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VULNERABILITY OF BANANA TREES VIA CLOSENESS AND RESIDUAL CLOSENESS PARAMETERS

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Abstract. One of the most important research topics about complex networks is examination of their vulnerability. Therefore, there are many studies in the literature about analyzing the robustness and reliability of networks using graph theoretical parameters. Among these parameters, the centrality parameters play an important role. The closeness parameters and its derivatives are widely discussed. In this study, the closeness parameter and the more sensitive parameter residual closeness which is based on closeness parameter have been considered. Furthermore, the closeness and residual closeness of banana tree structure have been calculated.

1. Introduction

With the developments in network science, computer science and its applications are developing rapidly. Many problems that can appear in real life can be modeled as a network and the system can be analyzed utilizing relationship between vertices and edges. Therefore, graph theory is an important scientific tool for determining vulnerability of a network. Analyzing a network in terms of vulnerability is one of the main purpose of the graph theory problems. Thus, utilizing graph theory parameters and techniques, a network can be investigated in terms of robustness and reliability from many researchers. One of the most important goal of network analysis is to research concept of centrality. There are several centrality parameters in the literature yet closeness centrality is one of the quite significant index that measures how capital position a node is in the network.

Closeness parameter have also changed to provide more sensitive approaches to network analyzing. First important closeness definition provided in [13]. Nevertheless, it can not suitable for disconnected graphs. The other closeness definition is given by Latora and Marchiori in [14]. The definition is formulated as $C(i) = \sum_{j \neq i} \frac{1}{d(i,j)}$. Here $d(i,j)$ denotes distance between vertices i and j. This new

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definition can be applied to disconnected graphs. Afterwards, Danglachev introduced new closeness definition which is varied Latora and Marchiori's definition to provide calculation and formulation convention [9]. Dangalachev's new closeness definition is $\sum_{j\neq i} \frac{1}{2^{d(i,j)}}$. Derived from Dangalchev's closeness definition, many vulnerability parameters have been appeared to measure resilience of a network. Among of these new indexes, *residual vertex and edge closeness* parameters calculate closeness value of a graph after vertex or edge extracted from a graph [9]. The concept of residual closeness, a more sensitive parameter based on this definition of closeness, also emerged meanwhile from Dangalchev again [9]. The all-important point here is to find how a vertex removed from the graph influences the vulnerability of the graph. In order to evaluate closeness value after vertex k is removing such as $C_k = . \sum_i \sum_{j\neq i} \frac{1}{2^{d_k(i,j)}}$ where $d_k(i,j)$ is distance between vertices i and j after removing vertex k. Then the vertex residual closeness ,denoted by R, defined as $R = \min_k\{C_k\}$.Another measure is *additional closeness* which determine maximal potential of graph's closeness via adding an edge. It can be referred the readers to get detailed information about these new sensitive parameters [1–6, 10–12, 15–17].

In this paper, the graph G is taken as a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of any vertex in $V(G)$, denoted by $N(v) = \{u \in V(G) : (uv) \in E(G)\}$. *The degree of a vertex* v denoted by $deg(v)$, is cardinality of its neighborhood. *The distance* between two vertices u and v is shortest path between them, denoted by $d(u,v)$. A vertex of degree one is called *pendant vertex* and its incident edge is called support edge [7].

In this work, we investigate results about closeness and residual closeness of Banana Trees. Banana tree is a structure introduced by Chen et al. [8] as obtained by linking one leaf of each of n copies of an k vertices star graph structure with a single root vertex that is distinct from all the stars and the tree is denoted by $B_{n,k}$.

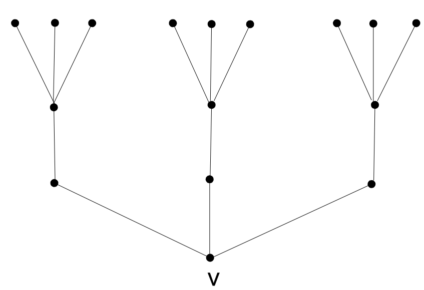

Figure 1. Banana Tree graph illustration with three copy of five vertices star graph, $B_{3,5}$

Calculating closeness value for huge graph structures is detailed process. In order to facilitate this process, it will be easier to use the method of splitting the graph into subgraphs in some structures, as in [11]. Utilizing this idea, we will denote root vertex as v, vertex 1_i be neighbour of vertex v in i^{th} copy of star graph, vertex 2_i is hub vertex of i^{th} copy of star graph and $3_i, ..., (3+j)_i$ are leaf vertices in banana graph where $1 \leq i \leq n$ and $1 \leq j \leq k-2$ in order to formulate closeness and residual closeness value for banana trees.

2. Closeness of Banana Trees

In this section we will get closeness value of Banana Tree structure. In order to ease of formulation, graph can be split into subforms and relationship between them. Next theorem gives us closeness value of Banana tree graph in terms of number of copy, denoted by n and number of vertices of star graph, denoted by k.

Theorem 2.1. *Let $B_{n,k}$ be banana tree with $nk+1$ vertices.The closeness value of $B_{n,k}$ is*

$$C(B_{n,k}) = \frac{16n(k^2+2k+2)+(n^2-n)(k+4)^2}{64}.$$

Proof. Due to form of Banana tree, graph can be splitted into three subforms such as $C(v)$ where v is root vertex, $C(S_k)$ and $C(S_k \sim S_k)$ where $C(v)$ is closeness value of vertex v, $C(S_k) = \frac{(k-1)(k+2)}{4}$ is closeness value of star graph with k vertices [9], $C(S_k \sim S_t)$ is closeness value of vertices in a copy of star graphs to other copies. Let 1_i be neighbour of root vertex v in i^{th} copy of star graph, 2_i is hub vertex of i^{th} copy of star graph and $3_i, ..., (3+j)_i$ are leaf vertices in banana graph where $1 \leq i \leq n$ and $1 \leq j \leq k-2$. Distace between v and 1_i is one, distance between v and 2_i is two and distance between v and all leaves notated by $3_i, ..., (3+j)_i$ is three for all $1 \leq i \leq n$ and $1 \leq j \leq k-2$. Therefore, closeness value of root vertex v is

$$C(v) = \sum_{\substack{i \in V(B_{n,k}) \\ i \neq v}} 2^{-d(i,v)} = (\frac{n}{2} + \frac{n}{2^2} + \frac{n(k-2)}{2^3})$$

and in order to calculate $C(S_k \sim S_k)$ value, without loss of generality, we can consider distance between vertices of 1^{st}*copy* of the S_k and 2^{st}*copy* of the S_k initially. Then, distance of vertex 1_1 to all vertices of 2^{st}*copy* of star graph is $A = \frac{1}{2^2} + \frac{1}{2^3} + \frac{(k-2)}{2^4}$ and distance of vertex 2_1 to all vertices of 2^{st}*copy* of star graph is $\frac{1}{2}A$ and distance of any leaf vertex to all vertices of 2^{st}*copy* of star graph is $\frac{1}{2^2}A$. Thus, we can formulate closeness value of vertices between first and second copy of star graphs as follows

$$\begin{aligned} C(S_k^{(1)} \sim S_k^{(2)}) &= A + \frac{1}{2}A + \frac{1}{2^2}A \\ &= A(1 + \frac{1}{2} + \frac{1}{2^2}) \\ &= \frac{1}{2^2}(1 + \frac{1}{2} + \frac{1}{2^2})^2. \end{aligned}$$

We will consider closeness value to every other vertices in both direction and there are n copies of star graph and also there are $n(n-1)$ relationship between star graph structures. Therefore, we obtain closeness value of banana tree

$$\begin{aligned} C(B_{n,k}) &= 2C(v) + nC(S_k) + n(n-1)(C(S_k \sim S_k)) \\ &= \frac{16n(k^2+2k+2) + (n^2-n)(k+4)^2}{64}. \end{aligned}$$

□

3. Residual Closeness of Banana Trees

In order to evaluate residual closeness value, a vertex will be removed from the graph and the minimum closeness value will be calculated after removing. Therefore, the most sensitive vertex will be determined in the graph. In the banana tree structure, we will obtain four distinct value after removing. These modification can be get from removing vertex v, leaf vertex of a banana graph, a center of an star graph and a leaf of star graph that is connected with vertex v. After determining the effect of these modifications on the graph in the next theorem, we will get residual closeness value of banana trees.

Theorem 3.1. *Let $B_{n,k}$ be banana tree with $nk+1$ vertices.The residual closeness value of $B_{n,k}$ is*

$$R = \frac{n(k-1)(k+2)}{4}.$$

Proof. First determine notation of vertices that will removed from the graph in order to evaluate residual closeness value. Let v be root vertex, 1_i be neighbour of vertex v in i^{th} copy of star graph, 2_i is hub vertex of i^{th} copy of star graph and $3_i, ..., (3+j)_i$ are leaf vertices in banana graph where $1 \leq i \leq n$ and $1 \leq j \leq k-2$. Therefore, we will get four different value after vertex removing.

- If root vertex v will be removed then
$$\begin{aligned} R_1 &= nC(S_k) \\ &= \frac{n(k-1)(k+2)}{4} \end{aligned} \tag{3.1}$$
- If an 1_i removed from the graph for any i where $1 \leq i \leq n$ then
$$R_2 = C(B_{n-1,k}) + C(S_{k-1}) \tag{3.2}$$
- If an 2_i removed from the graph for any i where $1 \leq i \leq n$ then
$$R_3 = C(B_{n-1,k}) + 2C(1_i \sim B_{n-1,k}). \tag{3.3}$$
Here the notation $1_i \sim B_{n-1,k}$ denote the closeness value of a vertex 1_i after modification
- Let any leaf be removed from the graph. Due to removing any leaf has same effect on residual value, we can choose one of them. Without loss of generality, let choose any 3_i as removed vertex, where $1 \leq i \leq n$.
$$R_4 = C(B_{n,k}) - 2C(3_i). \tag{3.4}$$
If we compare equations 3.1, 3.2, 3.3 and 3.4 , then it can be seen that the value comes from equation 3.1 is the minimum value. Since the value $C(B_{n-1,k})$ include at least n closeness of star graph value. Hence,
$$R = \frac{n(k-1)(k+2)}{4}.$$

□

4. Conclusion

In this article, we have calculated closeness value and residual closeness value of banana tree graphs. In order to evaluate closeness of a graph easily, the graph can be splitted into subgraphs if we know the closeness of the underlying graph. Utilizing this idea, we can split the banana tree into subgraphs and using closeness value of components and relation between them. In addition, we have considered residual closeness value of banana tree graph. In order to do this calculation, we acted from the thought of how much a change in the closeness value of a vertex removed from the graph would cause. We obtained four different values, and the minimum value we obtained among them was obtained by removing the root vertex from the graph.

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DIRAC'S LINEARIZATION APPLIED TO THE FUNCTIONAL, WITH MATRIX ASPECT, FOR THE TIME OF FLIGHT OF LIGHT

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Abstract. In this paper, we adopt a matrix treatment to solve the variational problem that consists of determining the physical path traveled by light between two points in a medium whose refractive index depends on a spatial coordinate. The considered treatment begins with the trivial repetition of the expression of the value of the considered functional, repetition expressed in the form of a matrix. Next, we adopt the trick (of Dirac) originally used as part of the construction of the dynamic equation of relativistic quantum mechanics, which allows us to rewrite the (now) matrix integrand in the expression of the value of the functional in terms of the sum of two (non-diagonal) matrices brought externally to the problem, which are determined based on some requirements. As a result of this development, we obtain two equivalent versions of Snell's law.

1. Introduction

In the context of the variational formulation of optics [1],[2], the functional T is defined, whose value $T[y]$, for an arbitrary curve y, corresponds to the time of flight of light along the referred curve, between two fixed points P_1 and P_2, located within a medium whose refractive index may depend, in the most general situation, on the spatial coordinates $x; y; z$. Suppose, for simplicity, that the medium traversed by light has an index of refraction that depends only on the spatial coordinate y; that is, $n(y)$. Therefore, the value of the functional $T[y]$ is written as [3],

$$T[y] \;=\; \frac{1}{c}\int_a^b n(y)\Big(1+\Big(y'(x)\Big)^2\Big)^{1/2}dx \;=\; \frac{1}{c}\int_a^b F\Big(x,y(x),y'(x)\Big)\,dx. \qquad (1.1)$$

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where c is the magnitude of speed of light in vacuum. The functional T above will have as an extremal a curve that can be traversed by light in a stationary[1] time, compared to the travel times corresponding to the other curves in the T domain.

The functional T, whose value is defined in (1.1), corresponds to the following Euler equation [4],

$$F_y \;-\; \frac{d}{dx}F_{y'} \;=\; 0. \tag{1.2}$$

In the next section we will give T a "matrix clothing", without changing its essence, which will be convenient.

On the other hand, the expression "Dirac linearization" is used here to indicate a simplified version [5], adapted to our problem, of the linearization of a second-order differential operator raised to the 1/2 power [6]. This idea[2] was used by Dirac to obtain a relativistic dynamic equation [8],[9] for specific quantum particles from a (non-linear) expression for their energy.

2. "Matrix Clothing" for $T[y]$ and Dirac's Trick

It is trivial to recognize that the factor that multiplies $n(y)$ in the integrand in (1.1) is the square root of the sum of two quadratic terms. This factor may receive a matrix clothing by multiplying each member in (1.1) by any matrix; in particular, by the identity matrix or by the square root of the identity; that is, we would have,

$$\left(\,1+\Big(y'(x)\Big)^2\,\right)^{1/2}\mathbf{I} \qquad \text{or} \qquad \left(\,1+\Big(y'(x)\Big)^2\,\right)^{1/2}\mathbf{I}^{1/2}, \tag{2.1}$$

with the difference that the left term in (2.1) corresponds to a diagonal matrix, but the right expression corresponds to a matrix that will not necessarily be diagonal[3]. As we will see, it would be more convenient to consider a matrix that could contain as many degrees of freedom as possible.

Then the expression (1.1) will receive such a "matrix clothing". However, it will only be about the matrix appearance of (1.1), since the functional T continues to match functions y with numbers $T[y]$, and not functions with matrices. In fact, such a matrix clothing will correspond to the trivial repetition of the same expression (1.1). In concrete terms, what we do is multiply (1.1) by the matrix $\mathbf{I}^{1/2}$, of order N, which in principle can have an arbitrary value. Soon,

$$\mathbf{T}[y] \;\equiv\; T[y]\,\mathbf{I}^{1/2} \;=\; \frac{1}{c}\int_a^b n(y)\,\Big(1+(y'(x))^2\Big)^{1/2}\mathbf{I}^{1/2}\,dx. \tag{2.2}$$

In the case of considering matrices of order $N=2$ we can write,

$$\mathbf{I}^{1/2} \;=\; \frac{1}{\sqrt{1+ab}}\begin{pmatrix} 1 & b \\ a & -1 \end{pmatrix}, \tag{2.3}$$

(with arbitrary a and b, but such that $ab \neq -1$) whose square corresponds precisely to the identity matrix $\mathbf{I}$; then we would explicitly have the expression,

$$\sqrt{1+ab}\;\mathbf{T}[y] \;=\; \begin{pmatrix} T[y] & b\,T[y] \\ a\,T[y] & -T[y] \end{pmatrix} =$$

[1] This can be a minimum, as in situations typically found in books.

[2] Already acknowledged by O. Heaviside who, according to [7], would have stated that: "... *the square root of a differential operator is intrinsic to Physics*".

[3] For there are non-diagonal matrices whose square is diagonal.

$$= \begin{pmatrix} \frac{1}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2}dx & \frac{b}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2}dx \\ \frac{a}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2}dx & -\frac{1}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2}dx \end{pmatrix} \quad (2.4)$$

which, as already said, corresponds to the trivial repetition of (1.1).

Expression (2.2)(2.2) can be linearized. It is easy to verify that for matrices of order 2, non-diagonal and, in general, dependent on y, which we represent here by $\mathbf{A}(y)$ and $\mathbf{B}(y)$, which must be properly defined, one can write[4],

$$\left(1+\left(y'(x)\right)^2\right)^{1/2}\mathbf{I}^{1/2} \;=\; \mathbf{A}(y) \;+\; y'(x)\,\mathbf{B}(y), \quad (2.5)$$

Provided that the following requirements are met,

$$\mathbf{A}^2(y) \;=\; \mathbf{I}, \quad \mathbf{B}^2(y) \;=\; \mathbf{I}, \quad \mathbf{A}(y)\mathbf{B}(y) \;+\; \mathbf{B}(y)\mathbf{A}(y) \;=\; \mathbf{0}, \quad (2.6)$$

Furthermore, it must be taken into account, for a mathematical consistency argument, that the sum of the matrix on the right side of (2.5) must be non-diagonal, since the matrix $\mathbf{I}^{1/2}$ has this characteristic, as indicated in (2.3).

Now we can rewrite the value of the functional T with "matrix clothing" as follows,

$$\mathbf{T}[y] \;=\; \frac{1}{c}\int_a^b n(y)\left(\mathbf{A}(y)+(y'(x))\,\mathbf{B}(y)\right)dx, \quad (2.7)$$

3. The "matrix clothing" for the Euler equation

The integrand in $\mathbf{T}[y]$, in the considered context, is written as,

$$\mathbf{F}\Big(x, y(x), y'(x)\Big) \;=\; n(y)\left(\mathbf{A}(y)+y'(x)\,\mathbf{B}(y)\right). \quad (3.1)$$

On the other hand, the matrix clothing for the corresponding Euler equation is obtained directly: multiplying the expression (1.2) by the matrix, $\mathbf{I}^{1/2}$, that is,

$$\mathbf{I}^{1/2}\left(F_y-\frac{d}{dx}F_{y'}\right) \;=\; \left(\mathbf{F}_y-\frac{d}{dx}\mathbf{F}_{y'}\right) \;=\; \mathbf{I}^{1/2}\,0 \;\equiv\; \mathbf{0}. \quad (3.2)$$

This clothing is purely formal and obviously trivial, without changing the nature of the initial problem: we have a functional T that assigns the number $T[y]$ to a function y, which belongs to the domain of T.

So, from (3.1) we get,

$$\mathbf{F}_y \;=\; n(y)\left(\mathbf{A}'(y)+y'(x)\,\mathbf{B}'(y)\right)+n'(y)\left(\mathbf{A}(y)+y'(x)\,\mathbf{B}(y)\right), \quad (3.3)$$

and,

$$\mathbf{F}_{y'} \;=\; n(y)\,\mathbf{B}(y), \quad (3.4)$$

From expression (3.4) we have that,

$$\frac{d}{dx}\,\mathbf{F}_{y'} \;=\; n(y)\,\mathbf{B}'(y)\,y'(x)+n'(y)\,y'(x)\,\mathbf{B}(y). \quad (3.5)$$

[4] This is essentially the trick used by Dirac [5], which we mentioned earlier.

Substituting (3.3) and (3.5) into Euler's equation (3.2) and after simplifications we obtain the following relation,

$$n(y)\mathbf{A}'(y) + n'(y)\mathbf{A}(y) \ = \ \mathbf{0}, \tag{3.6}$$

which corresponds to the development of the derivative of a product,

$$\Big(n(y)\ \mathbf{A}(y)\Big)' = \mathbf{0}, \qquad \Rightarrow \qquad n(y)\ \mathbf{A}(y) = \mathbf{C}, \tag{3.7}$$

where $\mathbf{C}$ is a constant matrix.
Note: An aspect not discussed here refers to the existence of possible symmetries associated with the functional $\mathbf{T}$, which would be revealed through its invariance under specific transformations. In a broader context, we would see that expression (3.7) can be obtained directly from Noether's Theorem [10].

4. Determination of matrices A and B

Note that expression (3.7) is independent of matrix $\mathbf{B}$; which we can take advantage of considering that $\mathbf{B}$ is constant, which will simplify its determination.

Now we explicitly write the non-diagonal matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{A}(y) = \begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{21}(y) & a_{22}(y) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{4.1}$$

The algebraic equations resulting from the requirements in (2.6) correspond, if placed in terms of the elements of $\mathbf{A}$ and $\mathbf{B}$, in (4.1), to the following:

$$\begin{aligned} a_{11}^2(y) \ + \ a_{12}(y)\, a_{21}(y) &= 1, \\ \Big(a_{11}(y) + a_{22}(y)\Big)\, a_{12}(y) &= 0, \\ \Big(a_{11}(y) + a_{22}(y)\Big)\, a_{21}(y) &= 0, \\ a_{22}^2(y) \ + \ a_{12}(y)\, a_{21}(y) &= 1, \end{aligned} \tag{4.2}$$

$$\begin{aligned} b_{11}^2 \ + \ b_{12}\, b_{21} &= 1, \\ \Big(b_{11} + b_{22}\Big)\, b_{12} &= 0, \\ \Big(b_{11} + b_{22}\Big)\, b_{21} &= 0, \\ b_{22}^2 \ + \ b_{12}\, b_{21} &= 1, \end{aligned} \tag{4.3}$$

$$\begin{aligned} 2a_{11}b_{11} + a_{12}b_{22} + a_{21}b_{12} &= 0, \\ a_{11}b_{12} + a_{12}b_{22} + a_{12}b_{11} + a_{22}b_{12} &= 0, \\ a_{21}b_{11} + a_{22}b_{21} + a_{11}b_{21} + a_{21}b_{22} &= 0, \\ 2a_{22}b_{22} + a_{21}b_{12} + a_{12}b_{21} &= 0, \end{aligned} \tag{4.4}$$

The groups of equations in (4.2)−(4.4) are solved with the elements of the following matrices,

$$\mathbf{A}(y) = \begin{pmatrix} \sin(\theta(y)) & \cos(\theta(y)) \\ \cos(\theta(y)) & -\sin(\theta(y)) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{4.5}$$

where $\theta(y)$ can be freely defined, with y taking an arbitrary value. For example: $\theta(y) = \text{arccot}(dy/dx)$, which is to say it corresponds to the angle between the coordinate direction y is the path of light.

From (3.7) and (4.5) we extract the two independent relations,

$$n(y)\,\sin(\theta(y)) = c_1 \qquad e \qquad n(y)\,\cos(\theta(y)) = c_2 \tag{4.6}$$

with y being free.

5. Conclusion

It is easy to recognize what might be called Dirac's "matrix trick" in the solution presented. In the problem analyzed here, it was possible to consider that one of the two matrices freely brought to the problem is constant. Note that the expression on the left in (4.6), with the definition of $\theta(y)$ given above, corresponds to Snell's law; in this case, the expression on the right, in the same expression, is spurious. But if we define the angle between the light path and a direction orthogonal to y, like x, as follows,

$$\theta(y) = \arctan\left(\frac{dy}{dx}\right),$$

then the expression of Snell's law is given by the one on the right, in (4.6), and the one on the left, in the same expression (4.6), would be spurious. Finally, the solution presented shows us that Dirac's trick can find application in other problems, which should be expected considering that the physical results do not depend on the mathematical tools used. As an illustration, the "Dirac linearization", as used here, can also be applied in the construction of a variant of the Feynman temporal propagator that, instead of using the action functional (and the Planck constant), the length functional of path is used (and the Compton wavelength of the considered particle).

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SOME NEW STABILITY RESULTS OF VOLTERRA INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. In this paper, we generalize two types of Volterra integral equations given on time scales and examine their Hyers-Ulam and Hyers-Ulam-Rassias stabilities. We also prove these stability results for the non-homogeneous nonlinear Volterra integral equation on time scales and provide an example to support these results. Moreover, we show that the general Volterra type integral equation given on time scales has the Hyers-Ulam-Rassias stability. Our results extend and improve some recent developments announced in the current literature.

1. Introduction

In 1940, the famous stability theory of the linear functional equation was introduced by Ulam [25]. Since then a series of mathematical questions related to this stability theory was collected in the book [25] and studied by Hyers [15] and improved by Rassias [17]. From then on, stabilities of many functional, differential and integral equations have been investigated, see [1, 4, 5, 9, 14, 16, 19, 20], and references therein.

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. The theory of time scales analysis has been rising fast and has attracted much interest. Therefore, many researchers have studied this issue [2, 11, 21, 22, 24]. The pioneer of this theory was Hilger [12]. He introduced this theory in 1988 with the inspiration to unify continuous and discrete calculus. Also, the stability analysis of dynamic equations has become an important topic both theoretically and practically because dynamic equations occur in many areas such as mechanics, physics and economics. For the introduction to the calculus on time scales and to the theory of dynamic equations on time scales, we recommend the books [6] and [7] by Bohner and Peterson.

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To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are Andras and Meszaros [4] and Hua, Li, Feng [13]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constants.

Yaseen [26] investigated the Hyers-Ulam-Rassias stability for the following class of Volterra integral equations on time scales

$$x(t) = \int_{t_0}^{t} f(t, s, x(s)) \Delta s, \qquad t \in \mathbb{T}. \tag{1.1}$$

In 2017, Hamza and Ghallab [3] showed that the non-homogeneous Volterra integral equation of the first kind on time scales

$$x(t) = f(t) + \int_{a}^{t} K(t, s) x(s) \Delta s, \qquad t \in [a, b] \cap \mathbb{T} \tag{1.2}$$

has the Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

Gachpazan and Baghani [10] discussed the Hyers-Ulam stability of the following non-homogeneous nonlinear Volterra integral equation

$$x(t) = f(t) + \varphi\left(\int_{a}^{t} K(t, s, x(s))\, ds\right), \quad t \in [a, b]. \tag{1.3}$$

Finally, in 2020, Reinfelds and Christian [18] studied Hyers–Ulam stability of general Volterra type integral equations on bounded time scales

$$x(t) = f\left(t, x(t), x(\sigma(t)), \int_{a}^{t} K(t, s, x(s), x(\sigma(s)) \Delta s\right), \qquad t \in [a, \infty) \cap \mathbb{T}. \tag{1.4}$$

Motivated by the above papers, we generalize two equations (1.1) and (1.2) given on time scales and examine their Hyers-Ulam and Hyers-Ulam-Rassias stabilities. We prove the existence and uniqueness of the solution and the stability results for equation (1.3) on time scales, and also provide an example to support these results. After that, we showed that equation (1.4), which is given on a time scale, has the Hyers-Ulam-Rassias stability.

2. Preliminaries on Time Scales

In this section, we present some basic notations, definitions and properties concerning the calculus on time scales, for more details the reader is referred to [6, 7].

As we said above, a time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. Since a time scale may or may not be connected, the concept of jump operator is useful to describe the structure of the time scale under consideration and is also used in defining the delta derivative. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

The jump operators allow the classification of points in a time scale $\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right dense if $\sigma(t) = t$, right scattered if $\sigma(t) > t$, left dense if $\rho(t) = t$, left scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$, and dense if

$\rho(t) = t = \sigma(t)$. If $\mathbb{T}$ has a left scattered maximum n, then $\mathbb{T}^\kappa = \mathbb{T}\backslash\{n\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right dense points in $\mathbb{T}$ and its left sided limits exist (finite) at every left dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The graininess function $\mu : \mathbb{T} \to [0, +\infty)$ is defined by $\mu(t) = \sigma(t) - t$. The function $f : \mathbb{T} \to \mathbb{R}$ is regressive if

$$1 + \mu(t)f(t) \neq 0 \qquad \text{for all } t \in \mathbb{T}^\kappa.$$

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and fix $t \in \mathbb{T}^\kappa$. The delta derivative (also Hilger derivative) $f^\Delta(t)$ exists if for every $\epsilon > 0$ there exists a neighbourhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$\left|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)\right| \leq \epsilon\, |\sigma(t) - s| \qquad \text{for all } s \in U.$$

If f is rd-continuous, then there is a function F such that $F^\Delta(t) = f(t)$ (see [6, 7]). In this case, we define the (Cauchy) delta integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all } r, s \in \mathbb{T}.$$

Let $\beta : \mathbb{T} \to \mathbb{R}$ be a regressive and rd-continuous function. The Cauchy initial value problem for linear equation

$$x^\Delta = \beta(t)x, \quad x(a) = 1, \quad a \in \mathbb{T}$$

has the unique solution $e_\beta(., a) : \mathbb{T} \to \mathbb{R}$ [6, 7]. More explicitly, the exponential function $e_\beta(., a)$ is given by

$$e_\beta(t, a) = \exp\left(\int_a^t \xi_{\mu(s)}(\beta(s))\Delta s\right) \quad \text{for } \ a, t \in \mathbb{T},$$

where

$$\xi_h(z) = \begin{cases} z, & h = 0; \\ \frac{1}{h}\log(1 + hz), & h > 0. \end{cases}$$

Let $|.|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the linear space of continuous functions $C(I_\mathbb{T}, \mathbb{R}^n)$ such that

$$\sup_{t \in I_\mathbb{T}} \frac{|x(t)|}{e_\beta(t, a)} < \infty,$$

and denote it by $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$. The space $C_\beta(I_\mathbb{T}, \mathbb{R}^n)$, endowed with the Bielecki type norm

$$\|x\|_\beta = \sup_{t \in I_\mathbb{T}} \frac{|x(t)|}{e_\beta(t, a)},$$

is a Banach space (see [8, 23]).

3. Main results

Firstly, we generalize equation (1.1) and prove that it has the Hyers-Ulam-Rassias stability on time scales.

Consider the following non-linear Volterra integral equation

$$x(t) = f(t) + \int_a^t K\left(t, s, x(s), x(\sigma(s))\right) \Delta s, \qquad s, t \in I_{\mathbb{T}} = [a, \infty) \cap \mathbb{T}, \qquad (3.1)$$

where $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$, $K \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $x : I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function.

Theorem 3.1. *Let k_1, k_2, L_1 and L_2 are positive constants and assume that $K : I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which additionally satisfies*

$$|K(t, s, p, q) - K(t, s, \overline{p}, \overline{q})| \leq L_1 |p - \overline{p}| + L_2 |q - \overline{q}|$$

for $t, s \in I_{\mathbb{T}}$ and $p, q, \overline{p}, \overline{q} \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$. If a function $g \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ satisfies

$$\left| g(t) - f(t) - \int_a^t K\left(t, s, g(s), g(\sigma(s))\right) \Delta s \right| \leq \theta(t), \qquad t \in I_{\mathbb{T}}, \qquad (3.2)$$

where $\theta \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$ with

$$\int_a^t e_{L_1}\left(t, \sigma(s)\right) \theta(s) \Delta s \leq k_1 \theta(t) \text{ and } \int_a^t e_{L_2}\left(t, \sigma\left(\sigma(s)\right)\right) \theta(s) \Delta s \leq k_2 \theta(t),$$

then there exists a unique solution $u \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ of equation (3.1) such that

$$|g(t) - u(t)| \leq (1 + k_1 L_1 + k_2 L_2)\theta(t), \qquad t \in I_{\mathbb{T}}.$$

Proof. Set

$$l(t) = g(t) - f(t) - \int_a^t K\left(t, s, g(s), g(\sigma(s))\right) \Delta s, \qquad t \in I_{\mathbb{T}}.$$

Then, by (3.2), we have

$$|l(t)| \leq \theta(t).$$

Let $u \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ be the unique solution of equation (3.1). Hence, we obtain

$$\begin{aligned} & |g(t)-u(t)| \\ = & \left| l(t) + f(t) + \int_a^t K(t,s,g(s),g(\sigma(s)))\,\Delta s \; - f(t) - \int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s \right| \\ \leq & |l(t)| + \left| \int_a^t [K(t,s,g(s),g(\sigma(s))) - K(t,s,u(s),u(\sigma(s)))]\Delta s \right| \\ \leq & \theta(t) + \int_a^t |K(t,s,g(s),g(\sigma(s))) - K(t,s,u(s),u(\sigma(s)))|\,\Delta s \\ \leq & \theta(t) + L_1 \int_a^t |g(s)-u(s)|\,\Delta s + L_2 \int_a^t |g(\sigma(s)) - u(\sigma(s))|\,\Delta s. \end{aligned}$$

By using Gronwall's Inequality in [6, Section 6.1], we have that

$$\begin{aligned} |g(t)-u(t)| & \leq \theta(t) + L_1 \int_a^t e_{L_1}(t,\sigma(s))\,\theta(s)\Delta s + L_2 \int_a^t e_{L_2}(t,\sigma(\sigma(s)))\,\theta(s)\Delta s \\ & \leq \theta(t) + L_1 k_1 \theta(t) + L_2 k_2 \theta(t) \\ & \leq (1 + k_1 L_1 + k_2 L_2)\theta(t). \end{aligned}$$

This shows that equation (3.1) has the Hyers-Ulam-Rassias stability.

Corollary 3.2. *Under the assumptions of Theorem 3.1, if we take θ as a constant function then we say that equation (3.1) has the Hyers-Ulam stability.*

Secondly, by generalizing equation (1.2), the Hyers-Ulam and Hyers-Ulam-Rassias stabilities on time scales are proved. The Volterra integral equation examined at this stage is as follows:

$$x(t) = f(t) + \lambda \int_a^t K(t,s)\,x(s)\Delta s, \qquad t \in I_{\mathbb{T}} = [a,b] \cap \mathbb{T}, \tag{3.3}$$

where $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$, $\lambda \in \mathbb{R}$, $K \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}}, \mathbb{R})$ and $x : I_{\mathbb{T}} \to \mathbb{R}$ is the unknown function.

Theorem 3.3. *The integral equation (3.3) on $I_{\mathbb{T}}$ has the Hyers-Ulam-Rassias stability, that is, for a fixed function $\omega \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$ we have that for every $\psi \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$ with*

$$\left| \psi(t) - f(t) - \lambda \int_a^t K(t,s)\,\psi(s)\Delta s \right| \leq \omega(t),$$

for which there exist constants P and M with $|\lambda M P| < 1$ such that $\int_a^t \omega(s)\Delta s \leq P\omega(t)$ and $|K(t,s)| \leq M$, $\forall t,s \in I_{\mathbb{T}}$, then there exists a unique $\varphi \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$ such that $[\Psi\varphi](t) = \varphi(t)$ and $|\psi(t) - \varphi(t)| \leq C\omega(t)$ for some $C > 0$.

Proof. Consider the following iterative scheme

$$\psi_n(t) := f(t) + \lambda \int_a^t K(t,s)\, \psi_{n-1}(s)\Delta s, \qquad n = 1, 2, 3, ... \tag{3.4}$$

for $t \in I_{\mathbb{T}}$ with $\psi_0(t) = \psi(t)$. We prove that $\{\psi_n(t)\}_{n\in\mathbb{N}}$ converges uniformly to the unique solution of equation (3.3) on $I_{\mathbb{T}}$. We write $\psi_n(t)$ as a telescoping sum

$$\psi_n(t) = \psi_0(t) + \sum_{i=1}^{n} \left[\psi_i(t) - \psi_{i-1}(t)\right],$$

so

$$\lim_{n\to\infty} \psi_n(t) = \psi_0(t) + \sum_{i=1}^{\infty} \left[\psi_i(t) - \psi_{i-1}(t)\right], \qquad \forall t \in I_{\mathbb{T}}. \tag{3.5}$$

By mathematical induction, it is easy to see that the following estimate

$$|\psi_n(t) - \psi_{n-1}(t)| \le (\lambda MP)^{n-1}\omega(t) \tag{3.6}$$

holds for each $n \in \mathbb{N}$ and all $t \in I_{\mathbb{T}}$. For $n = 1$, we have

$$|\psi_1(t) - \psi(t)| \le \omega(t).$$

Hence, estimate (3.6) holds for $n = 1$. Assume that estimate (3.6) is true for $n = k \ge 1$. We have

$$\begin{aligned} |\psi_{k+1}(t) - \psi_k(t)| &\le \lambda \int_a^t |K(t,s)|\, |\psi_k(s) - \psi_{k-1}(s)|\, \Delta s \\ &\le \lambda M \int_a^t (\lambda MP)^{k-1}\omega(s)\Delta s \\ &\le (\lambda MP)^k \omega(t), \end{aligned}$$

hence estimate (3.6) is valid for $n = k + 1$. This shows that estimate (3.6) is true for all $n \ge 1$ on $I_{\mathbb{T}}$. We see that

$$|\psi_i(t) - \psi_{i-1}(t)| \le (\lambda MP)^{i-1}\omega(t),$$

and

$$\sum_{i=1}^{\infty} (\lambda MP)^{i-1}\omega(t) = \frac{\omega(t)}{1 - \lambda MP}.$$

Applying Weierstrass M-Test, we conclude that the infinite series

$$\sum_{i=1}^{\infty} \left[\psi_i(t) - \psi_{i-1}(t)\right]$$

converges uniformly on $t \in I_{\mathbb{T}}$. Thus from (3.5), the sequence $\{\psi_n(t)\}_{n\in\mathbb{N}}$ converges uniformly on $I_{\mathbb{T}}$ to some $\varphi(t) \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$. Next, we show that the limit of the sequence $\varphi(t)$ is the exact solution of (3.3). For all $t \in I_{\mathbb{T}}$ and each $n \ge 1$, we have

$$\left| \int_a^t K(t,s)\, \psi_n(s)\Delta s - \int_a^t K(t,s)\, \varphi(s)\Delta s \right| \le M \int_a^t |\psi_n(s) - \varphi(s)|\, \Delta s.$$

Taking the limits as $n \to \infty$ we see that the right hand side of the above inequality tends to zero and so

$$\lim_{n\to\infty} \int_a^t K(t,s)\,\psi_n(s)\Delta s = \int_a^t K(t,s)\,\varphi(s)\Delta s, \qquad \forall\, t \in I_{\mathbb{T}}.$$

By letting $n \to \infty$ on both sides of (3.4) we conclude that $\varphi(t)$ is the unique solution of (3.3). Then there exists a positive integer N such that $|\psi_N(t) - \varphi(t)| \leq \omega(t)$. Hence

$$\begin{aligned}
|\psi - \varphi| &\leq |\psi(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)| \\
&\leq |\psi(t) - \psi_1(t)| + |\psi_1(t) - \psi_2(t)| + ... + |\psi_{N-1}(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)| \\
&\leq \sum_{i=1}^{N} |\psi_{i-1}(t) - \psi_i(t)| + |\psi_N(t) - \varphi(t)| \\
&\leq \sum_{i=1}^{N} (\lambda MP)^{i-1}\omega(t) + |\psi_N(t) - \varphi(t)| \\
&\leq \sum_{i=1}^{\infty} (\lambda MP)^{i-1}\omega(t) + \omega(t) \\
&\leq \frac{1}{1-\lambda MP}\omega(t) + \omega(t) = \left(1 + \frac{1}{1-\lambda MP}\right)\omega(t) \ = C.\omega(t),
\end{aligned}$$

which shows that (3.3) has Hyers-Ulam-Rassias stability on $I_{\mathbb{T}}$.

Corollary 3.4. *Under the assumptions of Theorem 3.1, if we take ω as a constant function then we obtain the Hyers-Ulam stability result of equation (3.3).*

Thirdly, we generalize equation (1.3) to the time scale and show that it has Hyers-Ulam and Hyers-Ulam-Rassias stabilities. Consider the following Volterra integral equation

$$x(t) = f(t) + \varphi\left(\int_a^t K(t,s,x(s))\,\Delta s\right), \qquad t \in I_{\mathbb{T}} = [a,b] \cap \mathbb{T}, \tag{3.7}$$

where $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$, $K \in C_{rd}(I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n, \mathbb{R}^n)$, $x : I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function and φ is a bounded linear transformation on $I_{\mathbb{T}}$.

Theorem 3.5. *Let $f \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$, $K : I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be jointly continuous in its first and third variables and rd-continuous in its second variable, $L : I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\gamma > 1$ and $\beta(s) = L(s)\gamma$, where $\gamma > \|\varphi\|$. If*

$$|K(t,s,p) - K(t,s,q)| \leq L(s)\,|p-q|\,, \qquad p,q \in \mathbb{R}^n, \qquad s < t,$$

$$m = \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \varphi\left(\int_a^t K(t,s,0)\,\Delta s\right)\right| < \infty,$$

then the integral equation (3.7) has a unique solution $x \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$.

Proof. Consider the Banach space $C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$. To prove the result, we define an operator $F : C_\beta(I_{\mathbb{T}}, \mathbb{R}^n) \to C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ by

$$[Fx](t) = f(t) + \varphi\left(\int_a^t K(t, s, x(s))\,\Delta s\right). \tag{3.8}$$

The fixed point of F will be solution to (3.7). Thus we want to prove that there exists a unique x such that $Fx = x$. For any $u, v \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|Fu - Fv\|_\beta &= \sup_{t\in I_{\mathbb{T}}} \frac{|[Fu](t) - [Fv](t)|}{e_\beta(t,a)} \\
&\le \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left|\varphi\left(\int_a^t [K(t,s,u(s)) - K(t,s,v(s))]\Delta s\right)\right| \\
&\le \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \|\varphi\| \int_a^t L(s)\,|u(s) - v(s)|\,\Delta s \\
&= \|\varphi\| \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \frac{|u(s) - v(s)|}{e_\beta(s,a)} \Delta s \\
&\le \|\varphi\| \|u - v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \Delta s \\
&\le \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s) e_\beta(s,a) \Delta s \\
&= \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a) \Delta s \\
&= \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta \sup_{t\in I_{\mathbb{T}}} \left[1 - \frac{1}{e_\beta(t,a)}\right] \\
&\le \frac{\|\varphi\|}{\gamma} \|u - v\|_\beta .
\end{aligned}$$

Next, we show that $F : C_\beta(I_{\mathbb{T}}, \mathbb{R}^n) \to C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$. Let $x \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$. Taking norms, we get

$$\begin{aligned}
\|Fx\|_\beta &= \|Fx - F0 + F0\|_\beta \le \|Fx - F0\|_\beta + \|F0\|_\beta \\
&\le \frac{\|x\|_\beta}{\gamma} \|\varphi\| + m < \infty.
\end{aligned}$$

As $\frac{\|\varphi\|}{\gamma} < 1$, we see that F is a contraction self mapping on $C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ and so Banach's fixed point theorem applies, yielding the existence of a unique fixed point x of F.

Theorem 3.6. *Under the assumptions of Theorem* 3.5 *the equation* $Fx = x$, *where* $F \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ *is defined as in (3.8) has the Hyers-Ulam-Rassias stability, that is, for a fixed function* $\psi \in C_{rd}(I_{\mathbb{T}}, \mathbb{R})$ *we have that for every* $x \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ *with*

$\|x - Fx\|_\beta \leq \psi(t)$ *there exists a unique* $x_0 \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ *such that* $Fx_0 = x_0$ *and* $\|x - x_0\|_\beta \leq C\psi(t)$.

Proof. From Theorem 3.5, there exists a unique $x_0 \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ such that $Fx_0 = x_0$. Then, we have

$$\begin{aligned} \|x - x_0\|_\beta &\leq \|x - Fx\|_\beta + \|Fx - x_0\|_\beta \\ &\leq \psi(t) + \|Fx - Fx_0\|_\beta \\ &\leq \psi(t) + \frac{\|x - x_0\|_\beta}{\gamma} \|\varphi\|. \end{aligned}$$

Hence, we obtain

$$\|x - x_0\|_\beta \leq C\psi(t),$$

where $C = \left[1 - \frac{\|\varphi\|}{\gamma}\right]^{-1} > 0$.

Remark. *If we take* $\varphi(x)$ *as a constant function and consider the complete metric space* $(C[a,b], \|.\|_\infty)$*, then we get Theorem 4.1 in* [10].

Example 3.1. *Consider the scalar integral equation*

$$x(t) = t^4 + \varphi\left(\int_a^t (s + \sigma(s)) \left[x(s)^2 + 1\right]^{\frac{1}{2}} \Delta s\right), \qquad a, t \in I_{\mathbb{T}}, \quad a \geq 0.$$

We claim that this integral equation has a unique solution for arbitrary $\mathbb{T}$ *and the equation* $Fx = x$ *has the Hyers-Ulam Rassias stability.*

Proof. We will use Theorem 3.5 and make use of the fact that $K(t, s, p) = (s + \sigma(s)) \left[p^2 + 1\right]^{\frac{1}{2}}$ has a bounded partial derivative with respect to p everywhere. Consider

$$\begin{aligned} |K(t, s, p) - K(t, s, q)| &= \left|(s + \sigma(s)) \left[p^2 + 1\right]^{\frac{1}{2}} - (s + \sigma(s)) \left[q^2 + 1\right]^{\frac{1}{2}}\right| \\ &\leq (s + \sigma(s)) \sup_{r \in \mathbb{R}} \left|\frac{r}{\left[r^2 + 1\right]^{\frac{1}{2}}}\right| |p - q| \\ &\leq (s + \sigma(s)) |p - q|. \end{aligned}$$

We here used the mean value theorem. So, we have that $L(s) = s + \sigma(s)$. For choices of $\gamma = 2$ and $\|\varphi\| = \frac{2}{3}$, we have $\beta(s) = 2(s + \sigma(s))$. Using Bernoulli's Inequality in [18, p.42] we get

$$e_\beta(t, a) \geq 1 + t^2 - a^2,$$

which is followed by estimate $m < \infty$. The result now follows from Theorem 3.5.

If a function $x \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ satisfies the inequality $\|x - Fx\|_\beta \leq \psi(t)$ where $\psi(t)$ non-negative function then Theorem 3.6 implies that there exists a unique x_0 such that $Fx_0 = x_0$ and

$$\|x - x_0\|_\beta \leq \frac{3}{2}\psi(t).$$

From Theorem 3.6, we obtain the following result related to the Hyers-Ulam stability.

Corollary 3.7. *If* ψ *is a constant function, then equation (3.7) has the Hyers-Ulam stability.*

Finally, we showed Hyers-Ulam-Rassias stability of equation (1.4) on time scales.

Theorem 3.8. *If $x_0 \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ is a solution of the Volterra type integral equation (1.4) and*

$$M\left(1+\frac{1+q\gamma}{1-r\gamma}+\frac{1}{\gamma}\right)<1,$$

then the Volterra type integral equation (1.4) is Hyers-Ulam-Rassias stable, that is, for a fixed function $\psi \in C_\beta(I_{\mathbb{T}}, \mathbb{R})$ we have that for every $x \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ with

$$\sup_{t\in I_{\mathbb{T}}} \frac{\left|x(t)-f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s))\Delta s\right)\right|}{e_\beta(t,a)} = \|x-Fx\|_\beta \le \psi(t),$$

there exists a unique $x_0 \in C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$ such that $Fx_0 = x_0$ and $\|x-x_0\|_\beta \le C\psi(t)$.

Proof. According to Theorem 3.1 in [18], there is a unique solution x_0 to the Volterra type integral equation (1.4) in the Banach space $C_\beta(I_{\mathbb{T}}, \mathbb{R}^n)$. Therefore we have

$$\begin{aligned} \|x-x_0\|_\beta &\le \|x-Fx\|_\beta + \|Fx-Fx_0\|_\beta \\ &\le \psi(t) + M\left(1+\frac{1+q\gamma}{1-r\gamma}+\frac{1}{\gamma}\right)\|x-x_0\|_\beta . \end{aligned}$$

So we get

$$\|x-x_0\|_\beta \le C\psi(t),$$

where

$$C = \left(1-M\left(1+\frac{1+q\gamma}{1-r\gamma}+\frac{1}{\gamma}\right)\right)^{-1}.$$

Remark. *Choosing $\psi(t)=\epsilon$ in Theorem 3.8, yields Theorem 3.3 in* [18].

4. Conclusion

In this paper, the Hyers-Ulam stability and Hyers-Ulam-Rassias stability theorems for four types of Volterra integral equations on time scales were investigated. Additionally, a numerical example to support the study was given. The theorems proved here generalize some recent results given in [3, 10, 18, 26].

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THE EIGENVALUES AND EIGENVECTORS OF THE 5D DISCRETE FOURIER TRANSFORM NUMBER OPERATOR REVISITED

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Abstract. A systematic analytic approach to the evaluation of the eigenvalues and eigenvectors of the 5D discrete number operator $\mathcal{N}_5$ is formulated. This approach is essentially based on the use of the symmetricity of 5D discrete Fourier transform operator Φ_5 with respect to the discrete reflection operator P_d.

1. Introduction

Let me begin by recalling first that the eigenfunctions of the classical Fourier inegral transform (FIT), associated with the eigenvalues i^n, are explicitly given as

$$\psi_n(x) := H_n(x)\exp(-x^2/2), \quad n = 0, 1, 2, ..., \tag{1.1}$$

where $H_n(x)$ are Hermite polynomials. The functions $\psi_n(x)$ are usually referred to as *Hermite functions* in the mathematical literature, whereas in quantum mechanics they emerge as eigenfunctions of the Hamiltonian for the linear harmonic oscillator, which is a self-adjoint differential operator of the second order (see, for example, [1]). It is well known that the functions $\psi_n(x)$ are either symmetric or antisymmetric with respect to the reflection operator P, defined on the full real line $x \in \mathbb{R}$ as $P\,x = -\,x$; that is,

$$P\,\psi_n(x) = \psi_n(-x) = (-1)^n\,\psi_n(x). \tag{1.2}$$

Recall also that the discrete (finite) Fourier transform (DFT) based on N points is represented by an $N \times N$ unitary symmetric matrix Φ with entries

$$\Phi_{kl} = N^{-1/2}\,q^{kl}, \qquad k, l \in \mathbb{Z}_N := \{0, 1, 2 \ldots, N-1\}, \tag{1.3}$$

where $q = \exp(2\pi\mathrm{i}/N)$ is a primitive N-th root of unity and N is an arbitrary integer (see, for example, [2]-[7]). The discrete analogue of the above mentioned

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reflection operator P, associated with the DFT operator (1.3), is represented by the $N \times N$ matrix

$$P_d := C^{\intercal} J_N \equiv J_N\, C\,, \tag{1.4}$$

where C is the *basic circulant permutation* matrix with entries $C_{kl} = \delta_{k,l-1}$ and J_N is the $N \times N$ 'backward identity' permutation matrix with ones on the secondary diagonal (see [8], pages 26 and 28, respectively). Note that the matrix of the discrete reflection operator (1.4) can be partitioned as

$$P_d = \begin{bmatrix} 1 & 0_{N-1} \\ 0^{\intercal}_{N-1} & J_{N-1} \end{bmatrix}, \tag{1.5}$$

where 0_m and $0^{\intercal}_n$ are m-row and n-column zero vectors, respectively.

It is readily verified that the DFT operator (1.3) is P_d-symmetric, that is, the commutator $[\Phi, P_d] := \Phi\, P_d - P_d\, \Phi = 0$. Therefore, similar to the continuous case (1.2), the eigenvectors of the DFT operator Φ should be either P_d-symmetric or P_d-antisymmetric.

The purpose of this work is to discuss some additional findings concerning symmetry properties of two finite-dimensional intertwining operators with the DFT matrix 1.3. These operators are represented by matrices A and $A^{\intercal}$ of the same size $N \times N$ such that the intertwining relations

$$A\,\Phi = \mathrm{i}\,\Phi A, \qquad A^{\intercal}\,\Phi = -\,\mathrm{i}\,\Phi\, A^{\intercal}, \tag{1.6}$$

are valid. The explicit form of the matrices A and $A^{\intercal}$ is

$$A = X + \mathrm{i}Y = X + D\,, \qquad A^{\intercal} = X - \mathrm{i}Y = X - D\,, \tag{1.7}$$

where $X = diag\,(\mathsf{s}_0, \mathsf{s}_1, ..., \mathsf{s}_{N-2}, \mathsf{s}_{N-1})$, $\mathsf{s}_n := 2\sin(2\pi n/N)$, $n \in \mathbb{Z}_N$, and $Y = -\,\mathrm{i}\,\mathrm{D} = \mathrm{i}\,(\mathrm{C}^{\intercal} - \mathrm{C})$. The operators X and Y are Hermitian and play the role of finite-dimensional analogs of the operators of the coordinate and momentum in quantum mechanics, respectively.

The intertwining operators A and $A^{\intercal}$ have emerged in a paper [9] devoted to the problem of finding the eigenvectors of the DFT operator Φ. They can be interpreted as discrete analogs of the quantum harmonic oscillator lowering and raising operators $\mathbf{a} = 2^{-1/2}\Big(x + \frac{d}{dx}\Big)$ and $\mathbf{a}^{\dagger} = 2^{-1/2}\Big(x - \frac{d}{dx}\Big)$; their algebraic properties had been studied in detail in [10] -[12]. In particular, it was shown in [12] that the operators A and $A^{\intercal}$ form a cubic algebra $\mathcal{C}_q$ with q a root of unity. This algebra is intimately related to the two other well-known realizations of the cubic algebra: the Askey-Wilson algebra [13]–[16] and the Askey-Wilson-Heun algebra [17]. Note also that from the intertwining relations (1.6) it follows at once that the operator $\mathcal{N} := A^{\intercal}A$ commutes with the DFT operator Φ, that is, $[\mathcal{N}\,, \Phi] = 0$. The discrete number operator $\mathcal{N}$ and the DFT operator Φ thus have the same eigenvectors and one can employ the former for finding an explicit form of the eigenvectors of the latter (see [9] for a more detailed discussion of this point).

This idea that the discrete number operator $\mathcal{N}$ is the one that really governs the eigenvectors of the DFT operator Φ, was first successfully tested in [18] by considering the particular case of the 5D DFT operator Φ_5. But the explicit form of the 4 nonzero eigenvalues $\lambda_k, 1 \leq k \leq 4$, of the discrete number operator $\mathcal{N}_5$ have been found in [18] by using *Mathematica*. So it is the main goal of this work to formulate a systematic analytic approach to the evaluation of the above-mentioned eigenvalues λ_k without resorting to the help of any computer programs.

The lay out of the paper is as follows. In section 2 a detailed account is given on how one can construct a P_d-symmetrized basis in the eigenspace $\mathcal{H}_5$ of the discrete number operator $\mathcal{N}_5$, in terms of the eigenvectors of either the operator X_5, or the operator Y_5. In section 3 it is shown that the eigenspace $\mathcal{H}_5$ with thus symmetrized basis splits into two 3D and 2D subspaces $\mathcal{H}_3$ and $\mathcal{H}_2$; this remarkable fact is used then to find desired explicit forms of the eigenvalues and eigenvectors of the discrete number operator $\mathcal{N}_5$. Finally, section 4 briefly outlines some further research directions of interest.

2. 5D operators X_5 and Y_5 in the P_d-symmetrized basis

This section begins by a quotation from [12]: *It is a remarkable fact that the operators X and Y are "classical" operators with nice spectral properties.* For the 5D operator $X_5 = diag\,(\mathsf{s}_0, \mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3, \mathsf{s}_4)$, it is obvious because the spectrum of X_5 is

$$\lambda_n \,=\, \mathsf{s}_n = \mathrm{i}(q^{-\,n} - q^n), \qquad n \in \mathbb{Z}_5, \tag{2.1}$$

where $q = \exp(2\pi\mathrm{i}/5)$ and we introduced for brevity $\mathsf{s}_n := 2\sin(2\pi n/5)$. This indicates that the spectrum (2.1) belongs to the class of the Askey–Wilson spectra of the type

$$\lambda_n \,=\, C_1 q^n + C_2 q^{-\,n} + C_0\,. \tag{2.2}$$

The eigenvectors of the operator X_5 are represented by the Euclidean 5-column orthonormal vectors e_k with the components $(e_k)_l = \delta_{kl}$, $k, l \in \mathbb{Z}_5$, that is,

$$X_5\, e_k = \mathsf{s}_k\, e_k\,. \tag{2.3}$$

The spectrum of the matrix Y_5 belongs to the same Askey–Wilson family since the operators X_5 and Y_5 are unitary equivalent, $Y_5 = \Phi X_5 \Phi^\dagger$, and hence isospectral [12]. Note that the spectrum of X_5 is simple, i.e., it is nondegenerate. Also, from the unitary equivalence of the operators X_5 and Y_5 it follows that the eigenvectors of the latter operator are of the form

$$Y_5\, \epsilon_k = \mathsf{s}_k\, \epsilon_k\,, \qquad \epsilon_n := \Phi\, e_n = 5^{-1/2}\Big(1, q^n, q^{2n}, q^{3n}, q^{4n}\Big)^{\mathsf{T}}. \tag{2.4}$$

Let me draw attention now to the remarkable symmetry between the operators X_5 and Y_5: the operator X_5 is two-diagonal in the eigenbasis of the operator Y_5,

$$X_5\, \epsilon_n = \mathrm{i}\,(\epsilon_{n-1} - \epsilon_{n+1})\,, \tag{2.5}$$

whereas the operator Y_5 is similarly two-diagonal in the eigenbasis of the operator X_5,

$$Y_5\, e_n = \mathrm{i}\,(e_{n+1} - e_{n-1})\,. \tag{2.6}$$

Remark. *It may also be worth mentioning here that the N-column eigenvectors of the operator Y for a general N,*

$$\epsilon_n = \Phi\, e_n = \sum_{k=0}^{N-1} \Phi_{kn}\, e_k = N^{-1/2}\Big(1, q^n, q^{2n}, \ldots, q^{(N-1)n}\Big)^{\mathsf{T}}, \tag{2.7}$$

form an orthonormal basis in the N-dimensional complex plane $\mathbb{C}^N$ and are frequently used therefore as building blocks of the discrete Fourier transform in applications (see, for example, p.130 in [19]*, where the ϵ_n referred to as discrete trigonometric functions).*

Since the operators X and Y generate a particular algebra, associated with the DFT operator Φ for arbitrary integer values of N, one should use the eigenvectors of either the operator X, or the operator Y, as the most convenient basis for finding explicit forms of the eigenvectors of the operator Φ. But we know that the eigenvectors of the operator Φ should be either P_d-symmetric, or P_d-antisymmetric, whereas the eigenvectors of both the operators X and Y do not reveal any symmetry property of this type. The point is that the reflection operator P_d acts in the same way on both the eigenvectors e_n and ϵ_m, that is,

$$P_d\, e_n = e_{N-n}, \quad e_N = e_0\,, \qquad P_d\, \epsilon_n = \epsilon_{N-n}, \quad \epsilon_N = \epsilon_0\,. \tag{2.8}$$

Hence, the reflection operator P_d does not transform the eigenvectors e_0 and ϵ_0, and acts similarly by *cyclic permutation* on the other eigenvectors e_m and ϵ_n, with $1 \leq m, n \leq N-1$. To overcome this type of obstacle on the way of finding the eigenvectors of the operator Φ, one thus needs to find first some P_d-symmetric bases, associated with both of the operators X and Y. This can be achieved as follows.

Returning now to the case of the 5D operators X_5 and Y_5, let us consider first unit column-vectors $\widetilde{e}_n, n \in \mathbb{Z}_5$, defined in terms of the eigenvectors e_n of the operator X_5 as

$$\widetilde{e}_0 = e_0, \qquad \widetilde{e}_k = \frac{1}{\sqrt{2}}\,(e_k - e_{5-k}),\ k = 1,2, \qquad \widetilde{e}_l = \frac{1}{\sqrt{2}}\,(e_l + e_{5-l}),\ l = 3,4. \tag{2.9}$$

The explicit componentwise forms of the thus P_d-symmetrized column-vectors $\widetilde{e}_n$ are

$$\widetilde{e}_0 = (1,0,0,0,0)^{\intercal}, \quad \widetilde{e}_1 = \frac{1}{\sqrt{2}}\,(0,1,0,0,-1)^{\intercal}, \quad \widetilde{e}_2 = \frac{1}{\sqrt{2}}\,(0,0,1,-1,0)^{\intercal},$$

$$\widetilde{e}_3 = \frac{1}{\sqrt{2}}\,(0,0,1,1,0)^{\intercal}, \quad \widetilde{e}_4 = \frac{1}{\sqrt{2}}\,(0,1,0,0,1)^{\intercal}. \tag{2.10}$$

The interrelation (2.9) between the vectors $\widetilde{e}_n$ and the eigenvectors e_n of the operator X_5 can be written in the compact form as $\widetilde{e}_k = T e_k$, $k \in \mathbb{Z}_5$, where the unitary matrix T is

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix},$$

$$T^{-1} = T^{\intercal}\,, \quad T\,T^{-1} = T^{-1}T = I_5. \tag{2.11}$$

Note that from the geometric point of view the matrix T represents simply a product of two rotations by the same angle $\alpha = \pi/4$ in the 14- and 23-planes of the 5D-space, that is, $T = R_{14}(\pi/4)R_{23}(\pi/4)$, where

$$R_{14}(\pi/4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & 0 & 0 & \sin\frac{\pi}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\sin\frac{\pi}{4} & 0 & 0 & \cos\frac{\pi}{4} \end{bmatrix},$$

$$R_{23}(\pi/4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ 0 & 0 & -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

Similarly, let us introduce now 5 orthonormal column-vectors $\widetilde{\epsilon}_n, n \in \mathbb{Z}_5$, defined in terms of the eigenvectors ϵ_n of the operator Y_5 as $\widetilde{\epsilon}_k = T\epsilon_k,\ k \in \mathbb{Z}_5$, with the same matrix T as in (2.11). Then the explicit forms of these P_d-symmetrized 5-vectors $\widetilde{\epsilon}_n$ are

$$\widetilde{\epsilon}_0 = \epsilon_0 = \frac{1}{\sqrt{5}}(1,1,1,1,1)^{\mathsf{T}}, \qquad \widetilde{\epsilon}_1 = \frac{1}{\sqrt{2}}\left(\epsilon_1 - \epsilon_4\right) = \frac{\mathrm{i}}{\sqrt{10}}(0, \mathsf{s}_1, \mathsf{s}_2, -\mathsf{s}_2, -\mathsf{s}_1)^{\mathsf{T}},$$

$$\widetilde{\epsilon}_2 = \frac{1}{\sqrt{2}}\left(\epsilon_2 - \epsilon_3\right) = \frac{\mathrm{i}}{\sqrt{10}}(0, \mathsf{s}_2, -\mathsf{s}_1, \mathsf{s}_1, -\mathsf{s}_2)^{\mathsf{T}},$$

$$\widetilde{\epsilon}_3 = \frac{1}{\sqrt{2}}\left(\epsilon_2 + \epsilon_3\right) = \frac{\mathrm{i}}{\sqrt{10}}(\mathsf{c}_0, \mathsf{c}_2, \mathsf{c}_1, \mathsf{c}_1, \mathsf{c}_2)^{\mathsf{T}},$$

$$\widetilde{\epsilon}_4 = \frac{1}{\sqrt{2}}\left(\epsilon_1 + \epsilon_4\right) = \frac{\mathrm{i}}{\sqrt{10}}(\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_2, \mathsf{c}_1)^{\mathsf{T}}, \tag{2.13}$$

where $\mathsf{c}_n := 2\cos\left(2\pi n/5\right)$.

It remains only to recall that if $Z_{kl} := (e_k, Ze_l)$ represent the matrix elements of the operator (matrix) Z in the basis e_n, then the matrix $\tilde{Z} := TZT^{-1}$ represents the matrix elements of the same operator Z in the basis $\widetilde{e}_n = Te_n$. Hence the explicit forms of the operators X_5 and Y_5 in the P_d-symmetrized basis $\widetilde{e}_n$ are

$$\tilde{X}_5 = TX_5T^{-1} = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathsf{s}_1 \\ 0 & 0 & 0 & \mathsf{s}_2 & 0 \\ 0 & 0 & \mathsf{s}_2 & 0 & 0 \\ 0 & \mathsf{s}_1 & 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0_{33} & x_{32} \\ x_{32}^{\mathsf{T}} & 0_{22} \end{bmatrix}, \qquad x_{32} := \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{s}_1 \\ \mathsf{s}_2 & 0 \end{bmatrix},$$

$$\tilde{D}_5 = TD_5T^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ \sqrt{2} & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0_{33} & d_{32} \\ -d_{32}^{\mathsf{T}} & 0_{22} \end{bmatrix}, \qquad d_{32} := \begin{bmatrix} 0 & -\sqrt{2} \\ -1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{2.14}$$

where 0_{nn} is the $n \times n$ zero matrix.

Finally, from (2.14) it follows that the intertwining operators A_5 and A_5^{T} in the basis $\widetilde{e}_n$ can be partitioned as

$$\tilde{A}_5 = \widetilde{X}_5 + \widetilde{D}_5 = \begin{bmatrix} 0_{33} & -a_{32}(s) \\ a_{32}^{\mathsf{T}}(-s) & 0_{22} \end{bmatrix}, \qquad a_{32}(s) := \begin{bmatrix} 0 & \sqrt{2} \\ 1 & \mathsf{s}_1 \\ \mathsf{s}_2 - 1 & -1 \end{bmatrix},$$

$$\tilde{A}_5^{\intercal} = \widetilde{X}_5 - \widetilde{D}_5 = \begin{bmatrix} 0_{33} & a_{32}(-s) \\ -a_{32}^{\intercal}(s) & 0_{22} \end{bmatrix}, \quad a_{32}(-s) := \begin{bmatrix} 0 & \sqrt{2} \\ 1 & -\mathsf{s}_1 \\ -\mathsf{s}_2 - 1 & -1 \end{bmatrix}. \tag{2.15}$$

To close this section, it may be worth drawing attention now to the particular manner in which the P_d-symmetrization modifies explicit forms of the DFT eigenvectors f_k, $k \in \mathbb{Z}_5$ in the basis $\widetilde{e}_n$. It is obvious that the partitioning of the intertwining operators $\tilde{A}_5$ and $\tilde{A}_5^{\intercal}$ of the form (2.15) leads to the need to appropriately split the DFT eigenvectors f_k, $k \in \mathbb{Z}_5$ in the basis $\widetilde{e}_n$ into two components, that is, to represent the vectors $\widetilde{f}_k := T f_k$ as

$$\widetilde{f}_k = (\eta_k, \xi_k)^{\intercal}, \qquad \eta_k := (x_0, x_1, x_2), \qquad \xi_k := (x_3, x_4)\,. \tag{2.16}$$

Since every symmetric 5D DFT eigenvector $f^{(s)}$ in the basis e_n is of the form $f^{(s)} = (a, b, c, c, b)^{\intercal}$, whereas every antisymmetric 5D DFT eigenvector $f^{(a)}$ in the same basis e_n is of the form $f^{(a)} = (0, b, c, -c, -b)^{\intercal}$, it turns out that

$$\widetilde{f}^{(s)} := T f^{(s)} = (a, \sqrt{2}b, \sqrt{2}c, 0, 0)^{\intercal}, \quad \widetilde{f}^{(a)} := T f^{(a)} = -\sqrt{2}\,(0, 0, 0, c, b)^{\intercal}. \tag{2.17}$$

This means that all 5D DFT eigenvectors $\widetilde{f}_k$ in the basis $\widetilde{e}_n$ are either of the η-type (that is, with vanishing lower component ξ_k), or of the ξ-type (with the upper component $\eta_k = 0$).

3. DFT number operator N_5 in the P_d-symmetrized basis

Having defined explicitly the matrices A_5 and $A_5^{\intercal}$ in the P_d-symmetrized basis $\widetilde{e}_n$ in the previous section, it is not hard to evaluate that the discrete number operator $\mathcal{N}_5 = A_5^{\intercal} A_5$ in the same basis $\widetilde{e}_n$ is of the following form

$$\tilde{\mathcal{N}}_5 = \tilde{A}_5^{\intercal}\,\tilde{A}_5 = \begin{bmatrix} \mathcal{N}_3 & 0_{32} \\ 0_{32}^{\intercal} & \mathcal{N}_2 \end{bmatrix}, \tag{3.1}$$

where 0_{32} is the 3×2 zero matrix and $\mathcal{N}_3$ and $\mathcal{N}_2$ are 3×3 and 2×2 full Hermitian matrices,

$$\mathcal{N}_3 := \begin{bmatrix} 2 & -\sqrt{2}\,\mathsf{s}_1 & -\sqrt{2} \\ -\sqrt{2}\,\mathsf{s}_1 & 3 - \mathsf{c}_2 & \mathsf{c}_1\mathsf{s}_2 - 1 \\ -\sqrt{2} & \mathsf{c}_1\mathsf{s}_2 - 1 & 2(\mathsf{s}_2 + 2) - \mathsf{c}_1 \end{bmatrix},$$

$$\mathcal{N}_2 := \begin{bmatrix} 2(2 - \mathsf{s}_2) - \mathsf{c}_1 & \mathsf{c}_1\mathsf{s}_2 + 1 \\ \mathsf{c}_1\mathsf{s}_2 + 1 & 5 - \mathsf{c}_2 \end{bmatrix}, \tag{3.2}$$

respectively. Thus the Fock space $\mathcal{H}_5$ of all eigenvectors of the discrete number operator $\mathcal{N}_5$ in the P_d-symmetrized basis $\widetilde{e}_n$ splits into two 3D and 2D subspaces $\mathcal{H}_3$ and $\mathcal{H}_2$; the operator $\tilde{\mathcal{N}}_5$ represents in the eigenspace $\mathcal{H}_5$ the direct sum of the operators $\mathcal{N}_3$ and $\mathcal{N}_2$, that is, $\tilde{\mathcal{N}}_5 = \mathcal{N}_3 \oplus \mathcal{N}_2$.

One clarifying remark must be made at this point in connection with (3.1). The point is that this formula reveals that the discrete number operator $\tilde{\mathcal{N}}_5$ in the P_d-symmetrized basis $\widetilde{e}_n$ has 12 zero matrix elements, whereas its counterpart $\mathcal{N}_5$ in the basis of the eigenvectors e_n is represented by a 5D matrix with 25 nonzero entries. Note that it was possible to formulate such a remarkable transformation of the full matrix $\mathcal{N}_5$ into the *sparse matrix* $\tilde{\mathcal{N}}_5$ only because of the P_d-symmetricity of the DFT operator Φ_5. Recall then that the well-known Fast Fourier Transform algorithm of Cooley and Tukey is based essentially on a factorization of the Fourier matrix into a product of *sparse matrices* (see, for example, [19, 20]). Thus it becomes clear now that Cooley and Tukey had been able to construct so ingeniously their highly efficient implementation of the DFT only because of the P_d-symmetricity of the Fourier matrix, although they had never employed explicitly this fundamental symmetry property of the Fourier matrix.

From (3.1) it is evident that the eigenvectors and eigenvalues of the operator $\tilde{\mathcal{N}}_5$ may be now defined in terms of the eigenvectors and eigenvalues of the operators $\mathcal{N}_3$ and $\mathcal{N}_2$ from the separate subspaces $\mathcal{H}_3$ and $\mathcal{H}_2$, respectively. In order to proceed to this task under consideration, let me start first with the operator $\mathcal{N}_2$.

Lemma 3.1. *An arbitrary 2×2 Hermitian matrix of the form $M = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ can be written as a linear combination of the 2D identity matrix I_2 and the 2×2 traceless matrix M',*

$$M = u\, I_2 + M' = u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} v & b \\ b & -v \end{bmatrix}, \tag{3.3}$$

where $2u = a + d$ and $2v = a - d$. The eigenvalues of the matrix M' are equal to

$$\lambda_{1,2} = \pm \Big(- \det M' \Big)^{1/2} = \pm\, (v^2 + b^2)^{1/2}, \tag{3.4}$$

whereas the eigenvalues of the matrix M are

$$\mu_{1,2} = \lambda_{1,2} + (a + d)/2 = (a + d)/2 \,\pm\, (v^2 + b^2)^{1/2}. \tag{3.5}$$

Proof. Since

$$\det (M' - \lambda\, I_2) = \det \begin{bmatrix} v - \lambda & b \\ b & -v - \lambda \end{bmatrix} = \lambda^2 - v^2 - b^2\,, \tag{3.6}$$

the eigenvalues $\lambda_{1,2}$ of the matrix M' are roots of the quadratic equation $\lambda^2 - v^2 - b^2 = 0$; hence $\lambda_{1,2} = \pm\, (v^2 + b^2)^{1/2}$ and formula (3.4) is proved. Then from (3.3) it follows at once that the eigenvalues of the matix M are equal to $\mu_{1,2} = \lambda_{1,2} + (a + d)/2$ and formula (3.5) is proved as well. ☐

Evidently, the matrix $\mathcal{N}_2$ is of the same type as the matrix M from the lemma above, with $a = 2(2 - \mathsf{s}_2) - \mathsf{c}_1$, $b = \mathsf{c}_1\mathsf{s}_2 + 1$ and $d = 5 - \mathsf{c}_2$. This means that in this particular case $u = 5 - \mathsf{s}_2$, $v = \mathsf{c}_2 - \mathsf{s}_2 = \mathsf{c}_2\, b$ and $v^2 + b^2 = 2\sqrt{5}\,(\mathsf{s}_2 + 2\mathsf{c}_1) = (\mathsf{s}_1)^2\, b^2$. Thus from (3.5) it follows that the eigenvalues of the matrix $\mathcal{N}_2$ are

$$\mu_1 = 5 - \mathsf{s}_2 + \mathsf{s}_1\, b = 5 + \mathsf{s}_2\,(\mathsf{s}_2 + \mathsf{c}_1), \qquad \mu_2 = 5 - \mathsf{s}_2 - \mathsf{s}_1\, b = \mathsf{s}_1\,(\mathsf{s}_1 + \mathsf{c}_2). \tag{3.7}$$

Remark. *The equation which is solved to find eigenvalues of $n \times n$ matrix M is usually interpreted as the equation for finding roots of the characteristic polynomial in λ of degree n,*

$$p_n(\lambda) := \det (\lambda\, I - M) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n\,, \tag{3.8}$$

I is the $n \times n$ identity matrix and the coefficient c_k is $(-1)^k$ times the sum of the determinants of all of the principal $k \times k$ minors of M (in particular, $c_1 = -\operatorname{trace}(M)$ and $c_n = (-1)^n \det M$). The lemma 3.1 has been employed in order to reduce the characteristic equation $p_2(\lambda) = \lambda^2 + c_1\lambda + c_2 = 0$ for the matrix $\mathcal{N}_2$ to the readily solvable equation for the matrix $\mathcal{N}_2'$, which is of the form $p_2(\lambda) = \lambda^2 + c_2 = 0$.

Having defined the eigenvalues μ_1 and μ_2 of the matrix $\mathcal{N}_2$, it is not hard to find eigenvectors of $\mathcal{N}_2$, associated with those eigenvalues (3.7). Indeed, note first that

$$\mathcal{N}_2 = (5-\mathsf{s}_2)\,I_2 + \mathcal{N}_2', \qquad \mathcal{N}_2' = (1+\mathsf{c}_1\,\mathsf{s}_2)\begin{bmatrix} \mathsf{c}_2 & 1 \\ 1 & -\mathsf{c}_2 \end{bmatrix}, \tag{3.9}$$

where the eigenvalues of the matrix $\begin{bmatrix} \mathsf{c}_2 & 1 \\ 1 & -\mathsf{c}_2 \end{bmatrix}$ are $\pm\,\mathsf{s}_1$. Therefore to find the eigenvectors of the matrix $\mathcal{N}_2$, it is sufficient to determine the eigenvectors of the much simpler matrix $\begin{bmatrix} \mathsf{c}_2 & 1 \\ 1 & -\mathsf{c}_2 \end{bmatrix}$. So one readily derives that

$$\begin{bmatrix} \mathsf{c}_2 & 1 \\ 1 & -\mathsf{c}_2 \end{bmatrix}\begin{pmatrix} \mathsf{c}_1 \\ 1+\mathsf{s}_2 \end{pmatrix} = \mathsf{s}_1\begin{pmatrix} \mathsf{c}_1 \\ 1+\mathsf{s}_2 \end{pmatrix},$$
$$\begin{bmatrix} \mathsf{c}_2 & 1 \\ 1 & -\mathsf{c}_2 \end{bmatrix}\begin{pmatrix} 1+\mathsf{s}_2 \\ -\mathsf{c}_1 \end{pmatrix} = -\,\mathsf{s}_1\begin{pmatrix} 1+\mathsf{s}_2 \\ -\mathsf{c}_1 \end{pmatrix}. \tag{3.10}$$

Thus explicit forms of the two linearly independent eigenvectors of the operator $\mathcal{N}_2$, associated with the eigenvalues (3.7), are

$$\varphi_1 := (\mathsf{c}_1, 1+\mathsf{s}_2)^{\mathsf{T}}, \qquad \varphi_2 := (1+\mathsf{s}_2, -\,\mathsf{c}_1)^{\mathsf{T}}, \tag{3.11}$$

respectively. Note that the vectors φ_1 and φ_2 are essentially the same as the down-components of the antisymmetric eigenvectors $\widetilde{f}_1 = Tf_1$ and $\widetilde{f}_3 = Tf_3$ of the discrete number operator $\widetilde{\mathcal{N}}_5$ in the P_d-symmetrized basis $\widetilde{e}_n$, where f_1 and f_3 have been already derived in [18] by employing *Mathematica*; that is

$$\widetilde{f}_1 = (0_3, \varphi_1)^{\mathsf{T}}, \qquad \widetilde{f}_3 = (0_3, \varphi_1)^{\mathsf{T}}. \tag{3.12}$$

Turning now to the case of the matrix $\mathcal{N}_3$, one may likewise employ the polynomial $p_3(\lambda) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$ in order to find first the eigenvalues of $\mathcal{N}_3$. It turns out that the determinant of the matrix $\mathcal{N}_3$ is equal to zero,

$$\det\mathcal{N}_3 = \begin{vmatrix} 2 & -\sqrt{2}\,\mathsf{s}_1 & -\sqrt{2} \\ -\sqrt{2}\,\mathsf{s}_1 & 3-\mathsf{c}_2 & \mathsf{c}_1\mathsf{s}_2 - 1 \\ -\sqrt{2} & \mathsf{c}_1\mathsf{s}_2 - 1 & 2(\mathsf{s}_2+2) - \mathsf{c}_1 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & 0 & 0 \\ -\sqrt{2}\,\mathsf{s}_1 & \mathsf{s}_1 - 2\,\mathsf{c}_2 & -\,\mathsf{s}_2 - 1 \\ -\sqrt{2} & 3(\mathsf{c}_2 - \mathsf{s}_1) - \mathsf{c}_1 & (\mathsf{s}_2+1)^2 \end{vmatrix}$$
$$= 2\begin{vmatrix} \mathsf{s}_1 - 2\,\mathsf{c}_2 & -\,\mathsf{s}_2 - 1 \\ 3(\mathsf{c}_2 - \mathsf{s}_1) - \mathsf{c}_1 & (\mathsf{s}_2+1)^2 \end{vmatrix} = 2(\mathsf{c}_2 - \mathsf{s}_1)\begin{vmatrix} -1 & -1 \\ \mathsf{s}_2 + 1 & \mathsf{s}_2 + 1 \end{vmatrix} = 0\,. \tag{3.13}$$

Hence the characteristic equation for the matrix $\mathcal{N}_3$ reduces to the form

$$\lambda(\lambda^2 + c_1\lambda + c_2) = 0\,. \tag{3.14}$$

Consequently, one of the eigenvalues of the matrix $\mathcal{N}_3$ is $\lambda_0 = 0$, whereas the two remaining eigenvalues of $\mathcal{N}_3$ are roots of the quadratic equation

$$\lambda^2 + c_1\lambda + c_2 = 0\,, \tag{3.15}$$

where the coefficient $c_1 = -\,\mathrm{trace}\,(\mathcal{N}_3) = -2\,(5+\mathsf{s}_2)$ and the coefficient c_2, which represents the sum of the determinants of the three principal 2×2 minors of $\mathcal{N}_3$, is readily evaluated to be $c_2 = 10 + (4\,\mathsf{s}_2 + 3)(\mathsf{s}_1)^2$. So one concludes that

$$\lambda_{1,2} = -\frac{\mathsf{c}_1}{2} \pm \sqrt{\frac{(\mathsf{c}_1)^2}{4} - \mathsf{c}_2} \;=\; 5 + \mathsf{s}_2 \,\pm (\mathsf{c}_1\,\mathsf{s}_2 - 1)\,\mathsf{s}_1, \tag{3.16}$$

upon taking into account the readily verified identity $2(2-\mathsf{s}_1)=(\mathsf{s}_2+\mathsf{c}_2)^2$.

Quite similar to the case of the matrix $\mathcal{N}_2$, the knowledge of the explicit forms of the eigenvalues for the matrix $\mathcal{N}_3$ essentially simplifies the task of defining the appropriate eigenvectors of $\mathcal{N}_3$ for each of those eigenvalues. Indeed, by looking for solutions of the equation $\mathcal{N}_3\,\phi_\lambda=\lambda\,\phi_\lambda$ in the form $\phi_\lambda=(x_0,x_1,x_2)^{\mathsf{T}}$, one arrives simply at a system of three homogeneous equations

$$\begin{array}{cccccl} (2-\lambda)\,x_0 & - & \sqrt{2}\,\mathsf{s}_1\,x_1 & - & \sqrt{2}\,x_2 & =0,\\ -\sqrt{2}\,\mathsf{s}_1\,x_0 & + & (3-\mathsf{c}_2-\lambda)\,x_1 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_2 & =0,\\ -\sqrt{2}\,x_0 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_1 & + & [2(\mathsf{s}_2+2)-\mathsf{c}_1-\lambda]\,x_2 & =0, \end{array} \tag{3.17}$$

for the components of the 3D column-vector ϕ_λ.

1°. In the case of $\lambda_0=0$ the system (3.17) reduces to

$$\begin{array}{cccccl} \sqrt{2}\,x_0 & - & \mathsf{s}_1\,x_1 & - & x_2 & =0,\\ -\sqrt{2}\,\mathsf{s}_1\,x_0 & + & (3-\mathsf{c}_2)\,x_1 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_2 & =0,\\ -\sqrt{2}\,x_0 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_1 & + & [2(\mathsf{s}_2+2)-\mathsf{c}_1]\,x_2 & =0. \end{array} \tag{3.18}$$

Eliminating the component x_0 by adding to the second equation in (3.18) the first one, multiplied by s_1, one arrives at the relation $x_1=(1+\mathsf{s}_2)\,x_2$. Substituting this relation back into the first equation enables one to express the component x_0 via the component x_2 as

$$\sqrt{2}\,x_0=\mathsf{s}_1\,x_1+x_2=(\mathsf{s}_1-2\,\mathsf{c}_2)\,x_2.$$

Taking into account that the system (3.17) defines the eigenvector ϕ_0 up to the multiplication by an arbitrary constant factor, one thus concludes that

$$\phi_0=\Big(\mathsf{s}_1-2\,\mathsf{c}_2,\sqrt{2}\,(1+\mathsf{s}_2),\sqrt{2}\,\Big)^{\mathsf{T}}. \tag{3.19}$$

2°. In the case of $\lambda_1=5+\mathsf{s}_2+(\mathsf{c}_1\,\mathsf{s}_2-1)\,\mathsf{s}_1=5+\mathsf{s}_2\,(\mathsf{s}_2-\mathsf{c}_1)$, the system of equations (3.17) reduces to

$$\begin{array}{cccccl} [\mathsf{c}_1(\mathsf{s}_2+1)-5]\,x_0 & - & \sqrt{2}\,\mathsf{s}_1\,x_1 & - & \sqrt{2}\,x_2 & =0,\\ -\sqrt{2}\,\mathsf{s}_1\,x_0 & + & [\mathsf{c}_1(\mathsf{s}_2+2)-3]\,x_1 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_2 & =0,\\ -\sqrt{2}\,x_0 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_1 & - & (\mathsf{c}_2\,\mathsf{s}_1+3)\,x_2 & =0, \end{array} \tag{3.20}$$

As in the previous case of $\lambda_0=0$, one eliminates the component x_0 by adding to the second equation in (3.20) the third one, multiplied by $-\,\mathsf{s}_1$. This leads to the relation

$$a\,x_1+b\,x_2=0,\qquad a=\sqrt{5}\,\mathsf{s}_2+3\,\mathsf{c}_1-5,\qquad b=(4-\mathsf{c}_1)\,\mathsf{s}_1+3\,\mathsf{c}_2-2, \tag{3.21}$$

interconnecting the components x_1 and x_2. It turns out that the coefficients a and b in the relation (3.21) have a common factor,

$$a=\epsilon\,(2\,\mathsf{s}_2-2\mathsf{c}_1+3),\qquad b=\epsilon\,(2\,\mathsf{s}_2+1),\qquad \epsilon=-\,\mathsf{c}_2\,(\mathsf{c}_2\,\mathsf{s}_1+3). \tag{3.22}$$

Eliminating this common factor from the relation (3.21) reduces it to the simpler form,

$$(2\,\mathsf{s}_2-2\mathsf{c}_1+3)\,x_1+(2\,\mathsf{s}_2+1)\,x_2=0, \tag{3.23}$$

from which it follows at once that $x_1=-\,(2\,\mathsf{s}_2+1)$ and $x_2=2\,(\mathsf{s}_2-\mathsf{c}_1)+3$. Substituting these values of x_1 and x_2 into the first equation in (3.20), one finally finds that the component $x_0 = \sqrt{2}\,\mathsf{c}_1$. Thus the eigenvector ϕ_1, associated with the eigenvalue λ_1, has the form

$$\phi_1 = \Big(\sqrt{2}\,\mathsf{c}_1, -2\,\mathsf{s}_2 - 1,\, 2\,(\mathsf{s}_2 - \mathsf{c}_1) + 3\Big)^{\mathsf{T}}. \tag{3.24}$$

3°. Finally, in the case of $\lambda_2 = 5 + \mathsf{s}_2 - (\mathsf{c}_1\,\mathsf{s}_2 - 1)\,\mathsf{s}_1 = \mathsf{s}_1\,(\mathsf{s}_1 - \mathsf{c}_2)$, the system of equations (3.17) reduces to

$$\begin{array}{rcccccl} \mathsf{c}_2\,(\mathsf{s}_1+1)\,x_0 & - & \sqrt{2}\,\mathsf{s}_1\,x_1 & - & \sqrt{2}\,x_2 & = 0, \\ -\sqrt{2}\,\mathsf{s}_1\,x_0 & + & (\mathsf{c}_2\,\mathsf{s}_1+1)\,x_1 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_2 & = 0, \\ -\sqrt{2}\,x_0 & + & (\mathsf{c}_1\mathsf{s}_2-1)\,x_1 & - & \mathsf{c}_1^2\,(\mathsf{s}_1+\mathsf{s}_1)\,x_2 & = 0. \end{array} \tag{3.25}$$

As in the previous case of λ_1, one eliminates the component x_0 by adding to the second equation in (3.25) the third one, multiplied by $-\,\mathsf{s}_1$. This leads to the relation $x_1 = x_2$, which then enables one to find from the first equation in (3.25) that $x_0 = -\sqrt{2}\,\mathsf{c}_1\,x_1$. Thus the eigenvector ϕ_2, associated with the eigenvalue λ_2, has the form

$$\phi_2 = \Big(-\sqrt{2}\,\mathsf{c}_1,\, 1,\, 1\Big)^{\mathsf{T}}. \tag{3.26}$$

It remains only to add that the vectors ϕ_0, ϕ_1 and ϕ_2, associated with the eigenvalues λ_0, λ_1 and λ_2, are essentially the same as the up-components of the three symmetric eigenvectors $\widetilde{f}_0 = Tf_0$, $\widetilde{f}_4 = Tf_4$ and $\widetilde{f}_2 = Tf_2$ of the discrete number operator $\widetilde{\mathcal{N}}_5$ in the P_d-symmetrized basis $\widetilde{e}_n$, where f_0, f_2 and f_4 have been already derived in [18] by employing *Mathematica*; that is,

$$\widetilde{f}_0 = (\phi_0, 0_2)^{\mathsf{T}}, \qquad \widetilde{f}_2 = (\phi_2, 0_2)^{\mathsf{T}}, \qquad \widetilde{f}_4 = (\phi_1, 0_2)^{\mathsf{T}}. \tag{3.27}$$

4. Concluding remarks

To conclude this work, the following should be recalled first. Recently it has been proved that the 'position' and 'momentum' DFT operators X and Y form a special case of the Askey-Wilson algebra $AW(3)$ [12]. So it would be appropriate to use the eigenvectors of either X, or Y, as a basis in the eigenspace of the discrete number operator $\mathcal{N}$, that governs the eigenvectors of the DFT operator Φ. In this work it is shown that in the case of DFT this technique of employing the 'position' e_n and 'momentum' ϵ_n eigenvectors for resolving an eigenvalue problem for the discrete number operator $\mathcal{N}$ is not applicable, unless those eigenvectors are being symmetrized with respect to the discrete reflection operator P_d. Therefore the P_d-symmetrization operator T is found and a remarkable fact is established: it turns out that the matrix of the discrete number operator $\mathcal{N}_5$ in the P_d-symmetrized basis $\widetilde{e}_n = Te_n$ has only half of the number of the nonzero entries of the same matrix in the initial basis e_n. This *sparsealization* of the discrete number operator $\mathcal{N}_5$ is shown to be essentially helpful for finding explicit forms of the eigenvalues and eigenvectors of the operator $\mathcal{N}_5$. Finally, I believe that just a bit more time is needed now to resolve an eigenvalue problem for the DFT number operator $\mathcal{N}$ of a general dimension N.

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