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Certain Curvature Conditions on (k,μ)-Paracontact Metric Spaces

Pakize Uygun[1]*, Süleyman Dirik[2], Mehmet Atçeken[3], Tuğba Mert[4]

Abstract
The aim of this paper is to classify (k,μ)-paracontact metric spaces satisfying certain curvature conditions. We present the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $R\cdot W_6=0$, $R\cdot W_7=0$, $R\cdot W_8=0$ and $R\cdot W_9=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized. Also, several results are obtained.

Keywords: (k,μ)-Paracontact Manifold, η-Einstein manifold, Riemannian curvature tensor
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[1] *Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, 68100, Aksaray, Turkey, ORCID: 0000-0001-8226-4269*
[2] *Department of Mathematics, Faculty of Arts and Sciences, Amasya University, 05100, Amasya, Turkey, ORCID: 0000-0001-9093-1607*
[3] *Department of Mathematics, Faculty of Arts and Sciences, Aksaray University, 68100, Aksaray, Turkey, ORCID: 0000-0002-1242-4359*
[4] *Department of Mathematics, Faculty of Sciences, Sivas Cumhuriyet University, 58140, Sivas, Turkey, ORCID: 0000-0001-8258-8298*
***Corresponding author**: pakizeuygun@hotmail.com
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1. Introduction

Paracontact manifolds are smooth manifolds of dimension $(2n+1)$ equipped with a 1-form η, a vector field ξ and a field of endomorphisms of tangent spaces ϕ such that $\eta(\xi)=1$, $\phi^2=I-\eta\otimes\xi$ and ϕ induces an almost paracomplex structure by kernel of η [1]. On the other hand, if the manifold is equipped with a pseudo-Riemannian metric g of signature $(n+1,n)$ satisfying

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y),\;\; d\eta(X,Y)=g(X,\phi Y),$$

(M,η) becomes a contact manifold and (ϕ,ξ,η,g) is said to be a paracontact metric structure on M. In 1985, Kaneyuki and Williams initiated the perspective of paracontact geometry [5]. Zamkovoy performed a thorough study of paracontact metric Manifolds. [15]. Recently, B. Cappeletti-Montano, I. Küpeli Erken and C. Murathan introduced a new type of paracontact geometry so-called paracontact metric $(k,\mu)-$space, where k and μ are constant [4].

M. M. Tripathi and P. Gupta studied T-curvature tensors in semi-Riemannian manifolds. They defined T-conservative semi-Riemannian manifolds and give necessary and sufficient tensor on a Riemannian manifolds to be T-conservative. They proved that every T-flat semi-Riemannian manifold is Einstein. They also gave the conditions for semi-Riemannian manifold to be T-flat [8]. Since then several geometers studied curvature conditions and obtain various important properties [2, 6], [9]-[13].

The object of this paper is to study properties of the some certain curvature tensor in a $(k,\mu)-$paracontact metric manifold. In the present paper we survey $R\cdot W_6=0$, $R\cdot W_7=0$, $R\cdot W_8=0$ and $R\cdot W_9=0$, where W_6, W_7, W_8 and W_9 denote curvature tensors of manifold, respectively.

2. Preliminaries

An $(2n+1)$-dimensional manifold M is called to have an paracontact structure if it admits a $(1,1)-$tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions [5]:

(i) $\phi^2 X = X - \eta(X)\xi$, for any vector field $X \in \chi(M)$, the set of all differential vector fields on M,

(ii) $\eta(\xi) = 1$, $\eta \circ \phi = 0$, $\phi\xi = 0$.

An almost paracontact structure is called to be normal if and only if the $(1,2)-$type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$. An almost paracontact manifold equipped with a pseudo-Riemannian metric g so that

$$g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X) \tag{2.1}$$

for all vector fields $X,Y \in \chi(M)$ is said almost paracontact metric manifold, where signature of g is $(n+1,n)$. An almost paracontact structure is called to be a paracontact structure if $g(X,\phi Y) = d\eta(X,Y)$ with the associated metric g [15]. We now define a $(1,1)$ tensor field h by $h = \frac{1}{2}L_\xi \phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi = -\phi h, \qquad h\xi = 0, \qquad Trh = Tr.\phi h = 0. \tag{2.2}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X \tag{2.3}$$

for any $X \in \chi(M)$[15]. For a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, if ξ is a killing vector field or equivalently, $h = 0$, then it is called a K-paracontact manifold.

An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [15].

$$(\widetilde{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X$$

for all $X,Y \in \chi(M)$ [15]. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.4}$$

for all $X,Y \in \chi(M)$, but this is not a sufficient condition for a para-contact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true[3].

A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$,where a,b are smooth functions on M. If $b = 0$, then the manifold is also called Einstein and if $a = 0$, then it is called special type of η-Einstein manifolds [14].

A paracontact metric manifold is said to be a $(k,\mu)-$paracontact manifold if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{2.5}$$

for all $X,Y \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ [16].

In particular, if $\mu = 0$, then the paracontact metric $(k,\mu)-$manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k\eta(Y)X - k\eta(X)Y \tag{2.6}$$

for all $X,Y \in \chi(M)$. Though the geometric behavior of paracontact metric $(k,\mu)-$spaces is different according as $k < -1$, or $k > -1$, but there are some common results for $k < -1$ and $k > -1$[4].

Lemma 2.1. *There does not exist any paracontact $(k,\mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$ [4].*

In a paracontact metric $(k,\mu)-$manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, the following relation hold :

$$h^2=(k+1)\phi^2, \text{ for } k\neq -1, \tag{2.7}$$

$$(\widetilde{\nabla}_X\phi)Y=-g(X-hX,Y)\xi+\eta(Y)(X-hX), \tag{2.8}$$

$$S(X,Y) = [2(1-n)+n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)+[2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \tag{2.9}$$

$$S(X,\xi)=2nk\eta(X), \tag{2.10}$$

$$QY = [2(1-n)+n\mu]Y+[2(n-1)+\mu]hY+[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \tag{2.11}$$

$$Q\xi=2nk\xi, \tag{2.12}$$

$$Q\phi-\phi Q=2[2(n-1)+\mu]h\phi \tag{2.13}$$

for any vector fields X,Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1},g), respectively[4].

The concept of W_6-curvature tensor was defined by [7]. W_6-curvature tensor, W_7-curvature tensor, W_8-curvature tensor and W_9-curvature tensor, of a $(2n+1)$-dimensional Riemannian manifold are, respectively, defined as

$$W_6(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-g(X,Y)QZ], \tag{2.14}$$

$$W_7(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)QX-g(Y,Z)QX], \tag{2.15}$$

$$W_8(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Y)Z], \tag{2.16}$$

$$W_9(X,Y)Z=R(X,Y)Z+\frac{1}{2n}[S(X,Y)Z-g(Y,Z)QX], \tag{2.17}$$

for all $X,Y,Z\in\chi(M)$ where, $\chi(M)$ is set of all vector spaces [7].

3. Certain Curvature Conditions on (k,μ)-Paracontact metric spaces

We will provide the significant themes of this work in this part.

Let M be $(2n+1)-$dimensional $(k,\mu)-$paracontact metric manifold and we explain W_6 curvature tensor from (2.14), we have

$$W_6(X,Y)\xi=k(g(X,Y)\xi-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY). \tag{3.1}$$

Putting $X=\xi$, in (3.1), we get

$$W_6(\xi,Y)\xi=k(\eta(Y)\xi-Y)-\mu hY. \tag{3.2}$$

In (2.15) choosing $Z=\xi$ and using (2.5), we obtain

$$W_7(X,Y)\xi = k\eta(X)Y + \frac{1}{2n}\eta(Y)QX + \mu(\eta(Y)hX - \eta(X)hY). \tag{3.3}$$

It follows

$$W_7(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.4}$$

In the same way, putting $Z=\xi$ in (2.16) and using (2.5), we have

$$W_8(X,Y)\xi = \frac{1}{2n}S(X,Y)\xi - k\eta(X)Y + \mu(\eta(Y)hX - \eta(X)hY). \tag{3.5}$$

In (2.16), choosing $X=\xi$, we get

$$W_8(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.6}$$

In (2.17), choosing $Z=\xi$, we obtain

$$W_9(X,Y)\xi \quad = \quad k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \frac{1}{2n}(S(X,Y)\xi - \eta(Y)QX). \tag{3.7}$$

In(3.7) it follows

$$W_9(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.8}$$

In (2.5), we arrive

$$R(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY), \tag{3.9}$$

choosing $Z=\xi$, in (3.9)

$$R(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.10}$$

Theorem 3.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_6 semi-symmetric if and only if M is an Einstein manifold.*

Proof. Suppose that M is a W_6 semi-symmetric. This implies that

$$\begin{aligned}(R(X,Y)W_6)(U,W)Z \quad &= \quad R(X,Y)W_6(U,W)Z - W_6(R(X,Y)U,W)Z \\ &\quad -W_6(U,R(X,Y)W)Z - W_6(U,W)R(X,Y)Z = 0,\end{aligned} \tag{3.11}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X=Z=\xi$ in (3.11), making use of (3.1) and (3.9), for $A=\frac{1}{2n}$, we have

$$\begin{aligned}(R(\xi,Y)W_6)(U,W)\xi \quad &= \quad R(\xi,Y)(k(g(Y,W)\xi - \eta(U)W) + \mu(\eta(W)hU \\ &\quad -\eta(U)hW)) - W_6(k(g(Y,U)\xi - \eta(U)Y) \\ &\quad +\mu(g(hY,U)\xi - \eta(U)hY), W)\xi \\ &\quad -W_6(U, k(g(Y,W)\xi - \eta(W)Y) \\ &\quad +\mu(g(hY,W)\xi - \eta(W)hY)\xi \\ &\quad -W_6(U,W)(k(\eta(Y)\xi - Y) - \mu hY) = 0.\end{aligned} \tag{3.12}$$

Taking into account (3.1) and (3.2) in (3.12), we have

$$\begin{aligned}&kW_6(U,W)Y + \mu W_6(U,W)hY + k\mu(\eta(W)g(Y,hU)\xi \\ &-g(Y,W)hU) + \mu^2(1+k)(\eta(W)g(Y,U)\xi \\ &-\eta(U)g(Y,W)\xi) + k\mu(g(hY,U)W - g(hY,W)hU) \\ &+\mu k(g(hY,U)hW - g(hY,W)U) + \mu^2(g(hY,U)hW \\ &-g(hY,W)hU) + k^2(g(Y,W)\eta(U)\xi - g(Y,W)U) \\ &+k\mu(g(Y,U)hW + g(U,W)hY) + k^2(g(Y,U)W \\ &-g(U,W)Y) = 0.\end{aligned} \tag{3.13}$$

Putting (2.10), (2.14), choosing $U=\xi$ and taking inner product with $\xi \in \chi(M)$ in (3.13), we arrive

$$AkS(W,Y)+A\mu S(W,hY)+k^2g(W,Y)+k\mu g(W,hY)=0. \tag{3.14}$$

Using (2.7) and replacing hY of Y in (3.14), we get

$$AkS(W,hY)+A\mu(1+k)S(W,Y)-2nkA(1+k)g(W,hY)+k\mu(1+k)g(W,Y)=0. \tag{3.15}$$

From (3.14) and (3.15), we have

$$S(W,Y)=2nkg(W,Y).$$

So, M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an Einstein manifold, i.e. $S(W,Y)=2nkg(W,Y)$, then from equations (3.15), (3.14), (3.13), (3.12) and (3.11) we obtain M is a W_6 semi-symmetric. Which verifies our assertion. □

Theorem 3.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_7 semi-symmetric if and only if M is an $\eta-$Einstein manifold.*

Proof. Assume that M is a W_7 semi-symmetric. This yields to

$$\begin{aligned}(R(X,Y)W_7)(U,W)Z &= R(X,Y)W_7(U,W)Z-W_7(R(X,Y)U,W)Z\\ &\quad -W_7(U,R(X,Y)W)Z-W_7(U,W)R(X,Y)Z=0,\end{aligned} \tag{3.16}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X=Z=\xi$ in (3.16) and using (3.3), (3.9), (3.10), for $A=-\frac{1}{2n}$, we obtain

$$\begin{aligned}(R(\xi,Y)W_7)(U,W)\xi &= R(\xi,Y)(k\eta(U)W-A\eta(W)QU+\mu(\eta(W)hU\\ &\quad -\eta(U)hW))-W_7(k(g(Y,U)\xi-\eta(U)Y)\\ &\quad +\mu(g(hY,U)\xi-\eta(U)hY),W)\xi\\ &\quad -W_7(U,kg(Y,W)\xi-\eta(W)Y)\\ &\quad +\mu(g(hY,W)\xi-\eta(W)hY)\xi\\ &\quad -W_7(U,W)k(\eta(Y)\xi-Y)-\mu hY)=0.\end{aligned} \tag{3.17}$$

Taking into account that (3.4) and (3.9) in (3.17), we get

$$\begin{aligned}&kW_7(U,W)Y+\mu W_7(U,W)hY+k\mu(\eta(U)g(hY,W)\xi\\ &-g(Y,W)hU)+\mu^2(1+k)(\eta(W)g(Y,U)\xi\\ &-\eta(U)g(Y,W)\xi)-Ak(S(Y,U)\eta(W)\xi+\eta(W)\eta(U)QY)\\ &+A\mu(2nk\eta(W)\eta(U)hY-S(hY,U)\eta(W)\xi)\\ &+k^2(\eta(U)g(Y,W)\xi-\eta(W)g(Y,U)\xi)+k\mu(g(Y,U)hW\\ &-g(hY,W)U)+\mu^2(g(hY,U)hW-g(hY,W)hU)\\ &+\mu(kg(hY,U)W-A\eta(U)\eta(W)QhY)+k^2(g(Y,W)\eta(U)\xi\\ &+2nA\eta(U)\eta(W)Y)+k^2(g(Y,U)W-g(Y,W)U)=0.\end{aligned} \tag{3.18}$$

Putting $U=\xi$ and using (3.3) in (3.18), we get

$$AS(Y,W)+\mu S(W,hY)+2kg(Y,W)-2nkAg(Y,W)+\mu g(W,hY)=0. \tag{3.19}$$

Replacing hY of Y in (3.19) and making use of (2.7), we have

$$\begin{aligned}&AS(Y,hW)+\mu(1+k)S(Y,W)-2nk\mu(1+k)\eta(Y)\eta(W)\\ &-2nkAg(Y,hW)+\mu(1+k)g(Y,hW)-\mu(1+k)\eta(Y)\eta(W)=0.\end{aligned} \tag{3.20}$$

From (3.19), (3.20) and by using (2.9), for the sake of brevity, we set

$$\begin{aligned}p_1 &= (2nkA^2-2kA+\mu^2(1+k))[2(n-1)+\mu]+(A\mu+2nkA\mu-2k\mu)[2(1-n)+n\mu],\\ p_2 &= (A^2-\mu^2(1+k))[2(n-1)+\mu]+(2k\mu-2nkA\mu-A\mu),\\ p_3 &= (A\mu+2nkA\mu-2k\mu)[2(n-1)+n(2k-\mu)]-\\ &\quad (\mu^2(1+k)(2n+1))[2(n-1)+\mu]\end{aligned}$$

we conclude

$$p_2S(Y,W) = p_1g(Y,W) + p_3\eta(Y)\eta(W).$$

Thus, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an $\eta-$Einstein manifold, i.e. $p_2S(Y,W) = p_1g(Y,W) + p_3\eta(Y)\eta(W)$, then from equations (3.20), (3.19), (3.18), (3.17) and (3.16) we obtain M is a W_7 semi-symmetric. □

Theorem 3.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_8 semi-symmetric if and only if M is an $\eta-$Einstein manifold..*

Proof. Suppose that M is a W_8 semi-symmetric. This implies that

$$\begin{aligned}(R(X,Y)W_8)(U,W)Z &= R(X,Y)W_8(U,W)Z - W_8(R(X,Y)U,W)Z \\ &\quad -W_8(U,R(X,Y)W)Z - W_8(U,W)R(X,Y)Z = 0,\end{aligned} \tag{3.21}$$

for any $X,Y,U,W,Z \in \chi(M)$. Setting $X = Z = \xi$ in (3.21) and making use of (3.5), (3.9), (3.10), for $A = -\frac{1}{2n}$, we obtain

$$\begin{aligned}(R(\xi,Y)W_8)(U,W)\xi &= R(\xi,Y)(-k\eta(U)W - AS(U,W)\xi + \mu(\eta(W)hU \\ &\quad -\eta(U)hW)) - W_8(k(g(Y,U)\xi - \eta(U)Y) \\ &\quad +\mu(g(hY,U)\xi - \eta(U)hY),W)\xi \\ &\quad -W_8(U,k(g(Y,W)\xi - \eta(W)Y) \\ &\quad +\mu(g(hY,W)\xi - \eta(W)hY))\xi \\ &\quad -W_8(U,W)(k(\eta(Y)\xi - Y) - \mu hY) = 0.\end{aligned} \tag{3.22}$$

Inner product both sides of (3.22) by $Z \in \chi(M)$ and using of (3.5), (3.6) and (3.9), we get

$$\begin{aligned}&kg(W_8(U,W)Y,Z) + \mu g(W_8(U,W)hY,Z) + \mu^2(1+k)(\eta(W)\eta(Z)g(Y,U) \\ &-\eta(U)\eta(Z)g(Y,W)) + Ak(\eta(Y)\eta(Z)S(U,W) - \eta(Z)\eta(W)S(U,Y)) \\ &+A\mu(g(hY,Z)S(U,W) - \eta(W)\eta(Z)S(hY,U)) + Ak(S(U,W)g(Y,Z) \\ &-S(U,W)\eta(Y)\eta(Z)) + k^2(g(Y,U)g(W,Z) + g(Y,W)g(U,Z)) \\ &+\mu^2(g(hY,U)g(hW,Z) - g(hY,W)g(hU,Z)) + k\mu(g(hY,U)g(W,Z) \\ &-g(hY,W)g(U,Z)) - A(\mu S(hY,W)\eta(U)\eta(Z) + kS(Y,W)\eta(U)\eta(Z)) \\ &+k\mu(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z)) - k(\eta(W)\eta(Z)g(Y,U) \\ &+\eta(U)\eta(Z)g(Y,hW)) = 0.\end{aligned} \tag{3.23}$$

Making use of (2.7), (2.16) and choosing $W = Y = e_i$, ξ, $1 \le i \le n$, for orthonormal basis of $\chi(M)$ in (3.23), we have

$$\begin{aligned}&kS(U,Z) + \mu S(U,hZ) + (kAr + 2nA\mu(1+k)[2(n-1)+\mu] \\ &-2nk^2 + \mu^2(1+k))g(U,Z) + k\mu(1-2n)g(U,hZ) \\ &-(2nk^2A + \mu^2(1+k)(2n+1) + k^2 + Akr \\ &+2nA\mu(1+k)[2(n-1)+\mu] + 2nkA\mu)\eta(U)\eta(Z) = 0.\end{aligned} \tag{3.24}$$

In (3.24), hZ of Z, we arrive

$$\begin{aligned}&kS(U,hZ) + \mu(1+k)S(U,Z) - 2nk\mu(1+k)\eta(U)\eta(Z) \\ &+(kAr + 2nA\mu(1+k)[2(n-1)+\mu] - 2nk^2 \\ &+\mu^2(1+k))g(U,hZ) + k\mu(1-2n)(1+k)g(U,Z) \\ &-k\mu(1-2n)(1+k)\eta(U)\eta(Z) = 0.\end{aligned} \tag{3.25}$$

From (3.24), (3.25) and by using (2.9), for the sake of brevity, we set

$$\begin{aligned}p_1 &= (kAr + 2nA\mu(1+k)[2(n-1)+\mu] - 2nk^2 + \mu^2(1+k)), \\ p_2 &= k\mu(1-2n), \\ p_3 &= -(2nk^2A + \mu^2(1+k)(2n+1) + k^2 + Akr + 2nA\mu(1+k)[2(n-1)+\mu] + 2nkA\mu),\end{aligned}$$

we conclude

$$\begin{aligned}
q_1 &= (p_2\mu(1+k)-kp_1)[2(n-1)+\mu]+(kp_2-p_1\mu)[2(1-n)+n\mu],\\
q_2 &= (k^2-\mu^2(1+k))[2(n-1)+\mu]+(p_1\mu-kp_2),\\
q_3 &= (kp_2-p_1\mu)[2(n-1)+n(2k-\mu)]-(p_3k+2nk\mu^2(1+k)+p_2\mu(1+k))[2(n-1)+\mu],
\end{aligned}$$

$$q_2S(U,Z)=q_1g(U,Z)+q_3\eta(U)\eta(Z),$$

So, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an $\eta-$Einstein manifold, i.e. $q_2S(U,Z)=q_1g(U,Z)+q_3\eta(U)\eta(Z)$, then from equations (3.25), (3.24), (3.23), (3.22) and (3.21) we get M is a W_8 semi-symmetric. □

Theorem 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_9 semi-symmetric if and only if M is an Einstein manifold.*

Proof. Assume that M is a W_9 semi-symmetric. This means that

$$\begin{aligned}
(R(X,Y)W_9)(U,W,Z) &= R(X,Y)W_9(U,W)Z-W_9(R(X,Y)U,W)Z\\
&\quad -W_9(U,R(X,Y)W)Z-W_9(U,W)R(X,Y)Z=0,
\end{aligned} \tag{3.26}$$

for any $X,Y,U,W,Z\in\chi(M)$. Setting $X=Z=\xi$ in (3.26) and making use of (3.9), (3.7), for $A=\frac{1}{2n}$, we obtain

$$\begin{aligned}
(R(\xi,Y)W_9)(U,W)\xi &= R(\xi,Y)(k(\eta(W)U-\eta(U)W)+\mu(\eta(W)hU\\
&\quad -\eta(U)hW)+A(S(U,W)\xi-\eta(W)QU))\\
&\quad -W_9(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi\\
&\quad -\eta(U)hY,W)\xi-W_9(U,k(g(Y,W)\xi-\eta(W)Y)\\
&\quad +\mu(g(hY,W)\xi-\eta(W)hY))\xi\\
&\quad -W_9(U,W)(k(\eta(Y)\xi-Y)-\mu hY)=0.
\end{aligned} \tag{3.27}$$

Using (3.7), (3.8), (3.9) in (3.27), we get

$$\begin{aligned}
& kW_9(U,W)Y+\mu W_9(U,W)hY+k\mu(\eta(W)g(Y,hU)\xi\\
& -\eta(U)g(Y,hW)\xi)+\mu^2(1+k)(\eta(W)g(Y,U)\xi\\
& -\eta(U)g(Y,W)\xi)+k^2(g(Y,U)W-g(Y,W)U)\\
& +kA(\eta(U)S(Y,W)\xi-\eta(W)\eta(U)QY)\\
& +A\mu(\eta(U)S(hY,W)\xi+2nk\eta(U)\eta(W)hY)\\
& +k\mu(g(Y,U)hW-g(Y,W)hU)+k\mu(g(hY,U)W\\
& -g(hY,W)U)+A\mu(S(U,hY)\eta(W)\xi-\eta(W)\eta(U)QhY)\\
& +\mu^2(g(hY,U)hW+g(hY,W)hU)-A\mu(S(U,W)hY\\
& +S(hY,U)\eta(W)\xi)+Ak(2nk\eta(W)\eta(U)Y-S(U,W)Y)=0.
\end{aligned} \tag{3.28}$$

Making use of (2.17), (2.1) and choosing $U=\xi$, in (3.28), we have

$$kS(Y,W)+\mu S(hY,W)-2nk^2g(Y,W)-2nk\mu g(hY,W)=0. \tag{3.29}$$

Replacing hY of Y in (3.29) and taking into account (2.7), we arrive

$$\begin{aligned}
& kS(Y,hW)+\mu(1+k)S(Y,W)-2nk\mu(1+k)\eta(Y)\eta(W)\\
& -2nk^2g(W,hY)-2nk\mu(1+k)g(Y,W)\\
& +2nk\mu(1+k)\eta(W)\eta(Y)=0.
\end{aligned} \tag{3.30}$$

From (3.29), (3.30) and by using (2.7), we have

$$S(Y,W)=2nkg(Y,W).$$

This tell us, M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an Einstein manifold, i.e. $S(Y,W)=2nkg(Y,W)$, then from equations (3.26), (3.27), (3.28) and (3.30), we obtain M is a W_9 semi-symmetric. Which verifies our assertion. □

Example 3.5. *We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3,\ z\neq 0\}$, where (x,y,z) are standart coordinates of $\mathbb{R}^3$. The vector fields*

$$e_1=\frac{\partial}{\partial x},\quad e_2=4z^2\frac{\partial}{\partial x}+\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_2) &= g(e_1,e_3)=g(e_2,e_3)=0,\\ g(e_1,e_1) &= g(e_2,e_2)=1,\quad g(e_3,e_3)=-1 \end{aligned}$$

Let η be the 1-form defined by $\eta(X)=g(X,e_1)$ for any $X\in\chi(M)$. Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1)=0,\qquad \phi(e_3)=-e_2,\qquad \phi(e_2)=-e_3.$$

Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we get

$$[e_3,e_1]=0,\quad [e_1,e_2]=0,\quad [e_2,e_3]=-8ze_1.$$

Then we have

$$\eta(e_1)=g(e_1,e_1)=1,\quad \phi^2X=X-\eta(X)e_1,\quad g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y),$$

for any $X,Y\in\chi(M)$. Hence, (ϕ,ξ,η,g) defines a paracontact metric structure on M for $e_1=\xi$.

The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_XY,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\ &\quad -g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]). \end{aligned}$$

Using the above formula, we obtain.

$$\begin{aligned} \nabla_{e_1}e_1 &= 0, & \nabla_{e_2}e_1 &= -4ze_3, & \nabla_{e_3}e_1 &= -4ze_2,\\ \nabla_{e_1}e_2 &= -4ze_3, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_3}e_2 &= 4ze_1,\\ \nabla_{e_1}e_3 &= -4ze_2, & \nabla_{e_2}e_3 &= -4ze_1, & \nabla_{e_3}e_3 &= 0. \end{aligned}$$

Comparing the above relations with $\nabla_Xe_1=-\phi X+\phi hX$, we get

$$he_2=-(4z+1)e_2,\quad he_3=-(4z+1)e_3,\quad he_1=0.$$

Using the formula $R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$, we calculate the following:

$$\begin{aligned} R(e_2,e_1)e_1 &= \left[\frac{1}{(4z-1)^2}-1\right]\{\eta(e_1)e_2-\eta(e_2)e_1\}+\left[\frac{1}{(4z-1)^3}-\frac{16z^2+1}{4z+1}\right]\{\eta(e_1)he_2-\eta(e_2)he_1\}\\ &= -16z^2e_2 \end{aligned}$$

$$\begin{aligned} R(e_3,e_1)e_1 &= \left[\frac{1}{(4z-1)^2}-1\right]\{\eta(e_1)e_3-\eta(e_3)e_1\}+\left[\frac{1}{(4z-1)^3}-\frac{16z^2+1}{4z+1}\right]\{\eta(e_1)he_3-\eta(e_3)he_1\}\\ &= -16z^2e_3 \end{aligned}$$

$$\begin{aligned} R(e_2,e_3)e_1 &= \left[\frac{1}{(4z-1)^2}-1\right]\{\eta(e_3)e_2-\eta(e_2)e_3\}+\left[\frac{1}{(4z-1)^3}-\frac{16z^2+1}{4z+1}\right]\{\eta(e_3)he_2-\eta(e_2)he_3\}\\ &= 0. \end{aligned}$$

By the above expressions of the curvature tensor and using (2.9), we conclude that M is a generalized $(k,\mu)-$paracontact metric manifold with $k=\left[\frac{1}{(4z-1)^2}-1\right]$ and $\mu=\left[\frac{1}{(4z-1)^3}-\frac{16z^2+1}{4z+1}\right]$.

4. Conclusion

The aim of this paper is to classify (k,μ)-paracontact metric spaces satisfying certain curvature conditions. We present the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $R\cdot W_6=0$, $R\cdot W_7=0$, $R\cdot W_8=0$ and $R\cdot W_9=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized. Also, several results are obtained.

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Homotopies of 2-Algebra Morphisms

İbrahim İlker Akça[1]*, Ummahan Ege Arslan[2]

Abstract
In [1] it is defined the notion of 2-algebra as a categorification of algebras, and shown that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras. In this paper we define the notion of homotopy for 2-algebras and we explore the relations of crossed module homotopy and 2-algebra homotopy.

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[1] *Department of Mathematics and Computer Science, Faculty of Science, Eskişehir Osmangazi University, Eskişehir, Turkey, ORCID: 0000-0003-4269-498X*
[2] *Department of Mathematics and Computer Science, Faculty of Science, Eskişehir Osmangazi University, Eskişehir, Turkey, ORCID: 0000-0002-2995-0718*
***Corresponding author**: i.ilkerakca@gmail.com
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1. Introduction

A crossed module [20] $\mathscr{A} = (\partial : C \longrightarrow R)$ of commutative algebras is given by an algebra morphism $\partial : C \longrightarrow R$ together with an action $\cdot$ of R on C such that the relations below hold for each $r \in R$ and each $c, c^{'} \in C$,

$$\begin{aligned} \partial(r \cdot c) &= r\partial(c) \\ \partial(c) \cdot c^{'} &= cc^{'}. \end{aligned}$$

Group crossed modules were firstly introduced by Whitehead in [21],[22]. They are algebraic models for homotopy 2-types, in the sense that [5],[15] the homotopy category of the model category [6],[9] of group crossed modules is equivalent to the homotopy category of the model category [11] of pointed 2-types: pointed connected spaces whose homotopy groups π_i vanish, if $i \geq 3$. The homotopy relation between crossed module maps $\mathscr{A} \longrightarrow \mathscr{A}^{'}$ was given by Whitehead in [22], in the contex of "homotopy systems" called free crossed complexes.

In [2] it is addressed the homotopy theory of maps between crossed modules of commutative algebras. It is proven that if $\mathscr{A}$ and $\mathscr{A}^{'}$ are crossed modules of algebras without any restriction on $\mathscr{A}$ and $\mathscr{A}^{'}$ then the crossed module maps $\mathscr{A} \longrightarrow \mathscr{A}^{'}$ and their homotopies give a groupoid.

In [1] it is defined the notion of 2-algebra as a categorification of algebras, and shown that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras. In this paper we define the notion of homotopy for 2-algebras. This definition is essentially a special case of 2-natural transformation due to Gray in [12]. And we explore the relations between the crossed module homotopies and 2-algebra homotopies. Similar results are given [13] by İçen for 2-groupoids.

2. Preliminaries

In [1] it is defined the notion of 2-algebra as a categorification of algebras, and shown that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras.

2.1 2-algebras

Definition 2.1. *A weak 2-algebra consists of*

· a 2-module A equipped with a functor $\bullet : A \times A \longrightarrow A$, which is defined by $(x,y) \mapsto x \bullet y$ and bilinear on objects and defined by $(f,g) \mapsto f \bullet g$ on morphisms satisfying interchange law, i.e.,

$$(f_1 \bullet g_1) \circ (f_2 \bullet g_2) = (f_1 \circ f_2) \bullet (g_1 \circ g_2)$$

· $k-$bilinear natural isomorphisms

$$\alpha_{x,y,z} : (x \bullet y) \bullet z \longrightarrow x \bullet (y \bullet z)$$

$$l_x : 1 \bullet x \longrightarrow x$$

$$r_x : x \bullet 1 \longrightarrow x$$

such that the following diagrams commute for all objects $w,x,y,z \in A_0$.

$$\begin{tikzcd}
((w\bullet x)\bullet y)\bullet z \arrow[r, "\alpha_{w\bullet x,y,z}"] \arrow[d, "\alpha_{w,x,y}\bullet 1_z"'] & (w\bullet x)\bullet(y\bullet z) \arrow[dr, "\alpha_{w,x,y\bullet z}"] & \\
(w\bullet(x\bullet y))\bullet z \arrow[r, "\alpha_{w,x\bullet y,z}"'] & w\bullet((x\bullet y)\bullet z) \arrow[r, "1_w\bullet\alpha_{x,y,z}"'] & w\bullet(x\bullet(y\bullet z))
\end{tikzcd}$$

$$\begin{tikzcd}
(x\bullet 1)\bullet y \arrow[r, "\alpha_{x,1,y}"] \arrow[dr, "r_x\bullet 1_y"'] & x\bullet(1\bullet y) \arrow[d, "1_x\bullet l_y"] \\
 & x\bullet y
\end{tikzcd}$$

A strict 2-algebra is the special case where $\alpha_{x,y,z}$, l_x, r_x are all identity morphisms. In this case we have

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

$$1 \bullet x = x, x \bullet 1 = x$$

Strict 2-algebra is called commutative strict 2-algebra if $x \bullet y = y \bullet x$ for all objects $x,y \in A_0$ and $f \bullet g = g \bullet f$ for all morphisms $f,g \in A_1$.

In the rest of this paper, the term 2-algebra will always refer to a commutative strict 2-algebra. A homomorphism between 2-algebras should preserve both the 2-module structure and the $\bullet$ functor.

Definition 2.2. *Given 2-algebras A and A', a homomorphism*

$$F : A \longrightarrow A'$$

consists of

· a linear functor F from the underlying 2-module of A to that of A', and

· a bilinear natural transformation

$$F_2(x,y) : F_0(x) \bullet F_0(y) \longrightarrow F_0(x \bullet y)$$

· an isomorphism $F : 1' \longrightarrow F_0(1)$ where 1 is the identity object of A and $1'$ is the identity object of A', such that the following diagrams commute for $x,y,z \in A_0$,

$$\begin{tikzcd}
(F(x)\bullet F(y))\bullet F(z) \arrow[r, "F_2\bullet 1"] \arrow[d, "\alpha_{F(x),F(y),F(z)}"'] & F(x\bullet y)\bullet F(z) \arrow[r, "F_2"] & F((x\bullet y)\bullet z) \arrow[d, "F(\alpha_{x,y,z})"] \\
F(x)\bullet(F(y)\bullet F(z)) \arrow[r, "1\bullet F_2"'] & F(x)\bullet F(y\bullet z) \arrow[r, "F_2"'] & F(x\bullet(y\bullet z)).
\end{tikzcd}$$

$$\begin{tikzcd}
1'\bullet F(x) \arrow[r, "{l'}_{F(x)}"] \arrow[d, "F_0\bullet 1"'] & F(x) \\
F(1)\bullet F(x) \arrow[r, "F_2"'] & F(1\bullet x). \arrow[u, "F(l_x)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(x)\bullet 1' \arrow[r, "{r'}_{F(x)}"] \arrow[d, "1\bullet F_0"'] & F(x) \\
F(x)\bullet F(1) \arrow[r, "F_2"'] & F(x\bullet 1). \arrow[u, "F(r_x)"']
\end{tikzcd}$$

Definition 2.3. *2-algebras and homomorphisms between them give the category of 2-algebras denoted by* ***2Alg*** *.*

Therefore if $A=(A_0,A_1,s,t,e,\circ,\bullet)$ is a 2-algebra, A_0 and A_1 are algebras with this $\bullet$ bilinear functor. Thus we can take that 2-algebra is a 2-category with a single object say $*$, and A_0 collections of its 1-morphisms and A_1 collections of its 2-morphisms are algebras with identity.

2.2 Crossed modules

Crossed modules have been used widely and in various contexts since their definition by Whitehead [23] in his investigations of the algebraic structure of relative homotopy groups. We recalled the definition of crossed modules of commutative algebras given by Porter [20].

Let R be a k-algebra with identity. A pre-crossed module of commutative algebras is an R-algebra C together with a commutative action of R on C and a morphism

$$\partial : C \longrightarrow R$$

such that for all $c\in C$, $r\in R$

CM1) $\partial(r \blacktriangleright c) = r\partial c.$

This is a crossed R-module if in addition for all $c,c'\in C$

CM2) $\partial c \blacktriangleright c' = cc'.$

The last condition is called the Peiffer identity. We denote such a crossed module by (C,R,∂).

A morphism of crossed modules from (C,R,∂) to (C',R',∂') is a pair of k-algebra morphisms $\phi : C \longrightarrow C'$, $\psi : R \longrightarrow R'$ such that

$$\partial'\phi = \psi\partial \qquad \text{and} \qquad \phi(r \blacktriangleright c) = \psi(r) \blacktriangleright \phi(c).$$

Thus we get a category $\mathbf{XMod}_k$ of crossed modules (for fixed k).

Examples of Crossed Modules

1. Any ideal I in R gives an inclusion map, $inc : I \longrightarrow R$ which is a crossed module. Conversely given an arbitrary R-module $\partial : C \longrightarrow R$ one easily sees that the Peiffer identity implies that ∂C is an ideal in R.

2. Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ sending everything in M to the zero element of R is a crossed module. Conversely: If (C,R,∂) is a crossed module, $\partial(C)$ acts trivially on $\ker\partial$, hence $\ker\partial$ has a natural $R/\partial(C)$-module structure.

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

3. Let be $\mathscr{M}(C)$ multiplication algebra. Then $(C,\mathscr{M}(C),\mu)$ is multiplication crossed module. $\mu : C \to \mathscr{M}(C)$ is defined by $\mu(r) = \delta_r$ with $\delta_r(r') = rr'$ for all $r,r' \in C$, where δ is multiplier $\delta : C \to C$ such that for all $r,r' \in C$, $\delta(rr') = \delta(r)r'$. Also $\mathscr{M}(C)$ acts on C by $\delta \blacktriangleright r = \delta(r)$.(See [3] for details).

In [20] Porter states that there is an equivalence of categories between the category of internal categories in the category of k-algebras and the category of crossed modules of commutative k-algebras. In the following theorem, it is given a categorical presentation of this equivalence.

Theorem 2.4. *[1] The category of crossed modules* $\mathbf{XMod}_k$ *is equivalent to that of* 2*-algebras,* ***2Alg****.*

Proof. Let $A = (A_0,A_1,s,t,e,\circ,\bullet)$ be a 2-algebra consisting of a single object say $*$ and an algebra A_0 of 1-morphisms and an algebra A_1 of 2-morphisms and $\partial = t|_{Kers}$ algebra homomorphism by $\partial : Kers \longrightarrow A_0, \partial(h) = t(h)$. Then $(Kers,A_0,\partial)$ is a crossed module.

Let $\mathscr{A} = (A_0,A_1,s,t,e,\circ,\bullet)$ and $\mathscr{A}' = (A_0',A_1',s',t',e',\circ',\bullet')$ be 2-algebras and $F = (F_0,F_1) : \mathscr{A} \longrightarrow \mathscr{A}'$ be a 2-algebra morphism. Then $F_0 : A_0 \longrightarrow A_0'$ and $F_1 : A_1 \longrightarrow A_1'$ are the k-algebra morphisms. For $f_1 = F_1|_{Kers} : Kers \longrightarrow Kers'$ and $f_0 = F_0 : A_0 \longrightarrow A_0'$, (f_1,f_0) map is a crossed module morphism $(Kers,A_0,\partial) \longrightarrow (Kers',A_0',\partial')$. So it is got a functor

$$\Gamma : \mathbf{2Alg} \longrightarrow \mathbf{XMod}_k.$$

Conversely, let (G,C,∂) be a crossed module of algebras. For $s,t : G \rtimes C \to C$ and $e : C \to G \rtimes C$ by $s(g,c) = c, t(g,c) = \partial(g)+c, e(c) = (0,c)$ and

the compositions

$$(g,c)\bullet(h,d) = (c \blacktriangleright h + d \blacktriangleright g + gh, cd)$$

$$(g,c)\circ(g',\partial(g)+c) = (g+g',c)$$

such that $t(g,c) = s(g',\partial(g)+c) = \partial(g)+c$, it is constructed a 2-algebra $\mathscr{A} = (C,G \rtimes C,s,t,e,\circ,\bullet)$ consists of the single object say $*$ and the k-algebra C of 1-morphisms and the k-algebra $G \rtimes C$ of 2-morphisms. Let (G,C,∂) and (G',C',∂') be crossed modules and $f = (f_1,f_0) : (G,C,\partial) \longrightarrow (G',C',\partial')$ be a crossed module morphism. For

$$\begin{array}{llll} F_1 : & G \rtimes C & \longrightarrow & G' \rtimes C' \\ & (g,c) & \longmapsto & F_1(g,c) = (f_1(g), f_0(c)) \end{array}$$

and

$$\begin{array}{llll} F_0 : & C & \longrightarrow & C' \\ & c & \longmapsto & F_0(c) = f_0(c). \end{array}$$

$F = (F_1,F_0)$ is a 2-algebra morphism from $(C,G \rtimes C,s,t,e,\circ,\bullet)$ to $(C',G' \rtimes C',s',t',e',\circ',\bullet')$. Thus it is got a functor

$$\Psi : \mathbf{XMod}_k \longrightarrow \mathbf{2Alg}.$$

□

3. Homotopies of Crossed Modules and 2-Algebras

The notion of homotopy for morphisms of crossed modules over commutative algebras is given in [2]. In this section, we explain the relation between homotopies for crossed modules over commutative algebras and homotopies for 2-algebras. The formulae given below are playing important role in our study.

Definition 3.1. *[2] Let $\mathscr{A} = (E,R,\partial)$ and $\mathscr{A}' = (E',R',\partial')$ be crossed modules and $f_0 : R \longrightarrow R'$ be an algebra morphism. An f_0-derivation $s : R \longrightarrow E'$ is a k-linear map satisfying for all $r,r' \in R$,*

$$s(rr') = f_0(r) \blacktriangleright s(r') + f_0(r') \blacktriangleright s(r) + s(r)s(r').$$

Let $f=(f_1,f_0)$ be a crossed module morphism $\mathscr{A}\longrightarrow\mathscr{A}'$ and s be an f_0-derivation. If $g=(g_1,g_2)$ is defined as (where $e\in E$ and $r\in R$)

$$\begin{aligned} g_0(r) &= f_0(r)+(\partial' s)(r)\\ g_1(e) &= f_1(e)+(s\partial)(e),\end{aligned}$$

then g is also crossed module morphism $\mathscr{A}\longrightarrow\mathscr{A}'$. In such a case we write $f\xrightarrow{(f_0,s)}g$, and say that (f_0,s) is a homotopy connecting f to g.

If (f_0,s) and (g_0,s') are homotopies connecting f to g and g to u respectively, then $(f_0,s+s')$ is a homotopy connecting f to u, where $s+s':R\longrightarrow E'$ is an f_0-derivation defined by $(s+s')(r)=s(r)+s'(r)$.

The notion of homotopy for 2-algebras is essentially a special case of 2-natural transformation due to Gray in [12].

Definition 3.2. *Let $\mathbf{A}=(A_0,A_1,s,t,e,\circ,\bullet)$ and $\mathbf{A}'=(A_0',A_1',s',t',e',\circ',\bullet')$ be 2-algebras and let $F=(F_1,F_0)$ and $G=(G_1,G_0)$ be 2-algebra morphisms $\mathbf{A}\longrightarrow\mathbf{A}'$. A k-algebra morphism $\mu:A_0\longrightarrow A_1'$ satisfying the following conditions is called a homotopy connecting F to G:*
1) $s'\mu=F_0$
2) $t'\mu=G_0$
3) $F_1\circ'\mu t=\mu s\circ' G_1$. In such a case we write $F\xrightarrow{\mu}G$.

Theorem 3.3. *Let $\mathscr{A}=(A_0,A_1,s,t,e,\circ,\bullet)$, $\mathscr{A}'=(A_0',A_1',s',t',e',\circ',\bullet')$ be 2-algebras, $F=(F_1,F_0)$, $G=(G_1,G_0)$ and $U=(U_1,U_0)$ be 2-algebra morphisms $\mathscr{A}\longrightarrow\mathscr{A}'$ and μ be a homotopy connecting F to G, μ' be a homotopy connecting G to U. Then the map $\mu*\mu':A_0\longrightarrow A_1$ defined by $(\mu*\mu')(x)=\mu(x)+\mu'(x)-e'(t'\mu)(x)$ is a homotopy connecting F to U.*

Proof. We first show that $\mu*\mu'$ is an algebra morphism. Since μ and μ' are algebra morphisms, $\mu(x\bullet x')=\mu(x)\bullet'\mu(x')$ and $\mu'(x\bullet x')=\mu'(x)\bullet'\mu'(x')$ for all $x,x'\in A_0$. Then we get

$$\begin{aligned} (\mu*\mu')(x\bullet x') &= \mu(x\bullet x')+\mu'(x\bullet x')-e'(t'\mu)(x\bullet x')\\ &= \mu(x\bullet x')+\mu'(x\bullet x')-e'(G_0)(x\bullet x')\\ &= \mu(x\bullet x')\circ'\mu'(x\bullet x')\\ &= (\mu(x)\bullet'\mu(x'))\circ'(\mu'(x)\bullet'\mu'(x'))\\ &= (\mu(x)\circ'\mu'(x))\bullet'(\mu(x')\circ'\mu'(x')) \qquad \text{(interchange law)}\\ &= (\mu(x)+\mu'(x)-e'(G_0)(x))\bullet'(\mu(x')+\mu'(x')-e'(G_0)(x'))\\ &= (\mu*\mu')(x)\bullet'(\mu*\mu')(x').\end{aligned}$$

For all $x\in A_0$

$$\begin{aligned} s'(\mu*\mu')(x) &= s'(\mu(x)+\mu'(x)-e'G_0(x))\\ &= s'\mu(x)+s'\mu'(x)-s'e'G_0(x)\\ &= F_0(x)+G_0(x)-G_0(x)\\ &= F_0(x),\end{aligned}$$

$$\begin{aligned} t'(\mu*\mu')(x) &= t'(\mu(x)+\mu'(x)-e'G_0(x))\\ &= t'\mu(x)+t'\mu'(x)-t'e'G_0(x)\\ &= G_0(x)+U_0(x)-G_0(x)\\ &= U_0(x),\end{aligned}$$

and since $F_1\circ'\mu t=\mu s\circ' G_1$ and $G_1\circ'\mu' t=\mu' s\circ' U_1$, we get

$$\begin{aligned} F_1\circ'\mu t\circ'\mu' t &= \mu s\circ' G_1\circ'\mu' t\\ &= \mu s\circ'\mu' s\circ' U_1.\end{aligned}$$

Thus, we get

$$\begin{aligned} F_1\circ'(\mu*\mu')t &= F_1\circ'(\mu t\circ'\mu' t)\\ &= (\mu s\circ'\mu' s)\circ' U_1\\ &= (\mu*\mu')s\circ' U_1.\end{aligned}$$

Therefore $\mu*\mu':A_0\longrightarrow A_1$ is a homotopy connecting F to U. □

Theorem 3.4. *Let $\Gamma :: \mathbf{2Alg} \longrightarrow: \mathbf{XMod_k}$ be the functor as mentioned in Teorem 1.4 and μ be homotopy connecting F to G. Then*

$$\begin{array}{rclcl} \Gamma(\mu)=h & : & A_0 & \longrightarrow & Kers^{'} \\ & & x & \longmapsto & h(x)=\mu(x)-e^{'}(s^{'}\mu)(x) \end{array}$$

is a homotopy of corresponding crossed module morphisms.

Proof. We first show that h is an f_0-derivation where $f_0 : A_0 \longrightarrow A_0^{'}$ defined by $f_0(x) = F_0(x)$. For $x,x^{'} \in A_0$,

$$\begin{array}{rcl} f_0(x) \blacktriangleright h(x^{'}) & & \\ +f_0(x^{'}) \blacktriangleright h(x)+h(x)\bullet^{'} h(x^{'}) & = & F_0(x) \blacktriangleright (\mu(x^{'})-e^{'}(s^{'}\mu)(x^{'})) \\ & & +F_0(x^{'}) \blacktriangleright (\mu(x)-e^{'}(s^{'}\mu)(x)) \\ & & +(\mu(x)-e^{'}(s^{'}\mu)(x))\bullet^{'} (\mu(x^{'})-e^{'}(s^{'}\mu)(x^{'})) \\ & = & e^{'}(F_0(x))\bullet^{'} (\mu(x^{'})-e^{'}F_0(x^{'})) \\ & & +e^{'}(F_0(x^{'}))\bullet^{'} (\mu(x)-e^{'}F_0(x))+\mu(x)\bullet^{'} \mu(x^{'}) \\ & & -\mu(x)\bullet^{'} e^{'}F_0(x^{'})-e^{'}F_0(x)\bullet^{'} \mu(x^{'})+e^{'}F_0(x)\bullet^{'} e^{'}F_0(x^{'}) \\ & = & \mu(x\bullet x^{'})-e^{'}(s^{'}\mu)(x\bullet x^{'}) \\ & = & h(x\bullet x^{'}). \end{array}$$

Therefore h is an f_0-derivation.

Now we show that

$$\begin{array}{rcl} g_0(x) & = & f_0(x)+\partial^{'}h(x) \\ g_1(n) & = & f_1(n)+h\partial(n) \end{array}$$

for $x \in A_0$ and $n \in Kers$.

$$\begin{array}{rcl} \partial^{'}h(x) & = & \partial^{'}(\mu(x)-e^{'}f_0(x)) \\ & = & \partial^{'}(\mu(x))-\partial^{'}(e^{'}f_0(x)) \\ & = & (t^{'}\mu)(x)-(t^{'}e^{'})f_0(x) \\ & = & g_0(x)-f_0(x) \end{array}$$

and we get $g_0(x) = f_0(x)+\partial^{'}h(x)$.

Since $A_1 \simeq Kers \rtimes A_0$, we take $a=(n,x)$ for $a \in A_1$ where $n=a-es(a) \in Kers$ and $x=s(a) \in A_0$. We define $\mu^* : A_0 \longrightarrow Kers^{'} \rtimes A_0^{'}$, as $\mu^*(x) = (\mu(x)-e^{'}s^{'}(\mu(x)),s^{'}\mu(x))$ and $h^* : A_0 \longrightarrow Kers^{'} \rtimes A_0^{'}$, as $h^*(x) = (h(x),F_0(x))$. Therefore

$$\begin{tikzcd}[row sep=huge, column sep=huge]
A_1 \cong Ker(s) \rtimes A_0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "(G_1,G_0)"'] \arrow[d, shift left, "(F_1,F_0)"] & A_0 \arrow[l, bend right, "e"'] \arrow[d, shift right, "G_0"'] \arrow[d, shift left, "F_0"] \arrow[dl, "\mu*"] \\
A_1' \cong Ker(s') \rtimes A_0' \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] & A_0' \arrow[l, bend left, "e'"]
\end{tikzcd}$$

for $(F_1,F_0)(n,x),(\mu^*t)(n,x) \in A_1 \simeq Kers^{'} \rtimes A_0^{'}$ such that $t(F_1,F_0)(n,x) = s(\mu^*t)(n,x)$, we have $(F_1,F_0)(n,x)\circ^{'} \mu^*t(n,x) = (F_1(n)+\mu t(n),F_0(x))$ and $-(F_1,F_0)(n,x) = (-F_1(n),t^{'}F_1(n)+F_0(x))$ and then, since

$$(F_1,F_0)(n,x)\circ^{'} \mu^*t(n,x) = \mu^*s(n,x)\circ^{'} (G_1,G_0)(n,x)$$

we have

$$\begin{array}{rcl} \mu^*t(n,x) & = & -(F_1,F_0)(n,x)\circ^{'} \mu^*s(n,x)\circ^{'} (G_1,G_0)(n,x) \\ & = & (-F_1(n)+h(x)+G_1(n),t^{'}F_1(n)+F_0(x)) \end{array}$$

and

$$-e^{'}F_0t(n,x) = (0,t^{'}f_1(n)+f_0(x)).$$

Hence we get

$$\begin{aligned}\mu^*t(n,x)-e^{'}F_0t(n,x) &= (I_{t^{'}F_1(n)+F_0(x)}\circ \mu t)(n,x)\\ &= \mu^*t(n,x).\end{aligned}$$

Then

$$\begin{aligned}h^*(t(n,x)) &= \mu^*(t(n,x))-e^{'}(s^{'}\mu^*)(t(n,x))\\ &= \mu^*t(n,x)-e^{'}F_0t^*(n,x)\\ &= \mu^*t(n,x)\\ &= (-F_1(n)+h(x)+G_1(n),t^{'}F_1(n)+F_0(x)) \qquad (1)\end{aligned}$$

and

$$\begin{aligned}h^*(t(n,x)) &= h^*(\partial(n)+x))\\ &= (h(\partial(n)+x)),f_0(\partial(n)+x))\\ &= (h(\partial(n))+h(x),f_0(\partial(n))+f_0(x))\\ &= (h(\partial(n))+h(x),t^{'}F_1(n)+F_0(x)). \qquad (2)\end{aligned}$$

Therefore from (1) and (2) we have

$$h(\partial(n))+h(x) = -F_1(n)+h(x)+G_1(n)$$

and

$$h(\partial(n)) = -F_1(n)+G_1(n).$$

Then

$$g_1(n) = f_1(n)+h\partial(n).$$

Hence

$$\begin{array}{rccc} h: & A_0 & \longrightarrow & Kers^{'}\\ & x & \longmapsto & h(x)=\mu(x)-e^{'}F_0(x)\end{array}$$

is a homotopy connecting $f=(f_1,f_0):(Kers \xrightarrow{\partial} A_0)\longrightarrow (Kers^{'} \xrightarrow{\partial^{'}} A_0^{'})$ to $g=(g_1,g_0):(Kers \xrightarrow{\partial} A_0)\longrightarrow (Kers^{'} \xrightarrow{\partial^{'}} A_0^{'})$.

Let $F \xrightarrow{\mu} G$ and $G \xrightarrow{\mu^{'}} H$. Then we have

$$\begin{aligned}\Gamma(\mu*\mu^{'})(x) &= (\mu*\mu^{'})(x)-e^{'}(s^{'}\mu*\mu^{'})(x)\\ &= \mu(x)+\mu^{'}(x)-e^{'}(t^{'}\mu)(x)-e^{'}(s^{'}\mu)(x)\\ &= \mu(x)+\mu^{'}(x)-e^{'}(s^{'}\mu^{'})(x)-e^{'}(s^{'}\mu)(x)\\ &= (\mu(x)-e^{'}(s^{'}\mu)(x))+(\mu^{'}(x)-e^{'}(s^{'}\mu^{'})(x))\\ &= \Gamma(\mu)(x)+\Gamma(\mu^{'})(x)\end{aligned}$$

for all $x\in A_0$. □

Theorem 3.5. *Let $\Psi:\mathbf{XMod_k}\longrightarrow \mathbf{2Alg}$ be the functor as mentioned in Theorem 1.4 and h be homotopy connecting $f:(G,C,\partial)\longrightarrow (G^{'},C^{'},\partial^{'})$ to $g:(G,C,\partial)\longrightarrow (G^{'},C^{'},\partial^{'})$. Then*

$$\begin{array}{rcccc} \Psi(h)=\mu & : & C & \longrightarrow & G^{'}\rtimes C^{'}\\ & & x & \longmapsto & \mu(x)=(h(x),f_0(x))\end{array}$$

is a homotopy of corresponding 2-algebra morphisms.

Proof. We first show that μ is an algebra morphism. For $x,x^{'}\in C$

$$\begin{array}{rcl}\mu(xx^{'}) & = & (h(xx^{'}),f_0(xx^{'}))\\ & = & (f_0(x)\blacktriangleright h(x^{'})+f_0(x^{'})\blacktriangleright h(x)+h(x)h(x^{'}),\ f_0(x)f_0(x^{'}))\\ & = & (h(x),f_0(x))(h(x^{'}),f_0(x^{'}))\\ & = & \mu(x)\mu(x^{'}).\end{array}$$

Now we show that
1) $s^{'}\mu=F_0$ 2) $t^{'}\mu=G_0$ 3) $(f_1,f_0)\circ'\mu t=\mu s\circ'(g_1,g_0)$
1)For all $x\in C$,

$$\begin{array}{rcl}s^{'}\mu(x) & = & s^{'}(h(x),f_0(x))\\ & = & f_0(x)=F_0(x),\end{array}$$

2)For all $x\in C$,

$$\begin{array}{rcl}t^{'}\mu(x) & = & t^{'}(h(x),f_0(x))\\ & = & t^{'}(h(x))+f_0(x)\\ & = & \partial^{'}h(x)+f_0(x)\\ & = & g_0(x)=G_0(x),\end{array}$$

3)For all $x\in C,a\in G$, since $t^{'}(f_1(a),f_0(x))=\partial^{'}f_1(a)+f_0(x)$,

$$\begin{array}{rcl}s^{'}(\mu t(a,x)) & = & s^{'}(\mu(\partial(a)+x))\\ & = & s^{'}(h(\partial(a)+x),f_0(\partial(a)+x))\\ & = & f_0(\partial(a)+x)\\ & = & f_0(\partial(a))+f_0(x)\\ & = & \partial^{'}f_1(a)+f_0(x)\end{array}$$

then $t^{'}(f_1(a),f_0(x))=s^{'}(\mu t(a,x))$ and (f_1,f_0) , μt are composable pairs. Also since

$$\begin{array}{rcl}t^{'}(\mu s(a,x)) & = & t^{'}(\mu(x))=t^{'}(h(x),f_0(x))\\ & = & \partial^{'}(h(x))+f_0(x)\\ & = & g_0(x)\end{array}$$

and $s^{'}(g_1(a),g_0(x))=g_0(x)$ then $t^{'}(\mu s)=s^{'}(g_1,g_0)$ and $\mu s,(g_1,g_0)$ are composable pairs.
Therefore we get

$$(f_1(a),f_0(x))\circ^{'}\mu t(a,x) \quad = \quad (f_1(a)+h(\partial(a)+x),f_0(x))$$

and

$$\mu s(a,x)\circ^{'}(g_1(a),g_0(x)) \quad = \quad (f_1(a)+h(\partial(a)+x),f_0(x)).$$

Then $(f_1,f_0)\circ^{'}\mu t=\mu s\circ^{'}(g_1,g_0)$. So

$$\begin{array}{rrcl}\mu: & C & \longrightarrow & G^{'}\rtimes C^{'}\\ & c & \longmapsto & \mu(x)=(h(x),f_0(x))\end{array}$$

is a homotopy connecting $F=((f_1,f_0),f_0)$ to $G=((g_1,g_0),g_0)$.
Let $f\xrightarrow{h}g$ and $g\xrightarrow{h^{'}}u$. Then we have

$$\begin{array}{rcl}\Psi(h+h^{'})(x) & = & ((h+h^{'})(x),f_0(x))\\ & = & (h(x)+h^{'}(x),f_0(x))\\ & = & (h(x),f_0(x))+(h^{'}(x),g_0(x))-(0,g_0(x))\\ & = & \Psi(h)(x)+\Psi(h^{'})(x)-e^{'}(t^{'}(\Psi)(h))(x)\\ & = & (\Psi(h)*\Psi(h))(x).\end{array}$$

□

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On the Inner-Product Spaces of Complex Interval Sequences

Halise Levent[1]*, Yılmaz Yılmaz[2], Hacer Bozkurt[3]

Abstract
In recent years, there has been increasing interest in interval analysis. Thanks to interval numbers, many real world problems have been modeled and analyzed. Especially, complex intervals have an important place for interval-valued data and interval-based signal processing. In this paper, firstly we introduce the notion of a complex interval sequence and we present the complex interval sequence spaces $\mathbb{I}(w)$ and $\mathbb{I}(l_p)$, $1 \leq p < \infty$. Secondly, we show that these sequence spaces have an algebraic structure called quasilinear space. Further, we construct an inner-product on $\mathbb{I}(l_2)$ and we show that $\mathbb{I}(l_2)$ is an inner-product quasilinear space.

Keywords: Complex interval, Complex interval sequence, Consolidate space, Inner product quasilinear spaces
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[1] *Department of Mathematics, Faculty of Science and Arts, İnönü University, Malatya, Turkey, ORCID:0000-0002-7139-361X*
[2] *Department of Mathematics, Faculty of Science and Arts, İnönü University, Malatya, Turkey, ORCID:0000-0003-1484-782X*
[3] *Department of Mathematics, Faculty of Science and Arts, Batman University, Batman, Turkey, ORCID: 0000-0002-2216-2516*
***Corresponding author**: halisebanazili44@gmail.com
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1. Introduction

In many real life situations, it is very difficult to deal with a process with reliable information about the properties of the expected variations. This has naturally led to an increased interest in intervals. Because the most ideal way to represent the loss of information is to use intervals.

An interval x is the compact-convex subset of real numbers and x is denoted by $x = [\underline{x}, \overline{x}]$ where $\underline{x}$ and $\overline{x}$ are the left and right endpoints of x, respectively [1]. Further, if $\underline{x} = \overline{x}$ then we say that x is a degenerate interval and it can be shown by $\{x\}$ or $[x, x]$. The set of all real intervals is denoted by $\mathbb{I}_{\mathbb{R}}$.

The idea of using intervals has been highly preferred by many researchers recently [1]-[4]. The interval sequence spaces have been studied by many authors [5, 6]. Also, we presented the notion of a complex interval which is significant for interval-valued data and interval-based signal processing in [7]. A complex interval is defined by

$$X = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]$$

where $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are real intervals and $i = \sqrt{-1}$ is the complex unit. $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are called real and imaginary part of X, respectively. Further, $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are called real and imaginary part of X, respectively.

In this work, we introduce the notion of complex interval sequence and we analyze some sequence spaces of the complex intervals, e.g., $\mathbb{I}(w)$ and $\mathbb{I}(l_p)$, $1 \leq p < \infty$. However, each element of these sequence spaces does not have an inverse according to the addition operation. These sequence spaces are not a linear space and the algebraic structure on these spaces is called as "quasilinear space". In 1986, Aseev defined the concept of quasilinear space [8]. Further, he present an approach for analysis of set-valued functions. This work has motivated a lot of authors the introduce new results on set-valued analysis [9]-[12]. Let us give the definition:

A set $\mathscr{X}$ is called a quasilinear space on field $\mathbb{K}$ if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real or complex numbers are defined in it in such a way that the following conditions hold for any elements $x,y,z,v \in X$ and any $\alpha,\beta \in \mathbb{K}$:

$x \preceq x$,

$x \preceq z$ if $x \preceq y$ and $y \preceq z$,

$x = y$ if $x \preceq y$ and $y \preceq x$,

$x+y = y+x$,

$x+(y+z) = (x+y)+z$,

there exists an element (zero) $\theta \in \mathscr{X}$ such that $x+\theta = x$,

$\alpha(\beta x) = (\alpha\beta)x$,

$\alpha(x+y) = \alpha x+\alpha y$,

$1x = x$,

$0x = \theta$,

$(\alpha+\beta)x \preceq \alpha x+\beta x$,

$x+z \preceq y+v$ if $x \preceq y$ and $z \preceq v$,

$\alpha x \preceq \alpha y$ if $x \preceq y$.

The most popular examples are $\Omega(E)$ and $\Omega_C(E)$ which are defined as the sets of all non-empty closed bounded and non-empty convex closed bounded subsets of any normed linear space E, respectively. Both are a quasilinear space with the inclusion relation"$\subseteq$", the algebraic sum operation

$$A+B = \overline{\{a+b : a \in A,\ b \in B\}}$$

where the closure is taken on the norm topology of E and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

Actually, $\Omega_C(\mathbb{R})$ is the set $\mathbb{I}_\mathbb{R}$ and for $x,y \in \mathbb{I}_\mathbb{R}$ and $\lambda \in \mathbb{R}$, the Minkowski sum and scalar multiplication operations are defined by

$$x+y = [\underline{x},\overline{x}]+\left[\underline{y},\overline{y}\right] = [\underline{x}+\underline{y},\overline{x}+\overline{y}]$$

and

$$\lambda x = \begin{cases} [\lambda\underline{x},\lambda\overline{x}] & , \quad \lambda \geq 0 \\ [\lambda\overline{x},\lambda\underline{x}] & , \quad \lambda < 0, \end{cases}$$

respectively. Further, the product of two intervals $x = [\underline{x},\overline{x}]$ and $y = \left[\underline{y},\overline{y}\right]$ is given by

$$x \cdot y = [\underline{x},\overline{x}]\left[\underline{y},\overline{y}\right] = [\min S, \max S] \tag{1.1}$$

where $S = \{\underline{x}\underline{y},\underline{x}\overline{y},\overline{x}\underline{y},\overline{x}\overline{y}\}$, [1].

The Minkowski sum and scalar multiplication on $\mathbb{I}_\mathbb{C}$ are defined by

$$\begin{aligned} X+Y &= \left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]+\left[\underline{y_r},\overline{y_r}\right]+i\left[\underline{y_s},\overline{y_s}\right] \\ &= \left[\underline{x_r}+\underline{y_r},\overline{x_r}+\overline{y_r}\right]+i\left[\underline{x_s}+\underline{y_s},\overline{x_s}+\overline{y_s}\right] \\ &= \left\{a+ib : a \in \left[\underline{x_r}+\underline{y_r},\overline{x_r}+\overline{y_r}\right],\ b \in \left[\underline{x_s}+\underline{y_s},\overline{x_s}+\overline{y_s}\right]\right\} \end{aligned}$$

and

$$\begin{aligned} \lambda X &= \lambda\left[\underline{x_r},\overline{x_r}\right]+i\left(\lambda\left[\underline{x_s},\overline{x_s}\right]\right) \\ &= \left\{\lambda a+i\lambda b : a \in \left[\underline{x_r},\overline{x_r}\right],\ b \in \left[\underline{x_s},\overline{x_s}\right]\right\} \end{aligned}$$

on $\mathbb{I}_\mathbb{C}$ where $i = \sqrt{-1}$ and $\lambda \in \mathbb{C}$. Further, the relation

$$X \preceq Y \text{ iff } \left[\underline{x_r},\overline{x_r}\right] \subseteq \left[\underline{y_r},\overline{y_r}\right] \text{ and } \left[\underline{x_s},\overline{x_s}\right] \subseteq \left[\underline{y_s},\overline{y_s}\right]$$

is a partial order relation on $\mathbb{I}_\mathbb{C}$. Thus, $\mathbb{I}_\mathbb{C}$ is a quasilinear space [7].

2. Preliminaries

Let us start with some main definitions, notions and theorems.

Suppose that $\mathscr{X}$ is a quasilinear space and $\mathscr{Y} \subseteq \mathscr{X}$. Then $\mathscr{Y}$ is called a *subspace of* $\mathscr{X}$ whenever $\mathscr{Y}$ is a quasilinear space with the same partial order and the restriction to $\mathscr{Y}$ of the operations on $\mathscr{X}$. $\mathscr{Y}$ is subspace of a quasilinear space $\mathscr{X}$ if and only if for every $x,y \in \mathscr{Y}$ and $\alpha, \beta \in \mathbb{K}$, $\alpha x + \beta y \in \mathscr{Y}$. Proof of this theorem is quite similar to its classical linear space analogue. Let $\mathscr{Y}$ be a subspace of a quasilinear space $\mathscr{X}$ and suppose each element x in $\mathscr{Y}$ has an inverse in $\mathscr{Y}$. Then the partial order on $\mathscr{Y}$ is determined by the equality. In this case $\mathscr{Y}$ is a linear subspace of $\mathscr{X}$, [14].

An element x in a quasilinear space $\mathscr{X}$ is said to be *symmetric* if $-x = x$ and $\mathscr{X}_{sym}$ denotes the set of all symmetric elements. Also, $\mathscr{X}_r$ stands for the set of all regular elements of $\mathscr{X}$ while $\mathscr{X}_s$ stands for the sets of all singular elements and zero in $\mathscr{X}$. Further, it can be easily shown that $\mathscr{X}_r$, $\mathscr{X}_{sym}$ and$\mathscr{X}_s$ are subspaces of $\mathscr{X}$. They are called *regular, symmetric* and *singular subspaces* of $\mathscr{X}$, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of $\mathscr{X}$ is a linear space while the singular one is nonlinear at all. Further, $\mathbb{I}_{\mathbb{C}}$ is a closed subspace of $\Omega(\mathbb{C})$, [13].

A real-valued function $\|.\|$ on the quasilinear space $\mathscr{X}$ is called a *norm* if the following conditions hold:

$$\|x\| > 0 \text{ if } x \neq 0, \tag{2.1}$$

$$\|x+y\| \leq \|x\| + \|y\|, \tag{2.2}$$

$$\|\alpha x\| = |\alpha| \, \|x\|, \tag{2.3}$$

$$\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \tag{2.4}$$

$$\begin{aligned} &\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in \mathscr{X} \text{ such that} \\ &\qquad x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon \text{ then } x \preceq y, \end{aligned} \tag{2.5}$$

here x, y, x_ε are arbitrary element in $\mathscr{X}$ and α is any scalar. A quasilinear space $\mathscr{X}$ with a norm defined on it, is called *normed quasilinear space*, [8].

For a normed linear space E, a norm on $\Omega(E)$ is defined by

$$\|A\|_\Omega = \sup_{a \in E} \|a\|_E \ .$$

Hence $\Omega_C(E)$ and $\Omega(E)$ are normed quasilinear spaces. A norm on $\mathbb{I}_{\mathbb{R}}$ is defined by

$$\|x\| = \|[\underline{x}, \overline{x}]\| = \sup_{t \in [\underline{x}, \overline{x}]} |t| \, .$$

Moreover, $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space with the norm

$$\begin{aligned} \|X\|_{\mathbb{I}_{\mathbb{C}}} &= \sup\{|z| : z \in X\} \\ &= \sup\{|a+ib| : a \in [\underline{x_r}, \overline{x_r}], b \in [\underline{x_s}, \overline{x_s}]\}, \end{aligned}$$

for $X = [\underline{x_r}, \overline{x_r}] + i [\underline{x_s}, \overline{x_s}]$, [12].

Now we will give the notion of consolidate quasilinear space defined in [12]. Thanks to this definition, we were able to give a representation to every element in a quasilinear space and we were able to define an inner-product quasilinear space.

Definition 2.1. *[12] Let $\mathscr{X}$ be a quasilinear space and $y \in \mathscr{X}$. The floor of y is the set of all regular elements y of $\mathscr{X}$ such that $x \preceq y$. It is denoted by $F_y^{\mathscr{X}}$ and $F_y^{\mathscr{X}} \subset \mathscr{X}$. Hence $F_y^{\mathscr{X}} = \{x \in \mathscr{X}_r : x \preceq y\}$.*

Definition 2.2. *[12] A quasilinear space $\mathscr{X}$ is called* consolidate *or* Solid-Floored *whenever*

$$\sup_{\preceq}\{x \in \mathscr{X}_r : x \preceq y\} = \sup_{\preceq} F_y^{\mathscr{X}}$$

exists and

$$y = \sup_{\preceq}\{x \in \mathscr{X}_r : x \preceq y\}$$

for each $y \in \mathscr{X}$. Otherwise, $\mathscr{X}$ is called a non-consolidate quasilinear space.

From above example immediately we can see that $\mathbb{I}_{\mathbb{R}}$ is consolidate while $(\mathbb{I}_{\mathbb{R}})_s$ is not. Analogous results are also true for the spaces $\mathbb{I}_{\mathbb{C}}$ and $(\mathbb{I}_{\mathbb{C}})_s$.

Definition 2.3. *[13] Let $\mathscr{X}$ be a consolidate quasilinear space. A mapping $\langle\ ,\ \rangle : \mathscr{X} \times \mathscr{X} \to \Omega(\mathbb{K})$ is called an inner product on $\mathscr{X}$ if for any $x,y,z \in \mathscr{X}$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied :*

If $x,y \in \mathscr{X}_r$ then $\langle x,y\rangle \in \Omega_C(\mathbb{K})_r \equiv \mathbb{K}$,

$\langle x+y,z\rangle \subseteq \langle x,z\rangle + \langle y,z\rangle$,

$\langle \alpha x,y\rangle = \alpha \langle x,y\rangle$,

$\langle x,y\rangle = \overline{\langle y,x\rangle}$,

$\langle x,x\rangle \geq 0$ for $x \in \mathscr{X}_r$ and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$,

$\|\langle x,y\rangle\|_{\Omega} = \sup\left\{\|\langle a,b\rangle\|_{\Omega} : a \in F_x^{\mathscr{X}}, b \in F_y^{\mathscr{X}}\right\}$,

if $x \preceq y$ and $u \preceq v$ then $\langle x,u\rangle \subseteq \langle y,v\rangle$,

if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in \mathscr{X}$ such that
$x \preceq y + x_\varepsilon$ and $\langle x_\varepsilon, x_\varepsilon\rangle \subseteq S_\varepsilon(\theta)$ then $x \preceq y$.

A quasilinear space with an inner product is called as an *inner-product quasilinear space*.
$\mathscr{X}$ is a linear Hilbert space, then the space $\Omega(\mathscr{X})$ is a Hilbert quasilinear space with the inner product defined by

$$\langle A,B\rangle_{\Omega} = \overline{\{\langle a,b\rangle_{\mathscr{X}} : a \in A, b \in B\}}$$

for $A,B \in \Omega(\mathscr{X})$. Especially, the inner product on $\Omega(\mathbb{C})$ given by

$$\langle A,B\rangle_{\Omega} = \{\langle a,b\rangle_{\mathbb{C}} : a \in A, b \in B\}. \tag{2.6}$$

If $A,B \in \mathbb{I}_{\mathbb{C}}$ then the inner-product (2.6) is equivalent to the following:

$$\langle A,B\rangle = [\underline{a_1},\overline{a_1}]\,[\underline{b_1},\overline{b_1}] + [\underline{a_2},\overline{a_2}]\,[\underline{b_2},\overline{b_2}] + i([\underline{a_2},\overline{a_2}]\,[\underline{b_1},\overline{b_1}] - [\underline{a_1},\overline{a_1}]\,[\underline{b_2},\overline{b_2}])$$

where $A = [\underline{a_1},\overline{a_1}] + i\,[\underline{a_2},\overline{a_2}]$, $B = [\underline{b_1},\overline{b_1}] + i\,[\underline{b_2},\overline{b_2}]$ and the product of two intervals is given in (1.1). Namely, the above equality is the reduction of the inner-product on $\Omega(\mathbb{C})$ to the inner-product on $\mathbb{I}_{\mathbb{C}}$.

3. Complex Interval Sequence Spaces

In this section, firstly we present the complex interval sequence spaces $\mathbb{I}(w)$ and $\mathbb{I}(l_p)$, $1 \leq p < \infty$ and we show that these spaces are the normed quasilinear spaces. Later, we construct a set-valued inner-product on $\mathbb{I}(l_2)$.

The sequence $X = (X_i)_{i=1}^{\infty}$ is called as complex interval sequence if $X_i \in \mathbb{I}_{\mathbb{C}}$, $i = 1, 2, ...$ The set $\mathbb{I}(l_w)$ denotes the set of all complex interval sequences $X = (X_i)_{i=1}^{\infty}$. The addition and multiplication operations $\mathbb{I}(w)$ are defined by

$$\begin{aligned} X + Y &= (X_1, X_2, ...) + (Y_1, Y_2, ...) \\ &= (X_1 + Y_1, X_2 + Y_2, ...) \end{aligned}$$

and

$$\alpha X = \alpha(X_1, X_2, ...) = (\alpha X_1, \alpha X_2, ...),$$

respectively where $X_i + Y_i$ is the sum of two complex intervals and αX_i is the multiplication of a complex interval with the scalar α. Further, the partial order relation on $\mathbb{I}(w)$ is that

$$X \ll Y \text{ iff } X_i \preceq Y_i,\ i = 1, 2, ...$$

where the relation "$\preceq$" is the partial order relation on $\mathbb{I}_{\mathbb{C}}$. Thus, $\mathbb{I}(l_w)$ is a quasilinear space with the above operations and the partial order relation.

For $1 \leq p < \infty$, $\mathbb{I}(l_p)$ is the set of all complex interval sequences $X = (X_i)_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p < \infty.$$

The space $\mathbb{I}(l_p)$ is a quasilinear space with the operations and the partial order relation on $\mathbb{I}(l_w)$. Really, for $X, Y \in \mathbb{I}(l_p)$ we write that by the Minkowski inequality

$$\begin{aligned} \sum_{i=1}^{\infty}(\|X_i + Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} &\leq \sum_{i=1}^{\infty}(\|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p + \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} \\ &\leq (\sum_{i=1}^{\infty}\|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} + (\sum_{i=1}^{\infty}\|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} < \infty. \end{aligned}$$

Further,

$$\sum_{i=1}^{\infty} \|\lambda X_i\|_{\mathbb{I}_{\mathbb{C}}}^p = |\lambda|^p (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p) < \infty$$

for $X \in \mathbb{I}(l_p)$ and $\lambda \in \mathbb{C}$.

Proposition 3.1. *$\mathbb{I}(l_p)$, $1 \leq p < \infty$ is a normed quasilinear space with the norm defined by*

$$\|X\| = (\sum_{i=1}^{\infty} \|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p}.$$

Proof. It is obvious that $\|X\| \geq 0$ for any $X \in \mathbb{I}(l_p)$. Further, for any $X, Y \in \mathbb{I}(l_p)$ and $\lambda \in \mathbb{C}$ by the triangle inequality and Minkowski inequality we write that

$$\|X + Y\| = \sum_{i=1}^{\infty}(\|X_i + Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} \leq (\sum_{i=1}^{\infty}\|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} + (\sum_{i=1}^{\infty}\|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = \|X\| + \|Y\|$$

and

$$\|\lambda X\| = (\sum_{i=1}^{\infty}\|\lambda X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = |\lambda| (\sum_{i=1}^{\infty}\|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = |\lambda| \|X\|.$$

Let us assume that $X \ll Y$ for any $X, Y \in \mathbb{I}(l_p)$. Then $\|X_i\|_{\mathbb{I}_{\mathbb{C}}} \leq \|Y_i\|_{\mathbb{I}_{\mathbb{C}}}$ for $i = 1, 2, ...$ since $X_i \preceq Y_i$, $i = 1, 2, ...$ and $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space. This implies that

$$\|X\| = (\sum_{i=1}^{\infty}\|X_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} \leq (\sum_{i=1}^{\infty}\|Y_i\|_{\mathbb{I}_{\mathbb{C}}}^p)^{1/p} = \|Y\|.$$

Now suppose that there exists an element $X^{\varepsilon} \in \mathbb{I}(l_p)$ such that $X \ll Y + X^{\varepsilon}$ and $\|X^{\varepsilon}\| \leq \varepsilon$ for any $\varepsilon > 0$. Then we have that $X_i \subseteq Y_i + X_i^{\varepsilon}$ and for $i = 1, 2, ...$ and

$$\|X^{\varepsilon}\| = (\sum_{i=1}^{\infty} \|X_i^{\varepsilon}\|_{\mathbb{I}_{\mathbb{C}}}^{p})^{1/p} \leq \varepsilon.$$

Hence, we obtain that $\|X_i\|_{\mathbb{I}_{\mathbb{C}}} < \varepsilon$ for $i = 1, 2, ...$ By the fourth condition of norm on $\mathbb{I}_{\mathbb{C}}$ we write that $X_i \preceq Y_i$ for $i = 1, 2, ...$ and so $X \ll Y$. □

Example 3.2. *Let us take the complex interval sequence $X = (X_k)_{k=1}^{\infty}$ given as follows:*

$$(X_k)_{k=1}^{\infty} = (\frac{1}{2^k} + i[0, \frac{1}{2^k}])_{k=1}^{\infty} = (\frac{1}{2} + i[0, \frac{1}{2}], \frac{1}{2^2} + i[0, \frac{1}{2^2}], ...).$$

We can say that $X = (X_k)_{k=1}^{\infty} \in \mathbb{I}(l_2)$ since

$$\begin{aligned}\|X\|^2 &= \sum_{k=1}^{\infty} \left\| \frac{1}{2^k} + i[0, \frac{1}{2^k}] \right\|_{\mathbb{I}_{\mathbb{C}}}^2 \\ &= \sum_{k=1}^{\infty} (\sup\{|a + ib| : a = \frac{1}{2^k}, b \in [0, \frac{1}{2^k}]\})^2 \\ &= \sum_{k=1}^{\infty} (\frac{1}{2^{2k}} + \frac{1}{2^{2k}}) = 2\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{2} \frac{1}{1 - 1/4} = 2/3.\end{aligned}$$

Hence, the norm of the sequence $X = (X_k)_{k=1}^{\infty}$ is that

$$\|X\| = (\sum_{k=1}^{\infty} \|X_k\|_{\mathbb{I}_{\mathbb{C}}}^2)^{1/2} = \sqrt{\frac{2}{3}}.$$

Among the $\mathbb{I}(l_p)$ spaces, $\mathbb{I}(l_2)$ has an important place. Because $\mathbb{I}(l_2)$ is an inner-product quasilinear space. Before we construct an inner-product on $\mathbb{I}(l_2)$, we must show that it is a consolidate space.

Lemma 3.3. *The space $\mathbb{I}(l_p)$, $1 \leq p < \infty$ is a consolidate quasilinear space.*

Proof. To complete the proof we will show that

$$X = \sup_{"\ll"}\{Y \in (\mathbb{I}(l_p))_r : Y \ll X\}.$$

If $Y \ll X$ for $Y \in (\mathbb{I}(l_p))_r$ then we write that $Y_i \preceq X_i$ for $i = 1, 2, ...$ and $X_i \in \mathbb{I}_{\mathbb{C}}$. Since $\mathbb{I}_{\mathbb{C}}$ is a consolidate quasilinear space, we obtain that

$$X_i = \sup F_{X_i} = \sup\{Y_i \in \mathbb{I}_{\mathbb{C}} : Y_i \preceq X_i\}$$

for each $i = 1, 2, ...$ This means that $\sup F_X = X$ for $X = (X_i)_{i=1}^{\infty} \in \mathbb{I}(l_p)$. □

Theorem 3.4. *The quasilinear space $\mathbb{I}(l_2)$ with the inner-product*

$$\langle X, Y \rangle = \sum_{i=1}^{\infty} \langle X_i, Y_i \rangle_{\mathbb{I}_{\mathbb{C}}} \tag{3.1}$$

is an inner-product quasilinear space where

$$\begin{aligned}\langle X_i, Y_i \rangle_{\mathbb{I}_{\mathbb{C}}} &= \left\langle \left[\underline{x_i^r}, \overline{x_i^r}\right] + i\left[\underline{x_i^s}, \overline{x_i^s}\right], \left[\underline{y_i^r}, \overline{y_i^r}\right] + i\left[\underline{y_i^s}, \overline{y_i^s}\right] \right\rangle \\ &= \left[\underline{x_i^r}, \overline{x_i^r}\right] + \left[\underline{y_i^s}, \overline{y_i^s}\right] + i(\left[\underline{x_i^s}, \overline{x_i^s}\right]\left[\underline{y_i^r}, \overline{y_i^r}\right] - \left[\underline{x_i^r}, \overline{x_i^r}\right]\left[\underline{y_i^s}, \overline{y_i^s}\right]).\end{aligned}$$

Proof. Firstly, we will show that the equality (3.1) is well-defined, i.e., $\langle X,Y\rangle \in \Omega(\mathbb{C})$:

By the Hölder and Schwartz inequalities we observe that

$$\begin{aligned}\|\langle X,Y\rangle\| &= \left\|\sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}}\right\| \leq \sum_{i=1}^{\infty}\left\|\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}}\right\|_{\Omega}\\ &\leq \sum_{i=1}^{\infty}(\|X_i\|_{\mathbb{I}_\mathbb{C}}\|Y_i\|_{\mathbb{I}_\mathbb{C}}) \leq (\sum_{i=1}^{\infty}\|X_i\|^2_{\mathbb{I}_\mathbb{C}})^{1/2}(\sum_{i=1}^{\infty}\|Y_i\|^2_{\mathbb{I}_\mathbb{C}})^{1/2}\\ &= \|X\|\,\|Y\|\end{aligned}$$

for $X,Y \in \mathbb{I}(l_2)$. This means that the set $\langle X,Y\rangle$ is bounded. Now let us take a sequence $(X_n)_{n=1}^{\infty}$ in the set $\langle X,Y\rangle$ such that $X_n \to X_0$. Then $\{X_n\} \to \{X_0\}$ for $n=1,2,...$ in $\Omega(\mathbb{C})$ since $X_n \in \langle X,Y\rangle$ for $n=1,2,\ldots$. Further, we can say that $\langle X_n,Y_n\rangle \to \langle X,Y\rangle$. The Lemma 4-a in [8] implies $\{X_0\} \subseteq \langle X,Y\rangle$. Consequently, we obtain that $X_0 \in \langle X,Y\rangle$.

1. If $X,Y \in (\mathbb{I}(l_2))_r$ then

$$\langle X,Y\rangle = \sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} \in (\Omega(\mathbb{C}))_r \equiv \mathbb{C}$$

since $X_i,Y_i \in \mathbb{C}$ for $i=1,2,\ldots$.

2. By the second condition of inner-product on $\mathbb{I}_\mathbb{C}$ we write that

$$\begin{aligned}\langle X+Y,Z\rangle &= \sum_{i=1}^{\infty}\langle X_i+Y_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}\\ &\subseteq \sum_{i=1}^{\infty}(\langle X_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}+\langle Y_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}})\\ &= \sum_{i=1}^{\infty}\langle X_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}+\sum_{i=1}^{\infty}\langle Y_i,Z_i\rangle_{\mathbb{I}_\mathbb{C}}\\ &= \langle X,Z\rangle+\langle Y,Z\rangle.\end{aligned}$$

3. By the third condition of inner-product on $\mathbb{I}_\mathbb{C}$ we obtain that

$$\langle \alpha X,Y\rangle = \sum_{i=1}^{\infty}\langle \alpha X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \sum_{i=1}^{\infty}\alpha\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \alpha\sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \alpha\langle X,Y\rangle.$$

Further, it can be easily shown that $\langle X,\alpha Y\rangle = \bar{\alpha}\langle X,Y\rangle$.

4. By the fourth condition of inner-product on $\mathbb{I}_\mathbb{C}$,

$$\langle X,Y\rangle = \sum_{i=1}^{\infty}\langle X_i,Y_i\rangle_{\mathbb{I}_\mathbb{C}} = \langle X,Y\rangle = \sum_{i=1}^{\infty}\langle Y_i,X_i\rangle_{\mathbb{I}_\mathbb{C}} = \langle Y,X\rangle.$$

5.

$$\langle X,X\rangle = \{0\} \Leftrightarrow \sum_{i=1}^{\infty}\langle X_i,X_i\rangle_{\mathbb{I}_\mathbb{C}} = \{0\} \Leftrightarrow X_i = \theta, i=1,2,... \Leftrightarrow X = \theta.$$

and for any $X \in (\mathbb{I}(l_2))_r$ we write that

$$\langle X,X\rangle = \sum_{i=1}^{\infty}\langle X_i,X_i\rangle_{\mathbb{I}_\mathbb{C}} = \sum_{i=1}^{\infty}|X_i|^2 \geq 0.$$

6.

$$
\begin{aligned}
\|\langle X,Y\rangle\|_\Omega &= \left\| \sum_{i=1}^{\infty} \langle X_i, Y_i\rangle_{\mathbb{I}_\mathbb{C}} \right\|_\Omega \\
&= \sup\{|z| : z \in \sum_{i=1}^{\infty} \langle X_i, Y_i\rangle_{\mathbb{I}_\mathbb{C}}\} \\
&= \sup\{|z| : z \in \sum_{i=1}^{\infty} (\left[\underline{x_i^r}, \overline{x_i^r}\right] + \left[\underline{y_i}^s, \overline{y_i^s}\right] + i(\left[\underline{x_i}^s, \overline{x_i^s}\right] \left[\underline{y_i^r}, \overline{y_i^r}\right] - \left[\underline{x_i^r}, \overline{x_i^r}\right] \left[\underline{y_i}^s, \overline{y_i^s}\right]) \\
&= \sup\{|\langle x,y\rangle| : x \in F_X, y \in F_Y\}.
\end{aligned}
$$

7. If $X \ll Y$ and $Z \ll T$ then $X_i \preceq Y_i$ and $Z_i \preceq T_i$ for $i = 1,2,\ldots$. By the seventh condition of inner-product on $\mathbb{I}_\mathbb{C}$ we write that $\langle X_i, Z_i\rangle_{\mathbb{I}_\mathbb{C}} \subseteq \langle Y_i, T_i\rangle_{\mathbb{I}_\mathbb{C}}$ for $i = 1,2,\ldots$ and so $\langle X,Z\rangle \subseteq \langle Y,T\rangle$.

8. Suppose that for any $\varepsilon > 0$ there exists an element $X^\varepsilon \in \mathbb{I}(l_2)$ such that $X \ll Y + X^\varepsilon$ and $\langle X^\varepsilon, X^\varepsilon\rangle \subseteq S_\varepsilon(\theta)$. Then we say that $X_i \subseteq Y_i + X_i^\varepsilon$ for $i = 1,2,\ldots$. By the hypothesis we write that

$$\sum_{i=1}^{\infty} \langle X_i^\varepsilon, X_i^\varepsilon\rangle_{\mathbb{I}_\mathbb{C}} \subseteq S_\varepsilon(\theta).$$

Since $\mathbb{I}_\mathbb{C}$ is an inner-product quasilinear space, if $X_i \subseteq Y_i + X_i^\varepsilon$ for $i = 1,2,\ldots$ and $\|X_i^\varepsilon\|_\Omega \leq \varepsilon$ then $X_i \subseteq Y_i$ for $i = 1,2,\ldots$ This implies $X \ll Y$.

□

4. Conclusion

In this paper, we have presented the notion of complex interval sequence and some important complex interval sequence spaces. In this way, we brought a new perspective to sequence spaces with the help of interval analysis and quasilinear functional analysis. We also have defined the inner product function on the complex interval sequence space $\mathbb{I}(l_2)$, which is one of the most important sequence spaces. Thus, by using quasilinear functional analysis techniques, we have introduced a new type of space to the literature.

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On the Global of the Difference Equation $x_{n+1}=\frac{\alpha x_{n-m}+\eta x_{n-k}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}(x_{n-k}+x_{n-l})}$

M. A. El-Moneam[1]*

Abstract
In this article, we consider and discuss some properties of the positive solutions to the following rational nonlinear DE $x_{n+1}=\frac{\alpha x_{n-m}+\eta x_{n-k}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}(x_{n-k}+x_{n-l})}$, $n=0,1,...,$ where the parameters $\alpha,\beta,\gamma,\delta,\eta\in(0,\infty)$, while m,k,l are positive integers, such that $m<k<l$ and the initial conditions $x_{-m},...,x_{-k},...,x_{-l},...,$
$x_{-1},...,x_0$ are arbitrary positive real numbers, we will give also, some numerical examples to illustrate our results.

Keywords: Difference equations, Equilibrium,Oscillates, Globally asymptotically stable, Prime period two solution, Rational difference equations, Qualitative properties of solutions of difference equations
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[1] *Mathematics Department, Faculty of Science,*
Jazan University, Kingdom of Saudi Arabia, ORCID: https://orcid.org/0000-0002-1676-2662
***Corresponding author**: maliahmedibrahim@jazanu.edu.sa
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1. Introduction

The study of the solution of nonlinear rational sequence of high order is quite challenging and rewarding. Every dynamical system $b_{n+1}=f(b_n)$ determines DE and vice versa. An interesting class of nonlinear DE is the class of solvable DEs, and one of the interesting problems is to find equations that belong to this class and to solve them in closed form or in explicit form [1]-[14], [16]-[26]. Note that most of these Eq. often show increasingly complex behavior such as the existence of a bounded. The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. The applications of these difference equations can be found on the economy, biology and so on. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. The aim of this paper is to investigate some qualitative behavior of the solutions of the nonlinear DE

$$x_{n+1}=\frac{\alpha x_{n-m}+\eta x_{n-k}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}\left(x_{n-k}+x_{n-l}\right)},\qquad n=0,1,2,... \tag{1.1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ \delta,\ \eta\in(0,\infty)$, while $m,\ k,\ l,$ are positive integers, such that $m<k<l$ and the initial conditions $x_{-m},...,x_{-k},...,x_{-l},...,x_{-1},...,x_0$ are arbitrary positive real numbers. Equation (1.1) has been discussed in [15], when $m=1,\ k=2$ and $l=4$, and in [28], when $\delta=0$, where some global behavior of the more general nonlinear rational Eq. (1.1), we need the following well-known definitions and results [29]-[34].

Definition 1.1. *A difference equation of order $(k+1)$ is of the form*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{-k}), \quad n = 0, 1, 2, ... \tag{1.2}$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 1.2. *Let $\widetilde{x} \in (0,\infty)$ be an equilibrium point of the difference equation (1.2). Then*
(i) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if x_{-k}, ..., x_{-1},$x_0 \in (0,\infty)$ with $|x_{-k}-\widetilde{x}| + ... + |x_{-1}-\widetilde{x}| + |x_0-\widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if x_{-k}, ..., x_{-1}, $x_0 \in (0,\infty)$ with $|x_{-k}-\widetilde{x}| + ... + |x_{-1}-\widetilde{x}| + |x_0-\widetilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called a global attractor if for every x_{-k}, ..., x_{-1}, $x_0 \in (0,\infty)$ we have

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called unstable if it is not locally stable.

Definition 1.3. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.*

Definition 1.4. *We say that a sequence $\{x_n\}_{n=-l}^{\infty}$ is bounded and persisting if , there exists positive constants m and M such that*

$$m \leq x_n \leq M, \quad for\ all \quad n \geq -k.$$

Definition 1.5. *A positive semicycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $x_l, x_{l+1}, ..., x_m$ all greater than or equal to $\tilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that*

$$either \quad l = -k \quad or \quad l > -k \quad and \quad x_{l-1} < \tilde{x},$$

and

$$either \quad m = \infty \quad or \quad m < \infty \quad and \quad x_{m+1} < \tilde{x}.$$

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $x_l, x_{l+1}, ..., x_m$ all less than $\tilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$either \quad l = -k \quad or \quad l > -k \quad and \quad x_{l-1} \geq \tilde{x},$$

and

$$either \quad m = \infty \quad or \quad m < \infty \quad and \quad x_{m+1} \geq \tilde{x}.$$

Definition 1.6. *The linearized Eq. of Eq. (1.2) about the equilibrium point $\widetilde{x}$ is the linear Eq.*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Now, assume that the characteristic Eq. associated with Eq. (1.3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{1.4}$$

where

$$p_i = \partial F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x}) / \partial x_{n-i}.$$

Theorem 1.7. *Let $p_i \in R$, $i=1,2,\ldots,$ and $k \in \{0,1,2,\ldots\}$, then*

$$\sum_{i=1}^{k}|p_i|<1,$$

is sufficient condition for asymptotic stability of difference equation

$$x_{n+k}+p_1x_{n+k-1}+\ldots..+p_kx_n=0, \quad n=0,1,2,\ldots$$

Theorem 1.8 (The Linearized Stability Theorem)**.**
Suppose that F is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\widetilde{x}$. Then the following statements are true.
(i) If all roots of the characteristic equation (1.4) of the linearized equation (1.3) have an absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.
(ii) If at least one root of Eq.(1.4) has an absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.

2. Change of Variables

By using the change of variables $x_n=\left(\frac{\beta}{\gamma}\right)^{\frac{1}{3}}y_n$, the equation (1.1) reduces to the following difference equation

$$y_{n+1}=\frac{ry_{n-m}+ty_{n-k}+sy_n}{1+y_{n-k}y_{n-l}\left(y_{n-k}+y_{n-l}\right)}, \qquad n=0,1,2,\ldots \tag{2.1}$$

where $r=\frac{\alpha}{\beta}>0$, $s=\frac{\delta}{\beta}>0$, $t=\frac{\eta}{\beta}>0$, and the initial conditions $y_{-l},\ldots,y_{-k},\ldots,y_{-m},\ldots,y_{-l},y_0\in(0,\infty)$. In the next section, we shall study the global behavior of Eq. (2.1).

3. The Dynamics of Eq. (2.1)

The equilibrium points $\tilde{y}$ of Eq. (2.1) are the positive solutions of equation

$$\tilde{y}=\frac{[r+s+t]\tilde{y}}{1+2\tilde{y}^3}. \tag{3.1}$$

Thus $\tilde{y}_1=0$, is always an equilibrium point of the equation (2.5). If $(r+s+t)>1$, then the only positive equilibrium point $\tilde{y}_2$ of equation (2.1) is given by

$$\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}. \tag{3.2}$$

Let us introduce a continuous function $F:(0,\infty)^4\to(0,\infty)$, which is defined by

$$F(v_0,v_1,v_2,v_3)=\frac{rv_0+sv_1+tv_2}{1+v_2^2v_3+v_2v_3^2}. \tag{3.3}$$

Consequently, we get

$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_0}=\frac{r}{1+v_2^2v_3+v_2v_3^2},$$
$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_1}=\frac{s}{1+v_2^2v_3+v_2v_3^2},$$

$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_2}=\frac{t(1+v_2^2v_3+v_2v_3^2)-(rv_0+sv_1+tv_2)\,(2v_2v_3+v_3^2)}{(1+v_2^2v_3+v_2v_3^2)^2},$$

$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_3}=\frac{-(rv_0+sv_1+tv_2)(v_2^2+2v_2v_3)}{(1+v_2^2v_3+v_2v_3^2)^2}.$$

At $\tilde{y}_1=0$, we have $\frac{\partial F(0,0,0,0)}{\partial v_0}=r$, $\frac{\partial F(0,0,0,0)}{\partial v_1}=s$, $\frac{\partial F(0,0,0,0)}{\partial v_2}=t$, $\frac{\partial F(0,0,0,0)}{\partial v_3}=0$, and the linearized equation of Eq. (2.1) about $\tilde{y}_1=0$, is the equation

$$z_{n+1}-\rho_0 z_n-\rho_1 z_{n-m}-\rho_2 z_{n-k}=0, \tag{3.4}$$

where $\rho_0=s$, $\rho_1=r$, $\rho_2=t$. At $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$, we have

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_0}=\frac{r}{1+2\tilde{y}_2^3}=\frac{r}{1+([r+s+t]-1)}=\frac{r}{[r+s+t]},$$

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_1}=\frac{s}{1+2\tilde{y}_2^3}=\frac{s}{1+([r+s+t]-1)}=\frac{s}{[r+s+t]},$$

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_2}=\frac{2t-3([r+s+t]-1)}{2[r+s+t]},$$

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_3}=\frac{-3([r+s+t]-1)}{2[r+s+t]}.$$

And the linearized equation of Eq. (2.1) about $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ is the equation

$$z_{n+1}-\rho_0 z_n-\rho_1 z_{n-m}-\rho_2 z_{n-k}-\rho_3 z_{n-l}=0, \tag{3.5}$$

where $\rho_0=\frac{s}{[r+s+t]}$, $\rho_1=\frac{r}{[r+s+t]}$, $\rho_2=\frac{2t-3([r+s+t]-1)}{2[r+s+t]}$, $\rho_3=\frac{-3([r+s+t]-1)}{2[r+s+t]}$.

Theorem 3.1. *(i) If $[r+s+t]<1$, then the equilibrium point $\tilde{y}_1=0$ is locally asymptotically stable.*
(ii) If $[r+s+t]>1$, then the equilibrium point $\tilde{y}_1=0$ is unstable.
(iii) If $[r+s+t]>1$, $2t>3([r+s+t]-1)$, then the equilibrium point $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ is unstable.

Proof. With reference to Theorem 1.1, we deduce from Eq. (3.4) that $|\rho_0|+|\rho_1|+|\rho_2|=[r+s+t]<1$, and then the proof of parts (i), (ii) follow. Also, from Eq. (3.5) we deduce for $[r+s+t]>1$ that $|\rho_0|+|\rho_1|+|\rho_2|+|\rho_3|=1+\frac{3([r+s+t]-1)}{[r+s+t]}>1$, and hence the proof of part (iii) follows. □

Theorem 3.2. *Assume that $[r+s+t]>1$, and let $\{y_n\}_{n=-l}^{\infty}$ be a solution of Eq. (2.1) such that*

$$\begin{aligned}&y_{-l},y_{-l+2},\dots,y_{-l+2n},\dots,y_{-k},y_{-k+2},\dots,y_{-k+2n},\dots,\\&y_{-m+1},y_{-m+3},\dots,y_{-m+2n+1},\dots,y_0\geq\tilde{y}_2\\&\textit{and}\\&y_{-l+1},y_{-l+3},\dots,y_{-l+2n+1},\dots,y_{-k+1},y_{-k+3},\dots,\\&y_{-k+2n+1},\dots,y_{-m},y_{-m+2},\dots,y_{-m+2n},\dots,y_{-1}<\tilde{y}_2.\end{aligned} \tag{3.6}$$

Then $\{y_n\}_{n=-l}^{\infty}$ oscillates about $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ with a semicycle of length one.

Proof. Assume that (3.6) holds. Then

$$y_1=\frac{ry_{-m}+sy_0+ty_{-k}}{1+y_{-k}y_{-l}(y_{-k}+y_{-l})}<\frac{ry_{-m}+sy_0+ty_{-k}}{1+2\tilde{y}_2^3}<\frac{[r+s+t]\tilde{y}_2}{1+([r+s+t]-1)}=\tilde{y}_2,$$

and

$$y_2=\frac{ry_{-m+1}+sy_1+ty_{-k+1}}{1+y_{-k+1}y_{-l+1}(y_{-k+1}+y_{-l+1})}\geq\frac{ry_{-m+1}+sy_1+ty_{-k+1}}{1+2\tilde{y}_2^3}\geq\frac{[r+s+t]\tilde{y}_2}{1+([r+s+t]-1)}=\tilde{y}_2,$$

and hence the proof follows by induction.

Theorem 3.3. *Assume that* $[r+s+t] < 1$, *then the equilibrium point* $\tilde{y}_1 = 0$ *of Eq. (2.1) is globally asymptotically stable.*

Proof. We have shown in Theorem 3 that if $[r+s+t] < 1$ then the equilibrium point $\tilde{y}_1 = 0$ is locally asymptotically stable. It remains to show that $\tilde{y}_1 = 0$ is a global attractor. To this end, let $\{y_n\}_{n=-l}^{\infty}$ be a solution of Eq. (2.1). It suffics to show that $\lim_{n\to\infty} y_n = 0$. Since

$$0 \le y_{n+1} = \frac{ry_{n-m}+sy_n+ty_{n-k}}{1+y_{n-k}y_{n-l}(y_{n-k}+y_{n-l})} \le ry_{n-m}+sy_n+ty_{n-k} < y_{n-k}.$$

Then we have $\lim_{n\to\infty} y_n = 0$. This completes the proof. □

Theorem 3.4. *Assume that* $[r+s+t] > 1$, *then Eq. (2.1) possesses an unbounded solution.*

Proof. With the aid of Theorem 3.3, we have

$$\begin{aligned} y_{2n+2} &= \frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{1+y_{-k+2n+1}\, y_{-l+2n+1}(y_{-k+2n+1}+y_{-l+2n+1})} > \frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{1+2\tilde{y}_2^3} \\ &> \frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{1+([r+s+t]-1)} = \frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{[r+s+t]}, \end{aligned}$$

and

$$\begin{aligned} y_{2n+3} &= \frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{1+y_{-k+2n+2}\, y_{-l+2n+2}(y_{-k+2n+2}+y_{-l+2n+2})} \le \frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{1+2\tilde{y}_2^3} \\ &\le \frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{1+([r+s+t]-1)} = \frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{[r+s+t]}. \end{aligned}$$

From which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty \quad and \quad \lim_{n\to\infty} y_{2n+1} = 0.$$

Hence, the proof of Theorem 3.4 is now completed. □

Theorem 3.5. *(1) If m is odd, and k,l are even, Eq. (2.1) has prime period two solution if* $(r-[s+t]) < 1$ *and has not prime period two solution if* $(r-[s+t]) \ge 1$.
(2) If m is even and k,l are odd, Eq. (2.1) has not prime period two solution.
(3) If all m,k,l are even, Eq. (2.1) has prime period two solution.
(4) If all m,k,l are odd, Eq. (2.1) has prime period two solution if $(r-[s+t]) > 1$, *and has not prime period two solution if* $(r-[s+t]) \le 1$.
(5) If m,k are even and l is odd, Eq. (2.1) has not prime period two solution.
(6) If m,k are odd and l is even, Eq. (2.1) has prime period two solution if $(r-[s+t]) > 1$, *and has not prime period two solution if* $(r-[s+t]) \le 1$.
(7) If m,l are odd and k is even, Eq. (2.1) has prime period two solution if $(r-[s+t]) > 1$, *and has not prime period two solution if* $(r-[s+t]) \le 1$.
(8) If m,l are even and k is odd, Eq. (2.1) has not prime period two solution.

Proof. Assume that there exists distinct positive solutions

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

of prime period two of Eq. (2.1).
(1) If m is odd, and k,l are even, then $y_{n+1} = y_{n-m}$ and $y_n = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{r\phi+[s+t]\,\psi}{1+2\psi^3}, \quad \psi = \frac{r\psi+[s+t]\,\phi}{1+2\phi^3}.$$

Consequently, we have

$$0 < 2\phi\psi(\phi+\psi) = 1-(r-[s+t]). \tag{3.7}$$

We deduce that (3.7) is always true if $(r-[s+t]) < 1$ and hence Eq. (2.1) has prime period two solution. If $(r-[s+t]) \geq 1$, we have a contradiction, and hence Eq. (2.1) has not prime period two solution.
(2) If m is even, and k,l are odd, then $y_n = y_{n-m}$,and $y_{n+1} = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s+t]\,\psi}{1+2\phi^3}, \quad \psi = \frac{[r+s+t]\,\phi}{1+2\psi^3}.$$

Consequently, we have

$$0 < 2\,(\phi+\psi)\,(\phi^2+\psi^2) = -\left([r+s+t]+1\right). \tag{3.8}$$

Since $[r+s+t] > 0$, we have a contradiction. Hence Eq. (2.1) has not prime period two solution.
(3) If all m,k,l are even, then $y_n = y_{n-m} = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s+t]\,\psi}{1+2\psi^3}, \quad \psi = \frac{[r+s+t]\,\phi}{1+2\phi^3}.$$

Consequently, we get

$$0 < 2\phi\psi(\phi+\psi) = [r+s+t]+1. \tag{3.9}$$

Since $[r+s+t] > 0$, the formula (3.14) is always true. Hence Eq. (2.1) has prime period two solution.
(4) If all m,k,l are odd, then $y_{n+1} = y_{n-m} = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{r\phi+s\psi}{1+2\phi^3}, \quad \psi = \frac{r\psi+s\phi}{1+2\psi^3}.$$

Consequently, we get

$$0 < 2(\phi+\psi)(\phi^2+\psi^2) = (r-[s+t])-1. \tag{3.10}$$

If $(r-[s+t]) > 1$, the formula (15) is always true, and hence Eq. (2.1) has prime period two solution. If $(r-[s+t]) \leq 1$, we have a contradiction and hence Eq. (2.1) has not prime period two solution.
(5) If m,k are even, and l is odd, then $y_n = y_{n-k} = y_{n-m}$, and $y_{n+1} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s+t]\,\psi}{1+\psi^2\phi+\psi\phi^2}, \quad \psi = \frac{[r+s+t]\,\phi}{1+\phi^2\psi+\phi\psi^2}.$$

Consequently, we have

$$0 < \phi\psi(\phi+\psi) = -([r+s+t]+1). \tag{3.11}$$

Since $[r+s+t] > 0$, we have a contradiction. Hence Eq. (2.1) has not a prime period two solution.
(6) If m,k are odd, and l is even, then $y_{n+1} = y_{n-m} = y_{n-k}$, and $y_n = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+t]\,\phi+s\psi}{1+\phi^2\psi+\phi\psi^2}, \quad \psi = \frac{[r+t]\,\psi+s\phi}{1+\psi^2\phi+\psi\phi^2}.$$

Consequently, we have

$$0 < \phi\psi(\phi+\psi) = ([r+t]-s)-1. \tag{3.12}$$

If $([r+t]-s) > 1$,the formula (3.17) is always true, and hence Eq. (2.1) has prime period two solution. If $([r+t]-s) \leq 1$, we have a contradiction. Hence Eq.(2.5) has not a prime period two solution.
(7) If m,l are odd, and k is even, then $y_{n+1} = y_{n-m} = y_{n-l}$, and $y_n = y_{n-k}$. It follows from Eq. (2.1) that

$$\phi = \frac{r\phi+[s+t]\,\psi}{1+\psi^2\phi+\psi\phi^2}, \quad \psi = \frac{r\psi+[s+t]\,\phi}{1+\phi^2\psi+\phi\psi^2},$$

which give the same results of case (6).
(8) If m,l are even, and k is odd, then $y_n = y_{n-m} = y_{n-l}$, and $y_{n+1} = y_{n-k}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s]\,\psi+t\phi}{1+\psi^2\phi+\psi\phi^2}, \quad \psi = \frac{[r+s]\,\phi+t\psi}{1+\phi^2\psi+\phi\psi^2},$$

which give the same results of case (5). Hence the proof of Theorem 3.5 is now completed. □

4. Numerical Examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq. (2.1).

Example 4.1. *Figure 4.1, shows that the solution of Eq. (2.1) is bounded if* $x_{-3}=1,\ x_{-2}=2,\ x_{-1}=3,\ x_0=4,\ m=1,\ k=2,\ l=3,\ r=0.1,\ s=0.2,\ t=0.3,\ i.e\ [r+s+t]<1.$

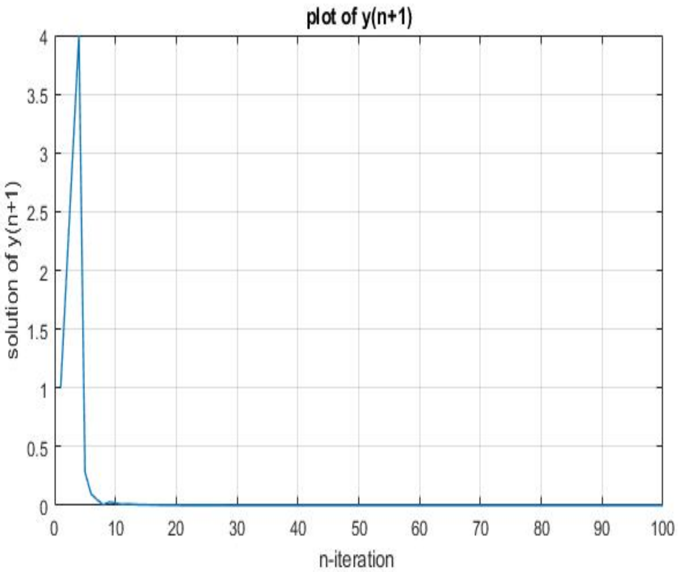

Figure 4.1. The solution of Eq. (2.1) is bounded.

Example 4.2. *Figure 4.2, shows that the solution of Eq. (2.1) is unbounded if* $x_{-3}=1,\ x_{-2}=2,\ x_{-1}=3,\ x_0=4,\ m=1,\ k=2,\ l=3,\ r=1,\ s=2,\ t=3,\ i.e\ [r+s+t]>1.$

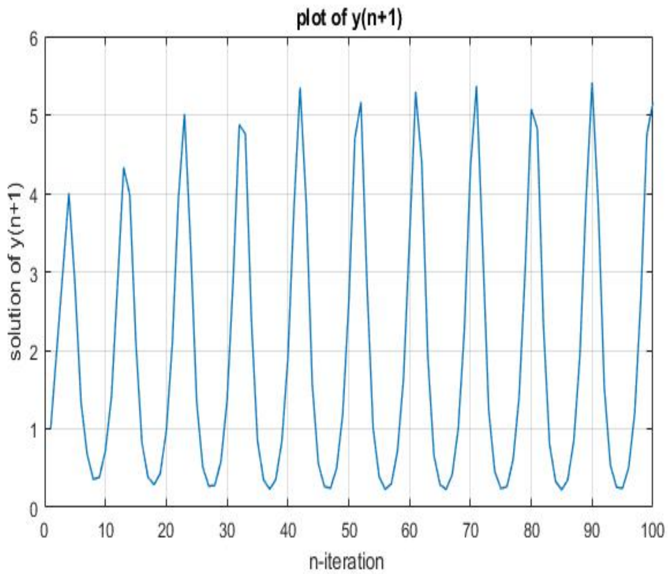

Figure 4.2. The solution of Eq. (2.1) is unbounded.

Example 4.3. *Figure 4.3, shows that Eq. (2.1) is globally asymptotically stable if* $x_{-4}=1,\ x_{-3}=2,\ x_{-2}=3,\ x_{-1}=4,\ x_0=5,\ m=2,\ k=3,\ l=4,\ r=0.1,\ s=0.5,\ t=0.2,\ i.e\ [r+s+t]<1.$

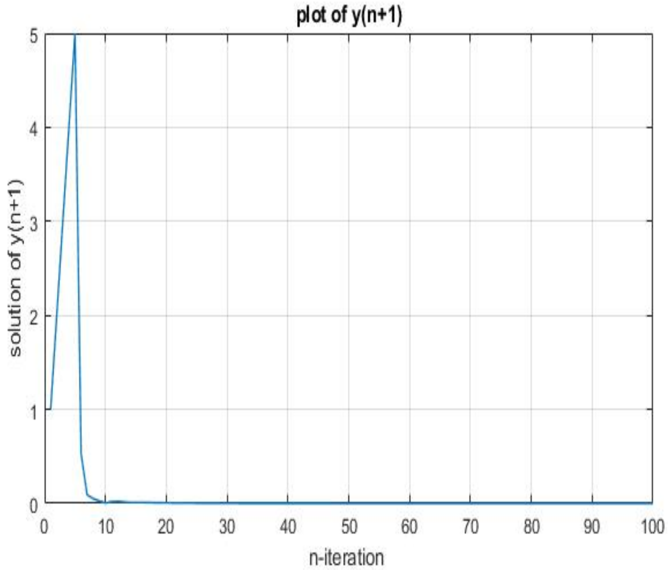

Figure 4.3. The solution of Eq. (2.1) is globally asymptotically stable.

Example 4.4. *Figure 4.4, shows that Eq. (2.1) has no positive prime period two solutions if* $x_{-3} = 1,\ x_{-2} = 2,\ x_{-1} = 3,\ x_0 = 4,\ m = 2,\ k = 1,\ l = 3,\ r = 100,\ s = 300,\ t = 400.$

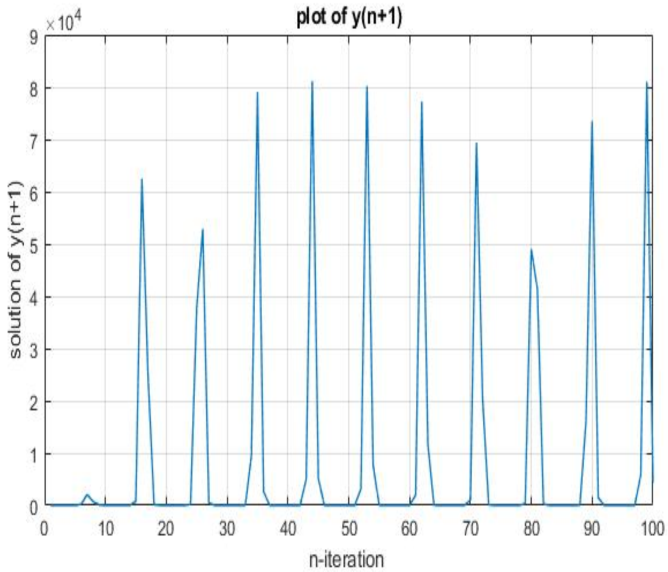

Figure 4.4. The solution of Eq. (2.1) is globally asymptotically stable.

5. Conclusions

In this article, we have shown that Eq. (2.1) has two equilibrium points $\tilde{y}_1 = 0$ and $\tilde{y}_2 = \left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$. If $[r+s+t] < 1$, we have proved that $\tilde{y}_1 = 0$ is globally asymptotically stable, while if $[r+s+t] > 1$, the solution of Eq. (2.1) oscillates about the point $\tilde{y}_2 = \left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ with a semicycle of length one. When $[r+s+t] > 1$, we have proved that the solution of Eq. (2.1) is unbounded. The periodicity of the solution of Eq. (2.1) has been discussed in details in Theorem 3.5.

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On Weakly 1-Absorbing Primary Ideals of Commutative Semirings

Mohammad Saleh[1]*, Ibaa Murra[2]

Abstract
Let R be a commutative semiring with $1 \neq 0$. In this paper, we study the concept of weakly 1-absorbing primary ideal which is a generalization of 1-absorbing ideal over commutative semirings . A proper ideal I of a semiring R is called a weakly 1-absorbing primary ideal if whenever nonunit elements $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$. A number of results concerning weakly 1-absorbing primary ideals and examples of weakly 1-absorbing primary ideals are given. An ideal is called a subtractive ideal I of a semiring R is an ideal such that if $x, x+y \in I$, then $y \in I$. Subtractive ideals or k-ideals are helpful in proving in many results related to ideal theory over semirings.

Keywords:
1-absorbing primary ideal, 2-absorbing primary ideal, Prime ideal, Weakly 1-absorbing primary ideal, Weakly 2-absorbing primary ideal, Weakly prime ideal, Weakly primary, Weakly primary ideal
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[1] *Department of Mathematics, Birzeit University, P.O.Box 14, West Bank, Palestine, ORCID: 0000-0002-4254-2540*
[2] *Department of Mathematics, Birzeit University, Ramalla, West Bank, Palestine, ORCID: xxxxxx*
***Corresponding author**: msaleh@birzeit.edu
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1. Introduction

The algebraic structure of semirings, that are considered as a generalization of rings, plays an important role in different branches of mathematics, especially in applied sciences and computer engineering. For general references on semiring theory one may refer to [1],[4],[13] and [16].
The first formal definition of semirings was introduced by H.S Vandiver in 1934 [20] "Note on a simple type of algebra in which cancelation law of addition does not holds".
In this paper we need a special kind of ideals that was defined by Henriksen [14] in 1958 which is called k-ideal or subtractive ideals. A subtractive ideal I of a semiring R is an ideal such that if $x, x+y \in I$, then $y \in I$.

Since prime and primary ideals have key roles in commutative semiring theory, many authors have studied generalizations of prime and primary ideals. One of the generalization of that concept is 2-absorbing ideals.
In 2012, Darani [12] introduced the connotation of a 2-absorbing ideal of a commutative semiring. A proper ideal I of a semiring R is said to be a 2-absorbing primary ideal if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$, or $bc \in I$, or $ac \in I$.
In [8], the concept of weakly 1-absorbing primary ideal which is a generalization of 1-absorbing ideal was introduced. A proper ideal I of a ring R is called a weakly 1-absorbing primary ideal if whenever nonunit elements $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$ and studied n number of results concerning weakly 1-absorbing primary ideals and examples of weakly

1-absorbing primary ideals .
We assume throughout this paper that all semirings are commutative with unity $1 \neq 0$. We start by recalling some background material. By a proper ideal I of R, we mean an ideal I of R with $I \neq R$. Let I be a proper ideal of R. Before we state some results, let us introduce some notation and terminology. By $\sqrt{I}$, we mean the radical of R, that is, $\{a \in R \mid an \in I\}$ for some positive integer $n\}$. In particular, $\sqrt{0}$ denotes the set of all nilpotent elements of R. We define $Z_I(R) = \{r \in R \mid rs \in I$ for some $s \in R \setminus I\}$. A semiring R is called a reduced semiring if it has no non-zero nilpotent elements; i.e., $\sqrt{0} = 0$. For two ideals I and J of R, the residual division of I and J is defined to be the ideal $(I : J) = \{a \in R \mid aJ \subseteq I\}$. Let R be a commutative semiring with identity and M a unitary R-semimodule. Then $R(+)M = R \bigoplus M$(direct sum) with coordinate-wise addition and multiplication $(a,m)(b,n) = (ab, an + bm)$ is a commutative semiring with identity called the idealization of M. A semiring R is called a quasilocal semiring if R has exactly one maximal ideal. As usual we denote Z and Z_n by the semiring of integers and the semiring of integers modulo n.
In this paper, we introduce the concept of (weakly) 1-absorbing ideal of a semiring R. A proper ideal I of a semiring R is called a weakly 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$, and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$. A proper ideal I of a semiring R is called 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$, and $abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$. It is clear that a 1-absorbing primary ideal of R is a weakly 1-absorbing primary ideal of R. However, since 0 is always weakly 1-absorbing primary, a weakly 1-absorbing primary ideal of R needs not be a 1-absorbing primary ideal of R. Among many results, we show (Theorem 2.5) that if a proper ideal I of R is a weakly 1-absorbing ideal of R such that $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is 1-absorbing primary ideal of R. We show (Theorem 2.6) that if R is a reduced semiring, and I is a weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is a prime ideal of R. If I is a proper nonzero ideal of a von-Neumann regular semiring R, then we show (Theorem 2.7) that I is a weakly 1-absorbing primary ideal of R if and only if I is a 1-absorbing primary ideal of R if and only if I is a primary ideal of R. We show (Theorem 2.8) that if R is a nonquasilocal semiring, and I be a proper ideal of R such that $ann(i) = \{r \in R \mid ri = 0\}$ is not a maximal ideal of R for every element $i \in I$, then I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R. If I is a proper ideal of a reduced divided semiring R, then we show (Theorem 2.11) that I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R. If I is a weakly 1-absorbing primary of a semiring R that is not a 1-absorbing primary ideal of R, then we give (Theorem 3.4) sufficient conditions so that $I^3 = 0$ (i.e., $I \subseteq \sqrt{I}$). In Theorem 3.2, we obtain some equivalent conditions for weakly 1-absorbing primary ideals of u-semirings. In (Theorem4.1), a characterization of weakly 1-absorbing primary ideals in $R = R_1 \times R_2$, where R_1 and R_2 are commutative semirings with identity that are not semifields is given. If $R_1, R_2, ..., R_n$ are commutative semirings with identity for some $2 \leq n < \infty$, and let $R = R_1 \times ..... \times R_n$, then it is shown in (Theorem 4.2) that every proper ideal of R is a weakly 1-absorbing primary ideal of R if and only if $n = 2$ and R_1, R_2 are semifields. For a weakly 1-absorbing primary ideal of a semiring R, we show (Theorem 4.8) that $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$ for every multiplicatively closed subset S of R that is disjoint from I, and we show that the converse holds if $S \cap Z(R) = \phi$ and $S \cap Z_I(R) = \phi$.

2. Properties of Weakly 1 -absorbing Primary Ideals

In this section, we will define some basic properties of weakly 1-absorbing primary ideals in a commutative semi-ring R.

Definition 2.1. *Let R be a commutative semiring, and I a proper ideal of R. We call I a weakly 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$.*

Definition 2.2. *Let R be a commutative semiring, and I a proper ideal of R. We call I a 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$ and $abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$.*

It is clear that every 1-absorbing primary ideal of a semiring R is a weakly 1-absorbing primary ideal of R.
The following example shows that the converse is not true.

Example 2.3. *1. $I = \{0\}$ is a weakly 1-absorbing primary ideal of $R = Z_6$ that is not a 1-absorbing primary of R. Indeed, $2.2.3 \in I$, but neither $2.2 \in I$ nor $3 \in \sqrt{I}$.*

2. Let $J = \{0,6\}$ as an ideal of Z_{12}, and let $R = Z_{12}(+)J$. Then an ideal $I = \{(0,0),(0,6)\}$ is a weakly 1-absorbing primary ideal of R. Observe that $abc \in I$ for some $a,b,c \in R \mid I$ if and only if $abc = (0,0)$. However, it is not a 1-absorbing primary ideal of R. Indeed; $(2,0)(2,0)(3,0) \in I$, but neither $(2,0)(2,0) \in I$ nor $(3,0) \in \sqrt{I}$.

We begin with the following trivial result:

Theorem 2.4. *Let be a proper ideal of a commutative semiring R. Then the following statements hold.*

1. *If I is a weakly prime ideal, then I is a weakly 1-absorbing primary ideal.*

2. *If I is a weakly primary ideal, then I is a weakly 1-absorbing primary ideal.*

3. *If I is a 1-absorbing primary ideal, then I is a weakly 1-absorbing primary ideal.*

4. *If I is a weakly 1-absorbing primary ideal, then I is a weakly 2-absorbing primary ideal.*

5. *If R/I is an semi-integral domain, then I is a weakly 1-absorbing primary ideal if and only if I is a 1-absorbing primary ideal of R.*

6. *Let R be a quasilocal semiring with maximal ideal $\sqrt{0}$. Then every proper ideal of R is a weakly 1-absorbing primary ideal of R.*

Theorem 2.5. *Let R be a semiring and I be a weakly 1-absorbing primary ideal of R. If $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is a 1-absorbing ideal primary of R.*
In particular, If I a weakly 1-absorbing primary ideal of R that is not a 1-absorbing ideal primary of R, then is not a maximal ideal of R.

Proof. Suppose that $\sqrt{I}$ is a maximal ideal of R. Then I is a semiprimary ideal of R. by [21] since I. Now, assume nonunit elements $a,b,c \in R$ and $abc \in I$. Assume ab not belong I . Since I is primary ideal, we have for some positive integer m, we have $c \in \sqrt{I}$. Hence, I is 1-absorbing primary ideal. □

Theorem 2.6. *Let R be a reduced semiring. If I is a nonzero weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is a prime ideal of R. In particular, if $\sqrt{Ii}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is a 1-absorbing primary ideal of R.*

Proof. Proof: Suppose that $0 \neq ab \in \sqrt{I}f$, for some $a,b \in R$. We may assume that a,b are nonunit. Then there exists an even positive integer $n = 2m(m \geq 1)$ such that $(ab)^n \in I$. Since $\sqrt{0} = \{0\}$, we have $(ab)^n \neq 0$. Hence, $0 \neq a^m a^m b^n \in I$. Thus, $a^m a^m = a^n \in I$ or $b^n \in \sqrt{I}$, and therefore $\sqrt{I}$ is a weakly prime ideal of R. Since R is reduced and $I \neq \{0\}$, we conclude that $\sqrt{I}$ is a prime ideal of R by [2] . The proof of the "in particular" statement : by Theorem 2, $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is a 1-absorbing ideal primary of R. □

Recall that a commutative semiring R is called a von-Neumann regular semiring if and only if for every $x \in R$, there is a $Y \in y$ such that $x^2 y = x$. It is known that a commutative semiring R is a von-Neumann regular semiring if and only if for each $x \in R$, there is an idempotent $e \in R$ and a unit $u \in R$ such that $x = eu$. We have the following result.

Theorem 2.7. *Let R be a von-Neumann regular semiring and I be a nonzero ideal of R. Then the following statements are equivalent.*

1. *I is a weakly 1-absorbing primary ideal of R.*

2. *I is a primary ideal of R.*

3. *I is a 1-absorbing ideal primary of R.*

Proof. $(1) \Rightarrow (2)$. R is a von-Neumann regular semiring, we know that R is reduced. Hence $\sqrt{I}$ is a prime ideal of R by Theorem 2.6. Since every prime ideal of a von-Neumann regular semiring is maximal, we conclude that $\sqrt{I}$ is a maximal ideal of R. Hence I is a primary ideal of R by Theorem 2.5.
$(2) \Rightarrow (3)$. Let nonunit elements $a,b,c \in R$, and $abc \in I$. Assume ab not belong I. Since I is a primary ideal, we have $c^m \in I$ for some positive integer m, so $c \in \sqrt{I}$. Thus, I is a 1-absorbing primary ideal.
$(3) \Rightarrow (1)$. Let nonunit elements $a,b,c \in R$, and $0 \neq abc \in I$. Since I is a 1-absorbing primary ideal, we have $ab \in I$, or $c \in \sqrt{I}$. Now, if a,b and $c \neq 0$, then $0 \neq abc \in I$. As a result I is a weakly 1-absorbing primary ideal. □

Theorem 2.8. *Let R be a non-quasilocal semiring and I be a k-ideal of R such that $ann(i) = \{r \in R \mid ri = 0\}$ is not a maximal ideal of R for every element $i \in I$. Then I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R.*

Proof. If I is a weakly primary ideal of R, then I is a weakly 1-absorbing primary ideal of R by Theorem 2.4. Now, suppose that I is a weakly 1-absorbing primary k-ideal of R and suppose that $0 \neq ab \in I$ for some elements $a,b \in R$. We show that $a \in I$ or $b \in \sqrt{I}$. We may assume that a,b are nonunit elements of R. Let $ann(ab) = \{c \in R \mid cab = 0\}$. Since $ab \neq 0$, $ann(ab)$ is a proper ideal of R. Let L be a maximal ideal of R such that $ann(ab) \subseteq L$. Since R is a non-quasilocal semiring, there is a maximal ideal M of R such that $M \neq L$. Let $m \in M \setminus L$. Hence m not belong to $ann(ab)$, and $0 \neq mab \in I$. Since I is a weakly 1-absorbing primary ideal of R, we have $ma \in I$ or $b \in \sqrt{I}$. If $b \in \sqrt{I}$, then we are done. Hence assume that b not belong to $\sqrt{I}$.

Hence $ma \in I$. Since m not belong to L and L is a maximal ideal of R, we conclude that m not belong to $J(R)$. Hence there exists an $r \in R$ such that $1+rm$ is a nonunit element of R. Suppose that $1+rm$ not belong to $ann(ab)$. Hence $0 \neq (1+rm)ab \in I$. Since I is a weakly 1-absorbing primary k-ideal of R and b not belong to $\sqrt{I}$, we conclude that $(1+rm)a = a+rma \in I$. Since $rma \in I$, we have $a \in I$ and we are done. Suppose that $1+rm \in ann(ab)$. Since $ann(ab)$ is not a maximal ideal of R and $ann(ab) \subseteq L$, there is a $w \in L \setminus ann(ab)$. Hence $0 \neq wab \in I$. Since I is a weakly 1-absorbing primary k-ideal of R and b not belong to $\sqrt{I}$, we conclude that $wa \in I$. Since $1+rm \in ann(ab) \subseteq L$ and $w \in L \setminus ann(ab)$, we have $1+rm+w$ is a nonzero nonunit element of L. Hence $0 \neq (1+rm+w)ab \in I$. Since I is a weakly 1-absorbing primary k-ideal of R and b not belong $\sqrt{I}$, we conclude that $(1+rm+w)a = a+rma+wa \in I$. Since $rma, wa \in I$, we conclude that $a \in I$. □

In light of the proof of Theorem 2.8, we have the following result.

Theorem 2.9. *Let I be a weakly 1-absorbing primary k-ideal of R such that for every nonzero element $i \in I$, there exists a nonunit $w \in R$ such that $wi \neq 0$, and $w+u$ is a nonunit element of R for some unit $u \in R$. Then I is a weakly primary k-ideal of R.*

Proof. Suppose that $0 \neq ab \in I$ and b not belong to $\sqrt{I}$ for some $a,b \in R$. We may assume that a,b are nonunit elements of R. Hence there is a nonunit $w \in R$ such that $wab \neq 0$ and $w+u$ is a nonunit element of R for some unit $u \in R$. Since $0 \neq wab \in I$ and b not belong to $\sqrt{I}$ and I is a weakly 1-absorbing primary k-ideal of R, we conclude that $wa \in I$.

Since $(w+u)ab \in I$ and I is a weakly 1-absorbing primary k-ideal of R and b not belong $\sqrt{I}$, we conclude that $(w+u)a = wa+ua \in I$. Since $wa \in I$ and $wa+ua \in I$, we conclude that $ua \in I$. Since u is a unit, we have $a \in I$. □

Corollary 2.10. *Let R be a semiring and $A = R[x]$. Suppose that I is a weakly 1-absorbing primary k-ideal of A. Then I is a weakly primary k-ideal of A.*

Proof. Since $xi \neq 0$ for every nonzero $i \in I$ and $x+1$ is a nonunit element of A, we are done by Theorem 2.9. □

Recall that a semiring R is called divided if for every prime ideal P of R and for every $x \in R \setminus P$, we have $x \mid p$ for every $p \in P$. We have the following result.

Theorem 2.11. *Let R be a reduced divided semiring and I be a proper ideal of R. Then the following statements are equivalent:*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *I is a weakly primary ideal of R.*

Proof. $(1) \Rightarrow (2)$. Suppose that $0 \neq ab \in I$ for some $a,b \in R$ and b not belong to $\sqrt{I}$. We may assume that a,b are nonunit elements of R. Since $\sqrt{I}$ is a prime ideal of R by Theorem 2.6, we conclude that $a \in \sqrt{I}$. Since R is divided, we conclude that $b \mid a$. Thus $a = bc$ for some $c \in R$. Observe that c is a nonunit element of R as b not belong to $\sqrt{I}$ and $a \in \sqrt{I}$. Since $0 \neq ab = bcb \in I$ and I is weakly 1-absorbing primary, and b not belong to $\sqrt{I}$, we conclude that $bc = a \in I$. Thus I is a weakly primary ideal of R.

$(2) \Rightarrow (1)$. It is clear by Theorem 2.4. □

Recall that a semiring R is called a chained semiring if for every $x,y \in R$, we have $x \mid y$ or $y \mid x$. Every chained semiring is divided. So, if R is a reduced chained semiring, then a proper ideal I of R is a weakly 1-absorbing primary ideal if and only if it is a weakly primary ideal of R.

Theorem 2.12. *Let R be a semiDedekind domain and I be a nonzero proper ideal of R. Then I is a weakly 1-absorbing primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R.*

Proof. $(\rightarrow)$. Suppose that I is a weakly 1-absorbing primary ideal of R. Then $\sqrt{I}$ is a prime ideal of R by Theorem 2.6.

$(\leftarrow)$. Suppose $\sqrt{I}$ is a prime ideal of R. Since R is a semiDedekind domain, it is well known that every nonzero prime ideal of R is a maximal ideal of R. Thus $\sqrt{I}$ is a maximal ideal of R. Hence I is a primary ideal of R, and thus I is 1-absorbing primary ideal of R. □

3. Characterizations of Weakly 1-absorbing Primary Ideals in u-semirings

In this section, we will study some characterizations of weakly 1-absorbing primary ideals in u-semirings

Definition 3.1. *If an ideal of R contained in a finite union of ideals must be contained in one of those ideals, then R is said to be a u-semiring.*

Theorem 3.2. *Let R be a commutative u-semiring, and I a proper ideal of R. Then the following statements are equivalent.*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *For every nonunit elements $a,b \in R$ with ab not belong to I, $(I:ab) = (0:ab)$, or $(I:ab) \subseteq \sqrt{I}$.*
3. *For every nonunit element $a \in R$, and every ideal I_1 of R with $I_1 \nsubseteq \sqrt{I}$. If $(I:aI_1)$ is a proper ideal of R, then $(I:aI_1) = (0:aI_1)$, or $(I:aI_1) \subseteq (I:a)$.*
4. *For every ideals I_1, I_2 of R with $I_1 \nsubseteq \sqrt{I}$. If $(I:I_1I_2)$ is a proper ideal of R, then $(I:I_1I_2) = (0:I_1I_2)$, or $(I:I_1I_2) \subseteq (I:I_2)$.*
5. *For every ideals I_1, I_2, I_3 of R with $0 \neq I_1I_2I_3 \subseteq II_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.*

Proof. $(1) \Rightarrow (2)$. Suppose that I is a weakly 1-absorbing primary ideal of R, ab not belong to I for some nonunit elements $a,b \in R$ and $c \in (I:ab)$. Then $abc \in I$. Since ab not belong to I, c is nonunit. If $abc = 0$, then $c \in (0:ab)$. Assume that $0 \neq abc \in I$. Since I is weakly 1-absorbing primary, we have $c \in \sqrt{I}$. Hence we conclude that $(I:ab) \subseteq (0:ab) \cup \sqrt{I}$. Since R is a u-semiring, we obtain that $(I:ab) = (0:ab)$ or $(I:ab) \subseteq \sqrt{I}$.
$(2) \Rightarrow (3)$. If $aI_1 \subseteq I$, then we are done. Suppose that $aI_1 \nsubseteq I$ for some nonunit element $a \in R$ and $c \in (I:aI_1)$. It is clear that c is nonunit. Then $acI_1 \subseteq I$. Now $I_1 \subseteq (I:ac)$. If $ac \in I$, then $c \in (I:a)$. Suppose that ac not belong to I. Hence $(I:ac) = (0:ac)$ or $(I:ac) \subseteq \sqrt{I}$ by 2. Thus $I_1 \subseteq (0:ac)$ or $I_1 \subseteq \sqrt{I}$. Since $I_1 \nsubseteq I$ by hypothesis, we conclude $I_1 \subseteq (0:ac)$; i.e. $c \in (0:aI_1)$. Thus $(I:aI_1) \subseteq (0:aI_1) \cup (I:a)$. Since R is a u-semiring, we have $(I:aI_1) = (0:aI_1)$ or $(I:aI_1) \subseteq (I:a)$.

$(3) \Rightarrow (4)$. If $I_1 \subseteq \sqrt{I}$, then we are done. Suppose that $I_1 \nsubseteq \sqrt{I}$ and $c \in (I:I_1I_2)$. Then $I_2 \subseteq (I:cI_1)$. Since $(I:I_1I_2)$ is proper, c is nonunit. Hence $I_2 \subseteq (0:cI_1)$ or $I_2 \subseteq (I:c)$ by 2.6. If $I_2 \subseteq (0:cI_1)$, then $c \in (I:I_1I_2)$. If $I_2 \subseteq (I:c)$, then $c \in (I:I_2)$. So, $(I:I_1I_2) \subseteq (0:I_1I_2) \cup (I:I_2)$ which implies that $(I:I_1I_2) = (0:I_1I_2)$, or $(I:I_1I_2) \subseteq (I:I_2)$, as needed.

$(4) \Rightarrow (5)$. It is clear.

$(5) \Rightarrow (1)$. Let $a,b,c \in R$ be nonunit elements and $0 \neq abc \in I$. Put $I_1 = aR$, $I_2 = bR$, and $I_3 = cR$. Then 1 is now clear by 5 □

Definition 3.3. *Let I be a weakly 1-absorbing primary ideal of R and a,b,c be nonunit elements of R. We call (a,b,c) a 1-triple-zero of I if $abc = 0$, ab not belong to I, and c not belong to $\sqrt{I}$.*

Observe that if I is a weakly 1-absorbing primary ideal of R that is not 1- absorbing primary, then there exists a 1-triple-zero (a,b,c) of I for some nonunit elements $a,b,c \in R$.

Theorem 3.4. *Let I be a weakly 1-absorbing primary k-ideal of R, and (a,b,c) be a 1-triple-zero of I. Then*

1. *$abI = 0$.*
2. *If a,b not belong to $(I:c)$, then $bcI = acI = aI^2 = bI^2 = cI^2 = 0$.*
3. *If a,b not belong to $(I:c)$, then $I^3 = 0$.*

Proof.
1. Suppose that $abI \neq 0$. Then $abx \neq 0$ for some nonunit $x \in I$. Hence $0 \neq ab(c+x) \in I$. Since ab not belong to I, $(c+x)$ is nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R and ab not belong to I, we conclude that $(c+x) \in \sqrt{I}$. Since $x \in I$, we have $c \in \sqrt{I}$, a contradiction. Thus $abI = 0$.
2. Suppose that $bcI \neq 0$. Then $bcy \neq 0$ for some nonunit element $y \in I$. Hence $0 \neq bcy = b(a+y)c \in I$. Since b not belong to $(I:c)$, we conclude that $a+y$ is a nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R and $ab \in I$ and $by \in I$, we conclude that $b(a+y)$ not belong to I, and hence $c \in \sqrt{I}$, a contradiction. Thus $bcI = 0$. We show that $acI = 0$. Suppose that $acI \neq 0$. Then $acy \neq 0$ for some nonunit element $y \in I$. Hence $0 \neq acy = a(b+y)c \in I$. Since a not belong to $(I:c)$, we conclude that $b+y$ is a nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R and ab not belong to I and $ay \in I$, we conclude that $a(b+y)$ not belong to I, and hence $c \in \sqrt{I}$, a contradiction. Thus $bcI = 0$. We show that $acI = 0$. Suppose that $acI \neq 0$. Then $acy \neq 0$ for some nonunit element $y \in I$. Hence $0 \neq acy = a(b+y)c \in I$. Since a not belong to $(I:c)$, we conclude that $b+y$ is a nonunit element of R. Since I is a

weakly 1-absorbing primary k-ideal of R and ab not belong to I and $ay \in I$, we conclude that $a(b+y)$ not belong to I, and hence $c \in \sqrt{I}$, a contradiction.

Thus $acI = 0$. Now we prove that $aI^2 = 0$. Suppose that $axy \neq 0$ for some $x, y \in I$. Since $abI = 0$ by (1) and $acI = 0$ by (2), $0 \neq axy = a(b+x)(c+y) \in I$.

Since ab not belong to I, we conclude that $c+y$ is a nonunit element of R. Since a not belong to $(I : c)$, we conclude that $b+x$ is a nonunit element of R. Since I is a weakly 1-absorbing Primary k-ideal of R, we have $a(b+x) \in I$ or $(c+y) \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $aI^2 = 0$. We show $bI^2 = 0$. Suppose that $bxy \neq 0$ for some $x, y \in I$. Since $abI = 0$ by (1) and $bcI = 0$ by (2), $bxy = b(a+x)(c+y) \in I$. Since ab not belong to I, we conclude that $c+y$ is a nonunit element of R. Since b not belong to $(I : c)$, we conclude that $a+x$is a nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R, we have $b(a+x) \in I$ or $(c+y) \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction.Thus $bI^2 = 0$. We show $cI^2 = 0$.

Suppose that $cxy \neq 0$ for some $x, y \in I$. Since $acI = bcI = 0$ by (2), $0 \neq cxy = (a+x)(b+y)c \in I$. Since a, b not belong to $(I : c)$, we conclude that $a+x$ and $b+y$ are nonunit elements of R. Since I is a weakly 1-absorbing primary k-ideal of R, we have $(a+x)(b+y) \in I$ or $c \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $cI^2 = 0$.

3. Assume that $xyz \neq 0$ for some $x, y, z \in I$. Then $0 \neq xyz = (a+x)(b+y)(c+z) \in I$ by (1) and (2). Since ab not belong to I, we conclude $c+z$ is a nonunit element of R. Since a, b not belong to $(I : c)$, we conclude that $a+x$ and $b+y$ are nonunit elements of R. Since I is a weakly 1-absorbing primary k-ideal of R, we have $(a+x)(b+y) \in I$ or $c+z \in \sqrt{I}$. Since $x, y, z \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $I^3 = 0$.

□

Theorem 3.5. *1. Let I be a weakly 1-absorbing primary k-ideal of a reduced semiring R. Suppose that I is not a 1-absorbing ideal primary ideal of R and (a, b, c) is a 1-triple-zero of I such that a, b not belong to $(I : c)$. Then $I = 0$.*

2. Let I be a nonzero weakly 1-absorbing primary k-ideal of a reduced semiring R. Suppose that I is not a 1-absorbing ideal primary ideal of R and (a, b, c) is a 1-triple-zero of I. Then $ac \in I$ or $bc \in I$.

Proof. 1. Since a, b not belong to $(I : c)$, then $I^3 = 0$ by Theorem 3.4. Since R is reduced, we conclude that $I = 0$.

2. Suppose that neither $ac \in I$ nor $bc = 0$. Then $I = 0$ by (1), a contradiction, since I is a nonzero ideal of R by hypothesis. Hence if (a, b, c) is a 1-triple-zero of I, then $ac \in I$ or $bc \in I$.

□

Theorem 3.6. *Let I be a weakly 1-absorbing primary ideal of R. If I is not a weakly primary ideal of R, then there exist an irreducible element $x \in R$ and a nonunit element $y \in R$ such that $xy \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Furthermore, if $ab \in I$ for some nonunit elements $a, b \in R$ such that neither $a \in I$ nor $b \in \sqrt{I}$, then a is an irreducible element of R.*

Proof. Suppose that I is not a weakly primary ideal of R. Then there exist nonunit elements $x, y \in R$ such that $0 \neq xy \in I$ with x not belong to I, y not belong to $\sqrt{I}$. Suppose that x is not an irreducible element of R. Then $x = cd$ for some nonunit elements $c, d \in R$. Since $0 \neq xy = cdy \in I$ and I is weakly 1-absorbing primary and y not belong to $\sqrt{I}$, we conclude that $cd = x \in I$, a contradiction. Hence x is an irreducible element of R. □

In general, the intersection of a family of weakly 1-absorbing primary ideals need not be a weakly 1-absorbing primary ideal.

Example 3.7. *consider the semiring $R = Z_6$. Then $I = (2)$ and $J = (3)$ are clearly weakly 1-absorbing primary ideals of Z_6 but $I \cap J = 0$ is not a weakly 1-absorbing primary ideal of R.*

However, we have the following result.

Proposition 3.8. *Let $\{I_i : i \in \wedge\}$ be a collection of weakly 1-absorbing primary ideals of R such that $Q = \sqrt{I_i} = \sqrt{I_j}$ for every distinct $i, j \in \wedge$. Then $I = \cap_{i \in \wedge} I_i$ is a weakly 1-absorbing primary ideal of R.*

Proof. Suppose that $0 \neq abc \in I = \cap_{i \in \wedge} I_i$ for nonunit elements $a, b, c \in R$ and ab not belong to I. Then for some $k \in \wedge$, $0 \neq abc \in I_k$ and ab not belong to I_k. It implies that $c \in \sqrt{I_k} = Q = \sqrt{I}$. □

Proposition 3.9. *Let I be a weakly 1-absorbing primary ideal of R and c be a nonunit element of $R \setminus I$. Then $(I : c)$ is a weakly primary ideal of R.*

Proof. Suppose that $0 \neq ab \in (I:c)$ for some nonunit $c \in R \setminus I$ and assume that a not belong to $(I:c)$. Hence b is a nonunit element of R. If a is unit, then $b \in (I:c) \subseteq \sqrt{(I:c)}$, and we are done. So assume that a is a nonunit element of R. Since $0 \neq abc = acb \in I$ and ac not belong to I and I is a weakly 1-absorbing primary ideal of R, we conclude that $b \in \sqrt{I} \subseteq \sqrt{(I:c)}$. Thus, $(I:c)$ is a weakly primary ideal of R. □

4. Characterization for Weakly 1-absorbing Primary Ideal of $R = R_1 \times R_2$

The next theorem gives a characterization for weakly 1-absorbing primary ideals of $R = R_1 \times R_2$ where R_1 and R_2 are commutative semirings with identity that are not semifields

Theorem 4.1. *Let R_1 and R_2 be commutative semirings with identity that are not semifields, and let $R = R_1 \times R_2$ and I be a a nonzero proper ideal of R. Then the following statements are equivalent.*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *$I = I_1 \times R_2$ for some primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some primary ideal I_2 of R_2.*
3. *I is a 1-absorbing primary ideal of R.*
4. *I is a primary ideal of R_1.*

Proof. $(1) \Rightarrow (2)$. Suppose that I is a weakly 1-absorbing primary ideal of R. Then I is of the form $I_1 \times I_2$ for some ideals I_1 and I_2 of R_1 and R_2, respectively. Assume that both I_1 and I_2 are proper. Since I is a nonzero ideal of R, we conclude that $I_1 \neq 0$ or $I_2 \neq 0$. We may assume that $I_1 \neq 0$. Let $0 \neq c \in I_1$ Then $0 \neq (1,0)(1,0)(c,1) = (c,0) \in I_1 \times I_2$. It implies that $(1,0)(1,0) \in I_1 \times I_2$ or $(c,1) \in \sqrt{(I_1 \times I_2)} = \sqrt{I_1} \times \sqrt{I_2}$, that is $I_1 = R_1$ or $I_2 = R_2$, a contradiction. Thus either I_1 or I_2 is a proper ideal. Without loss of generality, assume that $I = I_1 \times R_2$ for some proper ideal I_1 of R_1. We show that I_1 is a primary ideal of R_1. Let $ab \in I_1$ for some $a, b \in R_1$. We can assume that a and b are nonunit elements of R_1. Since R_2 is not a semifield, there exists a nonunit nonzero element $x \in R_2$. Then $0 \neq (a,1)(1,x)(b,1)\ I_1 \times R_2$ which implies that either $(a,1)(1,x) \in I_1 \times R_2$ or $(b,1) in \sqrt{I_1 \times R_2} = \sqrt{I}_1 \times R_2$; i.e., $a \in I_1$ or $b \in \sqrt{I_1}$.
$(2) \Rightarrow (3)$. Since I is a primary ideal of R, I is a 1-absorbing primary ideal of R by [[9], Theorem (1)].
$(3) \Rightarrow (4)$ Since I a 1-absorbing primary ideal of R and R is not a quasilocal semring, we conclude that I is a primary ideal of R by [9, Theorem(3)].
$(4) \Rightarrow (1)$ Let nonunit elements $a, b, c \in R$, and $0 \neq abc \in I$. Assume ab not belong to I. Since I is primary ideal, we have $c^m \in I$ for some positive integer m, so $c \in \sqrt{I}$. So I is a weakly 1-absorbing primary ideal. □

Theorem 4.2. *Let $R_1, \ldots, R_n$ be commutative semirings with $1 \neq 0$ for some $2 \leq n < \infty$, and let $R = R_1 \times \ldots\ldots \times R_n$. Then the following statements are equivalent.*

1. *Every proper ideal of R is a weakly 1-absorbing primary ideal of R.*
2. *$n = 2$ and R_1, R_2 are semifields.*

Proof. $(1) \Rightarrow (2)$. Suppose that every proper ideal of R is a weakly 1-absorbing primary ideal. Without loss of generality, we may assume that $n = 3$. Then $I = R_1 \times \{0\} \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $a \in R_1$, we have$(0,0,0) \neq (1,0,1)(1,0,1)(a,1,0) = (a,0,0) \in I$, but neither $(1,0,1)(1,0,1) \in I$ nor $(a,1,0) \in \sqrt{I}$, a contradiction. Thus $n = 2$. Assume that R_1 is not a semifield. Then there exists a nonzero proper ideal A of R_1. Hence $I = A \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $a \in A$, we have $(0,0) \neq (1,0)(1,0)(a,1) = (a,0) \in I$, but neither$(1,0)(1,0) \in I$ nor $(a,1) \in \sqrt{I}$, a contradiction. And, assume that R_2 is not a semifield. Then there exists a nonzero proper ideal B of R_2. Hence $I = B \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $b \in B$, we have $(0,0) \neq (1,0)(1,0)(b,1) = (a,0) \in I$, but neither $(1,0)(1,0) \in I$ nor $(a,1) \in \sqrt{I}$, a contradiction. Hence $n = 2$ and R_1, R_2 are semifields.

$(2) \Rightarrow (1)$. Suppose that $n = 2$ and R_1, R_2 are semifields. Then R has exactly three proper ideals, i.e., $\{(0,0)\}, \{0\} \times R_2$ and $R_1 \times \{0\}$ are the only proper ideals of R. Hence it is clear that each proper ideal of R is a weakly 1-absorbing primary ideal of R. □

Since every semiring that is a product of a finite number of fields is a von-Neumann regular semiring, in light of Theorem 4 and Theorem 14 we have the following result.

Corollary 4.3. *Let $R_1, \ldots, R_n$ be commutative semirings with $1 \neq 0$ for some $2 \leq n < \infty$, and let $R = R_1 \times \ldots.. \times R_n$. Then the following statements are equivalent.*

1. Every proper ideal of R is a weakly 1-absorbing primary ideal of R.

2. Every proper ideal of R is a weakly primary ideal of R.

3. $n = 2$ and R_1, R_2 are semifields, and hence $R = R_1 \times R_2$ is a von-Neumann regular semiring.

Theorem 4.4. *Let R_1 and R_2 be commutative semirings and $f : R_1 \to R_2$ be a semiring homomorphism with $f(1) = 1$. Then the following statements hold:*

1. *Suppose that f is a monomorphism and $f(a)$ is a nonunit element of R_2 for every nonunit element $a \in R_1$ and J is a weakly 1-absorbing primary ideal of R_2. Then $f^{(-1)}(J)$ is a weakly 1-absorbing primary ideal of R_1.*

2. *If f is an epimorphism and I is a weakly 1-absorbing primary ideal of R_1 such that $Ker(f) \subseteq I$, then $f(I)$ is a weakly 1-absorbing primary ideal of R_2.*

Proof. . (1) Let $0 \neq abc \in f^{(-1)}(J)$ for some nonunit elements $a, b, c \in R$. Since $Ker(f) = 0$, we have $0 \neq f(abc) = f(a)f(b)f(c) \in J$, where $f(a), f(b), f(c)$ are nonunit elements of R_2 by hypothesis. Hence $f(a)f(b) \in J$ or $f(c) \in \sqrt{J}$. Hence $ab \in f^{(-1)}(J)$ or $c \in \sqrt{(f^(-1)(J))} = f^{(-1)}(\sqrt{J})$. Thus $f^{(-1)}(J)$ is a weakly 1-absorbing primary ideal of R_1.

Let $0 \neq xyz \in f(I)$ for some nonunit elements $x, y, z \in R$. Since f is onto, there exists nonunit elements $a, b, c \in I$ such that $x = f(a), y = f(b), z = f(c)$. Then $f(abc) = f(a)f(b)f(c) = xyz \in f(I)$. Since $Ker(f) \subseteq I$, we have $0 \neq abc \in I$. It follows $ab \in I$ or $c \in \sqrt{I}$. Thus $xy \in f(I)$ or $z \in f(\sqrt{I})$. Since f is onto and $Ker(f) \subseteq I$, we have $f(\sqrt{I}) = \sqrt{(f(I))}$. Thus we are done. □

Example 4.5. *Let$A = K[x, y]$, where K is a semifield, $M = (x, y)A$, and $B = A_M$. Note that B is a quasilocal semiring with maximal ideal M_M. Then $I = xM_M = (x^2, xy)B$ is a 1-absorbing primary ideal of B and $\sqrt{I} = xB$. However $xy \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Thus I is not a primary ideal of B. Let $f : B \times B \to B$ such that $f(x, y) = x$. Then f is a semiring homomorphism from $B \times B$ onto B such that $f(1, 1) = 1$. However, $(1, 0)$ is a nonunit element of $B \times B$ and $f(1, 0) = 1$ is a unit of B. Thus f does not satisfy the hypothesis of 4.4. Now $f^{(-1)}(I) = I \times B$ is not a weakly 1-absorbing ideal of $B \times B$ by 4.1.*

Theorem 4.6. *Let I be a proper ideal of R. Then the following statements hold.*

1. *If J is a proper ideal of a semiring R with $J \subseteq I$ and I is a weakly 1-absorbing primary ideal of R, then I/J is a weakly 1-absorbing primary ideal of R/J.*

2. *If J is a proper ideal of a semiring R with $J \subseteq I$ such that $U(R/J) = \{a + J \mid a \in U(R)\}$. If J is a 1-absorbing primary ideal of R and I/J is a weakly 1-absorbing primary ideal of R/J, then I is a 1-absorbing primary ideal of R.*

3. *If $\{0\}$ is a 1-absorbing primary ideal of R and I is a weakly 1-absorbing primary ideal of R, then I is a 1-absorbing primary ideal of R.*

4. *If J is a proper ideal of a ring R with $J \subseteq I$ such that $U(R/J) = \{a + J \mid a \in U(R)\}$. If J is a weakly 1-absorbing primary ideal of R and I/J is a weakly 1-absorbing primary ideal of R/J, then I is a weakly 1-absorbing primary ideal of R.*

Proof.

1. Consider the natural epimorphism $\pi : R \to R/J$. Then $\pi(I) = I/J$. So we are done by Theorem 1.

2. Suppose that $abc \in I$ for some nonunit elements $a, b, c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \sqrt{J} \subseteq \sqrt{I}$ as J is a 1-absorbing primary ideal of R. Now assume that abc not belong to J. Then $J \neq (a + J)(b + J)(c + J) \in I/J$, where $a + J, b + J, c + J$ are nonunit elements of R/J by hypothesis. Thus $(a + J)(b + J) \in I/J$ or $(c + J) \in \sqrt{(I/J)}$. Hence $ab \in I$ or $c \in \sqrt{I}$.

3. The proof follows from (2).

4. Suppose that $0 \neq abc \in I$ for some nonunit elements $a, b, c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \sqrt{J} \subseteq \sqrt{I}$ as J is a weakly 1-absorbing primary ideal of R. Now assume that abc not belong to J. Then $J \neq (a + J)(b + J)(c + J) \in I/J$, where $a + J, b + J, c + J$ are nonunit elements of R/J by hypothesis. Thus $(a + J)(b + J) \in I/J$ or $(c + J) \in \sqrt{(I/J)}$. Hence $ab \in I$ or $c \in \sqrt{I}$.

□

Proposition 4.7. *1. Let R_1 and R_2 be commutative semirings and $f : R_1 \to R_2$ be a ring homomorphism with $f(1) = 1$ such that R_2 is not a quasilocal semiring, then $f(a)$ is a nonunit element of R_2 for every nonunit element $a \in R_1$ and J is a 1-absorbing primary ideal of R_2. Then $f^{(-1)}(J)$ is a 1-absorbing primary ideal of R_1.*

2. *Let I and J be proper ideals of a semiring R with $I \subseteq J$. If J is a 1-absorbing primary ideal of R, then J/I is a 1-absorbing primary ideal of R/I. Furthermore, assume that if R/I is a quasilocal semiring, then $U(R/I) = a+I \mid a \in U(R)$. If J/I is a 1-absorbing primary ideal of R/I, then J is a 1-absorbing primary ideal of R.*

3. *Let R be a semiring and $A = R[x]$. Then a proper ideal I of R is a 1-absorbing primary ideal of R if and only if $(I[x]+xA)/xA$ is a 1-absorbing primary ideal of A/xA, since R is semiring-isomorphic to A/xA.*

Theorem 4.8. *Let S be a multiplicatively closed subset of R, and I a proper ideal of R. Then the following statements hold.*

1. *If I is a weakly 1-absorbing primary ideal of R such that $I \cap S = \phi$, then $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$.*

2. *If $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$ such that $S \cap Z(R) = \phi$ and $S \cap Z_I(R) = \phi$, then I is a weakly 1-absorbing primary ideal of R.*

Proof. 1. Suppose that $0 \neq \frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{(-1)}I$ for some nonunit $a,b,c \in R \setminus S$, $s_1,s_2,s_3 \in S$ and $\frac{a}{s_1}\frac{b}{s_2}$ not belong to $S^{(-1)}I$. Then $0 \neq uabc \in I$ for some $u \in S$. Since I is weakly 1-absorbing primary and uab not belong to I, we conclude $c \in \sqrt{I}$. Thus $\frac{c}{s_3} \in S^{(-1)}\sqrt{I} = \sqrt{(S^{(-1)}I)}$. Thus $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$.

2. Suppose that $0 \neq abc \in I$ for some nonunit elements $a,b,c \in R$. Hence $0 \neq \frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{(-1)}I$ as $S \cap Z(R) = \phi$. Since $S^{(-1)}I$ is weakly 1-absorbing primary, we have either $\frac{a}{1}\frac{b}{1} \in S^{(-1)}I$, or $\frac{c}{1} \in \sqrt{S^{(-1)}I} = S^{-1}\sqrt{I}$. If $\frac{a}{1}\frac{b}{1} \in S^{(-1)}I$, then $uab \in I$ for some $u \in S$. Since $S \cap Z_I(R) = \phi$, we conclude that $ab \in I$. If $\frac{c}{1} \in S^{-1}\sqrt{I}$, then $(tc)^n \in I$ for some positive integer $n \geq 1$ and $t \in S$. Since t^n not belong to $Z_I(R)$, we have $c^n \in I$, i.e., $c \in \sqrt{I}$. Thus I is a weakly 1-absorbing primary ideal of R.

□

Definition 4.9. *Let I be a weakly 1-absorbing primary ideal of R and $I_1I_2I_3 \subseteq I$ for some proper ideals I_1,I_2,I_3 of R. If (a,b,c) is not 1-triple zero of I for every $a \in I_1$, $b \in I_2, c \in I_3$, then we call I a free 1-triple zero with respect to $I_1I_2I_3$.*

Theorem 4.10. *Let I be a weakly 1-absorbing primary ideal of R and J be a proper ideal of R with $abJ \subseteq I$ for some $a,b \in R$. If (a,b,j) is not a 1-triple zero of I for all $j \in J$and ab not belong to I, then $J \subseteq \sqrt{I}$.*

Proof. Suppose that $J \nsubseteq \sqrt{I}$. Then there exists $c \in J \setminus \sqrt{I}$. Then $abc \in abJ \subseteq I$. If $abc \neq 0$, then it contradicts our assumption that ab not belong to I and c not belong to $\sqrt{I}$. Thus $abc = 0$. Since (a,b,c) is not a 1-triple zero of I and ab not belong to I, we conclude $c \in \sqrt{I}$, a contradiction. Thus $J \subseteq \sqrt{I}$.

□

Theorem 4.11. *Let I be a weakly 1-absorbing primary ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some proper ideals I_1,I_2,I_3 of R. If I is free 1-triple zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.*

Proof. Suppose that I is free 1-triple zero with respect to $I_1I_2I_3$, and $0 \neq I_1I_2I_3 \subseteq I$. Assume that $I_1I_2 \nsubseteq I$. Then there exist $a \in I_1, b \in I_2$ such that ab not belong to I. Since I is a free 1-triple zero with respect to $I_1I_2I_3$, we conclude that (a,b,c) is not a 1-triple zero of I for all $c \in I_3$. Thus $I_3 \subseteq \sqrt{I}$ by Theorem 4.10. □

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