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A New Analytic Solution Method for a Class of Generalized Riccati Differential Equations

Adil Mısır[1]

[1] *Department of Mathematics, Faculty of Science, Gazi University, Teknikokullar, 06500 Ankara, Türkiye*

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Abstract

We give a useful and practicable solution method for the general Riccati differential equation of the form $w'(x) = p(x) + q(x)w(x) + r(x)w^2(x)$. In order to get the general solution many authors have been interested this type equation. They show that if there exists some relation about the coefficients $p(x)$, $q(x)$, and $r(x)$ then the general solution of this equation can be given in a closed form. We also determine some relations between these coefficients and find the general solutions to the given equation. Finally, we give some examples to illustrate the importance of the presented method.

1. Introduction

The general Riccati differential equation (GRDE) is a well-known first-order nonlinear type of differential equation that arises not only a whole range of mathematics but also physics and have many applications in different areas of science. Riccati differential equation was named after the Italian mathematician Jacopo Francesco Riccati [1]. In particular, the GRDE is given by

$$w'(x) = p(x) + q(x)w(x) + r(x)w^2(x), \tag{1.1}$$

where we assume that $w,\ p,\ q,\ r \in C(\mathbb{R},\mathbb{R})$ are real functions and the integral $\int q(x)\,dx$ exists. In case $r(x) = 0$, the GRDE reduces a first-order linear ordinary differential equation of the form

$$w'(x) = p(x) + q(x)w(x)$$

and its general solution can be expressed in closed form as

$$w(x) = \exp\left(\int q(x)\,dx\right)\left[\int p(x)\exp\left(-\int q(x)\,dx\right)dx + \text{ constant}\right].$$

Similarly in case $p(x) = 0$, the GRDE reduces a first-order ordinary differential equation and called Bernoulli differential equation of the form

$$w'(x) = q(x)w(x) + r(x)w^2(x),$$

and general solution can be expressed in closed form as

$$w(x) = \exp\left(-\int q(x)\,dx\right)\left[-\int r(x)\exp\left(\int q(x)\,dx\right)dx + \text{ constant}\right]^{-1}.$$

Thus, in this paper, we consider the case $p(x)r(x) \neq 0$ for all x. Because the GRDE has many application areas in fields of applied science, the solutions of the GRDE play a significant role see [2]. For instance, optimal control, random processes, diffusion problems, stochastic

Email address and ORCID number: **adil.misir@gmail.com, 0000-0002-4552-0769 (A. Mısır)**
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realization theory, robust stabilization, network synthesis, and more recently, financial mathematics [3–5], Kalman filtering systems such as orbiting satellites [3,6]. Additionally, it is well known that the GRDE of the form

$$w'(x)+p(x)+w^2(x)=0 \tag{1.2}$$

plays an important role in studying qualitative analysis of the second order linear differential equation of the form

$$\phi''(x)+p(x)\phi(x)=0. \tag{1.3}$$

In fact, if Eq. (1.3) has a positive solution $\phi(x)$ on an interval I, then the function $w(x)=\phi'(x)/\phi(x)$ is a solution of Eq. (1.2). The substitution $w(x)=\phi'(x)/\phi(x)$ for the Eq. (1.3) is embedded in the Picone identity and it can be considered a link between the so-called Riccati technique and variational technique in the oscillation theory of Eq. (1.3) [7–9].
It is well known that there is not a general method for solving method for the GRDE, but recently, there have been several papers which have presented methods for solving of the GRDE under certain conditions [10–14].
Let $w_0=w_0(x)$ be a particular solution of the GRDE, then the general solution of the Eq. (1.1) can be written as:

$$w(x)=w_0(x)+\Phi(x)\left[C-\int r(x)\Phi(x)\,dx\right]^{-1},$$

where

$$\Phi(x)=\exp\left(\int\left[2r(x)w_0(x)+q(x)\right]dx\right),$$

and C is an arbitrary constant, see [12].
The aim of this paper is to find a general solution to the GRDE by using the relations between the coefficients $p(x)$, $q(x)$, and $r(x)$ for which the Eq. (1.1) can be solved in closed form.
It is well known that if $r(x)\neq 0$ for all x, the substitution

$$w(x)=-\frac{y'(x)}{r(x)y(x)} \tag{1.4}$$

into the GRDE, Eq. (1.1) can always be reduced to the second-order linear ordinary differential equation of the form

$$y''(x)-\left(\frac{r'(x)}{r(x)}+q(x)\right)y'(x)+p(x)r(x)y(x)=0. \tag{1.5}$$

As we mentioned above, in general, for any real functions $p(x)$, $q(x)$, and $r(x)$ the Eq. (1.1) cannot be solved in closed form. However, if there exist some specified relations between these coefficient functions, then Eq. (1.1) can be transformed into a second order linear ordinary differential equation, which can be easily solved, for example see [15–17].
In this paper, we treat a special case of the GRDE Eq. (1.1) where the functions $p(x)$ and $r(x)$ are not identically zero for all x. More precisely, we consider the case where the functions have the following relations for all $x\geq x_0$

$$r(x)\exp\left(\int q(x)\,dx\right)=\alpha,\quad -p(x)\exp\left(-\int q(x)\,dx\right)=\beta,$$

where α and β are some real constants. We shall also use the obtained results to provide the solution of the linear second order ordinary differential equation corresponding to the considered GRDE. As far as the author is aware, the explicit solution of the class of ordinary differential equations considered here does not exist in the literature.

2. Solution Method

In order to be able to solve the GRDE there are some concepts which need to be introduced as given in [18].
In this section, we give the general solution of a class of GRDE. The following theorem gives a relationship between the GRDE and the homogeneous systems of first order differential equations.

Theorem 2.1. *Assume that p, q, and r are real functions and the integral $\int q(x)\,dx$ exists. Then the GRDE Eq. (1.1) has a solution $u(x)$, without zeros for $x\geq x_0$ iff the homogeneous system of first order differential equations*

$$z'(x)=\mathbf{A}(x)\cdot z(x) \tag{2.1}$$

has a solution $z(x)$. Where $z(x)=\begin{pmatrix} y(x)\\ \xi(x)\end{pmatrix}$ and

$$\mathbf{A}(x)=\begin{pmatrix} 0 & r(x)\exp\left(\int q(x)\,dx\right)\\ -p(x)\exp\left(-\int q(x)\,dx\right) & 0\end{pmatrix}. \tag{2.2}$$

Proof. Let $w(x)$ be a solution of Eq. (1.1) and let $w(x) = -\frac{y'(x)}{r(x)y(x)}$. Then $y(x)$ satisfies the second order linear differential equation Eq. (1.5)

$$y''(x) - \left(\frac{r'(x)}{r(x)} + q(x)\right) y'(x) + p(x)r(x)y(x) = 0.$$

Multiplying Eq. (1.5) by the integrating factor $(r(x))^{-1}\exp(-\int q(x)dx)$ for the first two term, we obtain

$$\left[(r(x))^{-1}\exp\left(-\int q(x)dx\right)y'(x)\right]' + p(x)\exp\left(-\int q(x)dx\right)y(x) = 0.$$

Hence, if we let $(r(x))^{-1}\exp(-\int q(x)dx)y'(x) = \xi(x)$ and $z(x) = \begin{pmatrix} y(x) \\ \xi(x) \end{pmatrix}$, then $z(x)$ is a solution of the homogeneous system of first order differential equations, Eq. (2.1) with $\mathbf{A}(x)$ is given (2.2). □

The following theorem summarizes the present study:

Theorem 2.2. *Assume that $p(x), q(x)$, and $r(x)$ hold the relations*

$$r(x)\exp\left(\int q(x)dx\right) = \alpha, \; -p(x)\exp\left(-\int q(x)dx\right) = \beta. \tag{2.3}$$

Then the general solution of Eq. (1.1) is given

$$w(x) = \begin{cases} \frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{1 - C\exp\left(2\sqrt{\alpha\beta}x\right)}{1 + C\exp\left(2\sqrt{\alpha\beta}x\right)}\right) & ; if\ \alpha\beta > 0, \\ \frac{\sqrt{-\alpha\beta}}{r(x)}\left(\frac{\sin\left(\sqrt{-\alpha\beta}x - C\cos\left(\sqrt{-\alpha\beta}x\right)\right)}{\cos\left(\sqrt{-\alpha\beta}x\right) + C\sin\left(\sqrt{-\alpha\beta}x\right)}\right) & ; if\ \alpha\beta < 0, \end{cases}$$

where C is any real constant.

Proof. If the conditions of (2.3) are fulfilled, the homogeneous system of first order differential equations Eq. (2.1) becomes a first order homogeneous system with constant coefficients

$$z'(x) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \cdot z(x). \tag{2.4}$$

Then, the eigenvalues of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ are $r_1 = -\sqrt{\alpha\beta}$ and $r_2 = \sqrt{\alpha\beta}$. If $\alpha\beta > 0$ the eigenvalues of the coefficient matrix $\mathbf{A}$ are real constants such as $r_1 = -\sqrt{\alpha\beta}$ and $r_2 = \sqrt{\alpha\beta}$. Similarly if $\alpha\beta < 0$ the eigenvalues of the coefficient matrix $\mathbf{A}$ are complex constants such as $r_1 = -i\sqrt{-\alpha\beta}$ and $r_2 = i\sqrt{-\alpha\beta}$. In case, $\alpha\beta > 0$, $\Phi(x) = \begin{pmatrix} \exp\left(-\sqrt{\alpha\beta}x\right) & \exp\left(\sqrt{\alpha\beta}x\right) \\ -\sqrt{\frac{\beta}{\alpha}}\exp\left(-\sqrt{\alpha\beta}x\right) & \sqrt{\frac{\beta}{\alpha}}\exp\left(\sqrt{\alpha\beta}x\right) \end{pmatrix}$ is a fundamental matrix of Eq. (2.4). Then general solution of Eq. (2.4)

$$z(x) = \Phi(x)\cdot\mathbf{C} = \begin{pmatrix} \exp\left(-\sqrt{\alpha\beta}x\right) & \exp\left(\sqrt{\alpha\beta}x\right) \\ -\sqrt{\frac{\beta}{\alpha}}\exp\left(-\sqrt{\alpha\beta}x\right) & \sqrt{\frac{\beta}{\alpha}}\exp\left(\sqrt{\alpha\beta}x\right) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where c_1 and c_2 are real constants. Thus the general solution of Eq. (1.5) is

$$y(x) = c_1\exp\left(-\sqrt{\alpha\beta}x\right) + c_2\exp\left(\sqrt{\alpha\beta}x\right).$$

Therefore,

$$w(x) = -\frac{y'(x)}{r(x)y(x)} = \frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{c_1\exp\left(-\sqrt{\alpha\beta}x\right) - c_2\exp\left(\sqrt{\alpha\beta}x\right)}{c_1\exp\left(-\sqrt{\alpha\beta}x\right) + c_2\exp\left(\sqrt{\alpha\beta}x\right)}\right).$$

When $c_1 = 0$, the function $w(x) = -\frac{\sqrt{\alpha\beta}}{r(x)}$ is a solution of the Eq. (1.1). When $c_1 \neq 0$ we can divide the numerator and denominator by $c_1 e^{-\sqrt{\alpha\beta}x}$ to get that

$$w(x) = \frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{1 - C\exp\left(2\sqrt{\alpha\beta}x\right)}{1 + C\exp\left(2\sqrt{\alpha\beta}x\right)}\right),$$

is general solution of the Eq. (1.1), where $C = \frac{c_2}{c_1}$ is any real constant. Thus the proof of the first part is complete. Similarly if $\alpha\beta < 0$

$$\Phi(x) = \begin{pmatrix} \cos\left(\sqrt{-\alpha\beta}x\right) & \sin\left(\sqrt{-\alpha\beta}x\right) \\ -\frac{\sqrt{-\alpha\beta}}{\beta}\sin\left(\sqrt{-\alpha\beta}x\right) & \frac{\sqrt{-\alpha\beta}}{\alpha}\cos\left(\sqrt{-\alpha\beta}x\right) \end{pmatrix}$$

is a fundamental matrix of Eq. (2.4). Then general solution of Eq. (2.4)

$$z(x)=\Phi(x)\cdot\mathbf{C}=\begin{pmatrix}\cos\left(\sqrt{-\alpha\beta}x\right) & \sin\left(\sqrt{-\alpha\beta}x\right)\\ -\frac{\sqrt{-\alpha\beta}}{\beta}\sin\left(\sqrt{-\alpha\beta}x\right) & \frac{\sqrt{-\alpha\beta}}{\alpha}\cos\left(\sqrt{-\alpha\beta}x\right)\end{pmatrix}\cdot\begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

where c_1 and c_2 are real constants. Thus the general solution of Eq. (1.5) is

$$y(x)=c_1\cos\left(\sqrt{-\alpha\beta}x\right)+c_2\sin\left(\sqrt{-\alpha\beta}x\right).$$

Therefore,

$$w(x)=-\frac{y'(x)}{r(x)y(x)}=\frac{\sqrt{-\alpha\beta}}{r(x)}\left(\frac{c_1\sin\left(\sqrt{-\alpha\beta}x\right)-c_2\cos\left(\sqrt{-\alpha\beta}x\right)}{c_1\cos\left(\sqrt{-\alpha\beta}x\right)+c_2\sin\left(\sqrt{-\alpha\beta}x\right)}\right).$$

When $c_1=0$, the function $w(x)=-\frac{\sqrt{-\alpha\beta}}{r(x)}\tan\left(\sqrt{-\alpha\beta}x\right)$ is a solution of the Eq. (1.1). When $c_1\neq 0$

$$w(x)=\frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{\sin\left(\sqrt{-\alpha\beta}x\right)-C\cos\left(\sqrt{-\alpha\beta}x\right)}{\cos\left(\sqrt{-\alpha\beta}x\right)+C\sin\left(\sqrt{-\alpha\beta}x\right)}\right),$$

is general solution of the Eq. (1.1), where C is any real constant. Thus the proof is complete. □

Remark 2.3. *If the functions $p(x)$, $q(x)$, and $r(x)$ are constants such as $p(x)=a$, $q(x)=b$, and $r(x)=c$. Then, the conditions of (2.3) are fulfilled as $\alpha=c$ and $\beta=-a$ for $b=0$ and we can use the Theorem 2.2 for the general solution of Eq. (1.1). But when $b\neq 0$, the conditions of (2.3) not satisfied. In general case $a,b,c\in\mathbb{R}$ and $ac\neq 0$, the Eq. (1.1) becomes a first-order separable ordinary differential equation which is defined by*

$$\frac{dw}{a+bw+cw^2}=dx.$$

Based on the integral involving the rational algebraic functions of the form

$$\int\frac{dw}{a+bw+cw^2}=\begin{cases}\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2cw+b}{\sqrt{4ac-b^2}}\right) & ;\ if\ 4ac-b^2>0,\\ \frac{2}{\sqrt{b^2-4ac}}\ln\left|\frac{2cw+b-\sqrt{b^2-4ac}}{2cw+b+\sqrt{b^2-4ac}}\right| & ;\ if\ 4ac-b^2<0,\\ -\frac{2}{2cw+b} & ;\ if\ 4ac-b^2=0,\end{cases}$$

in view of this, the general solution of Eq. (1.1) is given in a closed form by

$$w(x)=\begin{cases}\frac{1}{2c}\left[-b+\sqrt{4ac-b^2}\tan\left(\frac{1}{2}\sqrt{4ac-b^2}x+C\right)\right] & ;\ if\ 4ac-b^2>0,\\ \frac{\sqrt{b^2-4ac}}{2c}\left(\frac{1+C\exp\left(\frac{\sqrt{b^2-4ac}}{2}x\right)}{1-C\exp\left(\frac{\sqrt{b^2-4ac}}{2}x\right)}\right) & ;\ if\ 4ac-b^2<0,\\ -\frac{1}{2c}\left(b+\frac{2}{x+C}\right) & ;\ if\ 4ac-b^2=0,\end{cases}$$

where C is an arbitrary constant.

3. Some Examples

Here, we illustrate some examples to consider some special cases. In these examples, we assume that the above conditions are satisfied and the general solutions of the GRDE are obtained easily.

Example 3.1. *Consider the first-order nonlinear differential equation for $x\geq x_0>0$*

$$w'(x)=\frac{4}{x^2}-\frac{2}{x}w(x)+x^2w^2(x). \tag{3.1}$$

For this equation the conditions of (2.3) are satisfied with $p(x)=\frac{4}{x^2}$, $q(x)=-\frac{2}{x}$, and $r(x)=-x^2$. Thus, by Theorem 2.2 $\alpha\beta=-4<0$ and the general solution of Eq. (3.1) is obtained as

$$w(x)=\frac{2}{x^2}\left(\frac{\sin 2x-C\cos 2x}{\cos 2x+C\sin 2x}\right),$$

where C is an arbitrary constant.

Example 3.2. *Consider the first-order nonlinear differential equation*

$$w'(x) = \frac{9e^{x\arctan x}}{\sqrt{x^2+1}} + (\arctan x)\, w(x) + \left(4e^{-x\arctan x}\sqrt{x^2+1}\right) w^2(x). \tag{3.2}$$

For this equation the conditions of (2.3) *are satisfied with* $p(x) = \frac{9e^{x\arctan x}}{\sqrt{x^2+1}}, q(x) = \arctan x$, *and* $r(x) = 4e^{-x\arctan x}\sqrt{x^2+1}$. *Thus, by Theorem* 2.2 $\alpha\beta = -36 < 0$ *and the general solution of Eq.* (3.2) *is obtained as*

$$w(x) = \frac{3e^{x\arctan x}}{2\sqrt{x^2+1}} \left(\frac{\sin 6x - C\cos 6x}{\cos 6x + C\sin 6x}\right),$$

where C *is an arbitrary constant.*

Example 3.3. *Consider the first-order nonlinear differential equation for* $x \geq x_0 > 0$

$$w'(x) = x^x e^{-x} + (\ln x)\, w(x) - \left(x^{-x}e^x\right) w^2(x). \tag{3.3}$$

Note that the conditions of (2.3) *are satisfied with* $\alpha = \beta = -1$, $p(x) = x^x e^{-x}$, $q(x) = \ln x$, *and* $r(x) = -x^{-x}e^x$. *Thus, by Theorem* 2.2 $\alpha\beta = 1 > 0$ *and general solution of Eq.* (3.3) *is*

$$w(x) = -x^x e^{-x}\left(\frac{1 - Ce^{2x}}{1 + Ce^{2x}}\right),$$

where C *is any constant.*

Example 3.4. *Consider the first-order nonlinear differential equation*

$$w'(x) = 1 + 5w(x) + 9w^2(x). \tag{3.4}$$

For this equation $p(x) = 1$, $q(x) = 5$, *and* $r(x) = 9$ *are constant functions and conditions of* (2.3) *not satisfied. Thus, we can not use the Theorem* 2.2 *for the general solution of the Eq.* (3.4). *But we can use the Remark* 2.3 *for the general solution of equation, Eq.* (3.4) *and the general solution obtained as*

$$w(x) = \frac{1}{6}\left(\frac{1 + C\exp\left(\frac{3}{2}x\right)}{1 - C\exp\left(\frac{3}{2}x\right)}\right),$$

where $4ac - b^2 = -9 < 0$ *and* C *is any real constant.*

Example 3.5. *Consider the first-order nonlinear differential equation*

$$w'(x) = 1 + 4w(x) + 4w^2(x). \tag{3.5}$$

If we use the Remark 2.3 *for the equation Eq.* (3.5) *we get the general solution as*

$$w(x) = -\frac{1}{4}\left(2 + \frac{1}{x+C}\right),$$

where $4ac - b^2 = 0$ *and* C *is any real constant.*

4. Conclusion

In this paper, we have obtained the general solution of a class of first-order nonlinear ordinary differential equation, which called GRDE. We have converted the GRDE into a homogeneous system of first-order differential equations. In order to do this we use two well-known transformations as explained above. The first transformation converts the nonlinear first-order ordinary differential equation Eq. (1.1) to a linear second-order ordinary differential equation Eq. (1.5). The second one converts Eq. (1.5) to a homogeneous system of first order differential equations Eq. (2.1). Then, by using the fact of the Section 2, we give general solution of a particular class of GRDE. Examples were given here for each case demonstrate the present method.

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The Geometry of Bézier Curves in Minkowski 3−Space

Ayşe Yılmaz Ceylan[1]

[1]*Department of Mathematics, Faculty of Science, Akdeniz University, 07058 Antalya, Türkiye*

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Abstract

The scope of this paper is to look at some aspects of the differential geometry of Bézier curves in Minkowski space. For that purpose, we firstly introduce Frenet Bézier curve in Minkowski 3-space. Especially, we investigate the Serret-Frenet frame, curvature and torsion of the Frenet Bézier curves at all points. Moreover, we give the Frenet apparatus of these curves at the end points.

1. Introduction and Background

Let E_1^3 be the three dimensional Minkowski space with the metric $< dx, dx >= dx_1^2 + dx_2^2 - dx_3^2$ where x_1, x_2, x_3 denotes the canonical coordinates in E^3. An arbitrary vector x is said to be spacelike if $< x, x >> 0$ or $x = 0$, timelike if $< x, x >< 0$ and lightlike or null if $< x, x >= 0$. The norm is defined by $\|x\| = \sqrt{|< x, x >|}$ for $x \in E_1^3$. A regular curve in E_1^3 is called locally spacelike, timelike or null, if all its velocity vectors are spacelike, timelike or null, respectively [1]. For any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3, the inner product is the real number $< x, y >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $x \wedge_{IL} y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)$. See for more information on Minkowski space in [1, 2].
Bézier curves are represented by Pierre Bézier in 1968. Bézier curves are essential among the curves since they are applicable to computer graphics and related areas. See for more detailed information in [3, 4]. Recently, the geometry of Bézier curves have been investigated by many researechers due to the fact that they have several important properties. Incesu and GÃ¼rsoy studied the curvatures and principal form of the Bézier curve in [5]. Georgiev worked on the shapes of planar and cubic Bézier curve in [6, 7].
In the theory of curves in the Minkowski space, one of the interesting problem is the characterization of a regular curve. In [8], Georgiev studied on the geometry of the spacelike Bézier curve. He also examined the spacelike Bézier surfaces in Minkowski 3−space in [9]. Chalmoviansky, Pokorna studied quadratic and planar cubic spacelike Bézier curves in Minkowski 3−space in [10, 11]. In [12], Ugail, Marquaez and Yılmaz handled the conditions of timelike and spacelike Bézier surfaces. The Serret-Frenet frames, curvatures and torsion of the timelike and spacelike Bézier curves were calculated at the end points in [13–16]. Our aim in this paper is to investigate the timelike and spacelike Bézier curve of degree m at all points.
A classical Bézier curve of degree m with control points p_j is defined as

$$b(t) = \sum_{j=0}^{m} p_j B_j^m(t), t \in [0, 1] \tag{1.1}$$

where

$$B_{j,m}(t) = \begin{cases} \frac{m!}{(m-j)!j!}(1-t)^{m-j}t^j, & \text{if } 0 \le j \le m \\ 0, & \text{otherwise} \end{cases}$$

are called the Bernstein basis functions of degree m. The polygon formed by joining the control points $p_0, p_1, ..., p_m$ in the specified order is called the Bézier control polygon.
If a curve is differentiable at its each point in an open interval, in this case a set of orthogonal unit vectors can be obtained. And these unit vectors are called Frenet frame. The rates of these frame vectors along the curve define curvatures of the curves. The set of these vectors and curvatures of a curve, is called Frenet apparatus of the curve.

***Email address and ORCID number*:** **ayilmazceylan@akdeniz.edu.tr, 0000-0002-8051-2879 (A. Yılmaz Ceylan)**
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Theorem 1.1 ([14]). *Let $\vec{u}, \vec{v}$ and $\vec{w}$ vectors in E_1^3. Then*

(i) $< u \wedge_{IL} v, w >= -\det(u, v, w)$,
(ii) $(u \wedge_{IL} v) \wedge_{IL} w = - < u, w > v + < v, w > u$,
(iii) $< u \wedge_{IL} v, u >= 0$ *and* $< u \wedge_{IL} v, v >= 0$,
(iv) $< u \wedge_{IL} v, u \wedge_{IL} v >= - < u, u >< v, v > +(< u, v >)^2$.

Let β be a curve in E_1^3. Then β is called timelike (resp. spacelike, null) at t, if the tangent vector $\beta'(t)$ is a timelike (resp.cspacelike, null) vector.

Theorem 1.2 ([17]). *For a regular curve β with speed $v = \frac{ds}{dt}$, and curvature $\kappa > 0$,*

(i) β is spacelike non-unit speed curve, then the derivative formula of Frenet frame is as follows:

$$T' = v\kappa N,$$
$$N' = v(-\delta\kappa T + \tau B),$$
$$B' = v\tau N.$$

(ii) β is timelike non-unit speed curve, then the derivative formula of Frenet frame is as follows:

$$T' = v\kappa N,$$
$$N' = v(\kappa T + \tau B),$$
$$B' = v\tau N.$$

Theorem 1.3. *Let $\vec{u}$ and $\vec{v}$ be vectors in Minkowski 3−space.*

(i) If $\vec{u}$ and $\vec{v}$ are future pointing (or past pointing) timelike vectors, then $\vec{u} \wedge_{IL} \vec{v}$ is a spacelike vector, $< \vec{u}, \vec{v} >= -\|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\cosh\theta$ and $\|\vec{u} \wedge_{IL} \vec{v}\| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\sinh\theta$ where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.
(ii) If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|< \vec{u}, \vec{v} >| < \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}$, then $\vec{u} \wedge_{IL} \vec{v}$ is timelike vector, $< \vec{u}, \vec{v} >= \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\cos\theta$ and $\|\vec{u} \wedge_{IL} \vec{v}\| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.
(iii) If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|< \vec{u}, \vec{v} >| > \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}$, then $\vec{u} \wedge_{IL} \vec{v}$ is timelike vector, $< \vec{u}, \vec{v} >= -\|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\cosh\theta$ and $\|\vec{u} \wedge_{IL} \vec{v}\| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}\sinh\theta$ where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.
(iv) If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|< \vec{u}, \vec{v} >| = \|\vec{u}\|_{IL}\|\vec{v}\|_{IL}$, then $\vec{u} \wedge_{IL} \vec{v}$ is lightlike.

See more [1, 2, 18, 19].

Theorem 1.4 ([8, 14]). *Let $b(t)$ be a Bézier curve. If all the vectors of the Bézier control polygon is spacelike (timelike), then $b(t)$ is spacelike (timelike) curve.*

Definition 1.5. *Timelike Bézier curves and spacelike Bézier curves with spacelike or timelike normal vectors are called Frenet Bézier curves.*

2. Main Results

2.1. Timelike Bézier curves

In this section, we give Serret-Frenet frame, curvature and torsion of timelike Bézier curves.

Theorem 2.1. *Let $b(t)$ be a timelike Bézier curve and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ is given by*

$$T(t) = \frac{\sum\limits_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\left(-\sum\limits_{j,i=0}^{m-1} B_j^{m-1}(t)B_i^{m-1}(t) < \triangle p_j, \triangle p_i >\right)^{\frac{1}{2}}}, \tag{2.1}$$

$$N(t) = -\frac{\sum\limits_{j=0}^{m-1}\sum\limits_{i=0}^{m-2}\sum\limits_{k=0}^{m-1} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-1}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i) \wedge_{IL} \triangle p_k}{\|\sum\limits_{j=0}^{m-1}\sum\limits_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL} \|\sum\limits_{k=0}^{m-1} B_k^{m-1}(t)\triangle p_k\|_{IL}}, \tag{2.2}$$

$$B(t) = \frac{\sum\limits_{j=0}^{m-1}\sum\limits_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)}{\|\sum\limits_{j=0}^{m-1}\sum\limits_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}}, \tag{2.3}$$

$$\kappa(t) = \frac{m-1}{m} \frac{\|\sum\limits_{j=0}^{m-1}\sum\limits_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}}{\|\sum\limits_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}^3}, \tag{2.4}$$

$$\tau(t) = -\frac{m-2}{m}\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-3} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-3}(t)det(\triangle p_j, \triangle^2 p_i, \triangle^3 p_k)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}^2}, \tag{2.5}$$

where $\triangle p_j$ are in the same cone, $\triangle p_j = p_{j+1} - p_j$, $\triangle^2 p_j = \triangle p_{j+1} - \triangle p_j$ and $\triangle^3 p_j = \triangle^2 p_{j+1} - \triangle^2 p_j$.

Proof. Since all the vectors $\triangle p_j$ are timelike vectors, the norm of $\triangle p_j$ is

$$\|\triangle p_j\|_{IL} = \sqrt{-<\triangle p_j, \triangle p_j>} \tag{2.6}$$

for $t \in [0,1]$. The tangent vector is calculated as:

$$\begin{aligned} T(t) &= \frac{b'(t)}{\|b'(t)\|_{IL}} \\ &= \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}}. \end{aligned} \tag{2.7}$$

From the equation (2.6) and (2.7), the equation (2.1) is handled.
The binormal vector is obtained by

$$\begin{aligned} B(t) &= \frac{b'(t)\wedge_{IL} b''(t)}{\|b'(t)\wedge_{IL} b''(t)\|_{IL}} \\ &= \frac{(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)}{\|(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)\|_{IL}}. \end{aligned}$$

Since the tangent $\mathbf{T}$ of the timelike Bézier curve is timelike, $\mathbf{N}$ and $\mathbf{B}$ are spacelike vectors, the principal normal vector $\mathbf{N}$ is provided by

$$\begin{aligned} N(t) &= -B(t)\wedge_{IL} T(t) \\ &= -\frac{(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)}{\|(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)\|_{IL}} \wedge_{IL} \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}}. \end{aligned}$$

The curvature of timelike Bézier curve is

$$\begin{aligned} \kappa(t) &= \frac{\|b'(t)\wedge_{IL} b''(t)\|_{IL}}{\|b'(t)\|_{IL}^3} \\ &= \frac{m-1}{m}\frac{(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_i)}{\|\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j\|_{IL}^3}. \end{aligned}$$

and the torsion of timelike Bézier curve is

$$\begin{aligned} \tau(t) &= \frac{<b'(t)\wedge_{IL} b''(t), b'''(t)>}{\|b'(t)\wedge_{IL} b''(t)\|_{IL}} \\ &= \frac{m-2}{m}\frac{<\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j \wedge_{IL} \sum_{i=0}^{m-2} B_i^{m-2}(t)\triangle^2 p_j, \sum_{k=0}^{m-3} B_k^{m-3}(t)\triangle^3 p_k>}{\|(\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j)\wedge_{IL}(\sum_{j=0}^{m-2} B_j^{m-2}(t)\triangle^2 p_i)\|_{IL}}. \end{aligned}$$

□

From the Theorem 1.3 and Theorem 2.1, the following results can be handled.

Corollary 2.2 ([16]). *Let $b(t)$ be a timelike Bézier curve and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ at $t=0$ is given by*

$$T(0)=\frac{\triangle p_0}{\sqrt{-<\triangle p_0,\triangle p_0>}},$$
$$N(0)=\frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\coth\theta-\frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc h\theta,$$
$$B(0)=\frac{\triangle p_0\wedge_{IL}\triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sinh\theta},$$
$$\kappa(0)=\frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sinh\theta}{\|\triangle p_0\|_{IL}^2},$$
$$\tau(0)=-\frac{m-2}{m}\frac{\det(\triangle p_0,\triangle p_1,\triangle p_2)}{\|\triangle p_0\wedge_{IL}\triangle p_1\|_{IL}^2},$$

where θ is the angle between $\triangle p_0$ and $\triangle p_1$.

Corollary 2.3 ([16]). *Let $b(t)$ be a timelike Bézier curve and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ at $t=1$ is given by*

$$T(1)=\frac{\triangle p_{m-1}}{\sqrt{-<\triangle p_{m-1},\triangle p_{m-1}>}},$$
$$N(1)=\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc h\theta-\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\coth\theta,$$
$$B(1)=-\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sinh\theta},$$
$$\kappa(1)=\frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sinh\theta}{\|\triangle p_{m-1}\|_{IL}^2},$$
$$\tau(1)=\frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},$$

where θ is the angle between $\triangle p_{m-2}$ and $\triangle p_{m-1}$.

2.2. Spacelike Bézier Curves

In this section, we calculate Serret-Frenet frame, curvature and torsion of spacelike Bézier curves with spacelike and timelike normals.

2.2.1. Spacelike Bézier Curves with Spacelike normal

In this subsection, we calculate Frenet apparatus of a spacelike Bézier curve with spacelike normal.

Theorem 2.4. *Let $b(t)$ be a spacelike Bézier curve with spacelike normal and p_j are control points. The Serret-Frenet frame **T,N,B**, curvature κ and torsion τ of $b(t)$ is given by*

$$T(t)=\frac{\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j}{(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>)^{\frac{1}{2}}}, \tag{2.8}$$

$$N(t)=-\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-1}B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-1}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\wedge_{IL}\triangle p_k}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}\|\sum_{k=0}^{m-1}B_k^{m-1}(t)\triangle p_k\|_{IL}}, \tag{2.9}$$

$$B(t)=\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}}, \tag{2.10}$$

$$\kappa(t)=\frac{m-1}{m}\frac{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}}{\|\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j\|_{IL}^3}, \tag{2.11}$$

$$\tau(t) = -\frac{m-2}{m}\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-3} B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-3}(t)det(\triangle p_j, \triangle^2 p_i, \triangle^3 p_k)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2} B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j \wedge_{IL} \triangle^2 p_i)\|_{IL}^2}, \tag{2.12}$$

where $\triangle p_j$ are in the same cone.

Proof. Since all the vectors $\triangle p_j$ are spacelike vectors, the norm of $\triangle p_j$ is

$$\|\triangle p_j\|_{IL} = \sqrt{<\triangle p_j, \triangle p_j>}. \tag{2.13}$$

for $t \in [0,1]$. The tangent vector is calculated as:

$$\begin{aligned} T(t) =& \frac{b'(t)}{\|b'(t)\|_{IL}} \\ =& \frac{\sum_{j=0}^{m-1} B_j^{m-1}(t)\triangle p_j}{(\sum_{j,i=0}^{m-1} B_j^{m-1}(t)B_i^{m-1}(t) <\triangle p_j, \triangle p_i>)^{\frac{1}{2}}}. \end{aligned} \tag{2.14}$$

From the equation (2.13) and (2.14), the equation (2.8) is handled.
Since the tangent **T, N** spacelike and **B** is timelike, **N** is given by the equation

$$\mathbf{N} = \mathbf{B} \wedge_{IL} \mathbf{T}.$$

The rest of the proof is similar to Theorem 2.1. □

From the Theorem 1.3 and Theorem 2.4, the following results can be seen easily.

Corollary 2.5 ([13]). *Let $b(t)$ be a spacelike Bézier curve with spacelike normal and p_j are control points. The tangent vector **T** of $b(t)$ at $t=0$ is given by*

$$T(0) = \frac{\triangle p_0}{\sqrt{<\triangle p_0, \triangle p_0>}},$$

*If the inequality $|<\triangle p_0, \triangle p_1>|_{IL} < \|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, **N,B**, κ and τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned} N(0) =& \frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc\theta - \frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\cot\theta, \\ B(0) =& \frac{\triangle p_0 \wedge_{IL} \triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sin\theta}, \\ \kappa(0) =& \frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sin\theta}{\|\triangle p_0\|_{IL}^2}, \\ \tau(0) =& -\frac{m-2}{m}\frac{\det(\triangle p_0, \triangle p_1, \triangle p_2)}{\|\triangle p_0 \wedge_{IL} \triangle p_1\|_{IL}^2}, \end{aligned}$$

*and if the inequality $|<\triangle p_0, \triangle p_1>|_{IL} > \|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, **N,B**, κ and τ of $b(t)$ at $t=0$ is given by*

$$\begin{aligned} N(0) =& \frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc h\theta + \frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\coth\theta, \\ B(0) =& \frac{\triangle p_0 \wedge_{IL} \triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sinh\theta}, \\ \kappa(0) =& \frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sinh\theta}{\|\triangle p_0\|_{IL}^2}, \\ \tau(0) =& -\frac{m-2}{m}\frac{\det(\triangle p_0, \triangle p_1, \triangle p_2)}{\|\triangle p_0 \wedge_{IL} \triangle p_1\|_{IL}^2}, \end{aligned}$$

where θ is the angle between $\triangle p_0$ and $\triangle p_1$.

Corollary 2.6 ([13]). *Let $b(t)$ be a spacelike Bézier curve with spacelike normal and p_j are control points. The tangent vector **T** of $b(t)$ at $t=1$ is given by*

$$T(1) = \frac{\triangle p_{m-1}}{\sqrt{<\triangle p_{m-1}, \triangle p_{m-1}>}}.$$

If the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}<\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ *holds for* $\triangle p_{m-2}$ *and* $\triangle p_{m-1}$*, **N,B**,* κ *and* τ *of* $b(t)$ *at* $t=1$ *is given by*

$$\begin{aligned}
N(1) &= -\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc\theta+\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\cot\theta,\\
B(1) &= -\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sin\theta},\\
\kappa(1) &= \frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sin\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1) &= \frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

and if the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}>\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ *holds for* $\triangle p_{m-2}$ *and* $\triangle p_{m-1}$*, **N,B**,* κ *and* τ *of* $b(t)$ *at* $t=1$ *is given by*

$$\begin{aligned}
N(1) &= -\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc h\theta-\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\coth\theta,\\
B(1) &= -\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sinh\theta},\\
\kappa(1) &= \frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sinh\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1) &= \frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

where θ *is the angle between* $\triangle p_{m-2}$ *and* $\triangle p_{m-1}$*.*

2.2.2. Spacelike Bézier curves with timelike normal

In this subsection, we calculate Frenet apparatus of a spacelike Bézier curve with timelike normal.

Theorem 2.7. *Let* $b(t)$ *be a spacelike Bézier curve with timelike normal and* p_j *are control points. The Serret-Frenet frame **T,N,B**, curvature* κ *and torsion* τ *of* $b(t)$ *is given by*

$$T(t)=\frac{\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j}{\left(\sum_{j,i=0}^{m-1}B_j^{m-1}(t)B_i^{m-1}(t)<\triangle p_j,\triangle p_i>\right)^{\frac{1}{2}}}, \tag{2.15}$$

$$N(t)=\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-1}B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-1}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\wedge_{IL}\triangle p_k}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}\|\sum_{k=0}^{m-1}B_k^{m-1}(t)\triangle p_k\|_{IL}}, \tag{2.16}$$

$$B(t)=\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}}, \tag{2.17}$$

$$\kappa(t)=\frac{m-1}{m}\frac{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}}{\|\sum_{j=0}^{m-1}B_j^{m-1}(t)\triangle p_j\|_{IL}^3}, \tag{2.18}$$

$$\tau(t)=-\frac{m-2}{m}\frac{\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}\sum_{k=0}^{m-3}B_j^{m-1}(t)B_i^{m-2}(t)B_k^{m-3}(t)\det(\triangle p_j,\triangle^2 p_i,\triangle^3 p_k)}{\|\sum_{j=0}^{m-1}\sum_{i=0}^{m-2}B_j^{m-1}(t)B_i^{m-2}(t)(\triangle p_j\wedge_{IL}\triangle^2 p_i)\|_{IL}^2}, \tag{2.19}$$

where $\triangle p_i$ *and* $\triangle p_j$ *are in the same cone.*

Proof. Since the tangent **T, B** spacelike and **N** is timelike, **N** is given by the equation

$$\mathbf{N}=\mathbf{B}\wedge_{IL}\mathbf{T}.$$

The rest of the proof is similar to Theorem 2.4. □

From the Theorem 1.3 and Theorem 2.7, the following results can be obtained.

Corollary 2.8 ([15]). *Let $b(t)$ be a spacelike Bézier curve with timelike normal and p_j are control points. The tangent vector $\boldsymbol{T}$ of $b(t)$ at $t=0$ is given by*

$$T(0)=\frac{\triangle p_0}{\sqrt{<\triangle p_0,\triangle p_0>}}.$$

If the inequality $|<\triangle p_0,\triangle p_1>|_{IL}<\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, $\boldsymbol{N},\boldsymbol{B}$, κ and τ of $b(t)$ at $t=0$ is given by

$$\begin{aligned}
N(0)&=-\frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc\theta+\frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\cot\theta,\\
B(0)&=\frac{\triangle p_0\wedge_{IL}\triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sin\theta},\\
\kappa(0)&=\frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sin\theta}{\|\triangle p_0\|_{IL}^2},\\
\tau(0)&=-\frac{m-2}{m}\frac{\det(\triangle p_0,\triangle p_1,\triangle p_2)}{\|\triangle p_0\wedge_{IL}\triangle p_1\|_{IL}^2},
\end{aligned}$$

and if the inequality $|<\triangle p_0,\triangle p_1>|_{IL}>\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}$ holds for $\triangle p_0$ and $\triangle p_1$, $\boldsymbol{N},\boldsymbol{B}$, κ and τ of $b(t)$ at $t=0$ is given by

$$\begin{aligned}
N(0)&=-\frac{\triangle p_1}{\|\triangle p_1\|_{IL}}\csc h\theta-\frac{\triangle p_0}{\|\triangle p_0\|_{IL}}\coth\theta,\\
B(0)&=\frac{\triangle p_0\wedge_{IL}\triangle p_1}{\|\triangle p_0\|_{IL}\|\triangle p_1\|_{IL}\sinh\theta},\\
\kappa(0)&=\frac{m-1}{m}\frac{\|\triangle p_1\|_{IL}\sinh\theta}{\|\triangle p_0\|_{IL}^2},\\
\tau(0)&=-\frac{m-2}{m}\frac{\det(\triangle p_0,\triangle p_1,\triangle p_2)}{\|\triangle p_0\wedge_{IL}\triangle p_1\|_{IL}^2},
\end{aligned}$$

where θ is the angle between $\triangle p_0$ and $\triangle p_1$.

Corollary 2.9 ([15]). *Let $b(t)$ be a spacelike Bézier curve with timelike normal and p_j are control points. The tangent vector $\boldsymbol{T}$ of $b(t)$ at $t=1$ is given by*

$$T(1)=\frac{\triangle p_{m-1}}{\sqrt{<\triangle p_{m-1},\triangle p_{m-1}>}}.$$

If the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}<\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ holds for $\triangle p_{m-2}$ and $\triangle p_{m-1}$, $\boldsymbol{N},\boldsymbol{B}$, κ and τ of $b(t)$ at $t=1$ is given by

$$\begin{aligned}
N(1)&=\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc\theta-\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\cot\theta,\\
B(1)&=-\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sin\theta},\\
\kappa(1)&=\frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sin\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1)&=\frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

and if the inequality $|<\triangle p_{m-2},\triangle p_{m-1}>|_{IL}>\|\triangle p_{m-2}\|_{IL}\|\triangle p_{m-1}\|_{IL}$ holds for $\triangle p_{m-2}$ and $\triangle p_{m-1}$, $\boldsymbol{N},\boldsymbol{B}$, κ and τ of $b(t)$ at $t=1$ is given by

$$\begin{aligned}
N(1)&=\frac{\triangle p_{m-2}}{\|\triangle p_{m-2}\|_{IL}}\csc h\theta+\frac{\triangle p_{m-1}}{\|\triangle p_{m-1}\|_{IL}}\coth\theta,\\
B(1)&=-\frac{\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}}{\|\triangle p_{m-1}\|_{IL}\|\triangle p_{m-2}\|_{IL}\sinh\theta},\\
\kappa(1)&=\frac{m-1}{m}\frac{\|\triangle p_{m-2}\|_{IL}\sinh\theta}{\|\triangle p_{m-1}\|_{IL}^2},\\
\tau(1)&=\frac{m-2}{m}\frac{\det(\triangle p_{m-1},\triangle p_{m-2},\triangle p_{m-3})}{\|\triangle p_{m-1}\wedge_{IL}\triangle p_{m-2}\|_{IL}^2},
\end{aligned}$$

where θ is the angle between $\triangle p_{m-2}$ and $\triangle p_{m-1}$.

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Theorems of Second Korovkin Type with respect to Triangular A-Statistical Convergence

Selin Çınar[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Sinop University, 57000 Sinop, Türkiye*

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Abstract

This article is a continuation of our previous works. We mainly investigate a Korovkin type theorem for double sequences of positive linear operators defined in the space of all 2π-periodic and real valued continuous functions on the real two-dimensional space with help of the concept of triangular A-statistical convergence, which is a kind of statistical convergence for double real sequences. Also, we analyze the rate of convergence of double operators in this via modulus of continuity.

1. Introduction

Fast [**?**] (independently, Steinhaus [**?**]) introduced the concept of statistical convergence, which is an advantageous approach. This concept is studied in various fields and its generalization and properties are investigated. Bardaro et al. [**?**], introduced the concept of triangular A-statistical convergence which is a variant of statistical convergence in 2015. This new convergence offers another perspective as it is not comparable to statistical convergence. In addition, there are other studies in the literature [**?, ?, ?, ?**].

The Korovkin type theorem has an important place in approximation theory as it enables us to check convergence with minimum calculations [**?**]. This theorem has been studied by many mathematicians in different spaces and with various types of convergence, with the aim of obtaining more general results [**?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?**].

Let $C^*\left(\mathbb{R}^2\right)$ stands for the space of all 2π-periodic and continuous functions on $\mathbb{R}^2$.

Our main aim in this study is to present a theorem of Korovkin type on $C^*\left(\mathbb{R}^2\right)$ in the light of the triangular A-statistical convergence given by Bardaro et al.

Before proceeding we recall some notation on the paper.

A double sequence $x=(x_{m,n})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$, the set of all natural numbers, such that $|x_{m,n}-\imath|<\varepsilon$ whenever $m,n>N$, where $\imath$ is called the Pringsheim limit of x and denoted by $P-\lim x=\imath$ (see [**?**]). We shall call such an x, as P-convergent. By a bounded double sequence we mean there exists a $H>0$ such that $|x_{m,n}|\leq H$ for all $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. It is worthy of note that unlike the single sequences, the double sequence does not have to be bounded. Let $A=(a_{k,l,m,n})$ be a four-dimensional summability matrix. For a given double sequence $x=(x_{m,n})$, the A-transform of x, denoted by $Ax:=((Ax)_{k,l})$, is given by

$$(Ax)_{k,l}=\sum_{(m,n)\in\mathbb{N}^2}a_{k,l,m,n}x_{m,n}$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$.

If two dimensional matrix transformation of a given $x=(x_{m,n})$ sequence preserve $(Ax)_{k,l}$ limit, that is $P-\lim x=\imath$ whenever $P-\lim(Ax)_{k,l}=\imath$ then the matrix $A=(a_{k,l,m,n})$ is called a regular matrix.

Let's remember a four dimensional matrix $A=(a_{k,l,m,n})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The well establish characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions or RH-regularity (see, [**?, ?**]) state that a four dimensional matrix $A=(a_{k,l,m,n})$ is RH-regular iff

Email address and ORCID number: **scinar@sinop.edu.tr, 0000-0002-6244-6214** (S. Çınar)
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(i) $P-\lim_{k,l} a_{k,l,m,n}=0$ for each $(m,n)\in\mathbb{N}^2$,

(ii) $P-\lim_{k,l}\sum_{(m,n)\in\mathbb{N}^2} a_{k,l,m,n}=1$,

(iii) $P-\lim_{k,l}\sum_{m\in\mathbb{N}}\left|a_{k,l,m,n}\right|=0$ for each $n\in\mathbb{N}$,

(iv) $P-\lim_{k,l}\sum_{n\in\mathbb{N}}\left|a_{k,l,m,n}\right|=0$ for each $m\in\mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2}\left|a_{k,l,m,n}\right|$ is P-convergent for each $(k,l)\in\mathbb{N}^2$,

(vi) there exist finite $A,B>0$ such that $\sum_{m,n>B}\left|a_{k,l,m,n}\right|<A$ holds for every $(k,l)\in\mathbb{N}^2$.

Firstly let $A=(a_{k,l,m,n})$ be a non-negative RH-regular summability matrix, and let $K\subset\mathbb{N}^2$. Then A-density of K is given as below

$$\delta_A^2(K):=P-\lim_{k,l}\sum_{(m,n)\in K} a_{k,l,m,n}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. Now recall the definiton of A-statistical convergence by considering the concept of A-density. A real double sequence $x=(x_{m,n})$ is said to be A-statistically convergent to a number L if, for every $\varepsilon>0$,

$$\delta_A^2\left(\{(m,n)\in\mathbb{N}^2:|x_{m,n}-\iota|\geq\varepsilon\}\right)=0.$$

At this state, we can show it as $st_A^2-\lim x=\iota$. Also, while $P-\lim x=\iota$, $st_A^2-\lim x=\iota$ is true but when $st_A^2-\lim x=\iota$ is not always $P-\lim x=\iota$. Furhermore, the double sequence does not require to be bounded when $st_A^2-\lim x=\iota$ is exist.
It is worth noting that now with the special choices of the A matrix in concept of A-statistical convergence for double sequences, the following relations are obtained. If one replaces the matrices A the double Cesáro matrix, then A-statistical convergence coincides to the statistical convergence i.e., $st^2_{C(1,1)}-\lim x=st^2-\lim x$ [**?**].

2. Triangular Statistical Convergence

Let $x=(x_{m,n})$ be a double sequence and suppose that $x=(x_{m,n})$ is neither A-statistical convergent nor convergent in the Pringsheim's sense. On the question of whether a different convergence is considered in such a case, Bardaro et al. introduced the notion of triangular A-statistical convergence in [**?**]. First, consider the regular matrix for double sequences [**?**].
The Silverman-Toeplitz conditions, which have an important place in the literature for the regular characterization of the two-dimensional matrix transformation, are as follows (see, for instance, [**?**]).

(i) $\|A\|=\sup_m\sum_{n=1}^{\infty}|a_{m,n}|<\infty$,

(ii) $\lim_m a_{m,n}=0$ for each $n\in\mathbb{N}$,

(iii) $\lim_m\sum_{n=1}^{\infty}a_{m,n}=1$.

Let $A=(a_{m,n})$ be a nonnegative regular summability matrix, K denotes the set $\{(m,n)\in\mathbb{N}^2:n\leq m\}$ and K_m is the m-section of K, i.e., the set of all $n\in\mathbb{N}$ such that $(m,n)\in K$, then we define triangular A-density of K by

$$\delta_A^T(K):=\lim_m\sum_{n\in K_m}a_{m,n}$$

provided that the limit on the right-hand side exists [**?**].
Also,

(i) $\delta_A^T(\mathbb{N}^2)=1$,

(ii) if $K\subset L$ then $\delta_A^T(K)\leq\delta_A^T(L)$,

(iii) if K has triangular A-density then $\delta_A^T(\mathbb{N}^2/K)=1-\delta_A^T(K)$,

triangular A-density has the above properties.

Definition 2.1 ([**?**]). *Let $A=(a_{m,n})$ be a nonnegative regular summability matrix. The number sequence $x=(x_{m,n})$ is triangular A-statistically convergent to ι provided that for every $\varepsilon>0$*

$$\lim_m\sum_{n\in K_m(\varepsilon)}a_{m,n}=0,$$

where $K_m(\varepsilon)=\{n\in\mathbb{N}:n\leq m,\ |x_{m,n}-\iota|\geq\varepsilon\}$ also written as $st_A^T-\lim_m x_{m,n}=\iota$.

The case in which $A=C_1$ the Cesaro matrix of order one reduces to the triangular statistical convergence i.e., $st_A^T-\lim x=st_{C_1}^T-\lim x$. Triangular density $\delta^T(K)$ is given by

$$\delta^T(K)=\lim_m\frac{1}{m}|K_m|$$

or equivalently

$$\delta^T(K) = \lim_m (C_1 \chi_{K_m}(n))_m = \lim_m \sum_{n=1}^{\infty} c_{m,n} \chi_{K_m}(n)$$

if it exists. The number sequence $x = (x_{m,n})$ is triangular statistically convergent to ι provided that for every $\varepsilon > 0$, the set $K := K_m(\varepsilon) := \{n \in \mathbb{N} : n \leq m,\ |x_{m,n} - \iota| \geq \varepsilon\}$ if $\delta^T(K_m(\varepsilon)) = 0$; then we can write $st^T - \lim_m x_{m,n} = \iota$.

Let st_A^T be the set of all triangular A-statistically convergent sequences. As we mentioned before, triangular A-statistical convergence is a variant of statistical convergence. Here we give examples showing that these two convergences cannot be compared.

Example 2.2. *Let $A = C_1$ and*

$$x_{m,n} = \begin{cases} 2, & m = n = j^2 \\ \frac{j}{3(j+1)}, & m = 2j,\ n = 2j+1 \\ \frac{2j}{3(j+2)}, & m = 2j-1,\ n = 2(j+1) \\ 0, & otherwise \end{cases},\ j \in \mathbb{N}.$$

$x = (x_{m,n})$ be given as above. For every $\varepsilon > 0$,

$$\frac{1}{m} |\{n \in \mathbb{N} : n \leq m,\ |x_{m,n} - 0| \geq \varepsilon\}| = \begin{cases} \frac{1}{j^2}, & m = j^2 \\ 0, & otherwise \end{cases},\ j \in \mathbb{N}$$

clearly,

$$\lim_m \frac{1}{m} |\{\ n \in \mathbb{N} : n \leq m,\ |x_{m,n} - 0| \geq \varepsilon\}| = 0.$$

So, we obtain $st_{C_1}^T - \lim_m x_{m,n} = 0$. Nevertheless, $x = (x_{m,n})$ is not Pringsheim's and $C(1,1)$-statistically convergent.

Example 2.3. *Take $A = C(1,1)$ and*

$$x_{m,n} = \begin{cases} \sqrt{mn}, & m = n = j^2 \\ \frac{3}{mn}, & otherwise \end{cases},\ j \in \mathbb{N}.$$

$x = (x_{m,n})$ be given as above. Obviously, $st_{C(1,1)}^2 - \lim_{m,n} x_{m,n} = 0$ but x is not Pringsheim's and triangular statistically convergent.

Example 2.4. *Let $A = C_1$ and*

$$x_{m,n} = \begin{cases} -2, & m = n = j^2 \\ 0, & otherwise \end{cases},\ j \in \mathbb{N}.$$

$x = (x_{m,n})$ be given as above. Similarly, $st_{C_1}^T - \lim_m x_{m,n} = 0$ and $st_{C(1,1)}^2 - \lim_{m,n} x_{m,n} = 0$.

Example 2.5. *Let $A = C_1$ and*

$$x_{m,n} = \begin{cases} 1, & m = n = j^2 \\ \frac{j}{2j+1}, & m = 2j+1,\ n = 2j-1 \\ \frac{j}{4j+2}, & m = 2j,\ n = 2(j+1) \\ k, & m = j^2,\ n = j^2+1 \\ 0, & otherwise \end{cases},\ j \in \mathbb{N}.$$

$x = (x_{m,n})$ be given as above. So, we can easily see that $st_{C_1}^T - \lim_m x_{m,n} = 0$. Neither $x = (x_{m,n})$ is Pringsheim's and $C(1,1)$-statistically convergent nor bounded.

Remark 2.6. *(i) Triangular statistical convergence and statistical convergence are incompatible; i.e., $st_A^T \nsubseteq st_A^2$ and $st_A^2 \nsubseteq st_A^T$.*
(ii) A P-convergent double sequence is A-statistically convergent and triangular A-statistically convergent to the same value but the inverse implications are not true, i.e., $st_A^2 \nsubseteq c^2$ and $st_A^T \nsubseteq c^2$.

3. A Korovkin-Type Approximation Theorem

In this section using the concept of triangular A-statistical convergence for double sequence and test function $1, sins, coss, sint, cost$, we provide a Korovkin type theorem for positive linear operators on the space $C^*(\mathbb{R}^2)$.

If a function f on $\mathbb{R}^2$ has a 2π-period, then, for all $(s,t) \in \mathbb{R}^2$,

$$f(s,t) = f(s + 2k\pi, t) = f(s, t + 2k\pi)$$

holds for $k = 0, \pm 1, \pm 2, \ldots$. This space is equipped with the supremum norm

$$\|f\|_{C^*(\mathbb{R}^2)} = \sup_{(s,t) \in \mathbb{R}^2} |f(s,t)|,\ \left(f \in C^*\left(\mathbb{R}^2\right)\right).$$

Theorem 3.1 ([**?**]). *Let $A = (a_{k,l,m,n})$ be a non-negative RH-regular summability matrix and let $(L_{m,n})$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$. Then, for all $f \in C^*\left(\mathbb{R}^2\right)$*

$$st_A^2 - \lim \|L_{m,n}(f) - f\|_{C^*(\mathbb{R}^2)} = 0$$

iff the following statements hold:

$$st_A^2 - \lim \|L_{m,n}(f_r) - f_r\|_{C^*(\mathbb{R}^2)} = 0, \; r = 0,1,2,3,4,$$

where $f_0(s,t) = 1$, $f_1(s,t) = \sin s$, $f_2(s,t) = \sin t$, $f_3(s,t) = \cos s$ and $f_4(s,t) = \cos t$.

Theorem 3.2. *Let $A = (a_{m,n})$ be a nonnegative regular summability matrix and $(L_{m,n})$ be a double sequence of positive linear operators from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$. Then, for all $f \in C^*\left(\mathbb{R}^2\right)$*

$$st_A^T - \lim_m \|L_{m,n}(f) - f\|_{C^*(\mathbb{R}^2)} = 0 \tag{3.1}$$

iff the following statements hold:

$$st_A^T - \lim_m \|L_{m,n}(f_r) - f_r\|_{C^*(\mathbb{R}^2)} = 0, \; \text{for every } r = 0,1,2,3,4 \tag{3.2}$$

where $f_0(s,t) = 1$, $f_1(s,t) = \sin s$, $f_2(s,t) = \sin t$, $f_3(s,t) = \cos s$ and $f_4(s,t) = \cos t$.

Proof. Under the hypotheses, since 1, *sins*, *coss*, *sint* and *cost* belong to $C^*\left(\mathbb{R}^2\right)$, the necessity is clear. Suppose that (**??**) hold and let $f \in C^*\left(\mathbb{R}^2\right)$ and D, F be closed subinterval of length 2π of $\mathbb{R}$. Fix $(s,t) \in D \times F$. As in the proof of Theorem 2.1 in [**?**], it follows from the continuity of f that

$$|f(u,v) - f(s,t)| < \varepsilon + \frac{2M_f}{\sin^2 \frac{\delta}{2}} \varphi(u,v)$$

which gives,

$$\begin{aligned}
|L_{m,n}(f;s,t) - f(s,t)| \leq & L_{m,n}(|f(u,v) - f(s,t)|;s,t) + |f(s,t)|\,|L_{m,n}(f_0;s) - f_0(s,t)| \\
\leq & \left| L_{m,n}\left(\varepsilon + \frac{2M_f}{\sin^2 \frac{\delta}{2}} \varphi(u,v);s,t\right)\right| + M_f\,|L_{m,n}(f_0;s) - f_0(s,t)| \\
\leq & \left(\varepsilon + M_f\right)|L_{m,n}(f_0;s) - f_0(s,t)| + \frac{M_f}{\sin^2 \frac{\delta}{2}} \{2\,|L_{m,n}(f_0;s) - f_0(s,t)| \\
& + |\sin x|\,|L_{m,n}(f_1;s,t) - f_1(s,t)| + |\sin y|\,|L_{m,n}(f_2;s,t) - f_2(s,t)| \\
& + |\cos x|\,|L_{m,n}(f_3;s,t) - f_3(s,t)| + + |\cos t|\,|L_{m,n}(f_4;s,t) - f_4(s,t)|\} + \varepsilon \\
< & \varepsilon + N \sum_{r=0}^{4} |L_{m,n}(f_r;s) - f_r(s,t)|
\end{aligned}$$

where $M_f = \|f\|_{C^*(\mathbb{R}^2)}$, $\varphi(u,v) = \sin^2 \frac{u-s}{2} + \sin^2 \frac{v-t}{2}$ and $N := \varepsilon + M_f + \frac{2M_f}{\sin^2 \frac{\delta}{2}}$. Then, taking supremum over $(s,t) \in \mathbb{R}^2$, we obtain

$$\|L_{m,n}(f) - f\|_{C^*(\mathbb{R}^2)} < \varepsilon + N \sum_{r=0}^{4} \|L_{m,n}(f_r) - f_r\|_{C^*(\mathbb{R}^2)}. \tag{3.3}$$

Now given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$, and define

$$\begin{aligned}
D_m &:= \left\{ n \in \mathbb{N} : n \leq m, \; \|L_{m,n}(f) - f\|_{C^*(\mathbb{R}^2)} \geq \varepsilon' \right\}, \\
D_m^r &:= \left\{ n \in \mathbb{N} : n \leq m, \; \|L_{m,n}(f_r) - f_r\|_{C^*(\mathbb{R}^2)} \geq \frac{\varepsilon' - \varepsilon}{5N} \right\}, \; r = 0,1,2,3,4.
\end{aligned}$$

It is easy see that from (**??**)

$$D_m \subseteq \bigcup_{r=0}^{4} D_m^r.$$

Hence, we may write

$$\sum_{n \in D_m} a_{m,n} \leq \sum_{m=0}^{4} \sum_{n \in D_m^r} a_{m,n}.$$

Now taking the limit $m \to \infty$, (**??**) yield the result. □

Example 3.3. *We consider the following the double sequence of Fejer operators on $C^*\left(\mathbb{R}^2\right)$*

$$\sigma_{m,n}(f;s,t)=\frac{1}{(m\pi)(n\pi)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)F_m(u)F_n(v)\,dudv \tag{3.4}$$

where $F_m(u)=\frac{\sin^2\frac{m(u-s)}{2}}{2\sin^2\frac{u-s}{2}}$ *and* $\frac{1}{\pi}\int_{-\pi}^{\pi}F_m(u)\,du=1$. *Analyze this*

$$\begin{aligned}&\sigma_{m,n}(f_0;s,t)=f_0(s,t),\ \sigma_{m,n}(f_1;s,t)=\frac{m-1}{m}f_1(s,t),\\&\sigma_{m,n}(f_2;s,t)=\frac{n-1}{n}f_2(s,t),\ \sigma_{m,n}(f_3;s,t)=\frac{m-1}{m}f_3(s,t),\\&\sigma_{m,n}(f_4;s,t)=\frac{n-1}{n}f_4(s,t).\end{aligned} \tag{3.5}$$

Let $A=C_1$ and define a double sequence $(u_{m,n})$ by

$$u_{m,n}=\begin{cases}1, & m=n=k^2\\ \frac{k}{3(k+1)}, & m=2k+1,\ n=2k-1\\ \frac{k}{2(k+1)}, & m=2k,\ n=2(k+1)\\ 0, & otherwise\end{cases},\ k\in\mathbb{N}. \tag{3.6}$$

In this case, observe that

$$st_{C_1}^T-\lim_m u_{m,n}=0. \tag{3.7}$$

*Nevertheless, the sequence $(u_{m,n})$ is not statistically convergent. Also using (**??**) and (**??**), we define the following double positive linear operators on $C^*\left(\mathbb{R}^2\right)$ as follows:*

$$L_{m,n}(f;s,t)=(1+u_{m,n})\,\sigma_{m,n}(f;s,t). \tag{3.8}$$

*Then, observe that the double sequence of positive linear operators $(L_{m,n})$ defined by (**??**) satisfy all hypotheses of Theorem **??**. Therefore, by (**??**) and (**??**), we have, for all $f\in C^*\left(\mathbb{R}^2\right)$,*

$$st_A^T-\lim_m\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}=0.$$

*Since $(u_{m,n})$ is not statistically convergent, the Theorem **??** does not work for our operators $(L_{m,n})$ defined by (**??**).*

Example 3.4. *Fejer operators be the same in Example **??**. Now let $A=C(1,1)$ and define a double sequence $(\beta_{m,n})$ by*

$$\beta_{m,n}=\begin{cases}\sqrt{mn}, & m=n=k^2,\\ \frac{1}{mn} & otherwise.\end{cases} \tag{3.9}$$

Obviously

$$st_{C(1,1)}^2-\lim_{m,n}\beta_{m,n}=0. \tag{3.10}$$

*Combing (**??**) and (**??**), we define the following positive linear operators on $C\left(\mathbb{R}^2\right)$ as follows:*

$$L_{m,n}(f;s,t)=(1+\beta_{m,n})\,\sigma_{m,n}(f;s,t). \tag{3.11}$$

*So, by the Theorem **??** and (**??**), we are seeing this*

$$st_A^2-\lim_{m,n}\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}=0.$$

*Also, since $(\beta_{m,n})$ is not triangular statistical convergent, here we can explain that the Korovkin theorem in triangular statistical sense does not work for operators defined by (**??**).*

4. Rate of Triangular A-Statistical Convergence

Definition 4.1 ([?]). *Let $A=(a_{m,n})$ be a nonnegative regular summability matrix and let (α_m) be a positive non-increasing sequence. A double sequence $x=(x_{m,n})$ is triangular A-statistically convergent to a number $\imath$ with the rate of $o(\alpha_m)$ if for every $\varepsilon>0$,*

$$\lim_m\frac{1}{\alpha_m}\sum_{n\in K_m(\varepsilon)}a_{m,n}=0,$$

where

$$K_m(\varepsilon):=\{\,n\in\mathbb{N}:n\le m,\ |x_{m,n}-\imath|\ge\varepsilon\}.$$

We may write

$$x_{m,n}-\imath=st_A^T-o(\alpha_m)\ \ as\ \ m\to\infty.$$

Definition 4.2 ([**?**]). *Let $A=(a_{m,n})$ and (α_m) be the same as in Definition* **??**. *Then, a double sequence $x=(x_{m,n})$ is triangular A-statistically bounded with the rate of $O(\alpha_m)$ if for every $\varepsilon>0$,*

$$\sup_m \frac{1}{\alpha_m}\sum_{n\in L_m(\varepsilon)} a_{m,n}<\infty,$$

where

$$L_m(\varepsilon):=\{\, n\in\mathbb{N}: n\le m,\ |x_{m,n}|\ge\varepsilon\,\}.$$

In this case, we write $x_{m,n}=st_A^T-O(\alpha_m)$ as $m\to\infty$.

We now use the modulus of continuity $\omega(f;\delta)$, expressed as below:

$$\omega(f;\delta):=\sup\left\{|f(u,v)-f(s,t)|:(u,v),(s,t)\in\mathbb{R}^2,\sqrt{(u-s)^2+(v-t)^2}\le\delta\right\}$$

where $f\in C^*\left(\mathbb{R}^2\right)$ and $\delta>0$. We will use the fundamental inequality to obtain our main result, for all $f\in C^*\left(\mathbb{R}^2\right)$ and for $\lambda,\delta>0$,

$$\omega(f;\lambda\delta)\le(1+[\lambda])\,\omega(f;\delta) \tag{4.1}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.
To obtain our main result we require the following theorem.

Theorem 4.3. *Let $(L_{m,n})$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into itself and let $A=(a_{m,n})$ be a nonnegative regular summability matrix, and let (α_m) and (β_m) be positive non-increasing sequences. Then, for all $f\in C^*\left(\mathbb{R}^2\right)$,*

$$\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}=st_A^T-o(\gamma_m),\ \text{as } m\to\infty,\ \text{with } \gamma_m:=\max\{\alpha_m,\beta_m\} \text{ for each } m\in\mathbb{N}$$

provided that the following conditions hold:
(i) $\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}=st_A^T-o(\alpha_m)$ as $m\to\infty$, with $f_0(u,v)=1$,
(ii) $\omega(f;\delta_{m,n})=st_A^T-o(\beta_m)$ as $m\to\infty$, where $\delta_{m,n}:=\sqrt{\|L_{m,n}(\Psi)\|_{C^*(\mathbb{R}^2)}}$ with $\Psi(u,v)=\sin^2\frac{u-s}{2}+\sin^2\frac{v-t}{2}$ for each $(s,t),(u,v)\in\mathbb{R}^2$.
Also, analogue results holds when the symbol "o" is replaced by "O".

Proof. To express it, we first assume that $(s,t)\in[-\pi,\pi]\times[-\pi,\pi]$ and $f\in C^*\left(\mathbb{R}^2\right)$ be fixed, and that (i) and (ii) hold. Let $\delta>0$. Also, it is as in the the proof Theorem 9 in [**?**]. Using the definition of modulus of continuity and the linearity and the positivity of the operators $L_{m,n}$ for all $(m,n)\in\mathbb{N}^2$, we get

$$\begin{aligned}|L_{m,n}(f;s,t)-f(s,t)|\le&L_{m,n}(|f(u,v)-f(s,t)|;s,t)+|f(s,t)|\,|L_{m,n}(f_0;s,t)-f_0(s,t)|\\ \le&\omega(f;\delta)L_{m,n}(f_0,s,t)+\pi^2\frac{\omega(f;\delta)}{\delta^2}L_{m,n}(\Psi;s,t)+|f(s,t)|\,|L_{m,n}(f_0,s,t)-f_0(s,t)|.\end{aligned}$$

Taking supremum over (s,t) on the both-sides of the above inequality and $\delta:=\delta_{m,n}:=\sqrt{\|L_{m,n}(\Psi)\|_{C^*(\mathbb{R}^2)}}$, then we get

$$\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}\le\omega(f;\delta)\,\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}+\left(1+\pi^2\right)\omega(f;\delta)+M\,\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)} \tag{4.2}$$

where the quantity $M:=\|f\|_{C^*(\mathbb{R}^2)}$ is a finite number since $f\in C^*\left(\mathbb{R}^2\right)$. Then, given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D_m&:=\left\{\,n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}\ge\varepsilon\right\},\\ D_m^1&:=\left\{\,n\in\mathbb{N}:n\le m,\ \omega(f;\delta)\,\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}\ge\frac{\varepsilon}{3}\right\},\\ D_m^2&:=\left\{n\in\mathbb{N}:n\le m,\ \omega(f;\delta)\ge\frac{\varepsilon}{3\left(1+\pi^2\right)}\right\},\\ D_m^3&:=\left\{n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}\ge\frac{\varepsilon}{3M}\right\}.\end{aligned}$$

Then, thanks to (**??**) that $D_m\subset D_m^1\cup D_m^2\cup D_m^3$. Also, defining

$$\begin{aligned}D_m^4&:=\left\{\,n\in\mathbb{N}:n\le m,\ \omega(f;\delta)\ge\sqrt{\frac{\varepsilon}{3}}\right\},\\ D_m^5&:=\left\{\,n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}\ge\sqrt{\frac{\varepsilon}{3}}\right\},\end{aligned}$$

we have $D_m^1\subset D_m^4\cup D_m^5$, which yields

$$D_m\subseteq\bigcup_{r=2}^{5}D_m^r.$$

Therefore, since $\gamma_m := \max\{\alpha_m, \beta_m\}$, we get the result for all $m \in \mathbb{N}$,

$$\frac{1}{\gamma_m}\sum_{n\in D_m} a_{m,n} \leq \frac{1}{\beta_m}\sum_{n\in D_m^2} a_{m,n} + \frac{1}{\alpha_m}\sum_{n\in D_m^3} a_{m,n} + \frac{1}{\beta_m}\sum_{n\in D_m^4} a_{m,n} + \frac{1}{\alpha_m}\sum_{n\in D_m^5} a_{m,n}. \tag{4.3}$$

Letting $m \to \infty$ on both sides of (**??**), we get

$$\lim_{m\to\infty}\frac{1}{\gamma_m}\sum_{n\in D_m} a_{m,n} = 0.$$

Thus ends the proof. □

Now, having experienced from Theorem **??**, we can introduce the ordinary rates of convergence of a sequence of positive linear operators defined on the space $C^*\left(\mathbb{R}^2\right)$. Firstly, we should point out if we choose $\alpha_m = \beta_m = 1$ for all $m \in \mathbb{N}$, then Theorem **??** is get from Theorem **??** at once. So our theorem gives us the rate of triangular A-statistical convergence in Theorem **??**.

5. An Application to Theorem ??

Let $A = (a_{m,n})$ be a nonnegative regular summability matrix. Then, we consider the following operators defined by (**??**) on $C^*\left(\mathbb{R}^2\right)$:

$$L_{m,n}(f;s,t) = (1+u_{m,n})\,\sigma_{m,n}(f;s,t). \tag{5.1}$$

Then, we take $A = C_1 := (c_{m,n})$, the Cesáro matrix. Then, setting $(\alpha_m) = \left(\frac{1}{\sqrt{m}}\right)$, we get, for any $\varepsilon > 0$,

$$\frac{1}{\alpha_m}\sum_{n:|u_{i,j}|\geq\varepsilon} c_{m,n} = \sqrt{m}\sum_{n:|u_{m,n}|\geq\varepsilon}\frac{1}{m} \leq \frac{2\sqrt{m}}{m} = \frac{2}{\sqrt{m}}. \tag{5.2}$$

Taking the limit as $m \to \infty$ in (**??**), we get, for any $\varepsilon > 0$,

$$\lim_{m}\frac{1}{\alpha_m}\sum_{n:|u_{m,n}|\geq\varepsilon} c_{m,n} = 0$$

which gives,

$$u_{m,n} = st_A^T - o\left(\frac{1}{\sqrt{m}}\right) \text{ as } m \to \infty. \tag{5.3}$$

Also, observe that

$$\begin{aligned}
L_{m,n}(f_0;s,t) &= (1+u_{m,n}),\\
L_{m,n}(f_1;s,t) &= (1+u_{m,n})\frac{m-1}{m}f_1(s,t),\\
L_{m,n}(f_2;s,t) &= (1+u_{m,n})\frac{n-1}{n}f_2(s,t),\\
L_{m,n}(f_3;s,t) &= (1+u_{m,n})\frac{m-1}{m}f_3(s,t),\\
L_{m,n}(f_4;s,t) &= (1+u_{m,n})\frac{n-1}{n}f_4(s,t),
\end{aligned}$$

where $f_0(s,t) = 1$, $f_1(s,t) = \sin s$, $f_2(s,t) = \sin t$, $f_3(s,t) = \cos s$ and $f_4(s,t) = \cos t$. Since $\|L_{m,n}(f_0) - f_0\|_{C^*(\mathbb{R}^2)} = u_{m,n}$, we obtain from (**??**)

$$\|L_{m,n}(f_0) - f_0\|_{C(\mathbb{R}^2)} = st_A^T - o(\alpha_m) \text{ as } m \to \infty. \tag{5.4}$$

Now, we calculate the quantity $L_{m,n}(\Psi;s,t)$, where $\Psi(u,v) = \sin^2\frac{u-s}{2} + \sin^2\frac{v-t}{2}$. After some calculations, we have

$$L_{m,n}(\Psi;s,t) = \frac{1+u_{m,n}}{2}\left(\frac{1}{m}+\frac{1}{n}\right).$$

So, we get $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|_{C^*(\mathbb{R}^2)}} = \sqrt{\frac{1+u_{m,n}}{2}\left(\frac{1}{m}+\frac{1}{n}\right)}$. In this case, setting $(\beta_m) = \left(\frac{1}{\sqrt[4]{m}}\right)$, we have, for any $\varepsilon > 0$,

$$\frac{1}{\beta_m}\sum_{n:|\delta_{m,n}|\geq\varepsilon} c_{k,l,m,n} = \sqrt[4]{m}\sum_{n:|\delta_{m,n}|\geq\varepsilon}\frac{1}{m} \leq \frac{2\sqrt[4]{m}}{m} = \frac{2}{\sqrt[4]{m^3}}$$

which gives that

$$\lim_{m}\frac{1}{\beta_m}\sum_{n:|\delta_{m,n}|\geq\varepsilon} c_{k,l,m,n} = 0.$$

Hence, we obtain $\delta_{m,n} = st_{C_1}^T - o(\frac{1}{\sqrt[4]{m}})$ as $m \to \infty$. By the uniform continuity of f on $\mathbb{R}^2$, we can write as follows:

$$\omega(f;\delta_{m,n}) = st_{C_1}^T - o(\frac{1}{\sqrt[4]{m}}) \text{ as } m \to \infty. \tag{5.5}$$

Then, the sequence of positive linear operators $(L_{m,n})$ satisfy all hypotheses of Theorem **??** from (**??**) and (**??**). So, we have, for all $f \in C^*\left(\mathbb{R}^2\right)$,

$$\|L_{m,n}(f) - f\|_{C^*(\mathbb{R}^2)} = st_{C_1}^T - o(\frac{1}{\sqrt[4]{m}}) \text{ as } m \to \infty.$$

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On the Spectrum of the Non-Selfadjoint Differential Operator with an Integral Boundary Condition and Negative Weight Function

Nimet Çoşkun[1] and Merve Görgülü[1*]

[1]*Department of Mathematics, Kamil Özdağ Faculty of Science, Karamanoğlu Mehmetbey University, Karaman, Türkiye*
[*]*Corresponding author*

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Abstract

In this paper, we shall study the spectral properties of the non-selfadjoint operator in the space $L^2_\rho(\mathbb{R}_+)$ generated by the Sturm-Liouville differential equation

$$-y'' + q(x)y = \omega^2 \rho(x)y, \quad x \in \mathbb{R}_+$$

with the integral type boundary condition

$$\int_0^\infty G(x)y(x)dx + \gamma y'(0) - \theta y(0) = 0$$

and the non-standard weight function

$$\rho(x) = -1$$

where $|\gamma| + |\theta| \neq 0$. There are an enormous number of papers considering the positive values of $\rho(x)$ for both continuous and discontinuous cases. The structure of the weight function affects the analytical properties and representations of the solutions of the equation. Differently from the classical literature, we used the hyperbolic type representations of the fundamental solutions of the equation to obtain the spectrum of the operator. Moreover, the conditions for the finiteness of the eigenvalues and spectral singularities were presented. Hence, besides generalizing the recent results, Naimark's and Pavlov's conditions were adopted for the negative weight function case.

1. Introduction

Differential equations, particularly the ones with integral boundary conditions, have been inevitable tools in modeling natural phenomena such as thermodynamics, liquid flow, and demographics, see [1]. Modeling the vibration of a loaded string, equations of gas dynamics, and the theory of shock waves are a few quite interesting examples of a vast research area in mathematical physics that makes use of boundary value problems with a boundary condition involving spectral parameters in it [2]. Therefore in this paper we will focus on Sturm-Liouville operator generated by well-known one dimensional Schrödinger equation

$$-y'' + q(x)y = \omega^2 \rho(x)y, \quad x \in \mathbb{R}_+ \tag{1.1}$$

where ω is a spectral parameter and ρ is the weight function under the integral boundary condition.
The utility stemmed from the interconnection of studies on direct and inverse problems with the methods of solving many problems in mathematical analysis, keeps this research area vigorous [3–7]. This productive and efficient subject area, originated by the pioneer work of

Email addresses and ORCID numbers: **cannimet@kmu.edu.tr, 0000-0001-9753-0101 (N. Çoşkun), mervegorgulu@kmu.edu.tr, 0000-0002-0565-8034 (M. Görgülü)**
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Naimark dealing with the singular non-self-adjoint problem for $\rho(x)=1$, finds itself specialized sub-areas governing different but connected techniques, for example, cases considering positive weight [8–13], non-continuous weight [14–17], sign-changing weight [18–20] as well as discrete cases [21–28]. Especially, the spectral singularities of the non-selfadjoint problem under the integral boundary condition has been investigated in [9, 10].
At first, Gasymov's approach in considering the sign-changing weight function for the inverse problem of the Sturm-Liouville type operator yielded different results from the previous literature [18]. Besides the fact that the appearance of these weight functions enriched the study area with applications in physics, the analytical difficulties arising from the negative sign made the problem even more attractive.
In the Sturm-Liouville problem, hyperbolic-type solutions obtained depending on the negative weight function cause some analytical difficulties, as well as the necessity of re-evaluation of conventional techniques. In this paper, the spectral properties of the non-selfadjoint singular Sturm-Liouville type operator, under the integral boundary condition and the non-standard weight function $\rho(x)=-1$ shall be analyzed. We engage with this problem owing to the deficiency in the studies investigating the requirements of the analytical solutions of Sturm-Liouville equation in distinct regions.
Let us also remark that, while the transformation chosen for the eigenparameter determines the analytical properties of the Jost solutions in discrete problems; the structure of the weight function affects the Jost solution in differential case. Hence, based on this idea, this paper may also lay the groundwork for new research topics in both inverse and direct problems. This paper has also a crucial importance since this is the first study which considers the negative value of a weight function for singular non-selfadjoint operators under the integral boundary condition. Therefore, we adopt the recent results to the negative weight function case and obtain new results which might give rise to the new research topics.
This article is structured as follows: Section 2 presents the general solution to (1.1) subject to the integral boundary condition in terms of the fundamental solutions to the boundary value problem (1.1) with negative weight function. Later in the same section, we obtain resolvent set in terms of these solutions. In Section 3, more general theorems for eigenvalues and spectral singularities concerning some additional and more strict conditions on the potential function are provided.

Notation. *Let ω be a complex parameter. In this paper, for the complex left half-plane, we set the notation $\mathbb{C}_{left}:=\{\omega\in\mathbb{C}:\operatorname{Re}\omega<0\}$. As usual topological relatives, we use $\overline{\mathbb{C}}_{left}$ for its completion, and $\partial\mathbb{C}_{left}$ for its boundary set. We denote number of elements in a set A with $\#A$ and the linear Lebesgue measure of a Lebesgue measurable set A with $\mu(A)$.*

2. Solutions of the problem

In this part, we present some preliminary results for the negative weight function case which can be deduced similar to the theorems and techniques in [4–6, 8, 9].
Let $\mathscr{T}$ be the operator in $L^2_\rho(\mathbb{R}_+)$ generated by the differential equation

$$-y''+q(x)y=\omega^2\rho(x)y,\quad x\in\mathbb{R}_+ \tag{2.1}$$

with the integral boundary condition

$$\int_0^\infty G(x)y(x)\,dx+\gamma y'(0)-\theta y(0)=0 \tag{2.2}$$

and the non-standard weight function

$$\rho(x)=-1,\quad x\in\mathbb{R}_+ \tag{2.3}$$

where γ,θ are complex numbers with $|\gamma|+|\theta|\neq 0$, and ω is spectral parameter. Note that q and G are complex valued functions, such that $G\in L^1_\rho(\mathbb{R}_+)\cap L^2_\rho(\mathbb{R}_+)$, and q satisfies the following condition:

$$\int_0^\infty s|q(s)|\,ds<\infty. \tag{2.4}$$

Let us denote by $S(x,\omega)$ and $C(x,\omega)$, the solutions of (2.1) subject to the initial conditions

$$\begin{aligned} S(0,\omega)&=0,\\ C(0,\omega)&=1,\\ \left.\frac{\partial}{\partial x}S(x,\omega)\right|_{x=0}&=1,\\ \left.\frac{\partial}{\partial x}C(x,\omega)\right|_{x=0}&=0. \end{aligned}$$

Consider the case $q(x)\equiv 0$. Then, (2.1) takes the form

$$y''=\omega^2 y,\quad x\in\mathbb{R}_+.$$

Thus, $S(x,\omega)$ and $C(x,\omega)$ can be represented by the hyperbolic type functions

$$\begin{aligned} S(x,\omega)&=\frac{\sinh\omega x}{\omega},\\ C(x,\omega)&=\cosh\omega x. \end{aligned}$$

Using the results of [4] and constant coefficients method, one can easily show that the fundamental solutions $S(x,\omega)$ and $C(x,\omega)$ have the Volterra type integral representations as

$$S(x,\omega)=\frac{\sinh\omega x}{\omega}+\int_0^x\frac{\sinh\omega(x-t)}{\omega}q(t)S(t,\omega)dt,$$

and

$$C(x,\omega)=\cosh\omega x+\int_0^x\frac{\sinh\omega(x-t)}{\omega}q(t)S(t,\omega)dt.$$

Moreover, both functions $S(\cdot,\omega)$ and $C(\cdot,\omega)$ are entire in ω. They are also analytic on $\overline{\mathbb{C}}_{left}$. Existence and uniqueness results of the solutions $S(x,\omega)$ and $C(x,\omega)$ can also be proven analogous to [4]. Also, Wronskian of the solutions $S(x,\omega)$ and $C(x,\omega)$ can be written as

$$W[S(x,\omega),C(x,\omega)]=-1,\quad \omega\in\mathbb{C}.$$

Now, let us denote the Jost solution of the operator $\mathscr{T}$ by $e(x,\omega)$ which is the solution of (2.1) satisfying the asymptotic condition

$$\lim_{x\to\infty}e(x,\omega)e^{-\omega x}=1,\quad \omega\in\mathbb{C}_{left}. \tag{2.5}$$

Under the condition (2.4), this solution can be found as

$$e(x,\omega)=e^{\omega x}+\int_x^\infty K(x,s)e^{\omega s}ds, \tag{2.6}$$

where the kernel K is uniquely determined by the potential function q such that $K(x,.)\in L_1(0,\infty)$ and it is continuously differentiable with respect to its arguments.
On the same manner with [4], we deduce that the Jost solution $e(\cdot,\omega)$ is analytic in $\mathbb{C}_{left}$ and continuous on $\overline{\mathbb{C}}_{left}$ from the validity of the inequality

$$|K(x,s)|\le c\int_{\frac{x+s}{2}}^\infty|q(\tau)|d\tau,\quad x\le s<\infty, \tag{2.7}$$

for any constant $c>0$ independent of the variables x and s.
Denote by $g(x,\omega)$, another solution of (2.1) satisfying the asymptotic conditions

$$\begin{aligned}&\lim_{x\to\infty}g(x,\omega)e^{\omega x}=1,\quad \omega\in\overline{\mathbb{C}}_{left},\\&\lim_{x\to\infty}g_x(x,\omega)e^{\omega x}=-\omega,\quad \omega\in\overline{\mathbb{C}}_{left}.\end{aligned} \tag{2.8}$$

By the help of the asymptotic identities (2.5) and (2.8), the Wronskian of e and g can be found as

$$W[e(x,\omega),g(x,\omega)]=-2\omega,\quad \omega\in\overline{\mathbb{C}}_{left}, \tag{2.9}$$

which concludes that e and g form the fundamental system of solutions for (2.1) on $\partial\mathbb{C}_{left}$.
For the complex parameter ω, define the functions

$$N(\omega)=\int_0^\infty G(s)e(s,\omega)ds+\gamma e_x(0,\omega)-\theta e(0,\omega),$$

$$M(\omega)=\int_0^\infty G(s)g(s,\omega)ds+\gamma g_x(0,\omega)-\theta g(0,\omega),$$

and for $t\in\mathbb{R}_+$,

$$u(t,\omega)=\frac{-1}{2\omega}\left\{g(t,\omega)\int_t^\infty G(s)e(s,\omega)ds-e(t,\omega)\int_t^\infty G(s)g(s,\omega)ds+M(\omega)e(t,\omega)\right\}.$$

Clearly, resolvent operator of $\mathscr{T}$ can be obtained as

$$R_\omega(\mathscr{T})\phi=\int_0^\infty\mathscr{G}(x,t;\omega)\phi(t)dt,\quad \phi\in L^2_\rho(\mathbb{R}_+).$$

Here we set the notation $\mathscr{G}(x,t;\omega)$ for the Green's function of $\mathscr{T}$ defined as

$$\mathscr{G}(x,t;\omega)=\mathscr{G}^{(1)}(x,t;\omega)+\mathscr{G}^{(2)}(x,t;\omega),$$

with the functions

$$\mathscr{G}^{(1)}(x,t;\omega) := \frac{-e(x,\omega)\,u(t,\omega)}{N(\omega)},$$

$$\mathscr{G}^{(2)}(x,t;\omega) := -\begin{cases} \frac{e(x,\omega)u(t,\omega)}{2\omega}, & 0 \le t < x, \\ \frac{e(t,\omega)u(x,\omega)}{2\omega}, & x \le t < \infty. \end{cases}$$

Hence, the resolvent set $R_\omega(\mathscr{T})$ is given by

$$R_\omega(\mathscr{T}) = \left\{\eta : \eta = \omega^2, \operatorname{Re}\omega < 0, N(\omega) \neq 0\right\}.$$

3. Spectrum of $\mathscr{T}$

In this section in order to deal with the quantitative structure of the spectrum of $\mathscr{T}$, we will investigate the sets of zeros of the function N on left half plane and on its boundary, respectively. Let us denote these sets by

$$Z_1 := \left\{\omega : \omega \in \mathbb{C}_{left}, N(\omega) = 0\right\},$$
$$Z_2 := \left\{\omega : \omega \in \partial\mathbb{C}_{left}, N(\omega) = 0\right\}.$$

Recall that the multiplicity of a zero in the region $\overline{\mathbb{C}}_{left}$ is called the multiplicity of the corresponding eigenvalue and spectral singularity of the operator [7,8]. According to this definition, Z_3 denotes the set of all the accumulation points of Z_1 and Z_4 denotes the set of all zeros of N in $\overline{\mathbb{C}}_{left}$ with infinite multiplicity.
Notice that the set of eigenvalues of $\mathscr{T}$ is related to the set of zeros Z_1

$$\sigma_d(\mathscr{T}) = \left\{z : z = \omega^2, \omega \in Z_1\right\}, \tag{3.1}$$

and the set of spectral singularities $\mathscr{T}$ is related to the set of zeros Z_2

$$\sigma_{ss}(\mathscr{T}) = \left\{z : z = \omega^2, \omega \in Z_2\right\} \setminus \{0\}. \tag{3.2}$$

Within the same circle of ideas in the proofs of the theorems from [4,8,9], by the classical definition of spectrum of a differential operator we obtain that the set $\sigma_c(\mathscr{T})$ defined as

$$\sigma_c(\mathscr{T}) = \{z : z = i\tau, \tau \ge 0\}$$

is the continuous spectrum of $\mathscr{T}$.

Lemma 3.1. *Suppose $G \in L^1_\rho(\mathbb{R}_+) \cap L^2_\rho(\mathbb{R}_+)$ and (2.4) holds, then*

(i) Z_1 is bounded, $\#Z_1$ is at most countable, and Z_3 is a subset of a bounded interval of $\partial\mathbb{C}_{left}$,
(ii) Z_2 is a compact set with $\mu(Z_2) = 0$.

Proof. Using the inequality (2.6) and the expression of $N(\omega)$, it can be easily seen that $N(\omega)$ is analytic with respect to ω in $\mathbb{C}_{left}$ and continuous on the imaginary axis. Also, it yields the asymptotic

$$N(\omega) = \gamma\omega + \theta + o(1), \quad \omega \in \overline{\mathbb{C}}_{left}, |\omega| \to \infty, \tag{3.3}$$

for $|\gamma| + |\theta| \neq 0$. The boundedness of the sets Z_1 and Z_2 follows from (3.3). Hence, the proof of part (a) results from analicity of $N(\omega)$ in $\mathbb{C}_{left}$ and continuity on the imaginary axis. For the part (b), we shall consider the boundary uniqueness theorems of analytic functions [29]. Using these theorems, we get that Z_2 is a closed set and $\mu(Z_2) = 0$. □

The following theorem can be stated easily using (3.1), (3.2) and Lemma 3.1 :

Theorem 3.2. *Suppose $G \in L^1_\rho(\mathbb{R}_+) \cap L^2_\rho(\mathbb{R}_+)$ and (2.4) holds. Then,*

(i) $\sigma_d(\mathscr{T})$ is bounded, $\#\sigma_d(\mathscr{T})$ is at most countable, and the set of its limit points is contained in a bounded interval of $\partial\mathbb{C}_{left}$.
(ii) $\sigma_{ss}(\mathscr{T})$ is a bounded set with zero measure.

From now on, we will consider the spectral properties of $\mathscr{T}$ under more strict conditions on the potential. Firstly we consider the Naimark's condition

$$\int_0^\infty e^{\varepsilon\tau}(|q(\tau)| + |G(\tau)|)\,d\tau < \infty, \tag{3.4}$$

for any $\varepsilon > 0$, which enables us to use analytic continuation properties of the Jost function for the proof.

Theorem 3.3. *Suppose the condition (3.4) holds true. Then $\mathscr{T}$ possesses finitely many eigenvalues and spectral singularities and each one has finite multiplicity.*

Proof. (2.7) and (3.4) make it clear that

$$|K(x,s)| \leq Ae^{-\frac{\varepsilon(x+s)}{2}}, \tag{3.5}$$

for arbitrary positive constant A. Considering the expression of $N(\omega)$ and (3.5), it is clear that $N(\omega)$ continues analytically from $\mathbb{C}_{left}$ to the right half-plane $\{\omega : \mathrm{Re}\omega < \frac{\varepsilon}{4}\}$. As a consequence, the limit points of the zeros of $N(\omega)$ in $\mathbb{C}_{left}$ cannot lie in the imaginary axis. From the results of Lemma 3.1, we can see that the sets Z_1 and Z_2 are bounded and both have a finite number of elements. Also taking into account the analicity of $N(\omega)$ for $\{\omega : \mathrm{Re}\omega < \frac{\varepsilon}{4}\}$, we deduce that the zeros of $N(\omega)$ in $\mathbb{C}_{left}$ are of finite number and they are of finite multiplicity, which concludes the assertion of theorem. □

However, there is more strict condition for the potential called Pavlov's condition which pushes us to use new methods to prove the finiteness of the sets $\sigma_d(\mathscr{T})$ and $\sigma_{ss}(\mathscr{T})$. Let the following integral condition holds true:

$$\int_0^\infty \exp(\varepsilon\tau^\delta)(|q(\tau)|+|G(\tau)|)\,d\tau < \infty, \quad \frac{1}{2} \leq \delta < 1 \tag{3.6}$$

for any $\varepsilon > 0$. Clearly, $N(\omega)$ is analytic in the complex left-half plane $\mathbb{C}_{left}$ and continuous on the imaginary axis. Nevertheless, analytic continuation property does not hold from the left-half plane to the right-half plane. We will also benefit from the subsequent relations between the sets Z_1, Z_2, Z_3 and Z_4 in order to verify the following theorem which can be inferred directly from the boundary uniqueness theorems of analytic functions [29]:

$$Z_1 \cap Z_4 = \emptyset, \quad Z_3 \subset Z_4 \subset Z_2, \tag{3.7}$$

and

$$\mu(Z_3) = \mu(Z_4) = 0.$$

Theorem 3.4. *If the condition for the potential (3.6) holds to be true, then $Z_4 = \emptyset$.*

Proof. Using Lemma 3.1., we obtain that

$$\left|\int_{-\infty}^{-T} \frac{\ln|N(\omega)|}{1+\omega^2}d\omega\right| < \infty, \quad \left|\int_T^\infty \frac{\ln|N(\omega)|}{1+\omega^2}d\omega\right| < \infty, \tag{3.8}$$

for sufficiently large values of $T > 0$. Moreover, $N(\omega)$ is analytic in $\mathbb{C}_{left}$, all its derivatives are continuous up to the imaginary axis and

$$\left|N^{(r)}(\omega)\right| \leq C_r, \quad \omega \in \overline{\mathbb{C}}_{left}, \quad r = 1,2,\ldots, \quad |\omega| < 2T, \tag{3.9}$$

where

$$C_r := c\int_0^\infty s^r |K(0,s)|\,ds. \tag{3.10}$$

If we make use of (3.8), (3.9) and Pavlov's theorem, we get

$$\int_0^\omega \ln t(s)\,d\mu(Z_{4,s}) > -\infty, \tag{3.11}$$

where $t(s) := \inf_r \frac{C_r s^r}{r!}$ for $s \geq 0$, and $\mu(Z_{4,s})$ is the linear Lebesgue measure of the $s-$neighborhood of Z_4 [8,9]. We can also estimate C_r from above

$$C_r = c\int_0^\infty s^r |K(0,s)|\,ds \leq c\int_0^\infty s^r e^{-\frac{\varepsilon}{4}s}\,ds \leq Bb^r r^r r!, \tag{3.12}$$

for constants B and b depending on c and δ. When estimate (3.12) is substituted in the definition of t implies that

$$t(s) = \inf_r \frac{C_r s^r}{r!} \leq B\inf_r\{b^r s^r r^r\} \leq Be^{-s^{-1}e^{-1}b^{-1}}. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}}\,d\mu(Z_{4,s}) < \infty. \tag{3.14}$$

Then the inequality $\frac{\delta}{1-\delta} \geq 1$, together with (3.14) ensures that, for arbitrary s, $\mu(Z_{4,s}) = 0$ or $Z_4 = \emptyset$. □

Theorem 3.5. *Suppose that the condition (3.6) holds true. Then $\mathscr{T}$ possesses finitely many eigenvalues and spectral singularities and each one is of finite multiplicity..*

Proof. It would clearly have been necessary to show that $N(\omega)$ acquires finitely many zeros with finite multiplicities in $\overline{\mathbb{C}}_{left}$. When we applied the previous theorem, the relation (3.7) just amounts to saying that $Z_3 = \emptyset$. That is to say, the bounded sets Z_1 and Z_2 cannot possess accumulation points. Therefore, the zeros of $N(\omega)$ in $\overline{\mathbb{C}}_{left}$ are finitely many. The fact $Z_4 = \emptyset$ concludes that these zeros are of finite multiplicity. □

4. Conclusion

In this paper, we investigated the spectrum of the operator constructed by the help of differential Sturm-Liouville type operator and negative valued weight function. The specific feature of this study is that we obtain the spectrum using the hyperbolic type fundamental solutions. We also impose an integral boundary condition and this also effects the structure of the Naimark's and Pavlov's conditions. There are so many papers considering the trigonometric type fundamental solutions. Also, this paper is the differential analog of the hyperbolic type problems in discrete operators. Therefore, we bring a novel viewpoint to the recent papers and this paper may lay the goundwork for future studies.

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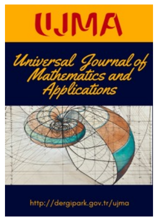

Qualitative Analysis of a Nicholson-Bailey Model in Patchy Environment

Rizwan Ahmed[1*] and Shehraz Akhtar[2]

[1]*Department of Mathematics, Air University Multan Campus, Multan, Pakistan*
[2]*Department of Mathematics, The Islamia University of Bahawalpur, Rahim Yar Khan Campus, Rahim Yar Khan, Pakistan*
[*]*Corresponding author*

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Abstract

We studied a host-parasite model qualitatively. The host-parasitoid model is obtained by modifying the Nicholson-Bailey model so that the number of hosts that parasitoids cannot attack is fixed. We investigate the effect of the presence of a refuge on the local stability and bifurcation of models. Topological classification of equilibria is achieved with the implementation of linearization. Furthermore, Neimark-Sacker bifurcation is explored using the bifurcation theory of normal forms at interior steady-state. The bifurcation in the model is controlled by implementing two control strategies. The theoretical studies are backed up by numerical simulations, which show the conclusions and their importance. A low rate of escaping of a host may lead to instability.

1. Introduction

The Nicholson-Bailey model [1] was proposed by Nicholson and Bailey in 1935 to model and study a biological system involving two insects: a host and a parasitoid. The parasitoid is a free-living adult parasite that lays eggs on the host larvae, and these eggs may survive to give birth to the next generation. The parasitoid hosts die, and the non-parasitoid hosts produce their offspring. There are some unnatural suppositions in the Nicholson-Bailey model, for instance, a homogeneous environment, a constant searching efficiency, and the reproductive rate of the host. These assumptions produce unstable positive fixed points for all the parametric values and lead to oscillations in the Nicholson-Bailey model at low parasitoid densities. By relaxing the homogeneous environment assumption and assuming a patchy environment, a proportion of the host population could hide away or refuge and be secure from the attack of parasitoids. Therefore, a modified Nicholson-Bailey model has been proposed by MP Hassell [2] and is given as

$$\begin{cases} H_{t+1} = r(1-\gamma)H_t + r\gamma H_t \exp(-aP_t), \\ P_{t+1} = e\gamma H_t(1-\exp(-aP_t)), \end{cases} \tag{1.1}$$

where H_t is the population size of the host in generation t and P_t is the parasitoid population size in generation t, r refers to the reproductive rate of the host, a to the efficiency with which the parasitoid searches for a host, and e represents the average number of viable eggs laid by a parasitoid on a single host, γ is the percentage of hosts that are vulnerable to parasitoids, and $1-\gamma$ shows how many are safe from parasities when they are in a refuge. It is evident to see that if we take $\gamma = 1$ in the system (1.1), then we retrieve the classical Nicholson-Bailey model

$$\begin{cases} H_{t+1} = rH_t \exp(-aP_t), \\ P_{t+1} = eH_t(1-\exp(-aP_t)), \end{cases} \tag{1.2}$$

where r, a, and e are positive constants. The parameters r, a, and e have the same biological interpretations as those in the previous model (1.2). Unfortunately, this classical model failed to produce a stable equilibrium. Several authors attempted to modify the model in order to achieve a more realistic and stable system.

***Email addresses and ORCID numbers*:** **rizwanahmed488@gmail.com, 0000-0002-4900-0488 (R. Ahmed), shahraz5768@gmail.com, 0000-0002-9212-6887 (S. Akhtar)**
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Another way to model the effect of a refugee can be achieved by sheltering a certain quantity of hosts denoted as H_0, which are immune to being attacked by parasitoids, and another modification [3] in the Nicholson-Bailey model is given by

$$\begin{cases} H_{t+1} = rH_0 + r(H_t - H_0)\exp(-aP_t), \\ P_{t+1} = e(H_t - H_0)(1-\exp(-aP_t)). \end{cases} \tag{1.3}$$

The mathematical modeling of population dynamics has been developed as a significant area of research within the last decade. The mathematical models described by exponential difference equations are extensively used to study population dynamics [16]. Nonlinear difference equations appear naturally in mathematical modeling as they provide a more flexible framework to model different biological systems' dynamics [4, 5]. These equations are the discrete-time counterparts of differential equations, which are used extensively in engineering and the biological sciences [6, 7]. The study of the consequences of the hiding behavior of host on the dynamics of host-parasitoid systems can be recognized as a major issue in applied mathematics and theoretical ecology. Some of the empirical and theoretical work have investigated the effect of host refuges and drawn a conclusion that the refuges used by host have a stabilizing effect on the considered interactions and host extinction can be prevented by the addition of refuges.
A complete examination of the qualitative behavior of models given by nonlinear difference equations, including local and global stability, bifurcation analysis, and chaos control, may be found in [8–15]. Q. Din [16] examined the qualitative behavior of the model (1.3). Specifically, the author examined the boundedness and persistence, the presence and uniqueness of steady-state, the local and global stability of the unique positive fixed point, and the rate of convergence of all solutions that converge to the fixed point for the model (1.3).
The motivation of our work is to study the impact of the refuge effect on the host population in the modified Nicholson-Bailey model. In this research, we investigate the qualitative behavior of the model (1.1) by identifying the unique positive fixed point, the parametric conditions for the local stability of the unique positive fixed point, and the presence of the Neimark-Sacker bifurcation at the positive fixed point, and by implementing the control strategies to control the Neimark-Sacker bifurcation in the model (1.1). In the end, some numerical examples are provided, followed by a necessary discussion on the qualitative behavior of the model (1.1).
The following describes the structure of the paper:
The derivation of a necessary and sufficient condition for the local asymptotic stability of the fixed point of the model (1.1) is given in Section 2. The Neimark-Sacker bifurcation at the unique positive fixed point is the subject of Section 3. In Section 4, two control techniques are employed to control the bifurcation in the model. The dependence of the model on the parameters γ and r is illustrated in Section 5. Section 6 has some final observations.

2. Local Stability of Positive Fixed Point

It is simple that $(0,0)$ and $(H_*,P_*) = \left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{ae(r-1)}, \frac{1}{a}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)$ are the fixed points of the system (1.1). Also, for $r>1$ and $\gamma > \frac{r-1}{r}$, (H_*,P_*) is the unique positive fixed point of system (1.1). The system will have to be linearized for stability analysis using the variational matrix at the fixed point (H_*,P_*). For the fixed point, $(H_*,P_*) = \left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{ae(r-1)}, \frac{1}{a}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)$, the variational matrix is

$$J(H_*,P_*) = \begin{bmatrix} 1 & -\frac{r(1-r(1-\gamma))}{e(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \\ \frac{e(r-1)}{r} & \frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \end{bmatrix}.$$

The characteristic polynomial of the variational matrix is given by

$$C(z) = z^2 - \left(1 + \frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)z + \frac{r(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right). \tag{2.1}$$

The following lemma is very important for both the topological categorization of the fixed points and the determination of the criteria that are necessary as well as sufficient for the local stability of the fixed points.

Theorem 2.1 ([17]). *Let $C(z) = z^2 - A\lambda + B$, and $C(1) > 0$ with z_1, z_2 be the roots of $C(z) = 0$. Then the following results hold:*

(i) $|z_1| < 1$ and $|z_2| < 1$ iff $C(-1) > 0$ and $C(0) < 1$.
(ii) $|z| < 1$ and $|z| > 1$, or $|z| > 1$ and $|z| < 1$ iff $C(-1) < 0$.
(iii) $|z_1| > 1$ and $|z_2| > 1$ iff $C(-1) > 0$ and $C(0) > 1$.
(iv) $z_1 = -1$ and $z_2 \neq 1$ iff $C(-1) = 0$ and $C(0) \neq \pm 1$.
(v) z_1 and z_2 are complex and $|z_1| = 1$ and $|z_2| = 1$ iff $A^2 - 4B < 0$ and $C(0) = 1$.

By using simple computations, we have

$$C(1) = \left(\ln(\frac{r\gamma}{1-r(1-\gamma)})\right)(1-r(1-\gamma)),$$

$$C(-1) = 2 + \frac{\left(\ln(\frac{r\gamma}{1-r(1-\gamma)})\right)(1-r(1-\gamma))(r+1)}{(r-1)},$$

$$C(0) = \frac{r\left(\ln(\frac{r\gamma}{1-r(1-\gamma)})\right)(1-r(1-\gamma))}{(r-1)}.$$

Notice that for all $r>1$ and $\gamma>\frac{r-1}{r}$, we have $C(1)>0$ and $C(-1)>0$. Therefore, cases (ii) and (iv) of Theorem 2.1 are not possible. It means that (H_*,P_*) in the system (1.1) is not a saddle point because case (ii) of Theorem 2.1 is not true and period-doubling bifurcation is not possible because case (iv) of Theorem 2.1 is not true.

Theorem 2.2. *Suppose that $r>1$, and $\gamma>\frac{r-1}{r}$. The unique fixed point (H_*,P_*) of the system (1.1) is*

(i) stable iff

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))<r-1,$$

(ii) unstable iff

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))>r-1,$$

(iii) non-hyperbolic iff

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))=r-1, \tag{2.2}$$

and

$$\left(1+\left(\frac{\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{r-1}\right)(1-r(1-\gamma))\right)^2-4\left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)(1-r(1-\gamma))}{(r-1)}\right)<0. \tag{2.3}$$

3. Bifurcation Analysis

In this section, we use bifurcation theory to investigate the Neimark-Sacker bifurcation at (H_*,P_*), using γ as the bifurcation parameter in the system (1.1). The existence of the Neimark-Sacker bifurcation ensures that dynamically invariant closed curves are produced. We refer to [18–23] for the relevant literature concerning the bifurcation analysis of such types of discrete dynamical systems.

We are looking for conditions on the system (1.1) that will allow us to have a non-hyperbolic point $(H*,P_*)$ with a pair of complex conjugate eigenvalues that have modulus values that are equal to one for $J(H*,P*)$. The characteristic polynomial (2.1) has complex roots $z_{1,2}$ with $|z_{1,2}|=1$ in the following region

$$\Theta=\left\{(r,\gamma):r>1,\gamma>\frac{r-1}{r},\quad (2.2)\text{ and }(2.3)\text{ are satisfied}\right\}.$$

We select γ as a bifurcation parameter. When parameters vary in a local region of Θ, the system's unique positive fixed point (1.1) undergoes Neimark-Sacker bifurcation. We consider the following perturbation of the system (1.1):

$$\begin{bmatrix}H_{t+1}\\P_{t+1}\end{bmatrix}=\begin{bmatrix}r(1-\gamma-\delta)H_t+r(\gamma+\delta)H_t exp(-aP_t)\\e(\gamma+\delta)H_t(1-exp(-aP_t))\end{bmatrix}, \tag{3.1}$$

where $|\delta|\ll 1$ is used as a small perturbation parameter.

We now consider the transformation $u_{t+1}=H_{t+1}-H_*$, $v_{t+1}=P_{t+1}-P_*$ to transfer the fixed point (H_*,P_*) of the system (1.1) to origin:

$$\begin{bmatrix}u_{t+1}\\v_{t+1}\end{bmatrix}=\begin{bmatrix}1 & -\frac{r(1-r(1-\gamma-\delta))}{e(r-1)}\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)\\ \frac{e(r-1)}{r} & \frac{(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{(r-1)}\end{bmatrix}\begin{bmatrix}u_t\\v_t\end{bmatrix}+\begin{bmatrix}f_1(u_t,v_t)\\f_2(u_t,v_t)\end{bmatrix}, \tag{3.2}$$

where

$$\begin{aligned}f_1(u_t,v_t)=&-(a(1-r(1-\gamma-\delta)))u_tv_t+\left(\frac{ar(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{2e(r-1)}\right)v_t^2\\&+\frac{1}{2}(a^2(1-r(1-\gamma-\delta)))u_tv_t^2-\left(\frac{a^2r(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{6e(r-1)}\right)v_t^3,\end{aligned}$$

and

$$\begin{aligned}f_2(u_t,v_t)=&\left(\frac{ae(1-r(1-\gamma-\delta))}{r}\right)u_tv_t-\left(\frac{a(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{2(r-1)}\right)v_t^2\\&-\left(\frac{a^2e(1-r(1-\gamma-\delta))}{2r}\right)u_tv_t^2+\left(\frac{a^2(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{6(r-1)}\right)v_t^3.\end{aligned}$$

The characteristic polynomial of the linearized part of (3.2) evaluated at the fixed point $(0,0)$ of (3.1) is given by

$$z^2-p(\delta)z+q(\delta)=0, \tag{3.3}$$

where

$$p(\delta) = 1 + \frac{(1-r(1-\gamma-\delta))}{(r-1)} \ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right),$$
$$q(\delta) = \frac{r(1-r(1-\gamma-\delta))}{(r-1)} \ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right).$$

The roots of (3.3) are

$$z_{1,2} = \frac{p(\delta)}{2} \pm \frac{i}{2}\sqrt{4q(\delta)-p^2(\delta)}$$

satisfying

$$|z_{1,2}| = \sqrt{q(\delta)},$$

and

$$\left(\frac{d|z_{1,2}|}{d\delta}\right)_{\tilde{\gamma}=0} = \frac{\sqrt{r}\left(1-r+r\gamma\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)}{2\gamma\sqrt{(r-1)(1-r(1-\gamma))\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}} > 0.$$

We also have $p(0) = \left(1 + \left(\frac{\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{r-1}\right)(1-r(1-\gamma))\right)$ and $(r,\gamma) \in \Theta$ which means $p(0) \neq \pm 2, 0, 1$. So $z_1^n, z_2^n \neq 1$ for all $n = 1,2,3,4$ at $\delta = 0$. Thus the roots of equation (3.3) do not lie in the unit circle intersection with the coordinate axes when $\delta = 0$. We use the following transformation to get the canonical form of the linearized part of (3.2) at $\delta = 0$.

$$\begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{e} & 0 \\ \frac{1-r}{2r} & -\frac{\sqrt{4-(1+\frac{1}{r})^2}}{2} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix}. \tag{3.4}$$

Under the transformation (3.4), the system (3.2) becomes

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1+r}{2r} & -\frac{\sqrt{4-(1+\frac{1}{r})^2}}{2} \\ \frac{\sqrt{4-(1+\frac{1}{r})^2}}{2} & \frac{1+r}{2r} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} F(x_t,y_t) \\ G(x_t,y_t) \end{bmatrix}, \tag{3.5}$$

where

$$\begin{aligned} F(x,y) =& \frac{a(r-1)(1+3r+4r^2(-1+\gamma))}{8r^2}x^2 - \frac{a(4-(1+\frac{1}{r})^2)}{8}y^2 \\ &+ \frac{a(1+r+2r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{4r^2}xy + \frac{a^2(-1+r)^2(1+5r+6r^2(-1+\gamma))}{48r^3}x^3 \\ &- \frac{a^2}{48}(4-(1+\frac{1}{r})^2)^{3/2}y^3 + \frac{a^2(-1-2r+3r^2)(1+r+2r^2(-1+\gamma)))}{16r^3}xy^2 \\ &+ \frac{a^2(-1+r)(1+3r+4r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{16r^3}x^2y + O((|x|+|y|)^4), \end{aligned}$$

and

$$\begin{aligned} G(x,y) =& -\frac{a(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{8r^2\sqrt{-1-2r+3r^2}}x^2 + \frac{a(1+r)\sqrt{-1-2r+3r^2}}{8r^2}y^2 \\ &- \frac{a(1+r)(1+r+2r^2(-1+\gamma))}{4r^2}xy - \frac{a^2(-1+r)^2(1+r)(1+5r+6r^2(-1+\gamma))}{48r^3\sqrt{-1-2r+3r^2}}x^3 \\ &+ \frac{a^2(-1-3r+r^2+3r^3)}{48r^3}y^3 - \frac{a^2(-1-3r+r^2+3r^3)(1+r+2r^2(-1+\gamma))}{16r^3\sqrt{-1-2r+3r^2}}xy^2 \\ &- \frac{a^2(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{16r^3}x^2y + O((|x|+|y|)^4). \end{aligned}$$

We define the real number L, which analyzes the direction of the closed invariant curve in a system undergoing Neimark-Sacker bifurcation [24].

$$L = \left(\left[-Re\left(\frac{(1-2z_1)z_2^2}{1-z_1}\eta_{20}\eta_{11}\right) - \frac{1}{2}|\eta_{11}|^2 - |\eta_{02}|^2 + Re(z_2\eta_{21})\right]\right)_{\delta=0},$$

where

$$
\begin{aligned}
\eta_{20} &= \frac{1}{8}\left[F_{xx} - F_{yy} + 2G_{xy} + i(G_{xx} - G_{yy} - 2F_{xy})\right],\\
\eta_{11} &= \frac{1}{4}\left[F_{xx} + F_{yy} + i(G_{xx} + G_{yy})\right],\\
\eta_{02} &= \frac{1}{8}\left[F_{xx} - F_{yy} - 2G_{xy} + i(G_{xx} - G_{yy} + 2F_{xy})\right],\\
\eta_{21} &= \frac{1}{16}\left[F_{xxx} + F_{xyy} + G_{xxy} + G_{yyy} + i(G_{xxx} + G_{xyy} - F_{xxy} - F_{yyy})\right],
\end{aligned}
$$

and

$$
\begin{aligned}
F_{xx} &= \frac{a(-1+r)(1+3r+4r^2(-1+\gamma))}{4r^2}, \quad F_{yy} = -\frac{1}{4}a\left(4-\left(1+\frac{1}{r}\right)^2\right),\\
F_{xy} &= \frac{a(1+r+2r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{4r^2}, \quad F_{xxx} = \frac{a^2(-1+r)^2(1+5r+6r^2(-1+\gamma))}{8r^3},\\
F_{yyy} &= -\frac{1}{8}a^2\left(4-\left(1+\frac{1}{r}\right)^2\right)^{\frac{3}{2}}, \quad F_{xyy} = \frac{a^2(-1-2r+3r^2)(1+r+2r^2(-1+\gamma))}{8r^3},\\
F_{xxy} &= \frac{a^2(-1+r)(1+3r+4r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{8r^3}, \quad G_{xx} = -\frac{a(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{4r^2\sqrt{-1-2r+3r^2}},\\
G_{yy} &= \frac{a(1+r)\sqrt{-1-2r+3r^2}}{4r^2}, \quad G_{xy} = -\frac{a(1+r)(1+r+2r^2(-1+\gamma))}{4r^2},\\
G_{xxx} &= -\frac{a^2(-1+r)^2(1+r)(1+5r+6r^2(-1+\gamma))}{8r^3\sqrt{-1-2r+3r^2}}, \quad G_{yyy} = \frac{a^2(-1-3r+r^2+3r^3)}{8r^3},\\
G_{xyy} &= -\frac{a^2(-1-3r+r^2+3r^3)(1+r+2r^2(-1+\gamma))}{8r^3\sqrt{-1-2r+3r^2}}, \quad G_{xxy} = -\frac{a^2(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{8r^3}.
\end{aligned}
$$

Due to the above calculations, we have the following theorem for the existence and direction of Neimark-Sacker bifurcation.

Theorem 3.1. *Assume that $(r,\gamma)\in\Theta$. If $L\neq 0$, then the system (1.1) experiences Neimark-Sacker bifurcation at the unique positive fixed point (H_*,P_*) when the parameter γ differs in a small neighborhood of Θ. Moreover, if $L<0$, then an attracting closed invariant curve bifurcates from the fixed point (H_*,P_*), and if $L>0$, then a repelling closed invariant curve bifurcates from the fixed point (H_*,P_*).*

4. Chaos Control

Controlling bifurcation in discrete models has recently fascinated the interest of many researchers, and practical approaches are being used in a variety of fields, including cardiology, physics laboratories, laser and plasma systems, biochemistry, turbulence, communications, mechanical and chemical engineering [25,26].
We use the state feedback control technique [7,19,27–29] to stabilize the unstable fixed point of the system (1.1). We consider the controlled system in compliance with (1.1) as follows:

$$
\begin{cases} H_{t+1} = r(1-\gamma)H_t + r\gamma H_t\exp(-aP_t) - U_t,\\ P_{t+1} = e\gamma H_t(1-\exp(-aP_t)), \end{cases} \tag{4.1}
$$

where $U_t = h(H_t - H_*) + p(P_t - P_*)$ is the feedback control and p, h are feedback gains. The variational matrix of the system (4.1) evaluated at (H_*,P_*) is given by

$$
J_C(H_*,P_*) = \begin{bmatrix} 1-h & -p-\frac{r(1-r(1-\gamma))}{e(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\\ \frac{e(r-1)}{r} & \frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \end{bmatrix}.
$$

The characteristic equation corresponding to $J_C(H_*,P_*)$ is given by

$$
z^2 - \left(1-h+\frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)z + \frac{pe(r-1)}{r} - \left(\frac{(h-r)(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right) = 0. \tag{4.2}
$$

If z_1 and z_2 are roots of the system (4.2), then we have

$$
z_1 + z_2 = 1-h+\frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right), \tag{4.3}
$$

and

$$
z_1 z_2 = \frac{pe(r-1)}{r} - \left(\frac{(h-r)(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right). \tag{4.4}
$$

To get marginal lines of stability, we assume $z_1 = \pm 1$ and $z_1 z_2 = 1$ which implies $|z_{1,2}| \leq 1$. If we assume that $z_1 z_2 = 1$ then (4.4) gives

$$L_1 : \left(\frac{(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) \right) h - \left(\frac{e(r-1)}{r} \right) p = \frac{r(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) - 1. \tag{4.5}$$

Next, if we assume that $z_1 = 1$ then (4.3) and (4.4) implies

$$L_2 : \left(1 - \frac{(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) \right) h + \left(\frac{e(r-1)}{r} \right) p = -(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right). \tag{4.6}$$

If we assume that $z_1 = -1$ then (4.3) and (4.4) yields

$$L_3 : \left(1 + \frac{(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) \right) h - \left(\frac{e(r-1)}{r} \right) p = 2 + \frac{(r+1)(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right). \tag{4.7}$$

It is easy to see that the triangular area bounded by the straight lines L_1, L_2, and L_3 have stable eigenvalues.
Next, we use a hybrid control technique [22, 30–33] to control the chaotic behavior of (1.1) at fixed point (H_*, P_*) due to Neimark-Sacker bifurcation. We consider the following controlled system associated with the system (1.1):

$$\begin{cases} H_{t+1} = \alpha(r(1-\gamma)H_t + r\gamma H_t \exp(-aP_t)) + (1-\alpha)H_t, \\ P_{t+1} = \alpha e\gamma H_t(1-\exp(-aP_t)) + (1-\alpha)P_t, \end{cases} \tag{4.8}$$

where $0 < \alpha \leq 1$. The fixed points of the controlled system (4.8) and the original system (1.1) are the same.
By using theorem (2.1), we have the following result for local asymptotic stability of fixed point (H_*, P_*) of the controlled system (4.8).

Theorem 4.1. *Let $r > 1$ and $\gamma > \frac{r-1}{r}$. The unique positive fixed point (H_*, P_*) of the controlled system (4.8) is locally asymptotically stable iff*

$$\frac{(1-\alpha)(r-1) + \alpha(1+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{(r-1)} < 1.$$

Proof. The variational matrix of the system (4.8) at the fixed point (H_*, P_*) is

$$J_C(H_*, P_*) = \begin{bmatrix} 1 & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where

$$J_{12} = -\frac{\alpha r(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{e(r-1)},$$

$$J_{21} = \frac{\alpha e(r-1)}{r},$$

$$J_{22} = \frac{r-1+\alpha(1-r) + (\alpha + \alpha r(\gamma-1)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1}.$$

The characteristic polynomial of $J_C(H_*, P_*)$ is

$$F_C(z) = z^2 - \left(2 - \alpha + \frac{(\alpha + \alpha r(\gamma-1)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1} \right) z + K,$$

where

$$K = \frac{r-1+\alpha(1-r) + \alpha(1+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1}.$$

By simple computations,

$$F_C(1) = \alpha^2(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) > 0,$$

$$F_C(-1) = 4 - 2\alpha + \frac{\alpha(2+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1} > 0,$$

and

$$F_C(0) = \frac{(1-\alpha)(r-1) + \alpha(1+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{(r-1)}.$$

□

5. Numerical Examples

Some interesting numerical examples are provided in this section to strengthen our theoretical findings on different qualitative characteristics of the model (1.1).

5.1. Neimark-Sacker bifurcation by using γ as bifurcation parameter

Setting the parameters $r = 2, a = 4, e = 1$ and initial condition $H_0 = 0.5, P_0 = 0.2$ for the system (1.1), the bifurcation value is $\gamma \approx 0.698976$ and the fixed point is $(H_*, P_*) \approx (0.628216, 0.314108)$. The eigenvalues of $J(H_*, P_*)$ are $z_{1,2} = .75 \pm 0.661438i$ having $|z_{1,2}| = 1$ which confirms that the system (1.1) undergoes Neimark-Sacker bifurcation at (H_*, P_*). It is observed that the fixed point is locally asymptotically stable for $\gamma < 0.698976$, and the fixed point is unstable for $\gamma \geq 0.698976$ due to the Neimark-Sacker bifurcation as shown in Figure 5.1.

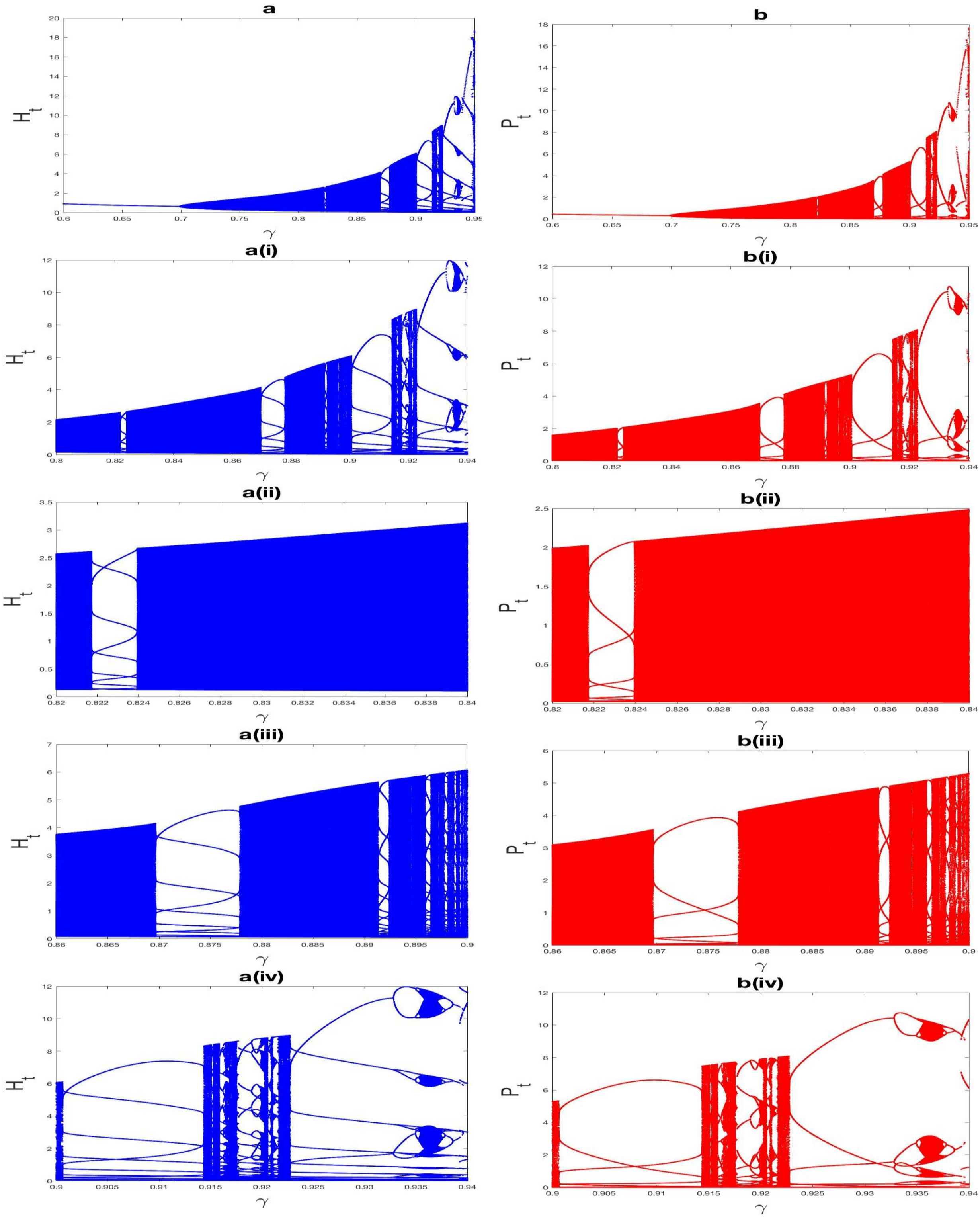

Figure 5.1: Bifurcation diagrams for system (1.1) and their amplifications.

The closed invariant curves and periodic orbits are observed for $\gamma \geq 0.698976$ as shown in Figure 5.2.

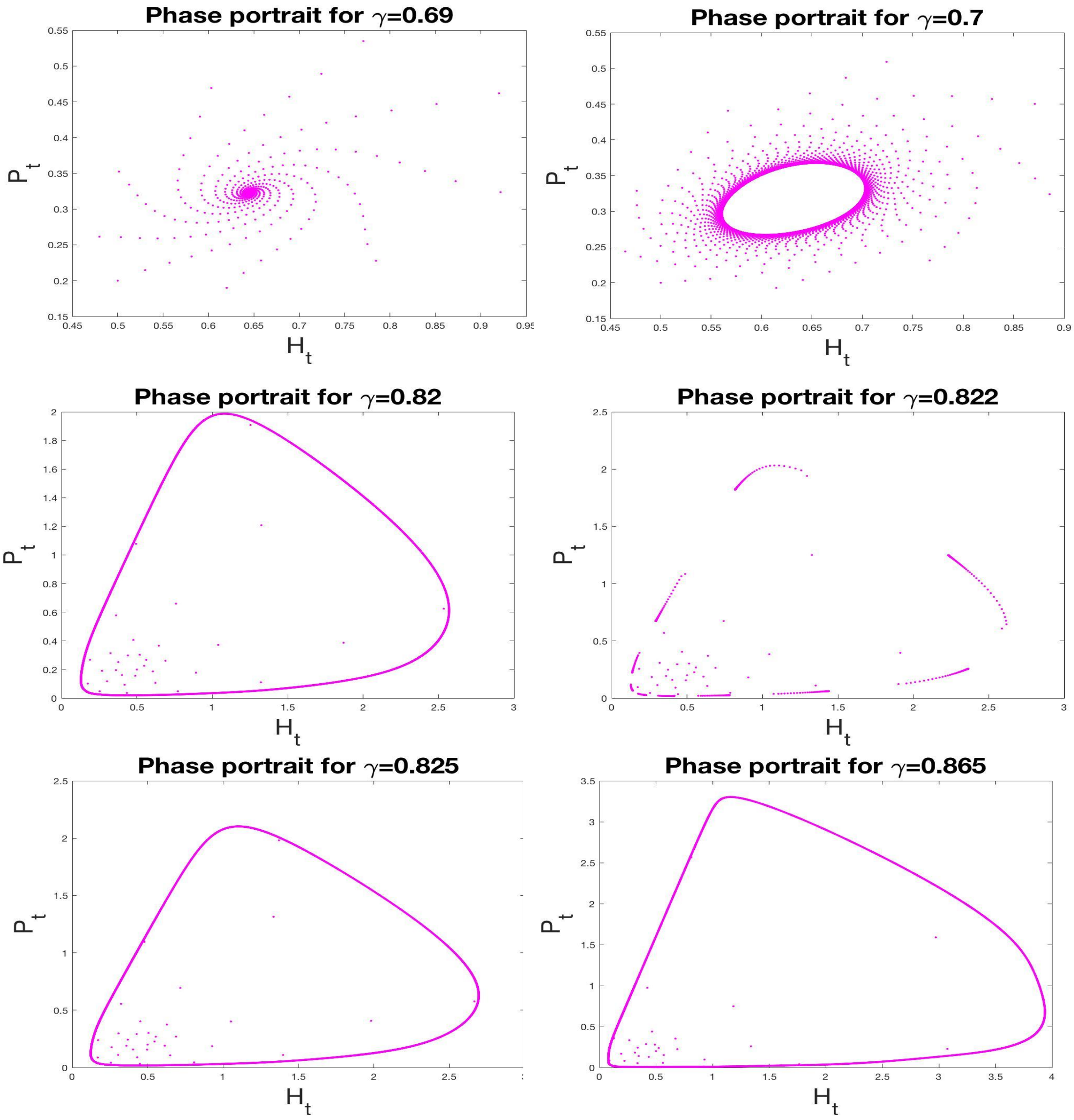

Figure 5.2: Phase portraits for system (1.1) for different values of γ.

Figure 5.3 displays the maximum Lyapunov exponent which affirms the stability and bifurcation regions obtained for the system (1.1).

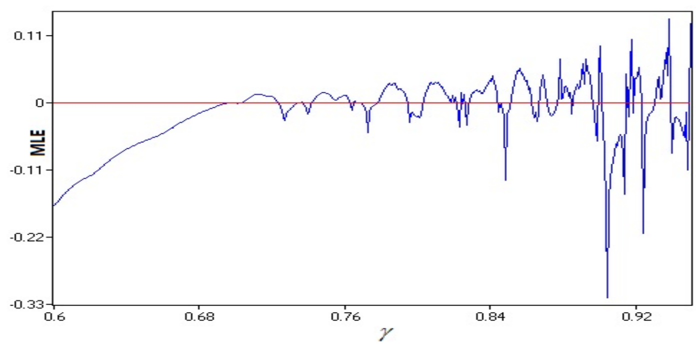

Figure 5.3: Maximum Lyapunov exponent for system (1.1).

5.2. Feedback control method

Setting the parameters $r = 2, a = 4, e = 1, \gamma = 0.7$ and initial condition $H_0 = 0.5, P_0 = 0.2$ for the system (4.1), the unique positive fixed point of the system (1.1) is unstable and the marginal stability lines for the controlled system (4.1) are

$$L_1 : h = 0.9977945005p - 0.9955890011,$$

$$L_2 : h = -1.002215271p - 1.004430524,$$

and

$$L_3 : h = 0.3330879170p + 2.333824167.$$

Figure 5.4 depicts the stable triangular area bounded by the marginal lines L_1, L_2, and L_3 for the controlled system (4.1).

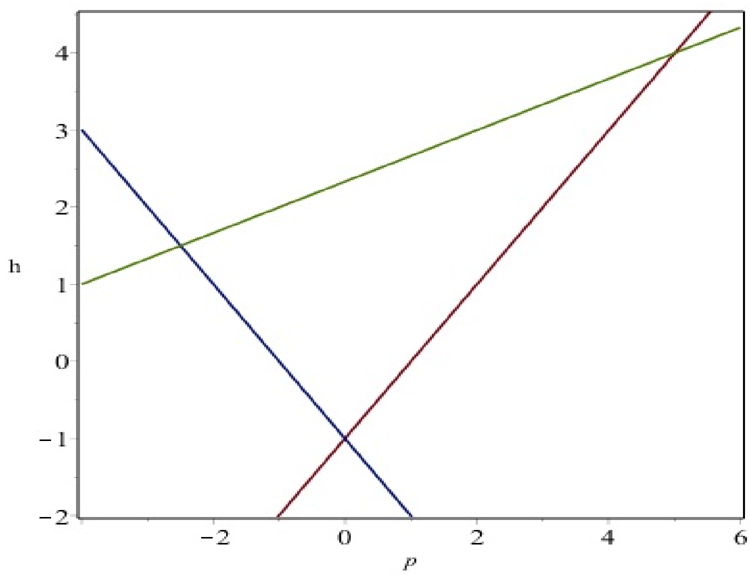

Figure 5.4: Stability region for controlled system (4.1).

5.3. Hybrid control method

Setting the parameters $r = 2, a = 4, e = 1$ and initial condition $H_0 = 0.5, P_0 = 0.2$ for the system (4.8), the bifurcation diagrams for H_t are displayed against the bifurcation parameter α in Figure 5.5, for different values of γ. These graphs show that the fixed point (H_*, P_*) of the controlled system (4.8) is locally asymptotically stable for a wide range of the control parameter α.

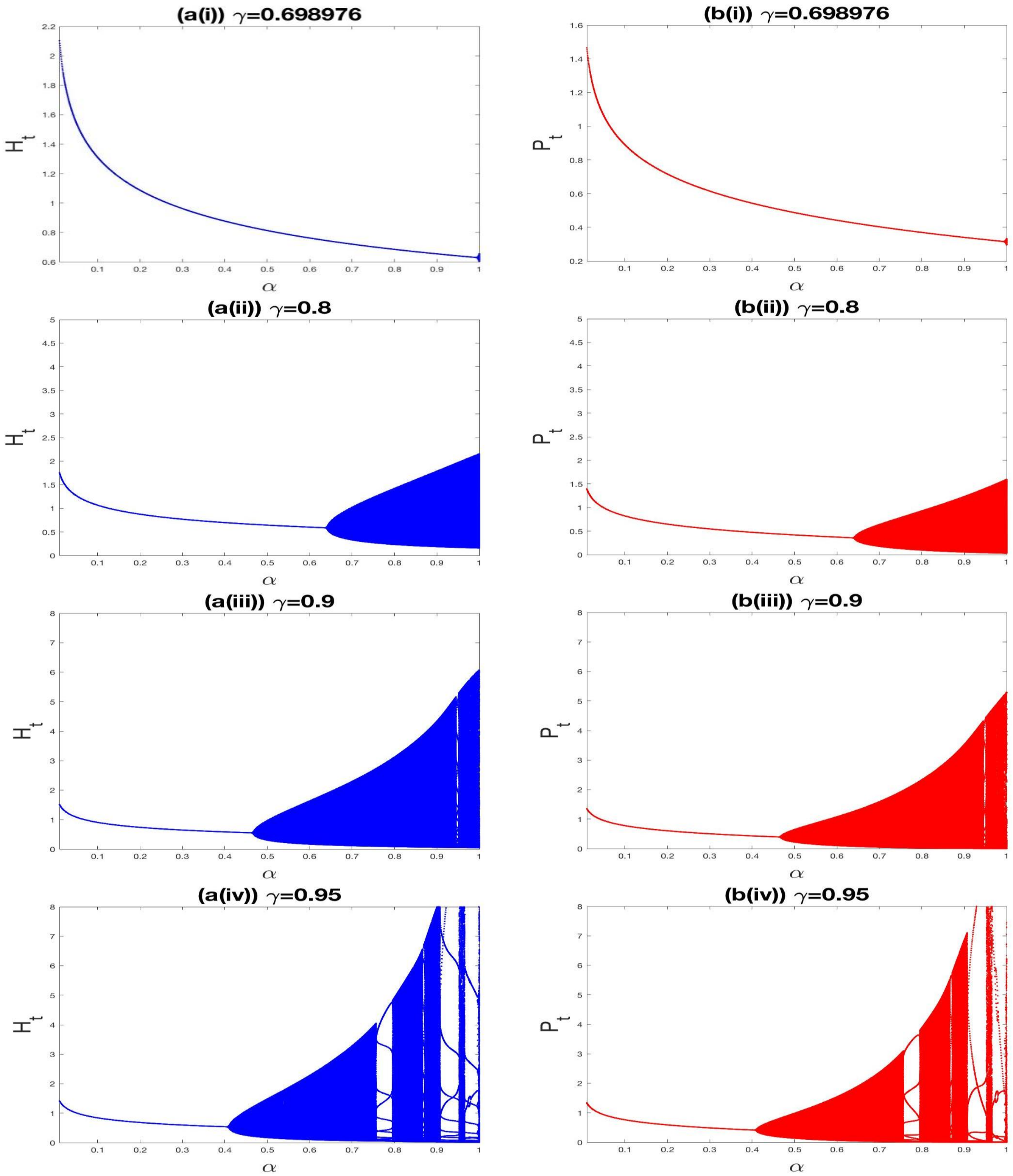

Figure 5.5: Bifurcation diagrams for controlled system (4.8) for different values of γ.

Furthermore, bifurcation diagrams for H_t are displayed against the bifurcation parameter γ in Figure 5.6 for different values of α. These graphs confirm that the bifurcation is delayed in the controlled system (4.8) compared to the original system (1.1).

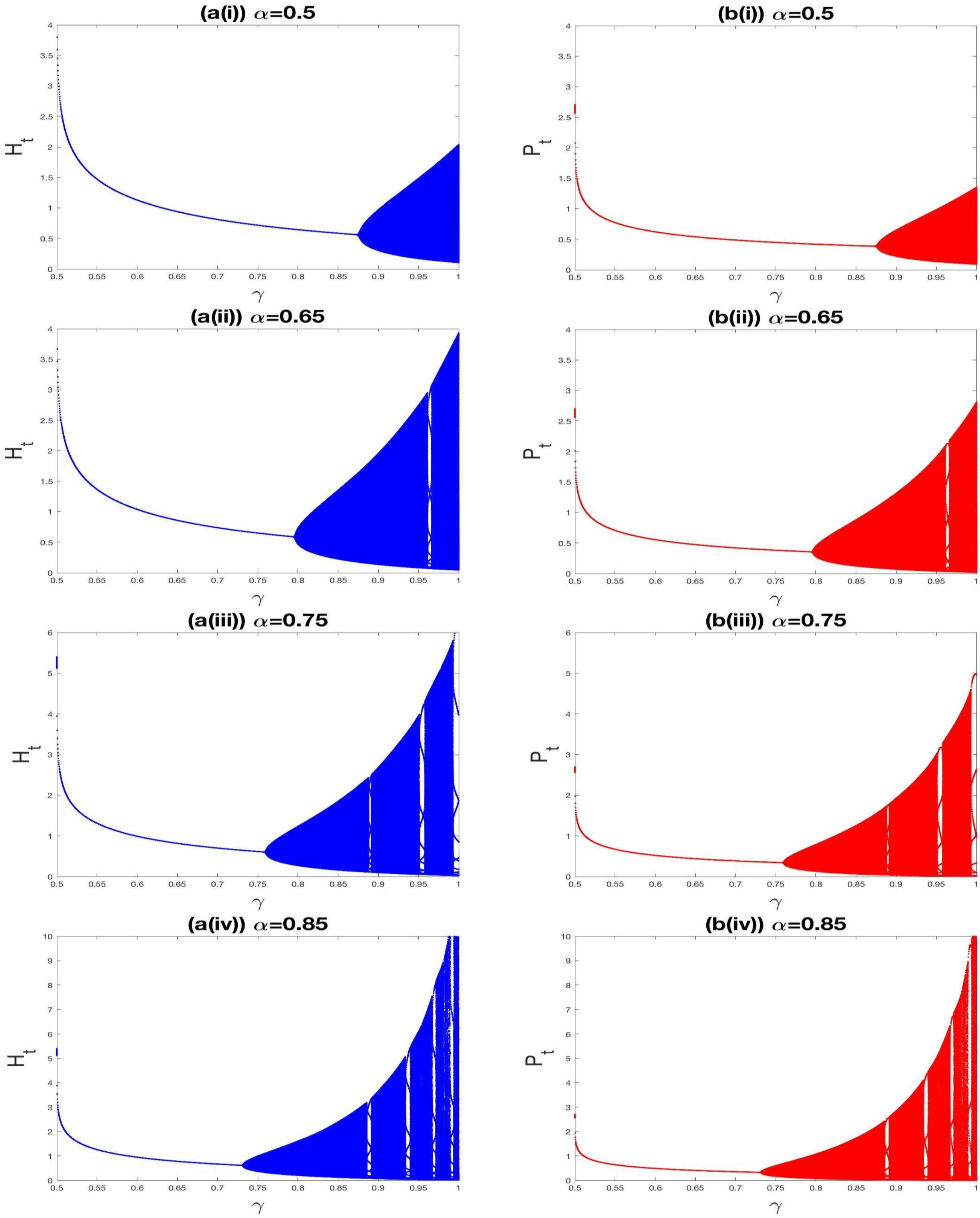

Figure 5.6: Bifurcation diagrams for controlled system (4.8) for different values of α.

5.4. Sensitive dependence on the initial conditions

Figure 5.7 shows two perturbed trajectories in blue and red colors to highlight the sensitivity of the system (1.1) to initial conditions. The two trajectories are initially overlapping and indistinguishable, but after a few iterations, the difference between them grows fast. With initial values $(H_0, P_0) = (0.5, 0.2)$ and $(H_0, P_0) = (0.50001, 0.20001)$, Figure 5.7 illustrates a sensitive dependence on the initial conditions for the system (1.1).

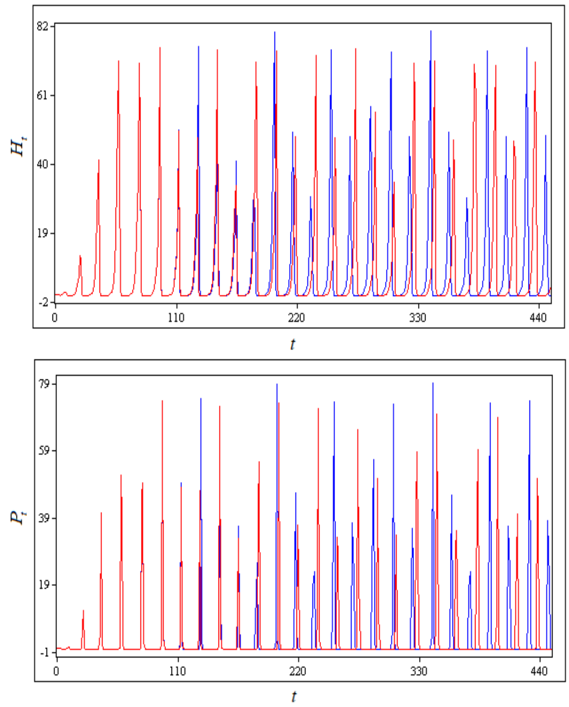

Figure 5.7: Sensitivity to initial conditions of the system (1.1).

6. Conclusion

The qualitative analysis of a host-parasitoid system (1.1) is carried out. The system (1.1) is a modification in the classical Nicholson-Bailey model, which is achieved by relaxing the uniform environment assumption with the patchy environment in which the number of hosts safe from attack by the parasitoid is fixed. The unique positive steady state of the system (1.1) is found to be

$$(H_*,P_*)=\left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{ae(r-1)},\frac{1}{a}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right).$$

The unique positive steady-state (H_*,P_*) is topologically classified by linearization. The local stability of the steady-state (H_*,P_*) is characterized by the following set of inequalities;

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))<r-1,\quad r>1,\quad \gamma>\frac{r-1}{r}.$$

The necessary and sufficient parametric conditions are derived for the local stability of the steady-state (H_*,P_*). In addition, sufficient conditions (2.2), (2.3) and $r>1$, $\gamma>\frac{r-1}{r}$ are derived for the steady-state (H_*,P_*) to be non-hyperbolic. The Neimark-Sacker bifurcation is carried out using the theory of normal forms by taking γ as a bifurcation parameter. The state feedback control and hybrid control strategies are used to stabilize the unstable steady state of the system. Finally, numerous numerical examples have been presented to illustrate the significance of the bifurcation parameter γ and the reproductive rate r of the host in the model (1.1). We show that the presence of a safe refuge, where a portion of the host is in a safe refuge from predation, has a stabilizing effect on the model. It is clear, therefore, that γ, the percentage of hosts that are vulnerable to parasitoids, can have a crucial impact on the stability of a host-parasitoid interaction. A small rate of escaping of a host, $1-\gamma$, may lead to instability.

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