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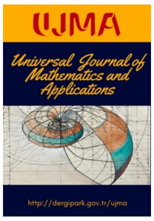

Weighted Ostrowski's Type Integral Inequalities for Mapping Whose Second Derivative is Bounded

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Abstract

The aim of this paper is to concentrate on the domain of L_∞, L_p, and L_1 norms of inequalities and their applications for some special weight functions. For different weights some previous results are recaptured. Applications are also discussed.

1. Introduction

In 1938, Ostrowski [1] established the interesting integral inequality for differentiable mappings with bounded derivative.

Lemma 1.1. *Let $f:[\breve{a},\hat{c}]\to\mathbb{R}$ be continuous on $[\breve{a},\hat{c}]$ and differentiable on $(\breve{a},\hat{c})$ and assume $\left|f'(\hat{s})\right|\le M$ for all $\hat{s}\in(\breve{a},\hat{c})$. Then the inequality*

$$|S(f;\breve{a},\hat{c})|\le\left[\left(\frac{\breve{a}-\hat{c}}{2}\right)^2+\left(\hat{s}-\frac{\breve{a}+\hat{c}}{2}\right)^2\right]\frac{M}{\breve{a}-\hat{c}} \tag{1.1}$$

holds for all $\hat{s}\in[\breve{a},\hat{c}]$. The constant $\frac{1}{4}$ is the best possible.

Then Cerone [2], Dragomir et al. [3] and Sarıkaya et al. [4] also worked on this inequality. A. Qayyum et al. [5–9] worked on generalization of Ostrowski's type inequalities. Different authors worked on the generalization of Ostrowski's type inequalities that are [10], [11] and [12]. Some latest work done by S. Fahad et al. [13]. Further works done by Iftikhar et al. [14], Mustafa et al. [15] and J. Amjad et al. [16].
Let the functional $S(f;\omega;\breve{a},\hat{c})$ via weighted version represent the deviation of $f(\hat{s})$ over $[\breve{a},\hat{c}]$ defined as:

$$S(f;\omega;\breve{a},\hat{c})=f(\hat{s})-M(f;\omega;\breve{a},\hat{c}), \tag{1.2}$$

where $f(\hat{s})$ is continuous function and $M(f;\omega;\breve{a},\hat{c})$ is weighted integral mean defined as:

$$M(f;\omega;\breve{a},\hat{c})=\frac{1}{\hat{c}-\breve{a}}\int_{\breve{a}}^{\hat{c}}f(\hat{y})\omega(\hat{y})d\hat{y}. \tag{1.3}$$

We suppose a weight function $\omega:(\breve{a},\hat{c})\to[0,\infty)$ is integrable on $[0,\infty)$ such that

$$\int_{\breve{a}}^{\hat{c}}\omega(\hat{y})d\hat{y}<\infty. \tag{1.4}$$

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We define m, m_1, m_2 and notations μ and σ as:

$$m(\breve{a},\hat{c}) = \int_{\breve{a}}^{\hat{c}} \omega(\hat{y})d\hat{y}, \quad m_1(\breve{a},\hat{c}) = \int_{\breve{a}}^{\hat{c}} \hat{y}\omega(\hat{y})d\hat{y}, \tag{1.5}$$

$$m_2(\breve{a},\hat{c}) = \int_{\breve{a}}^{\hat{c}} \hat{y}^2\omega(\hat{y})d\hat{y}, \quad \mu(\breve{a},\hat{c}) = \frac{m_1(\breve{a},\hat{c})}{m(\breve{a},\hat{c})}, \tag{1.6}$$

$$\sigma^2(\breve{a},\hat{c}) = \frac{m_2(\breve{a},\hat{c})}{m(\breve{a},\hat{c})} - \mu^2(\breve{a},\hat{c}) \tag{1.7}$$

2. Main Result

Lemma 2.1. *Let $f : [\breve{a},\hat{c}] \to R$ be continuous on $[\breve{a},\hat{c}]$ and twice differentiable mapping on $(\breve{a},\hat{c})$, then the following weighted Peano kernel, define $\mathbb{k}(.,.) : [\breve{a},\hat{c}]^2 \to \mathbb{R}$ as:*

$$\mathbb{k}(\hat{s},\hat{y}) = \begin{cases} \frac{\Phi}{\Phi+\Psi}\frac{1}{\hat{s}-\breve{a}}\int_{\breve{a}}^{\hat{y}}(\hat{y}-\breve{u})\,\omega(\breve{u})\,d\breve{u}, & if \quad \breve{a} \le \hat{y} \le \hat{s} \\ \frac{\Psi}{\Phi+\Psi}\frac{1}{\hat{c}-\hat{s}}\int_{\hat{c}}^{\hat{y}}(\hat{y}-\breve{u})\,\omega(\breve{u})\,d\breve{u}, & if \quad \hat{s} < \hat{y} \le \hat{c}, \end{cases} \tag{2.1}$$

where $\Phi, \Psi \in \mathbb{R}$ are non negative and both are non zero at the same time, $\forall\, \hat{y} \in [\breve{a},\hat{c}]$, $\hat{s} \in [\breve{a},\hat{c}]$ and ω is weight function as stated in (1.4). Before we state and prove our main result, we will prove the following identity by using integration by parts techniques, moments and notations. Then the following weighted integral identity

$$\tau(\omega;\hat{s};\Phi,\Psi) = \int_{\breve{a}}^{\hat{c}} \mathbb{k}(\hat{s},\hat{y})f''(\hat{y})d\hat{y} \tag{2.2}$$

$$= \breve{a}_1 f(\hat{s}) + \frac{1}{\Phi+\Psi} \times \left[\left(\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})] + \frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]\right)f'(\hat{s}) + (\{\Phi M(f;\omega,\breve{a},\hat{s}) + \Psi M(f;\omega,\hat{s},\hat{c})\})\right] \tag{2.3}$$

holds, here

$$\breve{a}_1 = \frac{-1}{\Phi+\Psi}\left(\frac{\Phi m(\breve{a},\hat{s})}{(\hat{s}-\breve{a})} + \frac{\Psi m(\hat{c},\hat{s})}{(\hat{c}-\hat{s})}\right),$$

and $M(f;\omega,\breve{a},\hat{c})$ is weighted integral mean as defined in (1.3).

Proof. From (2.1), we have

$$\int_{\breve{a}}^{\hat{c}} \mathbb{k}(\hat{s},\hat{y})f''(\hat{y})d\hat{y} = \frac{\Phi}{(\Phi+\Psi)(\hat{s}-\breve{a})}\int_{\breve{a}}^{\hat{s}}\int_{\breve{a}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}f''(\hat{y})d\hat{y} + \frac{\Psi}{(\Phi+\Psi)(\hat{c}-\hat{s})}\int_{\hat{s}}^{\hat{c}}\int_{\hat{c}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}f''(\hat{y})d\hat{y}.$$

After some calculations, we get

$$\int_{\breve{a}}^{\hat{c}} \mathbb{k}(\hat{s},\hat{y})f''(\hat{y})d\hat{y} = \frac{1}{\Phi+\Psi}\left[-\left(\frac{\Phi m(\breve{a},\hat{s})}{(\hat{s}-\breve{a})} + \frac{\Psi m(\hat{s},\hat{c})}{(\hat{c}-\hat{s})}\right)f(\hat{s}) + \left(\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})] + \frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]\right)f'(\hat{s}) \right.$$
$$\left. + \frac{\Phi}{\hat{s}-\breve{a}}\int_{\breve{a}}^{\hat{s}}\omega(\hat{y})f(\hat{y})d\hat{y} + \frac{\Psi}{(\hat{c}-\hat{s})}\int_{\hat{s}}^{\hat{c}}\omega(\hat{y})f(\hat{y})d\hat{y}\right],$$

here the integration by parts formula has been utilised on the separate interval $[\breve{a},\hat{s}]$ and $(\hat{s},\hat{c}]$.
Simlification of the expressions readily produces the identity as stated in (2.2), $\forall\, \hat{s} \in [\breve{a},\hat{c}]$. □

Theorem 2.2. *Let $f : [\breve{a},\hat{c}] \to R$ be continuous on $[\breve{a},\hat{c}]$ and twice differentiable mapping on $(\breve{a},\hat{c})$, whose second derivative $f'' : [\breve{a},\hat{c}]^2 \to R$ is bounded on $(\breve{a},\hat{c})$, then following weighted integral inequalities*

$$|\tau(\omega;\hat{s};\Phi,\Psi)| \le \begin{cases} \frac{\|f''\|_\infty}{2(\Phi+\Psi)}\left[\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}\left([\hat{s}-\mu(\breve{a},\hat{s})]^2 + \sigma^2(\breve{a},\hat{s})\right) + \frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}\left([\hat{s}-\mu(\hat{s},\hat{c})]^2 + \sigma^2(\hat{s},\hat{c})\right)\right] & for\ f'' \in L_\infty[\breve{a},\hat{c}] \\ \frac{\omega(\hat{s})\|f''\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1} + \Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & for\ f'' \in L_p[\breve{a},\hat{c}] \\ \frac{\gamma\|f''\|_1}{2(\Phi+\Psi)}\left[1 + \frac{|\varpi|}{\gamma}\right] & for\ f'' \in L_1[\breve{a},\hat{c}], \end{cases}$$

hold for $\forall\,\hat{y}\in[\breve{a},\hat{c}]$, $\hat{s}\in[\breve{a},\hat{c}]$ *and* ω *is weight function as stated in* (1.4), *and* $\Phi,\Psi\in\mathbb{R}$ *are non negative and both are non zero at the same time. Here* $\frac{1}{p}+\frac{1}{\hat{q}}=1,\ (p>1)$,

$$\gamma=\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left[\Phi m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\ +\Psi m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right]$$

and

$$\varpi=\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left[\Phi m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\ -\ \Psi m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right].$$

Proof. Take the modulus of (2.2)

$$|\tau(\omega;\hat{s};\Phi,\Psi)|=\left|\int_{\breve{a}}^{\hat{c}}\Bbbk(\hat{s},\hat{y})f''(\hat{y})d\hat{y}\right|\le\int_{\breve{a}}^{\hat{c}}|\Bbbk(\hat{s},\hat{y})|\left|f''(\hat{y})\right|d\hat{y},\tag{2.4}$$

here we use properties of the integral and modulus

$$|\tau(\varpi;\ddot{z};\varepsilon,\delta)|\le\left\|f''\right\|_\infty\int_{\breve{a}}^{\hat{c}}|\Bbbk(\hat{s},\hat{y})|\,d\hat{y}.$$

By using (2.1) we prove

$$\int_{\breve{a}}^{\hat{c}}|\Bbbk(\hat{s},\hat{y})|\,d\hat{y}=\int_{\breve{a}}^{\hat{s}}\int_{\breve{a}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}d\hat{y}+\int_{\hat{s}}^{\hat{c}}\int_{\hat{c}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}d\hat{y}$$

by using the techniques of J. Roummeliotis et al. [17],

$$\int_{\breve{a}}^{\hat{s}}\int_{\breve{a}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}d\hat{y}+\int_{\hat{s}}^{\hat{c}}\int_{\hat{c}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}d\hat{y}=\frac{1}{2}\int_{\breve{a}}^{\hat{c}}(\hat{s}-\hat{y})^2\,\omega(\hat{y})\,d\hat{y}$$

after some calculation we get

$$\int_{\breve{a}}^{\hat{c}}|\Bbbk(\hat{s},\hat{y})|\,d\hat{y}=\frac{\Phi m(\breve{a},\hat{s})}{2(\Phi+\Psi)(\hat{s}-\breve{a})}\left[[\hat{s}+\mu(\breve{a},\hat{s})]^2+\sigma^2(\breve{a},\hat{s})\right]+\frac{\Psi m(\hat{s},\hat{c})}{2(\Phi+\Psi)(\hat{c}-\hat{s})}\left[[\hat{s}+\mu(\hat{s},\hat{c})]^2+\sigma^2(\hat{s},\hat{c})\right].$$

From above, first inequality is obtained.
Further, using Holder's Inequality, we have for $f''\in L_p[\breve{a},\hat{c}]$, from (2.4)

$$|\tau(\omega;\hat{s};\Phi,\Psi)|\le\left\|f''\right\|_p\left(\int_{\breve{a}}^{\hat{c}}|\Bbbk(\hat{s},\hat{y})|^{\hat{q}}\,d\hat{y}\right)^{\frac{1}{\hat{q}}},$$

here $\frac{1}{p}+\frac{1}{\hat{q}}=1,\ (p>1)$,
by using Mean Value Theorem, we get

$$\left(\int_{\breve{a}}^{\hat{c}}|\Bbbk(\hat{s},\hat{y})|^{\hat{q}}\,d\hat{y}\right)^{\frac{1}{\hat{q}}}=\frac{1}{2(\Phi+\Psi)(2\hat{q}+1)^{\frac{1}{\hat{q}}}}\left(\Phi^{\hat{q}}(\hat{s}-\hat{y})^{\hat{q}+1}\omega(\hat{s})\right)^{\frac{1}{\hat{q}}}+\frac{1}{2(\Phi+\Psi)(2\hat{q}+1)^{\frac{1}{\hat{q}}}}\left(\Psi^{\hat{q}}(\hat{s}-\hat{y})^{\hat{q}+1}\omega(\hat{s})\right)^{\frac{1}{\hat{q}}},$$

so the second inequality is obtained.
Finally, for $f''\in L_1[\breve{a},\hat{c}]$ we have from (2.4)

$$|\tau(\omega;\hat{s};\Phi,\Psi)|\le\sup_{\hat{y}\in[\breve{a},\hat{c}]}|\Bbbk(\hat{s},\hat{y})|\left\|f''\right\|_1.$$

By using (2.1), we prove

$$\begin{aligned}\sup_{\hat{y}\in[\breve{a},\hat{c}]}|\Bbbk(\hat{s},\hat{y})|=&\frac{1}{(\Phi+\Psi)}\max\left\{\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})],\frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]\right\}\\=&\frac{1}{2(\Phi+\Psi)(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left(\Phi m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\right.\\&\left.+\Psi m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right)\left[1+\frac{\left|\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}[\Phi m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]-\Psi m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]]\right|}{\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}[\Phi m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]+\Psi m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]]}\right].\end{aligned}$$

Hence (2.2) is proved. □

Remark 2.3. *From* (2.2) *and* (1.2)

$$(\Phi+\Psi)\,\tau(\omega;\hat{s};\Phi,\Psi)=\Phi S(f;\varpi;\breve{a},\hat{s})+\Psi S(f;\varpi;\hat{s},\hat{c})$$

and using triangular inequality in (2.2), *we get*

$$|(\varepsilon+\delta)\,\tau(\varpi;\ddot{z};\varepsilon,\delta)|\leq\begin{cases}\frac{\Phi m(\breve{a},\hat{s})\|f''\|_{\infty,[\breve{a},\hat{s}]}}{2(\hat{s}-\breve{a})}\left([\hat{s}-\mu(\breve{a},\hat{s})]^2+\sigma^2(\breve{a},\hat{s})\right)\\ +\frac{\Psi m(\hat{s},\hat{c})\|f''\|_{\infty,[\hat{s},\hat{c}]}}{2(\hat{c}-\hat{s})}\left([\hat{s}-\mu(\hat{s},\hat{c})]^2+\sigma^2(\hat{s},\hat{c})\right) & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{\omega(\hat{s})\left\|f''\right\|_{p,[\breve{a},\hat{s}]}}{2}\left(\frac{\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}}{2\hat{q}+1}\right)^{\frac{1}{\hat{q}}}\\ +\frac{\omega(\hat{s})\left\|f''\right\|_{p,[\hat{s},\hat{c}]}}{2}\left(\frac{\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}}{2\hat{q}+1}\right)^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\gamma}{2}\left\|f''\right\|_{1,[\breve{a},\hat{s}]}+\frac{|\varpi|}{2}\left\|f''\right\|_{1,[\hat{s},\hat{c}]} & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}\tag{2.5}$$

Remark 2.4. *Since we may write* (2.2) *as*

$$\begin{aligned}\Phi M(f;\omega;\breve{a},\hat{s})+\Psi M(f;\omega;\hat{s},\hat{c})&=\Phi M(f;\omega;\breve{a},\hat{s})+\frac{\Psi}{\hat{c}-\hat{s}}\left(\int_{\breve{a}}^{\hat{c}}\omega(\breve{u})f(\breve{u})d\breve{u}-\int_{\breve{a}}^{\hat{s}}\omega(\breve{u})f(\breve{u})d\breve{u}\right)\\&=\left[\Phi+\Psi\left(\frac{\hat{s}-\breve{a}}{\hat{c}-\hat{s}}\right)\right]M(f;\omega;\breve{a},\hat{s})+\Psi\left(\frac{\hat{c}-\breve{a}}{\hat{c}-\hat{s}}\right)M(f;\omega;\breve{a},\hat{c}).\end{aligned}$$

Thus, the identity

$$\tau(\omega;\hat{s};\Phi,\Psi)=$$
$$\breve{a}_1 f(\hat{s})+\frac{1}{\Phi+\Psi}\left[\left(\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})]^2+\frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]^2\right)f'(\hat{s})\;+\left(1-\frac{\Psi\lambda}{\Phi+\Psi}\right)M(f;\omega;\breve{a},\hat{s})+\frac{\Psi\lambda}{\Phi+\Psi}M(f;\omega;\breve{a},\hat{c})\right],$$

same as $[\breve{a},\hat{c}]$ and $M(f;\omega;\breve{a},\hat{c})$ is also fixed.

Corollary 2.5. *Let the conditions of Theorem 2 hold. Then the results for* $\Phi=\Psi$

$$|\tau(\omega;\hat{s};\Phi,\Psi)|\leq\begin{cases}\frac{\left\|f''\right\|_\infty}{4}\left[\frac{m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}\left([\hat{s}-\mu(\breve{a},\hat{s})]^2f'(\hat{s})\;\;-f(\hat{s})+\sigma^2(\breve{a},\hat{s})\right)\right.\\ \left.+\frac{m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}\left([\hat{s}-\mu(\hat{s},\hat{c})]^2f'(\hat{s})\;\;-f(\hat{s})+\sigma^2(\hat{s},\hat{c})\right)\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{\omega(\hat{s})\left\|f''\right\|_p}{4(2\hat{q}+1)^{\frac{1}{\hat{q}}}}\left[(\hat{s}-\breve{a})^{\hat{q}+1}+(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\zeta\left\|f'\right\|_1}{4}\left[1+\frac{|\eta|}{\zeta}\right] & \text{for } f''\in L_1[\breve{a},\hat{c}],\end{cases}\tag{2.6}$$

here

$$\begin{aligned}\tau(\omega;\hat{s};\Phi,\Phi)=&\frac{-1}{2}\left(\frac{m(\breve{a},\hat{s})}{(\hat{s}-\breve{a})}+\frac{m(\hat{c},\hat{s})}{(\hat{c}-\hat{s})}\right)f(\hat{s})+\frac{1}{2}\left[\left(\frac{m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})]^2+\frac{m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]^2\right)f'(\hat{s})\right.\\&\left.+(\{M(f;\omega,\breve{a},\hat{s})+M(f;\omega,\hat{s},\hat{c})\})\right],\end{aligned}$$

$$\zeta=\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left[m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\;+m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right]$$

and

$$\eta=\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left[m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\;-\;m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right].$$

Proof. The result is readily obtained on allowing $\Phi=\Psi$ in (2.2) so that the left hand side is $\tau(\omega;\hat{s};\Phi,\Phi)$ from (2.2). □

Corollary 2.6. *According to Theorem 2, then mid point* $\left(\hat{s}=\breve{A}=\frac{\breve{a}+\hat{c}}{2}\right)$, *inequality from* (2.2)

$$\left|\tau\left(\omega;\breve{A};\Phi,\Psi\right)\right|\leq\begin{cases}\frac{\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left[\frac{\Phi m(\breve{a},\breve{A})}{\breve{A}-\breve{a}}\left(\left[\breve{A}-\mu(\breve{a},\breve{A})\right]^2+\sigma^2\left(\breve{a},\breve{A}\right)\right)\right.\\ \left.+\frac{\Psi m(\breve{A},\hat{c})}{\hat{c}-\breve{A}}\left(\left[\breve{A}-\mu(\breve{A},\hat{c})\right]^2+\sigma^2\left(\breve{A},\hat{c}\right)\right)\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{\omega(\breve{A})\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}\left(\breve{A}-\breve{a}\right)^{\hat{q}+1}+\Psi^{\hat{q}}\left(\hat{c}-\breve{A}\right)^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\psi\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[1+\frac{|\varkappa|}{\psi}\right] & \text{for } f''\in L_1[\breve{a},\hat{c}],\end{cases}\tag{2.7}$$

here

$$\psi=\frac{1}{2(\Phi+\Psi)(\breve{A}-\breve{a})(\hat{c}-\breve{a})}\left[\Phi m(\breve{a},\breve{A})(\hat{c}-\breve{a})\;[\breve{A}-\mu(\breve{a},\breve{A})]\;+\Psi m(\breve{A},\hat{c})(\hat{c}-\breve{a})[\breve{A}-\mu(\breve{A},\hat{c})]\right]$$

and

$$\varkappa=\frac{1}{(\breve{A}-\breve{a})(\hat{c}-\breve{a})}\left[\left|\Phi m(\breve{a},\breve{A})(\hat{c}-\breve{a})\;[\breve{A}-\mu(\breve{a},\breve{A})]\;-\Psi m(\breve{A},\hat{c})(\hat{c}-\breve{a})[\breve{A}-\mu(\breve{A},\hat{c})]\right]\right|.$$

Proof. Placing $\left(\hat{s}=\breve{A}=\frac{\breve{a}+\hat{c}}{2}\right)$ in (2.2) and (2.2) produces the results as stated in (2.7). □

Corollary 2.7. *When the conditions of Theorem 2 hold and (2.7) is evaluated at $\Phi=\Psi$, then we get*

$$|\tau(\omega;\breve{A};\Phi,\Phi)|\leq\begin{cases}\frac{\|f''\|_\infty}{4}\left[\frac{m(\breve{a},\breve{A})}{\breve{A}-\breve{a}}\left([\breve{A}-\mu(\breve{a},\breve{A})]^2+\sigma^2(\breve{a},\breve{A})\right)+\frac{m(\breve{A},\hat{c})}{\hat{c}-\breve{A}}\left([\breve{A}-\mu(\breve{A},\hat{c})]^2+\sigma^2(\breve{A},\hat{c})\right)\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{\omega(\breve{A})\|f''\|_p}{4(2\hat{q}+1)^{\frac{1}{\hat{q}}}}\left[(\breve{A}-\breve{a})^{\hat{q}+1}+(\hat{c}-\breve{A})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\psi\|f''\|_1}{4}\left[1+\frac{|\varkappa|}{\psi}\right] & \text{for } f''\in L_1[\breve{a},\hat{c}],\end{cases} \tag{2.8}$$

here

$$\psi=\frac{1}{(\breve{A}-\breve{a})(\hat{c}-\breve{a})}\left[m(\breve{a},\breve{A})(\hat{c}-\breve{a})\;[\breve{A}-\mu(\breve{a},\breve{A})]+m(\breve{A},\hat{c})(\hat{c}-\breve{a})[\breve{A}-\mu(\breve{A},\hat{c})]\right]$$

and

$$\varkappa=\frac{1}{(\breve{A}-\breve{a})(\hat{c}-\breve{a})}\left[m(\breve{a},\breve{A})(\hat{c}-\breve{a})\;[\breve{A}-\mu(\breve{a},\breve{A})]-m(\breve{A},\hat{c})(\hat{c}-\breve{a})[\breve{A}-\mu(\breve{A},\hat{c})]\right].$$

Proof. Putting $\Phi=\Psi$; in (2.7) we get (2.8). □

Remark 2.8. *For $\varpi(\hat{s})=1$ in (2.2), (2.5), (2.6), (2.7), and in (2.8) we get A. Qayyum et al.'s result [7].*

2.1. Applications for some special means:

Now we discuss applications for some special means by taking different weight.

Remark 2.9. *For Uniform (legendre) mean:*

Let $\varpi(\hat{s})=1$ put in (2.2) and in (2.2), we get A. Qayyum et al.'s results [7].

Remark 2.10. *For Logarithm mean:*

Let

$$\omega(\hat{y})=\ln(1/\hat{y});\quad \breve{a}=0,\ \hat{c}=1,$$

put in (1.7), we get

$$\mu(0,1)=\frac{\int_0^1\hat{y}\ln(1/\hat{y})\,d\hat{y}}{\int_0^1\ln(1/\hat{y})\,d\hat{y}}=\frac{1}{4}$$

and

$$\sigma^2(0,1)=\frac{\int_0^1\hat{y}^2\ln(1/\hat{y})\,d\hat{y}}{\int_0^1\ln(1/\hat{y})\,d\hat{y}}-(\mu(0,1))^2=\frac{7}{144},$$

put in (2.2), then the inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\int_{0}^{1}\ln(1/\hat{y})f(\hat{y})d\hat{y}-f(\hat{s})+\left(\hat{s}-\frac{1}{4}\right)f^{'}(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{\ln(1/\hat{y})\left\|f''\right\|_{\infty}}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[\left(\hat{s}-\frac{1}{4}\right)^{2}+\frac{7}{144}\right] & \text{for } f''\in L_{\infty}[\breve{a},\hat{c}]\\ \frac{\ln(1/\hat{y})\left\|f''\right\|_{p}}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_{p}[\breve{a},\hat{c}]\\ \frac{\left\|f''\right\|_{1}}{2(\Phi+\Psi)}\left[\Phi\ln(1/\hat{y})\left(\hat{s}-\frac{1}{4}\right)+\Psi\ln(1/\hat{y})\left(\hat{s}-\frac{1}{4}\right)\ \left|\Phi\ln(1/\hat{y})\left(\hat{s}-\frac{1}{4}\right)-\Psi\ln(1/\hat{y})\left(\hat{s}-\frac{1}{4}\right)\right|\right] & \text{for } f''\in L_{1}[\breve{a},\hat{c}].\end{cases}$$

The mid point reflecting if the optimum point $\hat{s}=\mu(0,1)=\frac{1}{4}$ is near to the origin.

Remark 2.11. *For Jacobi mean:*

Let

$$\omega(\hat{y})=1/\sqrt{\hat{y}};\quad \breve{a}=0,\quad \hat{c}=1,$$

we have

$$\mu(0,1)=\frac{\int_{0}^{1}\sqrt{\hat{y}}d\hat{y}}{\int_{0}^{1}1/\sqrt{\hat{y}}d\hat{y}}=\frac{1}{3}$$

and

$$\sigma^{2}(0,1)=\frac{\int_{0}^{1}\hat{y}\sqrt{\hat{y}}d\hat{y}}{\int_{0}^{1}1/\sqrt{\hat{y}}d\hat{y}}-\left(\frac{1}{3}\right)^{2}=\frac{4}{45}.$$

Then

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\int_{0}^{1}\frac{f(\hat{y})}{\sqrt{\hat{y}}}d\hat{y}-f(\hat{s})+(\hat{s}-1)f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{1/\sqrt{\hat{y}}\left\|f''\right\|_{\infty}}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[1+(\hat{s}-1)^{2}\right] & \text{for } f''\in L_{\infty}[\breve{a},\hat{c}]\\ \frac{1/\sqrt{\hat{y}}\left\|f''\right\|_{p}}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_{p}[\breve{a},\hat{c}]\\ \frac{\left\|f''\right\|_{1}}{2(\Phi+\Psi)}\left[\Phi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{3}\right)+\Psi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{3}\right)\ \left|\Phi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{4}\right)-\Psi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{4}\right)\right|\right] & \text{for } f''\in L_{1}[\breve{a},\hat{c}].\end{cases}$$

The optimum point $\hat{s}=\mu(0,1)=\frac{1}{3}$ is moved to the left of midpoint.

Remark 2.12. *For Chebyshev mean:*

Let

$$\omega(\hat{y})=1/\sqrt{1-\hat{y}^{2}};\quad \breve{a}=-1,\ \hat{c}=1,$$

mean

$$\mu(-1,1)=0$$

and

$$\sigma^{2}(-1,1)=\frac{1}{2}.$$

Hence, Chebyshev weighted inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\frac{1}{\pi}\int_{-1}^{1}\frac{f(\hat{y})}{\sqrt{1-\hat{y}}}d\hat{y}-f(\hat{s})+\hat{s}f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{1/\sqrt{1-\hat{y}^2}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[\hat{s}^2+\frac{1}{2}\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{1/\sqrt{1-\hat{y}^2}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[\hat{s}\Phi/\sqrt{1-\hat{y}^2}+\hat{s}\Psi/\sqrt{1-\hat{y}^2}\ \left|\Phi\hat{s}/\sqrt{1-\hat{y}^2}-\Psi\hat{s}/\sqrt{1-\hat{y}^2}\right|\right] & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}$$

The optimum point $\hat{s}=\mu(-1,1)=0$ is at the midpoint of the interval.

Remark 2.13. *For Laguerre mean:*

Let

$$\omega(\hat{y})=e^{-\hat{y}};\quad \breve{a}=0,\quad \hat{c}=\infty,$$

such that

$$\mu(0,\infty)=1$$

and

$$\sigma^2(0,\infty)=1$$

then inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\int_{0}^{\infty}e^{-\hat{y}}f(\hat{y})d\hat{y}-f(\hat{s})+(\hat{s}-1)f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{e^{-\hat{y}}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[1+(\hat{s}-1)^2\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{e^{-\hat{y}}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[\Phi e^{-\hat{y}}(\hat{s}-1)+\Psi e^{-\hat{y}}(\hat{s}-1)\ \left|\Phi e^{-\hat{y}}(\hat{s}-1)-\Psi e^{-\hat{y}}(\hat{s}-1)\right|\right] & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}$$

The optimum sample point is deduced $\hat{s}=1$.

Remark 2.14. *For Hermite mean:*

Let

$$\omega(\hat{y})=e^{-\hat{y}^2};\quad \breve{a}=-\infty,\quad \hat{c}=\infty,$$

then

$$\mu(-\infty,\infty)=0$$

and

$$\sigma^2(-\infty,\infty)=\frac{1}{2}.$$

Then inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-\hat{y}^2}f(\hat{y})d\hat{y}-f(\hat{s})+\hat{s}f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{e^{-\hat{y}^2}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[\frac{1}{2}+\hat{s}^2\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{e^{-\hat{y}^2}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\|f''\|_1}{2(\Phi+\Psi)}\left[\Phi\hat{s}e^{-\hat{y}^2}+\Psi\hat{s}e^{-\hat{y}^2}\ \left|\Phi\hat{s}e^{-\hat{y}^2}-\Psi\hat{s}e^{-\hat{y}^2}\right|\right] & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}$$

An optimum sampling point is $\hat{s}=0$.

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Singular Perturbations of Multibrot Set Polynomial[†]

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Abstract

We will give a complete description of the dynamics of the rational map $N_{F_{M_c}}(z) = \frac{3z^4-2z^3+c}{4z^3-3z^2+c}$ where c is a complex parameter. These are rational maps $N_{F_{M_c}}$ arising from Newton's method. The polynomial of Newton iteration function is obtained from singularly perturbed of the Multibrot set polynomial.

1. Introduction

Singular perturbations arise in all areas of dynamical systems from ODSs to discrete dynamical systems. There is a wide range of examples of singular perturbations in these areas [1–3]. For a simple example, suppose that we are applying Newton's method to find the roots of a complex polynomial equation $P(z) = z^2 - c$. The Newton iteration function is given by $N_P(z) = z - \frac{P(z)}{P'(z)} = \frac{z}{2} + \frac{c}{2z}$ when $c = 0$, the polynomial P has multiple roots at 0 and Newton iteration function is $N_P(z) = \frac{z}{2}$. In this case, of course, all orbits of $N_P(z) = \frac{z}{2}$ tend to be the unique root at 0. However, when $c \neq 0$, the degree of N_P jumps from 1 to 2, and dynamical behavior of N_P become excited. Moreover, instead of a fixed point at the origin, after perturbation, there is a pole at the origin, most orbits of N_P still do convergence to one of the two roots of P, that is $\pm\sqrt{c}$ but points on the straight line passing through the origin perpendicular to the line segment connecting $\pm\sqrt{c}$ have orbits that do not convergence to these roots. Rather all orbits on these lines behave chaotically, so the dynamical behavior is more complicated in this case.

In recent years, much attention has been paid to families of rational maps that arise as singular perturbation of polynomials. These are families of rational maps that depend on a parameter λ and have the property that, when $\lambda = 0$, the map involved is a polynomial of degree n, but for all other parameters, the maps are rational with a higher degree. When the parameter λ becomes non-zero, the dynamics of these maps are explored. Most of the studies of these singular perturbed rational maps have centered on families of the form $F_\lambda(z) = z^n + \frac{\lambda}{z^d}$ where $\lambda \in \mathbb{C}$, n, and d are positive integers [4]. A singular perturbation means that we have a complex analytic map which is the new map F_{M_c} obtained by multiplying Multibrot set polynomial $M_c(z) = z^n + c$ and a simple polynomial $P(z) = z - 1$ so that $F_{M_c}(z) = (z^n + c)(z-1)$ where c is a complex parameter and $n > 2$. In this study, specifically we consider the case when Newton's method is applied to the polynomial family $F_{M_c(z)} = (z^3 + c)(z-1)$. The dynamics of such a perturbation are very exciting for the following reasons:

1. they are non-polynomial examples,
2. their dynamical behavior is changed dramatically when the parameter c is non-zero quite small.

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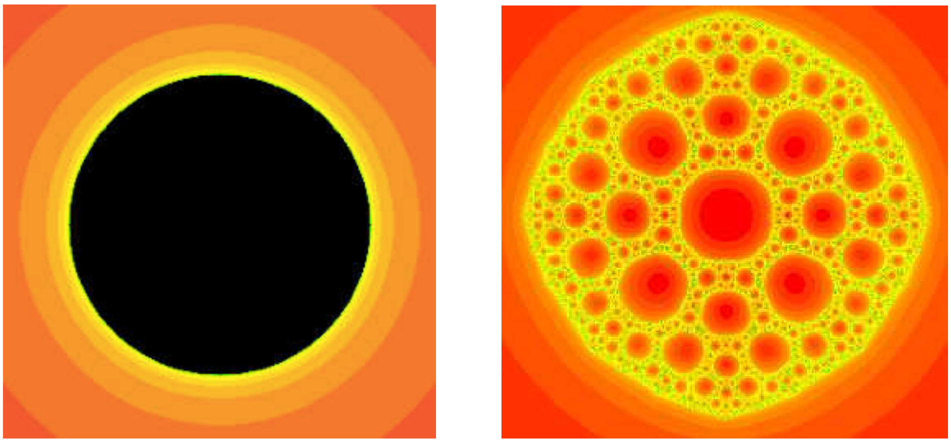


(a) $\lambda = 0$. **(b)** $\lambda \neq 0$.

Figure 1.1: How the dynamics is explodes when the degree is changed. For more detail [4].

In complex dynamics, the most important object in the dynamical plane is the *Julia set* of F, which we denote by $J(F)$. From an analytic viewpoint, the Julia set is the set of points at which the family of iterates on the map fails to be a normal family in the sense of Montel. There are many other equivalent definitions of the Julia set such as the Julia set is the closure of the set of repelling periodic points of F. Equivalently, the Julia set is also the boundary of the set of points whose orbits escape to ∞. From a dynamic point of view, the Julia set is the set of points on which the map is chaotic. The complement of the Julia set is called the *Fatou set*. This is where the dynamical behavior is relatively tame [2, 3, 5, 6].

The aim of this paper is to investigate the dynamics and the Julia sets of Newton iteration function, $N_{F_{M_c}}(z)$, applied to the polynomial $F_{M_c}(z) = (z^n + c)(z-1)$. We shall pay attention to one special critical point and see how the orbit of this point affects the dynamics of the Newton iteration map.

Newton's method is the best known iterative method for finding roots(real or complex) of a function f. It is the iterating function $N_f(z) = z - \frac{f(z)}{f'(z)}$ by starting with some initial approximation z_0 and defining the $n+1$ approximation by $z_{n+1} = N_f(z_n)$. Whether the function $f(z)$ is a polynomial or a rational function, then the iteration function N_f will be a rational map of the form $N(z) = N_f(z) = \frac{p(z)}{q(z)}$ where p and q are polynomials. So the dynamics of Newton's method become more difficult even when applied to polynomials in one variable. Iteration of Newton's method function often allows one to find the roots of the corresponding polynomial, but this is not always the case. The orbit of a point z_0 is the set of iterates of the function f which gives the sequence $\{z_0, N_f(z_0), N_f^2(z_0), N_f^3(z_0), \ldots\}$. This sequence hopefully converges to a root, ζ, of f. That certainly happens most of the time but other things might happen. For instance, if a function is not differentiable at the root such as considering the function $f(x) = x^{1/3}$, this function is not differentiable at the root x=0 and $|N_f'(0)| > 1$, then all sequences tend to ∞. Thus we may have no convergence if there is no differentiability. In some cases, the convergence of Newton's method is guaranteed by Kantorovitch theorem [7].

We shall think of Newton's iterating function as being defined on the whole the Riemann sphere, i.e. the complex numbers with the point at infinity adjoined, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The orbit of a point ζ could converge to a cycle, or it could wander chaotically about Riemann sphere, or it could behave in other ways. A point $\zeta \in \mathbb{C}$ is called a *periodic point* of period n if $N_f^n(\zeta) = \zeta$ and $N_f^k(\zeta) \neq \zeta$ for all $k < n$, where $k, n \in \mathbb{N}$. The least such integer n is called the *period* and the orbit of ζ is then an n-cycle. If $n = 1$, we say that ζ is a fixed point of N_f and, as is well known, such points correspond to the roots of f. A point ζ is *eventually periodic* if $N^n(\zeta) = N^{n+k}(\zeta)$ for positive integers n and k. If ζ is a periodic point of period n, then the derivative $\lambda = (N_f^n)'(\zeta)$ is called the *eigenvalue* of the periodic point ζ. It follows from the chain rule that λ is the product of the derivatives of N_f at each point on the orbit of ζ. Hence λ is an invariant of the orbit. A periodic orbit is called *attracting* if $|\lambda| < 1$, *super-attracting* if $|\lambda| = 0$, *repelling* if $|\lambda| > 1$, and *neutral* if $|\lambda| = 1$. Using Taylor's series for $N_f(\zeta)$, it can be shown that $N_f(\zeta)$ will be linearly convergent at an attracting fixed point and at least quadratically convergent at a super-attracting fixed point. Recall that the sequence $\{\zeta_n\}$ *convergence linearly* to w if, for sufficiently large n, $|\zeta_{n+1} - w| < t|\zeta_n - w|$, where $0 < c < 1$, where $0 < t < 1$, and *convergence quadratically* if, for sufficiently large n, $|\zeta_{n+1} - w| < t|\zeta_n - w|^2$, for some constant t. The point at ∞ is always a repelling fixed point with derivative $d/(d-1)$, where d is the degree of f, so large values of ζ will tend to move away from infinity under iteration [3]. A point is a *critical point* if the derivative of the map vanishes at this point. Critical points of N_f are solutions of $N_f'(\zeta) = 0$, i.e., zeroes and inflection points of f. The critical point is non-degenerate if $N_f''(\zeta) \neq 0$ and it is degenerate if $N_f''(\zeta) = 0$. For example, $f(x) = x^n$ has a *degenerate critical point* at 0 when $n > 2$, but has a *non-degenerate* when $n = 2$. Note that degenerate critical points may be maxima, minima, or saddle points as in the case of $f(x) = x^3$ [4, 6].

Theorem 1.1 (Julia)**.** *For any holomorphic map of the extended complex plane to itself, an attracting periodic cycle must attract at least one critical point* [8].

Theorem 1.2 (P. Fatou)**.** *Every attracting cycle for a polynomial or a rational function attracts at least one critical point* [4].

Theorem 1.3 (By The Riemann Hurtwitz Relation)**.** *A non-constant rational map with degree d has exactly* $2d-2$ *critical points in* $\mathbb{C}_\infty$*, counted with multiplicity* [8].

The critical points play a dominant role in determining the structure of the Julia set of rational iteration. In this paper, we will point out the case where the value of parameter c becomes non-zero, and when it happens, how the dynamical behavior changes strikingly.
We are interested in the dynamics of Newton's iteration map, N_f on the Riemann sphere. We can always conjugate N_f by an invertible linear(Möbius) transformation T, so the orbits of N_f will be essentially the same as the orbits of $T \circ N_f \circ T^{-1}$. On the Riemann sphere, the point at infinity is like any other point. In order to determine whether infinity is a fixed point of N_f and to find its eigenvalue there, we can conjugate N_f by the transformation $z \to \frac{1}{z}$ that interchanges 0 and ∞. Therefore the behavior of $N_f(z)$ at ∞ is the same as the behavior of $\frac{1}{N_f(1/z)}$ at 0.

The basin of attraction of a fixed point υ of the map N_f is the set $\left\{z| \lim_{n\to\infty} N_f^n(z) = \upsilon\right\}$, i.e., the set of all points whose orbits converge to υ under the iteration of N_f. This basin may have infinitely many components, and the immediate basin of attraction is the connected component containing the fixed point υ. The rational map N_f divides the Riemann sphere into two invariant sets, the Julia set, $J(N_f)$, and its complement. As mentioned earlier, the Julia set consists of points for which the dynamical behavior under iteration of N_f is complicated. Points in the complement of the Julia set will normally converge to a fixed point or an attracting cycle. This complement could also contain a Siegel disk or Herman ring in which the iterations are locally like an irrational rotation of a disk or an annulus.
A few basic facts about the Newton basin:

- The rational map N_f divides the Riemann sphere into two invariant sets, the *Julia set*, $J(N_f)$, and its complement.
- The points in the complement of the Julia set will normally converge to a fixed point, that could be infinity, or to an attractive cycle.
- $J(N_f)$ is the closure of the repelling periodic points.
- $J(N_f)$ is non-empty.
- $J(N_f)$ is completely invariant under N_f; i.e. $N_f(J(N_f)) = J(N_f) = N_f^{-1}(J(N_f))$.
- $J(N_f)$ is the boundary of the basin of attraction of each fixed point or attractive cycle: this guarantees that if there are more than two roots, $J(N_f)$ will be a fractal set.
- If $v \in J(N_f)$, then the closure of $\left\{z|N_f^n(z) = v \text{ for some non-negative integer} n\right\}$, the backward iterates of v, is the whole of $J(N_f)$.

It is well known that the Julia set is an unstable set. Iterates of points close to the Julia set will move away from that set. Hence Newton's method is very sensitive to initial conditions when the initial point is near the Julia set. Nearby points could converge to different roots or might not converge at all. If you start with a point actually on the Julia set, the iterates will also be on the Julia set because Julia set is a completely invariant set. As it is mentioned above, unfortunately, Newton's map does not converge to a root for every initial point. But the orbit could converge to an attractive cycle, rather than to a root.

2. The Dynamics of the Rational Map

In this section we consider the dynamics of the perturbed map which is a special class of rational functions, namely those obtained from Newton's method as applied to a polynomial of the form $F_{M_c(z)} = (z^3+c)(z-1)$. We are interested in the collection of Newton iteration maps given by $N_{F_{M_c}}$ as their dynamical properties are related to the non-degenerate free critical point.

Proposition 2.1. *Infinity is a repelling fixed point for Newton's method applied to $F_{M_c}(z) = (z^3+c)P(z)$ where $P(z) = z-1$ and c is any constant.*

Proof. Newton's method function is the rational map:

$$N_{F_{M_c}}(z) = z - \frac{M_c(z)P(z)}{M_c'(z)P(z) + M_c(z)P'(z)} = \frac{3z^4 - 2z^3 + c}{4z^3 - 3z^2 + c},$$

∞ is a fixed point, since $\lim_{z\to\infty} N_{F_{M_c}}(z) = \infty$. To determine its nature, we map ∞ to 0 via $g(z) = \frac{1}{z}(= v)$: the conjugate function G is given by $g \circ N_{F_{M_c}} = G \circ g$ thus we obtain

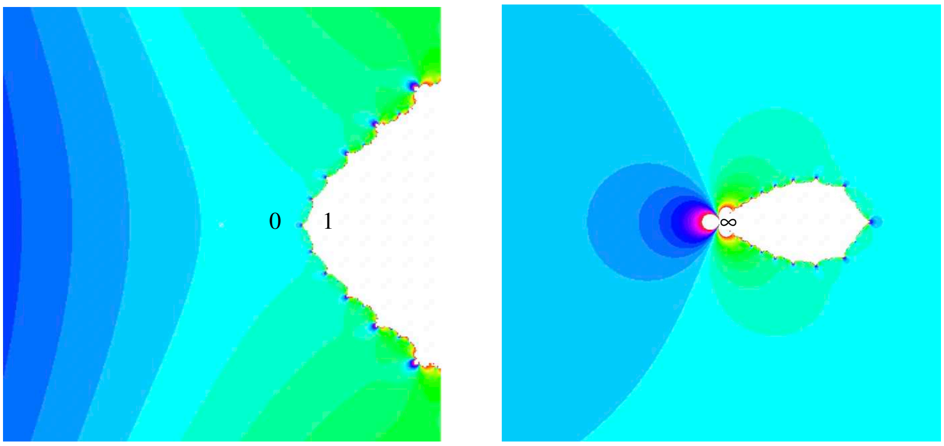
$$G(v) = g\left(N_{F_{M_c}}(\frac{1}{v})\right) = \frac{1}{N_{F_{M_c}}(\frac{1}{v})} = \frac{4v - 3v^2 + 4v^4}{3 - 2v + cv^4}.$$

∞ is a repelling fixed point, since $G(0) = 0$ and $|G'(0)| > 1$. □

Before examining the dynamics of F_{M_c} when c is small, we will consider the dynamics of the case $c = 0$.

2.1. The dynamics of $F_{M_0} = z^3(z-1)$

The Newton iterating function of F_{M_0} is a rational map of the form $N_{F_{M_0}}(z) = \frac{3z^4 - 2z^3}{4z^3 - 3z^2}$. The finite fixed points of $N_{F_{M_0}}(z)$ are 0 and 1 which are an attracting fixed point and a super-attracting fixed point, respectively. In addition, ∞ is a repelling fixed point. In Figures 2.1a and 2.1b, the computer graphics pictures illustrate $N_{F_{M_0}}(z)$ on the dynamical plane. Each color in the picture belongs to a finite root of $N_{F_{M_0}}(z)$. In Figure 2.1a, the area from blue to turquoise is the basin of attraction for the attracting fixed point 0 and the white area is the attracting basin for the super-attracting fixed point 1 of $N_{F_{M_0}}(z)$. In Figure 2.1b, the same basins are shown when viewed from infinity. It is the simple case $c = 0$ for Newton iteration that has decorations on the Julia set on the boundary of the basin; rather this boundary is a simple closed curve passing through ∞.



(a) **(b)**

Figure 2.1

One of the most important goals of Newton's method is to approximate the roots of a function - for which the convergence of the initial values is an important matter in dynamical systems. In Figure 2.1a and 2.1b, the speed of convergence for Newton's map of the function $z^3(z-1)$ is clearly observed. The critical orbits play a dominant role in determining the structure of the Julia sets in dynamical systems. Points $0,1$ and $1/2$ are critical points of $N_{F_{M_0}}$.
The aim of this paper is to draw attention to the case where the value of the parameter c becomes non-zero but quite small. When this happens, the dynamical behavior changes dramatically. We will now describe those changes.

2.2. The dynamics of $F_{M_c}(z)=(z^3+c)(z-1)$ for $c\neq 0$

We will deal with the value of c being different from zero but rather small. When we applied Newton's method to the polynomial $F_{M_c}(z)=(z^3+0.001)(z-1)$ obtained the rational map,

$$N(z)=N_{F_{M_{0.001}}}(z)=\frac{3z^4-2z^3+0.001}{4z^3-3z^2+0.001}.$$

∞ is a repelling fixed point and the real roots -0.1 and 1 are super-attracting fixed points of N. In addition to this, the complex roots are $0.05\pm 0.0866025i$ for the rational maps N with the parameter $c=0.001$. The points $0.05\pm 0.0866025i,-0.1,0,1$ and $1/2$ are critical points for N. Critical points $0,1/2$ and 1 are common critical points for the maps $N_{F_{M_{0.001}}}$ and $N_{F_{M_0}}$ with different critical values and also they are non-degenerate critical points. In addition to this, the common critical point 1 is a super-attracting fixed point for the maps $N_{F_{M_{0.001}}}$ and $N_{F_{M_0}}$.

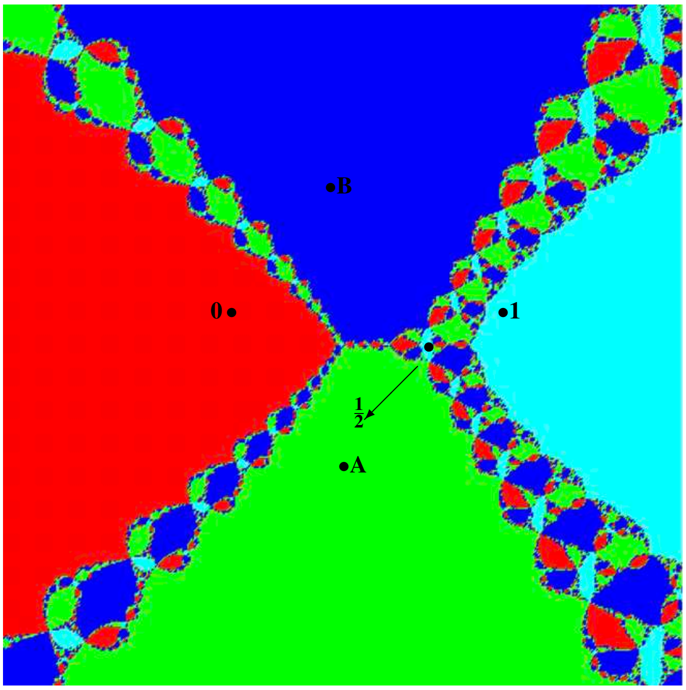
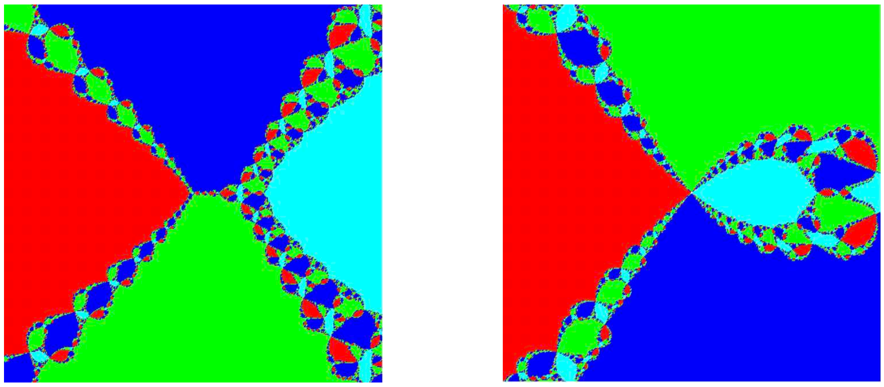


Figure 2.2: $A=0.05-0.0866025i$ and $B=0.05+0.0866025i$.

In Figure 2.2, the computer graphics picture illustrates how points behave under iteration of $N(z)$ in a dynamical plane. First of all, we will make clear the fact that we are considering the complex plane, the x-axis is the real direction and the y-axis is the imaginary direction.

Newton's map, N, for the polynomial $F_{M_{0.001}}(z) = z^4 - z^3 + (0.001)z - 0.001$ has degree 4. Since the function has four roots, the graph of the complex plane is divided into four parts, each of which is a basin of attraction for a root. Colors indicate to which of the four roots a given starting point converges to the finite roots of Newton's map which are contained in the Fatou set. The turquoise area is the basin of super-attracting fixed point for the map N, $\mathscr{A}_N(1) = \{z \in \mathbb{C} : N^n(z) \to 1, n \to \infty\}$. The boundary of the Newton basin including decorations is the Julia set, $\mathscr{A}_N(1) = \mathscr{J}(N_F)$, on which the dynamics of Newton iteration map are chaotic. The free critical point $1/2$ lies in the real axis and in a pre-image of the immediate basin of 1. Every root can be connected ∞ within its basin of attraction. Note importantly that there are no black regions in the basins, so Newton's map does not fail anywhere on that basin. The decorations on the boundary of the four immediate basins correspond to their pre-images. In addition, the immediate basin of attraction is a connected component containing the fixed points of N. It is no longer just a simple closed curve as in the case $c = 0$.



(a) **(b)**

Figure 2.3: The parameter plane pictures for the view from 0 and ∞ for the parameter $c = 0.001$.

Theorem 2.2 (P. Fatou). *The immediate basin of an attracting fixed point or cycle of N contains at least one critical point of N.*

In this paper, Newton's iteration map has free critical points that determine the fate of orbit in the complex dynamical behavior of N. The vital effect in the formation of this situation is in parameter c. When the parameter c takes a non-zero value, which is quite small, a dramatical change in the dynamics of the iteration map is observed. The importance of the periodic point in this change is seen in Figure 2.4. The parameter value of c after changing the parameter from 0 to any constant on a circle in a complex plane we see the periodic channels leading to ∞. In order to explain this situation, we change the parameter c from real to complex. For instance, in Figures 2.4-2.5, the value of parameter $c = 0.001 + 0.001i$. In Figure 2.4, the four roots of the function $F_{M_{0.001+0.001i}} : z \to (z^3 + 0.001 + 0.001i)(z-1)$ are $-0.100008 - 0.00303118i$, $0.0471496 - 0.0845998i$, $0.0528568 + 0.08863i$, $1. - 0.000998994i$. These are finite fixed points of Newton's iteration which are contained in the Fatou set. Since the function has four roots, the graph of the complex plane is divided into four parts, each of which is a basin for a root. The boundary of the basin is the chaotic part of Newton's fractal which is the Julia set. By the definition of Julia set, Newton's method does not converge on the boundary points, but it is chaotic. The Newton iteration functions for the values c have critical points 1 and $1/2$. In Figure 2.4, the green area goes to infinity and contains the free critical point. In Figure 2.5 the same area view from the point ∞.

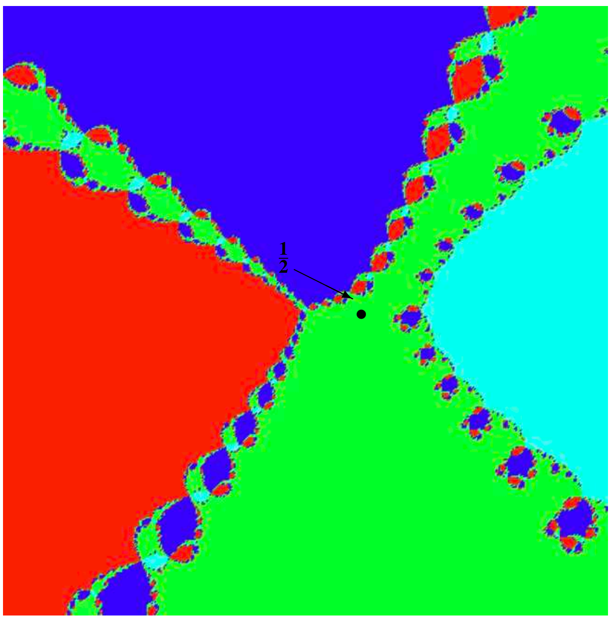


Figure 2.4: Dynamical plane view from 0.

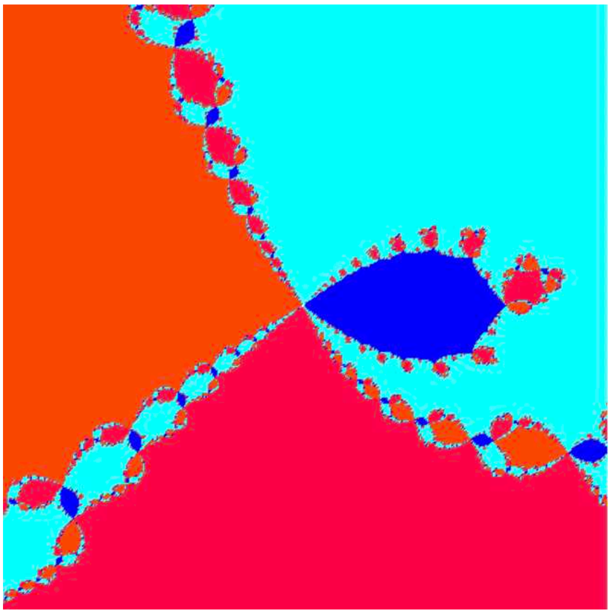


Figure 2.5: Dynamical plane view from ∞.

Corollary 2.3. *The non-degenerate free critical point plays a vital role in determining the dynamics of the rational map which arising in complex Newton's method is applied to polynomial family* $F_{M_c}(z) = (z^3 + c)(z-1)$*, where c is a complex (or non-complex) parameter.*

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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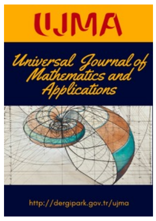

On a Rational $(P+1)$th Order Difference Equation with Quadratic Term

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Abstract

In this paper, we derive the forbidden set and determine the solutions of the difference equation that contains a quadratic term

$$x_{n+1} = \frac{x_n x_{n-p}}{a x_{n-(p-1)} + b x_{n-p}}, \quad n \in \mathbb{N}_0,$$

where the parameters a and b are real numbers, p is a positive integer and the initial conditions $x_{-p}, x_{-p+1}, \cdots, x_{-1}, x_0$ are real numbers.

1. Introduction

In [1], the authors determined the forbidden set, introduced an explicit formula for the solutions and discussed the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k+1}}{b x_{n-k+1} + c x_{n-k}}, \quad n \in \mathbb{N}_0,$$

where a,b,c are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are real numbers.
In [2], the second author studied the global behavior and introduced an explicit formula for the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{-b x_n + c x_{n-k-1}}, \quad n \in \mathbb{N}_0,$$

where a,b,c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \cdots, x_{-1}, x_0$ are real numbers.
In [3], the author determined the forbidden set, introduced an explicit formula for the solutions and discussed the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{b x_n - c x_{n-k-1}}, \quad n \in \mathbb{N}_0,$$

where a,b,c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \cdots, x_{-1}, x_0$ are real numbers.
In [4], Abo-Zeid determined the forbidden set and studied the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{b x_n + c x_{n-k-1}}, \quad n \in \mathbb{N}_0,$$

where a,b,c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \cdots, x_{-1}, x_0$ are real numbers.
For more on difference equations, one can see [5–28] and the references therein.

In this paper we generalize the solutions of the nonlinear rational difference equations presented in [5] and [10], which were established through a mere application of the induction principle.

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2. Main Results

In this section, we investigate the solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-p}}{a x_{n-(p-1)} + b x_{n-p}}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where the parameters a and b are real numbers, p is a positive integer and the initial conditions $x_{-p}, x_{-p+1}, \cdots, x_{-1}, x_0$ are real numbers.
The transformation

$$u_n = \frac{x_{n-1}}{x_n}, \; with \; u_{-i} = \frac{x_{-i-1}}{x_{-i}}, \quad i = \overline{0,(p-1)}, \tag{2.2}$$

reduces equation (2.1) into the difference equation

$$u_{n+1} = \frac{a}{u_{n-p+1}} + b, \quad n \in \mathbb{N}_0.$$

Suppose that

$$u_m^{(j)} = u_{pm+j}\,, \; j = \overline{1,p} \text{ and } m \geq -1.$$

Then, we can write

$$u_m^{(j)} = \frac{a}{u_{m-1}^{(j)}} + b, \quad m \in \mathbb{N}_0. \tag{2.3}$$

Let

$$u_m^{(j)} = \frac{z_{m+1}}{z_m}, \quad m \geq -1. \tag{2.4}$$

Then, equation (2.3) becomes

$$z_{m+1} - b z_m - a z_{m-1} = 0, \quad m \in \mathbb{N}_0. \tag{2.5}$$

with initial condition $z_{-1} = 1$, $z_0 = u_{-1}^{(j)}$.
Throughout this paper, we denote $b^2 + 4a$ by Δ.

2.1. Case $\Delta > 0$

In this subsection, we have that $b^2 > -4a$. Suppose that

$$\phi_j = \frac{\lambda_+^j - \lambda_-^j}{\lambda_+ - \lambda_-}, \quad j \in \mathbb{N}_0,$$

where λ_+ and λ_- are the roots of the equation $\lambda^2 - b\lambda - a = 0$.
Let

$$\gamma_{-i}(j) = a x_{-i}\phi_j + x_{-i-1}\phi_{j+1}, \quad i = \overline{0,(p-1)}.$$

Using equalities (2.2) and (2.4), we can write

$$\begin{aligned} x_{pm+p} &= \frac{1}{\prod_{i=1}^{p} u_{pm+i}} x_{pm} = x_0 \prod_{i=1}^{p} \frac{\gamma_{-p+i}(0)}{\gamma_{-p+i}(m+1)} \\ &= \frac{\nu}{\prod_{i=1}^{p} \gamma_{-p+i}(m+1)}, \; m \in \mathbb{N}_0, \end{aligned}$$

where $\nu = \prod_{i=0}^{p} x_{-i}$.
It follows that

$$\begin{aligned} x_{pm+t} &= \frac{1}{\prod_{i=1}^{t} u_{pm+i}} x_{pm} = \frac{\nu}{\prod_{i=1}^{p} \gamma_{-p+i}(m)} \cdot \frac{\prod_{i=1}^{t} \gamma_{-p+i}(m)}{\prod_{i=1}^{t} \gamma_{-p+i}(m+1)} \\ &= \frac{\nu}{\prod_{i=1}^{t} \gamma_{-p+i}(m+1) \prod_{i=t+1}^{p} \gamma_{-p+i}(m)}, \; m \in \mathbb{N}_0, \text{ and } t = \overline{1,p}. \end{aligned}$$

Using the above arguments, we obtain the following result:

Theorem 2.1. *Let $\{x_n\}_{n=-p}^{\infty}$ be a well defined solution for equation (2.1). Then*

$$x_n = \begin{cases} \dfrac{\nu}{\gamma_{-p+1}(\frac{n+p-1}{p}) \prod_{j=2}^{p} \gamma_{-p+j}(\frac{n-1}{p})}, & n = 1, p+1, \ldots, \\ \dfrac{\nu}{\prod_{i=1}^{2} \gamma_{-p+i}(\frac{n+p-2}{p}) \prod_{j=3}^{p} \gamma_{-p+j}(\frac{n-2}{p})}, & n = 2, p+2, \ldots, \\ \vdots & \vdots \\ \dfrac{\nu}{\prod_{i=1}^{p-1} \gamma_{-p+i}(\frac{n+1}{p}) \gamma_0(\frac{n-p+1}{p})}, & n = p-1, 2p-1, \ldots, \\ \dfrac{\nu}{\prod_{i=1}^{p} \gamma_{-p+i}(\frac{n}{p})}, & n = p, 2p, \ldots, \end{cases}$$

where $\nu = \prod_{i=0}^{p} x_{-i}$, $\gamma_{-j}(m) = a x_{-j}\phi_m + x_{-j-1}\phi_{m+1}$, $j = \overline{0,(p-1)}$ and $m \geq -1$.

Consider the two sets

$$\mathbb{D}_1 = \left\{ (v_0, v_1, \cdots, v_p) \in \mathbb{R}^{p+1} : \frac{v_0}{(-1)^p(\lambda_+/a)^p} = \frac{v_1}{(-1)^{p-1}(\lambda_+/a)^{(p-1)}} = \cdots = \frac{v_{p-1}}{-\lambda_+/a} = v_p \right\},$$

$$\mathbb{D}_2 = \left\{ (v_0, v_1, \cdots, v_p) \in \mathbb{R}^{p+1} : \frac{v_0}{(-1)^p(\lambda_-/a)^p} = \frac{v_1}{(-1)^{p-1}(\lambda_-/a)^{(p-1)}} = \cdots = \frac{v_{p-1}}{-\lambda_-/a} = v_p \right\}.$$

Theorem 2.2. *The two sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are invariant sets for equation (2.1).*

Proof. Let $(x_0, x_{-1}, \cdots, x_{-p}) \in \mathbb{D}_2$. We show that $(x_n, x_{n-1}, \cdots, x_{n-p}) \in \mathbb{D}_2$ for each $n \in \mathbb{N}$. The proof is by induction on n. The point $(x_0, x_{-1}, \cdots, x_{-p}) \in \mathbb{D}_2$ implies

$$\frac{x_0}{(-1)^p\lambda_-^p/a^p} = \frac{x_{-1}}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}} = \cdots = \frac{x_{-(p-1)}}{(-1)\lambda_-/a} = x_{-p}.$$

Now for $n = 1$, we have

$$\begin{aligned} x_1 &= \frac{x_0 x_{-p}}{a x_{-(p-1)} + b x_{-p}} = \frac{((-1)^{p-1}\lambda_-^{p-1}/a^{p-1})x_{-(p-1)}(-a/\lambda_-)x_{-(p-1)}}{a x_{-(p-1)} + b(-a/\lambda_-)x_{-(p-1)}} \\ &= \frac{(-1)^p}{a^{p-1}} \frac{\lambda_-^{p-2} x_{-(p-1)}}{1 - \frac{b}{\lambda_-}} = \frac{(-1)^p \lambda_-^p}{a^p} x_{-(p-1)}. \end{aligned}$$

Then we have

$$\frac{x_1}{(-1)^p\lambda_-^p/a^p} = \frac{x_0}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}} = \cdots = \frac{x_{-(p-2)}}{(-1)\lambda_-/a} = x_{-(p-1)}.$$

This implies that $(x_1, x_0, \cdots, x_{-p+1}) \in \mathbb{D}_2$. Suppose now that $(x_n, x_{n-1}, \cdots, x_{n-p}) \in \mathbb{D}_2$. That is

$$\frac{x_n}{(-1)^p\lambda_-^p/a^p} = \frac{x_{n-1}}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}} = \cdots = \frac{x_{n-(p-1)}}{(-1)\lambda_-/a} = x_{n-p}.$$

Then

$$\begin{aligned} x_{n+1} &= \frac{x_n x_{n-p}}{a x_{n-(p-1)} + b x_{n-p}} = \frac{((-1)^{p-1}\lambda_-^{p-1}/a^{p-1})x_{n-(p-1)}(-a/\lambda_-)x_{n-(p-1)}}{a x_{n-(p-1)} + b(-a/\lambda_-)x_{n-(p-1)}} \\ &= \frac{(-1)^p}{a^{p-1}} \frac{\lambda_-^{p-2} x_{n-(p-1)}}{1 - \frac{b}{\lambda_-}} = \frac{(-1)^p \lambda_-^p}{a^p} x_{n-(p-1)}. \end{aligned}$$

This implies that

$$\frac{x_{n+1}}{(-1)^p\lambda_-^p/a^p} = \frac{x_n}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}} = \cdots = \frac{x_{n-(p-2)}}{(-1)\lambda_-/a} = x_{n-(p-1)}.$$

That is $(x_{n+1}, x_n, \cdots, x_{n-p+1}) \in \mathbb{D}_2$. Then $(x_n, x_{n-1}, \cdots, x_{n-p}) \in \mathbb{D}_2$ for each $n \in \mathbb{N}$. Therefore, $\mathbb{D}_2$ is an invariant set for equation (2.1). By similar way, we can show that $\mathbb{D}_1$ is an invariant set for equation (2.1). This completes the proof. □

Theorem 2.3. *Assume that $\{x_n\}_{n=-p}^{\infty}$ is a well defined solution of equation (2.1). Then the following statements are true:*

1. *If $a + b > 1$, then the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero.*
2. *If $a + b < 1$, then the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.*

Proof. We can write $\phi_j = \lambda_+^j \frac{(1-(\frac{\lambda_-}{\lambda_+})^j)}{\sqrt{b^2+4a}}$.

1. If $a + b > 1$, then $\lambda_+ > 1$. That is $\phi_m \to \infty$ as $m \to \infty$. Then $|\gamma_{-j}(m)| = |a x_{-j}\phi_j + x_{-j-1}\phi_{m+1}| \to \infty$ as $m \to \infty$, $j = \overline{0, (p-1)}$. This implies that for each $t = \overline{1, p}$, we have

$$|x_{pm+t}| = \left| \frac{v}{\prod_{i=1}^{t} \gamma_{-p+i}(m+1) \prod_{i=t+1}^{p} \gamma_{-p+i}(m)} \right| \to 0 \text{ as } m \to \infty.$$

Therefore, the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero. For (2), it is enough to note that $\lambda_+ < 1$ when $a + b < 1$.

This completes the proof. □

Theorem 2.4. *Assume that $a + b = 1$, then every well defined solution $\{x_n\}_{n=-p}^{\infty}$ of equation (2.1) converges to a finite limit.*

Proof. When $a+b=1$, we have $\lambda_+=1$. Then

$$\gamma_{-p+i}(m)=ax_{-p+j}\phi_m+x_{-p+j-1}\phi_{m+1}\to\frac{ax_{-p+j}+x_{-p+j-1}}{1+a}\text{ as } m\to\infty,\ j=\overline{0,(p-1)}.$$

This implies that for each $t=\overline{1,p}$, we have

$$x_{pm+t}=\frac{\nu}{\prod_{i=1}^{t}\gamma_{-p+i}(m+1)\prod_{j=t+1}^{p}\gamma_{-p+j}(m)}\to\frac{(1+a)^p\nu}{\prod_{j=1}^{p}(ax_{-p+j}+x_{-p+j-1})}\text{ as } m\to\infty.$$

Therefore, the solution $\{x_n\}_{n=-p}^{\infty}$ of equation (2.1) converges to

$$\frac{(1+a)^p\nu}{\prod_{j=1}^{p}(ax_{-p+j}+x_{-p+j-1})}\text{ as } m\to\infty.$$

This completes the proof. □

2.2. Case $\Delta=0$

During this subsection, we assume that $b^2=-4a$. When $b^2=-4a$, the solution of equation (2.5) is

$$z_m=\frac{1}{2}\left(\frac{b}{2}\right)^m(2z_0(1+m)-bm),\ m\geq-1.$$

It follows that

$$\begin{aligned}u_{pm+j}&=\frac{b}{2}\frac{(m+1)b-2u_{-p+j}(2+m)}{mb-2u_{-p+j}(1+m)}\\&=\frac{b}{2}\frac{(m+1)bx_{-p+j}-2x_{-p+j-1}(2+m)}{mbx_{-p+j}-2x_{-p+j-1}(1+m)},\quad 1\leq j\leq p.\end{aligned}$$

If we set $\beta_{-p+j}(m)=mbx_{-p+j}-2x_{-p+j-1}(1+m)$, then we can write

$$u_{pm+j}=\frac{b}{2}\frac{\beta_{-p+j}(m+1)}{\beta_{-p+j}(m)},\quad 1\leq j\leq p. \tag{2.6}$$

Using equalities (2.2) and (2.6), we obtain the following result:

Theorem 2.5. *Let $\{x_n\}_{n=-p}^{\infty}$ be a well defined solution of equation (2.1). If $b^2+4a=0$, then*

$$x_n=\begin{cases}(-2)^p(\frac{2}{b})^n\frac{\nu}{\beta_{-p+1}(\frac{n+p-1}{p})\prod_{j=2}^{p}\beta_{-p+j}(\frac{n-1}{p})}, & n=1,p+1,...,\\ (-2)^p(\frac{2}{b})^n\frac{\nu}{\prod_{i=1}^{2}\beta_{-p+i}(\frac{n+p-2}{p})\prod_{j=3}^{p}\beta_{-p+j}(\frac{n-2}{p})}, & n=2,p+2,...,\\ \vdots & \vdots\\ (-2)^p(\frac{2}{b})^n\frac{\nu}{\prod_{i=1}^{p-1}\beta_{-p+i}(\frac{n+1}{p})\beta_0(\frac{n-p+1}{p})}, & n=p-1,2p-1,...,\\ (-2)^p(\frac{2}{b})^n\frac{\nu}{\prod_{i=1}^{p}\beta_{-p+i}(\frac{n}{p})}, & n=p,2p,...,\end{cases} \tag{2.7}$$

where $\nu=\prod_{i=0}^{p}x_{-i}$, $\beta_{-j}(m)=mbx_{-j}-2x_{-j-1}(1+m)$, $j=\overline{0,(p-1)}$ and $m\geq-1$.

Theorem 2.6. *Assume that $\{x_n\}_{n=-p}^{\infty}$ is a well defined solution of equation (2.1). The following statements are true:*

1. *If $b\geq2$ then the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero.*
2. *If $b<2$ then the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.*

Proof. The solution formula (2.7) can be written in the form

$$x_{pm+t}=(-2)^p\left(\frac{2}{b}\right)^{pm+t}\frac{\nu}{\prod_{i=1}^{t}\beta_{-p+i}(m+1)\prod_{j=t+1}^{p}\beta_{-p+j}(m)},\quad t=\overline{1,p}. \tag{2.8}$$

Clear that $\beta_{-p+i}(m)$ are unbounded, $i=\overline{1,p}$.

1. If $b\geq2$, then $\frac{2}{b}\leq1$ and the result follows.
2. If $b<2$, then $(\frac{2}{b})^{pm+t}\to\infty$ as $m\to\infty$ for all $t=\overline{1,p}$.
   Using formula (2.8), we can write for $t=1$

$$\begin{aligned}\mid x_{pm+1}\mid&=\mid(-2)^p(\frac{2}{b})^{pm+1}\frac{\nu}{\beta_{-p+1}(m+1)\prod_{j=2}^{p}\beta_{-p+j}(m)}\mid\\&=\mid(-2)^p\mid\frac{(\frac{2}{b})^{pm+1}}{m^p(1+\frac{1}{m})}\times\mid\frac{\nu}{(bx_{-p+1}-2x_{-p}\frac{2+m}{1+m})\prod_{j=2}^{p}(bx_{-p+j}-2x_{-p+j-1}\frac{1+m}{m})}\mid.\end{aligned}$$

Using L'Hospital's rule we can show that

$$\frac{(\frac{2}{b})^{pm+1}}{m^p(1+\frac{1}{m})} \to \infty \text{ as } m \to \infty.$$

This implies that $\mid x_{pm+1} \mid \to \infty$ as $m \to \infty$. Similarly, $\mid x_{pm+t} \mid \to \infty$ as $m \to \infty$, $2 \le t \le p$. Therefore, the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.

This completes the proof. □

2.3. Case $\Delta < 0$

During this subsection, we assume that $b^2 < -4a$. When $b^2 < -4a$, the solution of equation (2.5) is

$$z_m = \frac{(-a)^{\frac{m}{2}}}{\sin\theta}\left(z_0\sin(m+1)\theta - \sqrt{-a}\sin m\theta\right), \quad m \ge -1.$$

It follows that

$$u_{pm+j} = \sqrt{-a}\frac{\alpha_{-p+j}(m+1)}{\alpha_{-p+j}(m)}, \quad j = \overline{1,p}, \tag{2.9}$$

where $\theta = \arctan\left(\frac{\sqrt{-b^2-4a}}{b}\right)$, $sin\theta = \frac{\sqrt{-b^2-4a}}{2\sqrt{-a}}$ and $\alpha_{-p+j}(m) = x_{-p+j}\sqrt{-a}\sin m\theta - x_{-p+j-1}\sin(m+1)\theta$, $j = \overline{1,p}$, and $m \ge -1$. Using equalities (2.2) and (2.9), we obtain the following result:

Theorem 2.7. *Let $\{x_n\}_{n=-p}^{\infty}$ be a well defined solution of equation (2.1). If $b^2+4a<0$, then*

$$x_n = \begin{cases} \frac{(-1)^p\sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\alpha_{-p+1}(\frac{n+p-1}{p})\prod_{j=2}^{p}\alpha_{-p+j}(\frac{n-1}{p})}, & n = 1, p+1, \ldots, \\ \frac{(-1)^p\sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\prod_{i=1}^{2}\alpha_{-p+i}(\frac{n+p-2}{p})\prod_{j=3}^{p}\alpha_{-p+j}(\frac{n-2}{p})}, & n = 2, p+2, \ldots, \\ \vdots & \vdots \\ \frac{(-1)^p\sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\prod_{i=1}^{p-1}\alpha_{-p+i}(\frac{n+1}{p})\alpha_0(\frac{n-p+1}{p})}, & n = p-1, 2p-1, \ldots, \\ \frac{(-1)^p\sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\prod_{i=1}^{p}\alpha_{-p+i}(\frac{n}{p})}, & n = p, 2p, \ldots, \end{cases} \tag{2.10}$$

where $\nu = \prod_{i=0}^{p} x_{-i}$, $\alpha_{-j}(m) = x_{-j}\sqrt{-a}\sin m\theta - x_{-j-1}\sin(m+1)\theta$, $j = \overline{0,(p-1)}$ and $m \ge -1$.

Theorem 2.8. *Assume that $(x_n)_{n=-p}^{\infty}$ is a well defined solution of equation (2.1). The following statements are true:*

1. *Let $a=-1$ and if $\theta = \frac{l}{M}\pi$ is a rational multiple of π (with $0<l<\frac{M}{2}$), then $\{x_n\}_{n=-p}^{\infty}$ is periodic with prime period pM (if lp is even) or prime period $2pM$ (if lp is odd).*
2. *If $-1<a<0$, then the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.*
3. *If $a<-1$, then the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero.*

Proof. We can write the solution (2.10) as

$$x_{pm+t} = \frac{(-1)^p\sin^p\theta}{(\sqrt{-a})^{pm+t}}\frac{\nu}{\prod_{i=1}^{t}\alpha_{-p+i}(m+1)\prod_{j=t+1}^{p}\alpha_{-p+j}(m)}, \tag{2.11}$$

where $t = \overline{1,p}$ and $m \ge -1$.

1. Suppose that $a=-1$ and let $\theta = \frac{l}{M}\pi$ be a rational multiple of π (with $0<l<\frac{M}{2}$). Then for each $i = \overline{1,p}$, we have

$$\begin{aligned} \alpha_{-i}(m+M) &= x_{-i}\sin(m+M)\theta - x_{-i-1}\sin(m+M+1)\theta, \\ &= x_{-i}\sin(m\theta+M\theta) - x_{-i-1}\sin((m+1)\theta+M\theta), \\ &= x_{-i}\sin(m\theta+l\pi) - x_{-i-1}\sin((m+1)\theta+l\pi), \\ &= (-1)^l\alpha_{-i}(m). \end{aligned}$$

Then for each $t = \overline{1,p}$, we have

$$\begin{aligned} x_{pm+pM+t} &= (-1)^p\sin^p\theta\frac{\nu}{\prod_{i=1}^{t}\alpha_{-p+i}(m+M+1)\prod_{j=t+1}^{p}\alpha_{-p+j}(m+M)} \\ &= (-1)^{pl}x_{pm+t}. \end{aligned}$$

Therefore, if lp is even, then the solution $\{x_n\}_{n=-p}^{\infty}$ is periodic with prime period pM and if lp is odd, then the solution $\{x_n\}_{n=-p}^{\infty}$ is periodic with prime period $2pM$. (2) and (3) are directly obtained using (2.11).

This completes the proof. □

2.4. The forbidden sets

In this subsection, we introduce the forbidden sets of equation (2.1).

Theorem 2.9. *The following statements are true:*

1. *If* $b^2+4a>0$*, then the forbidden set of equation (2.1) can be written as*

$$
\begin{aligned}
F_1 = & \bigcup_{i=0}^{p}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-i}=0\right\}\cup \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+1}=-\frac{1}{a}\frac{\phi_{m+1}}{\phi_m}u_{-p}\right\}\cup \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+2}=-\frac{1}{a}\frac{\phi_{m+1}}{\phi_m}u_{-p+1}\right\}\cup \\
& \vdots \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{0}=-\frac{1}{a}\frac{\phi_{m+1}}{\phi_m}u_{-1}\right\}.
\end{aligned}
$$

2. *If* $b^2+4a=0$*, then the forbidden set of equation (2.1) can be written as*

$$
\begin{aligned}
F_2 = & \bigcup_{i=0}^{p}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-i}=0\right\}\cup \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+1}=\frac{2(1+m)}{mb}u_{-p}\right\}\cup \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+2}=\frac{2(1+m)}{mb}u_{-p+1}\right\}\cup \\
& \vdots \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{0}=\frac{2(1+m)}{mb}u_{-1}\right\}.
\end{aligned}
$$

3. *If* $b^2+4a<0$*, then the forbidden set of equation (2.1) can be written as*

$$
\begin{aligned}
F_3 = & \bigcup_{i=0}^{p}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-i}=0\right\}\cup \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+1}=\frac{\sin(m+1)\theta}{\sqrt{-a}\sin m\theta}u_{-p}\right\}\cup \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+2}=\frac{\sin(m+1)\theta}{\sqrt{-a}\sin m\theta}u_{-p+1}\right\}\cup \\
& \vdots \\
& \bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},\ldots,u_{-p})\in\mathbb{R}^{p+1}: u_{0}=\frac{\sin(m+1)\theta}{\sqrt{-a}\sin m\theta}u_{-1}\right\}.
\end{aligned}
$$

3. Illustrative Examples

Example 3.1. *Figure 3.1 shows that, if* $p=7$*,* $a=0.2$ *and* $b=1\,(\Delta>0\,\text{and}\,a+b>1)$*, then a solution* $\{x_n\}_{n=-7}^{\infty}$ *of equation (2.1) with* $x_{-7}=-4, x_{-6}=-5, x_{-5}=-3$*,* $x_{-4}=-8.2$*,* $x_{-3}=5$*,* $x_{-2}=3$*,* $x_{-1}=6.2$ *and* $x_0=-7$ *converges to zero.*

Example 3.2. *Figure 3.2 shows that, if* $p=4$*,* $a=0.1$ *and* $b=0.7\,(\Delta>0\,\text{and}\,a+b<1)$*, then a solution* $\{x_n\}_{n=-4}^{\infty}$ *of equation (2.1) with* $x_{-4}=-1, x_{-3}=-3, x_{-2}=-5.9, x_{-1}=-3$ *and* $x_0=-12.2$ *is unbounded.*

Example 3.3. *Figure 3.3 shows that, if* $p=7$*,* $a=-1$ *and* $b=2\,(\Delta=0)$*, then a solution* $\{x_n\}_{n=-7}^{\infty}$ *of equation (2.1) with* $x_{-7}=-2$*,* $x_{-6}=-5$*,* $x_{-5}=-3$*,* $x_{-4}=-12.2$*,* $x_{-3}=5$*,* $x_{-2}=3$*,* $x_{-1}=6.2$ *and* $x_0=-5$ *converges to zero.*

Example 3.4. *Figure 3.4 shows that, if* $p=7$*,* $a=-1/4$ *and* $b=1\,(\Delta=0\,\text{and}\,b<2)$*, then a solution* $\{x_n\}_{n=-7}^{\infty}$ *of equation (2.1) with* $x_{-7}=-4$*,* $x_{-6}=-5.3$*,* $x_{-5}=-1.3$*,* $x_{-4}=-9.2$*,* $x_{-3}=6$*,* $x_{-2}=13$*,* $x_{-1}=6.2$ *and* $x_0=-5$ *is unbounded.*

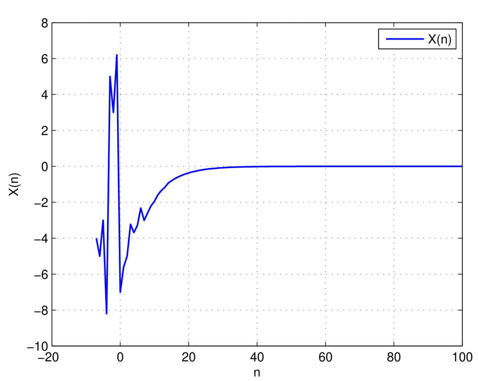

Figure 3.1: Equation $x_{n+1} = \frac{x_n x_{n-7}}{0.2x_{n-6}+x_{n-7}}$.

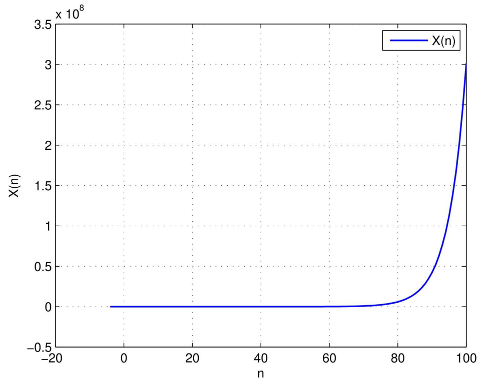

Figure 3.2: Equation $x_{n+1} = \frac{x_n x_{n-4}}{0.1x_{n-3}+0.7x_{n-4}}$.

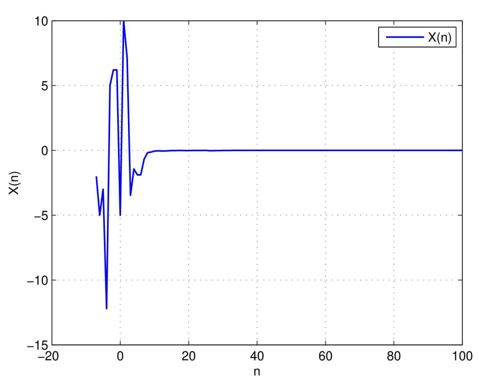

Figure 3.3: Equation $x_{n+1} = \frac{x_n x_{n-7}}{-x_{n-6}+2x_{n-7}}$.

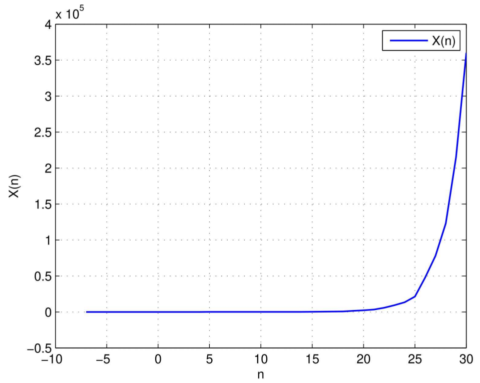

Figure 3.4: Equation $x_{n+1} = \frac{x_n x_{n-7}}{-0.25x_{n-6}+x_{n-7}}$.

Example 3.5. *Figure 3.5 shows that, if* $p = 4$, $a = -1$ *and* $b = \sqrt{3}\,(\Delta < 0\, and\, l\, p\, is\, even)$, *then a solution* $\{x_n\}_{n=-4}^{\infty}$ *of equation (2.1) with* $x_{-4} = -2$, $x_{-3} = -5$, $x_{-2} = 3$, $x_{-1} = 2.2$ *and* $x_0 = 5$ *is periodic with prime period 24.*

Example 3.6. *Figure 3.6 shows that, if* $p = 7$, $a = -1$ *and* $b = 1\,(\Delta < 0\, and\, l\, p\, is\, odd)$, *then a solution* $\{x_n\}_{n=-7}^{\infty}$ *of equation (2.1) with* $x_{-7} = -1$, $x_{-6} = -7$, $x_{-5} = -4$, $x_{-4} = -12.2$, $x_{-3} = 5$, $x_{-2} = 3$, $x_{-1} = 6.2$ *and* $x_0 = -5$ *is periodic with prime period 42.*

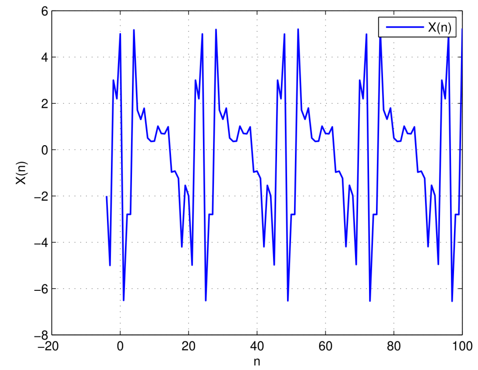

Figure 3.5: Equation $x_{n+1} = \frac{x_n x_{n-4}}{-x_{n-3}+\sqrt{3}x_{n-4}}$.

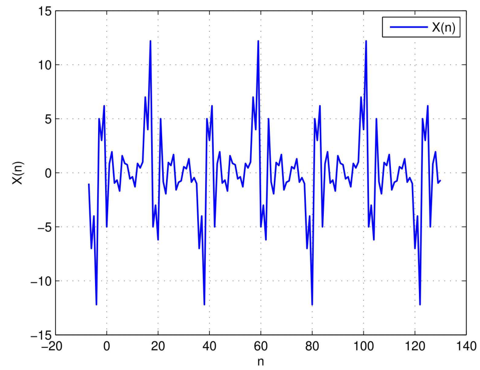

Figure 3.6: Equation $x_{n+1} = \frac{x_n x_{n-7}}{-x_{n-6}+x_{n-7}}$.

Example 3.7. *Figure 3.7 shows that, if* $p = 3$, $a = 0.3$ *and* $b = 0.7\,(\Delta > 0\, and\, a+b = 1)$, *then a solution* $\{x_n\}_{n=-3}^{\infty}$ *of equation (2.1) with initial conditions* $x_{-3} = 1$, $x_{-2} = -2$, $x_{-1} = 1$ *and* $x_0 = 0.7$ *converges to*

$$\frac{(1.3)^3((1)(-2)(1)(0.7))}{\prod_{j=1}^{3}(0.3x_{-3+j}+x_{-4+j})} \simeq 3.738.$$

Example 3.8. *Figure 3.8 shows that, if* $p = 5$, $a = 0.2$ *and* $b = 0.8\,(\Delta > 0\, and\, a+b = 1)$, *then a solution* $\{x_n\}_{n=-5}^{\infty}$ *of equation (2.1) with initial conditions* $x_{-5} = -2$, $x_{-4} = -1$, $x_{-3} = 0.5$, $x_{-2} = 0.8$, $x_{-1} = 0.7$ *and* $x_0 = -0.8$ *converges to*

$$\frac{(1.2)^5((-2)(-1)(0.5)(0.8)(0.7)(-0.8))}{\prod_{j=1}^{5}(0.2x_{-5+j}+x_{-6+j})} \simeq -1.681.$$

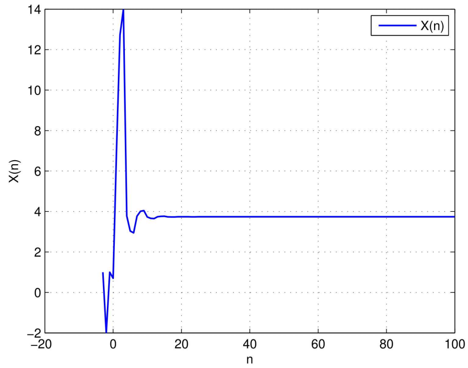

Figure 3.7: Equation $x_{n+1} = \frac{x_n x_{n-3}}{0.3x_{n-2}+0.7x_{n-3}}$.

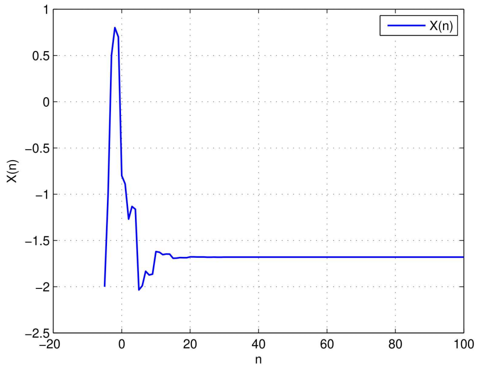

Figure 3.8: Equation $x_{n+1} = \frac{x_n x_{n-7}}{0.2x_{n-6}+0.8x_{n-7}}$.

Conclusion

In this study, we mainly obtained the solutions and introduced the forbidden sets of the difference equation that contains a quadratic term

$$x_{n+1} = \frac{x_n x_{n-p}}{ax_{n-(p-1)} + bx_{n-p}}, \quad n \in \mathbb{N}_0,$$

where the parameters a and b are real numbers, p is a positive integer and the initial conditions $x_{-p}, x_{-p+1}, \cdots, x_{-1}, x_0$ are real numbers. Also, we showed that the behavior of the solutions depends on the relation between a and b. That is if $\{x_n\}_{n=-p}^{\infty}$ is a solution of that equation, it may be converge to finite limit, unbounded or periodic with a certain period that depends on p. The mentioned difference equation may be generalized to a more complicated one that may has a complicated behavior.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Binomial Transform for Quadra Fibona-Pell Sequence and Quadra Fibona-Pell Quaternion

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Abstract

The main object of the study is to consider the binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion. In the paper, which consists of two parts in terms of the results found, the first step was taken for the sequence by defining the binomial transform for the quadra Fibona-Pell sequence in the first part and then finding the recurrence relation of this new binomial transform. Then, the Binet formula, generating function and various sum formulas of the sequence were found. In the second part, the binomial transform is applied for the quadra Fibona-Pell quaternion, which was discussed in a thesis before. Similar results in the first section are covered in the quaternion binomial transform.

1. Introduction

Number sequences such as Fibonacci, Pell, quadra Fibona-Pell, and their generalized versions are widely used in the literature. Fibonacci numbers form a sequence defined by the following recurrence relation: $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots$. The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.It has become known as Binet's formula $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geq 0$. The Pell numbers are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. The first few terms of the sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378 \cdots$. See [1] for detailed information on Fibonacci and Pell number sequences.

In [2], the author examined sequence with fourth-order recurrence relation and discussed a new sequence of fourth-order formed by the roots of the characteristic equation of both Fibonacci and Pell number sequences. Later, different authors worked with similar integer sequences with the same logic, see [3, 4]. In addition, we can come across many studies on integer sequences with fourth-order recurrence relation in the literature, see [5]. Quadra Fibona-Pell sequence as follows in [2]:

$$W_n = 3W_{n-1} - 3W_{n-3} + W_{n-4} \tag{1.1}$$

for $n \geq 4$, with initial values $W_0 = W_1 = 0, W_2 = 1, W_3 = 3$ and W_n is $n-$th quadra Fibona-Pell sequence. Note that here, the roots of the characteristic equation of W_n are the roots of the characteristic equations of both Fibonacci and Pell sequences, so $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ (α, β are the roots of the characteristic equation of Fibonacci numbers and γ, δ are the roots of the characteristic equation of Pell numbers). The Binet formula for the quadra Fibona-Pell sequence is given by

$$W_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} - \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for $n \geq 0$. Besides that generating function for the quadra Fibona-Pell sequence is

$$W(x) = \frac{x^2}{x^4 + 3x^3 - 3x + 1}.$$

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Normed division algebra, nowadays which is so important topic consists of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$. Quaternions are non-commutative normed division algebra over the real numbers, even it looks like things are going to be done with quaternions. For $a_0, a_{1,} a_{2,} a_3 \in \mathbb{R}$, a quaternion is defined by

$$e = a_0 + a_1 i + a_2 j + a_3 k$$

where i, j and k are unit vectors which verifies the following rules

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.2}$$

From equation (1.2), we get

$$ij = -ji = k, \quad jk = -kj = i, \quad jk = -kj = i.$$

You can find detailed information about quaternions from [6–8].

2. Binomial Transform of Quadra Fibona-Pell Sequence

It is possible to find different articles in the literature on binomial transforms of sequences as [9]. Actually, one of these was studied by Chen [10] and later was studied by Falcon in [11]. In [12], given a sequence $A = \{a_1, a_2, \cdots\}$, its binomial transform B is the sequence $B(A) = \{b_n\}$ defined as follows:

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i. \tag{2.1}$$

Also, detailed information about binomial transform can be found in [13, 14]. Some authors considered special binomial sequences which are based on fourth-order recurrence relations, for example binomial transform of quadrapell sequences in [15].
In this part of the study, with similar logic, we apply the binomial transform of quadra Fibona-Pell sequence. When the binomial transform of the quadra Fibona-Pell sequence, which has a fourth-order recurrence relation, is made, some additional identities, especially the generating function, Binet formula and sum formulas, will be found for the new sequence obtained.

Definition 2.1. *Let W_n be the $n-$th Quadra Fibona-Pell sequence. Then the binomial transform of quadra Fibona-Pell sequence is*

$$b_n = \sum_{i=0}^{n} \binom{n}{i} W_i. \tag{2.2}$$

Lemma 2.2. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (W_i + W_i).$$

Proof. By the help of (2.2), we get

$$b_{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} W_i = \sum_{i=1}^{n+1} \binom{n+1}{i} W_i + W_0.$$

Also $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$ and $\binom{n}{n+1} = 0$, we get

$$b_{n+1} = \sum_{i=1}^{n+1} \left[\binom{n}{i-1} + \binom{n}{i} \right] W_i + W_0.$$

Thus we obtain that

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (W_i + W_{i+1}).$$

This completes the proof. □

From Lemma 2.2, we can give the following result for the binomial transform of quadra Fibona-Pell sequence.

Corollary 2.3. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} W_{i+1}.$$

The recurrence relation of the binomial transforms of quadra Fibona-Pell sequence is obtained below.

Theorem 2.4. *Let b_n is the binomial transform of quadra Fibona-Pell sequence. b_n states the following recurrence relation*

$$b_{n+3} = 7b_{n+2} - 15b_{n+1} + 10b_n - 2b_{n-1} \tag{2.3}$$

for $n \geq 4$, where $b_0 = b_1 = 0, b_2 = 1$ and $b_3 = 6$.

Proof. Using Lemma 2.2 we get,

$$b_{n+3} = K_1 b_{n+2} + L_1 b_{n+1} + M_1 b_n + N_1 b_{n-1}.$$

Then, if we solve the system of equations

$$\begin{aligned} n = 1 &\Rightarrow b_4 = K_1 b_3 + L_1 b_2 + M_1 b_1 + N_1 b_0, \\ n = 2 &\Rightarrow b_5 = K_1 b_4 + L_1 b_3 + M_1 b_2 + N_1 b_1, \\ n = 3 &\Rightarrow b_6 = K_1 b_5 + L_1 b_4 + M_1 b_3 + N_1 b_2, \\ n = 4 &\Rightarrow b_7 = K_1 b_6 + L_1 b_5 + M_1 b_4 + N_1 b_3, \end{aligned}$$

by considering Definition 2.1, we deduce

$$K_1 = 7, L_1 = -15, M_1 = 10, N_1 = -2.$$

which is completed the proof. □

The generating function of the new binomial transform is found below.

Theorem 2.5. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. The generating function of the related binomial transform is*

$$b(x) = \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4},$$

where $b_0 = b_1 = 0, b_2 = 1$ and $b_3 = 6$.

Proof. Assume that

$$b(x) = \sum_{i=0}^{\infty} b_i x^i$$

is the generating function of the binomial transform for W_n. Then

$$\begin{aligned} b(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ 7xb(x) &= 7b_0 x + 7b_1 x^2 + 7b_2 x^3 + 7b_3 x^4 + \cdots \\ 15x^2 b(x) &= 15b_0 x^2 + 15b_1 x^3 + 15b_2 x^4 + 15b_3 x^5 + \cdots \\ 10x^3 b(x) &= 10b_0 x^3 + 10b_1 x^4 + 10b_2 x^5 + 10b_3 x^6 + \cdots \\ 2x^4 b(x) &= 2b_0 x^4 + 2b_1 x^5 + 2b_2 x^6 + 2b_3 x^7 + \cdots . \end{aligned}$$

Since, from equation (2.3), we obtain

$$(1 - 7x + 15x^2 - 10x^3 + 2x^4) b(x) = x^2 - x^3$$

and hence, the generating function for the binomial transform of the b_n is

$$b(x) = \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}.$$

□

Another formula that is essential to find other results of the binomial transform is the Binet formula, which is provided below.

Theorem 2.6. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. The Binet formula for b_n is*

$$b_n = \frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \tag{2.4}$$

for $n \geqslant 0$.

Proof. Note that the generating function is $W(x) = \frac{x^2 - x^3}{1-7x+15x^2-10x^3+2x^4}$. It is easily seen that $1 - 7x + 15x^2 - 10x^3 + 2x^4 = (2x^2 - 4x + 1)(x^2 - 3x + 1)$. So we can rewrite $W(x)$ as

$$\begin{aligned} \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4} &= \frac{x}{2x^2 - 4x + 1} - \frac{x}{x^2 - 3x + 1} \\ &= \sum_{n=0}^{\infty} \frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \sum_{n=0}^{\infty} \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \\ &= \sum_{n=0}^{\infty} \left(\frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \right) \end{aligned}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. Using roots in quadra Fibona-Pell sequence, we get

$$b_n = \frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}$$

the result. □

In the most general case, the following result was found for the series expansions.

Theorem 2.7. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$\sum_{n=0}^{\infty} b_{mn+s}x^n = b_s\left(\frac{1}{A_1}+\frac{1}{B_1}\right) - b_{s-m}\left(\frac{2^m}{A_1}+\frac{1}{B_1}\right)x + C_1$$

for all $n \in \mathbb{N}$ and $m, s \in \mathbb{N}$, $s > m$,

$$\begin{aligned}
A_1 &= (1-(\gamma+1)^m x)(1-(\delta+1)^m x),\\
B_1 &= (1-\alpha^{2m}x)(1-\beta^{2m}x),\\
C_1 &= \frac{1}{A_1}\left(\frac{\gamma^{2s}-\delta^{2s}}{\gamma-\delta}\right) - \frac{1}{B_1}\left(\frac{(\gamma+1)^s-(\delta+1)^s}{\gamma-\delta}\right) - \frac{2^m}{A_1}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right)x + \frac{1}{B_1}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right)x.
\end{aligned}$$

Proof. Again from equation (2.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{mn+s}x^n &= \sum_{n=0}^{\infty}\left(\frac{(\gamma+1)^{mn+s}-(\delta+1)^{mn+s}}{\gamma-\delta} - \frac{\alpha^{2(mn+s)}-\beta^{2(mn+s)}}{\alpha-\beta}\right)x^n\\
&= \frac{(\gamma+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\gamma+1)^m x)^n - \frac{(\delta+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\delta+1)^m x)^n - \frac{\alpha^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^{2m}x)^n + \frac{\beta^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\beta^{2m}x)^n
\end{aligned}$$

with the help of sum formula, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{mn+s}x^n =& \frac{(\gamma+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\gamma+1)^m x}\right) - \frac{(\delta+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\delta+1)^m x}\right) - \frac{\alpha^{2s}}{\alpha-\beta}\left(\frac{1}{1-\alpha^{2m}x}\right) + \frac{\beta^{2s}}{\alpha-\beta}\left(\frac{1}{1-\beta^{2m}x}\right)\\
=& \frac{\frac{(\gamma+1)^s-(\delta+1)^s}{\gamma-\delta} - ((\gamma+1)(\delta+1))^m\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right)x}{1-((\gamma+1)^m+(\delta+1)^m)x+((\gamma+1)(\delta+1))^m x^2} - \frac{\frac{\alpha^{2s}-\beta^{2s}}{\alpha-\beta} - (\alpha\beta)^{2m}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right)x}{1-(\alpha^{2m}+\beta^{2m})x+(\alpha\beta)^{2m}x^2}\\
=& \frac{1}{A_1}\left(\frac{(\gamma+1)^s-(\delta+1)^s}{\gamma-\delta}\right) - \frac{((\gamma+1)(\delta+1))^m}{A_1}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right)x\\
&- \frac{1}{B_1}\left(\frac{\alpha^{2s}-\beta^{2s}}{\alpha-\beta}\right) + \frac{(\alpha\beta)^{2m}}{B_1}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right)x,
\end{aligned}$$

if necessary arrangements are made, then we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{mn+s}x^n =& \frac{(\alpha-\beta)}{A_1}\left(\frac{(\gamma+1)^s-(\delta+1)^s}{(\gamma-\delta)(\alpha-\beta)}\right) - \frac{(\gamma-\delta)}{B_1}\left(\frac{\alpha^{2s}-\beta^{2s}}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&- \frac{((\gamma+1)(\delta+1))^m(\alpha-\beta)}{A_1}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{(\gamma-\delta)(\alpha-\beta)}\right)x + \frac{(\alpha\beta)^{2m}(\gamma-\delta)}{B_1}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
=& \frac{1}{A_1}\left(\frac{b_s((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(\alpha^{2s}-\beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&- \frac{1}{B_1}\left(\frac{(\alpha-\beta)((\gamma+1)^s-(\delta+1)^s)-b_s((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&- \frac{((\gamma+1)(\delta+1))^m}{A_1}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(\alpha^{2(s-m)}-\beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
&+ \frac{(\alpha\beta)^{2m}}{B_1}\left(\frac{(\alpha-\beta)((\gamma+1)^{s-m}-(\delta+1)^{s-m})-b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
=& b_s\left(\frac{1}{A_1}+\frac{1}{B_1}\right) - b_{s-m}\left(\frac{2^m}{A_1}+\frac{1}{B_1}\right)x + C_1.
\end{aligned}$$

Hence the result is obvious. □

Now let's get a grand total formula that includes all the sum results that deal with the sum formulas of $mk+s$ terms.

Theorem 2.8. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$\sum_{k=0}^{n} b_{mk+s} = \frac{2^m}{A_2}(b_{mn+s}-b_{s-m}) + \frac{1}{B_2}(b_{mn+s}+b_{s-m}+b_{mn+m+s}-b_s) + \frac{1}{A_2}(b_s-b_{mn+m+s}) + C_2$$

for all $n \in \mathbb{N}$ *and* $m, s \in \mathbb{Z}, s > m$, *where*

$$\begin{aligned}
A_2 =& (\gamma+1)^m(\delta+1)^m - ((\gamma+1)^m + (\delta+1)^m) + 1 \\
B_2 =& \alpha^{2m}\beta^{2m} - \left(\alpha^{2m} + \beta^{2m}\right) + 1 \\
C_2 =& \frac{2^m}{A_2}\left(\frac{\alpha^{2(mn+s)} - \beta^{2(mn+s)}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+s} - (\delta+1)^{mn+s}}{\gamma-\delta}\right) - \frac{2^m}{A_2}\left(\frac{\alpha^{2(s-m)} - \beta^{2(s-m)}}{\alpha-\beta}\right) \\
&+ \frac{1}{B_2}\left(\frac{(\gamma+1)^{s-m} - (\delta+1)^{s-m}}{\gamma-\delta}\right) - \frac{1}{A_2}\left(\frac{\alpha^{2(mn+m+s)} - \beta^{2(mn+m+s)}}{\alpha-\beta}\right) + \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+m+s} - (\delta+1)^{mn+m+s}}{\gamma-\delta}\right) \\
&+ \frac{1}{A_2}\left(\frac{\alpha^{2s} - \beta^{2s}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{s} - (\delta+1)^{s}}{\gamma-\delta}\right).
\end{aligned}$$

Proof. From (2.4), we get

$$\begin{aligned}
\sum_{k=0}^{n} b_{mk+s} =& \sum_{k=0}^{n}\left(\frac{(\gamma+1)^{mk+s} - (\delta+1)^{mk+s}}{\gamma-\delta} - \frac{\alpha^{2(mk+s)} - \beta^{2(mk+s)}}{\alpha-\beta}\right) \\
=& \frac{(\gamma+1)^s}{\gamma-\delta}\left(\frac{\left(((\gamma+1)^m)^{n+1} - 1\right)((\delta+1)^m - 1)}{((\gamma+1)^m - 1)((\delta+1)^m - 1)}\right) - \frac{(\delta+1)^s}{\gamma-\delta}\left(\frac{\left(((\delta+1)^m)^{n+1} - 1\right)((\gamma+1)^m - 1)}{((\delta+1)^m - 1)((\gamma+1)^m - 1)}\right) \\
&- \frac{\alpha^{2s}}{\alpha-\beta}\left(\frac{\left((\alpha^{2m})^{n+1} - 1\right)(\beta^{2m} - 1)}{(\alpha^{2m} - 1)(\beta^{2m} - 1)}\right) + \frac{\beta^{2s}}{\alpha-\beta}\left(\frac{\left((\beta^{2m})^{n+1} - 1\right)(\alpha^{2m} - 1)}{(\beta^{2m} - 1)(\alpha^{2m} - 1)}\right) \\
=& \frac{\left\{\begin{array}{c} (\gamma+1)^m(\delta+1)^m\left(\dfrac{(\gamma+1)^{mn+s} - (\delta+1)^{mn+s}}{\gamma-\delta}\right) - (\gamma+1)^m(\delta+1)^m\left(\dfrac{(\gamma+1)^{s-m} - (\delta+1)^{s-m}}{\gamma-\delta}\right) \\ -\left(\dfrac{(\gamma+1)^{mn+m+s} - (\delta+1)^{mn+m+s}}{\gamma-\delta}\right) + \left(\dfrac{(\gamma+1)^{s} - (\delta+1)^{s}}{\gamma-\delta}\right) \end{array}\right\}}{(\gamma+1)^m(\delta+1)^m - ((\gamma+1)^m + (\delta+1)^m) + 1} \\
&- \frac{\left\{\begin{array}{c} (\alpha\beta)^{2m}\left(\dfrac{\alpha^{2(mn+s)} - \beta^{2(mn+s)}}{\alpha-\beta}\right) + (\alpha\beta)^{2m}\left(\dfrac{\alpha^{2(s-m)} - \beta^{2(s-m)}}{\alpha-\beta}\right) \\ + \left(\dfrac{\alpha^{2(mn+m+s)} - \beta^{2(mn+m+s)}}{\alpha-\beta}\right) - \left(\dfrac{\alpha^{2s} - \beta^{2s}}{\alpha-\beta}\right) \end{array}\right\}}{\alpha^{2m}\beta^{2m} - (\alpha^{2m} + \beta^{2m}) + 1}.
\end{aligned}$$

As a result of calculations, we obtained

$$\begin{aligned}
\sum_{k=0}^{n} b_{mk+s} =& \frac{(\gamma+1)^m(\delta+1)^m}{A_2}\left(\frac{b_{mn+s}((\gamma+1)-(\delta+1))(\alpha-\beta) + ((\gamma+1)-(\delta+1))(\alpha^{2(mn+s)} - \beta^{2(mn+s)})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{(\alpha\beta)^{2m}}{B_2}\left(\frac{(\alpha-\beta)((\gamma+1)^{mn+s} - (\delta+1)^{mn+s}) - b_{mn+s}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{(\gamma+1)^m(\delta+1)^m}{A_2}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta) + ((\gamma+1)-(\delta+1))(\alpha^{2(s-m)} - \beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{(\alpha\beta)^{2m}}{B_2}\left(\frac{(\alpha-\beta)((\gamma+1)^{s-m} - (\delta+1)^{s-m}) - b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{1}{A_2}\left(\frac{b_{mn+m+s}((\gamma+1)-(\delta+1))(\alpha-\beta) + ((\gamma+1)-(\delta+1))(\alpha^{2(mn+m+s)} - \beta^{2(mn+m+s)})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{1}{B_2}\left(\frac{(\alpha-\beta)((\gamma+1)^{mn+m+s} - (\delta+1)^{mn+m+s}) - b_{mn+m+s}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{1}{A_2}\left(\frac{b_{s}((\gamma+1)-(\delta+1))(\alpha-\beta) + ((\gamma+1)-(\delta+1))(\alpha^{2s} - \beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right).
\end{aligned}$$

Note that if we substitute the roots of the characteristic equation,

$$\begin{aligned}\sum_{k=0}^{n} b_{mk+s} =& \frac{2^m}{A_2} b_{mn+s} + \frac{1}{B_2} b_{mn+s} + \frac{2^m}{A_2}\left(\frac{\alpha^{2(mn+s)}-\beta^{2(mn+s)}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+s}-(\delta+1)^{mn+s}}{\gamma-\delta}\right) - \frac{2^m}{A_2} b_{s-m} + \frac{1}{B_2} b_{s-m} \\ &+ \frac{2^m}{A_2}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right) + \frac{1}{B_2}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right) - \frac{1}{A_2} b_{mn+m+s} + \frac{1}{B_2} b_{mn+m+s} \\ &- \frac{1}{A_2}\left(\frac{\alpha^{2(mn+m+s)}-\beta^{2(mn+m+s)}}{\alpha-\beta}\right) + \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+m+s}-(\delta+1)^{mn+m+s}}{\gamma-\delta}\right) + \frac{1}{A_2} b_s - \frac{1}{B_2} b_s \\ &+ \frac{1}{A_2}\left(\frac{\alpha^{2s}-\beta^{2s}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{s}-(\delta+1)^{s}}{\gamma-\delta}\right) \\ =& \frac{2^m}{A_2}\left(b_{mn+s}-b_{s-m}\right) + \frac{1}{B_2}\left(b_{mn+s}+b_{s-m}+b_{mn+m+s}-b_s\right) + \frac{1}{A_2}\left(b_s-b_{mn+m+s}\right) + C_2.\end{aligned}$$

We get the result. □

3. Binomial Transform of Quadra Fibona-Pell Quaternions

In this section, we give the binomial transform of quadra Fibona-Pell quaternion sequence and obtain some certain identities related to this binomial transform. In [16], quaternion state of the Fibonacci-Pell sequence was investigated and here, Binet Formula, generating function, sum formulas are obtained. In [17], the results for the new sequence obtained by applying binomial transform to the quaternion sequence of the Horadam sequence, which is an integer sequence with a quadratic recurrence relation, are found. In [18], both quaternion and binomial transform are examined simultaneously for the first time.
In [16], let W_n be the quadra Fibona-Pell sequence, then

$$QW_n = W_n + W_{n+1}i + W_{n+2}j + W_{n+3}k$$

is called a quadra Fibona-Pell quaternion, containing the initial values of

$$\begin{aligned}QW_0 =& j+3k, \\ QW_1 =& i+3j+9k, \\ QW_2 =& 1+3i+9j+24k, \\ QW_3 =& 3+9i+24j+62k.\end{aligned}$$

Let QW_n be the $n-th$ quadra Fibona-Pell quaternion. Then the binomial transform of quadra Fibona-Pell sequence is

$$bq_n = \sum_{i=0}^{n}\binom{n}{i} QW_i. \tag{3.1}$$

Let us give a Lemma as a first step to find the recurrence relation of the binomial transform.

Lemma 3.1. *Let b_n be the binomial transform of quadra Fibona-Pell quaternion. Then*

$$bq_{n+1} = \sum_{i=0}^{n}\binom{n}{i}\left(QW_i + QW_{i+1}\right).$$

Proof. Notice that the equation (2.1)

$$\begin{aligned}bq_n &= \sum_{i=0}^{n}\binom{n}{i} QW_i, \\ bq_{n+1} &= \sum_{i=0}^{n+1}\binom{n+1}{i} QW_i, \\ bq_{n+1} &= \sum_{i=1}^{n+1}\binom{n+1}{i} QW_i + QW_0.\end{aligned}$$

Since $\binom{n}{i}+\binom{n}{i-1}=\binom{n+1}{i}$ and $\binom{n}{n+1}=0$, we get

$$\begin{aligned}bq_{n+1} &= \sum_{i=1}^{n+1}\left[\binom{n}{i}+\binom{n}{i-1}\right] QW_i + QW_0 \\ &= \sum_{i=1}^{n+1}\binom{n}{i} QW_i + \sum_{i=1}^{n+1}\binom{n}{i-1} QW_i + QW_0 \\ &= \sum_{i=0}^{n}\binom{n}{i} QW_i + \sum_{i=0}^{n}\binom{n}{i} QW_{i+1} \\ &= \sum_{i=0}^{n}\binom{n}{i}\left(QW_i + QW_{i+1}\right).\end{aligned}$$

□

Theorem 3.2. *The binomial transform of quadra Fibona-Pell quaternion states following recurrence relation*

$$bq_{n+3} = 7bq_{n+2} - 15bq_{n+1} + 10bq_n - 2bq_{n-1} \tag{3.2}$$

for $n \geq 4$, *where*

$$\begin{aligned} bq_0 &= j + 3k, \\ bq_1 &= i + 4j + 12, \\ bq_2 &= 1 + 5i + 16j + 45k, \end{aligned}$$

and

$$bq_3 = 6 + 21i + 61j + 164k.$$

Proof. Using Lemma 3.1 we get,

$$bq_{n+3} = K_2 bq_{n+2} + L_2 bq_{n+1} + M_2 bq_n + N_2 bq_{n-1}.$$

If we take $n = 1,2,3,4$ we take the system,

$$\begin{aligned} n = 1 &\Rightarrow bq_4 = K_2 bq_3 + L_2 bq_2 + M_2 bq_1 + N_2 bq_0, \\ n = 2 &\Rightarrow bq_5 = K_2 bq_4 + L_2 bq_3 + M_2 bq_2 + N_2 bq_1, \\ n = 3 &\Rightarrow bq_6 = K_2 bq_5 + L_2 bq_4 + M_2 bq_3 + N_2 bq_2, \\ n = 4 &\Rightarrow bq_7 = K_2 bq_6 + L_2 bq_5 + M_2 bq_4 + N_2 bq_3. \end{aligned}$$

By considering Cramer's rule for the system, we obtain

$$K_2 = 7, L_2 = -15, M_2 = 10, N_2 = -2$$

which is completed the proof. □

Theorem 3.3. *Let* bq_n *be the binomial transform of quadra Fibona-Pell quaternion sequences. The generating function of the related binomial transform is*

$$bq(x) = \frac{bq_0 + (bq_1 - 7bq_0)x + (bq_2 - 7bq_1 + 15bq_0)x^2 + (bq_3 - 7bq_2 + 15bq_1 - 10bq_0)x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}$$

where $bq_0 = j + 3k$, $bq_1 = i + 4j + 12$, $bq_2 = 1 + 5i + 16j + 45k$ *and* $bq_3 = 6 + 21i + 61j + 164k$.

Proof. Assume that

$$bq(x) = \sum_{i=0}^{\infty} bq_i x^i$$

is the generating function of the binomial transform for QW_n.Then

$$\begin{aligned} bq(x) &= bq_0 + bq_1 x + bq_2 x^2 + bq_3 x^3 + \cdots \\ 7xbq(x) &= 7bq_0 x + 7bq_1 x^2 + 7bq_2 x^3 + 7bq_3 x^4 + \cdots \\ 15x^2 bq(x) &= 15bq_0 x^2 + 15bq_1 x^3 + 15bq_2 x^4 + 15bq_3 x^5 + \cdots \\ 10x^3 bq(x) &= 10bq_0 x^3 + 10bq_1 x^4 + 10bq_2 x^5 + 10bq_3 x^6 + \cdots \\ 2x^4 bq(x) &= 2bq_0 x^4 + 2bq_1 x^5 + 2bq_2 x^6 + 2bq_3 x^7 + \cdots \end{aligned}$$

Since from equation (3.2), we obtain

$$(1 - 7x + 15x^2 - 10x^3 + 2x^4)bq(x) = j + 3k + (i - 3j - 9k)x + (1 - 2i + 3j + 6k)x^2 + (-1 + \hat{\imath} - j - k)x^3$$

and hence the generating function for the binomial transform of the bq_n is

$$bq(x) = \frac{bq_0 + (bq_1 - 7bq_0)x + (bq_2 - 7bq_1 + 15bq_0)x^2 + (bq_3 - 7bq_2 + 15bq_1 - 10bq_0)x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}.$$

Finally, we can give the following result. □

Now let's find the Binet formula, which we will use for the identities. For this, let the following equations be given

$$\begin{aligned} A &= -10bq_0 + 15bq_1 - 7bq_2 + bq_3, \\ B &= -3bq_0 + 10bq_1 - 6bq_2 + bq_3, \\ C &= -10bq_0 + 24bq_1 - 13bq_2 + 2bq_3, \\ D &= 4bq_0 - 10bq_1 + 6bq_2 - bq_3. \end{aligned} \tag{3.3}$$

Theorem 3.4. *Let bq_n be the binomial transform of quadra Fibona-Pell quaternion sequences. Binet formula for the related binomial transform is*

$$bq_n = \frac{P(\gamma+1)^n - R(\delta+1)^n}{\gamma-\delta} + \frac{S\alpha^{2n} - T\beta^{2n}}{\alpha-\beta} \tag{3.4}$$

for $n \geqslant 0$, where

$$\begin{aligned} A+B(\gamma+1) &= P, \\ A+B(\delta+1) &= R, \\ C+D\left(\alpha^2\right) &= S, \\ C+D\left(\beta^2\right) &= T. \end{aligned}$$

Proof. Assume that, from the previous theorem

$$bq(x) = \frac{Ax+B}{2x^2-4x+1} + \frac{Cx+D}{x^2-3x+1}.$$

When the denominator is equal

$$\begin{aligned} B+D &= bq_0, \\ A+2C &= bq_3 - 7bq_2 + 15bq_1 - 10bq_0, \\ B-3A-4C+2D &= bq_2 - 7bq_1 + 15bq_0, \\ A-3B+C-4D &= bq_1 - 7bq_0. \end{aligned}$$

the equation is obtained (3.3). When the values are replaced, we get

$$\begin{aligned} A &= -10(j+3k) + 15(i+4j+12k) - 7(1+5i+16j+45k) + 6+21i+61j+164k \\ &= i-j-k-1, \\ B &= -3(j+3k) + 10(i+4j+12k) - 6(1+5i+16j+45k) + 6+21i+61j+164k \\ &= i+2j+5k, \\ C &= -10(j+3k) + 24(i+4j+12k) - 13(1+5i+16j+45k) + 2(6+21i+61j+164k) \\ &= -1+i+k, \\ D &= 4(j+3k) - 10(i+4j+12k) + 6(1+5i+16j+45k) - (6+21i+61j+164k) \\ &= -i-j-2k. \end{aligned}$$

Finally, when necessary calculations are taken

$$\begin{aligned} bq(x) &= \frac{Ax+B}{2x^2-4x+1} + \frac{Cx+D}{x^2-3x+1} \\ &= \frac{Ax}{2x^2-4x+1} + \frac{B}{2x^2-4x+1} + \frac{Cx}{x^2-3x+1} + \frac{D}{x^2-3x+1} \\ &= \frac{A(\gamma+1)^n - A(\delta+1)^n + B(\gamma+1)^{n+1} - B(\delta+1)^{n+1}}{\gamma-\delta} + \frac{C\alpha^{2n} - C\beta^{2n} + D\alpha^{2n+2} - D\beta^{2n+2}}{\alpha-\beta} \\ &= \frac{(\gamma+1)^n(A+B(\gamma+1)) - (\delta+1)^n(A+B(\delta+1))}{\gamma-\delta} + \frac{\alpha^{2n}\left(C+D\alpha^2\right) - \beta^{2n}\left(C+D\beta^2\right)}{\alpha-\beta} \end{aligned}$$

we find the result

$$bq_n = \frac{P(\gamma+1)^n - R(\delta+1)^n}{\gamma-\delta} + \frac{S\alpha^{2n} - T\beta^{2n}}{\alpha-\beta}$$

where

$$\begin{aligned} A+B(\gamma+1) &= P, \\ A+B(\delta+1) &= R, \\ C+D\alpha^2 &= S, \\ C+D\beta^2 &= T. \end{aligned}$$

□

Now, we can give the following result.

Theorem 3.5. *Let bq_n be the binomial transform of quadra Fibona-Pell quaternion sequences. Then*

$$\sum_{n=0}^{\infty} bq_{mn+s}x^n = b_s\left(\frac{1}{E_1}-\frac{1}{F_1}\right) - b_{s-m}\left(\frac{2^m}{E_1}-\frac{1}{G_1}\right)x + H_1$$

for all $n \in \mathbb{N}$ and $m, s \in \mathbb{N}, s > m$,

$$\begin{aligned}
E_1 &= (1-(\gamma+1)^m x)(1-(\delta+1)^m x),\\
G_1 &= (1-\alpha^{2m}x)(1-\beta^{2m}x),\\
H_1 &= \frac{1}{E_1}\left(\frac{S\gamma^{2s}-T\delta^{2s}}{\gamma-\delta}\right) + \frac{1}{G_1}\left(\frac{P(\gamma+1)^s - R(\delta+1)^s}{\gamma-\delta}\right) - \frac{2^m}{E_1}\left(\frac{S\alpha^{2(s-m)}-T\beta^{2(s-m)}}{\alpha-\beta}\right)x - \frac{4^m}{G_1}\left(\frac{P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}}{\gamma-\delta}\right)x.
\end{aligned}$$

Proof. Again from equation (3.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} bq_{mn+s}x^n &= \sum_{n=0}^{\infty}\left(\frac{P(\gamma+1)^{mn+s}-R(\delta+1)^{mn+s}}{\gamma-\delta} + \frac{S\alpha^{2(mn+s)}-T\beta^{2(mn+s)}}{\alpha-\beta}\right)x^n\\
&= \frac{P(\gamma+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\gamma+1)^m x)^n - \frac{R(\delta+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\delta+1)^m x)^n + \frac{S\alpha^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^{2m}x)^n - \frac{T\beta^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\beta^{2m}x)^n\\
&= \frac{P(\gamma+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\gamma+1)^m x}\right) - \frac{R(\delta+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\delta+1)^m x}\right) + \frac{S\alpha^{2s}}{\alpha-\beta}\left(\frac{1}{1-\alpha^{2m}x}\right) - \frac{T\beta^{2s}}{\alpha-\beta}\left(\frac{1}{1-\beta^{2m}x}\right)\\
&= \frac{1}{\gamma-\delta}\,\frac{(P(\gamma+1)^s - R(\delta+1)^s) - (P(\gamma+1)^s(\delta+1)^m - R(\delta+1)^s(\gamma+1)^m)x}{1-((\gamma+1)^m+(\delta+1)^m)x + ((\gamma+1)(\delta+1))^m x^2}\\
&\quad + \frac{1}{\alpha-\beta}\,\frac{(S\alpha^{2s}-T\beta^{2s}) - (S\alpha^{2s}\beta^{2m} - T\beta^{2s}\alpha^{2m})x}{1-(\alpha^{2m}+\beta^{2m})x + (\alpha\beta)^{2m}x^2}\\
&= \frac{1\,(\alpha-\beta)}{E_1}\left(\frac{P(\gamma+1)^s-R(\delta+1)^s}{(\gamma-\delta)(\alpha-\beta)}\right) + \frac{1\,(\gamma-\delta)}{G_1}\left(\frac{S\alpha^{2s}-T\beta^{2s}}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&\quad - \frac{((\gamma+1)(\delta+1))^m(\alpha-\beta)}{E_1}\left(\frac{P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}}{(\gamma-\delta)(\alpha-\beta)}\right)x - \frac{(\alpha\beta)^{2m}(\gamma-\delta)}{G_1}\left(\frac{S\alpha^{2(s-m)}-T\beta^{2(s-m)}}{(\alpha-\beta)(\gamma-\delta)}\right)x.
\end{aligned}$$

If necessary arrangements are made, then we get

$$\begin{aligned}
\sum_{n=0}^{\infty} bq_{mn+s}x^n &= \frac{1}{E_1}\left(\frac{b_s((\gamma+1)-(\delta+1))(\alpha-\beta) + ((\gamma+1)-(\delta+1))(S\alpha^{2s}-T\beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&\quad + \frac{1}{G_1}\left(\frac{(\alpha-\beta)(P(\gamma+1)^s-R(\delta+1)^s) - b_s((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&\quad - \frac{((\gamma+1)(\delta+1))^m}{E_1}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta) + ((\gamma+1)-(\delta+1))(S\alpha^{2(s-m)}-T\beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
&\quad - \frac{(\alpha\beta)^{2m}}{G_1}\left(\frac{(\alpha-\beta)\left(P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}\right) - b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
&= b_s\left(\frac{1}{E_1}-\frac{1}{G_1}\right) - b_{s-m}\left(\frac{2^m}{E_1}-\frac{1}{G_1}\right)x + H_1.
\end{aligned}$$

Hence the result is obvious. □

Theorem 3.6. *Let bq_n be the binomial transform of quadra Fibona-Pell quaternion sequences. Then*

$$\sum_{k=0}^{n} bq_{mk+s} = \frac{2^m}{E_2}(b_{mn+s}-b_{s-m}) - \frac{1}{G_2}(b_{mn+s}-b_{s-m}-b_{mn+m+s}+b_s) - \frac{1}{E_2}(b_{mn+m+s}+b_s) + H_2$$

for all $n \in \mathbb{N}$ and $m, s \in \mathbb{Z}$, $s > m$,

$$\begin{aligned}
E_2 &= (\gamma+1)^m(\delta+1)^m - ((\gamma+1)^m + (\delta+1)^m) + 1,\\
G_2 &= \alpha^{2m}\beta^{2m} - \left(\alpha^{2m}+\beta^{2m}\right) + 1,\\
H_2 &= \frac{2^m}{E_2}\left(\frac{S\alpha^{2(mn+s)}-T\beta^{2(mn+s)}}{\alpha-\beta}\right) + \frac{1}{G_2}\left(\frac{P(\gamma+1)^{mn+s}-R(\delta+1)^{mn+s}}{\gamma-\delta}\right) - \frac{2^m}{E_2}\left(\frac{S\alpha^{2(s-m)}-T\beta^{2(s-m)}}{\alpha-\beta}\right)\\
&\quad - \frac{1}{G_2}\left(\frac{P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}}{\gamma-\delta}\right) - \frac{1}{E_2}\left(\frac{S\alpha^{2(mn+m+s)}-T\beta^{2(mn+m+s)}}{\alpha-\beta}\right) - \frac{1}{G_2}\left(\frac{P(\gamma+1)^{mn+m+s}-T(\delta+1)^{mn+m+s}}{\gamma-\delta}\right)\\
&\quad + \frac{1}{E_2}\left(\frac{S\alpha^{2s}-T\beta^{2s}}{\alpha-\beta}\right) + \frac{1}{G_2}\left(\frac{P(\gamma+1)^s-R(\delta+1)^s}{\gamma-\delta}\right).
\end{aligned}$$

Proof. From (3.4), we get

$$\begin{aligned}
\sum_{k=0}^{n} bq_{mk+s} =& \sum_{k=0}^{n}\left(\frac{P(\gamma+1)^{mk+s}-R(\delta+1)^{mk+s}}{\gamma-\delta}+\frac{S\alpha^{2(mk+s)}-T\beta^{2(mk+s)}}{\alpha-\beta}\right)\\
=&\frac{P(\gamma+1)^{mn+m+s}(\delta+1)^m-P(\gamma+1)^{mn+m+s}-P(\gamma+1)^s(\delta+1)^m+P(\gamma+1)^s}{(\gamma-\delta)((\gamma+1)^m(\delta+1)^m-((\gamma+1)^m+(\delta+1)^m)+1)}\\
&-\frac{R(\delta+1)^{mn+m+s}(\gamma+1)^m-R(\delta+1)^{mn+m+s}-R(\delta+1)^s(\gamma+1)^m+R(\delta+1)^s}{(\gamma-\delta)((\gamma+1)^m(\delta+1)^m-((\gamma+1)^m+(\delta+1)^m)+1)}\\
&+\frac{S\alpha^{2(mn+m+s)}\beta^{2m}-S\alpha^{2(mn+m+s)}-S\alpha^{2s}\beta^{2m}+S\alpha^{2s}}{(\alpha-\beta)(\alpha^{2m}\beta^{2m}-(\alpha^{2m}+\beta^{2m})+1)}\\
&-\frac{T\beta^{2(mn+m+s)}\alpha^{2m}-T\beta^{2(mn+m+s)}-T\beta^{2s}\alpha^{2m}+T\beta^{2s}}{(\alpha-\beta)(\alpha^{2m}\beta^{2m}-(\alpha^{2m}+\beta^{2m})+1)}.
\end{aligned}$$

As a result of calculations, we obtained

$$\begin{aligned}
\sum_{k=0}^{n} bq_{mk+s} =& \frac{(\gamma+1)^m(\delta+1)^m}{E_2}\left(\frac{b_{mn+s}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(mn+s)}-T\beta^{2(mn+s)})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&+\frac{(\alpha\beta)^{2m}}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{mn+s}-R(\delta+1)^{mn+s})-b_{mn+s}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{(\gamma+1)^m(\delta+1)^m}{E_2}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(s-m)}-T\beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{(\alpha\beta)^{2m}}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{s-m}-R(\delta+1)^{s-m})-b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{1}{E_2}\left(\frac{b_{mn+m+s}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(mn+m+s)}-T\beta^{2(mn+m+s)})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{1}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{mn+m+s}-R(\delta+1)^{mn+m+s})-b_{mn+m+s}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&+\frac{1}{E_2}\left(\frac{b_s((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2s}-T\beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&+\frac{1}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{s}-R(\delta+1)^{s})-b_s((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
=&\frac{2^m}{E_2}(b_{mn+s}-b_{s-m})-\frac{1}{G_2}(b_{mn+s}-b_{s-m}-b_{mn+m+s}+b_s)-\frac{1}{E_2}(b_{mn+m+s}+b_s)+H_2.
\end{aligned}$$

□

4. Conclusion

Our aim in this study is to study the binomial transform for quadra Fibona-Pell sequence and its binomial transform of quaternion sequence. In the article, the binomial transform of the sequence is found in the first part, and then the results related to this transform are mentioned. In the second part, similar results were obtained by binomial transform of quadra Fibona-Pell quaternion sequence, which was found before.

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Periodic Korovkin Theorem via P_p^2-Statistical $\mathscr{A}$-Summation Process

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Abstract

In the current research, we investigate and establish Korovkin-type approximation theorems for linear operators defined on the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method. We demonstrate with an example how our theory is more strong than previously studied. Additionally, we research the rate of convergence of positive linear operators defined on this space.

1. Introduction and Preliminaries Notations

Before starting with the presentation of the definitions which will be used to prove approximation theorems, we recall the well-known notions.
A double sequence $x = (x_{ij})$ is convergent to L in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j > N$ and denoted by $P - \lim_{i,j} x_{ij} = L$ (see [1]). A double sequence is bounded if there exists a positive number M such that $|x_{ij}| \le M$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. As it is known that every single convergent sequence (in the usual sense) is bounded, while a convergent double sequence need not to be bounded.
Let us turn our attention to statistical convergence and power series method for double sequences.
Moricz [2] proposed and investigated the idea of statistical convergence for double sequences, which may be restated in terms of natural density. Let $E \subset \mathbb{N}^2$ be a two-dimensional subset of positive integers and let $E_{m,n} = \{(i, j) \in E : i \le m,\ j \le n\}$. Then the two-dimensional analogue of natural density can be defined as follows:

$$\delta_2(E) := P - \lim_{m,n} \frac{1}{mn} |E_{mn}|$$

if it exists. The number sequence $x = (x_{ij})$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set $E := E_{mn}(\varepsilon) := \{i \le m,\ j \le n :\ |x_{ij} - L| \ge \varepsilon\}$ has natural density zero; in that case we write $st_2 - \lim_{i,j} x_{ij} = L$. For all that a statistically convergent sequence need not be convergent in light of the above.
It is obvious that a double sequence that is P-convergent statistically converges to the same value, but the opposite is not always true. Additionally, Moricz [2] characterized the statistical convergence for double sequences as follows:
A double sequence $x = (x_{ij})$ is statistically convergent to L if and only if there exists a set $E \subset \mathbb{N}^2$ such that the natural density of E is 1 and

$$P - \lim_{\substack{i,j\to\infty \\ \text{and } (i,j)\in E}} x_{ij} = L.$$

Let (p_{ij}) be a double sequence of nonnegative numbers with $p_{00} > 0$ and such that the following power series

$$p(t, s) := \sum_{i,j=0}^{\infty} p_{ij} t^i s^j$$

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has radius of convergence R with $R\in(0,\infty]$ and $t,s\in(0,R)$. If for all $t,s\in(0,R)$,

$$\lim_{t,s\to R^-}\frac{1}{p(t,s)}\sum_{i,j=0}^{\infty}p_{ij}t^is^jx_{ij}=L$$

then we say that the double sequence $x=(x_{ij})$ is convergent to L in the sense of power series method and denoted by $P_p^2-\lim x_{ij}=L$ ([3]). Keep in mind that the method is regular if and only if

$$\lim_{t,s\to R^-}\frac{\sum_{i=0}^{\infty}p_{i\nu}t^i}{p(t,s)}=0 \text{ and } \lim_{t,s\to R^-}\frac{\sum_{j=0}^{\infty}p_{\mu j}s^j}{p(t,s)}=0, \text{ for any } \mu,\upsilon, \tag{1.1}$$

hold [3].

Remark 1.1. *In case of $R=1$, if $p_{ij}=1$ and $p_{ij}=\frac{1}{(i+1)(j+1)}$, the power series methods coincide with Abel summability method and logarithmic summability method, respectively. In the case of $R=\infty$ and $p_{ij}=\frac{1}{i!j!}$, the power series method coincides with Borel summability method.*

Here and throughout the paper power series method is always assumed to be regular.
Ünver and Orhan [4] have recently introduced P_p-density of $E\subset\mathbb{N}_0$ and the definition of P_p-statistical convergence for single sequences. A natural question is what about statistical convergence or P_p-statistical convergence of the sequence. Hence, they showed that statistical convergence and P_p-statistical convergence are incompatible. In view of their work, Yıldız, Demirci and Dirik [5] have introduced the definitions of P_p^2-density of $F\subset\mathbb{N}_0^2=\mathbb{N}_0\times\mathbb{N}_0$ and P_p^2-statistical convergence for double sequences:

Definition 1.2 ([5]). *Let $F\subset\mathbb{N}_0^2$. If the limit*

$$\delta_{P_p}^2(F):=\lim_{t,s\to R^-}\frac{1}{p(t,s)}\sum_{(i,j)\in F}p_{ij}t^is^j$$

exists, then $\delta_{P_p}^2(F)$ is called the P_p^2-density of F. Note that, from the definition of a power series method and P_p^2-density it can be established that $0\leq\delta_{P_p}^2(F)\leq 1$ whenever it exists.

Definition 1.3 ([5]). *Let $x=(x_{ij})$ be a double sequence. Then x is said to be statistically convergent with respect to power series method (P_p^2-statistically convergent) to L if for any $\varepsilon>0$*

$$\lim_{t,s\to R^-}\frac{1}{p(t,s)}\sum_{(i,j)\in F_\varepsilon}p_{ij}t^is^j=0$$

where $F_\varepsilon=\left\{(i,j)\in\mathbb{N}_0^2:\left|x_{ij}-L\right|\geq\varepsilon\right\}$, that is $\delta_{P_p}^2(F_\varepsilon)=0$ for any $\varepsilon>0$. In this case we write $st_{P_p}^2-\lim x_{ij}=L$.

Let $A=[a_{klmn}]$, $k,l,m,n\in\mathbb{N}$, be a four-dimensional infinite matrix. The A-transform of a given double sequence $x=(x_{mn})$ is given by

$$(Ax)_{kl}=\sum_{(m,n)\in\mathbb{N}^2}a_{klmn}x_{mn},\quad k,l\in\mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$ and denoted by $Ax:=((Ax)_{kl})$. If the A-transform of x exists for all $k,l\in\mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P-\lim_{p,q}\sum_{m=1}^{p}\sum_{n=1}^{q}a_{klmn}x_{mn}=y_{kl} \text{ and } P-\lim_{k,l}y_{kl}=L$$

then we say that a sequence x is A-summable to L. A two-dimensional matrix transformation is referred to as regular in summability theory if it converts each convergent sequence into one with the same limit.
Now consider a sequence of four-dimensional infinite matrices with non-negative real elements $\mathscr{A}:=\left(A^{(i,j)}\right)=\left(a_{klmn}^{(i,j)}\right)$. For a given double sequence of real numbers, $x=(x_{mn})$ is said to be *$\mathscr{A}$-summable* to L if

$$P-\lim_{k,l}\sum_{(m,n)\in\mathbb{N}^2}a_{klmn}^{(i,j)}x_{mn}=L$$

uniformly in i and j.
$\mathscr{A}$-summability is the A-summability for four-dimensional infinite matrix if $A^{(i,j)}=A$, four-dimensional infinite matrix. Some results regarding matrix summability method for double sequences may be found in the papers [6, 7]. $C^*(\mathbb{R}^2)$ stands for the space of all continuous functions on $\mathbb{R}^2$ that are real valued and have a period of 2π. If a function $h\in C^*(\mathbb{R}^2)$, then

$$h(x,y)=h(x+2k\pi,y)=h(x,y+2k\pi), \text{ for all } (x,y)\in\mathbb{R}^2,$$

holds for $k=0,\pm1,\pm2,\ldots$. In what follows, this space is equipped with the supremum norm

$$\|f\|_*=\sup_{(x,y)\in\mathbb{R}^2}|h(x,y)|,\ \left(h\in C^*\left(\mathbb{R}^2\right)\right).$$

A sequence $\mathbb{L} := (L_{mn})$ of positive linear operators from $C^*(\mathbb{R}^2)$ into itself is referred to as an $\mathscr{A}$-summation process on $C^*(\mathbb{R}^2)$ if $(L_{mn}h)$ is $\mathscr{A}$-summable to h for every $h \in C^*(\mathbb{R}^2)$, i.e.,

$$P - \lim_{k,l} \left\| S_{klij}^{\mathbb{L}} h - h \right\|_* = 0, \text{ uniformly in } i, j$$

where for all $k,l,i,j \in \mathbb{N}$, $h \in C^*(\mathbb{R}^2)$ the series

$$S_{klij}^{\mathbb{L}} h := \sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn} h \tag{1.2}$$

and it is assumed that the series in (1.2) absolutely convergent for each $i,j,k,l \in \mathbb{N}$ and h.
For the rate of convergence, we need to recall the following modulus of continuity of h. Let $h \in C^*(\mathbb{R}^2)$, then

$$w(h;\gamma) = \sup\left\{ |h(u,v) - h(x,y)| : (u,v),(x,y) \in \mathbb{R}^2 \text{ and } \sqrt{(u-x)^2 + (v-y)^2} \le \gamma \right\}$$

for $\gamma > 0$. This definition yields the following basic property for $h \in C^*(\mathbb{R}^2)$.
For any $a > 0$,

$$w(h;a\gamma) \le (1 + [a])\, w(h;\gamma)$$

where $[a]$ is defined to be the greatest integer less than or equal to a.
The paper of Korovkin [8] is an important issue. It can help us to understand the nature of approximation of sequences. This approximation problem has a rich history associated with the names of the different convergence methods on some spaces in the theory. For some recent research works in this direction, see [9–21]. In this paper, we investigate and establish Korovkin-type approximation theorems for linear operators defined on the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method. We demonstrate with an example how our theory is more strong than previously studied. Additionally, we research the rate of convergence of positive linear operators defined on this space.

2. The Second Theorem of Korovkin Type

The aim of this section is to deal with approximation of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method.
Our main result is the following.

Theorem 2.1. *Let $\mathscr{A} = \left(A^{(i,j)}\right)$ be a sequence of four-dimensional infinite matrices. Let $\mathbb{L} = (L_{mn})$ be a sequence of positive linear operators acting from $C^*(\mathbb{R}^2)$ into itself. Assume that (1.2) holds. Then, for all $h \in C^*(\mathbb{R}^2)$*

$$st_{P_p}^2 - \lim \left\| S_{klij}^{\mathbb{L}} h - h \right\|_* = 0 \quad \text{uniformly in } i \text{ and } j \tag{2.1}$$

if and only if

$$st_{P_p}^2 - \lim \left\| S_{klij}^{\mathbb{L}} h_r - h_r \right\|_* = 0 \quad \text{uniformly in } i \text{ and } j \quad (r = 0,1,2,3,4) \tag{2.2}$$

where $h_0(x,y) = 1$, $h_1(x,y) = \sin x$, $h_2(x,y) = \sin y$, $h_3(x,y) = \cos x$ and $h_4(x,y) = \cos y$.

Proof. Since 1, $\sin x$, $\sin y$, $\cos x$ and $\cos y$ belong to $C^*(\mathbb{R}^2)$, the necessity is clear. Suppose now that (2.2) holds. Let $h \in C^*(\mathbb{R}^2)$ and I, J be closed subinterval of length 2π of $\mathbb{R}$. Fix $(x,y) \in I \times J$. As in the proof of Theorem 2.1 in [22], it follows from the continuity of h that

$$|h(u,v) - h(x,y)| < \varepsilon + \frac{2M_h}{\sin^2 \frac{\delta}{2}} \varphi(u,v)$$

which gives,

$$\begin{aligned}
\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h;x,y)-h(x,y)\right| &\le \sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}\left(\left|h(u,v)-h(x,y)\right|;x,y\right)+\left|h(x,y)\right|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
&\le \sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}\left(\varepsilon+\frac{2M_h}{\sin^2\frac{\delta}{2}}\varphi(u,v);x,y\right)+M_h\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
&\le \varepsilon+(\varepsilon+M_h)\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
&\quad +\frac{M_h}{\sin^2\frac{\delta}{2}}\left\{\begin{array}{l}
2\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
+\left|\sin x\right|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_1;x,y)-h_1(x,y)\right| \\
+\left|\sin y\right|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_2;x,y)-h_2(x,y)\right| \\
+\left|\cos x\right|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_3;x,y)-h_3(x,y)\right| \\
+\left|\cos y\right|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_4;x,y)-h_4(x,y)\right|
\end{array}\right\} \\
&\le \varepsilon+N\sum_{r=0}^{4}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_r;x)-h_r(x,y)\right|
\end{aligned}$$

where $M_h=\|f\|_*$, $\varphi(u,v)=\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}$ and $N:=\varepsilon+M_h+\frac{2M_h}{\sin^2\frac{\delta}{2}}$. Then, taking supremum over $(x,y)\in\mathbb{R}^2$, we obtain

$$\left\|S_{klij}^{\mathbb{L}}h-h\right\|_*\le\varepsilon+N\sum_{r=0}^{4}\left\|S_{klij}^{\mathbb{L}}h_r-h_r\right\|_*. \tag{2.3}$$

Now given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$, and define

$$\begin{aligned}
D &:= \left\{(k,l):\left\|S_{klij}^{\mathbb{L}}h-h\right\|_*\ge r\right\}, \\
D_r &:= \left\{(k,l):\left\|S_{klij}^{\mathbb{L}}h_r-h_r\right\|_*\ge\frac{r-\varepsilon}{5N}\right\},\ r=0,1,2,3,4.
\end{aligned}$$

It is easy see that from (2.3)

$$D\subseteq\bigcup_{r=0}^{4}D_r.$$

Hence, we may write

$$\delta_{P_p}^2(D)\le\sum_{r=0}^{4}\delta_{P_p}^2(D_r).$$

Then, according to (2.2), we have

$$\delta_{P_p}^2(D)=0,$$

and hence

$$st_{P_p}^2-\lim\left\|S_{klij}^{\mathbb{L}}h-h\right\|_*=0\quad\text{uniformly in } i \text{ and } j$$

which is the desired result. □

3. An example

Now, we give an example that our theorem (Theorem 2.1) is stronger than Theorem 9 in [23].

Example 3.1. *Now assume that* $\mathscr{A}=\left(A^{(i,j)}\right)$ *is a sequence of four-dimensional infinite matrices defined by* $a_{klmn}^{(i,j)}=\frac{1}{kl}$ *if* $i\le m\le k+i-1$, $j\le n\le l+j-1$ *and* $a_{klmn}^{(i,j)}=0$ *otherwise. Let us consider the double sequence of Fejer operators on* $C^*\left(\mathbb{R}^2\right)$ *where*

$$L_{mn}(h;x,y)=\frac{1}{(m\pi)(n\pi)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}h(u,v)F_m(u)F_n(v)\,du\,dv \tag{3.1}$$

where $F_m(u) = \frac{\sin^2 \frac{m(u-x)}{2}}{2\sin^2 \frac{u-x}{2}}$ *and* $\frac{1}{\pi}\int_{-\pi}^{\pi} F_m(u)\,du = 1$. *Let* (p_{ij}) *be defined as follows*

$$p_{ij} = \begin{cases} 0, & i \text{ and } j \text{ even} \\ 1, & i \text{ or } j \text{ odd} \end{cases},$$

and take the sequence (x_{ij}) *defined by*

$$x_{ij} = \begin{cases} ij, & i \text{ and } j \text{ even} \\ 0, & i \text{ or } j \text{ odd} \end{cases}. \tag{3.2}$$

It is easy to see that

$$st^2_{P_p} - \lim x_{ij} = 0. \tag{3.3}$$

However, the sequence (x_{ij}) *neither statistically convergent to* 0 *nor Pringsheim convergent. Now using (3.1) and (3.2), we define the following double positive linear operators* $\mathbb{T} = (T_{mn})$ *on* $C^*(\mathbb{R}^2)$ *as follows:*

$$T_{mn}(f;x,y) = (1+x_{mn})L_{mn}(f;x,y). \tag{3.4}$$

We now claim that

$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h_r - h_r \right\|_* = 0 \quad \text{uniformly in } i \text{ and } j, \ (r = 0,1,2,3,4). \tag{3.5}$$

Observe that $L_{mn}(h_0;x,y) = h_0(x,y)$, $L_{mn}(h_1;x,y) = \frac{m-1}{m}h_1(x,y)$, $L_{mn}(h_2;x,y) = \frac{n-1}{n}h_2(x,y)$, $L_{mn}(h_3;x,y) = \frac{m-1}{m}h_3(x,y)$, $L_{mn}(h_4;x,y) = \frac{n-1}{n}h_4(x,y)$. *So, we can see,*

$$\left\| S^{\mathbb{T}}_{klij} h_0 - h_0 \right\|_* = \left\| \sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} \frac{1}{kl}(1+x_{mn}) - 1 \right\|_* \leq \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} x_{mn}.$$

It is well known that if a sequence is convergent, its arithmetic mean will also converge to the same value. Thus, by virtue of P^2_p*-statistical convergence and thanks to (3.3) it is clear that*

$$st^2_{P_p} - \lim \left(\sup_{i,j} \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} x_{mn} \right) = 0, \tag{3.6}$$

and hence

$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h_0 - h_0 \right\|_* = 0, \ \text{uniformly in } i \text{ and } j,$$

which guarantees that (3.5) holds true for $r = 0$. *Also, we compute*

$$\left\| S^{\mathbb{T}}_{klij} h_1 - h_1 \right\|_* = \left\| \sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} \frac{1}{kl}(1+x_{mn})\frac{m-1}{m}h_1 - h_1 \right\|_* \leq \left| \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} \frac{m-1}{m} - 1 \right| + \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} \frac{x_{mn}(m-1)}{m}.$$

Since $st^2_{P_p} - \lim \left(\sup_{i,j} \left(\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1} \frac{m-1}{m} - 1 \right) \right) = 0$ *and from (3.6) we have,*

$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h_1 - h_1 \right\|_* = 0, \ \text{uniformly in } i \text{ and } j.$$

So (3.5) valid for $r = 1$. *Likewise, we have*

$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h_2 - h_2 \right\|_* = 0, \ \text{uniformly in } i \text{ and } j,$$
$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h_3 - h_3 \right\|_* = 0, \ \text{uniformly in } i \text{ and } j,$$
$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h_4 - h_4 \right\|_* = 0, \ \text{uniformly in } i \text{ and } j.$$

So, our claim (3.5) is valid for each $r = 0,1,2,3,4$. *Then, observe that the double sequence* $\mathbb{T} = (T_{mn})$ *defined by (3.4) satisfy all hypotheses of Theorem 2.1. Hence, we have, for all* $f \in C^*(\mathbb{R}^2)$,

$$st^2_{P_p} - \lim \left\| S^{\mathbb{T}}_{klij} h - h \right\|_* = 0.$$

Also, since (x_{ij}) *is not statistically convergent to* 0, (T_{mn}) *does not satisfy Theorem 9 in [23].*

4. Rates of Convergence

In this section, via $\mathscr{A}$-summation process via statistical convergence with respect to power series method, we study the rates of convergence of a double sequence of positive linear operators mapping acting from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$ by means of the modulus of continuity.
We have the following result.

Theorem 4.1. *Let $\mathscr{A}=\left(A^{(i,j)}\right)$ be a sequence of four-dimensional infinite matrices. Let $\mathbb{L}=(L_{mn})$ be a double sequence of positive linear operators moving from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$. Suppose that (1.2) and the following conditions provided:*
(i) $st^2_{P_p}-\lim\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*=0$, uniformly in i and j,
(ii) $st^2_{P_p}-\lim w(h;\gamma)=0$, uniformly in i and j,
where $\gamma:=\gamma^{(i,l)}_{(j,k)}:=\sqrt{\left\|S^{\mathbb{L}}_{klij}\varphi\right\|_*}$ with $\varphi(u,v)=\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}$. Then we have, for all $h\in C^*\left(\mathbb{R}^2\right)$,

$$st^2_{P_p}-\lim\left\|S^{\mathbb{L}}_{klij}h-h\right\|_*=0,\ \text{uniformly in } i \text{ and } j.$$

Proof. To prove this, we firstly suppose that $(x,y)\in[-\pi,\pi]\times[-\pi,\pi]$ and $h\in C^*\left(\mathbb{R}^2\right)$ be fixed, and that Let (i) and (ii) be provided.. Let γ be a positive number. As in the proof of Theorem 9 in [23], since the function h is continious, the following inequality is obtained:

$$|h(u,v)-h(x,y)|\leq\left(1+\pi^2\frac{\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}}{\gamma^2}\right)w(h;\gamma).$$

Using the definition of modulus of continuity and since the operators L_{mn} is linear and the positive, we have

$$\begin{aligned}\left|S^{\mathbb{L}}_{klij}(h;x,y)-h(x,y)\right|&=\left|\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h;x,y)-h(x,y)\right|\\&\leq\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}\left(|h(u,v)-h(x,y)|;x,y\right)+|h(x,y)|\left|\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h_0;x,y)-h_0(x,y)\right|\\&\leq w(h;\gamma)\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h_0;x,y)+\pi^2\frac{w(h;\gamma)}{\gamma^2}\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(\varphi;x,y)\\&\quad+|h(x,y)|\left|\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h_0;x,y)-h_0(x,y)\right|\end{aligned}$$

where $\varphi(u,v)=\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}$. If supremum over (x,y) is taken on both sides of the above inequality and is chosen $\gamma:=\gamma^{(i,l)}_{(j,k)}:=\sqrt{\left\|S^{\mathbb{L}}_{klij}\varphi\right\|_*}$, then we obtain

$$\left\|S^{\mathbb{L}}_{klij}h-h\right\|_*\leq w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*+\left(1+\pi^2\right)w\left(h;\gamma^{(i,l)}_{(j,k)}\right)+M_h\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_* \tag{4.1}$$

where $M_h:=\|h\|_*$. Now, given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D&:=\left\{(k,l):\left\|S^{\mathbb{L}}_{klij}h-h\right\|_*\geq\varepsilon\right\},\\D_1&:=\left\{(k,l):w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*\geq\frac{\varepsilon}{3}\right\},\\D_2&:=\left\{(k,l):w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\geq\frac{\varepsilon}{3\left(1+\pi^2\right)}\right\},\\D_3&:=\left\{(k,l):\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*\geq\frac{\varepsilon}{3M_h}\right\}.\end{aligned}$$

Then, it follows from (4.1) that $D\subset D_1\cup D_2\cup D_3$. Also, defining

$$\begin{aligned}D_4&:=\left\{(k,l):w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\geq\sqrt{\frac{\varepsilon}{3}}\right\},\\D_5&:=\left\{(k,l):\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*\geq\sqrt{\frac{\varepsilon}{3}}\right\},\end{aligned}$$

we have $D_1\subset D_4\cup D_5$, which yields

$$D\subseteq\bigcup_{i=2}^{5}D_i.$$

Hence, we may write

$$\delta_{P_p}^2(D) \leq \sum_{r=0}^{5} \delta_{P_p}^2(D_r).$$

Using the hypothesis (i) and (ii), we get

$$\delta_{P_p}^2(D) = 0,$$

and hence

$$st_{P_p}^2 - \lim \left\| S_{klij}^{\mathbb{L}} h - h \right\|_* = 0, \text{ uniformly in } i \text{ and } j.$$

Therefore, the proof is completed. □

5. Conclusion

The paper contains Korovkin-type approximation theorem and the rate of convergence for linear operators defined on the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method. Also, it is demonstrated with an example how the new theory is more stronger than previously studied.

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Author's contributions

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