C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

Series A1: Mathematics and Statistics

VOLUME: 71 **Number: 4** **YEAR: 2022**

Faculty of Sciences, Ankara University
06100 Beşevler, Ankara-Turkey
ISSN 1303-5991 e-ISSN 2618-6470

C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

Series A1: Mathematics and Statistics

Volume:71 **Number: 4** **Year:2022**

This Journal is published four issues in a year by the Faculty of Sciences, University of Ankara. Articles and any other material published in this journal represent the opinions of the author(s) and should not be construed to reflect the opinions of the Editor(s) and the Publisher(s).

Correspondence Address:
COMMUNICATIONS EDITORIAL OFFICE
Ankara University, Faculty of Sciences,
06100 Tandoğan, ANKARA – TURKEY
Tel: (90) 312-2126720 Fax: (90) 312-2235000
e-mail: commun@science.ankara.edu.tr
http://communications.science.ankara.edu.tr/index.php?series=A1

Print:
Ankara University Press
İncitaş Sokak No:10 06510 Beşevler
ANKARA – TURKEY
Tel: (90) 312-2136655

C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES DE
L'UNIVERSITE D'ANKARA

Series A1:Mathematics and Statistics

Volume: 71 **Number: 4** **Year: 2022**

Research Articles

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 899–906 (2022)
DOI:10.31801/cfsuasmas.981876
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: August 16, 2021; Accepted: April 22, 2022

STOCHASTIC INTEGRATION WITH RESPECT TO A CYLINDRICAL SPECIAL SEMI-MARTINGALE

Saeed HASHEMI SABABE

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, CANADA
Young Researchers and Elite Club, Malard Branch, Islamic Azad University, Malard, IRAN

Abstract. In this research, we introduce the stochastic integration with respect to a cylindrical special semi-martingale, which is a specific case of general integration, with specific properties of special semi-martingales.

1. Introduction

Cylindrical semi-martingales play a key role in application, specially in stochastic partial differential equations. Among the wide class of cylindrical semi-martingales, cylindrical Brownian motions are used widely as models in stochastic analysis [3, 5, 8, 9, 11, 14, 18, 19]. Although Brownian motions work as good models, motivation of using other classes of cylindrical semi-martingales appears in recent research. Interesting examples of such a view can be found in [1, 2, 6, 12, 13, 15]. In spite of the fact that most of the past articles have an applied view to extend the concepts and utilities the stochastic integration, none of these works considers stochastic integration with respect to cylindrical special semi-martingales.

In this work, our main objective is to introduce a theory of stochastic integration for cylindrical special semi-martingales, which are a particular family of semi-martingales with complex behavior in relation with the measure of the space, defined on. P is a special semi-martingale if P can be decomposed into $P = M + A$ where M is a local martingale and A a process with predictable finite variation, with $A_0 = 0$. Such a decomposition is then unique and is called canonical decomposition.

2020 *Mathematics Subject Classification.* 60H05, 60B11, 60G44, 47D06.
Keywords. Cylindrical martingale, special martingales, stochastic integration, Banach space.
S.hashemi@ualberta.ca, Hashemi_1365@yahoo.com
0000-0003-1167-5006.

On the other hand, for a Banach space $\mathscr{X}$, the cylindrical σ-algebra is defined to be the coarsest σ-algebra, i.e. the one with the fewest measurable sets, such that every continuous linear function on $\mathscr{X}$ is a measurable function. That is important to note that in general, the cylindrical σ-algebra is not the same as the Borel σ-algebra on $\mathscr{X}$, which is the coarsest σ-algebra that contains all open subsets of $\mathscr{X}$.

In the following, we study the cylindrical special semi-martingale $M : \mathscr{X}^* \to \mathcal{S}^{\mathrm{SP}}$ from the dual of a separable Banach space $\mathscr{X}$ to the space of special semi-martingales. Moreover, we define the integral of a progressive process with respect to a cylindrical special semi-martingale.

2. Preliminaries

Let $\mathscr{X}, \mathscr{Y}$ be two Banach spaces. We will denote the space of all bilinear operators from $\mathscr{X} \times \mathscr{Y}$ to $\mathbb{R}$ as $\mathscr{B}(\mathscr{X}, \mathscr{Y})$. Note that for a continuous $b \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$ there exists an operator $\mathcal{B} \in \mathscr{L}(\mathscr{X}, \mathscr{Y}^*)$ such that

$$b(x, y) = \langle \mathcal{B}x, y\rangle = \mathcal{B}x(y), \quad x \in X, y \in Y. \tag{1}$$

An operator $\mathcal{B} : \mathscr{X} \to \mathscr{X}^*$ is called self-adjoint, if for each $x, y \in \mathscr{X}$

$$\langle \mathcal{B}x, y\rangle = \langle \mathcal{B}y, x\rangle.$$

and is called positive, if $\mathcal{B}$ is self-adjoint and $\mathcal{B}_x(x) = \langle \mathcal{B}x, x\rangle \geq 0$ for all $x \in \mathscr{X}$.

Recall that if $\mathcal{B} : \mathscr{X} \to \mathscr{X}^*$ is a positive self-adjoint operator, then the Cauchy-Schwartz inequality holds for the bilinear form $\langle \mathcal{B}x, y\rangle$. In a natural way in functional analysis, the norm of $\mathcal{B}$ is defined as

$$\|\mathcal{B}\| = \sup_{x \in \mathscr{X}, \|x\|=1} |\langle \mathcal{B}x, x\rangle| \tag{2}$$

Note that if $\mathscr{X}$ is a Hilbert space, then (2) would be coincides with the induced norm of the inner product defined on $\mathscr{X}$.

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $\mathscr{X}$ a Banach space. A function $f : \Omega \to \mathscr{X}$ is called simple if there exist $x_1, x_2, \ldots x_n \in \mathscr{X}$ and $E_1, E_2, \ldots, E_n \in \mathscr{F}$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \neq E_i$. A function $f : \Omega \to \mathscr{X}$ is called strong measurable if there exists a sequence of simple functions (f_n) with $\lim_n \|f_n - f\| = 0$, μ-almost everywhere. A function $f : \Omega \to \mathscr{X}$ is called scalar measurable if for each $x^* \in \mathscr{X}^*$ the numerical function $x^* f$ is strong measurable.

Further we will need the following lemma.

Lemma 1. *[15, Proposition 32] Let (S, Σ) be a measurable space, H be a separable Hilbert space, $f : S \to \mathscr{L}(\mathscr{H})$ be a scalar measurable self-adjoint operator-valued function. Let $F : \mathbb{R} \to \mathbb{R}$ be locally bounded measurable. Then $F(f) : S \to \mathscr{L}(\mathscr{H})$ is a scalar measurable self-adjoint operator-valued function.*

That is trivial to think about the square root of a positive operator. It would be appreciated if the square root drops us in to a Hilbert space, even in a special case.

Lemma 2. *[19, Lemma 2.4] Let $\mathscr{X}$ be a reflexive separable Banach space, $\mathcal{B} : \mathscr{X} \to \mathscr{X}^*$ be a positive operator. Then there exists a separable Hilbert space $\mathscr{H}$ and an operator $\mathcal{B}^{1/2} : \mathscr{X} \to \mathscr{H}$ such that $\mathcal{B} = \mathcal{B}^{1/2*}\mathcal{B}^{1/2}$.*

A scalar-valued process M is called a continuous local martingale if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \infty$ almost surely as $n \to \infty$ and $1_{\tau_n>0}M^{\tau_n}$ is a continuous martingale.

We denote by $\mathscr{M}$ and $\mathscr{M}^{\text{loc}}$ the class of continuous and continuous local martingales, respectively. It is well known that $\mathscr{M}^{\text{loc}}$ is a vector space with respect to usual operations. Several topologies can be defined on $\mathscr{M}^{\text{loc}}$, for example UCP, which is based on convergence in probability, or Emery topology [4, 7]. Although, we can define a norm on $\mathscr{M}^{\text{loc}}$ as

$$\|M\|_{\mathscr{M}^{\text{loc}}} = \sum_{n=1}^{\infty} 2^{-n} E[1 \wedge \sup_{t\in[0,n]} |M_t|]. \tag{3}$$

It can be seen that the topology induced by the norm in (3) in coincides with the UCP and Emery topology (because of the continuity property). That is proved in several articles that $\mathscr{M}^{\text{loc}}$ equipped with the norm (3) is a complete metric space.

Let $\mathscr{X}$ be a Banach space. In general, a cylindrical semi-martingale on $\mathscr{X}$ is a continuous linear mapping $\varphi : \mathscr{X}^* \to S^0$, where S^0 denotes the space of real semi-martingales with respect to a common stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq 1}, P)$, endowed with the Emery topology. The general case is studied before in literature. (see for example [10]). As a special case, a continuous linear mapping $M : \mathscr{X}^* \to \mathscr{M}^{\text{loc}}$ is called a cylindrical continuous local martingale.

In the following, we interested to study the continuous linear mapping $M : X^* \to \mathcal{S}$ where $\mathcal{S}$ is the collection of locally integrable semi-martingales. Our motivation comes from the collection of particular type of martingales, called as *Special Semi-martingales* $\mathcal{S}^{\text{SP}}$, coincides with $\mathcal{S}$.

A processes $P = M + A$ which can be decomposed, by Doob decomposition, into a local martingale M and a predictable cádlag locally finite variation process A is known as special semimartingales. On the space of special semimartingales, we can define p-norm for $p > 0$ as follows and denote the semimartingales with finite p norm by $\mathbb{H}^p$:

$$\|P\|_{\mathbb{H}^p} = \left(E\left[[M,M]_\infty^{p/2} + (\int_0^\infty |dA|)^p\right]\right)^{1/p}.$$

One of the most interesting properties of special semi-martingales is compatibility of integration with the canonical decomposition in the construction of the stochastic integrals. That is, for a special semi-martingale $P = M + A$ and a predictable process ξ we have

$$\int \xi dP = \int \xi dM + \int \xi dA$$

3. Cylindrical Special Martingales

In this section, we define the notion of a cylindrical special martingale and integration with respect to a cylindrical special martingale.

Definition 1. *Let $\mathscr{X}$ be a Banach space. A continuous linear mapping $P : \mathscr{X}^* \to \mathcal{S}^{\mathrm{SP}}$ is called a cylindrical continuous special martingale. In this way, $Px^* = Mx^* + Ax^*$, where Mx^* is a local martingale and Ax^* is a finite variation process, for any $x^* \in X^*$,*

For a cylindrical continuous special martingale P and a stopping time τ, one can define $P^\tau : \mathscr{X}^* \to \mathcal{S}^{\mathrm{SP}}$ by $P^\tau x^*(t) = Px^*(t \wedge \tau)$. Clearly P^τ is also a cylindrical continuous special martingale.

We expect that our definition of a cylindrical continuous special martingale be a generalization of a cylindrical continuous local martingale. A characteristic property of a local martingale is its quadratic variation. Thanks to the finite variation part of P, which has the zero quadratic variation, we can easily define the quadratic variation $[[P]]$ of P similar to the quadratic variation of mapping to its local martingale part M.

Recall that If M is a continuous local martingale with values in a Hilbert space, then it is well known that it has a classical quadratic variation $[M]$ in the sense that there exists an a.s. unique increasing continuous process $[M]$ starting at zero such that $\|M\|^2 - [M]$ is a continuous local martingale again.

Definition 2. *Let $P : \mathscr{X}^* \to \mathcal{S}^{\mathrm{SP}}$ be a linear mapping. The quadratic variation $[[P]]$ of P is defined as*

$$[[P]]_t = \sup_{N \in \mathbb{N}} \sum_{n=1}^{N} \sup_m ([Mx_m^*]_{t_i+1} - [Mx_m^*]_{t_i}), \quad t \geq 0,$$

where Mx^* is the local martingale part of Px^* and the limit is taken over all rational partitions $0 = t_0 < \cdots < t_N = t$ and $(x_m^*)_{m \geq 1}$ is a dense subset of the unit ball in X^*.

Note that existence of $(x_m^*)_{m \geq 1}$ follows from the separability of $\mathscr{X}^*$. For a cylindrical special semi-martingale P on a Banach space $\mathscr{X}$, one can think about covariance $[Px^*, Py^*]_t$ for any $x^*, y^* \in X^*$. However, by the ineffectiveness of finite variation part A of P, we have $[Px^*, Py^*]_t = [Mx^*, My^*]_t$. Therefore, by the polar decomposition, there exists a process $Q_P : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathscr{X}^*, \mathscr{X}^{**})$ such that for almost surly $t > 0$

$$[Px^*, Py^*]_t = \int_0^t Q_P x^*(y^*) d[[P]]_s, \quad x^*, y^* \in X^*.$$

The process Q_P is self-adjoint and $\|Q_P(t)\| = 1$.

Let $\mathscr{X}, \mathscr{Y}$ be two Banach spaces. For any $x^* \in \mathscr{X}^*, y \in \mathscr{Y}$, we can define the linear operator $x^* \circledast y \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ such that $x^* \circledast y : x \mapsto x^*(x)y$. Using the defined operator, the process $\phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathscr{H}, \mathscr{X})$ is called elementary progressive with respect to the filtration $\mathscr{F} = (\mathscr{F}_t)_{t \in R_+}$ if it is of the form

$$\phi(t, \omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbf{1}_{(t_{n-1}, t_n] \times B_{mn}}(t, \omega) \sum_{k=1}^{K} v_k \circledast x_{kmn},$$

where $0 \le t_0 < \cdot < t_n < \infty$, for each $n = 1, \cdot, N$ the sets $B_{1n}, \ldots, B_{Mn} \in \mathscr{F}_{t_{n-1}}$ and vectors $v_1, \ldots, v_K$ are orthogonal.

For each elementary progressive ϕ we define the stochastic integral with respect to $\mathscr{X} \in \mathscr{M}^{\mathrm{sp}}_{\mathrm{var}}(\mathscr{H})$ as an element of $L_0(\Omega; C_b(\mathbb{R}_+; \mathscr{X}))$ as

$$\int_0^t \phi(s)\, \mathrm{d}P(s) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mathbf{1}_{B_{mn}} \sum_{k=1}^{K} (M(t_n \wedge t) v_k - M(t_{n-1} \wedge t) v_k + \mathrm{V}_n(A) v_k) x_{kmn}, \tag{4}$$

where $V_n(A)$ is the total variation of process A in the n-th interval, $[t_{n-1}, t_n]$, and C_b is the set of all continuous and bounded mappings. This is usual to use the notation $\phi \cdot P$ for the process $\int_0^\cdot \phi(s)\, \mathrm{d}P(s)$.

Clearly, the definition in (4) is a generalization of integration with respect to a cylindrical local martingale.

Lemma 3. *For all progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathscr{H}, \mathbb{R})$ with $\phi Q_P^{1/2} \in L^2(\mathbb{R}_+, [[P]]; \mathscr{L}(\mathscr{H}, \mathbb{R}))$ we have*

$$\left[\int_0^\cdot \phi\, \mathrm{d}P\right]_t = \int_0^t \phi(s) Q_P(s) \phi^*(s)\, \mathrm{d}[[P]]_s. \tag{5}$$

Proof. Note that our definition of quadratic variation for cylindrical special semi-martingales P is reduced to its local martingale part M. Therefore, the proof is similar to the proof of [13, Theorem 14.7.4]. □

It is important to note that for any (t, ω) in $\mathbb{R}_+ \times \Omega$, the mapping $Q_P(t, \omega)$ is a positive mapping from $\mathscr{X}^*$ to $\mathscr{X}^{**}$. Therefore, there exists a Hilbert space $\mathscr{H}$ such that $Q_P^{1/2}(t, \omega)$ maps $\mathscr{X}^*$ to $\mathscr{H}$ and $Q_P(t, \omega) = Q_P^{1/2*}(t, \omega) Q_P^{1/2}(t, \omega)$. Moreover, $\phi(t, \omega) Q_P^{1/2}(t, \omega)$ is an operator and we may think about $(\phi(t, \omega) Q_P^{1/2}(t, \omega))^* = Q_P^{1/2}(t, \omega)^* \phi(t, \omega)^*$. On the other hand, $\phi(t, \omega)$ is an operator from $\mathscr{H}$ to $\mathbb{R}$ and

$\phi(t,\omega)^*$ is well defined. Breaking the Q_P appears in (5) to its roots and have an inner product scheme can make a transparent illustration of the idea behind the lemma.

Theorem 1. *Let $\mathscr{H}$ be a Hilbert space and $P \in \mathscr{M}^{\mathrm{sp}}_{\mathrm{var}}(\mathscr{H})$. Let $\phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathscr{H}, \mathscr{X})$ be such that $\phi^* x^*$ is progressively measurable for each $x^* \in \mathscr{X}^*$, and assume $\phi(\omega)Q_P(\omega)\phi^*(\omega)x^*(x^*) \in L^1_{\mathrm{loc}}(\mathbb{R}_+, [[P]](\omega))$, for all $x^* \in \mathscr{X}^*, \omega \in \Omega$. Set $M := \phi \cdot P$ by*

$$Mx^*(t) := \int_0^t \phi^* x^* \,\mathrm{d}P, \quad x^* \in \mathscr{X}^*. \tag{6}$$

If $\|\phi Q_P \phi^*\|_\infty < \infty$ then $M \in \mathscr{M}^{\mathrm{sp}}_{\mathrm{var}}(\mathscr{X})$.

Proof. It is clear that for each $x^* \in \mathscr{X}^*$, mapping Mx^* is a continuous local martingale. We need just to show theta the mapping $x^* \mapsto Mx^*$ is continuous in the UPC topology. Fix $T > 0$ and set Ω_0 be a subset of Ω such that for almost every $\omega \in \Omega_0$ we have

$$t \mapsto \langle \phi(t,\omega)Q_N(t,\omega)^*\phi(t,\omega)^* x^*, x^*\rangle \in \mathscr{L}^1(0,T).$$

Therefore, we have a bounded operator and there exists a constant C such that

$$\|\langle \phi(\cdot,\omega)Q_N(\cdot,\omega)^*\phi(\cdot,\omega)^* x^*, y^*\rangle\|_{L^1(0,T,[[N]](\omega))} \le C\|x^*\|\,\|y^*\|.$$

Moreover, we have

$$[Mx^*]_t = \int_0^t \langle \phi(s)Q_P\phi^*(s)x^*, x^*\rangle \,\mathrm{d}[[P]], \quad \text{for all } x^* \in \mathscr{X}^*.$$

Note that $\|\phi(s){Q_P}^{1/2}\|_\infty < \infty$ by definition of ϕ and Q_P. Now let (x_n^*) be a sequence in $\mathscr{X}^*$ and $\lim_{n\to\infty} x_n = x$. We have

$$\begin{aligned}
&\|[Mx_n^*]_t - [Mx_n]_t\| \\
&= \left\| \int_0^t \langle \phi(s)Q_P\phi^*(s)x_n^*, x_n^*\rangle \,\mathrm{d}[[P]] - \int_0^t \langle \phi(s)Q_P\phi^*(s)x^*, x^*\rangle \,\mathrm{d}[[P]] \right\|_1 \\
&= \left\| \int_0^t \langle \phi(s)Q_P\phi^*(s)x_n^*, x_n^*\rangle - \langle \phi(s)Q_P\phi^*(s)x^*, x^*\rangle \,\mathrm{d}[[P]] \right\|_1 \\
&\le \|\phi(s)Q_P\phi^*(s)\|_\infty \|x_n - x\| \to 0
\end{aligned}$$

□

Corollary 1. *Let M be the cylindrical continuous local martingale defined in Theorem 1. Then we have*

$$[[M]]_t = \int_0^t \|\phi(s)Q_P(s)\phi^*(s)\| \,\mathrm{d}[[P]], \quad t \ge 0.$$

Proof. To prove the equivalence it suffices to observe that

$$\begin{aligned}[][M]]_t &= \lim \sum_{j=1}^{J} \sup_{x^* \in \mathscr{X}^*, \|x^*\|=1} ([Px^*]_{t_j} - [Px^*]_{t_{j-1}}) \\ &= \lim \sum_{j=1}^{J} \sup_{x^* \in \mathscr{X}^*, \|x^*\|=1} \int_{t_{j-1}}^{t_j} \langle \phi(s) Q_P(s) \phi^*(s) x^*, x^* \rangle \, \mathrm{d}[[P]]_s \\ &= \int_0^t \|\phi(s) Q_P(s) \phi^*(s)\| \, \mathrm{d}[[P]]_s. \end{aligned}$$

The limit takes when the partition of $0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$ becomes refined, when n tends to infinity. Note that the space $\mathscr{X}^*$ is assumed to be a separable space which helps us to justify the last equation. □

Corollary 2. *Let M be the cylindrical continuous local martingale defined in Theorem 1. Then we have*

$$\phi(s) Q_P(s) \phi^*(s) = Q_M(s) \|\phi(s) Q_P(s) \phi^*(s)\|$$

Proof. By the Corollary 1, we have

$$\begin{aligned} [[M]]_t &= \int_0^t \|\phi(s) Q_P(s) \phi^*(s)\| \, \mathrm{d}[[P]] \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}[[P]]}[[M]]_t = \frac{\mathrm{d}}{\mathrm{d}[[P]]} \left(\int_0^t \|\phi(s) Q_P(s) \phi^*(s)\| \, \mathrm{d}[[P]] \right) \\ &\Rightarrow \mathrm{d}[[M]]_s = \|\phi(s) Q_N(s) \phi^*(s)\| \, \mathrm{d}[[P]]_s. \end{aligned} \tag{7}$$

In the other way,

$$\begin{aligned} [Mx^*, My^*]_t &= \int_0^t \langle Q_P(s) \phi^*(s) x^*, \phi^*(s) y^* \rangle \, \mathrm{d}[[P]]_s \\ &= \int_0^t \langle \phi(s) Q_P(s) \phi^*(s) x^*, y^* \rangle \, \mathrm{d}[[P]]_s. \end{aligned} \tag{8}$$

Replacing (7) in (8) implies the statement. □

Conclusion

The stochastic integration with respect to a cylindrical Semi-martingale is studied before in general case. In this research, we specified the general case to special semi-martingales and used their specific properties to refine the definition. Since the case of semi-martingales would be studied in relation with the Banach space and some convergence theorems, our refined definition would affect the convergence accuracy.

Declaration of Competing Interests The author declare that he has no competing interest.

Acknowledgements A part of this research was carried out while the first author was visiting the University of Alberta.

References

[1] Brzézniak, Z., Van Neerven, J.M.A.M., Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem, *Studia Math.,* 143(1) (2000), 43-74.

[2] Criens, D., Cylindrical martingale problems associated with Lévy generators, *J. Theoret. Probab.,* 32(3) (2019), 1306–1359. https://doi.org/10.48550/arXiv.1706.06049

[3] Di Girolami, C., Fabbri, G., Russo, F., The covariation for Banach space valued processes and applications, *Metrika,* 77(1) (2014), 51-104. https://doi.org/10.48550/arXiv.1301.5715

[4] Emery, M., Une Topologie Sur L'espace Des Semimartingales, Sém. Probab. XIII. Univ. Strasbourg, 260–280, Lecture Notes in Math. 721, Springer, 1979.

[5] Fonseca-Mora, C.A., Semimartingales on duals of nuclear spaces, *Electron. J. Probab.,* 25(36) (2020). https://doi.org10.1214/20-EJP444

[6] Hashemi Sababe, S., Yazdi M., Shabani, M.M., Reproducing kernel Hilbert space based on special integrable semimartingales and stochastic integration, *Korean J. Math.,* 29(3) (2021), 639–647. https://doi.org/10.11568/kjm.2021.29.3.639

[7] Jacod, J., Shiryaev, A.N., Limit Theorems for Stochastic Processes, Springer, 2003.

[8] Kalinichenko, A.A., An approach to stochastic integration in general separable Banach spaces, *Potential Anal.,* 50(4) (2019), 591–608. https://doi.org/10.1007/s11118-018-9696-4

[9] Kalton, N.J., Weis, L.W.,The H^∞-calculus and square function estimates, *Nigel J. Kalton Selecta,* 1 (2016), 715-764. https://doi.org/10.48550/arXiv.1411.0472

[10] Kardaras, C., On the closure in the Emery topology of semimartingale wealth-process sets, *Ann. Appl. Probab.,* 23(4) (2013), 1355–1376. http://dx.doi.org/10.1214/12-AAP872

[11] Kumar, U., Riedle, M., The stochastic Cauchy problem driven by a cylindrical Lévy process, *Electron. J. Probab.,* 25(10), (2020). https://doi.org/10.48550/arXiv.1803.04365

[12] Memin, J., Espaces de semimartingales et changement de probabilité, *Z. Wahrsch. Verw. Gebiete,* 52(1) (1980), 9–39. https://doi.org/10.1007/BF00534184

[13] Métivier, M., Pellaumail, J., Stochastic Integration, Probability and Mathematical Statistics, Academic Press, 1980,

[14] Mnif, M., Pham, H., Stochastic optimization under constraints, *Stochastic Process. Appl.,* 93 (2001), 149-180. https://doi.org/10.1016/S0304-4149(00)00089-2

[15] Ondreját, M., Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces, *Czechoslovak Math. J.,* 55(130) (2005), 1003–1039. https://doi.org/10.1007/s10587-005-0084-z

[16] Rudin, W., Real and Complex Analysis, McGraw-Hill Book Co., 1987.

[17] Suchanecki, Z., Weron, A., Decomposability of cylindrical martingales and absolutely summing operators, *Math. Z.,* 185(2) (1984), 271–280. https://doi.org/10.1007/BF01181698

[18] Sun, X., Xie, L., Xie, Y., Pathwise uniqueness for a class of SPDEs driven by cylindrical -stable processes, *Potential Anal.,* 53(2) (2020), 659–675. https://doi.org/10.1007/s11118-019-09783-x

[19] Veraar, M., Yaroslavtsev, I., Cylindrical continuous martingales and stochastic integration in infinite dimensions, *Electron. J. Probab.,* 21(59) (2016). https://doi.org/10.1214/16-EJP7

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 907–918 (2022)
DOI:10.31801/cfsuasmas.1070240
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: February 8, 2022; Accepted: April 27, 2022

LOCAL T_0 AND T_1 QUANTALE-VALUED PREORDERED SPACES

Samed ÖZKAN[1] and Mesut GÜNEŞ[2]

[1]Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, TURKEY
[2]Ministry of National Education, Nevşehir, TURKEY

Abstract. In this paper, we characterize explicitly the separation properties T_0 and T_1 at a point p in the topological category of quantale-valued preordered spaces and investigate how these characterizations are related. Moreover, we prove that local T_0 and T_1 quantale-valued preordered spaces are hereditary and productive.

1. Introduction

Classical separation axioms of topology have been extended to topological category by several authors. Baran [2], in 1991, introduced these axioms in a set-based topological category in terms of initial, final structures and discreteness. He defined separation properties first locally, i.e., at a point p [4], then they are expanded to point-free concepts. Using local lower separation axioms, Baran [2,3] introduced the notion of (strongly) closedness in set-based topological categories that creates closure operators in sense of Dikranjan and Giuli [16] in some well-known topological categories **Conv** (the category of convergence spaces and filter convergence maps) [11], **Lim** (the category of limit spaces and filter convergence maps) [9], **Prord** (the category of preordered sets and monotone maps) [12] and **SUConv** (the category of semiuniform convergence spaces and uniformly continuous maps) [14]. The other significant use of these concepts to define the notions of Hausdorffness [5], compactness, perfectness [9], connectedness [10], regular, completely regular, normal objects [7,8] in set-based topological categories.

A topological space defines a preorder (reflexive and transitive) relation, and a topology can be obtained from a preorder relation on a set [17,20]. This indicates

2020 *Mathematics Subject Classification.* 54B30, 54A05, 54D10, 18B35, 06F07.
Keywords. Topological category, quantale-valued preorder, local T_0 objects, local T_1 objects.
ozkans@nevsehir.edu.tr-Corresponding author; 0000-0003-3063-6168
mesut-gunes@hotmail.com; 0000-0001-6822-7605

a connection between topology and order. Domain theory that was introduced by Dana Scott in the 1960s, is a branch of order theory which studies special kinds of partially ordered sets generally named as domains. In computer science, this field is used to establish denotational semantics, particularly for functional programming languages [18,29]. Domain theory is closely related to topology and formalizes the intuitive principles of convergence and approximation in a general way.

With the advancement of lattice theory, distinct mathematical frameworks have been studied with lattice structures including lattice-valued topology [15], quantale-valued approach space [23,24,28], quantale-valued metric space [25], lattice-valued convergence space [22] and lattice-valued preordered space [15]. This motivates us to study local T_0 and T_1 separation axioms in quantale-valued preordered spaces.

The main purpose of this paper is to give an explicit characterization for the local T_0 and T_1 separation axioms in the category of quantale-valued preordered spaces as well as to examine the relationship between them and to investigate their some invariance properties.

2. Preliminaries

Recall [24] that a partially ordered set $(L, \leq)$ is called a complete lattice if all subsets of L have both infimum ($\bigwedge$) and supremum ($\bigvee$). For any complete lattice, the bottom element and top element is denoted by $\bot$ and $\top$, respectively.

Definition 1. *Let $(L, \leq)$ be a complete lattice. We identify*

(1) $\alpha \lhd \beta$ *(the well-below relation) if $\forall X \subseteq L$ such that $\beta \leq \bigvee X$ there exists $\delta \in X$ such that $\alpha \leq \delta$.*
(2) $\alpha \prec \beta$ *(the well-above relation) if $\forall X \subseteq L$ such that $\bigwedge X \leq \alpha$ there exists $\delta \in X$ such that $\delta \leq \beta$.*

Definition 2. *A complete lattice $(L, \leq)$ is called a completely distributive iff for any $\alpha \in L$, $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$.*

Definition 3. *The triple $(L, \leq, *)$ is called a quantale if the following conditions are satisfied.*

(1) *$(L, \leq)$ is a complete lattice.*
(2) *$(L, *)$ is a semi group.*
(3) *$(\bigvee_{i\in I} \alpha_i) * \beta = \bigvee_{i\in I}(\alpha_i * \beta)$ and $\beta * (\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta * \alpha_i)$ for all $\alpha_i, \beta \in L$,*

Note that if $(L, *)$ is a commutative semi group, then the quantale $(L, \leq, *)$ is named as commutative and if for all $\alpha \in L$, $\alpha * \top = \top * \alpha = \alpha$, then it is called integral.

We denote a quantale by $\mathsf{L} = (L, \leq, *)$ if it is integral and commutative, where $(L, \leq)$ is completely distributive.

A quantale $\mathsf{L} = (L, \leq, *)$ is named as a value quantale if $(L, \leq)$ is completely distributive lattice such that $\forall \alpha, \beta \lhd \top$, $\alpha \vee \beta \lhd \top$ [19].

Definition 4. *[25, 30] Let $X \neq \emptyset$ be a set. A map $\mathsf{R} : X \times X \to \mathsf{L} = (L, \leq, *)$ is called an L-preorder relation on X and the pair (X, R) is called an L-preordered space if it satisfies*

1. *reflexivity, i.e., for all $x \in X$, $\mathsf{R}(x, x) = \top$,*
2. *transitivity, i.e., for all $x, y, z \in X$, $\mathsf{R}(x, y) * \mathsf{R}(y, z) \leq \mathsf{R}(x, z)$.*

Note that an L-preordered space (X, R) is named as an L-equivalence space (X, R) if for all $x, y \in X$, $\mathsf{R}(x, y) = \mathsf{R}(y, x)$ (symmetry). Also, (X, R) is called separated L-preordered space if $x = y$ whenever $\mathsf{R}(x, y) = \top$.

A map $f : (X, \mathsf{R}_X) \to (Y, \mathsf{R}_Y)$ is called an L-order preserving map if for all $x_1, x_2 \in X$, $\mathsf{R}_X(x_1, x_2) \leq \mathsf{R}_Y(f(x_1), f(x_2))$.

Let L**-Prord** denotes the category whose objects are L-preordered spaces and morphisms are L-order preserving mappings.

Example 1. (i) *For $\mathsf{L} = 2 = (\{0, 1\}, \leq, \wedge)$, **2-Prord** $\cong$ **Prord**, where **Prord** is the category of preordered sets and order preserving maps.*

(ii) *For $\mathsf{L} = ([0, \infty], \geq, +)$ (Lawvere's quantale), $[0, \infty]$**-Prord** $\cong \infty$**qMet**, where ∞**qMet** is the category of extended quasi metric spaces and non-expansive maps.*

(iii) *For $\mathsf{L} = (\triangle^+, \leq, *)$ (distance distribution functions quantale defined in [24]), then $\triangle^+$**-Prord** $\cong$ **ProbqMet**, where **ProbqMet** is the category of probabilistic quasi metric spaces and non-expansive maps [19].*

Note that in some literature, L-preordered space is often called a continuity space if L is a value quantale (see [19]), an L-metric space (see [25]) and an L-category (see [21]).

Recall [1], let E be a category, **Set** be the category of sets and functions and $\mathsf{U} : \mathsf{E} \to$ **Set** be a functor. U is called topological or E is called topological category on **Set** if

(i) U is amnestic and faithful (i.e., concrete),
(ii) U consists of small fibers,
(iii) Every U-source has a unique initial lift.

In addition, a topological functor is said to be normalized if constant objects, i.e., subterminals, have a unique structure.

Note that the forgetful functor $\mathsf{U} : \mathsf{L}$**-Prord** $\to$ **Set** is topological (see [21]) and it is also normalized.

Lemma 1. *[21] Let (X_i, R_i) be a collection of L-preordered spaces. A source $(f_i : (X, \mathsf{R}) \to (X_i, \mathsf{R}_i))_{i \in I}$ is initial in L**-Prord** iff $\forall a, b \in X$,*

$$\mathsf{R}(a, b) \quad = \quad \bigwedge_{i \in I} \mathsf{R}_i(f_i(a), f_i(b)).$$

Lemma 2. *[21] Let X be a non-empty set and (X, R) be an L-preordered space. For all $a, b \in X$,*

(i) *The discrete* L*-preorder structure on* X *is stated by*

$$\mathsf{R}_{dis}(a,b) = \begin{cases} \top, & a = b, \\ \bot, & a \neq b. \end{cases}$$

(ii) *The indiscrete* L*-preorder structure on* X *is stated by*

$$\mathsf{R}_{ind}(a,b) \quad = \quad \top.$$

3. Local T_0 and T_1 Objects

Let X be a set, $p \in X$ be a point and $X \vee_p X$ be the wedge product of X at p [2], i.e., two separate copies of X identified at p.

In the wedge $X \vee_p X$, a point x is represented as x_k if it lies in the k-th component for $k = 1, 2$.

Definition 5. *[2] Let* $X \vee_p X$ *be the wedge product at* p *and* X^2 *be the cartesian product of* X*.*

(1) $\mathsf{A}_p : X \vee_p X \to X^2$ *(the principal p-axis mapping) is given by*

$$\mathsf{A}_p(x_1) = (x, p) \text{ and } \mathsf{A}_p(x_2) = (p, x).$$

(2) $\mathsf{S}_p : X \vee_p X \to X^2$ *(the skewed p-axis mapping) is given by*

$$\mathsf{S}_p(x_1) = (x, x) \text{ and } \mathsf{S}_p(x_2) = (p, x).$$

(3) $\nabla_p : X \vee_p X \to X$ *(the fold mapping at p) is given by*

$$\nabla_p(x_1) = \nabla_p(x_2) = x.$$

Definition 6. *Let* (X, τ) *be topological space and* $p \in X$*. For each point* $x \neq p$*, there exists an open set* A *such that* $p \in A$*,* $x \notin A$ *or (resp. and) there exists an open set* B *such that* $x \in B$*,* $p \notin B$*, then* (X, τ) *is said to be* T_0 *(resp.* T_1*) at* p *[2, 6].*

Theorem 1. *Let* (X, τ) *be topological space and* $p \in X$*. Then* (X, τ) *is* T_0 *(resp.* T_1*) at* p *iff the initial topology induced by* $\{\mathsf{A}_p$ *(resp.* $\mathsf{S}_p) : X \vee_p X \to (X^2, \tau_*)$ *and* $\nabla_p : X \vee_p X \to (X, P(X))\}$ *is discrete, where* τ_* *is the product topology on* X^2*.*

Proof. The proofs are given in [6]. □

Definition 7. *[2] Let* $\mathsf{U} : \mathsf{E} \to \mathbf{Set}$ *be topological functor,* $X \in Ob(\mathsf{E})$ *with* $\mathsf{U}(X) = B$ *and* $p \in B$*.*

(i) X *is* $\overline{T_0}$ *at* p *provided that the initial lift of the* U*-source* $\{\mathsf{A}_p : B \vee_p B \to \mathsf{U}(X^2) = B^2$ *and* $\nabla_p : B \vee_p B \to \mathsf{UD}(B) = B\}$ *is discrete, where* D *is the discrete functor that is a left adjoint to* U*.*

(ii) X *is* T_1 *at* p *provided that the initial lift of the* U*-source* $\{\mathsf{S}_p : B \vee_p B \to \mathsf{U}(X^2) = B^2$ *and* $\nabla_p : B \vee_p B \to \mathsf{UD}(B) = B\}$ *is discrete.*

Remark 1. (1) *Separation axioms* $\overline{T}_0$ *at* p *and* T_1 *at* p *are used to identify the notions of (strong) closedness in arbitrary set-based topological categories [2, 3].*

(2) *In* **Top** *(the category of topological spaces and continuous mappings), by Theorem 1,* $\overline{T}_0$ *at* p *and* T_1 *at* p *reduce to Definition 6 [2].*

(3) *A topological space* X *is* T_i, $i = 0, 1$ *if and only if* X *is* T_i, $i = 0, 1$, *at* p *for all points* p *in* X *([6], Theorem 1.5(5)).*

(4) *Let* $\mathsf{U} : \mathsf{E} \to \mathbf{Set}$ *be a topological functor,* X *an object in* E *and* $p \in \mathsf{U}(X)$ *be a retract of* X, *i.e., the initial lift* $h : \overline{1} \to X$ *of the* U*-source* $p : 1 \to \mathsf{U}(X)$ *is a retract, where* 1 *is the terminal object in* **Set**. *Then if* X *is* $\overline{T_0}$ *(resp.* T_1*), then* X *is* $\overline{T_0}$ *at* p *(resp.* T_1 *at* p*) but the converse of implication is not true, in general ([4], Theorem 2.6).*

(5) *Specially, if* $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$ *is normalized, then* $\overline{T_0}$ *and* T_1 *imply* $\overline{T_0}$ *at* p *and* T_1 *at* p, *respectively. ([4], Corollary 2.7).*

Theorem 2. *An* L*-preordered space* (X, R) *is* $\overline{T_0}$ *at* p *iff* $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$ *for all* $x \in X$ *distinct from* p.

Proof. Assume (X, R) is $\overline{T_0}$ at p and $x \in X$ with $x \neq p$. Let R_{dis} be the discrete L-preorder relation on X and for $i = 1, 2$, $\pi_i : X^2 \to X$ be the projection maps. For x_1, $x_2 \in X \vee_p X$,

$$\begin{aligned} \mathsf{R}(\pi_1 \mathsf{A}_p(x_1), \pi_1 \mathsf{A}_p(x_2)) &= \mathsf{R}(\pi_1(x, p), \pi_1(p, x)) = \mathsf{R}(x, p) \\ \mathsf{R}(\pi_2 \mathsf{A}_p(x_1), \pi_2 \mathsf{A}_p(x_2)) &= \mathsf{R}(\pi_2(x, p), \pi_2(p, x)) = \mathsf{R}(p, x) \\ \mathsf{R}_{dis}(\nabla_p(x_1), \nabla_p(x_2)) &= \mathsf{R}_{dis}(x, x) = \top \end{aligned}$$

Since (A, R) is $\overline{T_0}$ and $x_1 \neq x_2$, by Definition 7 and Lemmas 1, 2,

$$\begin{aligned} \bot &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{A}_p(x_1), \pi_1 \mathsf{A}_p(x_2)), \mathsf{R}(\pi_2 \mathsf{A}_p(x_1), \pi_2 \mathsf{A}_p(x_2)), \mathsf{R}_{dis}(\nabla_p(x_1), \nabla_p(x_2))\} \\ &= \bigwedge \{\mathsf{R}(x, p), \mathsf{R}(p, x), \top\} \\ &= \mathsf{R}(x, p) \wedge \mathsf{R}(p, x) \end{aligned}$$

Hence, we have $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$.

Conversely, let R' be the initial L-preorder relation on $X \vee_p X$ induced by $\mathsf{A}_p : X \vee_p X \to \mathsf{U}(X^2, \mathsf{R}^2) = X^2$ and $\nabla_p : X \vee_p X \to \mathsf{U}(X, \mathsf{R}_{dis}) = X$, where R^2 is the product structure on X^2 induced by the projection maps π_1 and π_2.

Assume that the condition holds, i.e., for all $x \in X$ distinct from p, $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$. Let v and w be any points in the wedge.

(1) If $v = w$, then $\mathsf{R}'(v, w) = \top$.

(2) If $v \neq w$ and $\nabla_p v \neq \nabla_p w$, then $\mathsf{R}_{dis}(\nabla_p v, \nabla_p w) = \bot$. By Lemma 1,

$$\begin{aligned} \mathsf{R}'(v, w) &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{A}_p v, \pi_1 \mathsf{A}_p w), \mathsf{R}(\pi_2 \mathsf{A}_p v, \pi_2 \mathsf{A}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bot \end{aligned}$$

(3) Suppose $v \neq w$ and $\nabla_p v = \nabla_p w$. It follows that $\nabla_p v = x = \nabla_p w$ for some points $x \in X$ with $x \neq p$. We must have $v = x_1$ and $w = x_2$ or $v = x_2$ and $w = x_1$ since $v \neq w$.

(a) If $v = x_1$ and $w = x_2$, then

$$\begin{aligned} \mathsf{R}(\pi_1 \mathsf{A}_p v, \pi_1 \mathsf{A}_p w) &= \mathsf{R}(x, p) \\ \mathsf{R}(\pi_2 \mathsf{A}_p v, \pi_2 \mathsf{A}_p w) &= \mathsf{R}(p, x) \\ \mathsf{R}_{dis}(\nabla_p v, \nabla_p w) &= \mathsf{R}_{dis}(x, x) = \top \end{aligned}$$

and it follows that

$$\begin{aligned} \mathsf{R}'(v, w) &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{A}_p v, \pi_1 \mathsf{A}_p w), \mathsf{R}(\pi_2 \mathsf{A}_p v, \pi_2 \mathsf{A}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bigwedge \{\mathsf{R}(x, p), \mathsf{R}(p, x), \top\} \\ &= \mathsf{R}(x, p) \wedge \mathsf{R}(p, x) \end{aligned}$$

By the assumption $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$, we get $\mathsf{R}'(v, w) = \bot$.

(b) Similarly, if $v = x_2$ and $w = x_1$, then $\mathsf{R}'(v, w) = \bot$.

Consequently, for all v, w in the wedge $X \vee_p X$, we obtain

$$\mathsf{R}'(v, w) = \begin{cases} \top, & v = w \\ \bot, & v \neq w \end{cases}$$

By Lemma 2, R' is the discrete L-preorder relation on the wedge. Hence, by Definition 7, (X, R) is $\overline{T_0}$ at p. □

Theorem 3. *An* L*-preordered space* (X, R) *is* T_1 *at* p *iff* $\mathsf{R}(x, p) = \bot = \mathsf{R}(p, x)$ *for all* $x \in X$ *distinct from* p.

Proof. Assume that (X, R) is T_1 at p and $x \in X$ with $x \neq p$. Let $v = x_1$, $w = x_2 \in X \vee_p X$. Note that,

$$\begin{aligned} \mathsf{R}(\pi_1 \mathsf{S}_p v, \pi_1 \mathsf{S}_p w) &= \mathsf{R}(\pi_1(x, x), \pi_1(p, x)) = \mathsf{R}(x, p) \\ \mathsf{R}(\pi_2 \mathsf{S}_p v, \pi_2 \mathsf{S}_p w) &= \mathsf{R}(\pi_2(x, x), \pi_2(p, x)) = \mathsf{R}(x, x) = \top \\ \mathsf{R}_{dis}(\nabla_p v, \nabla_p w) &= \mathsf{R}_{dis}(x, x) = \top \end{aligned}$$

where R_{dis} is the discrete L-preorder relation on X and for each $i = 1, 2$, $\pi_i : X^2 \to X$ is the projection map. Since $v \neq w$ and (X, R) is T_1 at p, by Definition 7 and Lemmas 1, 2,

$$\begin{aligned} \bot &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{S}_p v, \pi_1 \mathsf{S}_p w), \mathsf{R}(\pi_2 \mathsf{S}_p v, \pi_2 \mathsf{S}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bigwedge \{\mathsf{R}(x, p), \top\} \\ &= \mathsf{R}(x, p) \end{aligned}$$

Similarly, if $v = x_2$, $w = x_1 \in X \vee_p X$, then by Lemma 1, we have

$$\bot = \bigwedge \{\mathsf{R}(p, x), \top\} = \mathsf{R}(p, x)$$

Conversely, let R' be the initial L-preorder relation on $X \vee_p X$ induced by $\mathsf{S}_p : X \vee_p X \to \mathsf{U}(X^2, \mathsf{R}^2) = X^2$ and $\nabla_p : X \vee_p X \to \mathsf{U}(X, \mathsf{R}_{dis}) = X$, where R^2 is the product structure on X^2 induced by the projection maps π_1 and π_2.

Assume that for all $x \in X$ distinct from p, $\mathsf{R}(x,p) = \bot = \mathsf{R}(p,x)$. Let v and w be any points in the wedge.

(1) If $v = w$, then $\mathsf{R}'(v,w) = \top$.

(2) If $v \neq w$ and $\nabla_p v \neq \nabla_p w$, then $\mathsf{R}_{dis}(\nabla_p v, \nabla_p w) = \bot$ since R_{dis} is the discrete structure. By Lemma 1,

$$\begin{aligned} \mathsf{R}'(v,w) &= \bigwedge \{\mathsf{R}(\pi_1 \mathsf{S}_p v, \pi_1 \mathsf{S}_p w), \mathsf{R}(\pi_2 \mathsf{S}_p v, \pi_2 \mathsf{S}_p w), \mathsf{R}_{dis}(\nabla_p v, \nabla_p w)\} \\ &= \bot \end{aligned}$$

(3) Suppose $v \neq w$ and $\nabla_p v = \nabla_p w$. It follows that we must have $v = x_1$ and $w = x_2$ or $v = x_2$ and $w = x_1$.

If $v = x_1$ and $w = x_2$, then by Lemma 1,

$$\mathsf{R}'(v,w) = \bigwedge \{\mathsf{R}(x,p), \top\} = \mathsf{R}(x,p)$$

By the assumption $\mathsf{R}(x,p) = \bot = \mathsf{R}(p,x)$, we get $\mathsf{R}'(v,w) = \bot$.

Similarly, we obtain $\mathsf{R}'(v,w) = \bot$ for $v = x_2$ and $w = x_1$.

Hence, for all $v, w \in X \vee_p X$, we have

$$\mathsf{R}'(v,w) = \begin{cases} \top, & v = w \\ \bot, & v \neq w \end{cases}$$

By Lemma 2, it follows that R' is the discrete L-preorder relation on the wedge. Consequently, by Definition 7, (X, R) is T_1 at p. □

Example 2. *Let $*$ be a binary operation identified as $\forall \alpha, \beta \in [0,1]$, $\alpha * \beta = (\alpha - 1 + \beta) \vee 0$ and $\mathsf{L} = ([0,1], \leq, *)$ be a triangular norm (Lukasiewicz t-norm) [26], where the bottom and top elements are $\bot = 0$ and $\top = 1$, respectively. Let $X = \{a, b, c\}$ and an L-preorder relation $\mathsf{R} : X \times X \to \mathsf{L}$ defined by*

$$\mathsf{R}(v,w) = \begin{cases} \top, & v = w \\ \frac{1}{2}, & (v,w) = (a,c) \\ \bot, & \textit{otherwise.} \end{cases}$$

Clearly, (X, R) is an L-preordered space. By Theorem 2, (X, R) is $\overline{T}_0$ at p for all $p \in X$, and by Theorem 3, (X, R) is T_1 at b but it is neither T_1 at a nor at c.

Remark 2. (1) *By Theorems 2 and 3, if an L-preordered space (X, R) is T_1 at p, then it is $\overline{T}_0$ at p. But in general, the converse is not true (see previous Example).*

(2) *In an arbitrary set-based topological category, $\overline{T_0}$ at p and T_1 at p objects may be equivalent, for example, in* **Prox** *(the category of proximity spaces and p-maps) [27],* **CP** *(the category of pairs and pair preserving maps) [3],*

Born *(the category of bornological spaces and bounded maps)* [3], **SULim** *(the category of semiuniform limit spaces and uniformly continuous maps)* [13], *Remark 3.6.*

4. Hereditary and Productive Properties

Definition 8. *Let (X, R) be an L-preordered space and $A \subset X$. A subspace (A, R_A) is defined by $\mathsf{R}_A(x, y) = \mathsf{R}(x, y)$ for all $x, y \in A$, where R_A is the initial L-preorder structure on A induced by the inclusion map $i : A \to X$.*

Theorem 4. *Let (X, R) be an L-preordered space, $A \subset X$ and $p \in A$.*

(i) *If (X, R) is $\overline{T}_0$ at p, then (A, R_A) is $\overline{T}_0$ at p.*
(ii) *If (X, R) is T_1 at p, then (A, R_A) is T_1 at p.*

Proof. (i) Suppose that $p \in A$ and (X, R) is $\overline{T}_0$ at p. By Theorem 2, $\mathsf{R}(x, p) \wedge \mathsf{R}(p, x) = \bot$ for $x \in A \subset X$ with $x \neq p$. By Definition 8, we have $\mathsf{R}_A(x, p) = \mathsf{R}(x, p)$ and $\mathsf{R}_A(p, x) = \mathsf{R}(p, x)$ for $x, p \in A \subset X$. It follows that $\mathsf{R}_A(x, p) \wedge \mathsf{R}_A(p, x) = \bot$. Hence, by Theorem 2, the subspace (A, R_A) is also $\overline{T}_0$ at p.
(ii) Similarly, let $p \in A$ and (X, R) be T_1 at p. By Theorem 3 and Definition 8, we have $\mathsf{R}_A(x, p) = \mathsf{R}(x, p) = \bot = \mathsf{R}(p, x) = \mathsf{R}_A(p, x)$ for $x, p \in A \subset X$ with $x \neq p$. Hence, by Theorem 3, the subspace (A, R_A) is also T_1 at p.

□

Theorem 5. *Let (X_i, R_i) be an L-preordered space for each $i \in I$ and (X, R) be the product of the spaces $\{(X_i, \mathsf{R}_i) : i \in I\}$, where $X = \prod_{i \in I} X_i$ and for all $x, y \in X$, $\mathsf{R}(x, y) = \bigwedge_{i \in I} \mathsf{R}_i(\pi_i(x), \pi_i(y))$. For all $i \in I$, the L-preordered space (X_i, R_i) is isomorphic to a subspace of the product space (X, R).*

Proof. Suppose that (X_i, R_i) is an L-preordered space for each $i \in I$ and (X, R) is the product space. Firstly, we choose a fixed point z_j in X_j for each $j \in I$ with $j \neq i$. Let $A = \{z_1\} \times \{z_2\} \times ... \times \{z_{i-1}\} \times X_i \times \{z_{i+1}\} \times ... \subset X$. Then, (A, R_A) is a subspace of the product space (X, R), where $\mathsf{R}_A(x, y) = \mathsf{R}(x, y)$ for all $x, y \in A$. Clearly, i-th projection map $\pi_i : (A, \mathsf{R}_A) \to (X_i, \mathsf{R}_i)$ defined by for $a_i \in X_i$, $\pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) = a_i$ is bijective. For all $(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)$, $(z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...) \in A$, we have

$$
\begin{aligned}
& \mathsf{R}_A((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\
= \; & \mathsf{R}((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\
= \; & \bigwedge_{j \neq i} \{\mathsf{R}_i(a_i, b_i), \mathsf{R}_j(z_j, z_j) = \top\} \\
\leq \; & \mathsf{R}_i(a_i, b_i) \\
= \; & \mathsf{R}_i(\pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), \pi_i(z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...))
\end{aligned}
$$

and it follows that π_i is an L-order preserving map.

On the other hand, let $f_i : (X_i, \mathsf{R}_i) \to (A, \mathsf{R}_A)$ be function defined by $f_i(a_i) = (z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)$ for $a_i \in X_i$. Then, we have

$$\begin{aligned} (\pi_i \circ f_i)(a_i) &= \pi_i(f_i(a_i)) \\ &= \pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) \\ &= a_i \\ &= 1_{X_i}(a_i) \end{aligned}$$

and

$$\begin{aligned} (f_i \circ \pi_i)(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) &= f_i(\pi_i(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...)) \\ &= f_i(a_i) \\ &= (z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) \\ &= 1_A(z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...) \end{aligned}$$

It follows that $f_i = (\pi_i)^{-1}$ since $\pi_i \circ f_i = 1_{X_i}$ and $f_i \circ \pi_i = 1_A$.

For all $a_i, b_i \in X_i$, we obtain

$$\begin{aligned} \mathsf{R}_i(a_i, b_i) &= \bigwedge_{j \neq i} \{\mathsf{R}_i(a_i, b_i), \mathsf{R}_j(z_j, z_j) = \top\} \\ &= \mathsf{R}((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\ &= \mathsf{R}_A((z_1, z_2, ..., z_{i-1}, a_i, z_{i+1}, ...), (z_1, z_2, ..., z_{i-1}, b_i, z_{i+1}, ...)) \\ &= \mathsf{R}_A(f_i(a_i), f_i(b_i)) \leq \mathsf{R}_A(f_i(a_i), f_i(b_i)) \end{aligned}$$

and it follows that f_i is an L-order preserving map.

Consequently, L-preordered space (X_i, R_i) and the subspace (A, R_A) are isomorphic. □

Theorem 6. *Let $\{(X_i, \mathsf{R}_i) : i \in I\}$ be a collection of L-preordered spaces and (X, R) be the product space, where $X = \prod_{i \in I} X_i$ and $\mathsf{R}(x, y) = \bigwedge_{i \in I} \mathsf{R}_i(\pi_i(x), \pi_i(y))$ for $x, y \in X$. Let $p = (p_i)_{i \in I}$ be a point in X.*

(i) *(X, R) is $\overline{T}_0$ at p iff (X_i, R_i) is $\overline{T}_0$ at p_i for each $i \in I$.*
(ii) *(X, R) is T_1 at p iff (X_i, R_i) is T_1 at p_i for each $i \in I$.*

Proof. (i) Assume that the product space (X, R) is $\overline{T}_0$ at p. By Theorem 5, for each $i \in I$, (X_i, R_i) is isomorphic to a subspace of (X, R) and by Theorem 4, a subspace of a local $\overline{T}_0$ L-preordered space is $\overline{T}_0$ at p. Since (X, R) is $\overline{T}_0$ at p, it follows that (X_i, R_i) is $\overline{T}_0$ at p_i for each $i \in I$.

Conversely, suppose that (X_i, R_i) is $\overline{T}_0$ at p_i for each $i \in I$. Let $x = (x_i)_{i \in I}$ be a point in X with $x \neq p = (p_i)_{i \in I}$. Since $x \neq p$, there exists $i_0 \in I$ such that $x_{i_0} \neq p_{i_0}$. By the assumption L-preordered space $(X_{i_0}, \mathsf{R}_{i_0})$ is $\overline{T}_0$ at p and by Theorem 2, we have $\mathsf{R}_{i_0}(x_{i_0}, p_{i_0}) \wedge \mathsf{R}_{i_0}(p_{i_0}, x_{i_0}) = \bot$. It follows that

$$\mathsf{R}(x, p) = \bigwedge_{i \in I} \{\mathsf{R}_i(x_i, p_i)\} \leq \mathsf{R}_{i_0}(x_{i_0}, p_{i_0})$$

and

$$\mathsf{R}(p,x)=\bigwedge_{i\in I}\{\mathsf{R}_i(p_i,x_i)\}\leq \mathsf{R}_{i_0}(p_{i_0},x_{i_0})$$

Since $\mathsf{R}_{i_0}(x_{i_0},p_{i_0})\wedge \mathsf{R}_{i_0}(p_{i_0},x_{i_0})=\bot$, we get $\mathsf{R}(x,p)\wedge \mathsf{R}(p,x)=\bot$. Hence, by Theorem 2, the product space (X,R) is $\overline{T}_0$ at p.

(ii) Similarly, suppose that the product space (X,R) is T_1 at p. By the assumption and Theorems 4 and 5, we have (X_i,R_i) is T_1 at p_i for each $i\in I$.

Conversely, assume that (X_i,R_i) is T_1 at p_i for each $i\in I$. Let $x\in X$ with $x\neq p$. Then, there exists $i_0\in I$ such that $x_{i_0}\neq p_{i_0}$. By the assumption L-preordered space $(X_{i_0},\mathsf{R}_{i_0})$ is T_1 at p and by Theorem 3, we have $\mathsf{R}_{i_0}(x_{i_0},p_{i_0})=\mathsf{R}_{i_0}(p_{i_0},x_{i_0})=\bot$. It follows that

$$\begin{aligned}\mathsf{R}(x,p) &= \bigwedge\{\mathsf{R}_1(x_1,p_1),\mathsf{R}_2(x_2,p_2),...,\mathsf{R}_{i_0-1}(x_{i_0-1},p_{i_0-1}),\\ &\qquad \mathsf{R}_{i_0}(x_{i_0},p_{i_0})=\bot,\mathsf{R}_{i_0+1}(x_{i_0+1},p_{i_0+1}),...\}\\ &= \bot\end{aligned}$$

and similarly,

$$\begin{aligned}\mathsf{R}(p,x) &= \bigwedge\{\mathsf{R}_1(p_1,x_1),...,\mathsf{R}_{i_0}(p_{i_0},x_{i_0})=\bot,...\}\\ &= \bot\end{aligned}$$

Consequently, by Theorem 3, we get the product space (X,R) is T_1 at p.

□

Author Contribution Statements The authors jointly worked on the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgements The authors are thankful to the editor and referees for their valuable comments and suggestions which helped very much in improving the paper. This work was supported by Research Fund of the Nevşehir Hacı Bektaş Veli University. (Project Number: TEZ21F1)

References

[1] Adamek, J., Herrlich, H., Strecker, G. E., Abstract and Concrete Categories, Pure and Applied Mathematics, John Wiley & Sons, New York, 1990.

[2] Baran, M., Separation properties, *Indian J. Pure Appl. Math.*, 23 (1991), 333–341.

[3] Baran, M., The notion of closedness in topological categories, *Comment. Math. Univ. Carolin.*, 34(2) (1993), 383–395.

[4] Baran, M., Generalized local separation properties, *Indian J. Pure Appl. Math.*, 25(6) (1994), 615–620.

[5] Baran, M., Altındiş, H., T_2 objects in topological categories, *Acta Math. Hungar.*, 71(1-2) (1996), 41–48. https://doi.org/10.1007/BF00052193

[6] Baran, M., Separation properties in topological categories, *Math. Balkanica*, 10(1) (1996), 39–48.
[7] Baran, M., T_3 and T_4-objects in topological categories, *Indian J. Pure Appl. Math.*, 29(1) (1998), 59–70.
[8] Baran, M., Completely regular objects and normal objects in topological categories, *Acta Math. Hungar.*, 80(3) (1998), 211–224. https://doi.org/10.1023/A:1006550726143
[9] Baran, M., Compactness, perfectness, separation, minimality and closedness with respect to closure operators, *Appl. Categ. Structures*, 10(4) (2002), 403–415. https://doi.org/10.1023/A:1016388102703
[10] Baran, M., Kula, M., A note on connectedness, *Publ. Math. Debrecen*, 68 (2006), 489–501.
[11] Baran, M., Closure operators in convergence spaces, *Acta Math. Hungar.*, 87(1-2) (2000), 33-45. https://doi.org/10.1023/A:1006768916033
[12] Baran, M., Al-Safar, J., Quotient-reflective and bireflective subcategories of the category of preordered sets, *Topology and its Applications*, 158(15) (2011), 2076-2084. https://doi.org/10.1016/j.topol.2011.06.043
[13] Baran, M., Kula, S., Erciyes, A., T_0 and T_1 semiuniform convergence spaces, *Filomat*, 27(4) (2013), 537–546. https://doi.org/10.2298/FIL1304537B
[14] Baran, M., Kula, S., Baran, T. M., Qasim, M., Closure Operators in Semiuniform Convergence Spaces, *Filomat*, 30(1) (2016), 131-140. https://doi.org/10.2298/FIL1601131B
[15] Denniston, J. T., Melton, A., Rodabaugh, S. E., Solovyov, S. A., Lattice-valued preordered sets as lattice-valued topological systems, *Fuzzy Sets and Systems*, 259 (2015), 89–110. https://doi.org/10.1016/j.fss.2014.04.022
[16] Dikranjan, D., Giuli, E., Closure operators I, *Topology and its Applications*, 27(2) (1987), 129–143. https://doi.org/10.1016/0166-8641(87)90100-3
[17] Dikranjan, D., Tholen, W., Categorical Structure of Closure Operators: With Applications to Topology, Algebra and Discrete Mathematics, Kluwer Academic Publishers, Dordrecht, 1995.
[18] Duquenne, V., Latticial structures in data analysis, *Theoretical Computer Science*, 217 (1999), 407–436.
[19] Flagg, R. C., Quantales and continuity spaces, *Algebra Universalis*, 37(3) (1997), 257–276. https://doi.org/10.1007/s000120050018
[20] Goubault-Larrecq, J., Non-Hausdorff Topology and Domain Theory, Cambridge University Press, Cambridge, 2013. https://doi.org/10.1017/CBO9781139524438
[21] Hofmann, D., Seal, G. J., Tholen, W., Monoidal Topology: A Categorical Approach to Order, Metric, and Topology, Cambridge University Press, Cambridge, 2014.
[22] Jäger, G., A category of L-fuzzy convergence spaces, *Quaest. Math.*, 24(4) (2001), 501–517. https://doi.org/10.1080/16073606.2001.9639237
[23] Jäger, G., Probabilistic approach spaces, *Math. Bohem.*, 142(3) (2017), 277–298. https://doi.org/10.21136/MB.2017.0064-15
[24] Jäger, G., Yao, W., Quantale-valued gauge spaces, *Iran. J. Fuzzy Syst.*, 15(1) (2018), 103–122. https://doi.org/10.22111/IJFS.2018.3581
[25] Jäger, G., The Wijsman structure of a quantale-valued metric space, *Iran. J. Fuzzy Syst.*, 17(1) (2020), 171–184. https://doi.org/10.22111/IJFS.2020.5118
[26] Klement, E. P., Mesiar, R., Pap, E., Triangular Norms, Springer, Dordrecht, 2000.
[27] Kula, M., Maraşlı, T., Özkan, S., A note on closedness and connectedness in the category of proximity spaces, *Filomat*, 28(7) (2014), 1483-1492. https://doi.org/10.2298/FIL1407483K
[28] Qasim, M., Özkan, S., The notions of closedness and D-connectedness in quantale-valued approach spaces, *Categ. Gen. Algebr. Struct. Appl.*, 12(1) (2020), 149–173. https://doi.org/10.29252/CGASA.12.1.149

[29] Scott, D. S., Domains for Denotational Semantics, *Proc. 9th. Int. Coll. on Automata, Languages and Programming*, (Aarhus, 1982), 577–610, Lecture Notes in Comput. Sci., 140, Springer, Berlin-New York, 1982. https://doi.org/10.1007/BFb0012801

[30] Zhang, Q. Y., Fan, L., Continuity in quantitative domains, *Fuzzy Sets and Systems*, 154(1) (2005), 118–131. https://doi.org/10.1016/j.fss.2005.01.007

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 919–929 (2022)
DOI:10.31801/cfsuasmas.1076462
ISSN 1303-5991 E-ISSN 2618-6470

ANKARA UNIVERSITY FACULTY OF SCIENCES 1948 COMMUNICATIONS SERIES A1

Research Article; Received: February 20, 2022; Accepted: May 1, 2022

SHARP WEAK BOUNDS FOR p-ADIC HARDY OPERATORS ON p-ADIC LINEAR SPACES

Amjad HUSSAIN[1], Naqash SARFRAZ[2] and Ferit GURBUZ[3]

[1]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, PAKISTAN
[2]Department of Mathematics University of Kotli Azad Jammu and Kashmir, PAKISTAN
[3]Hakkari University, Faculty of Education, Department of Mathematics Education, Hakkari 30000, TURKEY

Abstract. The current paper establishes the sharp weak bounds of p-adic fractional Hardy operator. Furthermore, optimal weak type estimates for p-adic Hardy operator on central Morrey space are also acquired.

1. Introduction

For every non-zero rational number x there is a unique $k = k(x) \in \mathbb{Z}$ such that $x = p^k s/t$, where $p \geq 2$ is a fixed prime number which is coprime to $s, t \in \mathbb{Z}$. We define a mapping $|.|_p : \mathbb{Q} \to \mathbb{R}_+$ as follows:

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1}$$

The p-adic norm $|\cdot|_p$ undergoes many properties of the usual real norm $|\cdot|$ with an additional non-Archimedean property,

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}. \tag{2}$$

The field of p-adic numbers, denoted by $\mathbb{Q}_p$, is the completion of rational numbers with respect to the p-adic norm $|\cdot|_p$. A p-adic number $x \in \mathbb{Q}_p$ can be written in the formal power series as (see [30]):

$$x = p^k(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + ...) \tag{3}$$

2020 *Mathematics Subject Classification.* 42B35, 26D15, 46B25, 47G10.
Keywords. Sharp bounds, boundedness, p-adic weak type spaces, p-adic fractional Hardy operator.
ahabbasi123@yahoo.com; 0000-0002-5840-0846
naqashawesome@gmail.com; 0000-0002-0705-8462
feritgurbuz@hakkari.edu.tr-Corresponding author; 0000-0003-3049-688X.

where $\alpha_i, k \in \mathbb{Z}, \alpha_0 \neq 0, \alpha_i \in \{0, 1, 2, ..., p-1\}, i = 1, 2, \cdots$. The p-adic norm ensures the convergence of series (3) in $\mathbb{Q}_p$, because $|p^k \alpha_i p^i|_p \leq p^{-k-i}$.

The n-dimensional vector space $\mathbb{Q}_p^n$, $n \geq 1$, consists of tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$ and $j = 1, 2, \ldots, n$. The norm on this space is given by

$$|\mathbf{x}|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

In non-Archimedean geometry, the ball and its boundary are defined, respectively, as:

$$B_k(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leq p^k\}, \; S_k(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^k\}.$$

For convenience we denote $B_k(\mathbf{0})$and $S_k(\mathbf{0})$ by B_k and S_k, respectively.

The local compactness and commutativity of the group $\mathbb{Q}_p^n$ under addition implies the existence of Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$, such that

$$\int_{B_0} d\mathbf{x} = |B_0|_H = 1,$$

where the notation $|B|_H$ refers to the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. Furthermore, it is not hard to see that $|B_k(\mathbf{a})|_H = p^{nk}$, $|S_k(\mathbf{a})|_H = p^{nk}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

Let $w(\mathbf{x})$ be a nonnegative locally integrable function on $\mathbb{Q}_p^n$ and $w(E)$ the weighted measure of measurable subset $E \subset \mathbb{Q}_p^n$, that is $w(E) = \int_E w(x)dx$ respectively. The space of all complex-valued functions f with norm conditions:

$$\|f\|_{L^r(w;\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^r w(\mathbf{x}) d\mathbf{x} \right)^{1/r} < \infty,$$

is denoted by $L^r(w, \mathbb{Q}_p^n), (0 < r < \infty)$, and is known as weighted Lebesgue space. Note that $L^r(1, \mathbb{Q}_p^n) = L^r(\mathbb{Q}_p^n)$.

In [22], authors have defined the weighted p-adic weak Lebesgue space $L^{r,\infty}(w; \mathbb{Q}_p^n)$ by

$$\|f\|_{L^{r,\infty}(w,\mathbb{Q}_p^n)} = \sup_{\mu > 0} \mu w \Big(\{\mathbf{x} \in \mathbb{Q}_p^n : |f(\mathbf{x})| > \mu\} \Big)^{1/r} < \infty.$$

When $w = 1$, we get the weak Lebesgue space $L^{r,\infty}(\mathbb{Q}_p^n)$ defined in [32]. Next, we give the relevant p-adic function spaces.

Definition 1. *[34] Suppose $1 < r < \infty$ and $\mu \in \mathbb{R}$. The p-adic space $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ is the set of all measurable functions $f \colon \mathbb{Q}_p^n \to \mathbb{R}$ which satisfy*

$$\|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |f(\mathbf{x})|^r d\mathbf{x} \right)^{1/r} < \infty.$$

When $\mu = -1/r$, then

$\dot{B}^{r,\mu}(\mathbb{Q}_p^n) = L^r(\mathbb{Q}_p^n)$. It is easy to see that $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ is reduced to $\{0\}$ whenever $\mu < -1/r$.

Definition 2. *[35] Suppose $\mu \in \mathbb{R}$ and $1 < r < \infty$. The p-adic space $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ is defined as*

$$W\dot{B}^{r,\mu}(\mathbb{Q}_p^n) = \{f : \|f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} |B_\gamma|_H^{-\mu-1/r} \|f\|_{WL^r(B_\gamma)},$$

and $\|f\|_{WL^r(B_\gamma)}$ is the local p-adic L^r-norm of $f(x)$ restricted to the ball B_γ, that is

$$\|f\|_{WL^r(B_\gamma)} = \sup_{\mu>0} |\{\mathbf{x} \in B_\gamma : |f(\mathbf{x})| > \mu\}|^{1/r}.$$

Evidently, if $\mu = -1/r$, then $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n) = L^{r,\infty}(\mathbb{Q}_p^n)$. Also, $\dot{B}^{r,\mu}(\mathbb{Q}_p^n) \subseteq W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ for $-1/r < \mu < 0$ and $1 \le r < \infty$.

In the last several decades, a growing interest to p-adic models have been seen because p-adic analysis is a natural base for development of various models of ultrametric diffusion energy landscape [4]. It also attracts great deal of interest towards quantum mechanics [30], theoretical biology [11], quantum gravity [1,7], string theory [31], spin glass theory [3,26]. In [4], it was shown that the p-adic analysis can be efficiently applied both to relaxation in complex speed systems and processes combined with the relaxation of a complex environment. Besides, the applications of p-adic analysis can be found in harmonic analysis and pseudo-differential equations, see for example [5,9,10,21,28,29].

The one-dimensional Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0, \tag{4}$$

has been introduced by Hardy in [18] for measurable functions $f : \mathbb{R}^+ \to \mathbb{R}^+$. This operator satisfies the inequality

$$\|Hf\|_{L^r(\mathbb{R}^+)} \le \frac{r}{r-1}\|f\|_{L^r(\mathbb{R}^+)}, \quad 1 < r < \infty, \tag{5}$$

where the constant $r/(r-1)$ is sharp.

In [12], Faris has proposed an extension of the Hardy operator H on higher dimensional Euclidean space $\mathbb{R}^n$ by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{|\mathbf{t}|\le|\mathbf{x}|} f(\mathbf{t})d\mathbf{t}. \tag{6}$$

where $|\mathbf{x}| = (\sum_{i=1}^n x_i^2)^{1/2}$ for $\mathbf{x} = (x_1, \cdots, x_n)$. In addition, Christ and Grafakos [8] have obtained the exact value of the norm of (6). For more details related to Hardy type operators and, in particular, to boundedness of these operators, we refer to publications [6,13,19,23,24,27,36,39].

On the other hand, the fractional Hardy operator is obtained by merely writing $|\cdot|^{n-\alpha}$ $(0 \le \alpha < n)$ instead of $|\cdot|^n$ with in (6). The weak type estimates for the fractional Hardy type operators has also spotlighted many researchers in the past, see for example [2, 13, 15, 16, 20, 37, 38].

In what follows, the higher dimensional fractional Hardy operator in the p-adic field

$$H_\alpha^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{t}|_p \le |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t}, \qquad \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}.$$

has been defined and studied for $0 \le \alpha < n$ and $f \in L_{\text{loc}}(\mathbb{Q}_p^n)$ in [33]. When $\alpha = 0$, the operator H_α^p transfers to the p-adic Hardy operator (see [14]). Fu et al. in [14] have acquired the optimal bounds of p-adic Hardy operator on Lebesgue spaces. For more details, we refer the publications [17, 22, 25, 34] and the references therein.

The purpose of the current paper is to study the sharp weak bounds for fractional Hardy operator in the p-adic field on p-adic Lebesgue space. Moreover, we also discuss the optimal weak type estimates for Hardy operator in the p-adic field on central Morrey spaces.

2. Sharp weak bounds for p-adic fractional Hardy Operator on Lebesgue spaces

Our main result for this section is as follows.

Theorem 1. *Suppose* $0 < \alpha < n$ *and* $n + \gamma > 0$. *If* $f \in L^1(\mathbb{Q}_p^n)$, *then*

$$\|H_\alpha^p f\|_{L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma; \mathbb{Q}_p^n)} \le C \|f\|_{L^1(\mathbb{Q}_p^n)},$$

where the constant

$$C = \left(\frac{1 - p^{-n}}{1 - p^{-(n+\gamma)}} \right)^{(n-\alpha)/(n+\gamma)}$$

is optimal.

Proof. We have

$$\begin{aligned} |H_\alpha^p f(\mathbf{x})| &= \left| \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{t}|_p \le |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t} \right| \\ &\le |\mathbf{x}|_p^{-(n-\alpha)} \|f\|_{L^1(\mathbb{Q}_p^n)}. \end{aligned} \tag{7}$$

Let $C_1 = \|f\|_{L^1(\mathbb{Q}_p^n)}$, then

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f(\mathbf{x})| > \mu\} \subset \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x}|_p < (C_1/\mu)^{1/(n-\alpha)}\}.$$

Thus,

$$
\begin{aligned}
&\|H_\alpha^p f\|_{L^{(n+\gamma)/(n-\alpha),\infty}(|x|_p^\gamma;\mathbb{Q}_p^n)}\\
\leq&\sup_{\mu>0}\mu\biggl(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f(\mathbf{x})|>\mu\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)}\\
\leq&\sup_{\mu>0}\mu\biggl(\int_{\mathbb{Q}_p^n}\chi_{\left\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<\left(C_1/\mu\right)^{1/(n-\alpha)}\right\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)}\\
=&\sup_{\mu>0}\mu\biggl(\int_{|\mathbf{x}|_p<\left(C_1/\mu\right)^{1/(n-\alpha)}}|\mathbf{x}|_p^\gamma d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)}\\
=&\sup_{\mu>0}\mu\biggl(\sum_{j=-\infty}^{\log_p\left(C_1/\mu\right)^{1/(n-\alpha)}}\int_{S_j}|\mathbf{x}|_p^\gamma d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{\mu>0}\mu\biggl(\sum_{j=-\infty}^{\log_p\left(C_1/\mu\right)^{1/(n-\alpha)}}p^{j(n+\gamma)}d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)}\\
=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\sup_{\mu>0}\mu\left(\frac{C_1}{\mu}\right)\\
\leq&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\|f\|_{L^1(|\mathbf{x}|_p^\beta)}.
\end{aligned}
\tag{8}
$$

To show that the constant

$$\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)},$$

appeared in (8) is optimal, we proceed as, consider

$$f_0(\mathbf{x})=\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p\leq 1\}}(\mathbf{x}),$$

then

$$\|f_0\|_{L^1(\mathbb{Q}_p^n)}=1.$$

Also,

$$
\begin{aligned}
H_\alpha^p f_0(\mathbf{x})=&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{t}|_p\leq|\mathbf{x}|_p}f_0(\mathbf{t})d\mathbf{t}\\
=&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{t}|_p\leq|\mathbf{x}|_p}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{t}|_p\leq 1\}}(\mathbf{t})d\mathbf{t}\\
=&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\begin{cases}\int_{|\mathbf{t}|_p\leq|\mathbf{x}|_p}d\mathbf{t}, & |\mathbf{x}|_p\leq 1;\\ \int_{|\mathbf{t}|_p\leq 1}d\mathbf{t}, & |\mathbf{x}|_p>1.\end{cases}
\end{aligned}
$$

Since $|B_{\log_p|\mathbf{x}|_p}|_H=|\mathbf{x}|_p^n|B_0|_H$, therefore,

$$H_\alpha^p f_0(\mathbf{x})=\begin{cases}|\mathbf{x}|_p^{\alpha}, & |\mathbf{x}|_p\leq 1;\\ |\mathbf{x}|_p^{\alpha-n}, & |\mathbf{x}|_p>1.\end{cases}$$

Now,

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f_0(\mathbf{x})|>\mu\}=\{|\mathbf{x}|_p\leq 1:|\mathbf{x}|_p^{\alpha}>\mu\}\cup\{|\mathbf{x}|_p>1:|\mathbf{x}|_p^{\alpha-n}>\mu\}.$$

Since $0<\alpha<n$, therefore, when $\mu\geq 1$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f_0(\mathbf{x})|>\mu\}=\emptyset,$$

and when $0<\mu<1$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f_0(\mathbf{x})|>\mu\}=\{\mathbf{x}\in\mathbb{Q}_p^n:(\mu)^{1/\alpha}<|\mathbf{x}|_p<(1/\mu)^{1/n-\alpha}\}.$$

Ultimately we are down to:

$$\begin{aligned}
&\|H_\alpha^p f_0\|_{L^{(n+\gamma)/(n-\alpha))),\infty}(|\mathbf{x}|_p^{\gamma};\mathbb{Q}_p^n)}\\
=&\sup_{0<\mu<1}\mu\left(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:(\mu)^{1/\alpha}<|\mathbf{x}|_p<(1/\mu)^{1/(n-\alpha)}\}}(\mathbf{x})|\mathbf{x}|_p^{\gamma}d\mathbf{x}\right)^{(n-\alpha)/(n+\gamma)}\\
=&\sup_{0<\mu<1}\mu\left(\int_{(\mu)^{1/\alpha}<|\mathbf{x}|_p<(1/\mu)^{1/(n-\alpha)}}|\mathbf{x}|_p^{\gamma}d\mathbf{x}\right)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\mu<1}\mu\left(\sum_{j=\log_p\mu^{1/\alpha}+1}^{\log_p\mu^{1/(\alpha-n)}}p^{j(n+\gamma)}\right)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\mu<1}\mu\left(\frac{p^{(\log_p\mu^{1/\alpha}+1)(n+\gamma)}-p^{(\log_p\mu^{1/(\alpha-n)}+1)(n+\gamma)}}{1-p^{(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\mu<1}\mu\left(\frac{\mu^{(n+\gamma)/\alpha}-\mu^{(n+\gamma)/(\alpha-n)}}{p^{-(n+\gamma)}-1}\right)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\mu<1}\left(\frac{1-\mu^{(n+\gamma)/\alpha}\mu^{(n+\gamma)/(n-\alpha)}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\\
=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\sup_{0<\mu<1}\left(1-\mu^{(n+\gamma)/\alpha}\mu^{(n+\gamma)/(n-\alpha)}\right)^{(n-\alpha)/(n+\gamma)}\\
=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\\
=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\|f_0\|_{L^1(\mathbb{Q}_p^n)}.
\end{aligned}\tag{9}$$

We thus conclude from (8) and (9) that

$$\|H_\alpha^p\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma;\mathbb{Q}_p^n)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/q}.$$

□

3. Optimal Weak Type Estimates for p-adic Hardy Operator on Weak Central Morrey Spaces

In the current section we investigate the boundedness of p-adic Hardy operator on p-adic weak central Morrey spaces. It is shown the constant obtained in this case is also optimal.

Theorem 2. *Suppose $-1/r \le \mu < 0$, $1 \le r < \infty$ and if $f \in \dot{B}^{r,\mu}(\mathbb{Q}_p^n)$, then*

$$\|H^p f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)},$$

and the constant 1 is optimal.

Proof. Applying Hölder's inequality, we obtain

$$\begin{aligned}|H^p f(\mathbf{x})| \le& \frac{1}{|\mathbf{x}|_p^n}\left(\int_{B(0,|\mathbf{x}|_p)} |f(\mathbf{t})|^r d\mathbf{t}\right)^{1/r}\left(\int_{B(0,|\mathbf{x}|_p)} d\mathbf{t}\right)^{1/r'}\\ =&|\mathbf{x}|_p^{n\mu}\|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.\end{aligned}$$

Let $C_2 = \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}$. Since $\mu < 0$, we have

$$\begin{aligned}\|H^p f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le& \sup_{\gamma\in\mathbb{Z}}\sup_{y>0} y|B_\gamma|_H^{-\mu-1/r}\left|\{\mathbf{x}\in B_\gamma : C_2|\mathbf{x}|_p^{n\mu} > y\}\right|^{1/r}\\ =& \sup_{\gamma\in\mathbb{Z}}\sup_{y>0} y|B_\gamma|_H^{-\mu-1/r}\left|\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (y/C_2)^{1/n\mu}\}\right|^{1/r}.\end{aligned}$$

If $\gamma \le \log_p(y/C_2)^{1/n\mu}$, then for $\mu < 0$, we obtain

$$\begin{aligned}&\sup_{y>0}\sup_{\gamma\le\log_p(y/C_2)^{1/n\mu}} y|B_\gamma|_H^{-\mu-1/r}\left|\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (y/C_2)^{1/n\mu}\}\right|^{1/r}\\ &\le \sup_{y>0}\sup_{\gamma\le\log_p(y/C_2)^{1/n\mu}} tp^{-\gamma n\mu}\\ &= C_2\\ &\le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.\end{aligned}$$

If $\gamma > \log_p(y/C_2)^{1/n\mu}$, then for $\mu + 1/r > 0$, we get

$$\begin{aligned}
&\sup_{y>0} \sup_{\gamma>\log_p(y/C_2)^{1/n\mu}} y|B_\gamma|_H^{-\mu-1/r}|\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (y/C_2)^{1/n\mu}\}|^{1/r} \\
&\le \sup_{y>0} \sup_{\gamma>\log_p(y/C_2)^{1/n\mu}} yp^{-\gamma n(\mu+1/r)}(y/C_2)^{1/r\mu} \\
&= C_2 \\
&\le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Therefore,

$$\|H^p f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}. \tag{10}$$

Conversely, to prove that the constant 1 is optimal, consider

$$f_0(\mathbf{x}) = \chi_{\{|\mathbf{x}|_p \le 1\}}(\mathbf{x}),$$

then,

$$\|f_0\|_{\dot{B}^{q,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} \chi_{\{|\mathbf{x}|_p\le 1\}}(\mathbf{x})d\mathbf{x}\right)^{1/r}.$$

If $\gamma < 0$, then

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma<0}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} d\mathbf{x}\right)^{1/r} = \sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma<0}} p^{-n\gamma\mu} = 1,$$

since $\mu < 0$. If $\gamma \ge 0$, then using the condition that $\mu + 1/r > 0$, we have

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma\ge 0}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_0} d\mathbf{x}\right)^{1/r} = \sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma\ge 0}} p^{-n\gamma(\mu+1/r)} = 1.$$

Therefore,

$$\|f_0\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = 1.$$

Moreover,

$$H^p f_0(\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}|_p \le 1; \\ |\mathbf{x}|_p^{-n}, & |\mathbf{x}|_p > 1, \end{cases}$$

which implies that $|H^p f_0(\mathbf{x})| \le 1$. Next, in order to construct weak central Morrey norm we divide our analysis into following two cases:
Case 1. When $\gamma \le 0$, then

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\le 1} y|\{\mathbf{x} \in B_\gamma : 1 > y\}|^{1/r} = p^{n\gamma/r},$$

and

$$\|H^p f_0\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\le 0} |B_\gamma|_H^{-\mu-1/r}\|f\|_{WL^r(B_\gamma)} = \sup_{\gamma\le 0} p^{-n\gamma\mu} = 1 = \|f_0\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.$$

Case 2. When $\gamma > 0$, we have

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\leq 1} y|\{\mathbf{x} \in B_{\mathbf{0}} : 1 > y\} \cup \{1 < |\mathbf{x}|_p < p^\gamma : |\mathbf{x}|_p^{-n} > y\}|^{1/r}.$$

For further analysis, this case is further divided into the following subcases:
Case 2(a). If $1 < \gamma < \log_p y^{-1/n}$, then

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\leq 1} y\{1 + p^{n\gamma} - 1\}^{1/r} = \sup_{0<t\leq 1} tp^{n\gamma/r}.$$

Case 2(b). If $1 < \log_p y^{-1/n} < \gamma$, then:

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\leq 1} y(1 + y^{-1} - 1)^{1/r} = \sup_{0<y\leq 1} y^{1-1/r}.$$

Now, for $1 \leq r < \infty$ and $-1/r \leq \mu < 0$, from case 2(a) and 2(b), we obtain

$$\begin{aligned}
&\|H^p f_0\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \\
&= \max\left\{\sup_{0<y\leq 1} \sup_{1<\gamma\leq \log_p(1/y)^{-1/n}} yp^{-n\gamma\mu}, \sup_{0<y\leq 1} \sup_{1<\log_p(1/y)^{-1/n}<\gamma} y^{1-1/r}p^{-n\gamma(\mu+1/r)}\right\} \\
&= \max\left\{\sup_{0<y\leq 1} t^{1+\mu}, \sup_{0<y\leq 1} y^{1+\mu}\right\} \\
&= 1 = \|f_0\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}. \qquad (11)
\end{aligned}$$

Finally, using (10) and (11), we arrive at:

$$\|H\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)\to W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = 1.$$

□

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement The authors are thankful to the referees for making valuable suggestions leading to the better presentations of this paper.

References

[1] Aref'eva, I. Ya., Dragovich, B., Frampton, P. H., Volovich, I. V., The wave function of the universe and p-adic gravity, *Internat. J. Modern Phys. A,* 6(24) (1991), 4341–4358. https://doi.org/10.1142/S0217751X91002094

[2] Asim, M, Hussain, A., Sarfraz, N., Weighted variable Morrey–Herz estimates for fractional Hardy operators, *J. Ineq. Appl.,* 2022(2) (2022), 12 pp. https://doi.org/10.1186/s13660-021-02739-z

[3] Avetisov, V. A., Bikulov, A. H., Kozyrev, S. V., Application of p-adic analysis to models of breaking of replica symmetry, *J. Phys. A: Math. Gen.,* 32(50) (1999), 8785–8791. https://doi.org/10.1088/0305-4470/32/50/301

[4] Avetisov, V. A., Bikulov, A. H., Kozyrev, S. V., Osipov, V. A., p-adic models of ultrametric diffusion constrained by hierarchical energy landscaapes, *J. Phys. A:Math.Gen.,* 35(2) (2002), 177–189. https://doi.org/10.1088/0305-4470/35/2/301

[5] Bandaliyev, R. A., Volosivets, S. S., Hausdorff operator on weighted Lebesgue and grand Lebesgue p-adic spaces, *p-Adic Numbers Ultrametric Anal. Appl.,* 11(2) (2019), 114–122. https://doi.org/10.1134/S207004661902002X

[6] Bliss, G. A., An integral inequality, *J. London Math. Soc.,* 5(1) (1930), 40–46. https://doi.org/10.1112/jlms/s1-5.1.40

[7] Brekke, L., Freund, Peter, G. O., p-adic numbers in Physics, *Phys. Rep.,* 233(1) (1993), 1–66. https://doi.org/10.1016/0370-1573(93)90043-D

[8] Christ, M., Grafakos, L., Best constants for two nonconvolution inequalities, *Proc. Amer. Math. Soc.*, 123(6) (1995), 1687–1693. https://doi.org/10.1090/S0002-9939-1995-1239796-6

[9] Chuong, N. M., Egorov, Yu. V., Khrennikov, A., Meyer, Y., Mumford, D., Harmonic, wavelet and p-adic analysis. Including papers from the International Summer School held at Quy Nhon University of Vietnam, Quy Nhon, June 10–15, 2005. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. x+381 pp. ISBN: 978-981-270-549-5; 981-270-549-X. https://doi.org/10.1142/6373

[10] Chuong, N. M., Hung, H. D., Maximal functions and weighted norm inequalities on local fields, *Appl. Comput. Harmon. Anal.*, 29(3) (2010), 272–286. https://doi.org/10.1016/j.acha.2009.11.002

[11] Dubischar, D., Gundlach, V.M., Steinkamp, O., Khrennikov, A., A p-adic model for the process of thinking disturbed by physiological and information noise, *J. Theor. Biol.*, 197(4) (1999), 451–467. DOI: 10.1006/jtbi.1998.0887

[12] Faris, W. G., Weak Lebesgue spaces and quantum mechanical binding, *Duke Math. J.*, 43(2) (1976), 365–373. DOI: 10.1215/S0012-7094-76-04332-5

[13] Fu, Z. W., Grafakos, L., Lu, S. Z., Zhao, F. Y., Sharp bounds for m-linear Hardy and Hilbert operators, *Houston. J. Math.,* 38(1) (2012), 225–244.

[14] Fu, Z. W., Wu, Q. Y., Lu, S. Z., Sharp estimates of p-adic Hardy and Hardy-Littlewood-Pólya operators, *Acta Math. Sin. (Engl. Ser.)*, 29(1) (2013), 137–150. https://doi.org/10.1007/s10114-012-0695-x

[15] Gao, G., Zhao, F. Y., Sharp weak bounds for Hausdorff operators, *Anal Math.*, 41(3) (2015), 163–173. https://doi.org/10.1007/s10476-015-0204-4

[16] Gao, G., Hu, X., Zhang, C., Sharp weak estimates for Hardy-type operators, *Ann. Funct. Anal.,* 7(3) (2016), 421–433. https://doi.org/10.1215/20088752-3605447

[17] Gao, G., Zhong, Y., Some estimates of Hardy operators and their commutators on Morrey-Herz spaces, *J. Math. Inequal.,* 11(1) (2017), 49–58. DOI: 10.7153/jmi-11-05

[18] Hardy, G. H., Note on a theorem of Hilbert, *Math. Z.*, 6(3-4) (1920), 314–317. https://doi.org/10.1007/BF01199965

[19] Ho, K.-P., Hardy's inequality on Hardy–Morrey spaces, *Georg. Math. J.*, 26(3) (2019), 405–413. https://doi.org/10.1515/gmj-2017-0046

[20] Hussain A., Asim, M., Aslam, M., Jarad, F., Commutators of the fractional Hardy operator on weighted variable Herz-Morrey spaces, *J. Funct. Spaces,* (2021), Art. ID 9705250, 10 pp. https://doi.org/10.1155/2021/9705250

[21] Hussain, A., Sarfraz, N., The Hausdorff operator on weighted p-adic Morrey and Herz type spaces, *p-Adic Numbers Ultrametric Anal. Appl.*, 11(2) (2019), 151–162. https://doi.org/10.1134/S2070046619020055

[22] Hussain, A., Sarfraz, N., Optimal weak type estimates for p-adic Hardy operators, *p-Adic Numbers Ultrametric Anal. Appl.,* 12(1) (2020), 29–38. https://doi.org/10.1134/S2070046620010033
[23] Hussain, A., Ahmed, M., Weak and strong type estimates for the commutators of Hausdorff operator, *Math. Inequal. Appl.,* 20(1) (2017), 49–56. DOI: 10.7153/mia-20-04
[24] Hussain, A., Gao, G., Multidimensional Hausdorff operators and commutators on Herz-type spaces, *J. Inequal. Appl.*, 2013(594) (2013), 12 pp. https://doi.org/10.1186/1029-242X-2013-594
[25] Liu, R.H., Zhou, J., Sharp estimates for the p-adic Hardy type operator on higher-dimensional product spaces, *J. Inequal. Appl.*, 2017(219) (2017), 13 pp. https://doi.org/10.1186/s13660-017-1491-z
[26] Parisi, G., Sourlas, N., p-adic numbers and replica symmetry, *Eur. Phys. J. B Condens. Matter Phys.*, 14(3) (2000), 535–542. https://doi.org/10.1007/s100510051063
[27] Persson, L.-E., Samko, S. G., A note on the best constants in some hardy inequalities, *J. Math. Inequal.*, 9(2) (2015), 437–447. DOI:10.7153/jmi-09-37
[28] Sarfraz, N., Gürbüz, F., Weak and strong boundedness for p-adic fractional Hausdorff operator and its commutator, *Int. J. Nonlinear Sci. Numer. Simul.*, 2021 (2021), 12 pp. https://dx.doi.org/10.1515/ijnsns-2020-0290
[29] Sarfraz, N., Aslam, M., Some weighted estimates for the commutators of p-adic Hardy operator on two weighted p-adic Herz-type spaces. *AIMS Math.,* 6(9) (2021), 9633–9646. DOI:10.3934/math.2021561
[30] Vladimirov, V. S., Volovich, I. V., Zelenov, E. I., p-adic Analysis and Mathematical Physics, Series on Soviet and East European Mathematics, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 1994, xx+319 pp. ISBN: 981-02-0880-4. https://doi.org/10.1142/1581
[31] Vladimirov, V. S., Volovich, I. V., p-adic quantum mechanics, *Commun. Math. Phy.*, 123 (1989), 659–676. https://doi.org/10.1007/BF01218590
[32] Volosivets, S. S., Weak and strong estimates for rough Hausdorff type operator defined on p-adic linear space, *p-Adic Numbers Ultrametric Anal. Appl.*, 9(3) (2017), 236–241. https://doi.org/10.1134/S2070046617030062
[33] Wu, Q.Y., Boundedness for commutators of fractional p-adic Hardy operator, *J. Inequal. Appl.*, 2012(293) (2012), 12pp. https://doi.org/10.1186/1029-242X-2012-293
[34] Wu, Q. Y., Mi, L., Fu, Z. W., Boundedness of p-adic Hardy operators and their commutators on p-adic central Morrey and BMO spaces, *J. Funct. Spaces Appl.,* (2013), Art. ID 359193, 10 pp. https://doi.org/10.1155/2013/359193
[35] Wu, Q. Y., Fu, Z. W., Hardy-Littlewood-Sobolev inequalities on p-adic central Morrey spaces, *J. Funct. Spaces*, (2015), Art. ID 419532, 7 pp. https://doi.org/10.1155/2015/419532
[36] Xiao, J., L^p and BMO bounds of weighted Hardy-Littlewood averages, *J. Math. Anal. Appl.*, 262(2) (2001), 660–666. https://doi.org/10.1006/jmaa.2001.7594
[37] Yu, H., Li, J., Sharp weak bounds for n-dimensional fractional Hardy operators, *Front. Math. China*, 13(2) (2018), 449–457. https://doi.org/10.1007/s11464-018-0685-0
[38] Zhao, F. Y., Lu, S. Z., The best bound for n-dimensional fractional Hardy operator, *Math. Inequal Appl.*, 18(1) (2015), 233–240. DOI: 10.7153/mia-18-17
[39] Zhao, F. Y., Fu, Z. W., Lu, S. Z., Endpoint estimates for n-dimensional Hardy operators and their commutators, *Sci. China Math.*, 55(10) (2012), 1977–1990. https://doi.org/10.1007/s11425-012-4465-0

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 930–943 (2022)
DOI:10.31801/cfsuasmas.1014919
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 26, 2021; Accepted: April 13, 2022

DOMINATOR SEMI STRONG COLOR PARTITION IN GRAPHS

Praba VENKATRENGAN[1], Swaminathan VENKATASUBRAMANIAN[2]
and Raman SUNDARESWARAN[3]

[1]Department of Mathematics, Shrimati Indira Gandhi College, Tiruchirappalli, Tamilnadu, INDIA
[2]Ramanujan Research Center in Mathematics, Saraswathi Narayanan College, Madurai, Tamilnadu, INDIA
[3]Sri Sivasubramaniya Nadar College of Engineering, Chennai, Madurai, Tamilnadu, INDIA

Abstract. Let $G = (V, E)$ be a simple graph. A subset S is said to be Semi-Strong if for every vertex v in V, $|N(v) \cap S| \leq 1$, or no two vertices of S have the same neighbour in V, that is, no two vertices of S are joined by a path of length two in V. The minimum cardinality of a semi-strong partition of G is called the semi-strong chromatic number of G and is denoted by $\chi_s G$. A proper colour partition is called a dominator colour partition if every vertex dominates some colour class, that is , every vertex is adjacent with every element of some colour class. In this paper, instead of proper colour partition, semi-strong colour partition is considered and every vertex is adjacent to some class of the semi-strong colour partition.Several interesting results are obtained.

1. Introduction

Let $G = (V, E)$ be a finite, undirected graph. We follow standard definitions of graph theory [2, 8]. A proper vertex coloring of a graph is defined as coloring the vertices of a graph such that no two adjacent vertices are colored using same color. A subset S of a graph $G = (V, E)$ is said to be a dominting set if every vertex not in S is adjacent to at least one vertex of $V - S$. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G [9, 10]. S. M. Hedetniemi [11, 12] introduced and discussed the concept of dominator coloring and dominator partition of graphs. S.Arumugam et.al. discussed further in dominator coloring in graphs [1]. The combination of the two most important fields in graph

2020 *Mathematics Subject Classification.* 05C69, 05C15.
Keywords. Dominator coloring, semi strong color partition, semi-strong coloring.
prabasigc@yahoo.co.in; 0000-0003-0171-3777
swaminathan.sulanesri@gmail.com; 0000-0002-5840-2040
sundareswaranr@ssn.edu.in-Corresponding author; 0000-0002-0439-695X

theory namely, Coloring and domination have a lot of research results. A dominator coloring of a graph G is a proper coloring, such that every vertex of G dominates at least one color class (possibly its own class). Gera et. al. [6] defined dominator colouring in a graph G as a proper colour partition in which every vertex dominates some color class. The dominator chromatic number of G, denoted by $\chi_d(G)$, is the minimum number of colors among all dominator colorings of G. Gera researched further in [7] on dominator coloring and safe clique partitions. Kazemi proposed the concept of total dominator coloring in graphs and studied its properties [15]. A proper coloring, such that each vertex of the graph is adjacent to every vertex of some (other) color class. For more results on the total dominator coloring, refer to [14,16]. M. Chellali and F. Maffray discussed Dominator colorings in some classes of graphs [4]. In 2015, Merouane et al. [17] proposed the dominated coloring which is defined as a proper coloring such that every color class is dominated by at least one vertex. The dominated chromatic number of G, denoted by $\chi_{dom}(G)$, is the minimum number of colors among all dominated colorings of G. For comprehensive results of coloring and domination in graphs, color class domination in graphs introudced and studied in detail. refer to [5, 20, 21]. As a generalization of strong set introduced by Claude Berge [3], E.Sampathkumar defined semi-strong sets [18] in a graph. In a simple graph G, a subset S of the vertex set $V(G)$ of G is called a semi-strong set of G if $|N[v] \cap S| \leq 1$ for v in $V(G)$. E.Sampathkumar also introduced Chromatic partition of a graph [19] and studied its properties. Also, G. Jothilakshmi et al studied (k,r) - Semi Strong Chromatic Number of a Graph [13]. Instead of proper color partition, semi-strong partition [18] of $V(G)$ is considered and domination property that every vertex dominates semi-strong color class is added. The minimum cardinality of such a partition is found for some classes of graphsand bounds are obtained. Interesting results in this new concepts are derived.

Definition 1. *A subset S of $V(G)$ is called a maximal semi-strong set of G if S is semi-strong and no proper super set of S is semi-strong. The maximum cardinality of a maximal semi-strong set of G is called semi-strong number of G and is denoted by $ss(G)$.*

Definition 2. *A **dominator coloring** of a graph G is a proper coloring in which each vertex of the graph dominates every vertex of some color class.*

Definition 3. *A semi-strong coloring of G is called a* **dominator semi-strong color partition** *of G if every vertex of G dominates an element of the partition. The minimum cardinality of such a partition is called the* **dominator semi-strong color partition number** *of G and is denoted by $\chi_s^d(G)$.*

Since the trivial partition is a semi-strong coloring of G, the existence of dominator semi-strong color partition is guaranteed in any graph.

2. $\chi_s^d(G)$ for Some Well-Known Graphs

Observation 1. *(i)* $\chi_s^d(K_n) = \chi_d(K_n) = n$.
(ia) $\chi_s^d(K_n - e) = n$ *(since $K_n - e$ has a full degree vertex).*
(ii) $\chi_s^d(K_{1,n}) = n + 1$, $\chi_s(K_{1,n}) + \gamma(K_{1,n}) = n + 1$.
(iii) $\chi_d(K_{m,n}) = 2 < \chi_s^d(K_{m,n})$ *if* $m \leq n$ *and* $n \geq 3$.

Remark 1. *Let* $\Pi = \{V_1, V_2, \ldots, V_k\}$ *be a dominator semi-strong color partition of* G*. A vertex* $u \in V$ *can dominate* V_i *if and only if* $|V_i| = 1$.

Theorem 1. *For any Path* P_n, $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, $n \geq 2$.

Proof. Let P_n be a path on n vertices.

Case 1: $n = 4k$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_4\}$, $V_{k+2} = \{v_8\}$, $\ldots$, $V_{2k+1} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k + 1 = \lceil \frac{n}{2} \rceil + 1$.

Let Π_1 be a χ_s^d-partition of P_n. The maximum cardinality of an element of Π_1 is at most $2k$. There are at least $2k$ singletons to dominate $4k$ elements, since no single element can dominate two elements of a set which are at a distance 2. Therefore $|\Pi| \geq 2k + 1$. Therefore $\chi_s^d(P_{4k}) = 2k + 1 = \lceil \frac{n}{2} \rceil + 1$.

Case 2: Let $n = 4k + 1$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_{4k+1}\}$, $V_{k+2} = \{v_4\}$, $V_{k+3} = \{v_8\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k + 2 = \lceil \frac{n}{2} \rceil + 1$.

Arguing as in case 1, $\chi_s^d(P_{4k+1}) \geq 2k+2 = \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, where $n = 4k + 1$.

Case 3: Let $n = 4k + 2$, $k \geq 0$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v_{4k+2}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_{4k+1}\}$, $V_{k+2} = \{v_4\}$, $V_{k+3} = \{v_8\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k+2 = \lceil \frac{n}{2} \rceil + 1$.

Arguing as in case 1, $\chi_s^d(P_{4k+2}) \geq 2k+2 = \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, where $n = 4k + 2$.

Case 4: Let $n = 4k + 3$, $k \geq 0$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}, V_{2k+3}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v_{4k+2}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_k = \{v_{4k-3}\}$, $V_{k+1} = \{v_{4k-3}\}$, $V_{k+2} = \{v_{4k+1}\}$, $V_{k+3} = \{v_4\}$, $V_{k+4} = \{v_8\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$, $V_{2k+3} = \{v_{4k+3}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(P_n) \leq 2k + 3 = \lceil \frac{n}{2} \rceil + 1$.

Arguing as in case 1, $\chi_s^d(P_{4k+3}) \geq 2k+3 = \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(P_n) = \lceil \frac{n}{2} \rceil + 1$, where $n = 4k + 3$. □

Theorem 2. $\chi_s^d(C_n) = \lceil \frac{n}{2} \rceil + 1$, $n \geq 3$.

Proof. Let C_n be a cycle on n vertices.

Case 1: $n = 4k$, $k \geq 1$

Let $V(C_n) = \{v_1, v_2, \ldots, v_{4k}\}$. Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}, V_2 = \{v_1\}, V_3 = \{v_5\}, \ldots, V_{k+1} = \{v_{4k-3}\}, V_{k+2} = \{v_4\}$, $V_{k+3} = \{v_8\}$, $\ldots$, $V_{2k+1} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of C_n. Therefore $\chi_s^d(C_n) \leq |\Pi| = 2k + 1 = \frac{4k}{2} + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k$ singletons and no single element can dominate a two element set which are at a distance 2. Therefore $\chi_s^d(C_{4k}) \geq \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(C_{4k}) = \lceil \frac{n}{2} \rceil + 1$.

Case 2: Let $n = 4k + 1$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_{k+1} = \{v_{4k-3}\}$, $V_{k+2} = \{v_{4k+1}\}$, $V_{k+3} = \{v_4\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of P_n. Therefore $\chi_s^d(C_{4k+1}) \leq |\Pi| = 2k + 2 = \lceil \frac{4k+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k$ singletons and no single element can dominate a two element set which are at a distance 2. Therefore $\chi_s^d(C_{4k+1}) \geq \lceil \frac{n}{2} \rceil + 1$ and hence $\chi_s^d(C_{4k+1}) = \lceil \frac{n}{2} \rceil + 1$.

Case 3: Let $n = 4k + 2$, $k \geq 1$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}$, $V_2 = \{v_{4k+1}, v_{4k+2}\}, V_3 = \{v_1\}, V_4 = \{v_5\}, \ldots, V_{k+2} = \{v_{4k-3}\}, V_{k+3} = \{v_4\}, V_{k+4} = \{v_8\}, \ldots, V_{2k+2} = \{v_{4k}\}$. Then Π is a dominator semi-strong color partition of C_n. Therefore $\chi_s^d(C_{4k+2}) \leq |\Pi| = 2k + 2 + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k$ singletons and no single element can dominate a two element set which are at a distance 2. Any doubleton must consist of consecutive vertices. Therefore $\chi_s^d(C_{4k+2}) \geq \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(C_{4k+2}) = \lceil \frac{n}{2} \rceil + 1$.

Case 4: Let $n = 4k + 3$, $k \geq 0$

Let $\Pi = \{V_1, V_2, \ldots, V_{2k}, V_{2k+1}, V_{2k+2}, V_{2k+3}\}$ where $V_1 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-1}, v_{4k+2}\}$, $V_2 = \{v_1\}$, $V_3 = \{v_5\}$, $\ldots$, $V_{k+1} = \{v_{4k-3}\}$, $V_{k+2} = \{v_{4k+1}\}$, $V_{k+3} = \{v_4\}$, $\ldots$, $V_{2k+2} = \{v_{4k}\}$, $V_{2k+3} = \{v_{4k+3}\}$. Then Π is a dominator semi-strong color partition of C_n. Therefore $\chi_s^d(C_{4k+3}) \leq |\Pi| = 2k+3 = \lceil \frac{4k+3}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1$.

There are at least $2k + 1$ singletons and no single element can dominate a two element set which are at a distance 2. Any doubleton set must consist of consecutive vertices. Therefore $\chi_s^d(C_{4k+3}) \geq \lceil \frac{n}{2} \rceil + 1$. Therefore $\chi_s^d(C_{4k+3}) = \lceil \frac{n}{2} \rceil + 1$. □

Theorem 3. *For Complete bi-partite graph* $K_{m,n}, \chi_s^d(K_{m,n}) = max\{m, n\} + 1$.

Proof. Let V_1, V_2 be the partite sets of $K_{m,n}$.

Case 1: Let $m < n$.
Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$.
Let $\Pi = \{\{u_1, v_1\}, \ldots, \{u_{m-1}, v_{m-1}\}, \{u_m\}, \{v_m\}, \ldots, \{v_n\}\}$. Then each of $v_1, v_2, \ldots, v_n$ dominates $\{u_m\}$, and each of $u_1, u_2, \ldots, u_{m-1}$ dominates $\{v_n\}$. Therefore Π is a dominator semi-strong color partition of $K_{m,n}$.
Therefore $\chi_s^d(K_{m,n}) \leq |\Pi| = m + n - (m-1) = n + 1$.

No two elements of V_1 can belong to an element of Π. Also no two elements of V_2 can belong to an element of Π. Any element of V_1 dominates all elements of V_2. So is the case with V_2. Therefore Π must consist of at least one singleton from V_1 and one singletons from V_2. Therefore $\chi_s^d(K_{m,n}) \geq m - 1 + 2 + (n - m) = n + 1$.
Therefore $\chi_s^d(K_{m,n}) = n + 1 = max\{m, n\} + 1$.
Case 2: Let $m = n$

Let $\Pi = \{\{u_1, v_1\}, \ldots, \{u_{m-1}, v_{m-1}\}, \{u_m\}, \ldots, \{v_n\}\}$. Proceeding as in case 1, $\chi_s^d(K_{m,n}) = m + 1 = max\{m, n\} + 1$. □

Corollary 1. $\chi_s^d(K_{1,n}) = n + 1$.

Theorem 4. $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) = m + max\{a_1, a_2, \ldots, a_m\}$.

Proof. Let $a_1 \leq a_2 \leq \ldots \leq a_m$. Let $V(K_m(a_1, a_2, \ldots, a_m)) = \{u_1, u_2, \ldots, u_m, v_{1,1}, v_{1,2}, \ldots, v_{1,a_1}, \ldots, v_{m,1}, \ldots, v_{m,a_m}\}$. Let $\Pi = \{\{u_1\}, \ldots, \{u_m\}, \{v_{1,1}, v_{2,1}, \ldots, v_{m,1}\}, \ldots, \{v_{1,a_1}, v_{2,a_1}, \ldots, v_{m,a_1}\}, \{v_{2,a_2}, v_{3,a_2}, \ldots, v_{m,a_2}\}, \ldots, \{v_{m,a_m}\}\}$. Then $|\Pi| = m + a_m = m + max\{a_1, a_2, \ldots, a_m\}$.

Therefore $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) \leq m + max\{a_1, a_2, \ldots, a_m\}$. Any χ_s^d-partition must contain $u_1, u_2, \ldots, u_m$ as singletons for dominating the pendent vertices. Further no two pendent vertices at any u_i, $1 \leq i \leq m$ can belong to an element of the partition. Therefore $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) \geq m + max\{a_1, a_2, \ldots, a_m\}$. Therefore $\chi_s^d(K_m(a_1, a_2, \ldots, a_m)) = m + max\{a_1, a_2, \ldots, a_m\}$. □

Let G be the graph shown in Figure 1

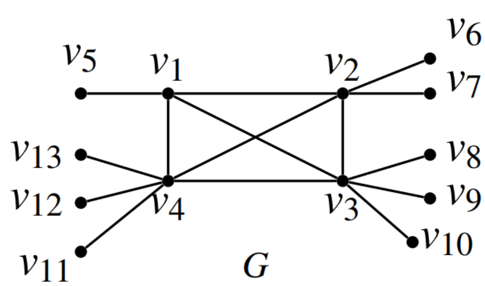

FIGURE 1. $G = K_4(1, 2, 3, 3)$ with $\chi_s^d(G) = 7$

Let $\Pi = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5, v_6, v_8, v_{11}\}, \{v_7, v_9, v_{12}\}, \{v_{10}, v_{13}\}\}$. Then Π is a χ_s^d-partition of G. Therefore $\chi_s^d(G) = |\Pi| = 4 + 3 = 7$.

Theorem 5. $\chi_s^d(K_{a_1,a_2,\ldots,a_m}) = a_1 + a_2 + \ldots + a_m$ *if* $m \geq 3$.

Proof. Let $m \geq 3$. Then any vertex of $K_{a_1,a_2,\ldots,a_m}$ is a common vertex of two vertices. Hence no two vertices can be included in an element of a χ_s^d-partition. Hence $\chi_s^d(K_{a_1,a_2,\ldots,a_m}) = a_1 + a_2 + \ldots + a_m$ if $m \geq 3$. □

Theorem 6. $\chi_s^d(P) = 7$ *where* P *is the Petersen graph.*

Proof. Consider the graph in Figure 2. Let $V(P) = \{v_1, v_2, \ldots, v_{10}\}$.
Let $\Pi = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}, \{v_6, v_9\}, \{v_7\}, \{v_8\}, \{v_{10}\}\}$. Then Π is a dominator semi-strong color partition of P. Therefore $\chi_s^d(P) \leq 7$.

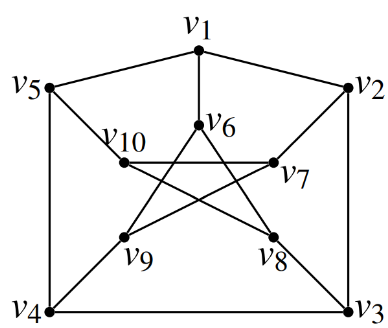

FIGURE 2. Petersen Graph

In any χ_s^d-partition of P, no three-element set can appear. Since for any three element set of P, there exists a vertex which is adjacent to two of the element of that set. Any three 2 element sets must have three singletons for domination. Hence the remaining one element must appear as a singleton. Therefore $\chi_s^d(P) \geq 7$. Therefore $\chi_s^d(P) = 7$. □

Remark 2. *(i)* $1 \leq \chi_s^d(G) \leq n$.
(ii) $\chi_s^d(G) = 1$ *if and only if* $G = K_1$.

Observation 2. *Let* G *be a graph with full degree vertex. Then* $\chi_s^d(G) = |V(G)|$.

Proof. Let Π be a χ_s^d-partition of G. Let $V_1 \in \Pi$. If $|V_1| \geq 2$, then any two points of V_1 are adjacent with full degree vertex, a contradiction. Therefore $|V_1| = 1$. Therefore $\chi_s^d(G) = |V(G)|$. □

Corollary 2. $\chi_s^d(W_n) = n$.

Corollary 3. $\chi_s^d(F_n) = n$.

3. Main Results

Theorem 7. *$max\{\chi_s(G), \gamma(G)\} \leq \chi_s^d(G) \leq \chi_s(G) + \gamma(G)$.*

Proof. Since any χ_s^d-partition of G is a χ_s-partition of G, $\chi_s(G) \leq \chi_s^d(G)$. Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ where $k = \chi_s^d(G)$ be a χ_s^d-partition of G. Let $x_i \in V_i$, $1 \leq i \leq k$. Let $S = \{x_1, x_2, \ldots, x_k\}$. Let $v \in V - S$. Then v dominates some color class, say V_i. Therefore v is adjacent with x_i. Therefore $\{x_1, x_2, \ldots, x_k\}$ dominates G. That is, S is a dominating set of G. That is, $\gamma(G) \leq |S| = k = \chi_s^d(G)$. Therefore $max\{\chi_s(G), \gamma(G)\} \leq \chi_s^d(G)$.

Let $\Pi = \{V_1, V_2, \ldots, V_k\}$ be a χ_s-coloring of G. Assign colors $\chi_s(G)+1, \ldots, \chi_s(G) + \gamma(G)$ to the vertices of a minimum dominating set of G, leaving the rest of the vertices colored as before. Then the resulting partition is a dominator semi-strong color partition of G. Therefore, $\chi_s^d(G) \leq |\Pi| + \gamma(G) = \chi_s(G) + \gamma(G)$. □

Remark 3. *The set S need not be a minimum dominating set. For example, when $G = P_6$, $\chi_s^d(G) = 4$. But $\gamma(P_6) = 2$.*

Theorem 8. *Let a, b be positive integers with $a \leq b$. Then there exists a graph G such that $\chi_d(G) = a$ and $\chi_s^d(G) = b$.*

Proof. When $a = b$, $\chi_d(K_a) = \chi_s^d(K_a) = a$. Let $a < b$. Let $G = K_{a_1, a_2, \ldots, a_k}$ where $k = a$. Then $\chi_d(G) = a$. Choose $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \ldots + a_k = b$. Then $\chi_s^d(G) = b$. □

Theorem 9. *$\chi_s^d(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $\chi_s^d(G) = 2$. Suppose $\chi_s^d(G) = 2$. Let $\Pi = \{V_1, V_2\}$ be a χ_s^d-partition of G. Suppose $|V_1| \geq 2$. Then any vertex of V_2 dominates V_1 unless $|V_2| = 1$. If $|V_2| > 1$, then it is a contradiction. Therefore $|V_2| = 1$. Similarly, $|V_1| = 1$. Therefore $G = K_2$. □

Corollary 4. *Suppose T is a tree of order $n \geq 2$. Then $\chi(T) = 2$. $\chi_s^d(T) = \chi(T)$ if and only if $\chi_s^d(T) = 2$. That is if and only if $G = K_2$.*

Theorem 10. *Let G be a connected unicyclic graph. Then $\chi_s^d(G) = \chi(G)$ if and only if $G = C_3$.*

Proof. If G is a cycle, then $\chi_s^d(G) = \chi(G)$ if and only if $G = C_3$. Suppose G contains C_{2n}. Then $\chi(G) = 2$, but $\chi_s^d(G) \geq 3$, a contradiction. Therefore G contains an odd cycle C_{2n+1}. Then $\chi(G) = 3$. If there exists a path attached with a vertex of C_{2n+1}, then $\chi_s^d(G) \geq 4$, a contradiction. Therefore G is a cycle. Since $\chi_s^d(G) = \chi(G)$, $G = C_3$. □

Theorem 11. *Let G be a connected graph. Then $\chi_s^d(G) = n$ if and only if either G has a full degree vertex or $N(G) = K_n$.*

Proof. Let $\chi_s^d(G) = n$. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Then $\Pi = \{\{u_1\}, \{u_2\}, \ldots, \{u_n\}\}$ is a χ_s^d-partition of G. Let $diam(G) = k \geq 3$. Let u and v be the end vertices of a diametrical path. Let $u = u_1, u_2, \ldots, u_{k+1} = v$. Then u and v have no common adjacent vertex. Therefore $\Pi_1 = \{\{u, v\}, \ldots, \{u_n\}\}$. Then u dominates $\{u_2\}$ and v dominates $\{u_k\}$. Also $\{u, v\}$ is dominated by a single vertex. Therefore Π_1 is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq n - 1$, a contradiction. Therefore $diam(G) \leq 2$.

Suppose u_1 and u_2 are adjacent and u_1u_2 is not the edge of a triangle. Then $\{u_1, u_2\}$ can be taken as an element of a dominator semi-strong color partition of G with all other vertices as singletons. If u_1 is adjacent with some u_i, $i \geq 3$ and u_2 is adjacent with some u_j, $j \neq \{1, 2\}$, then $\chi_s^d(G) \leq n - 1$, a contradiction. Therefore if $|V(G)| \geq 4$ and $diam(G) \leq 2$ and u_1u_2 is an edge such that u_1 and u_2 have separate adjacent vertices, then u_1u_2 is the edge of a triangle. In such case, $N(G) = K_n$. Suppose u_1 is adjacent with some vertex u_3 and u_2 is not adjacent with any vertex of G other than u_1. Suppose u_3 is adjacent with some vertex u_4. If u_1 is not adjacent with u_4, then $\Pi_2 = \{\{u_1, u_3\}, \{u_2\}, \{u_4\}, \ldots, \{u_n\}\}$ is a dominator semi-strong color partition of G, a contradiction. If u_3 is adjacent with u_1, then u_4 is also adjacent with u_1. Therefore G is a connected graph with a full degree vertex.

Suppose G has no full degree vertex. Then the case that only one of u_1, u_2 which are adjacent, has some other adjacent vertex does not hold. Therefore both u_1 and u_2 have different adjacent vertices. Therefore u_1u_2 is the edge of a triangle. Therefore $diam(G) \leq 2$ and when u_1u_2 is an edge, then u_1u_2 is the edge of a triangle. Therefore $N(G) = K_n$. The converse is obvious. □

Remark 4. *Let G be the graph given in Figure 3.*

Then $G = N(G)$, $N(G)$ is not complete and G has no full degree vertex. Therefore $\chi_s^d(G) = 4$ and $\chi_s(G) = 3$.

Remark 5. *Let G be the graph shown in Figure 4.*

Then $N(G) = K_5 - \{e\}$. G has a full degree vertex and hence $\chi_s^d(G) = 5$ eventhough $N(G)$ is not complete. Hence $\chi_s(G) = 4$ and $\chi_s^d(G) = 5$.

Remark 6. *Let G be a complete multipartite graph $K_{a_1, a_2, \ldots, a_n}$, $n \geq 3$. Then G has no full degree vertex. $\chi_s^d(G) = n$ and hence $N(G) = K_n$.*

Observation 3. *Let G be a cycle C_n with pendent vertex attached with exactly one vertex of C_n. Then* $\chi_s^d(G) = \begin{cases} \chi_s^d(C_n) + 1 & \text{if } n \not\equiv 1 \ (mod\ 4) \\ \chi_s^d(C_n) & \text{otherwise} \end{cases}$

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Let u_{n+1} be a pendent vertex attached with u_1.

Case 1: Let $n = 4k$.

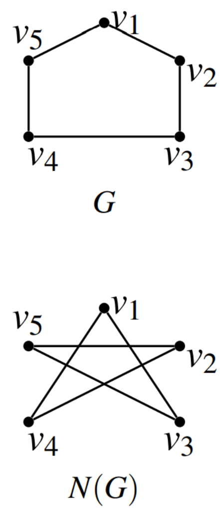

FIGURE 3. $G = N(G) = C_5$

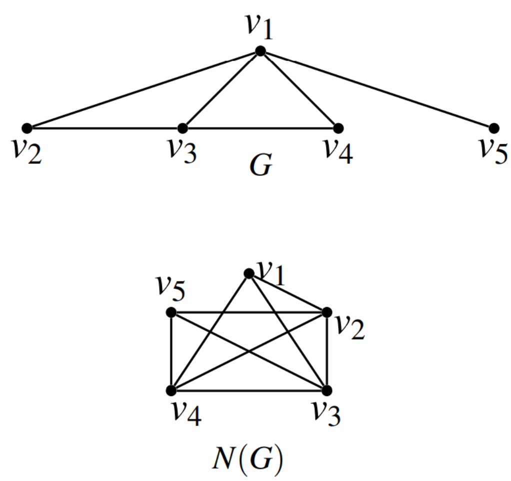

FIGURE 4. G and $N(G)$

Let $\Pi = \{\{u_{4k+1}, u_3, u_4, u_7, u_8, \ldots, u_{4k-5}, u_{4k-4}, u_{4k-1}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k-3}\}, \{u_{4k-2}\}, \{u_{4k}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq 1 + 2k + 1 = 2k + 2 = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

There are at least $2k$ singletons and no single element can dominate a 2 element set whose elements are at distance 2. Also for the pendent vertex either it appears as a singleton or its support appears as a singleton. Therefore $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil + 2$. Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Case 2: Let $n = 4k + 1$.

Let $\Pi = \{\{u_{4k+2}, u_3, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k+1}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq 1 + k + 1 + k = 2k + 2 = \lceil \frac{n}{2} \rceil + 1 = \chi_s^d(C_n)$.

If $\chi_s^d(G) < \lceil \frac{n}{2} \rceil + 1$, then removing the pendent vertex we get that $\chi_s^d(C_n) < \lceil \frac{n}{2} \rceil + 1$, a contradiction. Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 1 = \chi_s^d(C_n)$.

Case 3: Let $n = 4k + 2$.

Let $\Pi = \{\{u_{4k+3}, u_3, u_4, u_7, u_8, \ldots, u_{4k-5}, u_{4k-4}, u_{4k-1}, u_{4k}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k-3}\}, \{u_{4k-2}\}, \{u_{4k+1}\}, \{u_{4k+2}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Arguing as in case 1, we get that $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil + 2$.

Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Case 4: Let $n = 4k + 3$.

Let $\Pi = \{\{u_{4k+4}, u_3, u_4, u_7, u_8, \ldots, u_{4k-1}, u_{4k}\}, \{u_1\}, \{u_2\}, \{u_5\}, \{u_6\}, \ldots, \{u_{4k+1}\}, \{u_{4k+2}\}, \{u_{4k+3}\}\}$. Then Π is a dominator semi-strong color partition of G. Therefore $\chi_s^d(G) \leq |\Pi| = 1 + k + 1 + k + 2 = 2k + 4 = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$.

Arguing as in case 1, we get that $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil + 2$.

Therefore $\chi_s^d(G) = \lceil \frac{n}{2} \rceil + 2 = \chi_s^d(C_n) + 1$. □

Proposition 1. *If $diam(G) \leq 2$, then $\chi_s^d(G) \geq \lceil \frac{n}{2} \rceil$, where $|V(G)| = n$.*

Proof. Let G be a connected graph and $diam(G) \leq 2$. If $diam(G) = 1$, then $G = K_n$ and $\chi_s^d(G) = n \geq \lceil \frac{n}{2} \rceil$. Suppose $diam(G) = 2$. Then $\chi_s^d(G) \geq \chi_s(G) \geq \lceil \frac{n}{2} \rceil$ [?]. □

Remark 7. *The converse of the above proposition need not be true.*
For: $\chi_s^d(C_n) = \lceil \frac{n}{2} \rceil + 1 > \lceil \frac{n}{2} \rceil$ for all $n \geq 3$. When $n \geq 6$, $diam(C_n) \geq 3$.

Definition 4. *$C_m(a_1, a_2, \ldots, a_m)$ is the graph obtained from the cycle C_m by attaching a_i (≥ 1) pendent vertices at the vertex u_i of C_m, $1 \leq i \leq m$.*

Proposition 2. *$\chi_s^d(C_m(a_1, a_2, \ldots, a_m)) = m + max\{a_1, a_2, \ldots, a_m\}$.*

Proof. The proof follows on the same line as the proof of the Theorem 4. □

Theorem 12. *Let G be a connected graph. Then $\chi_s^d(G) = n - 1$, where $|V(G)| = n$ if and only if $n \geq 4$. When $n = 4$, $G = P_4$ or C_4. When $n = 5$, G is one of the ten graphs P_5, C_5, $D_{1,2}$ or G_i, $(1 \leq i \leq 7)$ given in Figure 5. When $n \geq 6$, there exist two vertices say u_1, u_2 such that u_1 and u_2 may be either adjacent or independent and there exist u_i, $(3 \leq i \leq n)$ adjacent with u_1 and not with u_2, there exist u_j, $(j \neq i)$, $(3 \leq k \leq n)$ such that u_r and u_s are adjacent with u_k and u_1 may*

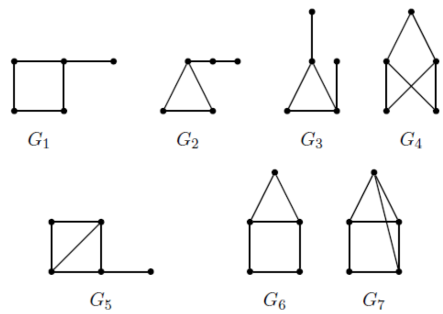

FIGURE 5. A set of graphs $G_1, G_2, G_3, G_4, G_5, G_5, G_7$ with $n = 5$ and $\chi_s^d(G) = n - 1$

be adjacent with any u_k, $(k \neq j)$, u_2 may be adjacent with any u_k, $(k \neq i)$ but u_1 and u_2 are not together adjacent with any u_k.

Proof. Let G be a connected graph. Let $\chi_s^d(G) = n - 1$. Let $\Pi = \{\{u_1, u_2\}, \{u_3\}, \{u_4\}, \ldots, \{u_n\}\}$ be a χ_s^d-partition of G.

Case 1: u_1 and u_2 are adjacent.

Let u_i, $3 \leq i \leq n$, be such that u_i is not adjacent with both u_1 and u_2. That is, either u_i is adjacent with u_1 and not with u_2 or u_i is adjacent with u_2 and not with u_1 or u_i is not adjacent with both u_1 and u_2. Since Π is a χ_s^d-partition, there exist some u_i, $3 \leq i \leq n$ adjacent with u_1 and some u_j, $j \neq i$, $3 \leq j \leq n$, adjacent with u_2. Then u_i, u_2 have a common vertex u_1 and u_j, u_1 have a common vertex u_2. Any two of the vertices $u_3, \ldots, u_n$ have a common vertex that is, $d(u_r, u_s) \leq 2$. Let $n \geq 6$. Suppose u_r and u_s are adjacent, $r \neq s$, $r, s \notin \{1, 2\}$, $3 \leq r, s \leq n$. Then there exist u_k, $3 \leq k \leq n$, $k \neq \{r, s\}$ such that u_i, u_j, u_k form a triangle. If u_r and u_s are independent, then there exist u_k, $3 \leq k \leq n$, $k \neq \{i, j\}$ such that u_r, u_s, u_k form a path of length 2. If $n = 5$, then only one vertex is left other than u_1, u_2, u_i, u_j, and the graph is either P_5 or $D_{1,2}$ or C_5, a contradiction.

Subcase 1: $n = 3$

Then $G = P_3$ or K_3. Then $\chi_s^d(G) = 3$, a contradiction. Therefore $n \geq 4$.

Subcase 2: $n = 4$

Then $G = P_4$, C_4, K_4, $K_{1,3}$, $K_4 - \{e\}$. When $G = K_4, K_{1,3}, K_4 - \{e\}$, G has a full degree vertex. Therefore $\chi_s^d(G) = 4$, a contradiction. Hence $G = P_4$ or C_4.

Subcase 3: $n = 5$

Then $G = P_5$, C_5, K_5, $K_{1,4}$, $K_5 - \{e\}$, $K_5 - \{e_1, e_2\}$ or one of the following graphs shown in Figure 6:

Therefore $\chi_s^d(G) = 4$ if $G = P_5, C_5, D_{1,2}$ or one of the following graphs shown in Figure 7:

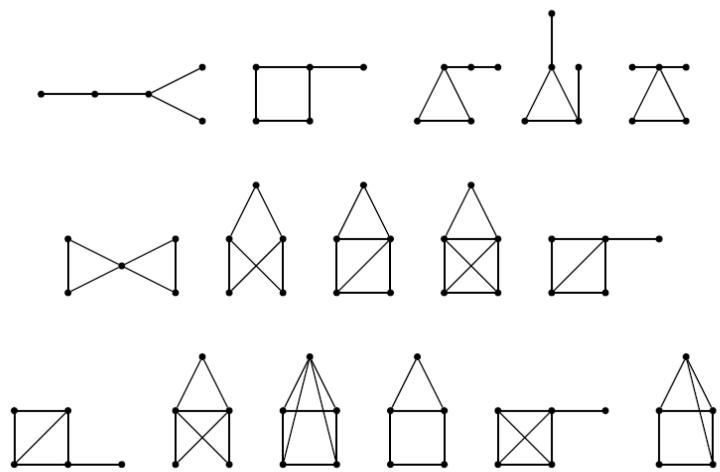

FIGURE 6. A set of graphs $G_1, G_2, G_3, G_4, G_5, G_5, G_7$ with $n = 5$

Case 2: u_i and u_j are independent.

Let u_i, $3 \leq i \leq n$, be not adjacent with both u_1 and u_2. That is, either u_i is adjacent with u_1 and not with u_2 or u_i is adjacent with u_2 and not with u_1 or u_i is not adjacent with both u_1 and u_2. Then Π is a χ_s^d-partition, there exist some u_i, $3 \leq i \leq n$ adjacent with u_1 and some u_j, $j \neq i$, $3 \leq j \leq n$, adjacent with u_2. Then u_i, u_2 have a common vertex u_1 and u_j, u_1 have a common vertex u_2. Any two of the vertices $u_3, \ldots, u_n$ have a common vertex that is, $d(u_r, u_s) \leq 2$. Let $n \geq 6$. Suppose u_r and u_s are adjacent, $r \neq s$, $r, s \notin \{1, 2\}$, $3 \leq r, s \leq n$. Then there exist u_k, $3 \leq k \leq n$, $k \neq \{r, s\}$ such that u_r, u_s, u_k form a triangle. If u_r and u_s are independent, then there exist u_k, $3 \leq k \leq n$, $k \neq \{r, s\}$ such that u_r, u_s, u_k form a path of length 2. If $n = 5$, then only one vertex is left other than u_1, u_2, u_i, u_j, and the graph is either P_5 or a contraction.

□

4. Conclusion

In this paper, a study of dominator semi-strong partition and the parameter $\chi_s^d(G)$ is initiated. Further study can be made on the complexity of the parameter and Nordhaus-Gaddum type results for $\chi_s^d(G)$.

Author Contribution Statements The authors have made equal contributions in this work.

Declaration of Competing Interests The author declares that there are no conflicts of interest about the publication of this paper.

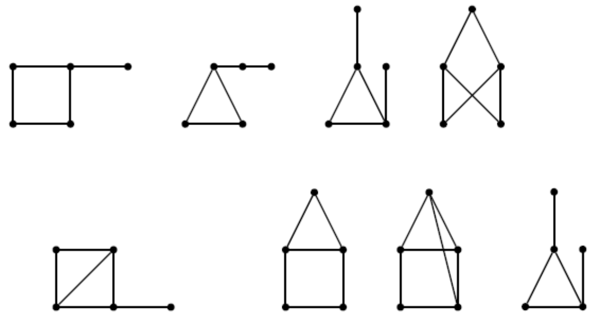

FIGURE 7. A set of graphs $G_1, G_2, G_3, G_4, G_5, G_5, G_7$ with $n = 5$

Acknowledgements The authors are thankful to the referees for making valuable suggestions leading to the better presentations of this paper.

REFERENCES

[1] Arumugam, S., Bagga, J., Chandrasekar, K.R., On dominator colorings in graphs, *Proc. Indian Acad. Sci. (Math Sci.),* 122(4) (2012), 561–578.
[2] Berge, C., Graphs and Hyper Graphs, North Holland, Amsterdam, 1973.
[3] Chartrand, G., Salehi, E., Zhang, P., The partition dimension of a graph, *Aequationes Math.,* 59 (2000), 45-54. https://doi.org/10.1007/PL00000127
[4] Chellali, M., Maffray, F., Dominator colorings in some classes of graphs, *Graphs and Combinatorics,* 28 (2012), 97–107. https://doi.org/10.1007/s00373-010-1012-z
[5] Chitra, S., Gokilamani and Swaminathan, V., Color Class Domination in Graphs, Mathematical and Experimental Physics, Narosa Publishing House, 2010, 24–28.
[6] Gera, R., On dominator coloring in graphs, *Graph Theory Notes, N.Y.*, 52 (2007), 25–30.
[7] Gera, R., Horton, S., Rasmussen, C., Dominator colorings and safe clique partitions, *Congr. Num.,* 181 (2006), 19-32.
[8] Harary, F., Graph Theory, Addison-Wesley Reading, MA, 1969.
[9] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., Fundamentals of Domination in Graphs, Marcel Dekker Inc., 1998.
[10] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., Domination in Graphs: Advanced Topics, Marcel Dekker Inc., 1998.
[11] Hedetniemi, S.M., Hedetniemi, S.T., Laskar, R., McRae, A.A., Blair, J.R.S., Dominator Colorings of Graphs, 2006, Preprint.
[12] Hedetniemi, S.M., Hedetniemi, S.T., Laskar, R., McRae, A.A., Wallis, C.K., Dominator partitions of graphs, *J. Combin. Systems Sci.,* 34(1-4) (2009), 183-192.
[13] Jothilakshmi, G., Pushpalatha, A.P., Suganthi, S., Swaminathan, V., (k,r)-Semi strong chromatic number of a graph, *International Journal of Computer Applications,* 21(2) (2011), 7-11.
[14] Kazemi, A.P., Total dominator chromatic number of a graph, *Trans. Comb.*, 4 (2015), 57-68.
[15] Kazemi, A.P., Total dominator coloring in product graphs, *Util. Math.,* 94 (2014), 329-345.

[16] Kazemi, A.P., Total dominator chromatic number and Mycieleskian graphs, *Util. Math.,* 103 (2017), 129-137.

[17] Merouane, H.B., Haddad, M., Chellali, M., Kheddouci, H., Dominated colorings of graphs, *Graphs and Combinatorics,* 31 (2015), 713-727. https://doi.org/10.1007/s00373-014-1407-3

[18] Sampathkumar, E., Pushpa Latha, L., Semi-strong chromatic number of a graph, *Indian Journal of Pure and Applied Mathematics,* 26(1) (1995), 35-40.

[19] Sampathkumar, E., Venkatachalam, C.V., Chromatic partition of a graph, *Discrete Mathematics,* 74 (1989), 227–239. https://doi.org/10.1016/S0167-5060(08)70311-X

[20] Venkatakrishnan, Y.B., Swaminathan,V., Color class domination number of middle graph and center graph of $K_{1,n}$, C_n, P_n, *Advanced Modeling and Optimization,* (12) (2010), 233–237.

[21] Venkatakrishnan, Y.B., Swaminathan, V., Color class domination numbers of some classes of graphs, *Algebra and Discrete Mathematics*, 18(2) (2014), 301-305.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 944–953 (2022)
DOI:10.31801/cfsuasmas.1123739
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: January 07, 2022; Accepted: June 8, 2022

IDEAL CONVERGENCE OF A SEQUENCE OF CHEBYSHEV RADII OF SETS

Hüseyin ALBAYRAK

Department of Statistics, Süleyman Demirel University, Isparta, TURKEY

ABSTRACT. In this paper, we investigate the diameters, Chebyshev radii, Chebyshev self-radii and inner radii of a sequence of sets in the normed spaces. We prove that if a sequence of sets is $\mathcal{I}$-Hausdorff convergent to a set, the sequence of Chebyshev radii of that sequence is $\mathcal{I}$-convergent. Similar relations are showed for the sequence of diameters, Chebyshev self-radii and inner radii of that sequence.

1. INTRODUCTION

The concept of statistical convergence, which is a generalization of the ordinary convergence of sequences, was first introduced by Fast [3] and Stainhaus [13], independently. Fridy [4,5] contributed greatly to the development of the theory of statistical convergence. In 2000, Kostyrko et al [7] introduced ideal convergence, which is a generalization of statistical convergence. Recently the ideal convergence theory continues to be popularly studied (see [9,10]). On the other hand, Hausdorff convergence of a sequence of sets, which is defined by the Hausdorff distance, corresponds to the uniform convergence of the sequence of distance (see [2,6,8]). The theory of statistical convergence and the theory of ideal convergence were combined with the theory of convergence of sequences of sets by Nuray and Rhoades [11] and by Talo and Sever [14], respectively.

In [12], Papini and Wu examined Kuratowski convergence and Hausdorff convergence of sequences of sets in Banach spaces. They showed that if a sequence of sets is Hausdorff convergent then the sequences of diameters, Chebyshev radii, Chebyshev self-radii, and inner radii, respectively, of this sequence are convergent.

2020 *Mathematics Subject Classification.* 40A05, 40A35, 54A20.
Keywords. Chebyshev radius, Hausdorff convergence, $\mathcal{I}$-convergence, $\mathcal{I}$-Hausdorff convergence.
huseyinalbayrak@sdu.edu.tr
0000-0001-8275-089X.

In this study, by generalizing some of the results in [12], we show that if a sequence $(A_n)_{n\in\mathbb{N}}$ of sets is $\mathcal{I}$-Hausdorff convergent to a set A then the sequence of Chebyshev radii of A_n's is $\mathcal{I}$-convergent to the Chebyshev radius of A. We give similar relations for diameter, relative Chebyshev radius, Chebyshev self-radius and inner radius.

2. Preliminaries

Let $(X, \|\cdot\|)$ be normed space. We denote the family of all nonempty closed subsets, the family of all nonempty closed and bounded subsets and the family of all nonempty closed, convex and bounded subsets of X by $\mathrm{Cl}(X)$, $\mathcal{B}(X)$ and $\mathcal{C}(X)$, respectively.

The *distance* $d(x, A)$ from a point $x \in X$ to a subset A of X is defined to be

$$d(x, A) = \inf_{a\in A} \|x - a\| .$$

The set A is said to be *bounded* if $\mathrm{diam}(A) < \infty$, where *diameter* $\mathrm{diam}(A)$ of a nonempty set A in a normed space $(X, \|\cdot\|)$ is defined by

$$\mathrm{diam}(A) = \sup_{a_1,a_2\in A} \|a_1 - a_2\| .$$

The open ball with centre $x \in X$ and radius $\delta > 0$ is the set

$$S(x, \delta) = \{y \in X : \|x - y\| < \delta\}.$$

Hausdorff distance of sets $A, B \subseteq X$ is defined as

$$H(A, B) = \max \{h(A, B), h(B, A)\}$$

where $h(A, B) = \sup_{a\in A} d(a, B)$, or equivalently

$$H(A, B) = \inf \{\varepsilon > 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}$$

where $A^\varepsilon = \bigcup_{a\in A} \{x \in X : \|x - a\| < \varepsilon\} = \{x \in X : d(x, A) < \varepsilon\}$ is the ε-enlargement of A.

Briefly, we recall some of basic notations in the theory of $\mathcal{I}-$convergence and we refer readers to [7,8] for more details. A family $\mathcal{I} \subseteq 2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ is said to be an *ideal* in $\mathbb{N}$ if $\emptyset \in \mathcal{I}$, and $A \cup B \in \mathcal{I}$ for each $A, B \in \mathcal{I}$, and $B \in \mathcal{I}$ for each $A \in \mathcal{I}$ such that $B \subseteq A$ (see [8]). An ideal is called *proper* if $\mathbb{N} \notin \mathcal{I}$, and a proper ideal is called *admissible* if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Obviously, an admissible ideal includes all finite subset of $\mathbb{N}$ (see [7]).

The definition of ideal convergence for real numbers is as follows: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Let $\mathcal{I}$ be any ideal on $\mathbb{N}$. If for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\} \in \mathcal{I}$$

then (x_n) is said to be ideal convergent (briefly, $\mathcal{I}$-convergent) to x_0. Then we write $\mathcal{I} - \lim x_n = x_0$ (see [7]).

Define $\mathcal{I}_f = \{A \subset \mathbb{N} : \text{the set } A \text{ has finite number of elements}\}$. Then $\mathcal{I}_f$-convergence and classical convergence is equivalent to each other. Similarly, if we denote $\mathcal{I}_d = \{A \subset \mathbb{N} : \text{the set } A \text{ has natural density zero}\}$, then $\mathcal{I}_d$-convergence and statistical convergence is equivalent to each other. We note that the ideals $\mathcal{I}_f$ and $\mathcal{I}_d$ are admissible.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. We say that the sequence (A_n) is *$\mathcal{I}$-Hausdorff convergent* to the set A if

$$\left\{n \in \mathbb{N} : \sup_{x\in X} |d(x, A_n) - d(x, A)| \geq \varepsilon\right\} \in \mathcal{I}$$

for every $\varepsilon > 0$, or if $\mathcal{I} - \lim H(A_n, A) = 0$, i.e., for every $\varepsilon > 0$

$$\{n \in \mathbb{N} : H(A_n, A) \geq \varepsilon\} \in \mathcal{I}$$

or equivalently

$$\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon \text{ or } h(A, A_n) \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $A_n \xrightarrow{\mathcal{I}\text{-}H} A$ (see [14]).

Now, we list some definitions of radii and centers associated with these radii (see [1,12,15]). Let A be a bounded subset of X and $Y \subseteq X$.

$R(x, A) = \sup_{a\in A} \|a - x\| \quad (x \in X)$	
$R_Y(A) = \inf_{y\in Y} R(y, A)$	: Relative Chebyshev radius of A in Y
$\quad\quad = \inf_{y\in Y} \sup_{a\in A} \|a - y\|$	
$R(A) = R_X(A)$	: Chebyshev radius of A
$R_A(A)$	: Chebyshev self-radius of A
$R'(A) = \sup_{a\in A} \inf_{x\notin A} \|x - a\|$	: Inner radius of A
$Z_Y(A) = \{y \in Y : R(y, A) = R_Y(A)\}$	: Relative Chebyshev center set of A in Y
$Z(A) = \{x \in X : R(x, A) = R(A)\}$	: Chebyshev center set of A
$Z_A(A) = \{a \in A : R(a, A) = R_A(A)\}$	: Chebyshev self center set of A
$Z'(A) = \{a \in A : R(a, A) = R'(A)\}$	: Inner center set of A

Example 1. *Consider the normed space $\left(\mathbb{R}^2, \|\cdot\|_1\right)$ where $\|\cdot\|_1$ is the ℓ_1 norm (aka the taxicab norm). Let A be a square whose vertices are on the points $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$, and let $Y = \left\{(x, y) \in \mathbb{R}^2 : x = 3\right\}$. We have the following results:*

$R(A)$	$= 2$	$Z(A)$	$= \{(0,0)\}$
$R_A(A)$	$= 3$	$Z_A(A)$	$= \{(-1,0), (1,0), (0,-1), (0,1)\}$
$R'(A)$	$= 0$	$Z'(A)$	$= \emptyset$
$R_Y(A)$	$= 5$	$Z_Y(A)$	$= \{(3,0)\}$

Lemma 1. *Let $A \in \mathcal{B}(X)$, $Y \subseteq X$ and $\varepsilon > 0$. Then the following is provided:*

(i) $\operatorname{diam}(A^{\varepsilon}) \leq \operatorname{diam}(A) + 2\varepsilon$

(ii) $R(x, A^{\varepsilon}) \leq R(x, A) + \varepsilon$ *for every* $x \in X$

(iii) $R_Y(A^{\varepsilon}) \leq R_Y(A) + \varepsilon$

(iv) $R(A^{\varepsilon}) \leq R(A) + \varepsilon$

(v) $R_{A^{\varepsilon}}(A^{\varepsilon}) \leq R_A(A) + \varepsilon$

Proof. (i)

$$\alpha_1, \alpha_2 \in A^{\varepsilon} \Longrightarrow \exists a_1, a_2 \in A \text{ such that } \|\alpha_1 - a_1\| < \varepsilon \text{ and } \|\alpha_2 - a_2\| < \varepsilon$$

Then, for every $\alpha_1, \alpha_2 \in A^{\varepsilon}$ we have

$$\begin{aligned} \|\alpha_1 - \alpha_2\| &\leq \|\alpha_1 - a_1\| + \|a_1 - a_2\| + \|\alpha_2 - a_2\| \\ &< \|a_1 - a_2\| + 2\varepsilon \\ &\leq \sup_{a_1, a_2 \in A} \|a_1 - a_2\| + 2\varepsilon \\ &= \operatorname{diam}(A) + 2\varepsilon \end{aligned}$$

and so

$$\operatorname{diam}(A^{\varepsilon}) = \sup_{\alpha_1, \alpha_2 \in A^{\varepsilon}} \|\alpha_1 - \alpha_2\| \leq \operatorname{diam}(A) + 2\varepsilon.$$

(ii)

$$\alpha \in A^{\varepsilon} \Longrightarrow \exists a \in A \text{ such that } \|\alpha - a\| < \varepsilon$$

Let $x \in X$. For every $\alpha \in A^{\varepsilon}$ we have

$$\begin{aligned} \|\alpha - x\| &\leq \|\alpha - a\| + \|a - x\| \\ &< \|a - x\| + \varepsilon \\ &\leq \sup_{a \in A} \|a - x\| + \varepsilon \\ &= R(x, A) + \varepsilon \end{aligned}$$

and so

$$R(x, A^{\varepsilon}) = \sup_{\alpha \in A^{\varepsilon}} \|\alpha - x\| \leq R(x, A) + \varepsilon.$$

(iii) From (ii), we have $R(y, A^{\varepsilon}) \leq R(y, A) + \varepsilon$ for every $y \in Y$. Then we get

$$\begin{aligned} \inf_{y \in Y} R(y, A^{\varepsilon}) &\leq \inf_{y \in Y} R(y, A) + \varepsilon \\ R_Y(A^{\varepsilon}) &\leq R_Y(A) + \varepsilon. \end{aligned}$$

(iv) It is easily obtained by taking $Y = X$ in (iii).

(v) From (ii), we have

$$R(a, A^{\varepsilon}) \leq R(a, A) + \varepsilon$$

for every $a \in A$, and so

$$\inf_{a\in A} R(a, A^{\varepsilon}) \leq \inf_{a\in A} R(a, A) + \varepsilon.$$

From the fact that

$$\inf_{\alpha\in A^{\varepsilon}} R(\alpha, A^{\varepsilon}) \leq \inf_{a\in A} R(a, A^{\varepsilon}),$$

we get

$$\begin{aligned} \inf_{\alpha\in A^{\varepsilon}} R(\alpha, A^{\varepsilon}) &\leq \inf_{a\in A} R(a, A) + \varepsilon \\ R_{A^{\varepsilon}}(A^{\varepsilon}) &\leq R_A(A) + \varepsilon. \end{aligned}$$

□

We cannot give similar results above for the inner radius, i.e., the inequality $R'(A^{\varepsilon}) \leq R'(A) + \varepsilon$ may not be satisfied. Such as, if we take $\varepsilon = \frac{3}{2}$ in Example 1, we get

$$R'(A^{\varepsilon}) = \frac{5}{2} \nleq R'(A) + \varepsilon = 0 + \frac{3}{2}.$$

Also, we cannot say a general upper bound for the difference $R'(A^{\varepsilon}) - R'(A)$. For example, in the Euclidean space $\mathbb{R}^2$, let the set A be a spiral with $r = \theta$ ($0 \leq \theta \leq 2n\pi$, $n \in \mathbb{N}$) polar equation. Let's take $\varepsilon > \pi$. Then we have $R'(A) = 0$ and $R'(A^{\varepsilon}) \geq (2n-1)\pi$. Thus the difference $R'(A^{\varepsilon}) - R'(A)$ depends not only on ε but also on n.

3. Main Results

For a sequence of closed and bounded sets, we show that $\mathcal{I}$-Hausdorff convergence implies $\mathcal{I}$-convergence of the sequence of Chebyshev radii (diameters, relative Chebyshev radii and Chebyshev self-radii, respectively) of this sequence. If the sets are convex as an additional condition, this proposition is also true for the sequence of inner radii.

Proposition 1. *Let $A, A_n \in \mathcal{B}(X)$ ($n \in \mathbb{N}$) and $Y \subseteq X$. If $A_n \xrightarrow{\mathcal{I}-H} A$ then the following hold:*

(i) $\mathcal{I} - \lim \operatorname{diam}(A_n) = \operatorname{diam}(A)$
(ii) $\mathcal{I} - \lim R_Y(A_n) = R_Y(A)$
(iii) $\mathcal{I} - \lim R(A_n) = R(A)$
(iv) $\mathcal{I} - \lim R_{A_n}(A_n) = R_A(A)$

Proof. (i) Let $\varepsilon > 0$. From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ we have

$$L(\varepsilon) := \left\{ n \in \mathbb{N} : H(A_n, A) \geq \frac{\varepsilon}{3} \right\} \in \mathcal{I}.$$

For every $n \in \mathbb{N} \setminus L(\varepsilon)$ we have

$$A \subseteq A_n^{\varepsilon/3} \text{ and } A_n \subseteq A^{\varepsilon/3}.$$

Then

$$\begin{aligned} A \quad \subseteq \quad A_n^{\varepsilon/3} &\Longrightarrow \operatorname{diam}(A) \leq \operatorname{diam}\left(A_n^{\varepsilon/3}\right) \leq \operatorname{diam}(A_n) + \frac{2\varepsilon}{3} \\ &\Longrightarrow \operatorname{diam}(A) - \operatorname{diam}(A_n) \leq \frac{2\varepsilon}{3} \\ A_n \quad \subseteq \quad A^{\varepsilon/3} &\Longrightarrow \operatorname{diam}(A_n) \leq \operatorname{diam}\left(A^{\varepsilon/3}\right) \leq \operatorname{diam}(A) + \frac{2\varepsilon}{3} \\ &\Longrightarrow \operatorname{diam}(A_n) - \operatorname{diam}(A) \leq \frac{2\varepsilon}{3} \end{aligned}$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$. Hence we get

$$\begin{aligned} \{n \in \mathbb{N} : |\operatorname{diam}(A_n) - \operatorname{diam}(A)| \geq \varepsilon\} &\subseteq L(\varepsilon) \in \mathcal{I} \\ \{n \in \mathbb{N} : |\operatorname{diam}(A_n) - \operatorname{diam}(A)| \geq \varepsilon\} &\in \mathcal{I} \end{aligned}$$

for every $\varepsilon > 0$. Consequently, we obtain $\mathcal{I} - \lim \operatorname{diam}(A_n) = \operatorname{diam}(A)$.

(ii) Let Y be any subset of X. From the triangle inequality, we have

$$\|a_n - y\| - \|a - y\| \leq \|a_n - a\| \tag{1}$$

$$\|a - y\| - \|a_n - y\| \leq \|a_n - a\| \tag{2}$$

where $y \in Y, a_n \in A_n$ and $a \in A$. Then, from (1)

$$\begin{aligned} \inf_{a \in A} (\|a_n - y\| - \|a - y\|) &\leq \inf_{a \in A} \|a_n - a\| \\ \|a_n - y\| - \sup_{a \in A} \|a - y\| &\leq \inf_{a \in A} \|a_n - a\| \\ \sup_{a_n \in A_n} \|a_n - y\| - \sup_{a \in A} \|a - y\| &\leq \sup_{a_n \in A_n} \inf_{a \in A} \|a_n - a\| \end{aligned}$$

$$\begin{aligned} R_Y(A_n) - R_Y(A) &= \inf_{y \in Y} \sup_{a_n \in A_n} \|a_n - y\| - \inf_{y \in Y} \sup_{a \in A} \|a - y\| \\ &\leq \sup_{a_n \in A_n} \inf_{a \in A} \|a_n - a\| = h(A_n, A) \end{aligned} \tag{3}$$

and similarly, from (2)

$$\begin{aligned} R_Y(A) - R_Y(A_n) &= \inf_{y \in Y} \sup_{a \in A} \|a - y\| - \inf_{y \in Y} \sup_{a_n \in A_n} \|a_n - y\| \\ &\leq \sup_{a \in A} \inf_{a_n \in A_n} \|a_n - a\| = h(A, A_n). \end{aligned} \tag{4}$$

Take $\varepsilon > 0$. From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$, we have

$$L(\varepsilon) := \{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon \text{ or } h(A, A_n) \geq \varepsilon\} \in \mathcal{I}.$$

From (3) and (4), we get

$$R_Y(A_n) - R_Y(A) \leq h(A_n, A) < \varepsilon,$$
$$R_Y(A) - R_Y(A_n) \leq h(A, A_n) < \varepsilon$$

and so

$$|R_Y(A_n) - R_Y(A)| < \varepsilon$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$. Hence we get

$$\begin{aligned} \{n \in \mathbb{N} : |R_Y(A_n) - R_Y(A)| \geq \varepsilon\} &\subseteq L(\varepsilon) \in \mathcal{I} \\ \{n \in \mathbb{N} : |R_Y(A_n) - R_Y(A)| \geq \varepsilon\} &\in \mathcal{I} \end{aligned}$$

for every $\varepsilon > 0$. This means that $\mathcal{I} - \lim R_Y(A_n) = R_Y(A)$.

(iii) It is the special case of (ii), with $Y = X$.

(iv) Let $\varepsilon > 0$. From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ we have

$$L(\varepsilon) := \left\{ n \in \mathbb{N} : h(A_n, A) \geq \frac{\varepsilon}{2} \text{ or } h(A, A_n) \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

If $a_0 \in Z_A(A)$ then $a_0 \in A$ and

$$R(a_0, A) = \sup_{a \in A} \|a - a_0\| = R_A(A). \tag{5}$$

Take $n \in \mathbb{N} \setminus L(\varepsilon)$. From $h(A, A_n) < \frac{\varepsilon}{2}$ we have

$$\sup_{a \in A} d(a, A_n) < \frac{\varepsilon}{2}. \tag{6}$$

From the closeness of A_n there exists an $a_n^{(1)} \in A_n$ such that

$$\left\| a_0 - a_n^{(1)} \right\| < \frac{\varepsilon}{2}. \tag{7}$$

Also, there exists an $a_n^{(2)} \in A_n$ such that

$$\sup_{a_n \in A_n} \left\| a_n - a_n^{(1)} \right\| = \left\| a_n^{(2)} - a_n^{(1)} \right\|. \tag{8}$$

From $h(A_n, A) < \frac{\varepsilon}{2}$ we get

$$d\left(a_n^{(2)}, A\right) \leq \sup_{a_n \in A_n} d(a_n, A) < \frac{\varepsilon}{2} \tag{9}$$

and so

$$\left\| a_0 - a_n^{(2)} \right\| < R_A(A) + \frac{\varepsilon}{2}. \tag{10}$$

From (7) and (10) we obtain

$$\begin{aligned} R_{A_n}(A_n) \leq \left\| a_n^{(1)} - a_n^{(2)} \right\| &\leq \left\| a_n^{(1)} - a_0 \right\| + \left\| a_0 - a_n^{(2)} \right\| \\ &< R_A(A) + \varepsilon \end{aligned}$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$.
Similarly, it can be shown that

$$R_A(A) < R_{A_n}(A_n) + \varepsilon$$

for every $n \in \mathbb{N} \setminus L(\varepsilon)$.
Consequently, we get

$$\begin{aligned} \{n \in \mathbb{N} : |R_{A_n}(A_n) - R_A(A)| \geq \varepsilon\} &\subseteq L(\varepsilon) \in \mathcal{I} \\ \{n \in \mathbb{N} : |R_{A_n}(A_n) - R_A(A)| \geq \varepsilon\} &\in \mathcal{I} \end{aligned}$$

for every $\varepsilon > 0$, and so $\mathcal{I} - \lim R_{A_n}(A_n) = R_A(A)$.

□

Lemma 2. *(see [12, Lemma 1]) Let $A, B \in \mathcal{C}(X)$. If $R'(A) > 0$ and $H(A, B) < \dfrac{R'(A)}{2}$ then*

$$R'(B) \geq R'(A) - H(A, B) > 0.$$

As a result of the above lemma we can give the following corollary.

Corollary 1. *Let $A \in \mathcal{C}(X)$ and $\varepsilon > 0$. If $R'(A^\varepsilon) > 2\varepsilon$ then*

$$R'(A) \geq R'(A^\varepsilon) - \varepsilon$$

(That is, $R'(A^\varepsilon) \leq R'(A) + \varepsilon$). Of course, for the condition here to be satisfied, $R'(A) > \varepsilon$ must be.

Proposition 2. *Let $A, A_n \in \mathcal{C}(X)$ $(n \in \mathbb{N})$. If $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ then*

$$\mathcal{I} - \lim R'(A_n) = R'(A).$$

Proof. First let's assume that $R'(A) = 0$. Suppose that $\mathcal{I} - \lim R'(A_n) \neq 0$. Then there is an $\varepsilon_0 > 0$ such that

$$K(\varepsilon_0) := \{n \in \mathbb{N} : R'(A_n) \geq \varepsilon_0\} \notin \mathcal{I}.$$

From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ we have

$$L(\varepsilon_0) := \left\{n \in \mathbb{N} : H(A_n, A) \geq \frac{\varepsilon_0}{2}\right\} \in \mathcal{I}.$$

Then $(\mathbb{N} \setminus L(\varepsilon_0)) \cap K(\varepsilon_0) \neq \emptyset$ and so we have

$$H(A_n, A) < \frac{\varepsilon_0}{2} \leq \frac{1}{2} R'(A_n)$$

for every $n \in (\mathbb{N} \setminus L(\varepsilon_0)) \cap K(\varepsilon_0)$. From Lemma 2, we get

$$R'(A) > 0$$

and this is a contradiction. Therefore, $\mathcal{I}-\lim R'(A_n)=0=R'(A)$ holds.
Now let's assume that $R'(A)>0$. Let $0<\varepsilon<\dfrac{R'(A)}{3}$. From $A_n \overset{\mathcal{I}-H}{\longrightarrow} A$ we have

$$L(\varepsilon):=\{n\in\mathbb{N}: H(A_n,A)\geq\varepsilon\}\in\mathcal{I}.$$

Then we have

$$H(A_n,A)<\varepsilon<\frac{R'(A)}{3}<\frac{R'(A)}{2}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. From Lemma 2, we get

$$R'(A_n)\geq R'(A)-H(A_n,A)>R'(A)-\varepsilon \tag{11}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. We also have

$$H(A_n,A)<\varepsilon<\frac{1}{2}(R'(A)-\varepsilon)<\frac{R'(A_n)}{2}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. Again from Lemma 2, we get

$$R'(A)\geq R'(A_n)-H(A,A_n)>R'(A_n)-\varepsilon \tag{12}$$

for every $n\in\mathbb{N}\setminus L(\varepsilon)$. From (11) and (12) we obtain

$$\begin{aligned}\{n\in\mathbb{N}:|R'(A_n)-R'(A)|\geq\varepsilon\} &\subseteq L(\varepsilon)\in\mathcal{I}\\ \{n\in\mathbb{N}:|R'(A_n)-R'(A)|\geq\varepsilon\} &\in \mathcal{I}\end{aligned}$$

for every $\varepsilon>0$, and so $\mathcal{I}-\lim R'(A_n)=R'(A)$. □

Declaration of Competing Interests The author declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Amir, D., Chebyshev centers and uniform convexity, *Pacific J. Math.*, 77(1) (1978), 1–6.
[2] Castaing, C., Valadier, M., Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin-New York, 1977.
[3] Fast, H., Sur la convergence statistique, *Colloq. Math.*, 2 (1951), 241–244.
[4] Fridy, J. A., On statistical convergence, *Analysis*, 5(4) (1985), 301–313. https://doi.org/10.1524/anly.1985.5.4.301
[5] Fridy, J. A., Statistical limit points, *Proc. Amer. Math. Soc.*, 118(4) (1993), 1187–1192. https://doi.org/10.1090/S0002-9939-1993-1181163-6
[6] Hausdorff, F., Grundzüge der Mengenlehre, Chelsea Publishing Company, New York, 1949.
[7] Kostyrko, P., Šalát, T., Wilczyński, W., $\mathcal{I}$-convergence, *Real Anal. Exchange*, 26(2) (2000/2001), 669–686.
[8] Kuratowski, K., Topology. Vol. II, Academic Press, New York-London, 1968.
[9] Mohiuddine, S. A., Hazarika, B., Alghamdi, M. A., Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems, *Filomat*, 33(14) (2019), 4549–4560. https://doi.org/10.2298/FIL1914549M
[10] Mursaleen, M., Mohiuddine, S. A., On ideal convergence in probabilistic normed spaces, *Math. Slovaca*, 62(1) (2012), 49–62. https://doi.org/10.2478/s12175-011-0071-9
[11] Nuray, F., Rhoades, B. E., Statistical convergence of sequences of sets, *Fasc. Math.*, 49 (2012), 87–99.

[12] Papini, P. L., Wu, S., Nested sequences of sets, balls, Hausdorff convergence, *Note Mat.*, 35(2) (2015), 99–114. https://doi.org/10.1285/i15900932v35n2p99

[13] Steinhaus, H., Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, 2 (1951), 73–74.

[14] Talo, Ö., Sever, Y., On Kuratowski $\mathcal{I}$-convergence of sequences of closed sets, *Filomat*, 31(4) (2017), 899–912. https://doi.org/10.2298/FIL1704899T

[15] Ward, J. D., Chebyshev centers in spaces of continuous functions, *Pacific J. Math.*, 52 (1974), 283–287.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 954–967 (2022)
DOI:10.31801/cfsuasmas.1072728
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: February 13, 2022; Accepted: May 5, 2022

PARAMETER UNIFORM SECOND-ORDER NUMERICAL APPROXIMATION FOR THE INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING BOUNDARY LAYERS

Muhammet Enes DURMAZ[1], Musa CAKIR[2] and Gabil AMIRALI[3]

[1]Department of Information Technology, Kırklareli University, Kırklareli, TURKEY
[2]Department of Mathematics, Faculty of Science, Van Yüzüncü Yıl University, Van, TURKEY
[3]Department of Mathematics, Faculty of Art and Science, Erzincan Binali Yıldırım University, Erzincan, TURKEY

Abstract. The work handles a Fredholm integro-differential equation involving boundary layers. A fitted second-order difference scheme has been created on a uniform mesh utilizing interpolating quadrature rules and exponential basis functions. The stability and convergence of the proposed discretization technique are analyzed and one example is solved to display the advantages of the presented technique.

1. Introduction

In the study, we deal with singularly perturbed Fredholm integro-differential equation (SPFIDE) in the form:

$$Lv := L_1 v + \lambda \int_0^l M(x,\zeta)v(\zeta)d\zeta = f(x), \qquad 0 < x < 1, \tag{1}$$

$$v(0) = A, \quad v(l) = B, \tag{2}$$

where $L_1 v = -\varepsilon v''(x) + a(x)v(x)$, $0 < \varepsilon \ll 1$ is a singular perturbation parameter, λ is a given parameter. The functions $a(x) \geq \alpha > 0$, $f(x)$ and $M(x,\zeta)$ are considered to be sufficiently smooth and satisfy certain regularity criteria. The solution $v(x)$ of (1)-(2) has in general boundary layers near $x = 0$ and $x = l$.

2020 *Mathematics Subject Classification.* 65R20, 65L12, 65L11, 65L20.
Keywords. Finite difference method, Fredholm integro-differential equation, singular perturbation, uniform convergence.
menesdurmaz025@gmail.com-Corresponding author; 0000-0002-6216-1032
cakirmusa@hotmail.com; 0000-0002-1979-570X
gabilamirali@yahoo.com; 0000-0001-6585-7353.

Singularly perturbed problems (SPPs) are defined by a small parameter ε multiplying the highest order derivative term. The solution to them generally involves boundary or initial layers. To quote a few, the exact solutions of SPPs and their applications may be found in [15, 18, 21]. SPPs have a wide range of applications in the fields of population dynamics, nanofluid, neurobiology, fluid dynamics, viscoelasticity, heat transfer problems, simultaneous control systems and mathematical biology etc. It is worth noting that when a small ε parameter is multiplied with the derivative, the great majority of classic numerical techniques on uniform meshes are ineffective at solving issues unless the step-size of discretization is drastically reduced. Thus, as the perturbation parameter ε goes smaller, the truncation error becomes boundless. To solve SPPs numerically, general approaches are done with the fitted finite difference method and are widely utilized [9, 12, 19, 20].

Most engineering applications and scientific disciplines have been expressed by Fredholm integro-differential equations (FIDEs). Plasma physics, biomechanics, financial mathematics, artificial neural networks, oceanopraphy, epidemic models, electromagnetic theory, fluid mechanics, biological and population dynamics processes are amongst these (see, e.g., [5, 7, 13]). For this reason, several studies have been conducted on FIDEs. Solving problems of this type is quite difficult. Therefore, we require robust and consistent numerical methods [6, 8, 14, 16, 23, 26](see, as well as the references therein).

These investigations in relation to FIDEs are just in relation to regular situations. Numerical examination of SPFIDEs has not been widespread till recently. Finite difference schemes for solving linear SPFIDEs are constructed in [1, 2]. A second order numerical tecnique for solving FIDE with boundary layer is developed in [10, 11].

The goal of this work is to propose a uniform convergence numerical technique to solve linear second-order FIDEs with boundary layers. A numerical technique that uses suitable interpolating quadrature rules and exponential basis functions is proposed on a uniform mesh. Error estimates are acquired in the discrete maximum norm with regard to the perturbation parameter. To corroborate theoretical estimates, numerical experiments are conducted and the results are presented.

The rest of the contents is organized kind of following. In Section 2, some properties of solutions (1)–(3) are presented, as well as a finite difference scheme. In Section 3, the stability and convergence analysis of this scheme are shown. In Section 4, the numerical results of an example to verify the theoretical estimates are presented. Finally, the work ends with a summary of the conclusions in Section 5.

2. Discretization Techniques

We have mentioned certain analytical bounds here, which we will use later in our error analysis.

Lemma 1. *Let* $a, f \in C^2[0,l]$, $\frac{\partial^m M}{\partial x^m} \in C[0,l]^2$, $(m = 0,1,2)$ *and*

$$|\lambda| < \frac{\alpha}{\max\limits_{0\le x\le l} \int\limits_0^l |M(x,\zeta)|\, d\zeta}.$$

Then the solution $u(x)$ *of the problem (1)-(2) satisfies the following estimates:*

$$\|v\|_\infty \le C, \tag{3}$$

$$\left|v^{(k)}(x)\right| \le C\left\{1 + \varepsilon^{-\frac{k}{2}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right\}, \quad (k = 1,2), \quad 0 \le x \le 1. \tag{4}$$

Proof. The proof of Lemma 1 is by like approach as in [2,10,17]. □

Let ω_N be an equidistant mesh on $[0,l]$:

$$\omega_N = \{x_i = ih,\ i = 1,2,...,N-1,\ h = lN^{-1}\}, \quad \bar{\omega}_N = \omega_N \cup \{x_0 = 0,\ x_N = l\}.$$

We utilize the following difference approximations for any mesh function $q(x)$ defined on $\bar{\omega}_N$:

$$q_{x,i} = \frac{q_{i+1} - q_i}{h}, \qquad q_{\bar{x},i} = \frac{q_i - q_{i-1}}{h}, \qquad q_{\bar{x}x,i} = \frac{q_{x,i} - q_{\bar{x},i}}{h}.$$

For the equation (1), we begin with the following integral identity:

$$\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} Lv(x)\psi_i(x)dx = \frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\psi_i(x)dx, \qquad 1 \le i \le N-1, \tag{5}$$

with the basis functions

$$\psi(x) = \begin{cases} \psi_i^{(1)}(x) \equiv \frac{sinh\gamma_i(x - x_i)}{sinh\gamma_i h}, & x \in (x_{i-1}, x_i), \\ \psi_i^{(2)}(x) \equiv \frac{sinh\gamma_i(x_{i+1} - x)}{sinh\gamma_i h}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where

$$\gamma_i = \sqrt{\frac{a(x_i)}{\varepsilon}}, \qquad \chi_i = \frac{1}{h}\int\limits_{x_{i-1}}^{x_{i+1}} \psi_i(x)dx = \frac{2tanh(\gamma_i h/2)}{\gamma_i h}.$$

We should remark that the functions $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are the solutions to the following problems:

$$-\varepsilon\psi'' + a_i\psi = 0, \quad x_{i-1} < x < x_i, \quad \psi(x_{i-1}) = 0, \quad \psi(x_i) = 1,$$
$$-\varepsilon\psi'' + a_i\psi = 0, \quad x_i < x < x_{i+1} \quad \psi(x_i) = 1, \quad \psi(x_{i+1}) = 0.$$

By using the technique of the exact difference approximations [3, 4, 11, 24, 25] (see also [22], pp. 207-214), it follows that

$$-\frac{\varepsilon}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\psi_i(x)v''(x)dx+\frac{a_i}{\chi_i h}a_i\int\limits_{x_{i-1}}^{x_{i+1}}\psi_i(x)v(x)dx=$$

$$-\frac{\varepsilon}{\chi_i}\left\{1+a_i\varepsilon^{-1}\int\limits_{x_{i-1}}^{x_i}\psi_i^{(1)}(x)(x-x_i)dx\right\}v_{\bar{x}x,i}$$

$$+\frac{a_i}{\chi_i}\left\{h^{-1}\int\limits_{x_{i-1}}^{x_i}\psi_i^{(1)}dx+h^{-1}\int\limits_{x_i}^{x_{i+1}}\psi_i^{(2)}dx\right\}v_i=-\varepsilon\theta_i v_{\bar{x}x,i}+a_i v_i$$

where

$$\theta_i=\frac{a_i\rho^2}{4sinh^2\left(\sqrt{a_i}\frac{\rho}{2}\right)},\qquad \left(\rho=\frac{h}{\sqrt{\varepsilon}}\right). \tag{6}$$

Thus

$$\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\varepsilon v''(x)\,\psi_i(x)\,dx+\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}a(x)\,v(x)\,\psi_i(x)\,dx=-\varepsilon\theta_i v_{\bar{x}x,i}+a_i v_i + R_i^{(1)}, \tag{7}$$

with remainder term

$$R_i^{(1)}=\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\left[a(x)-a(x_i)\right]v(x)\psi_i(x)dx. \tag{8}$$

Furthermore, for the right-side in (5) we get

$$\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}f(x)\psi_i(x)dx=f_i+R_i^{(2)}, \tag{9}$$

with remainder term

$$R_i^{(2)}=\frac{1}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}\left[f(x)-f(x_i)\right]\psi_i(x)dx. \tag{10}$$

For integral term that include the kernel function, from (5), we have

$$\frac{\lambda}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}dx\psi_i(x)\int\limits_0^l M(x,\zeta)v(\zeta)d\zeta=\lambda\int\limits_0^l M(x_i,\zeta)v(\zeta)d\zeta$$

$$+\frac{\lambda}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int\limits_{x_{i-1}}^{x_{i+1}}\left(\int\limits_0^l \frac{\partial^2 M(\xi,\zeta)}{\partial \xi^2} v(\zeta)\,d\zeta\right) M_1(x,\xi)\,d\xi,$$

where

$$M_1(x,\xi) = T_1(x-\xi) - T_1(x_i-\xi) + (2h)^{-1}(x_{i+1}-\xi)(x_i-x),$$
$$T_1(\lambda)=\lambda, \quad \lambda\geq 0; \qquad T_1(\lambda)=0 \quad \lambda<0.$$

We computed by using composite trapezoidal integration with the remainder term in integral form for the second integral term in the left side of the identity of (5):

$$\int\limits_0^l M(x_i,\zeta)v(\zeta)d\zeta = \sum_{j=0}^N \hbar_j M_{ij} v_j + \frac{1}{2}\sum_{j=1}^N \int\limits_{x_{j-1}}^{x_j} (x_j-\xi)(x_{j-1}-\xi)\left(M(x_i,\xi)v(\xi)\right)'' d\xi,$$

where

$$\hbar_0 = \hbar_N = \frac{h}{2}, \qquad \hbar_i = h, \qquad 1\leq i\leq N-1.$$

Thus we get

$$\frac{\lambda}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int\limits_0^l M(x,\zeta)v(\zeta)d\zeta = \lambda\sum_{j=0}^N \hbar_j M_{ij} v_j + R_i^{(3)}, \tag{11}$$

with remainder term

$$R_i^{(3)} = \frac{\lambda}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}} dx\psi_i(x)\int\limits_{x_{i-1}}^{x_{i+1}}\left(\int\limits_0^l \frac{\partial^2 M(\xi,\zeta)}{\partial \xi^2} v(\zeta)\,d\zeta\right) M_1(x,\xi)\,d\xi$$
$$+\frac{1}{2}\lambda\sum_{j=1}^N \int\limits_{x_{j-1}}^{x_j} (x_j-\xi)(x_{j-1}-\xi)\left(M(x_i,\xi)v(\xi)\right)'' d\xi. \tag{12}$$

Combining (7), (9) and (11) in (5) we obtain the following difference scheme:

$$L_N v_i := -\varepsilon\theta_i v_{\bar{x}x,i} + a_i v_i + \lambda\sum_{j=0}^N \hbar_j M_{ij} v_j + R_i = f_i, \qquad 1\leq i\leq N-1, \tag{13}$$

with remainder term

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}, \tag{14}$$

where the remainder terms $R_i^{(1)}, R_i^{(2)}$ and $R_i^{(3)}$ are defined by (8), (10) and (12) respectively.

Based on (13) we achieve the following difference approximate for approximating (1)-(2):

$$L_N y_i := -\varepsilon\theta_i y_{\bar{x}x,i} + a_i y_i + \lambda\sum_{j=0}^N \hbar_j M_{ij} y_j = f_i, \qquad 1\leq i\leq N-1, \tag{15}$$

$$y_0 = A, \quad y_N = B, \tag{16}$$

where θ_i is defined by (6).

3. Error Analysis

For the error function $z_i = y_i - v_i$ $(i = 0, 1, ..., N)$ considering (13) and (15), we get

$$L_N z_i := R_i, \qquad 1 \leq i \leq N-1, \tag{17}$$

$$z_0 = 0, \quad z_N = 0, \tag{18}$$

where the remainder term R_i is defined by (14).

Theorem 1. *Let* $\frac{\partial^m M}{\partial x^m} \in C^2[0,l]^2$, $(m = 0,1,2)$, $M(x,0) = M(x,l) = 0$; $a, f \in C^2[0,l]$, $a'(0) = a'(l) = 0$, *and*

$$|\lambda| < \frac{\alpha}{\max\limits_{1\leq i\leq N} \sum\limits_{j=0}^{N} \hbar_j |M_{ij}|}.$$

Then for the error of the scheme (15)-(16), we have

$$||y - v||_{\infty,\bar{\omega}_N} \leq Ch^2.$$

Proof. Applying the discrete maximum principle to discrete problem (17) and (18), we get

$$\|z\|_{\infty,\bar{\omega}_N} \leq \alpha^{-1} \left\| R - \lambda \sum_{j=0}^{N} \hbar_j M_{ij} z_j \right\|_{\infty,\omega_N}$$

$$\leq \alpha^{-1} \|R\|_{\infty,\omega_N} + \alpha^{-1} |\lambda| \max_{1\leq i\leq N} \sum_{j=0}^{N} \hbar_j |M_{ij}| \, \|z\|_{\infty,\bar{\omega}_N}.$$

Hence

$$\|z\|_{\infty,\bar{\omega}_N} \leq \frac{\alpha^{-1} \|R\|_{\infty,\omega_N}}{1 - \alpha^{-1} |\lambda| \max\limits_{1\leq i\leq N} \sum\limits_{j=0}^{N} \hbar_j |M_{ij}|},$$

which leads to

$$\|z\|_{\infty,\overline{\omega}_N} \leq C \|R\|_{\infty,\omega_N}. \tag{19}$$

Now we estimate the remainder terms $R_i^{(1)}$, $R_i^{(2)}$ and $R_i^{(3)}$ separately. First we will show that, for $R_i^{(1)}$ the estimate

$$\left|R_i^{(1)}\right| \leq Ch^2, \tag{20}$$

holds. Using relations

$$v(x) = v(x_i) + (x - x_i)\, v'(\eta_i), \quad \eta_i \in (x_i, x),$$

$$a(x) = a(x_i) + (x - x_i)\, a'(x_i) + \frac{(x - x_i)^2}{2} a''(\xi_i), \quad \xi_i \in (x_i, x)$$

and

$$\int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)\, \psi_i(x)\, dx = 0,$$

we take

$$\begin{aligned} R_i^{(1)} &= \frac{1}{\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]\, v(x) \psi_i(x) dx = \frac{a'(x_i)\, v(x_i)}{\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)\, \psi_i(x)\, dx \\ &+ \frac{a'(x_i)}{\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, v'(\eta_i(x))\, \psi_i(x)\, dx \\ &+ \frac{1}{2\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, a''(\xi_i(x))\, v(x)\, \psi_i(x)\, dx \\ &\equiv \frac{a'(x_i)}{\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, v'(\eta_i(x))\, \psi_i(x)\, dx \\ &+ \frac{1}{2\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, a''(\xi_i(x))\, v(x)\, \psi_i(x)\, dx. \end{aligned} \tag{21}$$

Since $a \in C^2[0, l]$, $|v(x)| \leq C$ and $|x - x_i| \leq h$ for the second term in the right side of (21), we have

$$\begin{aligned} \frac{1}{2\chi_i h} \left| \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2\, a''(\xi_i(x))\, v(x)\, \psi_i(x)\, dx \right| &\leq \frac{Ch^2}{\chi_i h} \int\limits_{x_{i-1}}^{x_{i+1}} \psi_i(x)\, dx \\ &= O\left(h^2\right). \end{aligned} \tag{22}$$

Next, according to Lemma 1, we take the following inequality

$$\begin{aligned} |v'(\eta_i)| &\leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}\eta_i}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l - \eta_i)}{\sqrt{\varepsilon}}} \right) \right\} \\ &\leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l - x_{i+1})}{\sqrt{\varepsilon}}} \right) \right\}, \qquad 1 < i < N - 1. \end{aligned}$$

Hence, for the first term in the right side of (21), we have

$$\frac{1}{\chi_i h}\left|a'(x_i)\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2 v'(\eta_i(x))\,\psi_i(x)\,dx\right| \le \frac{C}{\chi_i h}\left|a'(x_i)\right|\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,dx$$
$$+\frac{C}{\sqrt{\varepsilon}\chi_i h}\left|a'(x_i)\right|\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}dx$$
$$+\frac{C}{\sqrt{\varepsilon}\chi_i h}\left|a'(x_i)\right|\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,e^{-\frac{\sqrt{\alpha}x_{i+1}}{\sqrt{\varepsilon}}}dx. \tag{23}$$

We can easily view that the first term in the right side of (23) is that $O(h^2)$. From $a'(0)=0$ and $xe^{-x}\le e^{-\frac{x}{2}}$, $(x\ge 0)$ for the second term of (21), we have

$$\left|\frac{C}{\sqrt{\varepsilon}\chi_i h}a'(x_i)\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}dx\right|$$
$$\le\frac{C}{\sqrt{\varepsilon}\chi_i h}\left|a''(\bar{\xi}_i)\right|e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}\int\limits_{x_{i-1}}^{x_{i+1}}(x-x_i)^2\psi_i(x)\,dx$$
$$\le Ch^2\frac{x_i}{\sqrt{\varepsilon}}e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}$$
$$\le Ch^2\frac{x_i}{x_{i-1}}\frac{x_{i-1}}{\sqrt{\varepsilon}}e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}$$
$$\le Ch^2 i\,(i-1)^{-1}e^{-\frac{\sqrt{\alpha}x_{i-1}}{2\sqrt{\varepsilon}}}$$
$$\le Ch^2,\qquad i>1.$$

The same evaluation is achieved for the third term in the right side of (23) from $a'(l)=0$, for $i<N-1$. Thus, identity (21) is proved for $i=2,3,...,N-2$.

Also for $i=1$, using relations

$$a(x)=a(x_1)+(x-x_1)\,a'(x_1)+\frac{(x-x_1)^2}{2}a''(\xi_1),\quad \xi_1\in(x_1,x)$$

and

$$v(x)=v(x_0)+\int\limits_{x_0}^{x}v'(\xi)\,d\xi,$$

we get

$$R_1^{(1)}=\frac{1}{\chi_1 h}a'(x_1)\int\limits_{x_0}^{x_2}(x-x_1)\left[\int\limits_{x_0}^{x}v'(\xi)\,d\xi\right]\psi_1(x)\,dx$$

$$+\frac{1}{2\chi_1 h}\int\limits_{x_0}^{x_2}(x-x_1)^2\,a''\left(\xi_1\left(x\right)\right)v\left(x\right)\psi_1\left(x\right)dx. \tag{24}$$

From (22), the second term in the right side of (24) will be $O(h^2)$. From $a'\left(0\right)$ and Lemma 1, we can evaluate the first as following

$$\left|\frac{a'\left(x_1\right)}{\chi_1 h}\int\limits_{x_0}^{x_2}(x-x_1)\left[\int\limits_{x_0}^{x}v'\left(\xi\right)d\xi\right]\psi_1\left(x\right)dx\right|\leq\left|a'\left(x_1\right)\right|h\int\limits_{x_0}^{x_2}\left|v'\left(x\right)\right|dx$$

$$\leq Cx_1 h\left|a''\left(\bar{\eta}_1\right)\right|\int\limits_{x_0}^{x_2}\left\{1+\frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right\}dx$$

$$\leq Ch^2\left\{h+\frac{1}{\sqrt{\varepsilon}}\int\limits_{x_0}^{x_2}e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}dx\right\}$$

$$\leq Ch^2\left\{h+\sqrt{\alpha}^{-1}\left(1-e^{\frac{2\sqrt{\alpha}h}{\sqrt{\varepsilon}}}\right)\right\}=O\left(h^2\right).$$

Thus,

$$\left|R_1^{(1)}\right|=O\left(h^2\right)$$

are proved. The proof of $\left|R_{N-1}^{(1)}\right|=O\left(h^2\right)$ is similar. So, the inequality (20) is proved.

Next, it is not difficult to see that, for $f\in C^2[0,l]$

$$\left|R_i^{(2)}\right|=O\left(h^2\right),\qquad 1\leq i\leq N-1. \tag{25}$$

Finally, for $R_i^{(3)}$ we have

$$\left|R_i^{(3)}\right|\leq\left|\frac{\lambda}{\chi_i h}\int\limits_{x_{i-1}}^{x_{i+1}}dx\psi_i(x)\int\limits_{x_{i-1}}^{x_{i+1}}\left(\int\limits_0^l\frac{\partial^2 M\left(\xi,\zeta\right)}{\partial\xi^2}v\left(\zeta\right)d\zeta\right)M_1\left(x,\xi\right)d\xi\right|$$

$$+\left|\frac{1}{2}\lambda\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left(M\left(x_i,\xi\right)v\left(\xi\right)\right)''d\xi\right|. \tag{26}$$

By virtue of boundedness of $\frac{\partial^2 M}{\partial x^2}$, $v\left(x\right)$ and $|M_1(x,\zeta)|\leq Ch$ the first term in the right side of (26) will be $O(h^2)$.

Rearranging the second term in the right side of (26) gives

$$\frac{1}{2}\left|\lambda\right|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left(x_j-\xi\right)\left(\xi-x_{j-1}\right)\left|\left(M\left(x_i,\xi\right)v\left(\xi\right)\right)\right|''d\xi$$

$$\leq \frac{1}{2}|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M''(x_i,\xi)||v(\xi)|d\xi$$
$$+|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M'(x_i,\xi)||v'(\xi)|d\xi$$
$$+\frac{1}{2}|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M(x_i,\xi)||v''(\xi)|d\xi. \tag{27}$$

Hence, from $|v(x)|\leq C$ and $\frac{\partial^2 M}{\partial x^2}\in C^2[0,l]$ for the first term on the right side (27) will be $O(h^2)$.
For the second term in the right side (27), we have the estimate

$$|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M'(x_i,\xi)||v'(\xi)|d\xi\leq|\lambda|h^2\int_0^l|M'(x_i,\xi)||v'(\xi)|d\xi$$
$$\leq|\lambda|h^2\int_0^l\{|M'(x_i,\xi)||v(\xi)|+|M(x_i,\xi)||v'(\xi)|\}d\xi.$$

From here using Lemma 1 it is obtained the estimate

$$|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M'(x_i,\xi)||v'(\xi)|d\xi$$
$$\leq C|\lambda|h^2\int_0^l\left(1+1/\sqrt{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)\right)d\xi$$
$$\leq Ch^2. \tag{28}$$

For the third term in the right side (27), by virtue of (4) for $k=2$, we have

$$\frac{1}{2}|\lambda|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M(x_i,\xi)||v''(\xi)|d\xi$$
$$\leq C\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})|M(x_i,\xi)|\left\{1+\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right\}d\xi$$
$$\leq Ch^2\left\{1+\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}|M(x_i,\xi)|\left(\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)\right\}.$$

Taking into account the relations (the partial derivatives are estimated at intermediate points, as required by the mean value theorem, as indicated by the bar.)

$$M(x_i,\xi)=M(x,0)+\frac{\partial\bar{M}}{\partial\xi}\xi,\qquad M(x,0)=0,$$

we get

$$Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}|M(x_i,\xi)|\,\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}\,d\xi=Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}\left|M(x_i,0)+\frac{\partial\bar{M}}{\partial\xi}\xi\right|\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}$$

$$\leq Ch^2\int\limits_0^l\frac{\xi}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}\,d\xi,$$

from which after taking into consideration $xe^{-x}\leq e^{-\frac{x}{2}}$, we obtain

$$Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}|M(x_i,\xi)|\,\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}\,d\xi\leq Ch^2\frac{1}{\sqrt{\alpha}}\int\limits_0^l\frac{1}{\sqrt{\varepsilon}}e^{-\frac{\sqrt{\alpha}\xi}{2\sqrt{\varepsilon}}}\,d\xi$$

$$=Ch^2\frac{2}{\alpha}\left(1-e^{-\frac{\sqrt{\alpha}l}{\sqrt{\varepsilon}}}\right)\leq Ch^2.$$

Analogously, after using the relation

$$M(x_i,\xi)=M(x_i,l)+\frac{\partial\bar{M}}{\partial\xi}(\xi-l),\qquad M(x,l)=0,$$

it is not difficult to confirm that

$$Ch^2\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}|M(x_i,\xi)|\,\frac{1}{\varepsilon}e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\,d\xi\leq Ch^2.$$

Therefore, we obtain

$$\frac{1}{2}\,|\lambda|\sum_{j=1}^{N}\int\limits_{x_{j-1}}^{x_j}(x_j-\xi)\,(\xi-x_{j-1})\,|M\,(x_i,\xi)|\,|v''\,(\xi)|\,d\xi\leq Ch^2. \tag{29}$$

Thus, it can be easily seen that the first term in the right side of (26) is that $O(h^2)$. In addition, after taking into account (28) and (29) we obtain

$$\left|R_i^{(3)}\right|\leq Ch^2. \tag{30}$$

From (20), (25) and (30), we have

$$|R_i|\leq Ch^2. \tag{31}$$

The bound (19) together with (31) finish the proof. □

4. Numerical Calculates

In this section, theoretical calculates are tested on one sample.
Our particular example is

$$Lv := -\varepsilon v''(x) + \left(2 - cos^2(\pi x)\right) v(x) + \frac{1}{2}\int_0^1 \left(e^{xsin(\pi\zeta)} - 1\right) v(\zeta) d\zeta = (1+x)^{-1}, \quad (0 < x < 1),$$

$$v(0) = 1, \qquad v(1) = 0.$$

The exact solution to this problem is unknown. For this reason, we estimate errors and calculate solutions using the double-mesh method, which compares the obtained solution to a solution computed on a mesh that is twice as fine. We introduce the maximum point-wise errors and the computed ε-uniform maximum point-wise errors as

$$e_\varepsilon^N = \max_i |y_i^{\varepsilon,N} - \tilde{y}_{2i}^{\varepsilon,2N}|_{\infty,\overline{\omega}_N}, \quad e^N = \max_\varepsilon e_\varepsilon^N,$$

where $\tilde{y}_{2i}^{\varepsilon,2N}$ is the approximate solution of the related method on the mesh

$$\tilde{\omega}_{2N} = \{x_{\frac{i}{2}} : i = 0, 1, ..., 2N\}, \quad x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad 0 \le i \le N-1.$$

We also describe the computed ε-uniform the rates of convergence and the rates of convergence as follows

$$p_\varepsilon^N = \frac{\ln\left(\frac{e_\varepsilon^N}{e_\varepsilon^{2N}}\right)}{\ln 2}, \quad p^N = \frac{\ln\left(\frac{e^N}{e^{2N}}\right)}{\ln 2}.$$

The rate of convergence of the difference approximation is significantly in agreement with the theoretical analysis, as shown in the Table 1.

5. Conclusion

In this paper, we described a new second-order difference scheme, which was constructed on the uniform mesh by using composite trapezoidal rule for integral term involving kernel function to solve linear FIDEs with singular perturbation. We tested the technique on one example with various values of ε and N to demonstrate the appropriateness of the method. Numerical investigations can be sustained for more sophisticated types such as partial integro-differential equations, nonlinear, delay form, higher dimensional, etc.

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

ε	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$
1	0,00873286	0,00220145	0,00055304	0,00013874	0,00003471
	1,988	1,993	1,995	1,999	
10^{-2}	0,01038533	0,00262165	0,00065997	0,00016591	0,00004165
	1,986	1,99	1,992	1,994	
10^{-4}	0,01110798	0,00281381	0,00071081	0,00017931	0,00004514
	1,981	1,985	1,987	1,99	
10^{-6}	0,01173901	0,00297572	0,00075327	0,00019055	0,00004817
	1,98	1,982	1,983	1,984	
10^{-8}	0,01168824	0,00296902	0,00075366	0,00019131	0,00004853
	1,977	1,978	1,978	1,979	
e^N	0,01173901	0,00297572	0,00075366	0,00019131	0,00004853
p^N	1,98	1,982	1,978	1,979	

TABLE 1. Maximum point-wise errors and convergence rates for various ε and N values.

Declaration of Competing Interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Amiraliyev, G. M., Durmaz, M. E., Kudu, M., Uniform convergence results for singularly perturbed Fredholm integro-differential equation, *J. Math. Anal.*, 9(6) (2018), 55–64.

[2] Amiraliyev, G. M., Durmaz, M. E., Kudu, M., Fitted second order numerical method for a singularly perturbed Fredholm integro-differential equation, *Bull. Belg. Math. Soc. Simon Steven.*, 27(1) (2020), 71–88. https://doi.org/10.36045/bbms/1590199305

[3] Amiraliyev, G. M., Durmaz, M. E., Kudu, M., A numerical method for a second order singularly perturbed Fredholm integro-differential equation, *Miskolc Math. Notes.*, 22(1) (2021), 37–48. https://doi.org/10.18514/MMN.2021.2930

[4] Amiraliyev, G. M., Mamedov, Y. D., Difference schemes on the uniform mesh for singularly perturbed pseudo-parabolic equations, *Turk. J. Math.*, 19 (1995), 207–222

[5] Brunner, H., Numerical Analysis and Computational Solution of Integro-Differential Equations, Contemporary Computational Mathematics-A Celebration of the 80th Birthday of Ian Sloan (J. Dick et al., eds.), Springer, Cham, 2018, 205–231. https://doi.org 10.1007/978-3-319-72456-0_11

[6] Chen, J., He, M., Zeng, T., A multiscale Galerkin method for second-order boundary value problems of Fredholm integro differential equation II: Efficient algorithm for the discrete linear system, *J. Vis. Commun. Image R.*, 58 (2019), 112–118. https://doi.org/10.1016/j.jvcir.2018.11.027

[7] Chen, J., He, M., Huang, Y., A fast multiscale Galerkin method for solving second order linear Fredholm integro-differential equation with Dirichlet boundary conditions, *J. Comput. Appl. Math.*, 364 (2020), 112352. https://doi.org/10.1016/j.cam.2019.112352

[8] Dehghan, M., Chebyshev finite difference for Fredholm integro-differential equation, *Int. J. Comput. Math.*, 85 (1) (2008), 123–130. https://doi.org/10.1080/00207160701405436

[9] Doolan, E. R., Miller, J. J. H., Schilders, W. H. A., Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.
[10] Durmaz, M. E., Amiraliyev, G. M., A robust numerical method for a singularly perturbed Fredholm integro-differential equation, *Mediterr. J. Math.*, 18(24) (2021), 1–17. https://doi.org/10.1007/s00009-020-01693-2
[11] Durmaz, M. E., Amiraliyev, G. M., Kudu, M., Numerical solution of a singularly perturbed Fredholm integro differential equation with Robin boundary condition, *Turk. J. Math.*, 46(1) (2022), 207–224. https://doi.org/10.3906/mat-2109-11
[12] Farrell, P. A., Hegarty, A. F., Miller, J. J. H., O'Riordan, E., Shishkin, G. I., Robust Computational Techniques for Boundary Layers, Chapman Hall/CRC, New York, 2000. https://doi.org/10.1201/9781482285727
[13] Jalilian, R., Tahernezhad, T., Exponential spline method for approximation solution of Fredholm integro-differential equation, *Int. J. Comput. Math.*, 97(4) (2020), 791–801. https://doi.org/10.1080/00207160.2019.1586891
[14] Jalius, C., Majid, Z. A., Numerical solution of second-order Fredholm integrodifferential equations with boundary conditions by quadrature-difference method, *J. Appl. Math.*, (2017). https://doi.org/10.1155/2017/2645097
[15] Kadalbajoo, M. K., Gupta, V., A brief survey on numerical methods for solving singularly perturbed problems, *Appl. Math. Comput.*, 217 (2010), 3641–3716. https://doi.org/10.1016/j.amc.2010.09.059
[16] Karim, M. F., Mohamad, M., Rusiman, M. S., Che-him, N., Roslan, R., Khalid, K., ADM for solving linear second-order Fredholm integro-differential equations, *Journal of Physics*, (2018), 995. https://doi.org/10.1088/1742-6596/995/1/012009
[17] Kudu, M., Amirali, I., Amiraliyev, G. M., A finite-difference method for a singularly perturbed delay integro-differential equation, *J. Comput. Appl. Math.*, 308 (2016), 379–390. https://doi.org/10.1016/j.cam.2016.06.018
[18] Miller, J. J. H., O'Riordan, E., Shishkin, G. I., Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore, 1996.
[19] Nayfeh, A. H., Introduction to Perturbation Techniques, Wiley, New York, 1993.
[20] O'Malley, R. E., Singular Perturbations Methods for Ordinary Differential Equations, Springer, New York, 1991. https://doi.org/10.1007/978-1-4612-0977-5
[21] Roos, H. G., Stynes, M., Tobiska, L., Numerical Methods for Singularly Perturbed Differential Equations, Springer-Verlaq, Berlin, 1996. https://doi.org/10.1007/978-3-662-03206-0
[22] Samarskii, A. A., The Theory of Difference Schemes(1st ed.), CRC Press, 2001. https://doi.org/10.1201/9780203908518
[23] Shahsavaran, A., On the convergence of Lagrange interpolation to solve special type of second kind Fredholm integro differential equations, *Appl. Math. Sci.*, 6(7) (2012), 343–348.
[24] Yapman, Ö., Amiraliyev, G. M., Amirali, I., Convergence analysis of fitted numerical method for a singularly perturbed nonlinear Volterra integro-differential equation with delay, *J. Comput. Appl. Math.*, 355(2019), 301309. https://doi.org/10.1016/j.cam.2019.01.026
[25] Yapman, Ö., Amiraliyev, G. M., A novel second–order fitted computational method for a singularly perturbed Volterra integro-differential equation, *Int. J. Comput. Math.*, 97(6) (2020), 1293–1302. https://doi.org/10.1080/00207160.2019.1614565
[26] Xue, Q., Niu, J., Yu, D., Ran, C., An improved reproducing kernel method for Fredholm integro-differential type two-point boundary value problems, *Int. J. Comput. Math.*, 95(5) (2018), 1015–1023. https://doi.org/10.1080/00207160.2017.1322201

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 968–977 (2022)
DOI:10.31801/cfsuasmas.1061084
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: January 21, 2022; Accepted: May 16, 2022

ON $\mathcal{F}$-COSMALL MORPHISMS

Berke KALEBOĞAZ and Derya KESKİN TÜTÜNCÜ

Department of Mathematics, Hacettepe University, Ankara, TURKEY

Abstract. In this paper, we first define the notion of $\mathcal{F}$-cosmall quotients for an additive exact substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$. We show that every $\mathcal{F}$-cosmall quotient is right minimal in some cases. We also give the definition of $\mathcal{F}$-superfluous quotients and we relate it the approximation of modules. As an application, we investigate our results in a pure-exact substructure $\mathcal{F}$.

1. Introduction

In [12], Ziegler introduced the partial morphisms by using model theory of modules. Then in [9], the partial morphisms was investigated by Monari Martinez in terms of systems of linear equations. But this algebraic definition of partial morphisms was not useful in the categorical studies of purity. Then in [4] Cortés-Izurdiaga, Guil Asensio, Kaleboğaz and Srivastava studied partial morphisms by using category theory. In [4], the authors defined partial morphisms by using pushout with respect to an additive exact substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$ and they call them $\mathcal{F}$-partial morphisms. They showed that the definition of $\mathcal{F}$-partial morphisms with the pure-exact substructure $\mathcal{F}$ in the category of right R-modules are coincide with the partial morphisms that defined by Ziegler in [12]. By using $\mathcal{F}$-partial morphisms they also define $\mathcal{F}$-small extension and gave an application of this definition to the pure-exact substructure $\mathcal{F}$ in the category of right modules over a ring and called it Ziegler small extension. As a dual notion of $\mathcal{F}$-partial morphisms, in [6] $\mathcal{F}$-copartial morphisms was defined by Kaleboğaz: a morphism $f : X \longrightarrow U$ is $\mathcal{F}$-copartial morphism with respect to a quotient map $p : Y \longrightarrow U$ if and only if $\mathrm{Ext}^1(f, -)$ transforms p in an $\mathcal{F}$-deflation. She studied the properties of $\mathcal{F}$-copartial morphisms and investigated the applications of $\mathcal{F}$-copartial morphisms to some exact substructures of $\mathcal{E}$ in the category of right R-modules.

Keywords. $\mathcal{F}$-cosmall quotients, right minimal morphisms, $\mathcal{F}$-superfluous quotients.
bkuru@hacettepe.edu.tr-Corresponding author; 0000-0002-4903-2244
keskin@hacettepe.edu.tr; 0000-0002-3459-1924.

In this paper, by using $\mathcal{F}$-copartial morphisms, we first introduce $\mathcal{F}$-cosmall quotients for any additive exact substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$ (see Definition 2). We also give a new characterization of $\mathcal{F}$-cosmall quotients (see Proposition 1). As an application to a pure-exact structure $\mathcal{F}$, we give the definition of pure-cosmall quotients and we say that pure-cosmall quotients are dual of Ziegler small extensions.

A morphism $p : M \longrightarrow N$ is called *right minimal* if any endomorphism $g : M \longrightarrow M$ with $pg = p$ is an isomorphism (see [1, page 6]). In [8], right minimal morphisms are studied by Keskin Tütüncü. In [8] the author dualizes some results in [3] and gets several useful results by investigating the relationship between $\mathrm{End}_R(N)$ and $\mathrm{End}_R(M)$ when there is a right minimal epimorphism $p : M \longrightarrow N$. The author also proves that there is an isomorphism between two rings $END_R^M(N)/J(END_R^M(N))$ and $END_R^N(M)/J(END_R^N(M))$ if there exists a right minimal epimorphism $p : M \longrightarrow N$ in [8, Theorem 2.6 (1)]. As a consequence of this result the structure of the endomorphism ring of a quasi-projective module and an automorphism-coinvariant module are explained. One of the main purposes of this paper is to give an example of right minimal morphisms. In Theorem 1, we prove that every $\mathcal{F}$-cosmall quotient $f : P \longrightarrow M$ with P an $\mathcal{F}$-projective object (projective objects with respect to $\mathcal{F}$-deflations) is right minimal. An application of this theorem to the pure-exact structure gives us the dual version of [3, Proposition 1.6]. Moreover, we give the definitions of $\mathcal{F}$-superfluous quotient and weakly $\mathcal{F}$-superfluous quotient (see Definition 5). Then we investigate the relation between $\mathcal{F}$-cosmall quotient and $\mathcal{F}$-superfluous quotient (see Proposition 2). And finally we relate to the existence of approximations of modules. In Theorem 2, we show that a weakly $\mathcal{F}$-superfluous quotient $p : Y \longrightarrow U$ with $\mathcal{F}$-projective Y is an $\mathcal{F}$-Proj-cover when $\mathcal{F}$-Proj is the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.

2. Results

Let $\mathcal{A}$ be an additive category and (i, p) be a pair of composable morphisms in $\mathcal{A}$:

$$A \xrightarrow{i} B \xrightarrow{p} C$$

If i is a kernel of p and p is a cokernel of i then (i, p) is called *kernel-cokernel pair in* $\mathcal{A}$. Let $\mathcal{E}$ be the class of kernel-cokernel pairs on $\mathcal{A}$. i is called an *admissible monomorphism* if there exists a morphism p such that $(i, p) \in \mathcal{E}$. Similarly, p is called an *admissible epimorphism* if there exists a morphism i such that $(i, p) \in \mathcal{E}$.

The class of kernel-cokernel pairs $\mathcal{E}$ is said to be an *exact structure on* $\mathcal{A}$ if it is closed under isomorphisms and satisfies the following conditions;

[E0] For every object $A \in \mathcal{A}$, the identity morphism 1_A is an admissible monomorphism.

[E0$^{\mathrm{op}}$] For every object $A \in \mathcal{A}$, the identity morphism 1_A is an admissible epimorphism.

[E1] The classes of admissible monomorphisms are closed under compositions.

[E1$^{\mathrm{op}}$] The classes of admissible epimorphisms are closed under compositions.

[E2] The pushout of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism, that is, for any admissible monomorphism $i : A \longrightarrow B$ and any morphism $f : A \longrightarrow B'$, there is a pushout diagram;

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B' & \xrightarrow{i'} & P \end{array}$$

with i' an admissible monomorphism.

[E2$^{\mathrm{op}}$] The pullback of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism, that is, for any admissible epimorphism $p : B \longrightarrow C$ and any morphism $g : B' \longrightarrow C$ there is a pullback diagram;

$$\begin{array}{ccc} Q & \xrightarrow{p'} & B' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{p} & C \end{array}$$

with p' an admissible epimorphism.

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ with an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$. Elements of $\mathcal{E}$ are called *short exact sequences*. Keller [7] uses *conflation*, *inflation* and *deflation* for what we call short exact sequence, admissible monomorphism and admissible epimorphism, respectively. Throughout the paper we also use this terminology. Let A be an object of $\mathcal{A}$. An *admissible quotient of* A is a quotient object U of an object A such that one (and any) quotient map $p : A \longrightarrow U$ is a deflation.

An *exact substructure* $\mathcal{F}$ of $\mathcal{E}$ is just an exact structure on $\mathcal{A}$ such that each conflation in $\mathcal{F}$ (that we shall call $\mathcal{F}$-conflation) is also a conflation in $\mathcal{E}$. Inflations, deflations and admissible quotient objects with respect to $\mathcal{F}$ will be called $\mathcal{F}$-inflations, $\mathcal{F}$-deflations and $\mathcal{F}$-admissible quotient objects, respectively.

We shall start with giving the definition of $\mathcal{F}$-copartial morphisms (respectively, $\mathcal{F}$-copartial isomorphisms) for an additive substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$. $\mathcal{F}$-copartial morphisms first introduced and investigated in [6] by Kaleboğaz as the dual notion of $\mathcal{F}$-partial morphism that are studied in [4].

For the rest of the paper, we fix an exact category of $(\mathcal{A}, \mathcal{E})$ and an additive exact substructure $\mathcal{F}$ of $\mathcal{E}$.

Definition 1. Let X, Y be objects of $\mathcal{A}$ and U an admissible quotient of Y with the quotient map $p : Y \longrightarrow U$.

Let $f : X \longrightarrow U$ be a morphism and consider the pullback of f along the quotient map p:

$$\begin{array}{ccc} Q & \xrightarrow{\overline{p}} & X \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & U \end{array}$$

Then:

(1) f is called an $\mathcal{F}$*-copartial morphism* from X to Y with codomain U if $\overline{p}$ is an $\mathcal{F}$-deflation.

(2) f is called an $\mathcal{F}$*-copartial isomorphism* from X to Y with codomain U if both $\overline{p}$ and $\overline{f}$ are $\mathcal{F}$-deflations.

Now we recall two lemmas from [6], without proofs, that we will use in the rest of the paper. The first lemma given below is a special case of the dual of Obscure Axiom in [2, Proposition 2.16] (see [6, Proposition 2.3]). The other one is one of the main properties of $\mathcal{F}$-copartial morphisms (see [6, Proposition 2.5(1)]).

Lemma 1. *Let X, Y, Z be objects of $\mathcal{A}$. If an $\mathcal{F}$-deflation $f : Z \longrightarrow Y$ factors through an deflation $p : X \longrightarrow Y$ as follows;*

$$\begin{array}{ccc} & & Z \\ & {\scriptstyle g}\swarrow & \downarrow{\scriptstyle f} \\ X & \xrightarrow{p} & Y \end{array}$$

then p is an $\mathcal{F}$-deflation too.

Lemma 2. *Let X, Y be objects of $\mathcal{A}$ and U, an admissible quotient of Y with the quotient morphism $p : Y \longrightarrow U$. Suppose that p is an $\mathcal{F}$-deflation. A morphism $f : X \longrightarrow U$ is an $\mathcal{F}$-deflation if and only if f is an $\mathcal{F}$-copartial isomorphism from X to Y with codomain U.*

As a consequence of this lemma, we can give the following corollary:

Corollary 1. *Let Y be an object of $\mathcal{A}$ and $g : Z \longrightarrow Y$ be any morphism with any object Z in $\mathcal{A}$. g is an $\mathcal{F}$-deflation if and only if g is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y.*

Proof. Let us take the pullback of g along 1_Y. Since 1_Y is an $\mathcal{F}$-deflation, g is an $\mathcal{F}$-deflation if and only if g is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y by Lemma 2. □

One of the aims of this paper is to give an example of right minimal morphisms. To attain our goal we shall first give the definition of $\mathcal{F}$-cosmall quotient morphisms. These morphisms are dual of $\mathcal{F}$-small extensions that are defined in [4, Definition 3.4].

Definition 2. Let the object Y of $\mathcal{A}$ be an admissible quotient of any object X with the quotient map $p' : X \longrightarrow Y$, U be an admissible quotient of X and $p : Y \longrightarrow U$ be a deflation.

(1) We shall say that Y is *$\mathcal{F}$-cosmall in U over X* if for any $\mathcal{F}$-copartial morphism $g : Z \longrightarrow Y$ from any object Z to X with codomain Y, the following holds:

pg is an $\mathcal{F}$-copartial isomorphism from Z to X with codomain U implies that g is an $\mathcal{F}$-copartial isomorphism from Z to X with codomain Y.

(2) We shall say that Y is *$\mathcal{F}$-cosmall in U* if Y is $\mathcal{F}$-cosmall in U over Y. Namely, the deflation p' is the identity morphism of Y.

With the notion of $\mathcal{F}$-cosmall object which is defined above, now we can define $\mathcal{F}$-cosmall quotient morphisms as in the following:

Definition 3. Let $p : Y \longrightarrow U$ be a deflation. If Y is $\mathcal{F}$-cosmall in U then the deflation $p : Y \longrightarrow U$ is called an *$\mathcal{F}$-cosmall quotient*.

Namely, if Y is $\mathcal{F}$-cosmall in U over Y then p is an $\mathcal{F}$-cosmall quotient.

Here we will give a characterization of $\mathcal{F}$-cosmall quotient which will be used in the rest of the paper.

Proposition 1. *Let $p : Y \longrightarrow U$ be a deflation. p is an $\mathcal{F}$-cosmall quotient if and only if for any morphism $g : Z \longrightarrow Y$ for any object Z such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U, g is an $\mathcal{F}$-deflation.*

Proof. Let Z be an object of $\mathcal{A}$ and $g : Z \longrightarrow Y$ be a morphism such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. We will show that g is an $\mathcal{F}$-deflation. If we take pullback of g along 1_Y, then we get the following commutative diagram:

$$\begin{array}{ccc} Q & \xrightarrow{h} & Z \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{1_Y} & Y \end{array}$$

Since 1_Y is an $\mathcal{F}$-deflation, h is an $\mathcal{F}$-deflation. Therefore, g is an $\mathcal{F}$-copartial morphism from Z to Y with codomain Y. As p is an $\mathcal{F}$-cosmall quotient, g is also an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y. Then, by Corollary 1, g is an $\mathcal{F}$-deflation.

For the converse, to show that p is an $\mathcal{F}$-cosmall quotient, let us take an $\mathcal{F}$-copartial morphism $g : Z \longrightarrow Y$ from Z to Y with codomain Y such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. By assumption, g is an $\mathcal{F}$-deflation. By Corollary 1, g is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y. Therefore, p is an $\mathcal{F}$-cosmall quotient. □

Let R be a ring, Y and Z be right R-modules and $f : Y \longrightarrow Z$ be an epimorphism. Recall that, f is called *pure epimorphism* if $\mathrm{Hom}_R(M, f) : Hom_R(M, Y) \longrightarrow$

$\mathrm{Hom}_R(M, Z)$ is an epimorphism for all finitely presented right R-modules M. Let X be the kernel of f with the inclusion $u : X \longrightarrow Y$. Then by the theorem of Fieldhouse [5] and Warfield [10], f is pure epimorphism if and only if X is pure in Y (u is a pure monomorphism) in the sense that the natural homomorphism $X \otimes_R N \longrightarrow Y \otimes_R N$ derived from the inclusion map $u : X \longrightarrow Y$ is a monomorphism for all left R-modules N. Then, the conflation $X \longrightarrow Y \longrightarrow Z$ is said to be a *pure conflation* if f is a pure epimorphism (or u is a pure monomorphism). The class of all pure conflations is exact substructure of exact structure of the class of all conflations from [2, Exercise 5.6]. $\mathcal{F}$-copartial morphisms (respectively, $\mathcal{F}$-copartial isomorphisms) with respect to a pure-exact substructure $\mathcal{F}$ in the category of right R-modules are called *copartial morphisms (respectively, copartial isomorphisms)*, (see [6]). Here we will define pure-cosmall quotient morphisms as an application of $\mathcal{F}$-cosmall quotient with respect to a pure-exact substructure $\mathcal{F}$ in the category of right R-modules.

Definition 4. Let Y and U be right R-modules. An epimorphism $p : Y \longrightarrow U$ is called a *pure-cosmall quotient* if Y is pure-cosmall in U, that means, for any right R-module Z, any copartial morphism $g : Z \longrightarrow Y$ from Z to Y with codomain Y, the following holds:

If pg is a copartial isomorphism from Z to Y with codomain U then g is a copartial isomorphism from Z to Y with codomain Y.

Corollary 2. *Let Y and U be right R-modules, $p : Y \longrightarrow U$ be a deflation. p is a pure-cosmall quotient if and only if for any right R-module Z, any morphism $g : Z \longrightarrow Y$ such that pg is a copartial isomorphism from Z to Y with codomain U is a pure epimorphism.*

Pure-cosmall quotients are the dual of Ziegler small extensions that are introduced in [4] and are studied in [3]. In [3], the authors proved that every Ziegler small extension $u : M \longrightarrow E$ with E being pure-injective is a left minimal morphism. Now we proceed to extend dual of this result to any exact substructure $\mathcal{F}$. We will show that $\mathcal{F}$-cosmall quotient morphisms are right minimal under a condition. So the following theorem gives us an example of right minimal morphisms.

Let P be an object of $\mathcal{A}$ and $p : Y \longrightarrow Z$ be a deflation. Recall that, P is said to be *p-projective* (or projective with respect to p) if for each morphism $f : P \longrightarrow Z$ there exist a morphism $g : P \longrightarrow Y$ with $pg = f$. P is said to be a *projective object in* $\mathcal{A}$ if it is projective with respect to each deflation. Projective objects with respect to $\mathcal{F}$-deflations will be called *$\mathcal{F}$-projective objects*.

Theorem 1. *Every $\mathcal{F}$-cosmall quotient $f : P \longrightarrow M$ with P being an $\mathcal{F}$-projective object is right minimal.*

Proof. Let $g : P \longrightarrow P$ be a morphism such that $fg = f$. Now we will show that g is an isomorphism. If we consider the pullback of f along fg we get the following commutative diagram;

$$\begin{array}{ccc} Q & \xrightarrow{h_2} & P \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle fg} \\ P & \xrightarrow{f} & M \end{array}$$

Since $fg = f$, the identity map 1_P satisfies that $fg1_P = f1_P$. Then by the universal property of pullback, there exist $\alpha : P \longrightarrow Q$ such that $h_1\alpha = 1_P$ and $h_2\alpha = 1_P$. By Lemma 1, h_1 and h_2 are both $\mathcal{F}$-deflations. Therefore, fg is an $\mathcal{F}$-copartial isomorphism from P to P with codomain M. Since f is an $\mathcal{F}$-cosmall quotient, g is an $\mathcal{F}$-deflation by Proposition 1. So it is an epimorphism.

Now, using that P is an $\mathcal{F}$-projective, we get that there exists $h : P \longrightarrow P$ such that $gh = 1_P$. Then $f = f1_P = fgh = fh$. By using the previous argument we conclude that h is an epimorphism. Then as $hgh = h = 1_p h$, we get that $hg = 1_p$. Therefore, g is a monomorphism. So g is an isomorphism. □

Corollary 3. *Every pure-cosmall quotient $f : P \longrightarrow M$ with P being a pure-projective right R-module is right minimal.*

Now we will give the definition of $\mathcal{F}$-superfluous and weakly $\mathcal{F}$-superfluous quotients.

Definition 5. Let X and Y be objects of $\mathcal{A}$.

(1) An *$\mathcal{F}$-superfluous quotient* is an $\mathcal{F}$-deflation $p : X \longrightarrow Y$ such that for any object of Z in $\mathcal{A}$ and any morphism $\alpha : Z \longrightarrow X$ the following holds:

$p\alpha$ is an $\mathcal{F}$-deflation implies that α is an $\mathcal{F}$-deflation.

(2) A *weakly $\mathcal{F}$-superfluous quotient* is an $\mathcal{F}$-deflation $p : X \longrightarrow Y$ such that for any object of Z in $\mathcal{A}$ and any morphism $\alpha : Z \longrightarrow X$ the following holds:

$p\alpha$ is an $\mathcal{F}$-deflation implies that α is a deflation.

Remark 1. (1) If $\mathcal{A}$ is the category of right R-modules and $\mathcal{E}$ is the abelian exact structure, then $\mathcal{E}$-superfluous quotient morphism is coincide with the small epimorphism that is recalled in [8, Example 2.2(2)].

(2) If $\mathcal{A}$ is the category of right R-modules and $\mathcal{F}$ is the pure-exact structure, then $\mathcal{F}$-superfluous quotient morphism is coincide with the $\mathbb{S}$-superfluous epimorphism for $\mathbb{S}$ being the class of finitely presented modules that is introduced in [11].

Now we give the relation between $\mathcal{F}$-cosmall quotient and $\mathcal{F}$-superfluous quotient.

Proposition 2. *Let $p : Y \longrightarrow U$ be a deflation. p is an $\mathcal{F}$-superfluous quotient if and only if p is an $\mathcal{F}$-deflation and $\mathcal{F}$-cosmall quotient.*

Proof. Suppose that p is an $\mathcal{F}$-superfluous quotient. So p is an $\mathcal{F}$-deflation. Now we will show that p is an $\mathcal{F}$-cosmall quotient. Let us take an object Z and a morphism $g : Z \longrightarrow Y$ such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. Now if we take the pullback of pg along p we get the following commutative diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\overline{p}} & Z \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle pg} \\ Y & \xrightarrow{p} & U \end{array}$$

By Lemma 2, pg is an $\mathcal{F}$-deflation. Then g is an $\mathcal{F}$-deflation by the definition of $\mathcal{F}$-superfluous quotient. Therefore, by Proposition 1, p is an $\mathcal{F}$-cosmall quotient.

For the converse, assume that p is an $\mathcal{F}$-deflation and $\mathcal{F}$-cosmall quotient. To show that p is an $\mathcal{F}$-superfluous quotient let us take a morphism $\alpha : Z \longrightarrow Y$ such that $p\alpha$ is an $\mathcal{F}$-deflation. Now take the pullback of $p\alpha$ along p we get the following commutative diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\overline{p}} & Z \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle p\alpha} \\ Y & \xrightarrow{p} & U \end{array}$$

By Lemma 2, $p\alpha$ is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. Since p is an $\mathcal{F}$-cosmall quotient, α is an $\mathcal{F}$-deflation. Therefore p is an $\mathcal{F}$-superfluous quotient. □

Let $\mathcal{A}$ be any category and $\mathcal{X}$ be a class of objects in $\mathcal{A}$. Recall that, a morphism $\phi : X \longrightarrow Y$ in $\mathcal{A}$ is a $\mathcal{X}$-precover of Y if $X \in \mathcal{X}$ and for any morphism $f : Z \longrightarrow Y$ with $Z \in \mathcal{X}$, there is a morphism $g : Z \longrightarrow X$ such that $\phi g = f$. A $\mathcal{X}$-precover $\phi : X \longrightarrow Y$ is said to be a $\mathcal{X}$-cover if every morphism $g : X \longrightarrow X$ such that $\phi g = \phi$ is an isomorphism. It is clear that, an $\mathcal{X}$-cover is an $\mathcal{X}$-precover which is a right minimal morphism.

In the next result we will show that, under certain circumstances, a weakly $\mathcal{F}$-superfluous quotient $p : Y \longrightarrow U$ with Y being $\mathcal{F}$-projective is actually an $\mathcal{F}$-Proj-cover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.

Theorem 2. *Let $p : Y \longrightarrow U$ be a deflation. Consider the following assertions:*

(1) *p is an $\mathcal{F}$-superfluous quotient and Y is an $\mathcal{F}$-projective object.*
(2) *p is an $\mathcal{F}$-deflation, Y is an $\mathcal{F}$-projective and p is an $\mathcal{F}$-cosmall quotient.*
(3) *p is an $\mathcal{F}$-deflation, Y is an $\mathcal{F}$-projective and for any object X, each morphism $f : X \longrightarrow Y$ satisfying that pf is an $\mathcal{F}$-deflation, is a split epimorphism.*
(4) *p is an $\mathcal{F}$-Proj-cover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.*
(5) *p is a weakly $\mathcal{F}$-superfluous quotient with Y being $\mathcal{F}$-projective object.*

We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)*,* (2) $\Rightarrow$ (4)*,* (1) $\Rightarrow$ (5)*.*

If there exists an $\mathcal{F}$*-deflation* $\alpha : P \longrightarrow U$ *with* P *being an* $\mathcal{F}$*-projective object then* (4) $\Rightarrow$ (3)*.*

If there exists an $\mathcal{F}$*-superfluous quotient* $\alpha : P \longrightarrow U$ *with* P *being an* $\mathcal{F}$*-projective object then* (5) $\Rightarrow$ (1)*.*

Proof. (1) $\Leftrightarrow$ (2) Obvious from Proposition 2.

(1) $\Rightarrow$ (3) Let $f : X \longrightarrow Y$ be a morphism with pf being an $\mathcal{F}$-deflation. Since p is an $\mathcal{F}$-superfluous quotient, f is an $\mathcal{F}$-deflation. As Y is an $\mathcal{F}$-projective module, f is a split epimorphism.

(3) $\Rightarrow$ (1) It is clear since split epimorphisms are $\mathcal{F}$-deflations.

(2) $\Rightarrow$ (4) Since p is an $\mathcal{F}$-deflation, it is an $\mathcal{F}$-Proj-precover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$. As p is an $\mathcal{F}$-cosmall quotient, p is right minimal by Theorem 1. Therefore, p is an $\mathcal{F}$-Proj-cover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.

(1) $\Rightarrow$ (5) It is clear, since every $\mathcal{F}$-superfluous quotient is weakly $\mathcal{F}$-superfluous quotient.

(4) $\Rightarrow$ (3) Assume that there exists an $\mathcal{F}$-deflation $\alpha : P \longrightarrow U$ with P being an $\mathcal{F}$-projective object. Since p is an $\mathcal{F}$-Proj-precover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$, there exists $g : P \longrightarrow Y$ such that $pg = \alpha$. Since α is an $\mathcal{F}$-deflation, then p is also an $\mathcal{F}$-deflation by Lemma 1. Now let $f : X \longrightarrow Y$ be a morphism such that pf is an $\mathcal{F}$-deflation. Since Y is an $\mathcal{F}$-projective object then there exists $h : Y \longrightarrow X$ such that $pfh = p$. As p is an $\mathcal{F}$-Proj-cover then fh is an isomorphism. Therefore, f is split.

(5) $\Rightarrow$ (1) There exists an $\mathcal{F}$-superfluous quotient $\alpha : P \longrightarrow U$ with P is an $\mathcal{F}$-projective object. Since Y is $\mathcal{F}$-projective, there exists a morphism $w : Y \longrightarrow P$ such that $\alpha w = p$. Since p is an $\mathcal{F}$-deflation and α is an $\mathcal{F}$-superfluous then w is an $\mathcal{F}$-deflation. And α is an $\mathcal{F}$-deflation too by Lemma 1. As P is an $\mathcal{F}$-projective object then there exists $h : P \longrightarrow Y$ such that $wh = 1_P$. So w is an epimorphism. We get $ph = \alpha wh = \alpha 1_P = \alpha$. Then h is an $\mathcal{F}$-deflation as p is a weakly $\mathcal{F}$-superfluous. Then $hwh = h1_P = 1_P h$. Since h is epic $hw = 1_P$. So w is a monomorphism. Therefore, w is an isomorphism. By $\alpha w = p$ and α is an $\mathcal{F}$-superfluous quotient then p is an $\mathcal{F}$-superfluous quotient. □

Remark 2. Let $p : Y \longrightarrow U$ be a deflation with Y an $\mathcal{F}$-projective object of $\mathcal{A}$. From Theorem 2 (4)$\Rightarrow$(2), we can say that if p is an $\mathcal{F}$-Proj-cover of U for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$, then p is an $\mathcal{F}$-cosmall quotient. But Theorem 1 shows that p can be an $\mathcal{F}$-cosmall quotient map which is not an $\mathcal{F}$-Proj-cover (since here p need not be an $\mathcal{F}$-deflation). But p is always right minimal.

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements We would like to thank the referee for his/her careful reading of the manuscript.

References

[1] Auslander, M., Rieten, I., Smalø S.O., Represantation Theory of Artin Algebras Volume 36 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995.

[2] Bühler, T., Exact categories, *Expo. Math.*, 28(1) (2010), 1–69.

[3] Cortés-Izurdiaga, M., Guil Asensio, P.A., Keskin Tütüncü D., Srivastava, A.K., Endomorphism rings via minimal morphisms, *Mediterr. J. Math.*, 18(152) (2021), 16 pages. https://doi.org/10.1007/s00009-021-01802-9

[4] Cortés-Izurdiaga, M., Guil Asensio, P.A., Kaleboğaz, B., Srivastava, A.K., Ziegler partial morphisms in additive exact categories, *Bull. Math. Sci.*, 10(3) 2050012 (2020), 37 pages. https://doi.org/10.1142/S1664360720500125

[5] Fieldhouse, D. J., Pure theories, *Math. Ann.*, 184 (1969), 1-18.

[6] Kaleboğaz, B., $\mathcal{F}$-copartial morphisms, Accepted in Bull. Malaysian Math. Sci. Soc.

[7] Keller, B., Chain complexes and stable categories, *Manuscripta Math.*, 67(4) (1990), 379–417.

[8] Keskin Tütüncü, D., Subrings of endomorphism rings associated with right minimal morphisms, submitted.

[9] Monari Martinez, E., On Pure-Injective Modules, *Abelian Groups and Modules*, Proceedings of the Udine Conference CISM Courses and Lectures No. 287, Springer-Verlag, Vienna-New York (1984), 383-393. https://doi.org/10.1007/978-3-7091-2814-5_29

[10] Warfield, R. B., Purity and algebraic compactness for modules, *Pacific J. Math.*, 28 (1969), 699–719.

[11] Zhu, H.Y., Ding N. Q., $\mathbb{S}$-superfluous and $\mathbb{S}$-essential homomorphisms, *Acta Math. Sci.*, 29B(2) (2009), 391-401. https://doi.org/10.1016/S0252-9602(09)60038-2

[12] Ziegler, M., Model theory of modules, *Annals Pure Appl. Logic*, 26 (1984), 149-213.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 978–992 (2022)
DOI:10.31801/cfsuasmas.1037229
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: December 15, 2021; Accepted: May 8, 2022

ON THE ZEROS OF R-BONACCI POLYNOMIALS AND THEIR DERIVATIVES

Öznur ÖZTUNÇ KAYMAK[1] and Nihal ÖZGÜR[2]

[1]Information Technology Department, Izmir Democracy University, 35140 Karabağlar, Izmir, TÜRKİYE
[2]Department of Mathematics, Izmir Democracy University, 35140 Karabağlar, Izmir, TÜRKİYE

Abstract. The purpose of the present paper is to examine the zeros of R-Bonacci polynomials and their derivatives. We obtain new characterizations for the zeros of these polynomials. Our results generalize the ones obtained for the special case $r = 2$. Furthermore, we find explicit formulas of the roots of derivatives of R-Bonacci polynomials in some special cases. Our formulas are substantially simple and useful.

1. Introduction

The problem finding a convenient method to determine the zeros of a polynomial has a long history that dates back to the work of Cauchy [14]. Zeros of polynomials, which can be real or complex conjugate, have been perhaps among the most popular topics of study for centuries. When the historical development of polynomial studies have been examined, in 2000 BC, the ancient Babylon Tribe living in Mesopotamia stands out. This tribe knowing how to calculate positive roots is perhaps the best example. Some recent applications of the theory of polynomials with symmetric zeros can be found in [21]. This is a short review on the polynomials whose zeros are symmetric either to the real line or to the unit circle. These kind polynomials are very important in mathematics and physics (for more details see [21] and the references therein). On the other hand, the open problem of determining the exact number of zeros of a given polynomial on the unit circle was studied in [22]. Several classes of polynomials with symmetric zeros are also discussed in detail.

2020 *Mathematics Subject Classification.* 11B39, 12E10, 30C15.
Keywords. R-Bonacci polynomial, symmetric polynomial, complex polynomial, complex zeros.
oznur.oztunckaymak@idu.edu.tr; 0000-0003-3832-9947
nihal.ozgur@idu.edu.tr-Corresponding author; 0000-0002-8152-1830.

Fibonacci polynomials, a broad class of polynomials, were first described by Belgian mathematician Eugene Charles Catalan (1814-1894), German mathematician E. Jacobsthal and Lucas polynomials in 1970 by M. Bicknell. The starting point of this polynomial class is based on well-known Golden Ratio and Fibonacci numbers, which are still of great interest in the world of modern applied sciences and whose new applications are still found (see, for instance, [1]- [16] and [18]- [20]). For any positive real number x, the Fibonacci polynomials are defined by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x),$$

with initial values $F_0(x) = 0,\ F_1(x) = 1$. In [10], V. E. Hoggat and M. Bicknell are found explicitly the zeros of these polynomials using hyperbolic trigonometric functions. The symmetric polynomials of the zeros of Fibonacci polynomials were found by M. X. He, D. Simon and P. E. Ricci in [7]. Furthermore, in [8], the location and distribution of the zeros of the Fibonacci polynomials were determined. Fibonacci polynomials and their different properties have been examined (see, for example, [3], [24], [25], and the references therein).

In this paper our aim is to examine the zeros of R-Bonacci polynomials and their derivatives. R-Bonacci polynomials $R_n(x)$ are defined by the following recursive equation in [9] for any integer n and $r \geq 2$:

$$R_{n+r}(x) = x^{r-1}R_{n+r-1}(x) + x^{r-2}R_{n+r-2}(x) + \cdots + R_n(x), \tag{1}$$

with the initial values $R_{-k}(x) = 0,\ k = 0, 1, \cdots, r-2,\ R_1(x) = 1$. For $r = 2, 3$ in the recurrence relation (1), R-Bonacci polynomials become the so called Fibonacci and Tribonacci polynomials, respectively. Although, there are a large number of publications regarding to Fibonacci polynomials and their generalizations (see [7]-[9], [11] and [13]), the open expressions have not been found for the zeros of Tribonacci polynomials and their derivatives yet. Instead, numerical studies have been done more intensively in recent years. Zero attractors of these polynomials were obtained by W. Goh, M. X. He and P. E. Ricci in [6]. In [15], the number of the real roots of Tribonacci-coefficient polynomials were found. Recently, the smallest disc or annulus containing the zeros of Tribonacci polynomials have been examined by Ö. Öztunç Kaymak and an algorithm has given to use in other boundary problems in [12].

In this study, in order to determine the distribution of the zeros of R-Bonacci polynomials, we examine some properties of R-bonacci polynomials, a more general class of Fibonacci and Tribonacci polynomials. In Section 2, we consider some classes of R-Bonacci polynomials. We find the symmetric polynomials which are made up of the r^{th} order of the zeros of R-Bonacci polynomials. Using these symmetric polynomials, we determine the reference roots for the polynomials $R_{rn+p}(x)$ for $p = 0, 1$ and $n = 1$. So, we have generalized the results obtained for the special case $r = 2$ in [10].

On the other hand, there are several papers on the derivatives of the Fibonacci polynomials (see [4], [5], [17], [23] and the references therein). In Section 3, we study the roots of the derivatives of R-Bonacci polynomials. We obtain the most general symmetric polynomials which are made up of the r^{th} order of the zeros of derivatives of R-Bonacci polynomials. Using these symmetric polynomials, we find some formulas for the zeros of derivatives of R-Bonacci polynomials for some special values of t.

2. Zeros of Some Classes of R-Bonacci Polynomials

The general representations for R-Bonacci polynomials was given in [9] as

$$R_n(x) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{r}\right]} \binom{n-j-1}{j}_r x^{(r-1)(n-1)-rj}. \tag{2}$$

Here $r_{n,j} = \binom{n}{j}_r$ denotes the r-nomial coefficient and $[.]$ denotes the greatest integer function. In this section, we obtain the symmetric polynomials including the zeros of R-Bonacci polynomials. Before finding symmetric polynomial of the zeros of R-Bonacci polynomials, the following observation based on 2:

Observation 1. *The zeros of $R_n(x)$ and $R_n(xe^{\frac{2\pi}{r}i})$ are identical.*

To see the above observation, the following result is obtained by writing $xe^{\frac{2\pi}{r}i}$ instead of x in 2:

$$R_n(xe^{\frac{2\pi}{r}i}) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{r}\right]} r_{n,j} \left(xe^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)-rj}. \tag{3}$$

Then, the desired result is easily seen by taking a parenthesis $\left(e^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)}$ and we have

$$\begin{aligned} R_n(xe^{\frac{2\pi}{r}i}) &= \left(e^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)} \left(\begin{array}{c} r_{n,0}\ x^{(r-1)(n-1)} + r_{n,1}\ x^{(r-1)(n-1)-r} \\ + \cdots + r_{n,\left[\frac{(r-1)(n-1)}{r}\right]}\ x \end{array} \right) \\ &= \left(e^{\frac{2\pi i}{r}}\right)^{(r-1)(n-1)} R_n(x). \end{aligned}$$

By this observation, we can simply state that the zeros of R-Bonacci polynomials can be created by rotating the angle of $\frac{2\pi}{r}$ degrees in the complex plane. The zeros of $R_n(x)$ are same as $R_n(xe^{\frac{2\pi i}{r}})$, as they are with $R_n(xe^{-\frac{2\pi i}{r}})$. Thus, the zeros of $R_n(x)$ can be divided into r sets: $\{x_i\}$, $\left\{x_ie^{\frac{2\pi i}{r}}\right\}, \cdots, \left\{x_ie^{\frac{2\pi i}{r}}\right\}$. Here we refer to this set $\{x_i\}$ as a set of reference zeros. The zeros of the 20^{th} Tribonacci polynomial are seen in Figure 1. Notice that the zeros of this polynomial can be generated at an angle of 120 degrees with reference to the set $\{x_i\}$.

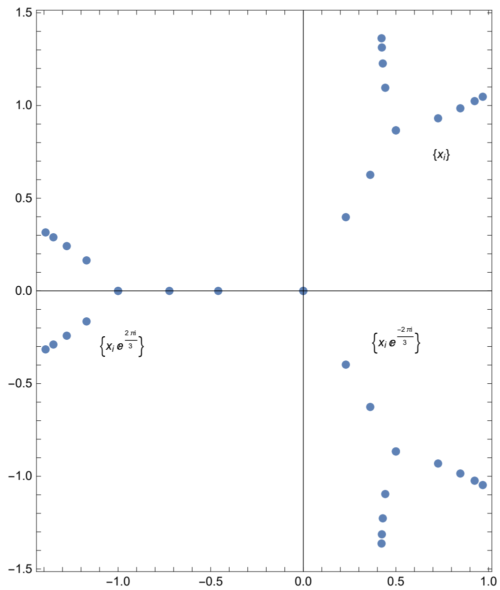

FIGURE 1. The zeros of $T_{20}(x)$

Our theorems are coincide with the ones obtained in [7] for $R = 2, 3$. Actually, Theorem 1 and Theorem 2 are the most generalized versions of the results obtained for Tribonacci and Fibonacci polynomials. For the definition of a symmetric polynomial one can see [7].

Theorem 1. *The most general form of the j^{th} symmetric polynomials consisting of over the r^{th} zeros of $R_{rn}(x)$ is as follows*:

$$\sigma_j\left(x_1^r, \cdots, x_{(r-1)n-1}^r\right) = (-1)^j \binom{rn-j-1}{j}_r. \tag{4}$$

Proof. It is known that the zeros of R-Bonacci polynomials lie in the argument $\frac{2\pi}{r}$ and hence the polynomial $R_{rn}(x)$ can be factorized as

$$R_{rn}(x) = x \prod_{k=1}^{(r-1)n-1} (x - x_k)\left(x - x_k e^{\frac{2\pi i}{r}}\right) \cdots \left(x - x_k e^{-\frac{2\pi i}{r}}\right).$$

If we rearrange this equation, we obtain

$$\begin{aligned} R_{rn}(x) \quad = \quad & x\{x^{r^2 n - rn - r} - \\ & x^{r^2 n - rn - 2r} \sum_{k=1}^{(r-1)n-1} x_k^r + x^{r^2 n - rn - 3r} \sum_{j \neq k} x_j^r x_k^r \end{aligned}$$

$$-x^{r^2n-rn-4r}\sum_{j\neq k\neq l}x_j^rx_k^rx_l^r+\cdots-\prod_{k=1}^{(r-1)n-1}x_k^r\}$$

$$=\left\{\sum_{j=0}^{(r-1)n-1}(-1)^jx^{(r-1)(rn-1)-rj}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^r\right\}\right\}$$

$$=\sum_{j=0}^{(r-1)n-1}(-1)^j\sigma_j\left(x_1^r,x_2^r,\cdots,x_{(r-1)n-1}^r\right)x^{(r-1)(rn-1)-rj}. \tag{5}$$

On the other hand by (2) we can write

$$R_{rn}(x)=\sum_{j=0}^{(r-1)n-1}\binom{rn-j-1}{j}_r x^{(r-1)(rn-1)-rj}. \tag{6}$$

Since the equations (5) and (6) are equal, we obtain the desired result (4). □

Corollary 1. *The following equations are satisfied by the zeros of* $R_{rn}(x)$ *:*

$$\sum_{k=1}^{(r-1)n-1}x_k^r=-\binom{rn-2}{1}_r. \tag{7}$$

Proof. By setting $j=1$ in the equation (4) desired result is obtained. □

Theorem 2. *The most general form of the* j^{th} *symmetric polynomials consisting of the* r^{th} *zeros of* $R_{rn+1}(x)$ *is as follows :*

$$\sigma_j\left(x_1^r,\cdots,x_{(r-1)n}^r\right)=(-1)^j\binom{rn-j}{j}_r. \tag{8}$$

Proof. By a similar way used in the proof of Theorem 1, we can write

$$R_{rn+1}(x)=\prod_{k=1}^{(r-1)n}(x-x_k)\left(x-x_ke^{\frac{2\pi i}{r}}\right)\cdots\left(x-x_ke^{-\frac{2\pi i}{r}}\right).$$

Then we get

$$\begin{aligned}R_{rn+1}(x) &= \{x^{r^2n-rn}-\\ &\quad x^{r^2n-rn-r}\sum_{k=1}^{(r-1)n}x_k^r+x^{r^2n-rn-2r}\sum_{j\neq k}x_j^rx_k^r\\ &\quad -x^{r^2n-rn-3r}\sum_{j\neq k\neq l}x_j^rx_k^rx_l^r+\cdots-\prod_{k=1}^{(r-1)n}x_k^r\}\\ &= \left\{\sum_{j=0}^{(r-1)n}(-1)^jx^{rn(r-1)-rj}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^r\right\}\right\}\end{aligned}$$

$$= \sum_{j=0}^{(r-1)n} (-1)^j \ \sigma_j \left(x_1^r, x_2^r, \cdots, x_{(r-1)n}^r \right) x^{rn(r-1)-rj}. \tag{9}$$

By putting $rn+1$ instead of n in (2), we find

$$R_{rn+1}(x) = \sum_{j=0}^{n(r-1)} \binom{rn-j}{j}_r x^{(r-1)rn-rj}. \tag{10}$$

It follows from the comparison (9) and (10), it is possible to write the desired result (8). □

Corollary 2. *The following equations are satisfied by the zeros of* $R_{rn+1}(x)$:

$$\sum_{k=1}^{(r-1)n} x_k^r = -\binom{rn-1}{1}_r. \tag{11}$$

Proof. If we set $j=1$ in the equation (8) then we get the equation (11). □

Now, using these symmetric polynomials, we obtain the reference roots of $R_{rn+p}(x)$ for $p=0,1$.

Theorem 3. *For* $p=0,1$ *and* $n=1$, *let* $x_j (1 \le j \le r)$ *be the reference zeros of* $R_{rn+p}(x)$. *Then we have*

$$x_j^r = -1. \tag{12}$$

Proof. Let $p=0$ or $p=1$ and let the set of the reference zeros of $R_{rn+p}(x)$ be $\{x_1, \cdots, x_r\}$. The other zeros of the polynomial $R_{rn+p}(x)$ will be generated by the argument $\frac{2\pi}{r}$ except the root $x=0$. For a fixed j, using the equations (11) and (7), we have

$$\sum_{k=1}^{r-1} x_k^r = x_1^r + x_2^r + \cdots + x_{r-1}^r$$

$$= x_j^r + \left(x_j e^{\frac{2\pi i}{r}}\right)^r + \left(x_j e^{\frac{4\pi i}{r}}\right)^r + \cdots + \left(x_j e^{\frac{2(r-2)\pi i}{r}}\right)^r = -(r-1)$$

and

$$\sum_{k=1}^{r-2} x_k^r = x_1^r + x_2^r + \cdots + x_{r-1}^r$$

$$= x_j^r + \left(x_j \ e^{\frac{2\pi i}{r}}\right)^r + \left(x_j \ e^{\frac{4\pi i}{r}}\right)^r + \cdots + \left(x_j e^{\frac{2(r-3)\pi i}{r}}\right)^r = -(r-2),$$

respectively. Rearranging the above equations, it can be easily seen that the reference roots of $R_{rn+p}(x)$ as in the equation (12). □

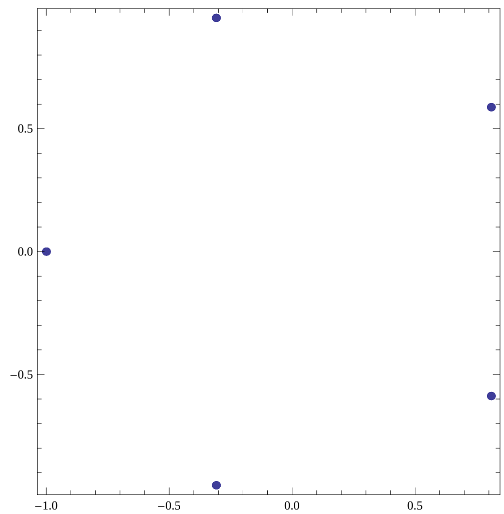

FIGURE 2. The zeros of $B_6(x)$

Example 1. *Let us consider the following* 5*-Bonacci polynomial*

$$B_6(x) = (x^5+1)^4.$$

Using (12)*, if we solve the equation* $x_j^5 = -1 (1 \leq j \leq 5)$*, the reference roots of the polynomial* $B_6(x)$ *are found as follows* (*see Figure* 2) :

$$x_1 = (-1), x_2 = (-1)^{\frac{1}{5}}, x_3 = -(-1)^{\frac{2}{5}}, x_4 = (-1)^{\frac{3}{5}}, x_5 = -(-1)^{\frac{4}{5}}.$$

3. Zeros of Derivatives of R-Bonacci Polynomials

Before we find the symmetric polynomials which are made up of the r^{th} order of the zeros of the derivatives of R-Bonacci polynomials $R_n^{(t)}(x)$, we write the algebraic representations of them. For any fixed n, using the equation (2), the algebraic representation of the derivative polynomial $R_n^{(t)}(x)$ is obtained as follows:

$$R_n^{(t)}(x) = \sum_{j=0}^{\left[\frac{(r-1)(n-1)}{rt}\right]} \binom{n-j-1}{j}_r ((r-1)(n-1)-rj)...((r-1)(n-1)-rj-t+1)x^{(r-1)(n-1)-rj-t}. \tag{13}$$

Now, we determine the symmetric polynomials for $R_{rn+p}^{(t)}(x)$ for special values of t. We give the following theorem.

Theorem 4. *Let* $k \in \mathbb{N}^+$*,* $p \in \{0, 1, \cdots, r-1\}$*. If we consider*

$$t = rk - (1-p)(r-1), \tag{14}$$

$$\mu = ((r-1)(rn+p-1)) \cdots (rn(r-1) - t + (p-1)r + (2-p)) \tag{15}$$

and

$$\eta = (r-1)n - \left(\frac{t + (1-p)(r-1)}{r} \right), \tag{16}$$

then the most general form of the symmetric polynomials consisting of the zeros of $R^{(t)}_{rn+p}(x)$ *is as follows*:

$$\sigma\left(x_1^r, ..., x_\eta^r\right) = \tag{17}$$

$$\frac{(-1)^j((r-1)(rn+p-1)-rj)...((r-1)(rn+p-1)-rj-t+1)}{\mu} \binom{rn+p-j-1}{j}_r .$$

Proof. It can be easily seen that

$$R^{(t)}_{rn+p}(x) = \mu \prod_{k=1}^{\eta} (x - x_k)\left(x - x_k e^{\frac{2\pi i}{r}}\right) \cdots \left(x - x_k e^{-\frac{2\pi i}{r}}\right),$$

where μ is a constant. Then we have

$$\begin{aligned} R^{(t)}_{rn+p}(x) &= \mu\{x^{r^2n-rn-(t+(1-p)(r-1))} - \\ &\quad x^{r^2n-rn-(t+(1-p)(r-1))-r}\sum_{k=1}^{\eta} x_k^r + x^{r^2n-rn-(t+(1-p)(r-1))-2r}\sum_{j\neq k} x_j^r\, x_k^r \\ &\quad - x^{r^2n-rn-(t+(1-p)(r-1))-3r}\sum_{j\neq k\neq l} x_j^r x_k^r x_l^r + \cdots - \prod_{k=1}^{\eta} x_k^r\} \\ &= \mu\left\{\sum_{j=0}^{\eta}(-1)^j x^{r^2n-rn-(t+(1-p)(r-1))-rj}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j} x_{l_i}^r\right\}\right\} \\ &= \mu\sum_{j=0}^{\eta}(-1)^j\sigma_j\left(x_1^r, x_2^r, \cdots, x_\eta^r\right)x^{(r-1)(rn+p-1)-rj-t}. \end{aligned} \tag{18}$$

By using the equation (13) and taking $rn+p$ instead of n we can write

$$R^{(t)}_{rn+p}(x) = \sum_{j=0}^{\left[\frac{(r-1)(rn+p-1)}{rt}\right]} \binom{rn+p-j-1}{j}_r \times$$

$$((r-1)(rn+p-1)-rj)...((r-1)(rn+p-1)-rj-t+1)x^{(r-1)(rn+p-1)-rj-t}. \tag{19}$$

Since the equations (18) and (19) are equal, then the proof follows. □

Corollary 3. *Let t and η be as in the equations (14) and (16), respectively. For $k \in \mathbb{N}^+$ and $p \in \{0, 1, \cdots, r-1\}$, the following equations are satisfied by the zeros* *of* $R^{(t)}_{rn+p}(x)$:

$$(i)\prod_{k=1}^{\eta}x_k^r = \tag{20}$$

$$\frac{(-1)^{\eta}t\ (t-1)...(1)}{((r-1)(rn+p-1))...(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-\eta-1}{\eta}_r.$$

and

$$(ii)\sum_{k=1}^{\eta}x_k^r = \tag{21}$$

$$-\frac{((r-1)(rn+p-1)-r)...((r-1)(rn+p-1)-r-t+1)}{((r-1)(rn+p-1))...(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-2}{1}_r.$$

Proof. In the equation (4), if we put $j=\eta$ and $j=1$ we obtain the desired results, respectively. □

Let

$$\upsilon_\eta = \tag{22}$$

$$\frac{(-1)^{\eta}t\ (t-1)\cdots(1)}{((r-1)(rn+p-1))\cdots(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-\eta-1}{\eta}_r$$

and

$$\psi_\eta = \tag{23}$$

$$-\frac{((r-1)(rn+p-1)-r)\cdots((r-1)(rn+p-1)-r-t+1)}{((r-1)(rn+p-1))\cdots(rn(r-1)-t+(p-1)r+(2-p))}\binom{rn+p-2}{1}_r.$$

Then we can give the following theorem.

Theorem 5. *For* $t=r(r-1)n-2r-(1-p)(r-1)$, $R^{(t)}_{rn+p}(x)$ *has* $r\left((r-1)n-\left(\frac{t+(1-p)(r-1)}{r}\right)\right)$ *roots and these roots are*

$$x_k=\left(\frac{\psi_2\pm\sqrt{\psi_2^2-4\upsilon_2}}{2}\right)^{\frac{1}{r}}e^{\frac{2k\pi i}{r}},(k=0,1,\cdots,r-1), \tag{24}$$

where υ_2 *and* ψ_2 *are defined by the equations (3) and (3), respectively.*

Proof. Since $R^{(r(r-1)n-2r-(1-p)(r-1))}_{rn+p}(x)$ is a polynomial of $r\left((r-1)n-\left(\frac{t+(1-p)(r-1)}{r}\right)\right)$-th degree then by using the equations (3) and (3) we have

$$\prod_{k=1}^{2}x_k^r = x_1^r x_2^r = \upsilon_2 \tag{25}$$

and

$$\sum_{k=1}^{2} x_k^r = x_1^r + x_2^r = \psi_2. \tag{26}$$

Since we know that $x_1^r = \frac{\upsilon_2}{x_2^r}$ it can be easily seen that

$$x_2^{2r} - \psi_2 \; x_2^r + \upsilon_2 = 0.$$

Solving this last equation of the second degree, the roots can be easily found. So the roots of $R_{rn+p}^{(t)}(x)$ must be as in the equation (24). □

Since we have Fibonacci and Tribonacci polynomials for $r = 2$ and $r = 3$, respectively, we can give the following corollaries.

Corollary 4. *Let $p \in \{0,1\}$ and $t = 2n - 5 + p$. The zeros of the polynomial $F_{2n+p}^{(t)}(x)$ can be formulized as follows*:

$$x_k = \left(\frac{\psi_2 \pm \sqrt{\psi_2^2 - 4\upsilon_2}}{2}\right)^{\frac{1}{2}} e^{k\pi i}, (k = 0, 1)$$

where υ_2 and ψ_2 are defined by the equations (3) and (3), respectively.

In [23], J. Wang proved the following equation for any fixed n

$$L_n^{(t)}(x) = nF_n^{(t-1)}(x), n \geq 1, \tag{27}$$

where $L_n(x)$ are Lucas polynomials. Hence the zeros of $L_n^{(t+1)}(x)$ and $F_n^{(t)}(x)$ are identical.

Corollary 5. *Let $p \in \{0,1,2\}$ and $t = 6n - 8 + 2p$. The zeros of the polynomial $T_{3n+p}^{(t)}(x)$ are*

$$x_k = \left(\frac{\psi_2 \pm \sqrt{\psi_2^2 - 4\upsilon_2}}{2}\right)^{\frac{1}{3}} e^{\frac{2k\pi i}{3}} (k = 0, 1, 2), \tag{28}$$

where υ_2 and ψ_2 are defined by the equations (3) and (3), respectively.

Now we give some examples.

Example 2. *Consider the zeros of the polynomial*

$$T_6^{(iv)}(x) = 5040x^6 + 3360x^3 + 144.$$

In the equation (28), writing $\psi_2 = 2/3$, $\upsilon_2 = 1/35$, we find the zeros of this polynomial as

$$x_k = \sqrt[3]{\frac{2/3 \pm \sqrt{(2/3)^2 - 4/35}}{2}} e^{\frac{2k\pi i}{3}}, (k = 0, 1, 2)$$

(*see Figure 3*).

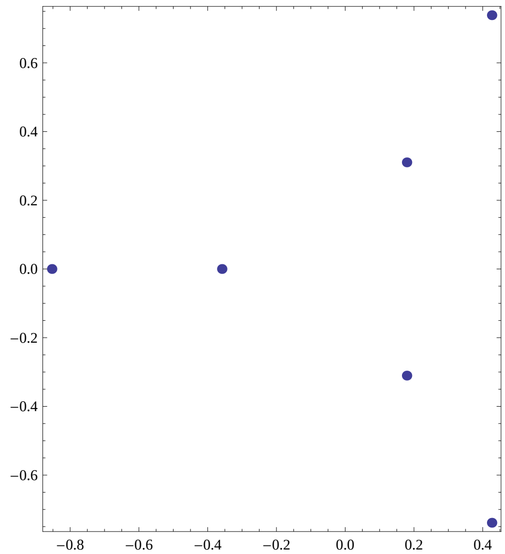

FIGURE 3. The roots of $T_6^{(iv)}(x)$.

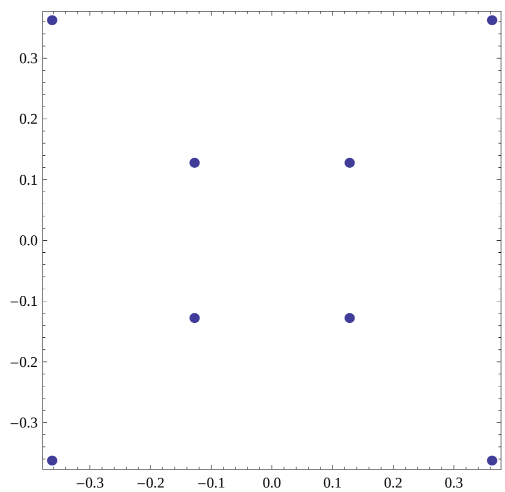

FIGURE 4. The roots of $Q_8^{(13)}(x)$

Example 3. *For $p=0$, $n=2$ and $r=4$, let us consider the polynomial*

$$Q_8^{(13)}(x) = 93405312000 + 88921857024000x^4 + 1267136462592000x^8.$$

Using the equations (3) and (3) we have

$$\prod_{k=1}^{2} x_k^4 = \frac{1}{13566} = \upsilon_2$$

and

$$\sum_{k=1}^{2} x_k^4 = -\frac{4}{57} = \psi_2.$$

Then the roots of $Q_8^{(13)}(x)$ *are generated by* x_k $(k = 0, 1, 2, 3)$. *By (24), the roots of the polynomial* $Q_8^{(13)}(x)$ *are obtained as*

$$x_1 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} = 0.127788 + 0.127788i,$$

and

$$x_2 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} = 0.36255 + 0.36255i$$

for $k = 0$,

$$x_3 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{\pi i}{2}} = -0.36255 + 0.36255i$$

and

$$x_4 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{\pi i}{2}} = -0.127788 + 0.127788i$$

for $k = 1$,

$$x_5 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\pi i} = -0.127788 - 0.127788i$$

and

$$x_6 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\pi i} = -0.36255 - 0.36255i,$$

for $k = 2$,

$$x_7 = \sqrt[4]{\frac{-\frac{4}{57} + \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{3\pi i}{2}} = 0.127788 - 0.127788i$$

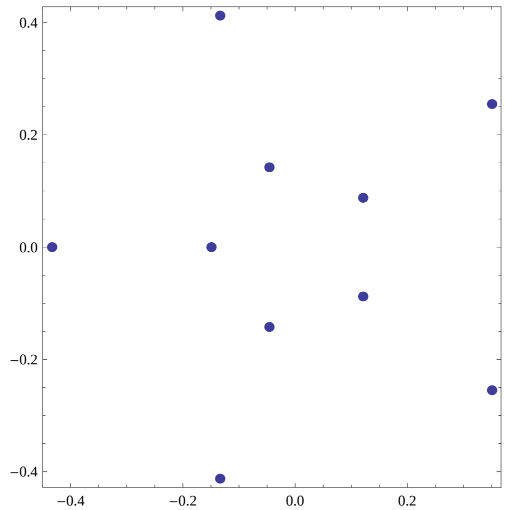

FIGURE 5. The roots of $B_8^{(18)}(x)$

and

$$x_8 = \sqrt[4]{\frac{-\frac{4}{57} - \sqrt{(-\frac{4}{57})^2 - \frac{4}{13566}}}{2}} e^{\frac{3\pi i}{2}} = 0.36255 - 0.36255i,$$

for $k = 3$ (*see Figure 4*).

Example 4. *Let us consider the* 5*-Bonacci polynomials* $B_8^{18}(x)$*. In this case, we have* $p = 3$*,* $n = 1$*,* $r = 5$ *and we obtain*

$$B_8^{18}(x) = 96035605585920000 + 12926008369442488320000x^5$$
$$+84019054401376174080000x^{10}.$$

The roots of this polynomial are found as follows (*see Figure 5*) :

$$x_k = \sqrt[5]{\frac{\psi_2 \pm \sqrt{\psi_2^2 - 4\upsilon_2}}{2}} e^{\frac{2k\pi i}{5}}, k = 0, 1, 2, 3, 4.$$

4. Conclusion and Future Work

In this paper, in order to obtain new formulas for the zeros of R-Bonacci polynomials and their derivatives, the most general form of the j^{th}symmetric polynomials consisting of over the r^{th} zeros of $R_n(x)$ and $R_{rn+p}^{(t)}(x)$ are given. Using some consequences of these symmetric polynomials, some explicit formulas for the zeros of these polynomials, which have been given in (12) and (24), are found. Although these formulas are simple, they are valuable because they formulate the zero values of many R-Bonacci polynomials, which is the most general form of the Fibonacci polynomials, and their derivatives.

Given the future studies on this topic, the zeros of the remaining R-Bonacci polynomials can be formulated using different methods. For this reason, it is thought that formulating the zeros of a R-Bonacci polynomial will increase the applicability of this problem in different engineering applications. In addition, this study is also thought to be a guide for formulating the zero locations of polynomials with unknown zero locations. Because this method is applicable for all polynomial classes.

Author Contribution Statements All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interests regarding the publication of this article.

Acknowledgements This work is supported by the Scientific Research Projects Unit of Balıkesir University under the project number Mat.BAP.2013.0001.

References

[1] Brousseau, A., Fibonacci statistics in conifers, *Fibonacci Quart.*, 7(4) (1969), 525–532.
[2] Carson, J., Fibonacci numbers and pineapple phyllotaxy, *The Two-Year College Mathematics Journal*, 9(3) (1978), 132–136. https://doi.org/10.2307/3026682
[3] Falcón, S., Plaza, Á., On k-Fibonacci sequences and polynomials and their derivatives, *Chaos, Solitons & Fractals*, 30(3) (2009), 1005-1019. https://doi.org/10.1016/j.chaos.2007.03.007
[4] Filipponi, P., Horadam, A. F., Derivative Sequences of Fibonacci and Lucas Polynomials, Applications of Fibonacci Numbers, Vol. 4 (Winston-Salem, NC, 1990), 99–108, Kluwer Acad. Publ., Dordrecht, 1991.
[5] Filipponi, P., Horadam, A., Second derivative sequences of Fibonacci and Lucas polynomials, *Fibonacci Quart.*, 31(3) (1993), 194–204.
[6] Goh, W., He, M. X., Ricci, P. E., On the universal zero attractor of the Tribonacci-related polynomials, *Calcolo*, 46(2) (2009), 95–129. https://doi.org/10.1007/s10092-009-0002-0
[7] He, M. X., Simon, D., Ricci, P. E., Dynamics of the zeros of Fibonacci polynomials, *Fibonacci Quart.*, 35(2) (1997), 160–168.
[8] He, M. X., Ricci, P. E., Simon, D., Numerical results on the zeros of generalized Fibonacci polynomials, *Calcolo*, 34 (1-4) (1997), 25–40.
[9] Hoggatt, V. E., Bicknell, M., Generalized Fibonacci polynomials, *Fibonacci Quart.*, 11(5) (1973), 457–465.
[10] Hoggatt, V. E., Bicknell, M., Roots of Fibonacci polynomials, *Fibonacci Quart.*, 11(3) (1973), 271–274.
[11] Öztunç Kaymak, Ö., R-Bonacci polynomials and Their Derivatives, Ph. D. Thesis, Balıkesir University, 2014.
[12] Öztunç Kaymak, Ö., Some remarks on the zeros of tribonacci polynomials, *Int. J. Anal. Appl.*, 16(3) (2018), 368-373. https://doi.org/10.28924/2291-8639-16-2018-368
[13] Koshy, T., Fibonacci and Lucas Numbers with Applications, Pure and Applied Mathematics, Wiley-Interscience, New York, 2001.
[14] Marden, M., Geometry of Polynomials, Second edition, Mathematical Surveys, No. 3 American Mathematical Society, Providence, R.I. 1966.

[15] Mátyás, F., Szalay, L., A note on Tribonacci-coefficient polynomials, *Ann. Math. Inform.* 38 (2011), 95–98.
[16] Mitchson, G. J., Phyllotaxis and the Fibonacci series, *Science*, 196 (1977), 270–275.
[17] Özgür, N. Y., Öztunç Kaymak, Ö., On the zeros of the derivatives of Fibonacci and Lucas polynomials, *Journal of New Theory*, 7 (2015), 22-28.
[18] Taş, N., Uçar, S., Özgür, N., Öztunç Kaymak, Ö., A new coding/decoding algorithm using Finonacci numbers, *Discrete Math. Algorithms Appl.*, 10(2) (2018), 1850028. https://doi.org/10.1142/S1793830918500283
[19] Taş, N., Uçar, S., Özgür, N., Pell coding and Pell decoding methods with some applications, *Contrib. Discrete Math.* 15(1) (2020), 52-66. https://doi.org/10.11575/cdm.v15i1.62606
[20] Uçar, S., Taş, N., Özgür, N. Y., A new application to coding theory via Fibonacci and Lucas numbers, *Mathematical Sciences and Applications E-Notes*, 7(1) (2019), 62–70.
[21] Vieira, R. S., Polynomials with Symmetric Zeros, In: Polynomials – Theory and Application, IntechOpen, 2019. https://doi.org/10.5772/intechopen.82728
[22] Vieira, R. S., How to count the number of zeros that a polynomial has on the unit circle?, *J Comp. Appl. Math.*, 384 (2021), Paper No. 113169, 11 pp. https://doi.org/10.1016/j.cam.2020.113169
[23] Wang, J., On the k-th derivative sequences of Fibonacci and Lucas polynomials, *Fibonacci Quart.*, 33(2) (1995), 174–178.
[24] Web, W. A., Parberry, E. A., Divisibility properties of Fibonacci polynomials, *Fibonacci Quart.*, 7(5) (1969), 457–463.
[25] Yuan, Y., Zhang, W., Some identities involving the Fibonacci polynomials, *Fibonacci Quart.*, 40(4) (2002), 314–318.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 993–1006 (2022)
DOI:10.31801/cfsuasmas.1013172
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 21, 2021; Accepted: May 18, 2022

SET-GENERATED SOFT SUBRINGS OF RINGS

Akın Osman ATAGUN[1] and Hüseyin KAMACI[2]

[1]Department of Mathematics, Faculty of Science and Arts, Kırşehir Ahi Evran University, 40100 Kırşehir, TURKEY
[2]Department of Mathematics, Faculty of Science and Arts, Yozgat Bozok University, 66100 Yozgat, TURKEY

Abstract. This paper focuses on the set-oriented operations and set-oriented algebraic structures of soft sets. Relatedly, in this paper, firstly some essential properties of α-intersection of soft set are investigated, where α is a non-empty subset of the universal set. Later, by using α-intersection of soft set, the notion of set-generated soft subring of a ring is introduced. The generators of soft intersections and products of soft subrings are given. Some related properties about generators of soft subrings are investigated and illustrated by several examples.

1. Introduction

Since the modeling of uncertain data in medical science, economics, sociology, environmental science, engineering and many other fields is very complex, it is difficult to successfully deal with them by classical methods. In the last century, many approaches that are useful in modeling uncertainties have been proposed. The fuzzy set theory [1,2], the interval mathematics [3], vague set theory [4] and rough set theory [5,6] and are favorable approaches to describing uncertain data, but each of these theories has its own difficulties in classifying data parametrically. To fill this gap, Molodtsov [7] proposed a completely new approach named soft set theory. This approach allowed the uncertain data frequently encountered in many areas to be classified parametrically, thereby providing a better representation of them. In the years following the budding of soft sets, the theoretical and practical aspects of these sets were discussed. Maji et al. conceptualized the some set operations of soft sets [8] and made further efforts to show the implementation of soft sets in

2020 *Mathematics Subject Classification.* 03E72, 05C25, 08A72.
Keywords. Soft sets, soft subrings of rings, generated soft subrings of rings.
aosman.atagun@ahievran.edu.tr; 0000-0002-2131-9980
huseyin.kamaci@bozok.edu.tr-Corresponding author; 0000-0002-0429-6713

decision making [9]. Ali et al. [10] introduced some new soft set operations such as the restricted difference, the restricted intersection, the extended intersection and the restricted union. Çağman and Enginoğlu [11] revisited some basic operations of soft sets to make them more efficient in some cases. In [12–14], the authors studied the operations of difference and symmetric difference of soft sets. Aygün and Kamacı [15] developed some functional operations of soft sets and then demonstrated their efficiency in handling decision making problems. Also, they defined XOR and XNOR products of soft sets and derived new soft algebraic structures by using these soft set products [16]. Çağman and Enginoğlu [17] introduced the soft matrices representing soft sets and their handy operations to create a soft max-min decision making procedure which can be successfully applied to the problems containing uncertainties. In [18–21], the researchers discussed specific kinds of soft matrices and construct new improved types of soft max-min decision making procedure. Moreover, the inverse types of soft matrices were investigated and their applications to decision making were presented [22,23]. Recently, the works on the operations of soft sets and soft matrices are progressing rapidly.
On the other hand, many algebraic structures based on the basic principles and operations of soft sets have been proposed. In 2007, Aktaş and Çağman [24] introduced the rudiments of soft groups and studied their basic properties. Uluçay et al. [25] studied soft representation of soft groups. Feng et al. [26] defined the concepts of soft subsemirings, soft semirings, soft semiring homomorphisms, soft ideals and idealistic soft semirings. In , the authors [27,28] introduced the fundamentals of soft rings and soft normed rings. In [29], Atagün and Sezgin discussed the algebraic soft substructures of rings and defined soft subring of a ring, soft ideal of a ring, soft submodule of a module and soft subfield of a field. Sezgin et al. [30] expanded the study of soft near-rings, especially according to the idealistic soft near-rings. Ostadhadi-Dehkordi and Shum [31] investigated regular and strongly regular relations on the soft hyperrings. Tahat et al. [32] discussed the characterizations of soft topological soft groups and soft rings. Karaaslan [33] investigated some outstanding properties of collection of soft sets over AG-groupoid, AG-band and AG*-groupoid. In [34], Yousafzai et al. introduced the notion of soft sets in an ordered AG-groupoid and they studied different type ideals and strongly regular elements. Zhan et al. [35] defined some new soft algebraic structures such as (M,N)-soft union hemiring and (M,N)-soft union h-ideal, which are generalisations of soft union hemiring and soft union h-ideal to tackle many uncertainty problems. Atagün and Sezgin [36] described the notions of soft N-subgroups, soft subnear-rings and soft ideals of near-rings and also derived the product operation and bi-intersection of soft N-groups, soft subnear-rings and soft ideals of near-rings. On the other hand, some authors developed soft topology in various aspects and discussed real life examples [37–39].
In [40], Sezer et al. argued that the set-oriented approaches based on inclusion of soft set can be extended the range of operations, algebraic structures, topological

structures, application aspects of soft sets. Thus, they defined the lower α-inclusion and upper α-inclusion of a soft set over the universal set U, where $\alpha \subseteq U$. Moreover, by using the upper α-inclusion of a soft set, they proposed the idea of upper α-semigroups for the soft sets. In [41], the authors made some analyzes with respect to group theory and showed that some subgroups of a group can be achieved easily by means of the notions of upper and lower α-inclusions of soft sets. They also demonstrated that a soft uni-group and a soft int-group can be derived by its lower α-subgroup and upper α-subgroup, respectively. In [40, 41], the authors focused on the α-oriented subgroup structures of soft sets. However, the α-oriented subring structure of soft sets is a gap in the literature. By filling this gap, both the theoretical aspects and practical aspects of the soft sets will be contributed. Relatedly, this paper aims to introduce soft subrings of a ring generated by the set α and to investigate their fundamental properties.

This paper is organized as follows. Section 2 recalls the rudiments of soft sets. Section 3 presents a detailed theoretical study for the α-intersection of a soft set. Section 4 introduces a new concept namely a soft subring of a ring generated by a set and gives many remarkable properties of this concept. Also, this section includes our main theorems, in which we examine generator sets under operations soft intersection and product. Some theoretical results are illustrated by several examples. Section 5 consists of the conclusions of the paper and the direction for future studies.

2. Preliminaries

In this section, we recall the rudiments of rings, soft sets and soft subrings.

By a ring, we mean an algebraic system $(\Re, +, .)$, where (the multiplication . will be omitted in formulas)

i) $(\Re, +)$ is a abelian group,
ii) $(\Re, .)$ is a semi-group,
iii) $a.(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in \Re$ (i.e., left and right distributive rules hold)

Throughout this paper, $\Re$ denotes a ring and the zero of $\Re$ is symbolized by $0_\Re$.

A subgroup S of $(\Re, +)$ with $SS \subseteq S$ is named a *subring* of $\Re$ and symbolized by $S < \Re$. Therefore, $S < \Re$ if and only if

i) $S \subseteq \Re$,
ii) $0_\Re \in S$,
iii) $a - b \in S$ for all $a, b \in S$,
iv) $ab \in S$ for all $a, b \in S$.

Molodtsov [7] described the soft set in the following manner:

Let U be a universal set and its power set be $P(U)$, $\mathcal{T}$ be a set of parameters and $\mathcal{X} \subseteq \mathcal{T}$.

Definition 1. *([7]) A pair $(\Psi, \mathcal{X})$ (or simply $\Psi_{\mathcal{X}}$) is termed to be a soft set over U, where Ψ is a mapping described by*

$$\Psi : \mathcal{X} \to P(U).$$

Stated in other words, a soft set over the universal set U can be considered as a parameterized family of the subsets of universal set U. For $t \in \mathcal{X}$, $\Psi(t)$ is the set of t-elements of the soft set $(\Psi, \mathcal{X})$, or simplistically the set of t-approximate elements of this soft set. To support this idea, Molodtsov presented various examples (see [7]). Indeed, there is a mutual correspondence among soft sets and binary relations as given in [42, 43]. Namely, let $\mathcal{T}$ and U be non-empty sets and suppose that σ refers to an arbitral binary relation between an element of $\mathcal{T}$ and an element of U. A set-valued function $\Psi : \mathcal{T} \to P(U)$ can be described as $\Psi(t) = \{u \in U \mid (t, u) \in \sigma\}$ for all $t \in \mathcal{T}$. Hence, the pair $(\Psi, \mathcal{T})$ is a soft set over U, which is derived from the relation σ.

Definition 2. *([8]) A soft set $(\Psi, \mathcal{X})$ over U is termed to be a null soft set symbolized by $\Phi_{\mathcal{X}}$, if for all $t \in \mathcal{X}$, $\Psi(t) = \emptyset$ (null set).*

Definition 3. *([8]) A soft set $(\Psi, \mathcal{X})$ over U is termed to be an absolute soft set, if for all $t \in \mathcal{X}$, $\Psi(t) = U$.*

Note that we denote the absolute soft set $(\Psi, \mathcal{X})$ over U by $\mathcal{U}_{\mathcal{X}}$ throughout this paper.

Definition 4. *([10]) The relative complement of a soft set $(\Psi, \mathcal{X})$ is symbolized by $(\Psi, \mathcal{X})^c$ and is defined as $(\Psi, \mathcal{X})^c = (\Psi^c, \mathcal{X})$, where $\Psi^c : \mathcal{X} \to P(U)$ is a mapping given by $\Psi^c(t) = U \setminus \Psi(t)$ for all $t \in \mathcal{X}$.*

Definition 5. *([8, 10]) Let $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ be two soft sets over the universal set U.*

- **a):** *The restricted intersection of $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ is denoted and defined as $(\Psi, \mathcal{X}) \Cap (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$, where $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y} \neq \emptyset$ and $\Theta(t) = \Psi(t) \cap \Upsilon(t)$ for all $t \in \mathcal{Z}$.*
- **b):** *The extended intersection of $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ is denoted and defined as $(\Psi, \mathcal{X}) \sqcap (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$, where $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ and for all $t \in \mathcal{Z}$*

$$\Theta(t) = \begin{cases} \Psi(t), \text{ if } t \in \mathcal{X} \setminus \mathcal{Y} \\ \Upsilon(t), \text{ if } t \in \mathcal{Y} \setminus \mathcal{X} \\ \Psi(t) \cap \Upsilon(t), \text{ if } t \in \mathcal{X} \cap \mathcal{Y} \end{cases}$$

- **c):** *The union intersection of $(\Psi, \mathcal{X})$ and $(\Upsilon, \mathcal{Y})$ is denoted and defined as $(\Psi, \mathcal{X}) \Cup (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$, where $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y} \neq \emptyset$ and $\Theta(t) = \Psi(t) \cup \Psi(t)$ for all $t \in \mathcal{Z}$.*

d): *The extended union of* $(\Psi, \mathcal{X})$ *and* $(\Upsilon, \mathcal{Y})$ *is denoted and defined as* $(\Psi, \mathcal{X}) \sqcup (\Upsilon, \mathcal{Y}) = (\Theta, \mathcal{Z})$*, where* $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ *and for all* $t \in \mathcal{Z}$

$$\Theta(t) = \begin{cases} \Psi(t), & \text{if } t \in \mathcal{X} \setminus \mathcal{Y} \\ \Upsilon(t), & \text{if } t \in \mathcal{Y} \setminus \mathcal{X} \\ \Psi(t) \cup \Upsilon(t), & \text{if } t \in \mathcal{X} \cap \mathcal{Y} \end{cases}$$

In 2010, Çağman and Enginoğlu [11] redescribed the approximate function Ψ of soft set $(\Psi, \mathcal{X})$ from $\mathcal{T}$ to $P(U)$ such that $\Psi(t) = \emptyset$ if $t \notin \mathcal{X}$. Thus, they revisited the operations of intersection and union of soft sets as follows:

Definition 6. *([11]) Let* $\Psi_{\mathcal{X}}$ *and* $\Upsilon_{\mathcal{Y}}$ *be soft sets over* U*. Then,*

a): *the soft union of* $\Psi_{\mathcal{X}}$ *and* $\Upsilon_{\mathcal{Y}}$*, denoted by* $\Theta_{\mathcal{Z}} = \Psi_{\mathcal{X}} \widetilde{\cup} \Upsilon_{\mathcal{Y}}$*, is defined as* $\Theta(t) = \Psi(t) \cup \Upsilon(t)$ *for all* $t \in \mathcal{T}$*.*

b): *the soft intersection of* $\Psi_{\mathcal{X}}$ *and* $\Upsilon_{\mathcal{Y}}$*, denoted by* $\Theta_{\mathcal{Z}} = \Psi_{\mathcal{X}} \widetilde{\cap} \Upsilon_{\mathcal{Y}}$*, is defined as* $\Theta(t) = \Psi(t) \cap \Psi(t)$ *for all* $t \in \mathcal{T}$*.*

For more details, it can be reviewed the concepts in [11].

The following definition first introduced the soft substructures of an algebraic structure to the literature.

Definition 7. *([29]) Let* S *be a subring of* $\Re$ *and* (Ψ, S) *be a soft set over* $\Re$*. If for all* $t, v \in S$*,*

s1) $\Psi(t - v) \supseteq \Psi(t) \cap \Psi(v)$,

s2) $\Psi(tv) \supseteq \Psi(t) \cap \Psi(v)$,

then it is said to be a soft subring of $\Re$ *and symbolized by* $(\Psi, S) \widetilde{<} \Re$ *or simplistically* $\Psi_S \widetilde{<} \Re$*.*

Proposition 1. *([29]) If* $\Psi_S \widetilde{<} \Re$*, then* $\Psi(0) \supseteq \Psi(t)$ *for all* $t \in S$*.*

Theorem 1. *([29]) If* $\Psi_{S_1} \widetilde{<} \Re$ *and* $\Upsilon_{S_2} \widetilde{<} \Re$*, then* $\Psi_{S_1} \Cap \Upsilon_{S_2} \widetilde{<} \Re$*.*

Definition 8. *([26]) Let* $(\Psi, \mathcal{X})$ *be soft set over* U*. Then, the set*

$$supp(\Psi, \mathcal{X}) = \{t \in \mathcal{X} | \ \Psi(t) \neq \emptyset\}$$

is said to be the support of the soft set $(\Psi, \mathcal{X})$*. A soft set* $(\Psi, \mathcal{X})$ *is called non-null if* $supp(\Psi, \mathcal{X}) \neq \emptyset$*.*

3. Some Aspects on α-Intersection of Soft Sets

In this section, we present some theoretical findings for the α-intersection of soft sets.

Definition 9. *([44]) Let* $(\Psi, \mathcal{X})$ *be a soft set over* U *and* $\emptyset \neq \alpha \subseteq U$*. Then, the subset of* $\mathcal{X}$ *given by*

$$(\Psi, \mathcal{X})^{\cap \alpha} = \{t \in \mathcal{X} | \ \Psi(t) \cap \alpha \neq \emptyset\}$$

is called the α*-intersection of* $(\Psi, \mathcal{X})$*.*

It seen that if $\alpha = U$ and $\Psi(t) \neq \emptyset$ for all $t \in \mathcal{X}$, then $(\Psi, \mathcal{X})^{\cap U} = \mathcal{X}$.

Proposition 2. *Let $(\Psi, \mathcal{X})$ be a soft set over U and let $\emptyset \neq \alpha \subseteq U$. Then*

i) $(\Psi, \mathcal{X})^{\cap\alpha} \subseteq supp(\Psi, \mathcal{X})$.
ii) *If $\alpha \subseteq \Psi(t)$ for all $t \in \mathcal{X}$, then $(\Psi, \mathcal{X})^{\cap\alpha} = supp(\Psi, \mathcal{X}) = \mathcal{X}$.*
iii) *If $\Psi(t) \neq \emptyset$ and $\Psi(t) \subseteq \alpha$ for all $t \in \mathcal{X}$, $(\Psi, \mathcal{X})^{\cap\alpha} = supp(\Psi, \mathcal{X}) = \mathcal{X}$.*
iv) *If $(\Psi, \mathcal{X}) = \mathcal{U}_{\mathcal{X}}$, then $(\Psi, \mathcal{X})^{\cap\alpha} = supp(\Psi, \mathcal{X}) = \mathcal{X}$.*
v) *If $(\Psi, \mathcal{X}) = \Phi_{\mathcal{X}}$ or $supp(\Psi, \mathcal{X}) = \emptyset$, then $(\Psi, \mathcal{X})^{\cap\alpha} = \emptyset$.*

Proof. The proof of (i) is seen from the Definitions 8 and 9.
(ii) Since $\alpha \subseteq \Psi(t)$ for all $t \in \mathcal{X}$ and $\emptyset \neq \alpha \subseteq U$, then $\Psi(t) \neq \emptyset$ for all $t \in \mathcal{X}$ and $supp(\Psi, \mathcal{X}) = \mathcal{X}$. Under the assumption $\Psi(t) \cap \alpha \neq \emptyset$ for all $t \in \mathcal{X}$, then $supp(\Psi, \mathcal{X}) \subseteq (\Psi, \mathcal{X})^{\cap\alpha}$. Hence the equality obtained from (i).
The proof of (iii) is similar to proof of (ii). The rest of the proof is easily seen. □

Proposition 3. *Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha \subseteq U$. Then*

i) *If $\alpha \subseteq \beta$, then $(\Psi, \mathcal{X})^{\cap\alpha} \subseteq (\Psi, \mathcal{X})^{\cap\beta}$.*
ii) $(\Psi^c, \mathcal{X})^{\cap\alpha} = \{t \in \mathcal{X} |\ \alpha \setminus \Psi(t) \neq \emptyset\}$.
iii) $(\Psi, \mathcal{X})^{\cap(U\setminus\alpha)} = \{t \in \mathcal{X} |\ \Psi(t) \setminus \alpha \neq \emptyset\}$.
iv) $(\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)} = \{t \in \mathcal{X} |\ U \setminus (\Psi(t) \cup \alpha) \neq \emptyset\}$.

Proof. If $\alpha \subseteq \beta$, then $\Psi(t) \cap \alpha \neq \emptyset$ implies $\Psi(t) \cap \beta \neq \emptyset$. Hence the proof of (i) is done. The rest of proof is obtained using algebraic operations, easily. □

Proposition 4. *Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha \subseteq U$. If $(\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)} = \emptyset$ then $(\Psi, \mathcal{X})^{\cap\alpha} \cup (\Psi^c, \mathcal{X})^{\cap\alpha} \cup (\Psi, \mathcal{X})^{\cap(U\setminus\alpha)} = \mathcal{X}$.*

Proof. Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha \subseteq U$. We assume that $(\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)} = \emptyset$. Then, by Proposition 3 (iv), we have $\Psi(t) \cup \alpha = U$ for all $t \in \mathcal{X}$. Hence, the proof is obvious from Definition 9 and Proposition 3 (ii) and (iii). □

Proposition 5. *Let $(\Psi, \mathcal{X})$ be a soft set over U and let $\emptyset \neq \alpha \subsetneq U$. If $\Psi(t) \cup \alpha \neq U$ for all $t \in \mathcal{X}$, then*

i) $supp(\Psi, \mathcal{X}) \subseteq (\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)}$.
ii) $(\Psi, \mathcal{X})^{\cap\alpha} \subseteq (\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)}$.

Proof. *(i)* Let $t \in supp(\Psi, \mathcal{X})$. Since $\Psi(t) \cup \alpha \neq U$ for all $t \in \mathcal{X}$, then $U \setminus (\Psi(t) \cup \alpha) \neq \emptyset$, which implies $t \in (\Psi^c, \mathcal{X})^{\cap(U\setminus\alpha)}$ by Proposition 3 (iv).
(ii) It is seen from the assertion (i) and Proposition 2 (i). □

Proposition 6. *Let $(\Psi, \mathcal{X})$ be a soft set over U and $\emptyset \neq \alpha, \beta \subseteq U$. Then*

i) $(\Psi, \mathcal{X})^{\cap\alpha} \cap (\Psi, \mathcal{X})^{\cap\beta} \subseteq (\Psi, \mathcal{X})^{\cap(\alpha\cup\beta)}$. *Here the equality does not hold in general, even if $\alpha \cap \beta = \emptyset$.*
ii) $(\Psi, \mathcal{X})^{\cap\alpha} \cup (\Psi, \mathcal{X})^{\cap\beta} = (\Psi, \mathcal{X})^{\cap(\alpha\cup\beta)}$.
iii) $(\Psi, \mathcal{X})^{\cap(\alpha\cap\beta)} \subseteq (\Psi, \mathcal{X})^{\cap\alpha} \cap (\Psi, \mathcal{X})^{\cap\beta}$.

Proof. *(i)* Let $t \in (\Psi,\mathcal{X})^{\cap\alpha} \cap (\Psi,\mathcal{X})^{\cap\beta}$. Then $\Psi(t)\cap\alpha \neq \emptyset$ and $\Psi(t)\cap\beta \neq \emptyset$, which implies $\Psi(t)\cap(\alpha\cup\beta) \neq \emptyset$. For the rest of the proof, we have the Example 1.
(ii) □

$$\begin{aligned}(\Psi,\mathcal{X})^{\cap\alpha} \cup (\Psi,\mathcal{X})^{\cap\beta} &= \{t \in supp(\Psi,\mathcal{X})|\ (\Psi(t)\cap\alpha \neq \emptyset) \vee (\Psi(t)\cap\beta \neq \emptyset)\}\\ &= \{t \in supp(\Psi,\mathcal{X})|\ \Psi(t)\cap(\alpha\cup\beta) \neq \emptyset)\}\\ &= (\Psi,\mathcal{X})^{\cap(\alpha\cup\beta)}\end{aligned}$$

(iii) Let $t \in (\Psi,\mathcal{X})^{\cap(\alpha\cap\beta)}$. Then $\Psi(t)\cap(\alpha\cap\beta) \neq \emptyset$, which implies $\Psi(t)\cap\alpha \neq \emptyset$ and $\Psi(t)\cap\beta \neq \emptyset$. Therefore $t \in (\Psi,\mathcal{X})^{\cap\alpha} \cap (\Psi,\mathcal{X})^{\cap\beta}$.

Example 1. *Let the universe $U = \{u_1,u_2,u_3,u_4,u_5,u_6\}$, the parameter set $\mathcal{T} = \{t_1,t_2,t_3,t_4,t_5,t_6,t_7\}$, and $\mathcal{X} = \{t_1,t_3,t_4,t_5\}$ and $\mathcal{Y} = \{t_1,t_2,t_3,t_4\}$ be two subsets of $\mathcal{T}$. Suppose that corresponding soft sets of $\mathcal{X}$ and $\mathcal{Y}$ are*

$(\Psi,\mathcal{X}) = \{(t_1,\{u_1,u_2\}),(t_3,\{u_1,u_4,u_5\}),(t_4,\{u_6\}),(t_5,\emptyset)\}$

and

$(\Upsilon,\mathcal{Y}) = \{(t_1,\{u_3,u_4\}),(t_2,\{u_1,u_2,u_5\}),(t_3,\{u_3,u_5\}),(t_4,\{u_1,u_5,u_6\})\}$.

If $\alpha = \{u_4,u_6\}$ and $\beta = \{u_1\}$, then it is seen that $(\Psi,\mathcal{X})^{\cap\alpha} = \{t_3,t_4\}$, $(\Psi,\mathcal{X})^{\cap\beta} = \{t_1,t_3\}$ and $(\Psi,\mathcal{X}))^{\cap(\alpha\cup\beta)} = \{t_1,t_3,t_4\}$. (Because, it is obtained that $\Psi(t_3)\cap\alpha = \{u_4\} \neq \emptyset$, $\Psi(t_4)\cap\alpha = \{u_6\} \neq \emptyset$, $\Psi(t_1)\cap\beta = \{u_1\} \neq \emptyset$, $\Psi(t_3)\cap\beta = \{u_1\} \neq \emptyset$, $\Psi(t_1)\cap(\alpha\cup\beta) = \{u_1\} \neq \emptyset$, $\Psi(t_3)\cap(\alpha\cup\beta) = \{u_1,u_4\} \neq \emptyset$ and $\Psi(t_3)\cap(\alpha\cup\beta) = \{u_6\} \neq \emptyset$). Thus, the proof of Proposition 6 (i) is completed. Since $\alpha\cap\beta = \emptyset$, we have $(\Psi,\mathcal{X})^{\cap(\alpha\cap\beta)} = \emptyset$.
If $\alpha = \{u_3,u_6\}$ and $\beta = \{u_5,u_6\}$ (i.e., $\alpha\cap\beta = \{u_6\}$), then it is seen that $(\Upsilon,\mathcal{Y})^{\cap\alpha} = \{t_1,t_3,t_4\}$, $(\Upsilon,\mathcal{Y})^{\cap\beta} = \{t_2,t_3,t_4\}$ and $(\Upsilon,\mathcal{Y})^{\cap(\alpha\cap\beta)} = \{t_4\}$. So, we have $(\Upsilon,\mathcal{Y})^{\cap(\alpha\cap\beta)} \subseteq (\Upsilon,\mathcal{Y})^{\cap\alpha} \cap (\Upsilon,\mathcal{Y})^{\cap\beta}$.

4. Set-Generated Soft Subrings of Rings

In this section, we propose the set-generated soft subrings of a ring by employing the α-intersection of soft sets. We also discuss some of the main properties and theoretical implications of this newly emerging soft algebraic structure.

Throughout this section, $\Re$ is a ring and $(\Psi,\Re)$ is a soft set over $\Re$. A subring S of $\Re$ denoted by $S < \Re$.

Definition 10. *Let $\Re$ be a ring, $\emptyset \neq \alpha \subseteq \Re$ and $(\Psi,\Re)$ be a soft set over $\Re$. If the soft set $(\Psi,(\Psi,\Re)^{\cap\alpha})$ is a soft subring of $\Re$, then this soft set is said to be a soft subring of $\Re$ generated by the set α and denoted by $\langle\Psi^{\cap\alpha}\rangle_\Re$. If the set $\alpha = \{t\}$, $\langle\Psi^{\cap\alpha}\rangle_\Re$ is a soft subring of $\Re$ generated by the element $t \in \Re$.*

As can be seen Definition 10, $\langle\Psi^{\cap\alpha}\rangle_{\Re}\tilde{<}\Re$ if and only if there exists at least an $\emptyset \neq \alpha \subseteq \Re$ such that $(\Psi,\Re)^{\cap\alpha}$ is a subring of $\Re$ and the conditions $s1, s2$ of the Definition 7 are satisfied for $S = (\Psi,\Re)^{\cap\alpha}$.

Example 2. *Given the ring $\Re = (\mathbb{Z}_6, +, .)$, a soft set $(\Psi,\Re)$ over $\Re$, where $\Psi : \Re \to P(\Re)$ is a set-valued function defined by $\Psi(0) = \{0,1,4,5\}, \Psi(1) = \{3\}, \Psi(2) = \{2\}, \Psi(3) = \{0,4,5\}, \Psi(4) = \{1,2\}$ and $\Psi(5) = \{3\}$. Let $\alpha = \{4,5\} \subseteq \Re$. Then, $(\Psi,\Re)^{\cap\alpha} = \{0,3\}$ is a subring of $\Re$ and the soft set $(\Psi,(\Psi,\Re)^{\cap\alpha}) = \{(0,\{0,1,4,5\}), (3,\{0,4,5\})\}$ satisfies the conditions $s1, s2$ of the Definition 7. (That is, $\Psi^{\cap\alpha}(t - v) \supseteq \Psi^{\cap\alpha}(t) \cap \Psi^{\cap\alpha}(v)$ and $\Psi^{\cap\alpha}(tv) \supseteq \Psi^{\cap\alpha}(t) \cap \Psi^{\cap\alpha}(v)$ for all $t, v \in \Re$). Hence $(\Psi,(\Psi,\Re)^{\cap\alpha}) = \langle\Psi^{\cap\alpha}\rangle_{\Re}\tilde{<}\Re$. If $\beta = \{4\}$ a single point set, then it is seen that $(\Psi,\Re)^{\cap\alpha} = (\Psi,\Re)^{\cap\beta}$ and then $\langle\Psi^{\cap\alpha}\rangle_{\Re} = \langle\Psi^{\cap\beta}\rangle_{\Re}$. Therefore, the soft set $\{(0,\{0,1,4,5\}, (3,\{0,4,5\})\}$ is a soft subring of $\Re$, generated by the element $4 \in \Re$.*

Let $(\Psi,\Re)$ be a soft set over $\Re$. Since $\{0_{\Re}\}$ is a subring of $\Re$, it is easily seen that $(\Psi,\{0_{\Re}\})\tilde{<}\Re$.

Definition 11. *The soft subring $(\Psi,\{0_{\Re}\})$ of $\Re$ is called a trivial soft subring of $\Re$ and denoted by $\langle 0_{\Re}\rangle_{\Psi}$.*

It is important to note that the soft sets $\langle 0_{\Re}\rangle_{\Psi}$ and $(\Psi,(\Psi,\Re)^{\cap\{0_{\Re}\}})$ are different, in general. In Example 2, $\langle 0_{\Re}\rangle_{\Psi} = (\Psi,\{0_{\Re}\}) = \{(0,\{0,1,4,5\})\}$ and $(\Psi,(\Psi,\Re)^{\cap\{0_{\Re}\}}) = \{(0,\{0,1,4,5\}),(3,\{0,4,5\})\}$. Furthermore, $(\Psi,(\Psi,\Re)^{\cap\{0_{\Re}\}})$ does not have to be a soft subring of $\Re$.

Proposition 7. *If $\langle\Psi^{\cap\alpha}\rangle_{\Re}\tilde{<}\Re$, then the generator α doesn't have to be unique. Furthermore, if $\langle\Psi^{\cap\{t\}}\rangle_{\Re} = \langle\Psi^{\cap\{v\}}\rangle_{\Re}$ for $t, v \in \Re$, then $\langle\Psi^{\cap\{t,v\}}\rangle_{\Re} = \langle\Psi^{\cap\{t\}}\rangle_{\Re}$.*

Proof. In the example 2, if we take $\eta = \{5\} \subseteq \Re$, then it is seen that $(\Psi,\Re)^{\cap\eta} = (\Psi,\Re)^{\cap\beta}$ and then $\langle\Psi^{\cap\eta}\rangle_{\Re} = \langle\Psi^{\cap\beta}\rangle_{\Re}$. Hence the generator doesn't have to be unique, even if it is a single point set. Now, let $\langle\Psi^{\cap\{t\}}\rangle_{\Re} = \langle\Psi^{\cap\{v\}}\rangle_{\Re}$ for $t, v \in \Re$. Consider the sets $\mathcal{X} = \{t \in \Re : \Psi(t) \cap \{t\} \neq \emptyset\} = \{t \in \Re : \Psi(t) \cap \{v\} \neq \emptyset\}$ and $\mathcal{Y} = \{t \in \Re : \Psi(t) \cap \{t,v\} \neq \emptyset\}$. Obviously $\mathcal{X} \subseteq \mathcal{Y}$. Let $t \in \mathcal{Y}$. Then,

$$\begin{aligned} \Psi(t) \cap \{t,v\} \neq \emptyset &\Rightarrow \Psi(t) \cap \{t\} \neq \emptyset \text{ or } \Psi(t) \cap \{v\} \neq \emptyset \\ &\Rightarrow t \in \mathcal{X} \text{ or } t \in \mathcal{X} \\ &\Rightarrow t \in \mathcal{X} \end{aligned}$$

Hence $\mathcal{Y} \subseteq \mathcal{X}$. Therefore, $(\Psi,\Re)^{\cap\{t\}} = (\Psi,\Re)^{\cap\{v\}} = (\Psi,\Re)^{\cap\{t,v\}}$, which implies that $\langle\Psi^{\cap\{t,v\}}\rangle_{\Re} = \langle\Psi^{\cap\{t\}}\rangle_{\Re}$. □

Proposition 8. *If $\langle\Psi^{\cap\alpha}\rangle_{\Re}\tilde{<}\Re$, then $\Psi(0_{\Re}) \cap \alpha \neq \emptyset$. But the reverse implication is not true, in general.*

Proof. Let $\langle\Psi^{\cap\alpha}\rangle_{\Re}\tilde{<}\Re$. Then the set $(\Psi,\Re)^{\cap\alpha} = \{t \in \Re : \Psi(t) \cap \alpha \neq \emptyset\}$ is a subring of $\Re$. Then $\Psi(0_{\Re}) \supseteq \Psi(t)$ for all $t \in (\Psi,\Re)^{\cap\alpha}$ by Proposition 1. Since $\Psi(t) \cap \alpha \neq \emptyset$ and $\Psi(0_{\Re}) \supseteq \Psi(t)$ for all $t \in (\Psi,\Re)^{\cap\alpha}$, then $\Psi(0_{\Re}) \cap \alpha \neq \emptyset$. For the rest of the proof, let $\lambda = \{0,1,5\} \subseteq \Re$ in Example 2. Then it is seen that $\Psi(0_{\Re}) \cap \lambda \neq \emptyset$, but

$(\Psi,\Re)^{\cap\lambda}=\{0,3,4\}$ is not a subring of $\Re$. Therefore, $(\Psi,(\Psi,\Re)^{\cap\lambda})$ is not a soft subring of $\Re$. □

Proposition 9. *Let* $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$ *and* $\langle\Psi^{\cap\beta}\rangle_{\Re}\widetilde{<}\Re$. *If* $\alpha\subseteq\beta$, *then* $\langle\Psi^{\cap\alpha}\rangle_{\Re}\subseteq\langle\Psi^{\cap\beta}\rangle_{\Re}$.

Proof. Let $t\in(\Psi,\Re)^{\cap\alpha}$. Then $\Psi(t)\cap\alpha\neq\emptyset$. Since $\alpha\subseteq\beta$, $\Psi(t)\cap\beta\neq\emptyset$. Hence $(\Psi,\Re)^{\cap\alpha}\subseteq(\Psi,\Re)^{\cap\beta}$. Therefore $(\Psi,(\Psi,\Re)^{\cap\alpha})\subseteq(\Psi,(\Psi,\Re)^{\cap\beta})$, which completes the proof. □

The following Theorem shows that Theorem 1 is also true for the operation soft intersection instead of restricted intersection when taking the soft set $(\Psi,\Re)$ instead of (Ψ,S).

Theorem 2. *If* $(\Psi,\Re)\widetilde{<}\Re$ *and* $(\Upsilon,\Re)\widetilde{<}\Re$, *then* $(\Psi,\Re)\widetilde{\cap}(\Upsilon,\Re)\widetilde{<}\Re$.

Proof. By Definition 6 $(\Psi,\Re)\widetilde{\cap}(\Upsilon,\Re)=(\Theta,\Re)$, where $\Theta(t)=\Psi(t)\cap\Upsilon(t)$ for all $t\in\Re$. Then for all $t,v\in\Re$,

$$\begin{aligned}\Theta(t-v)&=\Psi(t-v)\cap\Upsilon(t-v)\\&\supseteq(\Psi(t)\cap\Psi(v))\cap(\Upsilon(t)\cap\Upsilon(v))\\&=(\Psi(t)\cap\Upsilon(t))\cap(\Psi(v)\cap\Upsilon(v))\\&=\Theta(t)\cap\Theta(v),\\\Theta(tv)&=\Psi(tv)\cap\Upsilon(tv)\\&\supseteq(\Psi(t)\cap\Psi(v))\cap(\Upsilon(t)\cap\Upsilon(v))\\&=(\Psi(t)\cap\Upsilon(t))\cap(\Psi(v)\cap\Upsilon(v))\\&=\Theta(t)\cap\Theta(v).\end{aligned}$$

Therefore $(\Psi,\Re)\widetilde{\cap}(\Upsilon,\Re)\widetilde{<}\Re$. □

Now, some problems arise such that: Is the soft intersection of two set-generated soft subrings of $\Re$, again a set-generated soft subring of $\Re$? And, if $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{<}\Re$, $\langle\Psi^{\cap\beta}\rangle_{\Re}\widetilde{<}\Re$ such that $\langle\Psi^{\cap\alpha}\rangle_{\Re}\widetilde{\cap}\langle\Psi^{\cap\beta}\rangle_{\Re}=\langle\Psi^{\cap\xi}\rangle_{\Re}$, then can the subset ξ be expressed using α and β? The answer of the first problem is "No", we have the following example:

Example 3. *Given the ring* $\Re=(\mathbb{Z}_{12},+,.)$, *a soft set* $(\Psi,\Re)$ *over* $\Re$, *where* $\Psi:\Re\to P(\Re)$ *is a set-valued function defined by* $\Psi(0)=\{1,3,5,6,7,9,11\}$, $\Psi(1)=\{2,4\}$, $\Psi(2)=\{3,6,7,11\}$, $\Psi(3)=\{1,5,9\}$, $\Psi(4)=\{3,6,7,11\}$, $\Psi(5)=\{8,10\}$, $\Psi(6)=\{1,3,5,6,7,9,11\}$, $\Psi(7)=\{2,10\}$, $\Psi(8)=\{3,6,7,11\}$, $\Psi(9)=\{1,5,9\}$, $\Psi(10)=\{3,6,7,11\}$ *and* $\Psi(11)=\{2,8\}$. *Let* $\alpha=\{11\}$ *and* $\beta=\{5\}$. *Then,* $(\Psi,\Re)^{\cap\alpha}=\{0,2,4,6,8,10\}$ *and* $(\Psi,\Re)^{\cap\beta}=\{0,3,6,9\}$ *are subrings of* $\Re$ *and the soft sets*

$$(\Psi,(\Psi,\Re)^{\cap\alpha})=\left\{\begin{array}{l}(0,\{1,3,5,6,7,9,11\}),(2,\{3,6,7,11\}),(4,\{3,6,7,11\}),\\(6,\{1,3,5,6,7,9,11\}),(8,\{3,6,7,11\}),(10,\{3,6,7,11\})\end{array}\right\}$$

and

$$(\Psi, (\Psi, \Re)^{\cap\beta}) = \left\{ \begin{array}{l} (0, \{1,3,5,6,7,9,11\}), (3, \{1,5,9\}), \\ (6, \{1,3,5,6,7,9,11\}), (9, \{1,5,9\}) \end{array} \right\}$$

satisfy the conditions $s1, s2$ *of Definition 7. Hence* $\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{<} \Re$ *and* $\langle \Psi^{\cap\beta} \rangle_{\Re} \widetilde{<} \Re$. *Then,*

$\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{\cap} \langle \Psi^{\cap\beta} \rangle_{\Re} = \{(0, \{1,3,5,6,7,9,11\}), (6, \{1,3,5,6,7,9,11\})\} = (\Psi, S) \widetilde{<} \Re.$

But, there is no subset ξ *of* $\Re$ *such that* $(\Psi, S) = \langle \Psi^{\cap\xi} \rangle_{\Re}$.

Corollary 1. *The soft intersection of two set-generated soft subrings of* $\Re$ *is not a set-generated soft subring of* $\Re$*, in general.*

But, we have the following:

Theorem 3. *Let* $\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{<} \Re$ *and* $\langle \Psi^{\cap\beta} \rangle_{\Re} \widetilde{<} \Re$. *Then, either* $\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{\cap} \langle \Psi^{\cap\beta} \rangle_{\Re} = \langle 0_{\Re} \rangle_{\Psi}$ *trivial soft subring or if* $\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{\cap} \langle \Psi^{\cap\beta} \rangle_{\Re} = \langle \Psi^{\cap\xi} \rangle_{\Re}$*, then there exists* $\xi \subseteq \Re$ *such that* $\emptyset \neq \xi \subseteq \alpha \cup \beta$.

Proof. If $\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{\cap} \langle \Psi^{\cap\beta} \rangle_{\Re} = \langle 0_{\Re} \rangle_{\Psi}$, it is obvious. Assume that

$$\langle 0_{\Re} \rangle_{\Psi} \neq \langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{\cap} \langle \Psi^{\cap\beta} \rangle_{\Re} = \langle \Psi^{\cap\xi} \rangle_{\Re}.$$

Then $\langle \Psi^{\cap\xi} \rangle_{\Re} \widetilde{<} \Re$ by Theorem 2. Since $\langle \Psi^{\cap\alpha} \rangle_{\Re} \widetilde{\cap} \langle \Psi^{\cap\beta} \rangle_{\Re} = \langle \Psi^{\cap\xi} \rangle_{\Re}$, then we have

$$U \cap \alpha \neq \emptyset \wedge U \cap \beta \neq \emptyset \Leftrightarrow U \cap \xi \neq \emptyset$$

for $(t, U) \in \langle \Psi^{\cap\xi} \rangle_{\Re}$. The requirement (1) holds for:

i) $\alpha \subseteq \beta \Rightarrow \xi = \beta$,
ii) $\beta \subseteq \alpha \Rightarrow \xi = \alpha$,
iii) $\alpha \cap \beta = \emptyset \Rightarrow \xi = \alpha \cup \beta$,
iv) $\alpha \neq \beta$ and $\alpha \cap \beta \neq \emptyset \Rightarrow \xi = \alpha \cup \beta$.

Although the requirement (1) also holds for $\xi \supseteq \alpha \cup \beta$, it is enough to show existing $\xi \subseteq \Re$ such that $\emptyset \neq \xi \subseteq \alpha \cup \beta$ to complete the proof. □

Definition 12. *([29]) Let* $\Re_1$ *and* $\Re_2$ *be two rings, and* (Ψ, S_1) *and* (Υ, S_2) *be two soft subrings of* $\Re_1$ *and* $\Re_2$*, respectively. The product of soft subrings* (Ψ, S_1) *and* (Υ, S_2) *is defined as* $(\Psi, S_1) \times (\Upsilon, S_2) = (\Omega, S_1 \times S_2)$*, where* $\Omega(t, v) = \Psi(t) \times \Upsilon(v)$ *for all* $(t, v) \in S_1 \times S_2$.

Theorem 4. *([29]) If* $\Psi_{S_1} \widetilde{<} \Re_1$ *and* $\Upsilon_{S_2} \widetilde{<} \Re_2$*, then* $\Psi_{S_1} \times \Upsilon_{S_2} \widetilde{<} \Re_1 \times \Re_2$.

Theorem 4 leads to the problem: Is the product of two set-generated soft subrings of two rings, again a set-generated soft subring of the ring of product of rings? The answer is "Yes", we have the following:

Theorem 5. *Let $\Re_1$ and $\Re_2$ be two rings and let $(\Psi, \Re_1), (\Upsilon, \Re_2)$ be two soft sets over $\Re_1$ and $\Re_2$, respectively. If there exist $\alpha \subseteq \Re_1$ and $\beta \subseteq \Re_2$ such that $\langle \Psi^{\cap\alpha} \rangle_{\Re_1} \widetilde{<} \Re_1$ and $\langle \Upsilon^{\cap\beta} \rangle_{\Re_2} \widetilde{<} \Re_2$, then*

$$\langle \Psi^{\cap\alpha} \rangle_{\Re_1} \times \langle \Upsilon^{\cap\beta} \rangle_{\Re_2} = \langle \Theta^{\cap(\alpha\times\beta)} \rangle_{\Re_1\times\Re_2}.$$

Proof. Let $\langle \Psi^{\cap\alpha} \rangle_{\Re_1} \widetilde{<} \Re_1$ and $\langle \Upsilon^{\cap\beta} \rangle_{\Re_2} \widetilde{<} \Re_2$. Then $(\Psi, \Re_1)^{\cap\alpha}$ is a subring of $\Re_1$ and $(\Upsilon, \Re_2)^{\cap\beta}$ is a subring of $\Re_2$. So $(\Psi, \Re_1)^{\cap\alpha} \times (\Upsilon, \Re_2)^{\cap\beta}$ is a subring of $\Re_1 \times \Re_2$. Therefore, $\langle \Psi^{\cap\alpha} \rangle_{\Re_1} \times \langle \Upsilon^{\cap\beta} \rangle_{\Re_2} \widetilde{<} \Re_1 \times \Re_2$ by Theorem 4. Now, let $(t, \Psi(t)) \in \langle \Psi^{\cap\alpha} \rangle_{\Re_1}$ and $(v, \Upsilon(v)) \in \langle \Upsilon^{\cap\beta} \rangle_{\Re_2}$. Then $\Psi(t) \cap \alpha \neq \emptyset$ and $\Upsilon(v) \cap \beta \neq \emptyset$. Since

$$\Psi(t) \cap \alpha \neq \emptyset \wedge \Upsilon(v) \cap \beta \neq \emptyset \Leftrightarrow (\Psi(t) \times \Upsilon(v)) \cap (\alpha \times \beta) \neq \emptyset,$$

then we have

$$(t, \Psi(t)) \in \langle \Psi^{\cap\alpha} \rangle_{\Re_1} \wedge (v, \Upsilon(v)) \in \langle \Upsilon^{\cap\beta} \rangle_{\Re_2} \Leftrightarrow ((t, v), \Psi(t) \times \Upsilon(v)) \in \langle \Theta^{\cap(\alpha\times\beta)} \rangle_{\Re_1\times\Re_2}.$$

$(\Theta, \Re_1 \times \Re_2)$ is a soft set over $\Re_1 \times \Re_2$, where $\Theta : \Re_1 \times \Re_2 \to P(\Re_1 \times \Re_2)$ is a set-valued function defined by $\Theta(t, v) = \Psi(t) \times \Upsilon(v)$. Hence, the proof is completed. □

Example 4. *Over the ring $\Re_1 = (\mathbb{Z}_4, +, .)$, a soft set $(\Psi, \Re_1)$ given by $\Psi(0) = \{1, 2, 3\}$, $\Psi(1) = \{0\}$, $\Psi(2) = \{1, 3\}$, $\Psi(3) = \{2\}$. For $\alpha = \{3\}$, $(\Psi, \Re_1)^{\cap\alpha} = \{0, 2\}$ and $\langle \Psi^{\cap\alpha} \rangle_{\Re_1} = \{(0, \{1, 2, 3\}), (2, \{1, 3\})\} \widetilde{<} \Re_1$. Given the ring $\Re_2 = (\mathbb{Z}_6, +, .)$, a soft set $(\Upsilon, \Re_2)$ over $\Re_2$, defined by $\Upsilon(0) = \{0, 1, 2, 5\}$, $\Upsilon(1) = \{3, 4\}$, $\Upsilon(2) = \{4\}$, $\Upsilon(3) = \{0, 2\}$, $\Upsilon(4) = \{3\}$ and $\Upsilon(5) = \{4\}$. For $\beta = \{2\}$, $(\Upsilon, \Re_2)^{\cap\beta} = \{0, 3\}$ and $\langle \Upsilon^{\cap\beta} \rangle_{\Re_2} = \{(0, \{0, 1, 2, 5\}), (3, \{0, 2\})\} \widetilde{<} \Re_2$. $\langle \Psi^{\cap\alpha} \rangle_{\Re_1} \times \langle \Upsilon^{\cap\beta} \rangle_{\Re_2}$ is a soft set given by*

$$\left\{ \begin{array}{l} ((0,0), \{(1,0), (1,1), (1,2), (1,5), (2,0), (2,1), (2,2), (2,5), (3,0), (3,1), (3,2), (3,5)\}), \\ ((2,0), \{(1,0), (1,1), (1,2), (1,5), (3,0), (3,1), (3,2), (3,5)\}), \\ ((0,3), \{(1,0), (1,2), (2,0), (2,2), (3,0), (3,2)\}), \\ ((2,3), \{(1,0), (1,2), (3,0), (3,2)\}) \end{array} \right\}.$$

Now, let the soft set $(\Theta, \Re_1 \times \Re_2)$ over $\Re_1 \times \Re_2$, where $\Theta : \Re_1 \times \Re_2 \to P(\Re_1 \times \Re_2)$ is a set-valued function defined by $\Theta(t, v) = \Psi(t) \times \Upsilon(v)$. Then, for $\alpha \times \beta = \{(3, 2)\}$, it is easily seen that $\langle \Theta^{\cap(\alpha\times\beta)} \rangle_{\Re_1\times\Re_2} = \langle \Psi^{\cap\alpha} \rangle_{\Re_1} \times \langle \Upsilon^{\cap\beta} \rangle_{\Re_2}$.

5. Conclusions

In this paper, we are interested in the algebraic soft substructures of rings given in the article [29]. We introduced set-generated soft subrings of rings using non-empty subsets of rings. By theoretical directions, we applied some of the operations derived on soft sets to set-generated soft subrings. Moreover, we gave some relationships between the generators of soft subrings and studied their related various properties with assorted examples. To further this work, one could study the set-generated soft substructures of other algebraic structures such as fields, modules, vector spaces and algebras. Our future work will be based on the derivation of these algebraic structures and the investigation their application aspects.

Author Contribution Statements The authors contributed equally to this article.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement The authors are thankful to the referees for making valuable suggestions leading to the better presentations of this paper.

References

[1] Zadeh, L. A., Fuzzy sets, *Inf. Control,* 8 (1965), 338–353. http://dx.doi.org/10.1016/S0019-9958(65)90241-X

[2] Zadeh, L. A., Toward a generalized theory of uncertainty (GTU)-an outline, *Inf. Sci.,* 172 (2005), 1–40. https://doi.org/10.1016/j.ins.2005.01.017

[3] Gorzalzany, M. B., A method of inference in approximate reasoning based on interval-valued fuzzy sets, *Fuzzy Sets Syst.,* 21 (1987), 1–17. https://doi.org/10.1016/0165-0114(87)90148-5

[4] Gau, W. L., Buehrer, D. J., Vague sets, *IEEE Trans. Syst. Man Cybern.,* 23 (1993), 610–614. doi: 10.1109/21.229476

[5] Pawlak, Z., Rough sets, *Int. J. Comput. Inf. Sci.,* 11 (1982), 341–356. http://dx.doi.org/10.1007/BF01001956

[6] Pawlak, Z., Skowron, A., Rudiments of rough sets, *Inf. Sci.,* 177 (2007), 3–27. doi:10.1016/j.ins.2006.06.003

[7] Molodtsov, D., Soft set theory-first results, *Comput. Math. Appl.,* 37 (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

[8] Maji, P. K., Biswas, R., Roy, A. R., Soft set theory, *Comput. Math. Appl.,* 45 (2003), 555–562. https://doi.org/10.1016/S0898-1221(03)00016-6

[9] Maji, P. K., Roy, A. R., Biswas, R., An application of soft sets in a decision making problem, *Comput. Math. Appl.,* 44 (2002), 1077–1083. https://doi.org/10.1016/S0898-1221(02)00216-X

[10] Ali, M. I., Feng, F., Liu, X., Min, W. K., Shabir, M., On some new operations in soft set theory, *Comput. Math. Appl.,* 57 (2009), 1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009

[11] Çağman, N., Enginoğlu, S., Soft set theory and uni-int decision making, *Eur. J. Oper. Res.,* 207 (2010), 848–855. https://doi.org/10.1016/j.ejor.2010.05.004

[12] Kamacı, H., Similarity measure for soft matrices and its applications, *J. Intell. Fuzzy Syst.,* 36 (2019), 3061–3072. doi: 10.3233/JIFS-18339

[13] Kamacı, H., Atagün, A. O., Aygün, E., Difference operations of soft matrices with applications in decision making, *Punjab Univ. J. Math.,* 51 (2019), 1–21.

[14] Sezgin, A., Atagün, A. O., On operations of soft sets, *Comput. Math. Appl.,* 61 (2011), 1457–1467.https://doi.org/10.1016/j.camwa.2011.01.018

[15] Aygün, E., Kamacı, H., Some generalized operations in soft set theory and their role in similarity and decision making, *J. Intell. Fuzzy Syst.,* 36 (2019), 6537–6547. doi: 10.3233/JIFS-182924

[16] Aygün, E., Kamacı, H., Some new algebraic structures of soft sets, *Soft Comput.,* 25(13) (2021), 8609–8626. https://doi.org/10.1007/s00500-021-05744-y

[17] Çağman, N., Enginoğlu, S., Soft matrix theory and its decision making, *Comput. Math. Appl.,* 59 (2010), 3308–3314. https://doi.org/10.1016/j.camwa.2010.03.015

[18] Atagün, A. O., Kamacı, H., Oktay, O., Reduced soft matrices and generalized products with applications in decision making, *Neural Comput. Appl.,* 29 (2018), 445–456. https://doi.org/10.1007/s00521-016-2542-y
[19] Kamacı, H., Atagün, A. O., Sönmezoğlu, A., Row-products of soft matrices with applications in multiple-disjoint decision making, *Appl. Soft Comput.,* 62 (2018), 892–914. https://doi.org/10.1016/j.asoc.2017.09.024
[20] Kamacı, H., Atagün, A. O., Toktaş, E., Bijective soft matrix theory and multi-bijective linguistic soft decision system, *Filomat,* 32 (2018), 3799–3814. https://doi.org/10.2298/FIL1811799K
[21] Petchimuthu, S., Garg, H., Kamacı, H., Atagün, A. O., The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM, *Comput. Appl. Math.,* 39 (2020), Article Number 68. https://doi.org/10.1007/s40314-020-1083-2
[22] Kamacı, H., Saltık, K., Akız, H. F., Atagün, A. O., Cardinality inverse soft matrix theory and its applications in multicriteria group decision making, *J. Intell. Fuzzy Syst.,* 34 (2018), 2031–2049. doi: 10.3233/JIFS-17876
[23] Petchimuthu, S., Kamacı, H., The row-products of inverse soft matrices in multicriteria decision making, *J. Intell. Fuzzy Syst.,* 36 (2019), 6425–6441. doi: 10.3233/JIFS-182709
[24] Aktaş, H., Çağman, N., Soft sets and soft groups, *Inf. Sci.,* 177 (2007), 2726–2735. https://doi.org/10.1016/j.ins.2006.12.008
[25] Ulucay, V., Oztekin, O., Sahin, M., Olgun, N., Kargin, A., Soft representation of soft groups, *New Trend Math. Sci.,* 4(2) (2016), 23. http://dx.doi.org/10.20852/ntmsci.2016217001
[26] Feng, F., Jun, Y. B., Zhao, X., Soft semirings, *Comput. Math. Appl.,* 56 (2008), 2621–2628. https://doi.org/10.1016/j.camwa.2008.05.011
[27] Acar, U., Koyuncu, F., Tanay, B., Soft sets and soft rings, *Comput. Math. Appl.,* 59 (2010), 3458-3463. https://doi.org/10.1016/j.camwa.2010.03.034
[28] Uluçay, V., Şahin, M., Olgun, N., Soft normed rings, *SpringerPlus,* 5(1) (2016), 1–6. doi: 10.1186/s40064-016-3636-9
[29] Atagün, A. O., Sezer, A. S., Soft substructures of rings fields and modules, *Comput. Math. Appl.,* 61 (2011), 592–601. https://doi.org/10.1016/j.camwa.2010.12.005
[30] Sezgin, A., Atagün, A. O., Aygün, E., A note on soft near-rings and idealistic soft near-rings, *Filomat,* 25 (2011), 53–68. doi: 10.2298/FIL1101053S
[31] Ostadhadi-Dehkordi, S., Shum, K. P., Regular and strongly regular relations on soft hyperrings, *Soft Comput.,* 23 (2019), 3253–3260. https://doi.org/10.1007/s00500-018-03711-8
[32] Tahat, M. K., Sidky, F., Abo-Elhamayel, M., Soft topological soft groups and soft rings, *Soft Comput.,* 22 (2018), 7143–7156. https://doi.org/10.1007/s00500-018-3026-z
[33] Karaaslan, F., Some properties of AG*-groupoids and AG-bands under SI-product operation, *J. Intell. Fuzzy Syst.,* 36 (2019), 231–239. doi: 10.3233/JIFS-181208
[34] Yousafzaia, F., Khalaf, M. M., Alia, A., Arsham B., Saeidc, D., Non-associative ordered semigroups based on soft sets, *Commun. Algebra,* 47 (2019), 312–327. https://doi.org/10.1080/00927872.2018.1476524
[35] Zhan, J., Dudek, W. A., Neggers, J., A new soft union set: characterizations of hemirings, *Int. J. Mach. Learn. Cybern.,* 8 (2017), 525–535. https://doi.org/10.1007/s13042-015-0343-8
[36] Atagün, A. O., Sezgin, A., Soft subnear-rings, soft ideals and soft N-subgroups of near-rings, *Math. Sci. Lett.,* 7 (2018), 37–42. http://dx.doi.org/10.18576/msl/070106
[37] Riaz, M., Naeem, K., Aslam, M., Afzal, D., Almahdi, F. A. A., Jamal, S. S., Multi-criteria group decision making with Pythagorean fuzzy soft topology, *J. Intell. Fuzzy Syst.,* 39 (2020), 6703–6720. doi: 10.3233/JIFS-190854
[38] Riaz, M., Naim, Ç., Zareef, I., Aslam, M., N-soft topology and its applications to multi-criteria group decision making, *J. Intell. Fuzzy Syst.,* 36 (2019), 6521–6536. doi: 10.3233/JIFS-182919
[39] Riaz, M., Tehreim, S. T., On bipolar fuzzy soft topology with decision-making, *Soft Comput.,* 24 (2020), 18259–18272. https://doi.org/10.1007/s00500-020-05342-4

[40] Sezer, A. S., Çağman, N., Atagün, A. O., Ali, M. I., Türkmen, E., Soft intersection semigroups, ideals and bi-ideals; a new application on semigroup theory I, *Filomat,* 29 (2015), 917–946. doi: 10.2298/FIL1505917S

[41] Sezgin, A., Çağman, N., Çıtak, F., α-inclusions applied to group theory via soft set and logic, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.,* 68 (2019), 334–352. doi: 10.31801/cfsuasmas.420457

[42] Feng, F., Li, C. X., Davvaz, B., Ali, M. I., Soft sets combined with fuzzy sets and rough sets: a tentative approach, *Soft Comput.,* 14 (2010), 899–911. https://doi.org/10.1007/s00500-009-0465-6

[43] Feng, F., Liu, X. Y., Leoreanu-Fotea, V., Jun, Y. B., Soft sets and soft rough sets, *Inf. Sci.,* 181 (2011), 1125–1137. https://doi.org/10.1016/j.ins.2010.11.004

[44] Atagün, A. O., Kamacı, H., Decompositions of soft sets and soft matrices with applications in group decision making, *Scientia Iranica,* in press (2021). doi:10.24200/SCI.2021.58119.5575.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1007–1022 (2022)
DOI:10.31801/cfsuasmas.990670
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received:September 6, 2021; Accepted: May 25, 2022

CHEN INVARIANTS FOR RIEMANNIAN SUBMERSIONS AND THEIR APPLICATIONS

Mehmet GÜLBAHAR[1], Şemsi EKEN MERIÇ[2] and Erol KILIÇ[3]

[1]Department of Mathematics, Faculty of Science and Art, Harran University, Şanlıurfa, TURKEY
[2]Department of Mathematics, Faculty of Science and Art, Mersin University, Mersin, TURKEY
[3]Department of Mathematics, Faculty of Science and Art, İnönü University, Malatya, TURKEY

Abstract. In this paper, an optimal inequality involving the delta curvature is exposed. With the help of this inequality some characterizations about the vertical motion and the horizontal divergence are obtained.

1. Introduction

The celebrated divergence theorem states that divergence of a vector field indicates how much the vector spreads out from the certain point. In fluid kinematics, if a vector field X is considered as velocity of a fluid or a gas, then sign of $\operatorname{div}(X)$ describes the expansion or compression of flow. Therefore, the total expansion or compression of flow can be calculated by the help of divergence theorem so divergence is a useful tool to measuring the net flow of fluid diverging from a point or approaching a point. The first phenomenon is called as *horizontal divergence* and the other is called as *horizontal convergence*.

The continuity equation simple states that any matter can either be created or destroyed and implies for the atmosphere that its mass may be redistributed but can never be disappeared. Therefore, this equation gives us that

$$\operatorname{div}(U) = 0 \tag{1}$$

for any vector field $U = (u^1, u^2, u^3)$ on E^3. It can be written from (1) that

$$\operatorname{div}_H(U) + \frac{\partial u^3}{\partial z} = 0, \tag{2}$$

2020 *Mathematics Subject Classification.* 53B20,53C43,53C80.
Keywords. Curvature, Riemannian submersion, horizontal divergence.
mehmetgulbahar@harran.edu.tr-Corresponding author; 0000-0001-6950-7633
semsieken@hotmail.com; 0000-0003-2783-1149
erol.kilic@inonu.edu.tr; 0000-0001-7536-0404.

where $\mathrm{div}_H(U)$ is the horizontal divergence of U defined by

$$\mathrm{div}_H(U) = \frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}. \tag{3}$$

The equation given (2) is also known as the *continuity equation* in literature. Integrating (2), we have

$$\omega(p_1, p_0) \equiv u^3(p_1) - u^3(p_0) = -\int_{p_0}^{p_1} (\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y})dz, \tag{4}$$

where p_1 and p_0 is some pressure levels on the atmosphere. If we assume that p_0 is the surface pressure then $u^3(p_0) = 0$ and thus we get

$$\omega(p_1) = -\int_{p_0}^{p_1} (\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y})dz. \tag{5}$$

This formula tells us that w at a given pressure level is proportional to the integral of the horizontal divergence. Here, $\omega(p_1)$ is called the *vertical motion* at p_1. If $\omega(p) < 0$ at every point p then this statement is called *rising motion*, $\omega(p) > 0$ at every point p then this statement is called *descending motion*, (in this case, divergence is called convergence) in meteorology. There is no divergence and it is clear that there is a local maximum or minimum of w.

Beside these facts , B.-Y. Chen [7] initially introduced a new invariant the so-called delta curvature δ for an n-dimensional Riemannian manifold M by

$$\delta^k(p) = \tau(p) - (\inf \tau(\Pi_k))(p), \tag{6}$$

where $2 \leq k \leq n-1$, $\tau(p)$ is the scalar curvature at $p \in M$ and

$$(\inf \tau(\Pi_k))(p) = \inf\{\tau(\Pi_k) \,|\, \Pi_k \text{ is a } k\text{-plane section} \subset T_pM\}.$$

Furthermore, he gave a relation involving the delta curvature, the main intrinsic and extrinsic invariants of submanifolds in a real space form (cf. Lemma 3.2 in [7]). Then, this curvature drew attention of many authors and the notion of discovering simple basic relationships between intrinsic and extrinsic invariants of a submanifold becomes one of the most fundamental problems in submanifold theory (cf. [1, 3, 8, 10, 11, 19, 23, 24], etc.). Furthermore, various inequalities and their applications on Riemannian submersions were studied recently in [4, 12, 15, 22].

Apart from isometric immersions and submanifolds theory, Riemannian submersions have played a substantial role in differential geometry since this frame of maps also makes possible to compare geometrical properties between smooth manifolds. Besides the mathematical significance, Riemannian submersions have important physical and engineering aspects. There exist very nice applications of these mappings in the Kaluza-Klein theory [13, 16, 25], in the statical machine learning process [26], in the medical imaging [18], in the statical analysis [6], in the robotic theory [2, 20, 21].

Motivated by these facts, we firstly establish an optimal inequality involving the delta curvature for Riemannian manifolds admitting a Riemannian submersion.

Then, we investigate this inequality for some special cases. Finally, we obtain some results dealing the vertical motion and horizontal divergence.

2. Preliminaries

Let (M, g) be and n dimensional Riemannian manifold with Riemannian metric g. The sectional curvature, denoted $K_M(e_i \wedge e_j)$, of the plane section spanned by orthogonal unit vectors e_i and e_j at $p \in M$ is

$$K(e_i \wedge e_j) \equiv R\left(e_i, e_j, e_j, e_i\right) = R\left(e_j, e_i, e_i, e_j\right), \tag{7}$$

where R is the Riemann curvature tensor. Usually the sectional curvature $K(e_i \wedge e_j)$ is denoted by K_{ij}.

Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis for T_pM. In particular, the Ricci curvature Ric is defined by

$$\mathrm{Ric}\,(X) = \sum_{j=1}^{n} K\left(X \wedge e_j\right). \tag{8}$$

for each fixed e_i, $i \in \{1, ..., n\}$ we have

$$\mathrm{Ric}(e_i) = \sum_{j \neq i}^{n} K(e_i \wedge e_j).$$

The scalar curvature $\tau\,(p)$ at p is defined by

$$\tau\,(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j). \tag{9}$$

In particular, for a 2-dimensional Riemannian manifold, the scalar curvature is its Gaussian curvature.

Let Π_k be a k-plane section of T_pM and X a unit vector in Π_k. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of Π_k such that $e_1 = X$. The k-Ricci curvature of Π_k at X, denoted $\mathrm{Ric}_{\Pi_k}(X)$, is defined by [9]

$$\mathrm{Ric}_{\Pi_k}(X) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_k). \tag{10}$$

Thus for each fixed e_i, $i \in \{1, \ldots, k\}$ we get

$$\mathrm{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^{k} K(e_i \wedge e_j) = \sum_{j \neq i}^{k} K_{ij}. \tag{11}$$

We note that an *n-Ricci curvature* $\mathrm{Ric}_{T_pM}(e_i)$ is the usual *Ricci curvature* of e_i, denoted $\mathrm{Ric}\,(e_i)$. Thus for any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and for a fixed $i \in \{1, \ldots, n\}$, we have

$$\mathrm{Ric}_{T_pM}(e_i) \equiv \mathrm{Ric}\,(e_i) = \sum_{j \neq i}^{n} K_{ij}.$$

The scalar curvature $\tau(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau(\Pi_k) = \sum_{1\leq i<j\leq k} K(e_i \wedge e_j) = \sum_{1\leq i<j\leq k} K_{ij}, \tag{12}$$

where $\{e_1, ..., e_k\}$ is any orthonormal basis of the k-plane section Π_k. We note that

$$\tau(\Pi_k) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}^{k} K(e_i \wedge e_j) = \frac{1}{2}\sum_{i=1}^{k} \mathrm{Ric}_{\Pi_k}(e_i). \tag{13}$$

Given an orthonormal basis $\{e_1, ..., e_n\}$ for T_pM, $\tau_{1\cdots k}$ will denote the scalar curvature of the k-plane section spanned by $e_1, ..., e_k$.

The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is,

$$\tau(p) = \tau(T_pM).$$

Let (M, g) and $(B, \widetilde{g})$ be m and n dimensional Riemannian manifolds with Riemannian metrics g and $\widetilde{g}$, respectively. A smooth map $\pi : (M, g) \to (B, \widetilde{g})$ is called a *Riemannian submersion* if

i) π has maximal rank.
ii) The differential π_* preserves the lengths of horizontal vectors.

Now, let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemannian submersion. For any $b \in B$, $\pi^{-1}(b)$ is closed r-dimensional submanifold of M. The submanifolds $\pi^{-1}(b)$ are called *fibers*. A vector field tangent to fibers is called *vertical* and a vector field orthogonal to fibers is called *horizontal*. If we put

$$\mathcal{V}_p = \mathrm{kernel}(\pi_*) \tag{14}$$

at a point $p \in M$, then it can be obtained an integrable distribution $\mathcal{V}$ corresponding to the foliation of M determined by the fibres of π. The distribution $\mathcal{V}_p$ is called *vertical space* at $p \in M$.

Let $\mathcal{H}$ be a complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g. For any $p \in M$, the distribution $\mathcal{H}_p = (\mathcal{V}_p)^\perp$ is called *horizontal space* on M [17]. Thus, we have the following orthogonal decomposition:

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{15}$$

A vector field E on M is called *basic* if it is horizontal and $\pi-$ related to a vector field E_* on B i.e., $\pi_* E_p = E_{*\pi(p)}$ for all $p \in M$. Furthermore, it is known that if E and F are the basic vector fields respectively $\pi-$related to E_* and F_*, one has

$$g(E, F) \quad = \quad \widetilde{g}(E_*, F_*) \circ \pi. \tag{16}$$

Let h and v are the projections of $\Gamma(TM)$ onto $\Gamma(\mathcal{H})$ and $\Gamma(\mathcal{V})$, respectively. The *fundamental tensor fields* of π, denoted by A and T, are defined respectively by

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \tag{17}$$

$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{18}$$

for any $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M.

Now, let us define the following mappings:

$$\begin{aligned} T^{\mathcal{H}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) &\rightarrow \Gamma(\mathcal{H}), \\ (U, V) &\rightarrow T^{\mathcal{H}}(U, V) = h\nabla_U V, \end{aligned}$$

$$\begin{aligned} T^{\mathcal{V}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{H}) &\rightarrow \Gamma(\mathcal{V}), \\ (U, X) &\rightarrow T^{\mathcal{V}}(U, X) = v\nabla_U X, \end{aligned}$$

and

$$\begin{aligned} A^{\mathcal{H}} : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{V}) &\rightarrow \Gamma(\mathcal{H}), \\ (X, U) &\rightarrow A^{\mathcal{H}}(X, U) = h\nabla_X U, \end{aligned}$$

$$\begin{aligned} A^{\mathcal{V}} : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) &\rightarrow \Gamma(\mathcal{V}), \\ (X, Y) &\rightarrow A^{\mathcal{V}}(X, Y) = v\nabla_X Y, \end{aligned}$$

Then, it is clear from (17) and (18) that $T^{\mathcal{H}}$ is a symmetric operator on $\Gamma(\mathcal{V}) \times \Gamma(\mathcal{V})$ and $A^{\mathcal{V}}$ is an anti-symmetric operator on $\Gamma(\mathcal{H}) \times \Gamma(\mathcal{H})$. If (17) and (18) are taken into account in (15), we can write

$$\nabla_U V = T^{\mathcal{H}}(U, V) + v\nabla_U V, \tag{19}$$

$$\nabla_V X = h\nabla_V X + T^{\mathcal{V}}(V, X), \tag{20}$$

$$\nabla_X U = A^{\mathcal{H}}(X, U) + v\nabla_X U, \tag{21}$$

$$\nabla_X Y = h\nabla_X Y + A^{\mathcal{V}}(X, Y) \tag{22}$$

for any $U, V \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$.

Let $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ be an orthonormal basis on T_pM, where $\mathcal{V} = \mathrm{Span}\{U_1, \ldots, U_r\}$ and $\mathcal{H} = \mathrm{Span}\{X_1, \ldots, X_n\}$. The mean curvature vector field $\hbar(p)$ of any fibre is defined by

$$\mathcal{N}(p) = \frac{1}{r}\sum_{j=1}^{r} T^{\mathcal{H}}(U_j, U_j). \tag{23}$$

Note that each fiber is a minimal submanifold of M if and only if $\hbar(p) = 0$ for all $p \in M$. Furthermore, each fiber is called *totally geodesic* if both $T^{\mathcal{H}}$ and $T^{\mathcal{V}}$ vanish identically and it is called *totally umbilical* if

$$T^{\mathcal{H}}(U, V) = g\,(U, V)\,\hbar$$

for all $U, V \in \Gamma(\mathcal{V})$.

Now we recall the following Theorem [14]:

Theorem 1. *Let $\pi : (M, g) \rightarrow (B, \widetilde{g})$ be a Riemann submersion. Then the horizontal space $\mathcal{H}$ is an integrable distribution if and only if A vanishes identically.*

Remark 1. *As a consequence of Theorem 1, we see that both $A^{\mathcal{H}}$ and $A^{\mathcal{V}}$ are related to integrability of $\mathcal{H}$, that is, they are identically zero if and only if $\mathcal{H}$ is integrable.*

Let R, $\widetilde{R}$ and $\hat{R}$ are the curvature tensors on M, B and be the collection of all curvature tensors on fibers $\pi^{-1}(b)$ respectively, and $\check{R}(X,Y)Z$ be the horizontal lift of $\widetilde{R}_{\pi(b)}(\pi_{*p}X_b, \pi_{*p}Y_b)Z_b$ at any point $b \in M$ satisfying

$$\pi_*(\check{R}(X,Y)Z) = \widetilde{R}(\pi_*X, \pi_*Y)\pi_*Z.$$

Then, there exist the following relations between these tensors:

$$\begin{aligned} R(U,V,W,G) &= \hat{R}(U,V,W,G) + g\left((T^{\mathcal{H}}(U,G), T^{\mathcal{H}}(V,W)\right) \\ &\quad -g\left(T^{\mathcal{H}}(V,G), T^{\mathcal{H}}(U,W)\right), \end{aligned} \tag{24}$$

$$\begin{aligned} R(X,Y,Z,H) &= \check{R}(X,Y,Z,H) - 2g\left(A^{\mathcal{V}}(X,Y), A^{\mathcal{V}}(Z,H)\right) \\ &\quad +g\left(A^{\mathcal{V}}(Y,Z), A^{\mathcal{V}}(X,H)\right) - g\left(A^{\mathcal{V}}(X,Z), A^{\mathcal{V}}(Y,H)\right), \end{aligned} \tag{25}$$

$$\begin{aligned} R(X,V,Y,W) &= g\left(\left(\nabla_X T\right)(V,W), Y\right) + g\left(\left(\nabla_V A\right)(X,Y), W\right) \\ &\quad -g\left(T^{\mathcal{V}}(V,X), T^{\mathcal{V}}(W,Y)\right) \\ &\quad +g\left(A^{\mathcal{H}}(X,V), A^{\mathcal{H}}(Y,W)\right), \end{aligned} \tag{26}$$

for any $U,V,W,G \in \Gamma(\mathcal{V})$ and $X,Y,Z,H \in \Gamma(\mathcal{H})$. Note that the above equalities are known as *Gauss–Codazzi equations* for a Riemannian submersion. With the help of Gauss–Codazzi equations, we get the following relations between the sectional curvatures as follows:

$$\begin{aligned} K(U \wedge V) &= \hat{K}(U \wedge V) - \left\|T^{\mathcal{H}}(U,V)\right\|^2 \\ &\quad +g\left(T^{\mathcal{H}}(U,U), T^{\mathcal{H}}(V,V)\right), \end{aligned} \tag{27}$$

$$K(X \wedge Y) = \check{K}(\check{X} \wedge \check{Y}) + 3\|A^{\mathcal{V}}(X,Y)\|^2, \tag{28}$$

$$\begin{aligned} K(X \wedge V) &= -g\left(\left(\nabla_X T\right)(V,V), X\right) + \left\|T^{\mathcal{V}}(V,X)\right\|^2 \\ &\quad - \left\|A^{\mathcal{H}}(X,V)\right\|^2, \end{aligned} \tag{29}$$

where K, $\hat{K}$ and $\check{K}$ denote the sectional curvatures in M, any fiber $\pi^{-1}(b)$ and the horizontal distribution $\mathcal{H}$, respectively. The scalar curvatures of the vertical and horizontal spaces at a point $p \in M$ are given respectively by

$$\hat{\tau}(p) = \sum_{1 \le i < j \le r} \hat{K}(U_i, U_j) \tag{30}$$

and

$$\check{\tau}(p) = \sum_{1 \le i < j \le n} \check{K}(X_i, X_j). \tag{31}$$

Now, we recall the following definition of [5].

Definition 1. *Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemann submersion and X be a horizontal vector field on π. Then, horizontal divergence of X is defined by*

$$div_{\mathcal{H}}(X) = \sum_{i=1}^{n} g(\nabla_{X_i} X, X_i). \tag{32}$$

Lemma 1. *[14] Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemann submersion and $\{U_1, \dots, U_r\}$ be any orthonormal basis of $\Gamma(\mathcal{V})$. For any $E \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{H})$, we have*

$$g(\nabla_E \mathcal{N}, X) = \frac{1}{r} \sum_{j=1}^{r} g\left((\nabla_E T)(U_j, U_j), X \right). \tag{33}$$

As a consequence of Lemma 1, we obtain that

$$\mathrm{div}_{\mathcal{H}}(\mathcal{N}) = \frac{1}{r} \sum_{i=1}^{n} \sum_{j=1}^{r} g\left((\nabla_{X_i} T)(U_j, U_j), X_i \right). \tag{34}$$

3. An Optimal Inequality for Riemannian Submersions

We begin this section with the following algebraic lemma:

Lemma 2. *If $n > k \geq 2$ and $a_1, \dots, a_n, a$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i \right)^2 = (n - k + 1) \left(\sum_{i=1}^{n} a_i^2 + a \right), \tag{35}$$

then

$$2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a,$$

with equality holding if and only if

$$a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n.$$

Proof. By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i \right)^2 \leq (n-k+1)((a_1+a_2+\cdots+a_k)^2+a_{k+1}^2+\cdots+a_n^2). \tag{36}$$

From (35) and (36), we get

$$\sum_{i=1}^{n} a_i^2 + a \leq (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}^2 + \cdots + a_n^2.$$

The above equation is equivalent to

$$2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a.$$

The equality holds if and only if $a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n$. □

Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemannian submersion between Riemannian manifolds (M, g) and $(B, \widetilde{g})$. Suppose $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ be an orthonormal basis on T_pM, where $\mathcal{V} = \mathrm{Span}\{U_1, \ldots, U_r\}$ and $\mathcal{H} = \mathrm{Span}\{X_1, \ldots, X_n\}$. Then, we have

$$\|T^{\mathcal{H}}\|^2 = \sum_{i,j=1}^{r} g\left(T^{\mathcal{H}}(U_i, U_j), T^{\mathcal{H}}(U_i, U_j)\right), \tag{37}$$

$$\|T^{\mathcal{V}}\|^2 = \sum_{i=1}^{r}\sum_{j=1}^{n} g\left(T^{\mathcal{V}}(U_i, X_j), T^{\mathcal{V}}(U_i, X_j)\right), \tag{38}$$

$$\|A^{\mathcal{H}}\|^2 = \sum_{i=1}^{r}\sum_{j=1}^{n} g\left(A^{\mathcal{H}}(X_j, U_i), A^{\mathcal{H}}(X_j, U_i)\right), \tag{39}$$

$$\|A^{\mathcal{V}}\|^2 = \sum_{i,j=1}^{n} g\left(A^{\mathcal{V}}(X_i, X_j), A^{\mathcal{V}}(X_i, X_j)\right). \tag{40}$$

Putting (27) − (29), (34) and (37) − (40) in

$$\tau(p) = \sum_{1 \leq i < j \leq n} [K(U_i, U_j) + K(X_i, U_j) + K(X_i, X_j)],$$

we obtain the following lemma:

Lemma 3. *Let (M, g) and $(B, \widetilde{g})$ be a Riemannian manifolds admitting a Riemannian submersion $\pi : (M, g) \to (B, \widetilde{g})$. For any point $p \in M$, we have*

$$\begin{aligned} 2\tau(p) &= 2\hat{\tau}(p) + 2\check{\tau}(p) + r^2 \|\hbar(p)\|^2 - \left\|T^{\mathcal{H}}\right\|^2 + 3\left\|A^{\mathcal{V}}\right\|^2 \\ &\quad - r\ div_{\mathcal{H}}(\hbar(p)) + \|T^{\mathcal{V}}\|^2 - \|A^{\mathcal{H}}\|^2. \end{aligned} \tag{41}$$

Now, we are going to give an optimal inequality involving the $\delta-$curvature for Riemannian manifolds admitting a Riemannian submersion.

Theorem 2. *Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemannian submersion. Then, for each point $p \in M$ and each k-plane section $L_k \subset \mathcal{V}_p$ $(r > k \geq 2)$, we have*

$$\begin{aligned} \delta(k) &\leq \hat{\tau}(p) - \hat{\tau}(L_k) + \check{\tau}(p) + \frac{r^2(r-k)}{2(r-k+1)}\|\hbar\|^2 - \frac{r}{2}div_{\mathcal{H}}(\hbar(p)) \\ &\quad + \frac{3}{2}\|A^{\mathcal{V}}\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2. \end{aligned} \tag{42}$$

The equality of (42) holds at $p \in M$ if and only if $A^{\mathcal{H}}$ vanishes identically and the shape operators $S_{X_1}, \ldots, S_{X_n}$ of $\mathcal{V}_p$ take forms as follows:

$$S_{X_1} = \begin{pmatrix} \begin{matrix} T_{11}^1 & 0 & \cdots & 0 \\ 0 & T_{22}^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{kk}^1 \end{matrix} & 0 \\ 0 & \left(\sum_{i=1}^{k} T_{ii}^1\right) I_{r-k} \end{pmatrix}, \tag{43}$$

$$S_{X_s} = \begin{pmatrix} \begin{matrix} T_{11}^s & T_{12}^s & \cdots & T_{1k}^s \\ T_{12}^s & T_{22}^s & \cdots & T_{2k}^s \\ \vdots & \vdots & \ddots & \vdots \\ T_{1k}^s & T_{2k}^s & \cdots & -\sum_{i=1}^{k-1} T_{ii}^s \end{matrix} & 0 \\ 0 & 0_{n-k} \end{pmatrix}, \qquad s \in \{2, \ldots, n\}. \tag{44}$$

Proof. Let L_k be a k-plane section of $\mathcal{V}_p$. We choose an orthonormal basis $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ on T_pM such that $\mathcal{V} = \text{Span}\{U_1, \ldots, U_r\}$ and $\mathcal{H} = \text{Span}\{X_1, \ldots, X_n\}$. We write

$$T_{ij}^s = g(T^{\mathcal{H}}(U_i, U_j), X_s) \tag{45}$$

for any $i, j \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, n\}$. Suppose that the mean curvature vector $\hbar(p)$ is in the direction of X_1 and $X_1, ..., X_n$ diagonalize the shape operator S_{X_1}. If we put

$$\begin{aligned} \eta &= 2\tau(p) - 2\hat{\tau}(p) - 2\check{\tau}(p) - \frac{r^2(r-k)}{(r-k+1)}\|\hbar\|^2 + r\ div_{\mathcal{H}}(\hbar(p)) \\ &\quad -3\left\|A^{\mathcal{V}}\right\|^2 - \|T^{\mathcal{V}}\|^2 + \|A^{\mathcal{H}}\|^2 \end{aligned} \tag{46}$$

in (41), it follows that

$$r^2\|\hbar\|^2 = (n-k+1)(\eta + \|T^{\mathcal{H}}\|^2). \tag{47}$$

The equation (47) is equivalent to

$$\left(\sum_{i=1}^{r} T_{ii}^1\right)^2 = (n-k+1)\left(\eta + \sum_{i=1}^{r}\left(T_{ii}^1\right)^2 + \sum_{s=2}^{n}\sum_{i,j=1}^{r}\left(T_{ij}^s\right)^2\right). \tag{48}$$

Applying Lemma 2 to equation (48), we get

$$2\sum_{1\le i<j\le k} T_{ii}^{n+1} T_{jj}^{n+1} \ge \eta + \sum_{s=2}^{n}\sum_{i,j=1}^{r}\left(T_{ij}^s\right)^2. \tag{49}$$

On the other hand, we have from (41) that

$$\tau(L_k) = \hat{\tau}(L_k) + \sum_{1\le i<j\le k} T_{ii}^1 T_{jj}^1 + \sum_{s=2}^{n} \sum_{1\le i<j\le k} \left(T_{ii}^s T_{jj}^s - \left(T_{ij}^s\right)^2 \right). \tag{50}$$

From (49) and (50), we get

$$\begin{aligned} \tau(L_k) \quad &\ge \quad \hat{\tau}(L_k) + \frac{1}{2}\eta + \sum_{s=2}^{n}\sum_{j>k}\{(T_{1j}^s)^2 + (T_{2j}^s)^2 + \cdots + (T_{kj}^s)^2\} \\ &\quad + \frac{1}{2}\sum_{s=2}^{n}(T_{11}^s + T_{22}^s + \cdots + T_{kk}^s)^2 + \frac{1}{2}\sum_{s=2}^{n}\sum_{i,j>k}(T_{ij}^s)^2. \end{aligned} \tag{51}$$

In view of (51), we see that

$$\tau(\Pi_k) \ge \widetilde{\tau}(\Pi_k) + \frac{1}{2}\eta. \tag{52}$$

From (47) and (52), we obtain (42).

If the equality case of (42) holds, then we have $A^{\mathcal{H}}$ vanishes identically and

$$\begin{cases} T_{1j}^1 = T_{2j}^1 = T_{kj}^1 = 0, \qquad j = k+1, \ldots, r, \\ T_{ij}^s = 0, \qquad i, j = k+1, \ldots, r, \\ T_{11}^r + T_{22}^r + \cdots + T_{kk}^r = 0 \end{cases} \tag{53}$$

for $s = 2, \ldots, n$. Applying Lemma 2, we also have

$$T_{11}^1 + T_{22}^1 + \cdots + T_{kk}^1 = T_{ll}^1, \qquad l = k+1, \ldots, n. \tag{54}$$

Thus, with respect to a suitable orthonormal basis $\{X_1, \ldots, X_m\}$ on $\mathcal{H}_p$, the shape operator of $\mathcal{V}_p$ becomes of the form given by (43) and (44). The proof of the converse part is straightforward. □

In particular case of $k = 2$, we have the following:

Corollary 1. *Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemannian submersion. Then, for each point $p \in M$ and each plane section $L \subset \mathcal{V}_p$, we have*

$$\begin{aligned} \delta(2) \quad &\le \quad \hat{\tau}(p) - \hat{K}(L) + \check{\tau}(p) + \frac{r^2(r-2)}{2(r-1)}\|\hbar\|^2 - \frac{r}{2} div_{\mathcal{H}}(\hbar) \\ &\quad + \frac{3}{2}\|A^{\mathcal{V}}\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2. \end{aligned} \tag{55}$$

The equality of (55) holds at $p \in M$ if and only if $A^{\mathcal{H}}$ vanishes identically and the shape operators $S_{X_1}, \ldots, S_{X_n}$ of $\mathcal{V}_p$ take forms

$$S_{X_1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)\, I_{r-2} \end{pmatrix}, \tag{56}$$

$$S_{X_s}=\begin{pmatrix} c_s & d_s & 0 \\ d_s & -c_s & 0 \\ 0 & 0 & 0_{r-2} \end{pmatrix}, \qquad s\in\{2,\dots,n\}. \tag{57}$$

In particular case of $k=r-1$, we have the following

Corollary 2. *Let $\pi:(M,g)\to(B,\widetilde{g})$ be a Riemannian submersion. For each vertical unit vector U, we have*

$$\mathrm{Ric}_{\mathcal{V}}(U)\le \hat{\mathrm{Ric}}(U)+\check{\tau}(p)+\frac{r^2}{4}\|\hbar\|^2-\frac{r}{2}div_{\mathcal{H}}(\hbar)+\frac{3}{2}\|A^{\mathcal{V}}\|^2+\frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{58}$$

The equality case of (58) holds for all unit vectors $U\in\mathcal{V}_p$ if and only if $A^{\mathcal{H}}$ vanishes identically and we have either

(i) *if $r=2$, π has totally umbilical fibers at $p\in M$,*
(i) *if $r\neq 2$, π has totally geodesic fibers at $p\in M$.*

Proof. Let L_{r-1} be a $(r-1)$-plane section of $\mathcal{V}_p$. We get from Theorem 2 that

$$\begin{aligned}\delta(r-1) \quad \le \quad & \hat{\tau}(p)-\hat{\tau}(L_{r-1})+\check{\tau}(p)+\frac{r^2}{4}\|\hbar\|^2-\frac{r}{2}div_{\mathcal{H}}(\hbar) \\ & +\frac{3}{2}\|A^{\mathcal{V}}\|^2+\frac{1}{2}\|T^{\mathcal{V}}\|^2. \end{aligned} \tag{59}$$

Now, let U be a unit vertical vector field such that $U=U_r$. By a straightforward computation, we obtain (58).

The equality of (59) holds if and only if the forms of shape operators S_{X_s}, $s=1,\dots,n$, become

$$S_{X_1}=\begin{pmatrix} T^1_{11} & 0 & \cdots & 0 & 0 \\ 0 & T^1_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T^1_{(r-1)(r-1)} & 0 \\ 0 & 0 & \cdots & 0 & \left(\sum_{i=1}^{r-1}T^1_{ii}\right) \end{pmatrix}, \tag{60}$$

$$S_{X_s}=\begin{pmatrix} T^s_{11} & T^s_{12} & \cdots & T^r_{1(r-1)} & 0 \\ T^s_{12} & T^s_{22} & \cdots & T^s_{2(r-1)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T^s_{1(r-1)} & T^s_{2(r-1)} & \cdots & -\sum_{i=1}^{r-2}T^s_{ii} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad r\in\{2,\dots,n\}. \tag{61}$$

From (60) and (61), we see that the equality in (58) is valid for a unit vertical vector field $U = U_r$ if and only if

$$\begin{cases} T^s_{rr} = T^s_{11} + T^s_{22} + \cdots + T^s_{(r-1)(r-1)} \\ T^s_{1r} = T^s_{2r} = \cdots = T^s_{(r-1)r} = 0. \end{cases} \tag{62}$$

for $s \in \{1, \ldots, n\}$.

Assuming the equality case of (58) holds for all unit vertical vector fields, in view of (62), for each $s \in \{1, \ldots, n\}$, we have

$$\begin{cases} 2T^s_{ii} = T^s_{11} + T^s_{22} + \cdots + T^s_{rr}, \\ T^s_{ij} = 0, \quad i \neq j \end{cases} \tag{63}$$

for all $i \in \{1, ..., r\}$ and $s \in \{1, \ldots, n\}$. Thus, we have two cases, namely either $r = 2$ or $r \neq 2$. In the first case we see that π has totally umbilical fibers, while in the second case π has totally geodesic fibers. The proof of converse part is straightforward. □

Remark 2. *We note that (58) was also proved in [15] (see Theorem 4.1 in [15]). In Theorem 2, we gave a new proof for this inequality.*

4. Main Conclusions

In this section, we shall present a solution way with the help of differential geometry tools for the following natural problem:

"Which conditions should provide to the horizontal divergence or the convergence receives to the maximum value or minimum value?"

To obtain minimum or maximum values of the vertical motion (or horizontal divergence) it can be considered a Riemannian submersion on E^3 to E^2. Moreover, we can regard to different Riemannian submersions such as a Riemannian submersion on a three dimensional Riemannian manifold to two dimensional Riemannian manifold as

$$\pi : M^3 \to N^2. \tag{64}$$

It can also be considered globally in high dimensional Riemannian manifolds with taking a Riemannian submersion on m-dimensional Riemannian manifold to n-dimensional Riemannian manifold.

Taking into account of the continuity equation and (42), (55) and (58) inequalities, we get some result dealing minimum or maximum values of vertical motion for a manifold admitting a Riemannain submersion.

As a consequence of (42), we obtain the following:

Corollary 3. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion. Then we have*

$$\frac{r}{2}\omega(p) \quad \geq \quad \delta(k) - \frac{r^2(r-k)}{2(r-k+1)}\|\hbar\|^2 - \frac{3}{2}\|A^{\mathcal{V}}\|^2 - \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{65}$$

The vertical motion at a point p takes the minimum value if and only if $A^{\mathcal{H}}$ vanishes identically and the matrixes of shape operators of the vertical space of M take the form as (43) and (44).

As a consequence of (55), we obtain the followings:

Corollary 4. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion with integrable horizontal distribution. Then we have*

$$\frac{r}{2}\omega(p) \quad \geq \quad \delta(2) - \frac{r^2(r-2)}{2(r-1)}\|\hbar\|^2 - \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{66}$$

The vertical motion takes the minimum value if and only if the matrixes of shape operators $S_{x_1}, \ldots, S_{x_n}$ of the vertical space of M take the form as (56) and (57).

Corollary 5. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion with totally geodesic leaves and integrable horizontal distribution. Then we have*

$$\frac{r}{2}\omega(p) \quad = \quad \delta(2). \tag{67}$$

From (58), we get the followings:

Corollary 6. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion. For each vertical unit vector U, we have*

$$\frac{r}{2}\omega(p) \geq \mathrm{Ric}_{\mathcal{V}}(U) - \frac{r^2}{4}\|\hbar\|^2 - \frac{3}{2}\|A^{\mathcal{V}}\|^2 - \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{68}$$

The equality case of (68) holds for all unit vectors $U \in \mathcal{V}_p$ if and only if $A^{\mathcal{H}}$ vanishes identically and we have either

(i) *if $r = 2$, π has totally umbilical fibers at $p \in M$,*
(ii) *if $r \neq 2$, π has totally geodesic fibers at $p \in M$.*

Corollary 7. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion with totally geodesic fibers. For each vertical unit vector U, we have*

$$\frac{r}{2}\omega(p) = \mathrm{Ric}_{\mathcal{V}}(U) - \frac{3}{2}\|A^{\mathcal{V}}\|^2. \tag{69}$$

Now we shall mention some examples:

Example 1. *Consider the mapping $\pi : \mathrm{E}^5 \to \mathrm{E}^2$ which is defined by*

$$\pi(x_1, x_2, x_3, x_4, x_5) = \left(\frac{1}{\sqrt{2}}(x_1 + x_2), \frac{1}{\sqrt{2}}(x_3 + x_4)\right).$$

Then, it is clear that π is a Riemannian submersion and the Jacobian of π is equal to

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

The horizontal space and the vertical space are given by

$$\mathcal{H} = Span\{X_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2}, X_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}\}$$

and

$$\mathcal{V} = Span\{U_1 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}, U_2 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}, U_3 = \frac{\partial}{\partial x_5}\},$$

respectively. By a straightforward computation, we get the tensor fields A, T, $\mathrm{Ric}_{\mathcal{V}}$ *vanish and* $\omega(p) = div_{\mathcal{H}}(\hbar) = 0$ *from* (3). *Therefore,* π *is a trivial example satisfying Corollary 3-Corollary 7.*

Example 2. *(Example 5.1 in [15])*

Let us consider the Remannian submesion $\pi : M \to \mathrm{E}^3$ *defined by*

$$\pi(x_1, x_2, x_3, x_4, x_5) = (x_1 \cos x_3 + x_2 \sin x_3, x_4, x_5),$$

where M *is a non-flat submanifold of* E^5 *such that* $\cot x_3 = \frac{x_1}{x_2}$, $x_2 \neq 0$ *and* $x_3 \in \left(0, \frac{\pi}{2}\right)$. *Here, the horizontal space and the vertical space of* M *are given by*

$$\mathcal{H} = Span\{X_1 = \sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2}, X_2 = \frac{\partial}{\partial x_4}, X_3 = \frac{\partial}{\partial x_5}\}$$

and

$$\mathcal{V} = Span\{U_1 = -\cos x_3 \frac{\partial}{\partial x_1} + \sin x_3 \frac{\partial}{\partial x_2}, U_2 = \frac{\partial}{\partial x_3}\},$$

respectively. By straightforward computations, we have $T^{\mathcal{V}}(U_2, X_1) = -U_1$, $T^{\mathcal{H}}(U_1, U_2) = X_1$ *and the other components of operators* $T^{\mathcal{H}}$, $T^{\mathcal{V}}$, $A^{\mathcal{H}}$, $A^{\mathcal{V}}$ *vanish identically. Moreover, we have* $\hat{\mathrm{Ric}}(U_1) = 1$, $\mathrm{Ric}_{\mathcal{V}}(U_1) = \mathrm{Ric}_{\mathcal{V}}(U_2) = 0$ *and* $\omega(p) = 0$ *from* (3). *Considering these facts, we obtain the left hand side of 68 is equal to* 0 *and the right hand side of* (68) *is equal to* -1 *for* $U = U_1$. *This inequality also satisfies for* $U = U_2$. *This shows that the correctness of* (68) *and* π *is an example of Corollary 6.*

Author Contribution Statements The authors contributed equally to this study.

Declaration of Competing Interests The authors have no competing interests to declare.

Acknowledgment The authors are thankful to the editor İsmail Gök and reviewers for their constructive comments.

References

[1] Alegre, P., Chen, B. Y., Munteanu, M. I., Riemannian submersions, δ-invariants and optimal inequality, *Ann. Glob. Anal. Geom.*, 42 (2010), 317–331. https://doi.org/10.1007/s10455-012-9314-4

[2] Altafini, C., Redundant robotic chains on Riemannian submersions, *IEEE Robot. Autom.*, 20(2) (2004), 335–340. https://doi.org/10.1109/TRA.2004.824636

[3] Arslan, K., Ezentaş, R., Mihai, I, Murathan C., Özgür, C., B.Y Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds, *Bull. Inst. Math. Acad. Sin.*, 29(3) (2001), 231–242.

[4] Aytimur, H., Özgür, C., Sharp inequalities for anti-invariant Riemannian submersions from Sasakian space forms, *J. Geom. Phys.*, 166 (2021), 104251. https://doi.org/10.1016/j.geomphys.2021.104251

[5] Besse, A. L., Einstein Manifolds, Berlin-Heidelberg-New York, Spinger-Verlag, 1987.

[6] Bhattacharyaa, R., Patrangenarub, V., Nonparametic estimation of location and dispersion on Riemannian manifolds, *J. Statist. Plann. Inference*, 108 (2002), 23–35. https://doi.org/10.1016/S0378-3758(02)00268-9

[7] Chen, B. Y., Some pinching and classification theorems for minimal submanifolds, *Arch. Math.*, 60 (1993), 568–578. https://doi.org/10.1007/BF01236084

[8] Chen, B. Y., A Riemannian invariant and its applications to submanifold theory, *Results Math.*, 27 (1995), 17–28. https://doi.org/10.1007/BF03322265

[9] Chen, B. Y., Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasgow Math. J.*, 41 (1999), 33–41. https://doi.org/10.1017/S0017089599970271

[10] Chen, B. Y., Pseudo-Riemannian Geometry, δ-Invariants and Applications, World Scientific Publishing, Hackensack, NJ, 2011.

[11] Chen, B. Y., Dillen, F., Verstraelen, L., δ-invariants and their applications to centroaffine geometry, *Differ. Geom. Appl.*, 22, (2005) 341–354. https://doi.org/10.1016/j.difgeo.2005.01.008

[12] Eken Meriç, S., Gülbahar, M., Kılıç, E., Some inequalities for Riemannian submersions, *An. Stiint. Univ. Al. I. Cuza Iasi. Mat.*, 63 (2017), 471-482.

[13] Falcitelli, M., Ianus, S., Pastore, A. M., Visinescu, M., Some applications of Riemannian submersions in physics, *Rev. Roum. Phys.*, 48 (2003), 627–639.

[14] Falcitelli, M., Ianus, S., Pastore, A. M., Riemannian Submersions and Related Topics, World Scientific Company, 2004.

[15] Gülbahar, M., Eken Meriç, S., Kılıç, E., Sharp inequalities involving the Ricci curvature for Riemannian submersions, *Kragujevac J. Math.*, 42(2) (2017). https://doi.org/10.5937/KgJMath1702279G

[16] Kennedy, L. C., Some Results on Einstein Metrics on Two Classes of Quotient Manifolds, PhD thesis, University of California, 2003.

[17] Kobayashi, S., Submersions of CR-submanifolds, *Tohoku Math. J.*, 89 (1987), 95–100. https://doi.org/10.2748/tmj/1178228372

[18] Memoli F., Sapiro G., Thompson P., Implicit brain imaging, *Neuro Image*, 23 (2004), 179–188. https://doi.org/10.1016/j.neuroimage.2004.07.072

[19] Poyraz, N., Yaşar, E., Chen-like inequalities on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection, *Kragujevac J. Math.*, 40 (2016), 146–164. https://doi.org/10.5937/kgjmath1602146p

[20] Şahin, B., Riemannian submersions from almost Hermitian manifolds, *Taiwan. J. Math.*, 17 (2013), 629–659. https://doi.org/10.11650/tjm.17.2013.2191

[21] Şahin, B., Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications, Academic Press, 2017.

[22] Siddiqui, A. N., Chen inequalities for statistical submersions between statistical manifolds, *Int. J. Geom. Methods Mod. Phys.*, 18 (2021), 2150049. https://doi.org/10.1142/S0219887821500493

[23] Uddin, S., Solamy, F. R., Shahid, M. H., Saloom, A., B.-Y. Chen's inequality for bi-warped products and its applications in Kenmotsu manifolds, *Mediterr. J. Math.*, 15 (2018). https://doi.org/10.1007/s00009-018-1238-1

[24] Vilcu, G. E., On Chen invariants and inequalities in quaternionic geometry, *J. Inequal. Appl.*, 2013:66 (2013). https://doi.org/10.1186/1029-242X-2013-66

[25] Wang, H., Ziller, W., Einstein metrics on principal torus bundles, *J. Differ. Geo.*, 31 (1990), 215–248.

[26] Zhao, H., Kelly, A. R., Zhou, J., Lu, J., Yang, Y. Y., Graph attribute embedding via Riemannian submersion learning. *Comput. Vis. Image Underst.*, 115 (2011), 962–975. https://doi.org/10.1016/j.cviu.2010.12.005

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1023–1028 (2022)
DOI:10.31801/cfsuasmas.1051208
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: December 30, 2021; Accepted: June 2, 2022

PARITY OF AN ODD DOMINATING SET

Ahmet BATAL

Department of Mathematics, Izmir Institute of Technology, 35430, Urla, Izmir, TURKEY

Abstract. For a simple graph G with vertex set $V(G) = \{v_1, ..., v_n\}$, we define the closed neighborhood set of a vertex u as $N[u] = \{v \in V(G) \mid v$ is adjacent to u or $v = u\}$ and the closed neighborhood matrix $N(G)$ as the matrix obtained by setting to 1 all the diagonal entries of the adjacency matrix of G. We say a set S is odd dominating if $N[u] \cap S$ is odd for all $u \in V(G)$. We prove that the parity of an odd dominating set of G is equal to the parity of the rank of G, where the rank of G is defined as the dimension of the column space of $N(G)$. Using this result we prove several corollaries in one of which we obtain a general formula for the nullity of the join of graphs.

1. Introduction

Let $N[u]$ denote the *closed neighborhood set* of a vertex u in a simple graph G, i.e.;

$$N[u] = \{v \in V(G) \, |v \text{ is adjacent to } u \text{ or } v = u\}.$$

Then, we say a subset S of vertices is *odd (even) dominating* if $N[u] \cap S$ is odd (even) for all $u \in V(G)$. In general, for an arbitrary subset C of vertices, we say a set S is a C-*parity* set if $N[u] \cap S$ is odd for all $u \in C$ and even otherwise [2]. If there is a C-parity set for a given set C, we say that C is *solvable*. If there exists a C-parity set for every set C of vertices in a graph G, then we say G is *always solvable*.

Let n be the order of G, $V(G) = \{v_1, ..., v_n\}$ and W be a subset of $V(G)$. The column vector $\mathbf{x}_W = (x_1, ..., x_n)^t$, which is defined as $x_i = 1$ if $v_i \in W$ and $x_i = 0$ otherwise, is called the *characteristic vector* of W. The closed neighbourhood matrix $N = N(G)$ of a graph G is obtained by setting to 1 all the diagonal entries of the adjacency matrix of G. Equivalently, $N(G)$ is the matrix whose ith column

2020 *Mathematics Subject Classification.* 05C69.

Keywords. Lights out, all-ones problem, odd dominating set, parity domination, domination number.

ahmetbatal@iyte.edu.tr; 0000-0003-2869-6110.

is equal to $\mathbf{x}_{N[v_i]}$. It is easy to observe that S is a C-parity set if and only if

$$N(G)\mathbf{x}_S = \mathbf{x}_C \tag{1}$$

over the field $\mathbb{Z}_2$ [9], [10].

Let us denote the vectors whose components are all 0 and all 1 by $\mathbf{0}$ and $\mathbf{1}$, respectively. Then the following are equivalent. (*a1*) S is an odd dominating set, (*a2*) S is a $V(G)$-parity set, (*a3*) $N(G)\mathbf{x}_S = \mathbf{1}$. Similarly, (*b1*) S is an even dominating set, (*b2*) S is a $\emptyset$-parity set, (*b3*) $N(G)\mathbf{x}_S = \mathbf{0}$, are equivalent statements. Note that every graph has an even dominating set, which is $\emptyset$. On the other hand, it is proved by Sutner that every graph has an odd dominating set as well [9] (see also [6], [7], [8]).

Let $Ker(N)$ and $Col(N)$ denote the kernel and column space of N, respectively. Let $\nu(G) := dim(Ker(N(G))$ and $\rho(G) := dim(Col(N(G))$. We call $\nu(G)$, the *nullity* of G (Amin et al. [3] call it the parity dimension of G) and $\rho(G)$, the *rank* of G. We have $\nu(G) + \rho(G) = n$ by the rank nullity theorem.

From the matrix equation (1), we see that G is always solvable if and only if $\nu(G) = 0$. Moreover, $\nu(G) > 0$ if and only if G has a nonempty even dominating set.

We write $pr(a)$ to denote the parity function of a number a, i.e.; $pr(a) = 0$ if a is even and $pr(a) = 1$ if a is odd. In the case where A is a matrix, $pr(A)$ is the parity function of the sum of its entries. For a set S, we write $pr(S)$ to denote the parity function of the cardinality of S and say the *parity of* S instead of the parity of the cardinality of S. Note that $pr(S) = pr(\mathbf{x}_S)$. It was first noticed by Amin et al. [[1], Lemma 3], and follows immediately from Sutner's theorem, that for a given graph, the parity of all odd dominating sets are the same. Hence, the value of $pr(S)$, where S is an odd dominating set of a graph is independent of the particular odd dominating set S taken into account.

Our main result Theorem 1 states that the parity of an odd dominating set is equal to the parity of the rank of the graph.

2. Main Result

Lemma 1. *Let A be a $n \times n$, symmetric, invertible matrix over the field $\mathbb{Z}_2$ with diagonal entries equal to* 1. *Then $pr(A^{-1}) = pr(A) = pr(n)$.*

Proof. In the proof, all algebraic operations are considered over the field $\mathbb{Z}_2$. First of all, note that since A is a symmetric matrix with nonzero diagonal entries, we have

$$pr(A) = \sum_{i,j} A_{ij} = \sum_i A_{ii} = \sum_i 1 = pr(n).$$

Similarly,

$$pr(A^{-1}) = \sum_i (A^{-1})_{ii}.$$

On the other hand,

$$\begin{aligned} pr(n) = Tr(I) = Tr(AA^{-1}) \\ &= \sum_{i,j} A_{ij}(A^{-1})_{ij} \\ &= \sum_{i} A_{ii}(A^{-1})_{ii} \\ &= \sum_{i} (A^{-1})_{ii}. \end{aligned}$$

□

We call a vertex a *null vertex* of a graph G if it belongs to an even dominating set of G. Since the set of all characteristic vectors for even dominating sets of G is a subspace of the vector space of all binary n-tuples, if v is a null vertex of G, then precisely half of the even dominating sets of G contain v.

Lemma 2. *Let G be a graph and v be a null vertex of G. Then there exists an odd dominating set of G which does not contain v.*

Proof. Let R be an even dominating set containing v and S_1 be an odd dominating set of G. Assume S_1 contains v, otherwise we are done. Let S_2 be the symmetric difference of S_1 and R. Clearly S_2 is an odd dominating set which does not contain v. □

Let $G - v$ denote the graph obtained by removing a vertex v and all its incident edges from a graph G. The number $nd(v) := \nu(G - v) - \nu(G)$ is called the *null difference number*. It turns out that $nd(v)$ can be either -1, 0, or 1. Moreover, Ballard et al. proved the following lemma in [[5], Proposition 2.4.].

Lemma 3 ([5]). *Let v be a vertex of a graph G. Then v is a null vertex if and only if $nd(v) = -1$.*

Now we are ready to state our main result.

Theorem 1. *Let G be a graph and S be an odd dominating set of G. Then $pr(S) = pr(\rho(G))$. Equivalently, $pr(V(G)\backslash S) = pr(\nu(G))$.*

Proof. We prove the claim by applying induction on the nullity of the graph. Let n be the order of G. In the case where $\nu(G) = 0$, there exists a unique odd dominating set S such that $N\mathbf{x}_S = \mathbf{1}$. Note that N satisfies the conditions of Lemma 1. Hence, together with the rank nullity theorem, we have

$$pr(S) = pr(\mathbf{x}_S) = pr(N^{-1}\mathbf{1}) = pr(N^{-1}) = pr(N) = pr(n) = pr(\rho(G)).$$

Now assume that $\nu(G) > 0$ and the claim holds true for all graphs with nullity less than $\nu(G)$. Since $\nu(G)$ is nonzero, there exists a non-empty even dominating set. Hence, there exists a null vertex v of G. By Lemma 2, there is an odd dominating set S of G which does not contain v. Since S does not contain v, it is also an odd dominating set of the graph $G-v$. Moreover, by Lemma 3, $nd(v) = -1$. Hence, $\nu(G-v) = \nu(G) + nd(v) = \nu(G) - 1 < \nu(G)$. By the induction hypothesis $pr(S) = pr(\rho(G-v))$. On the other hand, using the rank nullity theorem we obtain $\rho(G-v) = n-1-\nu(G-v) = n-1-\nu(G)+1 = n-\nu(G) = \rho(G)$. We complete the proof by noting that all odd dominating sets in G have the same parity. □

3. Some Corollaries

Corollary 1. *Let G be an always solvable graph of order n. Then the odd dominating set of G has odd (even) cardinality if n is odd (even).*

Note that if every vertex of a graph G has even degree, then $V(G)$ itself is an odd dominating set. This, together with Theorem 1, gives the following.

Corollary 2. *If every vertex of a graph G has even degree, then $\nu(G)$ is even.*

Corollary 3. *If the number of even degree vertices of a tree T is at most one, then every odd dominating set of T has odd cardinality.*

Proof. Let n be the order of T. By [[3], Theorem 3] if every vertex of T has odd degree, then $\nu(T) = 1$. By the handshaking lemma, n must be even, hence $\rho(T)$ is odd. By [[3], Theorem 4], if exactly one vertex of T has even degree, then $\nu(T) = 0$. Since n must be odd, $\rho(T)$ is also odd. Hence in either case, every odd dominating set has odd cardinality by Theorem 1. □

Corollary 4. *Every odd dominating set of a graph G has an odd (even) number of vertices of odd degree if and only if $\nu(G)$ is odd (even). In particular, the odd dominating set of an always solvable graph has an even number of odd degree vertices.*

Proof. Observe that for any subsets A, B of $V(G)$, $pr(A \cap B) = \mathbf{x}_A^t \mathbf{x}_B$. In particular, $pr(A) = \mathbf{x}_A^t \mathbf{1}$. Let A^c be the complement of A in $V(G)$. Then we have $\mathbf{x}_{A^c} = \mathbf{x}_A + \mathbf{1}$. Now let S be an odd dominating set of G and D be the set of vertices with odd degree. Observe that $N\mathbf{1} = \mathbf{x}_{D^c}$. Therefore $N\mathbf{x}_{S^c} = N(\mathbf{x}_S + \mathbf{1}) = \mathbf{1} + \mathbf{x}_{D^c} = \mathbf{x}_D$. Then, $pr(D \cap S) = \mathbf{x}_D^t \mathbf{x}_S = (N\mathbf{x}_{S^c})^t \mathbf{x}_S = \mathbf{x}_{S^c}^t N \mathbf{x}_S = \mathbf{x}_{S^c}^t \mathbf{1} = pr(S^c)$. On the other hand, $pr(S^c) = pr(\nu(G))$ by Theorem 1. Hence, the result follows. □

We define the *join* $G_1 \oplus ... \oplus G_m$ of m pairwise disjoint graphs $G_1, ..., G_m$ as follows. We take the vertex set as $V(G_1 \oplus ... \oplus G_m) = \cup_{i=1}^m V(G_i)$ and the edge set as $E(G_1 \oplus ... \oplus G_m) = \cup_{i=1}^m E(G_i) \cup \{(u,v) \mid u \in V(G_k),\ v \in V(G_l)\ k,l \in \{1,...,m\}$ such that $k \neq l\}$. Then Amin et al. prove the following proposition in [[4], Corollary 6].

Proposition 1 ([4]). *$\nu(G_1 \oplus G_2) = \nu(G_1) + \nu(G_2)$ if either G_1 or G_2 has an odd dominating set of even cardinality, and $\nu(G_1 \oplus G_2) = \nu(G_1) + \nu(G_2) + 1$, otherwise.*

Together with Theorem 1, the above proposition implies the following.

$$\nu(G_1 \oplus G_2) = \nu(G_1) + \nu(G_2) + pr(\rho(G_1)\rho(G_2)). \quad (2)$$

Equivalently,

$$\rho(G_1 \oplus G_2) = \rho(G_1) + \rho(G_2) - pr(\rho(G_1)\rho(G_2)). \quad (3)$$

Equivalence of (2) and (3) follows from the rank nullity theorem.

Expressing the nullity/rank of $G_1 \oplus G_2$ as a single formula involving nullities/ranks of G_1 and G_2 as above enables us to extend this result and to write a formula for the nullity/rank of the join of arbitrary number of graphs as follows.

Proposition 2. *Let $\{G_1, ..., G_m\}$ be a collection of pairwise disjoint graphs. Let j be the number of graphs in $\{G_1, ..., G_m\}$ with odd rank. Then*

$$\nu(G_1 \oplus ... \oplus G_m) = \left\{ \begin{array}{cc} \sum_{i=1}^{m} \nu(G_i) & \text{if } j = 0 \\ \sum_{i=1}^{m} \nu(G_i) + j - 1 & \text{otherwise} \end{array} \right\}. \quad (4)$$

Equivalently,

$$\rho(G_1 \oplus ... \oplus G_m) = \left\{ \begin{array}{cc} \sum_{i=1}^{m} \rho(G_i) & \text{if } j = 0 \\ \sum_{i=1}^{m} \rho(G_i) - j + 1 & \text{otherwise} \end{array} \right\}. \quad (5)$$

Proof. We prove (5), then (4) follows from the rank nullity theorem. If $j = 0$, then all graphs have even rank and the result follows applying (3) successively. Now let $j \neq 0$. Without loss of generality, we can assume that the first j graphs have odd rank. Then, by (3), $\rho(G_1 \oplus G_2) = \rho(G_1) + \rho(G_2)$–1, which is odd. Hence, $\rho(G_1 \oplus G_2 \oplus G_3) = \rho(G_1) + \rho(G_2) - 1 + \rho(G_3)$–1 $= \rho(G_1) + \rho(G_2) + \rho(G_3)$–2, which is odd, and so on, yielding $\rho(G_1 \oplus G_2 \oplus \cdots \oplus G_j) = \rho(G_1) + \rho(G_2) + \cdots + \rho(G_j)$–$(j-1)$, which is odd. Since the rank of the joins of the $m - j$ even ones is the sum of the ranks (which is even), the join of all m of them is the sum of the ranks minus $(j-1)$. □

Declaration of Competing Interests The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements The author would like to thank the referees for their valuable suggestions which improved the clarity and quality of the paper.

References

[1] Amin, A. T., Slater, P. J., Neighborhood domination with parity restrictions in graphs, In *Proceedings of the Twenty-third Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1992)*, 91 (1992), 19–30.

[2] Amin, A. T., Slater, P. J., All parity realizable trees, *J. Combin. Math. Combin. Comput.*, 20 (1996), 53–63.

[3] Amin, A. T., Clark, L. H., Slater, P. J., Parity dimension for graphs, *Discrete Math.*, 187(1-3) (1998), 1–17. https://doi.org/10.1016/S0012-365X(97)00242-2

[4] Amin, A. T., Slater, P. J., Zhang, G. H., Parity dimension for graphs a linear algebraic approach, *Linear Multilinear Algebra*, 50(4) (2002), 327–342. https://doi.org/10.1080/0308108021000049293

[5] Ballard, L. E., Budge, E. L., Stephenson, D. R., Lights out for graphs related to one another by constructions, *Involve*, 12(2) (2019), 181–201. https://doi.org/10.2140/involve.2019.12.181

[6] Caro, Y., Simple proofs to three parity theorems, *Ars Combin.*, 42 (1996), 175–180.

[7] Cowen, R., Hechler, S. H., Kennedy, J. W., Ryba, A., Inversion and neighborhood inversion in graphs, *Graph Theory Notes N. Y.*, 37 (1999), 37–41.

[8] Eriksson, H., Eriksson, K., Sjöstrand, J., Note on the lamp lighting problem, *Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000)*, 27 (2001), 357–366. https://doi.org/10.1006/aama.2001.0739

[9] Sutner, K., Linear cellular automata and the Garden-of-Eden, *Math. Intelligencer*, 11(2) (1989), 49–53. https://doi.org/10.1007/BF03023823

[10] Sutner, K., The σ-game and cellular automata, *Amer. Math. Monthly*, 97(1) (1990), 24–34. https://doi.org/10.1080/00029890.1990.11995540

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1029–1043 (2022)
DOI:10.31801/cfsuasmas.1089480
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:March 18, 2022; Accepted: May 23, 2022

FAREY GRAPH AND RATIONAL FIXED POINTS OF THE EXTENDED MODULAR GROUP

Bilal DEMİR[1] and Mustafa KARATAŞ[2]

[1]Department of Mathematics Education, Necatibey Faculty of Education, Balıkesir University, Balıkesir, TURKEY
[2]Department of Mathematics, Institute of Science, Balıkesir University, Balıkesir, TURKEY

Abstract. Fixed points of matrices have many applications in various areas of science and mathematics. The extended modular group $\overline{\Gamma}$ is the group of 2×2 matrices with integer entries and determinant ± 1. There are strong connections between the extended modular group, continued fractions and Farey graph. The Farey graph is a graph with vertex set $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$. In this study we consider the elements in $\overline{\Gamma}$ that fix rationals. For a given rational number, we use its Farey neighbours to obtain the matrix representation of the element in $\overline{\Gamma}$ that fixes the given rational. Then we express such elements as words in terms of generators using the relations between the Farey graph and continued fractions. Finally we give the new block reduced form of these words which all blocks have Fibonacci numbers entries.

1. Introduction

The modular group $\Gamma = PSL(2, \mathbb{Z})$ is the projective special linear group of 2×2 matrices over the ring of integers with determinant one. This group is the quotient group $SL(2, \mathbb{Z})/\pm I$, hence each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the same element with its negative $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. The modular group acts on the upper half plane $\mathbb{H}$ via linear fractional transformations $z \to \frac{az+b}{cz+d}$. These transformations are orientation preserving isometries of $\mathbb{H}$. Modular group is generated by two elements;

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

2020 *Mathematics Subject Classification.* 20H10, 11B57.
Keywords. Extended modular group, fixed points, Farey sequence, Farey graph.
bdemir@balikesir.edu.tr-Corresponding author; 0000-0002-6638-6909
mustafa.karatas.9610@gmail.com; 0000-0002-1147-0169.

The presentation of Γ is;

$$\Gamma =< T, S : T^2 = S^3 = I >\approx \mathbb{Z}_2 * \mathbb{Z}_3,$$

the free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ where $S = TU = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Let us denote the set $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = -1 \right\}$. The corresponding transformations of elements in G are anti-automorphisms. Thus the extended modular group can be defined as $\overline{\Gamma} = PSL(2, \mathbb{Z}) \cup G$. Hence, the extended modular group is the projective linear group $PGL(2, \mathbb{Z})$ and isomorphic to the free product of two dihedral groups of order four and six amalgamated with the cyclic group of order 2 i.e.

$$\overline{\Gamma} =< T, S, R : T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I >\approx D_2 *_{\mathbb{Z}_2} D_3$$

where $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as a reflection map. So the modular group is normal in the extended modular group with index 2.

For each $V \in \overline{\Gamma}$; the number $z \in \mathbb{C} \cup \{\infty\}$ is called a fixed point of V if $V(z) = z$ where $V(z)$ is the corresponding transformation. There is a relation between the number fixed points and trace of V. Elements of $\overline{\Gamma}$ are classified according to the number of fixed points. There are five types of elements in $\overline{\Gamma}$. Now we list the certain types of elements.

If $V \in \Gamma$ then V has at most two fixed points. Also if;

- $|trV| > 2$, then there are two fixed points in $\mathbb{R} \cup \{\infty\}$ and V is called a hyperbolic element.
- $|trV| = 2$, then there is one fixed point in $\mathbb{R} \cup \{\infty\}$ and V is called a parabolic element.
- $|trV| < 2$, then there are two conjugate fixed points in $\mathbb{C} \cup \{\infty\}$ and V is called an elliptic element.

If $V \in G$ then it has either two fixed points in the real line or the fixed point set is a circle perpendicular to real line. Also if;

- $|trV| \neq 0$, then there is one fixed point in $\mathbb{R} \cup \{\infty\}$ and V is called a glide reflection.
- $|trV| = 0$, then the set of fixed points is a circle perpendecular to the real line and V is called a reflection.

For more information see [1, 2, 11].

There are impressive relations between the modular group and continued fractions. In [25], Rosen defined λ continued fractions for $\lambda \in \mathbb{R}$;

$$[r_0\lambda; r_1\lambda, ..., r_n\lambda] = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\ddots_{r_{n-1}\lambda - \frac{1}{r_n\lambda}}}}}$$

In this expansion, for $i \leq n$, $C_i = \frac{p_i}{q_i} = [r_0\lambda; r_1\lambda, ..., r_i\lambda]$ is called ith convergent of the expansion. And it can be seen by calculation $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. Owing to this viewpoint, Rosen revealed a criteria for membership problem for Hecke groups $H(\lambda)$, a general class of modular group. He proved that an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(\lambda)$ if and only if $\frac{a}{c}$ has a finite λ continued fraction expansion. For $\lambda = 1$ this expression is called integer continued fraction and related to the modular group, on the contrary the membership problem for the modular group is obvious because $\Gamma = PSL(2, \mathbb{Z})$. On the other hand, for $\lambda = 1$ it is possible to make connections between Rosen's fractions and the Farey sequence.

The Farey sequence of order n is a complete and ordered set of reduced rational numbers in the interval $[0, 1]$ which have the denominators not exceeding n.

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$

$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

It can be seen that if $\frac{a}{c}$ and $\frac{b}{d}$ appears one after another in some F_n then $ad - bc = \pm 1$. We called such rationals Farey neighbours. All Farey neighbours of a rational x is denoted by $\mathcal{N}(x)$. The Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ defined as;

$$\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$$

All Farey neigbours of a rational number can be obtained by Farey sum. More precisely if a rational $\frac{p}{q}$ first appears in F_n by the Farey sum of $\frac{a}{c}$ and $\frac{b}{d}$ in F_{n-1} i.e. $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d} = \frac{p}{q}$ then $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours of $\frac{p}{q}$. Here $\frac{a}{c}$ and $\frac{b}{d}$ are called the Farey parents of $\frac{p}{q}$, and conversely $\frac{p}{q}$ is called the Farey child of $\frac{a}{c}$ and $\frac{b}{d}$. If $\frac{a_i}{c_i}$ is a Farey neighbour of $\frac{p}{q}$ then $\frac{a_i}{c_i} \oplus \frac{p}{q}$ is also a Farey neighbour of $\frac{p}{q}$.

Observe that every F_n includes F_{n-1} and new members are obtained by Farey sum of its neighbours. For instance $\frac{1}{2} \in F_2$ is the Farey sum of $\frac{0}{1}$ and $\frac{1}{1}$ in F_1. This rule is known as the mediant rule. It should be noted that if the denominator of a Farey sum of two neighbours in F_{n-1} exceeds n then this will not be appear in F_n since the definition of Farey sequence. Definition of Farey sequence can be extended to $\mathbb{Q}_\infty$ by assuming $\infty = \frac{1}{0}$. Hence for a given rational $\frac{a}{c}$; it is known that $\frac{a}{c}$ has finite integer continued fraction expansion. In addition $\frac{b}{d}$ is the penultimate convergent of the integer continued fraction expansion of $\frac{a}{c}$. This

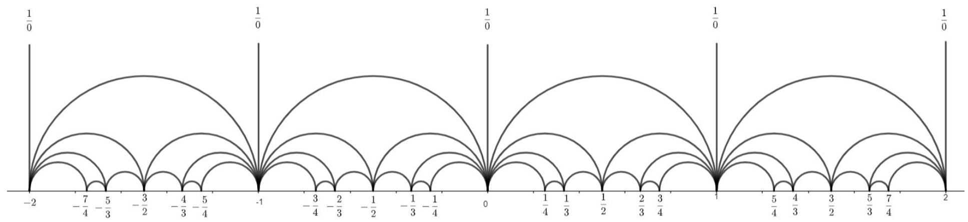

FIGURE 1. Farey graph

yields $ad-bc=\pm1$; in other words $\frac{a}{c}$ and $\frac{b}{d}$ are Farey neighbours. As a result $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$.

The Farey graph is a graph with vertex set $\mathbb{Q}_\infty$. And two reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent if and only if $ps-rq=\pm1$, i.e they are Farey neighbours. An edge between two vertices is drawn by a hyperbolic line in $\mathbb{H}$. The edges between $\frac{1}{0}=\infty$ and every integer a are vertical lines. To construct the graph, first join the vertices $\frac{1}{0}$, $\frac{0}{1}$ and $\frac{1}{1}$ and obtain a big triangle. Then by induction if the endpoints of a long edge are $\frac{a}{c}$ and $\frac{b}{d}$, the label of the third vertex of the triangle is $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$, see Figure 1.

2. MOTIVATION

There are numerous studies about modular and extended modular group in the literature, related to many branches of mathematics such as group theory, number theory automorphic functions, etc. Algebric structures of subgroups of modular and extended modular group and related topics are studied in [3, 4, 8, 17–21, 26, 31, 33, 34]. In recent years, many studies have contributed the theory of continued fractions related to the action of some subgroups of Möbius transformations. Series studied the relations between geodesics on the quotient of the hyperbolic plane by the modular group and continued fractions [28]. In [2], integer continued fraction expansions and geodesic expansions were studied from the perspective of graph theory. Short and Walker represented Rosen continued fractions by path in a class of graphs in hyperbolic geometry [30]. Same authors defined even integer continued fractions which all digits are even integers. And they studied connections between even integer continued fractions and the Farey graph [29].

The fixed points of automorphisms and anti-automorphisms of the extended complex plane have especially been of great interest in many fields of mathematics such as number theory, functional analysis, theory of complex functions, geometry and group theory [22, 24, 27]. Also fixed points of elements in $GL(2, \mathbb{R})$ in tropical algebra are discussed in [7]. In this study we focus on the fixed points of the elements in extended modular group $\overline{\Gamma}$.

Fixed points of an element $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$ can be calculated by solving the equation $\frac{az+b}{cz+d} = z$ i.e.

$$z = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c} \tag{1}$$

where $ad - bc = 1$ in other words the corresponding transformation $V(z)$ is automorphism. And similarly fixed points of an anti-automorphism are

$$z = \frac{a - d \pm \sqrt{(a+d)^2 + 4}}{2c} \tag{2}$$

where $ad - bc = -1$. The action of extended modular group on extended rational numbers $\mathbb{Q}_\infty$ is intriguing. This action is defined as;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}$ and the column vector $\begin{pmatrix} p \\ q \end{pmatrix}$ represents the rational number $\frac{p}{q}$. Fixed points of an element in $\overline{\Gamma}$ are rationals if and only if $a + d = 2$ or -2 for the equation (1) and $a + d = 0$ for the equation (2). This means that rational numbers are fixed only by parabolic or reflection elements.

In this study we establish relations between the Farey graph and elements of $\overline{\Gamma}$ that fixes a given rational $\frac{p}{q}$. Firstly we obtain matrix representation of the element fixing the rational $\frac{p}{q}$ via the Farey neighbours of $\frac{p}{q}$. Then, we consider the relations between paths in the Farey graph and integer continued fractions and obtain the element as a word of the generators U and T. Afterwards, we express this word in block reduced forms and new block reduced forms, related to Fibonacci numbers. Finally, we give some relevant examples of our results.

3. Matrix Representations of the Parabolic and Reflection Elements

In this section we obtain the parabolic and reflection elements in $\overline{\Gamma}$ as matrices that fix a given rational. To do this, we use Farey neighbours.

Theorem 1. *Let $z = \frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}, \frac{m}{k} \in \mathcal{N}(z)$ then the element*

$$V = \begin{pmatrix} ps - mq & pm - pr \\ qs - qk & pk - qr \end{pmatrix} \tag{3}$$

fixes the rational z.

Proof. Since $\frac{r}{s}, \frac{m}{k}$ are Farey neighbours of z, the elements $V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ and $V_2 = \begin{pmatrix} p & m \\ q & k \end{pmatrix}$ belong to $\overline{\Gamma}$. Furhermore V_1 and V_2 both send ∞ to $\frac{p}{q}$. As a result $V = V_2.V_1^{-1}$ is the desired element. □

Let $\frac{p}{q}$ and $\frac{r}{s}$ are adjacent such that $\frac{r}{s} < \frac{p}{q}$ then $ps - qr = 1$ otherwise -1. The trace of the element mentioned in (3) is $ps - mq + pk - qr$. By the fact that $\frac{r}{s}, \frac{m}{k}$ are Farey neighbours of $\frac{p}{q}$ we have $ps - qr = \pm 1$ and $pk - mq = \pm 1$. Hence $tr\,(V) = 0, \pm 2$ and we have proved the following corollary.

Corollary 1. *Let $z = \frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}, \frac{m}{k} \in \mathcal{N}\,(z)$. If $\frac{r}{s}$ and $\frac{m}{k}$ are at the same side of $\frac{p}{q}$ then the element in (3) is parabolic otherwise a reflection.*

We know that the fixed point set of a reflection map is a circle perpendicular to real axis. If the element V mentioned in (3) is a reflection then we know from [5] that V fixes the circle centered at $\left(\frac{ps-mq}{qs-qk}, 0\right)$ with radius $\frac{1}{|qs-qk|}$.

Example 1. *For the rational $\frac{8}{3}$ one can choose Farey neighbours as $\frac{5}{2}$ and $\frac{13}{5}$. Then, we have the parabolic element*

$$V = \begin{pmatrix} -23 & 64 \\ -9 & 25 \end{pmatrix}$$

fixes $\frac{8}{3}$. And if one chooses the neighbours as $\frac{5}{2}$ and $\frac{11}{4}$ then the reflection element

$$V' = \begin{pmatrix} -17 & 48 \\ -6 & 17 \end{pmatrix}$$

fixes not only $\frac{8}{3}$ but also the circle centered at $\left(\frac{17}{6}, 0\right)$ with radius $r = \frac{1}{6}$.

Suppose a Farey neighbour of $\frac{p}{q}$ is $\frac{r}{s}$. Then some other neighbours can be obtained by the mediant rule. The following two theorems based on this idea.

Theorem 2. *Let $\frac{p}{q} \in \mathbb{Q}_\infty$ then the parabolic element that fixes $\frac{p}{q}$ is*

$$V = \begin{pmatrix} \pm 1 - pq & p^2 \\ -q^2 & \pm 1 + pq \end{pmatrix}$$

Proof. Let $\frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}$ is a Farey neighbour of $\frac{p}{q}$. By the mediant rule we have another Farey neighbour $\frac{p+r}{q+s}$ on the same side with $\frac{r}{s}$. Using the same technique in the proof of Theorem 1 we have the element V as stated. Additionally the trace of the element V is ± 2 with determinant 1 which proves V is parabolic in Γ. □

Unlike the Theorem 1, Theorem 2 gives an algorithm to obtain a parabolic element that fixes a given rational, without using anything but the rational. Here we do similar things to obtain a reflection whose fixed circle includes a given rational.

Theorem 3. *Let $\frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}$ is a Farey parent of $\frac{p}{q}$. Then the reflection element in $\overline{\Gamma}$ that fixes $\frac{p}{q}$ is*

$$V = \begin{pmatrix} ps - pq + qr & p^2 - 2pr \\ 2qs - q^2 & -qr + qp - ps \end{pmatrix}$$

Proof. Suppose $\frac{p}{q} \in \mathbb{Q}_\infty$ and $\frac{r}{s}$ is a Farey parent of $\frac{p}{q}$. Another Farey parent of $\frac{p}{q}$ which is at the opposite side of $\frac{r}{s}$ can be obtained by the madiant rule. So we have this parent as $\frac{p-r}{q-s}$. The elements $V_1 = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ and $V_2 = \begin{pmatrix} p & p-r \\ q & q-s \end{pmatrix}$ belong to $\overline{\Gamma}$. Although one of them is automorphism, the other is anti-automorphism since the Farey parents are at the opposite side of $\frac{p}{q}$. Hence the element $V = V_2.V_1^{-1} \in \overline{\Gamma}$ fixes $\frac{p}{q}$. Since $trV = 0$, V is a reflection that the fixed point set is a circle that centered at $\left(\frac{ps-pq+qr}{2qs-q^2}, 0\right)$ with radius $r = \frac{1}{|2qs-q^2|}$ which proves the result. □

So far to this point, we have focused on Farey neighbours. Now observe all the Farey neighbours of a given reduced rational $\frac{p}{q}$. Suppose $\frac{r}{s}$ and $\frac{m}{k}$ are Farey parents of $\frac{p}{q}$ such that $\frac{r}{s} < \frac{p}{q} < \frac{m}{k}$. Then $\frac{p}{q}$ appears in F_q via $\frac{r}{s} \oplus \frac{m}{k} = \frac{p}{q}$. In other words, the hyperbolic line segment joining $\frac{r}{s}$ and $\frac{m}{k}$ covers all the neighbours. Consequently all neighbours of $\frac{p}{q}$ can be obtained by the mediant rule;

$$\frac{r}{s} < \frac{r}{s} \oplus \frac{p}{q} = \frac{p+r}{q+s} < \frac{p+r}{q+s} \oplus \frac{p}{q} = \frac{2p+r}{2q+s} < ... < \frac{p}{q} < ...\frac{p}{q} \oplus \frac{m}{k} = \frac{p+m}{q+k} < \frac{m}{k}$$

4. Farey Paths, Integer Continued Fractions and Blocks in Extended Modular Group $\overline{\Gamma}$

In this section, the relation between integer continued fractions and Farey paths is used to obtain the word form of the element in $\overline{\Gamma}$, which fixes a given rational number, in terms of generators. A path in a graph consists of consequtive adjacent vertices. So a Farey path $< v_1, v_2, ..., v_n >$ is a path such that $v_i = \frac{p_i}{q_i}$ for $i = 1, 2, ..., n$ are reduced rationals and since the consequtive v_i's are adjacent $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. The Farey graph is connected hence there is a natural distance between two rationals v and w that is $d(v, w)$, the minimum number of edges in any path from v to w in F_n. The distance of an integer to ∞ is $d(\infty, x) = 1$.

Lemma 1. *[25] Let $\frac{p}{q} = [r_0; r_1, r_2, ..., r_n]$ be a reduced rational number then;*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Theorem 4. *Let $\frac{p}{q}$ be a reduced rational and have an integer continued fraction expansion as $[r_0; r_1, r_2, ..., r_n]$, then the parabolic element fixing $\frac{p}{q}$ is*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0} \qquad (4)$$

Proof. Let $\frac{p}{q} = [r_0; r_1, r_2, ..., r_n]$. By Lemma 1, we have

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}\begin{pmatrix}p\\q\end{pmatrix}$$
$$= U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U\begin{pmatrix}1\\0\end{pmatrix}$$
$$= U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \begin{pmatrix}p\\q\end{pmatrix}$$

Since conjugacy preserves the trace we have

$$tr\left(U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}\right) = tr\,(U) = 2$$

which proves the element given in (4) is parabolic. □

We know from [9] that stabilizer of a point in Γ is an infinite cyclic group. So we can give the following corollary.

Corollary 2. *Let $\frac{p}{q} = [r_0; r_1, r_2, ..., r_n] \in \mathbb{Q}$; then for all $0 \neq k \in \mathbb{Z}$;*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U^k.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}$$

is a parabolic element in Γ whose fixed point is $\frac{p}{q}$.

Now we obtain a reflection element as a word in generators of $\overline{\Gamma}$ that fixes a given rational $\frac{p}{q}$.

Theorem 5. *Let $\frac{p}{q}$ be a reduced rational and have an integer continued fraction expansion as $[r_0; r_1, r_2, ..., r_n]$, then the reflection element in $\overline{\Gamma}$ fixing $\frac{p}{q}$ is*

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.RTU.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}$$

Proof. We have $RTU = \begin{pmatrix}1 & 1\\0 & -1\end{pmatrix}$ as a reflection map. Furthermore $RTU\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. The rest of the proof follows similar to the proof of Theorem 4. □

Example 2. *Choose the rational $\frac{8}{5}$. The integer continued fraction expansion of $\frac{8}{5}$ is*

$$\frac{8}{5} = 2 - \frac{1}{3 - \frac{1}{2}} = [2; 3, 2]\,.$$

Then the parabolic element fixing $\frac{8}{5}$ is

$$U^2TU^3TU^2TUTU^{-2}TU^{-3}TU^{-2} = \begin{pmatrix}-39 & 64\\-25 & 41\end{pmatrix}.$$

And the reflection element is

$$U^2TU^3TU^2TRTUTU^{-2}TU^{-3}TU^{-2} = \begin{pmatrix} -89 & 104 \\ -55 & 89 \end{pmatrix}$$

Here we mention about relations between paths in the Farey graph and integer continued fractions. The convergents of a certain continued fraction expansion of a reduced rational $\frac{p}{q} = [r_0; r_1, ..., r_n]$, are defined as $C_i = \frac{p_i}{q_i} = [r_0; r_1, ..., r_i]$ for $0 \leq i \leq n$, where $C_0 = \frac{p_0}{q_0} = \frac{r_0}{1}$ and $C_n = \frac{p_n}{q_n} = \frac{p}{q}$. Furthermore we know that $p_i.q_{i-1} - q_i.p_{i-1} = \pm 1$. Hence every consequtive pair C_i and C_{i-1} are Farey neighbours. Also, since $C_0 = r_0 \in \mathbb{Z}$ and every integer is adjacent to infinity with a vertical line, $< \infty, C_0, C_1, ..., C_{n-1}, C_n >$ is a path from ∞ to $\frac{p}{q}$. To sum up every integer fraction expansion of a rational $\frac{p}{q}$ is related to a path from ∞ to $\frac{p}{q}$. Moreover the shortest integer continued fraction of $\frac{p}{q}$ is related to a geodesic path from ∞ to $\frac{p}{q}$. In Theorem 4 and Theorem 5, the integer continued fraction expansion of a given rational is related to an element in $\overline{\Gamma}$ that fixes the rational. It is possible to make connections with Farey paths.

5. Block Reduced Forms in the Extended Modular Group $\overline{\Gamma}$

Every element in $\overline{\Gamma}$ can be expressed as a word of T, S and R denoted by $W(T, S, R)$. Consider the blocks

$$TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad TS^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Using these blocks every reduced word $W(T, S, R)$ in $\overline{\Gamma}$ where the sum of exponents of R is an even number can be expressed as;

$$S^i (TS)^{m_0} \left(TS^2\right)^{n_0} ... (TS)^{m_k} \left(TS^2\right)^{n_k} T^j,$$

and every reduced word $W(T, S, R)$ in $\overline{\Gamma}$ where the sum of exponents of R is an odd number can be expressed as;

$$S^i (TS)^{m_0} \left(TS^2\right)^{n_0} ... (TS)^{m_k} \left(TS^2\right)^{n_k} T^j R$$

Here $i = 0, 1, 2,\ j = 0, 1,\ m_0$ and n_k may be zero and other exponents are positive integers. This representetion is known as the *block reduced form* [13]. For example, the block reduced form of the word $W(T, S, R) = TSTSTSSTSST$ is $(TS)^2 . \left(TS^2\right)^2 T$. And the block reduced form of the word $W(T, S, R) = RTS^2RTS^2R$ is $(TS) . \left(TS^2\right)^2 R$. Trace classes of the modular group and extended modular group are studied in [6, 13] by using the block reduced form. In this section we give the block reduced form of the element in $\overline{\Gamma}$ fixing a given rational $\frac{p}{q}$.

Theorem 6. *Let $\frac{p}{q}$ be a reduced rational number and have an integer continued fraction expansion $[r_0; r_1, ..., r_n]$ then the block form of the parabolic element fixing $\frac{p}{q}$ is*

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}\left(TS^2\right)(TS)^{r_1-2}\left(TS^2\right)...(TS)^{r_{n-1}-2}\left(TS^2\right). \\ & (TS)^{r_n-1}\left(TS^2\right)^{-1}(TS)^{-r_n-1}\left(TS^2\right)(TS)^{-r_{n-1}-2}\left(TS^2\right). \\ & ...(TS)^{-r_1-2}\left(TS^2\right)(TS)^{-r_0-1} \end{aligned}$$

Proof. By Theorem 4, we know that

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.U.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}$$

fixes $\frac{p}{q}$. Considering $U = TS$ we have

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0}.T.(TS)^{r_1}.T...(TS)^{r_{n-1}}.T.(TS)^{r_n}.T \\ & (TS).T.(TS)^{-r_n}.T.(TS)^{-r_{n-1}}.T.....(TS)^{-r_1}.T.(TS)^{-r_0} \\ &= (TS)^{r_0-1}TS.T.TS.(TS)^{r_1-2}TS.T...TS(TS)^{r_{n-1}-2}TS. \\ & T.TS(TS)^{r_n-1}.T(TS).T.(TS)^{-r_n-1}TS.T.TS(TS)^{-r_n-2}. \\ & TS.T.....TS(TS)^{-r_1-2}TS.T.TS(TS)^{-r_0-1} \end{aligned}$$

Using the relations $T^2 = I$ and $\left(TS^2\right)^{-1} = ST$,

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}.\left(TS^2\right).(TS)^{r_1-2}.\left(TS^2\right)...\left(TS^2\right).(TS)^{r_{n-1}-2}. \\ & \left(TS^2\right).(TS)^{r_n-1}.\left(TS^2\right)^{-1}(TS)^{-r_n-1}.\left(TS^2\right).(TS)^{-r_{n-2}}. \\ & \left(TS^2\right).....\left(TS^2\right).(TS)^{-r_1-2}.\left(TS^2\right).(TS)^{-r_0-1} \end{aligned}$$

□

Theorem 7. *Let* $\frac{p}{q}$ *be a reduced rational number and have an integer continued fraction expansion* $[r_0; r_1, ..., r_n]$ *then the block form of the reflection element fixing* $\frac{p}{q}$ *is*

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}.\left(TS^2\right).(TS)^{r_1-2}.\left(TS^2\right).....(TS)^{r_{n-1}-2}.\left(TS^2\right). \\ & (TS)^{r_n}.\left(TS^2\right)^{-r_n-2}.(TS).\left(TS^2\right)^{-r_{n-1}-2}.(TS).... \\ & (TS).\left(TS^2\right)^{-r_1-2}.(TS)\left(TS^2\right)^{-r_0-1}.R \end{aligned}$$

Proof. From Theorem 5, the reflection element fixing $\frac{p}{q}$ is

$$U^{r_0}TU^{r_1}TU^{r_2}T...U^{r_n}T.RTU.TU^{-r_n}TU^{-r_{n-1}}T...U^{-r_1}TU^{-r_0}.$$

After substituting $U = TS$ in the word above, we have

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0}T(TS)^{r_1}T...(TS)^{r_{n-1}}T(TS)^{r_n}T \\ & RT(TS)T(TS)^{-r_n}T(TS)^{-r_{n-1}}T...(TS)^{-r_1}T(TS)^{-r_0} \\ &= (TS)^{r_0-1}TSTTS(TS)^{r_1-2}TSTTS...TS(TS)^{r_{n-1}-2}TS \\ & TTS(TS)^{r_n-1}TRST(TS)(TS)^{-r_n-2}TSTTS(TS)^{-r_{n-1}-2} \end{aligned}$$

$$TST...TS\,(TS)^{-r_1-2}\,TSTTS\,(TS)^{-r_0-1}$$

Since $(TR)^2 = (SR)^2 = I$ we obtain $TR = RT$ and $SR = RS^2$. Hence,

$$\begin{aligned} W(T,S,R) &= (TS)^{r_0-1}\left(TS^2\right)(TS)^{r_1-2}\left(TS^2\right)...(TS)^{r_{n-1}-2}\left(TS^2\right) \\ &\quad (TS)^{r_n}\left(TS^2\right)^{-r_n-2}(TS)\left(TS^2\right)^{-r_{n-1}-2}(TS)... \\ &\quad (TS)\left(TS^2\right)^{-r_1-2}(TS)\left(TS^2\right)^{-r_0-1}R \end{aligned}$$

□

We can obtain elements which fix a given rational $\frac{p}{q}$ in terms of TS and TS^2 by finding a path from ∞ to $\frac{p}{q}$ in the Farey graph. We explain this with an example:

Example 3. *Suppose the given rational is $\frac{-10}{3}$. Then one may choose the path $< \infty, -3, \frac{-13}{4}, \frac{-10}{3} >$, see Figure 2. We know the consequtive vertices in this path are consequtive convergents of the integer continued fraction expansion of the rational $\frac{-10}{3}$ i.e., $C_0 = -3$, $C_1 = \frac{-13}{4}$ and $C_2 = \frac{-10}{3}$. Hence, we obtain the integer continued fraction expansion as*

$$-3 - \cfrac{1}{4 - \frac{1}{1}} = [-3, 4, 1]$$

Using the values $r_0 = -3$, $r_1 = 4$ and $r_2 = 1$ in Theorem 6 we have the parabolic element fixing $\frac{-10}{3}$ in blocks TS and TS^2 as follows:

$$\begin{aligned} W(T,S,R) &= (TS)^{-4}\left(TS^2\right)(TS)^2\left(TS^2\right)(TS)^0\left(TS^2\right)^{-1}(TS)^{-2} \\ &\quad \left(TS^2\right)(TS)^{-6}.\left(TS^2\right)(TS)^2 \end{aligned}$$

We can reduce this word by the presentation of Γ as;

$$W(T,S,R) = S^2.\left(TS^2\right)^2.(TS)^2.\left(TS^2\right)^4.(TS)^3$$

For the reflection element fixing $\frac{-10}{3}$ we use Theorem 7;

$$\begin{aligned} W(T,S,R) &= (TS)^{-4}\left(TS^2\right)(TS)^2\left(TS^2\right)(TS)^1\left(TS^2\right)^{-3}(TS) \\ &\quad \left(TS^2\right)^{-6}(TS)\left(TS^2\right)^2 R \end{aligned}$$

The block reduced form of this word can be obtained by the relators of $\overline{\Gamma}$;

$$W(T,S,R) = S^2.\left(TS^2\right)^2.(TS)^3.\left(TS^2\right)^3.(TS)^3.\left(TS^2\right)^3.R$$

6. Fibonacci Sequence and New Block Reduced Forms

Jones and Thornton obtained relations between elements of extended modular group and Fibonacci numbers in [10]. Özgür defined two new sequences which are generalizations of Fibonacci and Lucas sequences for the Hecke group $H(\sqrt{q})$ where $q \geq 5$ prime [32]. Also there are some results for Modular group and Pell Fibonacci and Lucas numbers in [14–16, 23]. Koruoğlu and Şahin used a generalized version of Fibonacci sequence to get relations with extended Hecke groups $\overline{H}(\lambda)$ in [12]. In

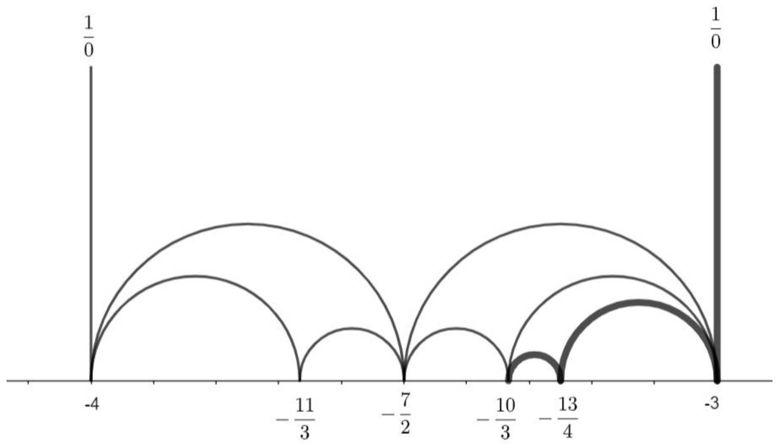

FIGURE 2. The path $< \infty, -3, \frac{-13}{4}, \frac{-10}{3} >$

same study they give an application to extended modular group $\overline{\Gamma}$. They considered the following elements:

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad h = TSR = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The k^{th} power of f and h are;

$$f^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix} \quad \text{and} \quad h^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$$

where f_k denotes the kth Fibonacci number. Hence every element in extended modular group can be expressed as a word in f and h. This reduced word called *New Block Reduced Form*. The relations between block reduced forms and new block reduced forms are;

$$TS = Rf = hR \tag{5}$$

$$TS^2 = Rh = fR \tag{6}$$

It is proved that every block reduced word has a New Block Reduced Form. From this viewpoint we can express the element given in Theorem 6 and Theorem 7 in new block reduced form. We explain this with an example.

In example 3 the parabolic element fixing $\frac{-10}{3}$ is;

$$S^2 \left(TS^2\right)^2 (TS)^2 \left(TS^2\right)^4 (TS)^3$$

Using the relations 5 and 6 and $S^2 = TfR$; we can write this word;

$$TfR.\,(Rh.fR)\,.\,(Rf.hR)\,.\,(Rh.fR.Rh.fR)\,.\,(Rf.hR.Rf)$$

Since $R^2 = I$ we have the block reduced form;

$$T.f.h.f^2.h^2.f.h.f^2.h.f = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & f_1\\ f_1 & f_2\end{pmatrix}\cdot\begin{pmatrix}f_2 & f_1\\ f_1 & 0\end{pmatrix}\begin{pmatrix}f_1 & f_2\\ f_2 & f_3\end{pmatrix}\begin{pmatrix}f_3 & f_2\\ f_2 & f_1\end{pmatrix}$$
$$\begin{pmatrix}0 & f_1\\ f_1 & f_2\end{pmatrix}\begin{pmatrix}f_2 & f_1\\ f_1 & 0\end{pmatrix}\begin{pmatrix}f_1 & f_2\\ f_2 & f_3\end{pmatrix}\begin{pmatrix}f_2 & f_1\\ f_1 & 0\end{pmatrix}\begin{pmatrix}0 & f_1\\ f_1 & f_2\end{pmatrix}$$

It is stated in the same example that the reflection element fixing $\frac{-10}{3}$ is;

$$S^2\left(TS^2\right)^2(TS)^3\left(TS^2\right)^3(TS)^3\left(TS^2\right)^3R$$

Following the same procedure above we have the new block reduced form of this word as;

$$T.f.h.\left(f^2.h\right)^4.f = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & f_1\\ f_1 & f_2\end{pmatrix}\begin{pmatrix}f_2 & f_1\\ f_1 & 0\end{pmatrix}\left[\begin{pmatrix}f_1 & f_2\\ f_2 & f_3\end{pmatrix}\begin{pmatrix}f_2 & f_1\\ f_1 & f_0\end{pmatrix}\right]^4\begin{pmatrix}0 & f_1\\ f_1 & f_2\end{pmatrix}$$

7. Conclusion

In this article, elements in the extended modular group $\overline{\Gamma}$ which fix rationals, are considered. Matrix representations of parabolic and reflection elements which fix a given rational are mentioned in Section 3 via Farey neighbours. In Section 4 relationship between Farey paths and elements of $\overline{\Gamma}$ which have rational fixed points, is established. And these elements obtained as words in generators U and T. Then, block reduced form of these words are given in Section 5. We use new block reduced forms in Section 6 to establish relations with Fibonacci numbers. As a summary of this work we give a final example, see Table 1.

Path	$< \infty, 0, \frac{1}{2}, \frac{3}{7} >$
ICF	$[0; -2, 3]$
W(U,T) for parabolic element	$T.U^{-2}.T.U^3.T.U.T.U^{-3}.T.U^2.T$
BRF for parabolic element	$\left(TS^2\right)^2.\left(TS\right)^2.\left(TS^2\right)^4.\left(TS\right)^2.T$
NBRF for parabolic element	$f.h^2.f^2.h.f.h^2.f.T$
W(U,T) for reflection element	$T.U^{-2}.T.U^3.T.R.T.U.T.U^{-3}.T.U^2.T$
BRF for reflection element	$\left(TS^2\right)^2.\left(TS\right)^3.\left(TS^2\right).\left(TS\right)^3.\left(TS^2\right)^2.T.R$
NBRF for reflection element	$f.h^2.f.h^3.f.h^2.f.T$

Table 1. Elements in $\overline{\Gamma}$ fixing $\frac{3}{7}$

Author Contribution Statements All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Declaration of Competing Interests On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgements The authors are grateful to anonymous reviewers for the evaluation of the paper and for valuable comments which have improved the paper. We would like to thank Dr. Mevhibe Kobak Demir from Balıkesir University for helping us to prepare figures. Thanks are also due to Professor Özden Koruoğlu for his valuable comments.

References

[1] Beardon, A. F., Algebra and Geometry, Cambridge University Press, 2005, https://dx.doi.org/10.1017/CBO9780511800436.

[2] Beardon, A. F., Hockman, M., Short, I., Geodesic continued fractions, *The Michigan Mathematical Journal, 61* (1) (2012), 133–150, https://dx.doi.org/10.1307/mmj/1331222851.

[3] Şahin, R., On the some normal subgroups of the extended modular group, *Applied Mathematics and Computation, 218* (3) (2011), 1025–1029, https://dx.doi.org/10.1016/j.amc.2011.03.074.

[4] Şahin, R., İkikardeş, S., Squares of congruence subgroups of the extended modular group, *Miskolc Mathematical Notes, 14* (3) (2013), 1031–1035, https://dx.doi.org/10.18514/MMN.2013.780.

[5] Demir, B., Özgür, N. Y., Koruoğlu, Ö., Relationships between fixed points and eigenvectors in the group $gl(2, r)$, *Fixed Point Theory and Applications, 2013* (1) (2013), 55, https://dx.doi.org/10.1186/1687-1812-2013-55.

[6] Fine, B., Trace classes and quadratic forms in the modular group, *Canadian Mathematical Bulletin, 37* (2) (1994), 202–212, https://dx.doi.org/10.4153/CMB-1994-030-1.

[7] Hayat, U., Farid, G., Karapinar, E., Relationships between tropical eigenvectors and tropical fixed points of the group gl (2, r), *Italian Journal of Pure and Applied Mathematics* (2020), 291.

[8] İkikardeş, S., Koruoğlu, O., Şahin, R., Cangül, I. N., One relator quotients of the extended modular group, *Advanced Studies in Contemporary Mathematics, 17* (2) (2008), 203–210.

[9] Jones, G., Singerman, D., Wicks, K., The modular group and generalized Farey graphs, *London Mathematical Society Lecture Note Series* (160) (1991), 316–341, https://dx.doi.org/10.1017/CBO9780511661846.006.

[10] Jones, G., Thornton, J., Automorphisms and congruence subgroups of the extended modular group, *Journal of the London Mathematical Society, 2* (1) (1986), 26–40, https://dx.doi.org/10.1112/jlms/s2-34.1.26.

[11] Jones, G. A., Singerman, D., Complex Functions: An Algebraic and Geometric Viewpoint, Cambridge University Press, 1987, https://dx.doi.org/10.1017/CBO9781139171915.

[12] Koruoğlu, Ö., Şahin, R., Generalized Fibonacci sequences related to the extended Hecke groups and an application to the extended modular group, *Turkish Journal of Mathematics, 34* (3) (2010), 325–332, https://dx.doi.org/10.3906/mat-0902-33.

[13] Koruoğlu, Ö., Şahin, R., İkikardeş, S., Trace classes and fixed points for the extended modular group, *Turkish Journal of Mathematics, 32* (1) (2008), 11–19, https://dx.doi.org/10.3906/mat-0609-3.

[14] Koruoğlu, Ö., Sarıca, Ş. K., Demir, B., Kaymak, A. F., Relationships between cusp points in the extended modular group and Fibonacci numbers, *Honam Math. J.* (2019), https://dx.doi.org/10.5831/HMJ.2019.41.3.569.

[15] Mushtaq, Q., Hayat, U., Horadam generalized Fibonacci numbers and the modular group, *Indian Journal of Pure and Applied Mathematics, 38* (5) (2007), 345.

[16] Mushtaq, Q., Hayat, U., Pell numbers, Pell–Lucas numbers and modular group, *Algebra Colloquium, 14* (2007), 97–102, https://dx.doi.org/10.1142/S1005386707000107.

[17] Newman, M., A complete description of the normal subgroups of genus one of the modular group, *American Journal of Mathematics, 86* (1) (1964), 17–24, https://dx.doi.org/10.2307/2373033.

[18] Newman, M., Free subgroups and normal subgroups of the modular group, *Illinois Journal of Mathematics, 8* (2) (1964), 262–265, https://dx.doi.org/10.1215/ijm/1256059670.

[19] Newman, M., Normal subgroups of the modular group which are not congruence subgroups, *Proceedings of the American Mathematical Society, 16* (4) (1965), 831–832, https://dx.doi.org/10.1090/S0002-9939-1965-0181618-X.

[20] Newman, M., Classification of normal subgroups of the modular group, *Transactions of the American Mathematical Society, 126* (2) (1967), 267–277, https://dx.doi.org/10.2307/1994453.

[21] Newman, M., Maximal normal subgroups of the modular group, *Proceedings of the American Mathematical Society, 19* (5) (1968), 1138–1144.

[22] Özdemir, N., İskender, B. B., Özgür, N. Y., Complex valued neural network with möbius activation function, *Communications in Nonlinear Science and Numerical Simulation, 16* (12) (2011), 4698–4703, https://dx.doi.org/10.1016/j.cnsns.2011.03.005.

[23] Rankin, R., Subgroups of the modular group generated by parabolic elements of constant amplitude, *Acta Arithmetica, 1* (18) (1971), 145–151.

[24] Ressler, W., On binary quadratic forms and the Hecke groups, *International Journal of Number Theory, 5* (08) (2009), 1401–1418, https://dx.doi.org/10.1142/S1793042109002730.

[25] Rosen, D., A class of continued fractions associated with certain properly discontinuous groups, *Duke Mathematical Journal, 21* (3) (1954), 549–563, https://dx.doi.org/10.1215/S0012-7094-54-02154-7.

[26] Şahin, R., İkikardeş, S., Koruoğlu, Ö., On the power subgroups of the extended modular group, *Turkish Journal of Mathematics, 28* (2) (2004), 143–152, https://dx.doi.org/10.3906/mat-0301-2.

[27] Schmidt, T. A., Sheingorn, M., Length spectra of the Hecke triangle groups, *Mathematische Zeitschrift, 220* (1) (1995), 369–397.

[28] Series, C., The modular surface and continued fractions, *Journal of the London Mathematical Society, 2* (1) (1985), 69–80, https://dx.doi.org/10.1112/jlms/s2-31.1.69.

[29] Short, I., Walker, M., Even-integer continued fractions and the Farey tree, In *Symmetries in Graphs, Maps, and Polytopes Workshop* (2014), Springer, pp. 287–300, https://dx.doi.org/10.1007/978-3-319-30451-9_15.

[30] Short, I., Walker, M., Geodesic Rosen continued fractions, *The Quarterly Journal of Mathematics* (2016), 1–31, https://dx.doi.org/doi.org/10.1093/qmath/haw025.

[31] Yılmaz, N., Cangul, I. N., The normaliser of the modular group in the Picard group, *Bulletin-Institue of Mathematics Academia Sinica, 28* (2) (2000), 125–130.

[32] Yılmaz Özgür, N., Generalizations of Fibonacci and Lucas sequences, *Note di Matematica, 21* (1) (2002), 113–125, https://dx.doi.org/10.1285/i15900932v21n1p113.

[33] Yılmaz Özgür, N., On the power subgroups of the modular group, *Ars Combinatoria, 89* (2) (2009), 112–125.

[34] Yılmaz Özgür, N., On the two-square theorem and the modular group, *Ars Combinatoria, 94* (1) (2010), 225–239.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1044–1058 (2022)
DOI:10.31801/cfsuasmas.937043
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: May 13, 2021; Accepted: May 18, 2022

SOME RESULTS ON PSEUDOSYMMETRIC NORMAL PARACONTACT METRIC MANIFOLDS

Mehmet ATÇEKEN[1] and Tuğba MERT[2]

[1]Department of Mathematics, University of Aksaray, 68100 Aksaray, TURKEY
[2]Department of Mathematics, University of Sivas Cumhuriyet, 58140 Sivas, TURKEY

Abstract. In this article, the M-projective and Weyl curvature tensors on a normal paracontact metric manifold are discussed. For normal paracontact metric manifolds, pseudosymmetric cases are investigated and some interesting results are obtained. We show that a semisymmetric normal paracontact manifold is of constant sectional curvature. We also obtain that a pseudosymmetric normal paracontact metric manifold is an $\eta-$Einstein manifold. Finally, we support our topic with an example.

1. Introduction

The notion of odd-dimensional manifolds with contact and almost contact structures was initiated by Boothby and Wang [1]. In [2], Sasaki and Hatakeyama reinvestigated them using tensor calculus. Tanno in [3] classified connected almost contact metric manifolds whose automorphism groups possess maximum dimension. For such manifolds, the sectional curvature of a plane section containing ξ is a constant named c. He showed that it can be divided into the following three classes.

- **Class-1**⇒ Homogeneous normal contact Riemannian manifolds with $c > 0$.

- **Class-2**⇒ Global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$.

2020 *Mathematics Subject Classification.* 53C15, 53C50.
Keywords. M-projective curvature tensor, Weyl curvature tensor, Einstein manifold, $\eta-$Einstein manifold.
mehmetatceken@aksaray.edu.tr; 0000-0002-1242-4359
tmert@cumhuriyet.edu.tr-Corresponding author; 0000-0001-8258-8298.

- **Class-3**⇒ A warped product space $\mathbb{R} \times_f C$ if $c < 0$.

It is well known that the manifolds of class-1 are characterized by admitting a Sasakian structure. In [4], Kenmotsu investigated the differential geometric properties of the manifolds of class-3. In general, these structures are not Sasakian [5].

In [6], S. Zamkovoy and G. Nakova reviewed the decomposition of almost contact metric manifolds in eleven classes. In addition to almost paracontact metric manifolds, K. Mandal and U.C De in [7], N. Özdemir, S. Aktay and M.solgun in [8] examined paracontact metric manifolds and obtained their various geometric properties. Also, in [9], H. Pandey and A. Kumar examined the anti-invariant submanifolds of almost paracontact manifolds. Similarly, J. Welyczko [10] studied Legendre curves on 3-dimensional normal paracontact metric manifolds.

After then, in [11], Pokhariyal and Mishra have introduced an $\mathcal{M}$-projective curvature tensor on a Riemannian manifold. The properties of the $\mathcal{M}$-projective curvature tensor in Sasakian and Kähler manifolds were developed by Ojha in [12]. He showed that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor, and concircular curvature tensor. $\mathcal{M}$-projective curvature tensor on different manifolds studied by many geometers such as Kenmotsu, Sasakian, and generalized Sasakian space form.

In [14], by using some tensors, invariant submanifolds of an almost Kenmotsu (κ, μ, ν)-space are characterized. Similarly, many authors have presented important work with various manifolds and some curvature tensors on them ([13], [15]- [18]).

Motivated by these ideas, we have attempted to study properties of the Weyl-conformal curvature tensor in a normal paracontact metric manifold. We think that some interesting results contribute differential geometry.

The present paper is organized as follows.

In section 2, we give the notations and preliminary results of normal paracontact metric manifolds that will be used later. In section 3, we show that a normal paracontact metric manifold satisfying $R(X,Y) \cdot R = 0$ if and only if it has constant sectional curvature and $R(X,Y) \cdot \mathcal{M} = 0$ implies that it η-Einstein manifold.

2. Preliminaries

An almost paracontact structure on a n-dimensional smooth manifold M is given by a $(1,1)$-type tensor field φ, a vector field ξ, and a 1-form η satisfying the following condition

$$\varphi^2 = I - \eta \otimes \xi, \ \ \eta(\xi) = 1. \tag{1}$$

As an immediate consequent $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the tensor φ has constant rank $n-1$. If an almost paracontact manifold is endowed with a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2}$$

for any $X, Y \in \Gamma(TM)$, then $M^n(\varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold, where $\Gamma(TM)$ is the set of the differentiable vector fields on M.
It follows that

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

The fundamental 2-form of the almost paracontact metric manifold is given by

$$\Phi(X, Y) = g(\varphi X, Y).$$

If $d\eta = \Phi$, then η becomes a contact form, that is, $\eta \wedge (d\eta)^n \neq 0$ and $M^n(\varphi, \xi, \eta, g)$ is said to be a paracontact metric manifold. Any such pseudo-Riemannian metric manifold is of signature $(\frac{n+1}{2}, \frac{n-1}{2})$ for $n = 2m + 1$. In this case, we have

$$(\nabla_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{3}$$

for any $X, Y \in \Gamma(TM)$, where ∇ denote the Levi-Civita connection on M. (1) and (3) imply that

$$\nabla_X \xi = \varphi X.$$

An almost paracontact structure is said to be normal if the tensor $N_\varphi - 2d\eta \oplus \xi = 0$ [13], where N_φ the Nijenhuis tensor of φ given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

For the sake of brevity, a normal paracontact metric manifold is said to be paracontact metric manifold [8].

A normal paracontact metric manifold M is of a constant sectional curvature c, then its Riemannian curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z &= \frac{c+1}{4}\Big\{g(Y, Z)X - g(X, Z)Y\Big\} \\ &+ \frac{c-1}{4}\Big\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &- g(Y, Z)\eta(X)\xi + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Y\Big\}, \end{aligned} \tag{4}$$

for any $X, Y, Z \in \Gamma(TM)$ [8].

For a $(0, k)$-type tensor field T and a $(0, 2)$-type tensor field A on a semi-Riemannian manifold (M, g), the Tachibana tensor $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(X_1, X_2, ..., X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, ...X_k) \\ &\quad -T(X_1, (X \wedge_A Y)X_2, ...X_k) \\ &\quad . \\ &\quad . \\ &\quad . \\ &- T(X_1, ...X_{k-1}, (X \wedge_A Y)X_k), \end{aligned} \tag{5}$$

for all $X_1, X_2, ...X_k, X, Y \in \Gamma(TM)$, where $X \wedge_A Y$ is an endomorphism defined by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{6}$$

A semi-Riemannian manifold (M, g) is pseudosymmetric if its the Riemannian curvature tensor R satisfies

$$R \cdot R = LQ(g, R), \tag{7}$$

where L is a function on M. Particularly, if $L = 0$, it is called a semisymmetric manifold.

On a normal paracontact metric manifold M^n, the following relations hold;

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{8}$$
$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \tag{9}$$
$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z) \tag{10}$$
$$S(X,\xi) = (1-n)\eta(X), \quad Q\xi = (1-n)\xi, \tag{11}$$

for any $X, Y, Z \in \Gamma(TM)$, where S and Q are, respectively, the Ricci tensor and Ricci operatory of M given by $g(QX, Y) = S(X, Y)$.

On the other hand, the Weyl-conformal curvature and M-projective curvature tensors play an important role in differential geometry as well as in relativity. The Weyl-conformal curvature tensor and M-projective curvature tensor of a Riemannian manifold (M^n, g), $n > 2$, are respectively, defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y\} \\ &+ \frac{\tau}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{M}(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\}, \end{aligned} \tag{13}$$

for any $X, Y, Z \in \Gamma(TM)$, where τ denote the scalar curvature of M.

A normal paracontact metric manifold M is called η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{14}$$

for any $X, Y \in \Gamma(TM)$, where a and b are arbitrary constants. If $b = 0$, then manifold is said to be Einstein.

If a normal paracontact manifold $M^n(\varphi, \eta, \xi, g)$ is an η-Einstein, from (11) and (14), we get $1 - n = a + b$, $\tau = na + b$, that is,

$$a = 1 + \frac{\tau}{n-1} \quad \textit{and} \quad b = -n - \frac{\tau}{n-1}.$$

Thus (14) takes form

$$S(X,Y) = g(X,Y)(1 + \frac{\tau}{n-1}) - (n + \frac{\tau}{n-1})\eta(X)\eta(Y). \tag{15}$$

Theorem 1. *An n-dimensional M-projectively flat normal paracontact metric manifold M^n is an Einstein manifold.*

Proof. Let us assume that normal paracontact metric manifold M^n is M-projectively flat, then from (8) and (13), we obtain

$$R(X,Y)Z = \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}.$$

Here replacing $Z = \xi$ and using (8), we obtain

$$\eta(X)Y - \eta(Y)X = \frac{1}{n-1}\{\eta(Y)X - \eta(X)Y\},$$

which implies that

$$QX = (1-n)X,$$

or

$$S(X,Y) = (1-n)g(X,Y), \tag{16}$$

for all $X, Y \in \Gamma(TM)$. □

Proposition 1. *If A normal paracontact metric manifold $M^n(\varphi, \eta, \xi, g)$ is Weyl-conformally flat, then it an η-Einstein manifold.*

Next, let us suppose that normal paracontact metric manifold M^n is Weyl-conformal flat, then from (12), we have

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\Big\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y\Big\} - \frac{\tau}{(n-1)(n-2)}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}, \end{aligned} \tag{17}$$

for any $X, Y, Z \in \Gamma(TM)$. Taking $Z = \xi$ and making use of (8) and (11), we have

$$\begin{aligned} \eta(X)Y &- \eta(Y)X = \frac{1}{n-2}\Big\{\eta(Y)QX - \eta(X)QY + (n-1)\eta(Y)X \\ &- (n-1)\eta(X)Y\Big\} - \frac{\tau}{(n-1)(n-2)}\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{18}$$

This implies that

$$(1+\frac{\tau}{n-1})(\eta(X)Y-\eta(Y)X)+\eta(Y)QX-\eta(X)QY=0.$$

It follows for $Y=\xi$,

$$QX=-(n+\frac{\tau}{n-1})\eta(X)\xi+(1+\frac{\tau}{n-1})X,$$

that is, the Weyl- projectively flat normal paracontact metric manifold is an η-Einstein. Thus we have

$$S(X,Y)=(1+\frac{\tau}{n-1})g(X,Y)-(n+\frac{\tau}{n-1})\eta(X)\eta(Y). \tag{19}$$

From (15) and (19), we have the following Proposition.

Proposition 2. *A normal paracontact metric manifold $M^n(\varphi,\eta,\xi,g)$ is an η-Einstein manifold if it is Weyl-projectively flat.*

3. Pseudosymmetric Normal Paracontact Metric Manifolds

In this section, we consider pseudosymmetric normal paracontact metric manifolds.

Theorem 2. *If a normal paracontact metric manifold $M^n(\varphi,\xi,\eta,g)$ is pseudosymmetric provided $L\neq -1$, then it is an η-Einstein manifold. Furthermore, it is a semisymmetric if and only if it has a constant sectional curvature* 1*.*

Proof. We suppose that n-dimensional normal paracontact metric manifold M^n is pseudosymmetric. Then from (7), we have

$$(R(X,Y)\cdot R)(U,V,Z)=LQ(g,R)(U,V,Z;X,Y),$$

for all $X,Y,Z,U,V\in\Gamma(TM)$. It follows that

$$\begin{aligned} R(X,Y)R(U,V)Z &- R(R(X,Y)U,V)Z-R(U,R(X,Y)V)Z \\ &- R(U,V)R(X,Y)Z=-L\{R((X\wedge_g Y)U,V)Z \\ &+ R(U,(X\wedge_g Y)V)Z+R(U,V)(X\wedge_g Y)Z\}. \end{aligned} \tag{20}$$

Putting $Y=Z=\xi$ in (20) and by virtue of (9), we have

$$\begin{aligned} R(X,\xi)R(U,V)\xi &- R(R(X,\xi)U,V)\xi-R(U,R(X,\xi)V)\xi \\ &- R(U,V)R(X,\xi)\xi=-L\{R(\eta(U)X-g(X,U)\xi,V)\xi \\ &+ R(U,\eta(V)X-g(X,V)\xi)\xi+R(U,V)(X-\eta(X)\xi)\}. \end{aligned}$$

after necessary arrangements are made, we conclude

$$\begin{aligned} R(U,V)X+g(X,V)U &- g(X,U)V=L\{g(X,U)V-g(X,V)U \\ &+ g(X,V)\eta(U)\xi-g(X,U)\eta(V)\xi-R(U,V)X\}. \end{aligned}$$

if both sides of this expression are multiplied by W, we have

$$g(R(U,V)X,W) + g(X,V)g(U,W)-g(X,U)g(V,W)$$

$$\begin{aligned} &= L\{g(X,U)g(V,W) - g(X,V)g(U,W) \\ &+ g(X,V)\eta(U)\eta(W) - g(X,U)\eta(V)\eta(W) \\ &- g(R(U,V)X,W)\}, \end{aligned} \tag{21}$$

for all $W \in \Gamma(TM)$. Here replacing $X = V = e_1, e_2, ..., e_{n-1}, e_n = \xi$ in (21) for the orthonormal basis of $\Gamma(TM)$ and by means of Ricci tensor, we get

$$\begin{aligned} S(U,W) &+ (n-1)g(U,W) = L\{(1-n)g(U,W) \\ &+ (n-1)\eta(U)\eta(W) - S(U,W)\}, \end{aligned}$$

After the necessary arrangements are made, we conclude

$$S(U,Z) + (n-1)g(U,Z) = L\Big\{(1-n)g(U,W) + (n-1)\eta(U)\eta(W) \\ -S(U,W)\Big\},$$

that is,

$$S(U,W) = (1-n)g(U,W) + (n-1)\frac{L}{L+1}\eta(U)\eta(W). \tag{22}$$

If it is a semisymmetric, then $L = 0$ and (21) takes form

$$R(U,V)X = g(X,U)V - g(X,V)U.$$

This tells us that M has a constant sectional curvature 1. Conversely, if it has a constant sectional curvature 1, then we have

$$\begin{aligned} (R(X,Y)R)(U,V,Z) &= R(X,Y)R(U,V)Z - R(R(X,Y)U,V)Z - R(U,R(X,Y)V)Z \\ &- R(U,V)R(X,Y)Z \\ &= R(X,Y)\{g(U,Z)V - g(V,Z)U\} - R(g(X,U)Y - g(Y,U)X,V)Z \\ &- R(U,g(X,V)Y - g(Y,V)X)Z - R(U,V)\{g(X,Z)Y - g(Y,Z)X\} \\ &= g(Z,U)\{g(X,V)Y - g(Y,V)X\} - g(V,Z)\{g(X,U)Y - g(Y,U)X\} \\ &- g(X,U)\{g(Y,Z)V - g(V,Z)Y\} + g(Y,U)\{g(X,Z)V - g(V,Z)X\} \\ &- g(X,V)\{g(U,Z)Y - g(Y,Z)U\} + g(Y,V)\{g(U,Z)X - g(X,Z)U\} \\ &- g(X,Z)\{g(U,Y)V - g(Y,V)U\} + g(Y,Z)\{g(U,X)V - g(V,X)U\} \\ &= 0. \end{aligned}$$

This completes the proof. □

Now, we will calculate $M(X,Y)\xi$ for later use. From (8)- (11), we obtain

$$\mathcal{M}(X,Y)\xi = \frac{1}{2}\{\eta(X)Y - \eta(Y)X\} + \frac{1}{2(n-1)}\{\eta(X)QY - \eta(Y)QX\}, \tag{23}$$

$$\mathcal{M}(\xi,Y)Z = \frac{1}{2}\{\eta(Z)Y - g(Y,Z)\xi\} - \frac{1}{2(n-1)}\{S(Y,Z)\xi - \eta(Z)QY\} \tag{24}$$

and

$$\begin{aligned}\eta(\mathcal{M}(X,Y)Z) &= \frac{1}{2(n-1)}\{\eta(Y)S(X,Z)-\eta(X)S(Y,Z)\} \\ &+ \frac{1}{2}\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\},\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{M}(\xi,X)Y &= \frac{1}{2}\{\eta(Y)X-g(X,Y)\xi\}+\frac{1}{2(n-1)}\{\eta(Y)QX \\ &- S(X,Y)\xi\}.\end{aligned} \tag{26}$$

Theorem 3. *A normal paracontact metric manifold M^n satisfying $\mathcal{M}\cdot R=0$ is an Einstein manifold.*

Proof. We suppose that $(\mathcal{M}(X,Y)\cdot R)(U,V,Z)=0$, for any $X,Y,Z,U,V\in\Gamma(TM)$. This implies that

$$\begin{aligned}\mathcal{M}(X,Y)R(U,V)Z &- R(\mathcal{M}(X,Y)U,V)Z-R(U,M(X,Y)V)Z \\ &- R(U,V)\mathcal{M}(X,Y)Z=0.\end{aligned} \tag{27}$$

Putting $Y=Z=\xi$ in (27), we obtain

$$\begin{aligned}\mathcal{M}(X,\xi)R(U,V)\xi &- R(\mathcal{M}(X,\xi)U,V)\xi-R(U,\mathcal{M}(X,\xi)V)\xi \\ &- R(U,V)\mathcal{M}(X,\xi)\xi=0.\end{aligned}$$

By using (9) and (24), we conclude

$$\begin{aligned}\frac{1}{2}g(X,V)U &+ \frac{1}{2(n-1)}S(X,V)U+\frac{1}{2}R(U,V)X \\ &+ \frac{1}{2(n-1)}R(U,V)QX=0.\end{aligned} \tag{28}$$

Taking the inner product with ξ, we reach

$$\begin{aligned}\eta(U)S(X,V) &+ (n-1)\eta(U)g(X,V)+(n-1)\{\eta(V)g(X,U) \\ &- \eta(U)g(X,V)\}+\eta(V)S(X,U)-\eta(U)S(X,V) \\ &= 0,\end{aligned}$$

that is,

$$S(X,U)=(1-n)g(X,U).$$

This proves our assertion. □

Definition 1. *A semi-Riemannian manifold (M,g) is said to be the mathcalMprojective pseudosymmetric if there exists a function L on M such that*

$$R\cdot\mathcal{M}=LQ(g,\mathcal{M}),$$

where R and $\mathcal{M}$ denote the Riemannian and $\mathcal{M}$- projectively curvature tensors of M. If $L=0$, it also called the $\mathcal{M}$-projectively semisymmetric.

Theorem 4. *An $\mathcal{M}$-projective pseudosymmetric normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$ is an η-Einstein manifold.*

Proof. Let us take $\mathcal{M}$-projective pseudosymmetric normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$. From (5), (6), we have

$$\begin{aligned} -L\Big\{M((X\wedge_g Y)U,V)Z &+ M(U,(X\wedge_g Y)V)Z + M(U,V)(X\wedge_g Y)Z\Big\} \\ &= R(X,Y)M(U,V)Z - M(R(X,Y)U,V)Z \\ &- M(U,R(X,Y)V)Z - M(U,V)R(X,Y)Z, \end{aligned} \tag{29}$$

for all $X,Y,U,V,Z \in \Gamma(TM)$. Setting $X=Z=\xi$ in (29), by using (23)-(26), we have

$$\begin{aligned} &- L\{\frac{1}{2}[g(Y,U)V - g(Y,V)U + g(V,Y)\eta(U)\xi - g(Y,U)\eta(V)\xi] \\ &+ \frac{1}{2(n-1)}[g(Y,U)QV - g(Y,V)QU + g(Y,V)\eta(U)Q\xi \\ &- g(Y,U)\eta(V)Q\xi] - \mathcal{M}(U,V)Y\} = \frac{1}{2}[g(Y,U)V - g(Y,V)U] \\ &+ \frac{1}{2(n-1)}[\eta(V)S(Y,U)\xi - \eta(U)S(V,Y)\xi + g(Y,U)QV \\ &- \eta(V)g(Y,U)Q\xi + \eta(U)g(Y,V)Q\xi - g(Y,V)QU] \\ &- \mathcal{M}(U,V)Y. \end{aligned}$$

If both sides of this equality are multiplied by W and by means of definition of the Ricci tensor, we obtain

$$\begin{aligned} &- L\Big\{\frac{1}{2}[g(Y,U)g(V,W) - g(Y,V)g(U,W) + g(V,Y)\eta(U)\eta(W) \\ &- g(Y,U)\eta(V)\eta(W)] + \frac{1}{2(n-1)}[g(Y,U)S(V,W) - g(Y,V)S(U,W) \\ &+ g(Y,V)\eta(U)S(\xi,W) - g(Y,U)\eta(V)S(\xi,W)] - g(\mathcal{M}(U,V)Y,W)\Big\} \\ &= \frac{1}{2}[g(Y,U)g(V,W) - g(Y,V)g(U,W)] \\ &+ \frac{1}{2(n-1)}\Big[\eta(V)S(Y,U)\eta(W) - \eta(U)S(V,Y)\eta(W) \\ &+ g(Y,U)S(V,W) - \eta(V)g(Y,U)S(\xi,W) + \eta(U)g(Y,V)S(\xi,W) \\ &- g(Y,V)S(U,W)\Big] - g(\mathcal{M}(U,V)Y,W). \end{aligned}$$

Here taking trace boht of sides for $Y = V = e_i$, for $1 \leq i \leq n$, in the last equality,

$$
\begin{aligned}
-L\sum_{i=1}^{n}\Big\{\frac{1}{2}[\epsilon_i g(e_i,U)g(e_i,W) &- \epsilon_i g(e_i,e_i)g(U,W)+\epsilon_i g(e_i,e_i)\eta(U)\eta(W)\\
-\epsilon_i g(e_i,U)\eta(e_i)\eta(W)] &+ \frac{1}{2(n-1)}[\epsilon_i g(e_i,U)S(e_i,W)-\epsilon_i g(e_i,e_i)S(U,W)\\
&+ \epsilon_i g(e_i,e_i)\eta(U)S(\xi,W)-\epsilon_i g(e_i,U)\eta(e_i)S(\xi,W)]\\
&- \epsilon_i g(M(U,e_i)e_i,W)\Big\}\\
&= \sum_{i=1}^{n}\epsilon_i\Big\{\frac{1}{2}[\epsilon_i g(e_i,U)g(e_i,W)-\epsilon_i g(e_i,e_i)g(U,W)]\\
&+ \frac{1}{2(n-1)}[\epsilon_i\eta(e_i)S(e_i,U)\eta(W)-\epsilon_i\eta(U)S(e_i,e_i)\eta(W)\\
&+ \epsilon_i g(e_i,U)S(e_i,W)-\epsilon_i\eta(e_i)g(e_i,U)S(\xi,W)\\
&+ \epsilon_i\eta(U)g(e_i,e_i)S(\xi,W)-\epsilon_i g(e_i,e_i)S(U,W)]\\
&- \epsilon_i g(M(U,e_i)e_i,W)\Big\}, \qquad (30)
\end{aligned}
$$

where ϵ_i is the signature $\{e_i\}$. On the other hand, by direct calculations, we have

$$\epsilon_i g(M(U,e_i)e_i,W) = \frac{1}{2(n-1)}\{n.S(U,W)-\tau.g(U,W)\}.$$

Making use of (30) and after the necessary arrangements are revised, we get

$$S(U,W) = \frac{(1-n)(n-1)+\tau}{2n-1}g(U,W)+\frac{n(1-n)-\tau}{(2n-1)(1+L)}\eta(U)\eta(W),$$

which proves the theorem. □

Definition 2. *A normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$ is said to be the Weyl-pseudosymmetric if there exists a function L on M such that*

$$R\cdot C = LQ(g,C),$$

where R and C denote the Riemannian and Weyl-conformal curvature tensors of M. If $L=0$, then it also called the Weyl-semisymmetric.

Now, we consider the Weyl-conformal curvature tensor of M^n given by (12) for later use.

$$
\begin{aligned}
C(X,Y)\xi &= \left(\frac{1-n-\tau}{(n-1)(n-2)}\right)(\eta(X)Y-\eta(Y)X)\\
&+ \frac{1}{n-2}(\eta(X)QY-\eta(Y)QX) \qquad (31)
\end{aligned}
$$

and

$$\begin{aligned} C(\xi, X)Y &= \left(\frac{1-n-\tau}{(n-1)(n-2)}\right)(\eta(Y)X - g(X,Y)\xi) \\ &+ \frac{1}{n-2}(\eta(Y)QX - S(X,Y)\xi). \end{aligned} \tag{32}$$

Theorem 5. *The Weyl-pseudosymmetric normal paracontact metric manifold $M^n(\varphi,\eta,\xi,g)$ is an η-Einstein manifold.*

Proof. Let $M^n(\varphi,\eta,\xi,g)$ be the Weyl-pseudosymmetric, then there is a function L such that

$$(R(X,Y)\cdot C)(U,V,Z) = LQ(g,C)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z \in \Gamma(TM)$. This implies that

$$\begin{aligned} R(X,Y)C(U,V)Z &- C(R(X,Y)U,V)Z - C(U,R(X,Y)V)Z \\ &- C(U,V)R(X,Y)Z = -L\Big\{C((X\wedge_g Y)U,V)Z \\ &+ C(U,(X\wedge_g Y)V)Z + C(U,V)(X\wedge_g Y)Z\Big\}. \end{aligned} \tag{33}$$

Here setting $X = Z = \xi$ in (33), we have

$$\begin{aligned} R(\xi,Y)C(U,V)\xi &- C(R(\xi,Y)U,V)\xi - C(U,R(\xi,Y)V)\xi \\ &- C(U,V)R(\xi,Y)\xi = -L\Big\{C((\xi\wedge_g Y)U,V)\xi \\ &+ C(U,(\xi\wedge_g Y)V)\xi + C(U,V)(\xi\wedge_g Y)\xi\Big\}. \end{aligned} \tag{34}$$

After the necessary calculations, we reach at

$$\begin{aligned} & \frac{1-n-\tau}{(n-1)(n-2)}\{g(Y,U)V - g(Y,V)U\} \\ + & \frac{1}{n-2}\{\eta(V)S(Y,U)\xi - \eta(U)S(V,Y)\xi \\ + & g(Y,U)QV - \eta(V)g(Y,U)Q\xi \\ + & \eta(U)g(Y,V)Q\xi - g(Y,V)QU\} - C(U,V)Y \\ = & -L\{\frac{1-n-\tau}{(n-1)(n-2)}(g(Y,U)V - g(Y,V)U - g(Y,U)\eta(V)\xi \\ + & g(Y,V)\eta(U)\xi) + \frac{1}{n-2}(g(Y,U)QV - g(Y,V)QU \\ - & \eta(V)g(Y,U)Q\xi + \eta(U)g(Y,V)Q\xi) - C(U,V)Y\}. \end{aligned} \tag{35}$$

If both sides of the equality are multiplied by W, we obtain

$$\frac{1-n-\tau}{(n-1)(n-2)}\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}$$

$$
\begin{aligned}
&+ \quad \frac{1}{n-2}\{\eta(V)S(Y,U)\eta(W) - \eta(U)S(V,Y)\eta(W) \\
&+ \quad g(Y,U)S(V,W) - \eta(V)g(Y,U)S(\xi,W) \\
&+ \quad \eta(U)g(Y,V)S(\xi,W) - g(Y,V)S(W,U)\} \\
&- \quad g(C(U,V)Y,W) \\
&= \quad -L\{\frac{1-n-\tau}{(n-1)(n-2)}(g(Y,U)g(V,W) - g(Y,V)g(U,W) \\
&- \quad g(Y,U)\eta(V)\eta(W) + g(Y,V)\eta(U)\eta(W)) \\
&+ \quad \frac{1}{n-2}(g(Y,U)S(V,W) - g(Y,V)S(U,W) \\
&- \quad \eta(V)g(Y,U)S(\xi,W) + \eta(U)g(Y,V)S(\xi,W)) \\
&- \quad g(C(U,V)Y,W)\}. \qquad (36)
\end{aligned}
$$

Putting $Y = V = e_1, e_2, ...e_{n-1}, e_n = \xi$ in (36) for the orthonormal basis of $\Gamma(TM)$ and taking into account definition of Ricci tensor, we have

$$
\begin{aligned}
&\frac{1-n-\tau}{(n-1)(n-2)}\sum_{i=1}^{n}\Big\{\epsilon_i\{g(e_i,U)g(e_i,W) - \epsilon_i g(e_i,e_i)g(U,W)\} \\
&+ \quad \frac{1}{n-2}\{\epsilon_i\eta(e_i)S(e_i,U)\eta(W) - \epsilon_i\eta(U)S(e_i,e_i)\eta(W) \\
&+ \quad \epsilon_i g(e_i,U)S(e_i,W) - \epsilon_i\eta(e_i)g(e_i,U)S(\xi,W) \\
&+ \quad \epsilon_i\eta(U)g(e_i,e_i)S(\xi,W) - \epsilon_i g(e_i,e_i)S(W,U)\} \\
&- \quad \epsilon_i g(C(U,e_i)e_i,W)\Big\} \\
&= \quad -L\{\frac{1-n-\tau}{(n-1)(n-2)}\sum_{i=1}^{n}\Big\{\epsilon_i(g(e_i,U)g(e_i,W) - \epsilon_i g(e_i,e_i)g(U,W) \\
&- \quad \epsilon_i g(e_i,U)\eta(e_i)\eta(W) + \epsilon_i g(e_i,e_i)\eta(U)\eta(W)) \\
&+ \quad \frac{1}{n-2}(\epsilon_i g(e_i,U)S(e_i,W) - \epsilon_i g(e_i,e_i)S(U,W) \\
&- \quad \epsilon_i\eta(e_i)g(e_i,U)S(\xi,W) + \epsilon_i\eta(U)g(e_i,e_i)S(\xi,W)) \\
&- \quad \epsilon_i g(C(U,e_i)e_i,W)\}\Big\}. \qquad (37)
\end{aligned}
$$

By using (11) and after the necessary abbreviations, (37) implies that

$$S(U,W) = (1 - \frac{\tau}{n-1})g(U,W) - (n + \frac{\tau}{n-1})\eta(U)\eta(W).$$

This proves our assertion. □

Now, we will give an non-trivial example for illustration our results.

Example 1. *Let us the 5-dimensional manifold*

$$M^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in R, \}$$

where (x_i) *denote the cartesian coordinate in* $\mathbb{R}^5$ *for* $1 \leq i \leq 5$*. Then the vector fields*

$$e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2}, e_3 = 2x_2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, e_4 = 2x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, e_5 = -2x_4\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}$$

are linearly independent at each point of M^5*. By* g*, we denote the semi-Riemannian metric tensor such that*

$$g(e_i, e_j) = \begin{cases} 1; & i = j = 1, 3, 4 \\ -1; & i = j = 2, 5 \\ 0; & i \neq j \end{cases}$$

Let η *be the 1-form defined by* $\eta(X) = g(X, e_1)$ *for all* $X \in \Gamma(TM)$*. Now, we definite the paracontact metric structure* φ *such that*

$$\varphi e_1 = 0, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = -e_2, \quad \varphi e_4 = -e_5, \quad \varphi e_5 = -e_4.$$

Then we can easily see that

$$\eta(e_5) = 1, \quad \varphi^2 X = X - \eta(X)\xi, \quad e_5 = \xi$$

and

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(\widetilde{M})$*. Thus* $M^5(\varphi, \eta, \xi, g)$ *defines an almost paracontact metric manifold. By* $\widetilde{\nabla}$*, we denote the Levi-Civita connection on* $\widetilde{M}$*. Then by direct calculations, we have non-zero the components*

$$[e_2, e_3] = 2e_1, \quad [e_3, e_4] = 2e_1, \quad [e_4, e_5] = -2e_1.$$

Let ∇ *be the Levi-Civita connection on* M*. Using the properties of paracontact metric structure and Kozsul formulae, we can observe the non-zero components*

$$\nabla e_2 e_1 = -e_3 = \varphi e_2, \quad \nabla e_3 e_1 = -e_2 = \varphi e_3, \quad \nabla e_4 e_1 = -e_5 = \varphi e_4, \quad \nabla e_5 e_1 = -e_4 = \varphi e_5$$

Thus one can easily verified

$$\widetilde{\nabla}_X e_1 = \varphi X,$$

for all $X \in \Gamma(TM)$ *This tells us that* $M^5(\varphi, \eta, \xi, g)$ *is a normal paracontact metric manifold with paracontact metric structure* (φ, η, ξ, g)*. By straightforward calculations, we can easily see that non-zero components of the Riemannian curvature tensor* R*,*

$$R(e_i, e_1)e_1 = -e_i, \quad 2 \leq i \leq 5.$$

This tell us that

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X,$$

for all $X, Y, Z \in \Gamma(TM)$*, that is,* $\widetilde{M}(\varphi, \eta, \xi, g)$ *is real space form with constant sectional curvature* 1*.*

Author Contribution Statements The authors have equal contribution to the preparation of the article.

Declaration of Competing Interests Authors have declared that no competing interests exist.

References

[1] Boothby, M., Wang, R. C., On contact manifolds, *Anna Math*, 68 (1958), 421-450.

[2] Sasaki, A., Hatakeyama, Y., On differentiable manifolds with certain structure which are closely related to almost contact structure, *Tohoku Math. J.*, 13 (1961), 281-294.

[3] Tanno, S., The automorphism groups of almost contact Riemannian manfifolds, *The Tohoku Math. J.*, 21 (1969), 21-38. DOI: 10.2748/tmj/1178243031

[4] Kenmotsu, K., A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, 24 (1972), 93-103.

[5] Marero, J. C., Chinea, D., On trans-Sasakian manifolds, *Proceedings of the XIV. th Spanish-Portuguese Conference on Mathematics. Uni. La. Laguna,* 1(3) (1990), 655-659.

[6] Zamkovoy, S., Nakova, G., The decomposition of almost paracontact metric manifolds in eleven classes revisited, *J. Geom.*, 109(18) (2018). https://doi.org/10.1007/s00022-018-0423-5

[7] Mandal, K., De, U. C., Some curvature properties of paracontact metric manifolds, *Advances in Pure and Applied Mathematics*, 9(3) (2018), 159-165. https://doi.org/10.1515/apam-2017-0064

[8] Özdemir, N., Aktay, Ş., Solgun, M., Almost paracontact structures obtained from $G^*_{2(2)}$ structures, *Turkısh Journal of Mathematics*, 42(6) (2018), 3025-3033. https://doi.org/10.3906/mat-1706-10

[9] Pandey, H., Kumar, A., Anti-Invariant submanifolds of almost paracontact manifolds, *Indian J. Pure Appl. Math.*, 16(6) (1985), 586-590.

[10] Welyczko, J., On Legendre curves in 3-dimensional normal almost paracontact metric manifolds, *Results. Math.*, 54 (2009), 377-387. DOI 10.1007/s00025-009-0364-2

[11] Pokhariyal, G. P., Mishra, R. S., The curvature tensor and their relativistic significances, *II. Yokohoma Mathematical journal*, 18 (1970), 105-108.

[12] Ojha, R. H., A note on the M-projective curvature tensor, *India J. Pure Applied Math.*, 8 (1975), 1531-1534.

[13] Li, D., Yin, J., Paracontact metric (κ, μ) manifold satisfying the Miao-Tam equation, *Advances in Mathematical Physics*, 6 (2021), 1-5. DOI: 10.1155/2021/6687223

[14] Atçeken, M., Yuca, G., Some results on invariant submanifolds of an almost Kenmotsu (κ, μ, ν)-space, *Honam Mathematical Journal,* 43(4) (2021), 655-665. https://doi.org/10.5831/HMJ.2021.43.4.655

[15] Atçeken, M. Some results on invariant submanifolds of Lorentz para-Kenmotsu manifolds, *Korean Journal of Mathematics,* 30(1) (2022), 175-185. http://dx.doi.org/10.11568/kjm.2022.30.1.175

[16] Atçeken, M., Mert, T., Characterizations for totally geodesic submanifolds of a K-paracontact manifold, *AIMS Math.,* 6(7) (2021), 7320-7332. http://dx.doi.org/10.3934/math.2021430

[17] Mert, T., Characterization of some special curvature tensor on almost $C(\alpha)-$manifold, *Asian Jour. of Math. and Com. Res.*, 29(1) (2022), 27-41.

[18] Mert, T., Atçeken, M., Almost $C(\alpha)-$manifold on W_0^*-curvature tensor, *App. Math. Sciences*, 15(15) (2021), 693-703. doi: 10.12988/ams.2021.916556

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1059–1079 (2022)
DOI:10.31801/cfsuasmas.1056890
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: January 12, 2022; Accepted: June 9, 2022

APPROXIMATION PROPERTIES OF BERNSTEIN'S SINGULAR INTEGRALS IN VARIABLE EXPONENT LEBESGUE SPACES ON THE REAL AXIS

Ramazan AKGÜN

Balikesir University, Faculty of Arts and Sciences, Department of Mathematics, Çağış Yerleşkesi, 10145 Balıkesir, TÜRKİYE

Abstract. In generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent $p(\cdot)$ defined on the real axis, we obtain several inequalities of approximation by integral functions of finite degree. Approximation properties of Bernstein singular integrals in these spaces are obtained. Estimates of simultaneous approximation by integral functions of finite degree in $L^{p(\cdot)}$ are proved.

1. Introduction

In this work we consider approximation properties of Bernstein's singular integrals for functions given in the variable exponent Lebesgue spaces $L^{p(x)}(\mathbb{R})$. This scale of function spaces were studied in detail in books Uribe-Fiorenza [15], Diening, Harjulehto, Hästö, Růžička [17] and Sharapudinov [40]. $L^{p(x)}(\mathbb{R})$ has many applications in several branches of mathematics such as elasticity theory [50], fluid mechanics [38], [37], differential operators [38], [18], nonlinear Dirichlet boundary value problems [32], nonstandard growth [50] and variational calculus. Variable exponent works started with W. Orlicz [35] and developed in many directions. For example, $L^{p(x)}(\mathbb{R})$ is a modular space ([33]) and under the condition $p^+ := esssup_{x\in\mathbb{R}} p(x) < \infty$, $L^{p(x)}(\mathbb{R})$ becomes a particular case of the Musielak-Orlicz spaces [33]. Starting from nineties, studies on $L^{p(x)}(\mathbb{R})$ has reached a positive momentum: See [32], [39], [20], [16] and many others.

2020 *Mathematics Subject Classification.* 41A17, 41A25, 41A28, 41A35, 42A27.

Keywords. Modulus of smoothness, simultaneous approximation, Bernstein singular integral, forward Steklov mean, mollifiers, Jackson inequality, entire integral functions of finite degree.

rakgun@balikesir.edu.tr; 0000-0001-6247-8518.

In variable exponent Lebesgue spaces on $[0,2\pi]$ (or $[0,1]$), some fundamental results corresponding to the approximation of function have been obtained by Sharapudinov [41–45]. Some results on approximation in $L^{p(x)}([0,2\pi])$ or other function classes can be seen e.g. in [1,3–6,8,9,19,21–25,27–30,48].

In this work, we aim to obtain simultaneous theorems on approximation by entire functions of finite degree in variable exponent Lebesgue spaces on the whole real axis $\mathbb{R}$.

Approximation by entire function of finite degree in the real axis started by the works of Bernstein [11,12], N. Wiener and R. Paley [36], N. I. Ahiezer [2], S. M. Nikolskii [34]. Note that an entire function of finite exponential type is merely an entire function of order 1 and finite type that in approximation theory, these often play an important role similar to trigonometric polynomials in the case of approximation of periodic functions.

Note that, some results on approximation by entire integral functions of finite degree were obtained by Ibragimov [26] and Taberski [46,47] in the classical Lebesgue spaces $L^p(\mathbb{R})$.

We can give some required definitions. We denote by P the class of exponents $p(x):\mathbb{R}\to[1,\infty)$ such that $p(x)$ is a measurable function and $p(x)$ satisfy conditions

$$1\le p_-:=essinf_{x\in\mathbb{R}}p(x),\quad p^+:=esssup_{x\in\mathbb{R}}p(x)<\infty. \tag{1}$$

We define $L^{p(\cdot)}:=L^{p(\cdot)}(\mathbb{R})$ as the set of all functions $f:\mathbb{R}\to\mathbb{C}$ such that

$$I_{p(\cdot)}\left(\frac{f}{\lambda}\right):=\int_{\mathbb{R}}\left|\frac{f(y)}{\lambda}\right|^{p(y)}dy<\infty \tag{2}$$

for some $\lambda>0$. The set of functions $L^{p(\cdot)}$, with norm

$$\|f\|_{p(\cdot)}:=\inf\left\{\eta>0:I_{p(\cdot)}\left(\frac{f}{\eta}\right)<1\right\}$$

is a Banach space.

For $p\in P$ we define its conjugate $p'(x):=\frac{p(x)}{p(x)-1}$ for $p(x)>1$ and $p'(x):=\infty$ for $p(x)=1$.

For $i\in\mathbb{N}$, all constants $\mathbf{c}_i$ (or $\mathbf{c}$) will be some positive numbers such that $\mathbf{c}_i$ will depend on main parameters of the problem. In some cases we will use temporaryly some generic constans $C,c>0$ for clarity (for example in statements of some theorems). We will give explicit constants in the proofs but these constants are not best constants.

Throughout this paper symbol $\mathfrak{A}\lesssim\mathcal{B}$ will mean that there exists a constant C depending only on unimportant parameters in question such that inequality $\mathfrak{A}\le \mathrm{C}\mathcal{B}$ holds.

Definition 1. *Let P^{Log} be a subclass ([17]) of P such that there exist constants $\mathbf{c}_1,\mathbf{c}_2>0$, $\mathbf{c}_3\in\mathbb{R}$ with properties*

$$|p(x)-p(y)|\ln(e+1/|x-y|)\le\mathbf{c}_1<\infty,\quad\forall x,y\in\mathbb{R}, \tag{3}$$

$$|p(x)-\mathbf{c}_3|\ln(e+|x|)\le \mathbf{c}_2<\infty, \quad \forall x\in\mathbb{R}. \tag{4}$$

2. Transference Result

Let C_0^∞ be class of infinitely times continuously differentiable functions ϕ with compact support $spt\phi:=\overline{\{x\in\mathbb{R}:\phi(x)\neq 0\}}$. Let $C(A)$ be the class of continuous functions defined on A. Define $\|f\|_{C(A)}:=\sup\{|f(x)|:x\in A\}$ for $f\in C(A)$.

For given $f\in L^{p(\cdot)}$ we can define an auxiliary function as follows: Define

$$F_f(u):=F_{f,G}(u):=\int_{\mathbb{R}} f(u+x)|G(x)|\,dx, \quad u\in\mathbb{R}, \tag{5}$$

where $G\in L^{p'(\cdot)}\cap C_0^\infty$ and $\|G\|_{p'(\cdot)}\le 1$. Also we set $\mathbf{c}_0:=\|G\|_{C(\mathbb{R})}$.

Theorem 1. *Let $p\in P^{Log}$ and $f,g\in L^{p(\cdot)}$. If*

$$\|F_{f,G}\|_{C(\mathbb{R})}\lesssim \|F_{g,G}\|_{C(\mathbb{R})},$$

with an absolute positive constant, then, we have following norm inequality

$$\|f\|_{p(\cdot)}\lesssim\|g\|_{p(\cdot)}$$

with a positive constant depending only on p.

3. Mollifiers and forward Steklov means in $L^{p(\cdot)}$

Definition 2. *Suppose that $0<\delta<\infty$ and $\tau\in\mathbb{R}$. We define ([44]) family of translated Steklov operators $\{\mathcal{S}_{\delta,\tau}f\}$, by*

$$\mathcal{S}_{\delta,\tau}f(x):=\frac{1}{\delta}\int_{x+\tau-\delta/2}^{x+\tau+\delta/2} f(t)\,dt, \quad x\in\mathbb{R} \tag{6}$$

for locally integrable function f defined on $\mathbb{R}$.

Let f and g be two real-valued measurable functions on $\mathbb{R}$. We define the convolution $f*g$ of f and g by setting $(f*g)(x)=\int_{\mathbb{R}}f(y)g(x-y)dy$ for $x\in\mathbb{R}$ for which the integral exists in $\mathbb{R}$.

The following result on mollifiers in variable exponent Lebesgue spaces is obtained by D. Cruz-Uribe and A. Fiorenza (see [14]).

Definition 3. *Let $\phi\in L^1(\mathbb{R})$ and $\int_{\mathbb{R}}\phi(t)\,dt=1$. For each $t>0$ we define $\phi_t(x)=\frac{1}{t}\phi\left(\frac{x}{t}\right)$. Such sequence $\{\phi_t\}$ will be called approximate identity. A function*

$$\phi^{\tilde{}}(x)=\sup_{|y|\ge|x|}|\phi(y)|$$

will be called radial majorant of ϕ. If $\phi^{\tilde{}}\in L^1(\mathbb{R})$, then, sequence $\{\phi_t\}$ will be called potential-type approximate identity.

Theorem 2. *([14]) Suppose $p \in P^{Log}$, $f \in L^{p(\cdot)}$, ϕ is a potential-type approximate identity. Then, for any $t > 0$,*

$$\|f * \phi_t\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

and

$$\lim_{t\to 0} \|f * \phi_t - f\|_{p(\cdot)} = 0$$

hold with a positive constant depend on p.

As a corollary of Theorem 1 we have

Theorem 3. *Suppose that $p \in P^{Log}$, $0 < \delta < \infty$ and $\tau \in \mathbb{R}$. Then, the family of operators $\{\mathcal{S}_{\delta,\tau} f\}$, defined by (6), is uniformly bounded (in δ and τ) in $L^{p(\cdot)}$, namely, for any $0 < \delta < \infty$ and $\tau \in \mathbb{R}$ norm inequality*

$$\|\mathcal{S}_{\delta,\tau} f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

holds with a positive constant depend on p.

As a corollary of Theorem 3 we get

Corollary 1. *Let $p \in P^{Log}$, $0 < \delta < \infty$, $f \in L^{p(\cdot)}$. If $\tau = \delta/2$ then,*

$$T_\delta f(x) := \mathcal{S}_{\delta,\delta/2} f(x) = \frac{1}{\delta} \int_0^\delta f(x+t)\, dt$$

and

$$\|T_\delta f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

holds with a positive constant depend on p.

4. Modulus of Smoothness and K-functional

If $f \in L^{p(\cdot)}$ and $0 \le \delta < \infty$, $r \in \mathbb{N}$, then

$$\Omega_r(f,\delta)_{p(\cdot)} := \|(I - T_\delta)^r f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}. \tag{7}$$

Here I is the identity operator. In what follows $W_r^{p(\cdot)}$, $r \in \mathbb{N}$, will be the class of functions $f \in L^{p(\cdot)}$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^{p(\cdot)}$.

Remark 1. *For $p \in P^{Log}$, f, $g \in L^{p(\cdot)}$ and $0 \le \delta < \infty$, the modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot)}$, has the following usual properties:*

(i) $\Omega_r(f,\delta)_{p(\cdot)}$ is non-negative; non-decreasing function of $\delta \ge 0$;
(ii) $\Omega_r(f+g,\cdot)_{p(\cdot)} \le \Omega_r(f,\cdot)_{p(\cdot)} + \Omega_r(g,\cdot)_{p(\cdot)}$;
(iii) $\lim_{\delta\to 0^+} \Omega_r(f,\delta)_{p(\cdot)} = 0$;
(iv) $\Omega_r(f,\delta)_{p(\cdot)} \lesssim \delta^r \left\|f^{(r)}\right\|_{p(\cdot)}$ for $r \in \mathbb{N}$, $f \in W_r^{p(\cdot)}$ and $\delta > 0$.

Indeed: (ii) follows from definition. (iii) is follow from (7), (3.4) and Theorem 3.1 of [7]. (iv) follows from Lemma 3.2 of [7]. (i) follows from Lemma 1 given below.

Definition 4. *Define, for $f \in L^{p(\cdot)}$, $p \in P^{Log}$, and $\delta > 0$,*

$$\left(\mathfrak{R}_\delta f\right)(\cdot) := \frac{2}{\delta}\int_{\delta/2}^{\delta}\left(\frac{1}{h}\int_0^h f(\cdot + t)\,dt\right)dh.$$

Remark 2. *Note that, for $0 < \delta < \infty$, $p \in P^{Log}$ we know from Corollary 1 that*

$$\left\|\mathfrak{R}_\delta f\right\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

and, hence, $f - \mathfrak{R}_\delta f \in L^{p(\cdot)}$ for $f \in L^{p(\cdot)}$.

We set $\mathfrak{R}_\delta^r f := \left(\mathfrak{R}_\delta f\right)^r$.

Lemma 1. *Let $0 < h \le \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$. Then*

$$\left\|\left(I - T_h\right) f\right\|_{p(\cdot)} \lesssim \left\|\left(I - T_\delta\right) f\right\|_{p(\cdot)} \tag{8}$$

holds with a positive constant depend on p.

Lemma 2. *Let $0 < \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$. Then*

$$\left\|\left(I - \mathfrak{R}_\delta\right) f\right\|_{p(\cdot)} \lesssim \left\|\left(I - T_\delta\right) f\right\|_{p(\cdot)}$$

holds with a positive constant depend on p.

Remark 3. *The function $\mathfrak{R}_\delta f$ is absolutely continuous and differentiable a.e. (almost everywhere) on $\mathbb{R}$ (see [43, (5.2) of Theorem 4]).*

The following lemma is obvious from definitions.

Lemma 3. *Let $0 < \delta < \infty$, $p \in P^{Log}$ and $f \in W_1^{p(\cdot)}$. Then*

$$\frac{d}{dx}\mathfrak{R}_\delta f = \mathfrak{R}_\delta \frac{d}{dx} f \quad \text{and} \quad \frac{d}{dx} T_\delta f = T_\delta \frac{d}{dx} f \tag{9}$$

a.e. on $\mathbb{R}$.

Lemma 4. *Let $0 < \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$ be given. Then*

$$\delta \left\|\frac{d}{dx}\mathfrak{R}_\delta f\right\|_{p(\cdot)} \lesssim \left\|\left(I - T_\delta\right) f\right\|_{p(\cdot)} \tag{10}$$

holds with a positive constant depend on p.

The following lemma can be proved using induction on r.

Lemma 5. *Let $0 < \delta < \infty$, $r - 1 \in \mathbb{N}$, $p \in P^{Log}$, and $f \in L^{p(\cdot)}$ be given. Then*

$$\frac{d^r}{dx^r}\mathfrak{R}_\delta^r f = \frac{d}{dx}\mathfrak{R}_\delta \frac{d^{r-1}}{dx^{r-1}}\mathfrak{R}_\delta^{r-1} f.$$

Modulus of smoothness $\left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)}$ and the K-functional

$$K_r\left(f, \delta; L^{p(\cdot)}, W_r^{p(\cdot)}\right)_{p(\cdot)} := \inf_{g \in W_r^{p(\cdot)}}\left\{\|f - g\|_{p(\cdot)} + \delta^r \left\|g^{(r)}\right\|_{p(\cdot)}\right\}$$

are equivalent:

Theorem 4. *If $r \in \mathbb{N}$, $p \in P^{Log}$, $f \in L^{p(\cdot)}$, and $\delta > 0$, then*

$$\frac{\left\| (I - T_\delta)^r f \right\|_{p(\cdot)}}{K_r \left(f, \delta; L^{p(\cdot)}, W_r^{p(\cdot)} \right)_{p(\cdot)}} \approx 1 \tag{11}$$

holds for some positive constants depend on p, r.

5. Results on Simultaneous Approximation

Let $\mathcal{G}_\sigma(X)$ be the subclass of entire integral functions $f(z)$ of exponential type $\leq \sigma$ that belonging to X and

$$A_\sigma(f)_X := \inf_g \{ \|f - g\|_X : g \in \mathcal{G}_\sigma(X) \}.$$

Let $\mathcal{C}$ be the class of bounded uniformly continuous functions defined on $\mathbb{R}$. We set $\mathcal{G}_{\sigma,\infty} := \mathcal{G}_\sigma(\mathcal{C})$ and $\mathcal{G}_{\sigma,p(\cdot)} := \mathcal{G}_\sigma\left(L^{p(\cdot)}\right)$.

Remark 4. *([10, definition given in (5.3)]) Let $\sigma > 0$, $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R})$,*

$$\vartheta(x) := \frac{2}{\pi} \frac{\sin(x/2)\sin(3x/2)}{x^2}$$

and

$$J(f, \sigma) = \sigma \int_{\mathbb{R}} f(x - u) \vartheta(\sigma u)\, du$$

be the delà Valèe Poussin operator ([10, definition given in (5.3)]). It is known (see (5.4)-(5.5) of [10]) that, if $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, then,

(i) $J(f, \sigma) \in \mathcal{G}_{2\sigma}(L^p(\mathbb{R}))$,

(ii) $J(g_\sigma, \sigma) = g_\sigma$ for any $g_\sigma \in \mathcal{G}_\sigma(L^p(\mathbb{R}))$,

(iii) $\|J(f, \sigma)\|_{L^p(\mathbb{R})} \leq \frac{3}{2} \|f\|_{L^p(\mathbb{R})}$,

(iv) $(J(f, \sigma))^{(r)} = J\left(f^{(r)}, \sigma\right)$ for any $r \in \mathbb{N}$ and $f \in W_r^p(\mathbb{R})$,

(v) $\|J\left(f, \frac{\sigma}{2}\right) - f\|_{L^p(\mathbb{R})} \to 0$ (as $\sigma \to \infty$) and hence

$$\| \left(J\left(f, \frac{\sigma}{2}\right)\right)^{(k)} - f^{(k)} \|_{L^p(\mathbb{R})} \to 0 \text{ as } \sigma \to \infty,$$

for $f \in W_r^p(\mathbb{R})$ and $1 \leq k \leq r$,

(vi) $\left\| f - J\left(f, \frac{\sigma}{2}\right) \right\|_{L^p(\mathbb{R})} \leq \frac{5\pi}{4} \frac{4^r}{\sigma^r} \|f^{(r)}\|_{L^p(\mathbb{R})}$ for $f \in W_r^p(\mathbb{R})$.

Theorem 5. *Let $p \in P^{Log}$, $\sigma > 0$, $r \in \mathbb{N}$ and $f \in W_r^{p(\cdot)}$. Then*

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} A_\sigma\left(f^{(r)}\right)_{p(\cdot)} \tag{12}$$

holds with a positive constant depend on p, r.

Theorem 6. *Let $p \in P^{Log}$, $\sigma > 0$, $k \in \mathbb{N}$, $r \in \{0\} \cup \mathbb{N}$ and $f \in W_r^{p(\cdot)}$. Then*

$$A_\sigma(f)_{p(\cdot)} \lesssim \frac{1}{\sigma^r} \Omega_k\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)}. \tag{13}$$

with positive constants depend on p, k, r.

Theorem 7. *Let $p \in P^{Log}$, $\sigma > 0$ and $g_\sigma \in \mathcal{G}_{\sigma,p(\cdot)}$. Then, Bernstein's inequality*

$$\| (g_\sigma)^{(r)} \|_{p(\cdot)} \lesssim \sigma^r \|g_\sigma\|_{p(\cdot)}$$

holds with a positive constant depend on p, r.

Definition 5. *[47, p.161] For $r,k \in \mathbb{N}$, $\sigma > 0$, we define*

$$g(\sigma, r, x) = \left(\frac{1}{x} \sin \frac{\sigma x}{2r}\right)^{2r}, \text{ and}$$

$$G(\sigma, r, k, \zeta) = \sum_{v=1}^{k} (-1)^{k-v} \frac{1}{v} \binom{k}{v} g\left(\sigma, r, \frac{\zeta}{v}\right).$$

For $r \geq \frac{1}{2}(k+2)$ we set

$$\gamma_{r,\sigma} := \int_{\mathbb{R}} \left(\frac{1}{t} \sin \frac{\sigma t}{2r}\right)^{2r} dt.$$

Let us introduce the Bernstein's singular integral ([47, p.161])

$$D_{\sigma,k} f(x) := \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} f(u) G(\sigma, r, k, u - x)\, dt \tag{14}$$

for $r,k \in \mathbb{N}$, $\sigma > 0$, and measurable complex valued f satisfying $\int_{\mathbb{R}} \frac{|f(u)|}{1+u^{2r}} du < \infty$.

Remark 5. *It is well known that, if $r,k \in \mathbb{N}$, $\sigma \in (0, \infty)$, $r \geq \frac{1}{2}(k+2)$, then $D_{\sigma,k} f \in \mathcal{G}_\sigma(L^p(\mathbb{R}))$ for $p \geq 1$.([47, p.161]).*

Lemma 6. *If $r \in \mathbb{N}$, $\sigma \in (0, \infty)$, then,*

(i) we have

$$\gamma_{r,\sigma} = \frac{\sigma^{2r-1}}{(2r)^{2r-1}} \int_{\mathbb{R}} \left(\frac{\sin v}{v}\right)^{2r} dv$$

(ii) (see, e.g. [13, (5)])

$$\int_{\mathbb{R}} \left(\frac{\sin v}{v}\right)^{2r} dv = \frac{2\pi}{(2r-1)!2^{2r}} \left\{(2r)^{2r-1} - \binom{2r}{1}(2r-2)^{2r-1} + \binom{2r}{2}(2r-4)^{2r-1} - \cdots\right\}$$

(iii) and, as a result,

$$\gamma_{r,\sigma} = \frac{\sigma^{2r-1}}{(2r)^{2r-1}} b_r$$

where b_r is the right hand side of equality in (ii), having r terms.

Define $\lceil a \rceil := \min\{n \in \mathbb{N} : n \geq a\}$ and $\lfloor \sigma \rfloor := \max\{n \in \mathbb{Z} : n \leq \sigma\}$. We will take $r := \lceil \frac{1}{2}(k+2) \rceil$ in the next Theorems.

Theorem 8. *Let $p \in P^{Log}$, $k \in \mathbb{N}$, $\sigma > 0$, $f \in W_k^{p(\cdot)}$, then*

$$\|f - D_{\sigma,k}f\|_{p(\cdot)} \lesssim \frac{1}{\sigma^k} \left\|f^{(k)}\right\|_{p(\cdot)} \tag{15}$$

holds with a positive constant depend on p, k.

Theorem 9. *Let $p \in P^{Log}$, $k \in \mathbb{N}$, $\sigma > 0$. If $f \in L^{p(\cdot)}$, then*

$$\|D_{\sigma,k}f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$$

holds with a positive constant depend on p, k.

Theorem 10. *Let $p \in P^{Log}$, $k \in \mathbb{N}$, $\sigma > 0$, $f \in L^{p(\cdot)}$, then*

$$\|f - D_{\sigma,k}f\|_{p(\cdot)} \lesssim \Omega_k\left(f, \frac{1}{\sigma}\right)_{p(\cdot)}$$

holds with a positive constant depend on p, k.

Corollary 2. *By Theorem 9, if $r,k \in \mathbb{N}$, $\sigma \in (0,\infty)$, $r \geq \frac{1}{2}(k+2)$, then $D_{\sigma,k}f \in \mathcal{G}_{\sigma,p(\cdot)}$ for $p \in P^{Log}$ and $f \in L^{p(\cdot)}$.*

Theorem 11. *Let $r \in \mathbb{N}$, $p \in P^{Log}$, $\sigma > 0$ and $f \in W_r^{p(\cdot)}$. Then for all $k = 0, 1, \ldots, r$, there exist positive constants depending only on k, r and $p(\cdot)$ such that*

$$\left\|f^{(k)} - (g_\sigma^*)^{(k)}\right\|_{p(\cdot)} \lesssim \frac{1}{\sigma^{r-k}} A_\sigma\left(f^{(r)}\right)_{p(\cdot)}$$

holds for any $g_\sigma^* \in \mathcal{G}_{\sigma,p(\cdot)}$ satisfying $A_\sigma(f)_{p(\cdot)} = \|f - g_\sigma^*\|_{p(\cdot)}$.

Theorem 12. *Let $r, s \in \mathbb{N}$, $p \in P^{Log}$ and $f \in W_r^{p(\cdot)}$. Then there exists a $\Phi \in \mathcal{G}_{2\sigma,p(\cdot)}$ such that for all $k = 0, 1, \ldots, r$ inequalities*

$$\left\|f^{(k)} - \Phi^{(k)}\right\|_{p(\cdot)} \lesssim \frac{1}{\sigma^{r-k}} \Omega_s\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)}$$

are hold with a positive constant depending only on k, r and $p(\cdot)$.

Definition 6. *Set $\sigma, \eta > 0$, $f \in L^1(\mathbb{R})$, $\Theta_\eta f(x,y) := f(x + \eta y)$ and*

$$B_\sigma f(x,t) := \int_{\mathbb{R}} \Theta_{\frac{2}{\sigma}} f(x,y)\, h(y,t)\, dy.$$

Remark 6. *The following theorem was poved in [31] for $\sigma = 2$ with three minor mistypes. For the sake of completeness here we will prove it when $\sigma > 0$.*

Theorem 13. *Suppose that $h(y,t)$, $y, t \in \mathbb{R}$, is positive measurable function with respect to y and*

$$\int_{\mathbb{R}} h(y,t)\, dy \lesssim 1, \quad \int_{\mathbb{R}} \left|y h_y'(y,t)\right| dy \lesssim 1$$

with constants independent of t*. If* $\sigma > 0$ *and* $f \in L^1(\mathbb{R})$*, then*

$$\sup_{t>0} |B_\sigma f(\cdot, t)| \lesssim Mf(\cdot)$$

for $t > 0$ *and a.e. on* $\mathbb{R}$ *where* Mf *is the Hardy-Littlewood maximal function of* f*.*

6. Proof of the Results

Let $C(A)$ be the class of continuous functions defined on A. For $r \in \mathbb{N}$, we define $C^r(A)$ consisting of every member $f \in C(A)$ such that the derivative $f^{(k)}$ exists and is continuous on A for $k = 1, ..., r$. We set $C^\infty(A) := \{f \in C^r(A) \text{ for any } r \in \mathbb{N}\}$. We denote by $C_c(A)$, the collection of real valued continuous functions on A and support of f is compact set in A. We define $C_c^r(A) := C^r(A) \cap C_c(A)$ for $r \in \mathbb{N}$ and $C_c^\infty(A) := C^\infty(A) \cap C_c(A)$. Let $L^p(A)$, $1 \le p \le \infty$ be the classical Lebesgue space of functions on A.

Definition 7. *([17]) Let* $\mathbb{N} := \{1, 2, 3, \cdots\}$ *be natural numbers and* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$*.*

(a) A family Q *of measurable sets* $E \subset \mathbb{R}$ *is called locally* N*-finite (*$N \in \mathbb{N}$*) if*

$$\sum_{E \in Q} \chi_E(x) \le N$$

almost everywhere in $\mathbb{R}$ *where* χ_U *is the characteristic function of the set* U*.*

(b) A family Q *of open bounded sets* $U \subset \mathbb{R}$ *is locally* 1*-finite if and only if the sets* $U \in Q$ *are pairwise disjoint.*

(c) Let $U \subset \mathbb{R}$ *be a measurable set and*

$$A_U f := \frac{1}{|U|} \int_U |f(t)| \, dt.$$

(d) For a family Q *of open sets* $U \subset \mathbb{R}$ *we define averaging operator by*

$$T_Q : L^1_{loc} \to L^0,$$

$$T_Q f(x) := \sum_{U \in Q} \chi_U(x) A_U f, \quad x \in \mathbb{R},$$

where L^0 *is the set of measurable functions on* $\mathbb{R}$*.*

(e) For a measurable set $A \subset \mathbb{R}$*, symbol* $|A|$ *will represent the Lebesgue measure of* A*.*

Theorem 14. *([17]) Suppose that* $p \in P^{Log}$*, and* $f \in L^{p(\cdot)}$*. If* Q *is* 1*-finite family of open bounded subsets of* $\mathbb{R}$ *having Lebesgue measure* 1*, then, the averaging operator* T_Q *is uniformly bounded in* $L^{p(\cdot)}$*, namely,*

$$\|T_Q f\|_{p(\cdot)} \le \mathbf{c}_4 \|f\|_{p(\cdot)}$$

holds with a positive constant $\mathbf{c}_4$ *depending only on* p*.*

We define $\langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) dx$ when integral exists. We will need the following Propositions.

Proposition 1. *([17]) Let $p \in P^{Log}$. Then*

$$\frac{1}{12\mathbf{c}_4} \|f\|_{p(\cdot)} \leq \sup_{g \in L^{p'(\cdot)} \cap C_0^\infty : \|g\|_{p'(\cdot)} \leq 1} \langle |f|, |g| \rangle \leq 2 \|f\|_{p(\cdot)}$$

holds for all $f \in L^{p(\cdot)}$.

Proposition 2. *(a) $C_c(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are dense subsets of $L^p(\mathbb{R})$, $1 \leq p < \infty$.(Theorems 17.10 and 23.59 of [49, p. 415 and p. 575]).*

(b) $C_c(\mathbb{R})$ contained $L^\infty(\mathbb{R})$ but not dense (Remark 17.11 of [49, p.416]) in $L^\infty(\mathbb{R})$.

Theorem 15. *Let $p \in P^{Log}$. In this case,*

(a) if $f \in L^{p(\cdot)}$, then, the function $F_f := F_{f,G}$ defined in (5) is a bounded, uniformly continuous function on $\mathbb{R}$,

(b) if $r \in \mathbb{N}$, and $f \in W_r^{p(\cdot)}$, then, $\frac{d^k}{du^k}(F_f)$ exists and

$$\tfrac{d^k}{du^k}(F_f) = F_{f^{(k)}}$$

for $k \in \{1, ..., r\}$.

Proof. (a) Since C_0^∞ is a dense subset of $L^{p(\cdot)}$, we consider functions $f \in C_0^\infty$ and its corresponding $F_{f,G}$ given in (5). For any $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ so that

$$|f(x+u_1) - f(x+u_2)| < \frac{\varepsilon}{1 + |sptG|}$$

for any $u_1, u_2 \in \mathbb{R}$ with $|u_1 - u_2| < \delta$, where $sptG$ is the support of the function $G \in L^{p'(\cdot)} \cap C_0^\infty$. Then, there holds inequality

$$|F_{f,G}(u_1) - F_{f,G}(u_2)| \leq \int_{\mathbb{R}} |f(x+u_1) - f(x+u_2)| |G(x)| dx$$

$$= \int_{sptG} |f(x+u_1) - f(x+u_2)| |G(x)| dx$$

$$\leq \sup_{x,u_1,u_2 \in sptG} |f(x+u_1) - f(x+u_2)| \|G\|_{1,sptG}$$

$$\leq \frac{\varepsilon}{1 + |sptG|}(1 + |sptG|) \|G\|_{p'(\cdot)} \leq \varepsilon$$

for any $u_1, u_2 \in \mathbb{R}$ with $|u_1 - u_2| < \delta$. Thus conclusion of Theorem 15 follows. For the general case $f \in L^{p(\cdot)}$ there exists an $g \in C_0^\infty$ so that

$$\|f - g\|_{p(\cdot)} < \frac{\xi}{4(1 + |sptG|)\mathbf{c}_0}$$

for any $\xi > 0$. Therefore

$$|F_{f,G}(u_1) - F_{f,G}(u_2)| = |F_{f,G}(u_1) - F_{g,G}(u_1)| + |F_{g,G}(u_1) - F_{g,G}(u_2)| +$$

$$+ |F_{g,G}(u_2) - F_{f,G}(u_2)| = |F_{f-g,G}(u_1)| + \frac{\xi}{2} + |F_{g-f,G}(u_2)|$$

$$\leq 2\left(1+|sptG|\right)\mathbf{c}_0\left\|f-g\right\|_{p(\cdot),\omega}+\frac{\xi}{2}<\xi.$$

As a result $F_{f,G}$ is uniformly continuous on $\mathbb{R}$.

(b) is follow from definitions. □

Proof of Theorem 1. Let $0\leq f,g\in L^{p(\cdot)}$. In this case there exists a constant $C>0$ such that

$$\left\|F_{f,G}\right\|_{C(\mathbb{R})}\leq C\left\|F_{g,G}\left(u\right)\right\|_{C(\mathbb{R})}=C\left\|\int_{\mathbb{R}}g\left(u+x\right)\left|G\left(x\right)\right|dx\right\|_{C(\mathbb{R})}$$

$$=C\sup_{u\in\mathbb{R}}\int_{\mathbb{R}}g\left(u+x\right)\left|G\left(x\right)\right|dx=C\sup_{u\in sptG}\int_{sptG}g\left(u+x\right)\left|G\left(x\right)\right|dx$$

$$\leq C\sup_{u\in sptG}\left\|g\left(u+\cdot\right)\right\|_{1,sptG}\left\|G\right\|_{\infty}\leq C\left(1+|sptG|\right)\mathbf{c}_0\left\|g\right\|_{p(\cdot)}.$$

On the other hand, for any $\varepsilon>0$ and appropriately chosen $\tilde{G}_{\varepsilon}\in L^{p'(\cdot)}$ with

$$\int_{\mathbb{R}}g\left(x\right)\tilde{G}_{\varepsilon}\left(x\right)dx\geq\frac{1}{12\mathbf{c}_4}\left\|g\right\|_{p(\cdot)}-\varepsilon,\qquad\left\|\tilde{G}_{\varepsilon}\right\|_{p'(\cdot)}\leq 1,$$

(see Proposition 1), one can find

$$\left\|F_{f,G}\right\|_{C(\mathbb{R})}\geq\left|F_{f,G}\left(0\right)\right|\geq\int_{\mathbb{R}}f\left(x\right)\left|G\left(x\right)\right|dx>\frac{1}{12\mathbf{c}_4}\left\|f\right\|_{p(\cdot)}-\varepsilon.$$

In the last inequality we take as $\varepsilon\to 0^{+}$ and obtain

$$\left\|F_{f,G}\right\|_{C(\mathbb{R})}>\frac{1}{12\mathbf{c}_4}\left\|f\right\|_{p(\cdot)}.$$

Combining these inequalities we get

$$\left\|f\right\|_{p(\cdot)}<12\mathbf{c}_4\left\|F_{f,G}\right\|_{C(\mathbb{R})}\leq 12\mathbf{c}_4C\left\|F_{g,G}\right\|_{C(\mathbb{R})}$$

$$\leq 12\mathbf{c}_4C\left(1+|sptG|\right)\mathbf{c}_0\left\|g\right\|_{p(\cdot)}.$$

For general case $f,g\in L^{p(\cdot)}$ we obtain

$$\left\|f\right\|_{p(\cdot)}<24\mathbf{c}_4\left(1+|sptG|\right)\mathbf{c}_0C\left\|g\right\|_{p(\cdot)}\tag{16}$$

and proof is finished. □

Remark 7. *Note that, in (16) constant depend on $|sptG|$ and $\|G\|_{\infty}$ but it is possible to avoid dependence on $|sptG|$ and $\|G\|_{\infty}$. To do so, we can change the definition of F_f with*

$$F_f\left(u\right):=\int_{\mathbb{R}}\mathcal{S}_{1,u}f(x)\left|G\left(x\right)\right|dx,\quad u\in\mathbb{R},$$

where $G\in L^{p'(\cdot)}\cap C_0^{\infty}$ and $\|G\|_{p'(\cdot)}\leq 1$. Now, boundedness of $\mathcal{S}_{1,u}f$ in $L^{p(\cdot)}$ for any $u\in\mathbb{R}$, and the same procedure give (16) with a constant does not depend on $|sptG|$ and $\|G\|_{\infty}$. Hence, constants in other results can be free of dependence on $|sptG|$ and $\|G\|_{\infty}$.

Proof of Lemma 1. Let $0 < h \le \delta < \infty$, $p \in P^{Log}$ and $f \in L^{p(\cdot)}$. Then, using (16) we get

$$\|(I - T_h) f\|_{p(\cdot)} < 24\mathbf{c}_4 \left\|F_{(I-T_h)f,G}\right\|_{C(\mathbb{R})} \le 24 \cdot 72\mathbf{c}_4 \left\|F_{(I-T_\delta)f,G}\right\|_{C(\mathbb{R})}$$

$$\le 1728\mathbf{c}_4 (1 + |sptG|)\, \mathbf{c}_0 \|(I - T_\delta) f\|_{p(\cdot)} .$$

□

Proof of Lemma 2. If $f \in L^{p(\cdot)}$, then, using generalized Minkowski's integral inequality and Lemma 1 we obtain

$$\|(I - \mathfrak{R}_\delta) f\|_{p(\cdot)} = \left\| \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(\frac{1}{h} \int_0^h (f(x+t) - f(x))\, dt \right) dh \right\|_{p(\cdot)}$$

$$= \left\| \frac{2}{\delta} \int_{\delta/2}^{\delta} (T_h f(x) - f(x))\, dh \right\|_{p(\cdot)} \le \frac{2}{\delta} \int_{\delta/2}^{\delta} \|T_\delta f - f\|_{p(\cdot)}\, dh$$

$$\le 1728\mathbf{c}_4 (1 + |sptG|)\, \mathbf{c}_0 \|T_\delta f - f\|_{p(\cdot)} \frac{2}{\delta} \int_{\delta/2}^{\delta} dh$$

$$= 1728\mathbf{c}_4 (1 + |sptG|)\, \mathbf{c}_0 \|(I - T_\delta) f\|_{p(\cdot)} .$$

□

Proof of Lemma 4. Using

$$\left\|F_{\delta(\mathfrak{R}_\delta f)',G}\right\|_{C(\mathbb{R})} = \left\|\delta \left(F_{(\mathfrak{R}_\delta f),G}\right)'\right\|_{C(\mathbb{R})} = \delta \left\|(\mathfrak{R}_\delta(F_{f,G}))'\right\|_{C(\mathbb{R})}$$

$$\le \cdots \le 2\left(37 + 146 \ln 2^{36}\right) \|(I - T_\delta)(F_{f,G})\|_{C(\mathbb{R})}$$

$$= 2\left(37 + 146 \ln 2^{36}\right) \left\|(F_{(I-T_\delta)f,G})\right\|_{C(\mathbb{R})}$$

we conclude from Transference Result that

$$\delta \left\|(\mathfrak{R}_\delta f)'\right\|_{p(\cdot)} \le \mathbf{c}_5 \|(I - T_\delta) f\|_{p(\cdot)} .$$

with $\mathbf{c}_5 := 24\mathbf{c}_4 (1 + |sptG|)\, \mathbf{c}_0 \left(37 + 146 \ln 2^{36}\right)$. □

Proof of Theorem 4. For $r = 1, 2, 3, \ldots$ we consider the operator

$$\mathcal{A}_\delta^r := I - (I - \mathfrak{R}_\delta^r)^r = \sum\nolimits_{j=0}^{r-1} (-1)^{r-j+1} \binom{r}{j} \mathfrak{R}_\delta^{r(r-j)}.$$

From the identity $I - \mathfrak{R}_\delta^r = (I - \mathfrak{R}_\delta) \sum_{j=0}^{r-1} \mathfrak{R}_\delta^j$ we find

$$\|(I - \mathfrak{R}_\delta^r) g\|_{p(\cdot)} \le \left(\sum_{j=0}^{r-1} \mathbf{c}_6^j \right) \|(I - \mathfrak{R}_\delta) g\|_{p(\cdot)}$$

with $\mathbf{c}_6 := 24\mathbf{c}_4 \left(1 + |sptG|\right)\mathbf{c}_0$. Therefore

$$\left\|\left(I - \mathfrak{R}_\delta^r\right) g\right\|_{p(\cdot)} \leq \left(1728\mathbf{c}_4 \left(1 + |sptG|\right)\mathbf{c}_0 \sum_{j=0}^{r-1} \mathbf{c}_6^j\right) \left\|\left(I - T_\delta\right) g\right\|_{p(\cdot)} \tag{17}$$
$$= \mathbf{c}_7 \left\|\left(I - T_\delta\right) g\right\|_{p(\cdot)}$$

when $0 < \delta < \infty$, $p \in P$ and $g \in L^{p(\cdot)}$. Since $\left\|f - \mathcal{A}_\delta^r f\right\|_{p(\cdot)} = \left\|\left(I - \mathfrak{R}_\delta^r\right)^r f\right\|_{p(\cdot)}$, recursive procedure gives

$$\left\|f - \mathcal{A}_\delta^r f\right\|_{p(\cdot)} = \left\|\left(I - \mathfrak{R}_\delta^r\right)^r f\right\|_{p(\cdot)} \leq \cdots \leq \mathbf{c}_7^r \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)}.$$

On the other hand, using Lemmas 5 and 4,

$$\delta^r \left\|\frac{d^r}{dx^r}\mathfrak{R}_\delta^r f\right\|_{p(\cdot)} = \delta^{r-1}\delta \left\|\frac{d}{dx}\mathfrak{R}_\delta \frac{d^{r-1}}{dx^{r-1}}\mathfrak{R}_\delta^{r-1} f\right\|_{p(\cdot)}$$
$$\leq \mathbf{c}_5 \delta^{r-1} \left\|\left(I - T_\delta\right) \frac{d^{r-1}}{dx^{r-1}}\mathfrak{R}_\delta^{r-1} f\right\|_{p(\cdot)} \leq \cdots \leq$$
$$\leq \mathbf{c}_5^{r-1}\delta \left\|\frac{d}{dx}\mathfrak{R}_\delta \left(I - T_\delta\right)^{r-1} f\right\|_{p(\cdot)} \leq \mathbf{c}_5^r \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)}.$$

Thus

$$K_r\left(f, \delta; L^{p(\cdot)}, W_r^{p(\cdot)}\right)_{p(\cdot)} \leq \left\|f - \mathcal{A}_\delta^r f\right\|_{p(\cdot)} + \delta^r \left\|\frac{d^r}{dx^r}\mathcal{A}_\delta^r f(x)\right\|_{p(\cdot)}$$
$$\leq \mathbf{c}_7^r \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)} + \sum_{j=0}^{r-1} \left|\binom{r}{j}\right| \delta^r \left\|\frac{d^r}{dx^r}\mathfrak{R}_\delta^{r(r-j)} f(x)\right\|_{p(\cdot)}$$
$$\leq \mathbf{c}_7^r \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)} + \mathbf{c}_5^r \sum_{j=0}^{r-1} \left|\binom{r}{j}\right| \left\|\left(I - T_\delta\right)^r \mathfrak{R}_\delta^{(r-j)} f\right\|_{p(\cdot)}$$
$$\leq \mathbf{c}_7^r \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)} + \mathbf{c}_5^r \sum_{j=0}^{r-1} \left|\binom{r}{j}\right| \mathbf{c}_6^{r-j} \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)}$$
$$\leq \mathbf{c}_8 \left\|\left(I - T_\delta\right)^r f\right\|_{p(\cdot)}$$

where

$$\mathbf{c}_8 := \max\left\{\mathbf{c}_7^r, \mathbf{c}_5^r \sum_{j=0}^{r-1} \left|\binom{r}{j}\right| \mathbf{c}_6^{r-j}\right\}.$$

For the reverse of the last inequality, when $g \in W_r^{p(\cdot)}$, we get

$$\Omega_r(f, \delta)_{p(\cdot)} \leq \left(1 + \mathbf{c}_6\right)^r \left\|f - g\right\|_{p(\cdot)} + \Omega_r(g, \delta)_{p(\cdot)}$$
$$\leq \left(1 + \mathbf{c}_6\right)^r \left\|f - g\right\|_{p(\cdot)} + 2^{-r}\mathbf{c}_6^r \delta^r \left\|g^{(r)}\right\|_{p(\cdot)}, \tag{18}$$

and taking infimum on $g \in W^r_{p(\cdot)}$ in (18) we obtain

$$\Omega_r(f,\delta)_{p(\cdot)} \le (1+\mathbf{c}_6)^r K_r\left(f,\delta;L^{p(\cdot)},W_r^{p(\cdot)}\right)_{p(\cdot)}.$$

□

Proof of Theorem 5. The following inequality

$$A_\sigma(f)_{C(\mathbb{R})} \le \left\| f - J\left(f,\frac{\sigma}{2}\right)\right\|_{C(\mathbb{R})} \le \frac{5\pi}{4}\frac{4^r}{\sigma^r}\|f^{(r)}\|_{C(\mathbb{R})}, \quad \forall f \in C^r(\mathbb{R})$$

known (see (vi) of Remark 4). Now using TR we find

$$\left\| f - J\left(f,\frac{\sigma}{2}\right)\right\|_{p(\cdot)} \le \frac{5\pi}{2}\frac{4^r\mathbf{c}_6}{\sigma^r}\|f^{(r)}\|_{p(\cdot)}, \quad \forall f \in W_r^{p(\cdot)}. \tag{19}$$

Let $r=1$. Suppose that

$$A_\sigma(f')_{p(\cdot)} = \|f' - g_\sigma^*(f')\|_{p(\cdot)}, \quad g_\sigma^*(f') \in \mathcal{G}_{\sigma,p(\cdot)}$$

and

$$\digamma(x) := \int_0^x g_\sigma^*(f')(t)\,dt.$$

Then $\digamma \in \mathcal{G}_\sigma$ ([26, p.397]). Setting

$$\varphi(x) = f(x) - \digamma(x)$$

one has

$$\|\varphi'\|_{p(\cdot)} = \|f' - g_\sigma^*(f')\|_{p(\cdot)} = A_\sigma(f')_{p(\cdot)}.$$

Thus

$$A_\sigma(f)_{p(\cdot)} = A_\sigma(f-\digamma)_{p(\cdot)} \overset{(19)}{\le} 10\pi\mathbf{c}_6\frac{1}{\sigma}\left\|(f-\digamma)'\right\|_{p(\cdot)}$$

$$= \frac{10\pi\mathbf{c}_6}{\sigma}\|f'-\digamma'\|_{p(\cdot)} = \frac{10\pi\mathbf{c}_6}{\sigma}\|f'-g_\sigma^*(f')\|_{p(\cdot)}$$

$$= 10\pi\mathbf{c}_6\frac{1}{\sigma}A_\sigma(f')_{p(\cdot)}.$$

Now, result follows from the last inequality:

$$A_\sigma(f)_{p(\cdot)} \le 10\pi\mathbf{c}_6\frac{1}{\sigma}A_\sigma(f')_{p(\cdot)} \le \cdots \le (10\pi\mathbf{c}_6)^r\frac{1}{\sigma^r}A_\sigma\left(f^{(r)}\right)_{p(\cdot)}.$$

□

Proof of Theorem 6. Let $p \in P^{Log}$, $\sigma > 0$, $k \in \mathbb{N}$, $r \in \{0\} \cup \mathbb{N}$ and $f \in W_r^{p(\cdot)}$. First we consider the case $r=0$. For every $g \in W_k^{p(\cdot)}$ we find

$$A_\sigma(f)_{p(\cdot)} \le A_\sigma(f-g)_{p(\cdot)} + A_\sigma(g)_{p(\cdot)}$$

$$\le \|f-g\|_{p(\cdot)} + \frac{5\pi}{2}\frac{4^k\mathbf{c}_6}{\sigma^k}\|f^{(k)}\|_{p(\cdot)}.$$

Taking infimum on g in the last inequality

$$A_{\sigma}(f)_{p(\cdot)} \leq \frac{5\pi}{2} 4^{k} \mathbf{c}_6 K_k \left(f, \delta; L^{p(\cdot)}, W_k^{p(\cdot)}\right)_{p(\cdot)}.$$

Now using (11)

$$A_{\sigma}(f)_{p(\cdot)} \leq \mathbf{c}_8 \frac{5\pi}{2} 4^{k} \mathbf{c}_6 \Omega_k \left(f, \frac{1}{\sigma}\right)_{p(\cdot)}.$$

In the second stage we consider the case $r \in \mathbb{N}$. In this case

$$A_{\sigma}(f)_{p(\cdot)} \leq (10\pi \mathbf{c}_6)^r \frac{1}{\sigma^r} A_{\sigma}\left(f^{(r)}\right)_{p(\cdot)}$$

$$\leq 5\pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2k-1} \frac{1}{\sigma^r} \Omega_k \left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)}.$$

□

Proof of Theorem 7. Let $p \in P^{Log}$, $\sigma > 0$ and $g_{\sigma} \in \mathcal{G}_{\sigma,p(\cdot)}$. Then, Bernstein's inequality

$$\| (g_{\sigma})^{(r)} \|_{C(\mathbb{R})} \leq \sigma^r \|g_{\sigma}\|_{C(\mathbb{R})}, \quad \forall g_{\sigma} \in \mathcal{G}_{\sigma,\infty}$$

and TR gives

$$\| (g_{\sigma})^{(r)} \|_{p(\cdot)} \leq \mathbf{c}_6 \sigma^r \|g_{\sigma}\|_{p(\cdot)}, \quad \forall g_{\sigma} \in \mathcal{G}_{\sigma,p(\cdot)}.$$

□

Proof of Theorem 8. Define for $k \in \mathbb{N}$ the classical modulus of smoothness of function $f \in C(\mathbb{R})$ of step $\delta > 0$ by

$$\omega_k (f, \delta)_{C(\mathbb{R})} := \sup_{|h| \leq \delta} \left\| \Delta_t^k f \right\|_{C(\mathbb{R})}$$

where $\Delta_t^k f(\cdot) := \left(I - \tilde{T}_h\right)^k f(\cdot)$, $\tilde{T}_h f(\cdot) := f(\cdot + h)$ and I is the identity operator. From (14), one can write

$$\|f - D_{\sigma,k} f\|_{C(\mathbb{R})} = \left\| \frac{(-1)^k}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=0}^{k} (-1)^{k-v} \binom{k}{v} f(x + vt)\, g(\sigma, r, t)\, dt \right\|_{C(\mathbb{R})}$$

$$\leq \frac{1}{\sigma^{2r-1} \frac{b_r}{(2r)^{2r-1}}} \int_{\mathbb{R}} \left\| \Delta_t^k f(x) \right\|_{C(\mathbb{R})} g(\sigma, r, t)\, dt \leq \frac{(2r)^{2r-1}}{b_r \sigma^{2r-1}} \int_{\mathbb{R}} \omega_k (f, t)_{C(\mathbb{R})}\, g(\sigma, r, t)\, dt$$

$$\leq \frac{(2r)^{2r-1} \sigma^k}{b_r \sigma^{2r-1}} \omega_k \left(f, \frac{1}{\sigma}\right)_{C(\mathbb{R})} \int_{\mathbb{R}} \left(t + \frac{1}{\sigma}\right)^k g(\sigma, r, t)\, dt$$

$$\leq \frac{(2r)^{2r-1} \sigma^k}{b_r \sigma^{2r-1}} \frac{1}{\sigma^k} \left\| f^{(k)} \right\|_{C(\mathbb{R})} \int_{\mathbb{R}} \left(t + \frac{1}{\sigma}\right)^k g(\sigma, r, t)\, dt$$

$$\leq \frac{(2r)^{2r-1}}{b_r\sigma^{2r-1}} \left\| f^{(k)} \right\|_{C(\mathbb{R})} \left\{ \frac{2^k}{\sigma^k} \int\limits_{|t|\leq\frac{1}{\sigma}} |g(\sigma,r,t)|\,dt + 2^k \int\limits_{|t|\geq\frac{1}{\sigma}} |t|^k\,|g(\sigma,r,t)|\,dt \right\}.$$

Using $r = \lceil \frac{1}{2}(k+2) \rceil$

$$\begin{aligned} &\frac{(2r)^{2r-1}2^k}{b_r\sigma^{2r-1}} \int_{|t|\geq 1/\sigma} |t|^k \left(\frac{1}{t}\sin\frac{\sigma t}{2r}\right)^{2r} dt \\ &\leq \frac{(2r)^{2r-1}2^k}{b_r\sigma^{2r-1}} \int_{|t|\geq 1/\sigma} \left(\frac{1}{t}\sin\frac{\sigma t}{2r}\right)^{2r-k} dt \\ &\leq \frac{(2r)^{2r-1}2^k}{b_r\sigma^{2r-1}} \frac{\sigma^{2r-k+1}}{(2r)^{2r-k+1}} \int_{\mathbb{R}} \left(\frac{\sin u}{u}\right)^2 dt = \frac{1}{\sigma^k}\frac{2^{2k}r^k}{b_r}\pi. \end{aligned}$$

On the other hand

$$\begin{aligned} &\frac{(2r)^{2r-1}}{b_r\sigma^{2r-1}}\frac{2^k}{\sigma^k} \int_{|t|\leq 1/\sigma} \left(\frac{1}{t}\sin\frac{\sigma t}{2r}\right)^{2r} dt \\ &\leq \frac{(2r)^{2r-1}}{b_r\sigma^{2r-1}}\frac{2^k}{\sigma^k} \int_{\mathbb{R}} \left(\frac{1}{t}\sin\frac{\sigma t}{2r}\right)^{2r} dt \\ &= \frac{(2r)^{2r-1}}{b_r^r\sigma^{2r-1}}\sigma^{2r-1}\frac{b_r}{(2r)^{2r-1}} = \frac{2^k}{\sigma^k}. \end{aligned}$$

Thus

$$\|f - D_{\sigma,k}f\|_{C(\mathbb{R})} \leq \left(\frac{2^{2k}r^k}{b_r} + 2^k\right)\frac{1}{\sigma^k}\left\|f^{(k)}\right\|_{C(\mathbb{R})}.$$

From this and TR we get

$$\|f - D_{\sigma,k}f\|_{p(\cdot)} \leq \mathbf{c}_6\left(\frac{2^{2k}r^k}{b_r} + 2^k\right)\frac{1}{\sigma^k}\left\|f^{(k)}\right\|_{p(\cdot)} = \mathbf{c}_6\mathbf{c}(k,r)\frac{1}{\sigma^k}\left\|f^{(k)}\right\|_{p(\cdot)}.$$

□

Proof of Theorem 9. Fixed $\sigma > 0$, we find

$$\|D_{\sigma,k}f\|_{C(\mathbb{R})} = \left\| \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} f(u)G(\sigma,r,k,u-x)\,du \right\|_{C(\mathbb{R})}$$

$$\left\| \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=1}^{k} (-1)^{k-v} \binom{k}{v} f(u)\,g\left(\sigma,r,\frac{u-x}{v}\right) du \right\|_{C(\mathbb{R})}$$

$$\leq \left\| \frac{(-1)^{k+1}}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=1}^{k} (-1)^{k-v} \binom{k}{v} f(x+vt)\,g(\sigma,r,t)\,vdt \right\|_{C(\mathbb{R})}$$

$$\leq \frac{k}{\gamma_{r,\sigma}} \int_{\mathbb{R}} \sum_{v=1}^{k} \left|\binom{k}{v}\right| \|f(x+vt)\|_{C(\mathbb{R})}\,g(\sigma,r,t)\,dt$$

$$\leq \|f\|_{C(\mathbb{R})} \sum_{v=1}^{k} \left| \binom{k}{v} \right| \frac{k}{\gamma_{r,\sigma}} \int_{\mathbb{R}} g(\sigma, r, t)\, dt \leq k2^k \|f\|_{C(\mathbb{R})}.$$

Now, transference result TR gives

$$\|D_{\sigma,k} f\|_{p(\cdot)} \leq k2^k \mathbf{c}_6 \|f\|_{p(\cdot)}.$$

□

Proof of Theorem 10. We can write

$$\|f - D_{\sigma,k} f\|_{p(\cdot)} = \left\| f - \mathcal{A}^k_{\frac{1}{\sigma}} f + \mathcal{A}^k_{\frac{1}{\sigma}} f - D_{\sigma,k} \mathcal{A}^k_{\frac{1}{\sigma}} f + D_{\sigma,k} \mathcal{A}^k_{\frac{1}{\sigma}} f - D_{\sigma,k} f \right\|_{p(\cdot)}$$

$$\leq \left\| f - \mathcal{A}^k_{\frac{1}{\sigma}} f \right\|_{p(\cdot)} + \left\| \mathcal{A}^k_{\frac{1}{\sigma}} f - D_{\sigma,k} \mathcal{A}^k_{\frac{1}{\sigma}} f \right\|_{p(\cdot)} + \left\| D_{\sigma,k} (\mathcal{A}^k_{\frac{1}{\sigma}} f - f) \right\|_{p(\cdot)}$$

$$\leq \mathbf{c}_7^k \Omega_k \left(f, \frac{1}{\sigma} \right)_{p(\cdot)} + \mathbf{c}_6 \mathbf{c}(k) \frac{1}{\sigma^k} \left\| (\mathcal{A}^k_{\frac{1}{\sigma}} f)^{(k)} \right\|_{p(\cdot)} + k2^k \mathbf{c}_6 \left\| \mathcal{A}^k_{\frac{1}{\sigma}} f - f \right\|_{p(\cdot)}$$

$$\leq \left(\mathbf{c}_7^k + \mathbf{c}_6 \mathbf{c}(k, r) \mathbf{c}_5^k \sum_{j=0}^{k-1} \left| \binom{k}{j} \right| \mathbf{c}_6^{k-j} + 2^k k \mathbf{c}_6 \mathbf{c}_7^k \right) \Omega_k \left(f, \frac{1}{\sigma} \right)_{p(\cdot)}$$

$$= \mathbf{c}_9 \Omega_k \left(f, \frac{1}{\sigma} \right)_{p(\cdot)}$$

and the result follows. □

Proof of Theorem 11. Let $q \in \mathcal{G}_\sigma$ and $A_\sigma \left(f^{(k)} \right)_{p(\cdot)} = \left\| f^{(k)} - q \right\|_{p(\cdot)}$. Then

$$\left\| f^{(k)} - (g^*_\sigma)^{(k)} \right\|_{p(\cdot)} \leq \left\| f^{(k)} - (J(f, \sigma))^{(k)} \right\|_{p(\cdot)} + \left\| (J(f, \sigma))^{(k)} - (g^*_\sigma)^{(k)} \right\|_{p(\cdot)}$$

$$\leq \left\| f^{(k)} - q \right\|_{p(\cdot)} + \left\| q - J\left(f^{(k)}, \sigma \right) \right\|_{p(\cdot)} + \left\| (J(f, \sigma) - g^*_\sigma)^{(k)} \right\|_{p(\cdot)}$$

$$\leq A_\sigma \left(f^{(k)} \right)_{p(\cdot)} + \left\| J\left(q - f^{(k)}, \sigma \right) \right\|_{p(\cdot)} + 2^k \mathbf{c}_6 \sigma^k \|J(f, \sigma) - g^*_\sigma\|_{p(\cdot)}$$

$$\leq (1 + 3\mathbf{c}_6) A_\sigma \left(f^{(k)} \right)_{p(\cdot)} + 2^k \mathbf{c}_6 \sigma^k \|J(f, \sigma) - J(g^*_\sigma, \sigma)\|_{p(\cdot)}$$

$$\leq (1 + 3\mathbf{c}_6) \frac{2\mathbf{c}_6 \left(5\pi 4^{r-1} \right)^r}{\sigma^{r-k}} A_\sigma \left(f^{(r)} \right)_{p(\cdot)} + 3\mathbf{c}_6^2 2^k \frac{2\mathbf{c}_6 \left(5\pi 4^{r-1} \right)^r}{\sigma^{r-k}} A_\sigma \left(f^{(r)} \right)_{p(\cdot)}$$

$$\leq \left(2\mathbf{c}_6 \left(5\pi 4^{r-1} \right)^r \right) \left(1 + 3\mathbf{c}_6 + 3\mathbf{c}_6^2 2^k \right) \frac{\sigma^k}{\sigma^r} A_\sigma \left(f^{(r)} \right)_{p(\cdot)} = \mathbf{c}_{10} \sigma^{k-r} A_\sigma \left(f^{(r)} \right)_{p(\cdot)}$$

and the proof of Theorem 11 is completed. □

Proof of Theorem 12. Let $g_\sigma^* \in \mathcal{G}_\sigma$, $A_\sigma(f)_{p(\cdot)} = \|f - g_\sigma^*\|_{p(\cdot)}$ and $\Phi = J(f,\sigma)$. Then

$$\begin{aligned}\|f - J(f,\sigma)\|_{p(\cdot)} &\le \|f - g_\sigma^* + g_\sigma^* - J(f,\sigma)\|_{p(\cdot)} \\ &= \|f - g_\sigma^* + J(g_\sigma^*,\sigma) - J(f,\sigma)\|_{p(\cdot)} \\ &\le A_\sigma(f)_{p(\cdot)} + 3\mathbf{c}_6 \|f - g_\sigma^*\|_{p(\cdot)} = (1+3\mathbf{c}_6) A_\sigma(f)_{p(\cdot)}\end{aligned}$$

and

$$\begin{aligned}\|f - J(f,\sigma)\|_{p(\cdot)} &\le (1+3\mathbf{c}_6) A_\sigma(f)_{p(\cdot)} \\ &\le (1+3\mathbf{c}_6)\, 5\pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2s-1} \frac{1}{\sigma^r} \Omega_s\left(f^{(r)}, 1/\sigma\right)_{p(\cdot)}.\end{aligned}$$

Now, from

$$\|f - g_\sigma^*\|_{p(\cdot)} \le \frac{\pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2s-1}}{\sigma^r} \Omega_s\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)}$$

we obtain

$$\|J(f,\sigma) - g_\sigma^*\|_{p(\cdot)} \le \frac{\mathbf{c}_{11}}{\sigma^r} \Omega_s\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)}$$

with

$$\mathbf{c}_{11} = \pi \mathbf{c}_8 (10)^r \pi^r \mathbf{c}_6^{r+1} 2^{2s-1} \left((1+3\mathbf{c}_6)\, 5 + 1\right).$$

Hence

$$\begin{aligned}\left\|f^{(k)} - (J(f,\sigma))^{(k)}\right\|_{p(\cdot)} &\le \left\|f^{(k)} - (g_\sigma^*)^{(k)}\right\|_{p(\cdot)} + \left\|(J(f,\sigma))^{(k)} - (g_\sigma^*)^{(k)}\right\|_{p(\cdot)} \\ &\le \mathbf{c}_{10} \sigma^{k-r} A_\sigma\left(f^{(r)}\right)_{p(\cdot)} + 2^k \mathbf{c}_6 \sigma^k \frac{\mathbf{c}_{11}}{\sigma^r} \Omega_s\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\cdot)} \\ &\le \left(\mathbf{c}_{10} \frac{5\pi \mathbf{c}_8}{2} 4^s \mathbf{c}_6 + 2^k \mathbf{c}_6 \mathbf{c}_{11}\right) \sigma^{k-r} \Omega_s\left(f^{(r)}, 1/\sigma\right)_{p(\cdot)}\end{aligned}$$

and the proof is completed. □

Proof of Theorem 13. Given $x \in \mathbb{R}$, let

$$\Gamma(y) := \int_0^y \Theta_{\frac{2}{\sigma}} f(x,u)\, du, \quad y > 0,$$

and $a, b > 0$. Integration by parts gives

$$\begin{aligned}\int_{-a}^{b} \Theta_{\frac{2}{\sigma}} f(x,y)\, h(y,t)\, dy &= \int_{-a}^{b} h(y,t)\, d\Gamma(y) \\ &= \Gamma(y) h(y,t) \,|_{-a}^{b} - \int_{-a}^{b} h_y'(y,t)\, \Gamma(y)\, dy.\end{aligned}$$

Since $\Gamma(y) \le |y| Mf(x)$ we obtain

$$\left|\int_{-a}^{b} \Theta_{\frac{2}{\sigma}} f(x,y)\, h(y,t)\, dy\right| \le Mf(x) \left(\int_{-a}^{b} \left|y h_y'(y,t)\right| dy + h(y,t) \,|_{-a}^{b}\right).$$

Now

$$\mathbf{c}_{12} \geq \int_{\mathbb{R}} h(y,t)\,dy \geq \int_{-a}^{b} h(y,t)\,dy = h(y,t)\mid_{-a}^{b} - \int_{-a}^{b} y h_y'(y,t)\,dy$$

gives

$$\left| \int_{-a}^{b} \Theta_{\frac{2}{\sigma}} f(x,y)\,h(y,t)\,dy \right| \leq (\mathbf{c}_{12} + 2\mathbf{c}_{13})\,Mf(x)$$

for any $t > 0$. The last inequality implies the result. □

Declaration of Competing Interests The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. Author was supported by Balikesir University Scientific Research Project 2019/61.

Acknowledgement The author is grateful to the referee's who provided helpful comments on the original version of this paper.

References

[1] Abdullaev, F., Chaichenko, S., Imashqızı, M., Shidlich, A., Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent, *Turk. J. Math.*, 44(1) (2020), 284-299. https://doi.org/10.3906/mat-1911-3

[2] Ackhiezer, N. I., Theory of Approximation, Fizmatlit, Moscow, 1965, English translation of 2nd edition, Frederick Ungar, New York, 1956.

[3] Akgün, R., Approximation of functions of weighted Lebesgue and Smirnov spaces, *Mathematica (Cluj)*, 54(77)(Special) (2012), 25-36.

[4] Akgün, R., Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces, *Proc. A. Razmadze Math. Inst.*, 152 (2010), 1-18.

[5] Akgün, R., Inequalities for one sided approximation in Orlicz spaces, *Hacet. J. Math. Stat.*, 40(2) (2011), 231-240.

[6] Akgün, R., Some convolution inequalities in Musielak Orlicz spaces, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 42(2) (2016), 279-291.

[7] Akgün, R., Ghorbanalizadeh, A., Approximation by integral functions of finite degree in variable exponent Lebesgue spaces on the real axis, *Turk. J. Math.*, 42(4) (2018), 1887-1903. https://doi.org/10.3906/mat-1605-26

[8] Avşar, A. H., Koç, H., Jackson and Stechkin type inequalities of trigonometric approximation in $A^{w,\theta}_{p,q(.)}$, *Turk. J. Math.*, 42(6) (2018), 2979-2993. https://doi.org/10.3906/mat-1712-84

[9] Avşar, A. H., Yildirir, Y. E., On the trigonometric approximation of functions in weighted Lorentz spaces using Cesaro submethod, *Novi Sad J. Math.*, 48(2) (2018), 41-54. https://doi.org/10.30755/NSJOM.06335

[10] Bardaro, C., Butzer, P. L., Stens, R. L., Vinti, G., Approximation error of the Whittaker cardinal series in terms of an averaged modulus of smoothness covering discontinuous signals, *J. Math. Anal. Appl.,* 316(1) (2006), 269-306. https://doi.org/10.1016/j.jmaa.2005.04.042

[11] Bernstein, S. N., Sur la meilleure approximation sur tout l'axe reel des fonctions continues par des fonctions entieres de degre n I, *Dokl. Acad. Sci. URSS (N.S.)*, 51 (1946), 331-334.

[12] Bernstein, S. N., Collected Works, Mir, Vol. I, Izdat. Akad. Nauk SSSR, Moscow, 1952, 11-104.

[13] Butler, R., On the evaluation of $\int_0^\infty \frac{\sin^m(t)}{t^m}dt$ by the trapezoidal rule, *Amer. Math. Monthly*, 67(6) (1960), 566-569.

[14] Cruz-Uribe, D., Fiorenza, A., Approximate identities in variable L_p spaces, *Math. Nachr.*, 280(3) (2007), 256-270.

[15] Cruz-Uribe, D., Fiorenza, A., Variable Lebesgue Spaces, Foundations and Harmonic Analysis, Birkhauser, Applied and Numerical Harmonic Analysis, 2013.

[16] Diening, L., Maximal function on generalized Lebesgue spaces $L^{p(x)}$, *Math. Ineq. & Appl.*, 7(2) (2004); 245-253. https://doi.org/10.7153/mia-07-27

[17] Diening, L., Harjulehto, P., Hästö, P., Růžička, M., Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2011.

[18] Diening, L., Růžička, M., Calderon-Zymund Operators on Generalized Lebesgue Spaces $L^{P(X)}$ and Problems Related to Fluid Dynamics, Preprint, Mathematische Fakültat, Albert-Ludwings-Universität Freiburg, 2002, 1-20.

[19] Dogu, A., Avsar, A. H., Yildirir, Y. E., Some inequalities about convolution and trigonometric approximation in weighted Orlicz spaces, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, 44 (1) (2018), 107-115.

[20] Fan, X., Zhao, D., On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, 263(2) (2001), 424-446. doi:10.1006/jmaa.2000.7617

[21] Guven, A., Israfilov, D. M., Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$*, J. Math. Ineq.* 4 (2) (2010), 285-299.

[22] Jafarov, S. Z., Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents, *Ukr. Math. J.*, 66(10) (2015), 1509-1518. https://doi.org/10.1007/s11253-015-1027-y

[23] Jafarov, S. Z., Approximation by trigonometric polynomials in subspace of variable exponent grand Lebesgue spaces, *Global J. Math. Sci.*, 8(2) (2016), 836-843.

[24] Jafarov, S. Z., Ul'yanov type inequalities for moduli of smoothness, *Appl. Math. E-Notes*, 12 (2012), 221-227.

[25] Jafarov, S. Z., S. M. Nikolskii type inequality and estimation between the best approximations of a function in norms of different spaces, *Math. Balkanica*, 21(1-2) (2007), 173-182.

[26] Ibragimov, I. I., Teoriya Priblizheniya Tselymi Funktsiyami(In Russian), The Theory of Approximation by Entire Functions, Elm, Baku, 1979, 468 pp.

[27] Israfilov, D. M., Testici, A., Approximation problems in the Lebesgue spaces with variable exponent, *J. Math. Anal. Appl.*, 459(1) (2018), 112-123. https://doi.org/10.1016/j.jmaa.2017.10.067

[28] Israfilov, D.M., Testici, A., Approximation by Faber-Laurent rational functions in Lebesgue spaces with variable exponent, *Indag. Math.*, 27(4) (2016), 914-922. https://doi.org/10.1016/j.indag.2016.06.001

[29] Israfilov, D. M., Yirtici, E., Convolutions and best approximations in variable exponent Lebesgue spaces, *Math. Reports,* 18(4) (2016), 497-508.

[30] Koc, H., Simultaneous approximation by polynomials in Orlicz spaces generated by quasi-convex Young functions, *Kuwait J. Sci.*, 43(4) (2016), 18-31.

[31] Kokilashvili, V., Nanobashvili, I., Boundedness criteria for the majorants of Fourier integrals summation means in weighted variable exponent Lebesgue spaces and application, *Georgian Math. J.*, 20(4) (2013), 721-727. https://doi.org/10.1515/gmj-2013-0038

[32] Kováčik, Z. O., Rákosnik, J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czech. Math. J.*, 41(116)(4) (1991), 592-618.

[33] Musielak, J., Orlicz Spaces and Modular Spaces, Lecture Notes in Mathematics, Springer, 1983.

[34] Nikolski, S. M., Inequalities for entire functions of finite degree and their application to the theory of differentiable functions of several variables, AMS Translation Series Two, 80 (1969), 1-38 *Trud. Steklov Math. Inst.*, 38 (1951), 211-278.

[35] Orlicz, W., Über konjugierte exponentenfolgen, *Studia Math.* 3 (1931), 200-212.

[36] Paley, R., Wiener, N., Fourier Transforms in the Complex Domain, American Mathematical Society, 1934.

[37] Rajagopal, K. R., Růžička, M., On the modeling elektroreological materials, *Mech. Res. Comm.*, 23(4) (1996), 401-407. https://doi.org/10.1016/0093-6413(96)00038-9

[38] Růžička, M., Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.

[39] Samko, S., Differentiation and Integration of Variable Order and the Spaces $L^{p(x)}$, In: Operator Theory for Complex and Hypercomplex Analysis (Mexico City, 1994), 203-219, Contemporary Math. 212, American Mathematical Society, Providence, RI, 1998.

[40] Sharapudinov, I. I., Some Questions in the Theory of Approximation in Lebesgue Spaces with Variable Exponent, (In Russian), Itogi Nauki Yug. Rossii Matematicheskii Monografiya, 5, Southern Institute of Mathematics of the Vladikavkaz Sceince Centre of the Russian Academy of Sciences and the Government of the Republic of North Ossetia-Alania, Vladikavkaz, 2012, 267 pp.

[41] Sharapudinov, I. I., Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials, *Izv. Math.*, 77(2) 2013, 407-434. https://doi.org/10.4213/im7808

[42] Sharapudinov, I. I., On direct and inverse theorems of approximation theory in variable Lebesgue and Sobolev spaces, *Azerbaijan J. Math.,* 4(1) (2014), 55-72.

[43] Sharapudinov, I. I., Approximation of functions in Lebesgue and Sobolev spaces with variable exponent by Fourier-Haar sums, *Sbornik Math.,* 205(1-2) (2014), 291-306. https://doi.org/10.4213/sm8274

[44] Sharapudinov, I. I., Some problems in approximation theory in the spaces $L^{p(x)}(E)$, (In Russian), *Analysis Math.*, 33(2) (2007), 135-153. https://doi.org/10.1007/s10476-007-0204-0

[45] Sharapudinov, I. I., The basis property of the Haar system in the space $L^{p(t)}([0,1])$ and the principle of localization in the mean, (In Russian), *Mat. Sbornik*, 130(172)(2) (1986), 275-283. https://doi.org/10.1070/SM1987v058n01ABEH003104

[46] Taberski, R., Approximation by entire functions of exponential type, *Demonstratio Math.*, 14 (1981), 151-181.

[47] Taberski, R., On exponential approximation of locally integrable functions, *Annales Sociatatis mathematicae Polonae, Series I: Commentationes mathematicae*, 32 (1992), 159-174.

[48] Volosivets, S. S., Approximation of functions and their conjugates in variable Lebesgue spaces, *Sbornik Math.*, 208(1) (2017), 44-59. https://doi.org/10.1070/SM8636

[49] Yeh, J., Real Analysis: Theory of Measure and Integration, 2nd edition, World Scientific, 2006.

[50] Zhikov, V. V., Averaging of functionals of the calculus of variations and elasticity theory, (In Russian), *Math. USSR-Izvestiya*, 50(4) (1986), 675-710. https://doi.org/10.1070/IM1987v029n01ABEH000958

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1080–1094 (2022)
DOI:10.31801/cfsuasmas.1017204
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: November 1, 2021; Accepted: June 16, 2022

SPECTRAL SINGULARITIES OF AN IMPULSIVE STURM–LIOUVILLE OPERATORS

Guler Basak OZNUR

Department of Mathematics, Gazi University, 06500 Ankara, Turkey

Abstract. In this paper, we handle an impulsive Sturm–Liouville equation with complex potential on the semi axis. The objective of this work is to examine some spectral properties of this impulsive Sturm–Liouville equation. By the help of a transfer matrix B, we obtain Jost solution of this problem. Furthermore, using Jost solution, we find Green function and resolvent operator of this equation. Finally, we consider two unperturbated impulsive Sturm–Liouville operators. We examine the eigenvalues and spectral singularities of these problems.

1. Introduction

The modeling of most of the problems encountered in the fields of mathematics, physics, mechanics and engineering in daily life is done with boundary value or initial value problems in applied mathematics and spectral analysis. Operator theory is used to solve these problems in spectral theory. First, many physicists and mathematicians studied the spectral theory of differential operators. The Sturm–Liouville operator, which is the equivalent of the one dimensional Schrödinger operator, has gained a wide place in the literature. Let us shortly give information about the literature of spectral theory of Sturm–Liouville operator. Spectral analysis of the nonself-adjoint Schrödinger operator was first investigated by Naimark in 1960 [20]. He proved that the spectrum of this operator consists of eigenvalues, continuous spectrum and spectral singularities. Furthermore, he discovered that the spectral singularities are poles of the resolvent operator's kernel on the continuous spectrum but not the eigenvalues of the operator. Kemp extended the results obtained by

2020 *Mathematics Subject Classification.* 34B09, 34B24, 34B37, 34K10, 34L05.

Keywords. Impulsive condition, Sturm–Liouville equation, eigenvalues, Jost function, spectral singularity, resolvent operator.

basakoznur@gazi.edu.tr

0000-0003-4130-5348.

Naimark to differential operators defined on the whole real axis [16]. Gasymov also extended these results to three-dimensional Schrödinger operators [12]. Then, Schwartz investigated the spectral singularities of a certain class of abstract linear operators in Hilbert space and proved that self-adjoint operators have no spectral singularity [23]. Furthermore, these equations were studied under different initial and boundary conditions by Pavlov, Guseinov and Bairamov et al. [7,9,10,14,22].

On the other hand, in some processes, instant changes are encountered due to external factors. These are short term sudden changes and can be neglected compared to the whole process. Ordinary differential equations are not sufficient to model these processes. For this reason, impulsive differential equations are used to explain these processes mathematically. Unlike the Schrödinger equation, differential equations with impulsive conditions do not have a long history in the literature. Impulsive differential systems were first studied by Myshkis and Mil'man [18]. After, these equations were investigated by Bainov, Simenov and Lakshmikantham [3,4]. Recently, many authors have examined impulsive differential equations in detail, because impulsive differential equations have been used in many scientific phenomena such as heart beat, population dynamics, atomic physics, mathematical economics, ecology, engineering, medicine and so forth [13,15,19]. Bairamov et al, Yardimci and Erdal investigated scattering analysis and spectral theory of different kinds of impulsive Sturm–Liouville equations [2,5,6,8,11,24]. Different from these studies, in this paper, we consider the Sturm–Liouville equation with complex valued potential and impulsive condition in matrix form. Therefore, it creates different perspective.

Let us introduce the Sturm–Liouville operator T in $L_2(0,\infty)$, generated by the equation

$$-\upsilon'' + q(z)\upsilon = \lambda^2\upsilon, \quad z \in [0, z_0) \cup (z_0, \infty) \tag{1}$$

with the boundary condition

$$(\eta_0 + \eta_1\lambda)\,\upsilon'(0) + (\zeta_0 + \zeta_1\lambda)\,\upsilon(0) = 0 \tag{2}$$

and the impulsive condition

$$\begin{bmatrix} \upsilon\left(z_0^+\right) \\ \upsilon'\left(z_0^+\right) \end{bmatrix} = B \begin{bmatrix} \upsilon\left(z_0^-\right) \\ \upsilon'\left(z_0^-\right) \end{bmatrix}, \qquad B = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}, \tag{3}$$

where β_i, η_j, ζ_j, $i = 1,2,3,4$, $j = 0,1$ are complex numbers such that $\det B \neq 0$ and $\eta_0\zeta_1 - \eta_1\zeta_0 \neq 0$, z_0 is a positive real constant and q is a complex valued function satisfying the following condition

$$\int_0^\infty (1+z)|q(z)|dz < \infty. \tag{4}$$

Throughout the paper, we will show impulsive boundary value problem (1)-(3) by ISBVP, shortly.

This paper is organized as follows: This study consists of five chapters including the introduction. In the next Section, we give basic solutions and definitions. Unlike other studies in the literature, we examine the effect of the impulsive condition on the Sturm–Liouville equation with complex potential in Section 3. We find the Jost solution of ISBVP (1)-(3). In Section 4, we obtain the set of eigenvalues and spectral singularities of (1)-(3). Also, we present an asymptotic equation to obtain the properties of eigenvalues. Then, we get the resolvent operator of the Sturm–Liouville operator T. Finally, we handle two different problems to apply our main results in Section 5.

2. Preliminaries

Let $S\left(z,\lambda^2\right)$ and $C\left(z,\lambda^2\right)$ be the fundamental solutions of (1) in the interval $[0,z_0)$ satisfying the initial conditions

$$S(0,\lambda^2)=0, \quad S'(0,\lambda^2)=1,$$
$$C(0,\lambda^2)=1, \quad C'(0,\lambda^2)=0.$$

It is evident that the solutions $S\left(z,\lambda^2\right)$ and $C\left(z,\lambda^2\right)$ are entire functions of λ and

$$W[S(z,\lambda^2),C(z,\lambda^2)]=-1, \qquad \lambda\in\mathbb{C},$$

where $W\left[\upsilon_1,\upsilon_2\right]$ denotes the Wronskian of the solutions υ_1 and υ_2 of the equation (1). The integral representations of $S\left(z,\lambda^2\right)$ and $C\left(z,\lambda^2\right)$ are well known in the literature as

$$S(z,\lambda^2)=\frac{\sin\lambda z}{\lambda}+\int_0^z Q(z,t)\frac{\sin\lambda t}{\lambda}dt \tag{5}$$

$$C(z,\lambda^2)=\cos\lambda z+\int_0^z R(z,t)\cos\lambda t dt, \tag{6}$$

where $Q(z,t)$ and $R(z,t)$ are expressed in terms of the potential function q [17].

On the other hand, $e(z,\lambda)$ is bounded solution of the equation (1) in the interval (z_0,∞) fulfilling the following condition

$$\lim_{z\to\infty} e(z,\lambda)e^{-i\lambda z}=1, \quad \lambda\in\overline{\mathbb{C}}_+:=\{\lambda\in\mathbb{C}:\mathrm{Im}\lambda\geq 0\}$$

and it has an integral representation

$$e(z,\lambda)=e^{i\lambda z}+\int_z^\infty K(z,t)e^{i\lambda t}dt, \quad \lambda\in\overline{\mathbb{C}}_+, \tag{7}$$

where $K(z,t)$ is defined by the potential function q [1]. The bounded solution $e(z,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+:=\{\lambda\in\mathbb{C}:\mathrm{Im}\lambda>0\}$ and continuous up to the real axis. Similarly, $e\left(z,-\lambda\right)$ is bounded solution of (1) in (z_0,∞) satisfying the condition

$$\lim_{z\to\infty} e(z,-\lambda)e^{i\lambda z}=1, \quad \lambda\in\overline{\mathbb{C}}_-:=\{\lambda\in\mathbb{C}:\mathrm{Im}\lambda\leq 0\}.$$

It is well known that

$$W\left[e\left(z,\lambda\right),e\left(z,-\lambda\right)\right]=-2i\lambda,\qquad \lambda\in\mathbb{R}\backslash\{0\}.$$

Furthermore, $\breve{e}(z,\lambda)$ is unbounded solution of (1) in (z_0,∞) subjecting the conditions [21]

$$\lim_{z\to\infty}\breve{e}(z,\lambda)e^{i\lambda z}=1,\qquad \lim_{z\to\infty}\breve{e}'(z,\lambda)e^{i\lambda z}=-i\lambda,\qquad \lambda\in\overline{\mathbb{C}}_+.$$

It is clear that

$$W\left[e\left(z,\lambda\right),\breve{e}\left(z,\lambda\right)\right]=-2i\lambda,\qquad z\in\left(z_0,\infty\right),\qquad \lambda\in\overline{\mathbb{C}}_+.$$

3. Solutions of Impulsive Sturm–Liouville Equation

By the help of linearly independent solutions (1), we will define the general solutions of (1) for $\lambda\in\mathbb{R}\backslash\{0\}$,

$$\Psi_1\left(z,\lambda\right)=\begin{cases}\upsilon_1^-\left(z,\lambda\right)=a^-(\lambda)S\left(z,\lambda^2\right)+b^-(\lambda)C\left(z,\lambda^2\right); & 0\le z<z_0\\ \upsilon_1^+\left(z,\lambda\right)=a^+(\lambda)e\left(z,\lambda\right)+b^+(\lambda)e\left(z,-\lambda\right); & z_0<z<\infty,\end{cases}\tag{8}$$

$$\Psi_2\left(z,\lambda\right)=\begin{cases}\upsilon_2^-\left(z,\lambda\right)=c^-(\lambda)S\left(z,\lambda^2\right)+d^-(\lambda)C\left(z,\lambda^2\right); & 0\le z<z_0\\ \upsilon_2^+\left(z,\lambda\right)=c^+(\lambda)e\left(z,\lambda\right)+d^+(\lambda)e\left(z,-\lambda\right); & z_0<z<\infty\end{cases}\tag{9}$$

and for $\lambda\in\overline{\mathbb{C}}_+\backslash\{0\}$,

$$\Psi_3\left(z,\lambda\right)=\begin{cases}\upsilon_3^-\left(z,\lambda\right)=f^-(\lambda)S\left(z,\lambda^2\right)+h^-(\lambda)C\left(z,\lambda^2\right); & 0\le z<z_0\\ \upsilon_3^+\left(z,\lambda\right)=f^+(\lambda)e\left(z,\lambda\right)+h^+(\lambda)\breve{e}\left(z,\lambda\right); & z_0<z<\infty,\end{cases}\tag{10}$$

respectively.
Using (3) and (8), we obtain

$$\begin{bmatrix}a^+(\lambda)\\ b^+(\lambda)\end{bmatrix}=N\begin{bmatrix}a^-(\lambda)\\ b^-(\lambda)\end{bmatrix},\tag{11}$$

where

$$N:=\begin{bmatrix}N_{11}(\lambda) & N_{12}(\lambda)\\ N_{21}(\lambda) & N_{22}(\lambda)\end{bmatrix}=L^-BM\tag{12}$$

such that

$$L=\begin{bmatrix}e\left(z_0,\lambda\right) & e\left(z_0,-\lambda\right)\\ e'\left(z_0,\lambda\right) & e'\left(z_0,-\lambda\right)\end{bmatrix}$$

and

$$M=\begin{bmatrix}S\left(z_0,\lambda^2\right) & C\left(z_0,\lambda^2\right)\\ S'\left(z_0,\lambda^2\right) & C'\left(z_0,\lambda^2\right)\end{bmatrix}.$$

Since $\det L=-2i\lambda$, in accordance with (12), we find that

$$\begin{aligned}N_{21}(\lambda)=\frac{i}{2\lambda}[&-e'(z_0,\lambda)\left(\beta_1S\left(z_0,\lambda^2\right)+\beta_2S'\left(z_0,\lambda^2\right)\right)\\ &+e\left(z_0,\lambda\right)\left(\beta_3S\left(z_0,\lambda^2\right)+\beta_4S'\left(z_0,\lambda^2\right)\right)]\end{aligned}\tag{13}$$

$$N_{22}(\lambda)=\frac{i}{2\lambda}[-e'(z_0,\lambda)\left(\beta_1 C\left(z_0,\lambda^2\right)+\beta_2 C'\left(z_0,\lambda^2\right)\right)$$
$$+e\left(z_0,\lambda\right)\left(\beta_3 C\left(z_0,\lambda^2\right)+\beta_4 C'\left(z_0,\lambda^2\right)\right)]. \tag{14}$$

Now, we shall consider the Jost solution of ISBVP (1)-(3) and denote by E. Thus, by using (8), the coefficients $a^+(\lambda)$ and $b^+(\lambda)$ turn into 1 and 0, respectively. For $\lambda\in\overline{\mathbb{C}}_+$, we write the following solution of (1)-(3)

$$E(z,\lambda)=\begin{cases}a^-(\lambda)S(z,\lambda^2)+b^-(\lambda)C(z,\lambda^2); & z\in[0,z_0)\\ e(z,\lambda); & z\in(z_0,\infty).\end{cases}$$

By the help of (11) and (12), we easily obtain the coefficients $a^-(\lambda)$ and $b^-(\lambda)$ as follows

$$a^-(\lambda)=\frac{N_{22}(\lambda)}{\det N},\qquad b^-(\lambda)=-\frac{N_{21}(\lambda)}{\det N}. \tag{15}$$

Let us consider the solution of (1)-(3) satisfying the boundary condition (2) and denote by F. By (2) and (9), the following can be easily seen

$$c^-(\lambda)=\left(\zeta_0+\zeta_1\lambda\right),\qquad d^-(\lambda)=\left(\eta_0+\eta_1\lambda\right).$$

For $\lambda\in\mathbb{R}\backslash\{0\}$, we will consider the following solution of ISBVP (1)-(3)

$$F(z,\lambda)=\begin{cases}-\left(\zeta_0+\zeta_1\lambda\right)S(z,\lambda^2)+\left(\eta_0+\eta_1\lambda\right)C(z,\lambda^2); & z\in[0,z_0)\\ c^+(\lambda)e(z,\lambda)+d^+(\lambda)e(z,-\lambda); & z\in(z_0,\infty).\end{cases}$$

From (3) and (12), we get

$$c^+(\lambda)=-\left(\zeta_0+\zeta_1\lambda\right)N_{11}(\lambda)+\left(\eta_0+\eta_1\lambda\right)N_{12}(\lambda) \tag{16}$$

$$d^+(\lambda)=-\left(\zeta_0+\zeta_1\lambda\right)N_{21}(\lambda)+\left(\eta_0+\eta_1\lambda\right)N_{22}(\lambda), \tag{17}$$

respectively.

Lemma 1. *For $\lambda\in\mathbb{R}\backslash\{0\}$, the Wronskian of the solutions $E\left(z,\lambda\right)$ and $F\left(z,\lambda\right)$ is given by*

$$W[E(z,\lambda),F(z,\lambda)]=\begin{cases}H\left(\lambda\right); & z\in[0,z_0)\\ 2i\lambda H\left(\lambda\right)\det N; & z\in(z_0,\infty),\end{cases}$$

where

$$H\left(\lambda\right):=\frac{\left(\zeta_0+\zeta_1\lambda\right)N_{21}(\lambda)-\left(\eta_0+\eta_1\lambda\right)N_{22}(\lambda)}{\det N}. \tag{18}$$

Proof. Using the definition of Wronskian for $z\in[0,z_0)$, we find

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right]=-\left(\zeta_0+\zeta_1\lambda\right)b^-(\lambda)-\left(\eta_0+\eta_1\lambda\right)a^-(\lambda).$$

By using (15), the following can be easily seen

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right]=H(\lambda)$$

for $z \in [0, z_0)$.
Similarly, we write

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right] = -2i\lambda d^{+}(\lambda), \quad z \in (z_0, \infty).$$

By the help of (17), it is clear that

$$W\left[E\left(z,\lambda\right),F\left(z,\lambda\right)\right] = 2i\lambda H\left(\lambda\right)\det N$$

for $z \in (z_0, \infty)$.
This completes the proof. □

Since H is composed of $e\left(z,\lambda\right), C\left(z,\lambda^2\right)$ and $S\left(z,\lambda^2\right)$, it is analytic in $\mathbb{C}_+$ and continuous up to the real axis.

4. Eigenvalues, Spectral Singularities And Resolvent Operator of T

From Lemma 1, a necessary and sufficient condition to investigate the eigenvalues and spectral singularities of the Sturm–Liouville operator T with impulsive condition (3) is to investigate the zeros of the function H.
The set of eigenvalues and spectral singularities of the operator T are defined as

$$\sigma_d\left(T\right) = \{\mu = \lambda^2 : \operatorname{Im}\lambda > 0 \text{ and } H(\lambda) = 0\},$$

$$\sigma_{ss}\left(T\right) = \{\mu = \lambda^2, \operatorname{Im}\lambda = 0, \lambda \neq 0 \text{ and } H(\lambda) = 0\},$$

respectively.

Theorem 1. *Under the condition* (4)*, the function* H *satisfies the following asymptotic equation*

$$H\left(\lambda\right) = \frac{\mu_1\beta_2\lambda^2}{\det N}\left(\frac{i}{4} + O\left(\frac{1}{\lambda}\right)\right), \quad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \to \infty,$$

where $\mu_1\beta_2 \neq 0$.

Proof. By means of (5)-(7), we easily find for $\lambda \in \mathbb{C}$

$$S'\left(z_0,\lambda^2\right) = \cos\lambda z_0 + Q\left(z_0,z_0\right)\frac{\sin\lambda z_0}{\lambda} + \int_0^{z_0} Q(z_0,t)\frac{\sin\lambda t}{\lambda}dt \tag{19}$$

$$C'\left(z_0,\lambda^2\right) = -\lambda\sin\lambda z_0 + R\left(z_0,z_0\right)\cos\lambda z_0 + \int_0^{z_0} R(z_0,t)\cos\lambda t dt \tag{20}$$

and for $\lambda \in \overline{\mathbb{C}}_+$

$$e'(z_0,\lambda) = i\lambda e^{i\lambda z_0} - K\left(z_0,z_0\right)e^{i\lambda z_0} + \int_{z_0}^{\infty} K_z(z_0,t)e^{i\lambda t}dt. \tag{21}$$

From (5)-(7), we get

$$S\left(z_0,\lambda^2\right)=\frac{e^{-i\lambda z_0}}{\lambda}\left(\frac{i}{2}+o\left(1\right)\right)$$
$$C\left(z_0,\lambda^2\right)=e^{-i\lambda z_0}\left(\frac{1}{2}+o\left(1\right)\right) \tag{22}$$
$$e\left(z_0,\lambda\right)=e^{i\lambda z_0}\left(1+o\left(1\right)\right),$$

where $\lambda\in\overline{\mathbb{C}}_+$ and $|\lambda|\to\infty$.
In a similar way, by using (19)-(21), we obtain for $\lambda\in\overline{\mathbb{C}}_+$ and $|\lambda|\to\infty$

$$S'\left(z_0,\lambda^2\right)=e^{-i\lambda z_0}\left(\frac{1}{2}+O\left(\frac{1}{\lambda}\right)\right)$$
$$C'\left(z_0,\lambda^2\right)=\lambda e^{-i\lambda z_0}\left(-\frac{i}{2}+O\left(\frac{1}{\lambda}\right)\right) \tag{23}$$
$$e'\left(z_0,\lambda\right)=\lambda e^{i\lambda z_0}\left(i+O\left(\frac{1}{\lambda}\right)\right).$$

By means of (22) and (23), it is obvious that $H\left(\lambda\right)$ satisfies the asymptotic equation given in Theorem 1. This completes the proof. □

Now, let us define another solution of (1)-(3)

$$G\left(z,\lambda\right)=\begin{cases}-\left(\zeta_0+\zeta_1\lambda\right)S(z,\lambda^2)+\left(\eta_0+\eta_1\lambda\right)C(z,\lambda^2); & z\in[0,z_0)\\ f^+(\lambda)e(z,\lambda)+h^+(\lambda)\breve{e}(z,\lambda); & z\in(z_0,\infty)\end{cases}$$

for all $\lambda\in\overline{\mathbb{C}}_+\backslash\{0\}$. By the help of (3), we obtain that

$$\begin{bmatrix}f^+(\lambda)\\h^+(\lambda)\end{bmatrix}=V\begin{bmatrix}-\left(\zeta_0+\zeta_1\lambda\right)\\\left(\eta_0+\eta_1\lambda\right)\end{bmatrix}, \tag{24}$$

where

$$V:=\begin{bmatrix}V_{11}(\lambda) & V_{12}(\lambda)\\V_{21}(\lambda) & V_{22}(\lambda)\end{bmatrix}=U^-BM \tag{25}$$

with

$$U=\begin{bmatrix}e\left(z_0,\lambda\right) & \breve{e}\left(z_0,\lambda\right)\\e'\left(z_0,\lambda\right) & \breve{e}'\left(z_0,\lambda\right)\end{bmatrix}. \tag{26}$$

From (25) and (26), the following equations can be found as

$$\begin{aligned}V_{21}(\lambda)=\frac{i}{2\lambda}[&-e'(z_0,\lambda)\left(\beta_1S\left(z_0,\lambda^2\right)+\beta_2S'\left(z_0,\lambda^2\right)\right)\\&+e\left(z_0,\lambda\right)\left(\beta_3S\left(z_0,\lambda^2\right)+\beta_4S'\left(z_0,\lambda^2\right)\right)]\end{aligned} \tag{27}$$

$$\begin{aligned}V_{22}(\lambda)=\frac{i}{2\lambda}[&-e'(z_0,\lambda)\left(\beta_1C\left(z_0,\lambda^2\right)+\beta_2C'\left(z_0,\lambda^2\right)\right)\\&+e\left(z_0,\lambda\right)\left(\beta_3C\left(z_0,\lambda^2\right)+\beta_4C'\left(z_0,\lambda^2\right)\right)].\end{aligned} \tag{28}$$

By using (24), the coefficients $f^{+}(\lambda)$ and $h^{+}(\lambda)$ must be as follows

$$f^{+}(\lambda) = -\left(\zeta_0 + \zeta_1\lambda\right) V_{11}(\lambda) + \left(\eta_0 + \eta_1\lambda\right) V_{12}(\lambda)$$

$$h^{+}(\lambda) = -\left(\zeta_0 + \zeta_1\lambda\right) V_{21}(\lambda) + \left(\eta_0 + \eta_1\lambda\right) V_{22}(\lambda).$$

By using (13), (14), (27) and (28), it is clear that

$$N_{21}(\lambda) = V_{21}(\lambda), \qquad N_{22}(\lambda) = V_{22}(\lambda).$$

Therefore, using (18), we rewrite $h^{+}(\lambda)$ as

$$h^{+}(\lambda) = -H(\lambda)\det N. \tag{29}$$

In view of (29), we obtain that

$$W[E(z,\lambda), G(z,\lambda)] = \begin{cases} H(\lambda); & z \in [0, z_0) \\ 2i\lambda H(\lambda)\det N; & z \in (z_0, \infty) \end{cases}$$

for $\lambda \in \overline{\mathbb{C}}_{+}\backslash\{0\}$.

Theorem 2. *Assume (4). Then the resolvent operator of T is defined by*

$$\mathbb{R}_{\lambda}\phi = \int_{0}^{\infty} R(z,t;\lambda)\phi(t)dt,$$

where

$$R(z,t;\lambda) = \begin{cases} \dfrac{E(z,\lambda)G(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}; & 0 \le t < z \\ \dfrac{G(z,\lambda)E(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}; & z \le t < \infty \end{cases}$$

is the Green function of (1)-(3) for $z \neq z_0$, $t \neq z_0$.

Proof. Let us consider the following equation

$$-\upsilon'' + q(z)\upsilon - \lambda^2\upsilon = \phi(z), \quad z \in [0, z_0) \cup (z_0, \infty). \tag{30}$$

By using the solutions $E(z,\lambda)$ and $G(z,\lambda)$, we write the solution of (30)

$$\phi(z,\lambda) = \theta_1(z)E(z,\lambda) + \theta_2(z)G(z,\lambda).$$

Using the method of variation of parameters, we get the coefficients $\theta_1(z)$ and $\theta_2(z)$ as follows

$$\theta_1(z) = k + \int_{0}^{z} \frac{\phi(t)G(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}dt$$

$$\theta_2(z) = m + \int_{z}^{\infty} \frac{\phi(t)E(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}dt,$$

where k and m are real numbers. Let us write the coefficients $\theta_1(z)$ and $\theta_2(z)$ in solution $\phi(z,\lambda)$

$$\phi(z,\lambda) = kE(z,\lambda) + \int_0^z \frac{\phi(t)G(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}dtE(z,\lambda)$$

$$+mG(z,\lambda) + \int_z^\infty \frac{\phi(t)E(t,\lambda)}{W[E(z,\lambda),G(z,\lambda)]}dtG(z,\lambda).$$

Since the solution $\phi(z,\lambda)$ is in $L_2(0,\infty)$, m becomes zero. In accordance with the boundary condition (2), we also find that k is equal to zero. The proof is completed. □

5. Unperturbated Impulsive Operators

In this section, we will investigate two unperturbated impulsive Sturm–Liouville operators.

Example 1. *Now, we consider the Sturm–Liouville operator T_0 in $L^2[0,\infty)$ corresponding to the following impulsive problem*

$$\begin{gathered} -\upsilon'' = \lambda^2\upsilon, \quad z \in [0,1)\cup(1,\infty) \\ (\eta_0+\eta_1\lambda)\,\upsilon'(0) + (\zeta_0+\zeta_1\lambda)\,\upsilon(0) = 0 \\ \begin{bmatrix}\upsilon(1^+)\\ \upsilon'(1^+)\end{bmatrix} = B\begin{bmatrix}\upsilon(1^-)\\ \upsilon'(1^-)\end{bmatrix}, \qquad B = \begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}, \end{gathered} \tag{31}$$

where γ_1,γ_2, η_j, ζ_j, $j=0,1$ are complex numbers such that $\eta_0\zeta_1-\eta_1\zeta_0\neq 0$ and $\gamma_1\gamma_2\neq 0$. Since $q=0$, it is evident that

$$e(z,\lambda) = e^{i\lambda z}, \quad C(z,\lambda^2) = \cos\lambda z, \quad S(z,\lambda^2) = \frac{\sin\lambda z}{\lambda}.$$

By using (18), we write that

$$\begin{aligned} H(\lambda) = \frac{ie^{i\lambda}}{2\lambda\det N}&[(\eta_0+\eta_1\lambda)(i\gamma_1\lambda\cos\lambda+\gamma_2\lambda\sin\lambda) \\ &+(\zeta_0+\zeta_1\lambda)(\gamma_2\cos\lambda - i\gamma_1\sin\lambda)]. \end{aligned} \tag{32}$$

To investigate the eigenvalues and spectral singularities of (31), we examine the zeros of H. Let us choose $\zeta_1=\eta_0=1$ and $\zeta_0=\eta_1=0$ in (32) for the simplicity. Therefore, we rewrite the equation (32)

$$H(\lambda) = \frac{ie^{i\lambda}}{2\det N}[i\gamma_1\cos\lambda + \gamma_2\sin\lambda - i\gamma_1\sin\lambda + \gamma_2\cos\lambda].$$

We obtain that

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + \frac{1}{2}\operatorname{Arg}\left(\frac{1+D}{1-D}\right) + k\pi, \quad k\in\mathbb{Z},$$

where $D = \dfrac{\gamma_1 - i\gamma_2}{\gamma_2 - i\gamma_1}$. *There appear three cases:*
Case1: Let $D = \dfrac{e^{i\theta}-1}{e^{i\theta}+1}$ *such that* $\theta \in \mathbb{R}$. *Since* $D = \dfrac{e^{i\theta}-1}{e^{i\theta}+1}$, *it is easily seen that* $\operatorname{Arg}\left(\dfrac{1+D}{1-D}\right) = \theta$ *and* $\left|\dfrac{1+D}{1-D}\right| = 1$. *Then, we find that*

$$\lambda_k = \frac{\theta}{2} + k\pi, \quad k \in \mathbb{Z}.$$

In this case, $\lambda_k \in \mathbb{R}\backslash\{0\}$, $k \in \mathbb{Z}$ *are the spectral singularities of* (31). *However, there is no eigenvalues.*
Case2: Let $\operatorname{Im} D \neq 0$.
2a: Let D *be purely imaginary. We obtain that*

$$\lambda_k = \frac{1}{2}\operatorname{Arg}\left(\frac{1+D}{1-D}\right) + k\pi, \quad k \in \mathbb{Z}.$$

In this case, similar with Case1, the ISBVP (31) *has no eigenvalues. But it has spectral singularity.*
2b: Assume $\operatorname{Re} D < 0$. *We get*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + \frac{1}{2}\operatorname{Arg}\left(\frac{1+D}{1-D}\right) + k\pi, \quad k \in \mathbb{Z}.$$

Since $0 < \left|\dfrac{1+D}{1-D}\right| < 1$, $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ *are the eigenvalues of* (31). *However, the operator* T_0 *doesn't have any spectral singularity.*
2c: For $0 < \operatorname{Re} D$, *the impulsive Sturm–Liouville boundary value problem* (31) *has no eigenvalues and spectral singularity.*
Case3: Let D *be a real number.*
3a: If $0 < D < 1$, *then* $1 < \left|\dfrac{1+D}{1-D}\right|$. *Similar to the Case2c, the eigenvalues and spectral singularity of* (31) *are not existing.*
3b: For $1 < D < \infty$, *we see that*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}.$$

Since $\lambda_k \in \mathbb{C}_-$, *there are no eigenvalues and spectral singularity.*
3c: Assume $-1 < D < 0$. *We obtain that*

$$\lambda_k = -\frac{i}{2}\ln\left(\frac{1+D}{1-D}\right) + k\pi, \quad k \in \mathbb{Z}.$$

Since $0 < \left|\dfrac{1+D}{1-D}\right| < 1$, *there exists eigenvalues but the problem* (31) *has no spectral singularty.*

3d: For $-\infty < D < 1$, we find that

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+D}{1-D}\right| + (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z},$$

where $0 < \left|\frac{1+D}{1-D}\right| < 1$. Hence, $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ are the eigenvalues of T_0. But this operator has no spectral singularity.

Example 2. *We investigate the Sturm–Liouville operator T_1 in $L^2[0,\infty)$ created by the following ISBVP*

$$\begin{gathered} -\upsilon'' = \lambda^2 \rho(z)\upsilon, \quad z \in [0,1) \cup (1,\infty) \\ (\eta_0 + \eta_1\lambda)\,\upsilon'(0) + (\zeta_0 + \zeta_1\lambda)\,\upsilon(0) = 0 \\ \begin{bmatrix} \upsilon(1^+) \\ \upsilon'(1^+) \end{bmatrix} = B \begin{bmatrix} \upsilon(1^-) \\ \upsilon'(1^-) \end{bmatrix}, \qquad B = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}, \end{gathered} \tag{33}$$

where τ_1, τ_2, η_j, ζ_j, $j = 0, 1$ are complex numbers, $\eta_0\zeta_1 - \eta_1\zeta_0 \neq 0$, $\tau_1\tau_2 \neq 0$ and ρ is density function defined as

$$\rho(z) = \begin{cases} \omega^2; & 0 \le z < 1 \\ 1; & 1 < z \end{cases}$$

such that $\omega \in \mathbb{C}\backslash\{-1, 0, 1\}$. It is evident that for this example

$$e(z,\lambda) = e^{i\lambda z}, \quad C\left(z,\lambda^2\right) = \cos(\lambda\omega z), \quad S\left(z,\lambda^2\right) = \frac{\sin(\lambda\omega z)}{\lambda\omega}.$$

From (18), we obtain that

$$\begin{aligned} H(\lambda) = \frac{ie^{i\lambda}}{2\lambda \det N}&[(\eta_0 + \eta_1\lambda)(i\tau_1\lambda\cos(\lambda\omega) + \tau_2\lambda\omega\sin(\lambda\omega)) \\ &+ (\zeta_0 + \zeta_1\lambda)(\tau_2\cos(\lambda\omega) - i\tau_1\frac{\sin(\lambda\omega)}{\omega})]. \end{aligned} \tag{34}$$

For the simplicity on calculations, if we choose $\zeta_1 = \eta_0 = 1$ and $\zeta_0 = \eta_1 = 0$ in (34), we get

$$H(\lambda) = \frac{ie^{i\lambda}}{2\det N}[i\tau_1\cos(\lambda\omega) + \tau_2\omega\sin(\lambda\omega) - i\tau_1\frac{\sin(\lambda\omega)}{\omega} + \tau_2\cos(\lambda\omega)].$$

We easily find that

$$\lambda_k = -\frac{i}{2\omega}\ln\left|\frac{1+P}{1-P}\right| + \frac{1}{2\omega}\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right], \quad k \in \mathbb{Z},$$

where $P = \frac{\tau_1\omega - i\tau_2\omega}{\tau_2\omega^2 - i\tau_1}$. Let $\omega = m + in$. We can write the real and imaginary parts of λ_k as follows

$$\operatorname{Re}\lambda_k = \frac{1}{2|\omega|^2}\left\{m\left[\operatorname{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right] - n\ln\left|\frac{1+P}{1-P}\right|\right\}$$

and

$$\mathrm{Im}\lambda_k = -\frac{1}{2\left|\omega\right|^2}\left\{m\ln\left|\frac{1+P}{1-P}\right| + n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right]\right\},$$

respectively.
It is evident that if

$$\left[m\ln\left|\frac{1+P}{1-P}\right| + n\left(\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right)\right] = 0$$

then the operator T_1 *has spectral singularities, and if*

$$\left[m\ln\left|\frac{1+P}{1-P}\right| + n\left(\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right)\right] < 0$$

then the operator T_1 *has eigenvalues.*
Case1: If $P = \dfrac{e^{i\theta}-1}{e^{i\theta}+1}$, $\theta \in \mathbb{R}$, *then* $\mathrm{Arg}\left(\dfrac{1+P}{1-P}\right) = \theta$ *and* $\left|\dfrac{1+P}{1-P}\right| = 1$. *We find that*

$$\lambda_k = \frac{\theta + 2k\pi}{2\omega}, \quad k \in \mathbb{Z}.$$

1a: Assume $\omega \in \mathbb{R}$. $\lambda_k \in \mathbb{R}\backslash\{0\}$, $k \in \mathbb{Z}$ *are spectral singularities of the operator* T_1 *but ISBVP* (33) *has no eigenvalues.*
2a: Assume $\omega \in \mathbb{C}$. *We write*

$$\mathrm{Im}\lambda_k = -\frac{1}{2\left|\omega\right|^2}\left[n\left(\theta + 2k\pi\right)\right], \quad k \in \mathbb{Z}.$$

If $n\left(\theta + 2k\pi\right) < 0$, *then* $\lambda_k \in \mathbb{C}_+$, $k \in \mathbb{Z}$ *are eigenvalues of this problem* (33). *Otherwise, the eigenvalues and spectral singularities of* (33) *are not existing.*
Case2: Let $\mathrm{Im}P \neq 0$.
2a: Let P *be purely imaginary. We write*

$$\lambda_k = \frac{1}{2\omega}\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right], \quad k \in \mathbb{Z}.$$

For $\omega \in \mathbb{R}$, $\lambda_k \in \mathbb{R}\backslash\{0\}$, $k \in \mathbb{Z}$ *are spectral singularities of the operator* T_1. *However, the problem* (33) *has no eigenvalues.*
If $\omega \in \mathbb{C}$, *then we find that*

$$\mathrm{Im}\lambda_k = -\frac{n}{2\left|\omega\right|^2}\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right) + 2k\pi\right], \quad k \in \mathbb{Z}.$$

It is easily seen that, for $n\left[\mathrm{Arg}\left(\dfrac{1+P}{1-P}\right) + 2k\pi\right] < 0$, *the impulsive Sturm–Liouville boundary value problem* (33) *has eigenvalues. Otherwise, the problem* (33) *has no eigenvalues and spectral singularities.*
2b: Assume $ReA < 0$. *For* $\omega \in \mathbb{R}$, *we get*

$$\mathrm{Im}\lambda_k = -\frac{m}{2\left|\omega\right|^2}\left(\ln\left|\frac{1+P}{1-P}\right|\right), \quad k \in \mathbb{Z}.$$

If $m>0$, then the operator T_1 has eigenvalues. Otherwise, there are no eigenvalues and spectral singularities of (33).
For $\omega\in\mathbb{C}$, we obtain that

$$\mathrm{Im}\lambda_k=-\frac{1}{2\left|\omega\right|^2}\left\{m\ln\left|\frac{1+P}{1-P}\right|+n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]\right\},\quad k\in\mathbb{Z}.$$

If $m>0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]<0$, then $\lambda_k\in\mathbb{C}_+$, $k\in\mathbb{Z}$ are eigenvalues of (33). *However, if $m<0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]>0$ then the operator T_1 has no eigenvalues and spectral singularities.*
2c: Assume $ReP>0$. Similar with case2b, if $\omega\in\mathbb{R}$ and $m<0$, then there exist eigenvalues of (33). *However, for $\omega\in\mathbb{R}$ and $m>0$, there are no eigenvalues and spectral singularities of ISBVP* (33).
Let $\omega\in\mathbb{C}$, it is clear that if $m<0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]<0$, then the problem (33) *has eigenvalues. If $m>0$ and $n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]>0$, then the eigenvalues and spectral singularities of* (33) *are not existing.*
Case3: Let P be a real number.
3a: For $0<P<1$, we find that

$$\lambda_k=-\frac{i}{2\omega}\ln\left(\frac{1+P}{1-P}\right)+\frac{k\pi}{\omega},\quad k\in\mathbb{Z}.$$

Assume $\omega\in\mathbb{R}$. If $m<0$, then the operator T_1 has eigenvalues. However, if $m>0$, then the problem (33) *does not have any spectral singularity and eigenvalues.*
Assume $\omega\in\mathbb{C}$. If $m<0$ and $n\left(2k\pi\right)<0$, then $\lambda_k\in\mathbb{C}_+$, $k\in\mathbb{Z}$ are eigenvalues of ISBVP (33) *but if $m>0$ and $n\left(2k\pi\right)>0$, then the operator T_1 has no eigenvalues and spectral singularity.*
3b: For $1<P<\infty$, it is evident that

$$\lambda_k=-\frac{i}{2\omega}\ln\left|\frac{1+P}{1-P}\right|+\frac{1}{2\omega}\left[\left(2k+1\right)\pi\right],\quad k\in\mathbb{Z}.$$

Let $\omega\in\mathbb{R}$. Similar with Case3a, for $m<0$, the problem (33) *has eigenvalues. Otherwise, the operator T_1 has no eigenvalues and spectral singularities.*
Let $\omega\in\mathbb{C}$. If $m<0$ and $n\left(2k+1\right)\pi<0$, then there exists eigenvalues of (33) *but if $m>0$ and $n\left(2k+1\right)\pi>0$, then there are no eigenvalues and spectral singularities.*
3c: For $-1<P<0$, we obtain

$$\lambda_k=-\frac{i}{2\omega}\ln\left(\frac{1+P}{1-P}\right)+\frac{k\pi}{2\omega},\quad k\in\mathbb{Z}.$$

Assume $\omega\in\mathbb{R}$. The operator T_1 has eigenvalues if and only if $m>0$.
Assume $\omega\in\mathbb{C}$. If $m>0$ and $n\left(2k\pi\right)<0$, then the problem (33) *has eigenvalues.*

But if $m<0$ and $n\left(2k\pi\right)>0$, then ISBVP (33) has no eigenvalues and spectral singularities.
3d: For $-\infty<P<1$, we get

$$\lambda_k=-\frac{i}{2\omega}\ln\left|\frac{1+P}{1-P}\right|+\frac{1}{2\omega}\left[\left(2k+1\right)\pi\right],\quad k\in\mathbb{Z}.$$

Let $\omega\in\mathbb{R}$. $\lambda_k\in\mathbb{C}_+$, $k\in\mathbb{Z}$ are eigenvalues of this example (33) if and only if $m>0$.
Let $\omega\in\mathbb{C}$. If $m>0$ and $n\left(2k+1\right)\pi<0$, then there exists eigenvalues of (33). If $m<0$ and $n\left(2k+1\right)\pi>0$, then the eigenvalues and spectral singularities of (33) are not existing.
Case4: Let ω be purely imaginary. We easily find that

$$\mathrm{Im}\lambda_k=-\frac{n}{2\left|\omega\right|^2}\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right],\quad k\in\mathbb{Z}.$$

The operator T_1 has spectral singularities if and only is

$$\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi=0.$$

The problem (33) has eigenvalues if and only if

$$n\left[\mathrm{Arg}\left(\frac{1+P}{1-P}\right)+2k\pi\right]<0.$$

Declaration of Competing Interests The author declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Agranovich, Z. S., Marchenko, V. A., The Inverse Problem of Scattering Theory, Pratt Institute Brooklyn, New York, 1963.

[2] Aygar, Y., Bairamov, E., Scattering theory of impulsive Sturm-Liouville equation in Quantum calculus, *Bull. Malays. Math. Sci. Soc.*, 42(6) (2019), 3247–3259. https://doi.org/10.1007/s40840-018-0657-2

[3] Bainov, D. D., Lakshmikantham, V., Simenov, P., Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[4] Bainov, D. D., Simenov, P. S., Impulsive Differential Equations: Periodic Solutions and Applications, Logman Scientific and Technical, England, 1993.

[5] Bairamov, E., Aygar, Y., Cebesoy, S., Investigation of spectrum and scattering function of impulsive matrix difference operators, *Filomat*, 33(5) (2019), 1301–1312. https://doi.org/10.2298/FIL1905301B

[6] Bairamov, E., Aygar, Y., Eren, B., Scattering theory of impulsive Sturm-Liouville equations, *Filomat*, 31(17) (2017), 5401–5409. https://doi.org/10.2298/FIL1717401B

[7] Bairamov, E., Aygar, Y., Koprubası, T., The spectrum of eigenparameter-dependent discrete Sturm-Liouville equations, *J. Comput. Appl. Math.*, 235(16) (2011), 4519–4523. https://doi.org/10.1016/j.cam.2009.12.037

[8] Bairamov, E., Aygar, Y., Oznur, G. B., Scattering properties of eigenparameter dependent impulsive Sturm-Liouville Equations, *Bull. Malays. Math. Sci. Soc.*, 43 (2019), 2769–2781. https://doi.org/10.1007/s40840-019-00834-5

[9] Bairamov, E., Cakar, O., Celebi, A. O., Quadratic pencil of Schrödinger operators with spectral singularities, *J. Math. Anal. Appl.*, 216 (1997), 303–320. https://doi.org/10.1006/jmaa.1997.5689

[10] Bairamov, E., Cakar, O., Krall, A. M., An eigenfunction expansion for a quadratic pencil of Schrödinger operator with spectral singularities, *J. Diff. Equat.*, 151 (1999), 268–289. https://doi.org/10.1006/jdeq.1998.3518

[11] Bairamov, E., Erdal, I., Yardimci, S., Spectral properties of an impulsive Sturm-Liouville operator, *J. Inequal. Appl.*, 191 (2018), 16 pp. https://doi.org/10.1186/s13660-018-1781-0

[12] Gasymov, M. G., Expansion in terms of the solutions of a scattering theory problem for the non-selfadjoint Schrödinger equation, *Soviet Math. Dokl.*, 9 (1968), 390–393.

[13] Guseinov, G. S., Boundary value problems for nonlinear impulsive Hamilton systems, *J. Comput. Appl. Math.*, 259 (2014), 780–789. http://dx.doi.org/10.1016/j.cam.2013.06.034

[14] Guseinov, G. S., On the concept of spectral singularities, *Pramana J. Phys.*, 73 (2009), 587–603.

[15] Guseinov, G. S., On the impulsive boundary value problems for nonlinear Hamilton systems, *Math. Methods Appl. Sci.*, 36(15) (2016), 4496–4503. https://doi.org/10.1002/mma.3877

[16] Kemp, R. R. D., A singular boundary value problem for a non-selfadjoint differential operator, *Canad. J. Math.*, 10 (1958), 447–462. https://doi.org/10.4153/CJM-1958-043-1

[17] Levitan, B. M., Sargsjan, I. S., Sturm-Liouville and Dirac Operators, Kluwer Academic Publisher Group, Dordrecht, 1991.

[18] Mil'man, V. D., Myshkis, A. D., On the stability of motion in the presence of impulses, *Sib. Math. J.*, 1 (1960), 233–237.

[19] Mukhtarov, F. S., Aydemir, K., Mukhtarov, O. S., Spectral analysis of one boundary value transmission problem by means of Green's function, *Electron J. Math. Anal. Appl.*, 2 (2014), 23–30. http://fcag-egypt.com/Journals/EJMAA/

[20] Naimark, M. A., Investigation of the spectrum and the expansion in eigenfunctions of a non-self adjoint operator of the second order on a semi axis, *Amer. Math. Soc. Transl.*, 16(2) (1960), 103–193.

[21] Naimark, M. A., Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space, World Scientific Publishing Co. Inc., River Edge, 1995.

[22] Pavlov, B. S., The non-selfadjoint Schrödinger operator, *Topics in Math. Phy.*, 1 (1967), 87–110.

[23] Schwartz, J. T., Some non-selfadjoint operator, *Commun. Pure Appl. Math.*, 13 (1960), 609–639.

[24] Yardimci, S., Erdal I., Investigation of an impulsive Sturm-Liouville operator on semi axis, *Hacet. J. Math. Stat.*, 48(5) (2019), 1409–1416. https://doi.org/10.15672/HJMS.2018.591

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1095–1120 (2022)
DOI:10.31801/cfsuasmas.1035344
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: December 10, 2021; Accepted: May 4, 2022

AN OVERVIEW TO ANALYTICITY OF DUAL FUNCTIONS

Olgun DURMAZ[1], Buşra AKTAŞ[2] and Osman KEÇİLİOĞLU[2]

[1]Department of Mathematics, University of Atatürk, Erzurum, TURKEY
[2]Department of Mathematics, University of Kırıkkale, Kırıkkale, TURKEY

Abstract. In this paper, the analyticity conditions of dual functions are clearly examined and the properties of the concept derivative are given in detail. Then, using the dual order relation, the dual analytic regions of dual analytic functions are constructed such that a collection of these regions forms a basis on D^n. Finally, the equivalent of the inverse function theorem in dual space is given by a theorem and proved.

1. Introduction

In 1873, W. K. Clifford originally introduced the theory of algebra of dual numbers as a tool for his geometrical researches. Clifford showed that they constitute an algebra but not a field because only dual numbers with real part not zero have an inverse element [1]. An ordered pair of real numbers $\overline{x} = (x, x^*)$ is called a dual number, where x and x^* are termed by real part and dual part of the dual number, respectively. Dual numbers may be formally stated by $\overline{x} = x + \varepsilon x^*$, where $\varepsilon = (0, 1)$ is entitled by dual unit satisfying the condition that $\varepsilon^2 = 0$. The algebra of dual numbers is derived from this description. If $x = y$, $x^* = y^*$ for $\overline{x} = x + \varepsilon x^*$ and $\overline{y} = y + \varepsilon y^*$, $\overline{x}$ and $\overline{y}$ are equal, and it is indicated as $\overline{x} = \overline{y}$. As for complex numbers, addition and product of two dual numbers are defined as follows, respectively:

$$(x + \varepsilon x^*) + (y + \varepsilon y^*) = x + y + \varepsilon (x^* + y^*),$$

$$(x + \varepsilon x^*) \cdot (y + \varepsilon y^*) = xy + \varepsilon (xy^* + x^* y).$$

The set of all dual numbers which is symbolized as D, i.e.,

$$\mathrm{D} = \left\{ \overline{x} = x + \varepsilon x^* \mid x, x^* \in \mathbb{R},\ \varepsilon^2 = 0 \right\}$$

2020 *Mathematics Subject Classification.* 32A38, 53A35, 53A40, 54A05.
Keywords. Dual numbers, topology, dual analytic functions.
olgun.durmaz@atauni.edu.tr; 0000-0002-0913-3307
baktas6638@gmail.com-Corresponding author; 0000-0002-1285-7250
okecilioglu@yahoo.com; 0000-0002-5149-5221.

is a commutative ring over the real numbers field according to the operators $+$ and $\cdot$. The unit element of multiplication operation $\cdot$ in D is the dual number $\overline{1} = (1, 0) = 1 + \varepsilon 0$. The dual number $\overline{x} = x + \varepsilon x^*$ that is divided by the dual number $\overline{y} = y + \varepsilon y^*$ providing $y \neq 0$ can be described as

$$\frac{\overline{x}}{\overline{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} = \frac{x}{y} + \varepsilon \left(\frac{x^* y - x y^*}{y^2} \right)$$

(see [1] and [2]). The dual number has a geometrical meaning which is discussed in detail in Yaglom [3]. It has contemporary applications within the curve design methods in computer aided geometric design and computer modeling of rigid bodies, linkages, robots, modelling human body dynamics, mechanism design, etc. [4]. The dual vectors were improved by A. P. Kotelnikov in the early part of the twentieth century [5]. After W. K. Clifford, E. Study applied dual numbers and dual vectors to his study on kinematics and line geometry [6]. There exist several articles with regard to algebraic study of dual numbers (see [1] and [2]). This nice notion was first performed by Kotelnikov to mechanics. Besides, the notion is often used in several fields of fundamental sciences such as astronomy, algebraic geometry, quantum mechanics and Riemannian geometry. For more details, we refer the reader to [3]-[12].

The set $\mathrm{D}^n = \left\{ \overrightarrow{\overline{x}} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n) \mid \overline{x}_i \in \mathrm{D}, 1 \leq i \leq n \right\}$ is a module over the ring D according to the operators

$$\overrightarrow{\overline{x}} + \overrightarrow{\overline{y}} = (\overline{x}_1 + \overline{y}_1, \overline{x}_2 + \overline{y}_2, ..., \overline{x}_n + \overline{y}_n)$$

and

$$\overline{\lambda} \overrightarrow{\overline{x}} = \left(\overline{\lambda} \overline{x}_1, \overline{\lambda} \overline{x}_2, ..., \overline{\lambda} \overline{x}_n \right).$$

This module is called D$-$module or dual space. The elements of D^n are called dual vectors and a dual vector $\overrightarrow{\overline{x}}$ can be expressed as

$$\overrightarrow{\overline{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x}^*,$$

where $\overrightarrow{x}$ and $\overrightarrow{x}^*$ are real vectors in $\mathbb{R}^n$ [1].

The dual function $\langle , \rangle_{\mathrm{D}} : \mathrm{D}^n \times \mathrm{D}^n \to \mathrm{D}$,

$$\left\langle \overrightarrow{\overline{x}}, \overrightarrow{\overline{y}} \right\rangle_{\mathrm{D}} = \langle \overrightarrow{x}, \overrightarrow{y} \rangle + \varepsilon \left(\langle \overrightarrow{x}, \overrightarrow{y}^* \rangle + \langle \overrightarrow{x}^*, \overrightarrow{y} \rangle \right)$$

is called dual inner product function on D^n, where the notation $\langle , \rangle$ is Euclidean inner product on $\mathbb{R}^n$.

Similar to dual inner product function, dual norm function $\|.\|_{\mathrm{D}} : \mathrm{D}^n \to \mathrm{D}$ is defined as follows:

$$\left\| \overrightarrow{\overline{x}} \right\|_{\mathrm{D}} = \begin{cases} 0 & , \overrightarrow{x} = \overrightarrow{0} \\ \| \overrightarrow{x} \| + \varepsilon \dfrac{\langle \overrightarrow{x}, \overrightarrow{x}^* \rangle}{\| \overrightarrow{x} \|} & , \overrightarrow{x} \neq \overrightarrow{0}, \end{cases}$$

where the notation $\|.\|$ is Euclidean norm on $\mathbb{R}^n$.

Given the vectors $\overrightarrow{e}_i = \left(\overline{\delta}_{i1}, \overline{\delta}_{i2}, ..., \overline{\delta}_{in}\right)$, where

$$\overline{\delta}_{ij} = \begin{cases} 1+\varepsilon 0 & , i=j \\ 0+\varepsilon 0 & , i\neq j \end{cases}, \ 1 \leq i,j \leq n,$$

the set $\left\{\overrightarrow{\overline{e}}_1, \overrightarrow{\overline{e}}_2, ..., \overrightarrow{\overline{e}}_n\right\}$ is standard basis of D^n. It turns out that every dual vector $\overrightarrow{\overline{x}} \in \mathrm{D}^n$ can be written in the form

$$\overrightarrow{\overline{x}} = \overline{x}_1 \overrightarrow{\overline{e}}_1 + \overline{x}_2 \overrightarrow{\overline{e}}_2 + ... + \overline{x}_n \overrightarrow{\overline{e}}_n,$$

where $\overrightarrow{\overline{e}}_i = \overrightarrow{e}_i + \varepsilon \overrightarrow{0}$ for $1 \leq i \leq n$.

Consider that $\overline{x} = x + \varepsilon x^*$ and $\overline{y} = y + \varepsilon y^*$ are dual numbers. The relation $\overline{x} <_{\mathrm{D}} \overline{y}$ (resp. $\overline{x} \leq_{\mathrm{D}} \overline{y}$) between these dual numbers is as follows (see [13], [14]):

1) Firstly, one compares the real parts of these dual numbers and must be $x < y$ (resp. $x < y$).

2) If the real parts of these dual numbers are the same, one compares their dual parts and must be $x^* < y^*$ (resp. $x^* \leq y^*$).

We can infer that there exist the following relations:

$$\overline{x} <_{\mathrm{D}} \overline{y} \Leftrightarrow x < y \text{ or } \left(x = y \text{ and } x^* < y^*\right)$$

and

$$\overline{x} \leq_{\mathrm{D}} \overline{y} \Leftrightarrow x < y \text{ or } \left(x = y \text{ and } x^* \leq y^*\right).$$

For the historical development of the term derivative, the expression "The derivative was first used, then discovered, and then studied and developed and finally defined." was used. The reason for using this expression is development process of the derivative starting with P. de Fermat in 1630s, continuing with I. Newton, J. L. Lagrange, G.W. Leibniz, A. L. Cauchy and reaching maturity in the 1870s with K. Weierstrass. The approaches to the derivative put forward by Leibniz and Newton were sufficient to find answers to the questions about the tangent of the curve and the velocity of the bodies. In fact, in the 19th century, this concept reached a consistent and solid foundation with the definition of derivative created by Cauchy using the term limit. It is well known that Cauchy put forward the first popularly acceptable account of the fundamental notions of the calculus. In order to prove the theorems related to the derivative, he used his own definitions. He described the derivative $\xi'(x)$ of a continuous function ξ as the limit when it exists, of the ratio $\dfrac{\xi(x+h)-\xi(x)}{h}$ as h went to zero. The instantaneous rate of change is entitled by the derivative. A comparison of the change in one quantity to the simultaneous change in a second quantity is expressed as a rate of change. Many of today's important problems in several fields such as engineering, biology, chemistry, physics, economics, involve finding the rate at which one quantity changes with respect to another, that is, they involve finding the derivative [15].

Topology is a mathematical discipline which originated at the turn of the 20th century. On the other hand, some isolated results about topology can be traced back several centuries. In mathematics, topology is interested in the properties of a geometric object which is preserved under continuous deformations including twisting, crumpling, stretching and bending. For many years, topology has been one of the most influential and exciting fields of research in modern mathematics. Topology is used for application fields such as physics, computer science, biology, robotics, fiber art, puzzles and games. Besides, topology has lots of applications in several branches of mathematics including differential equations, knot theory, dynamical systems, and Riemann surfaces in complex analysis. It also has some applications for describing the space-time structure of universe and analyzing many biological systems such as nanostructure and molecules, and in string theory in physics (see [16]- [29]).

In this paper, using the order relation on dual numbers, we obtain the topology on D^n denoted by $\overline{\tau}_{\overline{d}}$. Then, how the analyticity conditions of a dual function which is often expressed in other studies are obtained is given clearly. Making use of this topology, dual analytic areas of dual analytic functions are determined. Besides, inner and external operations on the set constituted by dual analytic functions are given. With the help of these operations, some properties regarding dual analytic functions are expressed and proved. The relations between the elements of dual space and real space which will be used to define the basic concepts of differential geometry are examined. The terms dual tangent space, dual directional derivative, dual vector field and dual tangent map which are the basic tools of differential geometry are given in detail. The concepts of injective function, surjective function, inverse function and diffeomorphism in dual space is firstly expressed in this study. Some theorems related to these terms are obtained and proved. The foundation of term surface in dual space is constituted via these terms [13].

2. On Dual Analytic Functions

Firstly, we shall study the concept of topology generating the basic structure of theory of curves and surfaces given by means of the expression of distance function in dual space. Previously, we talked about this basis [13]. After constructing a topology structure in dual space, we will determine the dual analytic regions of dual analytic functions by means of this topology.

Theorem 1. *Given the sets*

$$\begin{aligned} \overline{B}\left(\overline{a},\overline{r}\right) &= \{\overline{x}=x+\varepsilon x^*\in\mathrm{D}^n \mid \|x-a\|<r,\ x^*\in\mathbb{R}\} \\ &\quad \cup\left\{\overline{x}=x+\varepsilon x^*\in\mathrm{D}^n \mid \|x-a\|=r \ \textit{and}\ \frac{\langle x-a,x^*-a^*\rangle}{\|x-a\|}<r^*\right\} \\ &= U_1\cup U_2 \\ &= U_1\cup C_1\cup\ldots\cup C_k,\ \ (k\in I=\{1,2,\ldots\}) \end{aligned}$$

and

$$U_3=\left\{\overline{x}=x+\varepsilon x^*\in\mathrm{D}^n \mid x=a',\ m<x_1^*<n,\ x_{j+1}^*=c_j\in\mathbb{R},\ m,n\in[-\infty,\infty]\right\},$$

then a collection of all the sets $U_1, U_3, C_1, ..., C_k$ $(k \in I)$ *forms a basis* $\overline{\beta}$ *on* D^n, *where* $\overline{a} = a + \varepsilon a^* \in \mathrm{D}^n$, $r \in \mathbb{R}^+$, $r^* \in \mathbb{R}$ *and* $1 \leq j \leq n-1$.

Proof. It is enough to remark that two conditions given in definition of the term basis are satisfied.

i) It is easily seen that

$$\bigcup_{\overline{A} \in \overline{\beta}} \overline{A} = \mathrm{D}^n.$$

ii) The set $\overline{A}_1 \cap \overline{A}_2$ is an arbitrary union of some sets belonging to class $\overline{\beta}$ for all $\overline{A}_1, \overline{A}_2 \in \overline{\beta}$ expect for $\overline{A}_1 \cap \overline{A}_2 = \varnothing$. Now, let us show that this expression is correct. Suppose that $\overline{y}$ belongs to $\overline{A}_1 \cap \overline{A}_2$. Taking into account the sets $\overline{B}_1, \overline{B}_2, U_3'$ and U_3'', the following situations hold, where

$$\begin{aligned}
\overline{B}_1(\overline{a}_1, \overline{r}_1) &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_1\| < r_1, \ x^* \in \mathbb{R}^n\} \\
&\quad \cup \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_1\| = r_1 \text{ and } \frac{\langle x - a_1, x^* - a_1^* \rangle}{\|x - a_1\|} < r_1^*\right\} \\
&= U_1 \cup C_1' \cup ... \cup C_l',
\end{aligned}$$

$$\begin{aligned}
\overline{B}_2(\overline{a}_2, \overline{r}_2) &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_2\| < r_2, \ x^* \in \mathbb{R}^n\} \\
&\quad \cup \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a_2\| = r_2 \text{ and } \frac{\langle x - a_2, x^* - a_2^* \rangle}{\|x - a_2\|} < r_2^*\right\} \\
&= U_1' \cup C_1'' \cup ... \cup C_{l'}'',
\end{aligned}$$

$$U_3' = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = b', \ m_1 < x_1^* < n_1, \ x_{j+1}^* = c_j' \in \mathbb{R}, \ m_1, n_1 \in [-\infty, \infty]\right\}$$

and

$$U_3'' = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = b'', \ m_2 < x_1^* < n_2, \ x_{j+1}^* = c_j'' \in \mathbb{R}, \ m_2, n_2 \in [-\infty, \infty]\right\}.$$

1) Suppose that $\overline{y} \in U_3' \cap U_3''$. The following set can be written:

$$U_3' \cap U_3'' = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = a, \ m < x_1^* < n, \ x_{j+1}^* = c_j \in \mathbb{R}, \ m, n \in [-\infty, \infty]\right\} \in \overline{\beta},$$

where $y = b' = b'' = a$, $m < y_1^* < n$, $y_{j+1}^* = c_j' = c_j'' = c_j \in \mathbb{R}$, $m = \max\{m_1, m_2\}$ and $n = \min\{n_1, n_2\}$.

2) Assume that $\overline{y} \in U_1 \cap U_3''$. Hence, it is clear that $U_1 \cap U_3'' = U_3'' \in \overline{\beta}$.

3) Suppose that $\overline{y} \in C_l' \cap U_3''$ for any $l \in I$. In this case, the set $C_l' \cap U_3''$ can be written as

$$U_3^j = \left\{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = a_j, \ m_j < x_1^* < n_j, \ x_{j+1}^* = c_j' \in \mathbb{R}, \ m_j, n_j \in [-\infty, \infty]\right\}.$$

Therefore, $C_l' \cap U_3'' \in \overline{\beta}$.

4) Assume that $\overline{y} \in U_1 \cap U_1'$. The set $U_1 \cap U_1'$ can be written as an arbitrary union of the sets

$$U = \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a\| < r, \ x^* \in \mathbb{R}^n\}.$$

5) Suppose that $\overline{y} \in U_1 \cap C''_{l'}$ for any $l' \in I$. It is easy to check that $U_1 \cap C''_{l'} = C''_{l'} \in \overline{\beta}$.

6) Assume that $\overline{y} \in C'_l \cap C''_{l'}$ for any $l, l' \in I$. The set $C'_l \cap C''_{l'}$ is expressed as $C_l \in \overline{\beta}$, for $l \in I$ or an arbitrary union of the sets U_3 belonging to class $\overline{\beta}$.

With these conventions, we have

$$\overline{A}_1 \cap \overline{A}_2 = \bigcup_{\overline{A} \in \mathring{A} \subseteq \overline{\beta}} \overline{A}$$

for all $\overline{A}_1, \overline{A}_2 \in \overline{\beta}$ expect for $\overline{A}_1 \cap \overline{A}_2 = \varnothing$, where the class $\mathring{A}$ is a class of some sets belonging to the class $\overline{\beta}$. □

Definition 1. *The class $\overline{\beta}$ given in the above mentioned theorem is called dual basis on D^n. The topology obtained from this basis is symbolized as $\overline{\tau}_{\overline{d}}$. Each element of this topology is termed by dual open set.*

Theorem 2. *Suppose that the class of the sets*

$$\begin{aligned} \overline{U} &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - a\| < r, x^* \in \mathbb{R}^n\} \\ &= U \times \mathbb{R}^n \end{aligned}$$

belonging to the topology $\overline{\tau}_{\overline{d}}$ is symbolized as $\overline{\beta}_1$, where U is open set with respect to the standard topology of $\mathbb{R}^n$. Then the class $\overline{\beta}_1$ also constitutes a basis on D^n and the relationship between the topology $\overline{\tau}$ obtained from this basis and the topology $\overline{\tau}_{\overline{d}}$ is $\overline{\tau} \subseteq \overline{\tau}_{\overline{d}}$.

For example; let us study the topology $\overline{\tau}_{\overline{d}}$ on D. Assume that

$$\begin{aligned} \overline{B}(\overline{a}, \overline{r}) &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid |x - a| < r, x^* \in \mathbb{R}\} \\ &\quad \cup \left\{ \overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid |x - a| = r \text{ and } \frac{(x-a)(x^* - a^*)}{|x-a|} < r^* \right\} \\ &= U_1 \cup U_2 \\ &= U_1 \cup C_1 \cup C_2, \end{aligned}$$

where $\overline{a} = a + \varepsilon a^* \in \mathrm{D}$, $r \in \mathbb{R}^+$, $r^* \in \mathbb{R}$ and

$$\begin{aligned} U_2 &= \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid x = a + r,\ x^* < a^* + r^*\} \\ &\quad \cup \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid x = a - r,\ x^* > a^* - r^*\} \\ &= C_1 \cup C_2. \end{aligned}$$

Taking into consideration the set

$$U_3 = \{\overline{x} = x + \varepsilon x^* \in \mathrm{D} \mid x = a',\ m < x^* < n,\ m, n \in [-\infty, \infty]\},$$

the collection of the sets U_1, C_1, C_2 and U_3 forms a basis on D. The topology obtained from this basis is symbolized as $\overline{\tau}_{\overline{d}}$. Besides, the collection of the sets $U_1 \cup U_2$ and U_3 is also a basis on D and the topology generated by this basis is also $\overline{\tau}_{\overline{d}}$.

Observe that

$$B = \left\{ \widetilde{x} = (x, x^*) \in \mathbb{R}^2 \mid a < x < b,\ c < x^* < d,\ a, b, c, d \in \mathbb{R} \right\}.$$

The collection of all the sets B forms a basis on $\mathbb{R}^2$. If the topology generated by this basis is symbolized as τ_1, the relationship between $\overline{\tau}_{\overline{d}}$ and τ_1 is $\tau_1 \subseteq \overline{\tau}_{\overline{d}}$. On the other hand, if the topology derived from the collection of only the sets U_1 is symbolized as $\overline{\tau}$, then there exists the following relationship:

$$\overline{\tau} \subseteq \tau_1 \subseteq \overline{\tau}_{\overline{d}}.$$

Definition 2. *Let $\overline{x} = x + \varepsilon x^*$ be a dual variable. The function $\overline{\xi} : \mathrm{D} \to \mathrm{D}$ of the dual variable $\overline{x} = x + \varepsilon x^*$ is defined as follows:*

$$\overline{\xi}\left(\overline{x}\right) = \xi\left(x, x^*\right) + \varepsilon \xi^0\left(x, x^*\right),$$

where ξ and ξ^0 are real functions of the two real variables x and x^*.

In the following theorem, by eliminating the deficiencies in other studies, we shall discuss analyticity conditions of dual functions.

Theorem 3. *The dual function $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \mathrm{D}$, $\overline{\xi}\left(\overline{x} = x + \varepsilon x^*\right) = \xi\left(x, x^*\right) + \varepsilon \xi^0\left(x, x^*\right)$ is said to be analytic at the point $\overline{x} \in \overline{U}$ if and only if the functions ξ and ξ^0 have continuous partial derivatives ξ_x and ξ^0_x and there exist the equalities $\xi_{x^*} = 0$ and $\xi^0_{x^*} = \xi_x$, where $\xi_x = \dfrac{\partial \xi}{\partial x}$.*

Proof. Firstly, let the dual function $\overline{\xi}$ be analytic at the point $\overline{x} \in \overline{U}$. Thus, this assumption permits us to write the following relation:

$$\frac{d\overline{\xi}}{d\overline{x}} = \lim_{\overline{h} \to \overline{0}} \frac{\overline{\xi}\left(\overline{x} + \overline{h}\right) - \overline{\xi}\left(\overline{x}\right)}{\overline{h}}. \tag{1}$$

Observe that $\overline{x} = x + \varepsilon x^*$ and $\overline{h} = h + \varepsilon h^*$. By definition of dual variable functions and $\varepsilon^2 = 0$, the following equality holds:

$$\begin{aligned}
\frac{d\overline{\xi}}{d\overline{x}} &= \lim_{\overline{h} \to \overline{0}} \frac{\overline{\xi}\left(\overline{x} + \overline{h}\right) - \overline{\xi}\left(\overline{x}\right)}{\overline{h}} \\
&= \lim_{(h,h^*) \to (0,0)} \frac{\xi\left(x + h, x^* + h^*\right) + \varepsilon \xi^0\left(x + h, x^* + h^*\right) - \xi\left(x, x^*\right) - \varepsilon \xi^0\left(x, x^*\right)}{h + \varepsilon h^*} \\
&= \lim_{(h,h^*) \to (0,0)} \frac{\xi\left(x + h, x^* + h^*\right) - \xi\left(x, x^*\right)}{h} \\
&\quad + \lim_{(h,h^*) \to (0,0)} \varepsilon \begin{pmatrix} \frac{\xi^0(x+h, x^*+h^*) - \xi^0(x, x^*)}{h} \\ -\frac{h^*}{h^2}\left(\xi\left(x + h, x^* + h^*\right) - \xi\left(x, x^*\right)\right) \end{pmatrix} \\
&= \frac{\partial \xi}{\partial x} + \varepsilon \frac{\partial \xi^0}{\partial x}.
\end{aligned}$$

In view of equation (1), it is seen that the limit for $(h,h^*)\to(0,0)$ of real part of the expression $\dfrac{\overline{\xi}\left(\overline{x}+\overline{h}\right)-\overline{\xi}\left(\overline{x}\right)}{\overline{h}}$ is $\dfrac{\partial\xi}{\partial x}$. Then, it is easy to check that

$$\begin{aligned}&\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}\\ =\ &\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*+h^*\right)}{h}+\frac{\xi\left(x,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}.\end{aligned}\tag{2}$$

From the hypothesis and the equality (2), we have

$$\lim_{(h,h^*)\to(0,0)}\frac{\xi\left(x,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}=0.$$

If this limit exists and equals to zero, it is obvious from discussion that

$$\xi\left(x,x^*+h^*\right)-\xi\left(x,x^*\right)=0$$

such that $\xi\left(x,x^*\right)=\xi\left(x\right)$. Thus, the function ξ depends only on the variable x, i.e., $\dfrac{\partial\xi}{\partial x^*}=0$. It is well known from equation (1) that the limit for $(h,h^*)\to(0,0)$ of dual part of the expression $\dfrac{\overline{\xi}\left(\overline{x}+\overline{h}\right)-\overline{\xi}\left(\overline{x}\right)}{\overline{h}}$ is $\dfrac{\partial\xi^0}{\partial x}$. By some calculations, the following equality holds:

$$\begin{aligned}&\frac{\xi^0\left(x+h,x^*+h^*\right)-\xi^0\left(x,x^*\right)}{h}-\frac{\xi\left(x+h,x^*+h^*\right)-\xi\left(x,x^*\right)}{h}\frac{h^*}{h}\\ =\ &\frac{\xi^0\left(x+h,x^*+h^*\right)-\xi^0\left(x,x^*+h^*\right)}{h}\\ &+\frac{\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)}{h}-\frac{\xi\left(x+h\right)-\xi\left(x\right)}{h}\frac{h^*}{h}.\end{aligned}\tag{3}$$

From the hypothesis and the equality (3), we get

$$\lim_{h^*\to0}\left(\lim_{h\to0}\frac{h\left(\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)\right)-h^*\left(\xi\left(x+h\right)-\xi\left(x\right)\right)}{h^2}\right)=0.\tag{4}$$

Since the statement

$$\lim_{h\to0}\frac{h\left(\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)\right)-h^*\left(\xi\left(x+h\right)-\xi\left(x\right)\right)}{h^2}$$

has the indefiniteness $\left(\frac{0}{0}\right)$, we write the following equality:

$$\lim_{h^*\to0}\left(\lim_{h\to0}\frac{\left(\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)\right)-h^*\xi_x\left(x+h\right)}{2h}\right)=0.\tag{5}$$

From (5), we obtain

$$\xi^0\left(x,x^*+h^*\right)-\xi^0\left(x,x^*\right)=h^*\xi_x\left(x\right).$$

Therefore, it is possible to express that

$$\frac{\xi^0(x, x^* + h^*) - \xi^0(x, x^*)}{h^*} = \xi_x(x),$$

where $h^* \neq 0$. The limit of both sides of this identity for $h^* \to 0$ is $\xi^0_{x^*} = \xi_x$.

Conversely, suppose that the functions ξ and ξ^0 have continuous partial derivatives ξ_x and ξ^0_x and there are the equalities $\xi_{x^*} = 0$ and $\xi^0_{x^*} = \xi_x$. The expression of dual function $\overline{\xi}$ is simplified to the following form

$$\overline{\xi}(\overline{x}) = \xi(x) + \varepsilon\left(x^*\xi'(x) + \widetilde{\xi}(x)\right), \tag{6}$$

where $\xi \in C^2$, $\widetilde{\xi} \in C^1$. Given a point $\overline{x} \in \overline{U}$, we must show that the expression $\lim\limits_{\overline{h}\to\overline{0}} \frac{\overline{\xi}(\overline{x}+\overline{h}) - \overline{\xi}(\overline{x})}{\overline{h}}$ exists. From the equality (6), the derivative of the dual function $\overline{\xi}$ with respect to dual variable $\overline{x}$ can be expressed as follows:

$$I = \lim_{\overline{h}\to\overline{0}} \frac{\overline{\xi}(\overline{x}+\overline{h}) - \overline{\xi}(\overline{x})}{\overline{h}} = \lim_{(h,h^*)\to(0,0)} \left[\begin{array}{c} \frac{\xi(x+h)-\xi(x)}{h} \\ +\varepsilon\left(\begin{array}{c} x^*\left(\frac{\xi'(x+h)-\xi'(x)}{h}\right) + \frac{\widetilde{\xi}(x+h)-\widetilde{\xi}(x)}{h} \\ +\frac{h^*}{h}\xi'(x+h) - \frac{\xi(x+h)-\xi(x)}{h}\frac{h^*}{h} \end{array}\right) \end{array}\right].$$

From the hypothesis, we have

$$\begin{aligned} I_1 &= \lim_{(h,h^*)\to(0,0)} \frac{\xi(x+h)-\xi(x)}{h} = \xi'(x), \\ I_2 &= \lim_{(h,h^*)\to(0,0)} x^*\left(\frac{\xi'(x+h)-\xi'(x)}{h}\right) = x^*\xi''(x), \\ I_3 &= \lim_{(h,h^*)\to(0,0)} \frac{\widetilde{\xi}(x+h)-\widetilde{\xi}(x)}{h} = \widetilde{\xi}'(x), \\ I_4 &= \lim_{(h,h^*)\to(0,0)} \frac{h^*}{h}\xi'(x+h) - \frac{\xi(x+h)-\xi(x)}{h}\frac{h^*}{h} = 0 \end{aligned}$$

such that

$$I = I_1 + \varepsilon(I_2 + I_3 + I_4) = \xi'(x) + \varepsilon\left(x^*\xi''(x) + \widetilde{\xi}'(x)\right).$$

Thus, this obviously completes the proof of the theorem. □

We are now ready to state the following corollaries.

Corollary 1. *Theorem 3 implies that the derivative of dual function* $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \mathrm{D}$ *with respect to dual variable* $\overline{x}$ *is*

$$\frac{d\overline{\xi}}{d\overline{x}} = \lim_{\Delta\overline{x}\to\overline{0}} \frac{\overline{\xi}(\overline{x}+\Delta\overline{x}) - \overline{\xi}(\overline{x})}{\Delta\overline{x}}.$$

This limit is independent of the ratio $\frac{\Delta x^*}{\Delta x}$ *[30].*

Corollary 2. *Taking into account Theorem 3, the analyticity conditions of dual function* $\overline{\xi}:\overline{U}\subseteq \mathrm{D}\to \mathrm{D}$, $\overline{\xi}\left(\overline{x}\right)=\xi\left(x,x^{*}\right)+\varepsilon\xi^{0}\left(x,x^{*}\right)$ *are* $\dfrac{\partial\xi}{\partial x^{*}}=0$ *and* $\dfrac{\partial\xi^{0}}{\partial x^{*}}=\dfrac{\partial\xi}{\partial x}$. *Thus, the general representation of dual analytic functions is*

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x,x^{*}\right)+\varepsilon\xi^{0}\left(x,x^{*}\right)=\xi\left(x\right)+\varepsilon\left(x^{*}\xi'\left(x\right)+\widetilde{\xi}\left(x\right)\right),$$

where $\xi,\widetilde{\xi}:U\subseteq\mathbb{R}\to\mathbb{R}$ *and* $\xi\in C^{2}$, $\widetilde{\xi}\in C^{1}$. *In the proof of Theorem 3, it is clearly seen that the derivative of this analytic function* $\overline{\xi}$ *with respect to dual variable* $\overline{x}$ *is*

$$\frac{d\overline{\xi}}{d\overline{x}}=\frac{\partial\xi}{\partial x}+\varepsilon\frac{\partial\xi^{0}}{\partial x}=\xi'\left(x\right)+\varepsilon\left(x^{*}\xi''\left(x\right)+\widetilde{\xi}'\left(x\right)\right)$$

[30].

Now, based on Theorem 3, let us determine the analyticity conditions of dual function $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^{n}\to\mathrm{D}$,

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x_{1},...,x_{n},x_{1}^{*},...,x_{n}^{*}\right)+\varepsilon\xi^{0}\left(x_{1},...,x_{n},x_{1}^{*},...,x_{n}^{*}\right)=\xi+\varepsilon\xi^{0}.$$

The partial derivatives of dual function $\overline{\xi}$ at any dual point $\overline{a}\in\overline{U}\subseteq\mathrm{D}^{n}$ (if there exists) are

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_{i}}\left(\overline{a}\right)=\lim_{\Delta\overline{x}_{i}\to\overline{0}}\frac{\overline{\xi}\left(\overline{a}_{1},...,\overline{a}_{i}+\Delta\overline{x}_{i},...,\overline{a}_{n}\right)-\overline{\xi}\left(\overline{a}_{1},...,\overline{a}_{n}\right)}{\Delta\overline{x}_{i}},\ 1\leq i\leq n.$$

The above formula is simplified to the following form:

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_{i}}\left(\overline{a}\right)=\frac{d}{d\overline{x}_{i}}\overline{\xi}\left(\overline{a}_{1},...,\overline{x}_{i},...,\overline{a}_{n}\right)\mid_{\overline{x}_{i}=\overline{a}_{i}}=\lim_{\overline{x}_{i}\to\overline{a}_{i}}\frac{\overline{\mu}\left(\overline{x}_{i}\right)-\overline{\mu}\left(\overline{a}_{i}\right)}{\overline{x}_{i}-\overline{a}_{i}},$$

where $\overline{\mu}\left(\overline{x}_{i}\right)=\overline{\xi}\left(\overline{a}_{1},...,\overline{x}_{i},...,\overline{a}_{n}\right)$. When Theorem 3 is taken into consideration, one can check that if this limit exists, for $1\leq i\leq n$, then the functions ξ and ξ^{0} have continuous partial derivatives $\xi_{x_{i}}$ and $\xi^{0}_{x_{i}}$ at any dual point $\overline{a}\in\overline{U}$ and these relations $\dfrac{\partial\xi}{\partial x_{i}^{*}}=0$ and $\dfrac{\partial\xi^{0}}{\partial x_{i}^{*}}=\dfrac{\partial\xi}{\partial x_{i}}$ are satisfied. From Theorem 3, it is easy to see that the reverse exists. This result follows by proceeding as in the proof of the first assertion. Thus, these conventions permit us to write the following relation:

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_{i}}\left(\overline{a}\right)=\frac{\partial\xi}{\partial x_{i}}\left(a_{1},...,a_{n},a_{1}^{*},...,a_{n}^{*}\right)+\varepsilon\frac{\partial\xi^{0}}{\partial x_{i}}\left(a_{1},...,a_{n},a_{1}^{*},...,a_{n}^{*}\right).$$

Besides, the expression $\displaystyle\lim_{\Delta\overline{x}_{i}\to\overline{0}}\frac{\Delta\overline{\xi}}{\Delta\overline{x}_{i}}$ is independent of the ratio $\dfrac{\Delta x_{i}^{*}}{\Delta x_{i}}$. Note that the analyticity conditions of dual function $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^{n}\to\mathrm{D}$ are $\dfrac{\partial\xi}{\partial x_{i}^{*}}=0$ and $\dfrac{\partial\xi^{0}}{\partial x_{i}^{*}}=\dfrac{\partial\xi}{\partial x_{i}}$ $(1\leq i\leq n)$. In view of these equalities, we can write the following expressions:

$$\xi\left(x_{1},...,x_{n},x_{1}^{*},...,x_{n}^{*}\right)=\xi\left(x_{1},...,x_{n}\right)$$

and

$$\xi^0\left(x_1,...,x_n,x_1^*,...,x_n^*\right)=\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}\left(x_1,...,x_n\right),$$

where $\xi\in C^2$, $\widetilde{\xi}\in C^1$. By definition of the analyticity conditions of dual function $\overline{\xi}:\overline{U}\subseteq \mathrm{D}^n\to\mathrm{D}$, the general representation of these dual analytic functions is

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x_1,...,x_n\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}\left(x_1,...,x_n\right)\right). \tag{7}$$

The partial derivatives of this function with respect to dual variables $\overline{x}_j$ are

$$\frac{\partial\overline{\xi}}{\partial\overline{x}_j}=\frac{\partial\xi}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\xi}{\partial x_j\partial x_i}+\frac{\partial\widetilde{\xi}}{\partial x_j}\right)$$

$(1\leq j\leq n)$. Throughout this paper, the functions ξ and $\widetilde{\xi}$ will be considered as belonging to $C^\infty-$class. Note that the sets of the topology $\overline{\tau}$ mentioned in Theorem 2 is dual analytic regions of dual analytic functions. The set of dual analytic functions is symbolized as $C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$. Therefore, the following expression holds:

$$C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)=\left\{\overline{\xi}\mid\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D},\ \overline{\xi}\left(\overline{x}\right)=\xi\left(x\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}\left(x\right)\right)\right\}.$$

Given the dual functions $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}^m$, $\overline{\xi}=\left(\overline{\xi}_1,...,\overline{\xi}_m\right)$, we conclude that if the dual functions $\overline{\xi}_j:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D},(1\leq j\leq m)$ are dual analytic, then the dual function $\overline{\xi}$ is dual analytic. When the above information is taken into consideration, the following functions can be defined:

i) $+_C:C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)\times C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)\to C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$, for $\overline{\xi},\overline{\mu}\in C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$ and $\overline{x}\in\overline{U}\subseteq\mathrm{D}^n$, we have

$$\left(\overline{\xi}+_C\overline{\mu}\right)\left(\overline{x}\right)=\overline{\xi}\left(\overline{x}\right)+\overline{\mu}\left(\overline{x}\right)=\xi\left(x\right)+\mu\left(x\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\left(\xi+\mu\right)}{\partial x_i}+\widetilde{\xi}\left(x\right)+\widetilde{\mu}\left(x\right)\right).$$

ii) $\cdot_C:\mathrm{D}\times C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)\to C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$, for $\overline{\xi}\in C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$, $\overline{\lambda}=\lambda+\varepsilon\lambda^*\in\mathrm{D}$ and $\overline{x}\in\overline{U}\subseteq\mathrm{D}^n$, we have

$$\left(\overline{\lambda}\cdot_C\overline{\xi}\right)\left(\overline{x}\right)=\overline{\lambda}\cdot\overline{\xi}\left(\overline{x}\right)=\lambda\xi\left(x\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\left(\lambda\xi\right)}{\partial x_i}+\lambda\widetilde{\xi}\left(x\right)+\lambda^*\xi\left(x\right)\right).$$

iii) $\cdot_{1_C}:C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)\times C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)\to C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$, for $\overline{\xi},\overline{\mu}\in C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$ and $\overline{x}\in\overline{U}\subseteq\mathrm{D}^n$, we have

$$\left(\overline{\xi}\cdot_{1_C}\overline{\mu}\right)(\overline{x})=\overline{\xi}(\overline{x})\cdot\overline{\mu}(\overline{x})=\xi(x)\mu(x)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial(\xi\mu)}{\partial x_i}+\xi(x)\widetilde{\mu}(x)+\widetilde{\xi}(x)\mu(x)\right)$$

[31].

We are interested now to some properties regarding dual analytic functions.

Proposition 1. *Consider $\overline{\mu}:\overline{I}\subseteq \mathrm{D}\to\mathrm{D}^n$ and $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}$ are dual analytic functions, where the functions $\overline{\mu}$ and $\overline{\xi}$ are as below:*

$$\overline{\mu}\left(\overline{t}\right)=\mu(t)+\varepsilon\left(t^*\mu'(t)+\widetilde{\mu}(t)\right)$$

and

$$\overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}(x)\right)$$

such that the functions $\xi,\widetilde{\xi},\mu$ and $\widetilde{\mu}$ belong to $C^\infty-$class. If the functions $\overline{\xi}$ and $\overline{\mu}$ are dual analytic at the dual points $\overline{\mu}\left(\overline{t}\right)$ and $\overline{t}$, respectively, then the composition of $\overline{\mu}$ and $\overline{\xi}$,i.e., $\overline{\xi}\circ\overline{\mu}$ is dual analytic function. The derivative of this dual analytic function with respect to dual variable $\overline{t}$ is

$$\frac{d}{d\overline{t}}\left(\overline{\xi}\circ\overline{\mu}\right)\left(\overline{t}\right)=(\xi\circ\mu)'(t)+\varepsilon\left(\begin{array}{c} t^*(\xi\circ\mu)''(t)+\left\langle\widetilde{\mu}(t),\sum\limits_{i=1}^{n}\left(\frac{\partial\xi}{\partial x_i}\circ\mu\right)(t)\overrightarrow{e}_i\right\rangle' \\ +\left(\widetilde{\xi}\circ\mu\right)'(t)\end{array}\right),$$

where $(\xi\circ\mu)'(t)=\dfrac{d}{dt}(\xi\circ\mu)(t)$.

Theorem 4. *Let $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\ \mathrm{D}$ be dual analytic function. Then the following identity holds*

$$\frac{\partial^2\overline{\xi}}{\partial\overline{x}_k\partial\overline{x}_j}=\frac{\partial^2\overline{\xi}}{\partial\overline{x}_j\partial\overline{x}_k}\qquad(1\leq j,k\leq n),$$

for any dual point of $\overline{U}\subseteq\mathrm{D}^n$.

Proof. Let $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}$ be dual analytic function. From the equality (7), we can write

$$\overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}+\widetilde{\xi}(x)\right),$$

where $\xi,\widetilde{\xi}\in C^\infty$. The partial derivatives of dual function $\overline{\xi}$ with respect to dual variable $\overline{x}_j$ are

$$\begin{aligned}\frac{\partial\overline{\xi}}{\partial\overline{x}_j} &= \frac{\partial\xi}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\xi}{\partial x_j\partial x_i}+\frac{\partial\widetilde{\xi}}{\partial x_j}\right)\\ &= \frac{\partial\xi}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2\xi}{\partial x_i\partial x_j}+\frac{\partial\widetilde{\xi}}{\partial x_j}\right).\end{aligned}$$

The above formula are simplified to the following form

$$\frac{\partial \overline{\xi}}{\partial \overline{x}_j} = \mu(x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial \mu}{\partial x_i} + \widetilde{\mu}(x) \right),$$

where $\dfrac{\partial \xi}{\partial x_j} = \mu(x)$ and $\dfrac{\partial \widetilde{\xi}}{\partial x_j} = \widetilde{\mu}(x)$, i.e., $\mu, \widetilde{\mu} \in C^{\infty}$. Thus, we deduce that $\dfrac{\partial \overline{\xi}}{\partial \overline{x}_j} \in C\left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D}\right)$. In analogous to the derivative $\dfrac{\partial \overline{\xi}}{\partial \overline{x}_j}$, the partial derivatives of dual function $\dfrac{\partial \overline{\xi}}{\partial \overline{x}_j}$ with respect to dual variable $\overline{x}_k$ are

$$\begin{aligned} \frac{\partial^2 \overline{\xi}}{\partial \overline{x}_k \partial \overline{x}_j} &= \frac{\partial \mu}{\partial x_k} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 \mu}{\partial x_k \partial x_i} + \frac{\partial \widetilde{\mu}}{\partial x_k} \right) \\ &= \frac{\partial \mu}{\partial x_k} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 \mu}{\partial x_i \partial x_k} + \frac{\partial \widetilde{\mu}}{\partial x_k} \right), \end{aligned}$$

where $\dfrac{\partial \mu}{\partial x_k} = \dfrac{\partial^2 \xi}{\partial x_k \partial x_j} = \dfrac{\partial^2 \xi}{\partial x_j \partial x_k}$ and $\dfrac{\partial \widetilde{\mu}}{\partial x_k} = \dfrac{\partial^2 \widetilde{\xi}}{\partial x_k \partial x_j} = \dfrac{\partial^2 \widetilde{\xi}}{\partial x_j \partial x_k}$. Therefore, this yields

$$\frac{\partial^2 \overline{\xi}}{\partial \overline{x}_k \partial \overline{x}_j} = \frac{\partial^2 \xi}{\partial x_j \partial x_k} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial}{\partial x_i} \left(\frac{\partial^2 \xi}{\partial x_j \partial x_k} \right) + \frac{\partial^2 \widetilde{\xi}}{\partial x_j \partial x_k} \right). \tag{8}$$

On the other hand, it is easy to compute

$$\frac{\partial^2 \overline{\xi}}{\partial \overline{x}_j \partial \overline{x}_k} = \frac{\partial^2 \xi}{\partial x_j \partial x_k} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial}{\partial x_i} \left(\frac{\partial^2 \xi}{\partial x_j \partial x_k} \right) + \frac{\partial^2 \widetilde{\xi}}{\partial x_j \partial x_k} \right). \tag{9}$$

Comparing these two equations (8) and (9), we have $\dfrac{\partial^2 \overline{\xi}}{\partial \overline{x}_k \partial \overline{x}_j} = \dfrac{\partial^2 \overline{\xi}}{\partial \overline{x}_j \partial \overline{x}_k}$. Thus, this achieves the proof. □

Remark 1. *On the set* $\mathbb{R}^n \times \mathbb{R}^n = \{(x, x^*) \mid x, x^* \in \mathbb{R}^n\}$, *the equality, inner operation and external operation can be defined as follows:*

(*i*) *For any* $(x, x^*), (y, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$, *we get*

$$(x, x^*) = (y, y^*) \Leftrightarrow x = y \text{ and } x^* = y^*.$$

(*ii*) $+_1 : (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n$, *for* $(x, x^*), (y, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$, *we get*

$$(x, x^*) +_1 (y, y^*) = (x + y, x^* + y^*).$$

(*iii*) $\cdot_1 : \mathrm{D} \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n$, *for* $(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n$ *and* $\overline{\lambda} = \lambda + \varepsilon \lambda^* \in \mathrm{D}$, *we get*

$$\overline{\lambda} \cdot_1 (x, x^*) = (\lambda x, \lambda x^* + \lambda^* x).$$

According to the above operations, the set $(\mathbb{R}^n \times \mathbb{R}^n, +_1, \cdot_1)$ *constitutes a module over the set* $(\mathrm{D}, +, \cdot)$.

We are now ready to express the following theorem:

Theorem 5. *Let the sets* $(\mathbb{R}^n \times \mathbb{R}^n, +_1, \cdot_1)$ *and* $(\mathrm{D}^n, +, \cdot)$ *be modules over the set* $(\mathrm{D}, +, \cdot)$. *Then the function* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{D}^n$, $f(x, x^*) = x + \varepsilon x^*$ *is a (module) isomorphism.*

Proof. It is easy to check that f is bijective function. Now, for $(x, x^*), (y, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\overline{\lambda} = \lambda + \varepsilon\lambda^* \in \mathrm{D}$, the following equality can be written

$$\begin{aligned} f\left(\overline{\lambda} \cdot_1 (x, x^*) +_1 (y, y^*)\right) &= f(\lambda x + y, \lambda x^* + \lambda^* x + y^*) \\ &= \lambda x + y + \varepsilon(\lambda x^* + \lambda^* x + y^*) \\ &= (\lambda + \varepsilon\lambda^*)(x + \varepsilon x^*) + (y + \varepsilon y^*) \\ &= \overline{\lambda} f(x, x^*) + f(y, y^*) \end{aligned}$$

such that f is a (module) linear function. In view of these conventions, we deduce that f is a (module) isomorphism. This permits us to conclude the proof. □

Theorem 6. *The real vector space* $\mathbb{R}^n$ *is isomorphic to a subset of* D^n *defined as* $\overline{A} = \left\{ \overrightarrow{\overline{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{0} \mid \overrightarrow{x} \in \mathbb{R}^n \right\}$ *[32].*

Definition 3. *Let* $\{x_1, ..., x_n, x_1^*, ..., x_n^*\}$ *be coordinate functions of* $\mathbb{R}^{2n}$ *and* $\widetilde{p} = (p_1, ..., p_n, p_1^*, ..., p_n^*) \in \mathbb{R}^{2n}$. *Then we have*

$$\widetilde{x}_i = (x_i, x_i^*) : \mathbb{R}^{2n} \to \mathbb{R} \times \mathbb{R}, \ \widetilde{x}_i(\widetilde{p}) = (x_i(\widetilde{p}), x_i^*(\widetilde{p})),$$

where $x_i : \mathbb{R}^{2n} \to \mathbb{R}$, $x_i(\widetilde{p}) = p_i$ *and* $x_i^* : \mathbb{R}^{2n} \to \mathbb{R}$, $x_i^*(\widetilde{p}) = p_i^*$. *Since the function* $h_n : \mathrm{D}^n \to \mathbb{R}^{2n}$, $h_n(\overline{p}) = \widetilde{p}$ *is bijective function, we can write the following diagram:*

$$\begin{array}{ccc} \mathrm{D}^n & \xrightarrow{\overline{x}_i} & \mathrm{D} \\ h_n \downarrow & & \downarrow h_1 \\ \mathbb{R}^{2n} & \xrightarrow[\widetilde{x}_i]{} & \mathbb{R} \times \mathbb{R} \end{array}$$

such that dual coordinate functions $\overline{x}_i$ *can be stated by* $\overline{x}_i = h_1^{-1} \circ \widetilde{x}_i \circ h_n$. *Therefore, for dual coordinate functions* $\overline{x}_i$ $(1 \leq i \leq n)$, *we obtain*

$$\overline{x}_i(\overline{p}) = x_i(\widetilde{p}) + \varepsilon x_i^*(\widetilde{p}) = p_i + \varepsilon p_i^* = \overline{p}_i,$$

where $\overline{p} = (\overline{p}_1, ..., \overline{p}_n) \in \mathrm{D}^n$ *and* $\overline{p}_i = p_i + \varepsilon p_i^* \in \mathrm{D}$.

Definition 4. *Suppose that* $\overline{p} \in \mathrm{D}^n$ *is a dual point and* $\overrightarrow{\overline{x}} \in \mathrm{D}^n$ *is a dual vector. On the set*

$$T_{\overline{p}}\mathrm{D}^n = \{\overline{p}\} \times \mathrm{D}^n = \left\{ \left(\overline{p}, \overrightarrow{\overline{x}}\right) \mid \overrightarrow{\overline{x}} \in \mathrm{D}^n \right\},$$

equality, inner operation and external operation can be determined as follows:

(i) For any $\left(\overline{p},\overrightarrow{\overline{x}}\right)$ *and* $\left(\overline{q},\overrightarrow{\overline{y}}\right)$*, we have*

$$\left(\overline{p},\overrightarrow{\overline{x}}\right)=\left(\overline{q},\overrightarrow{\overline{y}}\right)\Leftrightarrow\overline{p}=\overline{q}\text{ and }\overrightarrow{\overline{x}}=\overrightarrow{\overline{y}}.$$

(ii) $\oplus:T_{\overline{p}}\mathrm{D}^n\times T_{\overline{p}}\mathrm{D}^n\to T_{\overline{p}}\mathrm{D}^n$*, for* $\left(\overline{p},\overrightarrow{\overline{x}}\right),\left(\overline{p},\overrightarrow{\overline{y}}\right)\in T_{\overline{p}}\mathrm{D}^n$*, we have*

$$\left(\overline{p},\overrightarrow{\overline{x}}\right)\oplus\left(\overline{p},\overrightarrow{\overline{y}}\right)=\left(\overline{p},\overrightarrow{\overline{x}}+\overrightarrow{\overline{y}}\right).$$

(iii) $\odot:\mathrm{D}\times T_{\overline{p}}\mathrm{D}^n\to T_{\overline{p}}\mathrm{D}^n$ *for* $\left(\overline{p},\overrightarrow{\overline{x}}\right)\in T_{\overline{p}}\mathrm{D}^n$ *and* $\overline{\lambda}\in\mathrm{D}$*, we have*

$$\overline{\lambda}\odot\left(\overline{p},\overrightarrow{\overline{x}}\right)=\left(\overline{p},\overline{\lambda}\overrightarrow{\overline{x}}\right).$$

Corollary 3. *Taking into account the operations* $\oplus$ *and* $\odot$ *defined on the set* $T_{\overline{p}}\mathrm{D}^n=\{\overline{p}\}\times\mathrm{D}^n=\left\{\left(\overline{p},\overrightarrow{\overline{x}}\right)\mid\overrightarrow{\overline{x}}\in\mathrm{D}^n\right\}$*, this set generates a module over the set* $(\mathrm{D},+,\cdot)$*. This module* $(T_{\overline{p}}\mathrm{D}^n,\oplus,(\mathrm{D},+,.),\odot)$ *is called dual tangent space and every element of this module is entitled by dual tangent vector.*

Corollary 4. *When above defined operations* $\oplus$ *and* $\odot$ *is taken into consideration, every element* $\overrightarrow{\overline{x}}_{\overline{p}}=\left(\overline{p},\overrightarrow{\overline{x}}\right)$ *of* $T_{\overline{p}}\mathrm{D}^n$ *can be expressed by*

$$\overrightarrow{\overline{x}}_{\overline{p}}=\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)\oplus\varepsilon\odot\left(\overline{p},\overrightarrow{x}^*+\varepsilon\overrightarrow{0}\right),$$

where $\overrightarrow{\overline{x}}=\overrightarrow{x}+\varepsilon\overrightarrow{x}^*\in\mathrm{D}^n$*.*

Corollary 5. *Let us define the sets*

$$\Phi=\left\{\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)\mid\overline{p}\in\mathrm{D}^n,\overrightarrow{x}\in\mathbb{R}^n\right\}$$

and

$$\Psi=\left\{(\widetilde{p},(x_1,...,x_n,0,...,0))\mid\widetilde{p}\in\mathbb{R}^{2n},x_i\in\mathbb{R}\right\}.$$

The inner operation on the set Φ *(resp.* Ψ*) is*

$$\begin{aligned}\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)+_\Phi\left(\overline{p},\overrightarrow{y}+\varepsilon\overrightarrow{0}\right)&=\left(\overline{p},\overrightarrow{x}+\overrightarrow{y}+\varepsilon\overrightarrow{0}\right),\\(\widetilde{p},(x_1,...,x_n,0,...,0))+_\Psi(\widetilde{p},(y_1,...,y_n,0,...,0))&=(\widetilde{p},(x_1+y_1,...,x_n+y_n,0,...,0))\end{aligned}$$

and for $\lambda\in\mathbb{R}$*, the external operation on the set* Φ *(resp.* Ψ*) is*

$$\begin{aligned}\lambda\cdot_\Phi\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)&=\left(\overline{p},\lambda\overrightarrow{x}+\varepsilon\overrightarrow{0}\right),\\\lambda\cdot_\Psi(\widetilde{p},(x_1,...,x_n,0,...,0))&=(\widetilde{p},(\lambda x_1,...,\lambda x_n,0,...,0))\end{aligned}$$

such that the sets $(\Phi,+_\Phi,\cdot_\Phi)$ *and* $(\Psi,+_\Psi,\cdot_\Psi)$ *are* n*-dimensional vector spaces over the field* $(\mathbb{R},+,\cdot)$*.*

With these conventions, the following theorem can be given:

Theorem 7. *Consider that*

$$\Phi=\left\{\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)\mid\overline{p}\in\mathrm{D}^n,\overrightarrow{x}\in\mathbb{R}^n\right\}$$

and

$$\Psi=\left\{\left(\widetilde{p},(x_1,...,x_n,0,...,0)\right)\mid\widetilde{p}\in\mathbb{R}^{2n},x_i\in\mathbb{R}\right\}.$$

Then the function $g:(\Phi,+_\Phi,\cdot_\Phi)\to(\Psi,+_\Psi,\cdot_\Psi)$, $g\left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)=(\widetilde{p},(x_1,...,x_n,0,...,0))$ *is a isomorphism.*

Corollary 6. *From Theorem 7 and* $\overrightarrow{x}=(x_1,...,x_n)\tilde{=}(x_1,...,x_n,0,...,0)$, *every dual vector* $\overrightarrow{\overline{x}}_{\overline{p}}=\left(\overline{p},\overrightarrow{\overline{x}}\right)\in T_{\overline{p}}\mathrm{D}^n$ *can be written as*

$$\begin{aligned}\overrightarrow{\overline{x}}_{\overline{p}} &= \left(\overline{p},\overrightarrow{\overline{x}}\right)\\ &= \left(\overline{p},\overrightarrow{x}+\varepsilon\overrightarrow{0}\right)\oplus\varepsilon\odot\left(\overline{p},\overrightarrow{x}^*+\varepsilon\overrightarrow{0}\right)\\ &= (\widetilde{p},\overrightarrow{x})\oplus\varepsilon\odot(\widetilde{p},\overrightarrow{x}^*)\\ &= \overrightarrow{x}_{\widetilde{p}}\oplus\varepsilon\odot\overrightarrow{x}^*_{\widetilde{p}}.\end{aligned}$$

For simplicity, throughout this paper, the operations $+$ *and* $\cdot$ *is used instead of the operations* $\oplus$ *and* $\odot$*, respectively. Thus, this means that*

$$\overrightarrow{\overline{x}}_{\overline{p}}=\overrightarrow{x}_{\widetilde{p}}+\varepsilon\overrightarrow{x}^*_{\widetilde{p}}.$$

Also, it is possible to write the following equality:

$$\overrightarrow{\overline{x}}_{\overline{p}}=\left(\overline{p},\overrightarrow{\overline{x}}\right)=\left(\overline{p},\overline{x}_1\overrightarrow{e}_1+...+\overline{x}_n\overrightarrow{e}_n\right)=\overline{x}_1\overrightarrow{e}_{1\overline{p}}+...+\overline{x}_n\overrightarrow{e}_{n\overline{p}},$$

where $\overrightarrow{e}_{i\overline{p}}=\left(\overline{p},\overrightarrow{e}_i+\varepsilon\overrightarrow{0}\right)$. *Moreover, the equation* $\overline{\lambda}_1\overrightarrow{e}_{1\overline{p}}+...+\overline{\lambda}_n\overrightarrow{e}_{n\overline{p}}=\overrightarrow{\overline{0}}_{\overline{p}}$ *can only be satisfied by* $\overline{\lambda}_i=\overline{0}$ *for* $1\leq i\leq n$. *Thus, the set* $\left\{\overrightarrow{e}_{1\overline{p}},...,\overrightarrow{e}_{n\overline{p}}\right\}$ *forms a basis of dual tangent space* $T_{\overline{p}}\mathrm{D}^n$.

Theorem 8. *Assume that* $\overline{\xi}\in C\left(\overline{U}\subseteq\mathrm{D}^n,\mathrm{D}\right)$ *and* $\overrightarrow{\overline{x}}_{\overline{p}}\in T_{\overline{p}}\mathrm{D}^n$. *The derivative of dual analytic function* $\overline{\xi}$ *in the direction of dual tangent vector* $\overrightarrow{\overline{x}}_{\overline{p}}$ *is*

$$\overrightarrow{\overline{x}}_{\overline{p}}\left[\,\overline{\xi}\,\right]=\frac{d}{d\overline{t}}\overline{\xi}\left(\overline{p}+\overline{t}\,\overrightarrow{\overline{x}}\right)_{|\overline{t}=\overline{0}}=\overrightarrow{x}_{\widetilde{p}}[\xi]+\varepsilon\left(\sum_{i=1}^n p_i^*\overrightarrow{x}_{\widetilde{p}}[\xi_{x_i}]+\overrightarrow{x}_{\widetilde{p}}\left[\widetilde{\xi}\right]+\overrightarrow{x}^*_{\widetilde{p}}[\xi]\right),$$

where $\overrightarrow{x}_{\widetilde{p}}[\xi]=\sum\limits_{i=1}^n\dfrac{\partial\xi}{\partial x_i}\left(x\left(\widetilde{p}\right)\right)x_i$.

Proof. The proof can be easily made using definitions of dual tangent vector and composition of dual analytic functions. $\square$

Theorem 9. *For* $\overline{\xi}, \overline{\mu} \in C\left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D}\right)$, $\overrightarrow{\overline{x}}_{\overline{p}}, \overrightarrow{\overline{y}}_{\overline{p}} \in T_{\overline{p}}\mathrm{D}^n$ *and* $\overline{\lambda} \in \mathrm{D}$, *the following equalities exist:*

(*i*) $\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi} + \overline{\mu}\ \right] = \overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right] + \overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\mu}\ \right]$

(*ii*) $\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\lambda} \cdot \overline{\xi}\ \right] = \overline{\lambda}\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]$

(*iii*) $\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi} \cdot \overline{\mu}\ \right] = \overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right]\overline{\mu}\left(\overline{p}\right) + \overline{\xi}\left(\overline{p}\right)\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\mu}\ \right]$

(*iv*) $\left(\overrightarrow{\overline{x}}_{\overline{p}} + \overrightarrow{\overline{y}}_{\overline{p}}\right)\left[\ \overline{\xi}\ \right] = \overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}\ \right] + \overrightarrow{\overline{y}}_{\overline{p}}\left[\ \overline{\xi}\ \right]$.

Definition 5. *A dual vector field* $\overline{X}$ *on* D^n *is a function that assigns to each dual point* $\overline{p}$ *of* D^n *a dual tangent vector* $\overrightarrow{\overline{X}}_{\overline{p}}$ *to* D^n *at* $\overline{p}$, *i.e.,* $\overline{X} : \mathrm{D}^n \to T\mathrm{D}^n$,

$$\overline{X}\left(\overline{p}\right) = \overrightarrow{\overline{X}}_{\overline{p}} = \overrightarrow{X}_{\widetilde{p}} + \varepsilon \overrightarrow{X}^*_{\widetilde{p}},$$

where $\overrightarrow{\overline{X}} = \overrightarrow{X} + \varepsilon \overrightarrow{X}^*$. *Suppose that* $\overline{a}_i : \overline{U} \subseteq \mathrm{D}^n \to \mathrm{D}$, $\overline{a}_i = a_i + \varepsilon a_i^0$ $(1 \leq i \leq n)$ *are dual analytic functions. When the dual vector field can be written in the form* $\overline{X}\left(\overline{x}\right) = \left(\overline{a}_1\left(\overline{x}\right), ..., \overline{a}_n\left(\overline{x}\right)\right)$, *the equality can be rearranged as follows:*

$$\overline{X}\left(\overline{x}\right) = X\left(x\right) + \varepsilon\left(\sum_{j=1}^{n} x_j^* X_{x_j} + \widetilde{X}\left(x\right)\right),$$

where $X\left(x\right) = \left(a_1\left(x\right), ..., a_n\left(x\right)\right)$, $\widetilde{X}\left(x\right) = \left(\widetilde{a}_1\left(x\right), ..., \widetilde{a}_n\left(x\right)\right)$ *and the functions* a_i *and* $\widetilde{a}_i$ *belong to* $C^{\infty}-$*class for* $1 \leq i \leq n$. *The set of dual analytic vector fields is symbolized as* $\chi\left(\mathrm{D}^n\right)$. *Hence, it is possible to write below expression:*

$$\chi\left(\mathrm{D}^n\right) = \left\{\overline{X} \mid \overline{X} : \mathrm{D}^n \to T\mathrm{D}^n,\ \overrightarrow{\overline{X}}_{\overline{p}} = \overrightarrow{X}_{\widetilde{p}} + \varepsilon \overrightarrow{X}^*_{\widetilde{p}}\right\}.$$

We are now ready to introduce that the inner and external operations on $\chi\left(\mathrm{D}^n\right)$ *is described as below:*

(*i*) $+ : \chi\left(\mathrm{D}^n\right) \times \chi\left(\mathrm{D}^n\right) \to \chi\left(\mathrm{D}^n\right)$, *for* $\overrightarrow{\overline{X}}, \overrightarrow{\overline{Y}} \in \chi\left(\mathrm{D}^n\right)$ *and* $\overline{p} \in \mathrm{D}^n$, *we have*

$$\left(\overline{X} + \overline{Y}\right)\left(\overline{p}\right) = \overrightarrow{\overline{X}}_{\overline{p}} + \overrightarrow{\overline{Y}}_{\overline{p}}.$$

(*ii*) $\cdot : \mathrm{D} \times \chi\left(\mathrm{D}^n\right) \to \chi\left(\mathrm{D}^n\right)$, *for* $\overrightarrow{\overline{X}} \in \chi\left(\mathrm{D}^n\right)$, $\overline{\lambda} \in \mathrm{D}$ *and* $\overline{p} \in \mathrm{D}^n$, *we have*

$$\left(\overline{\lambda} \cdot \overline{X}\right)\left(\overline{p}\right) = \overline{\lambda} \cdot \overline{X}\left(\overline{p}\right) = \overline{\lambda} \cdot \overrightarrow{\overline{X}}_{\overline{p}}.$$

In view of above mentioned operations, the set $\left(\chi\left(\mathrm{D}^n\right), +, \cdot\right)$ *forms a module over the set* $\left(\mathrm{D}, +, \cdot\right)$.

Now, suppose that $\overline{\xi} \in C\left(\overline{U} \subseteq \mathrm{D}^n, \mathrm{D}\right)$. The derivative of dual analytic function $\overline{\xi}$ in the direction of dual analytic vector field $\overline{X}$ is

$$\overline{X}\left[\ \overline{\xi}\ \right] = X\left[\xi\right] + \varepsilon\left(\sum_{j=1}^{n} x_j^*\left(X\left[\xi\right]\right)_{x_j} + X\left[\widetilde{\xi}\right] + \widetilde{X}\left[\xi\right]\right),$$

where $X[\xi]=\sum\limits_{i=1}^{n}\dfrac{\partial\xi}{\partial x_i}a_i$, such that $\overline{X}\left[\ \overline{\xi}\ \right]\in C\left(\overline{U}\subseteq \mathrm{D}^n,\mathrm{D}\right)$. We can infer that for $\overline{p}\in\mathrm{D}^n$,

$$\overline{X}_{\overline{p}}\left[\ \overline{\xi}\ \right]=X_{\widetilde{p}}[\xi]+\varepsilon\left(\sum_{j=1}^{n}x_j^*(\widetilde{p})\,(X[\xi])_{x_j}(\widetilde{p})+X_{\widetilde{p}}\left[\widetilde{\xi}\right]+\widetilde{X}_{\widetilde{p}}[\xi]\right).$$

Definition 6. *Suppose that* $\overline{\xi}:\overline{U}\subseteq\mathrm{D}^n\to\mathrm{D}^m$, $\overline{\xi}=\left(\overline{\xi}_1,...,\overline{\xi}_m\right)$ *is a dual analytic function. The function* $\overline{\xi}_{*\overline{p}}:T_{\overline{p}}\overline{U}\to T_{\overline{\xi}(\overline{p})}\mathrm{D}^m$ *is called a dual tangent map of the function* $\overline{\xi}$ *at the dual point* $\overline{p}$, *where*

$$\begin{aligned}\overline{\xi}_{*\overline{p}}\left(\overrightarrow{\overline{x}}_{\overline{p}}\right) &= \left(\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}_1\right],...,\overrightarrow{\overline{x}}_{\overline{p}}\left[\ \overline{\xi}_m\right]\right)|_{\overline{q}=\overline{\xi}(\overline{p})}\\ &= \xi_{*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)+\varepsilon\left(\sum_{j=1}^{n}p_j^*\xi_{x_j*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)+\widetilde{\xi}_{*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)+\xi_{*\widetilde{p}}\left(\overrightarrow{x}^*_{\widetilde{p}}\right)\right)\end{aligned}$$

and

$$\xi_{*\widetilde{p}}\left(\overrightarrow{x}_{\widetilde{p}}\right)=\left(\overrightarrow{x}_{\widetilde{p}}[\xi_1],...,\overrightarrow{x}_{\widetilde{p}}[\xi_m]\right).$$

In that case, it turns out that $\overline{\xi}_*:\chi(\ \mathrm{D}^n)\to\chi(\mathrm{D}^m)$,

$$\overline{\xi}_*\left(\overline{X}\right)=\xi_*(X)+\varepsilon\left(\sum_{j=1}^{n}x_j^*\left(\xi_*(X)\right)_{x_j}+\widetilde{\xi}_*(X)+\xi_*\left(\widetilde{X}\right)\right),$$

where $\xi_*(X)=(X[\xi_1],...,X[\xi_m])$.

Theorem 10. $\overline{\xi}_{*\overline{p}}:T_{\overline{p}}\mathrm{D}^n\to T_{\overline{\xi}(\overline{p})}\mathrm{D}^m$ *is a (module) linear map and the matrix (dual Jacobian matrix) corresponding to this linear map with respect to the bases* $\left\{\overrightarrow{\overline{e}}_{1\overline{p}},...,\overrightarrow{\overline{e}}_{n\overline{p}}\right\}$ *and* $\left\{\overrightarrow{\overline{e}}_{1\overline{q}},...,\overrightarrow{\overline{e}}_{m\overline{q}}\right\}$ *is*

$$\begin{aligned}\overline{J}\left(\overline{\xi},\overline{p}\right) &= \begin{bmatrix}\frac{\partial\xi_1}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_1}{\partial x_n}(\widetilde{p})\\ \frac{\partial\xi_2}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_2}{\partial x_n}(\widetilde{p})\\ \vdots & \ddots & \vdots\\ \frac{\partial\xi_m}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_m}{\partial x_n}(\widetilde{p})\end{bmatrix}+\varepsilon\begin{bmatrix}\frac{\partial\xi_1^0}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_1^0}{\partial x_n}(\widetilde{p})\\ \frac{\partial\xi_2^0}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_2^0}{\partial x_n}(\widetilde{p})\\ \vdots & \ddots & \vdots\\ \frac{\partial\xi_m^0}{\partial x_1}(\widetilde{p}) & \cdots & \frac{\partial\xi_m^0}{\partial x_n}(\widetilde{p})\end{bmatrix}\\ &= J(\xi,\widetilde{p})+\varepsilon J\left(\xi^0,\widetilde{p}\right)\\ &= J(\xi,\widetilde{p})+\varepsilon\left(\sum_{j=1}^{n}p_j^*J\left(\xi_{x_j},\widetilde{p}\right)+J\left(\widetilde{\xi},\widetilde{p}\right)\right),\end{aligned}$$

where $\dfrac{\partial\xi_j^0}{\partial x_i}=\sum\limits_{k=1}^{n}x_k^*\dfrac{\partial^2\xi_j}{\partial x_i\partial x_k}+\dfrac{\partial\widetilde{\xi}_j}{\partial x_i}$.

Remark 2. *Assume that* $\overline{\xi}:\overline{U}\subseteq \mathrm{D}^n\to \mathrm{D}$ *is dual analytic function, where* $\overline{U}=U\times\mathbb{R}^n$*. Then we know that*

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x_1,...,x_n\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial\xi}{\partial x_i}\left(x_1,...,x_n\right)+\widetilde{\xi}\left(x_1,...,x_n\right)\right).$$

The value of this function at the dual point $\overline{x}=\overline{p}$ *is*

$$\begin{aligned}\overline{\xi}\left(\overline{p}\right) &= \xi\left(x_1\left(\widetilde{p}\right),...,x_n\left(\widetilde{p}\right)\right)+\varepsilon\left(\begin{array}{c}\sum_{i=1}^{n}x_i^*\left(\widetilde{p}\right)\frac{\partial\xi}{\partial x_i}\left(x_1\left(\widetilde{p}\right),...,x_n\left(\widetilde{p}\right)\right)\\+\widetilde{\xi}\left(x_1\left(\widetilde{p}\right),...,x_n\left(\widetilde{p}\right)\right)\end{array}\right)\\ &= \xi\left(p_1,...,p_n\right)+\varepsilon\left(\sum_{i=1}^{n}p_i^*\frac{\partial\xi}{\partial x_i}\left(p_1,...,p_n\right)+\widetilde{\xi}\left(p_1,...,p_n\right)\right).\end{aligned}$$

As a result, the functions ξ *and* $\widetilde{\xi}$ *can be reduced to the functions defined from* $U\subseteq\mathbb{R}^n$ *to* $\mathbb{R}$ *such that these functions belong to* $C^\infty-$*class.*

Definition 7. *Assume that* $\overline{\xi}:\overline{U}\subseteq\mathrm{D}\to\mathrm{D}$, $\overline{\xi}\left(\overline{x}\right)=\xi\left(x\right)+\varepsilon\left(x^*\xi'\left(x\right)+\widetilde{\xi}\left(x\right)\right)$ *is a dual analytic function and* $\xi'\left(x\right)$ *is not zero for all* $x\in U\subseteq\mathbb{R}$*. If the equality* $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ *requires the equality* $\overline{x}_1=\overline{x}_2$ *for all* $\overline{x}_1,\overline{x}_2\in\overline{U}\subseteq\mathrm{D}$*, then the function* $\overline{\xi}$ *is called injective function.*

Theorem 11. *Assume that* $\overline{\xi}:\overline{U}\subseteq\mathrm{D}\to\mathrm{D}$, $\overline{\xi}\left(\overline{x}\right)=\xi\left(x\right)+\varepsilon\left(x^*\xi'\left(x\right)+\widetilde{\xi}\left(x\right)\right)$ *is a dual analytic function and* $\xi'\left(x\right)$ *is not zero for all* $x\in U\subseteq\mathbb{R}$*. Then* ξ *is injective function if and only if the dual analytic function* $\overline{\xi}$ *is injective function.*

Proof. Suppose that $\overline{\xi}:\overline{U}\subseteq\mathrm{D}\to\mathrm{D}$, $\overline{\xi}\left(\overline{x}\right)=\xi\left(x\right)+\varepsilon\left(x^*\xi'\left(x\right)+\widetilde{\xi}\left(x\right)\right)$ is a dual analytic function, $\xi'\left(x\right)$ is not zero for all $x\in U\subseteq\mathbb{R}$ and ξ is injective function. Assume that there exists the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ for all $\overline{x}_1,\overline{x}_2\in\overline{U}\subseteq\mathrm{D}$. From the definition of dual analytic functions, the following equality can be written:

$$\overline{\xi}\left(\overline{x}_1\right)=\xi\left(x_1\right)+\varepsilon\left(x_1^*\xi'\left(x_1\right)+\widetilde{\xi}\left(x_1\right)\right)=\xi\left(x_2\right)+\varepsilon\left(x_2^*\xi'\left(x_2\right)+\widetilde{\xi}\left(x_2\right)\right)=\overline{\xi}\left(\overline{x}_2\right)$$

which implies

$$\xi\left(x_1\right)=\xi\left(x_2\right)$$

and

$$x_1^*\xi'\left(x_1\right)+\widetilde{\xi}\left(x_1\right)=x_2^*\xi'\left(x_2\right)+\widetilde{\xi}\left(x_2\right).$$

From the hypothesis, since ξ is an injective function, it is clear that $x_1=x_2$. On the other hand, we have $\widetilde{\xi}\left(x_1\right)=\widetilde{\xi}\left(x_2\right)$, since $\widetilde{\xi}$ is a well-defined function. Hence, since $\xi'\left(x\right)$ is not zero for all $x\in U\subseteq\mathbb{R}$ and $\widetilde{\xi}\left(x_1\right)=\widetilde{\xi}\left(x_2\right)$, it is easily seen that $x_1^*=x_2^*$. That is to say, $\overline{x}_1=\overline{x}_2$. Therefore, $\overline{\xi}$ is an injective function.

Conversely, we shall prove this part of theorem by means of contrapositive method. Assume that ξ is not an injective function. That is to say, the equality $\xi\left(x_1\right)=\xi\left(x_2\right)$ requires the inequality $x_1\neq x_2$ for at least $x_1,x_2\in U\subseteq\mathbb{R}$.

We must show that dual analytic function $\overline{\xi}$ is not an injective function. It is enough to remark that the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ requires $\overline{x}_1 \neq \overline{x}_2$ for at least $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$. Suppose that there exists the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ for at least $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$. Thus, this gives rise to the relation

$$\overline{\xi}\left(\overline{x}_1\right)=\xi\left(x_1\right)+\varepsilon\left(x_1^* \xi'\left(x_1\right)+\widetilde{\xi}\left(x_1\right)\right)=\xi\left(x_2\right)+\varepsilon\left(x_2^* \xi'\left(x_2\right)+\widetilde{\xi}\left(x_2\right)\right)=\overline{\xi}\left(\overline{x}_2\right),$$

i.e.,

$$\xi\left(x_1\right)=\xi\left(x_2\right)$$

and

$$x_1^* \xi'\left(x_1\right)+\widetilde{\xi}\left(x_1\right)=x_2^* \xi'\left(x_2\right)+\widetilde{\xi}\left(x_2\right).$$

As already known, since ξ is not an injective function, the equality $\overline{\xi}\left(\overline{x}_1\right)=\overline{\xi}\left(\overline{x}_2\right)$ requires the expression $\overline{x}_1 \neq \overline{x}_2$ for at least $\overline{x}_1, \overline{x}_2 \in \overline{U} \subseteq \mathrm{D}$, that is, dual analytic function $\overline{\xi}$ is not an injective function. Therefore, the proof is completed. □

Definition 8. *Assume that* $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{V} \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ *is a dual analytic function and* $\xi'(x)$ *is not zero for all* $x \in U \subseteq \mathbb{R}$. *If there exists at least one* $\overline{x}=x+\varepsilon x^* \in \overline{U} \subseteq \mathrm{D}$ *satisfying the equality* $\overline{y}=\overline{\xi}(\overline{x})$ *for all* $\overline{y}=y+\varepsilon y^* \in \overline{V} \subseteq \mathrm{D}$, *then the dual analytic function* $\overline{\xi}$ *is called surjective function. That is to say, if* ξ *is surjective function and there exists* $x^* \in \mathbb{R}$ *satisfying the equality* $x^*=\dfrac{y^*-\widetilde{\xi}(x)}{\xi'(x)}$ *for all* $y^* \in \mathbb{R}$, *then dual analytic function* $\overline{\xi}$ *is called surjective function.*

Definition 9. *Assume that* $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{V} \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ *is a dual analytic function and* $\xi'(x)$ *is not zero for all* $x \in U \subseteq \mathbb{R}$. *If dual analytic function* $\overline{\xi}$ *is bijective function, there is only one function* $\overline{\mu}: \overline{V} \subseteq \mathrm{D} \rightarrow \overline{U} \subseteq \mathrm{D}$ *satisfying the equalities* $\left(\overline{\mu} \circ \overline{\xi}\right)(\overline{x})=\overline{I}(\overline{x})$ *and* $\left(\overline{\xi} \circ \overline{\mu}\right)(\overline{y})=\overline{I}(\overline{y})$, *where* $\overline{I}$ *is dual unit function. The function* $\overline{\mu}$ *is called inverse function of dual function* $\overline{\xi}$ *and the inverse function is symbolized as* $\overline{\mu}=\overline{\xi}^{-1}$.

Theorem 12. *Assume that* $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ *is a dual analytic function and* $\xi'(x)$ *is not zero for all* $x \in U \subseteq \mathbb{R}$. *If there exists inverse of dual function* $\overline{\xi}$, *it is expressed as*

$$\overline{\xi}^{-1}(\overline{y})=\xi^{-1}(y)+\varepsilon\left(y^*\left(\xi^{-1}\right)'(y)-\left(\widetilde{\xi} \circ \xi^{-1}\right)(y) \cdot\left(\xi^{-1}\right)'(y)\right),$$

where ξ^{-1} *is inverse of real function* ξ.

Proof. Suppose that $\overline{\xi}: \overline{U} \subseteq \mathrm{D} \rightarrow \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}, \overline{\xi}(\overline{x})=\xi(x)+\varepsilon\left(x^* \xi'(x)+\widetilde{\xi}(x)\right)$ is a dual analytic function, $\xi'(x)$ is not zero for all $x \in U \subseteq \mathbb{R}$ and there exists inverse of dual analytic function $\overline{\xi}$. Since the function ξ is bijective function, there is inverse of function ξ, i.e., ξ^{-1} such that $\left(\xi^{-1}\right)'(y) = \dfrac{1}{\xi'(x)}$ for all $x \in U \subseteq \mathbb{R}$. Thus, from the hypothesis, we have

$$\begin{aligned}\left(\overline{\xi}^{-1} \circ \overline{\xi}\right)(\overline{x}) &= \left(\xi^{-1} \circ \xi\right)(x) + \varepsilon \left(\begin{array}{c} x^* \left(\left(\xi^{-1} \circ \xi\right)'(x)\right) + \widetilde{\xi}(x) \left(\xi^{-1}\right)'(\xi(x)) \\ -\widetilde{\xi}\left(\xi^{-1}(\xi(x))\right) \left(\xi^{-1}\right)'(\xi(x)) \end{array}\right) \\ &= x + \varepsilon x^* \\ &= \overline{I}(\overline{x}),\end{aligned}$$

where $\overline{I} : \mathrm{D} \to \mathrm{D}$, $\overline{I}(\overline{x}) = x + \varepsilon x^*$. Similarly, we get $\left(\overline{\xi} \circ \overline{\xi}^{-1}\right)(\overline{y}) = \overline{I}(\overline{y})$. From the definition of equality in functions, we can write

$$\overline{\xi}^{-1} \circ \overline{\xi} = \overline{\xi} \circ \overline{\xi}^{-1} = \overline{I}.$$

Hence, this achieves the proof. $\square$

Corollary 7. *If there exists the inverse of dual analytic function* $\overline{\xi} : \overline{U} \subseteq \mathrm{D} \to \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}, \overline{\xi}(\overline{x}) = \xi(x) + \varepsilon\left(x^* \xi'(x) + \widetilde{\xi}(x)\right)$, *the inverse function is a dual analytic function expressed as follows:*

$$\overline{\xi}^{-1}(\overline{x}) = \xi^{-1}(x) + \varepsilon\left(x^* \left(\xi^{-1}\right)'(x) - \left(\widetilde{\xi} \circ \xi^{-1}\right)(x) \left(\xi^{-1}\right)'(x)\right).$$

The derivative of this dual analytic function with respect to dual variable $\overline{x}$ *is*

$$\frac{d\overline{\xi}^{-1}}{d\overline{x}} = \left(\xi^{-1}\right)'(x) + \varepsilon \left(\begin{array}{c} x^* \left(\xi^{-1}\right)''(x) - \left(\widetilde{\xi} \circ \xi^{-1}\right)(x) \left(\xi^{-1}\right)''(x) \\ -\widetilde{\xi}'\left(\xi^{-1}(x)\right) \left(\left(\xi^{-1}\right)'(x)\right)^2 \end{array}\right).$$

Definition 10. *Let* $\overline{\xi} : \overline{U} \subseteq \mathrm{D}^n \to \overline{V} \subseteq \mathrm{D}^n$ *be a dual analytic function, where*

$$\begin{aligned}\overline{\xi}(\overline{x}) &= \left(\overline{\xi}_1(\overline{x}), ..., \overline{\xi}_n(\overline{x})\right) \\ &= \left(\xi_1(x), ..., \xi_n(x)\right) + \varepsilon\left(\left(\sum_{i=1}^n x_i^* \frac{\partial \xi_1}{\partial x_i}, ..., \sum_{i=1}^n x_i^* \frac{\partial \xi_n}{\partial x_i}\right) + \left(\widetilde{\xi}_1(x), ..., \widetilde{\xi}_n(x)\right)\right).\end{aligned}$$

If there exists the inverse function $\overline{\xi}^{-1}$ *being dual analytic function, then the dual analytic function* $\overline{\xi}$ *is called dual diffeomorphism.*

Theorem 13. *Assume that* $\overline{\xi} : \mathrm{D}^n \to \mathrm{D}^n$, $\overline{\xi}(\overline{x}) = \left(\overline{\xi}_1(\overline{x}), ..., \overline{\xi}_n(\overline{x})\right) = \xi + \varepsilon\xi^0$ *is a dual analytic function. If* $rankJ(\xi, q) = n$ *for all* $q \in U$, *where* $\overline{q} = q + \varepsilon q^* \in \mathrm{D}^n$ *and* U *is open set in terms of standard topology of* $\mathbb{R}^n$, *then there is at least one dual open set* $\overline{U} \in \overline{\tau}$ *in* D^n *covering point* $\overline{q} \in \mathrm{D}^n$ *such that* $\overline{\xi}\mid_{\overline{U}}: \overline{U} \to \overline{\xi}\left(\overline{U}\right)$ *is dual diffeomorphism.*

Proof. Assume that $\overline{\xi} : \mathrm{D}^n \to \mathrm{D}^n$, $\overline{\xi}(\overline{x}) = \left(\overline{\xi}_1(\overline{x}), ..., \overline{\xi}_n(\overline{x})\right) = \xi + \varepsilon\xi^0$ is a dual analytic function and $rankJ(\xi, q) = n$ for all $q \in U$. We know that the functions ξ and $\widetilde{\xi}$ can be reduced to the functions defined from $U \subseteq \mathbb{R}^n$ to $\mathbb{R}$ such that these functions belong to $C^{\infty}-$class. Hence, the function $\overline{\xi}:\mathbb{D}^n\to\mathbb{D}^n$ can be expressed as

$$\overline{\xi}\left(\overline{x}\right)=\xi\left(x\right)+\varepsilon\left(\sum_{j=1}^{n}x_j^{*}\frac{\partial\xi}{\partial x_j}\left(x\right)+\widetilde{\xi}\left(x\right)\right).$$

Since the function $\xi:\mathbb{R}^n\to\mathbb{R}^n$ belongs to $C^{\infty}-$class and $rankJ\left(\xi,q\right)=n$ for all $q\in U$, $\xi\mid_U:U\to\xi\left(U\right)$ is a diffeomorphism.

Suppose that there is the equality $\overline{\xi}\left(\overline{p}\right)=\overline{\xi}\left(\overline{q}\right)$ for all $\overline{p},\overline{q}\in\overline{U}\subseteq\mathbb{D}^n$ $\left(p,q\in U\subseteq\mathbb{R}^n\right)$. Hence, we can write

$$\xi\left(p\right)+\varepsilon\left(\sum_{j=1}^{n}p_j^{*}\frac{\partial\xi}{\partial x_j}\left(p\right)+\widetilde{\xi}\left(p\right)\right)=\xi\left(q\right)+\varepsilon\left(\sum_{j=1}^{n}q_j^{*}\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right)\right),$$

which implies

$$\xi\left(p\right)=\xi\left(q\right)$$

and

$$\sum_{j=1}^{n}p_j^{*}\frac{\partial\xi}{\partial x_j}\left(p\right)+\widetilde{\xi}\left(p\right)=\sum_{j=1}^{n}q_j^{*}\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right).\tag{10}$$

Since $\xi\mid_U$ is injective function, we have $p=q$. From the equation (10), we get the following equality:

$$\left(p_1^{*}-q_1^{*}\right)\frac{\partial\xi}{\partial x_1}+...+\left(p_n^{*}-q_n^{*}\right)\frac{\partial\xi}{\partial x_n}=\left(0,...,0\right).$$

Since the set $\left\{\frac{\partial\xi}{\partial x_1},...,\frac{\partial\xi}{\partial x_n}\right\}$ is linearly independent, we have $p_i^{*}=q_i^{*}$ for $1\leq i\leq n$, i.e., $p^{*}=q^{*}$. That is to say, we can write $\overline{p}=p+\varepsilon p^{*}=q+\varepsilon q^{*}=\overline{q}$ such that the dual analytic function $\overline{\xi}\mid_{\overline{U}}$ is injective.

Now, let us show that there exists at least one $\overline{q}\in\overline{U}\subseteq\mathbb{D}^n$ satisfying the equality $\overline{p}=\overline{\xi}\left(\overline{q}\right)$ for all $\overline{p}\in\overline{\xi}\left(\overline{U}\right)\subseteq\mathbb{D}^n$. The equality

$$\overline{p}=p+\varepsilon p^{*}=\xi\left(q\right)+\varepsilon\left(\sum_{j=1}^{n}q_j^{*}\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right)\right)=\overline{\xi}\left(\overline{q}\right)$$

allows us to write

$$p=\xi\left(q\right)$$

and

$$p^{*}=\sum_{j=1}^{n}q_j^{*}\frac{\partial\xi}{\partial x_j}\left(q\right)+\widetilde{\xi}\left(q\right).\tag{11}$$

Since $\xi\mid_U$ is bijective function, there exists $q=\xi^{-1}\left(p\right)\in U\subseteq\mathbb{R}^n$. Expanding the equation (11), it is seen that the following linear equation system is obtained:

$$\begin{aligned}
q_1^* \frac{\partial \xi_1}{\partial x_1}(q) + \dots + q_n^* \frac{\partial \xi_1}{\partial x_n}(q) &= p_1^* - \widetilde{\xi}_1(q) \\
q_1^* \frac{\partial \xi_2}{\partial x_1}(q) + \dots + q_n^* \frac{\partial \xi_2}{\partial x_n}(q) &= p_2^* - \widetilde{\xi}_2(q) \\
&\vdots \\
q_1^* \frac{\partial \xi_n}{\partial x_1}(q) + \dots + q_n^* \frac{\partial \xi_n}{\partial x_n}(q) &= p_n^* - \widetilde{\xi}_n(q).
\end{aligned}$$

The matrix form of this linear equation system is

$$\begin{bmatrix} \frac{\partial \xi_1}{\partial x_1}(q) & \frac{\partial \xi_1}{\partial x_2}(q) & \cdots & \frac{\partial \xi_1}{\partial x_n}(q) \\ \frac{\partial \xi_2}{\partial x_1}(q) & \frac{\partial \xi_2}{\partial x_2}(q) & \cdots & \frac{\partial \xi_2}{\partial x_n}(q) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi_n}{\partial x_1}(q) & \frac{\partial \xi_n}{\partial x_2}(q) & \cdots & \frac{\partial \xi_n}{\partial x_n}(q) \end{bmatrix} \begin{bmatrix} q_1^* \\ q_2^* \\ \vdots \\ q_n^* \end{bmatrix} = \begin{bmatrix} p_1^* - \widetilde{\xi}_1(q) \\ p_2^* - \widetilde{\xi}_2(q) \\ \vdots \\ p_n^* - \widetilde{\xi}_n(q) \end{bmatrix}.$$

If we denote $[A]_{n\times n} = \left[\frac{\partial \xi_i}{\partial x_j}(q)\right]_{1\le i,j\le n}$ and $[B]_{n\times 1} = \left[p_i^* - \widetilde{\xi}_i(q)\right]_{1\le i\le n}$, the above matrix form can be rewritten as

$$[A]_{n\times n}[q^*]_{n\times 1} = [B]_{n\times 1}. \tag{12}$$

Since $rankJ(\xi, q) = n$ for all $q \in U \subseteq \mathbb{R}^n$, there exists an inverse of the matrix $[A]_{n\times n}$ such that $[q^*]_{n\times 1} = \left[A^{-1}\right]_{n\times n}[B]_{n\times 1}$. Therefore, there exists dual point $\overline{q} = q + \varepsilon q^* \in \overline{U} \subseteq \mathrm{D}^n$, that is, dual analytic function $\overline{\xi}\mid_{\overline{U}}$ is surjective.

With these conventions, there exists the inverse of dual analytic function $\overline{\xi}\mid_{\overline{U}}$ and this inverse function is $\overline{\mu} : \overline{\xi}\left(\overline{U}\right) \subseteq \mathrm{D}^n \to \overline{U} \subseteq \mathrm{D}^n$,

$$\overline{\mu}(\overline{y}) = \mu(y) + \varepsilon\left(\sum_{j=1}^{n} y_j^* \frac{\partial \mu}{\partial y_j} + \widetilde{\mu}(y)\right) = \mu + \varepsilon\mu^0,$$

where $\mu = (\xi\mid_U)^{-1}$ and $\widetilde{\mu}_i(y) = \left\langle -\widetilde{\xi}(\mu(y)), \nabla\mu_i(y)\right\rangle$ for $1 \le i \le n$. This fact can be verified as follows:

$$\left(\overline{\xi}\mid_{\overline{U}} \circ \overline{\mu}\right)(\overline{y}) = (\xi\mid_U \circ \mu)(y) + \varepsilon \begin{pmatrix} y_1^*\left(\frac{\partial \mu_1}{\partial y_1}\frac{\partial \xi}{\partial x_1}(\mu(y)) + \dots + \frac{\partial \mu_n}{\partial y_1}\frac{\partial \xi}{\partial x_n}(\mu(y))\right) \\ + \dots + \\ y_n^*\left(\frac{\partial \mu_1}{\partial y_n}\frac{\partial \xi}{\partial x_1}(\mu(y)) + \dots + \frac{\partial \mu_n}{\partial y_n}\frac{\partial \xi}{\partial x_n}(\mu(y))\right) \\ +\widetilde{\mu}_1(y)\frac{\partial \xi}{\partial x_1}(\mu(y)) + \dots + \widetilde{\mu}_n(y)\frac{\partial \xi}{\partial x_n}(\mu(y)) \\ +\widetilde{\xi}\mid_U(\mu(y)) \end{pmatrix}$$

$$
\begin{aligned}
&= \left(\xi\mid_U \circ\mu\right)(y)+\varepsilon\begin{pmatrix} y_1^*\dfrac{\partial\left(\xi\circ\mu\right)}{\partial y_1}+...+y_n^*\dfrac{\partial\left(\xi\circ\mu\right)}{\partial y_n} \\ -\widetilde{\xi}_1\left(\mu\left(y\right)\right)\dfrac{\partial\left(\xi\circ\mu\right)}{\partial y_1}-...-\widetilde{\xi}_n\left(\mu\left(y\right)\right)\dfrac{\partial\left(\xi\circ\mu\right)}{\partial y_n} \\ +\widetilde{\xi}\mid_U\left(\mu\left(y\right)\right)\end{pmatrix} \\
&= y+\varepsilon y^* \\
&= \overline{I}\left(\overline{y}\right).
\end{aligned}
$$

In analogous to $\left(\overline{\xi}\mid_{\overline{U}}\circ\overline{\mu}\right)\left(\overline{y}\right)=\overline{I}\left(\overline{y}\right)$, it is easy to check that $\left(\overline{\mu}\circ\overline{\xi}\mid_{\overline{U}}\right)\left(\overline{x}\right)=\overline{I}\left(\overline{x}\right)$. On the other hand, the dual function $\overline{\mu}=\left(\overline{\xi}\mid_{\overline{U}}\right)^{-1}$ is a dual analytic function, since $\dfrac{\partial\mu}{\partial y_i^*}=0$ and $\dfrac{\partial\mu^0}{\partial y_i^*}=\dfrac{\partial\mu}{\partial y_i}$ for $1\leq i\leq n$ and the functions μ and $\widetilde{\mu}$ belong to $C^\infty-$class. That is to say, $\overline{\xi}\mid_{\overline{U}}$ is dual diffeomorphism. □

3. Conclusion

The relation between some sets of the topology constituted in dual space and regions where dual analytic functions are analytic is explained in this paper. Besides, we can assert that it is possible to construct the concept of dual surface via the expression of the inverse function theorem in dual space.

Author Contribution Statements The authors have made substantial contributions to the analysis and interpretation of the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement The authors would like to express their thanks to the referees and editors for their valuable comments and suggestions.

References

[1] Clifford, W. K., Preliminary sketch of biquaternions, *Proc. London Math. Soc.*, 4 (1873), 381–395.
[2] Kandasamy, W. B. V., Smarandache, F., Dual Numbers, ZIP Publishing, Ohio, 2012.
[3] Yaglom, I. M., A Simple Non-Euclidean Geometry and Its Physical Basis, Springer-Verlag, New York, 1979.
[4] Li, S., Ge, Q. J., Rational Beizer line symmetric motions, *Trans. ASME J. Mech. Design*, 127(2) (2005), 222–226. https://doi.org/10.1115/DETC99/DAC-8654
[5] Kotelnikov, A. P., Screw Calculus and Some of Its Applications in Geometry and Mechanics, Annals of the Imperial University, Kazan, 1895.
[6] Study, E., Geometrie der Dynamen, Druck und Verlag von B.G. Teubner, Leipzig, 1903.
[7] Çöken, A. C., Görgülü, A., On the dual darboux rotation axis of the dual space curve, *Demonstratio Math.*, 35(2) (2002), 385–390. https://doi.org/10.1515/dema-2002-0219

[8] Ercan, Z., Yüce, S., On properties of the dual quaternions, *Eur. J. Pure Appl. Math.*, 4(2) (2011), 142–146.

[9] Pennestri, E., Stafenelli, R., Linear algebra and numerical algorithms using dual numbers, *Multibody Syst. Dyn.*, 18(3) (2007), 323–344. https://doi.org/10.1007/s11044-007-9088-9

[10] Soule, C., Rational K-Theory of The Dual Numbers of A ring of Algebraic Integers, Springer Lec. Notes 854, pp. 402-408, 1981.

[11] Veldkamp, G. R., On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, *Mechanism and Machine Theory 2*, 11 (1976), 141–156. https://doi.org/10.1016/0094-114X(76)90006-9

[12] Yang, A. T., Freudenstein, F., Application of dual number quaternion algebra to the analysis of spatial mechanisms, *Trans. ASME J. Appl. Mech.*, 31(2) (1964), 300–308. https://doi.org/10.1115/1.3629601

[13] Aktaş, B., Surfaces and Some Special Curves on These Surfaces in Dual Space (Turkish), Doctorial Dissertation, Kırıkkale University, Kırıkkale, 2020.

[14] Aktaş, B., Durmaz, O., Gündoğan, H., On the basic structures of dual space, *Facta Universitatis, Series: Mathematics and Informatics*, 35(1) (2020), 253–272. https://doi.org/10.22190/FUMI2001253A

[15] Zembat, İ. Ö., Özmantar, M. F., Bingölbali, E., Şandır, H., Delice, A., Tanımları ve Tarihsel Gelişimleriyle Matematiksel Kavramlar (1.baskı), Ankara:Pegem-Akademi, 2013.

[16] Croom, F. H., Principles of Topology, Saunders College Publishing, 1989.

[17] Richeson, D., Euler's Gem: The Polyhedron Formula and The Birth of Topology, Princeton University Press, 2008.

[18] Mashaghi, S., Jadidi, T., Koenderink, G., Mashaghi, A., Lipid nanotechnology, *Int. J. Mol. Sci.*, 14(2) (2013), 4242–4282. https://doi.org/10.3390/ijms14024242

[19] Adams, C., The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots, American Mathematical Society, 2004.

[20] Stadler, B. M. R., Stadler, P. F., Wagner, G. P., Fontana, W., The topology of the possible: formal spaces underlying patterns of evolutionary change, *Journal of Theoretical Biology*, 213(2) (2001), 241–274. https://doi.org/10.1006/jtbi.2001.2423

[21] Carlsson, G., Topology and data, *Bulletin (New Series) of the American Mathematical Society*, 46(2) (2009), 255–308. https://doi.org/10.1090/S0273-0979-09-01249-X

[22] Vickers, S., Topology via Logic, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 1996.

[23] Stephenson, C., Lyon, D., Hübler, A., Topological properties of a self assembled electrical network via ab initio calculation, *Scientific Reports,* 7 (2017).

[24] Cambou, A. D., Menon, N., Three dimensional structure of a sheet crumpled into a ball,*Proc. Natl. Acad. Sci. U.S.A.*, 108(36) (2011), 14741–14745. https://doi.org/10.1073/pnas.1019192108

[25] Yau, S., Nadis, S., The Shape of Inner Space, Basic Books, 2010.

[26] Craig, J. J., Introduction to Robotics: Mechanics and Control, 3rd Ed. Prentice-Hall, 2004.

[27] Farber, M., Invitation to Topological Robotics, European Mathematical Society, 2008.

[28] Horak, M., Disentangling topological puzzles by using knot theory, *Mathematics Magazine*, 79(5) (2006), 368–375. https://doi.org/10.1080/0025570X.2006.11953435

[29] Eckman, E., Connect the Shapes Crochet Motifs: Creative Techniques for Joining Motfis of all Shapes, Storey Publishing, 2012.

[30] Dimentberg, F. M., The Screw Calculus and its Applications to Mechanics, Foreing Technology Division, Wright-Patterson Air Force Base, Ohio, 1965.

[31] Durmaz, O., Aktaş, B., Gündoğan, H., New approaches on dual space, *Facta Universitatis, Series: Mathematics and Informatics*, 35(2) (2020), 437–458. https://doi.org/10.22190/FUMI2002437D

[32] Hacısalihoğlu, H. H., Hareket Geometrisi ve Kuaterniyonlar Teorisi, Gazi Üniversitesi Fen Fakültesi Yayınları, Ankara, 1983.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1121–1135 (2022)
DOI:10.31801/cfsuasmas.1086809
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: March 12, 2022; Accepted: June 8, 2022

(U,V)-LUCAS POLYNOMIAL COEFFICIENT RELATIONS OF THE BI-UNIVALENT FUNCTION CLASS

Arzu AKGÜL[1] and Timilehin Gideon SHABA[2]

[1]Department of Mathematics, Faculty of Arts and Science, Kocaeli University, Kocaeli, TURKEY
[2]Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, NIGERIA

Abstract. In geometric function theory, Lucas polynomials and other special polynomials have recently gained importance. In this study, we develop a new family of bi-univalent functions. Also we examined coefficient inequalities and Fekete-Szegö problem for this new family via these polynomials.

1. Introduction

Let $\mathfrak{A}$ denote the family of all functions $\theta(\xi)$ that are analytic in the unit disc $\mathfrak{U}=\{\xi:\xi\in C,|\xi|<1\}$ normalized by the conditions $\theta(0)=\theta'(0)-1=0$. Such a function $\theta(\xi)$ takes the form

$$\theta(\xi)=\xi+\sum_{r=2}^{\infty}n_r\xi^r \quad (\xi\in\mathfrak{U}). \tag{1}$$

Assume that $\mathcal{S}$ be the subclass of $\mathfrak{A}$ compose of univalent functions.

As a subclass of $\mathfrak{A}$, the class of bi-univalent functions was first presented by Lewin [18]. He indicated that $|n_2|\le 1.15$. After that, a lot of studies have been made about coefficient estimates. See for example [4, 10, 11, 14, 15, 27, 30–41]. According to the Koebe 1/4 theorem (see [12]), the range of every function $\theta\in\mathcal{S}$ contains the disc $d_\omega=\{\omega:|\omega|<0.25\}$, thus, for all $\theta\in\mathcal{S}$ with its inverse θ^{-1}, such that $\theta^{-1}(\theta(\xi))=\xi \quad (\xi\in\mathfrak{U})$ and $\theta(\theta^{-1}(\omega))=\omega, \quad (\omega:|\omega|<r_0(\theta); r_0(\theta)\ge 0.25)$ where $\theta^{-1}(\omega)$ is expressed as

$$\vartheta(\omega)=\omega-n_2\omega^2+(2n_2^2-n_3)\omega^3-(5n_2^3-5n_2n_3+n_4)\omega^4+\cdots. \tag{2}$$

2020 *Mathematics Subject Classification.* 30C45, 30C50.
Keywords. (U,V)-Lucas polynomial, bi-univalent analytic function, subordination.
akgulcagla@hotmail.com-Corresponding author; 0000-0001-7934-0339
shabatimilehin@gmail.com; 0000-0001-5881-9260.

Thus, a function $\theta \in \mathfrak{A}$ is said that bi-univalent in $\mathfrak{U}$, if both $\theta(\xi)$ and $\theta^{-1}(\omega)$ are univalent in $\mathfrak{U}$. Let we show the class of holomorphic and bi-univalent functions in $\mathfrak{U}$ by $\mathfrak{B}$.

It is known that some similar functions $\theta \in \mathcal{S}$ for instance the Koebe function $\kappa(\xi) = \xi/(1-\xi)^2$, its rotation function $\kappa_\varsigma(\xi) = \xi/(1-e^{i\varsigma}\xi)^2$, $\theta(\xi) = \xi - \xi^2/2$ and $\theta(\xi) = \xi/(1-\xi^2)$ are in $\mathfrak{B}$. Also some functions $\theta \in (\mathcal{S} \cap \mathfrak{B})$ contains $\theta(\xi) = \xi$, $\theta(\xi) = 1/2\log[(1-\xi)/(1-\xi)]$, $\xi/(1-\xi)$.

For the functions $h, H \in \mathfrak{A}$, The function h is said to be subordinate to H or H is said to be superordinate to h, if there exists a function η, analytic in $\mathfrak{U}$, with $\eta(0) = 0$ and $|\eta(z)| < 1$ and such that $h(\xi) = H(\eta(\xi))$. In such a case we write $h \prec H$ or $h(\xi) \prec H(\xi)$. If h is one-to-one, this $h \prec H$ iff $h(0) = H(0)$ and $h(U) \subset H(U)$. Babalola [9] studied the class $\mathfrak{L}_\sigma(\varphi)$ of σ-pseudo- starlike functions of order $\varphi(0 \leq \varphi < 1)$ which is own geometric conditions fulfill

$$\Re\left(\frac{\xi(\theta'(\xi))^\sigma}{\theta(\xi)}\right) > \varphi.$$

He discover that every pseudo-starlike functions are Bazilevič of type $\left(1-\frac{1}{\sigma}\right)$ order $\varphi^{\frac{1}{\sigma}}$ and univalent in $\mathfrak{U}$.

In recent years, theory and applications of Dickson, Fibonacci, Lucas, Chebyshev, Lucas-Lehmer polynomials in modern science have emerged as a very current subject. These polynomials are important in mathematics due to the fact that they can applicable to number theory, numerical analysis, combinatorics, and other fields. Nowadays, these polyinomials have been studied and different generalizations have been made by many authors: see [1–3,5–8].Also see [13,17,19–26,28,29,43–46].

We recall some important properties interested in which we use to construct our new class. Assume that polynomials with real coefficients are written by $U(x)$ and $V(x)$. By using the recurrence relation, the (U, V)-Lucas polynomials $\mathcal{L}_{U,V,t}(x)$ are explained [17] as

$$\mathcal{L}_{U,V,t}(x) = U(x)\mathcal{L}_{U,V,t-1}(x) + V(x)\mathcal{L}_{U,V,t-2}(x)\ (t \geq 2). \tag{3}$$

Also

$$\begin{aligned}
\mathcal{L}_{U,V,0}(x) &= 2,\\
\mathcal{L}_{U,V,1}(x) &= U(x),\\
\mathcal{L}_{U,V,2}(x) &= U^2(x) + 2V(x),\\
\mathcal{L}_{U,V,3}(x) &= U^3(x) + 3U(x)V(x).
\end{aligned} \tag{4}$$

The generating function of the (U, V)-Lucas polynomial sequence $\mathcal{L}_{U,V,t}(x)$ is expressed by [17]

$$\mathcal{K}_{\{\mathcal{L}_t(x)\}}(\xi) = \sum_{t=0}^{\infty} \mathcal{L}_{U,V,t}(x)\xi^t = \frac{2-U(x)\xi}{1-U(x)\xi-V(x)\xi^2}. \tag{5}$$

In the next section, using this polynomials as a tool, we define the family $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$ as follow:

Definition 1. *For $\beta \geq 0$, $\sigma \geq 1$, $|\gamma| \leq 1$ but $\gamma \neq 1$, a function $\theta \in \mathfrak{B}$ is called in the family $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$ if the following subordinations are satisfied:*

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^{\sigma}}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi)-1 \tag{6}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^{\sigma}}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega)-1. \tag{7}$$

Taking special values for β, γ and σ, the class $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$ reduces some exciting new families:

Remark 1. *For $\sigma = 1$, we get the new family $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,1;x)$. If $\theta \in \mathfrak{B}$, is in $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,1;x)$ then following condition fulfilles*

$$\frac{((1-\gamma)\xi)^{1-\beta}\theta'(\xi)}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi)-1 \tag{8}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}\vartheta'(w)}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega)-1. \tag{9}$$

Remark 2. *For $\beta = 0$, we obtain the new class*

$$\mathfrak{H}^{\mathfrak{B},0}(\gamma,\sigma;x) = \mathfrak{H}^{\mathfrak{B}}(\gamma,\sigma;x).$$

If $\theta \in \mathfrak{B}$ is in $\mathfrak{H}^{\mathfrak{B}}(\gamma,\sigma;x)$, then following condition fulfilles

$$\frac{\xi(1-\gamma)(\theta'(\xi))^{\sigma}}{\theta(\xi)-f(\gamma\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi)-1 \tag{10}$$

and

$$\frac{\omega(1-\gamma)(\vartheta'(w))^{\sigma}}{\vartheta(\omega)-\vartheta(\gamma\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega)-1. \tag{11}$$

Also,

(1) Choosing $\sigma = 1$ in the class $\mathfrak{H}^{\mathfrak{B}}(\gamma,\sigma;x)$ we have new family $\mathfrak{H}^{\mathfrak{B}}(\gamma,1;x) = \mathfrak{H}^{\mathfrak{B}}(\gamma;x)$. The class $\mathfrak{H}^{\mathfrak{B}}(\gamma;x)$ consists of the function $f \in \mathfrak{B}$ fulfilling

$$\frac{\xi(1-\gamma)\theta'(\xi)}{\theta(\xi)-\theta(\gamma\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi)-1 \tag{12}$$

and

$$\frac{\omega(1-\gamma)\vartheta'(w)}{\vartheta(\omega)-\vartheta(\gamma\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega)-1. \tag{13}$$

(2) Choosing $\gamma = 0$ in the class $\mathfrak{H}^{\mathfrak{B}}(\gamma, \sigma; x)$ we have the class $\mathfrak{H}^{\mathfrak{B}}(0, \sigma; x) = \mathfrak{H}^{\mathfrak{B}}(\sigma; x) = \mathcal{L}_{\Sigma}(\mathfrak{U}; x)$. The class $\mathcal{L}_{\Sigma}(\mathfrak{U}; x)$ was studied by Murugusundaramoorthy and Yalçin [20]. This class consists of the function $\theta \in \mathfrak{B}$ satisfying

$$\frac{\xi(\theta'(\xi))^{\sigma}}{\theta(\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{14}$$

and

$$\frac{\omega(\vartheta'(w))^{\sigma}}{\vartheta(\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{15}$$

(3) Choosing $\sigma = 2$ in the class $\mathfrak{H}^{\mathfrak{B}}(\sigma; x)$ we have the class

$$\mathfrak{H}^{\mathfrak{B}}(2; x) = \mathfrak{H}^{\mathfrak{B}}(x).$$

The class consists of the function $f \in \mathfrak{B}$ satisfying

$$\theta'(\xi)\frac{\xi\theta'(\xi)}{\theta(\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{16}$$

and

$$\vartheta'(\omega)\frac{\omega\vartheta'(w)}{\vartheta(\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{17}$$

Remark 3. *For $\beta = 1$, we have the new class $\mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$. If $\theta \in \mathfrak{B}$, is in $\mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$, then following condition fulfilles*

$$(\theta'(\xi))^{\sigma} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{18}$$

and

$$(\vartheta'(\omega))^{\sigma} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{19}$$

Also,

(1) *Choosing $\sigma = 1$ in the class $\mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$ we have the class*

$$\mathfrak{H}^{\mathfrak{B},1}(1; x).$$

This class consists of the function $\theta \in \mathfrak{B}$ satisfying

$$\theta'(\xi) \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{20}$$

and

$$\vartheta'(\omega) \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{21}$$

2. Main Theorems for the class $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$

Theorem 1. *Let $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$. Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| \begin{array}{c} U^2(x)\left[(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2}+2\sigma(1+\gamma)+(1+\gamma+\gamma^2)\right]+\sigma(2\sigma+1)\right. \\ \left. -[2\sigma+(\beta-1)(1+\gamma)]^2\right]-2V(x)[2\sigma+(\beta-1)(1+\gamma)]^2 \end{array} \right|}} \tag{22}$$

$$|n_3| \leq \frac{U^2(x)}{[2\sigma+(\beta-1)(1+\gamma)]^2}+\frac{|U(x)|}{|3\sigma+(\beta-1)(1+\gamma+\gamma^2)|}, \tag{23}$$

where $\beta \geq 0$, $\sigma \geq 1$ and $|\gamma| \leq 1$ but $\gamma \neq 1$.

Proof. Let $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$. Then, according to the Definition 1, for some holomorphic functions Φ, Υ such that $\Upsilon(0) = \Phi(0) = 0$, $|\Upsilon(\omega)| < 1$, $|\Phi(\xi)| < 1$, $(\xi, \omega \in \mathfrak{U})$, we can write

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^\sigma}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} = \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\Phi(\xi))-1$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^\sigma}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} = \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\Upsilon(\omega))-1,$$

by equivalence

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^\sigma}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} = -1+\mathcal{L}_{U,V,0}(x)+\mathcal{L}_{U,V,1}(x)\Phi(\xi)+\mathcal{L}_{U,V,2}(x)\Phi^2(\xi)+\cdots \tag{24}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^\sigma}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} = -1+\mathcal{L}_{U,V,0}(x)+\mathcal{L}_{U,V,1}(x)\Upsilon(\omega)+\mathcal{L}_{U,V,2}(x)\Upsilon^2(\omega)+\cdots \tag{25}$$

From (24) and (25), yields

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^\sigma}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} = 1+\mathcal{L}_{U,V,1}(x)y_1\xi+\left[\mathcal{L}_{U,V,1}(x)y_2+\mathcal{L}_{U,V,2}(x)y_1^2\right]\xi^2+\cdots \tag{26}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^\sigma}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} = 1+\mathcal{L}_{U,V,1}(x)\mu_1\omega+\left[\mathcal{L}_{U,V,1}(x)\mu_2+\mathcal{L}_{U,V,2}(x)\mu_1^2\right]\omega^2+\cdots \tag{27}$$

for $\xi, \omega \in \mathfrak{U}$, it known before that if

$$|\Phi(\xi)| = \left|\sum_{j=1}^{\infty} y_j \xi^j\right| < 1$$

and

$$|\Upsilon(\omega)| = \left|\sum_{j=1}^{\infty} \mu_j \omega^j\right| < 1,$$

thus

$$|y_j| < 1 \tag{28}$$

also

$$|\mu_j| < 1 \tag{29}$$

where $j \in \mathfrak{N} = \{1, 2, 3, \cdots\}$. If we compare corresponding coefficients in (26) and (27), then we have

$$[2\sigma + (\beta-1)(1+\gamma)]n_2 = \mathcal{L}_{U,V,1}(x)y_1, \tag{30}$$

$$[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]n_3+\left[\frac{(\beta-1)(\beta-2)}{2}(1+\gamma)^2+2\sigma(\sigma-1+(\beta-1)(1+\gamma))\right]n_2^2$$
$$= \mathcal{L}_{U,V,1}(x)y_2 + \mathcal{L}_{U,V,2}(x)y_1^2, \tag{31}$$

$$-[2\sigma + (\beta-1)(1+\gamma)]n_2 = \mathcal{L}_{U,V,1}(x)\mu_1, \tag{32}$$

$$\left[2(\beta-1)(1+\gamma+\gamma^2) + \frac{(\beta-1)(\beta-2)}{2}(1+\gamma)^2 + 2\sigma(\sigma+2+(\beta-1)(1+\gamma))\right]n_2^2$$
$$- [3\sigma + (\beta-1)(1+\gamma+\gamma^2)]n_3 = \mathcal{L}_{U,V,1}(x)\mu_2 + \mathcal{L}_{U,V,2}(x)\mu_1^2. \tag{33}$$

From (30) and (32)

$$y_1 = -\mu_1, \tag{34}$$

$$2[2\sigma + (\beta-1)(1+\gamma)]^2 n_2^2 = \mathcal{L}^2_{U,V,1}(x)(y_1^2 + \mu_1^2). \tag{35}$$

Summation of (31) and (33) gives

$$\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2)\right] + 2\sigma(2\sigma+1)\right] n_2^2$$
$$= \mathcal{L}_{U,V,1}(x)(y_2 + \mu_2) + \mathcal{L}_{U,V,2}(x)(y_1^2 + \mu_1^2). \tag{36}$$

Applying (35) in (36), yields

$$\Bigg\{\mathcal{L}^2_{U,V,1}(x)\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2)\right] + 2\sigma(2\sigma+1)\right]$$

$$\left. - 2\mathcal{L}_{U,V,2}(x)[2\sigma + (\beta-1)(1+\gamma)]^2 \right\} n_2^2 = \mathcal{L}_{U,V,1}^3(x)(y_2+\mu_2), \quad (37)$$

$$\left[U^2(x)\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2)\right] + 2\sigma(2\sigma+1)\right.\right.$$
$$\left.\left. - 2[2\sigma+(\beta-1)(1+\gamma)]^2\right] - 4[2\sigma+(\beta-1)(1+\gamma)]^2 V(x)\right] n_2^2 = \mathcal{L}_{U,V,1}^3(x)(y_2+\mu_2)$$

which gives desired result given by (1).

Hence, (31) minus (33) gives us

$$2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]n_3 + 2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]n_2^2 = \mathcal{L}_{U,V,1}(x)(y_2-\mu_2). \quad (38)$$

Then, by using (34) and (35) in (38), we get

$$n_3 = n_2^2 + \frac{\mathcal{L}_{U,V,1}(x)(y_2-\mu_2)}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]} \quad (39)$$

$$n_3 = \frac{\mathcal{L}_{U,V,1}^2(x)(y_1^2+\mu_1^2)}{2[2\sigma+(\beta-1)(1+\gamma)]^2} + \frac{\mathcal{L}_{U,V,1}(x)(y_2-\mu_2)}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]}. \quad (40)$$

Applying (4), we have

$$|n_3| \le \frac{U^2(x)}{[2\sigma+(\beta-1)(1+\gamma)]^2} + \frac{|U(x)|}{|3\sigma+(\beta-1)(1+\gamma+\gamma^2)|}.$$

Thus, the proof of our main theorem is completed. □

3. Corollaries

By specializing the parameters γ, β, σ, in Theorem 1, we get the following consequences.

Corollary 1. *Let $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},\beta}(\gamma, 1; x)$. Then*

$$|n_2| \le \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| \begin{array}{c} U^2(x)\left[(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 3(1+\gamma) + \gamma^2\right] + 3 - [2+(\beta-1)(1+\gamma)]^2\right] \\ -2V(x)[2+(\beta-1)(1+\gamma)]^2 \end{array} \right|}} \quad (41)$$

$$|n_3| \le \frac{U^2(x)}{[2+(\beta-1)(1+\gamma)]^2} + \frac{|U(x)|}{|3+(\beta-1)(1+\gamma+\gamma^2)|}. \quad (42)$$

Corollary 2. *Let $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},0}(\gamma,\sigma;x) = \mathfrak{H}^{\mathfrak{B}}(\gamma,\sigma;x)$. Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x)\left[(2\sigma-1)(\sigma-\gamma)-[2\sigma-(1+\gamma)]^2\right]-2V(x)[2\sigma-(1+\gamma)]^2\right|}}, \tag{43}$$

$$|n_3| \leq \frac{U^2(x)}{[2\sigma-(1+\gamma)]^2} + \frac{|U(x)|}{|3\sigma-(1+\gamma+\gamma^2)|}. \tag{44}$$

Corollary 3. *Let $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},0}(\gamma,1;x) = \mathfrak{H}^{\mathfrak{B}}(\gamma,1;x)$. Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x)(\gamma-\gamma^2)-2V(x)(\gamma^2-2\gamma+1)\right|}}, \tag{45}$$

$$|n_3| \leq \frac{U^2(x)}{(1-\gamma)^2} + \frac{|U(x)|}{|2-\gamma(\gamma+1)|}. \tag{46}$$

Corollary 4. *Choosing $\beta = 0$ and $\gamma = 0$ in Theorem 1, that is if $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B}}(\sigma;x)$, the results which we obtain reduce to Theorem 2.1 in [20].*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x)(-2\sigma^2-1+3\sigma)-2V(x)(2\sigma-1)^2|}}, \tag{47}$$

$$|n_3| \leq \frac{U^2(x)}{[2\sigma-1]^2} + \frac{|U(x)|}{|3\sigma-1|}. \tag{48}$$

Corollary 5. *Choosing $\beta = 0$, $\gamma = 0$ and $\sigma = 2$ in Theorem 1, $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B}}(2;x)$, our corollary coincides with the corollary 2.3 of Theorem 2.1 in [20].*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{3|U^2(x)+6V(x)|}}, \tag{49}$$

$$|n_3| \leq \frac{U^2(x)}{9} + \frac{|U(x)|}{5}. \tag{50}$$

Corollary 6. *Choosing $\beta = 0$, $\gamma = 0$ and $\sigma = 1$ in Theorem 1, $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B}}(1;x)$, our corollary coincides with the corollary 2.2 of Theorem 2.1 in [44].*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x)|}} = \sqrt{|U(x)|}, \tag{51}$$

$$|n_3| \leq U^2(x) + \frac{|U(x)|}{2}. \tag{52}$$

Corollary 7. *Let* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},1}(\sigma;x)$. *Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\sigma|U^2(x)(1-2\sigma)-8\sigma V(x)|}}, \tag{53}$$

$$|n_3| \leq \frac{U^2(x)}{4\sigma^2} + \frac{|U(x)|}{3\sigma}. \tag{54}$$

Corollary 8. *Let* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},1}(1;x)$. *Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x)+8V(x)|}}, \tag{55}$$

$$|n_3| \leq \frac{U^2(x)}{4} + \frac{|U(x)|}{3}. \tag{56}$$

Theorem 2. *For* $\beta \geq 0$, $\sigma \geq 1$, $|\gamma| \leq 1$ *but* $\gamma \neq 1$, *let* $\theta \in \mathfrak{A}$ *be in the class* $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$. *Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma+(\beta-1)(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta-2V(x)[2\sigma+(\beta-1)(1+\gamma)]^2|}, & |\chi-1| \geqq K. \end{cases}$$

Where

$$K = \frac{1}{|3\sigma+(\beta-1)(1+\gamma+\gamma^2)|}\left|\Delta - 2[2\sigma+(\beta-1)(1+\gamma)]^2\frac{V(x)}{U^2(x)}\right|$$

$$\Delta = (\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + 1 + \gamma + \gamma^2\right] + \sigma(2\sigma+1) - [2\sigma+(\beta-1)(1+\gamma)]^2.$$

Proof. From (37) and (38), we get

$$n_3 - \chi n_2^2 = \mathcal{L}_{U,V,1}(x)\left[\left(\zeta(\chi;x) + \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]}\right)y_2 + \left(\zeta(\chi;x) - \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]}\right)\mu_2\right]$$

where

$$\zeta(\chi;x) = \frac{\mathcal{L}^2_{U,V,1}(x)(1-\chi)}{\mathcal{L}^2_{U,V,1}(x)\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma\right.\right.}$$

$(1+\gamma) + (1+\gamma+\gamma^2) + 2\sigma(2\sigma+1) - 2\mathcal{L}_{U,V,2}(x)[2\sigma+(\beta-1)(1+\gamma)]^2$. Thus, according to (4), we have

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma+(\beta-1)(1+\gamma+\gamma^2)}, & 0 \leqq |\zeta(\chi;x)| \leqq \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]} \\ 2|\zeta(\chi;x)|\cdot|U(x)|, & |\zeta(\chi;x)| \geqq \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]} \end{cases}$$

hence, after some calculations, gives

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma+(\beta-1)(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta-2V(x)[2\sigma+(\beta-1)(1+\gamma)]^2|}, & |\chi-1| \geqq K. \end{cases}$$

□

By choosing special values for the parameters γ, β, σ, in Theorem 2, we get the following corollaries:

Corollary 9. *For $\sigma = 1$, let $\theta \in \mathfrak{H}^{\mathfrak{B},\beta}(\gamma, 1; x)$. Then*

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3+(\beta-1)(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K_1 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_1-2V(x)[2+(\beta-1)(1+\gamma)]^2|}, & |\chi-1| \geqq K_1. \end{cases}$$

Where

$$K_1 = \frac{1}{|3+(\beta-1)(1+\gamma+\gamma^2)|}\left|\Delta_1 - 2[2+(\beta-1)(1+\gamma)]^2\frac{V(x)}{U^2(x)}\right|$$

$$\Delta_1 = (\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2}+\gamma^2+3\gamma+3\right]+3-[2+(\beta-1)(1+\gamma)]^2.$$

Corollary 10. *For $\beta = 0$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(\gamma, \sigma; x)$. Then*

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma-(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K_2 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_2-2V(x)[2\sigma-(1+\gamma)]^2|}, & |\chi-1| \geqq K_2. \end{cases}$$

Where

$$K_2 = \frac{1}{|3\sigma-(1+\gamma+\gamma^2)|}\left|\Delta_2 - 2[2\sigma-(1+\gamma)]^2\frac{V(x)}{U^2(x)}\right|$$

$$\Delta_2 = (2\sigma-1)(\sigma-\gamma)-[2\sigma-(1+\gamma)]^2.$$

Corollary 11. *For $\sigma = 1$, $\beta = 0$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(\gamma, 1; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|2-\gamma(\gamma+1)|}, & |\chi - 1| \leqq K_3 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_3 - 2V(x)[1-\gamma]^2|}, & |\chi - 1| \geqq K_3. \end{cases}$$

Where

$$K_3 = \frac{1}{|2-\gamma(\gamma+1)|}\left|\Delta_3 - 2[1-\gamma]^2 \frac{V(x)}{U^2(x)}\right|$$

$$\Delta_3 = \gamma(1-\gamma).$$

Corollary 12. *For $\beta = 0$, $\gamma = 0$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(0, \sigma; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|3\sigma-1|}, & |\chi - 1| \leqq K_4 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_4 - 2V(x)(2\sigma-1)^2|}, & |\chi - 1| \geqq K_4. \end{cases}$$

Where

$$K_4 = \frac{1}{|3\sigma - 1|}\left|\Delta_4 - 2[2\sigma-1]^2 \frac{V(x)}{U^2(x)}\right|.$$

$$\Delta_4 = (2\sigma - 1)(1 - \sigma)$$

Corollary 13. *For $\sigma = 2$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(0, 2; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{5}, & \left(|\chi - 1| \leqq \frac{3}{5}\left|1 + 6\frac{V(x)}{U^2(x)}\right|\right) \\ \frac{|1-\chi|\cdot|U^3(x)|}{3|U^2(x)+6V(x)|}, & \left(|\chi - 1| \geqq \frac{3}{5}\left|1 + 6\frac{V(x)}{U^2(x)}\right|\right). \end{cases}$$

Corollary 14. *[44] For $\sigma \geq 1$, let $\theta \in \mathfrak{A}$ be in the class $\mathfrak{H}^{\mathfrak{B},0}(0, 1; x) = \mathfrak{H}^{\mathfrak{B}}(x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{2}, & \left(|\chi - 1| \leqq \frac{|V(x)|}{|U^2 x|}\right) \\ \frac{|1-\chi|\cdot|U^3(x)|}{2|V(x)|}, & \left(|\chi - 1| \geqq \frac{|V(x)|}{|U^2 x|}\right). \end{cases}$$

Corollary 15. *For $\beta = 1$, let $\theta \in \mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma}, & \left(|\chi - 1| \leqq K_5\right) \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_5 - 8\sigma^2 V(x)|}, & \left(|\chi - 1| \geqq K_5\right). \end{cases}$$

Where

$$K_5 = \frac{1}{|3\sigma|}\left|\Delta_5 - 8\sigma^2 \frac{V(x)}{U^2(x)}\right|$$

$$\Delta_5 = \sigma(1 - 2\sigma)$$

Corollary 16. *[8] For $\sigma = 1$, $\beta = 1$, let $\theta \in \mathfrak{H}^{\mathfrak{B},1}(1; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3}, & \left(|\chi - 1| \leqq \frac{1}{3}\left|1 + 8\frac{V(x)}{U^2(x)}\right|\right) \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)+8V(x)|}, & \left(|\chi - 1| \geqq \frac{1}{3}\left|1 + 8\frac{V(x)}{U^2(x)}\right|\right). \end{cases}$$

Author Contribution Statements All authors contributed equally to design and implementation of the research. They jointly analyzed the results and wrote the manuscriprt. They read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

Acknowledgements The authors are thankful to the editor and referee(s) for their valuable comments and suggestions.

References

[1] Akgül, A., (P, Q)-Lucas polynomial coefficient inequalities of the bi-univalent function class, *Turk. J. Math.*, 43 (2019), 2170–2176. https://doi.org/10.3906/mat-1903-38

[2] Akgül, A., On a family of bi-univalent functions related to the Fibonacci numbers, *Mathematica Moravica*, 26(1) (2022), 103–112. https://doi.org/10.5937/MatMor2201103A

[3] Akgül, A., Sakar, F. M., A new characterization of (P, Q)-Lucas polynomial coefficients of the bi-univalent function class associated with q-analogue of Noor integral operator, *Afrika Matematika*, 33(3) (2022), 1-12.

[4] Ali, R. M., Lee, S. K., Ravichandran V., Supramanian S., Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, 25(3) (2012), 344–351. https://doi.org/10.1016/j.aml.2011.09.012

[5] Altınkaya, Ş., Yalçın, S., On the (p, q)-Lucas polynomial coefficient bounds of the bi-univalent function class σ, *Boletin de la Sociedad Matemática Mexicana*, 25 (2019), 567-575. https://doi.org/10.1007/s40590-018-0212-z

[6] Altınkaya, Ş., Yalçın, S., The (p, q)-Chebyshev polynomial bounds of a general bi-univalent function class, *Boletin de la Sociedad Matemática Mexicana*, 26 (2019), 341–348.

[7] Al-Shbeil, I., Shaba, T. G., Cătaş, A., Second hankel determinant for the subclass of bi-univalent functions using q-Chebyshev polynomial and Hohlov operator, *Fractal and Fractional*, 6(4) (2022), 186. https://doi.org/10.3390/fractalfract6040186

[8] Altinkaya, Ş., Yalçın, S., Some application of the (p, q)-Lucas polynomials to the bi-univalent function class Σ, *Mathematical Sciences and Applications E-Notes*, 8(1) (2020), 134–141. https://doi.org/10.36753/MATHENOT.650271

[9] Babalola, K. O., On $\mathfrak{U}$-pseudo-starlike functions, *J. Class. Anal.*, 3(2) (2013), 137-147.

[10] Çağlar, M., Deniz, E., Srivastava, H. M., Second Hankel determinant for certain subclasses of bi-univalent functions, *Turk. J. Math.*, 41 (2017), 694-706. https://doi.org/10.3906/mat-1602-25

[11] Çağlar, M., Orhan H., Yagmur, N., Coefficient bounds for new subclasses of bi-univalent functions, *Filomat*, 27(7) (2013), 1165–1171. https://doi.org/10.2298/FIL1307165C

[12] Duren, P. L., Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Springer, New York, 1983.

[13] Filipponi, P., Horadam, A. F., Second derivative sequences of Fibonacci and Lucas polynomials, *Fibonacci Q*, 31(3) (1993), 194–204.

[14] Goswami, P., Alkahtani, B. S., Bulboaca T., Estimate for initial Maclaurin coefficients of certain subclasses of bi-univalent functions, *arXiv:1503.04644v1* (2015).

[15] Ibrahim, I. O., Shaba, T. G., Patil, A. B., On some subclasses of m-fold symmetricbi-univalent functions associated with the Sakaguchi type functions, *Earthline Journal of Mathematical Sciences*, 8(1) (2022), 1-15. https://doi.org/10.34198/ejms.8122.115

[16] Khan, B., Liu, Z. G., Shaba, T. G., Araci, S., Khan, N., Khan, M. G., Application of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomial, *Journal of Mathematics*, (2022), Article ID: 8162182. https://doi.org/10.1155/2022/8162182

[17] Lee, G. Y., Aşcı M., Some properties of the (p, q)-Fibonacci and (p, q)-Lucas polynomials, *J. Appl. Math.*, (2012), Article ID: 264842, 1-18. https://doi.org/10.1155/2012/264842

[18] Lewin, M., On a coefficient problem for bi-univalent functions, *Proc. Am. Math. Soc.*, 18(1) (1967), 63–68.

[19] Lupas, A., A guide of Fibonacci and Lucas polynomials, *Octagon Mathematics Magazine*, 7 (1999), 2-12.

[20] Murugusundaramoorthy, G., Yalçin, S., On λ-Pseudo bi-starlike functions related (p, q)-Lucas polynomial, *Libertas Mathematica (new series)*, 39(2) (2019), 79-88.

[21] Orhan, H., Arıkan H., Lucas polynomial coefficients inequalities of bi-univalent functions defined by the combination of both operators of Al-Oboudi and Ruscheweyh, *Afr. Mat.*, 32(3-4) (2021), 589–598. https://doi.org/10.1007/s13370-020-00847-5

[22] Orhan, H., Shaba, T. G., Çağlar, M., (P, Q)-Lucas polynomial coefficient relations of bi-univalent functions defined by the combination of Opoola and Babalola differential operator, *Afrika Matematika*, 33(1), (2022), 1-13. https://doi.org/10.1007/s13370-021-00953-y

[23] Özkoç A., Porsuk, A., A note for the (p, q)-Fibonacci and Lucas quaternion polynomials, *Konuralp J. Math.*, 5(2) (2017), 36-46.

[24] Patil, A. B., Shaba, T. G., On sharp Chebyshev polynomial bounds for general subclass of bi-univalent functions, *Applied Sciences*, 23 (2021), 109–117.

[25] Sakar, F. M., Estimate for initial Tschebyscheff polynomials coefficients on a certain subclass of bi-univalent functions defined by Sălăgean differential operator, *Acta Universitatis Apulensis*, 54 (2018), 45-54. https://doi.org/10.17114/j.aua.2018.54.04

[26] Shaba, T. G., Subclass of bi-univalent functions satisfying subordinate conditions defined by Frasin differential operator, *Turkish Journal of Inequalities*, 4(2) (2020), 50-58.

[27] Shaba, T. G., Patil, A. B., Coefficient estimates for certain subclasses of m-fold symmetric bi-univalent functions associated with pseudo-starlike functions, *Earthline Journal of Mathematical Sciences*, 6(2) (2021), 2581-8147. https://doi.org/10.34198/ejms.6221.209223

[28] Shaba, T. G., On some subclasses of bi-pseudo-starlike functions defined by Salagean differential operator, *Asia Pac. J. Math.*, 8(6) (2021), 1–11. https://doi:10.28924/apjm/8-6

[29] Shaba, T. G., Wanas, A. K., Coefficient bounds for a new family of bi-univalent functions associated with $(U, V)-$Lucas polynomials, *International Journal of Nonlinear Analysis and Applications*, 13(1), (2022), 615-626. http://dx.doi.org/10.22075/ijnaa.2021.23927.2639

[30] Srivastava, H. M., Bulut, S., Çağlar, M., Yağmur, N., Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat*, 27 (2013), 831–842. https://doi.org/10.2298/FIL1305831S

[31] Srivastava, H. M., Eker, S. S., Hamidi, S. G., Jahangiri, J. M., Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, *Bull. Iran. Math. Soc.*, 44 (2018), 149–157. https://doi.org/10.1007/s41980-018-0011-3

[32] Srivastava, H. M., Khan, S., Ahmad, Q. Z, Khan, N., Hussain, S., The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, *Stud. Univ. Babeş-Bolyai Math.*, 63 (2018), 419–436.

[33] Srivastava, H. M., Mishra, A. K., Das, M. K., The Fekete-Szego problem for a subclass of close-to-convex functions, *Complex Variables Theory Appl.*, 44 (2001), 145–163. https://doi.org/10.1080/17476930108815351

[34] Srivastava, H. M., Eker, S. S., Some applications of a subordination theorem for a class of analytic functions, *Appl. Math. Lett.*, 21 (2008), 394–399. https://doi.org/10.1016/j.aml.2007.02.032

[35] Srivastava, H. M., Mishra, A. K., Gochhayat, P., Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, 23(10) (2010), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009

[36] Srivastava, H. M., Eker, S. S., Ali, R. M., Coefficient bounds for a certain class of analytic and bi-univalent functions, *Filomat*, 29(8) (2015), 1839–1845. https://doi.org/10.2298/FIL1508839S

[37] Srivastava, H. M., Magesh, N., Yamini, J., Initial coefficient estimates for bi-λ-convex and bi-μ-starlike functions connected with arithmetic and geometric means, *Electron. J. Math. Anal. Appl.*, 2(2) (2014), 152–162.

[38] Srivastava, H. M., Sakar, F. M., Güney, H. Ö., Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination, *Filomat*, 34 (2018), 1313–1322. https://doi.org/10.2298/FIL1804313S

[39] Srivastava, H. M., Gaboury, S., Ghanim, F., Coefficient estimates for some general subclasses of analytic and bi-univalent functions, *Afr. Mat.,* 28 (2017), 693–706. https://doi.org/10.1007/s13370-016-0478-0

[40] Srivastava, H. M., Altinkaya, Ş., Yalçın, S., Certain subclasses of bi-univalent functions associated with the Horadam polynomials, *Iran. J. Sci. Technol. Trans. Sci.*, 43 (2019), 1873–1879. https://doi.org/10.1007/s40995-018-0647-0

[41] Tang, H., Srivastava, H. M., Sivasubramanian, S., Gurusamy, P., The Fekete-Szegö functional problems for some subclasses of m-fold symmetric bi-univalent functions, *J. Math. Inequal.*, 10(4) (2016), 1063–1092. https://doi.org/10.7153/jmi-10-85

[42] Vellucci, P., Bersani, A. M., The class of Lucas-Lehmer polynomials, *Rendiconti di Matematica*, 37 (2016), 43-62 .

[43] Wang, T., Zhang, W., Some identities involving Fibonacci, Lucas polynomials and their applications, *Bull. Math. Soc. Sci. Math. Roum.,* 5(1) (2012), 95-103.

[44] Wanas, A. K., Application of (M, N)-Lucas polynomials for holomorphic and bi-univalent functions, *Filomat*, 39(10) (2020), 3361–3368. https://doi.org/10.2298/FIL2010361W

[45] Wanas, A. K., Shaba, T. G., Horadam polynomials and their applications to certain family of bi-univalent functions defined by Wanas operator, *General Mathematics*, (2022), 103. https://doi.org/10.2478/gm-2021-0009

[46] Yalçın, S., Muthunagai, K., Saravanan, G., A subclass with bi-univalent involving (p, q)-Lucas polynomials and its coefficient bounds, *Boletin de la Sociedad Matemática Mexicana*, 26 (2020), 1015-1022. https://doi.org/10.1007/s40590-020-00294-z

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1136–1168 (2022)
DOI:10.31801/cfsuasmas.1067635
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: February 3, 2022; Accepted:June 6, 2022

APPROXIMATION PROPERTIES OF THE FRACTIONAL q-INTEGRAL OF RIEMANN-LIOUVILLE INTEGRAL TYPE SZÁSZ-MIRAKYAN-KANTOROVICH OPERATORS

Mustafa KARA

Department of Mathematics, Eastern Mediterranean University, Gazimagusa
99628 Mersin 10, NORTHERN CYPRUS

Abstract. In the present paper, we introduce the fractional q-integral of Riemann-Liouville integral type Szász-Mirakyan-Kantorovich operators. Korovkin-type approximation theorem is given and the order of convergence of these operators are obtained by using Lipschitz-type maximal functions, second order modulus of smoothness and Peetre's K-functional. Weighted approximation properties of these operators in terms of modulus of continuity have been investigated. Then, for these operators, we give a Voronovskaya-type theorem. Moreover, bivariate fractional q- integral Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators are constructed.The last section is devoted to detailed graphical representation and error estimation results for these operators.

1. Introduction

Approximation theory is a subject that serves as an important bridge between applied and pure mathematics. The approximation of functions by positive linear operators is an important research area in general mathematics. Especially, it plays an important role in mathematical analysis problems and in many fields of science. One of its most important advantages is that it provides powerful tools for application areas such as computer aided geometric design and numerical analysis. One of the best known of these operators is the Szász - Mirakyan operator (see [9] and [10]), which is generalizations of Bernstein polynomials to the infinite interval and defined as

2020 *Mathematics Subject Classification.* 41A25, 41A36, 47A58.
Keywords. Szasz-Mirakyan-Kantorovich operators, q-integral of Riemann-Liouville, Voronovskaya-type.

mustafa.kara@emu.edu.tr; 0000-0003-3091-5781.

$$S_n(f;x)=\sum_{k=0}^{n} s_{n,k}(x)\ f\left(\frac{k}{n}\right),$$

where $n\in\mathbb{N}$, $x\in[0,\infty)$ and $s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$. In literature, there are a lot of studies that involve Szàsz operators, Szàsz-Kantorovich operators and their generalizations. For instance, see [1]- [8] and [14]- [22]. Due to the rapid development

of the q-calculus, various generalizations of Szàsz Mirakyan operators involving q-integers have been introduced and approximation properties have been studied. Several researchers introduced and studied different generalizations of the q-Szász-Mirakjan operators in recent years ([28], [29], [19], [30], [41]). In [28], Mahmudov introduced and studied the following q-Szász-Mirakjan operators.

$$S_{n,q}(f;x)=\sum_{k=0}^{n} s_{n,k}(q;x)\ f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right),$$

where $s_{n,k}(q;x)=\frac{1}{E_q\left([n]_q x\right)}q^{\frac{k(k-1)}{2}}\frac{[n]_q^k x^k}{[k]_q!}.$

About contributions on Kantorovich type modication modified many times q-Szász-Mirakjan operators, so we refer to the papers [31]- [34]. Recently, Fractional calculus and its applications have been paid more and more attention. fractional calculus deals with the study of fractional degree derivative and integral operators on complex or real fields and their applications (see [23]- [27]). Mahmudov and Kara, introduced and discussed the fractional integral of Riemann-Liouville integral type Szász Mirakyan-Kantorovich operators as follows:

$$K_n^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}\alpha s_{n,k}(x)\int_0^1\frac{f\left(\frac{k+t}{n}\right)}{(1-t)^{1-\alpha}}dt, \tag{1}$$

where $s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$.The aim of the present paper is to construct the frac-

tional q-integral of Riemann-Liouville type Szász-Mirakyan-Kantorovich operators and discuss their approximation properties. The fractional q-integral of Riemann-Liouville type ([35]) is given by $\left(I_q^0 f\right)(t)=f(t)$ and

$$(I_q^{\alpha}f)(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x\frac{f(t)}{(x-qt)^{(1-\alpha)}}d_qt \qquad (\ \alpha>0).$$

We start by reminding the basic concepts and notations about fractional q-calculus.

2. Preliminaries

For $q \in (0,1)$,

$$[m]_q = \frac{1-q^m}{1-q}, \quad m \in \mathbb{R}$$

The q-analog of the power function $(n-m)^{(k)}$ with $k \in \mathbb{N}_0 := \{0,1,2,...\}$ is

$$(n-m)^{(0)} = 1, \qquad (n-m)^{(k)} = \prod_{i=0}^{k-1} \left(n - mq^i\right), \qquad k \in \mathbb{N}, n, m \in \mathbb{R}.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(n-m)^{(\gamma)} = \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{\gamma+i}}, \quad n \neq 0.$$

Note if $m = 0$, then $(n)^{(\gamma)} = n^\gamma$. We also use the natation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, \quad t \in \mathbb{R} \setminus \{0, -1, -2, ...\}.$$

Obviously, $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$.

For any $s, t > 0$, the q-beta function is defined by

$$B_q(s,t) = \int_0^1 u^{(s-1)} (1-qu)^{(t-1)} \, d_q u.$$

The q-beta function can be expressed by using the q-gamma function as follows:

$$B_q(s,t) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}.$$

The q-integral definition of the function h on the interval $[0,b]$ is given as:

$$(I_q h)(t) = \int_0^t h(s) d_q s = t(1-q) \sum_{i=0}^{\infty} h(tq^i) q^i, \quad t \in [0,b].$$

In q-calculus (see [36]) the following functions are well known as analogues of the exponential function:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \qquad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} q^{\frac{k(k-1)}{2}}, \quad |q| < 1.$$

3. Riemann-Liouville Type Szász-Mirakyan-Kantorovich Operators

Lemma 1. *([28]) Let $0 < q < 1$. we have*

$$S_{n,q}(t^4;x) = \frac{x^4}{q^2} + \left(3q + 2 + \frac{1}{q}\right)\frac{x^3}{[n]_q} + \left(3q^3 + 3q^2 + q\right)\frac{x^2}{[n]_q^2} + \frac{q^4 x}{[n]_q^3}.$$

Definition 1. *Let $q \in (0,1)$ and $\alpha > 0$. For $f \in C[0,\infty)$, Fractional q-integral of Riemann-Liouville type Szász-Mirakyan-Kantorovich operators can be defined by*

$$K_{n,q}^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} [\alpha]_q\, s_{n,k}(q;x) \int_0^1 f\left(\frac{q^{1-k}[k]_q + t}{[n]_q}\right)(1-qt)^{(\alpha-1)}\, d_q t, \qquad (2)$$

where $s_{n,k}(q;x) = \frac{1}{E_q\left([n]_q x\right)} q^{\frac{k(k-1)}{2}} \frac{[n]_q^k x^k}{[k]_q!}$ *and* $n \in \mathbb{N}$

if $\alpha = 1$ and $q = 1$, then the operator (2) reduces to classical Szász-Mirakyan - Kantorovich operators.

Due to the moments of the $K_{n,q}^{(\alpha)}$ operators plays significant role in our main results, we derive the following formula to obtain them.

Lemma 2. *Let $q \in (0,1)$ and $\alpha > 0$. Then for $x \in [0,\infty)$, we have*

$$K_{n,q}^{(\alpha)}(t^m;x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[\alpha]_q\, [n]_q^j\, B_q(m-j+1,\alpha)}{q^j\, [n]_q^m} S_{n,q}(t^j;x), \qquad (3)$$

where

$$S_{n,q}(f;x) = \sum_{k=0}^{n} s_{n,k}(q;x)\ f\left(\frac{[k]_q}{q^{(k-2)}[n]_q}\right)$$

and

$$B_q(a,b) = \int_0^1 x^{a-1}(1-qx)^{b-1}\, d_q x, \qquad a,b > 0.$$

Proof. From (2), we can write

$$K_{n,q}^{(\alpha)}(t^m;x) = \sum_{k=0}^{\infty} [\alpha]_q\, s_{n,k}(q;x) \int_0^1 \left(\frac{q^{1-k}[k]_q + t}{[n]_q}\right)^m (1-qt)^{(\alpha-1)}\, d_q t$$

$$= \sum_{k=0}^{\infty} [\alpha]_q\, s_{n,k}(q;x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^{(1-k)j}[k]_q^j}{[n]_q^m} \int_0^1 t^{(m-j)}(1-qt)^{(\alpha-1)}\, d_q t$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{[\alpha]_q [n]_q^j B_q(m-j+1,\alpha)}{q^j [n]_q^m} \sum_{k=0}^{\infty} s_{n,k}(q;x) \frac{[k]_q^j}{q^{(k-2)j} [n]_q^j}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{[\alpha]_q [n]_q^j B_q(m-j+1,\alpha)}{q^j [n]_q^m} S_{n,q}(t^j;x).$$

□

For $j = 0,1,2,3,4$ $(K_{n,q}^{\alpha}(t^j;x))$,the following can be written immediately.

Lemma 3. *Let $q \in (0,1)$,$\alpha > 0$ and $n \in \mathbb{N}$. Then for $x \in [0,\infty)$, we have*

$$\text{(i) } K_{n,q}^{(\alpha)}(1;x) = 1,$$

$$\text{(ii) } K_{n,q}^{(\alpha)}(t;x) = x + \frac{1}{[n]_q [\alpha+1]_q}$$

$$\text{(iii) } K_{n,q}^{(\alpha)}(t^2;x) = \frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_q [n]_q} x + \frac{x^2}{q},$$

$$\begin{aligned} \text{(iv) } K_{n,q}^{(\alpha)}(t^3;x) &= \frac{[3]_q [2]_q}{[\alpha+1]_q [\alpha+2]_q [\alpha+3]_q [n]_q^3} \\ &+ \left(\frac{3.[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{3}{[\alpha+1]_q [n]_q^2} + 1 \right) x \\ &+ \left(\frac{3}{q [n]_q [\alpha+1]_q} + \frac{2q^2+q}{q^3 [n]_q} \right) x^2 + \frac{x^3}{q^3}, \end{aligned}$$

$$\begin{aligned} \text{(v) } K_{n,q}^{(\alpha)}(t^4;x) &= \frac{[4]_q!}{[\alpha+1]_q [\alpha+2]_q [\alpha+3]_q [\alpha+4]_q [n]_q^4} \\ &+ \left(\frac{4[3]_q! + 6[\alpha+3]_q [2]_q! + [\alpha+2]_q [\alpha+3]_q \left\{4+[\alpha+1]_q\right\}}{[\alpha+1]_q [\alpha+2]_q [\alpha+3]_q [n]_q^3} \right) x \\ &+ \left(\frac{6[2]_q!}{q[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{4\left(2q^2+q\right)}{q^3 [\alpha+1]_q [n]_q^2} + \frac{\left(3q^3+3q^2+q\right)}{q^4 [n]_q^2} \right) x^2 \\ &+ \left(\frac{4}{q^3 [n]_q [\alpha+1]_q} + \frac{3q+2+\frac{1}{q}}{q^4 [n]_q} \right) x^3 + \frac{x^4}{q^6}. \end{aligned}$$

Proof. Since they have the same proof technique, we only give for $K_{n,q}^{(\alpha)}(t^2;x)$. Using recurrence formula(3) and Lemma 1, we get

$$K_{n,q}^{(\alpha)}(t^2;x) = \frac{[\alpha]_q B_q(3,\alpha)}{[n]_q^2} S_{n,q}(1;x) + \frac{2[n]_q [\alpha]_q B_q(2,\alpha)}{q [n]_q^2} S_{n,q}(t;x)$$

$$
\begin{aligned}
&+ \frac{[n]_q^2\,[\alpha]_q\,B_q(1,\alpha)}{q^2\,[n]_q^2} S_{n,q}(t^2;x) \\
&= \frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2} + \frac{2}{[\alpha+1]_q\,[n]_q}x + \left(\frac{x^2}{q} + \frac{x}{[n]_q}\right) \\
&= \frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2} + \frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_q\,[n]_q}x + \frac{x^2}{q}.
\end{aligned}
$$

□

We are now ready to present the central moments of the operators $K_{n,q}^{(\alpha)}$.

Lemma 4. *Let $q \in (0,1)$ and $\alpha > 0$. For every $x \in [0,\infty)$, there holds*

$$K_{n,q}^{(\alpha)}(t-x;x) = \frac{1}{[n]_q\,[\alpha+1]_q},$$

$$K_{n,q}^{(\alpha)}((t-x)^2;x) = \frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right),$$

$$
\begin{aligned}
&K_{n,q}^{(\alpha)}((t-x)^4;x) \\
&= \frac{[4]_q!}{[\alpha+1]_q\,[\alpha+2]_q\,[\alpha+3]_q\,[\alpha+4]_q\,[n]_q^4} \\
&+ \left(\frac{6\,[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^3} + \frac{4}{[\alpha+1]_q\,[n]_q^3} + \frac{1}{[n]_q^3}\right)x \\
&+ \left(\begin{array}{c} \frac{6[2]_q}{q[\alpha+1]_q[\alpha+2]_q[n]_q^2} + \frac{4\left(2q^2+q\right)}{[\alpha+1]_q q^3[n]_q^2} \\ + \frac{3q^3+3q^2+q}{q^4[n]_q^2} - \frac{6[2]_q-12[\alpha+2]_q}{[\alpha+1]_q[\alpha+2]_q[n]_q^2} - 4 \end{array}\right)x^2 \\
&+ \left(\begin{array}{c} \frac{4}{q^3[\alpha+1]_q[n]_q} + \frac{3q+2+\frac{1}{q}}{q^4[n]_q} - \frac{12}{q[\alpha+1]_q[n]_q} \\ - \frac{4\left(2q^2+q\right)}{q^3[n]_q} + \frac{6\left(2+[\alpha+1]_q\right)-4}{[\alpha+1]_q[n]_q} \end{array}\right)x^3 \\
&+ \left(\frac{1}{q^6} - \frac{4}{q^3} + \frac{6}{q} - 3\right)x^4.
\end{aligned}
$$

Proof. Since they have the same proof technique, we only give for $K_{n,q}^{(\alpha)}((t-x)^2;x)$. From the linearity property of $K_{n,q}^{(\alpha)}(t;x)$and Lemma 3, we get

$$
\begin{aligned}
&K_{n,q}^{(\alpha)}((t-x)^2;x) \\
&= \frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2} + \frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_q\,[n]_q}x + \frac{x^2}{q} - 2x\left(x + \frac{1}{[n]_q\,[\alpha+1]_q}\right) + x^2.
\end{aligned}
$$

□

Lemma 5. *Assume that the sequence* (q_n) *satisfy* $0 < q_n \leq 1$ *such that* $q_n \to 1$ *and* $q_n^n \to b \in [0,1]$ *as* $n \to \infty$. *For every* $\alpha > 0$ *and* $x \in [0,\infty)$, *there holds*

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x);x) = \frac{1}{\alpha+1}, \tag{4}$$

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^2;x) = x + x^2(1-b), \tag{5}$$

and

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^4;x) = 0. \tag{6}$$

Proof. Using explicit formula for moments (Lemma 4), we obtain as

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x);x) = \frac{1}{(\alpha+1)},$$

$$\begin{aligned} & \lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^2;x) \\ = & \lim_{n\to\infty} [n]_{q_n} \left(\frac{[2]_{q_n}}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2} + \frac{x}{[n]_{q_n}} + x^2\left(\frac{1}{q_n}-1\right)\right) \\ = & x + x^2(1-b) \end{aligned}$$

and

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^4;x) = 0.$$

□

In [28], Mahmudov gave the following formula for the moments of $S_{n,q}(t^m;x)$, which is a q-analogue of result of Beker [37].

Lemma 6. *[29]For* $0 < q < 1$ *and* $m \in \mathbb{N}$, *there holds*

$$S_{n,q}(t^m;x) = \sum_{j=1}^{m} a_{m,j}(q)\frac{x^j}{[n]_q^{m-j}} \tag{7}$$

where

$$a_{m+1,j}(q) = \frac{[j]_q a_{m,j}(q) + a_{m,j-1}(q)}{q^{j-2}}, \qquad m \geq 0, j \geq 1,$$
$$a_{0,0}(q) = 1, a_{m,0}(q) = 0, \quad m > 0, \quad a_{m,j}(q) = 0, \qquad m < j.$$

In particular $S_{n,q}(t^m;x)$ *is a polynomial of degree* m *without a constant term.*

Now we additionally need to give the following definitions for our main results:

1. $B_m[0,\infty) = \{f : [0,\infty) \to \mathbb{R}; |f(x)| \leq M_f(1+x^m)\}$,where M_f is constant depending on the function f.
2. $C_m[0,\infty) = B_m[0,\infty) \cap C[0,\infty)$.

3. $C_m^{\star}[0,\infty)=\left\{f: C_m[0,\infty): \lim\limits_{|x|\to\infty}\frac{|f(x)|}{1+x^m}<\infty\right\}$

The norm on the space $C_m^{\star}[0,\infty)$ is showed as $\|f(x)\|_m=\sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}$.

Lemma 7. *Let $m\in\mathbb{N}\cup\{0\}$, $0<q<1$ and $\alpha>0$ be fixed. Then, we have*

$$\left\|K_{n,q}^{(\alpha)}(1+x^m;x)\right\|_m\leq C_{m,j}(q,\alpha),\ n\in\mathbb{N}, \tag{8}$$

where $C_{m,j}(q,\alpha)$ is a positive constant. Morever, we have

$$\left\|K_{n,q}^{(\alpha)}(f;x)\right\|_m\leq C_{m,j}(q,\alpha)\|f\|_m,\quad n\in\mathbb{N}, \tag{9}$$

where $f\in C_m^{*}[0,\infty)$. Thus, for any $m\in\mathbb{N}\cup\{0\}$, $K_{n,q}^{(\alpha)}:C_m^{*}[0,\infty)\to C_m^{*}[0,\infty)$ is a linear positive operator.

Proof. For $m=0$, inequality (8) is obvious.

For $m\geq1$, combining Lemma (3) and inequality (7), we obtain as

$$\begin{aligned}
&\frac{1}{x^m+1}K_{n,q}^{(\alpha)}(1+t^m;x)\\
&=\frac{1}{x^m+1}+\frac{1}{x^m+1}K_{n,q}^{(\alpha)}(t^m;x)\\
&\frac{1}{x^m+1}+\frac{1}{x^m+1}\sum_{j=0}^{m}\frac{[\alpha]_q\,[n]_q^j\,B_q(m-j+1,\alpha)}{q^j\,[n]_q^m}\sum_{j_0=1}^{j}a_{j,j_0}(q)\frac{x^{j_0}}{n^{j-j_0}}\\
&\leq 1+k_{m,j}(q,\alpha)=C_{m,j}(q,\alpha).
\end{aligned}$$

$C_{m,j}(q,\alpha)$ is a positive constant with depend on q,m,j and α. Moreover,

$$\left\|K_{n,q}^{(\alpha)}(f;x)\right\|_m\leq\|f\|_m\left\|K_{n,q}^{(\alpha)}(1+t^m;x)\right\|_m \tag{10}$$

for every $f\in C_m^{*}[0,\infty)$. Therefore, from (8), we get

$$\left\|K_{n,q}^{(\alpha)}(f;x)\right\|_m\leq C_{m,j}(q,\alpha)\|f\|_m.$$

□

4. Direct Results

Let $C_B[0,\infty)$ denote the space of all real-valued continuous and bounded functions f on $[0,\infty)$. The norm on the space $C_B[0,\infty)$ is showed as

$$\|f\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|.$$

Then, the modulus of continuity of $f\in C_B[0,\infty)$ is given by

$$w(f,\delta)=\sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)}|f(x+h)-f(x)|.$$

Further, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in w^2}\left\{\|f-g\| + \delta\left\|g^{''}\right\|\right\} \qquad \delta > 0,$$

where $w^2 := \left\{g \in C_B\left[0,\infty\right) : g^{'}, g^{''} \in C_B\left[0,\infty\right)\right\}$. By Theorem 2.4 in [11], there exists an absolute constant $L > 0$ such that

$$K_2(f;\delta) \le L\omega_2\left(f;\sqrt{\delta}\right). \tag{11}$$

where $\delta > 0$ are absolute constant.

Here, $\omega_2\left(f;\delta\right)$ is the second order modulus of smoothness of $f \in C_B\left[0,\infty\right)$ and defined as

$$\omega_2\left(f;\delta\right) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} \left|f(x+2h) - 2f(x+h) + f(x)\right|$$

Lemma 8. *Let $f \in C_B\left[0,\infty\right)$, $0 < q < 1$ and $\alpha > 0$. Consider the operators*

$$^{*}K_{n,q}^{(\alpha)}(f;x) = K_{n,q}^{(\alpha)}(f;x) + f(x) - f\left(x + \frac{1}{[n]_q\left[\alpha+1\right]_q}\right). \tag{12}$$

Then, for all $g \in w^2$, we have

$$\begin{aligned} &\left|{}^{*}K_{n,q}^{(\alpha)}(g;x) - g(x)\right| \qquad (13)\\ \le\ &\left(\frac{[2]_q}{\left[\alpha+1\right]_q\left[\alpha+2\right]_q[n]_q^2} + \frac{x}{[n]} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q\left[\alpha+1\right]_q}\right)^2\right)\left\|g^{''}\right\|. \end{aligned}$$

Proof. From (12) we have

$$\begin{aligned} {}^{*}K_{n,q}^{(\alpha)}\left((t-x);x\right) &= K_{n,q}^{(\alpha)}\left((t-x);x\right) - \left(x + \frac{1}{[n]_q\left[\alpha+1\right]_q} - x\right)\\ &= K_{n,q}^{(\alpha)}(t;x) - xK_{n,q}^{(\alpha)}(1;x) - \left(x + \frac{1}{[n]_q\left[\alpha+1\right]_q}\right) + x = 0. \qquad (14) \end{aligned}$$

Let $x \in [0,\infty)$ and $g \in w^2$. Using the Taylor's formula,

$$g(t) - g(x) = (t-x)g^{'}(x) + \int_x^t (t-u)g^{''}(u)du, \tag{15}$$

Applying ${}^{*}K_{n,q}^{(\alpha)}$ and using (14), we can get

$${}^{*}K_{n,q}^{(\alpha)}(g;x) - g(x) ={}^{*} K_{n,q}^{(\alpha)}\left((t-x)g^{'}(x);x\right) +{}^{*} K_{n,q}^{(\alpha)}\left(\int_x^t (t-u)g^{''}(u)du;x\right)$$

$$= g^{'}(x)^{*}K_{n,q}^{(\alpha)}\left((t-x);x\right) + K_{n,q}^{(\alpha)}\left(\int_{x}^{t}(t-u)g^{''}(u)du;x\right)$$

$$-\int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\left[\alpha+1\right]_q}-u\right)g^{''}(u)du$$

$$= K_{n,q}^{(\alpha)}\left(\int_{x}^{t}(t-u)g^{''}(u)du;x\right)$$

$$-\int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\left[\alpha+1\right]_q}-u\right)g^{''}(u)du.$$

On the other hand, since

$$\int_{x}^{t}|t-u|\left|g^{''}(u)\right|du \leq \left\|g^{''}\right\|\int_{x}^{t}|t-u|\,du \leq (t-x)^2\left\|g^{''}\right\|$$

and

$$\left|\int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\left[\alpha+1\right]_q}-u\right)g^{''}(u)du\right|$$

$$\leq\left(\frac{1}{[n]_q\left[\alpha+1\right]_q}\right)^2\left\|g^{''}\right\|,$$

we conclude that

$$\left|{}^{*}K_{n,q}^{(\alpha)}(g;x)-g(x)\right|$$

$$=\left|K_{n,q}^{(\alpha)}\left(\int_{x}^{t}(t-u)g^{''}(u)du;x\right)-\int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\left[\alpha+1\right]_q}-u\right)g^{''}(u)du\right|$$

$$\leq\left\|g^{''}\right\|K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)+\left(\frac{1}{[n]_q\left[\alpha+1\right]_q}\right)^2\left\|g^{''}\right\|.$$

Finally, from Lemma 4, we can write

$$\left|{}^{*}K_{n,q}^{(\alpha)}(g;x)-g(x)\right|$$

$$\leq \left(\frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q [\alpha+1]_q}\right)^2 \right) \left\| g^{''} \right\|.$$

□

Theorem 1. *Let $f \in C_B[0,\infty)$, $0 < q < 1$ and $\alpha > 0$. Then, for every $x \in [0,\infty)$, there exists a constant $M > 0$ such that*

$$\left| K_{n,q}^{(\alpha)}(f;x) - f(x) \right| \leq M\omega_2\left(f; \sqrt{\delta_n^{(\alpha)}(x)}\right) + \omega\left(f; \beta_n^{(\alpha)}(x)\right)$$

where

$$\delta_n^{(\alpha)}(x) = \left(\frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q [\alpha+1]_q}\right)^2 \right) \left\| g^{''} \right\|$$

and

$$\beta_n^{(\alpha)}(x) = \left| \frac{1}{[n]_q [\alpha+1]_q} \right|.$$

Proof. It follows from Lemma (8), that

$$\begin{aligned}
&\left| K_{n,q}^{(\alpha)}(f;x) - f(x) \right| \\
&\leq \left| {}^{*}K_{n,q}^{(\alpha)}(f;x) - f(x) \right| + \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| \\
&\leq \left| {}^{*}K_{n,q}^{(\alpha)}(f-g;x) - (f-g)(x) \right| \\
&+ \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| + \left| {}^{*}K_{n,q}^{(\alpha)}(g;x) - g(x) \right| \\
&\leq \left| {}^{*}K_{n,q}^{(\alpha)}(f-g;x) \right| + \left| (f-g)(x) \right| \\
&+ \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| + \left| {}^{*}K_{n,q}^{(\alpha)}(g;x) - g(x) \right|.
\end{aligned}$$

Since boundedness of the ${}^{*}K_{n,q}^{(\alpha)}$ and using inequality (13), we get

$$\begin{aligned}
&\left| K_{n,q}^{(\alpha)}(f;x) - f(x) \right| \\
&\leq 4\left\| f-g \right\| + \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| \\
&+ \left(\frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q [\alpha+1]_q}\right)^2 \right) \left\| g^{''} \right\|
\end{aligned}$$

$$\leq 4\|f-g\|+\omega\left(f;\left|\frac{1}{[n]_q\,[\alpha+1]_q}\right|\right)+\delta_n^{(\alpha)}(x)\left\|g''\right\|.$$

Now, taking infimum on the right hand side over all $g\in w^2$ and using the property of Peetre's K-functional(11), we can get

$$\begin{aligned}\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| &\leq 4K_2\left(f;\delta_n^{(\alpha)}(x)\right)+\omega\left(f;\beta_n^{(\alpha)}(x)\right)\\ &\leq M\omega_2\left(f;\sqrt{\delta_n^{(\alpha)}(x)}\right)+\omega\left(f;\beta_n^{(\alpha)}(x)\right).\end{aligned}$$

□

Corollary 1. *Let $0<q_n<1$, $\alpha>0$. For any $A>0$ and $f\in C_B[0,\infty)$, then $K_{n,q_n}^{(\alpha)}(f;x)$ converges to uniformly f on $[0,A]$ if and only if $q_n\to 1$ as $n\to\infty$.*

Theorem 2. *Let $K_{n,q}^{(\alpha)}$ be the operators defined by (2), $0<q<1$, $\alpha>0$, $\rho\in(0,1]$ and D be any subset of the interval $[0,\infty)$. if $f\in C_B[0,\infty)$ is locally Lip(ρ) on D, i.e., if f satisfies the following inequality:*

$$|f(t)-f(x)|\leq C_{f,\rho}\,|t-x|^{\rho},\qquad t\in D \text{ and } x\in[0,\infty), \tag{16}$$

then for each $x\in[0,\infty)$, we have

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq C_{f,\rho}\left\{\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}+2d^{\rho}(x,D)\right\},$$

where $C_{f,\rho}$ is constant depending on f and ρ and $d(x,D)$ is the distance between x and D defined by

$$d(x,D)=\inf\{|t-x|:t\in D\}.$$

Proof. Let $\overline{D}$ denote the closure of D. Due to the features of infimum, there is at least a point $t_0\in\overline{D}$ such that $d(x,D)=|x-t_0|$. By the triangle inequality

$$|f(t)-f(x)|\leq|f(t)-f(t_0)|+|f(x)-f(t_0)|.$$

Applying $K_{n,q}^{(\alpha)}$ to the above inequality and using (16), we can get

$$\begin{aligned}&\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\\ &\leq K_{n,q}^{(\alpha)}\left(|f(t)-f(t_0)|;x\right)+K_{n,q}^{(\alpha)}\left(|f(x)-f(t_0)|;x\right)\\ &\leq C_{f,\rho}\left\{K_{n,q}^{(\alpha)}\left(|t-t_0|^{\rho};x\right)+|x-t_0|^{\rho}\right\}\\ &\leq C_{f,\rho}\left\{K_{n,q}^{(\alpha)}\left(|t-x|^{\rho}+|x-t_0|^{\rho};x\right)+|x-t_0|^{\rho}\right\}\\ &= C_{f,\rho}\left\{K_{n,q}^{(\alpha)}\left(|t-x|^{\rho};x\right)+2\,|x-t_0|^{\rho}\right\}.\end{aligned}$$

Choosing $a_1=\frac{2}{\rho}$ and $a_2=\frac{2}{2-\rho}$ and applying Hölder inequality, we have:

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|$$

$$\leq C_{f,\rho}\left\{\left(K_{n,q}^{(\alpha)}\left(|t-x|^{a_1\rho};x\right)\right)^{\frac{1}{a_1}}\left(K_{n}^{(\alpha)}\left(1^{a_2};x\right)\right)^{\frac{1}{a_2}}+2d^{\rho}(x,D)\right\}$$
$$\leq C_{f,\rho}\left\{\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}+2d^{\rho}(x,D)\right\}.$$

□

In [38], Lipcshitz type maximal function of the order ρ defined as

$$\phi_\rho\left(f;x\right)=\sup_{x,t\in[0,\infty),x\neq t}\frac{|f(t)-f(x)|}{|t-x|^{\rho}} \tag{17}$$

where $x\in[0,\infty)$ and $\rho\in(0,1]$. In the next theorem we obtain local direct estimate of the operators $K_{n,q}^{(\alpha)}$ by using (17).

Theorem 3. *Let $f\in C_B[0,\infty)$, $0<q<1$, $\alpha>0$ and $\rho\in(0,1]$. Then, for all $x\in[0,\infty)$, we have*

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\phi_\rho\left(f;x\right)\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}.$$

Proof. From the equation (17), we have

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\phi_\rho\left(f;x\right)K_{n,q}^{(\alpha)}\left(|t-x|^{\rho};x\right)$$

Applying the Hölder inequality with $a_1=\frac{2}{\rho}$ and $a_2=\frac{2}{2-\rho}$, we get

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\phi_\rho\left(f;x\right)\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}.$$

□

Theorem 4. *For $0<q<1$, $\alpha>0$, $f\in C_2[0,\infty)$, $w_{a+1}(f;\delta)$ is the modulus of continuity of f on the interval $[0,a+1]\subset[0,\infty)$, $a>0$. Then, we have*

$$\left\|K_{n,q}^{(\alpha)}(f;x)-f(x)\right\|_{C[0,a]}\leq 4N_f\left(1+a^2\right)\delta_n(x)+2w_{a+1}\left(f;\sqrt{\delta_n(x)}\right).$$

where $\sqrt{K_{n,q}^{(\alpha)}((t-x)^2;x)}$ given by Lemma 4 and $\|f\|_{C[0,a]}=\sup_{x\in[0,a]}|f(x)|$.

Proof. For $0\leq x\leq a$ and $a+1<t$, since $1<t-x$, we have

$$\begin{aligned}|f(t)-f(x)|&\leq M_f\left(x^2+t^2+2\right)\\&\leq M_f\left(2\left(t-x\right)^2+2+3x^2\right)\\&\leq M_f\left(t-x\right)^2\left(4+3x^2\right)\\&\leq 4M_f\left(t-x\right)^2\left(1+a^2\right).\end{aligned} \tag{18}$$

Also, for $0\leq x\leq a$ and $a+1\geq t$, we have

$$|f(t)-f(x)|\leq w_{a+1}\left(f;|t-x|\right)$$

$$\leq \left(1+\frac{|t-x|}{\delta}\right) w_{a+1}(f;\delta), \tag{19}$$

with $\delta > 0$.

For $0 \leq x \leq a$ and $t \geq 0$, combining (18) and (19) gives

$$\begin{aligned} & |f(t)-f(x)| \\ \leq \quad & 4M_f(t-x)^2\left(1+a^2\right)+\left(1+\frac{|t-x|}{\delta}\right) w_{a+1}(f;\delta), \end{aligned} \tag{20}$$

Applying Cauchy-Schwarz's inequality to the above inequality(20), we get

$$\begin{aligned} & \left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \\ & \leq K_{n,q}^{(\alpha)}(f;x)\left(|f(t)-f(x)|;x\right) \\ & \leq 4M_f\left(1+a^2\right) K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)+\left(1+\frac{\sqrt{K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)}}{\delta}\right) w_{a+1}(f;\delta) \\ & \leq 4M_f\left(1+a^2\right) K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)+2w_{a+1}\left(f;\delta_n(x)\right) \end{aligned}$$

on choosing $\delta := \delta_n(x) = \sqrt{K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)}$. □

5. Weighted Approximation

Theorem 5. *Let $q = q_n \in (0,1]$ such that $q_n \to 1$ as $n \to \infty$ and $\alpha > 0$. Then for each $f \in C_2^*[0,\infty)$, we have:*

$$\lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right\|_2 = 0.$$

Proof.

- Since the Korovkin type theorem on the weighted approximation([12]), we need to verify

$$\lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(t^m;x)-x^m\right\|_2 = 0, \qquad m = 0,1,2. \tag{21}$$

- For $m = 0$, obvious.
- For $m = 1$ and $m = 2$, using Lemma 3, we can write:

$$\begin{aligned} \lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(t;x)-x\right\|_2 &= \sup_{x\geq 0}\frac{\left|K_{n,q_n}^{(\alpha)}(t;x)-x\right|}{1+x^2} \\ &= \sup_{x\geq 0}\frac{1}{1+x^2}\left|\frac{1}{[n]_{q_n}[\alpha+1]_{q_n}}\right| \\ &= \frac{1}{[n]_{q_n}[\alpha+1]_{q_n}}\sup_{x\geq 0}\frac{1}{1+x^2} \\ &\leq \frac{1}{[n]_{q_n}[\alpha+1]_{q_n}} \to 0,\ n\to\infty \end{aligned}$$

and

$$\lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(t^2;x)-x^2\right\|_2$$
$$=\sup_{x\geq 0}\frac{\left|K_{n,q_n}^{(\alpha)}(t^2;x)-x^2\right|}{1+x^2}$$
$$=\sup_{x\geq 0}\frac{1}{1+x^2}\left|\frac{[2]_q}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2}+\frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_{q_n}[n]_{q_n}}x+\frac{x}{q_n}^2-x^2\right|$$
$$\leq\left(\frac{1}{q_n}-1\right)\sup_{x\geq 0}\frac{x^2}{1+x^2}+\frac{\left(2+[\alpha+1]_{q_n}\right)}{[\alpha+1]_{q_n}[n]_{q_n}}\sup_{x\geq 0}\frac{x}{1+x^2}$$
$$+\frac{[2]_q}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2}\sup_{x\geq 0}\frac{1}{1+x^2}$$
$$\leq\left(\frac{1}{q_n}-1\right)+\frac{\left(2+[\alpha+1]_{q_n}\right)}{[\alpha+1]_{q_n}[n]_{q_n}}+\frac{[2]_q}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2}\to 0,\ n\to\infty,$$

which implies that

$$\lim_{n\to\infty}\left\|K_{n,q}^{(\alpha)}(t^m;x)-x^m\right\|_2=0,\quad m=0,1,2.$$

□

In the next theorem, we present a weighted approximation theorem for $f\in C_2^*[0,\infty)$, where Doğru studied for classical Szász operators in [13].

Theorem 6. *Let $q=q_n\in(0,1]$ such that $q_n\to 1$ as $n\to\infty$ and $\alpha>0$. For each $f\in C_2^*[0,\infty)$ and $\beta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\geq 0}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}=0.$$

Proof. Let $x_0\in[0,\infty)$ be arbitrary but fixed. Then

$$\sup_{x\in[0,\infty)}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}$$
$$=\sup_{x\in[0,x_0]}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}+\sup_{x\in(x_0,\infty)}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}$$
$$\leq\left\|K_{n,q_n}^{(\alpha)}(f)-f\right\|_{C[0,x_0]}+\|f\|_2\sup_{x\in(x_0,\infty)}\frac{\left|K_{n,q_n}^{(\alpha)}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\beta}}$$

$$+ \sup_{x\in(x_0,\infty)} \frac{|f(x)|}{(1+x^2)^{1+\beta}}$$
$$= H_1 + H_2 + H_3.$$

Since $|f(x)| \leq N_f(1+x^2)$, we have

$$H_3 = \sup_{x\in(x_0,\infty)} \frac{|f(x)|}{\left(1+x^2\right)^{1+\beta}} \leq \sup_{x\in(x_0,\infty)} \frac{N_f}{\left(1+x^2\right)^{\beta}} \leq \frac{N_f}{\left(1+x_0^2\right)^{\beta}}.$$

Firstly, From Theorem (4), we have

$$H_1 \text{ goes to zero as } n \to \infty.$$

Secondly, by Theorem 5,

$$H_2 = \|f\|_2 \lim_{n\to\infty} \sup_{x\in(x_0,\infty)} \frac{\left|K_{n,q}^{(\alpha)}\left(1+t^2;x\right)\right|}{\left(1+x^2\right)^{1+\beta}}$$
$$= \sup_{x\in(x_0,\infty)} \frac{\left(1+x^2\right)}{\left(1+x^2\right)^{1+\beta}} \|f\|_2$$
$$= \sup_{x\in(x_0,\infty)} \frac{\|f\|_2}{\left(1+x^2\right)^{\beta}} \leq \frac{\|f\|_2}{\left(1+x_0^2\right)^{\beta}}.$$

Moreover, if we choose $x_0 > 0$ large enough, we can see that

$$H_2 \to 0 \text{ and } H_3 \to 0 \text{ as } n \to \infty,$$

Combining, H_1, H_2 and H_3, we get desired result. □

In the next theorem we obtain direct estimation in terms of weighted modulus of continuity. For every $f \in C_m^*[0,\infty)$ the weighted modulus of continuity defined as

$$\Omega_m(f,\delta) = \sup_{x\geq 0,\ 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^m}, \tag{22}$$

Lemma 9. *[39]If $f \in C_m^*[0,\infty), m \in \mathbb{N}$, then*

(i) $\Omega_m(f,\delta)$ is a monotone increasing function of δ,
(ii) $\lim_{\delta\to 0^+} \Omega_m(f,\delta) = 0$,
(iii) for any $\rho \in [0,\infty), \Omega_m(f,\rho\delta) \leq (1+\rho)\Omega_m(f,\delta)$.

In the next theorem, we express the approximation error of $K_{n,q}^{(\alpha)}$ by using Ω_m.

Theorem 7. *For $f \in C_m^*[0,\infty)$, we have*

$$\left\|K_{n,q}^{(\alpha)}(f) - f\right\|_{m+1} \leq N\Omega_m(f,(1/\sqrt{qn})),$$

where N is a constant independent of f and n.

Proof. From (22) and Lemma 9, we can write

$$\begin{aligned}|f(t)-f(x)| &\le (1+(x+|t-x|)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta)\\ &\le (1+(2x+t)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(\varphi,\delta).\end{aligned}$$

Then, we have

$$\begin{aligned}&\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\\ &\le K_{n,q}^{(\alpha)}\left|(f(t)-f(x)\right|;x)\\ &\le \Omega_m(f,\delta)\left(K_{n,q}^{(\alpha)}((1+(2x+t)^m);x)+K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)\frac{|t-x|}{\delta};x\right)\right).\\ &=\Omega_m(f,\delta)\left(K_{n,q}^{(\alpha)}\left(1+(2x+t)^m;x\right)+I_1\right).\end{aligned}$$

Applying Cauchy-Schwartz inequality to the I_1, we get

$$I_1\le (K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)^2;x\right))^{1/2}\left(K_{n,q}^{(\alpha)}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}.$$

Therefore,

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \tag{23}$$

$$\begin{aligned}\le\quad &\Omega_m(f,\delta)K_{n,q}^{(\alpha)}((1+(2x+t)^m);x)\\ &+\Omega_m(f,\delta)(K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)^2;x\right))^{1/2}\left(K_{n,q}^{(\alpha)}\left(\frac{|t-\tau|^2}{\delta^2};x\right)\right)^{1/2}.\end{aligned}\tag{24}$$

By Lemma 7 and Lemma 4,

$$\begin{aligned}K_{n,q}^{(\alpha)}(1+(2x+t)^m;x)&\le C_{m,j}(q,\alpha)\left(1+x^m\right),\\ (K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)^2;x\right))^{1/2}&\le C_{m,j}^1(q,\alpha)\left(1+x^m\right).\end{aligned}\tag{25}$$

and

$$\begin{aligned}\left(K_{n,q}^{(\alpha)}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2} &\le \frac{1}{\delta}\sqrt{\frac{[2]_q}{[\alpha+1]_q[\alpha+2]_q[n]_q^2}+\frac{x}{[n]}+x^2\left(\frac{1}{q}-1\right)}\\ &\le \frac{(2+x)}{\delta\sqrt{qn}}.\end{aligned}\tag{26}$$

Combining 23, 25 and 26, we have

$$\begin{aligned}&\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\\ &\le \Omega_m(f,\delta)\left(C_{m,j}(q,\alpha)\left(1+x^m\right)+C_{m,j}^1(q,\alpha)\frac{\left(1+x^m\right)(2+x)}{\delta\sqrt{qn}}\right)\end{aligned}$$

$$= \Omega_m(f,\delta)\left(C_{m,j}(q,\alpha)\left(1+x^m\right)+C^1_{m,j}(q,\alpha)C_1\frac{\left(1+x^{m+1}\right)}{\delta\sqrt{qn}}\right),$$

where

$$C_1 = \sup_{x\geq 0}\frac{\left(2+2x^m+x+2x^{m+1}\right)}{1+x^{m+1}}.$$

if we take $\delta = \left(1/\sqrt{q\,[n]_q}\right)$ in the above inequality, we obtain the desired result. □

Next result is a Voronovskaja type formula for the operators $K^{(\alpha)}_{n,q}(f;x)$.

6. Voronovskaja Type

Theorem 8. *Let $q = q_n \in (0,1]$ such that $q_n \to 1$, $q_n^n \to b$ as $n\to\infty$ and $\alpha > 0$. For any $f \in C_2^*[0,\infty)$ such that $f\,'$, $f\,''\in C_2^*[0,\infty)$ the following equality holds*

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left[K^{(\alpha)}_{n,q_n}(f;x)-f(x)\right]\\ =\;& \frac{1}{(\alpha+1)}f\,'(x)+\frac{1}{2}\left(x+x^2\left(1-b\right)\right)f\,''(x).\end{aligned}$$

Proof. By the Taylor's formula, we can write

$$f(t) = f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2 \tag{27}$$

where $r(t,x)$ is Peano form of remainder, $r(.,x)\in C_2^*[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$.

Applying $K^{(\alpha)}_{n,q}$ to the both sides of (27), we get

$$\begin{aligned}&[n]_{q_n}\left[K^{(\alpha)}_{n,q_n}(f;x)-f(x)\right]\\ =\;& f\,'(x)\,[n]_{q_n}K^{(\alpha)}_{n,q_n}\left((t-x)\,;x\right)+\frac{1}{2}f\,''(x)\,[n]_{q_n}K^{(\alpha)}_{n,q_n}\left((t-x)^2;x\right)\\ &+[n]_{q_n}K^{(\alpha)}_{n,q_n}\left(r(t,x)(t-x)^2;x\right).\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$K^{(\alpha)}_{n,q_n}\left(r(t,x)(t-x)^2;x\right) \leq \sqrt{K^{(\alpha)}_{n,q_n}\left(r^2(t,x);x\right)}\sqrt{K^{(\alpha)}_{n,q_n}\left((t-x)^4;x\right)}. \tag{28}$$

Observe that $r^2(t,x) = 0$ and $r^2(.,x)\in C_2^*[0,\infty)$.

Then, it follows from that Corollary (1),

$$\lim_{n\to\infty}[n]_{q_n}K^{(\alpha)}_{n,q_n}\left(r^2(t,x);x\right) = r^2(x,x) = 0. \tag{29}$$

Moreover, from (6), (28) and (29), we can obtain

$$\lim_{n\to\infty}K^{(\alpha)}_{n,q_n}\left(r(t,x)(t-x)^2;x\right) = 0 \tag{30}$$

Hence, combining (4), (5) and (30), we get

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left[K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right]\\ =\ &\frac{1}{(\alpha+1)}f^{\,'}(x)+\frac{1}{2}\left(x+x^2(1-b)\right)f^{\,''}(x).\end{aligned}$$

□

7. Bivariate Fractional q-Integral

In this section, we introduce the bivariate fractional q-integral of Riemann-Liouville integral type $K_{n,q}^{(\alpha)}(f;x)$ (2)as follows:

$$\begin{aligned}&K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)\\ &=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}[\alpha_1]_{q_1}[\alpha_2]_{q_2}s_{n_1,k_1}(q_1;x)s_{n_2,k_2}(q_2;y)\\ &\int_0^1\int_0^1 f\left(\frac{q_1^{1-k_1}[k_1]_{q_1}+t_1}{[n_1]_{q_1}},\frac{q_2^{1-k_2}[k_2]_{q_2}+t_2}{[n_2]_{q_2}}\right)(1-t_1)^{\alpha_1-1}(1-t_2)^{\alpha_2-1}d_{q_1}t_1d_{q_2}t_2\end{aligned}$$

where $(x,y)\in I^2=[0,\infty)\times[0,\infty)$ and $\alpha_1,\alpha_2>0$.

Fractional q-integral of Riemann-Liouville integral type $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(\cdot;x,y)$ can be rewritten as

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(\cdot;x,y)=K_{n_1,q_1}^{(\alpha_1)}(\cdot;x)\times K_{n_2,q_2}^{(\alpha_2)}(\cdot;y).$$

Lemma 10. *Let $e_{ij}(x,y)=x^iy^j$, $0<q_1,q_2<1$, $0\leq i+j\leq 2$ and $\alpha_1,\alpha_2>0$. For $(x,y)\in I^2=[0,\infty)\times[0,\infty)$, we have*

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{00};x,y)=1,$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{10};x,y)=x+\frac{1}{[n_1]_{q_1}[\alpha_1+1]_{q_1}},$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{01};x,y)=y+\frac{1}{[n_2]_{q_2}[\alpha_2+1]_{q_2}},$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{20};x,y)=\frac{[2]_{q_1}}{[\alpha_1+1]_{q_1}[\alpha_1+2]_{q_1}[n_1]_{q_1}^2}+\frac{\left(2+[\alpha_1+1]_{q_1}\right)}{[\alpha_1+1]_{q_1}[n_1]_{q_1}}x+\frac{x}{q_1}^2,$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{02};x,y)=\frac{[2]_{q_2}}{[\alpha_2+1]_{q_1}[\alpha_2+2]_{q_2}[n_2]_{q_2}^2}+\frac{\left(2+[\alpha_2+1]_{q_2}\right)}{[\alpha_2+1]_{q_2}[n_2]_{q_2}}y+\frac{y}{q_2}^2.$$

Remark 1. *According to above Lemma (10), we get*

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{10}-x;x,y)=\frac{1}{[n_1]_{q_1}[\alpha_1+1]_{q_1}},$$

$$K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(e_{01}-y;x,y) = \frac{1}{[n_2]_{q_2}[\alpha_2+1]_{q_2}},$$

$$K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}((e_{10}-x)^2;x,y) = \frac{[2]_{q_1}}{[\alpha_1+1]_{q_1}[\alpha_1+2]_{q_1}[n_1]^2_{q_1}} + \frac{x}{[n_1]_{q_1}} + x^2\left(\frac{1}{q_1}-1\right) = \delta^{(\alpha_1)}_{n_1}(q_1;x),$$

$$K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}((e_{01}-y)^2;x,y) = \frac{[2]_{q_2}}{[\alpha_2+1]_{q_2}[\alpha_2+2]_{q_2}[n_2]^2_{q_2}} + \frac{y}{[n_2]_{q_2}} + y^2\left(\frac{1}{q_2}-1\right) = \delta^{(\alpha_2)}_{n_2}(q_2;y).$$

In the next theorem, we obtain the uniform convergence of the bivariate q-Riemann-Liouville fractional integral type of q-Szász-Mirakyan-Kantorovich operators to the bivariate functions defined on $I^2=[0,\infty)\times[0,\infty)$.

Theorem 9. *Let $C(I^2)$ be the space of continuous bivariate function on $I^2=[0,\infty)\times[0,\infty)$ and $\alpha_1,\alpha_2>0$. Then for any $f\in C(I^2)$, we have*

$$\lim_{n_1,n_2\to\infty}\left\|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}f-f\right\|=0.$$

Proof. Using lemma 1, we get

$$\left\|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}e_{00}-e_{00}\right\|=0,\ \left\|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}e_{10}-e_{10}\right\|\to 0$$
$$\left\|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}e_{01}-e_{01}\right\|\to 0,\ \left\|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(e_{20}+e_{02})-(e_{20}+e_{02})\right\|\to 0$$
$$\text{as } n_1,n_2\to\infty$$

As a result, by Volkov's theorem [40], we get

$$\lim_{n_1,n_2\to\infty}\left\|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}f-f\right\|=0.$$

□

For bivariate real functions, modulus of continuity defined as

$$w(f;\delta_n,\delta_m)=\sup\left\{|f(t,s)-f(x,y)|:(t,s),x,y\in I^2,|t-x|\le\delta_n,|s-y|\le\delta_m\right\}.$$

Theorem 10. *Let $f\in C(I^2)$, $0<q_1,q_2<1$and $\alpha_1,\alpha_2>0$. Then for all $(x,y)\in I^2$, the inequality*

$$\left|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f;x,y)-f(x,y)\right|\le 4w\left(f;\delta^{(\alpha_1)}_{n_1}(q_1;x),\delta^{(\alpha_2)}_{n_2}(q_2;y)\right)$$

holds, where $\delta^{(\alpha_1)}_{n_1}(q_1;x),\delta^{(\alpha_2)}_{n_2}(q_2;y)$ are as in Remark 1.

Proof. By the positivity and linearity properties of the $K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}$, we can write

$$\left|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(f;x,y)-f(x,y)\right|\le K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}(|f(t,s)-f(x,y)|;x,y)$$

$$\leq w(f;\delta_1,\delta_2)\left(K_{n_1,q_1}^{(\alpha_1)}(1;x)+\frac{1}{\delta_1}K_{n_1,q_1}^{(\alpha_2)}(|t-x|;x)\right)$$
$$\times\left(K_{n_2,q_2}^{(\alpha_2)}(1;y)+\frac{1}{\delta_2}K_{n_2,q_2}^{(\alpha_2)}(|s-y|;y)\right)$$

Applying Cauchy-Schwarz inequality, we obtain

$$K_{n_1,q_1}^{(\alpha_1)}(|t-x|;x)\leq K_{n_1,q_1}^{(\alpha_1)}\left((t-x)^2;x\right)^{\frac{1}{2}}$$
$$K_{n_2,q_2}^{(\alpha_2)}(|s-y|;y)\leq K_{n_2,q_2}^{(\alpha_2)}\left((s-y)^2;y\right)^{\frac{1}{2}}$$

Choosing $\delta_1=\delta_{n_1}^{(\alpha_1)}(q_1;x)$ and $\delta_2=\delta_{n_2}^{(\alpha_2)}(q_2;y)$, we have desired result. □

Now, we are present some graphs and numerical results for $K_{n,q}^{(\alpha)}$ and $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}$ obtained by using Matlab.

8. Graphical Simulations

Example 1. *Consider $f(x)=x^3-9x^2-15x+9$ with $x\in[0,6]$. Here we take the value of $q\in\{0.75,0.85,0.95\}$,for $K_{100,q}^{(2)}$. The Figure 1 demonstrate the convergence of operators $K_{100,q}^{(2)}$ to $f(x)$ for increasing values of q and fixed α,n. Moreover, absolute error function $E_{100,q}^{(2)}(f;x)=\left|K_{100,q}^{(2)}(f;x)-f(x)\right|$ is illustrated in Figure 2. Then, numerical values of $E_{100,q}^{(2)}(f;x)$ at some points on the interval $[0,6]$ for $\{q\in 0.75,0.85,0.95\}$ are given in Table 1.*

Table 1. Estimation of the absolute error function $E_{100,q}^{(2)}$ with $f(x)=x^3-9x^2-15x+9$ for some values of x in $[0,6]$ and $q\in\{0.75,0.85,0.95\}$.

x	$E_{100,0.75}^{(2)}$	$E_{100,0.85}^{(2)}$	$E_{100,0.95}^{(2)}$
0	0.479	0.170	0.056
1	1.830	0.761	0.178
2	1.013	0.283	0.692
3	16.273	6.732	3.551
4	52.172	22.357	9.397
5	116.932	50.926	19.228
6	218.776	96.211	34.043

As we increase the value of q and fixed α and n, the approximation is good, i.e for the largest value of q, the error is minumum.

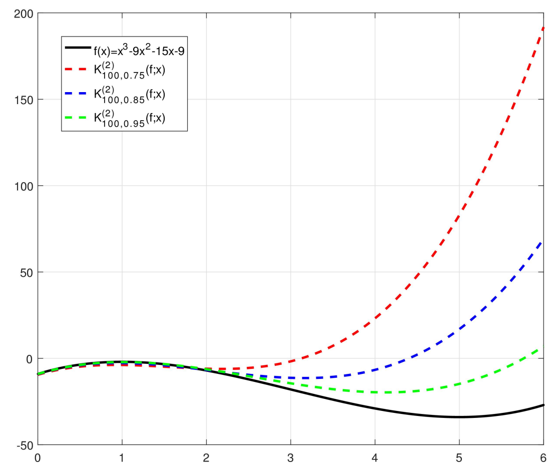

FIGURE 1. Approximation to $f(x) = x^3 - 9x^2 - 15x + 9$ by $K^{(2)}_{100,q}(f;x)$ for $q \in \{0.75, 0.85, 0.95\}$.

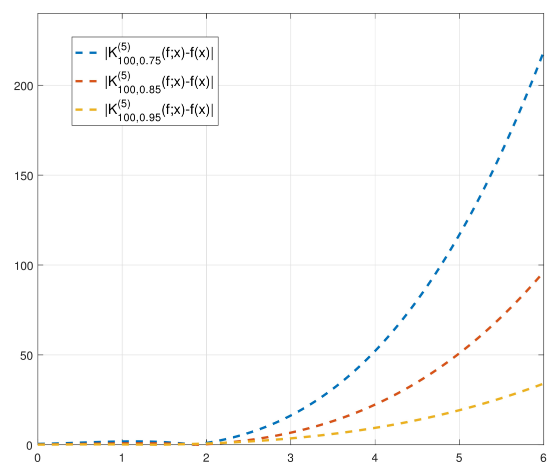

FIGURE 2. $E^{(2)}_{100,q}(f;x)$ for $f(x) = x^3 - 9x^2 - 15x + 9$ and $q = \{0.75, 0.85, 0.95\}$.

Example 2. *Let $f(x) = x^6$ with $x \in [0, 6]$. Here we take the value of $n \in \{10, 100\}$, $\alpha = 5$ and $q = 0.95$. The Figure 3 demonstrate the convergence of operators*

$K^{(5)}_{n,0.95}$ *to* $f(x)$ *for increasing values of* n*. Secondly, The absolute error function* $E^{(5)}_{n,0.95}\,(f;x) = \left|K^{(5)}_{n,0.95}(f;x) - f(x)\right|$ *is illustrated in Figure 4. Finally, numerical values of* $E^{(5)}_{n,0.95}\,(f;x)$ *at some points on the interval* $[0,6]$ *for* $n \in \{10,100\}$ *are given in Table 2.*

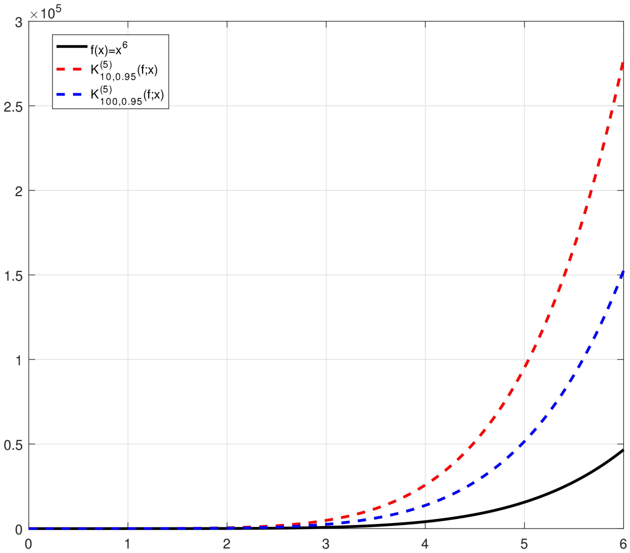

FIGURE 3. Approximation to $f(x) = x^6$ by $K^{(5)}_{n,0.95}(f;x)$ for $n \in \{10,100\}$.

TABLE 2. Estimation of the absolute error function $E^{(5)}_{n,0.95}$ with $f(x) = x^6$ for some values of x in $[0,6]$ and $n \in \{10,100\}$

x	$E^{(5)}_{10,0.95}$	$E^{(5)}_{100,0.95}$
1	8.57	3.06
2	401.68	164.39
3	4067.76	1758.63
4	21481.26	9566.43
5	78862.08	35778.98
6	229422.04	105421.79

As we increase the value of n *and fixed* α *and* q*, the approximation is good, i.e for the largest value of* n*, the error is minumum.*

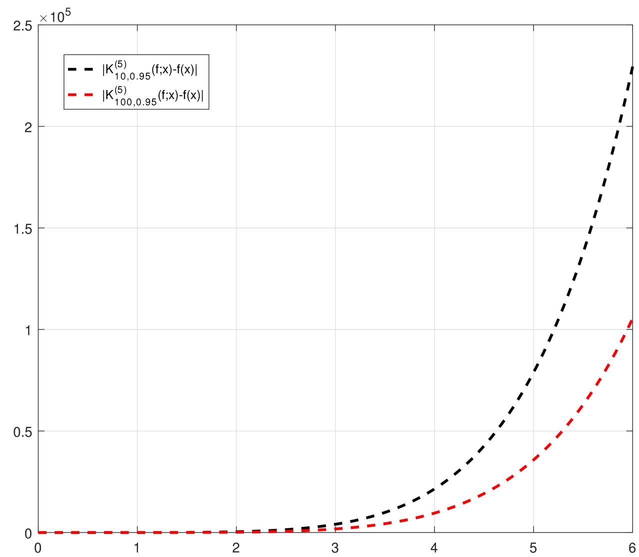

FIGURE 4. $E^{(5)}_{n,0.95}(f;x)$ for $n = \{10, 100\}$, $f(x) = x^6$.

Example 3. *Let* $f(x) = x^3 - 4x^2 + 2$ *with* $x \in [0, 5]$. *Here we take the value of* $\alpha \in \{0.1, 10\}$, $n = 150$ *and* $q = 0.95$. *The Figure 5 demonstrate the convergence of operators* $K^{(\alpha)}_{150,0.95}$ *to* $f(x)$ *for increasing values of* α. *Secondly, The absolute error function* $E^{(\alpha)}_{150,0.95}(f;x) = \left|K^{(\alpha)}_{n,q}(f;x) - f(x)\right|$ *is illustrated in Figure 6. Finally, numerical values of* $E^{(\alpha)}_{150,0.95}$ *at some points on the interval* $[3, 5]$ *for* $\alpha \in \{0.1, 10\}$ *are given in Table 3.*

TABLE 3. Estimation of the absolute error function $E^{(\alpha)}_{150,0.95}$ with $f(x) = x^3 - 4x^2 + 2$ for some values of x in $[3, 5]$ and $\alpha \in \{0.1, 10\}$

x	$E^{(0.1)}_{150,0.95}$	$E^{(10)}_{150,0.95}$
3	6.682	6.472
3.5	9.853	9.388
4	13.944	13.161
4.5	19.081	17.917
5	25.388	23.780

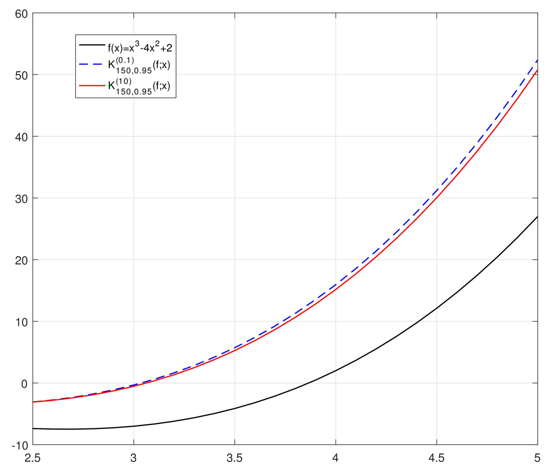

FIGURE 5. Approximation to $f(x) = x^3 - 4x^2 + 2$ by $K^{(\alpha)}_{150,0.95}(f;x)$ for $\alpha \in \{0.1, 10\}$.

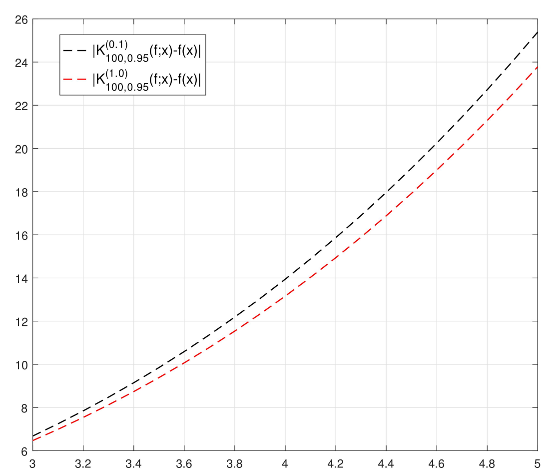

FIGURE 6. $E^{(\alpha)}_{150,0.95}(f;x)$ for $f(x) = x^3 - 4x^2 + 2$ and $\alpha \in \{0.1, 10\}$.

Now, we are present some graphs and numerical results for the convergence of bivariate fractional q-integral Riemann-Liouville integral type Szász-Mirakyan-Kantorovich operators $K^{(\alpha_1,\alpha_2)}_{n_1,n_2,q_1,q_2}$ by considering the function $f(x,y) = x + y$.

Example 4. *Consider,* $f(x,y)=x+y$ *with* $(x,y)\in[0,4]\times[0,4]$*. Here we take the value of* $n_1,n_2\in\{5,150\}$, $q_1=q_2=0.75$ *and* $\alpha_1=\alpha_2=0.1$ *. The Figure 7 explains the convergence of the operators* $K^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ *towards the function* $f(x,y)$ *for increasing values of* n_1,n_2*. Secondly, The absolute error function* $E^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}(f;x,y)=\left|K^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}(f;x,y)-f(x,y)\right|$ *is illustrated Figure 8. Finally numerical values of* $E^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ *at some points on the interval* $[0,4]\times[0,4]$ *for* $n_1,n_2\in\{5,150\}$ *are given in Table 4.*

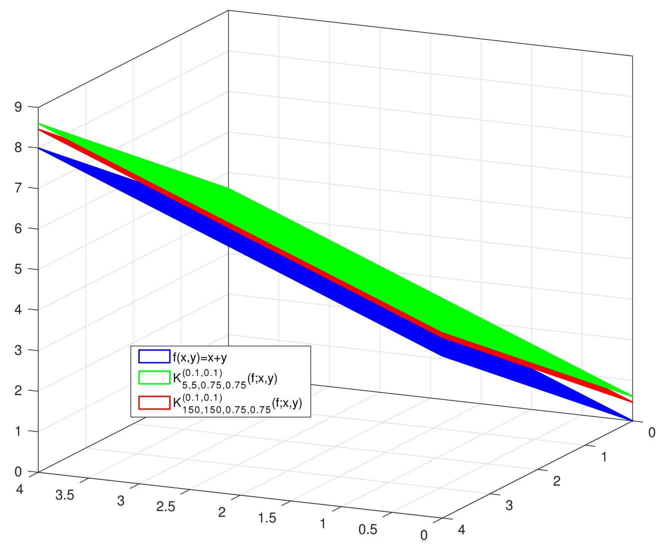

FIGURE 7. Convergence of the operators $K^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ to the function $f(x,y)=x+y$.

As we increase the value of n_1 *and* n_2 *and fixed* α_1, α_2,q_1 *and* q_2*, the approximation is good, i.e for the largest value of* n_1 *and* n_2 *and fixed* α_1, α_2,q_1 *and* q_2*, the error is minumum.*

Example 5. *Consider* $f(x,y)=x+y$ *with* $(x,y)\in[0,4]\times[0,4]$*. Here we take the value of* $\alpha_1,\alpha_1\in\{0.1,10\}$, $q_1=q_2=0.75$ *and* $n_1=n_2=5$*. The Figure 9 explains the convergence of the operators* $K^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}$ *towards the function* $f(x,y)$ *for increasing values of* $\alpha_1,\alpha_2\in\{0.1,10\}$*. Secondly, absolute error function*

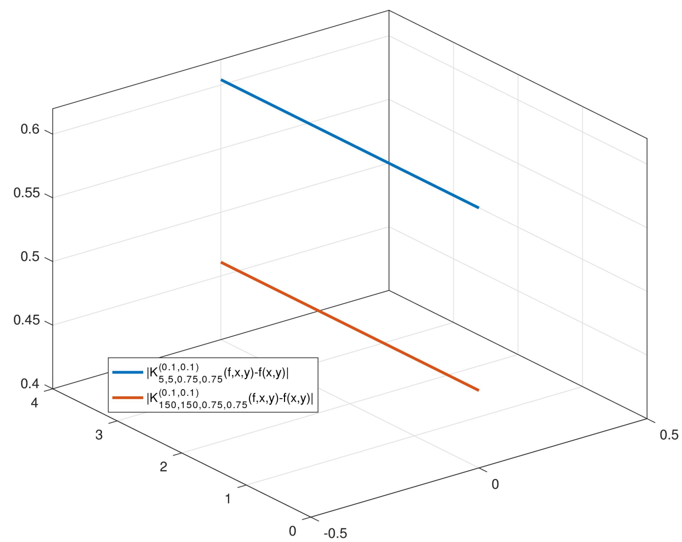

FIGURE 8. $\mathrm{E}^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ with $f(x,y)=x+xy+12y^2$ for $n_1,n_2\in\{5,150\}$ on the interval $[0,4]\times[0,4]$.

TABLE 4. Estimation of the absolute error function $E^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ with $f(x,y)=x+y$ for some values of (x,y) in $[0,4]\times[0,4]$ and $n_1,n_2\in\{5,150\}$.

x	y	$E^{(0.1,0.1)}_{5,5,0.75,0.75}$	$E^{(0.1,0.1)}_{150,150,0.75,0.75}$
0	0	0.604	0.461
0	0.5	0.604	0.461
0	1	0.604	0.461
0	1.5	0.604	0.461
0	2	0.604	0.461
0	2.5	0.604	0.461
0	3	0.604	0.461
0	3.5	0.604	0.461
0	4	0.604	0.461

$E^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}(f;x,y)=\left|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,,q_1,q_2}(f;x,y)-f(x,y)\right|$ *is illustrated Figure 10. Finally, numerical values of* $E^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}$ *at some points on the interval* $[0,4]\times[0,4]$ *for* $\alpha_1,\alpha_2\in\{0.1,10\}$ *are given in Table 5.*

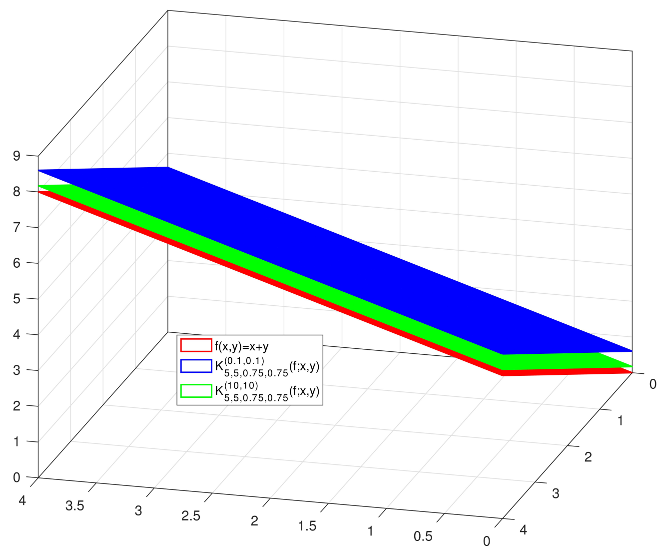

FIGURE 9. Convergence of the operators $K^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}(f;x,y)$ to the function $f(x,y)=x+y$.

TABLE 5. Estimation of the absolute error function $E^{(\alpha_1,\alpha_2)}_{50,50,0.75,0.75}$ with $f(x,y)=x+y$ for some values of (x,y) in $[0,4]\times[0,4]$ and $\alpha_1,\alpha_2\in\{0.1,10\}$.

x	y	$E^{(0.1,0.1)}_{50,50,0.75,0.75}$	$E^{(10,10)}_{50,50,0.75,0.75}$
0.1	0.1	0.461	0.131
0.1	0.5	0.461	0.131
0.1	1	0.461	0.131
0.1	1.5	0.461	0.131
0.1	2	0.461	0.131
0.1	2.5	0.461	0.131
0.1	3	0.461	0.131
0.1	3.5	0.461	0.131
0.1	4	0.461	0.131

As we increase the value of α_1 and α_2 and fixed q_1,q_2,n_1 and n_2, the approximation is good, i.e for the largest value of α_1 and α_2 and fixed q_1,q_2,n_1 and n_2, the error is minumum.

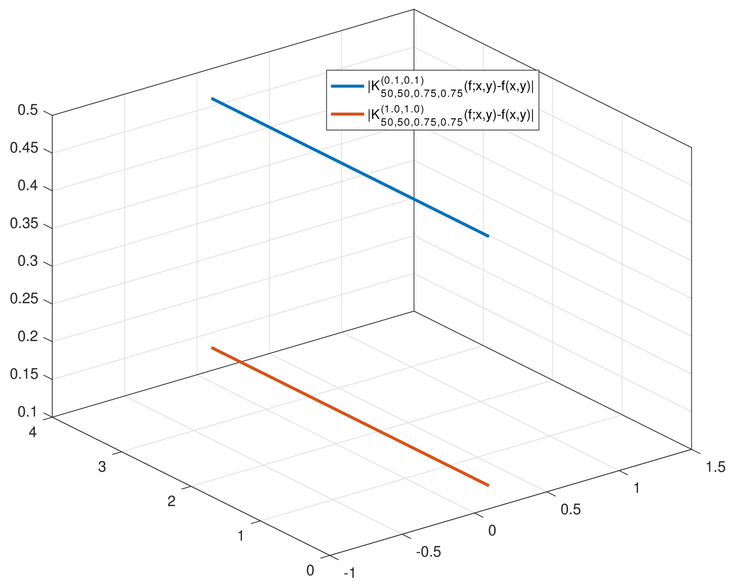

FIGURE 10. For some (x, y) points, error function $E^{(\alpha_1,\alpha_2)}_{50,50,0.75,0.75}$ with $f(x, y) = x + y$.

Example 6. *Consider $f(x) = x + y$ with $(x, y) \in [0, 4] \times [0, 4]$. Here we take the value of $q \in \{0.35, 0.75\}$, $n_1 = n_2 = 10$ and $\alpha_1 = \alpha_1 = 5$ for $K^{(\alpha_1,\alpha_2)}_{n_1,n_2,,q_1,q_2}$. The Figure 11 demonstrate the convergence of operators $K^{(5,5)}_{10,10,q_1,q_2}$ to $f(x, y)$ for increasing values of q_1 and q_2. Moreover, function of absolute error $E^{(5,5)}_{10,10,q_1,q_2}(f; x, y) = \left|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,,q_1,q_2}(f; x, y) - f(x, y)\right|)$ in is illustrated Figure 12. Then, numerical values of $E^{(5,5)}_{10,10,q_1,q_2}$ at some points on the interval $[0, 4]\times[0, 4]$ for $q_1, q_2 \in \{0.35, 0.75\}$ are given in Table 6.*

As we increase the value of q_1 and q_2 and fixed α_1, α_2, n_1 and n_2, the approximation is good, i.e for the largest value of q_1 and q_2 and fixed α_1, α_2, n_1 and n_2, the error is minumum.

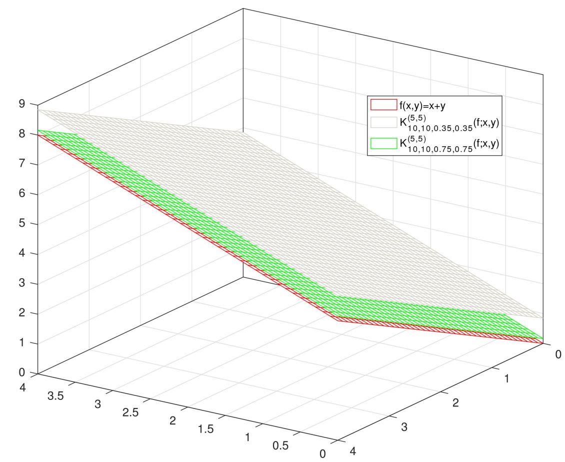

FIGURE 11. Approximation to $f(x,y) = x+y$ by $K^{(5,5)}_{10,10,q_1,q_2}$ $q_1, q_2 \in \{0.35, 0.75\}$.

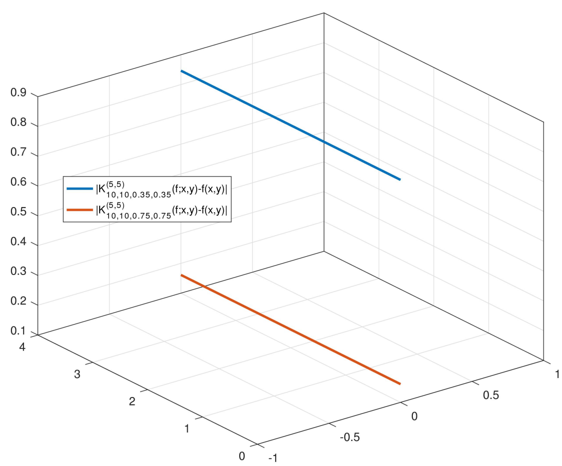

FIGURE 12. $E^{(2,2)}_{10,10,q_1,q_2}(f;x)$ for $f(x,y) = x+y$ and $q_1, q_2 = \{0.35, 0.75\}$.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Table 6. Estimation of the absolute error function $E^{(5,5)}_{10,10,q_1,q_2}$ with $f(x,y)=x+y$ for some values of x in $[0,4]\times[0,4]$ and $q_1,q_2\in\{0.35,0.75\}$.

x	y	$E^{(5,5)}_{10,10,0.35,0.35}$	$E^{(5,5)}_{10,10,0.75,0.75}$
0	0	0.847	0.161
0	0.5	0.847	0.161
0	1	0.847	0.161
0	1.5	0.847	0.161
0	2	0.847	0.161
0	2.5	0.847	0.161
0	3	0.847	0.161
0	3.5	0.847	0.161
0	4	0.847	0.161

References

[1] Ditzian, Z, Totik, V., Moduli of Smoothness, Springer Series in Computational Mathematics, New-York Springer, 1987.

[2] Aral, A., Limmam, L.M, Özsaraç, F., Approximation properties of Szász-Mirakjan-Mirakyan Kantorovich type operators, *Math. Methods Appl. Sci.*, 42(16) (2019), 5233-5240. https://doi:10.1002/mma.5280

[3] Duman, O., Özarslan, M.A., Vecchia, B.D., Modified Szász-Mirakjan-Kantorovich operators preserving linear functions, *Turk J Math.*, 33 (2009), 151-158. https://doi:10.3906/mat-0801-2

[4] Aral, A., Inoan, D., Rasa, I., On the generalized Szász-Mirakyan operators, *Results Math.*, 65 (2014), 441-452. https://doi:10.1007/s00025-013-0356-0

[5] Acar, T., Aral, A., Cárdenas-Morales, D., Garrancho, P., Szász-Mirakyan type operators which fix exponentials, *Results in Math.*, 72 (2017), 1393-1404. https://doi:10.1007/s00025-017-0665-9

[6] Acar, T., Aral, A., Gonska, H., On Szász-Mirakyan operators preserving e^{2ax}, $a>0$, *Mediterr. J. Math.*, 14(6) (2017). https://doi.org/10.1007/s00009-016-0804-7

[7] Gupta, V., Approximation with Positive Linear Operators and Linear Combinations, Springer International Publishing, 2017.

[8] Gupta, V., Aral, A., A note on Szász-Mirakyan-Kantorovich type operators preserving e^{-x}, *Positivity*, 22 (2018), 415-423. https://doi.org/10.1007/s11117-017-0518-5

[9] Otto, S., Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. of Standards,* 45(3) (1950), 239-245.

[10] Mirakjan, G.M., Approximation of continuous functions with the aid of polynomials, *In Dokl. Acad. Nauk SSSR*, 31 (1941), 201-205.

[11] Devore, R.A., Lorentz, G.G., Constructive Approximation, Springer-Verlang, New York-London, 1993.

[12] Gadjieva, A.D., A problem on the convergence of a sequence of positive linear operators on unbounded sets, and theorems that are analogous to P.P. Korovkin's theorem, *Doklady Akademii Nauk SSSR*, 218(5) (1974), 1001-1004.

[13] Doğru, O., Gadjieva, E., Ağırlıklı uzaylarda Szász tipinde operatörler dizisinin sürekli fonksiyonlara yaklaşımı, *II. Kızılırmak Uluslararası Fen Bilimleri Kongresi Bildiri Kitabı*, Kırıkkale, (1998), 29-37.

[14] Dhamija, M., Pratap, R., Deo, N., Approximation by Kantorovich form of modified Szasz-Mirakyan operators, *Appl. Math. Comput.*, 317 (2018), 109-120. https://doi.org/10.1016/j.amc.2017.09.004

[15] Gupta, V., Acu, A.M., On Baskakov-Szász-Mirakyan-type operators preserving exponential type functions, 22(3) (2018), 919-929. https://doi.org/10.1007/s11117-018-0553

[16] Mursaleen, M., Alotaibi, A., Ansari, K.J., On a Kantorovich variant of Szász- Mirakjan operators, *J. Funct. Spaces*, 2016. https://doi.org/10.1155/2016/1035253

[17] Acar, T., Gupta, V., Aral, A., Rate of convergence for generalized Szász operators, *Bull. Math. Sci.*, 1 (2011), 99-113. https://doi.org/10.1007/s13373-011-0005-4

[18] Agrawal, P.N., Gupta, V., Kumar, A.S., Kajla, A., Generalized Baskakov-Szász type operators, *Appl. Math. Comput.,* 236 (2014), 311-324. https://doi.org/10.1016/j.amc.2014.03.084

[19] Aral, A., A generalization of Szász-Mirakyan operators based on q-integers, *Math. Comput. Modelling*, 47(9-10) (2008), 1052-1062. https://doi.org/10.1016/j.amc.2014.03.084

[20] Finta, Z., Govil, N.K., Gupta, V., Some results on modified Szász-Mirakjan operators, *J. Math. Anal. Appl.,* 327(2) (2007), 1284-1296. https://doi.org/10.3906/mat-0801-2

[21] Mazhar, S.M., Totik, V., Approximation by modified Szász operators, *Acta Sci. Math.*, 49 (1985), 257-269.

[22] Totik, V., Approximation by Szász-Mirakjan-Kantorovich operators in $Lp(p > 1)$, *Analysis Mathematica*, 9(2) (1983), 147-167. https://doi.org/10.1007/bf01982010

[23] Dahmani, Z., Tabharit, L., Taf, S., New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.*, 2(3) (2010), 93-99.

[24] Katugompola, U.N., New approach generalized fractional integral, *Applied Math and Comp.*, 218(3) (2011), 860-865. https://doi.org/10.1016/j.amc.2011.03.062

[25] Latif, M.A., Hussain, S., New inequalities of Ostrowski type for co-ordineted convex functions via fractional integrals, *Journal of Fractional Calculus and Applications*, 2(9) (2012), 1-15.

[26] Romero, L.G., Luque, L.L., Dorrego, G.A., Cerutti, R.A., On the k-Riemann Liouville fractional derivative, *Int. J. Contemp. Math. Sciences*, 8(1) (2013), 41-51. http://dx.doi.org/10.12988/ijcms.2013.13004

[27] Tunc, M., On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat,* 27(4) (2013), 559–565. https://doi.org/10.2298/FIL1304559T

[28] Mahmudov, N.I., On q-Parametric Szász-Mirakjan operators, *Mediterr. J. Math.*, 7 (2010), 297–311. https://doi.org/10.1007/s00009-010-0037-0

[29] Mahmudov, N.I, Approximation properties of complex q-Szász-Mirakjan operators in compact disks, *Computers and Mathematics with Applications*, 60(6) (2010), 1784-1791. https://doi.org/10.1016/j.camwa.2010.07.009

[30] Aral, A., Gupta, V., The q-derivative and applications to q-Szász Mirakyan operators, *Calcolo*, 43(3) (2006), 151-170. https://doi.org/10.1007/s10092-006-0119-3

[31] Cai, Q., Zeng, X.M., Cui, Z., Approximation properties of the modification of Kantorovich type q-Szász operators, *J. Computational Analysis and Applications*, 15(1) (2013), 176-187.

[32] Gal, S., Mahmudov, N.I, Kara, M., Approximation by complex q-Szász-Kantorovich operators in compact disks, $q > 1$, *Complex Anal. Oper. Theory*, 7 (2013), 1853-1867. https://doi.org/10.1007/s11785-012-0257-3

[33] Örkövü, M., Doğru, O., q-Szász Mirakyan Kantorovich type operators preserving some test functions, *Appl. Math. Lett.*, 24(9) (2011), 1588-1593. https://doi.org/10.1016/j.aml.2011.04.001

[34] Mahmudov, N.I., Vijay, G., On certain q-analogue of Szász Kantorovich operators, *J. Appl. Math. Comput.*, 37 (2011), 407-419. https://doi.org/10.1007/s12190-010-0441-4

[35] Tariboon, J., Ntouyas, S.K., Agarwal, P., New concepts of fractional quantum calculusand applications to impulsive fractional q-difference equations, *Advance in Difference Equations*, 18 (2015). https://doi.org/10.1186/s13662-014-0348-8

[36] Kac, V., Cheung, P., Quantum Calculus, Universitext, New York, 2002.

[37] Becker, M., Global approximation theorems for *Szász*-Mirakjan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.*, 27(1) (1978), 127-142.

[38] Lenze, B., On Lipschitz type maximal functions and their smoothness spaces, *Nederl. Akad. Indag. Math.*, 91(1) (1988), 53-63.

[39] Lopez-Moreno, A.J., Weighted simultaneous approximation with Baskakov type operators, *Acta Mathematica Academiae Scientiarum Hungaricae*, 104 (2004), 143-151. https://doi.org/10.1023/B:AMHU.0000034368.81211.23

[40] Volkov, V.I., On the convergence of sequences of linear positive operators in the space of continuous functions of two variables, *Dokl. Akad. Nauk SSSR*, 115(1) (1957), 17-19.

[41] Gupta, V., Agarwal, R.P., Convergence Estimates in Approximation Theory, Springer International Publishing, 2014.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1169–1179 (2022)
DOI:10.31801/cfsuasmas.914887
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: April 12, 2021; Accepted: July 7, 2022

A STUDY ON MODELING OF RAT TUMORS WITH THE DISCRETE-TIME GOMPERTZ MODEL

Levent ÖZBEK

Department of Statistics, Faculty of Sciences, Ankara University, 06100 Ankara, TURKEY

Abstract. Cancer formation is one of the pathologies whose frequency has increased in the recent years. In the literature, the compartment models, which are non-linear, are used for such problems. In nonlinear compartment models, nonlinear state space models and the extended Kalman filter (EKF) are used to estimate the parameter and the state vector. This paper presents a discrete-time Gompertz model (DTGM) for the transfer of optical contrast agent, namely indocyanine green (ICG), in the presence of tumors between the plasma and extracellular extravascular space (EES) compartments. The DTGM, which is proposed for ICG and the estimation of ICG densities used in the vascular invasion of tumor cells of the compartments and in the measurement of migration from the intravascular area to the tissues, is obtained from the experimental data of the study. The ICG values are estimated online (recursive) using the DTGM and the adaptive Kalman filter (AKF) based on the experimental data. By employing the data, the results show that the DTGM in conjunction with the AKF provides a good analysis tool for modeling the ICG in terms of mean square error (MSE), mean absolute percentage error (MAPE) and R^2. When the results obtained from the compartment model used in the reference [9] are compared with the results obtained with the DTGM, the DTGM gives better results in terms of MSE, MAPE and R^2 criteria. The DTGM and the AKF compartment model require less numerical processing when compared to the EKF, which indicates that DTGM is a less complicated model. In the literature, EKF is used for such problems.

1. Introduction

In recent years the use of optical contrast agents and advanced medical imaging techniques to analyze and diagnose tissue abnormalities has become almost a standard procedure [1]. The existence of tumors is one of the main causes of tissue

2020 *Mathematics Subject Classification.* 62H12.
Keywords. Indocyanine green, tumor cell, discrete time Gompertz models, adaptive Kalman filter, estimation.
ozbek@science.ankara.edu.tr; 0000-0003-1018-3114.

abnormalities, and in [2] it is shown that tumor vessel permeability to macromolecular blood solutes correlates with tumor growth as well as vascular growth. ICG is a blood pool agent that binds to globulin proteins (predominantly albumin) in blood [3], and because of its ability to bind to plasma proteins, it behaves as a macromolecular contrast agent with a low or no vascular permeability. Once injected, ICG rapidly and completely binds to albümin. Its macromolecular behavior results in a slow leakage which permits application of a pharmacokinetic model that in return allows for the determination of individual vascular parameters, such as capillary permeability. Compartmental analysis is a method of bio-mathematical modeling which assumes that a biological system can be divided into a series of homogeneous compartments which interact by exchanging material. For compartmental models used in pharmacokinetics, the material concentration varies with time depending on individual pharmacokinetics parameters [4]. If the appropriate parameters are known, the concentration level in a particular compartment can be predicted by applying suitable pharmacokinetic equations. Thus, a robust method of identifying and estimating individual parameters is required. The parameter identification is a common nonlinear estimation problem. In essence, it is the problem of estimating a model parameter that occurs as a coefficient of a dynamic system state variable - either as a dynamic coefficient or as measurement sensitivity. When this estimation problem is solved simultaneously with the state estimation problem (via state vector augmentation), the linear model becomes nonlinear. The extended Kalman filter (EKF) is one of the most popular and intensively investigated estimation technique for the nonlinear state estimation. It consists of applying the standard Kalman filter equations to the first-order approximation of the nonlinear model of the last estimate [5]. This study addresses the most commonly used growth models, the DTGM to estimate the ICG level without resorting to nonlinear models. The growth curves are used for modelling the increase in the number of plants, bacteria or viruses in an environment. The rest of this article is organized as follows. In Section 2, information about The ICG Compartment Model is presented. In Section 3, the mathematical and computational methodologies of DTGM are specified and the mathematical equations, that are aimed to be used further in the study are given, and the modeling analysis and estimation results are also presented. Finally, Section 4 concludes the study.

2. The Icg Compartment Model

If there is a tumor in any tissue, the given ICG passes through the vessel into the tumor tissue area. There is also a return to the vein from the tumor tissue. In accordance with this physiological structure, a two-compartment model can be considered. In this compartment model, C_p indicates ICG concentration in the vessel, C_e indicates ICG concentration in tumor tissue. k_1 ratio is the ratio of ICG passing from the vessel to the tumor tissue, k_2 is the ratio of ICG passing from the tumor area into the vessel, and k_3 is the ratio of ICG passing from the plasma to the liver and kidney. Since the mentioned ratio is quite small, this ratio is ignored while creating the mathematical model. The ICG density in the tumor tissue tend to increase as the k_1 ratio increase of the ICG density in the vessel (since there are transitions from here) and tend to decrease as the k_2 ratio of its own density. Accordingly, the change in ICG density in the tumor tissue per unit time is expressed as in Equation 1,

$$\frac{dC_e(t)}{dt} = k_1 C_p(t) - k_2 C_\theta(t) \tag{1}$$

As mentioned above, its rate can be ignored, and the change in ICG density in the tumor tissue per unit time is defined as in Equation 2,

$$\frac{dC_p(t)}{dt} = -k_1 C_p(t) + k_2 C_e(t) \tag{2}$$

Because the ratio of ICG concentration, which is only in the vessel, is expected to be transferred to the tumor tissue per unit time. k_1 and k_2 show the permeability parameters mentioned before. According to the model, there is no information about the permeability parameters and there is no need for their estimates. When the differential equation system given by equations 1-2 are made discrete, nonlinear discrete time-state space model is obtained. In this model, both the parameter and the state vector are required to be estimated simultaneously. In the literature, the EKF is used for such problems [6]- [11].

3. Discrete-Time Gompertz Model

In this study, DTGM, one of the growth models, is used to estimate the ICG level without considering the nonlinear models.

The growth curves are used for modelling the increase in the number of plants, bacteria or viruses in an environment. Expressing the growth of an organism or an increase in the number of viruses temporally is called ”growth”. The identification of the complex growth process is aimed at using the growth curves [12]- [14]. DTGM is well known and widely used model in many sub-fields of biology [15]- [18]. Numerous parametrizations and re-parametrizations of the DTGM can be found in the literature [17]. DTGM was originally recommended to explain human mortality curves Gompertz [12], and it has been further used in the description of growth processes, for example, growing of bacterial colonies [15] and tumors [16]. The model, a stochastic version of the DTGM, can be transformed into a linear Gaussian state-space model for the convenient fitting to time-series data. In this study, ICG values are estimated online using the DTGM and the AKF based on the experimental data. By employing the data, the results show that the DTGM in conjunction with AKF provides a good analysis tool for modeling the ICG in terms of mean square error (MSE), mean absolute percentage error (MAPE), and R^2 . When the results obtained from the compartment model used in the reference [9] are compared with the results obtained with the DTGM, the DTGM gives better results in terms of MSE, MAPE and R^2 criteria. The DTGM and the AKF compartment model require less numerical processing when compared to the EKF, which indicates that DTGM is a less complicated model.

Let n_t denote ICG level at time t. The process model is as:

$$n_t = n_{t-1}exp(a + blnn_{t-1} + e_t) \tag{3}$$

where a and b are constants, and e_t is a random variable distributed as $e_t\ N(0, \sigma_1^2)$. The random variables $e_1, \ldots, e_n$ are assumed to be uncorrelated. On the logarithmic scale, the DTGM is a linear autoregressive time-series model of order 1 [AR(1) process] defined as equation 4.

$$y_t = y_{t-1} + a + by_{t-1} + e_t = a + cy_{t-1} + e_t \tag{4}$$

where, $y_t = \ln n_t$ and $c = b + 1$. For statistical properties of DTGM, see [18].

The model has a long history in density-dependence modeling see [19]- [21]. A freguently seen alternative is a stochastic version of the Moran-Ricker model [21], which uses n_{t-1} instead of $\ln n_{t-1}$ in the exponential function; in comparative data analysis studies, the Gompertz model has performed as well as the Moran-Ricker [22]. The probability distribution of n_{t-1} is a normal distribution with mean and variance that change as functions of time. If $-1 < c < 1$, the probability distribution of n_t eventually approaches a time-independent stationary distribution that is a normal distribution with a mean of $a/(1 - c)$ and a variance of $\sigma_1^2/\left(1 - c^2\right)$. The stationary distribution is the stochastic version of an equilibrium in the deterministic model, and is an important statistical manifestation of density dependence in the population growth model Dennis [18]. In equation 4, a is the intrinsic growth rate, b is the density-dependent influence [18].

3.1. **Mathematical and Computational Methodologies.** The optimum linear filtering and estimation methods introduced by Kalman [31] have been considered as one of the greatest achievements in estimation theory. Discrete-time linear state-space models and Kalman filtering (KF) have been employed since the 1960s, mostly in the control and signal processing areas. The KF has been extensively employed in many areas of estimation. The extensions and applications of discrete-time linear state-space models can be found in almost all disciplines [20]- [28]. In this study, KF has been used to estimate the time-varying parameter of the DTGM. KF is a recursive estimator to estimate the time-varying parameters. If $a = 0$ in Eq.(4), n_t takes the case counts observed until t and $y_t = lnn_t$. Then the equation

$$y_t = cy_{t-1} + e_t \tag{5}$$

is acquired. In the case that the parameter c in Eq.(5) is time-varying and presumed as random walk process, that is . Then state-space model,

$$y_t = c_t y_{t-1} + e_t \tag{6}$$

$$c_t = c_{t-1} + w_t \tag{7}$$

is obtained and w_t is distributed as $w_t\ N(0, \sigma_2^2)$. The random variables $w_1, w_2, \ldots, w_n$ are assumed to be uncorrelated. Here, the state variable is unobservable, time-varying, and can be estimated through AKF (explanation regarding AKF is given in the Appendix section). If this time-varying parameter is estimated using on-line AKF, the ICG level in times $t+1$,$t+2$, ... can be estimated via this online-estimated parameter. When the models given in equations (6) and (7) are compared with the state space model given in the Appendix, the following equations are obtained.

$$x_t = c_t, F_t = 1, G_t = 1, H_t = y_{t-1}, R_t = \sigma_1, Q_t = \sigma_2 \tag{8}$$

3.2. **Application of DTGM.** Details of the experimental setup, and how the data were collected can be found in [23]. Data is given in Table 1. Since this study deals with the collected data, here only a very brief discussion regarding the experiments is given in order to put more emphasis on the mathematical representation, along with parameter estimation. In the experiments, bolus injections of the optical contrast agent ICG were administered to the rat through the tail vein. The measurements were collected by placing the probe normal to the tumor surface and probing the whole tissue including plasma. After injection, ICG rapidly and completely binds to albumin, after which the kinetics of ICG are governed by the temporal dynamics of albumin in and between the vascular compartment and the EES.

3.3. **Estimation (AKF) Algorithm.** The steps of the AKF algorithm using to estimate the parameter in DTGM are as follows. The code is written in Matlab program for the estimation algorithm.

Step 1. Initial values $\hat{c}_0 = 0.9, P_0 = 1, R_t = \sigma_1 = \mathrm{std}\,(y_t)\,, t = 1, 2, \ldots, n, Q_t = \sigma_2 = 0.01, t = 1, 2, \ldots, n, \alpha = 1.0001$

Step 2. $\hat{c}_{t|t-1} = \hat{c}_{t-1}$ Predicted (a priori) state estimate

Step 3. $P_{t|-1} = \alpha\,(P_{t-1t-1} + \sigma_2)$ Predicted (a priori) estimate covariance

Step 4. $K_t = P_{t|t-1} y_{t-1} \left(y_{t-1} P_{t|t-1} y_{t-1} + \sigma_1\right)^{-1}$ Optimal Kalman gain

Step 5. $P_{t=} = [I - K_t y_{t-1}]\, P_{\text{tt-1}}$ Updated (a posteriori) estimate covariance

Step 6. $\hat{c}_t = \hat{c}_{t|t-1} + K_t \left(y_t - y_{t-1}\hat{c}_{t|t-1}\right)$ Updated (a posteriori) state estimate

In the experiment, the ICG concentration in the lump space, i.e. EES and plasma, was monitored for 500 seconds. According to the estimation results obtained by using the ICG level in DTGM, the MSE, MAPE, R^2 and values are calculated (see Table 2). These calculated values indicate that the compatibility of the model with real data is quite high. This tells us estimating the ICG level via DTGM is a reliable method. Since estimation using the AR(1) stochastic process does not require any other model assumption. As for AKF, utilizing only the observation in time and the preceding estimation is the most advantageous aspect of this method.

TABLE 1. Collected data.

Time	Observed value	Time	Observed value	Time	Observed value	Time	Observed value	Time	Observed value	Time	Observed value
0	-0.0237	45	0.623	90	0.749	135	0.682	180	0.603	225	0.532
1	-0.0269	46	0.632	91	0.752	136	0.677	181	0.596	226	0.533
2	-0.0282	47	0.645	92	0.753	137	0.678	182	0.594	227	0.534
3	-0.0251	48	0.65	93	0.751	138	0.683	183	0.594	228	0.533
4	-0.0249	49	0.66	94	0.747	139	0.684	184	0.591	229	0.527
5	-0.0224	50	0.665	95	0.739	140	0.685	185	0.592	230	0.531
6	-0.0236	51	0.671	96	0.741	141	0.688	186	0.586	231	0.528
7	-0.0243	52	0.682	97	0.743	142	0.686	187	0.587	232	0.527
8	-0.0232	53	0.685	98	0.748	143	0.688	188	0.591	233	0.525
9	0.0265	54	0.688	99	0.747	144	0.685	189	0.594	234	0.521
10	0.139	55	0.688	100	0.742	145	0.671	190	0.587	235	0.523
11	0.216	56	0.691	101	0.739	146	0.661	191	0.579	236	0.523
12	0.255	57	0.696	102	0.741	147	0.657	192	0.576	237	0.522
13	0.275	58	0.704	103	0.738	148	0.66	193	0.579	238	0.514
14	0.282	59	0.711	104	0.738	149	0.662	194	0.576	239	0.513
15	0.286	60	0.716	105	0.738	150	0.658	195	0.575	240	0.516
16	0.295	61	0.717	106	0.735	151	0.653	196	0.574	241	0.513
17	0.305	62	0.729	107	0.739	152	0.652	197	0.574	242	0.508
18	0.316	63	0.737	108	0.739	153	0.654	198	0.573	243	0.503
19	0.324	64	0.739	109	0.731	154	0.653	199	0.572	244	0.504
20	0.337	65	0.729	110	0.734	155	0.653	200	0.57	245	0.502
21	0.352	66	0.725	111	0.734	156	0.646	201	0.568	246	0.504
22	0.365	67	0.726	112	0.737	157	0.648	202	0.567	247	0.503
23	0.381	68	0.732	113	0.733	158	0.654	203	0.569	248	0.503
24	0.399	69	0.728	114	0.734	159	0.652	204	0.566	249	0.497
25	0.413	70	0.722	115	0.736	160	0.645	205	0.566	250	0.5
26	0.426	71	0.725	116	0.727	161	0.639	206	0.566	251	0.5
27	0.439	72	0.732	117	0.724	162	0.635	207	0.565	252	0.501
28	0.45	73	0.731	118	0.723	163	0.631	208	0.568	253	0.5
29	0.466	74	0.731	119	0.719	164	0.629	209	0.564	254	0.504
30	0.477	75	0.732	120	0.718	165	0.623	210	0.557	255	0.501
31	0.49	76	0.736	121	0.711	166	0.62	211	0.553	256	0.499
32	0.503	77	0.74	122	0.699	167	0.62	212	0.551	257	0.491
33	0.513	78	0.738	123	0.702	168	0.623	213	0.554	258	0.489
34	0.524	79	0.743	124	0.699	169	0.623	214	0.546	259	0.486
35	0.537	80	0.743	125	0.7	170	0.622	215	0.543	260	0.49
36	0.553	81	0.751	126	0.699	171	0.621	216	0.54	261	0.486
37	0.569	82	0.755	127	0.697	172	0.62	217	0.541	262	0.482
38	0.579	83	0.756	128	0.696	173	0.623	218	0.538	263	0.478
39	0.579	84	0.754	129	0.702	174	0.626	219	0.538	264	0.477
40	0.58	85	0.747	130	0.697	175	0.624	220	0.537	265	0.477
41	0.589	86	0.747	131	0.691	176	0.617	221	0.535	266	0.476
42	0.597	87	0.749	132	0.684	177	0.608	222	0.537	267	0.477
43	0.605	88	0.749	133	0.681	178	0.609	223	0.532	268	0.479
44	0.613	89	0.745	134	0.679	179	0.607	224	0.531	269	0.476
270	0.48	315	0.441	360	0.419	405	0.396	450	0.382	495	0.374
271	0.478	316	0.438	361	0.417	406	0.395	451	0.378	496	0.377
272	0.477	317	0.437	362	0.414	407	0.397	452	0.381	497	0.375
273	0.474	318	0.437	363	0.409	408	0.396	453	0.383	498	0.374
274	0.471	319	0.438	364	0.409	409	0.395	454	0.381	499	0.376
275	0.474	320	0.441	365	0.409	410	0.396	455	0.381	500	0.372
276	0.473	321	0.437	366	0.409	411	0.397	456	0.383	501	0.371
277	0.471	322	0.434	367	0.407	412	0.396	457	0.382	502	0.369
278	0.467	323	0.433	368	0.405	413	0.391	458	0.383	503	0.368
279	0.468	324	0.434	369	0.408	414	0.391	459	0.38	504	0.37
280	0.467	325	0.432	370	0.407	415	0.389	460	0.382		
281	0.464	326	0.433	371	0.406	416	0.391	461	0.378		
282	0.468	327	0.43	372	0.408	417	0.391	462	0.376		
283	0.462	328	0.43	373	0.409	418	0.391	463	0.38		
284	0.465	329	0.429	374	0.411	419	0.393	464	0.379		
285	0.465	330	0.432	375	0.405	420	0.394	465	0.379		
286	0.463	331	0.433	376	0.407	421	0.392	466	0.377		
287	0.462	332	0.434	377	0.409	422	0.393	467	0.376		
288	0.46	333	0.429	378	0.406	423	0.396	468	0.376		
289	0.462	334	0.427	379	0.407	424	0.393	469	0.378		
290	0.465	335	0.423	380	0.408	425	0.395	470	0.377		
291	0.461	336	0.423	381	0.407	426	0.391	471	0.379		
292	0.454	337	0.422	382	0.41	427	0.392	472	0.383		
293	0.452	338	0.424	383	0.406	428	0.389	473	0.38		
294	0.45	339	0.421	384	0.4	429	0.391	474	0.38		
295	0.452	340	0.423	385	0.396	430	0.387	475	0.378		
296	0.448	341	0.421	386	0.398	431	0.39	476	0.378		
297	0.45	342	0.418	387	0.399	432	0.391	477	0.38		
298	0.447	343	0.418	388	0.395	433	0.388	478	0.378		
299	0.446	344	0.419	389	0.393	434	0.384	479	0.377		
300	0.448	345	0.417	390	0.394	435	0.387	480	0.372		
301	0.442	346	0.419	391	0.393	436	0.385	481	0.373		
302	0.448	347	0.415	392	0.392	437	0.385	482	0.373		
303	0.447	348	0.419	393	0.397	438	0.386	483	0.372		
304	0.447	349	0.42	394	0.395	439	0.387	484	0.375		
305	0.446	350	0.419	395	0.398	440	0.388	485	0.374		
306	0.449	351	0.417	396	0.397	441	0.39	486	0.375		
307	0.447	352	0.416	397	0.395	442	0.389	487	0.373		
308	0.443	353	0.418	398	0.395	443	0.388	488	0.374		
309	0.441	354	0.416	399	0.397	444	0.387	489	0.375		
310	0.441	355	0.415	400	0.395	445	0.386	490	0.373		
311	0.437	356	0.417	401	0.397	446	0.384	491	0.374		
312	0.44	357	0.418	402	0.395	447	0.382	492	0.374		
313	0.439	358	0.417	403	0.394	448	0.382	493	0.375		
314	0.438	359	0.418	404	0.395	449	0.384	494	0.376		

TABLE 2. Calculated R^2, MSE, MAPE.

Model	MSE	R^2	MAPE
DTGM	0.0001	0.9973	5.7385
Compartment Models	0.0004	0.9826	19.2059

Figure 1 depicts the observed ICG concentration and the model fit obtained through the use of DTGM. Figure 2 depicts the observed ICG concentration and the model fit obtained through the use of compartment models [9]. Figure 3 depicts the observed ICG concentration and the model fit obtained through the use of compartment models and DTGM. It is clearly seen that the DTGM mathematical model provides a rather good fit to the observations, which indicates the correctness of the model.

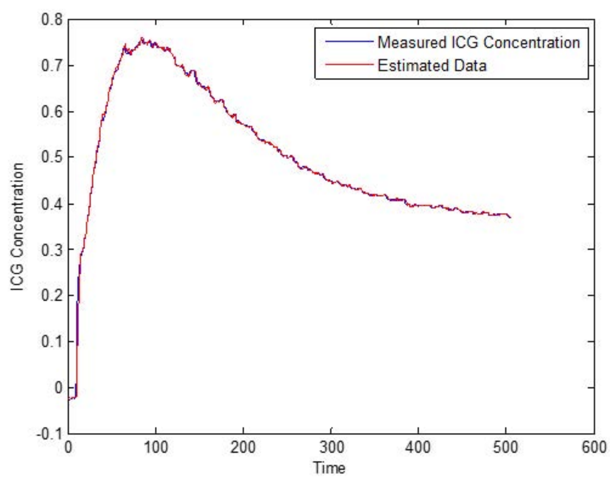

FIGURE 1. DTGM: Observed ICG concentration and the model fit.

4. CONCLUSION

In this study, we introduced a DTGM representing the metabolic elimination and transfer of ICG between compartments in rat tumors, and presented a method for the quantitative analysis of experimentally obtained ICG concentration data. This will be useful in the analysis of tumor cell behavior patterns in cancerous tissues. In this study, ICG concentration data have been estimated online using DTGM and AKF. The ICG concentration data is modeled with DTGM, and the time-varying parameters of the obtained AR(1) stochastic time series are estimated

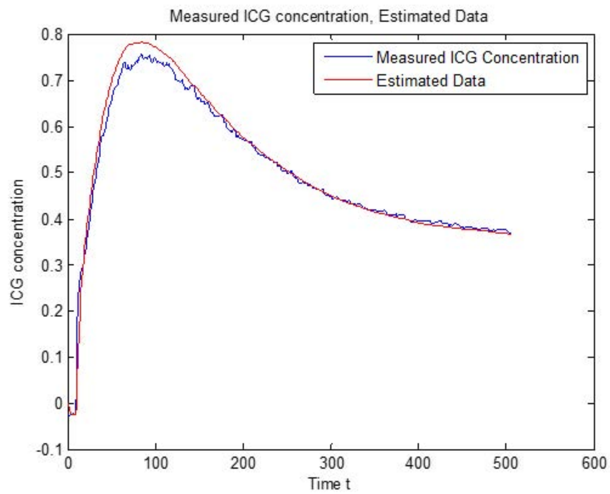

FIGURE 2. Compartment Models: Observed ICG concentration and the model fit.

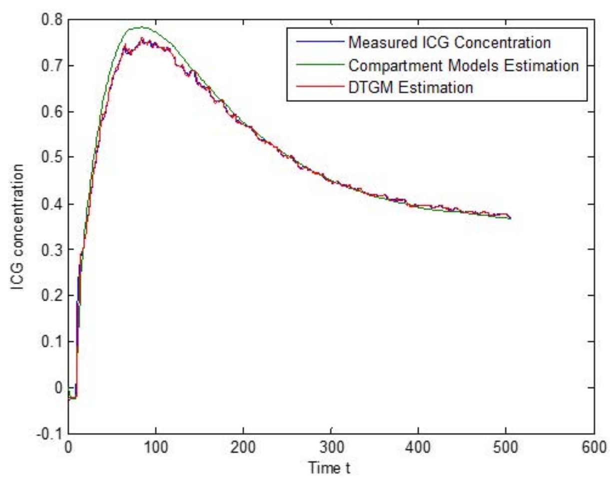

FIGURE 3. Compartment Models and DTGM: Observed ICG concentration and the model fit.

by the on-line AKF. The estimation by the acquired data shows that employing the DTGM model and the AKF in terms of MSE, MAPE, and R^2 provide efficient analysis for modeling the ICG concentration data. It is proposed that using the DTGM and the AKF will be appropriate. It is quite a simple method to model the ICG concentration time series data with the time-varying parameter AR(1) stochastic process and to estimate the time-varying parameter with the online AKF. When the results obtained from the compartment model used in the reference [9] are compared with the results obtained with the DTGM, the DTGM offers better results according to MSE, MAPE and R^2 criteria. The DTGM and the AKF compartment model require less numerical processing compared to the EKF, and DTGM is a simpler model. In the literature, the EKF is used for such problems. As far as we know no other method has been used before.

Appendix

State-Space Model and Adaptive Kalman Filter (AKF)

Let us consider a general discrete-time stochastic system represented by the state and measurement models given as:

$$x_{t+1} = F_t x_t + G_t w_t$$
$$y_t = H_t x_t + v_t$$

where x_t is an $n \times 1$ system vector, y_t is an $m \times 1$ observation vector, F_t is an $n \times n$ system matrix, H_t is an $m \times n$ matrix, w_t an $n \times 1$ vector of zero mean white noise sequence and v_t is an $m \times 1$ measurement error vector assumed to be a zero mean white sequence uncorrelated with the w_t sequence. The covariance matrices w_t and w_t are defined by $w_t \sim N(0, Q_t), v_t \sim N(0, R_t)$. The filtering problem is the problem of determining the best estimate of its x_t condition, given its observations $Y_t = (y_0, y_1, \ldots, y_t)$ [14 − 20]. When $Y_t = (y_0, y_1, \ldots, y_t)$ observations are given, the estimation of state x_t with

$$\hat{x}_t = E(x_t \mid y_0, y_1, \ldots, y_t) = E(x_t \mid Y_t)$$

and the covariance matrix of the error with

$$P_{t|t} = E\left[\left(x_t - \hat{x}_{t|}\right)\left(x_t - \hat{x}_{tt}\ \right)' \mid Y_t\right]$$

when $Y_{t-1} = (y_0, y_1, \ldots, y_{t-1})$ observations are given, the estimation of state x_t with $\hat{x}_{t|t-1} = E(x_t \mid y_0, y_1, \ldots, y_{t-1}) = E(x_t \mid Y_{t-1})$

and the covariance matrix of the error are shown with

$$P_{t|t-1} = E\left[\left(x_t - \hat{x}_{t|-1}\right)\left(x_t - \hat{x}_{t|t-1}\right)' \mid Y_{t-1}\right].$$

Let the initial state be assumed to have a normal distribution in the form of $x_0 \sim \mathrm{N}(\bar{x}_0, \mathrm{P}_0)$.

The optimum update equations for KF are,

$$\hat{x}_{t|-1} = F_{t-1}\hat{x}_{t-1}$$

$$P_{t|t-1} = F_{t-1}P_{t-4t-1}F'_{t-1} + G_{t-1}Q_{t-1}G'_{t-1}$$

$$K_t = P_{t|t-1} H_t' (H_t P_{t|t-1} H_t' + R_t)^{-1}$$

$$P_{t|t} = [I - K_t H_t] P_{t|t-1}$$
$$\hat{x}_t = \hat{x}_{t|t-1} + K_t \left(y_t - H_t \hat{x}_{t|t-1} \right)$$

In the above equations, $\hat{X}_{t|t-1}$ is the a priori estimation and $\hat{X}_t$ is the a posteriori estimation of x_t. Also, $P_{t|t-1}$ and $P_{t|t}$ are the covariance of a priori and a posteriori estimations respectively [24]- [33]. In some cases, divergence problems may ocur in the KF due to the incorrect installation of the model. In order to eliminate divergence in the KF, adaptive methods are used [5], [32], [33]. One of these is the use of the forgetting factor. A forgetting factor is proposed by [32].

$$P_{t|t-1} = \alpha \left(F_{t-1} P_{t-1|-1} F_{t-1}' + G_{t-1} Q_{t-1} G_{t-1}' \right)$$

Declaration of Competing Interests The author declare that there is no conflict of interest regarding the publication of this article.

References

[1] Tofts, P.S., Modeling tracer kinetics in dynamic Gd-DTPA MR imaging, *J. Magn. Reson. Imag.,* 7 (1997), 91-101. doi: 10.1002/jmri.1880070113

[2] Su, M.Y., Jao, J.C., Nalcioglu, O., Measurement of vascular volume fraction and blood-tissue permeability constants with a pharacokinetic model: studies in rat muscle tumors with dynamic Gd-DTPA enhanced MRI, *Magn. Reson. Med.,* 32 (1994), 714-724. doi: 10.1002/mrm.1910320606

[3] Ntziachristos, V., Yodh, A.G., Schnall, M., Chance, B., Concurrent MRI and diffuse optical tomography of breast after indocyanine green enhancement, *Proc. Natl. Acad. Sci. USA,* 97 (2000), 2767-2772. doi/10.1073/pnas.040570597

[4] Botsman, K., Tickle, K., Smith, J.D., A Bayesian formulation of the Kalman filter applied to the estimation of individual pharmacokinetic parameters, *Comput. Biomed. Res.,* 30 (1997), 83-93. doi/10.1006/cbmr.1997.1440

[5] Özbek, L., Efe, M., An adaptive extended Kalman filter with application to compartment models, *Communications In Statistics-Simulation and Computation,* 33(1) (2004), 145-158. doi/10.1081/SAC-120028438

[6] Alacam, B., Yazici, B., Chance, B., Extended Kalman filtering for the modeling and analysis of ICG pharmacokinetics in cancerous tumors using NIR optical methods, *IEEE Transactions on Biomedical Engineering,* 53(10) (2006), 1861-1871. doi:10.1109/TBME.2006.881796

[7] Alacam, B., Yazici, B., Intes, X., Nioka, S., Chance, B., Pharmacokinetic-rate images of indocyanine green for breast tumors using near-infrared optical methods, *Phys. Med. Biol.,* 53 (2008), 837-859. doi: 10.1088/0031-9155/53/4/002

[8] Alacam, B., Yazici, B., Direct reconstruction of pharmacokinetic-rate images of optical fluorophores from NIR measurements, *IEEE Transactions on Medical Imaging*, 28(9) (2009), 1337-1353. doi: 10.1109/TMI.2009.2015294

[9] Özbek, L., Efe, M., Babacan, E.K., Yazihan, N., Online estimation of capillary permeability and contrast agent concentration in rat tumors, *Hacettepe Journal of Mathematics and Statistics*, 39(2) (2010), 283-293.

[10] Gottam, O., Naik, N., Gambhirc, S., Parameterized level-set based pharmacokinetic fluorescence optical tomography using the regularized Gauss-Newton filter, *Journal of Biomedical Optics,* 24(3) (2019), 1-17. doi/10.1117/1.JBO.24.3.031010

[11] Gottam, O., Naik, N., Gambhirc, S., Pandey, P.K., RBF level-set based fully-nonlinear fluorescence photoacoustic pharmacokinetic tomography, *Inverse Problems in Science and Engineering,* doi/10.1080/17415977.2021.1982934

[12] Gompertz, B., On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies, *Philosophical Transactions of the Royal Society of London*, 115 (1825), 513-583.

[13] Bertalanffy, L., Problems of organic growth, *Nature,* 163 (1949), 156-158.

[14] Richards, F.A., Flexible growth function for empirical use, *Journal of Experimental Botany*, 10 (1959), 280-300.

[15] Zwietering, M.H., Jongenburger, I., Rombouts, F.M., Van't Riet, K., Modeling of the bacterial growth curve, *Appl Environ Microbiol*, 56(6) (1990), 1875-1881. doi: 10.1128/aem.56.6.1875-1881.1990

[16] Gerlee, P., The model muddle: in search of tumor growth laws, *Cancer research*, 73(8) (2013), 2407-2411. doi/10.1158/0008-5472.CAN-12-4355

[17] Tjorve, K.M.C., Tjorve, E., The use of Gompertz models in growth analyses, and new Gompertz-model approach: An addition to the Unified-Richards family, *PLoS One,* 12(6) (2017), e0178691. doi/10.1371/journal.pone.0178691

[18] Dennis, B., Ponciano, J.M., Subhash, R., Traper, L.M.L., Staples, D.F., Estimating density dependence, process noise and observation erros, *Ecological Monographs*, 76(3) (2006), 323-341. doi/10.1890/0012-9615.

[19] Reddingius, J., Gambling for existence: A discussion of some theoretical problems in animal population ecology, *Acta Biotheoretica,* 20 (1971), 1-208.

[20] Pollard, E., Lakhani, K.H., Rothery, P., The detection of density-dependence from a series of annual censuses, *Ecology*, 68 (1987), 2046-2055. doi: 10.2307/1939895

[21] Dennis, B., Taper, M.L., Density dependence in time series observations of natural populations: estimation and testing, *Ecological Monographs*, 64 (1994), 205-224. doi/10.2307/2937041

[22] Rotella, J.J., Ratti, J.T., Reese, K.P., Taper, M.L., Dennis, B., Long-term population analysis of Gray Partridge in eastern Washington, *Journal of Wildlife Management*, 60 (1996), 817-825. doi/10.2307/3802382

[23] Cuccia, D.J., Bevilacqua, F., Durkin, A.J., Merritt, S., Tromberg, B.J., Gulsen, G., Yu, H., Wang, J., Nalcioglu, O., In vivo quantification of optical contrast agent dynamics in rat tumors by use of diffuse optical spectroscopy with magnetic resonance imaging coregistration, *Appl. Opt.*, 42 (2003), 2940-2950. doi/10.1364/AO.42.002940

[24] Jazwinski, A.H., Stochastic Processes and Filtering Theory, Academic Press, 1970.

[25] Anderson, B.D.O., Moore, J.B., Optimal Filtering, Prentice Hall, 1979.

[26] Chui, C.K., Chen, G., Kalman Filtering with Real-time Applications, Springer Verlag, 1991.

[27] Ljung, L., Söderström T., Theory and Practice of Recursive Identification, The MIT Press, 1993.

[28] Chen, G., Approximate Kalman Filtering, World Scientific, 1993.

[29] Grewal, S.M., Andrews, A.P., Kalman Filtering: Theory and Practice, Prentice Hall, 1993.

[30] Özbek, L., Kalman Filtresi, Akademisyen Kitabevi, 2017.

[31] Kalman, R.E., A new approach to linear filtering and prediction problems, *Journal of Basic Engineering,* 82 (1960), 35-45. http://dx.doi.org/10.1115/1.3662552

[32] Özbek, L., Aliev, F.A., Comments on adaptive Fading Kalman filter with an application, *Automatica*, 34(12) (1998), 1663-1664.

[33] Efe, M., Özbek, L., Fading Kalman filter for manoeuvring target tracking, *Journal of the Turkish Statistical Assocation,* 2(3) (1999), 193-206.

http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1180–1203 (2022)
DOI:10.31801/cfsuasmas.1049893
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: December 28, 2021; Accepted:June 15, 2022

STUDY AND SUPPRESSION OF SINGULARITIES IN WAVE-TYPE EVOLUTION EQUATIONS ON NON-CONVEX DOMAINS WITH CRACKS

Cheikh SECK

Laboratoire d'Analyse Numérique et d'Informatique, BP 234, Université Gaston Berger de Saint-Louis, SÉNÉGAL
Département de Mathématiques de la FASTEF ex ENS de l'Université Cheikh Anta Diop, Dakar, SÉNÉGAL

ABSTRACT. One of the objectives of this paper is to establish the exact controllability for wave-type evolution equations on non-convex and/or cracked domains with non-concurrent support crack lines. Admittedly, we know that according to the work of Grisvard P., in domains with corners or cracks, the formulas of integrations by parts are subject to geometric conditions: the lines of cracks or their supports must be concurrent. In this paper, we have established the exact controllability for the wave equation in a domain with cracks without these additional geometric conditions.

1. Introduction

The presence of a crack in equipment (especially under pressure) requires, for obvious safety reasons, to know precisely its degree of harmfulness. When this crack propagates, under cyclic loading, it is important to evaluate and to quickly control the evolution of this degree of harmfulness and more concretely the residual life of the cracked structure.
In the works of the pioneers and precursors, not least Kondratiev [1], Grisvard [2], Moussaoui [3] and Niane [4], the control and removal of singularities were established in domains with corners or cracks.
Indeed, when these cracks propagate, under cyclic loading, it is important to evaluate and to quickly control the evolution of this degree of harmfulness and more concretely the residual life of the cracked structure. Thin plates and shells are widely used in aeronautics. Due to the significant stresses to which the structure

2020 *Mathematics Subject Classification.* 13P25, 93B03, 35B45, 93B05, 93C05.
Keywords. Control, controllability, norms estimations, singularities, cracks.
cheikh5.seck@ucad.edu.sn; 0000-0001-8185-5956.

of an aircraft is subjected in flight, for example, the appearance of small cracks is inevitable. Depending on the situation, these cracks are more or less dangerous; thus, certain cracks do not propagate, on the other hand, others present a certain risk.The risks alluded to earlier must, consequently, be curbed. So, once a crack has been detected, it is important to know if it can be dangerous or not? The safety of persons and that of the goods involved means repairing the work first and foremost. Notwithstanding, repairing all the cracks won't be necessary as, if the crack is not dangerous, it is no good repairing as it will be costly.

Accordingly, it is important to figure out whether or not the crack is dangerous, and whether it can be spread. Apart from extreme cases (very small or very large cracks), this diagnosis is not easy to make because even a small crack can spread brutally. It is very clear that the accuracy of this diagnosis is very important.

More recently, Seck [5], Bayili [6], taking inspiration from the exact controllability in Lipschitzian domains by Costabel [7], Niane [8] and Lions [9, 10], established results of exact controllability of the wave equation in non-regular Sobolev spaces. But, in all these works, the domains admit a crack or a corner or even cracks with condition of control: the lines of cracks are concurrent (or the supports of the lines of cracks are concurrent).

In this paper, without making additional assumptions and conditions on the crack lines and their supports, an exact controllability result was established for wave equation.

2. Reminders of Fundamental Results

2.1. **Problem position.** We denote by Ω an open polygonal uncracked, non convex and bounded of $\mathbb{R}^2$ and for $T > 0$, we denote by $Q_T = \Omega \times]0, T[$.

Let Γ the boundary of Ω, $\nu(x)$ the external unit normal at all points x (apart from the vertices) of Γ and Σ_T the lateral border of the cylinder Q_T.

Γ is the union of a finite number of closed line segments; the corresponding open segments are denoted Γ_j, $0 \leq j \leq N$ and S_{ij} the end common to Γ_j and Γ_i if it exists. We denote by ω_{ij} the measure of the angle made by Γ_j and Γ_i in S_{ij} towards the interior of Ω.

We denote by ν_j the unit normal vector outside Γ_j and τ_j the unit vector tangent to Γ_j and directed towards the vertex S_i. For x_0 any point of $\mathbb{R}^2$, we consider the function $m(x) = x - x_0$ and a partition of the border as follows:

$$\Gamma_0 = \{x \in \Gamma; m(x) \cdot \nu \geq 0\}, \quad \Gamma_0^* = \{x \in \Gamma; m(x) \cdot \nu < 0\},$$

and

$$\Sigma_0^* = \Gamma_0^* \times]0, T[.$$

Let $||.||$ be the Euclidean norm in $\mathbb{R}^2$ and introduce the following constants

$$R_0 = R(x_0) = \max_{x \in \overline{\Omega}} ||x - x_0||, \text{ and } T_0 = 2R(x_0).$$

Let also $f \in L^2(\Omega)$ and $y \in H_0^1(\Omega)$ be the unique solution of the homogeneous Dirichlet problem

$$(P1)\ \begin{cases} -\triangle y = f, \\ y_{|\Gamma} = 0. \end{cases} \tag{1}$$

In the space $H = L^2(\Omega)$, we consider A the operator defined by:

$$D(A) = \left\{ y \in H_0^1(\Omega); -\triangle y \in L^2(\Omega) \right\},$$
$$\forall A \in D(A), Ay = -\triangle y.$$

A: is a compact positive inverse self-adjoint operator see Brezis [11] and Hormander [12].
y is solution of (1)$(P1) \Longrightarrow y \in D(A)$.
Let $m+1$ be the number of non-convex angles of the $\partial\Omega$ boundary of the domain Ω having $m+1$ vertices $(S_i)_{0 \leq i \leq m}$.
It has been proved in Niane [4] that if $\bar{\omega}$ is an arbitrarily small part of Ω not meeting any vertex of cracks, there exist regular functions $(g_i)_{1 \leq i \leq m}$ with compact support in $\bar{\omega}$ such that for all $f \in L^2(\Omega)$, if $(\lambda_i)_{1 \leq i \leq m}$ are the coefficients of singularities of the problem $(P1)$ then the problem

$$(P2)\ \begin{cases} -\triangle \tilde{y} = f + u, \\ \tilde{y}_{|\Gamma} = 0. \end{cases} \tag{2}$$

admits a solution $\tilde{y} \in H^2(\Omega)$, with $u = -\sum_{i=0}^{m} \lambda_i h_i, \quad \lambda_i = \int_\Omega f w_i dx$ where w_i the singular functions Cf. Grisvard [2] and $< g_i, w_j >= \delta_{ij}$ Moussaoui [3] and Niane [4].
Let for $i \in \{0, ..., m\}, (r_i, \theta_i)$ represent the polar coordinates of a point M of Ω relatively to the vertex S_i with $r_i = || \overrightarrow{S_i M} ||$ Gilbert [6].

Remark 1. *The singular functions w_i are harmonic*

$$\begin{cases} -\triangle w_i = 0 \ sur \ \Omega, \\ \omega_i = 0 \qquad \partial\Omega \setminus \{x_i\}. \end{cases}$$

2.2. **Internal control of the homogeneous waves equation on a non-convex domain.** Let y: solution of the following homogeneous wave equation

$$(EOH): \begin{cases} y'' - \Delta y = 0 \qquad in\ Q_T, \\ \\ y = 0, \qquad in\ \Sigma_T, \\ \\ y(0) = y_0 \ \ y'(0) = y_1 \qquad in\ \Omega. \end{cases} \tag{3}$$

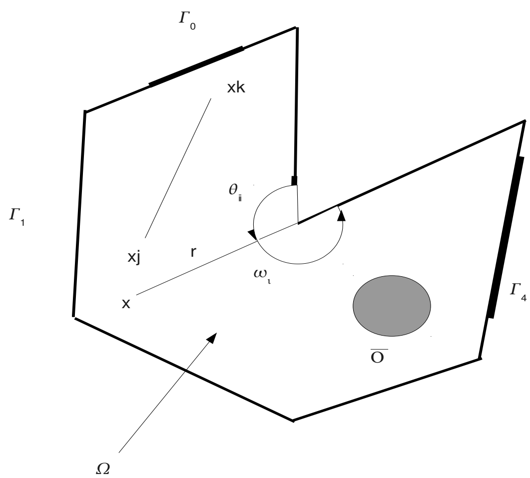

FIGURE 1. Non-convex cracked domain

$$(EOH) \Longleftrightarrow (EOH)' \begin{cases} -\Delta y = -y'' & in\ Q_T, \\ y = 0, & in\ \Sigma_T, \\ y(0) = y_0\ \ y'(0) = y_1 & in\ \Omega. \end{cases} \tag{4}$$

Let $(y_o, y_1) \in D(A) \times H_0^1(\Omega) \Longrightarrow$ the solution y of the equation (EOH) (3) verified $y \in C(O, T; D(A)) \cap C^1(0, T; H_0^1(\Omega)) \cap C^2(0, T; L^2(\Omega))$.

In addition, in Grisvard [2], the solution can be decomposed as folloow:
$y = y_R + \sum_{i=1}^m \lambda_i(t) S_i(t)$ with:
$\lambda_i(t) = \int_\Omega (-y'') w_i(t) dt$ and $S_i(t) = r^{\alpha_i} sin(\alpha_i \theta_i)$ with α_i : the singularity exponent defined by $\alpha_i = \frac{\pi}{w_i}, w_i$: the aperture angle at the vertex S_i.
As in the first part, we can, for any $t > 0$, add an internal check $u(t) = -\sum_{i=1}^m \lambda_i(t) g_i(t)$

of such that if $\hat{y}$ is the regularized solution of the equation

$$\begin{cases} -\Delta\tilde{y} = -\tilde{y}'' + u(t) \qquad in\ Q_T, \\ \\ \qquad \tilde{y} = 0 \qquad in\ \Sigma_T, \\ \\ \tilde{y}(0) = \tilde{y}_0,\ \ \tilde{y}'(0) = \tilde{y}_1 \qquad in\ \Omega. \end{cases} \tag{5}$$

then $\hat{y} \in H^2(\Omega)$.
In fact, $\hat{y} = 0$ on the edge Σ_T, the solution $\hat{y} \in C(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.
Let $\tilde{V}$ be a subspace of $H^1(\Omega)$ of admissible solutions for the problem $(EOH)'$ defined by

$$\tilde{V} = \{\hat{y} \in H^1(\Omega)/\ \hat{y}_{|\Sigma_T} = 0\}. \tag{6}$$

For continuity, let us state the following proposition:

Proposition 1. *The problem $(EOH)'$ (4) admits an unique solution $\hat{y}$ in the space $\tilde{V}$ and there exist a constant $C_T > 0$ such that*

$$||\hat{y}||_{C(0,T;H_0^1(\Omega))} \leq C_T \left[||\tilde{y_0}||_{H_0^1(\Omega)} + ||\tilde{y_1}||_{L^2(\Omega)}\right]^{\frac{1}{2}}. \tag{7}$$

Proof. Let A be the unbounded operator of $L^2(\Omega)$ previously defined. According to Spectral Theory and by Fourier transform, A is diagonalizable and there exists a countable Hilbertian basis of $L^2(\Omega)$ made up of eigenvectors $(z_k)_{k\in\mathbb{N}^*} \subset D(A)$ such that the sequence of eigenvalues $(\lambda_k)_{k\geq 1}$ of associated eigenvalues verify: $(\lambda_k) \nearrow +\infty$ and $\lambda_1 > 0$.

$$z_k \in H_0^1(\Omega), -\Delta z_k = \lambda_k z_k \tag{8}$$

The family $Z = (z_k)_{k\geq 1}$ Hilbert base of $L^2(\Omega)$ ie $\hat{y} \in L^2(\Omega) \Longrightarrow \hat{y} = \sum_{k\geq 1} \hat{y}_k z_k$ with $\hat{y}_k =< \hat{y}, z_k >_{L^2(\Omega)}$ and $\sum_{k\geq 1} z_k^2 < +\infty$. What's more $\|\hat{y}\|_{L^2(\Omega)} = \left(\sum_{k=1}^{+\infty} \hat{y}_k^2\right)^{\frac{1}{2}}$.

$$\hat{y} \in H_0^1(\Omega) \Longleftrightarrow \hat{y} = \sum_{k\geq 1} \hat{y}_k z_k,\ \sum_{k\geq 1} \lambda_k \hat{y}_k^2 < +\infty \ and\ ||\hat{y}||_{H_0^1(\Omega)} = \left(\sum_{k\geq 1} \lambda_k \hat{y}_k^2\right)^{\frac{1}{2}}. \tag{9}$$

So, if $\hat{y}$ is solution of $(EOH)'$ (4) then

$$\begin{cases} \hat{y}(t,x) = \sum_{k\geq 1} \hat{y}_k(t) z_k(x), \\ \\ \hat{y}_{0k}(x) = \sum_{k\geq 1} \hat{y}_{0k} z_k(x), \\ \\ \hat{y}_{1k}(x) = \sum_{k\geq 1} \hat{y}_{1k} z_k(x), \\ \\ \sum_{k\geq 1} \left(\hat{y}_k''(t) - \lambda_k \hat{y}_k(t)\right) z_k(x) = 0. \end{cases} \tag{10}$$

We multiply the relation (10) by the eigenfunctions z_k and integrate on the cylinder Q_T

$$\begin{cases} \hat{y}_k^{''}(t) - \lambda_k \hat{y}_k(t) = 0, \\ \quad \hat{y}_k(0) = \hat{y}_{0k}, \\ \quad \hat{y}_k(1) = \hat{y}_{1k}. \end{cases} \tag{11}$$

And, for all $k \geq 1$, the solution of (11) (see Lions [9, 10])is under the form

$$\hat{y}_k(t) = \hat{y}_{0k} cos(\sqrt{\lambda_k} t) + \hat{y}_{1k} \frac{sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}}, \tag{12}$$

So

$$\hat{y}_k(t,x) = \sum_{k \geq 1} \left(\hat{y}_{0k} cos(\sqrt{\lambda_k} t) + \hat{y}_{1k} \frac{sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \right) z_k(x). \tag{13}$$

Assume

$$\begin{aligned} ||\hat{y}||^2_{C(0,T;H_0^1(\Omega))} &= sup_{t \in [0,T]} ||\hat{y}(t,.)||^2_{H_0^1(\Omega)} \\ &= sup_{t \in [0,T]} \sum_{k \geq 1} |\lambda_k| |\hat{y}_k(t)|^2 \end{aligned}$$

$\Longrightarrow$

$$||\hat{y}||^2_{C(0,T;H_0^1(\Omega))} \leq \sum_{k \geq 1} |\lambda_k| sup_{t \in [0,T]} |\hat{y}_k(t)|^2 \tag{14}$$

Based on the relationship (13)

$$\begin{aligned} ||\hat{y}||^2_{C(0,T;H_0^1(\Omega))} &\leq 2. \sum_{k > 1} \lambda_k \left[\hat{y}_{0k}^2 + \frac{\hat{y}_{1k}^2}{\lambda_k} \right] \\ &\leq 2. \sum_{k > 1} \lambda_k \left[\hat{y}_{0k}^2 + \hat{y}_{1k}^2 \right] \end{aligned}$$

let's remember that

$$\hat{y}_0 \in H_0^1(\Omega) \Longleftrightarrow \begin{cases} \hat{y}_0(x) = \sum_{k>1} \hat{y}_{0k} z_k(x), \\ \sum_{k>1} \lambda_k \tilde{y}_{0k}^2 < +\infty \; and \\ ||\hat{y}_0||^2_{H_0^1(\Omega)} = \sum_{k>1} \lambda_k \hat{y}_{0k}^2. \end{cases} \tag{15}$$

and

$$\hat{y}_1 \in L^2(\Omega) \Longleftrightarrow \begin{cases} \hat{y}_1(x) = \sum_{k>1} \hat{y}_{1k} z_k(x), \\ \sum_{k>1} \lambda_k \hat{y}_{1k}^2 < +\infty \text{ and} \\ ||\hat{y}_1||^2_{H_0^1(\Omega)} = \sum_{k>1} \lambda_k \hat{y}_{1k}^2. \end{cases} \tag{16}$$

Therefore, we get that $\hat{y} \in C(0,T;H_0^1(\Omega))$ (1) with

$$||\hat{y}||_{C(0,T;H_0^1(\Omega))} \leq C_T \left(||\hat{y}_0||_{H_0^1(\Omega)} + ||\hat{y}_1||_{L^2(\Omega)} \right) \tag{17}$$

□

2.3. **Application to the removal of singularities.** Let $\tilde{y}$ regularized solution of the equation

$$(EOS): \begin{cases} \tilde{y}'' - \Delta \tilde{y} + \sum_{i=1}^m g_i \int_\Omega (\tilde{y}'') w_i dx = 0 \qquad in\ Q_T, \\ \tilde{y} = 0, \qquad in\ \Sigma_T, \\ \tilde{y}(0) = \tilde{y}_0,\ \ \tilde{y}'(0) = \tilde{y}_1 \qquad in\ \Omega. \end{cases} \tag{18}$$

It will then be a matter of showing that the solution $\tilde{y}$ of the equation (EOH) (3) is in $C(0,T;H^2(\Omega) \cap H_0^1(\Omega))$?
In general, it was proved in Grisvard [2] that the following wave equation

$$(EOS)_2: \begin{cases} \varphi'' - \Delta \varphi = f \in L^1(0,T;H_0^1(\Omega)), \\ \varphi = 0 \qquad in\ \Sigma_T, \\ \varphi(0) = \varphi_0,\ \ \varphi'(0) = \varphi_1 \qquad in\ \Omega, \\ (\varphi_0, \varphi_1) \in D(A) \times D(A^{\frac{1}{2}}). \end{cases} \tag{19}$$

admit a solution $\varphi \in C(0,T;D(A)) \cap C^1(0,T;H^1(\Omega)) \cap C(0,T;L^2(\Omega))$ and that this solution verifies the inequality:

$$||\varphi||_{C(0,T;D(A))} \leq K \left(||\varphi_0||_{D(A)} + ||\varphi_1||_{D(A^{\frac{1}{2}})} + ||f||_{H_0^1(\Omega)} \right), \tag{20}$$

called continuous dependence of the solution compared to the initial conditions and to the second member.
Let us apply this Grisvard result to the equation (EOS) (18); For this consider for

$\zeta \in C^2(0,T;L^2(\Omega))$, $\tilde{y} = y(\zeta)$ is solution of the equation

$$(EOS)_3 : \begin{cases} \tilde{y}''(\zeta) - \Delta\tilde{y}(\zeta) = -\sum_{i=1}^m g_i \int_\Omega \zeta'' w_i dx \qquad in\ Q_T, \\ \tilde{y}(\zeta) = 0 \qquad in\ \Sigma_T, \\ \tilde{y}(\zeta)(0) = \tilde{y}(\zeta_0),\ \ \tilde{y}(\zeta)'(0) = \tilde{y}(\zeta_1) \qquad in\ \Omega. \end{cases}$$

$(EOS)_3$ and the inequality (20) implies a priori that $y(\zeta) \in C(0,T;D(A))$ and that

$$||y(\zeta)||_{C(0,T;D(A))} \leq K_1 \left(|| \sum_{i=1}^m g_i \int_\Omega \zeta'' ||_{L^1(\Omega)} \right). \tag{21}$$

Consider the application $\Lambda : \zeta \longmapsto y(\zeta)$; Let us show that Λ is contracting ?

Let $\zeta_1 \longmapsto y(\zeta_1),\ \zeta_2 \longmapsto y(\zeta_2)$ and $\zeta = \zeta_1 - \zeta_2 \longmapsto y(\zeta)$.
Applying it to the equation $(EOS)_3$ we get

$$(EOS)_4 : \begin{cases} \tilde{y}''(\zeta) - \Delta\tilde{y}(\zeta) = -\sum_{i=1}^m g_i \left(\int_\Omega \zeta'' w_i dx\right) \qquad in\ Q_T, \\ \tilde{y}(\zeta) = 0 \qquad in\ \Sigma_T. \end{cases}$$

More $y(\zeta_1)(0) = y(\zeta_2)(0) = 0$ ($y(\zeta_1)$ and $y(\zeta_2)$ have the same initial conditions as y_0 and y_1).
From inequality (21) we deduce

$$\begin{aligned} ||y(\zeta)||_{C(0,T;D(A))} &\leq K_1 \left(|| \sum_{i=1}^m g_i \int_\Omega \zeta'' w_i ||_{L^1(\Omega)} dx \right), & (22) \\ &\leq K_2 \left(\sum_{i=1}^m ||g_i||.||w_i||.|| \int_\Omega \zeta'' dx|| \right), & (23) \\ &\leq K_3 \left(\sum_{i=1}^m ||g_i||.||w_i||||\zeta'||_{L^1(\Omega)}.mes(\Omega) \right), & (24) \\ &\leq K_4 \left(\sum_{i=1}^m ||g_i||_{H_0^1(\Omega)} ||w_i||_{L^1(\Omega)} ||\zeta||_{L^1(\Omega)} \right), & (25) \\ &\leq K_5 ||\zeta||_{L^1(\Omega)}. & (26) \end{aligned}$$

With the constant $K_5 = \sum_{i=1}^m ||g_i||_{H_0^1(\Omega)} ||w_i||_{L^1(\Omega)}$.
Let us show that $0 < K_5 < 1$ ie Λ is contracting ?

We know that the dual singular functions are such that:

$w_i = r^{-\alpha_i} sin(\alpha_i \theta_i)\eta_i + \zeta_i$ *with* $\alpha_i = \frac{\pi}{\omega_i}$ *and* $\omega_i > \pi$, η_ia truncation function in the neighborhood of the vertices of x_i and $\zeta_i \in H_0^1(\Omega)$ for all $i \in \{0, ..., m\}$.

The application Λ is Lipschitzian, let us show that it is contracting ie $0 < K_5 < 1$?

$$\begin{aligned} ||w_i|| &= ||r^{-\alpha_i} sin(\alpha_i \theta_i)\eta_i + \zeta_i||, && (27)\\ &\le \frac{1}{r^{\alpha_i}}||sin(\alpha_i\theta_i)\eta_i|| + ||\zeta_i||, && (28)\\ &\le \frac{1}{r^{\alpha_0}} + ||\zeta||. && (29) \end{aligned}$$

where $\alpha_0 = min_{i\in\{1,...,m\}}\alpha_i$ thus $\frac{1}{r^{\alpha_i}} < \frac{1}{r^{\alpha_0}}$.
The functions $(g_i)_{1\le i\le m}$ are compact support on $\bar{\omega}$ which is compact, so there is $g_0 = max_{1\le i\le m} g_i$ on $\bar{\omega}$ such that $||g_i|| \le ||g_0||$ for all i. Therefore

$$\sum_{i=1}^{m} ||g_i||||w_i|| \le m^2||g_0||\frac{1}{r^{\alpha_0}} + C_1 \textit{ with } C_1 > 1 \textit{ a constant}.$$

As a result,

$$0 < K_5 < m^2||g_0||\frac{1}{r^{\alpha_0}} + C_1.$$

A sufficient condition for Λ to be contracting is that

$$m^2||g_0||\frac{1}{r^{\alpha_0}} + C_1 < 1 \Longleftrightarrow r \ge e^{\frac{1}{\alpha_0}log\left(\frac{m^2||g_0||}{1-C_1}\right)}. \tag{30}$$

Remember that

$$r = ||\vec{S_iM}|| = ||x - x_i||$$

ie $M \neq S_i$, $\forall i \in \{1, ..., m\}$ on $\bar{\omega}$.
Hence if M is far from the top of the crack ie $r >> 1$ the application Λ is contracting. Thereby,

$$||y(\zeta)||_{C(0,T;D(A))} \le K_5||\zeta||_{C^2(0,T;L^2(\Omega))} \tag{31}$$

Therefore, if (30) holds then the application Λ is contracting and according to the Fixed Point Theorem $y(\zeta) = y(\zeta_1) - y(\zeta_2) = 0$ and y being continuous so ζ is unique.
Hence the equation

$$(EOS)_3 : \begin{cases} \tilde{y}''(\zeta) - \Delta\tilde{y}(\zeta) = \underbrace{-\sum_{i=1}^{m} g_i \int_\Omega (\tilde{y}''(\zeta))w_i dx}_{u(t)} & in\ Q_T, \\ \tilde{y}(\zeta) = 0 & in\ \Sigma_T, \\ \tilde{y}(\zeta)(0) = \tilde{y}(\zeta)_0,\ \ \tilde{y}(\zeta)'(0) = \tilde{y}(\zeta)_1 & in\ \Omega. \end{cases} \tag{32}$$

admits a unique solution $\tilde{y} \in C(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

Proposition 2. *The solution $\tilde{y}$ is the regularized solution, therefore the singularity coefficient $\tilde{\lambda}$ associated with it is null.*

Proof. Let $\tilde{\lambda}$ the singularity coefficient associated with $\tilde{y}$. By definition,

$$\begin{aligned} \tilde{\lambda} &= \int_\Omega u(t) w_i dx, && (33) \\ &= \int_\Omega \left(-\sum_{i=1}^{m} g_i \int_\Omega \tilde{y}'' w_i dx \right) .w_i dx, && (34) \\ &= -\int_\Omega \int_\Omega \sum_{i=1}^{m} g_i \tilde{y}'' w_i . w_i dx dx, && (35) \\ &= -\int_\Omega \int_\Omega \sum_{i=1}^{m} < g_i, w_i > \Delta\tilde{y} w_i dx dx. && (36) \end{aligned}$$

so $< g_i, w_i >= 1 \Longrightarrow$

$$\begin{aligned} \tilde{\lambda} &= -\int_\Omega \int_\Omega \sum_{i=1}^{m} \Delta\tilde{y} w_i dx dx, && (37) \\ &= -\int_\Omega \int_\Omega \sum_{i=1}^{m} \tilde{y} \Delta w_i dx dx && (38) \end{aligned}$$

because $\tilde{y}_{|\Sigma_T} = 0$.
We also know that the dual singular functions are harmonic ie $\Delta w_i = 0$ hence $\tilde{\lambda} = 0$ □

Remark 2. *The corrective term or internal control $u(t)$ depends on $\tilde{y}''$, therefore $\tilde{y}$.*

3. Use in the Implementation of the Hum Method

3.1. **Preliminaries.** Let y solution of wave equation

$$(EOH) : \begin{cases} y'' - \Delta y = 0 \quad in\ Q_T, \\ y = 0 \quad in\ \Sigma_T, \\ y(0) = y_0, \ y'(0) = y_1 \quad in\ \Omega. \end{cases}$$

For initial data y_0 and y_1 belonging respectively to $H_0^1(\Omega$ and $L^2(\Omega)$. Let also be the energy of (EOH) defined by

$$E_0 = \frac{1}{2} \left(||y_0||_{H_0^1(\Omega)} + ||y_1||_{H_0^1(\Omega)} \right). \quad (39)$$

We know that in a polygonal domain with corner, $(x - x_0).\nu\frac{\partial\varphi}{\partial\nu}$ is not always a square integrable on the edge near of corner. Grisvard [2] got around this difficulty by imposing drastic geometric conditions. And, in Seck [5] this result has been generalized with less constraints in non-regular Sobolev spaces. Also Niane [4] have shown, without geometric conditions, the exact controllability of the wave equation by combining a boundary control and an internal control on a small part whose support is in the vicinity of a vertex crack.

3.2. **Implementation of the HUM method.** Let us return to the equation of the following waves

$$(EOS)_5 : \begin{cases} \tilde{\varphi}'' - \Delta\tilde{\varphi} = u(\tilde{\varphi}) \qquad in\ Q_T, \\ \\ \tilde{\varphi} = 0 \qquad in\ \Sigma_T, \\ \\ \tilde{\varphi}(0) = \tilde{\varphi}_0,\ \ \tilde{\varphi}'(0) = \tilde{\varphi}_1 \qquad in\ \Omega. \end{cases} \tag{40}$$

From the above, with $u(\tilde{\varphi}) = \sum_{i=1}^m g_i \left(\int_\Omega \varphi'' w_i dx\right)$, the solution $\tilde{\varphi} \in H^2(\Omega)$. Indeed, we multiply the equation $(EOS)_5$ (40) by $m\nabla\tilde{y}$ and integrate by parts:

$$\int_{Q_T} (\tilde{\varphi}'' - \Delta\tilde{\varphi}) m\nabla\tilde{y} dxdt = \int_{Q_T} m\nabla\tilde{y} u(\tilde{\varphi}) dxdt, \tag{41}$$

$$= -m\nabla\tilde{y} \sum_{i=1}^m g_i \left(\int_{Q_T} \varphi'' w_i dxdt\right). \tag{42}$$

Assume

$$I = \int_{Q_T} (\tilde{\varphi}'' - \Delta\tilde{\varphi}) m\nabla\tilde{y} dxdt \tag{43}$$

$$= \underbrace{\int_{Q_T} \tilde{\varphi}'' m\nabla\tilde{y} dxdt}_{I_1} - \underbrace{\int_{Q_T} \Delta\tilde{\varphi} m\nabla\tilde{y} dxdt}_{I_2} \tag{44}$$

3.3. **Some integrations by parts.**

3.3.1. *First Term* I_1.

$$\begin{aligned} I_1 &= \int_{Q_T} \tilde{\varphi}'' m\nabla\tilde{y} dxdt, \\ &= \int_0^T \int_\Omega \tilde{\varphi}'' m(x)\nabla\tilde{y} dxdt, \\ &= \int_\Omega \tilde{\varphi}' m(x)\nabla\tilde{y} dx|_0^T - \int_0^T \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial^2\tilde{y}}{\partial t\partial x_k} dtdx, \end{aligned}$$

$$\begin{aligned}
&= \int_\Omega \tilde{\varphi}' m(x)\nabla\tilde{y}dx|_0^T - \int_0^T \left(\int_\Omega \left(\frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial}{\partial t}(\frac{\partial\tilde{y}}{\partial x_k})\right)dx\right)dt,\\
&= \int_\Omega \tilde{\varphi}' m(x)\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T - \int_0^T\int_\Omega \left(\frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial^2\tilde{y}}{\partial t\partial x_k}\right)dxdt,\\
&= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k}dx|_0^T - \int_0^T\int_\Omega m_k\frac{\partial^2\tilde{\varphi}}{\partial t\partial x_k}\frac{\partial\tilde{y}}{\partial t}dxdt.
\end{aligned}$$

Noting that: $N = 2, divm = \sum_{k=1}^2 \frac{\partial m_k}{\partial x_k} = 2$ and applying Green again we have:

$$\begin{aligned}
I_1 &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k}dx|_0^T - \int_0^T \left[-\int_\Omega \frac{\partial m_k}{\partial x_k}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dx + \underbrace{\int_{\partial\Omega} m_k\tilde{\varphi}_k\frac{\partial\tilde{y}}{\partial t}d\sigma}_{=0}\right]dt,\\
&= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt.
\end{aligned}$$

3.3.2. *Second Term I_2.*

$$\begin{aligned}
I_2 &= \int_{Q_T}\Delta\tilde{\varphi} m\nabla\tilde{y}dxdt,\\
&= \int_0^T\int_\Omega \Delta\tilde{\varphi} m\nabla\tilde{y}dxdt,\\
&= \int_0^T\left[\int_\Omega \nabla\tilde{\varphi}.\nabla(m_k\frac{\partial\tilde{\varphi}}{\partial x_k})dx - \int_{\partial\Omega}\frac{\partial\tilde{\varphi}}{\partial n}m_k\frac{\partial\tilde{y}}{\partial x_k}d\sigma\right]dt,\\
&= \int_0^T\left[\int_\Omega \frac{\partial\tilde{\varphi}}{\partial x_i}(m_k\frac{\partial\tilde{y}}{\partial x_k})dx - \int_{\partial\Omega}\frac{\partial\tilde{\varphi}}{\partial n}m_k\frac{\partial\tilde{y}}{\partial x_k}d\sigma\right]dt,\\
&= \int_0^T\left[\int_\Omega \frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dx + \underbrace{\int_\Omega \frac{\partial\tilde{\varphi}}{\partial x_i}m_k\frac{\partial^2\tilde{y}}{\partial x_i\partial x_k}dx}_{J} - \int_{\partial\Omega}\frac{\partial\tilde{\varphi}}{\partial n}m_k\frac{\partial\tilde{y}}{\partial x_k}d\sigma\right]dt.
\end{aligned}$$

Let's study the integral J:

$$\begin{aligned}
J &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial x_i}m_k\frac{\partial^2\tilde{y}}{\partial x_i\partial x_k}dx,\\
&= \int_\Omega m_k\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial^2\tilde{y}}{\partial x_i\partial x_k}dx,\\
&= \int_\Omega \frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dx - \int_{\partial\Omega} m_k n_k\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma.
\end{aligned}$$

By grouping together we get:

$$\begin{aligned} I_2 &= \int_0^T \Big[\int_\Omega \frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dx + \int_\Omega \frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dx \\ &- \int_{\partial\Omega} m_k n_k \frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma - \int_{\partial\Omega}\frac{\partial \tilde{\varphi}}{\partial n}m_k\frac{\partial \tilde{y}}{\partial x_k}d\sigma\Big]dt, \\ &= \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dxdt \\ &- \int_{\Sigma_T} m.n\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma dt - \int_{\Sigma_T}\frac{\partial \tilde{\varphi}}{\partial n}m_k\frac{\partial \tilde{y}}{\partial x_k}d\sigma dt. \end{aligned}$$

Back to $I = I_1 + I_2$ (43) and (45):

$$I = \int_\Omega \frac{\partial \tilde{\varphi}}{\partial t}m_k\frac{\partial \tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial \tilde{\varphi}}{\partial t}\frac{\partial \tilde{y}}{\partial t}dxdt + \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt$$
$$+ \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dxdt - \int_{\Sigma_T} m.n\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma dt - \underbrace{\int_{\Sigma_T}\frac{\partial \tilde{\varphi}}{\partial n}m_k.n_k\frac{\partial \tilde{y}}{\partial x_k}d\sigma dt}_{L}.$$

Also

$$\begin{aligned} L &= \frac{1}{2}\int_{\Sigma_T}\frac{\partial \tilde{\varphi}}{\partial n}m.n\nabla\tilde{y}d\sigma dt, \\ &= \frac{1}{2}\int_{\Sigma_T}\left(\frac{\partial \tilde{\varphi}}{\partial n}\right)m.nd\sigma dt. \end{aligned}$$

The two equations have the same initial and boundary conditions.
Let's study L ?

$$\frac{\partial \tilde{\varphi}}{\partial x_i} = \frac{\partial \tilde{\varphi}}{\partial n}.n_i + \frac{\partial \tilde{\varphi}}{\partial \tau_i},$$

Decomposition according to the normal and the tangential. However

$$\frac{\partial \tilde{\varphi}}{\partial n_i} = \frac{\partial \tilde{\varphi}}{\partial n}.n_i \Rightarrow \sum_i \frac{\partial \tilde{\varphi}}{\partial n_i} = \sum_i \frac{\partial \tilde{\varphi}}{\partial n}.n_i \Rightarrow \nabla\tilde{\varphi} = \frac{\partial \tilde{\varphi}}{\partial n}.n.$$

So we deduce that:

$$I = \int_\Omega \frac{\partial \tilde{\varphi}}{\partial t}m_k\frac{\partial \tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial \tilde{\varphi}}{\partial t}\frac{\partial \tilde{y}}{\partial t}dxdt + \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt +$$
$$\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial \tilde{y}}{\partial x_i}dxdt - \int_{\Sigma_T} m.n\frac{\partial \tilde{\varphi}}{\partial x_i}\frac{\partial \tilde{y}}{\partial x_i}d\sigma dt - \frac{1}{2}\int_{\Sigma_T}(\frac{\partial \tilde{\varphi}}{\partial n})^2 m.nd\sigma dt. \quad (45)$$

3.3.3. *Third Term I_3.*

$$\begin{aligned} I_3 &= \int_{Q_T}\left\{-m\nabla\tilde{y}\sum_{i=1}^{m} g_i\left(\int_\Omega \varphi'' w_i dx\right)dxdt\right\},\\ &= \int_0^T\int_\Omega\left\{-m\nabla\tilde{y}\sum_{i=1}^{m} g_i\left(\int_\Omega \varphi'' w_i dx\right)dxdt\right\},\\ &= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\underbrace{\sum_{i=1}^{m} < g_i, w_i >}_{\delta_{ii}=1}\tilde{\varphi}'' dxdxdt,\\ &= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\tilde{\varphi}'' dxdxdt,\\ &= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt - \underbrace{\int_{\Sigma_T}\int_{\partial\Omega} m\tilde{y}\tilde{\varphi}(\sigma)d\sigma dt}_{=0},\\ &= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt. \end{aligned}$$

Let's recap $I = I_3$(46)$\Longleftrightarrow$

$$\begin{aligned} &\int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k} dx|_0^T + \int_{Q_T} divm \frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t} dxdt + \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k} dxdt\\ &+ \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i} dxdt - \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i} d\sigma dt - \frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.n d\sigma dt\\ &= -\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt \end{aligned} \tag{46}$$

$\Longleftrightarrow$

$$\begin{aligned} \frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.n d\sigma dt &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t} m_k \frac{\partial\tilde{y}}{\partial x_k} dx|_0^T + \int_{Q_T} divm \frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t} dxdt\\ &+ \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k} dxdt + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i} dxdt\\ &- \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i} d\sigma dt + \int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi} dxdxdt. \end{aligned} \tag{47}$$

3.4. **Getting started with the HUM method.** For $x_0 \in \mathbb{R}^2$, assume

$$\Sigma_T^{0*} = \Gamma_0^* \times]0, T[\ ,\ \Sigma_T^{1*} = \Gamma_0 \times]0, T[.$$

Let $||.||$ the Euclidean in $\mathbb{R}^2$ and introduce the following constants.

$$R_0 = R(x_0) = \max_{x\in\overline{\Omega}} ||x - x_0||,\ T_0 = 2R(x_0).$$

Let us define in the same way the energies (see Lions [9, 10]) associated respectively with the systems $(EOS)_5$, (40) and (EOH) :

$$E(t,\tilde{\varphi}_0,\tilde{\varphi}_1)=\frac{1}{2}\left[\int_\Omega ||\nabla\tilde{\varphi}(t)||^2_{\mathbb{R}^2}dx+\int_\Omega(\frac{\partial\tilde{\varphi}}{\partial t}(t))^2dx\right],$$

$$E(t,\tilde{y}_0,\tilde{y}_1)=\frac{1}{2}\left[\int_\Omega ||\nabla\tilde{y}(t)||^2_{\mathbb{R}^2}dx+\int_\Omega(\frac{\partial\tilde{y}}{\partial t}(t))^2dx\right].$$

3.4.1. *Direct Inequality.* Back to the relationship (46)

$$\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.n d\sigma dt=\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T+\int_{Q_T}divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt+$$
$$+\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt+$$
$$-\int_{\Sigma_T}m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt+\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt \tag{48}$$

$$\Longleftrightarrow\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt-\int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt=\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T$$
$$+\int_{Q_T}divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt+\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\tilde{y}}{\partial x_k}dxdt$$
$$+\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt-\int_{\Sigma_T}m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt. \tag{49}$$

We know that:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\leq R_0.\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T \tag{50}$$

and noticing that: $|ab|\leq\frac{1}{2}(a^2+b^2)$ we have:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx\leq\frac{1}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dx. \tag{51}$$

Therefore:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\leq\frac{T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt. \tag{52}$$

Assume

$$\Sigma_T=\Sigma_T^{0*}\cup\Sigma_T^{1*},\qquad M_1=max_{1\leq i,k\leq 2}max_{x\in\bar{\mathcal{B}}_i}|\frac{\partial x_k}{\partial x_i}(x)|.$$

Consider an open ball $\mathcal{B}_i$ which does not meet any crack vertex ie $h\equiv\eta h$ (In the general case we can recover the domain Ω by a finite union of $\mathcal{B}_i$ ie $\Omega=\cup_{i=1}^{Ns}\mathcal{B}_i$).

$$\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt\quad=\quad\int_{B_i\times]0,T[}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt,$$

$$\leq \frac{M_1}{2}\int_{B_i\times]0,T[}\left[(\frac{\partial\tilde{\varphi}}{\partial x_i})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt,$$
$$\leq \frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt \quad (53)$$

Relationships (50), (51), and (53), we deduce:

$$\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt-\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i} m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt\leq R_0.\sum_{i=1}^{N_s}\int_{B_i}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T+$$
$$\frac{T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt+\sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt \quad (54)$$

$$\Longrightarrow \quad \frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt \leq R_0.\sum_{i=1}^{N_s}\int_{B_i}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T$$
$$+\frac{T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt$$
$$+\sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt$$
$$+\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i} m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt. \quad (55)$$

Therefore

$$\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i} m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt \leq R_0\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt$$
$$\leq R_0\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial x_k})^2+(\frac{\partial\tilde{y}}{\partial t})^2\right],$$
$$\leq R_0\sum_{i=1}^{N_s}\int_{B_i}\int_0^T\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial x_k})^2+(\frac{\partial\tilde{y}}{\partial t})^2\right],$$
$$\leq R_0\sum_{i=1}^{N_s}mes(B_i)\left[E(t,\tilde{\varphi}_0,\tilde{\varphi}_1)+E(t,\tilde{y}_0,\tilde{y}_1)\right]. \quad (56)$$

Starting from the fact that the energy associated with the wave equation is constant, we obtain:

$$\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i} m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt\leq 2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \quad (57)$$

So the relation (53) implies

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq R_0.\sum_{i=1}^{N_s}\int_{B_i}\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T\\ &+ \frac{T}{2}\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+ \sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt\\ &+ 2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \end{aligned} \tag{58}$$

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq \frac{T.R_0}{2}.\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+ \frac{T}{2}\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt\\ &+ \sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[||\nabla\tilde{\varphi}||^2_{L^2(\mathbb{R}^2)}+||\nabla\tilde{y}||_{L^2(\mathbb{R})^2}\right]dxdt\\ &+ 2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \end{aligned} \tag{59}$$

$$\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt \leq \frac{T.R_0+1}{2}.\sum_{i=1}^{N_s}\int_{B_i}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(||\nabla\tilde{\varphi}||_{L^2(\mathbb{R}^2)})^2\right]dxdt+$$
$$\sum_{i=1}^{N_s}\frac{M_1}{2}\int_{B_i\times]0,T[}\left[(\frac{\partial\tilde{y}}{\partial t})^2+(||\nabla\tilde{y}||_{L^2(\mathbb{R}^2)})^2\right]dxdt+2R_0\sum_{i=1}^{N_s}mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \tag{60}$$

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &\leq \left(\frac{T.R_0+1}{2}+\frac{M_1}{2}+2R_0\sum_{i=1}^{N_s}mes(B_i)\right)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1)\\ &\leq C_T^0(\Omega)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1.) \end{aligned} \tag{61}$$

From

$$\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||^2_{L^2(\Omega)}\leq C_T^0(\Omega)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \tag{62}$$

$$\text{where}\qquad C_T^0(\Omega)=\left(\frac{T.R_0+1}{2}+\frac{M_1}{2}+2R_0\sum_{i=1}^{N_s}mes(B_i)\right). \tag{63}$$

3.4.2. *Inverse Inequality.* Feedback on the relationship (47)

$$\begin{aligned}\frac{1}{2}\int_{\Sigma_T}(\frac{\partial\tilde{\varphi}}{\partial n})^2 m.nd\sigma dt &= \int_\Omega \frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T + \int_{Q_T} divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt \\ &+ \int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt + \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt \\ &- \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt + \int_0^T\int_\Omega\int_\Omega m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt. \quad (64)\end{aligned}$$

$$\begin{aligned}\int_{Q_T}\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt &= \sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt, \\ &\le \frac{1}{2}\sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}\left[(\frac{\partial\tilde{\varphi}}{\partial x_i})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt, \\ &\le \frac{1}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]\sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}, \\ &\le \frac{1}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]Ns.\int_0^T m(x)dt, \\ &\le \frac{T}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]Ns.m(x), \\ &\le \frac{T.R_0Ns}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right].\end{aligned}$$

We deduce that:

$$-\sum_{i=1}^{Ns}\int_{B_i}\int_0^T\frac{\partial m_k}{\partial x_i}.\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_k}dxdt \ge -\frac{T.R_0Ns}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2}+||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right]. \quad (65)$$

In addition, let us pose $M_2 = min_{x\in\bar{\Omega}}||m(x)||^2_{\mathbb{R}^2}$:

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T \ge M_2\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T,$$

therefore

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T dx \le \frac{1}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt \Rightarrow$$

$$-\frac{1}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt \le -\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T dx,$$

So

$$\int_\Omega\frac{\partial\tilde{\varphi}}{\partial t}m_k\frac{\partial\tilde{y}}{\partial x_k}dx|_0^T \ge -\frac{M_2.T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2+(\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt. \quad (66)$$

Also, from the relation (65) we deduce

$$-\sum_{i=1}^{N_s}\int_0^T\int_{B_i}\int_{B_i} m\nabla\tilde{y}\nabla\tilde{\varphi}dxdxdt \geq -2R_0\sum_{i=1}^{N_s} mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \tag{67}$$

We also know that,

$$\int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\partial\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt = \int_{Q_T}\frac{\partial m_k}{\partial x_k}\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}dxdt + \int_{Q_T} m_k\frac{\partial^2\tilde{\varphi}}{\partial x_i^2}\frac{\partial\tilde{y}}{\partial x_i}dxdt$$

$$\int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt = \underbrace{\int_{\Sigma_T^{0*}} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt}_{m.n<0} + \underbrace{\int_{\Sigma_T^{1*}} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt}_{m.n>0}$$

By grouping and reducing simultaneously we have:

$$\begin{aligned}\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||_{L^2(\Sigma_T)} \quad &\geq \quad -\frac{M_2.T}{2}\int_{Q_T}\left[(\frac{\partial\tilde{\varphi}}{\partial t})^2 + (\frac{\partial\tilde{y}}{\partial x_k})^2\right]dxdt \\ &- \quad \frac{T.R_0Ns}{2}\left[||\nabla\tilde{\varphi}||^2_{\mathbb{R}^2} + ||\nabla\tilde{y}||^2_{\mathbb{R}^2}\right] \\ &- \quad 2R_0\sum_{i=1}^{N_s} mes(B_i)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1) + \int_{Q_T} divm\frac{\partial\tilde{\varphi}}{\partial t}\frac{\partial\tilde{y}}{\partial t}dxdt \\ &+ \quad \int_{Q_T}\frac{\partial}{\partial x_k}(m_k\frac{\partial\tilde{\varphi}}{\partial x_i})\frac{\partial\tilde{y}}{\partial x_i}dxdt - \int_{\Sigma_T} m.n\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\tilde{y}}{\partial x_i}d\sigma dt \qquad \Longrightarrow\end{aligned}$$

$$\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||_{L^2(\Sigma_T)} \geq \left(-\frac{M_2.T}{2} - \frac{T.R_0Ns}{2} - 2R_0\sum_{i=1}^{N_s} mes(B_i) + 2.\frac{M_2.T}{2}\right)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1), \tag{68}$$

$$\geq \left(\frac{M_2.T}{2} - \frac{T.R_0Ns}{2} - 2R_0\sum_{i=1}^{N_s} mes(B_i)\right)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \tag{69}$$

By posing

$$C_T^1(\Omega) = \left(\frac{M_2.T}{2} - \frac{T.R_0Ns}{2} - 2R_0\sum_{i=1}^{N_s} mes(B_i)\right), \tag{70}$$

$$\frac{1}{2}||\frac{\partial\tilde{\varphi}}{\partial n}||_{L^2(\Sigma_T)} \geq C_T^1(\Omega)E(t,\tilde{\varphi}_0,\tilde{\varphi}_1). \tag{71}$$

3.5. **Exact Controllability Result.** Either the operator $\Lambda : H_0^1(\Omega)\times L^2(\Omega)$ Lions [9] defined by:

$$\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1) = (\tilde{y}'(0), -\tilde{y}(0)). \tag{72}$$

Indeed, we know that Grisvard [2]:

$$\Lambda \in \mathcal{L}\left[H_0^1(\Omega)\times L^2(\Omega), H^{-1}(\Omega)\times L^2(\Omega)\right] \qquad \text{and} \tag{73}$$

$$||\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1)||^2_{H^{-1}(\Omega)} = ||\tilde{y}(0)||^2_{L^2(\Omega)} + ||\tilde{y}'(0)||^2_{H^{-1}(\Omega)}. \tag{74}$$

Considering $\tilde{\varphi}_n \in C(0,T;D(A)) \cap C^1(0,T;D(A^{\frac{1}{2}})) \cap C^2(0,T;L^2(\Omega))$, and also $\tilde{\varphi}_{0n} \in D(A), \tilde{\varphi}_{1n} \in D(A^{\frac{1}{2}})$.
Assume $\zeta_n = \left[u(\tilde{\varphi})\right]\chi_{\bar{O}} = \left[\sum_{i=1}^m g_i\left(\int_\Omega \varphi'' w_i dx\right)\right]\chi_{\bar{O}}$ where O is an arbitrarily small part of the domain Ω not meeting any vertex of cracks.

Let $z_n \in C(0,T;H_0^2(\Omega)) \cap C^1(0,T;L^2(\Omega))$ solution of the following equation

$$(EOS)_6 : \begin{cases} z''_n - \Delta z_n = \zeta_n & in\ Q_T, \\ (z_n).\chi_{\bar{O}} = 0 & on\ \Sigma_T, \\ z_n(T) = z'_n(T) = 0 & in\ \Omega. \end{cases}$$

So we have (3.5):

$$\langle \Lambda(\tilde{\varphi}_{0n},\tilde{\varphi}_{1n}),(z_{0n},z_{1n})\rangle = \langle z'_n(0),\tilde{\varphi}_{0n}\rangle - \langle z_n(0),\tilde{\varphi}_{1n}\rangle. \tag{75}$$

By multiplying the equation $(EOS)_6$ by $\tilde{\varphi}_n$ and the equation (EOS) (18) by z_n we get:

$$-\int_{Q_T}(z''_n - \Delta z_n)\tilde{\varphi}_n dxdt + \int_{Q_T}(\tilde{\varphi}''_n - \Delta\tilde{\varphi}_n)z_n dxdt = \int_{Q_T}\sum_{i=1}^m ||g_i||\left(\int_\Omega ||\tilde{\varphi}''||.||w_i||dx\right)dt.$$

In particular on the open O:

$$\begin{aligned} -\int_{O\times]0,T[}(z''_n - \Delta z_n)\tilde{\varphi}_n dxdt &+ \int_{O\times]0,T[}(\tilde{\varphi}''_n - \Delta\tilde{\varphi}_n)z_n dxdt \\ &= \int_{O\times]0,T[}\sum_{i=1}^m ||g_i||\left(\int_O ||\tilde{\varphi}''||.||w_i||dx\right)dt, \end{aligned}$$

which is also written

$$\begin{aligned} \int_{O\times]0,T[}\sum_{i=1}^m ||g_i||\left(\int_O ||\tilde{\varphi}''||.||w_i||dx\right)dt &= -\langle\tilde{\varphi}_n, z'_n\rangle|_0^T + \langle\tilde{\varphi}'_n, z_n\rangle|_0^T \\ &- \underbrace{\int_{\partial O\times]0,T[}\left(\frac{\partial\tilde{\varphi}_n}{\partial\nu}\right)^2\tilde{\varphi}_n(\sigma)d\sigma dt}_{=0} \\ &- \underbrace{\int_{\partial O\times]0,T[}\left(\frac{\partial z_n}{\partial\nu}\right)^2 z_n(\sigma)d\sigma dt}_{=0}. \end{aligned}$$

Therefore

$$\begin{aligned}\int_{O\times]0,T[}\sum_{i=1}^{m}||g_i||\left(\int_O||\tilde{\varphi}''||.||w_i||dx\right)dt &= -\left\langle\tilde{\varphi}_n,z_n'\right\rangle|_0^T+\left\langle\tilde{\varphi}_n',z_n\right\rangle|_0^T,\\ &= \left\langle\Lambda(\tilde{\varphi}_{0n},\tilde{\varphi}_{1n}),(\tilde{\varphi}_{0n},\tilde{\varphi}_{1n})\right\rangle.\end{aligned}$$

Passing to the limit,

$$\langle\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1),(\tilde{\varphi}_0,\tilde{\varphi}_1)\rangle=\int_{O\times]0,T[}\sum_{i=1}^{m}||g_i||\left(\int_O||\tilde{\varphi}''||.||w_i||dx\right)dt.$$

But we also know that

$$\begin{aligned}\int_{O\times]0,T[}\sum_{i=1}^{m}||g_i||\left(\int_O||\tilde{\varphi}''||.||w_i||dx\right)dt &= \int_{O\times]0,T[}\sum_{i=1}^{m}\int_O||\tilde{\varphi}''||.||g_i||.||w_i||dxdt,\\ &\geq \int_{O\times]0,T[}.||\sum_{i=1}^{m}\int_O\tilde{\varphi}''dx||.|<g_i,w_i>|dxdt,\\ &\geq 2m\left[\frac{1}{2}\int_O||\tilde{\varphi}'|||_0^T dx\right].\end{aligned}$$

By covering the domain Ω by a disjoint finite union of openings O_i *ie* $\Omega=\bigcup_{i=1}^{m}O_i$ and $O_i\cap O_j=\emptyset$ if $i\neq j$.
Consequently, we deduce that:

$$\langle\Lambda(\varphi_0,\varphi_1),(\varphi_0,\varphi_1)\rangle\geq K_1(T-T_0)E_0. \tag{76}$$

Λ being linear, continuous and coercive on $H_0^1(\Omega)\times L^2(\Omega)$ for $T>T_0$, then according to a Classical Controllability Theorem, Λ is an isomorphism of $H_0^1(\Omega)\times L^2(\Omega)$ in $L^2(\Omega)\times H^{-1}(\Omega)$.
Let $(z_0,z_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, the following equation

$$\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1)=(z_1,-z_0),$$

admit a unique solution $(\tilde{\varphi}_0,\tilde{\varphi}_1)\in H_0^1(\Omega)\times L^2(\Omega)$ for all $T>T_0$.
Let us now consider $\tilde{\varphi}$ and z respective solutions of the equations $(EOS)_5$ and $(EOS)_6$ with as initial conditions:

$$\begin{aligned} z_0 &= \tilde{\varphi}_0,\\ z_1 &= \tilde{\varphi}_1,\\ \zeta_n &= \left(\sum_{i=1}^{m}g_i\left(\int_\Omega\tilde{\varphi}''w_idx\right)\right)\chi_O,\\ &\quad and\\ \varphi &= \begin{cases}\tilde{\varphi}.\chi_O & on\ \Sigma_T^{*0},\\ 0 & on\ \Sigma_T^{*1}.\end{cases}\end{aligned}$$

By a uniqueness of solutions theorem, we deduce that: $\tilde{\varphi} = z$ so therefore $z(T) = z'(T) = 0$.

Hence the result of exact controllability.

Remark 3. *This result does not depend on any geometrical condition: consequently the crack lines may not be concurrent; and, the exact controllability result has been proven.*

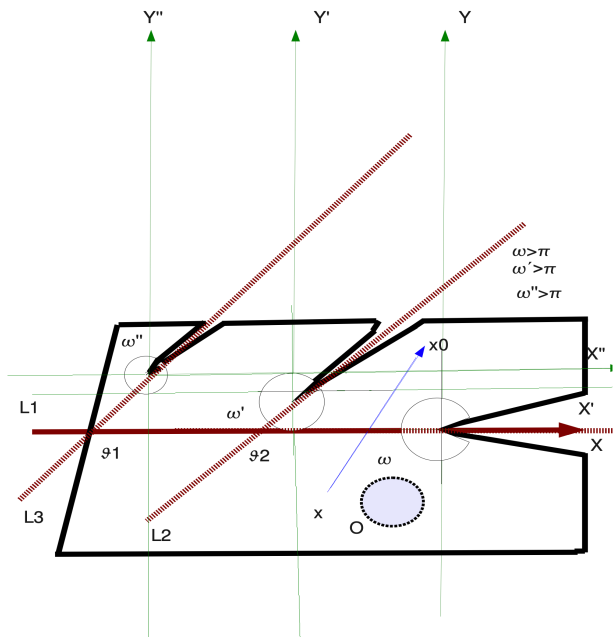

FIGURE 2. Non-convex domain with non-concurrent cracks

4. Conclusion and Perspectives

The presence of cracks, corners or angles in a mechanical device or materials always leads to the appearance of singularities. And, once the diagnosis of these cracks (desired or not) has been made, it is necessary to try to control them without

major geomeric constraints.
One of the objectives that we set ourselves, within the framework of this research paper, was assess the exact controllability of the wave equation in the cracked domains without constraints on the cracks. If anything, the formulas of integrations by parts (formulas of Green in the fields with corners and/or cracks) could be done (to our knowledge) only if the lines of cracks or their support were concurrent.
Based on recent work by Dauge [13, 14], Dauge [15] and Costabel [16], we were able to establish, without additional assumptions on the nature of the cracks or their support, the exact controllability of the wave equation with more cracks. Consequently its results were obtained on a non-convex polygonal domain with non-concurrent crack lines. From the results obtained in this paper, certain questions naturally emerge. Our goal is to no longer have constraining geometric conditions (”Closer” to reality).
When it comes to the perspectives, we have a double goal that we plan on achieving in the near future. Firstly, generalize in higher dimension the results obtained in this paper. And,secondly, make numerical simulations to support its theoretical results.

Declaration of Competing Interests The author declare that they have no conflicts of interest.

Acknowledgements The author thanks:

- Prof. Mary Teuw Niane, Glbert Bayili and Abdoulaye Sène for the numerous discussions, exchanges and suggestions before the finalization of this work.
- Colleagues and researchers who have reviewed this work.
- The members of the LANI laboratory where his work has been presented on several occasions.
- My colleagues from the Department of Mathematics of FASTEF ex ENS of Cheikh Anta Diop University of Dakar, Senegal.

References

[1] Kondratiev, V.A., Boundary value problems for elliptic equation in domain with conical or angular points, *Transactions Moscow Mat. Soc.,* (1967), 227-313.

[2] Grisvard, P., Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, Vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[3] Moussaoui, M., Singularités des solutions du problème mêlé, contrôlabilité exacte et stabilisation frontière, *Soc. Math. Appl. Indust.,* 1997.

[4] Niane, M.T., Bayili, G., Sène, A., Sène, A., Sy, M., Is it possible to cancel singularities in a domain with corners and cracks?, *C. R. Math. Acad. Sci.*, 343(2) (2006), 115-118. https://doi:10.1016/j.crma.2006.05.003

[5] Seck, C., Bayili, G., Séne, A., Niane, M.T., Contrôlabilité exacte de l'équation des ondes dans des espaces de Sobolev non réguliers pour un ouvert poygonal, *Afrika Matematika,* 23 (2012), 1-9. https://doi:10.1007/s13370-011-0001-6

[6] Gilbert, B., Nicaise, S., Stabilization of the wave equation in a polygonal domain with cracks, *Rev. Mat. Complut.,* 27(1) (2014), 259-289. https://doi: 10.1007/s13163-012-0113-z

[7] Costabel, M., On the limit Sobolev regularity for Dirichlet and Neumann problems on Lipschitz domains, *Math. Nachr.,* 292 (2019), 2165-2173. https://doi:10.1002/mana.201800077.

[8] Niane, M.T., Contrôlabilité spectrale élargie des systèmes distribués par une action sur une petite partie analytique arbitraire de la fontière, *C.R. Acad. Sci.,* Paris, 309(1), (1989), 335-340.

[9] Lions, J.L., Contrôlabilité Exacte, Perturbations et Stabilisation de SystÈMes DistribuÉS,Tome 2, Recherches en Mathématiques Appliquées, Research in Applied Mathematics, Volume 9, Perturbations, Masson, Paris, 1988.

[10] Lions, J.-L., Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, *Recherches en Mathématiques Appliquées ,Research in Applied Mathematics*, Volume 8, Paris,1988.

[11] Brezis, H., Analyse Fonctionnelle, Théorie et Applications, Masson, 1983.

[12] Hörmander, L., Linear Partial Differential Operators, Springer Verlag, Berlin, 1976.

[13] Dauge, M., Balac, S., Moitier, Z., Asymptotics for 2D whispering gallery modes in optical micro-disks with radially varying index, Arxiv: 2003.14315. https://doi:10.1093/imamat/hxab033.

[14] Dauge, M., Costabel, M., Hu, J.Q., Characterization of Sobolev spaces by their Fourier coefficients in axisymmetric domains, arXiv:2004.07216v1, 2020.

[15] Dauge, M., Initiation Into Corner Singularities, Course Given in the RICAM Special Semester on Computational Methods in Science and Engineering, October, 2016.

[16] Costabel, M., On the limit Sobolev regularity for Dirichlet and Neumann problems on Lipschitz domains, *Math. Nachr.,* 292 (2019), 2165-2173. arXiv: 1711.07179. https://doi:10.1002/mana.201800077

INSTRUCTIONS TO CONTRIBUTORS

Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics (Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.) is a single-blind peer reviewed open access journal which has been published biannually since 1948 by Ankara University, accepts original research articles written in English in the fields of Mathematics and Statistics. It will be published four times a year from 2022. Review articles written by eminent scientists can also be invited by the Editor.

The publication costs for Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics are covered by the journal, so authors do not need to pay an article-processing and submission charges. The PDF copies of accepted papers are free of charges and can be downloaded from the website. Hard copies of the paper, if required, are due to be charged for the amount of which is determined by the administration each year.

All manuscripts should be submitted via our online submission system https://dergipark.org.tr/en/journal/2457/submission/step/manuscript/new. Note that only two submissions per author per year will be considered. Once a paper is submitted to our journal, all co-authors need to wait 6 months from the submission date before submitting another paper to Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. Manuscripts should be submitted in the PDF form used in the peer-review process together with the COVER LETTER and the TEX file (Source File). In the cover letter the authors should suggest the most appropriate Area Editor for the manuscript and potential four reviewers with full names, universities and institutional email addresses. Proposed reviewers must be experienced researchers in your area of research and at least two of them should be from different countries. In addition, proposed reviewers must not be co-authors, advisors, students, etc. of the authors. In the cover letter, the author may enter the name of anyone who he/she would prefer not to review the manuscript, with detailed explanation of the reason. Note that the editorial office may not use these nominations, but this may help to speed up the selection of appropriate reviewers.

Manuscripts should be typeset using the LATEX typesetting system. Authors should prepare the article using the Journal's templates (commun.cls and commun.cst). Manuscripts written in AMS LaTeX format are also acceptable. A template of manuscript can be downloaded in tex form from the link https://dergipark.org.tr/en/download/journal-file/22173(or can be reviewed in pdf form). The title page should contain the title of the paper, full names of the authors, affiliations addresses and e-mail addresses of all authors. Authors are also required to submit their Open Researcher and Contributor ID (ORCID)'s which can be obtained from http://orcid.org as their URL address in the format http://orcid.org/xxxx-xxxx-xxxx-xxxx. Please indicate the corresponding author. Each manuscript should be accompanied by classification numbers from the Mathematics Subject Classification 2020 scheme. The abstract should state briefly the purpose of the research. The length of the Abstract should be between 50 to 5000 characters. At least 3 keywords are required. Formulas should be numbered consecutively in the parentheses. All tables must have numbers (TABLE 1) consecutively in accordance with their appearance in the text and a legend above the table. Please submit tables as editable text not as images. All figures must have numbers (FIGURE 1) consecutively in accordance with their appearance in the text and a caption (not on the figure itself) below the figure. Please submit figures as EPS, TIFF or JPEG format. Authors Contribution Statement, Declaration of Competing Interests and Acknowledgements should be given at the end of the article before the references. Authors are urged to use the communication.bst style in BibTeX automated bibliography. If manual entry is preferred for bibliography, then all citations must be listed in the references part and vice versa. Number of the references (numbers in squared brackets) in the list can be in alphabetical order or in the order in which they appear in the text. Use of the DOI is highly encouraged. Formal abbreviations of the journals can be used. The Editor may seek the advice of two, or three referees, depending on the response of the referees, chosen in consultation with appropriate members of the Editorial Board, from among experts in the field of specialization of the paper. The reviewing process is conducted in strict confidence and the identity of a referee is not disclosed to the authors at any point since we use a single-blind peer review process.

Declarations/Ethics:
With the submission of the manuscript authors declare that:

- All authors of the submitted research paper have directly participated in the planning, execution, or analysis of study;
- All authors of the paper have read and approved the final version submitted;
- The contents of the manuscript have not been submitted, copyrighted or published elsewhere and the visual-graphical materials such as photograph, drawing, picture, and document within the article do not have any copyright issue;
- The contents of the manuscript will not be copyrighted, submitted, or published elsewhere, while acceptance by the Journal is under consideration.
- The article is clean in terms of plagiarism, and the legal and ethical responsibility of the article belongs to the author(s). Author(s) also accept that the manuscript may go through plagiarism check using iThenticate software;
- The objectivity and transparency in research, and the principles of ethical and professional conduct have been followed. Authors have also declared that they have no potential conflict of interest (financial or non-financial), and their research does not involve any human participants and/or animals.

Research papers published in **Communications Faculty of Sciences University of Ankara** are archived in the Library of Ankara University and in Dergipark immediately following publication with no embargo.

Editor in Chief
http://communications.science.ankara.edu.tr
Ankara University, Faculty of Sciences
06100, Besevler - ANKARA TURKEY

C O M M U N I C A T I O N S

FACULTY OF SCIENCES
UNIVERSITY OF ANKARA

DE LA FACULTE DES SCIENCES
DE L'UNIVERSITE D'ANKARA

Series A1:Mathematics and Statistics

Volume: 71 Number: 4 Year:2022

Research Articles