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Study on the Applications of Semi-Analytical Method for the Construction of Numerical Solutions of the Burgers' Equation

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Abstract

In the present paper explores, the Burgers' Equation which is the considerable partial differential equation that emerges in nonlinear science. Also, Homotopy Analysis Method (HAM) has been implemented to Burgers' equation with given initial conditions. The efficieny of the proposed method is analyzed by using some illustrative problems. We are compared approximate solutions acquired via HAM with the exact solutions. As a result of comparisons, it is demonstrated that the gained solutions are in excellent agreement. Additionally, 2D-3D graphs and tables of attained results are drawn by means of a ready-made package program. The recent obtained results denote that HAM is extremely efficient technique. Nonlinear partial differential equations can be solved with the help of presented method.

1. Introduction

Nonlinear phenomena have important effects on different fields of applied mathematics and science. These phenomena frequently occur naturally in various fields of study such as in physics, applied mathematics and engineering. Many mathematicians and scientists have drawn too much attention to the extensive effectiveness of these equations. Because of this, most of methods have been constructed and applied to solve these problems [1–12]. Therefore, developing methods for finding the analytical and numerical solutions to these types of equations is very important. HAM is an essential semi-analytical method and has a considerable role among them [13, 14].
The Burgers' equation is considered one of the fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics, chemical reaction, heat conduction, elasticity and traffic flow.

Burgers' equation has the following general structure,

$$u_t + uu_x = vu_{xx}\,,\ x \in \mathbb{R},\ \ t > 0, \tag{1.1}$$

where v is a constant. It describes the kinematic viscosity. Mentioned equation is named inviscid Burgers' equation when $v = 0$. This equation has a significant role in gas dynamics. The equation (1.1) indicates the connection between diffusion and convection procedures.

Numerical solutions of the nonlinear partial equations have an important role in mathematics, physics, applied sciences, etc. Several analytical and semi-analytical methods have been developed to solve these problems. For the importance of these solutions, there are many studies are available in literature. In manuscript [15], authors analyzed the behaviors of forced KdV equation describing the free surface critical flow over a hole by finding the solution with the help of q-homotopy analysis transform technique (q-HATT). Also, the partial differential equations were transformed into nonlinear ordinary differential equations by introducing relevant similarity variables and approximate analytical solution was determined operating the homotopy analysis method [16]. Influence of different relevant parameters such as Deborah number, stratification, chemical reaction and variable thermophysical parameters on temperature, velocity and concentration distributions was shown to highlight the specifics of heat and mass transfer flow characteristics by authors [16]. The conformable fractional Adomian decomposition method (CFADM) and conformable fractional modified homotopy perturbation method (CFMHPM) redefined to gain the

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approximate-analytical solution of fractional partial differential equations (PDEs) by using conformable fractional derivative [17]. In the framework [18], the coupled mathematical model of the atmosphere-ocean system called El Nino-Southern Oscillation (ENSO) was analyzed with the aid of Adams-Bashforth numerical scheme. The non-linear regularized long wave (RLW) equation was solved by semi-inverse method [19]. Authors investigated novel solutions of fractional-order option pricing models and their fundamental mathematical analyses in [20]. The modified Laplace decomposition method (MLDM) defined in the sense of Caputo, Atangana-Baleanu and Caputo-Fabrizio (in the Riemann sense) operators was used in securing the approximate-analytical solutions of the nonlinear model in reference [21].

Some analytical, numerical and approximate analytical methods were investigated by considering time-fractional nonlinear Burger–Fisher equation (FBFE) [22]. $(1/G')$-expansion method, finite difference method (FDM) and Laplace perturbation method (LPM) were considered to solve the FBFE. Authors attained the analytical solution of the mentioned problem via $(1/G')$-expansion method [22]. In the mentioned paper, authors indicated that the finite difference method was a lower error level than the Laplace perturbation method. In our work, similarly, we demonstrated the effectiveness, validity and strength of our proposed method namely HAM to solve the Burgers' equation.

The goal of this study, we examine the equation in detail and HAM will be used to acquire new analytic solutions of the Burgers' equation. The method is powerful and effective and avoids the complexity involved in other purely numerical methods.

2. General Structure of Homotopy Analysis Method (HAM)

For the purpose of explaining the methodology of HAM, focus on the subsequent differential equation in a general form;

$$\mathrm{N}\left[\varphi\left(x,p\right)\right]=0,$$

where N is a nonlinear operator, φ is an unknown function. Then, x and p are independent variables. For convenience, all initial or boundary conditions are ignored. By using the present method, firstly, the one-parameter family of equations is constructed has the following form [1]

$$(1-q)\,L\left[\Phi\left(x,p;q\right)-\varphi_0\left(x,p\right)\right]=q\hbar H\left(x,p\right)\mathrm{N}\left[x,p;q\right], \tag{2.1}$$

where $q\in[0,1]$ is the embedding parameter, $\Phi\left(x,p;q\right)$ is an unknown function, $\varphi_0\left(x,p\right)$ is an initial guess of $\varphi\left(x,p\right)$, $\hbar\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator and $H\left(x,p\right)$ defines a non-zero auxiliary function.

Expressly when $q=0$ and $q=1$, equation (2.1) holds that

$$\Phi\left(x,p;0\right)=\varphi_0\left(x,p\right),\quad \Phi\left(x,p;1\right)=\varphi\left(x,p\right).$$

respectively [1]. Hence, defining

$$\varphi_n\left(x,p\right)=\frac{1}{n!}\frac{\partial^n\Phi\left(x,p;q\right)}{\partial q^n}\bigg|_{q=0}, \tag{2.2}$$

and expanding $\Phi\left(x,p;q\right)$ in Taylor series with respect to the embedding parameter q, we have

$$\Phi\left(x,p;q\right)=\varphi_0\left(x,p\right)+\sum_{n=1}^{+\infty}\varphi_n\left(x,p\right)q^n, \tag{2.3}$$

If the series (2.3) converge when the value $q=1$, one has

$$\varphi\left(x,p\right)=\varphi_0\left(x,p\right)+\sum_{n=1}^{+\infty}\varphi_n\left(x,p\right), \tag{2.4}$$

which is proven by Liao [1]. Describe the vectors

$$\vec{\varphi}_m=\left\{\varphi_0\left(x,p\right),\varphi_1\left(x,p\right),\cdots,\varphi_m\left(x,p\right)\right\}.$$

After essential mathematical operations and regulations have been performed, the nth-order deformation equation is obtained which has the following form:

$$L\left[\varphi_n\left(x,p\right)-\chi_n\varphi_{n-1}\left(x,p\right)\right]=\hbar\Re\left(\vec{\varphi}_{n-1}\right), \tag{2.5}$$

where

$$\Re\left(\vec{\varphi}_{n-1}\right)=\frac{1}{(n-1)!}\frac{\partial^{n-1}\mathrm{N}\left[\Phi\left(x,p;q\right)\right]}{\partial q^{n-1}}\bigg|_{q=0}, \tag{2.6}$$

and

$$\chi_n=\begin{cases}0, & n\leq 1,\\ 1, & n>1.\end{cases} \tag{2.7}$$

It should be indicated that the nth-order deformation equations form a set of linear differential equations which can be solved via help of a ready-made package program such as Maple, Mathematica, Matlab and others.

3. Implementation of the Method and Numerical Results

In this section of the paper, the Homotopy Analysis Method (HAM) is developed to acquire the approximate solutions of the Burgers' equation is used and we discuss the obtained results. Primarily, we describe the algorithm of the HAM as it applies to the Burgers' equation. By using nth-order deformation equation, we consider the following operator form for applying HAM to Burgers' equation (1.1),

$$L\left[u_n(x,t)-\chi_n u_{n-1}(x,t)\right]=\hbar H(x,t)R_n\left[\overrightarrow{u}_{n-1}(x,t)\right],\quad n\geq 1, \tag{3.1}$$

where $\hbar$ is a non-zero auxiliary parameter and $H(x,t)$ is a non-zero auxiliary function, respectively.

$$R_n\left[u_{n-1}(x,t)\right]=\frac{\partial u_{n-1}(x,t)}{\partial t}+u_{n-1}(x,t)\frac{\partial u_{n-1}(x,t)}{\partial x}-\nu\frac{\partial^2 u_{n-1}(x,t)}{\partial x^2}, \tag{3.2}$$

$$\chi_n=\begin{cases}0, & n\leq 1\\ 1, & n>1\end{cases},\quad H(x,t)=1. \tag{3.3}$$

Assuming the inverse of the operator exists which is denoted L^{-1} and putting the initial condition $u_n(x,0)$, we write (3.1) for $n=1,2,3,\ldots$ and yields the components $u_1,u_2,u_3,u_4,u_5,\ldots$ etc. of Burgers' equation's solutions by using (3.2)-(3.3). In view of the obtained components, we attain the approximate solution of initial value problem in series form by $u(x,t)=\sum\limits_{n=0}^{\infty}u_n(x,t)$. On the other hand, accuracy level could be improved via elegantly computing further components.

Example 3.1. Consider the subsequent Burgers' equation

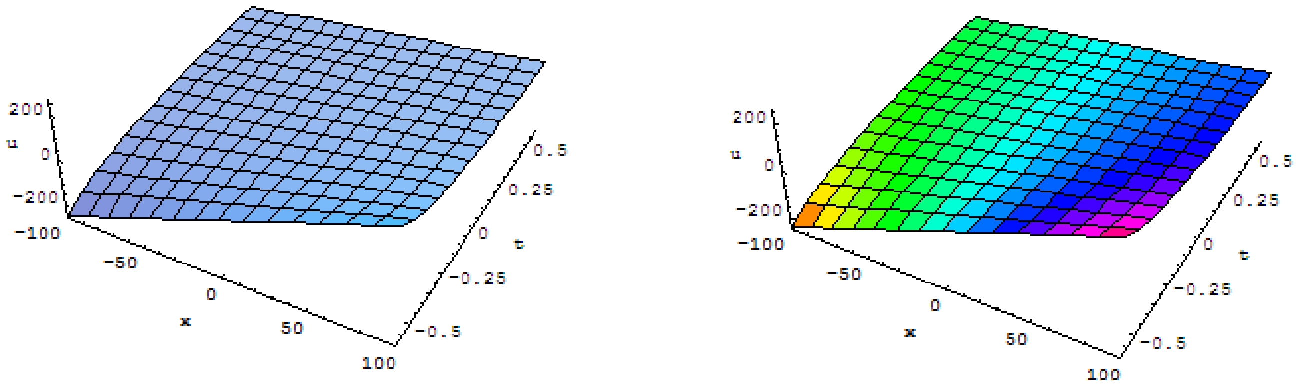
$$u_t+uu_x=u_{xx}, \tag{3.4}$$
$$u(x,0)=x.$$

By using the manner explained above, six components of the series were acquired of which $u(x,t)$ was evaluated to have the following expansion

$$u(x,t)=x+xt\hbar+xt\hbar(t\hbar+\hbar+1)+xt\hbar(t\hbar+\hbar+1)^2+xt\hbar(t\hbar+\hbar+1)^3+xt\hbar(t\hbar+\hbar+1)^4+\ldots \tag{3.5}$$

Also, the exact solution of the equation (3.4) is given by

$$u(x,t)=\frac{x}{1+t}.$$



(a) Exact solution function's graph of the equation (3.4)

(b) (3.5) approximate solution for $u(x,t)$ as a graphical sketch $(\hbar=-1)$

Figure 3.1: Exact and approximate solution function of Burgers' equation by means of HAM

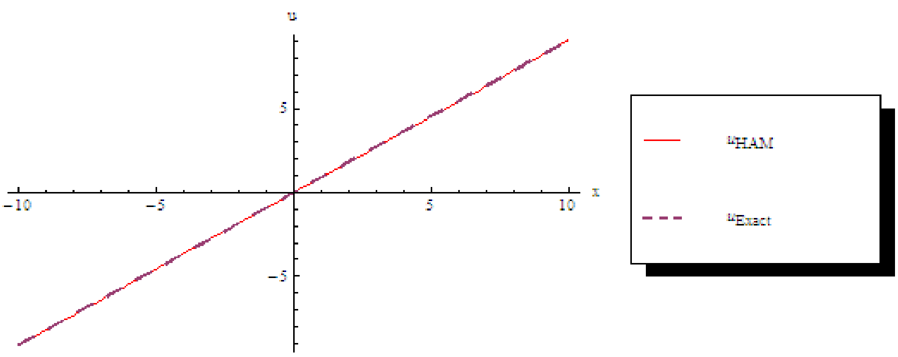


Figure 3.2: Comparison between HAM and exact solution when $\hbar=-1$

In this part of the work, we will compare the approximate solutions of Burgers' equation obtained by HAM with the exact solution. The results presented in Figure 3.1 and Figure 3.2, respectively, clearly demonstrate the good accuracy of the HAM with exact solution and the good agreement between HAM and exact solution of the equation. Additionally, we have the following results:

t/x	0.1	0.2	0.3	0.4	0.5
0.1	9.09091×10^{-8}	5.33333×10^{-6}	0.0000560769	0.000292571	0.00104167
0.2	1.81818×10^{-7}	0.0000106667	0.000112154	0.000585143	0.00208333
0.3	2.72727×10^{-7}	0.000016	0.000168231	0.000877714	0.003125
0.4	3.63636×10^{-7}	0.0000213333	0.000224308	0.00117029	0.00416667
0.5	4.54545×10^{-7}	0.0000266667	0.000280385	0.00146286	0.00520833

Table 1: Numerical solution's absolute error according to exact solution by using HAM when $\hbar=-1$

t	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$	$\hbar=-0.8$
0.1	0.0109227	0.00386764	0.00104167	0.000175073	0.0000106667
0.2	0.0218453	0.00773527	0.00208333	0.000350146	0.0000213333
0.3	0.032768	0.0116029	0.003125	0.000525219	0.000032
0.4	0.0436907	0.0154705	0.00416667	0.000700292	0.0000426667
0.5	0.0546133	0.0193382	0.00520833	0.000875365	0.0000533333

Table 2: Absolute error of Burgers' equation's numerical solution for different values of auxiliary parameter $\hbar$

The detailed results are demonstrated in Table 1 and Table 2. In Table 1, fix $\hbar=-1$, we compared the six component approximation of HAM with the exact solution. From Table 1, we see that numerical approximation by means of HAM show good agreement with the exact solution. In Table 1, it is seen that numerical solutions also obtained for six components by means of HAM are very convergent to exact solution. Furthermore, interval of the convergence can be found by drawing auxiliary parameter's graph on HAM solutions. In Figure 3.3, curve of the auxiliary parameter $\hbar$ is attained for series solution of the equation. As seen in this figure, interval of the convergence is approximately $-1.1\le\hbar\le-0.8$ for the series solution. The best value of the approximation on this interval can be found by giving values of $\hbar$.

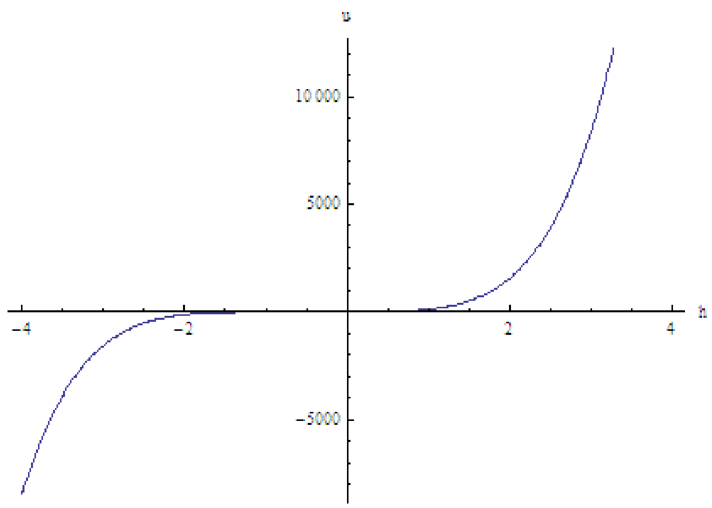


Figure 3.3: Graph of $\hbar$ curves of Burgers' equation for $\phi_6(x,t)$

Example 3.2. We focus on the following Burgers' equation

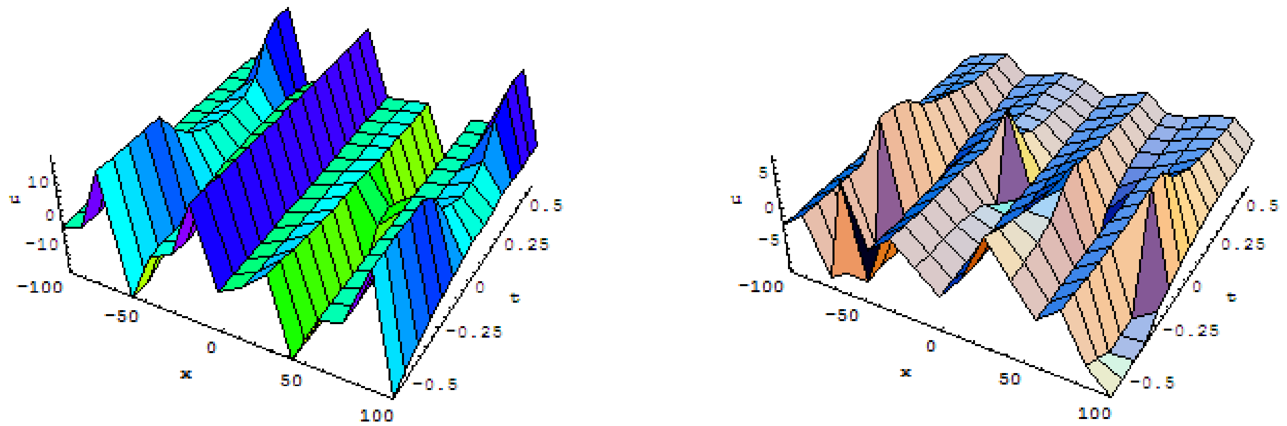
$$u_t+uu_x=u_{xx},\qquad (3.6)$$
$$u(x,0)=\frac{-2\cos x}{1+\sin x}.$$

By means of the process explained before, five components of the series were attained of which $u(x,t)$ was evaluated to have the following expansion

$$\begin{aligned}u(x,t)=&\frac{\text{-2cosx}}{\text{1+sinx}}-2\frac{\hbar t\text{cosx}}{(\text{1+sinx})^2}+\frac{\hbar t\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[-2-\hbar(2+t)+(-2+\hbar(-2+t))\sin x\right]}{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^5}\\&+\frac{\hbar t\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[-3\left(6+2\hbar(t+6)+\hbar^2\left(t^2+2t+6\right)\right)+\left[6-6\hbar(t-2)+\hbar^2\left(t^2-6t+6\right)\right]\cos 2x\right]}{6\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^7}\\&+\frac{8\left[-3-6\hbar+\hbar^2\left(t^2-3\right)\right]\sin x]}{6\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^7}+\ldots\end{aligned}\qquad (3.7)$$

Then, the exact solution of the equation (3.6) has the form

$$u(x,t) = \frac{-2e^{-t}\cos x}{1+e^{-t}\sin x}.$$



(a) Exact solution function's graph of the equation (3.6)

(b) (3.7) approximate solution for $u(x,t)$ as a graphical sketch $(\hbar = -1)$

Figure 3.4: Exact and approximate solution function of Burgers' equation by means of HAM

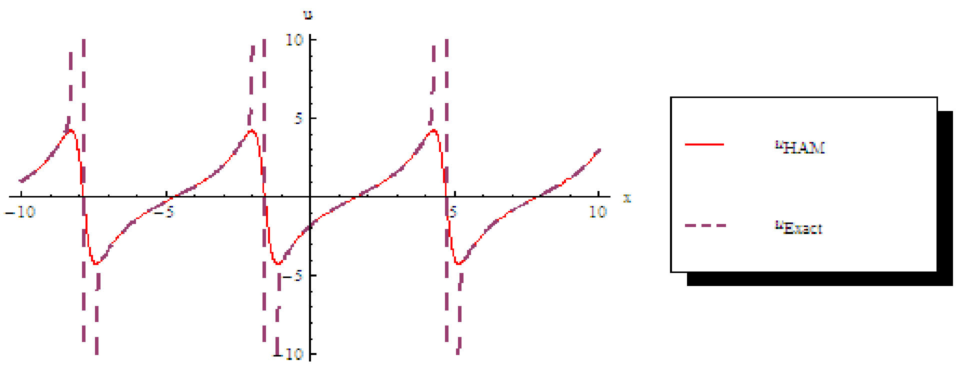


Figure 3.5: Comparison between HAM and exact solution when $\hbar = -1$

In this part of the manuscript, approximate solutions of the equation acquired by HAM are compared with the exact solution. The results presented in Figure 3.4 and 3.5, respectively, clearly show the good accuracy of the HAM with exact solution and the good agreement between HAM and exact solution. In addition, we get the following results:

The detailed results are demonstrated in Table 3-4. In Table 3, we compared the five step approximation of HAM with the exact solution. We see that numerical approximation by means of HAM show good agreement with the exact solution $u(x,t)$.

t/x	0.1	0.2	0.3	0.4	0.5
0.1	8.75091×10^{-8}	2.71323×10^{-6}	0.0000199575	0.0000814457	0.000240664
0.2	9.64863×10^{-8}	3.0651×10^{-6}	0.0000230798	0.0000963385	0.000290944
0.3	5.46078×10^{-8}	1.78415×10^{-6}	0.0000137918	0.0000590014	0.000182335
0.4	1.5428×10^{-8}	5.54887×10^{-7}	4.65233×10^{-6}	0.0000213409	0.0000700775
0.5	1.04609×10^{-8}	2.72352×10^{-7}	1.6068×10^{-6}	4.88699×10^{-6}	9.36752×10^{-6}

Table 3: Numerical solution's absolute error according to the exact solution by using HAM when $\hbar = -1$

t	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$	$\hbar=-0.8$
0.1	0.0126481	0.00176267	0.000240664	0.000112127	0.000394338
0.2	0.00480997	0.000128467	0.000290944	0.000147172	0.000500502
0.3	0.0012364	0.00071677	0.000182335	0.000135249	0.000559188
0.4	0.000334458	0.000802543	0.0000700775	0.0000985342	0.000572464
0.5	0.000957229	0.000712871	9.36752×10^{-6}	0.0000559068	0.000551245

Table 4: Absolute error of Burgers' equation's solution for different values of auxiliary parameter $\hbar$

In Table 3, it is seen that numerical solutions also obtained for five components by means of HAM are very convergent to the exact solution. Furthermore, in Figure 3.6, curve of the auxiliary parameter $\hbar$ is attained for series solution of the equation. As seen in this figure, interval of the convergence is approximately $-1.2\leq\hbar\leq-0.8$ for the series solution. The best value of the approximation on this interval can be found by giving values of $\hbar$. This situation is given in Table 4. $\hbar=-1$ is the best approximation to the exact solution as seen in the mentioned table.

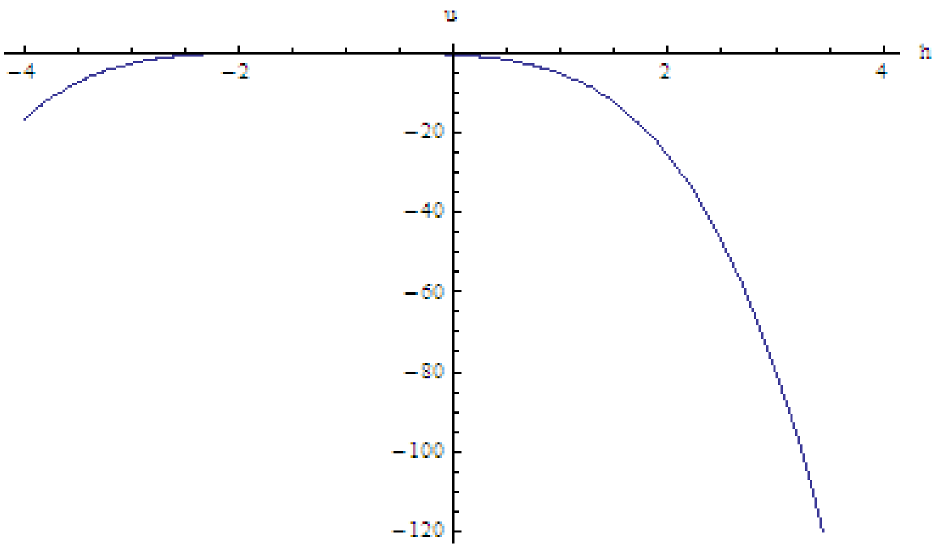


Figure 3.6: Graph of $\hbar$ curves of equation (3.6) for $\phi_5(x,t)$

4. Conclusion

In the present research, we investigate the numerical solution of Burgers' equations acquired by HAM. The application of the proposed technique has also been discussed. Besides, we draw two-three dimensional graphs and tables of this equation by use of a ready-made package program.

From the obtained results, it has been deduced that HAM is highly effective, credible and strong in the sense that finding analytical solutions. Thus the validity and flexibility of the proposed method are verified via all of these successful applications. The study demonstrates that the HAM algorithm is productive and can be used for many other complicated nonlinear partial differential equations in mathematical physics.

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Competing interests

There are no competing interests.

Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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Petrie Paths in Triangular Normalizer Maps

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Abstract

This study is devoted to investigate the Petrie paths in the normalizer maps and regular triangular maps corresponding to the subgroups $\Gamma_0(N)$ of the modular group Γ. We show that each regular triangular map admits a closed Petrie path. Thus, for each regular map, we find the Petrie length of the corresponding map.

1. Introduction

Let $PSL(2,\mathbb{R})$ denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{R}$ and $ad-bc = 1$. In terms of matrix representation, the elements of $PSL(2,\mathbb{R})$ correspond to the matrices $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,b,c,d \in \mathbb{R}$ and the determinant is 1. This is the automorphism group of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. The modular group Γ is the subgroup of $PSL(2,\mathbb{R})$ such that $a,b,c,d \in \mathbb{Z}$ [1]. $\Gamma_0(N)$, a well-known congruence subgroup of the modular group, consists of the transformations of Γ such that $N \mid c$.

Let $\mathscr{U}$ denote the space consisting of the points of upper half plane and the extended rational numbers. The normalizer of the congruence subgroups $\Gamma_0(N)$ of the modular group Γ in $PSL(2,\mathbb{R})$ is $\Gamma_B(N)$ which is simply called "the normalizer". We refer reader to [2–7] and references therein for results concerning the normalizer. It is known that the normalizer is a triangle group, and it acts transitively on the set of extended rational numbers $\hat{\mathbb{Q}} = \mathbb{Q}\cup\{\infty\}$ for $N = 2^\alpha 3^\beta$ with $\alpha \in \{0,2,4,6\}$ and $\beta \in \{0,2\}$ [8,9]. We denote the values of $N = 2^\alpha 3^\beta$ by N_1 for $\alpha \in \{0,2,4,6\}$ and $\beta \in \{0,2\}$. The normalizer maps are universal maps on $\mathscr{U}$ which are arose from the action of $\Gamma_B(N_1)$ on $\hat{\mathbb{Q}}$. Using the normalizer maps, regular triangular maps can be obtained by the quotients $\mathscr{M}_3^h/\Gamma_0(N_1)$. These regular maps are introduced in [8]. However, combinatorial properties of these maps has not been addressed. In this manner, to reveal the complete structure of the normalizer and the maps corresponding to the normalizer, we consider the Petrie paths in these maps. This paper investigates the Petrie paths in the normalizer maps which are the universal maps arising from the action of the normalizer on the extended rationals, and the regular triangular maps corresponding to the subgroups $\Gamma_0(N)$ which are the quotient maps of the normalizer maps.

This paper is organized as follows. Section 2 describes the normalizer $\Gamma_B(N)$ and its associated group structure which is important to construct regular maps. Section 3 introduces a brief information about the basic concept of the normalizer maps which is already addressed. Section 4 provides the results concerning the Petrie paths in the normalizer maps, where we concluded that the Petrie paths in the normalizer maps are related to well-known Fibonacci sequence. Section 5 investigates the Petrie paths in the regular triangular maps. Finally, Section 6 presents our conclusions.

2. The Normalizer

As described in [10], the normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ consists of the transformations corresponding to the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$$

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where all symbols represent integers, h is the largest divisor of 24 for which $h^2 \mid N$, $e > 0$ is an exact divisor of N/h^2, and the determinant is e. (We say that r is an exact divisor of s if $r \mid s$ and $(r, s/r) = 1$).
The following results are taken from [11] which characterizes the structure of the normalizer.

Theorem 2.1 ([11]). *Let $N = 2^\alpha 3^\beta$ and $\beta = 0$ or 2. Then, $\Gamma_B(N)$ is a triangle group if and only if $\alpha \leq 8$. In these cases*

$$\Gamma_B(N) \text{ has signature } \begin{cases} (0;2,3,\infty) & \text{if } \alpha = 0,2,4,6 \\ (0;2,4,\infty) & \text{if } \alpha = 1,3,5,7 \\ (0;2,\infty,\infty) & \text{if } \alpha = 8. \end{cases}$$

Theorem 2.2 ([11]). *Let $N = 2^\alpha 3^\beta$ and $\beta = 1$ or 3. Then $\Gamma_B(N)$ is a triangle group if and only if $\alpha = 0,2,4,6$. In these cases $\Gamma_B(N)$ has signature $(0;2,6,\infty)$.*

In this paper we only consider the case $\beta = 0,2$ and $\alpha = 0,2,4,6$ so that the normalizer maps and the regular maps corresponding to the quotients of the normalizer maps will be all triangular. Thus if $\beta = 0,2$ and $\alpha = 0,2,4,6$, then the normalizer $\Gamma_B(N_1)$ is the set of all transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ ch & d \end{pmatrix},\ ad - bc = 1,$$

where h is the largest divisor of 24 for which $h^2 \mid N_1$.

3. The Normalizer Maps

The information in this section is given in [8,9]. We denote the normalizer maps corresponding to $\Gamma_B(N_1)$ by $\mathscr{M}_3^h$. Vertices of $\mathscr{M}_3^h$ are the fractions $\dfrac{a}{ch}$ with $(a,c) = 1$ and two vertices, $\dfrac{a}{ch}$ and $\dfrac{b}{dh}$, are joined by an edge if and only if $ad - bc = \pm 1$. $\mathscr{M}_3^h$ has the following properties:

(1) There is a triangle with vertices $\dfrac{1}{0}, \dfrac{1}{h}, \dfrac{0}{h}$.
(2) $\Gamma_B(N_1)$ acts as a group of homomorphisms of $\mathscr{M}_3^h$.
(3) There is a triangle with vertices $\dfrac{a}{ch}, \dfrac{a+b}{(c+d)h}, \dfrac{b}{dh}$.

One can easily see that when $h = 1$, $\mathscr{M}_3^1$ is just the Farey map. When $h > 1$, the normalizer map $\mathscr{M}_3^h$ is the Farey map scaled by a factor of $1/h$ (see Fig. 3.1).

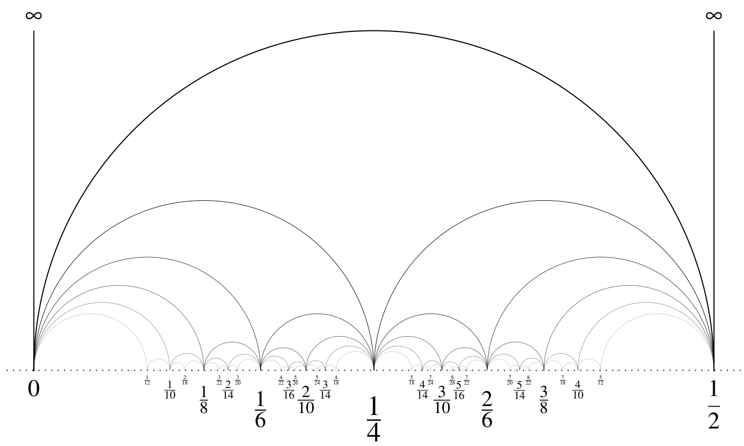


Figure 3.1: Part of the normalizer map $\mathscr{M}_3^2$.

It is easily seen that normalizer maps are all triangular maps each of which corresponds to a value of h with triangular faces given by (3). We refer reader to [8,9] for further details about normalizer maps and regular maps corresponding to quotients of normalizer maps.

4. Petrie Paths in the Normalizer Maps

A Petrie polygon in a map is a zig-zag path in the corresponding map. In other words, we start by choosing a vertex on the map. This vertex is the starting point of the polygon. On that vertex, we go along an edge to an adjacent vertex. On the last vertex we turn left and then go to another adjacent vertex. Then we turn right, and so on, but interchanging left and right consecutively. In a finite map, this procedure ends on the first vertex. Thus a path or polygon is obtained by this procedure. This path or polygon is called a Petrie path or Petrie polygon. The number of edges of a Petrie polygon in a map is called Petrie length of the map.

In this section, we investigate the Petrie paths in $\mathscr{M}_3^h$. Now consider a Petrie path in $\mathscr{M}_3^h$. By the transitivity (see [8]), first edge of the Petrie path can be chosen the edge from $E_1 = \frac{1}{0}$ to $E_2 = \frac{0}{h}$. Keeping in mind that left turns correspond to $ad-bc=-1$ and right turns correspond to $ad-bc=1$, the first left turn goes to the vertex $E_3 = \frac{1}{h}$. Then a right turn goes to the vertex $E_4 = \frac{1}{2h}$. As this procedure goes on, the following vertices of the Petrie path can be found as $\frac{a}{ch}, \frac{b}{dh}, \frac{a+b}{(c+d)h}$. Now, if we denote the elements of the well-known Fibonacci sequence by f_k, where $f_0=0, f_1=1$ and $f_k = f_{k-1}+f_k$, $k \geq 1$, then we can express the $(k+1)$th vertex of the Petrie path as $E_{k+1} = \frac{f_{k-1}}{hf_k}$. Thus the Petrie path in normalizer maps can be found as

$$\frac{1}{0}, \frac{0}{h}, \frac{1}{h}, \frac{1}{2h}, \frac{2}{3h}, \frac{3}{5h}, \ldots, \frac{f_{k-1}}{hf_k}, \ldots.$$

Note that if $h=1$, then the above Petrie path is exactly the Petrie path in the Farey map (see [12]).

Example 4.1. *The Petrie path in the normalizer map $\mathscr{M}_3^2$ is*

$$\frac{1}{0}, \frac{0}{2}, \frac{1}{2}, \frac{1}{4}, \frac{2}{6}, \frac{3}{10}, \ldots, \frac{f_{k-1}}{2f_k}, \ldots.$$

We close this section by the following results concerning the relation between the vertices of the Petri paths in normalizer maps.

Lemma 4.2. *The transformation corresponding to the matrix $T = \begin{pmatrix} 0 & 1/h \\ h & 1 \end{pmatrix}$ maps each vertex of the Petrie polygon in the normalizer maps $\mathscr{M}_3^h$ to the next vertex.*

Proof. The first vertex of the Petrie path is $E_1 = \frac{1}{0}$ and $(k+1)$th vertex is $\frac{f_{k-1}}{hf_k}$ such that $k \geq 1$. We use induction on k. So the second vertex can be found by taking $k=1$ as $E_2 = \frac{f_0}{hf_1} = \frac{0}{h}$. Now one can easily see that $T(E_1) = E_2$. Let us assume that $T(E_k) = E_{k+1}$ for each $k > 1$. We will show that $T(E_{k+1}) = E_{k+2}$.Thus

$$T(E_k) = \frac{f_{k-1}}{h(f_{k-2}+f_{k-1})}.$$

Since f_k is the Fibonacci sequence, we have $f_{k-2}+f_{k-1}) = f_k$. Using this equality, we have

$$T(E_k) = \frac{f_{k-1}}{hf_k}.$$

By the definition of $E_{k+1} = \frac{f_{k-1}}{hf_k}$, we have

$$T(E_{k+1}) = \begin{pmatrix} 0 & 1/h \\ h & 1 \end{pmatrix} \begin{pmatrix} f_{k-1} \\ hf_k \end{pmatrix} = \begin{pmatrix} f_k \\ h(f_{k-1}+f_k) \end{pmatrix} = \begin{pmatrix} f_k \\ hf_{k+1} \end{pmatrix} = E_{k+2}.$$

This completes the proof. □

Proposition 4.3. *The transformations corresponding to the matrices $T^k = \begin{pmatrix} f_{k-1} & f_k/h \\ hf_k & f_{k+1} \end{pmatrix}$ maps each vertex of the Petrie polygon in the normalizer maps $\mathscr{M}_3^h$ to the next vertex.*

Proof. Applying induction on k, using Lemma 4.2 and the properties of the Fibonacci sequence f_k, the proof follows. □

We denote the determinant $f_{k-1}f_{k+1} - f_k^2$ of the matrix T^k by $D(k) := f_{k-1}f_{k+1} - f_k^2$.

Proposition 4.4. *For $k \geq 1$, $D(k) = (-1)^k$.*

Proof. We use induction on k. So let $k=1$, then we have $D(1) = -1$. Now let $D(k) = (-1)^k$. We will show that $D(k+1) = (-1)^{k+1}$ holds. By the definition of $D(k)$, we have

$$D(k+1) = f_k f_{k+2} - f_{k+1}^2.$$

By the definition of the Fibonacci sequence, substituting $f_{k+2} = f_k + f_{k-1}$ in the above equality, we get

$$D(k+1) = f_k(f_k+f_{k-1}) - f_{k+1}^2 = f_k^2 + f_k f_{k-1} - f_{k+1}^2.$$

Again substituting $f_k = f_{k+1} - f_{k-1}$ in the above equality, it is obtained that

$$D(k+1) = f_k^2 + f_{k+1}^2 - f_{k-1}f_{k+1} - f_{k+1}^2 = -D(k) = -(-1)^k = (-1)^{k+1}.$$

The above equation leads to $D(k+1) = (-1)^{k+1}$. By the induction, the proof is completed. □

Corollary 4.5. *If $k \geq 1$ is even, then T^k is an element of the normalizer.*

Proof. By Proposition 4.4, if k is even, then $D(k) = 1$. Thus T^k has determinant 1 if k is even. By the definition of the normalizer T^k is an element of the normalizer. □

5. Petrie Paths in the Regular Triangular Maps

The regular triangular maps corresponding to the subgroups $\Gamma_0(N_1)$ of the modular group Γ are defined by the quotients $\mathscr{M}_3^h(N_1) = \mathscr{M}_3^h/\Gamma_0(N_1)$. The complete table of these regular triangular maps can be found in [8]. In this section we determine the Petrie polygons in $\mathscr{M}_3^h(N_1)$.

In [8], the set of vertices of the maps $\mathscr{M}_3^h(N_1)$ is $\{(a,ch) \mid a,c \in \mathbb{Z}_h, (a,c,h)=1\}/\sim$, where $(a,ch) \sim (h-a,(h-c)h)$. We denote any vertex of this kind by $\left[\frac{a}{ch}\right]$. Also, the vertices $\left[\frac{a}{ch}\right]$ and $\left[\frac{b}{dh}\right]$ is combined by an edge in $\mathscr{M}_3^h(N_1)$ if and only if $ad-bc \equiv \pm 1 \mod h$.

As in the previous section, we can define the Petrie path of the regular triangular maps $\mathscr{M}_3^h(N_1)$. The Petrie path has vertices

$$\left[\frac{1}{0}\right], \left[\frac{0}{h}\right], \left[\frac{1}{h}\right], \left[\frac{1}{2h}\right], \ldots, \left[\frac{f_{r-1}}{hf_r}\right], \ldots.$$

Now we are going to find Petrie polygons in $\mathscr{M}_3^h(N_1)$ explicitly for some low values of N_1. Further details about regular triangular maps $\mathscr{M}_3^h(N_1)$ can be found in [8].

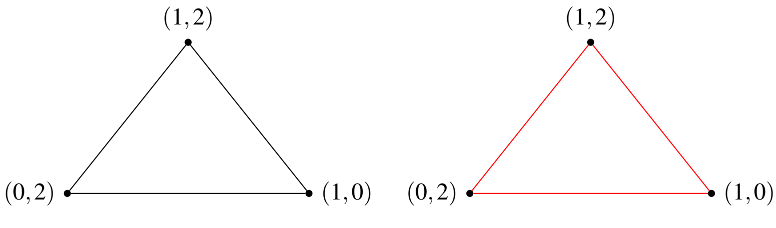
1) $N_1 = 4$. In this case $h=2$, then $\mathscr{M}_3^2(4)$ is the triangle with vertices

$$(1,0),(0,2),(1,2).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^2(4)$ are

$$\left[\frac{1}{0}\right], \left[\frac{0}{2}\right], \left[\frac{1}{2}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^2(4)$ (see Fig. 5.1a) and the Petrie polygon, edges of the polygon are highlighted in red, in the map (see Fig. 5.1b)



(a) $\mathscr{M}_3^2(4)$ **(b)** Petrie polygon in $\mathscr{M}_3^2(4)$

Figure 5.1: The map $\mathscr{M}_3^2(4)$ and Petrie polygon

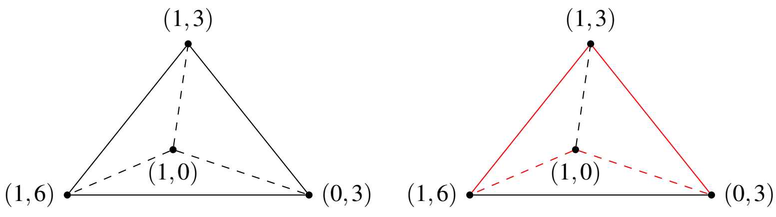
2) $N_1 = 9$. In this case $h=3$, then $\mathscr{M}_3^3(9)$ is the tetrahedron with vertices

$$(1,0),(0,3),(1,6),(1,3).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^3(9)$ are

$$\left[\frac{1}{0}\right], \left[\frac{0}{3}\right], \left[\frac{1}{3}\right], \left[\frac{1}{6}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^3(9)$ (see Fig. 5.2a) and the Petrie polygon in the map (see Fig. 5.2b)



(a) $\mathscr{M}_3^3(9)$ **(b)** Petrie polygon in $\mathscr{M}_3^3(9)$

Figure 5.2: The map $\mathscr{M}_3^3(9)$ and Petrie polygon

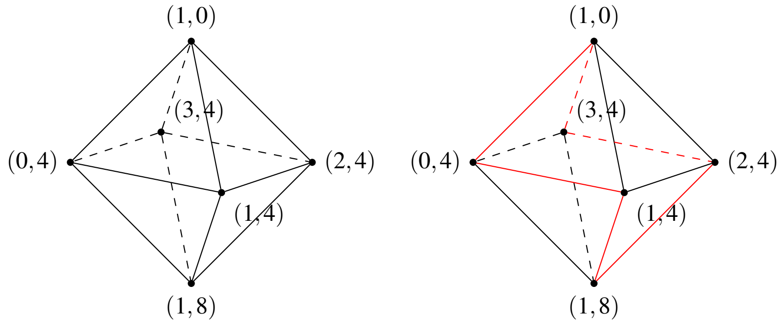
3) $N_1 = 16$. In this case $h = 4$, then $\mathscr{M}_3^4(16)$ is the octahedron with vertices

$$(1,0),(0,4),(1,8),(1,8),(2,4),(3,4).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^4(16)$ are

$$\left[\frac{1}{0}\right],\left[\frac{0}{4}\right],\left[\frac{1}{4}\right],\left[\frac{1}{8}\right],\left[\frac{2}{4}\right],\left[\frac{3}{4}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^4(16)$ (see Fig. 5.3a) and the Petrie polygon in the map (see Fig. 5.3b)



(a) $\mathscr{M}_3^4(16)$ **(b)** Petrie polygon in $\mathscr{M}_3^4(16)$

Figure 5.3: The map $\mathscr{M}_3^4(16)$ and Petrie polygon

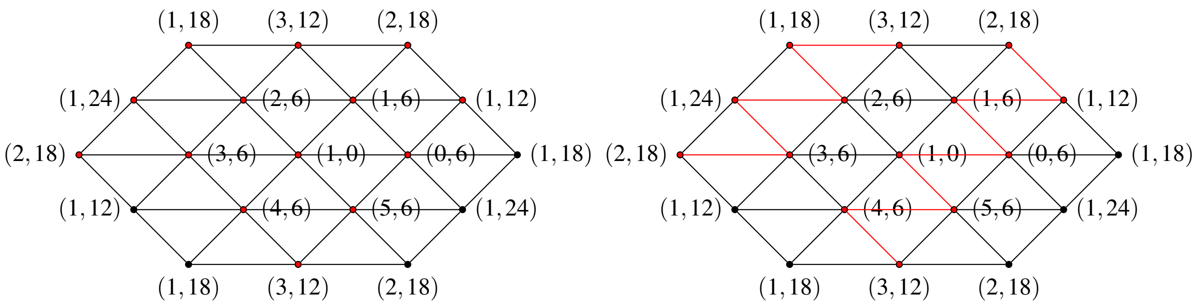
4) $N_1 = 36$. In this case $h = 6$, then $\mathscr{M}_3^6(36)$ is the map $\{3,6\}_{12}$ with vertices

$$(1,0),(0,6),(1,6),(1,12),(1,18),(1,24),(2,6),(2,18),(3,6),(3,12),(4,6),(5,6).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^6(36)$ are

$$\left[\frac{1}{0}\right],\left[\frac{0}{6}\right],\left[\frac{1}{6}\right],\left[\frac{1}{12}\right],\left[\frac{2}{18}\right],\left[\frac{3}{6}\right],\left[\frac{1}{24}\right],\left[\frac{2}{6}\right],\left[\frac{1}{18}\right],\left[\frac{3}{12}\right],\left[\frac{4}{6}\right],\left[\frac{5}{6}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^6(36)$ (see Fig. 5.4a) and the Petrie polygon in the map (see Fig. 5.4b).



(a) $\mathscr{M}_3^6(36)$ **(b)** Petrie polygon in $\mathscr{M}_3^6(36)$

Figure 5.4: The map $\mathscr{M}_3^6(36)$ and Petrie polygon

Now we are going to formulate Petrie lenghts of the Petrie polygons in the regular triangular maps $\mathscr{M}_3^h(N_1)$. In order to do that let us recall a well-known notion called Pisano period [13], and the notion of semi period of a Fibonacci sequence.

Definition 5.1 ([12]). *Let f_k be the Fibonacci sequence. The Pisano period $\pi(n)$ of f_k is defined to be the least positive integer m such that $f_{m-1} \equiv 1 \mod n$ and $f_m \equiv 0 \mod n$. The semi-period $\sigma(n)$ of f_k is defined to be the least positive integer m such that $f_{m-1} \equiv \pm 1 \mod n$ and $f_m \equiv 0 \mod n$.*

Example 5.2. *Let f_k be the Fibonacci sequence and $n = 5$. The elements of the Fibonacci sequence modulo 5 are*

$$0,1,1,2,3,0,3,3,1,-1,0,-1,-1,3,2,0,2,2,-1,1,0,1,1,2,3,\ldots.$$

So the Pisano period of f_k is $\pi(5) = 20$ and the semi period is $\sigma(5) = 10$.

Now we denote the Petrie polygon in the regular triangular maps $\mathscr{M}_3^h(N_1)$ by $\mathscr{P}(N_1)$, and we denote the Petrie length of $\mathscr{P}(N_1)$ by $\mathscr{L}(\mathscr{P})(N_1)$.

Theorem 5.3. $\mathscr{L}(\mathscr{P})(N_1) = \sigma(h)$.

Proof. By the definition, $\mathscr{P}(N_1)$ in $\mathscr{M}_3^h(N_1)$ has vertices

$$\left[\frac{1}{0}\right], \left[\frac{0}{h}\right], \left[\frac{1}{h}\right], \left[\frac{1}{2h}\right], \ldots, \left[\frac{f_{r-1}}{hf_r}\right], \ldots.$$

Also, by the definition, $f_{\sigma(h)-1} \equiv \pm 1 \mod h$ and $f_{\sigma(h)} \equiv 0 \mod h$. So the Petrie polygon $\mathscr{P}(N_1)$ is exactly the following polygon

$$\left[\frac{1}{0}\right] \to \left[\frac{0}{h}\right] \to \left[\frac{1}{h}\right] \to \left[\frac{1}{2h}\right] \to \ldots \to \left[\frac{f_{\sigma(h)-2}}{hf_{\sigma(h)-1}}\right] \to \left[\frac{f_{\sigma(h)-1}}{hf_{\sigma(h)}}\right] = \left[\frac{1}{0}\right].$$

Number of edges in this polygon is $\sigma(h)$. This completes the proof. □

Please see Table 1 for the complete table of the Petrie lengths of the regular triangular maps $\mathscr{M}_3^h(N_1)$.

N_1	h	Map	Petrie Length ($\sigma(h)$)
4	2	Triangle	3
9	3	Tetrahedron	4
16	4	Octahedron	6
36	6	$\{3,6\}$	12
64	8	Dual Dyck	12
144	12	$\{3,12\}$	24
576	24	$\{3,24\}$	24

Table 1: The complete table of the Petrie lengths of the regular triangular maps $\mathscr{M}_3^h(N_1)$

6. Conclusions

In this paper we addressed the problem of determining the Petrie paths in certain maps. The maps that are considered in this paper are the normalizer maps corresponding to the universal maps arising from the action of the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$ on $\hat{\mathbb{Q}}$, and the regular triangular maps corresponding to the subgroups $\Gamma_0(N)$ of the modular group which are the quotient maps of the normalizer maps. We completely determined the Petrie paths in these maps. We showed that the Petrie paths in the normalizer maps are strongly related to Fibonacci sequence. Using this result we managed to determine the Petrie paths in the regular triangular maps. Finally we showed that the Petrie paths in the regular triangular maps are related to Fibonacci sequence modulo an integer, h. As the regular triangular maps are finite maps, we found the Petrie lengths of these maps. Other than that, it would be interesting to determine the Petrie paths in the regular quadrilateral and hexagonal maps corresponding to the subgroups $\Gamma_0(N)$. However it needs further investigation to determine these maps, since currently it is not known whether Fibonacci sequence can be used in that case. Thus we keep this approach as a future research.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Q-Curvature Tensor on f-Kenmotsu 3-Manifolds

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Abstract

The object of the present paper is to consider f-Kenmotsu 3-manifolds fulfilling certain curvature conditions on Q-curvature tensor with the Schouten-van Kampen connection. Certain consequences of Q-curvature tensor on such manifolds bearing Ricci soliton in perspective of Schouten-van Kampen association are likewise displayed. In the last segment, examples are given.

1. Introduction

Let $\vec{M}$ be a $(2n+1)$-dimensional almost contact manifold with an almost contact metric structure $(\breve{\phi},\xi,\eta,g)$ [1]. We denote by $\vec{\Omega}$, the fundamental 2-form of $\vec{M}$ i.e., $\vec{\Omega}(\vec{X},\vec{Y}) = g(\vec{X},\breve{\phi}\vec{Y})$, $\vec{X},\vec{Y} \in \chi(\vec{M})$, where $\chi(\vec{M})$ being the Lie algebra of the differentiable vector fields on $\vec{M}$. Furthermore, we recall the following definitions [1,2].
The manifold $\vec{M}$ and its structure $(\breve{\phi},\xi,\eta,g)$ is said to be:
(i) normal if the almost complex structure defined on the product manifold $\vec{M}\times\Re$ is integrable (equivalently $[\breve{\phi},\breve{\phi}]+2d\eta\otimes\xi = 0$),
(ii) almost cosymplectic if $d\eta = 0$ and $d\breve{\phi} = 0$,
(iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\vec{\nabla}\breve{\phi} = 0$, $\vec{\nabla}$ being covariant differentiation with respect to the Levi-Civita connection).
Olszak and Rosca [3] contemplated normal locally conformal almost cosymplectic manifold and gave the geometric translation of f-Kenmotsu manifolds and its curvature tensors. Among others, they proved that a Riccisymmetric f-Kenmotsu manifold is an Einstein manifold.
The Schouten-van Kampen connection is quite possibly the most widely recognized connection acclimated to two or three necessary allocations on a differentiable manifold conceding with a relative connection [4,5]. Solov'ev has investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [6,7]. From that point, Olszak has contemplated the Schouten-van Kampen connection with an almost contact metric structure [8]. He has depicted a few classes of almost contact metric manifolds bearing the Schouten-van Kampen connection and closed some particular curvature properties of this connection on such manifolds.
Let $\vec{M}$ be a $(2n+1)$-dimensional Riemannian manifold. On the off chance that there exists a balanced correspondence between each facilitate neighborhood of $\vec{M}$ and an area in Euclidean space with the end goal that any geodesic of the Riemannian manifold compares to a straight line in the Euclidean space, at that point $\vec{M}$ is supposed to be locally projectively flat. For $n\geq 1$, $\vec{M}$ is locally projectively flat if and just if the notable projective curvature tensor P vanishes. Truth be told, P is projectively flat (i. e., P=0) if and just if the manifold is of consistent curvature [9]. ξ-conformally flat K-contact manifolds have been concentrated by Zhen et al. [10]. Yildiz et al. [11] considered f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection and demonstrated that such manifold is consistently ξ-projectively flat. The projective curvature tensor is characterized by [12]:

$$P(\vec{X},\vec{Y})\vec{Z} = \vec{R}(\vec{X},\vec{Y})\vec{Z} - \frac{1}{2n}\{\vec{Ric}(\vec{Y},\vec{Z})\vec{X} - \vec{Ric}(\vec{X},\vec{Z})\vec{Y}\}, \tag{1.1}$$

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where $\vec{Ric}$ is the Ricci tensor on $\vec{M}$.
A change in a $(2n+1)$-dimensional Reimannian manifold $\vec{M}$, which changes each geodesic circle of $\vec{M}$ into a geodesic circle of $\vec{M}$, is supposed to be a concircular change [13, 14]. A concircular change is consistently a conformal change [13]. It means a geodesic circle by a bend in $\vec{M}$ whose first curvature is steady and second arch is indistinguishably zero. Subsequently the geometry of concircular change is a speculation of intrusive geometry as in the difference in measurement is more broad than incited by a circle safeguarding diffeomorphism. A significant invariant of concircular transformation is the concircular curvature tensor C, characterized by [14]

$$C(\vec{X},\vec{Y})\vec{Z}=\vec{R}(\vec{X},\vec{Y})\vec{Z}-\frac{\vec{scal}}{2n(2n+1)}\{g(\vec{Y},\vec{Z})\vec{X}-g(\vec{X},\vec{Z})\vec{Y}\}, \tag{1.2}$$

for all $\vec{X},\vec{Y},\vec{Z}\in\chi(\vec{M})$, where $\vec{R}$ is the Reimannian curvature tensor and $\vec{scal}$ is the scalar curvature with respect to the Levi-Civita connection. An $(2n+1)$-dimensional Riemannian manifold $(\vec{M}^n,g)$, the Q-curvature tensor is defined as [15]

$$Q(\vec{X},\vec{Y})\vec{Z}=\vec{R}(\vec{X},\vec{Y})\vec{Z}-\frac{\breve{\psi}}{2n}\{g(\vec{Y},\vec{Z})\vec{X}-g(\vec{X},\vec{Z})\vec{Y}\}, \tag{1.3}$$

where $\breve{\psi}$ is an arbitrary scalar function. If $\breve{\psi}=\frac{\vec{scal}}{(2n+1)}$, then Q- curvature tensor reduces to concircular curvature tensor. Mantica and Suh [15] have studied pseudo-Q-symmetric Riemannian manifolds.
In a Riemannian manifold $(\vec{M},g)$, the metric g is called a Ricci soliton if [16]

$$\frac{1}{2}\mathfrak{L}_{\vec{V}}g+\vec{Ric}+\lambda g=0, \tag{1.4}$$

where $\mathfrak{L}$ is the Lie derivative, $\vec{Ric}$ the Ricci tensor, $\vec{V}$ a complete vector field on $\vec{M}$ and λ is a constant. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g$=-2$\vec{Ric}$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding if λ is negative, zero and positive respectively. A Ricci soliton with $\vec{V}$=0 is reduced to Einstein equation. During the last two decades, the geometry of Ricci solitons have been light up by the several mathematicians [17–19]. It has became more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904.
Our paper is structured as follows: After the introduction. In section 2 we recall the fundamental results of the Schouten-van Kampen connection and f-Kenmotsu 3-manifolds. In the portion 3 we review the thought of Ricci solition on f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection. In segment 4 we study ξ-Q flat f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection. We demonstrate the some results on f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection under the condition $\widetilde{Q}\cdot\widetilde{Ric}$=0, $\widetilde{Q}\cdot\widetilde{R}$=0, $\widetilde{Q}\cdot\widetilde{P}$=0, $\widetilde{Q}(\xi,\vec{X})\cdot\widetilde{Q}$=0 and $((\xi\wedge_{\widetilde{Ric}}\vec{X})\cdot\widetilde{Q})$=0 in the sections 5-9, respectively. In the last segment, we give the examples.

2. Preliminaries

Let $\vec{M}$ be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure $(\breve{\phi},\xi,\eta,g)$ satisfying

$$\breve{\phi}^2=I-\eta\otimes\xi,\;\;\eta(\xi)=1,\;\;\breve{\phi}\xi=0,\;\;\eta\circ\breve{\phi}=0,\;\;\eta(\vec{X})=g(\vec{X},\xi), \tag{2.1}$$

and

$$g(\breve{\phi}\vec{X},\breve{\phi}\vec{Y})=g(\vec{X},\vec{Y})-\eta(\vec{X})\eta(\vec{Y}), \tag{2.2}$$

for any vector fields $\vec{X},\vec{Y}\in\chi(\vec{M})$, where I is the identity of the tangent bundle $T\vec{M}$, $\breve{\phi}$ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor of $\vec{M}$. We say that $(\breve{\phi},\xi,\eta,g)$ is a f-Kenmotsu manifold [20, 21] if the covariant differentiation of $\breve{\phi}$ satisfies

$$(\nabla_{\vec{X}}\breve{\phi})\vec{Y}=f\{g(\breve{\phi}\vec{X},\vec{Y})\xi-\eta(\vec{Y})\breve{\phi}\vec{X}\}, \tag{2.3}$$

where $f\in C^\infty(\vec{M})$ such that $df\wedge\eta=0$. If $f=\alpha(\neq 0)=$ constant, then the manifold $(\vec{M},g)$ is an α-Kenmotsu manifold [21]. Kenmotsu manifold is an example of f-Kenmotsu manifold with f=1 [22, 23]. If f=0, then the manifold $(\vec{M},g)$ reduces to cosymplectic [21]. An f-Kenmotsu manifold is said to be regular if $f^2+\dot{f}\neq 0$, where $\dot{f}=\xi f$. For an f-Kenmotsu manifold from (2.3) it follows that

$$\nabla_{\vec{X}}\xi=f\{\vec{X}-\eta(\vec{X})\xi\}. \tag{2.4}$$

The condition $df\wedge\eta=0$ holds if dim $\vec{M}\geq 5$. In general this relation does not hold if dim $\vec{M}$=3 [23]. It is well-known that in a Riemannian 3-manifold.

$$\vec{R}(\vec{X},\vec{Y})\vec{Z}=g(\vec{Y},\vec{Z})\vec{Q}\vec{X}-g(\vec{X},\vec{Z})\vec{Q}\vec{Y}+\vec{Ric}(\vec{Y},\vec{Z})\vec{X}-\vec{Ric}(\vec{X},\vec{Z})\vec{Y}-\tfrac{\vec{scal}}{2}\{g(\vec{Y},\vec{Z})\vec{X}-g(\vec{X},\vec{Z})\vec{Y}\}. \tag{2.5}$$

In a f-Kenmotsu 3-manifold, we have [3].

$$\vec{R}(\vec{X},\vec{Y})\vec{Z}=(\tfrac{\vec{scal}}{2}+2f^2+2\dot{f})(\vec{X}\wedge\vec{Y})\vec{Z}-(\tfrac{\vec{scal}}{2}+3f^2+3\dot{f})\{\eta(\vec{X})(\xi\wedge\vec{Y})\vec{Z}+\eta(\vec{Y})(\vec{X}\wedge\xi)\vec{Z}\}, \tag{2.6}$$

$$\vec{Ric}(\vec{X},\vec{Y})=(\frac{\vec{scal}}{2}+f^2+\dot{f})g(\vec{X},\vec{Y})-(\frac{\vec{scal}}{2}+3f^2+3\dot{f})\eta(\vec{X})\eta(\vec{Y}), \tag{2.7}$$

where $\vec{scal}$ is the scalar curvature of $\vec{M}$. From (2.6) and (2.7) we obtain

$$\vec{R}(\vec{X},\vec{Y})\xi = -(f^2+\dot{f})[\eta(\vec{Y})\vec{X}-\eta(\vec{X})\vec{Y}], \tag{2.8}$$

$$\vec{Ric}(\vec{X},\xi) = -2(f^2+\dot{f})\,\eta(\vec{X}), \tag{2.9}$$

$$\vec{Ric}(\xi,\xi) = -2(f^2+\dot{f}), \tag{2.10}$$

$$\vec{Q}\xi = -2(f^2+\dot{f})\xi, \tag{2.11}$$

for any vector fields $\vec{X},\vec{Y}$ on $\vec{M}$.
On the other hand $\vec{H}$ and $\vec{V}$ are two complementary, orthogonal distributions on $\vec{M}$ such that dim$\vec{H}$=$n-1$, dim$\vec{V}$=1, and the distribution $\vec{V}$ is non-null. Thus $T\vec{M}$=$\vec{H}\oplus\vec{V}$, $\vec{H}\cap\vec{V}$=$\{0\}$ and $\vec{H}\perp\vec{V}$. Assume that ξ is a unit vector field and η is a linear form such that $\eta(\xi)$=1, $g(\xi,\xi)$=ε=±1 and

$$\vec{H} = \ker\eta, \quad \vec{V} = \mathrm{span}\{\xi\}. \tag{2.12}$$

For any $X\in T\vec{M}$, by $\vec{X}^h$ and $\vec{X}^v$ we denote the projections of $\vec{X}$ onto $\vec{H}$ and $\vec{V}$, respectively. Thus, we have $\vec{X}=\vec{X}^h+\vec{X}^v$ with

$$\vec{X}^h = \vec{X}-\eta(\vec{X})\xi, \quad \vec{X}^v = \eta(\vec{X})\xi. \tag{2.13}$$

The Schouten-van Kampen connection $\widetilde{\nabla}$ associated to the Levi-Civita connection $\vec{\nabla}$ and adapted to the pair of the distributions $(\vec{H},\vec{V})$ is defined by [5]

$$\widetilde{\nabla}_{\vec{X}}\vec{Y} = (\vec{\nabla}_{\vec{X}}\vec{Y}^h)^h + (\vec{\nabla}_{\vec{X}}\vec{Y}^v)^v. \tag{2.14}$$

From (2.13), we compute

$$(\vec{\nabla}_{\vec{X}}\vec{Y}^h)^h = \vec{\nabla}_{\vec{X}}\vec{Y}-\eta(\vec{\nabla}_{\vec{X}}\vec{Y})\xi-\eta(\vec{Y})\vec{\nabla}_{\vec{X}}\xi, \tag{2.15}$$

$$(\vec{\nabla}_{\vec{X}}\vec{Y}^v)^v = \eta(\vec{\nabla}_{\vec{X}}\vec{Y})\xi+\eta(\vec{\nabla}_{\vec{X}}\vec{Y})\xi, \tag{2.16}$$

which enables us to express the Schouten-van Kampen connection with help of the Levi-Civita connection in the following way [6]

$$\widetilde{\nabla}_{\vec{X}}\vec{Y} = \vec{\nabla}_{\vec{X}}\vec{Y}-\eta(\vec{Y})\vec{\nabla}_{\vec{X}}\xi+(\vec{\nabla}_{\vec{X}}\eta)(\vec{Y})\xi. \tag{2.17}$$

In view of the Schouten-van Kampen connection (2.17), many properties of some geometric objects connected with the distributions $\vec{H},\vec{V}$ can be characterized [6,7]. For example $\widetilde{\nabla}g=0, \widetilde{\nabla}\xi=0, \widetilde{\nabla}\eta=0$.

Proposition 2.1 ([24]). *Let $\vec{M}$ be a f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ we have*

$$\widetilde{\nabla}_{\vec{X}}\vec{Y} = \vec{\nabla}_{\vec{X}}\vec{Y}+f\{g(\vec{X},\vec{Y})\xi-\eta(\vec{Y})\vec{X}\}. \tag{2.18}$$

$$\widetilde{R}(\vec{X},\vec{Y})\vec{Z} = \vec{R}(\vec{X},\vec{Y})\vec{Z}+f^2\{g(\vec{Y},\vec{Z})\vec{X}-g(\vec{X},\vec{Z})\vec{Y}\}+\dot{f}\{g(\vec{Y},\vec{Z})\eta(\vec{X})\xi-g(\vec{X},\vec{Z})\eta(\vec{Y})\xi+\eta(\vec{Y})\eta(\vec{Z})\vec{X}-\eta(\vec{X})\eta(\vec{Z})\vec{Y}\}. \tag{2.19}$$

$$\widetilde{Ric}(\vec{Y},\vec{Z}) = \vec{Ric}(\vec{Y},\vec{Z})+(2f^2+\dot{f})g(\vec{Y},\vec{Z})+\dot{f}\eta(\vec{Y})\eta(\vec{Z}), \tag{2.20}$$

$$\ddot{Q}\vec{X} = \vec{Q}\vec{X}+(2f^2+\dot{f})\vec{X}+\dot{f}\eta(\vec{X})\xi, \tag{2.21}$$

$$\widetilde{scal} = \vec{scal}+6f^2+4\dot{f}, \tag{2.22}$$

where $\widetilde{R}$, $\vec{R}$, $\widetilde{Ric}$, $\vec{Ric}$, $\ddot{Q}$, $\vec{Q}$ and $\widetilde{scal}$, $\vec{scal}$ are consider as the Riemann curvature, Ricci tensors, Ricci operators and the scalar curvatures of the connection $\widetilde{\nabla}$ and $\vec{\nabla}$ respectively.

3. Ricci Soliton on f-Kenmotsu 3-Manifold with the Schouten-Van Kampen Connection

In this section, we study the nature of Ricci soliton on f-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$. Let $(\vec{M}^3,\phi,\xi,\eta,g)$ be a f-Kenmotsu 3-manifold with the Schouten-van Kampen connection, since $\widetilde{\nabla}g$=0 and $\widetilde{T}\neq 0$ then from [25], we have

$$(\widetilde{\mathfrak{L}}_{\vec{V}}g)(\vec{X},\vec{Y}) = g(\widetilde{\nabla}_{\vec{X}}\vec{V},\vec{Y})+g(\vec{X},\widetilde{\nabla}_{\vec{Y}}\vec{V}) = (\mathfrak{L}_{\vec{V}}g)(\vec{X},\vec{Y}), \tag{3.1}$$

where $\widetilde{\mathfrak{L}}$ denotes the Lie derivative on the manifold with respect to the Schouten-van Kampen connection. Thus from (1.4) we can write

$$(\widetilde{\mathfrak{L}}_{\vec{V}}g+2\widetilde{Ric}+2\lambda g)(\vec{X},\vec{Y}) = 0, \tag{3.2}$$

that is

$$g(\widetilde{\nabla}_{\vec{X}}\vec{V},\vec{Y})+g(\vec{X},\widetilde{\nabla}_{\vec{Y}}\vec{V})+2\widetilde{Ric}(\vec{X},\vec{Y})+2\lambda g(\vec{X},\vec{Y}) = 0, \tag{3.3}$$

Putting $\vec{V}$=ξ in (3.3) and using (2.4) we obtain

$$\widetilde{Ric}(\vec{X},\vec{Y}) = -(\lambda+f)g(\vec{X},\vec{Y})+f\eta(\vec{X})\eta(\vec{Y}) \tag{3.4}$$

In view of (2.20) and (3.4), we get

$$\vec{Ric}(\vec{X},\vec{Y}) = -(\dot{f}+2f^2+f+\lambda)g(\vec{X},\vec{Y})+(-\dot{f}+f)\eta(\vec{X})\eta(\vec{Y}) \tag{3.5}$$

Thus we can state the following:

Proposition 3.1. *A f-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$ admitting Ricci soliton then the manifold is an η-Einstein manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ and Levi-Civita connection $\vec{\nabla}$.*

Proposition 3.2. *A Ricci soliton on an f-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$ is always steady.*

Also from (3.4), we get

$$\widetilde{scal} = -2f-3\lambda. \tag{3.6}$$

In view of (2.22) and (3.6), one can easily bring out that

$$\lambda = -\frac{1}{3}(\vec{scal}+6f^2+4\dot{f}+2f). \tag{3.7}$$

We have the following:

Proposition 3.3. *A Ricci soliton on f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ is an expanding, steady or shrinking according as $\vec{scal} < -6f^2-4\dot{f}-2f$, $\vec{scal}$=$-6f^2-4\dot{f}-2f$ or $\vec{scal} > -6f^2-4\dot{f}-2f$.*

Proposition 3.4. *A Ricci soliton on α-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ is an expanding, steady or shrinking according as $\vec{scal} < -6\alpha^2-2\alpha$, $\vec{scal} = -6\alpha^2-2\alpha$ or $\vec{scal} > -6\alpha^2-2\alpha$.*

Proposition 3.5. *A Ricci soliton on cosymplectic 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$ is an expanding, steady or shrinking according as $\vec{scal} < 0$, $\vec{scal} = 0$ or $\vec{scal} > 0$.*

In [24], Yildiz et al. demonstrated that f-Kenmotsu 3-manifold is projectively flat with respect to the Schouten-van Kampen connection if and only if $\vec{M}$ is a Ricci-flat manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$. Therefore in perspective on this outcome and utilizing (3.4) we express the following:

Corollary 3.6. *A Ricci soliton on a projectively flat f-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$ is always steady.*

With the help of Theorem 6.1. of [24] and (3.4) we have the following:

Corollary 3.7. *A Ricci soliton on a conharmonically flat f-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$ is always steady.*

4. ξ-$\widetilde{Q}$ Flat f-Kenmotsu 3-Manifold with the Schouten-Van Kampen Connection

In this section, we consider ξ-$\widetilde{Q}$ flat f-Kenmotsu 3-manifold admitting the Schouten-van Kampen connection $\widetilde{\nabla}$. Now we state the following definitions and result:

Definition 4.1. *A f-Kenmotsu 3-manifold is said to be ξ-$\widetilde{Q}$ flat if $\widetilde{Q}(\vec{X},\vec{Y})\xi = 0$ on $\vec{M}$.*

Theorem 4.2. *A f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ is ξ-$\widetilde{Q}$ flat if and only if ψ=0.*

Proof. From (1.3) we have

$$\widetilde{Q}(\vec{X},\vec{Y})\xi = \widetilde{R}(\vec{X},\vec{Y})\xi - \frac{\breve{\psi}}{2}[\eta(\vec{Y})\vec{X} - \eta(\vec{X})\vec{Y}], \tag{4.1}$$

for any for any vector fields $\vec{X}$ and $\vec{Y} \in \chi(\vec{M})$. With the help of (2.6) and (2.19), equation (4.1) reduces

$$\widetilde{Q}(\vec{X},\vec{Y})\xi = -\frac{\breve{\psi}}{2}[\eta(\vec{Y})\vec{X} - \eta(\vec{X})\vec{Y}]. \tag{4.2}$$

This completes the proof. □

If $\breve{\psi}$=$\frac{\vec{scal}}{3}$ then Q-curvature tensor reduces to concircular curvature tensor. Thus keeping in mind Theorem 4.2 and making use of (1.2) we obtain the followings:

Corollary 4.3. *A f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ is ξ-concircularly flat if and only if the scalar curvature of the manifold is zero.*

Corollary 4.4. *A ξ-concircularly flat complete Einstein f-Kenmotsu 3-manifold is Ricci flat.*

Corollary 4.5. *A Ricci soliton on ξ-concircularly flat complete Einstein f-Kenmotsu 3-manifold is always steady.*

If $0 \neq f$=constant (we assume f=α) then $\dot{f}$=0. Thus we state the followings:

Corollary 4.6. *An α-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ is ξ-$\widetilde{Q}$ flat if and only if $\breve{\psi}$=0.*

Corollary 4.7. *In a ξ-$\widetilde{Q}$ flat α-Kenmotsu 3- manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ the Q-curvature tensor is equal to the Reimannian curvature tensor.*

Corollary 4.8. *In a ξ-$\widetilde{Q}$ flat α-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ the concircular curvature tensor is equal to the Reimannian curvature tensor.*

Corollary 4.9. *A Ricci soliton on ξ-concircularly flat α-Kenmotsu 3-manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$ is always shrinking.*

5. f-Kenmotsu 3-Manifolds Satisfying $\widetilde{Q} \cdot \widetilde{Ric}$=0 with the Schouten-Van Kampen Connection

In this section we restrict our study to f-Kenmotsu 3-manifolds satisfying $\widetilde{Q} \cdot \widetilde{Ric}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$. We conclude the following:

Theorem 5.1. *A f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{Q} \cdot \widetilde{Ric}$=0, then ether Q-curvature tensor is equal to the Riemannian curvature or the manifold is an η-Einstein manifold.*

Proof. Let $\vec{M}$ satisfies the condition $\widetilde{Q}(\xi,\vec{X}) \cdot \widetilde{Ric}$=0. So it implies that

$$\widetilde{Ric}(\widetilde{Q}(\xi,\vec{X})\vec{Y},\vec{Z}) + \widetilde{Ric}(\vec{Y},\widetilde{Q}(\xi,\vec{X})\vec{Z}) = 0, \tag{5.1}$$

for any $\vec{X},\vec{Y},\vec{Z}$ on $\vec{M}$. Using (1.3), (2.6) and (2.19) in (5.1), we have

$$\frac{\breve{\psi}}{2}\left\{g(\vec{X},\vec{Y})\widetilde{Ric}(\xi,\vec{Z}) - \widetilde{Ric}(\vec{X},\vec{Z})\eta(\vec{Y}) + g(\vec{X},\vec{Z})\widetilde{Ric}(\xi,\vec{Y}) - \widetilde{Ric}(\vec{X},\vec{Y})\eta(\vec{Z})\right\} = 0. \tag{5.2}$$

For $\vec{Z}$=ξ and keeping in mind (2.9) and (2.20), we obtain

$$\breve{\psi}\widetilde{Ric}(\vec{X},\vec{Y}) = 0, \tag{5.3}$$

which implies that either $\breve{\psi}$=0, or $\widetilde{Ric}(\vec{X},\vec{Y})$=0. Thus we have:
Case (i) In particular, if $\breve{\psi}$=0, and $\widetilde{Ric}(\vec{X},\vec{Y}) \neq 0$ then from (1.3) we get $Q(\vec{X},\vec{Y})\vec{Z} = \vec{R}(\vec{X},\vec{Y})\vec{Z}$.
Case (ii) Also if $\breve{\psi} \neq 0$ and $\widetilde{Ric}(\vec{X},\vec{Y})$=0, then from (2.20), the manifold is an η-Einstein manifold. This completes the proof. □

Again, if $\breve{\psi}$=$\frac{\vec{scal}}{3}$ then Q-curvature tensor reduces to concircular curvature tensor. So from Theorem 5.1 and making use of (1.2), we can mention the following:

Corollary 5.2. *A f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C} \cdot \widetilde{Ric}$=0 then either Q-curvature tensor is equal to concircular curvature tensor or the manifold is an η-Einstein manifold.*

Also, if $0 \neq f$=constant (we assume f=α), then $\dot{f}$=0. Thus we state the followings:

Corollary 5.3. *A f-Kenmotsu 3-manifolds satisfying $\widetilde{Q} \cdot \widetilde{Ric}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then ether the Q-curvature tensor is equal to the Riemannian curvature or the manifold is an η-Einstein manifold.*

Corollary 5.4. *An α-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C} \cdot \widetilde{Ric}$=0 then either Q-curvature tensor reduces to concircular curvature tensor or the manifold is an η-Einstein manifold.*

Again, in view of (5.3) and (3.4), we have the followings:

Corollary 5.5. *A Ricci soliton on f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{Q} \cdot \widetilde{Ric}$=0, then either the soliton is steady or Q-curvature tensor is equal to the Remannian curvature tensor.*

Corollary 5.6. *A Ricci soliton on f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C} \cdot \widetilde{Ric}$=0, then either the soliton is steady or concircular curvature tensor is equal to the Remannian curvature tensor.*

6. f-Kenmotsu 3-Manifolds Satisfying $\widetilde{Q}\cdot\widetilde{R}$=0 with the Schouten-Van Kampen Connection

At this stage we consider f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{Q}\cdot\widetilde{R}$=0 . Therefore we illustrate the following:

Theorem 6.1. *A f-Kenmotsu 3-manifolds satisfying $\widetilde{Q}\cdot\widetilde{R}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either Q-curvature tensor is equal to the Riemannian curvature, or it has the sectional curvature $-(f^2+\dot{f})$.*

Proof. Suppose that f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying

$$\widetilde{Q}(\xi,\vec{X})\widetilde{R}(\vec{Y},\vec{Z})\vec{U}=0. \tag{6.1}$$

Equation (6.1) can be written as

$$\widetilde{Q}(\xi,\vec{X})\widetilde{R}(\vec{Y},\vec{Z})\vec{U}-\widetilde{R}(\widetilde{Q}(\xi,\vec{X})\vec{Y},\vec{Z})\vec{U}-\widetilde{R}(\vec{Y},\widetilde{Q}(\xi,\vec{X})\vec{Z})\vec{U}-\widetilde{R}(\vec{Y},\vec{Z})\widetilde{Q}(\xi,\vec{X})\vec{U}=0, \tag{6.2}$$

for any vector fields $\vec{X}$, $\vec{Y}$, $\vec{Z}$ and $\vec{U}$ on $\vec{M}$. Using (1.3), (2.6) and (2.19) in (6.2), we obtain

$$\tfrac{\breve{\psi}}{2}[-g(\vec{X},\widetilde{R}(\vec{Y},\vec{Z})\vec{U})\xi+\eta(\widetilde{R}(\vec{Y},\vec{Z})\vec{U})-\eta(\vec{Y})\widetilde{R}(\vec{X},\vec{Z})\vec{U}-\eta(\vec{Z})\widetilde{R}(\vec{Y},\vec{X})\vec{U}-\eta(\vec{U})\widetilde{R}(\vec{Y},\vec{Z})\vec{X}]=0. \tag{6.3}$$

Taking the inner product with ξ of (6.3) and using (2.19) we get

$$\tfrac{\breve{\psi}}{2}[g(\vec{X},\widetilde{R}(\vec{Y},\vec{Z})\vec{U}+(f^2+\dot{f})\{g(\vec{Z},\vec{U})g(\vec{X},\vec{Y})-g(\vec{Y},\vec{U})g(\vec{X},\vec{Z})\}+\dot{f}\{g(\vec{X},\vec{Y})\eta(\vec{Z})\eta(\vec{U})-g(\vec{X},\vec{Z})\eta(\vec{Y})\eta(\vec{U})\}]=0. \tag{6.4}$$

It follows that either $\breve{\psi}$=0, or it has the sectional curvature $-(f^2+\dot{f})$.
This completes the proof. □

In particular, if $\breve{\psi}=\frac{\vec{scal}}{3}$ then Q-curvature tensor reduces to concircular curvature tensor. Therefore in view of the first result of the above Theorem 6.1 and making use of (1.2), we can mention the following:

Corollary 6.2. *If a f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{R}$=0 then either concircular curvature tensor is equal to the Riemannian curvature or it has the sectional curvature $-(f^2+\dot{f})$.*

Also with the help of (3.7) and Theorem 6.1, we conclude that:

Corollary 6.3. *If a f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{R}$=0 then either Ricci soliton is shrinking or it has the sectional curvature $-(f^2+\dot{f})$.*

If $0\neq f$=constant (we assume f=α), then $\dot{f}$=0. Thus we state the followings:

Corollary 6.4. *If an α-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{R}$=0 then either concircular curvature tensor is equal to the Riemannian curvature or it has the sectional curvature α^2.*

Corollary 6.5. *If an α-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{R}$=0 then either Ricci soliton is shrinking or it has the sectional curvature α^2.*

7. f-Kenmotsu 3-Manifolds Satisfying $\widetilde{Q}\cdot\widetilde{P}$=0 with the Schouten-Van Kampen Connection

We consider f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying the condition $\widetilde{Q}\cdot\widetilde{P}$=0. Then we have:

Theorem 7.1. *A f-Kenmotsu 3-manifolds satisfying $\widetilde{Q}\cdot\widetilde{P}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ is either the Q-curvature tensor is equal to the Riemannian curvature or it has the sectional curvature $\frac{1}{2}(\frac{\vec{scal}}{2}+f^2+2\dot{f})$.*

Proof. The condition $\widetilde{Q}(\xi,\vec{X})\widetilde{P}=0$ reflect that

$$(\widetilde{Q}(\xi,\vec{X})\widetilde{P})(\vec{Y},\vec{Z})\vec{U})=\widetilde{Q}(\xi,\vec{X})\widetilde{P}(\vec{Y},\vec{Z})\vec{U}-\widetilde{P}(\widetilde{Q}(\xi,\vec{X})\vec{Y},\vec{Z})\vec{U}-\widetilde{P}(\vec{Y},\widetilde{Q}(\xi,\vec{X})\vec{Z})\vec{U}-\widetilde{P}(\vec{Y},\vec{Z})\widetilde{Q}(\xi,\vec{X})\vec{U}=0, \tag{7.1}$$

for any vector fields $\vec{X}$, $\vec{Y}$, $\vec{Z}$ and $\vec{U}$ on $\vec{M}$. On the other hand from (1.3), we have

$$\widetilde{Q}(\xi,\vec{X})\widetilde{P}(\vec{Y},\vec{Z})\vec{U}=-\frac{\breve{\psi}}{2}\{g(\vec{X},\widetilde{P}(\vec{Y},\vec{Z})\vec{U})\xi-\eta(\widetilde{P}(\vec{Y},\vec{Z})\vec{U})\vec{X}\}, \tag{7.2}$$

$$\widetilde{P}(\widetilde{Q}(\xi,\vec{X})\vec{Y},\vec{Z})\vec{U}=-\frac{\breve{\psi}}{2}\{g(\vec{X},\vec{Y})\widetilde{P}(\xi,\vec{Y})\vec{Z}-\eta(\vec{Y})\widetilde{P}(\vec{X},\vec{Z})\vec{U}\}, \tag{7.3}$$

$$\widetilde{P}(\vec{Y},\widetilde{Q}(\xi,\vec{X})\vec{Z},\vec{U})=-\frac{\breve{\psi}}{2}\{g(\vec{X},\vec{Z})\widetilde{P}(\vec{Y},\xi)\vec{U}-\eta(\vec{Z})\widetilde{P}(\vec{Y},\vec{X})\vec{U}\}, \tag{7.4}$$

$$\widetilde{P}(\vec{Y},\vec{Z},\widetilde{Q}(\xi,\vec{X})\vec{U})=-\frac{\breve{\psi}}{2}\{g(\vec{X},\vec{U})\widetilde{P}(\vec{Y},\vec{Z})\xi-\eta(\vec{U})\widetilde{P}(\vec{Y},\vec{Z})\vec{X}\}. \tag{7.5}$$

Using (7.2), (7.3), (7.4) and (7.5) in (7.1), we get

$$\frac{\breve{\psi}}{2}\{-g(\vec{X},\widetilde{P}(\vec{Y},\vec{Z})\vec{U})\xi+\eta(\widetilde{P}(\vec{Y},\vec{Z})\vec{U})\vec{X}+g(\vec{X},\vec{Y})\widetilde{P}(\xi,\vec{Y})\vec{Z}-\eta(\vec{Y})\widetilde{P}(\vec{X},\vec{Z})\vec{U}+g(\vec{X},\vec{Z})\widetilde{P}(\vec{Y},\xi)\vec{U}-\eta(\vec{Z})\widetilde{P}(\vec{Y},\vec{X})\vec{U} \\ +g(\vec{X},\vec{U})\widetilde{P}(\vec{Y},\vec{Z})\xi-\eta(\vec{U})\widetilde{P}(\vec{Y},\vec{Z})\vec{X}\}=0. \tag{7.6}$$

Taking the inner product of (7.6) with ξ and using (1.1), (2.6), (2.8) and (2.19), which implies

$$\frac{\breve{\psi}}{2}\{g(\vec{X},\vec{R}(\vec{Y},\vec{Z})\vec{U})-\frac{1}{2}(\frac{s\vec{c}al}{2}+f^2+2\dot{f})(g(\vec{X},\vec{Y})g(\vec{Z},\vec{U})-g(\vec{X},\vec{Z})g(\vec{Y},\vec{U}))\}=0. \tag{7.7}$$

It is clear that either $\breve{\psi}$=0, or it has the sectional curvature $\frac{1}{2}(\frac{s\vec{c}al}{2}+f^2+2\dot{f})$. □

This leads to the proof of the Theorem 7.1.

For $\breve{\psi}=\frac{s\vec{c}al}{3}$ then Q-curvature tensor reduces to concircular curvature tensor. Therefore in view of the first result of the above Theorem 7.1 and use of (1.2), we can mention the following:

Corollary 7.2. *A f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{Ric}$=0 then either concircular curvature tensor is equal to the Remannian curvature tensor or it has the sectional curvature $\frac{1}{2}(f^2+2\dot{f})$.*

Again from Corollary 7.2, and (3.7), we have the following:

Corollary 7.3. *A f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{Ric}$=0 then either Ricci soliton is shrinking or it has the sectional curvature $\frac{1}{2}(f^2+2\dot{f})$.*

If $0\neq f$=constant (we assume $f=\alpha$), then $\dot{f}$=0. Thus we state the followings:

Corollary 7.4. *An α-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{Ric}$=0 then either concircular curvature tensor is equal to the Remannian curvature tensor or it has the sectional curvature $\frac{\alpha^2}{2}$.*

Corollary 7.5. *An α-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{C}\cdot\widetilde{Ric}$=0 then either Ricci soliton is shrinking or it has the sectional curvature $\frac{\alpha^2}{2}$.*

8. f-Kenmotsu 3-Manifolds Satisfying $\widetilde{Q}(\xi,\vec{X})\cdot\widetilde{Q}$=0 with the Schouten-Van Kampen Connection

In this section we study f-Kenmotsu 3-manifolds with the Schouten-van Kampen connection $\widetilde{\nabla}$ satisfying $\widetilde{Q}(\xi,\vec{X})\cdot\widetilde{Q}$=0 . We have the following:

Theorem 8.1. *A f-Kenmotsu 3-manifolds satisfying $\widetilde{Q}(\xi,\vec{X})\cdot\widetilde{Q}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either the Q-curvature tensor is equal to the Riemannian curvature or it has the sectional curvature -$(f^2+\dot{f})$.*

Proof. The condition $(\widetilde{Q}(\xi,\vec{X})\cdot\widetilde{Q})(\vec{Y},\vec{Z})\vec{U}$=0 implies that

$$\widetilde{Q}(\xi,\vec{X})\widetilde{Q}(\vec{Y},\vec{Z})\vec{U}-\widetilde{Q}(\widetilde{Q}(\xi,\vec{X})\vec{Y},\vec{Z})\vec{U}-\widetilde{Q}(\vec{Y},\widetilde{Q}(\xi,\vec{X})\vec{Z})\vec{U}-\widetilde{Q}(\vec{Y},\vec{Z})\widetilde{Q}(\xi,\vec{X})\vec{U}=0, \tag{8.1}$$

for any vector fields $\vec{X}$,$\vec{Y}$,$\vec{Z}$ and $\vec{U}$ on $\vec{M}$.

In view of (2.6) and (2.19), equation (1.3) reduces to

$$\widetilde{Q}(\vec{Y},\vec{Z})\vec{U}=\left\{\frac{s\vec{c}al}{2}+3f^2+2\dot{f}-\frac{\breve{\psi}}{2}\right\}[g(\vec{Z},\vec{U})\vec{Y}-g(\vec{Y},\vec{U})\vec{Z}] \\ -\left\{\frac{s\vec{c}al}{2}+3f^2+2\dot{f}\right\}[g(\vec{Z},\vec{U})\eta(\vec{Y})\xi-g(\vec{Y},\vec{U})\eta(\vec{Z})\xi+\eta(\vec{Z})\eta(\vec{U})\vec{Y}-\eta(\vec{Y})\eta(\vec{U})\vec{Z}]. \tag{8.2}$$

Then we have

$$\widetilde{Q}(\xi,\vec{Z})\vec{U}=-\frac{\breve{\psi}}{2}[g(\vec{Z},\vec{U})\xi-\eta(\vec{U})\vec{Z}], \tag{8.3}$$

$$\widetilde{Q}(\xi,\vec{X})\widetilde{Q}(\vec{Y},\vec{Z})\vec{U}=-\frac{\breve{\psi}}{2}[g(\vec{X},\widetilde{Q}(\vec{Y},\vec{Z})\vec{U}))\xi-\eta(\widetilde{Q}(\vec{Y},\vec{Z})\vec{U})\vec{X}], \tag{8.4}$$

$$\widetilde{Q}(\widetilde{Q}(\xi,\vec{X})(\vec{Y},\vec{Z})\vec{U}=-\frac{\breve{\psi}}{2}[g(\vec{X},\vec{Y})\widetilde{Q}(\xi,\vec{Z})\vec{U})-\eta(\vec{Y})\widetilde{Q}(\vec{X},\vec{Z})\vec{U}], \tag{8.5}$$

$$\widetilde{Q}(\vec{Y},\widetilde{Q}(\xi,\vec{X})\vec{Z})\vec{U}=-\frac{\breve{\psi}}{2}[g(\vec{X},\vec{Z})\widetilde{Q}(\vec{Y},\xi)\vec{U}-\eta(\vec{Z})\widetilde{Q}(\vec{Y},\vec{X})\vec{U}], \tag{8.6}$$

$$\widetilde{Q}(\vec{Y},\vec{Z})\widetilde{Q}(\xi,\vec{X})\vec{U}=-\frac{\breve{\psi}}{2}[g(\vec{X},\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\xi-\eta(\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\vec{X}]. \tag{8.7}$$

Using (8.4), (8.5), (8.6) and (8.7) in (8.1), we get

$$\frac{\breve{\psi}}{2}[-g(\vec{X},\widetilde{Q}(\vec{Y},\vec{Z})\vec{U}))\xi+\eta(\widetilde{Q}(\vec{Y},\vec{Z})\vec{U})\vec{X}+g(\vec{X},\vec{Y})\widetilde{Q}(\xi,\vec{Z})\vec{U})-\eta(\vec{Y})\widetilde{Q}(\vec{X},\vec{Z})\vec{U}+g(\vec{X},\vec{Z})\widetilde{Q}(\vec{Y},\xi)\vec{U}-\eta(\vec{Z})\widetilde{Q}(\vec{Y},\vec{X})\vec{U} \\ +g(\vec{X},\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\xi-\eta(\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\vec{X}]=0. \tag{8.8}$$

Taking the inner product of (8.8) with ξ, and using (8.2) and (8.3) we obtain

$$\frac{\breve{\psi}}{2}[g(\vec{X},\vec{R}(\vec{Y},\vec{Z})\vec{U})+(f^2+\dot{f})[g(\vec{X},\vec{Y})g(\vec{Z},\vec{Y})-g(\vec{X},\vec{Z})g(\vec{Y},\vec{U})]=0. \tag{8.9}$$

This implies that either $\breve{\psi}$=0, or it has the sectional curvature -$(f^2+\dot{f})$.
If $\breve{\psi}$=0, then from (1.3) we get $Q(\vec{X},\vec{Y})\vec{Z}=\vec{R}(\vec{X},\vec{Y})\vec{Z}$. This complete the proof. ☐

Further if $\breve{\psi}$=$\frac{\vec{scal}}{3}$ then Q-curvature tensor reduces to concircular curvature tensor. Therefore in view of Theorem 8.1 and use of (1.2), we have the followings:

Corollary 8.2. *A f-Kenmotsu 3-manifolds satisfying $\widetilde{C}(\xi,\vec{X})\cdot\widetilde{C}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either the concircular curvature tensor is equal to the Riemannian curvature or it has the sectional curvature -$(f^2+\dot{f})$.*

Corollary 8.3. *A f-Kenmotsu 3-manifolds satisfying $\widetilde{C}(\xi,\vec{X})\cdot\widetilde{C}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either Ricci soltion is shrinking or it has the sectional curvature -$(f^2+\dot{f})$.*

If $0\neq f$=constant (we assume f=α), then $\dot{f}$=0. Therefore, we have:

Corollary 8.4. *An α-Kenmotsu 3-manifolds satisfying $\widetilde{C}(\xi,\vec{X})\cdot\widetilde{C}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either the concircular curvature tensor is equal to the Riemannian curvature or it has the sectional curvature -α^2.*

Corollary 8.5. *An α-Kenmotsu 3-manifolds satisfying $\widetilde{C}(\xi,\vec{X})\cdot\widetilde{C}$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either Ricci soltion is shrinking or it has the sectional curvature -α^2.*

9. f-Kenmotsu 3-Manifolds Bearing Ricci Soliton Satisfying $((\xi\wedge_{\widetilde{Ric}}\vec{X})\cdot\widetilde{Q})$=0 with the Schouten-Van Kampen Connection

In this segment we study f-Kenmotsu 3-manifolds bearing Ricci soliton satisfying $((\xi\wedge_{\widetilde{Ric}}\vec{X})\cdot\widetilde{Q})$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$. Therefore, we have the following:

Theorem 9.1. *A f-Kenmotsu 3-manifolds bearing Ricci soliton satisfying $((\xi\wedge_{\widetilde{Ric}}\vec{X})\cdot\widetilde{Q})$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either Q-curvature tensor is equal to the Riemannian curvature or soliton is steady.*

Proof. The condition $((\xi\wedge_{\widetilde{Ric}}\vec{X})\cdot\widetilde{Q})(\vec{Y},\vec{Z})\vec{U}$=0 implies that

$$\widetilde{Ric}(\vec{X},\widetilde{Q}(\vec{Y},\vec{Z})\vec{U})\xi-\widetilde{Ric}(\xi,\widetilde{Q}(\vec{Y},\vec{Z})\vec{U})\vec{X}-\widetilde{Ric}(\vec{X},\vec{Y})\widetilde{Q}(\xi,\vec{Z})\vec{U} \\ +\widetilde{Ric}(\xi,\vec{Y})\widetilde{Q}(\vec{X},\vec{Z})\vec{U}-\widetilde{Ric}(\vec{X},\vec{Z})\widetilde{Q}(\vec{Y},\xi)\vec{U}+\widetilde{Ric}(\xi,\vec{Z})\widetilde{Q}(\vec{Y},\vec{X})\vec{U} \\ -\widetilde{Ric}(\vec{X},\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\xi+\widetilde{Ric}(\xi,\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\vec{X}=0. \tag{9.1}$$

Using (3.4) in (9.1), we get

$$-\lambda g(\vec{X},\widetilde{Q}(\vec{Y},\vec{Z})\vec{U})\xi+\lambda\eta(\widetilde{Q}(\vec{Y},\vec{Z})\vec{U})\vec{X}+\lambda g(\vec{X},\vec{Y})\widetilde{Q}(\xi,\vec{Z})\vec{U} \\ -\lambda\eta(\vec{Y})\widetilde{Q}(\vec{X},\vec{Z})\vec{U}+\lambda g(\vec{X},\vec{Z})\widetilde{Q}(\vec{Y},\xi)\vec{U}-\lambda\eta(\vec{Z})\widetilde{Q}(\vec{Y},\vec{X})\vec{U} \\ +\lambda g(\vec{X},\vec{U})\widetilde{Q}(\vec{Y},\vec{Z},\xi)-\lambda\eta(\vec{U})\widetilde{Q}(\vec{Y},\vec{Z})\vec{X}=0. \tag{9.2}$$

Taking the inner product of (9.2) with ξ and using (8.2) that implies

$$\left\{(\frac{\vec{scal}}{2}+3f^2+2\dot{f}-\frac{\breve{\psi}}{2}\right\}[-\lambda g(\vec{Z},\vec{U})g(\vec{X},\vec{Y})+3\lambda g(\vec{Y},\vec{U})g(\vec{X},\vec{Z})+3\lambda g(\vec{Z},\vec{U})\eta(\vec{Y}) \\ -3\lambda g(\vec{Y},\vec{U})\eta(\vec{Z})]-\left\{\frac{\vec{scal}}{2}+3f^2+2\dot{f}\right\}[-3\lambda g(\vec{Z},\vec{U})\eta(\vec{X})\eta(\vec{Y})+3\lambda g(\vec{Y},\vec{U})\eta(\vec{X})\eta(\vec{Z}) \\ -3\lambda g(\vec{X},\vec{Y})\eta(\vec{Z})\eta(\vec{U})+3\lambda g(\vec{X},\vec{Z})\eta(\vec{Y})\eta(\vec{U})+3\lambda g(\vec{Z},\vec{U})\eta(\vec{Y}) \\ -3\lambda g(\vec{Y},\vec{U})\eta(\vec{Z})]+\frac{\breve{\psi}}{2}[-3\lambda g(\vec{X},\vec{Y})g(\vec{Z},\vec{U})+3\lambda g(\vec{X},\vec{Y})\eta(\vec{Z})\eta(\vec{U}) \\ +3\lambda g(\vec{Z},\vec{U})\eta(\vec{X})\eta(\vec{Y})-3\lambda g(\vec{X},\vec{U})\eta(\vec{Z})\eta(\vec{Y})-3\lambda g(\vec{X},\vec{Z})\eta(\vec{Y})\eta(\vec{U}) \\ +3\lambda g(\vec{X},\vec{Z})g(\vec{Y},\vec{U})+3\lambda g(\vec{X},\vec{U})\eta(\vec{Y})\eta(\vec{Z})-3\lambda g(\vec{Y},\vec{U})\eta(\vec{Y})\eta(\vec{X}) \\ +3\lambda g(\vec{X},\vec{Z})\eta(\vec{Y})\eta(\vec{U})-3\lambda g(\vec{X},\vec{Y})\eta(\vec{Z})\eta(\vec{U})]=0. \tag{9.3}$$

For fix $\vec{U}$=ξ in (9.3) and on simplification, we get

$$3\lambda\breve{\psi}[g(\vec{X},\vec{Z})\eta(\vec{Y})-g(\vec{X},\vec{Y})\eta(\vec{Z})]=0. \tag{9.4}$$

This implies that either λ=0, or $\breve{\psi}$=0. If λ=0, and $\breve{\psi}\neq 0$, then the Ricci soliton is steady. Whereas if $\lambda\neq 0$ and $\breve{\psi}$=0, so from (1.3), we obtain $Q(\vec{X},\vec{Y})\vec{Z}$=$\vec{R}(\vec{X},\vec{Y})\vec{Z}$. This complete the proof. ☐

As per consequence if $\breve{\psi}$=$\frac{\vec{scal}}{3}$ then Q-curvature tensor reduces to concircular curvature tensor. Therefore in view of Theorem 9.1 and use of (1.2), we have the following:

Corollary 9.2. *A f-Kenmotsu 3-manifolds bearing Ricci soliton satisfying $((\xi\wedge_{\widetilde{Ric}}\vec{X})\cdot\widetilde{C})$=0 with the Schouten-van Kampen connection $\widetilde{\nabla}$ then either concircular curvature tensor is equal to the Riemannian curvature or Ricci soliton is steady.*

10. Examples

Example 10.1. *We consider the* 3*-dimensional manifold* $\vec{M}=\{(u,v,w)\in\Re^3, w\neq 0\}$, *where* (u,v,w) *are the standard coordinate in* $\Re^3$. *Let* $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ *be linearly independent vector fields at each point of* $\vec{M}$, *given by*

$$\vec{e}_1=\frac{1}{w}\frac{\partial}{\partial u},\quad \vec{e}_2=\frac{1}{w}\frac{\partial}{\partial v},\quad \vec{e}_3=-\frac{\partial}{\partial w}$$

are linearly independent at each point of $\vec{M}$. *Let* g *be the Riemannian metric defined*

$$g(\vec{e}_1,\vec{e}_2)=g(\vec{e}_2,\vec{e}_3)=g(\vec{e}_1,\vec{e}_3)=0,\quad g(\vec{e}_1,\vec{e}_1)=g(\vec{e}_2,\vec{e}_2)=g(\vec{e}_3,\vec{e}_3)=1.$$

and given by

$$g=w^2[du\otimes du+dv\otimes dv+\frac{1}{w^2}dw\otimes dw].$$

Let η *be the* 1*-form have the significance*

$$\eta(\vec{U})=g(\vec{U},\vec{e}_3)$$

for any $\vec{U}\in\Gamma(T\vec{M})$ *and* $\check{\phi}$ *be the* $(1,1)$*-tensor field defined by*

$$\check{\phi}\vec{e}_1=-\vec{e}_2,\ \ \check{\phi}\vec{e}_2=\vec{e}_1,\ \ \check{\phi}\vec{e}_3=0.$$

Making use of the linearity of $\check{\phi}$ *and* g *we have*

$$\eta(\vec{e}_3)=1,\quad \check{\phi}^2(\vec{U})=-\vec{U}+\eta(\vec{U})\vec{e}_3,\quad g(\check{\phi}\vec{U},\check{\phi}\vec{V})=g(\vec{U},\vec{V})-\eta(\vec{U})\eta(\vec{V}),$$

for any $\vec{U},\vec{W}\in\Gamma(T\vec{M})$. *Now we can easily calculate*

$$[\vec{e}_1,\vec{e}_2]=0,\quad [\vec{e}_1,\vec{e}_3]=-\frac{1}{w}\vec{e}_2,\quad [\vec{e}_2,\vec{e}_3]=-\frac{1}{w}\vec{e}_1.$$

The Riemannian connection $\vec{\nabla}$ *of the metric tensor* g *is given by the Koszul's formula, i. e.,*

$$2g(\vec{\nabla}_{\vec{U}}\vec{V},\vec{W})=\vec{U}(g(\vec{V},\vec{W}))+\vec{V}(g(\vec{W},\vec{X}))-\vec{W}(g(\vec{U},\vec{V}))-g(\vec{U},[\vec{V},\vec{W}])-g(\vec{V},[\vec{U},\vec{W}])+g(\vec{W},[\vec{U},\vec{V}]).$$

Making use of Koszul's formula we get the following:

$$\begin{array}{lll}\vec{\nabla}_{\vec{e}_2}\vec{e}_3=-\frac{1}{w}\vec{e}_2, & \vec{\nabla}_{\vec{e}_2}\vec{e}_2=\frac{1}{w}\vec{e}_3, & \vec{\nabla}_{\vec{e}_2}\vec{e}_1=0,\\ \vec{\nabla}_{\vec{e}_3}\vec{e}_3=0, & \vec{\nabla}_{\vec{e}_3}\vec{e}_2=0, & \vec{\nabla}_{\vec{e}_3}\vec{e}_1=0,\\ \vec{\nabla}_{\vec{e}_1}\vec{e}_3=-\frac{1}{w}\vec{e}_1, & \vec{\nabla}_{\vec{e}_1}\vec{e}_2=0, & \vec{\nabla}_{\vec{e}_1}\vec{e}_1=\frac{1}{w}\vec{e}_3.\end{array}$$

Consequently it is clear that $\vec{M}$ *satisfies the condition* $\vec{\nabla}_{\vec{U}}\xi=f\{\vec{U}-\eta(\vec{U})\xi\}$ *for* $\vec{e}_3=\xi$, *where* $f=-\frac{1}{w}$. *Thus we conclude that* $\vec{M}$ *leads to* f*-Kenmotsu manifold. Also* $f^2+\dot{f}=\frac{2}{w^2}\neq 0$. *That implies* $\vec{M}$ *is a regular* f*-Kenmotsu* 3*-manifold. Also the Schouten-van Kampen connection* $\widetilde{\nabla}$ *on* $\vec{M}$ *as follows*

$$\begin{array}{lll}\widetilde{\nabla}_{\vec{e}_2}\vec{e}_3=-(\frac{1}{w}+f)\vec{e}_2, & \widetilde{\nabla}_{\vec{e}_2}\vec{e}_2=(\frac{1}{w}+f)\vec{e}_3, & \widetilde{\nabla}_{\vec{e}_2}\vec{e}_1=0,\\ \widetilde{\nabla}_{\vec{e}_3}\vec{e}_3=0, & \widetilde{\nabla}_{\vec{e}_3}\vec{e}_2=0, & \widetilde{\nabla}_{\vec{e}_3}\vec{e}_1=0,\\ \widetilde{\nabla}_{\vec{e}_1}\vec{e}_3=-(\frac{1}{w}+f)\vec{e}_1, & \widetilde{\nabla}_{\vec{e}_1}\vec{e}_2=0, & \widetilde{\nabla}_{\vec{e}_1}\vec{e}_1=(\frac{1}{w}+f)\vec{e}_3.\end{array}$$

It is clear that for $\vec{e}_3=\xi$ *and* $f=-\frac{1}{w}$, *we get* $\widetilde{\nabla}_{\vec{e}_i}\vec{e}_j=0$ $(1\leq i,j\leq 3)$. *So the manifold* $\vec{M}$ *is a* f*-Kenmotsu* 3*-manifold with the Schouten-van Kampen connection* $\widetilde{\nabla}$. *Also one can seen that* $\widetilde{R}=0$. *Thus the manifold* $\vec{M}$ *is a flat manifold with respect to the Schouten-van Kampen connection* $\widetilde{\nabla}$. *Since a flat manifold is a Ricci-flat manifold with respect to the Schouten-van Kampen connection* $\widetilde{\nabla}$. *So from* (3.4), *we get* $\lambda=0$, *that is Ricci solition is always steady on regular* f*-Kenmotsu* 3*-manifold with the Schouten-van Kampen connection* $\widetilde{\nabla}$. *In case of Ricci soliton, from* (3.4) *it is sufficient to verify that*

$$\widetilde{Ric}(\vec{e}_i,\vec{e}_i)=-(\lambda+f)g(\vec{e}_i,\vec{e}_i)+f\eta(\vec{e}_i)\eta(\vec{e}_i),\ \ i=1,2,3. \tag{10.1}$$

It is clear that $\lambda=0$, *that is Ricci solition is always steady on regular* f*-Kenmotsu* 3*-manifold with the Schouten-van Kampen connection* $\widetilde{\nabla}$. *Hence Proposition* 3.2, *Corollary* 3.6 *and Corollary* 3.7 *are hold.*

Example 10.2. *We consider the* 3*-dimensional manifold* $\vec{M}=\{(u,v,w)\in\Re^3, w\neq 0\}$, *where* (u,v,w) *are the standard coordinate in* $\Re^3$. *Let* $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ *be linearly independent vector fields at each point of* $\vec{M}$, *given by*

$$\vec{e}_1=\sin^2 w\frac{\partial}{\partial u},\quad \vec{e}_2=\sin^2 w\frac{\partial}{\partial v},\quad \vec{e}_3=\sin w\frac{\partial}{\partial w}.$$

are linearly independent at each point of $\vec{M}$. *Let* g *be the Riemannian metric defined*

$$g(\vec{e}_1,\vec{e}_2)=g(\vec{e}_2,\vec{e}_3)=g(\vec{e}_1,\vec{e}_3)=0,\quad g(\vec{e}_1,\vec{e}_1)=g(\vec{e}_2,\vec{e}_2)=g(\vec{e}_3,\vec{e}_3)=1.$$

and given by

$$g = \sin^4 w[du\otimes du + dv\otimes dv + \frac{1}{\sin^2 w}dw\otimes dw].$$

Let η be the 1-form have the significance

$$\eta(\vec{U}) = g(\vec{U},\vec{e}_3)$$

for any $\vec{U}\in\Gamma(TM)$ and $\breve{\phi}$ be the $(1,1)$-tensor field defined by

$$\breve{\phi}\vec{e}_1 = -\vec{e}_2,\ \ \breve{\phi}\vec{e}_2 = \vec{e}_1,\ \ \breve{\phi}\vec{e}_3 = 0.$$

Making use of the linearity of $\breve{\phi}$ and g we have

$$\eta(\vec{e}_3) = 1,\quad \breve{\phi}^2(\vec{U}) = -\vec{U}+\eta(\vec{U})\vec{e}_3,\quad g(\breve{\phi}\vec{U},\breve{\phi}\vec{V}) = g(\vec{U},\vec{V})-\eta(\vec{U})\eta(\vec{V}),$$

for any $\vec{U},\vec{W}\in\Gamma(T\vec{M})$. Now we can easily calculate

$$[\vec{e}_1,\vec{e}_2] = 0,\quad [\vec{e}_1,\vec{e}_3] = -2\cos w\vec{e}_2,\quad [\vec{e}_2,\vec{e}_3] = -2\cos w\vec{e}_1.$$

The Riemannian connection $\vec{\nabla}$ of the metric tensor g is given by the Koszul's formula, that is.,

$$2g(\nabla_{\vec{U}}\vec{V},\vec{W}) = \vec{U}(g(\vec{V},\vec{W}))+\vec{V}(g(\vec{W},\vec{X}))-\vec{W}(g(\vec{U},\vec{V}))-g(\vec{U},[\vec{V},\vec{W}])-g(\vec{V},[\vec{U},\vec{W}])+g(\vec{W},[\vec{U},\vec{V}]).$$

Making use Koszul's formula we get the following:

$$\begin{array}{lll}
\vec{\nabla}_{\vec{e}_2}\vec{e}_3 = -2\cos w\vec{e}_2, & \vec{\nabla}_{\vec{e}_2}\vec{e}_2 = 2\cos w\vec{e}_3, & \vec{\nabla}_{\vec{e}_2}\vec{e}_1 = 0,\\
\vec{\nabla}_{\vec{e}_3}\vec{e}_3 = 0, & \vec{\nabla}_{\vec{e}_3}\vec{e}_2 = 0, & \vec{\nabla}_{\vec{e}_3}\vec{e}_1 = 0,\\
\vec{\nabla}_{\vec{e}_1}\vec{e}_3 = -2\cos w\vec{e}_1, & \vec{\nabla}_{\vec{e}_1}\vec{e}_2 = 0, & \vec{\nabla}_{\vec{e}_1}\vec{e}_1 = 2\cos w\vec{e}_3.
\end{array}$$

Consequently it is clear that $\vec{M}$ satisfies the condition $\vec{\nabla}_U\xi$ $=f\{\vec{U}-\eta(\vec{U})\xi\}$ for $\vec{e}_3$ $=\xi$, where f=-2$\cos w$. Thus we conclude that $\vec{M}$ leads to f-Kenmotsu manifold. Also $f^2+\dot{f}$=2$\cos w(2\cos w+\tan w)\neq 0$, which implies that $\vec{M}$ is a regular f-Kenmotsu 3-manifold
It is known that

$$\vec{R}(\vec{X},\vec{Y})\vec{Z} = \vec{\nabla}_{\vec{X}}\vec{\nabla}_{\vec{Y}}\vec{Z}-\vec{\nabla}_{\vec{Y}}\vec{\nabla}_{\vec{X}}\vec{Z}-\vec{\nabla}_{[\vec{X},\vec{Y}]}\vec{Z}.$$

Therefore, we find the component of curvature tensor as follows

$$\begin{array}{l}
\vec{R}(\vec{e}_2,\vec{e}_3)\vec{e}_3 = -2(\sin w+2\cos^2 w)\vec{e}_2,\quad \vec{R}(\vec{e}_3,\vec{e}_2)\vec{e}_2 = -2(\sin w+2\cos^2 w)\vec{e}_3,\\
\vec{R}(\vec{e}_1,\vec{e}_3)\vec{e}_3 = -2(\sin w+2\cos^2 w)\vec{e}_1,\quad \vec{R}(\vec{e}_3,\vec{e}_1)\vec{e}_1 = -2(\sin w+2\cos^2 w)\vec{e}_2,\\
\vec{R}(\vec{e}_3,\vec{e}_1)\vec{e}_2 = 0,\quad \vec{R}(\vec{e}_1,\vec{e}_2)\vec{e}_2 = -4\cos^2 w\vec{e}_1,\quad \vec{R}(\vec{e}_1,\vec{e}_2)\vec{e}_3 = 0,\\
\vec{R}(\vec{e}_2,\vec{e}_3)\vec{e}_1 = 0,\quad \vec{R}(\vec{e}_2,\vec{e}_1)\vec{e}_1 = 4\cos^2 w\vec{e}_3.
\end{array}$$

The Schouten-van Kampen connection $\widetilde{\nabla}$ on $\vec{M}$ is given by

$$\begin{array}{lll}
\widetilde{\nabla}_{\vec{e}_2}\vec{e}_3 = (-2\cos w-f)\vec{e}_2, & \widetilde{\nabla}_{\vec{e}_2}\vec{e}_2 = (-2\cos w-f)\vec{e}_3, & \widetilde{\nabla}_{\vec{e}_2}\vec{e}_1 = 0,\\
\widetilde{\nabla}_{\vec{e}_3}\vec{e}_3 = 0, & \widetilde{\nabla}_{\vec{e}_3}\vec{e}_2 = 0, & \widetilde{\nabla}_{\vec{e}_3}\vec{e}_1 = 0,\\
\widetilde{\nabla}_{\vec{e}_1}\vec{e}_3 = (-2\cos w-f)\vec{e}_1, & \widetilde{\nabla}_{\vec{e}_1}\vec{e}_2 = 0, & \widetilde{\nabla}_{\vec{e}_1}\vec{e}_1 = (-2\cos w-f)\vec{e}_3.
\end{array}$$

It is clear that for $\vec{e}_3$=ξ and $f=-2\cos w$, we get $\widetilde{\nabla}_{\vec{e}_i}\vec{e}_j$=0 $(1\leq i,j\leq 3)$. So the manifold $\vec{M}$ is a f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$. Also from above curvature component one can be seen that $\widetilde{R}$=0. Thus the manifold $\vec{M}$ is a flat manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$. Since a flat manifold is a Ricci-flat manifold with respect to the Schouten-van Kampen connection $\widetilde{\nabla}$.
In case of Ricci soliton, from (3.4) it is sufficient to verify that

$$\widetilde{Ric}(\vec{e}_i,\vec{e}_i) = -(\lambda+f)g(\vec{e}_i,\vec{e}_i)+f\eta(\vec{e}_i)\eta(\vec{e}_i),\ \ i=1,2,3. \tag{10.2}$$

It is clear that λ=0, that is Ricci solition is always steady on regular f-Kenmotsu 3-manifold with the Schouten-van Kampen connection $\widetilde{\nabla}$. Hence Proposition 3.2, Corollary 3.6 and Corollary 3.7 are hold.

11. Conclusion

In this study, we examine certain new curvature conditions of Q-curvature tensor on f-Kenmotsu 3-manifold admitting the Schouten-van Kampen connection $\widetilde{\nabla}$ and deduce some geometrical results. Also we explore the nature of Ricci soliton.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Numerical Stability of Runge-Kutta Methods for Differential Equations with Piecewise Constant Arguments with Matrix Coefficients

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Abstract

The paper discusses the analytical stability and numerical stability of differential equations with piecewise constant arguments with matrix coefficients. Firstly, the Runge-Kutta method is applied to the equation and the recurrence relationship of the numerical solution of the equation is obtained. Secondly, it is proved that the Runge-Kutta method can preserve the convergence order. Thirdly, the stability conditions of the numerical solution under different matrix coefficients are given by Padé approximation and order star theory. Finally, the conclusions are verified by several numerical experiments.

1. Introduction

With more and more research on delay differential equations [1–5], it has been widely used in various fields, such as population research [6,7], epidemiology [8,9], electrodynamics [10,11], neural network system [12,13] and so on. As a special class of delay differential equations, differential equations with piecewise constant arguments (EPCA) are difficult to solve accurately because of their complex structure. Therefore, a series of numerical methods are introduced, for example, Euler method [14], improved Euler method [15] and Runge-Kutta methods [16]. Meanwhile, scholars also paid more attention to the study of the properties of the numerical solution of EPCA. Gao [17] considered the numerical oscillation and non-oscillation for EPCA. It is proved that oscillation of the analytic solution is preserved by the Runge-Kutta methods under some conditions. The conditions under which the non-oscillation of analytic solution is preserved by the Runge-Kutta methods were obtained. In [18], Wang introduced the condition that Runge-Kutta methods was called $BN_f - stable$ and established two classes of Runge-Kutta methods which satisfied the condition. It is shown that a class of Runge-Kutta methods can preserve their original convergence order for EPCA but not the other class of Runge-Kutta methods. At the end of the paper, the asymptotic stability results of Runge-Kutta methods are obtained. In [19], a class of linear impulsive EPCA were considered. From the paper, it is proved that the θ-methods preserved stability of the equations. Stability conditions of Runge-Kutta methods were obtained. Liang et al. [20] considered the conditions of numerical stability of EPCA with matrix coefficients through the Runge-Kutta methods. The different types of matrix coefficient L are classified and discussed in [20]. Different from [20], we will consider a more complex equation and obtain new conclusions in this work.

This paper deals with the stability of the numerical solution of the following EPCA with matrix coefficients:

$$
\begin{aligned}
&x'(t) = Lx(t) + Mx\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\
&x(0) = x_0,
\end{aligned} \tag{1.1}
$$

where $[\cdot]$ designates the greatest-integer function, $L, M \in \mathbb{C}^{d\times d}$, L is nonsingular, and $x_0 \in \mathbb{C}^d$ is a given initial value. The general form of

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this type equation is

$$x'(t) = f(t, x(t), x(\alpha(t))), t \geq 0,$$
$$x(0) = x_0, \tag{1.2}$$

where the argument $\alpha(t)$ has intervals of constancy. Because the argument deviation of Eq. (1.1) is positive in $\left[n, n+\frac{1}{2}\right)$ and negative in $\left[n+\frac{1}{2}, n+1\right)$, Eq. (1.1) is also said to be of alternately advanced and retarded type.

2. Analytical Stability

Definition 2.1 ([21]). *A solution of Eq. (1.1) on $[0,\infty)$ is a function $x(t)$ satisfies the conditions:*
(i) $x(t)$ is continuous on $[0,\infty)$;
(ii) The derivative $x'(t)$ exists at each point $t \in [0,\infty)$, with the possible exception of the points $t = n+\frac{1}{2}, n = 1,2,\ldots$, where one-sided derivatives exist;
(iii) Eq. (1.1) is satisfied on $\left[0,\frac{1}{2}\right)$ and each interval $\left[n-\frac{1}{2}, n+\frac{1}{2}\right)$ for $n = 1,2,\ldots$

Theorem 2.2 ([21]). *Eq. (1.1) has a unique solution on $[0,\infty)$*

$$x(t) = M(T(t))M_0^{\left[t+\frac{1}{2}\right]}x_0, \tag{2.1}$$

where $T(t) = t - \left[t+\frac{1}{2}\right], M(t) = e^{Lt} + \left(e^{Lt} - I\right)L^{-1}M, M_0 = M\left(-\frac{1}{2}\right)^{-1}M\left(\frac{1}{2}\right).$

Definition 2.3. *The zero solution of Eq. (1.1) is asymptotically stable if any solution $x(t)$ of Eq. (1.1) satisfies*

$$\lim_{t\to\infty} x(t) = 0.$$

Lemma 2.4 ([21]). *The zero solution of Eq. (1.1) is asymptotically stable, if and only if the eigenvalues λ_j $(j = 1,\cdots,r)$ of the matrix M_0 satisfy the inequality $|\lambda_j| < 1$.*

From [22], we suppose that $\|\cdot\|$ denotes the matrix norm derived from a vector norm on $\mathbb{C}^d$ and $\mu[\cdot]$ denotes the logarithmic norm of the matrix which defined by

$$\mu[L] = \lim_{\Delta\to 0^+} \frac{\|I_d + \Delta L\| - 1}{\Delta},$$

where I_d is the $d \times d$ identity matrix.

Theorem 2.5. *The zero solution of Eq. (1.1) is asymptotically stable if*
(i) $\mu[L] < 0$;
(ii) $\|M\| < -\mu[L]$.

Proof. Suppose that $x\left(n-\frac{1}{2}\right) = c_n, x\left(n+\frac{1}{2}\right) = c_{n+1}$ for interval $\left[n-\frac{1}{2}, n+\frac{1}{2}\right]$. According to Eq. (1.1), we use the method of constant variation, let

$$x(t) = a(t)e^{Lt}, \tag{2.2}$$

so

$$x'(t) = a'(t)e^{Lt} + a(t)Le^{Lt}, \tag{2.3}$$

$$a\left(n-\frac{1}{2}\right) = c_n e^{-L\left(n-\frac{1}{2}\right)}, \tag{2.4}$$

then we substitute Eq. (2.3) into Eq. (1.1) and obtain

$$a'(t) = e^{-Lt}Mc_n, \tag{2.5}$$

integrate both sides of Eq. (2.5), we have

$$\int_{n-\frac{1}{2}}^{t} a'(t)dt = Mc_n \int_{n-\frac{1}{2}}^{t} e^{-Lt}dt, \tag{2.6}$$

i.e.,

$$a(t) - a\left(n-\frac{1}{2}\right) = \left(e^{-L\left(n-\frac{1}{2}\right)} - e^{-Lt}\right)L^{-1}Mc_n, \tag{2.7}$$

in other words

$$a(t) = \left(e^{-L\left(n-\frac{1}{2}\right)} - e^{-Lt}\right)L^{-1}Mc_n + c_n e^{-L\left(n-\frac{1}{2}\right)}, \tag{2.8}$$

therefore

$$x(t)=\left(e^{L\left(t-n+\frac{1}{2}\right)}-I\right)L^{-1}Mc_n+c_ne^{L\left(t-n+\frac{1}{2}\right)}. \tag{2.9}$$

Let $t=n+\frac{1}{2}$, we have

$$c_{n+1}=\left(e^L+\left(e^L-I\right)L^{-1}M\right)c_n, \tag{2.10}$$

i.e.,

$$\lambda=\frac{c_{n+1}}{c_n}=e^L+\left(e^L-I\right)L^{-1}M. \tag{2.11}$$

Now we just need to prove $\|\lambda\|<1$. From (2.11), we have

$$\|\lambda\|=\left\|e^L+\left(e^L-I\right)L^{-1}M\right\|\leq\left\|e^L\right\|+\left\|\left(e^L-I\right)L^{-1}\right\|\|M\|, \tag{2.12}$$

by the condition (i), we know that $\mu[L]\neq 0$, so

$$\left\|\left(e^L-I\right)L^{-1}\right\|=\left\|\int_0^1 e^{Ls}ds\right\|\leq\int_0^1\left\|e^{Ls}\right\|ds\leq\int_0^1 e^{\mu[L]s}ds=\frac{1}{\mu[L]}\left(e^{\mu[L]}-1\right).$$

Noting that $e^{\mu[L]}-1$ and $\mu[L]$ have the same sign, by the condition (ii), we have

$$\|\lambda\|\leq\left\|e^L\right\|+\left\|\left(e^L-I\right)L^{-1}\right\|\|M\|\leq e^{\mu[L]}+\frac{\|M\|}{\mu[L]}\left(e^{\mu[L]}-1\right)<e^{\mu[L]}-\left(e^{\mu[L]}-1\right)=1.$$

By Lemma 2.4, the proof is completed. □

3. Runge-Kutta Methods and Convergence

In this section, we consider the Runge-Kutta methods (A,B,C) to solve the given equation.The following is the form of the Butcher column of the Runge-Kutta methods:

$$\begin{array}{c|c} C & A \\ \hline & B^T \end{array}$$

where the matrix $A=\{a_{ij}\}$, the weight vector B^T with $B_1+B_2+\cdots+B_v=1$ and the knot vector C with $0\leq C_1\leq C_2\leq\cdots\leq C_v\leq 1$. Let $h=1/2m$ be a given stepsize with integer $m\geq 1$ and the gridpoints t_n be defined by $t_n=nh(n=0,1,2\cdots)$. Applying the Runge-Kutta methods to Eq. (1.2) leads to a numerical process of the following type, generating approximations $x_1,x_2,x_3,\cdots$ to the exact solution $x(t)$ of Eq. (1.2) at the gridpoints $t_n(n=1,2,3,\cdots)$

$$\begin{aligned} x_{n+1}&=x_n+h\sum_{i=1}^{v}B_i\left(Ly_i^{(n)}+Mz_i^{(n)}\right),\\ y_i^{(n)}&=x_n+h\sum_{j=1}^{v}a_{ij}\left(Ly_j^{(n)}+Mz_j^{(n)}\right),\quad i=1,2,\cdots,v, \end{aligned} \tag{3.1}$$

where x_n is the numerical approximation to $x(t)$ at $t_n, y_i^{(n)}$ and $z_i^{(n)}$ are the numerical approximations to $x(t_n+C_ih)$ and $x\left(\left[t_n+C_ih+\frac{1}{2}\right]\right)$, respectively. If we denote $n=2km+l$, $L(k)=\{0,1,\cdots,m-1\}$ for $k=0$ and $L(k)=\{-m,-m+1,\cdots,m-2,m-1\}$ for $k\geq 1$, then $z_i^{(2km+l)}$ can be defined as x_{2km} according to Definition 2.1. Let $Y^{(n)}=\left(\left(y_1^{(n)}\right)^T,\left(y_2^{(n)}\right)^T,\cdots,\left(y_V^{(n)}\right)^T\right)^T$, then Eq. (3.1) reduces to

$$\begin{aligned} x_{2km+l+1}&=x_{2km+l}+h\left(B^T\otimes L\right)Y^{(2km+l)}+hMx_{2km},\quad l\in L(k),\\ Y^{(2km+l)}&=(e\otimes I_d)x_{2km+l}+h(A\otimes L)Y^{(2km+l)}+h(Ae\otimes M)x_{2km}, \end{aligned} \tag{3.2}$$

where $e=(1,1,\cdots,1)^T$, $A=\left(a_{ij}\right)_{v\times v}$, $B=(B_1,B_2,\cdots,B_v)^T$ and $\otimes$ denotes the Kronecker product.
Therefore, we have

$$x_{2km+l+1}=R(Z)x_{2km+l}+\varphi(Z,Y)x_{2km},\quad l\in L(k), \tag{3.3}$$

where $Z=hL$, $Y=hM$, $\varphi(Z,Y)=\left(\mathrm{B}^T\otimes Z\right)\left(I_{vd}-A\otimes Z\right)^{-1}(Ae\otimes Y)+Y$ and $R(Z)=I_d+\left(\mathrm{B}^T\otimes Z\right)\left(I_{vd}-A\otimes Z\right)^{-1}(e\otimes I_d)$ is the stability function of the Runge-Kutta methods.
Let the Runge-Kutta methods be of order q, then there is a constant K such that for sufficiently small h [23]- [24],

$$\left|e^Z-R(Z)\right|\leq Kh^{q+1}, \tag{3.4}$$

we have

$$e^{-Lt_{2km+l+1}}x(t_{2km+l+1})=e^{-Lt_{2km+1}}x(t_{km+l})+\left(e^{-Lt_{2km+1}}-e^{-Lt_{2km+t+1}}\right)L^{-1}Mx(t_{2km}). \tag{3.5}$$

So

$$x\left(t_{2km+l+1}\right)=e^{Lh}x\left(t_{km+l}\right)+\left(e^{Lh}-I_d\right)L^{-1}Mx\left(t_{2km}\right),$$

i.e.,

$$x\left(t_{2km+l+1}\right)=e^{Z}x\left(t_{km+l}\right)+\left(e^{Z}-I_d\right)Z^{-1}Yx\left(t_{2km}\right). \tag{3.6}$$

Moreover

$$\begin{aligned}\varphi(Z,Y)&=\left(\mathrm{B}^T\otimes Z\right)\left(I_{vd}-A\otimes Z\right)^{-1}\left(Ae\otimes Y\right)+Y\\&=\left(\left(\mathrm{B}^T\otimes Z\right)\left(I_{vd}-A\otimes Z\right)^{-1}\left(Ae\otimes I_d\right)+I_d\right)Y\\&=\left(\mathrm{B}^T\otimes Z\right)\left(I_{vd}-A\otimes Z\right)^{-1}\left(\left(Ae\otimes I_d\right)+\left(I_{vd}-A\otimes Z\right)\left(e\otimes Z^{-1}\right)\right)Y\\&=\left(\mathrm{B}^T\otimes Z\right)\left(I_{vd}-A\otimes Z\right)^{-1}\left(e\otimes Z^{-1}\right)Y\\&=\left(R(Z)-I_d\right)Z^{-1}Y.\end{aligned}$$

From Eq. (3.4) and Eq. (3.6), if $x\left(t_{2km}\right)=x_{2km}$ and $x\left(t_{2km+l}\right)=x_{2km+l}$, then

$$\left\|x\left(t_{2km+l+1}\right)-x_{2km+l+1}\right\|=\left\|\left(e^{Z}-R(Z)\right)\left(x\left(t_{2km+l}\right)+Z^{-1}Yx\left(t_{2km}\right)\right)\right\|\le Kh^{q+1}\left(1+\left\|Z^{-1}Y\right\|\right)\max_{k-\frac{1}{2}\le t\le k+\frac{1}{2}}|x(t)|, \tag{3.7}$$

which implies that for Eq. (1.1) the Runge-Kutta method is also convergent of order q.

4. Numerical Stability

4.1. The general asymptotic stability

In this section, we will study the conditions of numerical stability for any initial value. We introduce the set Σ consisting of all pairs (L,M), which satisfies Theorem 2.5, i.e., $\Sigma=\left\{(L,M)\in\mathbb{C}^{d\times d}\times\mathbb{C}^{d\times d}:\mu[L]<0,\|M\|<-\mu[L]\right\}$. From Eq. (3.4), we obtain

$$\begin{pmatrix}x_{2km}\\x_{2km+1}\\\vdots\\x_{2km+m-1}\\x_{(2k+1)m}\end{pmatrix}=T_1\begin{pmatrix}x_{2km-1}\\x_{2km}\\\vdots\\x_{2km+m-2}\\x_{2km+m-1}\end{pmatrix},\begin{pmatrix}x_{2km-1}\\x_{2km}\\\vdots\\x_{2km+m-2}\\x_{2km+m-1}\end{pmatrix}=T_2\begin{pmatrix}x_{2km-2}\\x_{2km-1}\\\vdots\\x_{2km+m-3}\\x_{2kmm+m-2}\end{pmatrix},\cdots,\begin{pmatrix}x_{(2k-1)m+1}\\x_{(2k-1)m+2}\\\vdots\\x_{2km}\\x_{2km+1}\end{pmatrix}=T_m\begin{pmatrix}x_{(2k-1)m}\\x_{(2k-1)m+1}\\\vdots\\x_{2km-1}\\x_{2km}\end{pmatrix}.$$

where

$$T_1=\begin{pmatrix}0&I_d&0&\cdots&0\\0&0&I_d&\cdots&0\\\vdots&\vdots&\vdots&\ldots&\vdots\\0&0&0&\cdots&I_d\\0&\varphi(Z,Y)&0&\cdots&R(Z)\end{pmatrix},T_2=\begin{pmatrix}0&I_d&0&\cdots&0\\0&0&I_d&\cdots&0\\\vdots&\vdots&\vdots&\ldots&\vdots\\0&0&0&\cdots&I_d\\0&0&\varphi(Z,Y)&\cdots&R(Z)\end{pmatrix},$$

$$T_m=\begin{pmatrix}0&I_d&0&\cdots&0\\0&0&I_d&\cdots&0\\\vdots&\vdots&\vdots&\ldots&\vdots\\0&0&0&\cdots&I_d\\0&0&0&\cdots&\varphi(Z,Y)+R(Z)\end{pmatrix}.$$

Let $X_{2k}=\left(x_{2km}^T,x_{2km+1}^T,\cdots,x_{2km+m}^T\right)^T$ and $T=\prod_{i=1}^{m}T_i$, we obtain

$$X_{2k}=TX_{2k-1},\quad k=1,2,\cdots, \tag{4.1}$$

where

$$T=\begin{pmatrix}0&\cdots&0&B_{1,m+1}\\0&\cdots&0&B_{2,m+1}\\\vdots&\ldots&\vdots&\vdots\\0&\cdots&0&B_{m+1,m+1}\end{pmatrix},$$

$$B_{i,m+1}=I_d+\left(R(Z)^{i-1}-I_d\right)\left(I_d+Z^{-1}Y\right).$$

Definition 4.1. *Process (4.1) for Eq. (1.1) is called asymptotically stable at (L,M) if and only if for all $h = 1/2m$,*
(i) $I_{vd} - A \otimes Z$ is invertible,
(ii) for any given $x_i (1 \le i \le m)$, process (4.1) defines $X_{2k} (k = 1, 2 \cdots)$ that satisfy $X_{2k} \to 0$ for $k \to \infty$.

Definition 4.2. *The set of all pairs (L,M) at which the process (4.1) for Eq. (1.1) is asymptotically stable for all $h = 1/2m$ is called the asymptotical stability region denoted by S, i.e., $S = \left\{(L,M) : \rho\left(R(Z)^m + (R(Z)^m - I_d) Z^{-1} Y\right) < 1\right\}$.*

Lemma 4.3 ([20]). *If the Runge-Kutta method is A-stable and $\mu[L] < 0$, then for any integer m,*

$$\frac{1 - \|R(Z)^m\|}{\left\|(I_d - R(Z)^m) Z^{-1}\right\|} \ge \frac{1 - \|R(Z)\|}{\left\|(I_d - R(Z)) Z^{-1}\right\|}. \tag{4.2}$$

Theorem 4.4. *If the Runge-Kutta method is A-stable, for all Z with $\mu[Z] < 0$,*

$$\|R(Z)\| - \mu[Z] \left\|(I_d - R(Z)) Z^{-1}\right\| \le 1,$$

then $\Sigma \subset S$.

Proof. Let $(L,M) \in \Sigma$. By Lemma 4.3, we obtain

$$\begin{aligned}
\left\|R(Z)^m + (R(Z)^m - I_d) Z^{-1} Y\right\| &\le \|R(Z)^m\| + \left\|(R(Z)^m - I_d) Z^{-1}\right\| \|Y\| \\
&\le \frac{\|R(Z)^m\| (1 - \|R(Z)\|) + \left\|(R(Z)^m - I_d) Z^{-1}\right\| \|Y\| (1 - \|R(Z)\|)}{1 - \|R(Z)\|} \\
&= \frac{\|R(Z)^m\| (1 - \|R(Z)\|) + \left\|(R(Z)^m - I_d) Z^{-1}\right\| \|Y\| (1 - \|R(Z)\|)}{1 - \|R(Z)\|} - 1 + 1 \\
&= \frac{(\|R(Z)^m\| - 1)(1 - \|R(Z)\|) + (1 - \|R(Z)^m\|) \left\|(I_d - R(Z)) Z^{-1}\right\| \|Y\|}{1 - \|R(Z)\|} + 1 \\
&< \frac{(1 - \|R(Z)^m\|)\left(\|R(Z)\| - 1 - \mu[Z] \left\|(I_d - R(Z)) Z^{-1}\right\|\right)}{1 - \|R(Z)\|} + 1 \\
&\le 1,
\end{aligned}$$

therefore, $\rho\left(R(Z)^m + (R(Z)^m - I_d) Z^{-1} Y\right) < 1$. By Definition 4.2, the proof is completed. □

4.2. In the case of 2-norm and L being a normal matrix

In this section, we suppose that L is a normal matrix, i.e., $LL^* = L^*L$, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_d$ and $\|\cdot\|$ denotes the spectral norm, i.e.,

$$\|L\| = \max\left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } L^*L\right\}.$$

Lemma 4.5 ([20]). *If the Runge-Kutta method is A-stable and (L,M) satisfies Theorem 2.5, then $x_n \to 0$ as $n \to \infty$ if and only if*

$$|R(\aleph_i)| - \mathrm{Re}(\aleph_i) \left|\frac{1 - R(\aleph_i)}{\aleph_i}\right| \le 1, \quad i = 1, 2, \cdots, d,$$

where $\aleph_i = h\lambda_i$ and λ_i is an eigenvalue of L.

Let $\Sigma_1 = \{(L,M) : (L,M) \in \Sigma \text{ and } L \text{ is a normal matrix}\}$.

Theorem 4.6. *If the Runge-Kutta method is A-stable, then $\Sigma_1 \subset S$ if and only if for all $\aleph$ with $\mathrm{Re}(\aleph) < 0$,*

$$|R(\aleph)| - \mathrm{Re}(\aleph) \left|\frac{1 - R(\aleph)}{\aleph}\right| \le 1.$$

Proof. By Lemma 4.5, for A-stable Runge-Kutta method, $\Sigma_1 \subset S$ if and only if for $\aleph_i = h\lambda_i$ and λ_i being an eigenvalue of L,

$$|R(\aleph_i)| - \mathrm{Re}(\aleph_i) \left|\frac{1 - R(\aleph_i)}{\aleph_i}\right| \le 1, \quad i = 1, 2, \cdots, d,$$

then the proof is completed by the arbitrariness of L. □

4.3. In the case of 2-norm and L being a real symmetric matrix

In this section, we suppose that L is a real symmetric matrix and $\|\cdot\|$ denotes the spectral norm. Let

$$\Sigma_2 = \{(L,M) : (L,M) \in \Sigma \text{ and } L \text{ is a real symmetric matrix}\}.$$

Theorem 4.7. *If the Runge-Kutta method is A-stable, then $\Sigma_2 \subset S$ if and only if $0 \le R(\aleph) \le 1$ for all $\aleph < 0$.*

Proof. It is obvious that L is a normal matrix, in the view of Theorem 4.6, $\Sigma_2 \subset S$ if and only if for all $\aleph < 0$,

$$|R(\aleph)| - \mathrm{Re}(\aleph)\left|\frac{R(\aleph)-1}{\aleph}\right| \le 1,$$

the eigenvalues of real symmetric matrices must be real numbers, so we have

$$|R(\aleph)| - \mathrm{Re}(\aleph)\left|\frac{R(\aleph)-1}{\aleph}\right| = |R(\aleph)| + |R(\aleph)-1| \le 1,$$

which is equivalent to $0 \le R(\aleph) \le 1$. □

Corollary 4.8. *Suppose that the Runge-Kutta method is A-stable and $R(\aleph)$ is the (r,s)-Padé approximation to $e^{\aleph}$, then r is odd if and only if $0 < R(\aleph) \le e^{\aleph}$ for all $\aleph < 0$.*

Proof. According to Lemma 4.3 in [24], $0 < R(\aleph) \le e^{\aleph}$ for all $\aleph < 0$ if and only if the negative real axis is contained in a white sector in the left-half plane, which is the same as to r is odd. The proof is completed. □

Theorem 4.9. *If Runge-Kutta method is A-stable and $R(\aleph)$ is the (r,s)-Padé approximation to $e^{\aleph}$, then $\Sigma_2 \subset S$ if and only if r is odd.*

Proof. By Corollary 4.8, $0 < R(\aleph) \le e^{\aleph} < 1$ for all $\aleph < 0$ if and only if r is odd, then for A-stable Runge-Kutta method, $0 < R(\aleph) < 1$ for all $\aleph < 0$ if and only if r is odd. According to Theorem 4.7, the proof is completed. □

For the higher order Runge-Kutta methods, their stability conclusions are shown in Table 1.

	Gauss-Legendre	*Radau IA, IIA*	*Lobatto IIIA, IIIB*	*Lobatto IIIC*
(r,s)	(v,v)	$(v-1,v)$	$(v-1,v-1)$	$(v-2,v)$
$\Sigma_2 \subset S$	v is odd	v is even	v is even	v is odd

Table 1: The higher order Runge-Kutta methods

5. Numerical Experiments

In this section, we give some examples to verify the conclusions in the paper. Four Runge-Kutta methods are used: 2-Gauss-Legendre, 2-Radau IA, 3-Lobatto IIIB and 3-Lobatto IIIC. Their Butcher columns are listed as follows:
2-Gauss-Legendre:

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

2-Radau IA:

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

3-Lobatto IIIB:

$$
\begin{array}{c|ccc}
0 & \frac{1}{6} & -\frac{1}{6} & 0 \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 \\
1 & \frac{1}{6} & \frac{5}{6} & 0 \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

3-Lobatto IIIC:

$$
\begin{array}{c|ccc}
0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\
\frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\
1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

For Theorem 4.4, we consider the following equation:

$$
\begin{cases}
x'(t) = \begin{pmatrix} -15 & -6 & -9 \\ 1 & -3 & 0 \\ 1 & -1 & -10 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 0 & -1 \end{pmatrix} x\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\
x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\end{cases}
\tag{5.1}
$$

where L is a matrix with $\mu[L] \approx -2.4862 < 0$ and $\|M\| \approx 2.4495 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For all Z with $\mu[Z] \approx -0.0249 < 0$, we obtain Theorem 4.4 is satisfied. We plot the numerical solution of 2-Gauss-Legendre for Eq. (5.1) with $m = 50$ in Figure 5.1 and we obtain that the numerical solution is asymptotically stable.

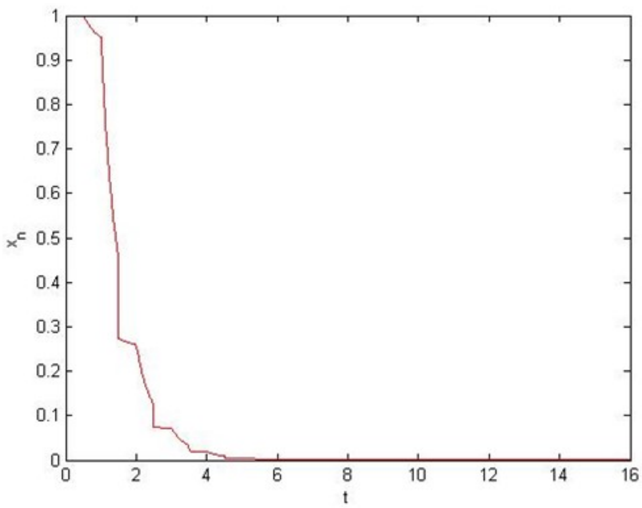

Figure 5.1: 2-Gauss-Legendre solution for Eq. (5.1) with $m = 50$.

For Theorem 4.6, we consider the following equation:

$$
\begin{cases}
x'(t) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} x\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\
x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{cases}
\tag{5.2}
$$

where L is a normal matrix with $\mu[L] = -1 < 0$ and $\|M\| \approx 0.7071 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For $\aleph = \begin{pmatrix} -0.02 \\ -0.01 \end{pmatrix}$ with $\mathrm{Re}(\aleph) = \begin{pmatrix} -0.02 \\ -0.01 \end{pmatrix} < 0$, we obtain Theorem 4.6 is satisfied. We plot the numerical solution of 2-Radau IA for Eq. (5.2) with $m = 50$ in Figure 5.2 and we obtain that the numerical solution is asymptotically stable.

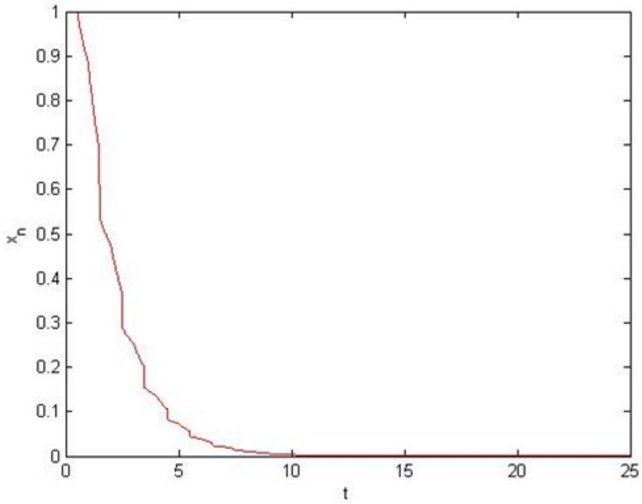

Figure 5.2: 2-Radau IA solution for Eq. (5.2) with $m = 50$.

For Theorem 4.7, we consider the following equation:

$$\begin{cases} x'(t) = \begin{pmatrix} -8 & 1 & 1 \\ 1 & -6 & 1 \\ 1 & 1 & -8 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 0 & 0 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix} x\left(\left[t + \frac{1}{2}\right]\right), t \geq 0, \\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{cases} \tag{5.3}$$

where L is a real symmetric matrix with $\mu[L] = -5 < 0$ and $\|M\| \approx 3.4338 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For $\aleph = \begin{pmatrix} -0.09 \\ -0.08 \\ -0.05 \end{pmatrix} < 0$ with $R(\aleph) \approx \begin{pmatrix} 0.9139 \\ 0.9231 \\ 0.9512 \end{pmatrix}$, which satisfies Theorem 4.7. We plot the numerical solution of 3-Lobatto IIIB for Eq. (5.3) with $m = 50$ in Figure 5.3 and we obtain that the numerical solution is asymptotically stable.

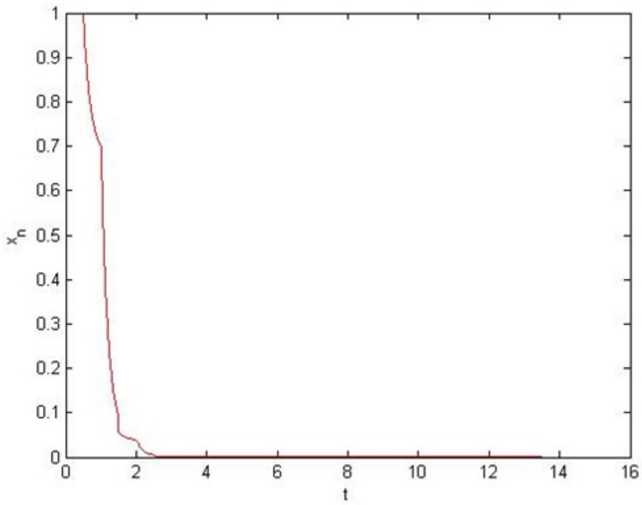

Figure 5.3: 3-Lobatto IIIB solution for Eq. (5.3) with $m = 50$.

For Theorem 4.9, we consider the following equation:

$$\begin{cases} x'(t) = \begin{pmatrix} -8 & -1 & -2 \\ -1 & -3 & -1 \\ -2 & -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} x\left(\left[t+\frac{1}{2}\right]\right), t \geq 0, \\ x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{cases} \tag{5.4}$$

where L is a real symmetric matrix with $\mu[L] \approx -1.1270 < 0$ and $\|M\| \approx 0.8660 < -\mu[L]$, i.e., L and M satisfy the conditions of Theorem 2.5. For $\aleph = \begin{pmatrix} -0.0887 \\ -0.0300 \\ -0.0127 \end{pmatrix} < 0$, $R(\aleph) \approx \begin{pmatrix} 0.9151 \\ 0.9704 \\ 0.9888 \end{pmatrix}$, $e^{\aleph} \approx \begin{pmatrix} 0.9151 \\ 0.9704 \\ 0.9888 \end{pmatrix}$, which satisfies Corollary 4.8. Therefore, Theorem 4.9 holds. We plot the numerical solution of 3-Lobatto IIIC for Eq. (5.4) with $m = 50$ in Figure 5.4 and we obtain that the numerical solution is asymptotically stable.

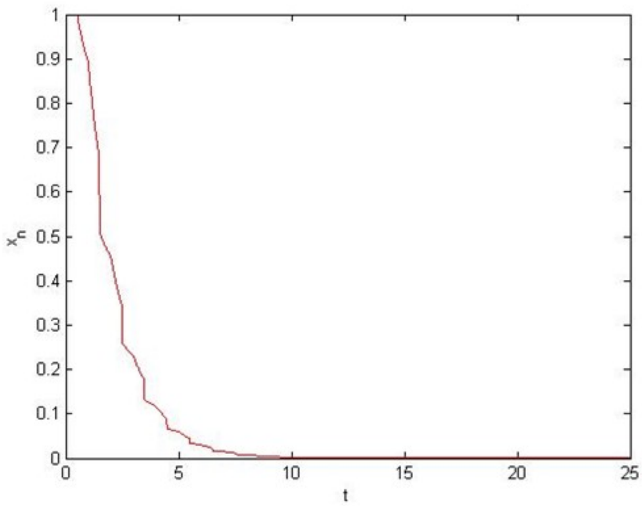

Figure 5.4: 3-Lobatto IIIC solution for Eq. (5.4) with $m = 50$.

6. Conclusion

In this paper, we consider the numerical stability of EPCA with matrix coefficients. For different types of matrix coefficients L, the corresponding stability conditions are obtained. In the future work, we will consider the nonlinear problems.

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Competing interests

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Authors contributions

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Almost Kaehlerian and Hermitian Structures on Four Dimensional Indecomposable Lie Algebras

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Abstract

It is known that from a given almost Hermitian structure on a simply connected Lie group, one can obtain left-invariant almost Hermitian structure on its Lie algebra. In this work, we consider Mubarakzyanov's classification of four-dimensional real Lie algebras and evaluate the existence of almost Hermitian structures on four dimensional decomposable real Lie algebras. In particular, we focus on almost Kaehlerian and Hermitian structures on these Lie algebras.

1. Introduction

An almost Hermitian manifold is an even dimensional Riemannian manifold (M,g) together with an almost complex structure J, $(J^2 = -Id)$ such that

$$g(JX,JY) = g(X,Y) \tag{1.1}$$

for all $X,Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M. The fundamental 2-form (or Kähler form) of an almost Hermitian manifold (M,g,J) is defined by

$$F(X,Y) = g(JX,Y), \tag{1.2}$$

for all $X,Y \in \mathfrak{X}(M)$. Also, the Nijenhuis tensor of M will be denoted by S, that is,

$$S(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \tag{1.3}$$

for $X,Y \in \mathfrak{X}(M)$. The covariant derivative ∇F of the Kähler form F, given with

$$(\nabla_X F)(Y,Z) = g((\nabla_X J)(Y),Z) \tag{1.4}$$

is a covariant tensor of degree 3 having the following symmetry properties [1]:

$$(\nabla_X F)(Y,Z) = -(\nabla_X F)(Z,Y) = -(\nabla_X F)(JY,JZ). \tag{1.5}$$

The space of those tensors possessing the same symmetries is a finite dimensional vector space, $\mathscr{W}$. Then $\mathscr{W}$ can be expressed as

$$\mathscr{W} = \{\alpha \in \otimes_3^0 T_pM | \alpha(X,Y,Z) = -\alpha(X,Z,Y) = -\alpha(X,JY,JZ)\} \tag{1.6}$$

for all $X,Y,Z \in \mathfrak{X}(M)$. In [1], almost Hermitian manifolds were classified depending on the space, $\mathscr{W}$, the covariant derivative of the fundamental 2-form belongs to. After writing the space $\mathscr{W}$ of tensors having the same properties as the covariant derivative of F, using the representation of the unitary group $U(n)$ on $\mathscr{W}$; $\mathscr{W}$ was written as a direct sum of four $U(n)$-irreducible subspaces. Thus there are 16 invariant subspaces of $\mathscr{W}$, each corresponding to a different class of almost Hermitian manifolds, as given in the following table:

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$\mathscr{K}$	$\nabla F = 0$
$\mathscr{W}_1 = \mathscr{N}\mathscr{K}$	$\nabla_X(F)(X,Y) = 0$ (or $3\nabla F = dF$)
$\mathscr{W}_2 = \mathscr{A}\mathscr{K}$	$dF = 0$
$\mathscr{W}_3 = \mathscr{S}\mathscr{K} \cap \mathscr{K}$	$\delta F = S = 0$ (or $\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = \delta F = 0$)
$\mathscr{W}_4$	$\nabla_X(F)(Y,Z) = \frac{-1}{2(n-1)}\{<X,Y> \delta F(Z) - <X,Z> \delta F(Y)$ $- <X,JY> \delta F(JZ) + <X,JZ> \delta F(JY)\}$
$\mathscr{W}_1 \oplus \mathscr{W}_2 = \mathscr{Q}\mathscr{K}$	$\nabla_X(F)(Y,Z) + \nabla_{JX}(F)(JY,Z) = 0$
$\mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{H}$	$S = 0$ (or$\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0$)
$\mathscr{W}_1 \oplus \mathscr{W}_3$	$\nabla_X(F)(X,Y) - \nabla_{JX}(F)(JX,Y) = \delta F = 0$
$\mathscr{W}_2 \oplus \mathscr{W}_4$	$\mathfrak{S}\{\nabla_X(F)(Y,Z) - \frac{1}{n-1}F(X,Y)\delta F(JZ)\} = 0$
$\mathscr{W}_1 \oplus \mathscr{W}_4$	$\nabla_X(F)(X,Y) = \frac{-1}{2(n-1)}\{\|X\|^2 \delta F(Y) - <X,Y> \delta F(X)$ $- <JX,Z> \delta F(JX)\}$
$\mathscr{W}_2 \oplus \mathscr{W}_3$	$\mathfrak{S}\{\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z)\} = \delta F = 0$
$\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_3 = \mathscr{S}\mathscr{K}$	$\delta F = 0$
$\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_4$	$\nabla_X(F)(Y,Z) + \nabla_{JX}(F)(JY,Z) = \frac{-1}{n-1}\{<X,Y> \delta F(Z)$ $- <X,Z> \delta F(Y) - <X,JY> \delta F(JZ) + <X,JZ> \delta F(JY)\}$
$\mathscr{W}_1 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{G}_1$	$\nabla_X(F)(X,Y) - \nabla_{JX}(F)(JX,Y) = 0$
$\mathscr{W}_2 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{G}_2$	$\mathfrak{S}\{\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z)\} = 0$
$\mathscr{W}$	No condition

Table 1: Defining relations for classes of almost Hermitian manifolds [1]

For example, the class $\mathscr{K}$, in which the covariant derivative of F is zero, is the class of Kähler manifolds. $\mathscr{W}_1$ corresponds to the class of nearly Kähler manifolds, $\mathscr{W}_2$ to the class of almost Kähler manifolds, etc. [1]. For the case dimension 4, the classification is induced to four subclasses as given in the following table:

$\mathscr{K}$	$\nabla F = 0$
$\mathscr{W}_2 = \mathscr{A}\mathscr{K}$	$dF = 0$
$\mathscr{W}_4 = \mathscr{H}$	$S = 0$
$\mathscr{W}$	No condition

Table 2: Almost Hermitian Manifolds of dimension 4 [1]

Here, the exterior derivative dF is defined as:

$$dF(X,Y,Z) = \mathfrak{S}(\nabla_X F)(Y,Z), \tag{1.7}$$

where $X,Y,Z \in \mathfrak{X}(M)$. One can see [2, 3] for more details. In the literature, there are many studies such as [4–6] that consider (almost)(para)contact structures on certain Lie algebras. In this work, by following a similar path to these studies, we will consider the classification as given in Table 2 since we focus the four dimensional almost Hermitian manifolds.

2. Four Dimensional Indecomposable Real Lie Algebras

Let G be a connected Lie group and $\mathfrak{g}$ be its Lie algebra. The almost Hermitian structures on G that we consider are invariant in the sense that the tensors g,J,F are left invariant tensors. By restricting the structures element to the left-invariant vector fields, we can directly obtain an almost Hermitian structure on the Lie algebra $\mathfrak{g}$, that will be denoted (g,J) again for convenience. In [7] and [8], four dimensional real Lie algebras are classified and with respect to this classification the decomposable Lie algebras are defined as follows with non zero commutators where $\{e_1,e_2,e_3,e_4\}$ is an orthonormal basis:

$$\begin{aligned}
\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1 &: [e_1,e_2] = e_1, \\
2\mathfrak{g}_{2,1} &: [e_1,e_2] = e_1, [e_3,e_4] = e_3, \\
\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 &: [e_2,e_3] = e_1, \\
\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1 &: [e_1,e_3] = e_1,\ [e_2,e_3] = e_1 + e_2, \\
\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1 &: [e_1,e_3] = e_1,\ [e_2,e_3] = e_2, \\
\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1 &: [e_1,e_3] = e_1,\ [e_2,e_3] = \alpha e_2, \quad (-1 \leq \alpha < 1,\ \alpha \neq 0) \\
\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1 &: [e_1,e_3] = \beta e_1 - e_2,\ [e_2,e_3] = e_1 + \beta e_2, \quad (\beta \geq 0)
\end{aligned}$$

If (g,J) is an almost Hermitian structure on a 4-dimensional Lie algebra $\mathfrak{g}$, then by the conditions $J^2 = -1$ and (1.1), the structure J has the form:

$$J(e_1) = ae_2 + be_3 + ce_4, \quad J(e_2) = -ae_1 + de_3 + ee_4, \quad J(e_3) = -be_1 - de_2 + fe_4, \quad J(e_4) = -ce_1 - ee_2 - fe_3, \tag{2.1}$$

where $a,b,c,d,e,f \in \mathbb{R}$ satisfy:

$$a^2+b^2+c^2=1,\ a^2+d^2+e^2=1,\ b^2+d^2+f^2=1,\ c^2+e^2+f^2=1,$$
$$bd+ce=0,\ ad-cf=0,\quad ae+bf=0,\quad ab+ef=0,\quad df-ac=0,\quad bc+de=0. \tag{2.2}$$

Also the derivative of the fundamental 2-form F, dF becomes:

$$dF(e_i,e_j,e_k)=-F([e_i,e_j],e_k)-F([e_j,e_k],e_i)-F([e_k,e_i],e_j). \tag{2.3}$$

Now, we study the existence of almost Kaehlerian and Hermitian structures on this algebras:

The algebra $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$:

Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$-tensor with $dF=0$ in $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$. By direct calculation, it can be seen that

$$dF(e_1,e_2,e_3)=dF(e_1,e_2,e_4)=0.$$

So, F has the form

$$F=a_{12}e^{12}+a_{23}e^{23}+a_{24}e^{24}+a_{34}e^{34}.$$

On the other other, by considering the equations (2.2), one can see that the structure $J(e_1)=e_2,\ J(e_2)=-e_1,\ J(e_3)=e_4,\ J(e_4)=-e_3$ has the 2- form F, as $F=e^{12}+e^{34}$ $(a_{12}=a_{34}=1,\ a_{23}=a_{24}=0)$, for which $dF=0$. Thus, **there exists an almost Kaehlerian structure in the class $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$.**

Now, we evaluate the existence of Hermitian structures on the algebra $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$. Let the tensor S given with (1.3) vanishes. Then considering the equations $S(e_i,e_j)=0$, it can be seen that the almost Hermitian structures with $J(e_1)=ae_2$, $J(e_2)=-ae_1, J(e_3)=fe_4,\quad J(e_4)=-fe_3$, with $a^2=f^2=1$ have Nijenhus tensors, that vanishes. Thus, **there exists Hermitian structures in the class $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$.**

The algebra $2\mathfrak{g}_{2,1}$:

Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$-tensor with $dF=0$ in the algebra $2\mathfrak{g}_{2,1}$. Then, by (2.3), the equations

$$dF(e_1,e_2,e_3)=dF(e_1,e_2,e_4)=dF(e_2,e_3,e_4)=0$$

imply $a_{13}=a_{14}=a_{23}=0$, repsectively. Thus F becomes

$$F=a_{12}e^{12}+a_{24}e^{24}+a_{34}e^{34}.$$

It can be seen that the structure $J(e_1)=e_2,\quad J(e_2)=-e_1,\quad J(e_3)=e_4,\quad J(e_4)=-e_3$ has the fundamental 2-form $F=e^{12}+e^{34}$ $(a_{12}=a_{34}=1,\ a_{24}=0$, for which $dF=0$. Thus, **there exists an almost Kaehlerian structure in the class $2\mathfrak{g}_{2,1}$.**

Assume the tensor S given with (1.3) vanishes. By the equations $S(e_1,e_2)=S(e_1,e_3)=S(e_1,e_4)=0$, we get $a^2=f^2=1,\ b=c=d=e=0$ in (2.2). Hence, we get the structures $J(e_1)=ae_2,\ J(e_2)=-ae_1,\ J(e_3)=fe_4,\ J(e_4)=-fe_3$, for which the Nijenhuis tensors vanish. Hence, **there exists Hermitian structures in the algebra $2\mathfrak{g}_{2,1}$.**

The algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$:

For the $(0,2)$tensor $F=\sum_{i,j}a_{ij}e^{ij}$ in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$, the equation $dF(e_2,e_3,e_4)=0$ implies $a_{14}=0$. Thus F has the form:

$$F=a_{12}e^{12}+a_{13}e^{13}+a_{23}e^{23}+a_{24}e^{24}+a_{34}e^{34}.$$

On the other hand, by considering the definitions of J and F, we see that the almost Hermitian structure given with $J(e_1)=e_3$, $J(e_2)=e_4,\ J(e_3)=-e_1,\ J(e_4)=-e_2$ has the fundamental 2-form

$$F=e^{13}+e^{24}\ (a_{13}=a_{24}=1,\ a_{12}=a_{23}=a_{34}=0).$$

Since $dF=0$, **there exists an almost Kaehlerian structure in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$.**

Let the tensor S given with (1.3) vanishes in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$. Then from the equations $S(e_1,e_2)=S(e_1,e_3)=S(e_2,e_3)=0$, and (2.2), we get $c^2=d^2=1,\ a=b=e=f=0$. Thus the almost Hermitian structures given with $J(e_1)=ce_4,\ J(e_2)=de_3,\ J(e_3)=-de_2,\ J(e_4)=-ce_1$, have Nijenhuis tensors S, with $S=0$. So, **there exists Hermitian structures in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$.**

The algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$:

Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$ tensor in the algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$. From the equations

$$dF(e_1,e_2,e_3)=dF(e1,e3,e4)=dF(e_2,e_3,e_4)=0,$$

we get $a_{12}=a_{14}=a_{24}=0$, respectively. Thus, the tensor F has the form

$$F=a_{13}e^{13}+a_{23}e^{23}+a_{34}e^{34}. \tag{2.4}$$

Also, by considering the definitions of J and the fundamental two form F, it can be seen that there exist no almost Hermitian structure with the fundamental 2-form given with (2.4). Thus, **there is no almost Kaehlerian structure on the algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$.**

Assume S is a tensor on the algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$, given with (1.3) and $S=0$. By considering $S(e_1,e_2)=S(e_2,e_3)=0$, $S(e_1,e_3)=0$ and the equations (2.2), we get $a^2=0$ and $a^2=1$. By this contradiction, **there is no Hermitian structure on the algebra** $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$.

The algebra $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$:

For the tensor $F=\sum_{i,j}a_{ij}e^{ij}=0$ in $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$, we get $a_{12}=a_{14}=a_{24}=0$, from the equations

$$dF(e_1,e_2,e_3)=dF(e_1,e_3,e_4)=dF(e_2,e_3,e_4)=0,$$

respectively. So, F has the form

$$F=a_{13}e^{13}+a_{23}e^{23}+a_{34}e^{34}. \tag{2.5}$$

However, it can be seen that there is no almost Hermitian structure with fundamental 2-form of the form (2.5). Thus, **there is no almost Kaehlerian structure in**$\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$.
Let S be a tensor in $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$ defined with (1.3) and $S=0$. From $S(e_2,e_3)=0$, we get $a^2=1$. So, by considering the equations (2.2), we get $b=c=d=e=0$, that implies $f^2=1$. Thus, the structure with

$$J(e_1)=ae_2,\quad J(e_2)=-ae_1,\quad J(e_3)=fe_4,\quad J(e_4)=-fe_3,$$

with $a^2=f^2=1$, has Nijenhuis tensor S, that vanishes. Thus, **there exist Hermitian structures on $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$.**

The algebra $\mathfrak{g}_{3,4}\oplus\mathfrak{g}_1$:
For the tensor $F=\sum_{i,j}a_{ij}e^{ij}=0$ in $\mathfrak{g}_{3,4}\oplus\mathfrak{g}_1$, we get $a_{12}=a_{24}=0$, from the equations $dF(e_1,e_2,e_3)=dF(e_2,e_3,e_4)=0$, respectively. Thus, F has the form,

$$F=a_{13}e^{13}+a_{14}e^{14}+a_{23}e^{23}+a_{34}e^{34}. \tag{2.6}$$

So, by considering the defining conditions of an almost Hermitian structure, it can be seen that the structure given with $J(e_1)=e_4$, $J(e_2)=e_3$, $J(e_3)=-e_2$, $J(e_4)=-e_1$ has fundamental 2-form

$$F=e^{14}+e^{23},\quad (a_{14}=a_{23}=1,\quad a_{13}=a_{34}=0).$$

Thus, **there exists almost Kaehlerian structure in $\mathfrak{g}_{3,4}\oplus\mathfrak{g}_1$.**
On the other hand, $\mathfrak{g}_{3,4}\oplus\mathfrak{g}_1$ agrees with a Hermitian structure. Indeed, it can be seen that the structure $J(e_1)=ae_2$, $J(e_2)=-ae_1$, $J(e_3)=fe_4$, $J(e_4)=-fe_3$ with $a^2=f^2=1$ have Nijenhuis tensor S, with $S=0$. Thus, **there exist Hermitian structures in** $\mathfrak{g}_{3,4}\oplus\mathfrak{g}_1$.

The algebra $\mathfrak{g}_{3,5}\oplus\mathfrak{g}_1$:
Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$ tensor in the algebra $\mathfrak{g}_{3,5}\oplus\mathfrak{g}_1$. From the equations $dF(e_1,e_2,e_3)=0$, $dF(e_1,e_3,e_4)=0$ and $dF(e_2,e_3,e_4)=0$, we get $\beta a_{12}=0$, $\beta a_{14}=a_{24}$ and $\beta a_{24}=-a_{14}$, respectively. However, this imply $\beta^2=-1$, which is contradiction since β is real number. So, **there is no almost Kaehlerian structure in** $\mathfrak{g}_{3,5}\oplus\mathfrak{g}_1$.
Assume, the tensor S given with (1.3) in the algebra $\mathfrak{g}_{3,5}\oplus\mathfrak{g}_1$ vanishes. After long but direct calculations of $S(e_i,e_j)$'s , one can see that the structures of the form

$$J(e_1)=ae_2,\quad J(e_2)=-ae_1,\quad J(e_3)=f(e_4),\quad J(e_4)=-fe_3,$$

for $a^2=f^2=1$, have Nijenhuis tensor that vanishes ($\beta\neq0$). Thus, There exist Hermitian structures in the algebra $\mathfrak{g}_{3,5}\oplus\mathfrak{g}_1$.

3. Conclusion

In this paper, the existences of Hermitian structures and almost Kaehlerian structures on four dimensional indecomposable real Lie algebras are investigated and so, the possible structures are stated as examples.

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Authors contributions

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