbar(t)} \leq N$ holds for $n, N \in \mathbb{R}^+$ and $t \in [a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{2}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{2}{p}}(\tau) \geq c_2\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau),$$

with $c_2 = \frac{(N+1)(n+1)}{N} - 2$.

Proof. Multiplying inequality (6) by inequality (8), we obtain

$$\frac{(N+1)(n+1)}{N}\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{2}{p}}(\tau). \tag{9}$$

Using the Minkowski inequality, on the right side of (9), we get

$$\begin{aligned} &\frac{(N+1)(n+1)}{N}\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \\ \leq\ &\left[\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau)\right]^2. \end{aligned}$$

Then, we have

$$\begin{aligned} &\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{2}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{2}{p}}(\tau) \\ \geq\ &\left[\frac{(N+1)(n+1)}{N} - 2\right]\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \end{aligned}$$

which is the desired result. □

Remark 3. *Applying Theorem 5 for $\phi(\tau)=\tau$ and $w(\theta)=1$, we obtain Theorem 2.3 in [9].*

Remark 4. *In Theorem 5, if we choose $\phi(\tau)=\tau$, $w(\theta)=1$ and $\ell=1$, we have Theorem 2.2 in [30].*

3. Other Fractional Integral Inequalities

Theorem 6. *Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$, $\ell>0$, $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ holds for $n,N\in\mathbb{R}^+$ and $t\in[a,\tau]$, then the following inequality for weighted fractional operators holds:*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_w^{\ell:\phi}\hbar\right)^{\frac{1}{q}}(\tau)\leq\left(\frac{N}{n}\right)^{\frac{1}{qp}}\left({}_{a+}\Im_w^{\ell:\phi}\wp^{\frac{1}{p}}.\hbar^{\frac{1}{q}}\right)(\tau).$$

Proof. Using the given condition $\frac{\wp(t)}{\hbar(t)}\leq N$, $t\in[a,\tau]$, it can be written

$$\begin{aligned}\wp(t)&\leq N\hbar(t)\\ N^{-\frac{1}{q}}\wp^{\frac{1}{q}}(t)&\leq\hbar^{\frac{1}{q}}(t).\end{aligned}\tag{10}$$

Multiplying both sides of (10) by $\wp^{\frac{1}{p}}(t)$, we can rewrite as follows

$$N^{-\frac{1}{q}}\wp(t)\leq\wp^{\frac{1}{p}}(t)\hbar^{\frac{1}{q}}(t)\tag{11}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Multiplying both sides of (11) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we have

$$\begin{aligned}&\frac{N^{-\frac{1}{q}}w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp(t)w(t)dt\\ \leq&\frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp^{\frac{1}{p}}(t)\hbar^{\frac{1}{q}}(t)w(t)dt.\end{aligned}$$

From weighted fractional operators, we obtain

$$N^{-\frac{1}{pq}}\left({}_{a+}\Im_w^{\ell:\phi}\wp\right)^{\frac{1}{p}}(\tau)\leq\left({}_{a+}\Im_w^{\ell:\phi}\wp^{\frac{1}{p}}.\hbar^{\frac{1}{q}}\right)^{\frac{1}{p}}(\tau).\tag{12}$$

On the contrary, as $n\leq\frac{\wp(t)}{\hbar(t)}$, it follows

$$n^{\frac{1}{p}}\hbar^{\frac{1}{p}}(t)\leq\wp^{\frac{1}{p}}(t).\tag{13}$$

Multiplying both sides of (13) by $\hbar^{\frac{1}{q}}(t)$ and using the relation $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$n^{\frac{1}{p}}\hbar(t)\leq\wp^{\frac{1}{p}}(t)\hbar^{\frac{1}{q}}(t).\tag{14}$$

Multiplying both sides of (14) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we get

$$n^{\frac{1}{pq}}\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar\right)^{\frac{1}{q}}(\tau)\leq\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{\frac{1}{p}}.\hbar^{\frac{1}{q}}\right)^{\frac{1}{q}}(\tau). \tag{15}$$

Conducting the product between (12) and (15), we have

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\right)^{\frac{1}{p}}(\tau)\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar\right)^{\frac{1}{q}}(\tau)\leq\left(\frac{N}{n}\right)^{\frac{1}{qp}}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{\frac{1}{p}}\hbar^{\frac{1}{q}}\right)(\tau).$$

where $\frac{1}{p}+\frac{1}{q}=1$. So the proof is completed. □

Theorem 7. *For $\ell>0$, $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)(\tau)$ and $\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{p}\right)(\tau)$ are finite reals for $\tau>a>0$. If $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ for $n,N\in\mathbb{R}^{+}$ and for all $t\in[a,\tau]$, then*

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau)\leq c_3\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp^{p}+\hbar^{p})\right)(\tau)+c_4\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp^{q}+\hbar^{q})\right)(\tau)$$

with $c_3=\frac{2^{p-1}N^{p}}{p(N+1)^{p}}$ and $c_4=\frac{2^{q-1}}{q(n+1)^{q}}$.

Proof. Using the hypothesis, we obtain the following inequality:

$$(N+1)^{p}\wp^{p}(t)\leq N^{p}(\wp+\hbar)^{p}(t). \tag{16}$$

Multiplying both sides of (16) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we have

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)(\tau)\leq\frac{N^{p}}{(N+1)^{p}}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp+\hbar)^{p}\right)(\tau). \tag{17}$$

For $t\in[a,\tau]$, since $0\leq n\leq\frac{\wp(t)}{\hbar(t)}$ holds we get

$$(n+1)^{q}\hbar^{q}(t)\leq(\wp+\hbar)^{q}(t). \tag{18}$$

Similarly, multiplying both sides of (18) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we can write

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{q}\right)(\tau)\leq\frac{1}{(n+1)^{q}}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp+\hbar)^{q}\right)(\tau). \tag{19}$$

Using the Young's inequality, we have

$$\wp(t)\hbar(t)\leq\frac{1}{p}\wp^{p}(t)+\frac{1}{q}\hbar^{q}(t), \tag{20}$$

again multiplying both sides of (20) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau)\leq\frac{1}{p}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)(\tau)+\frac{1}{q}\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{q}\right)(\tau). \tag{21}$$

Using (17) and (19) in (21), we obtain

$$\begin{aligned}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau) &\leq \frac{N^{p}}{p(N+1)^{p}}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp+\hbar)^{p}\right)(\tau) \\ &\quad+\frac{1}{q(n+1)^{q}}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp+\hbar)^{q}\right)(\tau).\end{aligned} \tag{22}$$

Using the inequality $(x+y)^r \leq 2^{r-1}(x^r+y^r)$, $r>1$, $x,y>0$ in (22), we have

$$\begin{aligned}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau) &\leq \frac{2^{p-1}N^{p}}{p(N+1)^{p}}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp^{p}+\hbar^{p})\right)(\tau) \\ &\quad+\frac{2^{q-1}}{q(n+1)^{q}}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp^{q}+\hbar^{q})\right)(\tau).\end{aligned}$$

This is the required result. □

Theorem 8. *For $\ell>0$, $p\geq 1$. Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)(\tau)$ and $\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{p}\right)(\tau)$ are finite reals for $\tau>a>0$. If $0<c<n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ for $n,N\in\mathbb{R}^{+}$ and for all $t\in[a,\tau]$, then*

$$\begin{aligned}\frac{N+1}{N-c}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp-c\hbar)^{p}\right)^{\frac{1}{p}}(\tau) &\leq \left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)^{\frac{1}{p}}(\tau)+\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{p}\right)^{\frac{1}{p}}(\tau) \\ &\leq \frac{n+1}{n-c}\left({}_{a+}\Im_{w}^{\ell:\phi}(\wp-c\hbar)^{p}\right)^{\frac{1}{p}}(\tau).\end{aligned}$$

Proof. Using the hypothesis $0<c<n\leq N$, we have

$$nc\leq Nc \implies nc+n\leq nc+N\leq Nc+N \implies (N+1)(n-c)\leq(n+1)(N-c).$$

It can be concluded that

$$\frac{N+1}{N-c}\leq\frac{n+1}{n-c}.$$

Also,

$$\begin{aligned}n\leq\frac{\wp(t)}{\hbar(t)}\leq N &\implies n-c\leq\frac{\wp(t)-c\hbar(t)}{\hbar(t)}\leq N-c \\ &\implies \frac{(\wp(t)-c\hbar(t))^{p}}{(N-c)^{p}}\leq\hbar^{p}(t)\leq\frac{(\wp(t)-c\hbar(t))^{p}}{(n-c)^{p}}.\end{aligned} \tag{23}$$

Multiplying both sides of (23) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we get

$$\begin{aligned}&\frac{w^{-1}(\tau)}{(N-c)^{p}\Gamma(\ell)}\int_{a}^{\tau}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}(\wp(t)-c\hbar(t))^{p}w(t)dt \\ \leq\;&\frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_{a}^{\tau}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\hbar^{p}(t)w(t)dt \\ \leq\;&\frac{w^{-1}(\tau)}{(n-c)^{p}\Gamma(\ell)}\int_{a}^{\tau}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}(\wp(t)-c\hbar(t))^{p}w(t)dt\end{aligned}$$

Then, we can write

$$\begin{aligned} \frac{1}{N-c}\left({}_{a+}\Im_w^{\ell:\phi}(\wp-c\hbar)^p\right)^{\frac{1}{p}}(\tau) &\leq \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \\ &\leq \frac{1}{n-c}\left({}_{a+}\Im_w^{\ell:\phi}(\wp-c\hbar)^p\right)^{\frac{1}{p}}(\tau). \end{aligned} \tag{24}$$

Again, we obtain

$$\frac{1}{N}\leq\frac{\hbar(t)}{\wp(t)}\leq\frac{1}{n}\Longrightarrow\frac{n-c}{nc}\leq\frac{\wp(t)-c\hbar(t)}{c\wp(t)}\leq\frac{N-c}{cN}$$

which implies

$$\left(\frac{N}{N-c}\right)^p(\wp(t)-c\hbar(t))^p\leq\wp^p(t)\leq\left(\frac{n}{n-c}\right)^p(\wp(t)-c\hbar(t))^p. \tag{25}$$

Repeating the same procedure with (25), we have

$$\begin{aligned} \frac{N}{N-c}\left({}_{a+}\Im_w^{\ell:\phi}(\wp-c\hbar)^p\right)^{\frac{1}{p}}(\tau) &\leq \left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \\ &\leq \frac{n}{n-c}\left({}_{a+}\Im_w^{\ell:\phi}(\wp-c\hbar)^p\right)^{\frac{1}{p}}(\tau). \end{aligned} \tag{26}$$

Adding (24) and (26), the required result is obtained. □

Theorem 9. *For $\ell>0$, $p\geq 1$. Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau>a>0$. If $0\leq a\leq\wp(t)\leq A$ and $0\leq b\leq\hbar(t)\leq B$, $t\in[a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau)+\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau)\leq c_5\left({}_{a+}\Im_w^{\ell:\phi}(\wp+\hbar)^p\right)^{\frac{1}{p}}(\tau) \tag{27}$$

with $c_5=\frac{A(a+B)+B(b+A)}{(a+B)(b+A)}$.

Proof. Under the given conditions, it follows that

$$\frac{1}{B}\leq\frac{1}{\hbar(t)}\leq\frac{1}{b}. \tag{28}$$

Considering the product of (28) and $0\leq a\leq\wp(t)\leq A$, we have

$$\frac{a}{B}\leq\frac{\wp(t)}{\hbar(t)}\leq\frac{A}{b}. \tag{29}$$

From (29), we get

$$\hbar^p(t)\leq\left(\frac{B}{a+B}\right)^p(\wp(t)+\hbar(t))^p \tag{30}$$

and

$$\wp^p(t)\leq\left(\frac{A}{b+A}\right)^p(\wp(t)+\hbar(t))^p. \tag{31}$$

Multiplying both sides of (30) and (31) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{p}\right)^{\frac{1}{p}}(\tau)\leq\frac{B}{a+B}\left({}_{a+}\Im_{w}^{\ell:\phi}\left(\wp+\hbar\right)^{p}\right)^{\frac{1}{p}}(\tau) \tag{32}$$

and

$$\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)^{\frac{1}{p}}(\tau)\leq\frac{A}{b+A}\left({}_{a+}\Im_{w}^{\ell:\phi}\left(\wp+\hbar\right)^{p}\right)^{\frac{1}{p}}(\tau). \tag{33}$$

respectively. The proof of (27) can be concluded by adding (32) and (33). □

Theorem 10. *Let $\wp,\hbar\in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_{w}^{\ell:\phi}\wp^{p}\right)(\tau)$ and $\left({}_{a+}\Im_{w}^{\ell:\phi}\hbar^{p}\right)(\tau)$ are positive reals for $\tau>a>0$. If $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$ for $n,N\in\mathbb{R}^{+}$ and for all $t\in[a,\tau]$, then*

$$\frac{1}{N}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau)\leq\frac{1}{(n+1)(N+1)}\left({}_{a+}\Im_{w}^{\ell:\phi}\left(\wp+\hbar\right)^{2}\right)(\tau)\leq\frac{1}{n}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau)$$

for $\ell>0$.

Proof. Using $0\leq n\leq\frac{\wp(t)}{\hbar(t)}\leq N$, we obtain

$$\hbar(t)(n+1)\leq\hbar(t)+\wp(t)\leq\hbar(t)(N+1). \tag{34}$$

Also, it follows that $\frac{1}{N}\leq\frac{\hbar(t)}{\wp(t)}\leq\frac{1}{n}$, which yields

$$\wp(t)\left(\frac{N+1}{N}\right)\leq\hbar(t)+\wp(t)\leq\wp(t)\left(\frac{n+1}{n}\right). \tag{35}$$

Evaluating the product between (34) and (35), we get

$$\frac{\wp(t)\hbar(t)}{N}\leq\frac{\left(\hbar(t)+\wp(t)\right)^{2}}{(n+1)(N+1)}\leq\frac{\wp(t)\hbar(t)}{n}. \tag{36}$$

Multiplying both sides of (36) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\begin{aligned}
&\frac{w^{-1}(\tau)}{N\Gamma(\ell)}\int_{a}^{\tau}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp(t)\hbar(t)w(t)dt\\
\leq\ &\frac{w^{-1}(\tau)}{(n+1)(N+1)\Gamma(\ell)}\int_{a}^{\tau}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\left(\hbar(t)+\wp(t)\right)^{2}w(t)dt\\
\leq\ &\frac{w^{-1}(\tau)}{n\Gamma(\ell)}\int_{a}^{\tau}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp(t)\hbar(t)w(t)dt.
\end{aligned}$$

Hence

$$\frac{1}{N}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau)\leq\frac{1}{(n+1)(N+1)}\left({}_{a+}\Im_{w}^{\ell:\phi}\left(\wp+\hbar\right)^{2}\right)(\tau)\leq\frac{1}{n}\left({}_{a+}\Im_{w}^{\ell:\phi}\wp\hbar\right)(\tau).$$

This completes the proof. □

Theorem 11. *Let $\wp, \hbar \in L[a,\tau]$ be two positive functions on $[0,\infty)$, such that $\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)(\tau)$ and $\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)(\tau)$ are finite reals for $\tau > a > 0$. If $0 < n \leq \frac{\wp(t)}{\hbar(t)} \leq N$ holds for $n, N \in \mathbb{R}^+$ and for all $t \in [a,\tau]$, then*

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) + \left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq 2\left({}_{a+}\Im_w^{\ell:\phi}\Psi^p(\wp,\hbar)\right)^{\frac{1}{p}}(\tau) \tag{37}$$

holds for $\ell > 0$ where $\Psi(\wp(t),\hbar(t)) = \max\left\{N\left[\left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t)\right], \frac{(N+n)\hbar(t)-\wp(t)}{n}\right\}$.

Proof. From the hypothesis $0 < n \leq \frac{\wp(t)}{\hbar(t)} \leq N$, we have

$$0 < n \leq N + n - \frac{\wp(t)}{\hbar(t)} \tag{38}$$

and

$$N + n - \frac{\wp(t)}{\hbar(t)} \leq N. \tag{39}$$

Hence, using (38) and (39), we get

$$\hbar(t) < \frac{(N+n)\hbar(t) - \wp(t)}{n} \leq h(\wp(t),\hbar(t)), \tag{40}$$

where $\Psi(\wp(t),\hbar(t)) = \max\left\{N\left[\left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t)\right], \frac{(N+n)\hbar(t)-\wp(t)}{n}\right\}$.

Using the hypothesis, it follows that $0 < \frac{1}{N} \leq \frac{\hbar(t)}{\wp(t)} \leq \frac{1}{n}$. In this way, we have

$$\frac{1}{N} \leq \frac{1}{N} + \frac{1}{n} - \frac{\hbar(t)}{\wp(t)} \tag{41}$$

and

$$\frac{1}{N} + \frac{1}{n} - \frac{\hbar(t)}{\wp(t)} \leq \frac{1}{n}. \tag{42}$$

From (41) and (42), we obtain

$$\frac{1}{N} \leq \frac{\left(\frac{1}{N}+\frac{1}{n}\right)\wp(t) - \hbar(t)}{\wp(t)} \leq \frac{1}{n},$$

which can be rewritten as

$$\begin{aligned} \wp(t) &\leq N\left(\frac{1}{N}+\frac{1}{n}\right)\wp(t) - N\hbar(t) \\ &= \left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t) \\ &\leq N\left[\left(\frac{N}{n}+1\right)\wp(t) - N\hbar(t)\right] \\ &\leq \Psi(\wp(t),\hbar(t)). \end{aligned} \tag{43}$$

We can write from (40) and (43)

$$\wp^p(t) \leq \Psi^p(\wp(t),\hbar(t)) \tag{44}$$

$$\hbar^p(t) \quad \leq \quad \Psi^p(\wp(t), \hbar(t)). \tag{45}$$

Multiplying both sides of (44) by $\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}w(t)$ and then integrating we obtain

$$\begin{aligned} &\frac{w^{-1}(\tau)}{\Gamma(\ell)}\int_a^\tau \phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\wp^p(t)w(t)dt \\ \leq\quad &\frac{w^{-1}(\tau)}{\Gamma(\ell)}\phi'(t)\left[\phi(\tau)-\phi(t)\right]^{\ell-1}\Psi^p(\wp(t),\hbar(t))w(t)dt. \end{aligned}$$

Accordingly, it can be written as

$$\left({}_{a+}\Im_w^{\ell:\phi}\wp^p\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}\Psi^p(\wp,\hbar)\right)^{\frac{1}{p}}(\tau)\,. \tag{46}$$

Using the same procedure as above, for (45), we have

$$\left({}_{a+}\Im_w^{\ell:\phi}\hbar^p\right)^{\frac{1}{p}}(\tau) \leq \left({}_{a+}\Im_w^{\ell:\phi}\Psi^p(\wp,\hbar)\right)^{\frac{1}{p}}(\tau)\,. \tag{47}$$

The required result (37) follows from (46) and (47). □

4. Conclusion

In this paper, first we gave different definitions of fractional integral operators and then we introduced the reverse Minkowski type inequalities using weighted fractional operators. The obtained results are an extension of some known results in the literature. Especially, we would like to emphasize that different types all integral inequalities can be obtained using this operators.

Author Contribution Statements All authors jointly worked on the results and they read and approved the final manuscript.

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References

[1] Abdeljawad, T., On conformable fractional calculus, *J. Comput. Appl. Math,* 279 (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016

[2] Akdemir, A.O., Butt, S.I., Nadeem, M., Ragusa, M.A., New general variants of Chebyshev type inequalities via generalized fractional integral operators, *Mathematics,* 9(2) (2021), 122. https://doi.org/10.3390/math9020122

[3] Bayraktar, B., Some integral inequalities of Hermite-Hadamard type for differentiable (s, m)-convex functions via fractional integrals, *TWMS Jour. App. Engin. Math.,* 10(3) (2020), 625-637.

[4] Belarbi, S., Dahmani, Z., On some new fractional integral inequalities, *J. Ineq. Pure and Appl. Math.,* 10(3) (2009), art. 86.

[5] Bougoffa, L., On Minkowski and Hardy integral inequalities, *J. Ineq. Pure & Appl. Math.,* 7(2) (2006), art. 60.

[6] Butt, S.I., Nadeem, M., Farid, G., On Caputo fractional derivatives via exponential s-convex functions, *Turkish Jour. Sci.,* 5(2) (2020), 140-146.

[7] Butt, S.I., Umar, M., Rashid, S., Akdemir, A.O., Chu, Y.M., New Hermite-Jensen-Mercer-type inequalities via k-fractional integrals, *Adv. Diff. Equat.,* 1 (2020), 1-24. https://doi.org/10.1186/s13662-020-03093-y

[8] Chinchane, V.L., Pachpatte, D.B., New fractional inequalities via Hadamard fractional integral, *Int. J. Funct. Anal.,* 5 (2013), 165-176.

[9] Dahmani, Z., On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, *Ann. Funct. Anal.,* 1(1) (2010), 51–58. https://doi.org/10.15352/afa/1399900993

[10] Ekinci, A., Özdemir, M.E., Some new integral inequalities via Riemann-Liouville integral operators, *App. Comp. Math.,* 3(18) (2019), 288–295.

[11] Ekinci, A., Özdemir, M.E., Set, E., New integral inequalities of Ostrowski type for $quasi-$convex functions with applications, *Turkish Jour. Sci.,* 5(3) (2020), 290-304.

[12] Hardy, G.H., Littewood, J.E., Pòlya, G., Inequalities, 2nd ed. Cambridge, England: Cambridge University Press, 1988, pp.30-32, 123, 146-150.

[13] Jarad, F., Abdeljawad, T., Shah, K., On the weighted fractional operators of a function with respect to another function, *Fractals,* 28(08) (2020), 2040011. https://doi.org/10.1142/S0218348X20400113

[14] Jarad, F., Uğurlu, U., Abdeljawad, T., Baleanu, D., On a new class of fractional operators, *Adv. Differ. Equ*, 2017(247) (2017), 1-16. https://doi.org/10.1186/s13662-017-1306-z

[15] Katugampola, U.N., New approach to generalized fractional integral, *Appl. Math. Comput.,* 218 (2011), 860-865. https://doi.org/10.3390/math10040573

[16] Katugampola, U.N., A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, 6 (2014), 1-15.

[17] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., A new definition of fractional derivative, *J. Comput. Appl. Math.,* 264 (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002

[18] Khan, T.U., Khan, M.A., Generalized conformable fractional integral operators, *J. Comput. Appl. Math.,* 346 (2019), 378–389. https://doi.org/10.1016/j.cam.2018.07.018

[19] Kızıl, Ş., Ardıç, M.A., Inequalities for strongly convex functions via Atangana-Baleanu integral operators, *Turkish Jour. Sci.,* 6(2) (2021), 96-109.

[20] Liouville, J., Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions, *Journal de l'École Polytechnique*, 13 (1832), 1–69.

[21] Miller, K.S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[22] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., Classical and New Inequalities in Analysis, Kluwer Academic Publishers, London, 1993.

[23] Mohammed, P.O., Abdeljawad, T., Alqudah, M.A., Jarad, F., New discrete inequalities of Hermite–Hadamard type for convex functions, *Adv. Diff. Equat.,* 1 (2021), 1-10. https://doi.org/10.1186/s13662-021-03290-3

[24] Mohammed, P.O., Abdeljawad, T., Kashuri, A., Fractional Hermite-Hadamard-Fejer inequalities for a convex function with respect to an increasing function involving a positive weighted symmetric function, *Symmetry,* 12(9) (2020), 1503. https://doi.org/10.3390/sym12091503

[25] Mohammed, P.O., Abdeljawad, T., Zeng, S., Kashuri, A., Fractional Hermite-Hadamard integral inequalities for a new class of convex functions, *Symmetry,* 12(9) (2020), 1485. https://doi.org/10.3390/sym12091485

[26] Özdemir, M.E., Dragomir, S.S., Yıldız, Ç., The Hadamard inequality for convex function via fractional integrals, *Acta Math. Sci.,* 33B(5) (2013), 1293-1299. https://doi.org/10.1016/S0252-9602(13)60081-8

[27] Podlubny, I., Fractional Differential Equations, Academic Press, San Diego, 1999.

[28] Sarıkaya, M.Z., Yaldız, H., On generalization integral inequalities for fractional inegrals, *Nihonkai Math. J.,* 25 (2014), 93-104.
[29] Set, E., Akdemir, A.O., Gözpınar, A., Jarad, F., Ostrowski type inequalities via new fractional conformable integrals, *AIMS Mathematics,* 4(6) (2019), 1684–1697. https://doi.org/10.3934/math.2019.6.1684
[30] Set, E., Özdemir, M.E., Dragomir, S.S., On the Hermite–Hadamard inequality and other integral inequalities involving two functions, *J. Ineq. Appl.,* 2010(148102) (2010), 1-9. https://doi.org/10.1155/2010/148102
[31] Sousa, J.V.D.C., de Oliveira, E.C., The Minkowski's inequality by means of a generalized fractional integral, *AIMS Mathematics,* 3(1) (2018), 131-142. https://doi.org/10.3934/Math.2018.1.131
[32] Yavuz, M., Özdemir, N., Comparing the new fractional derivative operators involving exponential and Mittag-Leffler kernel, *Discrete Contin. Dyn. Syst., Ser.,* 13(3) (2020), 995. https://doi.org/10.3934/dcdss.2020058

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ERRATUM TO: ZERO-BASED INVARIANT SUBSPACES IN THE BERGMAN SPACE

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Abstract. In this Erratum we would like to clarify statement and the proof of Theorem 2 in our paper: "Zero-based invariant subspaces in the Bergman space *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 67(1) (2018), 277-285."

1. Main Part

Theorem 2 in the paper [2] had been already proved in [1]. The citation of the Reference [1] was omitted in the original article [2]. The authors would like to correct this deficiency as follows:

Theorem 2 [1] Let M be a zero-based invariant subspace of $L^p_a(D)$, $0 < p < +\infty$. Then M is generated by its extremal function G, that is, $M = [G]$.

References

[1] Abkar, A., A Beuling-type theorem in Bergman spaces, *Turk. J. Math.*, 35 (2011), 711-716. doi:10.3906/mat-0911-115

[2] Bouabdallah, F., Bendaoud, Z., Zero-based invariant subspaces in the Bergman space, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 67(1) (2018), 277-285. doi:10.1501/Commua1_0000000849

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