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Asymptotic Bound for RSA Variant with Three Decryption Exponents

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Abstract

This paper presents a cryptanalysis attack on the RSA variant with modulus $N = p^r q$ for $r \geq 2$ with three public and private exponents (e_1, d_1), (e_2, d_2), (e_3, d_3) sharing the same modulus N where p and q are consider to be primes having the same bit size. Our attack shows that we get the private exponent $\sigma_1\sigma_2\sigma_3 < \left(\frac{r-1}{r+1}\right)^4$, which makes the modulus vulnerable to Coppersmith's attacks and can lead to the factorization of N efficiently where $d_1 < N^{\sigma_1}$, $d_2 < N^{\sigma_2}$, and $d_3 < N^{\sigma_3}$. The asymptotic bound of our attack is greater than the bounds for May [1], Zheng and Hu [2], and Lu et al. [3] for $2 \leq r \leq 10$ and greater than Sarkar's [4] and [5] bounds for $5 \leq r \leq 10$.

1. Introduction

The importance of keeping information secret cannot be overemphasized, especially in this digital era where intruders can easily eavesdrop on someone's information and get access to his private belongings. The Construction of strong encryption scheme(s) using complex mathematics provides confidentiality and privacy to our daily transactions and communication as they pass through insecure communication channels.
The most acceptable and widely used public key cryptosystem is the RSA cryptosystem which was invented in 1976 by Rivest, Shamir, and Adleman [6]. The security of RSA modulus $N = pq$ relies on the integer factorization problem and was first exploited using a private exponent attack by Wiener (1990) as reported in [7]. Other cryptanalysis attacks that led to the polynomial time factorization of the RSA modulus $N = pq$ can be found in [8, 9].
In order to improve the security of standard RSA modulus $N = pq$, various researchers proposed many variants. Prime power modulus $N = p^r q$ for $r \geq 2$ was among the RSA variants developed by Takagi using the Chinese remainder theorem showing that the decryption process is faster than the standard RSA [10]. Also, Boneh et al. presented a partial exposure attack where they proved that prime power modulus $N = p^r q$ can be efficiently factored if someone knows $\frac{1}{r+1}$ fraction of the most significant bits (MSBs) of the prime factors p [11].
The decryption exponent bound of [10] was improved from $d < N^{\frac{1}{2(r+1)}}$ to $d < N^{\frac{r}{(r+1)^2}}$ or $d < N^{\left(\frac{r-1}{r+1}\right)^2}$ by May [1] using the lattice-based technique. Sarkar [4] presented a small secret exponent attack on prime power modulus $N = p^r q$ for $r \geq 2$ where he improved the work of [1] for $r \leq 5$. Similarly, Sarkar improved his work [4] when $2 \leq r \leq 8$ as reported in [5] with a decryption exponent bound of $d < N^{\frac{1}{r+1} + \frac{3r-2\sqrt{3r+3}+3}{3(r+1)}}$. Lu et al. [3] proved that prime power modulus $N = p^r q$ when $r \geq 2$ can be factored efficiently when the decryption exponent bound $d < N^{\frac{r(r-1)}{(r+1)^2}}$. Moreover, Zheng and Hu [2] proposed a cryptanalysis lattice-based construction attack on prime power RSA modulus $N = p^r q$ for $r \geq 2$ with two decryption exponents where they have shown that N is insecure when $\delta_1\delta_2 < N^{\left(\frac{r-1}{r+1}\right)^3}$ where $d_1 < N^{\delta_1}$ and $d_2 < N^{\delta_2}$. By assuming $\delta_1 = \delta_2 = \delta$, [2] made comparisons with previous results of [1, 4] when $r \geq 4$.
In this paper, we employ a similar approach to [2] using lattice-based approach except that we utilize three pairs of public and private exponents (e_1, d_1), (e_2, d_2), and (e_3, d_3) of RSA variant $N = p^r q$ for $r \geq 2$ with three decryption exponents sharing common modulus N, and prove that the security of prime power moduli N can be broken and prime factors p and q can be factored in polynomial-time. We assume $d_1 = N^{\sigma_1}$, $d_2 = N^{\sigma_2}$ and $d_3 = N^{\sigma_3}$ to be the decryption exponents where $d_1 = d_2 = d_3 = d = \sigma$ for $0 < \sigma < 1$ and utilize generalized key

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equation $e_i d_i = 1 + k_i \phi(N)$, where $k_i \in \mathbb{Z}$ and $\phi(N) = p^{r-1}(p-1)(q-1)$ for the construction of three equations of the form

$$e_1 d_1 = 1 + k_1 p^{r-1}(p-1)(q-1), \tag{1.1}$$
$$e_2 d_2 = 1 + k_2 p^{r-1}(p-1)(q-1), \tag{1.2}$$
$$e_3 d_3 = 1 + k_3 p^{r-1}(p-1)(q-1), \tag{1.3}$$

for some positive integers k_1, k_2, k_3. Let e'_1, e'_2, e'_3 be the inverses of $e_1, e_2, e_3 \mod N$ respectively. Then we get:

$$e_1 e'_1 = z_1 N + 1, \tag{1.4}$$
$$e_2 e'_2 = z_2 N + 1, \tag{1.5}$$
$$e_3 e'_3 = z_3 N + 1, \tag{1.6}$$

for some positive integers z_1, z_2, z_3. In order to easily get the prime factors of N, we assume that inverses e'_1, e'_2, or e'_3 does not exist, we can then get the result through finding the $\gcd(e_1, N)$, $\gcd(e_2, N)$ and $\gcd(e_3, N)$. Multiplying equations (1.1) by e'_1 and (1.4) by d_1 respectively and equating them give

$$d_1 - e'_1 = [e'_1 k_1 (p-1)(q-1) - d_1 z_1 pq] p^{r-1}. \tag{1.7}$$

Similarly, for equations (1.2) and (1.5) we get the following equation

$$d_2 - e'_2 = [e'_2 k_2 (p-1)(q-1) - d_2 z_2 pq] p^{r-1} \tag{1.8}$$

Also, for equations (1.3) and (1.6), we get the following equation

$$d_3 - e'_3 = [e'_3 k_3 (p-1)(q-1) - d_3 z_3 pq] p^{r-1}. \tag{1.9}$$

Equations (1.7), (1.8) and (1.9) reduce to the following equations respectively

$$d_1 - e'_1 = 0 \mod p^{r-1}, \tag{1.10}$$
$$d_2 - e'_2 = 0 \mod p^{r-1}, \tag{1.11}$$
$$d_3 - e'_3 = 0 \mod p^{r-1}. \tag{1.12}$$

Applying method of [12] for solving multivariate linear equations modulo unknown divisor, we can estimate the unknown divisor of our attacks. Since the modulus is $N = p^r q$ for $r \geq 2$ and $q < p < 2q$. Multiplying by p^r gives $N < p^{r+1} < 2N$. Since $q \approx p \approx N^{\frac{1}{r+1}}$, we have $p^{r-1} \approx N^{\frac{r-1}{r+1}}$.

Moreover, the Coppersmith technique will be deployed in finding small roots of the constructed modular equations which can later be transformed into finding them over integers. This can be achieved through constructing a set of polynomials sharing common root modulo R to produce some integer linear combinations of the constructed polynomials' coefficient vectors whose norm is expected to be sufficiently small using the LLL algorithm. This enables us to get an asymptotic bound $\sigma < \left(\frac{r-1}{r+1}\right)^{\frac{4}{3}}$, where $d_1 < N^{\sigma_1}$, $d_2 < N^{\sigma_2}$, $d_3 < N^{\sigma_3}$. Also, we assume $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ in order to compare our results with the theoretical results of [1], [2], [3], [4] and [5], our work show that for $5 \leq r \leq 10$ we obtain better bounds.

The rest of the paper is organised as follows. In section 2, we give definitions of lattice and determinant, some important theorems and a lemma to be used in this research. Section 3 presents the major contributions of this paper where results are thoroughly discussed and comparisons of theoretical bounds with earlier reported bounds are presented. Finally, in Section 4 we conclude the paper.

2. Preliminaries

In this section, we define some basic terms that are found to be useful in this research work.

Definition 2.1 (Lattice)**.** *A lattice $\mathscr{L}$ is a discrete additive subgroup of $\mathbb{R}^m$. Let $b_1, \cdots, b_n \in \mathbb{R}^m$ be $n \leq m$ linearly independent vectors. The lattice generated by $\{b_1, \cdots, b_n\}$ is the set*

$$\mathscr{L} = \sum_{i=1}^{n} \mathbb{Z} b_i = \left\{ \sum_{i=1}^{n} x_i b_i | x_i \in \mathbb{Z} \right\}.$$

The set $B = \langle b_1, \cdots, b_n \rangle$ is called a lattice basis for $\mathscr{L}$. The lattice dimension is $dim(\mathscr{L}) = n$. If $n = m$ then $\mathscr{L}$ is said to be a full rank lattice.

A lattice $\mathscr{L}$ can be represented by a basis matrix. Given a basis B, a basis matrix M for the lattice generated by B is the $n \times m$ matrix defined by the rows of the set $b_1 ..., b_n$

$$M = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

It is often useful to represent the matrix M by B. A very important notion for the lattice $\mathscr{L}$ is the determinant [13].

Definition 2.2 (Determinant [13])**.** *Let $\mathscr{L}$ be a lattice generated by the basis $B = \langle b_1, ..., b_n \rangle$. The determinant of $\mathscr{L}$ is defined as*

$$det(\mathscr{L}) = \sqrt{det(BB^T)}.$$

If $n = m$, we have

$$det(\mathscr{L}) = \sqrt{det(BB^T)} = |det(B)|.$$

Theorem 2.3 ([2], [14]). *Let L be a lattice spanned by a basis $(b_1, b_2, \cdots, b_m)$. The Lenstra-Lenstra-Lovasz (LLL) algorithm outputs a reduced basis $(v_1, v_2, \cdots, v_m)$ of L in polynomial time that satisfies*

$$||V_1||, ||V_2||, \cdots, ||V_m|| \leq 2^{\frac{m(m-1)}{4(m+1-i)}} det(L)^{\frac{1}{(m+1-i)}}$$

for $1 \leq i \leq m$.

For $i = 3$, the above LLL equation becomes

$$||V_1||||V_2||||V_m|| \leq 2^{\frac{m(m-1)}{4(m-2)}} det(L)^{\frac{1}{(m-2)}}.$$

Lemma 2.4 ([15]). *Let $g(x_1, x_2, \cdots, x_n) \in \mathbb{Z}[x_1, x_2, \cdots, x_n]$ be an integer polynomial that is a sum of at most m monomials. Suppose that*

1. $g(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) \equiv 0(\mod R)$, *where* $|x_1^{(0)}|X_1, \cdots, |x_n^{(0)}| < X_n$,
2. $||g(x_1X_1, x_2X_2, \cdots, x_nX_n)|| < \frac{R}{\sqrt{m}}$.

   This can also be true over the integers $(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) = 0$.

Thus we can solve the polynomials derived from the LLL algorithm. Consider the three basis vectors by the LLL algorithm, the condition for finding common root over the integers is as follows

$$2^{\frac{m(m-1)}{4(m-2)}} det(L)^{\frac{1}{(m-2)}} < \frac{R}{\sqrt{m}},$$
$$2^{\frac{m(m-1)}{4(m-2)}} det(L) < R^{m-2} M^{-\frac{m-2}{2}},$$
$$det(L) < R^{m-2} M^{-\frac{m-2}{2}} 2^{-\frac{m(m-1)}{4(m-2)}}.$$

Since we usually have $m < R$, an error term ε is used on behalf of the small terms except R^m, then the above equation reduces to $det(L) < R^{m-\varepsilon}$.

We obtain a lower triangular basis matrix in our method all the time. The determinant can be calculated as $det(L) = N^{sN} X_1^{s_1} X_2^{s_2} X_3^{s_3}$ where s_i denotes the sum of the total exponents of X_i or N that appears on the diagonal. Hence we give the following condition

$$N^{sN} X_1^{s_1} X_2^{s_2} X_3^{s_3} < R^m. \tag{2.1}$$

3. Results

This section presents the major findings of this paper. The discussion is as follows:

To solve equations (1.10-1.12), we apply shift polynomials technique for a positive integer u as define below:

$$pj_1, pj_2, pj_3(x_1, x_2, x_3) = (x_1 - e'_1)^{j_1} (x_2 - e'_2)^{j_2} (x_3 - e'_3)^{j_3} N^{\max(u-j_1-j_2-j_3, 0)}$$

where $|x_1| < X_1,\ |x_2| < X_2,\ |x_3| < X_3$.

So all the polynomials $pj_1, pj_2, pj_3(x_1, x_2, x_3)$ share common root $(d_1, d_2, d_3) \mod p^{u(r-1)}$. The optimal condition for choosing the shift polynomials is given in [12], thus applying it in our case with three unknown private keys we have

$$0 \leq \sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 \leq \frac{r-1}{r+1} u.$$

When we consider a general case where $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, we get a more concise condition as

$$0 \leq j_1 + j_2 + j_3 \leq \left(\frac{r-1}{r+1}\right) \frac{u}{\sigma}.$$

Taking $u = r = 3$, we can search for integer linear combinations of all

$$pj_1, pj_2, pj_3(x_1X_1, x_2X_2, x_3X_3)$$

by the LLL algorithm and ensure that its norm is sufficiently small to satisfy the conditions of Lemma 2.4. Thus, we have

$$pj_1, pj_2, pj_3(x_1, x_2, x_3) = (x_1 - e'_1)^{j_1} (x_2 - e'_2)^{j_2} (x_3 - e'_3)^{j_3} N^{\max(u-j_1-j_2-j_3, 0)}.$$

Using the above equation, we derive the following monomials:

$p(i_1,i_2,i_3)$	1	x_1	x_2	x_3	x_1x_2	x_2x_3	x_1x_3	x_1^2	x_2^2	x_3^2	x_1^3	$x_1x_2x_3$	$x_1^2x_2$	$x_2^2x_3$	$x_2x_3^2$	x_2^3	x_3^3	x_1^4	$x_2^3x_3$	$x_2x_3^3$	$x_2^2x_3^2$
$p(0,0,0)$	N^3																				
$p(1,0,0)$	$*$	X_1N^2																			
$p(0,1,0)$	$*$	$*$	X_2N^2																		
$p(0,0,1)$	$*$	$*$	$*$	X_3N^2																	
$p(1,1,0)$	$*$	$*$	$*$	$*$	X_1X_2N																
$p(0,1,1)$	$*$	$*$	$*$	$*$	$*$	X_2X_3N															
$p(1,0,1)$	$*$	$*$	$*$	$*$	$*$	$*$	X_1X_3N														
$p(2,0,0)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_1^2N													
$p(0,2,0)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_2^2N												
$p(0,0,2)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_3^2N											
$p(3,0,0)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_1^3										
$p(1,1,1)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_1X_2X_3$									
$p(0,2,1)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_1^2X_2$								
$p(0,2,1)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_2^2X_3$							
$p(0,1,2)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_2X_3^2$						
$p(0,3,0)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_2^3					
$p(0,0,3)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_3^3				
$p(4,0,0)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	X_1^4			
$p(0,3,1)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_2^3X_3$		
$p(0,1,3)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_2X_3^3$	
$p(0,2,3)$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$X_2^2X_3^3$

Table 3.1: Monomials

Taking u as a given parameter, the dimension m of the full-rank lattice can be calculated which can further allow us to compute $\det(L)$. This can be computed by enumerating the exponential numbers of X_1, X_2, X_3 and N respectively from the lower triangular square matrix s depicted above. Thus we get

$$m = \sum_{\sigma_1 j_1+\cdots+\sigma_n j_n}^{1} 1 = \frac{u^n}{n!}\frac{\beta^n}{\sigma_1\cdots\sigma_n} + o(u^n), \quad \beta = \frac{r-1}{r+1}.$$

So, in our case $m = n = 3$, we have

$$m = \sum_{\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3}^{\frac{r-1}{r+1}u} 1 = \frac{1^3}{6}\frac{\left(\frac{r-1}{r+1}u\right)^3}{\sigma_1\sigma_2\sigma_3} = \frac{1}{6\sigma_1\sigma_2\sigma_3}\left(\frac{r-1}{r+1}u\right)^3 + o(u^3).$$

Also, to compute u_N we can use similar method as outlined in [2] and [12]. Thus, we have

$$\begin{aligned}
u_N &= \sum_{i_1+i_2+\cdots+j_n=0}^{s}\left(\sum_{i=j}^{n} j_i + n - 1\right)\left(u - \sum_{i=1}^{n} j_i\right) = \frac{u^{n+1}}{(n+1)!} + o(u^{n+1}),\\
u_N &= \frac{1}{4!}u^4 + o(u^4) = \frac{1}{24}u^4 + o(u^4),\\
u_n &= \sum_{\sigma_1+\sigma_2+\cdots\sigma_j n=0} jn = \frac{u^{n+1}}{(n+1)!}\frac{\beta^{n+1}}{\sigma_1\cdots\sigma_{i-1}\sigma_j^2\sigma_i+\sigma_n} + o(u^{n+1}),\\
u_1 &= \sum_{\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}^{\frac{r-1}{r+1}u} j_1 = \frac{1^4}{24}\frac{\left(\frac{r-1}{r+1}u\right)^4}{\sigma_1^2\sigma_2\sigma_3} = \frac{1}{24\sigma_1^2\sigma_2\sigma_3}\left(\frac{r-1}{r+1}u\right)^4 + o(u^4),\\
s_2 &= \sum_{\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}^{\frac{r-1}{r+1}u} j_2 = \frac{1^4}{24}\frac{\left(\frac{r-1}{r+1}u\right)^4}{\sigma_1\sigma_2^2\sigma_3} = \frac{1}{24\sigma_1\sigma_2^2\sigma_3}\left(\frac{r-1}{r+1}u\right)^4 + o(u^4),\\
s_3 &= \sum_{\sigma_1 u_1+\sigma_2 u_2+\sigma_3 u_3=0}^{\frac{r-1}{r+1}u} j_3 = \frac{1^4}{24}\frac{\left(\frac{r-1}{r+1}u\right)^4}{\sigma_1\sigma_2\sigma_3^2} = \frac{1}{24\sigma_1\sigma_2\sigma_3^2}\left(\frac{r-1}{r+1}u\right)^4 + o(u^4).
\end{aligned}$$

Since, we have $det(L) = N^{un}X_1^{u_1}X_2^{u_2}X_3^{u_3}$ for $X_1 = N^{\sigma_1}$, $X_2 = N^{\sigma_2}$, $X_3 = N^{\sigma_3}$ as mentioned above.The norms of the first three vectors can be sufficiently small only if the condition for finding the common root is fulfilled as derived from LLL-reduced basis. This can further be transformed using Lemma 2.4 into the corresponding polynomials with same root and lastly solve for the integers (d_1, d_2, d_3) We can now estimate σ_1, σ_2, σ_3. Using equation 2.1, we have

$$N^{\frac{1}{24}s^4+o(u^4)}N^{\sigma_1\frac{1}{24\sigma_1^2\sigma_2\sigma_3}\left(\frac{r-1}{r+1}u\right)^4+o(u^4)}N^{\sigma_2\frac{1}{24\sigma_1\sigma_2^2\sigma_3}\left(\frac{r-1}{r+1}u\right)^4+o(u^4)}N^{\sigma_3\frac{1}{24\sigma_1\sigma_2\sigma_3^2}\left(\frac{r-1}{r+1}u\right)^4+o(u^4)} < N^{\frac{r-1}{r+1}u\frac{1}{6\sigma_1\sigma_2\sigma_3}\left(\frac{r-1}{r+1}u\right)^3+o(u^3)}.$$

Taking $u \to \infty$ and omitting the lower term $o(u^3)$ gives the following result

$$\begin{aligned}
\frac{1}{24} + \frac{1}{24\sigma_1\sigma_2\sigma_3}\left(\frac{r-1}{r+1}\right)^4 + \frac{1}{24\sigma_1\sigma_2^2\sigma_3}\left(\frac{r-1}{r+1}\right)^4 + \frac{1}{24\sigma_1\sigma_2\sigma_3^2}\left(\frac{r-1}{r+1}\right)^4 &< \frac{1}{6\sigma_1\sigma_2\sigma_3}\left(\frac{r-1}{r+1}\right)^4\\
\sigma_1\sigma_2\sigma_3 &< \left(\frac{r-1}{r+1}\right)^4
\end{aligned}$$

In order to make comparison with other bounds, we assume $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ as shown in Table 3.2. It gives asymptotic bound of $\sigma < \left(\frac{r-1}{r+1}\right)^{\frac{4}{3}}$.

r	$\left(\frac{r-1}{r+1}\right)^{\frac{4}{3}}$	[1]	[2]	[3]	[4]	[5]
2	0.231	0.222	0.192	0.222	0.395	0.395
3	0.396	0.250	0.353	0.375	0.461	0.410
4	0.506	0.360	0.464	0.480	0.508	0.437
5	0.582	0.444	0.544	0.550	0.545	0.464
6	0.638	0.510	0.603	0.610	0.574	0.489
7	0.681	0.562	0.649	0.65	0.598	0.512
8	0.715	0.605	0.685	0.690	0.619	0.532
9	0.742	0.640	0.715	0.720	0.637	0.549
10	0.868	0.669	0.740	0.743	0.653	0.565

Table 3.2: Comparison of Bounds

From Table 3.2, one can observe that, our bound is better than [2], [4] and [5] for $r \geq 2$ and also better than all the compared bounds for $5 \leq r \leq 10$.

4. Conclusion

This paper shows that prime power RSA modulus $N = p^r q$ for $r \geq 2$ with three decryption exponents can be attacked using lattice-based attack through combinations of Coppersmith's and [12] lattice-base construction methods. We also showed that the modulus N is insecure if $d_1 < N^{\sigma_1}$, $d_2 < N^{\sigma_2}$ and $d_3 < N^{\sigma_3}$ which yielded asymptotic bound $\sigma < \left(\frac{r-1}{r+1}\right)^{\frac{4}{3}}$. Our results is an improvement on the work of [1], [2], [3], [4] and [5].

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Set-Valued Control of Cancer by Combination Chemotherapy

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Abstract

A mathematical model of ordinary differential equations is considered to analyze the pharmacokinetics of multi-chemotherapeutic drugs and their pharmacodynamic effects on homogeneous tumors. Set-valued analysis is used to design protocols of drug administration and applied to decrease tumor density under their carrying capacity of Gompertz growth and converge to zero.

1. Introduction

Several works were carried out on cancer control by the combination of multi-chemotherapeutic agents to have more effects on tumor cells, and their density [1]. Uses multi-objective optimization method to minimize the area under the curve of tumors as well as the side effects on the patient during chemotherapy [2]. Introduces an adaptive neural networks control approach, based on feedback linearization, in order to optimize chemotherapy regimens [3]. Develops optimal therapeutic strategies, subject to reducing tumor size and toxicity throughout treatment [4]. Employs swarm intelligence for optimization of cancer chemotherapy [5]. Uses evolutionary algorithms to minimize tumor and maximize patient survival time [6]. Applies genetic algorithms to eradicate tumor [7]. Computes the optimal doses of CAF (Cyclophosphamide, Adriamycin, and Fluorouracil) regimen for each patient suffering with breast cancer stage IIB in adjuvant chemotherapy [8]. Develops a mixed-integer program for combination chemotherapy optimization to reduce the number of cancer cells in the body [9]. Deals with the optimisation of multi-drug chemotherapy in order to better cope with the occurrence of drug-resistant cancer cells [10]. Subjects a multi-drug chemotherapy schedule optimisation problem to local optima network.
In this work, we adapt the set-valued analysis methods developed in the previous works [11–18], to approach a model of combined chemotherapy control in cancer, and make the solution viable on decreasing subset, with converging tumor density towards zero [11]. Investigates a general class of immunotherapy ODE models and gives some numerical examples [12]. Evokes viability and set-valued theories to provide chemotherapy protocol laws [13]. Illustrates the approach by two applications on anti-angiogrnic therapy and tumor-immune with chemotherapy [14]. Generalizes the method to anti-angiogenic therapy with chemotherapy [15]. Treats the problem of cancer control by chemotherapy through a general model in ordinary differential equation form of tumor dynamics [16]. Analysis a tuberculosis (TB) infection model with the treatment of four ordinary differential equations, namely, susceptible, latent, infected, and treated individuals [17]. Proposes an extension of the classical SEIR-type models to describe and control the spread of COVID-19 in Morocco [18]. Controls general class of ordinary differential equations that model the temporal evolution of diseases spread and applies the approach to a SIRS model for several diseases such that influenza and malaria.
The rest of this paper is organized as follows: Section 2 lunches the general model and states the associated viability problem. Section 3 approaches the problem with some tools of the set-valued analysis. Section 4 figures some numerical calculus of analytical results on a model example. Section 5 concludes the paper.

2. General Model and Problem Formulation

Pharmacokinetiks of chemotherapeutic drugs

$$u \in U = \prod_{i=1}^{n} \left[u_i^{\min}, u_i^{\max}\right],$$

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and their pharmacodynamics on tumor density

$$\tau \in \mathbb{R}_+ = [0,\infty),$$

are modeled by the coupled ordinary differential equations

$$\begin{aligned} \dot{\tau} &= \psi(\tau) - G(u)\tau, \text{ with } \tau(0) = \tau_0 \in \mathbb{R}_+^*, & (2.1a)\\ \dot{u} &= f(u,v), \text{ with } u(0) = u_0 \in U, & (2.1b) \end{aligned}$$

with the explicit expressions of the functions ψ and G in (2.1a)

$$\begin{aligned} \psi(\tau) &= -\xi\tau\ln\left(\frac{\tau}{\theta}\right), & (2.1c)\\ G(u) &= \sum_{1\leq i\leq n}\kappa_i u_i + \sum_{1\leq i<j\leq n}\kappa_{ij}u_i u_j, & (2.1d) \end{aligned}$$

where in (2.1c) ξ and θ are the parameters of the Gompertz growth function, and in (2.1d) κ_i is the effectiveness coefficient of the i-th drug, while κ_{ij} is the coefficient of the potentialization in drug cytotoxicity induced by the presence of i-th and j-th drugs.
And with the explicit expression of the vector function f in (2.1b)

$$f(u,v) = \left(-f_1 u_1 + \frac{v_1}{V_1}, \cdots, -f_n u_n + \frac{v_n}{V_n}\right)', \tag{2.1e}$$

where the parameters V_i are the volumes of distribution, and the parameters f_i are the elimination rates, and the input functions $v_i(t)$ are the protocol administration, associated to the compartments u_i.
We have to find input control function v, expressing the protocol administration, and satisfying the constraint

$$\forall t \in [0,\infty),\ v(t) \in V = \prod_{i=1}^{n}\left[f_i V_i u_i^{\min}, f_i V_i u_i^{\max}\right], \tag{2.2a}$$

by which the tumor density τ is as follows

$$\lim_{t\to\infty}\tau(t) = 0. \tag{2.2b}$$

We will formulate the control problem (2.2) in the framework of the viability theory [19].
To each real number $\alpha > 0$, we define the function

$$\psi_\alpha(\tau,u) = \psi(\tau) - G(u)\tau + \alpha\tau, \tag{2.3a}$$

where the functions ψ and G still given by (2.1c) and (2.1d) respectively, and we associate the subset

$$D_\alpha = \{(\tau,u) \in \mathbb{R}_+ \times U \mid \psi_\alpha(\tau,u) \leq 0\}. \tag{2.3b}$$

Proposition 2.1. *Let be α such that $(\tau_0,u_0) \in D_\alpha$.*
If the system (2.1) is globally viable in the subset D_α by a control $v : [0,\infty) \to V$, then v is a protocol in the sense of the problem (2.2).

Proof. Let $t \geq 0$.
By (2.1a) and (2.3) we have the differential inequality

$$\dot{\bar{\tau}}(t) = \psi(\bar{\tau}(t)) - G(\bar{u}(t))\bar{\tau}(t) \leq -\alpha\bar{\tau}(t),$$

and by applying Gronwall's Lemma we get the exponential estimate

$$0 \leq \bar{\tau}(t) \leq \tau_0\exp(-\alpha t),$$

then

$$\lim_{t\to\infty}\bar{\tau}(t) = 0.$$

□

3. Set-Valued Approach

We associate with the system (2.1), the regulation map F_α defined on the subset D_α (2.3b) in the following way

$$F_\alpha(\tau,u)=\{v\in V \mid (\psi(\tau)-G(u)\tau, f(u,v))'\in T_{D_\alpha}(\tau,u)\}, \tag{3.1a}$$

where

$$T_{D_\alpha}(\tau,u)=\left\{(\hat{\tau},\hat{u})\in\mathbb{R}\times\mathbb{R}^n \mid \liminf_{h\downarrow 0}\frac{d_{D_\alpha}(\tau+h\hat{\tau},u+h\hat{u})}{h}\right\}, \tag{3.1b}$$

stands for the tangent cone to the subset D_α at point (τ,u).

Lemma 3.1. *Let be α such that $(\tau_0,u_0)\in D_\alpha$.*
The system (2.1) is locally viable in the subset D_α, if and only if for all $(\tau,u)\in D_\alpha$ there exists $v_\alpha\in V$ such that

$$(\psi(\tau)-G(u)\tau, f(u,v_\alpha))'\in T_{D_\alpha}(\tau,u). \tag{3.2}$$

i.e., if and only if the regulation map F_α is strict.

Corollary 3.2. *Let be α such that $(\tau_0,u_0)\in D_\alpha$.*
If the regulation map F_α admits a single-valued selection v_α, then the system (2.1) is globally viable in the subset D_α by the protocol v_α.

Proof. Let be α such that $(\tau_0,u_0)\in D_\alpha$, and $v_\alpha : D_\alpha\to V$ a single-valued selection of the regulation map F_α.
According to the Lemma 3.1, the system (2.1) under the depending state control $v=v_\alpha(\tau,u)$, admits to a local viable solution $(\bar{\tau},\bar{u})$ in the subset D_α, over a maximal time interval $[0,\bar{t})$.
We have to prove that $\bar{t}\to\infty$:
As $\bar{\tau}$ is a non-negative decreasing function, then $\bar{\tau}(t)$ has a limit denoted by $\bar{\tau}(\bar{t})$ when $t\to\bar{t}^-$.
By (2.1b), (2.1e), and (2.2a) we have

$$\|\dot{\bar{u}}(t)\|\le\|f\|\|\bar{u}(t)\|+\|f\|\|u^{\max}\|,$$

then by applying Gronwall's Lemma we get the exponential estimate

$$\|\bar{u}(t)\|\le(\|u_0\|+\|u^{\max}\|)\exp(\|f\|t),$$

then $\bar{u}(t)$ has a limit denoted by $\bar{u}(\bar{t})$ when $t\to\bar{t}^-$.
Therefore

$$(\bar{\tau}(t),\bar{u}(t))\to(\bar{\tau}(\bar{t}),\bar{u}(\bar{t}))\text{ when } t\to\bar{t}^-,$$

and $(\bar{\tau}(\bar{t}),\bar{u}(\bar{t}))$ belongs to D_α because it is a closed subset.
Now, by considering $(\bar{\tau}(\bar{t}),\bar{u}(\bar{t}))$ as an initial state to the system (2.1), it follows that $(\bar{\tau},\bar{u})$ may be prolonged to a viable solution $(\bar{\tau},\bar{u})$ in D_α, starting at $(\bar{\tau}(\bar{t}),\bar{u}(\bar{t}))$ on some interval $[\bar{t},t^{\max})$ where $t^{\max}>\bar{t}$, which is in contradiction with the maximality of $\bar{t}$, then the solution $(\bar{\tau},\bar{u})$ becomes globally viable in D_α.
Finally the Proposition 2.1 confirms that v_α is a protocol. □

Now to give an explicit expression to the tangent cone T_{D_α} (3.1b), we appeal the following Lemma

Lemma 3.3. *If the function ψ_α (2.3a) is continuously differentiable on D_α, and admits a partial derivative $\partial\psi_\alpha$ strictly negative on D_α. Then for each $(\tau,u)\in D_\alpha$ the tangent directions $(\hat{\tau},\hat{u})$ of $T_{D_\alpha}(\tau,u)$ are characterized by*

$$\hat{u}_i\ge 0 \text{ if } u=u_i^{\min},\text{ for } i=1,\cdots,n, \tag{3.3a}$$
$$\hat{u}_i\le 0 \text{ if } u=u_i^{\max},\text{ for } i=1,\cdots,n, \tag{3.3b}$$
$$\dot{\psi}_\alpha(\tau,u)(\hat{\tau},\hat{u})\le 0,\text{ if } \psi_\alpha(\tau,u)=0. \tag{3.3c}$$

Corollary 3.4. *For each $(\tau,u)\in D_\alpha$ the tangent directions $(\hat{\tau},\hat{u})$ of $T_{D_\alpha}(\tau,u)$ are characterized by the inequality*

$$\dot{\psi}_\alpha(\tau,u)(\hat{\tau},\hat{u})\le 0,\text{ if } \psi_\alpha(\tau,u)=0. \tag{3.4}$$

Proof. Thanks to the expression (2.1e)

- If $u_i=u_i^{\min}$, then

$$\begin{aligned}-f_iu+\frac{v_i}{V_i} &= -f_iu_i^{\min}+\frac{v_i}{V_i}\\ &\ge -f_iu_i^{\min}+f_iu_i^{\min}\\ &\ge 0.\end{aligned}$$

- If $u_i=u_i^{\max}$, then

$$\begin{aligned}-f_iu+\frac{v_i}{V_i} &= -f_iu_i^{\max}+\frac{v_i}{V_i}\\ &\le -f_iu_i^{\max}+f_iu_i^{\max}\\ &\le 0.\end{aligned}$$

□

To give a useful expression of the regulation map F_α (3.1a), we set the functions h_α and ℓ_α by the expressions

$$h_\alpha(\tau,u) = \left(\frac{\partial_{u_1}\psi_\alpha(\tau,u)}{V_1},\cdots,\frac{\partial_{u_n}\psi_\alpha(\tau,u)}{V_n}\right)', \tag{3.5a}$$

$$\ell_\alpha(\tau,u) = (\psi(\tau)-G(u)\tau)\partial_\tau\psi_\alpha(\tau,u) - \sum_{1\le i\le n} f_i u_i \partial_{u_i}\psi_\alpha(\tau,u). \tag{3.5b}$$

Corollary 3.5. *The regulation map F_α is expressed explicitly on the subset D_α as*

$$F_\alpha(\tau,u) = \begin{cases} V & \text{if} \quad \psi_\alpha(\tau,u)<0, \\ V_\alpha(\tau,u) & \text{if} \quad \psi_\alpha(\tau,u)=0, \end{cases} \tag{3.6a}$$

with

$$V_\alpha(\tau,u) = \{v\in V \mid \langle h_\alpha(\tau,u),v\rangle + \ell_\alpha(\tau,u)\le 0\}. \tag{3.6b}$$

Proof. For all $(\tau,u)\in D_\alpha$ we have

$$\begin{aligned} \dot\psi_\alpha(\tau,u)(\psi(\tau)-G(u)\tau,f(u,v)) &= \langle\nabla\psi_\alpha(\tau,u),(\psi(\tau)-G(u)\tau,f(u,v))'\rangle \\ &= (\psi(\tau)-G(u)\tau)\partial_\tau\psi_\alpha(\tau,u) - \sum_{1\le i\le n} f_i u_i\partial_{u_i}\psi_\alpha(\tau,u) + \sum_{1\le i\le n} v_i\frac{\partial_{u_i}\psi_\alpha(\tau,u)}{V_i}, \end{aligned}$$

then by (3.5)

$$\dot\psi_\alpha(\tau,u)(\psi(\tau)-G(u)\tau,f(u,v)) = \langle h_\alpha(\tau,u),v\rangle + \ell_\alpha(\tau,u). \tag{3.7}$$

□

Proposition 3.6. *A single-valued selection of the regulation map F_α may be given on the subset D_α by the expression*

$$v_\alpha(\tau,u) = \pi_{V_\alpha(\tau,u)}(0), \tag{3.8}$$

where π denotes the operator of best approximation.

Remark 3.7. *As Lemma 3.1, the viability of the solution $(\bar\tau,\bar u)$ demands the necessary following condition, between initial tumor density $\bar\tau(0)$ and initial control $\bar u(0)$*

$$\frac{\psi(\bar\tau(0))}{\bar\tau(0)} < G(\bar u(0)). \tag{3.9}$$

To deal with this situation, we introduce the set-valued map

$$W_\beta(\tau,u) = \{v\in V \mid \langle h(\tau,u),v\rangle + \ell(\tau,u)\le -\beta\}, \tag{3.10a}$$

where β is a non-negative real number, and the functions h and ℓ are given by the expressions

$$h(\tau,u) = \left(\frac{\partial_{u_1}\Phi(\tau,u)}{V_1},\cdots,\frac{\partial_{u_n}\Phi(\tau,u)}{V_n}\right)', \tag{3.10b}$$

$$\ell(\tau,u) = (\psi(\tau)-G(u)\tau)\partial_\tau\Phi(\tau,u) - \sum_{1\le i\le n} f_i u_i\partial_{u_i}\Phi(\tau,u), \tag{3.10c}$$

and the function Φ is given by the expression

$$\Phi(\tau,u) = \psi(\tau)-G(u)\tau, \tag{3.10d}$$

where the functions ψ and G still given by (2.1c) and (2.1d) respectively.

Theorem 3.8. *Let be (τ_0,u_0) an initial state such that $\dfrac{\psi(\tau_0)}{\tau_0}\ge G(u_0)$.*
The minimal selection w_β of the set-valued map W_β

$$w_\beta(\tau,u) = \pi_{W_\beta(\tau,u)}(0), \tag{3.11}$$

controls the system (2.1) to a final state $(\bar\tau(\bar t),\bar u(\bar t))$ such that $\dfrac{\psi(\bar\tau(\bar t))}{\bar\tau(\bar t)} < G(\bar u(\bar t))$ (3.9), on the interval $[0,\bar t]$ where $\bar t > \dfrac{\Phi(\tau_0,u_0)}{\beta}$.

Proof. By dynamic equations (2.1a) and (2.1b) we have

$$\Phi(\bar\tau(\bar t),\bar u(\bar t)) = \Phi(\bar\tau(0),\bar u(0)) + \int_0^{\bar t}\dot\Phi(\bar\tau(s),\bar u(s))(\psi(\bar\tau(s))-G(\bar u(s))\bar\tau(s),f(\bar u(s),w_\beta(s)))\,\mathrm{d}s,$$

then by the formula (3.7) we get

$$\Phi(\bar\tau(\bar t),\bar u(\bar t)) = \Phi(\tau_0,u_0) + \int_0^{\bar t}[\langle h(\bar\tau(s),\bar u(s)),w_\beta(\bar\tau(s),\bar u(s))\rangle + \ell(\bar\tau(s),\bar u(s))]\,\mathrm{d}s,$$

since w_β is a single-valued selection of the set-valued map W_β then we have

$$\Phi(\bar\tau(\bar t),\bar u(\bar t)) \le \Phi(\tau_0,u_0) - \beta\bar t,$$

as $\beta\bar t > \Phi(\tau_0,u_0)$ it follows that $\Phi(\bar\tau(\bar t),\bar u(\bar t))<0$. □

4. Particular Model and Numerical Simulation

To give numerical simulations for the analytical results of the previous section, we consider the following model from the paper [20], which describes the phamacokinetiks of Etoposide drug $u_1 \in U_1 = [u_1^{\min}, u_1^{\max}]$ and Cisplatin drug $u_2 \in U_2 = [u_2^{\min}, u_2^{\max}]$, and their pharmacodynamics on tumor the density $\tau \in \mathbb{R}_+ = [0, \infty)$

$$\begin{aligned} \dot{\tau} &= \psi(\tau) - G(\tilde{u}_1, \tilde{u}_2)\tau, && (4.1a)\\ \dot{u}_1 &= f_1(u_1, v_1), && (4.1b)\\ \dot{u}_2 &= f_2(u_2, v_2), && (4.1c) \end{aligned}$$

where the explicit expressions of the functions ψ are G are given as follows

$$\begin{aligned} \psi(\tau) &= -\xi \tau \ln\left(\frac{\tau}{\theta}\right), && (4.1d)\\ G(\tilde{u}_1, \tilde{u}_2) &= \kappa_1 \tilde{u}_1 + \kappa_2 \tilde{u}_2 + \kappa_{12} \tilde{u}_1 \tilde{u}_2, && (4.1e) \end{aligned}$$

with

$$\tilde{u}_i = [u_i - u_i^{\min}] H[u_i - u_i^{\min}], \text{ for } i = 1, 2, \tag{4.1f}$$

where $H(\cdot)$ is the Heaviside's step function

$$H[u_i - u_i^{\min}] = \begin{cases} 1, & u_i \geq u_i^{\min}, \\ 0, & u_i < u_i^{\min}, \end{cases} \tag{4.1g}$$

and f_1, and f_2 are given as follows

$$\begin{aligned} f_1(u_1, v_1) &= -f_1 u_1 + \frac{v_1}{V_1}, && (4.1h)\\ f_2(u_2, v_2) &= -f_2 u_2 + \frac{v_2}{V_2}. && (4.1i) \end{aligned}$$

The numerical values of the model parameters are grouped in the Table 1.
For the non-advanced stage of tumor $\Phi(\tau_0, u_0) < 0$, we initiate the model (4.1) at the four states $(2,0,0)$, $(2,0.1,0)$, $(2,0,0.01)$, $(2,0.1,0.01)$, to compare between single and coupled effects of chemo-therapies on the tumor density in Figure 4.1, so by the protocols of Figure 4.3, while Figure 4.2 illustrates their corresponding pharmacokinetics, concerning the viability parameter α of (2.3b) we take 20 (without unit) as numerical value. In the following scheme we combine the numerical methods of Euler by step $\bar{h} > 0$ and Uzawa of parameter $\lambda \in \mathbb{R}_+^5$ to discretize and solve the model

$$\begin{cases} \dot{\tau} = \psi(\tau) - G(\tilde{u})\tau, \\ \dot{u} = f(u, v), \\ v = v_\alpha(\tau, v) \in F_\alpha(\tau, u), \\ t_0 \in \mathbb{R}_+, (\tau_0, u_0) \in D_\alpha. \end{cases} \tag{4.2}$$

1. Initialization
   (a) $t_0 \in \mathbb{R}_+$,
   (b) $(\tau_0, u_0) \in D_\alpha$,
   (c) $\lambda^0 \in \mathbb{R}_+^5$,
2. Iteration
   (a) $t_{n+1} = t_n + \bar{h}$,
   (b) $\begin{cases} \tau_{n+1} = \tau_n + \bar{h}\left(-\xi \tau_n \ln\left(\frac{\tau_n}{\theta}\right)\right), \\ u_1^{n+1} = u_1^n + \bar{h}\left(-f_1 u_1^n + \frac{v_1^n}{V_1}\right), \\ u_2^{n+1} = u_2^n + \bar{h}\left(-f_2 u_2^n + \frac{v_2^n}{V_2}\right), \end{cases}$
   (c) $\begin{cases} v_1^n = -\lambda_5^n h_\alpha^1(\tau_n, u_n) + \lambda_3^n - \lambda_1^n, \\ v_2^n = -\lambda_5^n h_\alpha^2(\tau_n, u_n) + \lambda_4^n - \lambda_2^n, \end{cases}$
   (d) $\begin{cases} \lambda_1^{n+1} = \max(\lambda_1^n + \sigma(v_1^n - v_1^{\max}), 0), \\ \lambda_2^{n+1} = \max(\lambda_2^n + \sigma(v_2^n - v_2^{\max}), 0), \\ \lambda_3^{n+1} = \max(\lambda_3^n - \sigma v_1^n, 0), \\ \lambda_4^{n+1} = \max(\lambda_4^n - \sigma v_2^n, 0), \\ \lambda_5^{n+1} = \max(\lambda_5^n + \sigma(h_\alpha^1(\tau_n, u_n) v_1^n + h_\alpha^2(\tau_n, u_n) v_2^n + \ell_\alpha(\tau_n, u_n), 0), \text{ with } 0 < \sigma < \dfrac{2}{\|h_\alpha(\tau, u)\|}. \end{cases}$

For the advanced stage of tumor $\Phi(\tau_0,u_0)\geq 0$, we choose $(0.5,0,0)$ as initial state to the model (4.1), and parameter $\beta=0.1$ (3.10a) (without unit). Tumor density in Figure 4.5 needs the minimal time $\bar{t}=6$ (by days) of Figure 4.8, before reaching the non-advanced stage $\Phi(\tau(\bar{t}),u(\bar{t}))<0$, so by the controls of Figure 4.7. We follow the preceding algorithm to approach the minimal selection (3.11) and analyze the model

$$\begin{cases} \dot{\tau}=\psi(\tau)-G(\tilde{u})\tau, \\ \dot{u}=f(u,v), \\ v=w_\beta(\tau,v)\in W_\beta(\tau,u), \\ t\in[t_0,\bar{t}], \\ t_0\in\mathbb{R}_+, \Phi(\tau_0,u_0)\geq 0, \end{cases} \tag{4.3}$$

with the both modifications on the initialization 1. (b) and the iteration 2. (d) to

1. (b) $\Phi(\tau_0,u_0)\geq 0$, and
2. (d) $\lambda_5^{n+1}=\max(\lambda_5^n+\sigma(h_1(\tau_n,u_n)v_1^n+h_2(\tau_n,u_n)v_2^n+\ell(\tau_n,u_n)+\beta,0)$, where $0<\sigma<\dfrac{2}{\|h(\tau,u)\|}$.

Parameter	Value	Unit	Description	Reference
ξ	0.006	d^{-1}	Gompertz growth parameter	[20]
θ	1	kg	Carrying capacity	[20]
k_1	10	$d^{-1}g^{-1}.\ell$	Coffecient of u_1 effectiveness	[20]
k_2	5	$d^{-1}g^{-1}.\ell$	Coffecient of u_2 effectiveness	[20]
k_{12}	2×10^4	$d^{-1}.g^{-2}.\ell^{-2}$	Coefficient of the cytotoxicity by u_1 and u_2	[20]
f_1	2	d^{-1}	Elimination rate of u_1	[20]
f_2	0.1	d^{-1}	Elimination rate of u_2	[20]
V_1	25	ℓ	Volume of distribution for u_1	[20]
V_2	40	ℓ	Volume of distribution for u_2	[20]
$u_1^{\max}$	5	$mg.\ell^{-1}$	Upper bound of u_1	[20]
$u_2^{\max}$	10	$mg.\ell^{-1}$	Upper bound of u_2	[20]
$u_1^{\min}$	10^{-4}	$g.\ell^{-1}$	Lower bound of u_1	[20]
$u_2^{\min}$	10^{-4}	$g.\ell^{-1}$	Lower bound of u_2	[20]

Table 1: Parameter Values with Units and Descriptions

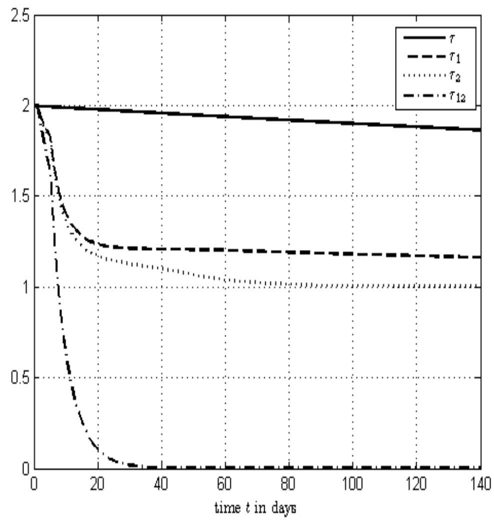

Figure 4.1: Tumors densities τ, τ_1, τ_2, and τ_{12}, under null-control $v=0$, single protocols v_α^1, v_α^2, and coupled protocol (v_α^1,v_α^2) respectively.

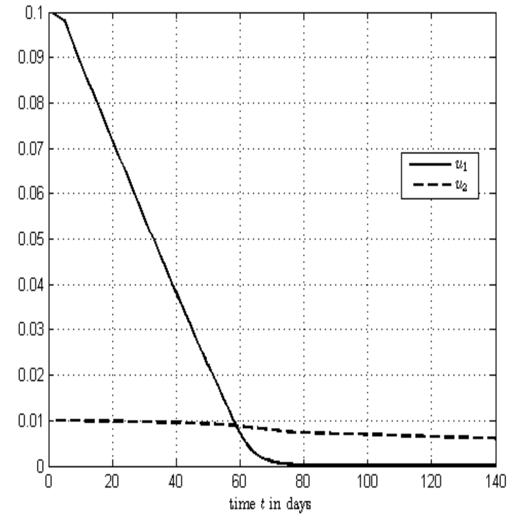

Figure 4.2: Pharmacokinetics u_1 of Etoposide and u_2 of Cisplatin

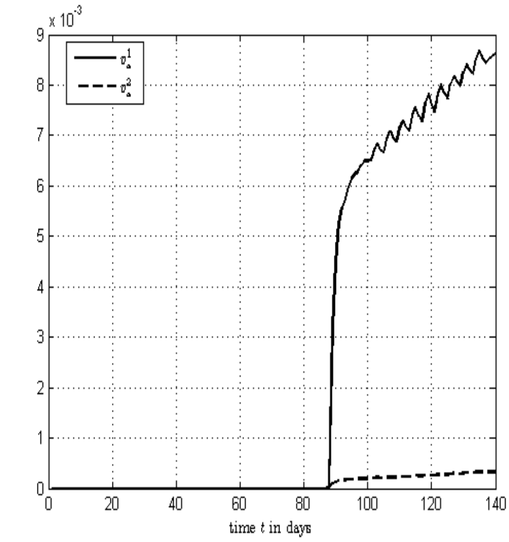

Figure 4.3: Etoposide v_α^1 and Cisplatin v_α^2 Protocols

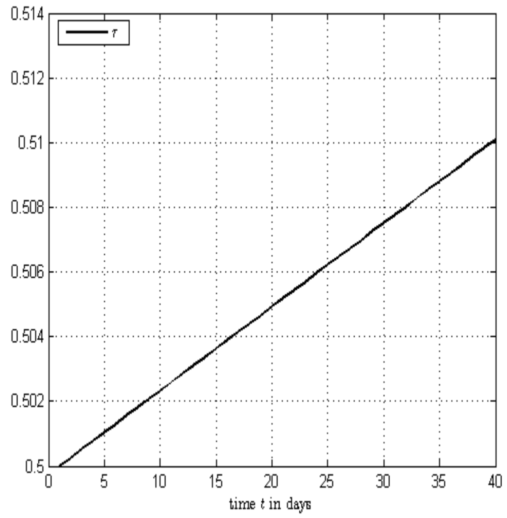

Figure 4.4: Tumor τ in Advanced Stage

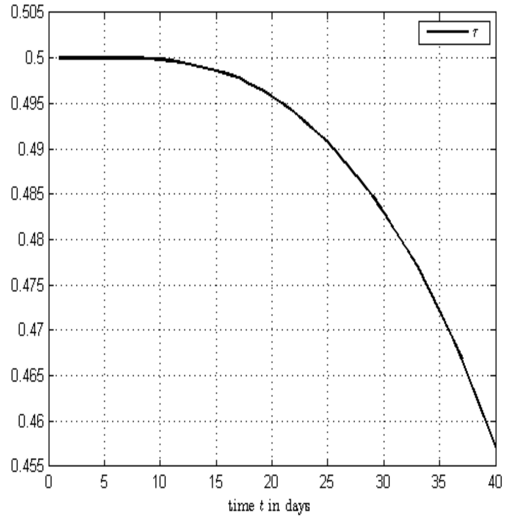

Figure 4.5: Tumor τ in Transition from Advanced Stage to Non-Advanced One

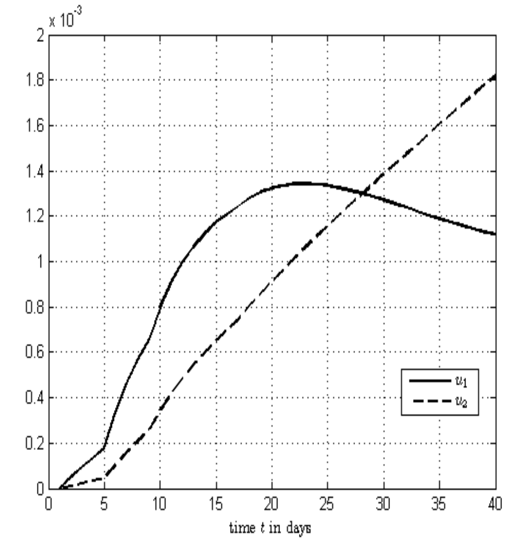

Figure 4.6: Pharmacokinetics u_1 of Etoposide and u_2 of Cisplatin for the Stages Transition

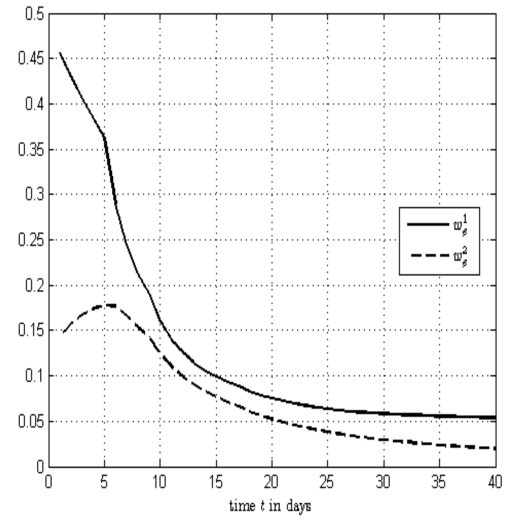

Figure 4.7: Etoposide w_β^1 and Cisplatin w_β^2 Controls of Stages Transition.

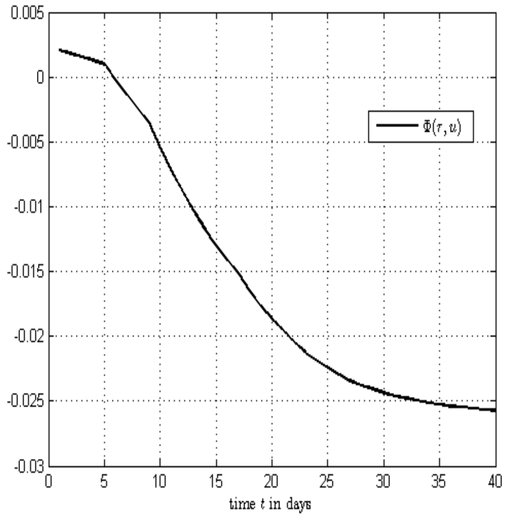

Figure 4.8: Sign of the Indicator Function Φ of the Tumor Stages and the Minimal Time $\bar{t}$

5. Conclusion

The control problem of the tumor density (2.2) is successfully approached by the set-valued analysis, the single-valued selection v_α (3.8) of the regulation map F_α (3.1a) controls the general model (2.1) to be globally viable in the subset D_α (2.3b), and strictly decreases the tumor density $\bar{\tau}$ under the carrying capacity $\theta = 1\,kg$ (2.1c) towards zero $\bar{\tau}(\infty) = 0\,kg$ (2.2b), under the exponential estimate $\bar{\tau}(t) \leq \tau_0 \exp(-\alpha t)$, for all $t \in [0,\infty)$. The protocols of the numerical model (4.2) given in Figure 4.3 are in feedback forms $v^i_\alpha = v^i_\alpha(\tau,u)$ for $i = 1,2$, and their combination provides a considerable reduction of the tumor density in Figure 4.1, where $\bar{\tau}_{12}(t) \ll \bar{\tau}_2(t) < \bar{\tau}_1(t) \ll \tau(t)$ for all $t \in [0,\infty)$, yet $\tau(\infty) = \theta \neq 0$ when there is no therapy, while $\bar{\tau}_{12}(\infty) = \bar{\tau}_2(\infty) = \bar{\tau}_1(\infty) = 0$, under mono-chemo-therapies v^1_α, and v^2_α, and multi-chemo-therapies (v^1_α, v^2_α) respectively. Nonetheless if the tumor density τ is in advanced stage $\Phi(\tau_0,u_0) \geq 0$, the minimal selection w_β (3.11) of the set-valued map W_β (3.10a) controls the general model (2.1) to the non-advanced stage $\Phi(\bar{\tau}(\bar{t}), \bar{u}(\bar{t})) < 0$ on $[0,\bar{t}]$, where the staging function Φ of cancer is given by (3.10d), which is in complete conformity with the numerical simulations of the specific model (4.3) figured by 4.4, 4.5, 4.6, 4.7, and 4.8.

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Training Data Generation for U-Net Based MRI Image Segmentation using Level-Set Methods

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Abstract

Image segmentation has been a well-addressed problem in pattern recognition for the last few decades. As a sub-problem of image segmentation, the background separation in biomedical images generated by magnetic resonance imaging (MRI) has also been of interest in the applied mathematics literature. Level set evolution of active contours idea can successfully be applied to MRI images to extract the region of interest (ROI) as a crucial preprocessing step for medical image analysis. In this study, we use the classical level set solution to create binary masks of various brain MRI images in which black color implies background and white color implies the ROI. We further used the MRI image and mask image pairs to train a deep neural network (DNN) architecture called U-Net, which has been proven to be a successful model for biomedical image segmentation. Our experiments have shown that a properly trained U-Net can achieve a matching performance of the level set method. Hence we were able to train a U-Net by using automatically generated input and label data successfully. The trained network can detect ROI in MRI images faster than the level-set method and can be used as a preprocessing tool for more enhanced medical image analysis studies.

1. Introduction

Since the deep neural networks started to perform human-level performance in the image classification tasks with ImageNet [1], many problems in computer vision have been solved by using application-specific DNN architectures. Moreover, today, it is also possible to find data-driven solutions to solve complex nonlinear partial differential equations using deep neural networks with a supervised learning approach. The recent related works [2], [3] and [4] shows the effectiveness of DNN based solution methods with some classical problems in the fields like fluid mechanics and quantum mechanics.

The most important contribution of the DNN based solution methods is to solve the related problems accurately without solving complex PDEs. Moreover, the trained DNNs make very fast predictions against previously unseen data during inference time. Hence, DNN based PDE solution methods are also more time-efficient than classical numerical and analytical solutions.

In this study, the MRI image segmentation problem is revisited. The aim is to propose a DNN based preprocessing framework that detects the region of interest in an MRI image. The idea is to find the boundary surrounding the corresponding organ in the image. We used a publicly available brain MRI image dataset [5] for the experiments.

We start with an efficient numerical solution called level-set methods. This method has been applied to various 3D computer graphics and 2D computer vision problems [6], [7]. In computer graphics, level-set methods effectively solve problems like surface reconstruction from unorganized noisy point clouds. In computer vision, this method has also been used for image segmentation problems in different digital image sources, like MRIs.

In the second phase of this study, we use the segmentation results of the renowned level-set method to train a deep neural network. Our architecture preference is the U-Net [8], a state-of-the-art DNN for image segmentation.

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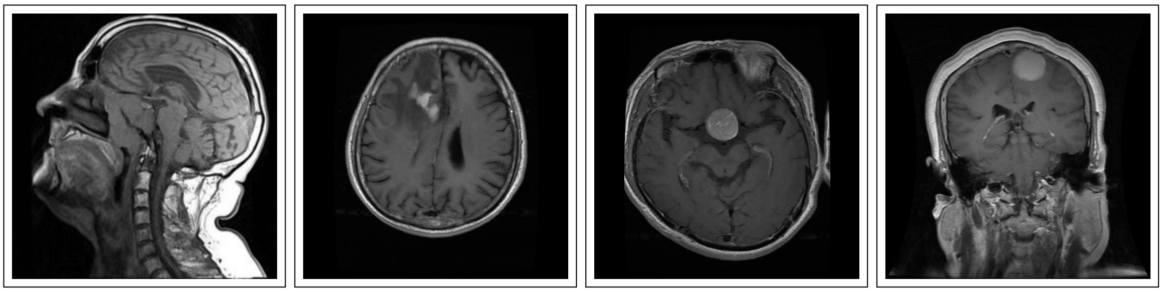
2. Material and Methods

2.1. Dataset

The data samples are the images from Brain Tumor Classification (MRI) dataset. The images are of size (256,256) RGB JPEG images. The dataset comprises four classes, three for different brain tumors called glioma, pituitary, and meningioma, and one for the normal brain. In this study, we do not address the tumor type classification problem. We instead concentrate on a more primer problem called ROI detection in brain MRI images.
In Figure 2.1, one sample for each class can be seen. The MRIs are taken from different segments and different orientations of the human skull. Hence, the sample regions in the images are not uniform in size, shape, and position. Moreover, since the images are obtained from different hardware, the images' dynamic range, resolution, and sharpness differ. These variations make the ROI detection problem a complex nonlinear problem that rule-based explicit programming approaches can not solve.



(a) No Tumor **(b)** Glioma **(c)** Pituitary **(d)** Meningioma

Figure 2.1: There are four basic classes in Brain Tumor Classification (MRI) dataset [5]. One sample selected from these classes and shown in the figure. (a) No tumor sample (b) Glioma tumor sample (c) Pituitory tumor sample (d) Meningioma tumor sample

To increase the number of samples in the dataset, we performed data augmentation by randomly combining the images and obtaining even more complex ROIs from plain brain MRIs. Some augmented samples can be seen in Figure 2.2. We primarily had 3172 images from the original dataset. After data augmentation, we increased the number of data to 6172.

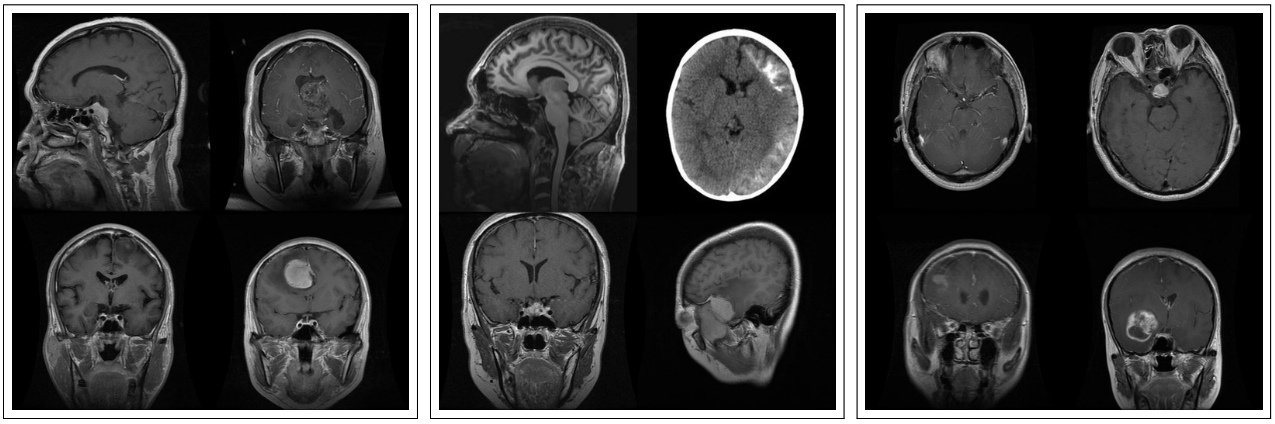


Figure 2.2: Augmented data samples

2.2. Implementation

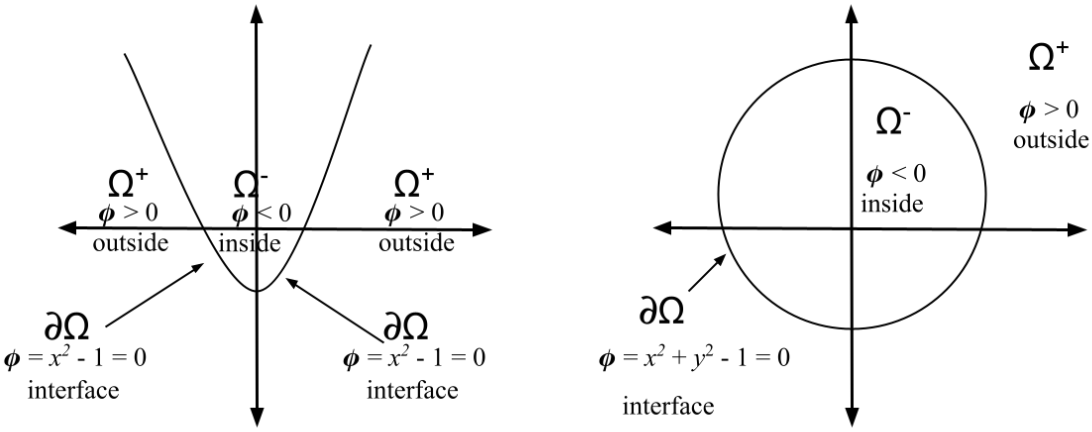
The implementation procedure comprises two main parts called Level-Set and UNet-MRI, which are publicly available at the corresponding GitHub repositories ([9], [10], respectively) for researchers and enthusiasts who want to reproduce the reported results. The U-Net repository [10] also includes trained model parameters to be directly used for ROI segmentation in MRIs.
For the U-Net training, the input images are the dataset introduced in Section 2.1. The label images are generated by using the level-set-based image segmentation procedure, which will be explained in Section 3 in detail.

3. Theory

3.1. Evolving boundaries

We will apply the level set method to an initial implicit boundary. The implicit functions make it possible to capture complex curves without even explicitly defining them analytically. The function $\phi(\overrightarrow{x})$ can be defined on $\mathbb{R}^n$ without loss of generality. The implicit representation idea can be depicted with the one-dimensional and two-dimensional cases in Figure 3.1. For $n=3$ the implicit function represents a 3D surface. For example, the implicit function $\phi(\overrightarrow{x}) = x^2+y^2+z^2-1$ represents the unit sphere boundary $\partial\Omega = \{\overrightarrow{x} \mid |\overrightarrow{x}| = 1\}$ with the

exterior region $\Omega^+ = \{\vec{x} \,|\, |\vec{x}| > 1\}$ and the interior region $\Omega^- = \{\vec{x} \,|\, |\vec{x}| < 1\}$, at $\phi(\vec{x}) = 0$ isocontour, i.e. the zero level set of ϕ. Accordingly, we will be capturing the zero level set of an evolving curve at each iteration.
We first initialize an implicit function $\phi(x)$, such that it represents a rectangular area that ensures the enclosure of ROI. We want the implicit boundary of the box to evolve in time such that eventually, it will represent the ROI. In order to capture the surface evolution in time, we add a temporal variable t to ϕ. Hence the zero level set of temporal ϕ becomes Equation (3.1).



(a) $\phi(x) = x^2 - 1$ **(b)** $\phi(x) = x^2 + y^2 - 1$

Figure 3.1: Implicit representations of two functions. (a) Implicitly defined function $\phi(x) = x^2 - 1$ defines the regions Ω^+, Ω^- and the boundary $\partial\Omega$ (b) Implicitly defined function $\phi(x) = x^2 + y^2 - 1$ defines the regions Ω^+, Ω^- and the boundary $\partial\Omega$

$$\phi(x(t), t) = 0 \tag{3.1}$$

In order to track the movement of the zero level set $\phi(x(t), t) = 0$, we have to take its derivative with respect to t. Since the implicit function represents the position, its average temporal change implies the velocity of each point in the computational domain. Considering the chain rule, derivative of Equation (3.1) becomes Equation (3.2).

$$\frac{\partial\phi}{\partial x(t)} \frac{\partial x(t)}{\partial t} + \frac{\phi}{\partial t} \quad = 0 \tag{3.2}$$

As a further interpretation, we know that $\partial\phi/\partial x$ is the gradient of the curve, i.e., $\nabla\phi$. By following the notation convention in [11], we can rewrite the equation as can be rewritten as Equation (3.3). This form is known as level set equation [12]. F is called the speed function, which will be defined over the computation domain by the gradient of the MRI image.

$$\phi_t + F|\nabla\phi| = 0 \tag{3.3}$$

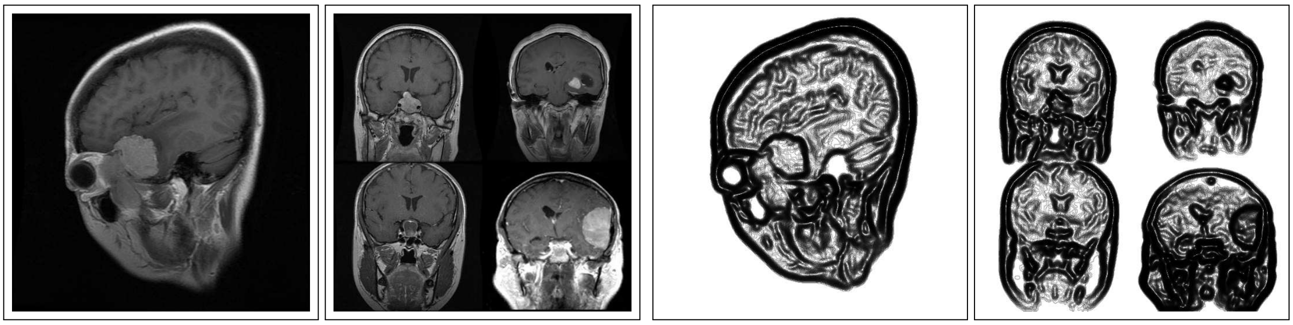
By using finite difference method, specifically forward differencing, the partial differential equation (3.3) can be reformulated as Equation (3.4). This final form is the evolution equation we will use through the iterations. At time t, ϕ' is the value of ϕ after next iteration at $t + \Delta t$.

$$\phi' = \phi + \Delta t F|\nabla\phi| \tag{3.4}$$

We want our initial surrounding box boundary to evolve so that it will eventually cover the ROI. Hence, we want the speed function F to be high outside the ROI and ideally zero at the ROI boundary. Concretely, deriving F from the edge features of the image is a convenient method. We can use the edge indicator function g in Equation (3.5), where ∇I is the gradient of the MRI image.

$$g(I) = \frac{1}{1 + ||\nabla I||^2} \tag{3.5}$$

Two samples of generated F speed function images can be seen in Figure 3.2. As a preprocessing step, we subtract the mean intensity value of the image from each pixel. It eliminates measurement noise in the dark areas and makes it possible to obtain clear speed function images. The method needs a stopping condition to end the iterations. We use the mean square error between consecutive images representing ϕ and ϕ' and stop iterations if this value is smaller than a predetermined threshold value. The overall procedure can be followed using Algorithm 1.



(a) Sample MRIs **(b)** Speed functions F

Figure 3.2: (a) Two sample MRI images from the dataset. (b) Speed function images generated from the sample MRI images in (a)

Algorithm 1 Level-set boundary evolution algorithm

I $\leftarrow$ an MRI from Dataset;
I $\leftarrow$ I - **MEAN**(I);
F $\leftarrow 1./(1.+||\nabla \mathrm{I}||^2)$;
$\phi \leftarrow$ Initial box surrounding ROI;
$\Delta \leftarrow$ A large value (e.g. 1e+15);
$\varepsilon \leftarrow$ A small value (e.g. 1e-15);
while $\Delta > \varepsilon$ **do**
  $\phi_{new} \leftarrow \phi + F|\nabla \phi|$;
  $\Delta \leftarrow$ **MSE**(ϕ_{new}, ϕ);
  $\phi \leftarrow \phi_{new}$;
end while
return ϕ;

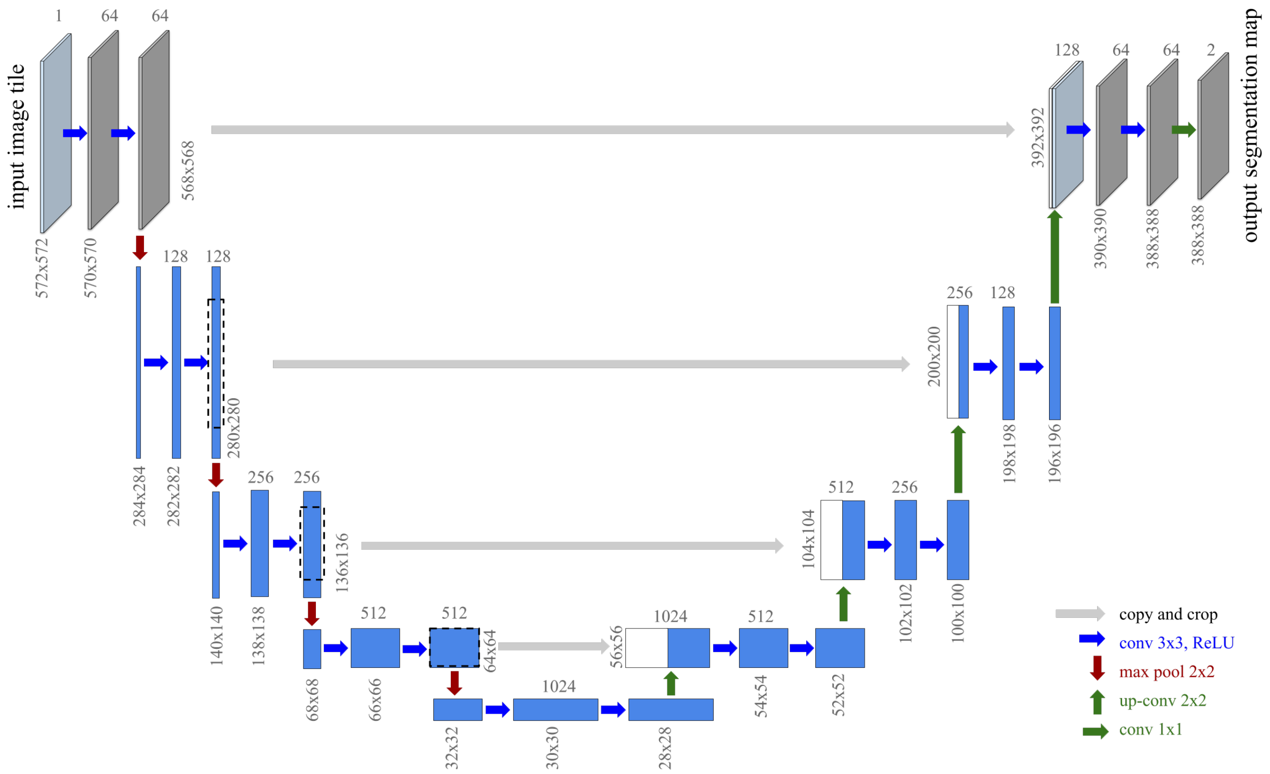


Figure 3.3: The re-depicted model architecture of the original U-Net model proposed in [8].

3.2. U-Net Training

The very essence of this work is to find a state-of-the-art alternative for the method described in Section 3.1. We chose the U-Net deep neural network model architecture, which has proven to be an effective model for image segmentation problems. The re-depicted model

architecture of the original model proposed in [8] can be seen in Figure 3.3. We used a slightly modified version of this architecture where we kept the input and output image sizes equal to 256 by 256. It resulted in a model with around 31 Million parameters.
The model training is performed by using the augmented dataset described in Section 2.1 as input. As output images, we use the mask images created by using the level set method, which is described in Section 3.1.
We used RMSProp [13] as an optimizer for the network training. Since the output is a binary image, we chose sparse categorical cross-entropy loss as the cost function. Different loss functions commonly used in deep learning are listed and compared in [14].
The data is split as %80 - %20 for training and validation, respectively. U-Net training does not need very long training epochs. Hence, we performed the training for ten epochs and achieved the best validation loss at the eighth epoch. We assured faster convergence by applying batch normalization [15], which prevents the neural network optimization deceleration due to covariate shift.
Covariate shift happens due to the complicated nature of deep neural networks. The input of each layer changes drastically as the parameters of the previous layers change. It lowers the adaptive learning rates, and hence the training eventually slows down. The related work [15] proposes Algorithm 2 which has been extensively used in deep learning literature recently.
The batch normalization is defined for each mini-batch since the RMSProp runs on mini-batches. An overview and comparison of different gradient descent procedures can be seen in [16].

Algorithm 2 Batch normalization (BN) applied to activation x over a mini-batch.

Input: Values of x over a mini-batch $\mathscr{B}$ where $\mathscr{B} = \{x_1, \dots, x_m\}$
Parameters to be learned: γ, β
Output: $\{y_i = BN_{\gamma,\beta}(x_i)\}$

1: $\mu_{\mathscr{B}} \leftarrow \frac{1}{m}\sum_{i=1}^{m} x_i$ #mini-batch mean
2: $\sigma^2_{\mathscr{B}} \leftarrow \frac{1}{m}\sum_{i=1}^{m}(x_i - \mu_{\mathscr{B}})^2$ #mini-batch variance, small ε prevents division by zero
3: $\hat{x}_i \leftarrow \frac{x_i - \mu_{\mathscr{B}}}{\sqrt{\sigma^2_{\mathscr{B}} + \varepsilon}}$ #normalize
4: $y_i \leftarrow \gamma\hat{x}_i + \beta \equiv BN_{\gamma,\beta}(x_i)$ #scale and shift

4. Results and Discussion

By using the update equation in Equation (3.4), the evolution of the initial boundary can be iterated. At each step, the exact boundary shape can be recovered by using the positive and negative regions,

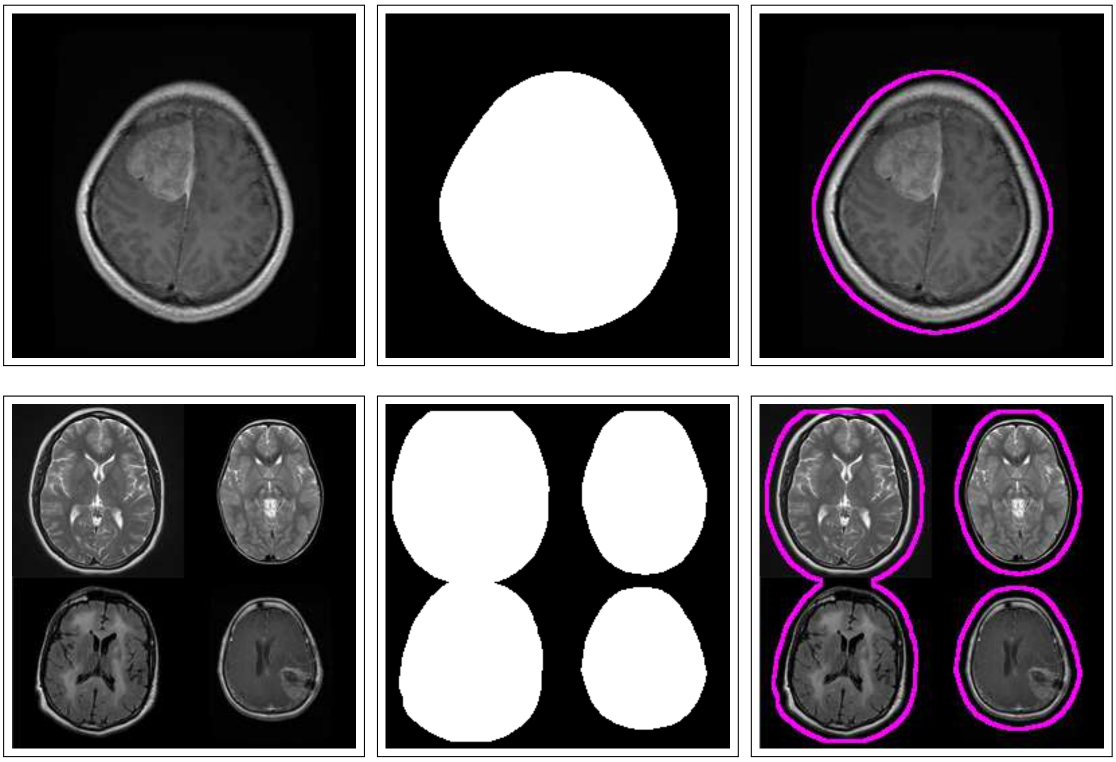


Figure 4.1: Left column: Sample MRIs from the dataset. **Middle Column:** Final state of the boundary after evolution iterations. The black and white colors represent the positive and negative regions, Ω^+ and Ω^-, of ϕ. **Right Column:** Shows the converged boundary (in magenta) on the original image.

Ω^+ and Ω^-, of ϕ. Three sample animated boundary evolutions can be seen in Animation 1, Animation 2 and Animation 3. Moreover, two selected boundary evolution results can be seen in Figure 4.1.

We applied the level-set-based image segmentation algorithm to all of the 6172 images in the dataset. Since the ROI size and complexity differs, the amount of iterations for these images varies. It took 7 minutes 53 seconds for Algorithm 1 to detect RIO in the dataset images. The populated mask images constitute a new dataset.

In the next step, the corresponding image pairs in the original and mask images are used to train a tailored version U-Net architecture [8]. We kept the model parameters of the lowest validation loss and generated mask images for the validation set. In Figure 4.2, mask generation results using both the level-set method and U-Net of two samples from the validation set can be seen. We observe that the U-Net generated mask image closely matches the level-set result.

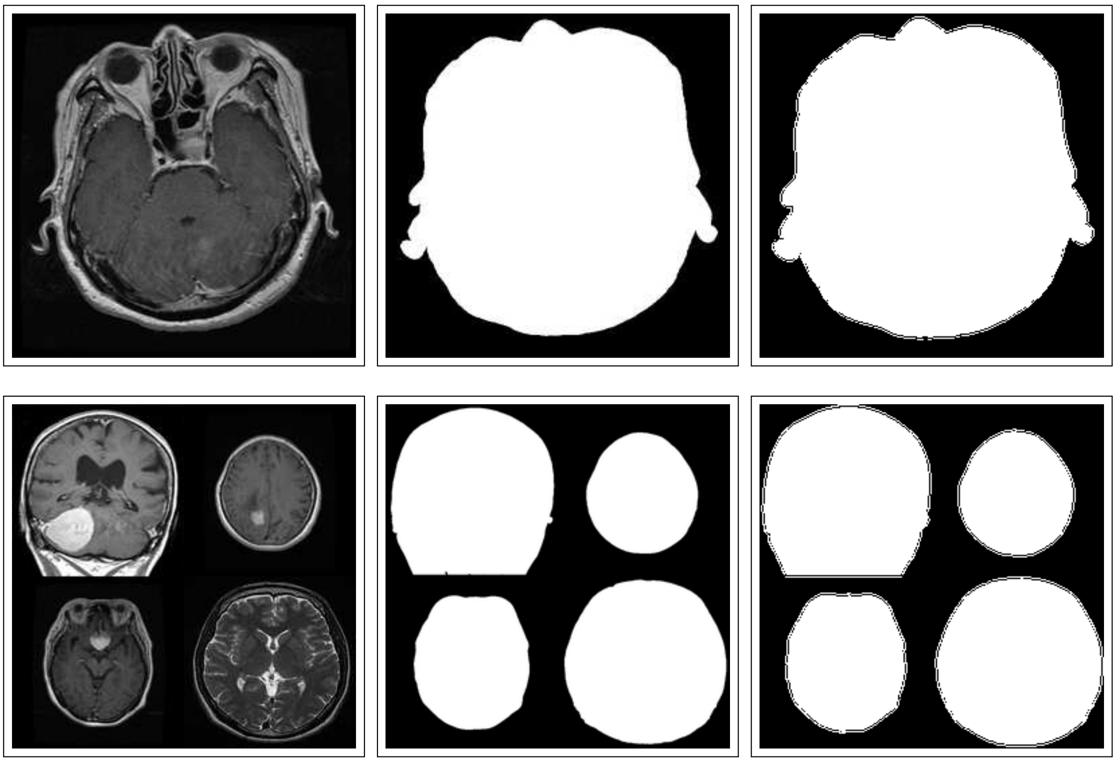


Figure 4.2: Left column: Sample MRIs from the dataset. **Middle Column:** The mask images representing the ROIs in sample images. **Right Column**: Shows the mask images generated by trained U-Net model.

To obtain a comparable processing time, we fed the network with all the images in the dataset. The trained U-Net DNN processed all the images in 1 minute 45 seconds, nearly five times faster than the level set method.

4.1. Conclusions

This study used a classical numerical solution for a specific image segmentation problem to create a training dataset for a deep neural network model. We chose the problem of ROI detection on MRIs since the ROI masks are essential for analyzing medical image analysis to optimize processing time [17], [18]. The ROIs on MRI slices can also be used for the 3D reconstruction of the organs [19].

We used the brain MRI dataset [5] which consists of MRI slices of tumourous and normal brains of different patients. We revisited the level-set solution to the image segmentation problem and successfully applied it to these MRIs. This numerical solution needs some preprocessing steps to obtain clear image gradients, which are to be used to specify the speed function F in the level set equation (3.3). By following the listed procedures in Algorithm 1, we created a mask image of the ROI in each MRI in the dataset.

We showed that similar segmentation masks could also be obtained using a U-Net once trained adequately on an appropriate training dataset. We used the dataset images and their generated ROI masks to train our U-Net model architecture till the model overfit and kept the best parameters where we achieved the lowest validation loss at the eighth epoch. The inference results showed that U-Net could create closely matched ROI segmentation masks for input MRI images in the validation dataset.

Using a DNN to solve the corresponding segmentation problem did have three significant advantages over the classical numerical solution using level-sets. First, the input images can directly be processed by the U-Net without any need for image preprocessing. Second, the DNN based solution method did not require numerical solutions to a partial differential equation like the level-set equation. Lastly, the trained U-Net model could process images five times faster than the level-set-based method, which is the most critical aspect of the proposed method for practical use since it can save a significant amount of time for vast datasets.

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Lacunary Statistical Convergence for Double Sequences on $\mathscr{L}-$ Fuzzy Normed Space

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Abstract

On $\mathscr{L}-$ fuzzy normed spaces, which is the generalization of fuzzy spaces, the notion of lacunary statistical convergence for double sequences which is a generalization of statistical convergence, are studied and developed in this paper. In addition, the definitions of lacunary statistical Cauchy and completeness for double sequences and related theorems are given on $\mathscr{L}-$ fuzzy normed spaces. Also, the relationship of lacunary statistical Cauchyness and lacunary statistical boundedness for double sequences with respect to $\mathscr{L}-$ fuzzy norm is shown.

1. Introduction

After the fuzzy set theory was introduced to the world of mathematics by Zadeh [1], this theory was developed and generalized by many different mathematicians such as intuitionistic fuzzy sets, which was developed by Atanassov [2]. Different convergence studies of sequences on these proposed spaces have received and continue to be of great interest in the mathematical community. The concept of statistical convergence [3]- [9] which can be accepted as a generalization of convergence in the classical sense, is also very important in the field of functional analysis, and together with this concept, statistical limitation, statistical Cauchy and statistical bounded sequences have been examined.
Many studies have been carried out in the fields fuzzy metric spaces [10], [11] and intuitionistic fuzzy metric spaces [12]- [15].
$\mathscr{L}-$ fuzzy normed spaces [16]- [18] are natural generalizations of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces, in which important work has been done on the theory of summability in this space [19]- [21], based on some logical algebraic structures.
To date, the types of convergence have been studied by many mathematician [22]- [29]. In particular, the characteristics of convergence types have been introduced to the mathematical community by Dündar [30]- [36].
The goal of the present study is to examine on $\mathscr{L}-$ fuzzy normed spaces the lacunary statistical convergence, which was initially introduced by Fridy, John Albert, and Cihan Orhan [37], [38]. Next, we give some results regarding lacunary statistical convergence of double sequences and investigate the relationship between lacunary statistical convergent, lacunary statistical Cauchy and lacunary statistical bounded sequences, which will be newly introduced on $\mathscr{L}-$ fuzzy normed spaces. We propose a relevant characterisation for lacunary statistically convergent for double sequences. Furthermore, we show an example where our convergence approach outperforms more than the traditional convergence on $\mathscr{L}-$ fuzzy normed spaces.

2. Preliminaries

Preliminaries on $\mathscr{L}-$ fuzzy normed spaces are presented in this section.

Definition 2.1 ([39]). *Assume that $K:[0,1]\times[0,1]\to[0,1]$ is a function that satisfies the following*

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1. $K(a,b)=K(b,a)$,
2. $K(K(a,b),c)=K(a,K(b,c))$,
3. $K(a,1)=K(1,a)=x$,
4. $a\leq b, c\leq d$ then $K(a,c)\leq K(b,d)$,

is known as a $t-$ norm.

Example 2.2 ([39]). *K_1, K_2 and K_3 are the functions that given with,*

$$K_1(a,b)=\min\{a,b\},$$
$$K_2(a,b)=ab,$$
$$K_3(a,b)=\max\{a+b-1,0\}$$

are the samples, which are well known of $t-$ norms.

Definition 2.3 ([39]). *Let $\mathscr{L}=(L,\preceq)$ be a complete lattice and let a set A be called the universe. An $L-$fuzzy set, on A is defined with a function*

$$X: A\to L.$$

On a set A, the family of all $L-$sets is denoted by L^A.

Two $L-$ sets on A intersect

$$(C\cap D)(x)=C(x)\wedge D(x)$$

for all $x\in A$. Similarly, union and intersection of a family $\{B_i : i\in I\}$ of $L-$ fuzzy sets is given by

$$\Big(\bigcup_{i\in I} B_i\Big)(x)=\bigvee_{i\in I} B_i(x)$$

and

$$\Big(\bigcap_{i\in I} B_i\Big)(x)=\bigwedge_{i\in I} B_i(x).$$

0_L and 1_L are the smallest and biggest elements of the full Lattice L, respectively. On a given lattice $(L,\preceq)$, we also employ the symbols $\succeq, \prec$, and $\succ$ in the obvious meanings.

Definition 2.4 ([39]). *Let $\mathscr{L}=(L,\preceq)$ be a complete lattice. Therefore, $t-$ norm is a function $\mathscr{K}: L\times L\to L$ that satisfies the following for all $a,b,c,d\in L$:*

1. $\mathscr{K}(a,b)=\mathscr{K}(b,a)$,
2. $\mathscr{K}(\mathscr{K}(a,b),c)=\mathscr{K}(a,\mathscr{K}(b,c))$,
3. $\mathscr{K}(a,1_L)=\mathscr{K}(1_L,a)=a$,
4. $a\preceq b$ and $c\preceq d$, then $\mathscr{K}(a,c)\preceq\mathscr{K}(b,d)$.

Definition 2.5 ([39]). *For sequences (a_n) and (b_n) on L such that $(a_n)\to a\in L$ and $(b_n)\to b\in L$, if the property that $\mathscr{K}(a_n,b_n)\to\mathscr{K}(a,b)$ satisfies on L, then a $k-$norm $\mathscr{K}$ on a complete lattice $\mathscr{L}=(L,\preceq)$ is called continuous.*

Definition 2.6 ([39]). *The function $\mathscr{N}: L\to L$ is defined as a negator on $\mathscr{L}=(L,\preceq)$ if,*

$N_1)$ $\mathscr{N}(0_L)=1_L$,

$N_2)$ $\mathscr{N}(1_L)=0_L$,

$N_3)$ $a\preceq b$ *implies* $\mathscr{N}(b)\preceq\mathscr{N}(a)$ *for all* $a,b\in L$.

If in addition,

$N_4)$ $\mathscr{N}(\mathscr{N}(a))=a$ *for all* $a\in L$.

Therefore, $\mathscr{N}$ is known as an involutive.

The mapping $\mathscr{N}_s:[0,1]\to[0,1]$, on the lattice $([0,1],\leq)$ defined as $\mathscr{N}_s(x)=1-x$ is a well known sample of an involutive negator. This type of negator are using in the notion of stansard fuzzy sets. In addition, with the order

$$(\mu_1,\nu_1)\preceq(\mu_2,\nu_2)\iff \mu_1\leq\mu_2\quad and\quad \nu_1\geq\nu_2$$

given the lattice $([0,1]^2,\preceq)$ with for all $i=1,2$, $(\mu_i,\nu_i)\in[0,1]^2$. Therefore, the function $\mathscr{N}_1:[0,1]^2\to[0,1]^2$,

$$\mathscr{N}_1(\mu,\nu)=(\nu,\mu)$$

in the sense of Atanassov, is known as a involutive negator. This type of negator are using in the notion of intuitionistic fuzzy sets.

Definition 2.7 ([39]). *Let $\mathscr{L}=(L,\preceq)$ be a complete lattice and V be a real vector space. $\mathscr{K}$ be a continuous $t-$norm on $\mathscr{L}$ and μ be an $L-$set on $V\times(0,\infty)$ satisfying the following*

(a) $\mu(a,t)\succ 0_L$ *for all* $a\in V, t>0$,
(b) $\mu(a,t)=1_L$ *for all* $t>0$ *if and only if* $a=\theta$,
(c) $\mu(\alpha a,t)=\mu(a,\frac{t}{|\alpha|})$ *for all* $a\in V, t>0$ *and* $\alpha\in\mathbb{R}-\{0\}$,
(d) $\mathscr{K}(\mu(a,t),\mu(b,s))\preceq\nu(a+b,t+s)$, *for all* $a,b\in V$ *and* $t,s>0$,
(e) $\lim_{t\to\infty}\mu(a,t)=1_L$ *and* $\lim_{t\to 0}\mu(a,t)=0_L$ *for all* $a\in V-\{\theta\}$,

(f) The functions $f_a : (0,\infty) \to L$ which is $f(t) = \mu(a,t)$ are continuous.
The triple $(V,\mu,\mathscr{K})$ is referred to as an $\mathscr{L}-$ fuzzy normed space or $\mathscr{L}-$ normed space in this context.

Definition 2.8 ([39]). *A sequence (a_n) is said to be Cauchy sequence in a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$ if, there exists $n_0 \in \mathbb{N}$ such that, for all $m,n > n_0$*

$$\mu(a_n - a_m,t) \succ \mathscr{N}(\varepsilon),$$

where $\mathscr{N}$ is a negator on $\mathscr{L}$, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$.

Definition 2.9. *A sequence $a = (a_n)$ is said to be bounded with respect to fuzzy norm in a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$, provided that, for each $r \in L - \{0_L, 1_L\}$ and $t > 0$,*

$$\mu(a_n,t) \succ \mathscr{N}(r),$$

for all $n \in \mathbb{N}$.

On $\mathscr{L}-$ fuzzy normed spaces, we'll look at statistical convergence. Before we continue, let's go through basic statistical convergence terms. If $K \subseteq \mathbb{N}$, the set of natural numbers, then $\delta\{A\}$ is the asymptotic density of A, is

$$\delta\{A\} := \lim_k \frac{1}{k}\left|\{n \le k : n \in A\}\right|$$

the limit exists the cardinality of the set A is given by $|A|$.
If the set $K(\varepsilon) = \{n \le k : |a_n - l| > \varepsilon\}$ has the asymptotic density zero, i.e.

$$\lim_k \frac{1}{k} \mid \{n \le k : |a_n - l| > \varepsilon\} \mid = 0,$$

then the sequence $a = (a_n)$ is known as a statistically convergent to the number l. In this case, we will write $st - \lim a = l$.
Despite the fact that every convergent sequence is statistically convergent to the same limit, the opposite of this is not necessarily true.

Definition 2.10 ([40]). *A sequence $a = (a_n)$ is statistically convergent to $l \in V$ with respect to μ fuzzy norm in a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$ if provided that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$,*

$$\delta\{n \in \mathbb{N} : \mu(a_n - l,t) \not\succ \mathscr{N}(\varepsilon)\} = 0$$

or equivalently

$$\lim_m \frac{1}{m}\{j \le m : \mu(a_n - l,t) \not\succ \mathscr{N}(\varepsilon)\} = 0.$$

In this case, we will write $st_{\mathscr{L}} - \lim a = l$.

Definition 2.11 ([40]). *A sequence $a = (a_k)$ is said to be statistically Cauchy with respect to fuzzy norm μ in a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$, if provided that*

$$\delta\{k \in \mathbb{N} : \mu(a_k - a_m,t) \not\succ \mathscr{N}(\varepsilon)\} = 0$$

for each $\varepsilon \in L - \{0_L\}$, $m \in \mathbb{N}$ and $t > 0$.

Definition 2.12 ([40]). *A sequence $a = (a_k)$ is said to be statistically bounded with respect to fuzzy norm μ in a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$ if provided that there exists $r \in L - \{0_L, 1_L\}$ and $t > 0$ such that*

$$\delta\{k \in \mathbb{N} : \mu(a_k,t) \not\succ \mathscr{N}(r)\} = 0$$

for each positive integer k.

3. Lacunary Statistical Convergence for Double Sequences on $\mathscr{L}-$Fuzzy Normed Space

In this section we define and study lacunary statistical convergence for double sequences on $\mathscr{L}-$ fuzzy normed space.

Definition 3.1. *By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.*
For any set $N \subseteq \mathbb{N}$, the number

$$\delta_\theta(N) = \lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : k \in N\}|$$

is called the θ density of the set N, provided the limit exists.
A sequence $a = (a_k)$ is said to be lacunary statistically convergent or S_θ convergent to a number ℓ provided that for each $\varepsilon > 0$,

$$\delta_\theta\{k \in \mathbb{N} : |a_k - \ell| \ge \varepsilon\}| = 0.$$

In other words, the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - \ell| \ge \varepsilon\}$ has $\theta-$ density zero. In this case the number ℓ is called lacunary statistical limit of the sequence $x = (x_k)$ and we write $S_\theta - \lim_{r\to\infty} x_k = \ell$ or $x_k \to \ell(S_\theta)$.

Definition 3.2. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then a sequence $a = (a_k)$ is lacunary statistically convergent to $l \in V$ with respect to μ fuzzy norm, provided that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$,*

$$\delta_\theta\{k \in \mathbb{N} : \mu(a_k - l,t) \not\succ \mathscr{N}(\varepsilon)\} = 0.$$

In this scenario, $S_\theta^{\mathscr{L}} - \lim x = l$.

[41] The double sequence $\theta = \{(k_r, l_s)\}$ is called double lacunary if there exist there exist two increasing integer sequence such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty, as \ \ r \to \infty$$

and

$$l_0 = 0, m_s = l_s - l_{s-1} \to \infty, as \ \ s \to \infty.$$

The intervals are determined by θ, $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < s \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r, l_{s-1} < s \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $u_s = \frac{l_s}{l_{s-1}}$.
Note that the double $\theta-$ density will be denoted by δ_{θ_2}.

Definition 3.3. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then a double sequence $a = (a_{mn})$ is lacunary statistically convergent to $l \in V$ with respect to ν fuzzy norm, provided that, for each $\varepsilon \in L - \{0_L\}$ and $t > 0$,*

$$\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

In this scenario, it is denoted by $S^{\mathscr{L}}_{\theta_2} - \lim a = l$.

Proposition 3.4. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\varepsilon \in L - \{0_L\}$ and $t > 0$:*

(a) $S^{\mathscr{L}}_{\theta_2} - \lim a = l$,
(b) $\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0$,
(c) $\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \succ \mathscr{N}(\varepsilon)\} = 1$,
(d) $S^{\mathscr{L}}_{\theta_2} - \lim \mu(a_{mn} - l, t) = 1_L$.

Theorem 3.5. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space and $a = (a_{mn})$ be a double sequence. If $\lim a = l$ in Pringsheim sense, then $S^{\mathscr{L}}_{\theta_2} - \lim a = l$.*

Proof. Let $\lim a = l$. Then, for every $\varepsilon \in L - \{0_L\}$ and $t > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$\mu(a_{mn} - l, t) \succ \mathscr{N}(\varepsilon),$$

for all $m, n \geq k_0$. Therefore,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\}$$

has at most finitely many terms. We can see right away that any finite subset of the natural numbers has double $\theta-$ density zero. Hence,

$$\delta_{\theta_2}\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - l, t) \nsucc \mathscr{N}(\varepsilon)\} = 0.$$

□

As shown in the following case, the converse of the theorem is not true.

Example 3.6. *Let $V = \mathbb{R}$ and $\mathscr{L} = (\mathscr{P}(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Define the function $\mu : \mathbb{R} \times (0, \infty) \to \mathscr{P}(\mathbb{R}^+)$ with*

$$\mu(x,t) = \{r \in \mathbb{R}^+ : \mid x \mid < \frac{t}{r}\}.$$

Then, $(\mathbb{R}, \mu, \mathscr{P}(\mathbb{R}^+))$ is a $\mathscr{L}-$ fuzzy normed space. On this space, consider the sequence $a = (a_{mn})$ given by the rule

$$a_{mn} = \begin{cases} 1, & for \ m \in (k_r - \ln(h_r), k_r] \ \ and \ \ n \in (l_s - \ln(m_s), l_s], r, s \in \mathbb{N}, \\ 0, & otherwise. \end{cases}$$

Then,

$$\lim_{r \to \infty} \delta_{\theta_2} = 0$$

which means $S^{\mathscr{L}}_{\theta_2} - \lim a = l \in \mathbb{R}$, while the sequence itself is not convergent.

Theorem 3.7. *Let $(V, \mu, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. If a double sequence $a = (a_{mn})$ is lacunary statistically convergent with respect to the $\mathscr{L}-$ fuzzy norm μ, then $S^{\mathscr{L}}_{\theta_2}-$ limit is unique.*

Proof. Suppose that $S^{\mathscr{L}}_{\theta_2} - \lim a = \ell_1$ and $S^{\mathscr{L}}_{\theta_2} - \lim a = \ell_2$, where $\ell_1 \neq \ell_2$. For any given $\varepsilon \in L - \{0_L\}$ and $t > 0$, we can choose a $r \in L - \{0_L\}$ such that

$$\mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

Define the following sets

$$K_1 = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - \ell_1, t)) \nsucc \mathscr{N}(r)\}$$

and

$$K_2 = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mu(a_{mn} - \ell_2, t)) \nsucc \mathscr{N}(r)\}$$

for any $t > 0$. Since for elements of the set $K(\varepsilon, t) = K_1(\varepsilon, t) \cup K_2(\varepsilon, t)$ we have

$$\mu(\ell_1 - \ell_2, t) \succeq \mathscr{K}(\mu(a_{mn} - \ell_1, \frac{t}{2}), \mu(a_{mn} - \ell_2, \frac{t}{2})) \succ \mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\varepsilon).$$

it can be concluded that $\ell_1 = \ell_2$. □

Theorem 3.8. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then, $S_{\theta_2}^{\mathscr{L}}-\lim a=\ell$ if and only if there exists a subset $K\subset\mathbb{N}\times\mathbb{N}$ such that $\delta_{\theta_2}(K)=1$ and $\mathscr{L}-\lim_{k,l\to\infty}a_{kl}=\ell$.*

Proof. Suppose that $S_{\theta_2}^{\mathscr{L}}-\lim a=l$. Let (ε_n) be a sequence in $L-\{0_L\}$ such that $\mathscr{N}(\varepsilon_n)\to 1_L$ in L increasingly, and for any $t>0$ and $j\in\mathbb{N}$, let

$$K(j)=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\mu(a_{kl}-l,t)\succ\mathscr{N}(\varepsilon_j)\}.$$

Then, observe that, for any $t>0$ and $j\in\mathbb{N}$,

$$K(j+1)\subset K(j).$$

Since $S_{\theta_2}^{\mathscr{L}}-\lim a=l$, it is obvious that

$$\delta_{\theta_2}\{K(j)\}=1,(j\in\mathbb{N}\text{ and }t>0).$$

Now, let (p_1,q_1) be an arbitrary number of $K(1)$. Then, there exist numbers $(p_2,q_2)\in K(2)$, $p_2>p_1,q_2>q_1$, such that for all $l>p_2,k>q_2$,

$$\frac{1}{h_rt_s}|\{(k,l)\in I_{r,s}:\mu(x_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_2)\}|>\frac{1}{2}.$$

Further, there is a number $(p_3,q_3)\in K(3),p_3>p_2,q_3>q_2$ such that for all $l>p_3,k>q_3$,

$$\frac{1}{h_rt_s}|\{(k,l)\in I_{r,s}:\mu(x_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_3)\}|>\frac{2}{3}$$

and so on. So, we can construct, by induction, an increasing index sequence increasing in both coordinates $(p_j,q_k)_{j,k\in\mathbb{N}}$ of the natural numbers such that $(q_j,q_j)\in K(j)$ and that the following statement holds for all $l>p_j,k>q_j$:

$$\frac{1}{h_rt_s}|\{(k,l)\in I_{r,s}:\mu(x_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_j)\}|>\frac{j-1}{j}.$$

Now, we construct an index sequence increasing in both coordinates as follows:

$$K:=\{(k,l)\in\mathbb{N}\times\mathbb{N}:1<l<p_1,1<k<q_1\}\cup\Big[\bigcup_{j\in\mathbb{N}}\{(k,l)\in K(j):p_j\le l<p_{j+1},q_j\le k<q_{j+1}\}\Big].$$

Hence, it follows that $\delta_{\theta_2}(K)=1$. Now, let $\varepsilon\succ 0_L$ and choose a positive integer j such that $\varepsilon_j\prec\varepsilon$. Such a number j always exists since $(\varepsilon_n)\to 0_L$. Assume that $l\ge p_j,k\ge q_j$ and $k,l\in K$. Then, by the definiton of K, there exists a number $d\ge j$ such that $p_d\le l<p_{d+1},q_d\le k<q_{d+1}$ and $(k,l)\in K(j)$. Hence, we have, for every $\varepsilon\succ 0_L$

$$\mu(a_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon_k)\succ\mathscr{N}(\varepsilon)$$

for all $l\ge p_j,k\ge q_j$ and $(k,l)\in K$ and this means

$$\mathscr{L}-\lim_{k,l\in K}a_{kl}=\ell.$$

Conversely, suppose that there exists an increasing index sequence $K=(a_{kl})_{k,l\in\mathbb{N}}$ of pairs of natural numbers such that $\delta_{\theta_2}(K)=1$ and $\mathscr{L}-\lim\limits_{k,l\in K}a_{kl}=\ell$. Then, for every $\varepsilon\succ 0_L$ there is a number n_0 such that for each $k,l\ge n_0$ the inequality $\mu(a_{kl}-\ell,t)\succ\mathscr{N}(\varepsilon)$ holds. Now, define

$$M(\varepsilon):=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\mu(a_{kl}-\ell,t)\not\succ\mathscr{N}(\varepsilon)\}.$$

Then, there exists an $n_0\in\mathbb{N}$ such that

$$M(\varepsilon)\subseteq\mathbb{N}\times\mathbb{N}-(K-\{(a_k,a_l):k,l\le n_0\}).$$

Since $\delta_{\theta_2}(K)=1$, we get $\delta_{\theta_2}\{(\mathbb{N}\times\mathbb{N})-(K-\{(a_k,a_l):k,l\le n_0\})\}=0$, which yields that $\delta_{\theta_2}\{M(\varepsilon)\}=0$. In other words, $S_{\theta_2}^{\mathscr{L}}-\lim a=l$. □

4. The Relationship Between Lacunary Statistical Double Cauchy and Lacunary Statistical Double Bounded Sequences

In this section, the notion of lacunary statistically double Cauchy and lacunary statistically double bounded sequences will be defined and relationship between them will be given.

Definition 4.1. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. Then, a sequence $a=(a_{mn})$ is said to be lacunary statistically double Cauchy with respect to $\mathscr{L}-$ fuzzy norm μ, if for every $\varepsilon\in L-\{0_L\}$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that for all $m,k\ge N$ and $n,l\ge M$ provided that*

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{kl},t)\not\succ\mathscr{N}(\varepsilon)\}=0.$$

Theorem 4.2. *Every lacunary statistically convergent double sequence is lacunary statistically double Cauchy.*

Proof. Let $a=(a_{mn})$ be a double sequence such that lacunary statistical convergent to ℓ with respect to $\mathscr{L}-$ fuzzy norm μ, in other saying $S_{\theta_2}^{\mathscr{L}}-\lim a=l$. For a given $\varepsilon>0$, choose $r>0$ such that,

$$\mathscr{K}(\mathscr{N}(r),\mathscr{N}(r))\succ\mathscr{N}(\varepsilon).$$

For $t>0$ we can write,

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-\ell,\frac{t}{2})\succ\mathscr{N}(r)\}.$$

Take $(p,q)\in A$. Obviously, $\mu(a_{pq}-\ell,\frac{t}{2})\succ\mathscr{N}(r)$. Also since,

$$\mu(\ell-a_{pq},\frac{t}{2})=\mu(a_{pq}-\ell,\frac{\frac{t}{2}}{|-1|})=\mu(a_{pq}-\ell,\frac{t}{2})\succ\mathscr{N}(\varepsilon)$$

we have

$$\begin{aligned}\mu(a_{mn}-x_{pq},t)&=\mu\big((a_{mn}-\ell)+(\ell-a_{pq}),\frac{t}{2}+\frac{t}{2}\big)\\&\succ\mathscr{K}\big(\mu(a_{mn}-\ell,\frac{t}{2}),(\nu(\ell-a_{pq},\frac{t}{2})\big)\\&\succ\mathscr{K}\big(\mathscr{N}(r),\mathscr{N}(r)\big)\\&\succ\mathscr{N}(\varepsilon).\end{aligned}$$

If we define a set $B=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{pq},t)\succ\mathscr{N}(\varepsilon)\}$, then $A\subseteq B$. Since $\delta_{\theta_2}(A)=1$, $\delta_{\theta_2}(B)=1$. Thus, the double theta density of complement of B equals to zero,i.e. $\delta_{\theta_2}(B^c)=0$, which means $a=(a_{mn})$ is lacunary statistical double Cauchy. ☐

Definition 4.3. *Let $(V,\mu,\mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space and $a=(a_{mn})$ be a double sequence. Then, $a=(a_{mn})$ is said to be lacunary statistically double bounded with respect to $\mathscr{L}-$ fuzzy norm μ, provided that there exists $r\in L-\{0_L,1_L\}$ and $t>0$ such that*

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},t)\not\succ\mathscr{N}(r)\}=0$$

for each positive integer m,n.

Theorem 4.4. *Every double bounded sequence on a $\mathscr{L}-$ fuzzy normed space $(V,\mu,\mathscr{K})$, is lacunary statistically double bounded.*

Proof. Let (a_{mn}) be a double bounded sequence on $(V,\mu,\mathscr{K})$. Then, there exist $t>0$ and $r\in L-\{0_L,1_L\}$ such that $\mu(a_{mn},t)\succ\mathscr{N}(r)$. In that case we have,

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},t)\not\succ\mathscr{N}(r)\}=\emptyset$$

which yields

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},t)\not\succ\mathscr{N}(r)\}=0.$$

Thus, (a_{mn}) is lacunary statistically bounded. ☐

However the converse of this theorem does not hold in general as seen in the example below.

Example 4.5. *Let $V=\mathbb{R}$ and $\mathscr{L}=(L,\leq)$ where L is the set of non-negative extended real numbers, that is $L=[0,\infty]$. Then, $0_L=0,1_L=\infty$. Define a $\mathscr{L}-$fuzzy norm ν on V by $\mu(x,t)=\frac{t}{|x|}$ for $x\neq 0$ and $\nu(0,t)=\infty$ for each $t\in(0,\infty)$. Consider the $t-$ norm $\mathscr{K}(a,b)=\min\{a,b\}$ on $\mathscr{L}$. Given the sequence,*

$$x_{mn}=\begin{cases}m+n, & \text{if m+n is a prime number},\\ \frac{1}{\tau(m+n)-2}, & \text{otherwise}\end{cases}$$

where, $\tau(m+n)$ denotes the number of positive divisors of $m+n$. Note that (x_{mn}) is not bounded since for each $t>0$ and $r\in L-\{0,\infty\}$, for any prime number $m+n$ such that $rt\leq m+n$ we have

$$\mu(x_{mn},t)=\mu(m+n,t)=\frac{t}{|m+n|}=\frac{t}{m+n}\not\succ\frac{1}{r}=\mathscr{N}(r).$$

However for $t=1$ and any non-prime integer $m+n$, $r=2$ satisfies

$$\mu(x_{mn},1)=\mu(\frac{1}{\tau(m+n)-2},1)=\frac{1}{|\frac{1}{\tau(m+n)-2}|}=|\tau(m+n)-2|>\frac{1}{2}=\mathscr{N}(r)$$

since $\tau(m+n)\neq 2$ for any non-prime $m+n$, and since the density of prime numbers converges zero by Prime Number Theorem we have,

$$\delta_{\theta_2}\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu(x_{jk},1)\not\succ\mathscr{N}(2)\}=0$$

suggesting that (x_{mn}) is lacunary statistically double bounded.

Theorem 4.6. *Every lacunary statistically double Cauchy sequence on a $\mathscr{L}-$fuzzy normed space $(V,\mu,\mathscr{K})$ is lacunary statistically double bounded.*

Proof. Let (a_{mn}) be a lacunary statistically double Cauchy on $(V,\mu,\mathscr{K})$. Then, for every $\varepsilon \in L-\{0_L\}$ and $t>0$, there exist $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that for all $m,k \geq N$ *and* $n,l \geq M$ provided that

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{kl},t)\nsucc\mathscr{N}(\varepsilon)\}=0.$$

Then,

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn}-a_{kl},t)\succ\mathscr{N}(\varepsilon)\}=1.$$

Consider a number $(m,n)\in\mathbb{N}\times\mathbb{N}$ such that $\mu(a_{mn}-a_{kl},1)\succ\mathscr{N}(\varepsilon)$. Then, for $t=2$

$$\mu(a_{mn},2)=\mu(a_{mn}-a_{kl}+a_{kl},2)\succ\mathscr{K}(\mu(a_{mn}-a_{kl},1),\mu(a_{kl},1))\succ\mathscr{K}(\mathscr{N}(\varepsilon),\nu(x_{kl},1)).$$

Say $r:=\mathscr{N}(\mathscr{K}(\mathscr{N}(\varepsilon),\mu(a_{kl},1)))$. Then,

$$\mu(a_{mn},2)\succ\mathscr{K}(\mathscr{N}(\varepsilon),\mu(a_{kl},1))=\mathscr{N}(r),$$

which implies

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},2)\succ\mathscr{N}(r)\}=1$$

or equivalently

$$\delta_{\theta_2}\{(m,n)\in\mathbb{N}\times\mathbb{N}:\mu(a_{mn},2)\nsucc\mathscr{N}(r)\}=0$$

giving lacunary statistically double boundedness of (a_{mn}). □

5. Conclusion

In this study, the properties of Lacunary statistical convergence for double sequences, which is a generalization of statistical convergence, are defined on L fuzzy spaces, which are a generalization of fuzzy spaces, and their properties are examined. Some characteristics of the lacunary statistical convergence of sequences within the context of the current investigation are examined on L-fuzzy normed spaces, a structure that provides a flexible frame- work that generalizes other structures like normed spaces, fuzzy normed spaces, and IF-normed spaces. As a result of this research, the concept of norm was emphasized on a broader concept, the topological vector space, by combining the lattice structure and the norm structure.

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Existence and Decay of Solutions for a Parabolic-Type Kirchhoff Equation with Variable Exponents

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Abstract

This paper deals with a parabolic-type Kirchhoff equation with variable exponents. Firstly, we obtain the global existence of solutions by Faedo-Galerkin method. Later, we prove the decay of solutions by Komornik's inequality.

1. Introduction

In this work, we study the following parabolic-type Kirchhoff equation with variable exponents

$$\begin{cases} \left(1+|u|^{p(x)-2}\right)u_t+\Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u = |u|^{q(x)-2}u, & \text{in } (x,t)\in\Omega\times(0,T), \\ u(x,t)=\frac{\partial u}{\partial v}(x,t)=0, & \text{on } x\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & \text{in } x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$ and

$$M(s)=1+s^{\gamma},\ \ \gamma\geq 1.$$

The variable exponents $p(.)$ and $q(.)$ are given as measurable functions on Ω satisfying

$$\begin{cases} 2\leq p^-\leq p(x)\leq p^+\leq p^*, \\ 2\leq q^-\leq q(x)\leq q^+\leq q^*, \end{cases}$$

where

$$\begin{cases} p^- = ess\inf_{x\in\Omega} p(x), & p^+ = ess\sup_{x\in\Omega} p(x), \\ q^- = ess\inf_{x\in\Omega} q(x), & q^+ = ess\sup_{x\in\Omega} q(x), \end{cases}$$

and

$$p^*,q^*=\begin{cases} \infty, & \text{if } n\leq 4, \\ \frac{2n}{n-4} & \text{if } n>4. \end{cases} \tag{1.2}$$

We also suppose that $p(.)$ and $q(.)$ satisfy the log-Hölder continuity condition:

$$|p(x)-p(y)|\leq -\frac{A}{\log|x-y|},$$

for a.e. $x,y\in\Omega$, $|x-y|<\delta$ with $A>0$, $0<\delta<1$.

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- Parabolic type equation: Many phenomena in physics lead up to problems that deal with parabolic type equations, such as; mathematical description of the reaction-diffusion or diffusion, population dynamic processes and heat transfer [1].
- Kirchhoff equation: The Kirchhoff equation is among the famous wave equation's model which describe small vibration amplitude of elastic strings. This equation has been introduced in 1876 by Kirchhoff [2].
- Variable exponent: The problems with variable exponents arises in many branches in sciences such as electrorheological fluids, nonlinear elasticity theory and image processing [3]-[5].

In [6], Wu et al. established the blow up of solutions with positive initial energy for the following equation

$$u_t - \Delta u = u^{p(x)}.$$

Later, some authors get new results for the same equation to blow up result (see [7]-[10]).
In [11], Qu et al. studied the fourth order parabolic equation as follows

$$u_t + \Delta^2 u = u^{p(x)}.$$

The authors studied the asymptotic behavior of solutions.
When there is no fourth-order term $\Delta^2 u$, (1.1) is reduced to the following equation

$$u_t - M\left(\|\nabla u\|^2\right)\Delta u + |u|^{m(x)-2}u_t = |u|^{r(x)-2}u.$$

Khaldi et al. [12] studied the global existence and stability of solutions.
Recently, problems with variable exponents have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability have been found ([13]-[17]).
In this work, we considered the existence and decay of solutions of the parabolic type Kirchhoff equation with variable exponents, motivated by above works. To our best knowledge, there is no research, related to the parabolic type Kirchhoff equation (1.1) with fourth-order term ($\Delta^2 u$) and variable exponent source term ($|u|^{q(x)-2}u$), hence, our work is the generalization of the above studies.
This work consists of four parts: Firstly, in part 2, we give some needed theories about Lebesgue and Sobolev space with variable-exponents. Then, in Section 3, we get the existence result by the Faedo-Galerkin method. Moreover, in Section 4, we obtain the decay of solutions by the Komornik's inequality.

2. Preliminaries

Throughout this work, we denote by $\|.\|_p$ the $L^p(\Omega)$ norm. Also, we give some needed theories about Lebesgue space and Sobolev space with variable-exponents (for detailed, see [4, 18, 19]).
Let $p:\Omega\to[1,\infty]$ be a measurable function. We introduce the Lebesgue space with variable exponent $p(.)$

$$L^{p(.)}(\Omega) = \left\{u:\Omega\to R \text{ measurable in } \Omega,\ \rho_{p(.)}(\lambda u) < \infty,\ \text{ for some } \lambda > 0\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

The norm, called Luxemburg's norm, is defined by

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(.)}(\Omega)$ is a Banach space.
Next we define the variable-exponent Sobolev space $W^{m,p(.)}(\Omega)$ as

$$W^{m,p(.)}(\Omega) = \left\{u\in L^{p(.)}(\Omega) \text{ such that } D^\alpha u \text{ exists and } D^\alpha u \in L^{p(.)}(\Omega),\ |\alpha|\le m\right\}.$$

Lemma 2.1. *[4]. If*

$$1 \le p_1 := ess\inf_{x\in\Omega} p(x) \le p(x) \le p_2 := ess\sup_{x\in\Omega} p(x) < \infty,$$

then we have

$$\min\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \le \rho_{p(.)}(u) \le \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\},$$

for any $u\in L^{p(.)}$.

Lemma 2.2. *(Hölder's inequality)[4]. Assume that $p,q,s\ge 1$ are measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \text{ for a.e. } y\in\Omega.$$

If $u\in L^{p(.)}(\Omega)$ and $v\in L^{q(.)}(\Omega)$, then $uv\in L^{s(.)}(\Omega)$ with

$$\|uv\|_{s(.)} \le c\|u\|_{p(.)}\|v\|_{q(.)}.$$

Lemma 2.3. *[4]. If $p:\Omega\to[1,\infty)$ is a measurable function satisfying (1.2) then the embedding $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^{p(.)}$ is continuous and compact.*

Lemma 2.4. *[20]. Let $\varphi:R^+\to R^+$ is a nonincreasing function and suppose that there are two constants $\alpha>0$ and $c>0$ such that*

$$\int_0^\infty \varphi^{\alpha+1}(s)\,ds\le c\varphi^\alpha(0)\,\varphi(s)\ \forall t\in R^+.$$

Then we have

$$\varphi(t)\le\varphi(0)\left(\frac{c+\alpha t}{c+\alpha c}\right)^{-1/\alpha}\quad\forall t\ge c.$$

3. Existence

In this part, we state and prove the global existence result. Now, let us introduce some functionals as follows:

$$E(t)=\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}\,dx,$$

$$I(t)=\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega|u|^{q(x)}\,dx.$$

Lemma 3.1. *Suppose that (1.2) holds. Then*

$$E'(t)=-\|u_t\|_2^2-\int_\Omega|u|^{p(x)-2}|u_t|^2\,dx\le0,\tag{3.1}$$

and

$$E(t)\le E(0).$$

Proof. We multiply the eq. (1.1) by u_t and integrate over Ω, we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}\,dx\right)\\ =\ &-\|u_t\|_2^2-\int_\Omega|u|^{p(x)-2}|u_t|^2\,dx,\end{aligned}$$

thus

$$E'(t)=-\|u_t\|_2^2-\int_\Omega|u|^{p(x)-2}|u_t|^2\,dx\le0.$$

A simple integration of (3.1) over $(0,T)$, yields

$$E(t)\le E(0).$$

□

Lemma 3.2. *Let assumption (1.2) holds. Further assume that $q_1>2(\gamma+1)$, $I(0)>0$ and*

$$\beta_1+\beta_2<1,$$

where

$$\beta_1=\max\left\{\alpha c_*^{q_1}\left(\frac{2q_1}{q_1-2}E(0)\right)^{(q_1-2)/2},\alpha c_*^{q_2}\left(\frac{mq_1}{q_1-m}E(0)\right)^{(q_2-2)/2}\right\},$$

$$\beta_2=\max\left\{\begin{array}{l}(1-\alpha)c_*^{q_1}\left(\frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(0)\right)^{(q_1-2(\gamma+1))/(2(\gamma+1))}\\ (1-\alpha)c_*^{q_2}\left(\frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(0)\right)^{(q_2-2(\gamma+1))/(2(\gamma+1))}\end{array}\right\},$$

with $0<\alpha<1$ and c_* is the best embedding constant of $H_0^2(\Omega)\hookrightarrow L^{q(.)}(\Omega)$. Then $I(t)>0$ for all $t\in[0,T]$.

Proof. Since $I(0)>0$, then by continuity there exists T_* such that

$$I(t)\geq 0,\ \forall t\in[0,T_*]. \tag{3.2}$$

Now, we have for all $t\in[0,T]$ that

$$\begin{aligned} E(t) &= \frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2 \\ &\quad +\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\geq \frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} \\ &\quad -\frac{1}{q_1}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}-I(t)\right) \\ &\geq \frac{q_1-2}{2q_1}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right) \\ &\quad +\frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\|\nabla u\|_2^{2(\gamma+1)}+\frac{1}{q_1}I(t). \end{aligned}$$

Using (3.2), we have

$$\frac{q_1-2}{2q_1}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right)+\frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\|\nabla u\|_2^{2(\gamma+1)}\leq E(t).$$

By the definition of E, we obtain

$$\begin{aligned} \|\Delta u\|_2^2+\|\nabla u\|_2^2 &\leq \frac{2q_1}{q_1-2}E(t) \\ &\leq \frac{2q_1}{q_1-2}E(0), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|\nabla u\|_2^{2(\gamma+1)} &\leq \frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(t) \\ &\leq \frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(0). \end{aligned} \tag{3.4}$$

On the other hand, by Lemma 2.1, we get

$$\begin{aligned} \int_\Omega |u|^{q(x)}dx &\leq \max\left\{\|u\|_{q(.)}^{q_1},\|u\|_{q(.)}^{q_2}\right\} \\ &= \alpha\max\left\{\|u\|_{q(.)}^{q_1},\|u\|_{q(.)}^{q_2}\right\}+(1-\alpha)\max\left\{\|u\|_{q(.)}^{q_1},\|u\|_{q(.)}^{q_2}\right\}. \end{aligned}$$

By the embedding of $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^{q(.)}(\Omega)$, we have

$$\begin{aligned} \int_\Omega |u|^{q(x)}dx &\leq \alpha\max\left\{c_*^{q_1}\|\Delta u\|_2^{q_1},c_*^{q_2}\|\Delta u\|_2^{q_2}\right\} \\ &\quad +(1-\alpha)\max\left\{c_*^{q_1}\|\nabla u\|_2^{q_1},c_*^{q_2}\|\nabla u\|_2^{q_2}\right\} \\ &\leq \alpha\max\left\{c_*^{q_1}\|\Delta u\|_2^{q_1-2},c_*^{q_2}\|\Delta u\|_2^{q_2-2}\right\}\|\Delta u\|_2^2 \\ &\quad +(1-\alpha)\max\left\{c_*^{q_1}\|\nabla u\|_2^{q_1-2(\gamma+1)},c_*^{q_2}\|\nabla u\|_2^{q_2-2(\gamma+1)}\right\}\|\nabla u\|_2^{2(\gamma+1)} \\ &\leq \alpha\max\left\{c_*^{q_1}\|\Delta u\|_2^{q_1-2},c_*^{q_2}\|\Delta u\|_2^{q_2-2}\right\}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right) \\ &\quad +(1-\alpha)\max\left\{c_*^{q_1}\|\nabla u\|_2^{q_1-2(\gamma+1)},c_*^{q_2}\|\nabla u\|_2^{q_2-2(\gamma+1)}\right\}\|\nabla u\|_2^{2(\gamma+1)}. \end{aligned}$$

By (3.3) and (3.4), we obtain

$$\int_\Omega |u|^{q(x)}dx\leq \beta_1\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right)+\beta_2\|\nabla u\|_2^{2(\gamma+1)}. \tag{3.5}$$

Since $\beta_1+\beta_2<1$, then

$$\int_\Omega |u|^{q(x)}dx<\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}. \tag{3.6}$$

This implies that

$$I(t)>0,\quad \forall t\in[0,T_*].$$

Repeating the above procedure, we can extend T_* to T. □

Theorem 3.3. *(Existence of weak solution). Suppose that (1.2) holds. Let $u_0 \in L^2(\Omega)$ be given. Then the problem (1.1) admits a weak local solution*

$$u \in L^{\infty}\left((0,T), H_0^2(\Omega)\right), \quad u_t \in L^2\left((0,T), L^2(\Omega)\right).$$

Proof. We shall use the Faedo-Galerkin method of approximation. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^2(\Omega)$ which forms a complete orthonormal system in $L^2(\Omega)$. Denote by

$$V_k = span\{v_1, v_2, ..., v_k\},$$

the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. After normalization, we get $\|v_l\| = 1$ and for any given integer k, we consider the approximate solution

$$u_k(t) = \sum_{l=1}^{k} u_{lk}(t) v_l,$$

where u_k are the solutions to the problem

$$\begin{aligned} & \left(u_k'(t), v_l\right) + \left(\Delta^2 u_k(t), v_l\right) \\ & - \left(M\left(\int_{\Omega} |\nabla u_k(t)|^2 dx\right) \Delta u_k(t), v_l\right) + \left(|u_k(t)|^{p(x)-2} u_k'(t), v_l\right) \\ = & \left(|u_k(t)|^{q(x)-2} u_k(t), v_l\right), \quad l = 1, 2, ..., k, \end{aligned} \tag{3.7}$$

$$u_k(0) = u_{0k} = \sum_{l=1}^{k} (u_k(0), v_l) v_l \to u_0 \text{ in } L^2(\Omega). \tag{3.8}$$

Note that we can solve the system (3.7) and (3.8) by Picard's iterative method for ordinary differential equations. Therefore, there exists a solution in $[0, T_*)$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0, T]$ for any given $T > 0$ by making use of the a priori estimates below. We multiply the equation (3.7) by $u_{lk}'(t)$ and summing over l from 1 to k, we have

$$\begin{aligned} & \frac{d}{dt}\left(\begin{array}{c} \frac{1}{2}\|\Delta u_k(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|^{2(\gamma+1)} \\ -\int_{\Omega} \frac{1}{q(x)} |u_k(t)|^{q(x)} dx \end{array}\right) \\ = & -\left\|u_{t,k}(t)\right\|_2^2 - \int_{\Omega} |u_k(t)|^{p(x)-2} \left|u_{t,k}(t)\right|^2 dx. \end{aligned} \tag{3.9}$$

Then

$$E'(u_k(t)) = -\left\|u_{t,k}(t)\right\|_2^2 - \int_{\Omega} |u_k(t)|^{p(x)-2} \left|u_{t,k}(t)\right|^2 dx \leq 0.$$

Integrating (3.9) over $(0, T)$, we get

$$\begin{aligned} & \frac{1}{2}\|\Delta u_k(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|^{2(\gamma+1)} - \int_{\Omega} \frac{1}{q(x)} |u_k(t)|^{q(x)} dx \\ & + \int_0^t \left\|u_{t,k}(s)\right\|_2^2 ds + \int_0^t \int_{\Omega} |u_k(s)|^{p(x)-2} \left|u_{t,k}(s)\right|^2 dx ds \\ \leq & E(0). \end{aligned} \tag{3.10}$$

Then, from (3.6), the inequality (3.10) becomes

$$\begin{aligned} & \frac{q_1-2}{2q_1} \sup_{t\in(0,T)} \|\Delta u_k(t)\|_2^2 + \frac{q_1-2}{2q_1} \sup_{t\in(0,T)} \|\nabla u_k(t)\|_2^2 \\ & + \frac{q_1 - 2(\gamma+1)}{2(\gamma+1) q_1} \sup_{t\in(0,T)} \|\nabla u_k(t)\|_2^{2(\gamma+1)} + \int_0^t \left\|u_{t,k}(s)\right\|_2^2 ds \\ & + \int_0^t \int_{\Omega} |u_k(s)|^{p(x)-2} \left|u_{t,k}(s)\right|^2 dx ds \\ \leq & E(0). \end{aligned} \tag{3.11}$$

From (3.11), we conclude that

$$\left\{ \begin{array}{l} \{u_k\} \text{ is uniformly bounded in } L^{\infty}\left([0,T], H_0^2(\Omega)\right), \\ \left\{u_k'\right\} \text{ is uniformly bounded in } L^2\left([0,T], L^2(\Omega)\right). \end{array} \right. \tag{3.12}$$

Furthermore, we have from Lemma 2.3 and (3.12) that

$$\begin{cases} \left\{ |u_k|^{q(x)-2} u_k \right\} \text{ is uniformly bounded in } L^\infty\left([0,T], L^2(\Omega)\right), \\ \left\{ |u_k|^{p(x)-2} u_k^{'} \right\} \text{ is uniformly bounded in } L^\infty\left([0,T], L^2(\Omega)\right). \end{cases} \tag{3.13}$$

By (3.12) and (3.13) we infer that there exist a subsequence of u_k and a function u such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly star in } L^\infty\left([0,T], H_0^2(\Omega)\right), \\ u_k^{'} \rightharpoonup u^{'} \text{ weakly star in } L^2\left([0,T], L^2(\Omega)\right), \\ |u_k|^{q(x)-2} u_k \rightharpoonup |u|^{q(x)-2} u \text{ weakly star in } L^\infty\left([0,T], L^2(\Omega)\right), \\ |u_k|^{p(x)-2} u_k^{'} \rightharpoonup |u|^{p(x)-2} u^{'} \text{ weakly star in } L^\infty\left([0,T], L^2(\Omega)\right). \end{cases} \tag{3.14}$$

By the Aubin-Lions compactness lemma (see [21]), we conclude from (3.14) that

$$u_k \rightharpoonup u \text{ strongly in } C\left([0,T], H_0^2(\Omega)\right),$$

yields

$$u_k \rightharpoonup u \text{ everywhere in } \Omega \times [0,T]. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{cases} |u_k|^{q(x)-2} u_k \rightharpoonup |u|^{q(x)-2} u \text{ weakly in } L^\infty\left([0,T], L^2(\Omega)\right), \\ |u_k|^{p(x)-2} u_k^{'} \rightharpoonup |u|^{p(x)-2} u^{'} \text{ weakly in } L^\infty\left([0,T], L^2(\Omega)\right). \end{cases}$$

Letting $k \to \infty$ and passing to the limit in (3.7) we have

$$\begin{aligned} & \left(u^{'}(t), v_l\right) + \left(\Delta^2 u(t), v_l\right) - \left(M\left(\int_\Omega |\nabla u(t)|^2 dx\right) \Delta u(t), v_l\right) \\ & + \left(|u(t)|^{p(x)-2} u_k^{'}(t), v_l\right), \\ = \; & \left(|u(t)|^{q(x)-2} u(t), v_l\right), \quad l = 1, 2, \ldots, k. \end{aligned}$$

Since $\{v_l\}_{l=1}^\infty$ is a basis of $H_0^2(\Omega)$, we deduce that u satisfies equation (1.1). From (3.14) and Lemma 3.1.7 of [22] with $B = L^2(\Omega)$ we infer that

$$u_k(0) \rightharpoonup u(0) \text{ weakly in } L^2(\Omega). \tag{3.16}$$

We get from (3.8) and (3.16) that $u(0) = u_0$. The proof of the Theorem is now finished. □

Theorem 3.4. *Let the assumptions of Lemma 3.2 hold. Then the local solution of (1.1) is global.*

Proof. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx, \\ &\geq \frac{q_1-2}{2q_1}\|\Delta u\|_2^2 + \frac{q_1-2}{2q_1}\|\nabla u\|_2^2 + \frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\|\nabla u\|_2^{2(\gamma+1)}, \end{aligned}$$

which implies that

$$\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)} \leq CE(t). \tag{3.17}$$

By Lemma 3.1, we get

$$\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)} \leq CE(0).$$

□

4. Decay

In this part, we state and prove the decay of solutions. Firstly, we give the following lemma.

Lemma 4.1. *Let the assumptions of Lemma 3.2 hold. Then*

$$\int_{\Omega} |u|^{p(x)}\, dx \leq cE(t),$$

where $c > 0$.

Proof.

$$\begin{aligned}\int_{\Omega} |u|^{p(x)}\, dx &= \max\left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\}, \\ &\leq \max\left\{ c_*^{p_1} \|\Delta u\|_2^{p_1}, c_*^{p_2} \|\Delta u\|_2^{p_2} \right\}, \\ &\leq \max\left\{ c_*^{p_1} \|\Delta u\|_2^{p_1-2}, c_*^{p_2} \|\Delta u\|_2^{p_2-2} \right\} \|\Delta u\|_2^2.\end{aligned}$$

Using (3.3), we have

$$\int_{\Omega} |u|^{p(x)}\, dx \leq cE(t).$$

□

Theorem 4.2. *Let the assumptions of Lemma 3.2 hold. Then*

$$E(t) \leq E(0) \left(\frac{c+rt}{c+rc} \right)^{-1/r}, \quad \forall t \geq c,$$

where $c > 0$.

Proof. Multiplying the equation (1.1) by $u(t)E^q(t)$ $(q>0)$ and then integrating over $\Omega \times (S,T)$, we get

$$\begin{aligned}&\int_S^T \int_{\Omega} E^q(t) \left[u\Delta^2 u + uu_t - u \left(M \left(\int_{\Omega} |\nabla u|^2\, dx \right) \Delta u + uu_t |u|^{p(x)-2} \right) \right] dxdt \\ &= \int_S^T E^q(t) \int_{\Omega} |u|^{q(x)}\, dxdt.\end{aligned}$$

Then

$$\begin{aligned}&\int_S^T \int_{\Omega} E^q(t) \left(|\Delta u|^2 + uu_t + |\nabla u|^2 + \|\nabla u\|_2^{2\gamma} |\nabla u|^2 + uu_t |u|^{p(x)-2} \right) dxdt \\ &= \int_S^T E^q(t) \int_{\Omega} |u|^{q(x)}\, dxdt.\end{aligned}$$

We adding and substracting the term

$$\int_S^T E^q(t) \int_{\Omega} \left(\beta_1 \left(|\Delta u|^2 + |\nabla u|^2 \right) + \beta_2 \|\nabla u\|_2^{2\gamma} |\nabla u|^2 \right) dxdt,$$

and use (3.5), we obtain

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega \left(|\Delta u|^2\right) dxdt \\
&+(1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u|^2 dxdt \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left(\|\nabla u\|_2^{2\gamma}|\nabla u|^2\right) dxdt \\
&+\int_S^T E^q(t)\int_\Omega (uu_t)\, dxdt \\
&+\int_S^T E^q(t)\int_\Omega \left(uu_t|u|^{p(x)-2}\right) dxdt \\
= &-\int_S^T E^q(t)\int_\Omega \left(\beta_1|\Delta u|^2+\beta_1|\nabla u|^2+\beta_2\|\nabla u\|_2^{2\gamma}|\nabla u|^2-|u|^{q(x)}\right) dxdt \\
\le\ & 0. \qquad (4.1)
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\xi\int_S^T E^q(t)\int_\Omega \left(\begin{array}{c}\frac{1}{2}|\Delta u|^2+\frac{1}{2}|\nabla u|^2 \\ +\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma}|\nabla u|^2-\frac{|u(t)|^{q(x)}}{q(x)}\end{array}\right) dxdt \\
\le\ & (1-\beta_1)\int_S^T E^q(t)\int_\Omega |\Delta u|^2 dxdt \\
&+(1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u|^2 dxdt \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \|\nabla u\|_2^{2\gamma}|\nabla u|^2 dxdt, \qquad (4.2)
\end{aligned}$$

where

$$\xi=\min\{(1-\beta_1),(1-\beta_2)\}.$$

By (4.1), (4.2) and the definition of $E(t)$, we otain

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt \ \le\ & -\int_S^T E^q(t)\int_\Omega uu_t dxdt \qquad (4.3)\\
&-\int_S^T E^q(t)\int_\Omega uu_t|u|^{p(x)-2} dxdt.
\end{aligned}$$

We estimate the terms on the right-hand side of (4.3). For the first term, we use the Young's inequality

$$AB\le\frac{\varepsilon}{\eta_1}A^{\eta_1}+\frac{1}{\eta_2\varepsilon^{\eta_2/\eta_1}}B^{\eta_2},\ \ A,B\ge 0,\ \varepsilon>0 \text{ and } \frac{1}{\eta_1}+\frac{1}{\eta_2}=1,$$

and get

$$-\int_S^T E^q(t)\int_\Omega uu_t dxdt\le\int_S^T E^q(t)\int_\Omega \left(\varepsilon c|u|^2+c_\varepsilon|u_t|^2\right) dxdt. \qquad (4.4)$$

We use again the Young's inequality to get

$$\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega uu_t|u|^{p(x)-2} dxdt \\
=\ & -\int_S^T E^q(t)\int_\Omega |u|^{(p(x)-2)/2}u_t|u|^{(p(x)-2)/2}u dxdt \\
\le\ & \int_S^T E^q(t)\int_\Omega \left(\varepsilon c|u|^{p(x)}+c_\varepsilon|u_t|^{p(x)-2}u_t^2\right) dxdt. \qquad (4.5)
\end{aligned}$$

By (4.4) and (4.5) the inequality (4.3) becomes

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt &\le \int_S^T E^q(t)\int_\Omega\left(\varepsilon c|u|^2+c_\varepsilon|u_t|^2\right)dxdt\\
&\quad+\int_S^T E^q(t)\int_\Omega\left(\varepsilon c|u|^{p(x)}+c_\varepsilon|u_t|^{p(x)-2}u_t^2\right)dxdt\\
&\le \varepsilon c\int_S^T E^q(t)\int_\Omega\left(|u|^2+|u|^{p(x)}\right)dxdt\\
&\quad+c_\varepsilon\int_S^T E^q(t)\int_\Omega\left(|u_t|^2+|u|^{p(x)-2}u_t^2\right)dxdt.
\end{aligned}$$

We use (3.17) , Lemma 4.1 and definition of $E^{'}(t)$ to obtain

$$\xi\int_S^T E^{q+1}(t)\,dt\le\varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\int_S^T E^q(t)\left(-E^{'}(t)\right)dt.$$

This implies

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt &\le \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\left(E^{q+1}(s)-E^{q+1}(T)\right)\\
&\le \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon E^q(0)E(s).
\end{aligned}$$

Choosing ε so small such that $\xi>\varepsilon c$, we arrive at

$$\int_S^T E^{q+1}(t)\,dt\le cE^q(0)E(s).$$

By taking $T\to\infty$, we obtain

$$\int_S^\infty E^{q+1}(t)\,dt\le cE^q(0)E(s).$$

Thus, Komornik's Lemma implies the desired result. □

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