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Research Article

Exact Solutions of Nonlinear Time Fractional Schrödinger Equation with Beta-Derivative

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Abstract: This article consists of Improved Bernoulli Sub-Equation Function Method (IBSEFM) to get the new solutions of nonlinear fractional Schrödinger equation described by beta-derivative. Foremost, it is dealt with derivative of Atangana. Secondly, basic properties of the IBSEFM are given. Finally, the proposed method has been applicated to the considered equation to get its new solutions. Moreover, the graphs of the obtained solutions are plotted via Mathematica. It is inferred from the results that IBSEFM is effectual technique for new solutions of nonlinear equations containing conformable derivatives.

Keywords: Atangana derivative, IBSEFM, Schrödinger equation

1. Introduction

Fractional equations are useful tool to determine numerous nonlinear phenomena of physics such as chaotic systems, heat transmission, diffusion, acoustic waves, viscoelasticity, plasma waves [12–17]. Lots of fractional operators have been defined, for instance: Riemann-Liouville, Caputo derivative [19], Caputo-Fabrizio [9], Jumarie's modified Riemann-Liouville [13], Atangana-Baleanu [4]. By the aid of these derivative operators, lots of techniques have been advanced which supply analytical solutions of fractional equations such as generalized Kudryashov [11], extended direct algebraic [20], IBSEFM [5, 6], modified trial equation method [18].

In [14] the definiton of conformable derivative is given and then using this derivative exact solutions of the time-heat differential equation have been investigated in [10]. In addition to this, a new definition of fractional derivative called beta-derivative is obtained in [4]. Several analytical methods are improved to get the exact solutions of fractional equations with beta-conformable time derivative [22–24].

The aim of this study is to get the exact solutions of nonlinear time fractional Schrödinger

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equation with beta-derivative using IBSEFM. Before the solution process we will give the basic properties of Atangana's conformable derivative and fundamental steps of proposed method in the rest of the paper.

2. Beta Derivative and It's Specifications

This section contains some essential concepts of beta derivative that have been utilized in this work.

Let $f(t)$ be a function defined for all non-negative t. Then, β-derivative of $f(t)$ of order β is given by in [1, 4] as

$$D^{\beta}\left(f(t)\right) = \frac{d^{\beta} f(t)}{dt^{\beta}} = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(t)}{\varepsilon},$$

where $0 < \beta \leq 1$. In fractional calculus, the β-derivative is known as the generalization of classical derivative and it's characteristics properties have been given in [1, 4]. Suppose that $u(t)$ and $v(t)$ are β-differentiable functions for all $t > 0$ and $\beta \in (0,1]$. Then

$i)$ $D^{\beta}\left(af(t) + bg(t)\right) = aD^{\beta}\left(f(t)\right) + bD^{\beta}\left(g(t)\right) \left(\forall\ a, b \in \mathbb{R}\right)$,

$ii)$ $D^{\beta}\left(f(t)g(t)\right) = g(t)D^{\beta}\left(f(t)\right) + f(t)D^{\beta}\left(g(t)\right)$,

$iii)$ $D^{\beta}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D^{\beta}(f(t)) - f(t)D^{\beta}(g(t))}{(g(t))^2}$,

$iv)$ $D^{\beta}\left(f(t)\right) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{df(t)}{dt}$.

It should be noted that these properties provide us an easy way to convert a nonlinear partial differential equation with β-derivative to a nonlinear ordinary differential equation of integer-order. There are many works with β-derivative in literature [2, 3].

3. Description of The Proposed Method

In this part, the fundamental properties of IBSEFM is given ([6–8]). There are five main steps of the IBSEFM below the following:

Step 1: Let us consider following equation with beta derivative for a function according to the two variables space x and time t;

$$P(u_0^A, D_t^{\beta} u, u_x, u_x x, \ldots) = 0, \tag{1}$$

here P involves $u(x,t)$ and partial derivatives. The goal is to exchange (1) to nonlinear ordinary differential equation with a suitable wave transformation as

$$u(x,t) = V(\eta), \quad \eta = mx - \frac{\gamma}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}, \tag{2}$$

m and γ are arbitrary constants. Using (2), (1) turns into the ordinary differential equation in the form

$$N(V, V', V'', \ldots) = 0, \tag{3}$$

where N is the function of $V, V', V'', \ldots$ and its derivatives with respect to η. Integrating (3) term to term, we acquire integration constants which may be determined then.

Step 2: We hypothesize that the solution of (3) may be presented below;

$$V(\eta) = \frac{\sum_{i=0}^{n} a_i Q^i(\eta)}{\sum_{j=0}^{m} b_j Q^j(\eta)} = \frac{a_0 + a_1 Q(\eta) + a_2 Q^2(\eta) + \ldots + a_n Q^n(\eta)}{b_0 + b_1 Q(\eta) + b_2 Q^2(\eta) + \ldots + b_m Q^m(\eta)}, \tag{4}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are coefficients which will be determined later. $m \neq 0, n \neq 0$ are chosen arbitrary according to the balance principle and considering the form of Bernoulli differential equation below the following;

$$Q'(\eta) = \sigma Q(\eta) + d Q^M(\eta), \; d \neq 0, \sigma \neq 0, \; M \in \mathbb{R} \setminus \{0, 1, 2\}, \tag{5}$$

here $Q(\eta)$ is a polynomial.

Step 3: The positive integer m, n, M (are different from zero) are found respect to the balance principle that is both nonlinear term and the highest order derivative term of (3). Substituting (4) and (5) into (3) an equation of polynomial $\Omega(Q)$ of Q is acquired below the following;

$$\Omega(Q(\eta)) = \alpha_s Q(\eta)^s + \ldots + \alpha_1 Q(\eta) + \alpha_0 = 0,$$

where α_i are coefficients that will be determined later.

Step 4: The coefficients of $\Omega(Q(\eta))$ which will give us an algebraic equations systems;

$$\alpha_i = 0, i = 0, \ldots, s.$$

Step 5: When we solve (5), we get the following two cases with respect to σ and d,

$$Q(\eta) = \left[\frac{-d e^{\sigma(\epsilon-1)\eta} + \epsilon\sigma}{\sigma e^{\sigma(\epsilon-1)\eta}}\right]^{\frac{1}{1-\epsilon}}, \quad d \neq \sigma, \tag{6}$$

$$Q(\eta) = \left[\frac{(\epsilon-1) + (\epsilon+1)\tanh\left(\sigma(1-\epsilon)\frac{\eta}{2}\right)}{1 - \tanh\left(\sigma(1-\epsilon)\frac{\eta}{2}\right)}\right], \; d = \sigma, \; \epsilon \in \mathbb{R}. \tag{7}$$

Using a complete discrimination system for polynomial of $Q(\eta)$, exact solutions of (1) are get via Wolfram Mathematica and categorize the exact solutions of (1). To achieve better results, 2D and 3D graphs of exact solutions might be plotted taking proper values of parameters.

4. Mathematical Analysis of The Model

Let us consider the nonlinear Schrödinger equation in β-derivative sense

$$i{}_0^A D_t^\beta u + p u_{xx} + q |u|^2 u = 0, \quad 0 < \beta \leq 1 \tag{8}$$

and apply the transformation

$$u(x,t) = e^{i\theta} U(\xi), \ \theta = \tau x + \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta, \quad \xi = x - \frac{2r\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta. \tag{9}$$

Here τ, λ and r are constants, using the basic properties of β-derivative and substituting (9) into (8), we get the following equation containing the real and imaginary part;

$$i\left[-2r\lambda\frac{dU}{d\xi} + 2p\tau\frac{dU}{d\xi}\right] + p\frac{d^2U}{d\xi^2} - \left(\lambda + p\tau^2\right)U + qU^3 = 0. \tag{10}$$

From the imaginary part of (10), $r = \frac{p\tau}{\lambda}$. Moreover, the real part of (10) is

$$pU'' - \left(\lambda + p\tau^2\right)U + qU^3 = 0. \tag{11}$$

When we reconsider (11) for balance principle between U'' and U^3, we get the relationship as follow;

$$M = n - m + 1. \tag{12}$$

(12) shows us the different cases of the solutions of (11) and we can obtain some analytical solutions. According to the balance, we consider $M = 3, m = 1, n = 3$ for (12) and the following equations hold:

$$U(\xi) = \frac{a_0 + a_1 Q(\xi) + a_2 Q^2(\xi) + a_3 Q^3(\xi)}{b_0 + b_1 Q(\xi)} \equiv \frac{\Upsilon(\xi)}{\Psi(\xi)}, \tag{13}$$

$$U'(\xi) = \frac{\Upsilon'(\xi)\Psi(\xi) - \Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)} \tag{14}$$

and

$$U''(\xi) = \frac{\Upsilon'(\xi)\Psi(\xi) - \Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)} - \frac{\left[\Upsilon(\xi)\Psi'(\xi)\right]'\Psi^2(\xi) - 2\Upsilon(\xi)[\Psi'(\xi)]^2\Psi(\xi)}{\Psi^4(\xi)}, \tag{15}$$

where $Q' = \sigma Q + dQ^3$, $a_3 \neq 0$, $b_1 \neq 0$, $\sigma \neq 0$, $d \neq 0$. Using (13)-(15) in (11), we get from coefficients of polynomial of Q as follow;

$$Q^0 : qa_0^3 - \lambda a_0 b_0^2 - p\tau^2 a_0 b_0^2 = 0,$$

$$Q^1 : 3qa_0^2 a_1 - \lambda a_1 b_0^2 + p\sigma^2 a_1 b_0^2 - p\tau^2 a_1 b_0^2 - 2\lambda a_0 b_0 b_1 - p\sigma^2 a_0 b_0 b_1 - 2p\tau^2 a_0 b_0 b_1 = 0,$$

$$\vdots$$

$$Q^7 : 3qa_3a_2^2 + 3qa_1a_3^2 + 15d^2pa_3b_0^2 + 9d^2pa_2b_0b_1 + 12dp\sigma a_3b_1^2 = 0,$$

$$Q^8 : 3qa_2a_3^2 + 21d^2pa_3b_0b_1 + 3d^2pa_2b_1^2 = 0,$$

$$Q^9 : qa_3^3 + 8d^2pa_3b_1^2 = 0.$$

Solving above the equation system of Q via Mathematica, the coefficients are obtained for $\sigma \neq d$:

Family1.

$$a_0 = -\frac{i\sqrt{2}\sqrt{p}\sigma b_0}{\sqrt{q}},\ a_1 = -\frac{i\sqrt{2}\sqrt{p}\sigma b_1}{\sqrt{q}}, a_2 = -\frac{2i\sqrt{2}d\sqrt{p}b_0}{\sqrt{q}}, a_3 = -\frac{2i\sqrt{2}d\sqrt{p}b_1}{\sqrt{q}},\ \tau = -\frac{\sqrt{-\lambda - 2p\sigma^2}}{\sqrt{p}}.$$

Substituting these coefficients along with (7) in (13), we obtain the following solution of (8) as follows;

$$q_1(x,t) = \frac{-1}{2}\exp\left\{-\frac{x\sqrt{-\lambda-2p\sigma^2}}{\sqrt{p}} + \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right\}\exp\left\{-2x\sigma + \frac{4rl\sigma\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\epsilon - \frac{d}{\sigma}\right\}.$$

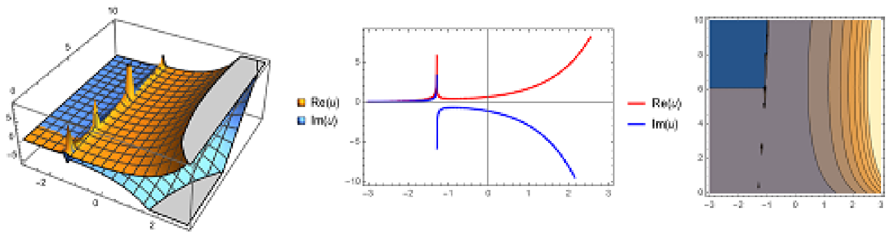

Figure 1: 3D-plots of $q_1(x,t)$ for the values $\beta = 0.5;\ d = 0.4;\ r = 0.1;\ \epsilon = 0.2;\ \lambda = 0.3;\ \sigma = 0.5;$ $p = 0.3;\ t = 0.4;\ -3 < x < 3,\ 0 < t < 10$, 2D-plots and contoursurfaces

Family2. For $\sigma \neq d$,

$$a_0 = -\frac{i\sqrt{-\lambda - p\tau^2}b_0}{\sqrt{q}},\ a_1 = -\frac{i\sqrt{-\lambda - p\tau^2}b_1}{\sqrt{q}},\ a_2 = \frac{2i\sqrt{2}d\sqrt{p}b_0}{\sqrt{q}}, a_3 = \frac{2i\sqrt{2}d\sqrt{p}b_1}{\sqrt{q}},\ \sigma = \frac{\sqrt{-\lambda - p\tau^2}}{\sqrt{2}\sqrt{p}}.$$

Substituting these coefficients along with (7) in (13), we obtain the following solution of (8) as follows;

$$q_2\left(x,t\right)=\frac{\exp\left\{ix\tau+\frac{i\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}\right\}\left(\lambda+p\tau^2\right)\left(2d^2\exp\left\{-\frac{2\sqrt{2}\sqrt{-\lambda-p\tau^2}\left(x-\frac{2r\lambda\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\right)}{\sqrt{p}}\right\}p+\epsilon^2\left(\lambda+p\tau^2\right)\right)}{\sqrt{\exp\left\{-2x\sigma+\frac{4rl\sigma\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}\epsilon-\frac{d}{\sigma}\right\}}}.$$

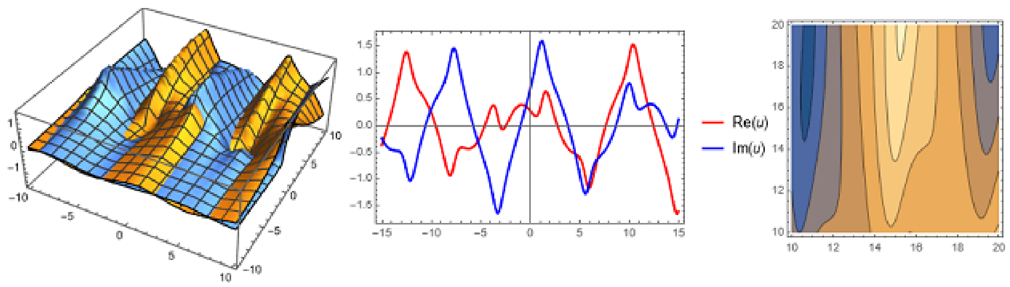

Figure 2: 3D- plots of $q_2(x,t)$ for the values $\beta=0.5$; $d=0.4$; $r=0.1$; $\epsilon=0.2$; $\lambda=0.3$; $\sigma=0.5$; $p=0.3$; $t=0.4$; $-10<x<10$, $-10<t<10$, 2D-plots and contoursurfaces

We can understand the characteristics of the solutions from the figures that for a few parameter values, the displayed numerical analysis acknowledges that the solutions are periodic wave shapes in exponential classes. According to the figures, one can see that the formats of exact solutions in two and three dimensional surfaces are similar to the physical meaning of results.

5. Conclusion

In this paper, the IBSEFM is applied for fractional Schrödinger equation in β-derivative. Using wave transformation the considered equation has been converted into the ordinary differential equation which can be solved according to the IBSEFM. By means of this method, exact solutions are obtained. Figures of all solutions according to the suitable parameters are plotted by showing the main characteristic physical properties of the solutions with the help of Wolfram Mathematica. It seems from the results that the more steps are developed and the better approximations are obtained. It is inferred from the conclusions that IBSEFM is simple, effective and powerful. Thus, in mathematical physics it is applicable to solve other nonlinear differential equations.

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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Research Article

On the Analytical Determination of Geometric Characterizations of Analytic Functions

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Abstract: As it is known, there are many sufficient conditions for the classification complex functions of one variable $f(z)$, which are analytic and univalent in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and are also normalized with $f(0) = 1 - f'(0) = 0$ which are also known as normalization conditions. In this sense, the main goal of present article is to derive some special sufficient conditions for $f(z)$ to be starlike of order 2^{-r} and convex of order 2^{-r} in $\mathcal{U}$, with r is a positive integer.

Keywords: Analytic function, convex function, starlike function, univalent function.

1. Introduction

Let's take $\mathcal{A}$ as the class of functions of the form

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + \ldots, \tag{1}$$

which are analytic in the open unit disc $\mathcal{U}$ and additionally satisfy normalization conditions $f(0) = 1 - f'(0) = 0$. If being univalent is imposed as an additional condition on the elements of class $\mathcal{A}$, class $\mathcal{S}$, which is a subclass of class $\mathcal{A}$, is obtained. The fact that a complex-valued function $f(z)$ is univalent in the unit disk $\mathcal{U}$ indicates that $w = f(z)$ for distinct z elements in $\mathcal{U}$ are also distinct [2]. In other words, the equation $f(z) = w$ has at most one root in the unit disk $\mathcal{U}$. Under these conditions, the regions $f(\mathcal{U})$ of the functions $f(z)$ belonging to the class $\mathcal{S}$ exhibits very interesting geometries. Moreover, these functions are classified using common geometries. We denoted by $\mathcal{S}^*$ the subclass of class $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which exhibit a starlike geometry with respect to the origin and also denote by $\mathcal{C}$ the subclass of class $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which exhibit a convex geometry. These two classes can be given

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analytically as

$$\mathcal{S}^* = \left\{ f(z) \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathcal{U} \right\} \tag{2}$$

and

$$\mathcal{C} = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathcal{U} \right\}, \tag{3}$$

respectively [2, 3, 6].

Furthermore, we denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which are satisfies the condition $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \ (0 \le \alpha < 1)$, and analytically this subclass is given as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \alpha \in [0,1), z \in \mathcal{U} \right\}. \tag{4}$$

Similarly, also we denote by $\mathcal{C}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions $f(z)$ in class $\mathcal{A}$, which are the satisfies the conditions $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \ (0 \le \alpha < 1)$. Analytically this subclass is given as

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \alpha \in [0,1), z \in \mathcal{U} \right\}. \tag{5}$$

Functions belonging to subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively, are called starlike of order α and convex of order α functions in the open unit disc $\mathcal{U}$. For proofs of analytical characterizations given so far, we refer to [1, 9, 12]. Since it is very difficult to classify analytic functions with different domains, it would be good to remember that the domain of definition is taken as the open unit disc $\mathcal{U}$ in the studies carried out in this field in the light of the Riemann mapping theorem. In this sense, the information given is only valid for the analytic $f(z)$ functions defined in the open unit disc $\mathcal{U}$ and satisfying the relevant conditions. Moreover, it is clear that the common geometric characterization of analytic functions belonging to a subclass cannot be generalized to all elements of the classes that cover this class. At this stage, it is clear from the subclass relationship that $\mathcal{C}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$ and $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ for $\alpha = 0$. When interpreting the subclass relationship given above, it will be helpful to remember that any convex region is also a starlike region with respect to every point. Since the functions belonging to the class $\mathcal{A}$ meet the normalization conditions, functions that are starlike according to the origin are mainly used in the studies in this field.

Theorem 1.1 *The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ satisfies Alexander duality relation*

$$zf'(z) \in \mathcal{S}^*(\alpha) \Leftrightarrow f(z) \in \mathcal{C}(\alpha), \ \ 0 \le \alpha < 1. \tag{6}$$

The basic argument provided by this theorem, also known as the Alexander's theorem that $f(z)$ is univalent and convex if and only if $zf'(z)$ can be univalent and convex [2]. In this case, it can be said immediately that f is convex, according to the argument of Alexander's theorem. This brilliant theorem, which is not difficult to prove, is used as a very useful mathematical tool in obtaining many results set forth in univalent function theory.

Definition 1.2 *Let's take $f(z)$ and $g(z)$ be analytic in the open unit disc $\mathcal{U}$. If there is an analytic function $w(z)$ in $\mathcal{U}$ that satisfies the conditions $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))$, then the $f(z)$ function is said to be subordinate to the $g(z)$ function and denoted as $g(z) \prec f(z)$ [2].*

Lemma 1.3 *(Jack's Lemma) Let the (none constant) function $w(z)$ be analytic in the open unit disc $\mathcal{U}$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0 \in \mathcal{U}$, then $c=\dfrac{z_0 w'(z_0)}{w(z_0)}$, where c is a real number and $c \geq 1$.*

It is well known that Jack's lemma is a very useful mathematical tool used in many applications in the theory of geometric functions [2, 5]. In this sense, we start by reminding that it is also used as a basic tool in the proof of our results.

2. Main Results

Theorem 2.1 *Let $f(z)$ be a function in class $\mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{3+2^{-r}}{2\left(1+2^{-r}\right)}, \ |z|<1 \tag{7}$$

for some $2+2^{-r}(r \in \mathbb{N})$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{\left(2+2^{-r}\right)(1-z)}{\left(2+2^{-r}\right)-z}, \ |z|<1 \tag{8}$$

and

$$\left|\frac{zf'(z)}{f(z)}-\frac{2+2^{-r}}{3+2^{-r}}\right| < \frac{2+2^{-r}}{3+2^{-r}}, \ |z|<1. \tag{9}$$

This implies that $f(z) \in \mathcal{S}^*\left(2^{-r}\right)$.

Proof As in many studies in this field, let's define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{\left(2+2^{-r}\right)(1-w(z))}{\left(2+2^{-r}\right)-w(z)}, \ w(z) \neq 2+2^{-r} \tag{10}$$

to prove the result of the theorem under the given conditions. It is clear that the function $w(z)$ is analytic in the open unit disc $\mathcal{U}$ and also $w(0)=0$. Thus, we need to prove that $|w(z)|<1$ in $\mathcal{U}$ according to the Jack's Lemma Lemma 1.3. Since

$$1+\frac{zf''(z)}{f'(z)}=\frac{\left(2+2^{-r}\right)\left(1-w(z)\right)}{\left(2+2^{-r}\right)-w(z)}-\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{\left(2+2^{-r}\right)-w(z)}, \tag{11}$$

we have that

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\Re\left(\frac{\left(2+2^{-r}\right)\left(1-w(z)\right)}{\left(2+2^{-r}\right)-w(z)}-\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{\left(2+2^{-r}\right)-w(z)}\right),$$

so $\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\dfrac{3+2^{-r}}{2\left(1+2^{-r}\right)}, |z|<1$ for some $2+2^{-r}(r\in\mathbb{N})$. At this stage, using the exponential form of the complex number provides ease of operation. Now, if there is a point z_0 in $\mathcal{U}$ such that $\max\limits_{|z|\leq|z_0|}=|w(z_0)|=1$, then $w(z_0)=e^{i\theta}$ and $c=\dfrac{z_0w'(z_0)}{w(z_0)}$, $c\geq1$ by Jack's Lemma 1.3. So, we have

$$\begin{aligned}1+\frac{z_0f''(z_0)}{f'(z_0)}&=\frac{\left(2+2^{-r}\right)\left(1-w(z_0)\right)}{\left(2+2^{-r}\right)-w(z_0)}-\frac{zw'(z_0)}{1-w(z_0)}+\frac{zw'(z_0)}{\left(2+2^{-r}\right)-w(z_0)}\\&=\left(2+2^{-r}\right)+\left(2+2^{-r}\right)\left(1-\left(2+2^{-r}\right)+c\right)\frac{1}{\left(2+2^{-r}\right)-e^{i\theta}}-\frac{c}{1-e^{i\theta}}.\end{aligned}$$

Thus, if follows that

$$\Re\left(\frac{1}{\left(2+2^{-r}\right)-e^{i\theta}}\right)=\frac{1}{2\left(2+2^{-r}\right)}+\frac{\left(2+2^{-r}\right)^2-1}{2\left(2+2^{-r}\right)\left(1+\left(2+2^{-r}\right)^2-2cos\theta\right)}$$

and so $\Re\left(\dfrac{1}{1-w(z_0)}\right)=\dfrac{1}{2}$. This implies that, for $2+2^{-r}\ (r\in\mathbb{N})$,

$$\begin{aligned}\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)&\geq\frac{3+2^{-r}}{2}+\frac{\left(3+2^{-r}\right)\left(1-\left(2+2^{-r}\right)+c\right)}{2\left(1+2^{-r}\right)}\\&\geq\frac{3+2^{-r}}{2}+\frac{\left(3+2^{-r}\right)\left(-2^{-r}\right)}{2\left(1+2^{-r}\right)}\\&=\frac{3+2^{-r}}{2\left(1+2^{-r}\right)}.\end{aligned}$$

This contradicts the hypothesis of our theorem. Therefore, there is no $z_0\in\mathcal{U}$ such that $|w(z_0)|=1$ for all $z\in\mathcal{U}$, that is

$$\frac{zf'(z)}{f(z)}\prec\frac{\left(2+2^{-r}\right)(1-z)}{\left(2+2^{-r}\right)-z},\ |z|<1.$$

Furthermore, since

$$w(z)=\frac{\left(2+2^{-r}\right)\left(\frac{zf'(z)}{f(z)}-1\right)}{\frac{zf'(z)}{f(z)}-\left(2+2^{-r}\right)}$$

and $|w(z)|<1$ $(|z|<1)$, we conclude that

$$\left|\frac{zf'(z)}{f(z)}-\frac{2+2^{-r}}{3+2^{-r}}\right|<\frac{2+2^{-r}}{3+2^{-r}},\ |z|<1,$$

which implies that $f(z)\in\mathcal{S}^*$. □

While $r\to\infty$ in the theorem, the following corollary due to Singh R. and Singh S. is obtained with a different calculation [11]. The proof is obtained directly from the proof of Theorem 2.1.

Corollary 2.2 *Let $f(z)$ be a function in class $\mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3}{2},\ |z|<1,$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{2(1-z)}{2-z},\ |z|<1,$$

and

$$\left|\frac{zf'(z)}{f(z)}-\frac{2}{3}\right|<\frac{3}{2},\ |z|<1.$$

Theorem 2.3 *Let $f(z)$ be a function in class $\mathcal{A}$. If $f(z)$ satisfies*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{7+3.2^{-r}}{2\left(2+2^{-r}\right)\left(3+2^{-r}\right)},\ |z|<1 \tag{12}$$

for some $1+2^{-r}(r\in\mathbb{N})$, then $f(z)\in\mathcal{C}\left(\frac{3+2^{-r}}{2(2+2^{-r})}\right)$.

Proof We define the function in $\mathcal{U}$ by

$$\frac{f(z)}{zf'(z)}=\frac{\left(2+2^{-r}\right)\left(1-w(z)\right)}{\left(2+2^{-r}\right)-w(z)},\ (w(z)\neq 2+2^{-r}, r\in\mathbb{N}), \tag{13}$$

so that $w(z)$ is analytic in $\mathcal{U}$ and $|w(z)|<1$. In this case, if the logarithmically derivative of both sides of equation (13) is taken and necessary simplifying are made,

$$1+\frac{zf''(z)}{f'(z)}=\frac{\left(2+2^{-r}\right)-w(z)}{\left(2+2^{-r}\right)\left(1-w(z)\right)}+\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{\left(2+2^{-r}\right)-w(z)}$$

and hence

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\Re\left(\frac{(2+2^{-r})-w(z)}{(2+2^{-r})(1-w(z))}+\frac{zw'(z)}{1-w(z)}+\frac{zw'(z)}{(2+2^{-r})-w(z)}\right)$$
$$>\frac{7+3.2^{-r}}{2(2+2^{-r})(3+2^{-r})}$$

for $1+2^{-r}(r\in\mathbb{N})$. Now, if there is a point z_0 in $\mathcal{U}$ such that $\max\limits_{|z|\leq|z_0|}=|w(z_0)|=1$, then $w(z_0)=e^{i\theta}$ and $c=\frac{z_0w'(z_0)}{w(z_0}$, $c\geq 1$ by Jack's Lemma 1.3. So, we have

$$\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)=\frac{1}{2}+\frac{1}{2(2+2^{-r})}-\frac{c\left((2+2^{-r})^2+1\right)}{2\left(1+(2+2^{-r})^2-2(2+2^{-r})\,cos\theta\right)}$$

and, for $1+2^{-r}>0$ $(r\in\mathbb{N})$,

$$\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\leq\frac{1}{2}+\frac{1}{2(2+2^{-r})}-\frac{1+2^{-r}}{2(3+2^{-r})}$$
$$=\frac{7+3.2^{-r}}{2(2+2^{-r})(3+2^{-r})}.$$

This contradicts the hypothesis of our theorem. Therefore, there is no $z_0\in\mathcal{U}$ such that $|w(z_0)|=1$ for all $z\in\mathcal{U}$, that is

$$\frac{f(z)}{zf'(z)}\prec\frac{(2+2^{-r})(1-z)}{(2+2^{-r})-z},\ |z|<1.$$

Furthermore, since

$$w(z)=\frac{(2+2^{-r})\left(1-\frac{zf'(z)}{f(z)}\right)}{1-(2+2^{-r})\frac{zf'(z)}{f(z)}}$$

and $|w(z)|<1$ $(|z|<1)$, we conclude that $f(z)\in\mathcal{C}\left(\frac{3+2^{-r}}{2(2+2^{-r})}\right)$. □

Corollary 2.4 *Letting $r\to\infty$ in Theorem 2.3, then $f(z)\in\mathcal{C}(3/4)$.*

3. Conclusion

As it is known, geometric functions basically aim to classify complex functions that are analytic on the open unit disk, provided that they meet some additional conditions such as being univalent and satisfying the normalization conditions. While doing this, a relationship is established between the analytical properties of the functions in question and the geometric properties of their images, as an interaction of analysis and geometry. If the function can be easily graphed, it will be fairly easy to classify the image set according to its geometric characterization. Unfortunately, that may not always be the case. In this case, there are many analytical methods that can be used in the literature. In this sense, we tried to give a different perspective to the conditions existing in the literature.

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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Research Article

Smarandache Curves According to Flc-frame in Euclidean 3-space

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Abstract: The paper investigates some special Smarandache curves according to Flc-frame in Euclidean 3-space. The Frenet and Flc frame vectors, curvature and torsion of the new constructed curves are expressed by means of the initial curve invariants. For the sake of comparison in view, an example for Smarandache curves according to both Frenet and Flc frame is also presented at the end of paper.

Keywords: Flc-frame, natural curvatures, polynomial curves, Smarandache curves.

1. Introduction

Characterizations of curves in classical differential geometry are generally expressed with the help of Frenet framework. However, the disadvantage of this frame is that the frame cannot be settled at points where the second derivative of the curve is zero. In this case, an alternative frame is needed. Bishop defined Bishop frame which we call alternative parallel frame in 1975 [5]. This frame is formed by rotating the normal vectors at a certain angle by keeping the tangent vector in the Frenet framework constant and can be defined including the points where the second derivative of the curve is zero. Even if the Bishop frame is suitable for applications, it is not an analytical frame. Recently, Dede has introduced a new framework called Flc (Frenet like curve) frame along a given polynomial curve, and provided some insight into the geometric meaning for the nth derivative of a given curve [8]. Calculations made according to this frame are easier than the Frenet frame and Bishop frame. The most important advantage of the Flc Frame is that it has less singular points compared to the Frenet frame. Thus, by hindering the sudden rotation of the tangent vector of the curve, the deformation that may occur on the surface is prevented, and the problem of sudden ruptures and bends on the surface are removed. The Smarandache curve is defined as the regular curve with the place vector generated by the Frenet vectors of a regular curve.

In Euclidean space, the first studies for this subject were given by Ali in [2]. Turgut and Yılmaz, described the Smarandache curves in Minkowski space [15]. Later, at either Euclidean

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or Minkowski space, some features of the Smarandache curves are investigated according to the Darboux frame, Bishop frame, alternative frame, q frame and Sabban frame, [1, 3, 4, 6, 7, 9, 11–14]. In this study, we introduce special Smarandache curves according to the new Flc frame in Euclidean 3-space. The Flc apparatus of each new curve are calculated and the graphs of the curves are also presented.

2. Preliminaries

In this section, we recall some basic concepts that we refer in the context of the paper. Let $\alpha : I \subset R \to R^3$ be a regular curve in E^3. The general forms of Frenet vectors and formulas are given as

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \qquad N(s) = B(s) \wedge T(s), \qquad B(s) = \frac{\alpha'(s) \wedge \alpha''(s)}{\|\alpha'(s) \wedge \alpha''(s)\|}, \tag{1}$$

$$\kappa(s) = \frac{\|\alpha'(s) \wedge \alpha''(s)\|}{\|\alpha'(s)\|^3}, \qquad \tau(s) = \frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s) \rangle}{\|\alpha'(s) \wedge \alpha''(s)\|^2}, \tag{2}$$

$$T'(s) = \nu\kappa(s)N(s), \qquad N'(s) = \nu\left(-\kappa(s)T(s) + \tau(s)B(s)\right), \qquad B'(s) = -\nu\tau(s)N(s), \tag{3}$$

where $\nu = \|\alpha'(s)\|$, κ is the curvature and τ is the torsion of the curve [10].

Moreover, a point $s_0 \in I$ is said to be a singular point of order 0 of the curve α, if $\alpha'(s_0)$ vanishes. Another point $s_1 \in I$ is said to be a singular point of order 1 if $\alpha''(s_1)$ vanishes. If $\alpha'(s_2) \wedge \alpha''(s_2) = 0$ that is the curvature vanishes at a point $s_2 \in I$, then s_2 is called an inflection point. Therefore as known to be the main disadvantage of the Frenet frame, it has inflection points and two type of singular points. However, recently, Dede introduced a new frame moving along a polynomial space curve of degree n and named it as Flc-frame. The vector elements of this new frame is defined as following;

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \qquad D_2(s) = D_1(s) \wedge T(s), \qquad D_1(s) = \frac{\alpha'(s) \wedge \alpha^{(n)}(s)}{\left\|\alpha'(s) \wedge \alpha^{(n)}(s)\right\|}, \tag{4}$$

where the prime (n) stands for the n^{th} derivative with respect to s [8]. The new vectors D_1 and D_2 are called as binormal-like and normal-like vectors, respectively. The curvatures of the Flc-frame d_1, d_2 and d_3 are defined as

$$d_1 = \frac{\langle T', D_2 \rangle}{\nu}, \qquad d_2 = \frac{\langle T', D_1 \rangle}{\nu}, \qquad d_3 = \frac{\left\langle {D_2}', D_1 \right\rangle}{\nu}. \tag{5}$$

The relationship between the Frenet and Frenet like frame (Flc) is given by

$$\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{6}$$

and the relations between the curvatures of two frames are

$$d_1 = \kappa cos\theta, \qquad d_2 = -\kappa sin\theta, \qquad \theta = arctan\left(-\frac{d_2}{d_1}\right), \qquad d_3 = \frac{d\theta}{\nu} + \tau, \tag{7}$$

where $\theta = \sphericalangle(N, D_2)$. Therefore, the local rate of change for the Flc-frame or namely the Frenet-like formulas can be expressed as in the following form

$$\begin{bmatrix} T' \\ D_2{}' \\ D_1{}' \end{bmatrix} = \nu \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}. \tag{8}$$

3. Smarandache Curves According to Flc Frame

Let us consider the curve $\beta(s) : I \subset R \to R^3$ as a regular polynomial curve in Euclidean space and denote $\{T(s),\ D_2(s),\ D_1(s)\}$ as its moving Flc frame. We define and consider the following Smarandache curves. Note that for simplicity we omit to denote the parameter s throughout the paper.

3.1. TD_2 Smarandache Curve

Definition 3.1 *The curve β_1 defined by the linear combination of two vectors T and D_2 is called the TD_2 Smarandache curve and is defined as;*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(T + D_2). \tag{9}$$

We examine the Flc frame invariants of the TD_2 Smarandache curve β_1 by means of the main curve β. To do so, we first differentiate (9) with respect to s and recall the relations given at (8) to get

$$\beta_1' = \frac{\nu}{\sqrt{2}}(-d_1 T + d_1 D_2 + (d_2 + d_3) D_1).$$

By taking the norm of above and considering the equations (1), we obtain the tangent vector T_{β_1} as;

$$T_{\beta_1} = \frac{-d_1 T + d_1 D_2 + (d_2 + d_3) D_1}{\sqrt{2{d_1}^2 + (d_2 + d_3)^2}}.$$

On the other hand, by recalling (8) again, the second order derivative of (9) with respect to s is given as

$$\beta_1'' = \eta_1 T + \eta_2 D_2 + \eta_3 D_1,$$

where

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \frac{-1}{\sqrt{2}} \begin{bmatrix} \nu^2 \left({d_1}^2 + d_2 d_3 + {d_2}^2\right) + (\nu d_1)' \\ \nu^2 \left({d_1}^2 + d_2 d_3 + {d_3}^2\right) - (\nu d_1)' \\ \nu^2 \left(d_1 d_2 - d_1 d_3\right) - (\nu d_2)' + (\nu d_3)' \end{bmatrix}.$$

Then, the cross product of first and second order derivatives is given

$$\beta_1' \wedge \beta_1'' = \zeta_1 T + \zeta_2 D_2 + \zeta_3 D_1,$$

where

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{\nu}{\sqrt{2}} \begin{bmatrix} (d_1 \eta_3 - \eta_2 (d_2 + d_3)) \\ (d_1 \eta_3 + \eta_1 (d_2 + d_3)) \\ -d_1 (\eta_1 + \eta_2) \end{bmatrix}.$$

Hence, we express the principal normal and the binormal vector field of β_1 as in the following;

$$N_{\beta_1} = \frac{\nu}{\sqrt{2}} \frac{(\zeta_2 (d_2 + d_3) - \zeta_3 d_1) T - (\zeta_1 (d_2 + d_3) + \zeta_3 d_1) D_2 + (\zeta_1 d_1 + \zeta_2 d_1) D_1}{\left(\sqrt{2{d_1}^2 + (d_2 + d_3)^2}\right)\left(\sqrt{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}\right)},$$

$$B_{\beta_1} = \frac{\zeta_1 T + \zeta_2 D_2 + \zeta_3 D_1}{\sqrt{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}}.$$

The third derivative of β_1 Smarandache curve is

$$\beta_1''' = \rho_1 T + \rho_2 D_2 + \rho_3 D_1,$$

where

$$\rho_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \nu {d_1}^2 \left(\nu^2 d_1 - 3\nu'\right) + d_1 \left(\nu^3 \left({d_2}^2 + {d_3}^2\right) - 3\nu^2 {d_1}' - \nu''\right) \\ -d_2 \left(3\nu\nu' (d_2 + d_3) + \nu^2 \left(3{d_2}' + 2{d_3}'\right)\right) - {d_2}'\nu^2 d_3 - 2\nu' {d_1}' - \nu {d_1}'' \end{array} \right),$$

$$\rho_2 = \frac{1}{\sqrt{2}} \left(\begin{array}{c} -\nu {d_1}^2 \left(\nu^2 d_1 + 3\nu'\right) - d_1 \left(\nu^3 \left({d_3}^2 + {d_2}^2\right) + 3\nu^2 {d_1}' - \nu''\right) \\ -\nu d_2 \left(\nu {d_3}' - 3\nu' d_3\right) - 3\nu'\nu {d_3}^2 - \nu^2 d_3 \left(2{d_2}' + 3{d_3}'\right) + \nu {d_1}'' + 2\nu' {d_1}' \end{array} \right),$$

$$\rho_3 = \frac{1}{\sqrt{2}} \left(\begin{array}{c} -\nu^3 {d_1}^2 (d_2 + d_3) - \nu d_1 \left(\nu \left({d_2}' - {d_3}'\right) + 3\nu' (d_2 - d_3)\right) - \nu^3 {d_2}^2 (d_2 + d_3) - \nu^3 {d_3}^3 \\ -d_2 \left(\nu^2 \left(\nu {d_3}^2 + 2{d_1}'\right) - \nu''\right) + \left(2\nu^2 {d_1}' + \nu''\right) d_3 + \nu \left({d_2}'' + {d_3}''\right) + 2\nu' \left({d_2}' + {d_3}'\right) \end{array} \right).$$

Hence, the Frenet curvatures κ and τ of β_1 are given as

$$\kappa_{\beta_1} = \frac{2\sqrt{2}\sqrt{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}}{\nu^3 \left(2{d_1}^2 + (d_2 + d_3)^2\right)^{\frac{3}{2}}}, \qquad \tau_{\beta_1} = \frac{\zeta_1 \rho_1 + \zeta_2 \rho_2 + \zeta_3 \rho_3}{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}.$$

Therefore, by using (6), the Flc apparatus of β_1 can be given by means of the Flc components of β as

$$T_{\beta_1} = \frac{-d_1 T + d_1 D_2 + (d_2 + d_3) D_1}{\mu/\sqrt{2}},$$

$$D_{2\beta_1} = \frac{1}{\mu\zeta}\left(\begin{array}{c} (\nu cos\theta_1 (\zeta_2 (d_2 + d_3) - \zeta_3 d_1) + \mu sin\theta_1 \zeta_1) T \\ -(\nu cos\theta_1 (\zeta_1 (d_2 + d_3) + \zeta_3 d_1) - \mu sin\theta_1 \zeta_2) D_2 \\ +(\nu cos\theta_1 (\zeta_1 d_1 + \zeta_2 d_1) + \mu sin\theta_1 \zeta_3) D_1 \end{array}\right),$$

$$D_{1\beta_1} = \frac{1}{\mu\zeta}\left(\begin{array}{c} -(\nu sin\theta_1 (\zeta_2 (d_2 + d_3) - \zeta_3 d_1) - \mu cos\theta_1 \zeta_1) T \\ +(\nu sin\theta_1 (\zeta_1 (d_2 + d_3) + \zeta_3 d_1) + \mu cos\theta_1 \zeta_2) D_2 \\ -(\nu sin\theta_1 (\zeta_1 d_1 + \zeta_2 d_1) - \mu cos\theta_1 \zeta_3) D_1 \end{array}\right),$$

where $\mu = \sqrt{4{d_1}^2 + 2(d_2 + d_3)^2}$, $\zeta = \sqrt{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}$, $\theta_1 = \sphericalangle (N_{\beta_1}, D_{2\beta_1})$ and

$$d_{1\beta_1} = \left(\frac{2\sqrt{2}\sqrt{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}}{\nu^3 \left(2{d_1}^2 + (d_2 + d_3)^2\right)^{\frac{3}{2}}}\right) cos\theta_1,$$

$$d_{2\beta_1} = -\left(\frac{2\sqrt{2}\sqrt{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}}{\nu^3 \left(2{d_1}^2 + (d_2 + d_3)^2\right)^{\frac{3}{2}}}\right) sin\theta_1,$$

$$d_{3\beta_1} = -\frac{d_{2\beta_1}' d_{1\beta_1} - d_{2\beta_1} d_{1\beta_1}'}{\nu \left(d_{1\beta_1}^2 + d_{2\beta_1}^2\right)} + \frac{\zeta_1 \rho_1 + \zeta_2 \rho_2 + \zeta_3 \rho_3}{{\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2}.$$

3.2. TD_1 Smarandache Curve

Definition 3.2 *The curve β_2 defined by the linear combination of two vectors T and D_1 is called the TD_1 Smarandache curve and is defined as:*

$$\beta_2(s) = \frac{1}{\sqrt{2}} (T + D_1). \tag{10}$$

We examine the Flc frame invariants of the TD_1 Smarandache curve β_2 by means of the main curve β. To do so, we first differentiate (10) with respect to s and recall the relations given at (8) to get

$$\beta_2' = \frac{\nu}{\sqrt{2}} (-d_2 T + (d_1 - d_3) \mathrm{D}_2 + d_2 \mathrm{D}_1).$$

By taking the norm of above and considering the equations (1), we obtain the tangent vector T_{β_2} as;

$$T_{\beta_2} = \frac{-d_2 T + (d_1 - d_3) \mathrm{D}_2 + d_2 \mathrm{D}_1}{\sqrt{2{d_2}^2 + (d_1 - d_3)^2}}.$$

On the other hand, by recalling (8) again, the second order derivative of (10) with respect to s is given as

$$\beta_2'' = \xi_1 T + \xi_2 D_2 + \xi_3 D_1,$$

where

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{-1}{\sqrt{2}} \begin{bmatrix} \nu^2\left({d_2}^2 + {d_1}^2 - d_1 d_3\right) + (\nu d_2)' \\ -\nu^2 d_2\left(d_1 + d_3\right) + \nu'\left(d_1 - d_3\right) + \nu\left({d_1}' - {d_3}'\right) \\ -\nu^2\left({d_2}^2 - d_1 d_3 + {d_3}^2\right) + (\nu d_2)' \end{bmatrix}.$$

Next, the cross product of first and second order derivatives is given

$$\beta_2' \wedge \beta_2'' = \chi_1 T + \chi_2 D_2 + \chi_3 D_1,$$

where

$$\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} = \frac{\nu}{\sqrt{2}} \begin{bmatrix} \xi_3\left(d_1 - d_3\right) - d_2 \xi_2 \\ d_2\left(\xi_1 + \xi_3\right) \\ \xi_1\left(d_1 + d_3\right) - d_2 \xi_2 \end{bmatrix}.$$

Hence, we express the principal normal and the binormal vector field of β_2 as in the following;

$$N_{\beta_2} = \frac{\nu}{\sqrt{2}} \frac{\left(\chi_2 d_2 - \chi_3\left(d_1 - d_3\right)\right) T - d_2\left(\chi_1 + \chi_3\right) D_2 + \left(\chi_1\left(d_1 - d_3\right) + \chi_2 d_2\right) D_1}{\left(\sqrt{2{d_2}^2 + \left(d_1 - d_3\right)^2}\right)\left(\sqrt{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}\right)},$$

$$B_{\beta_2} = \frac{\chi_1 T + \chi_2 D_2 + \chi_3 D_1}{\sqrt{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}}.$$

The third derivative of β_2 Smarandache curve is

$$\beta_2''' = \omega_1 T + \omega_2 D_2 + \omega_3 D_1,$$

where

$$\omega_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{l} \nu {d_1}^2\left(d_2 \nu^2 - 3\nu'\right) - \nu d_1\left(3\nu d_1 - 2\nu {d_3}' - 3\nu' d_3\right) + \nu^3 {d_2}^3 - 3\nu {d_2}^2 \nu' \\ +\left(\nu^2\left(\nu {d_3}^2 - 3{d_2}'\right) - \nu''\right) d_2 + {d_1}' \nu^2 d_3 - 2\nu' {d_2}' - \nu {d_2}'' \end{array} \right),$$

$$\omega_2 = \frac{1}{\sqrt{2}} \left(\begin{array}{l} -\nu^3 {d_1}^2\left(d_1 - d_3\right) - d_1\left(\nu^3\left({d_2}^2 + {d_3}^2\right) + 2{d_2}'\nu^2 + 3\nu d_2 \nu' - \nu''\right) + \nu^3 {d_2}^2 d_3 \\ -\nu d_2\left(\nu\left({d_1}' + {d_3}'\right) + 3\nu' d_3\right) + \nu^3 {d_3}^3 - d_3\left(2\nu^2 {d_2}' + \nu''\right) + \nu\left({d_1}'' - {d_3}''\right) + 2\nu'\left({d_1}' - {d_3}'\right) \end{array} \right),$$

$$\omega_3 = \frac{1}{\sqrt{2}} \left(\begin{array}{l} -\nu^3 d_2 {d_1}^2 + \nu d_1\left(\nu {d_3}' + 3\nu' d_3\right) - \nu {d_2}^2\left(\nu^2 d_2 + 3\nu'\right) - d_2\left(\nu^2\left(\nu {d_3}^2 + 3{d_2}'\right) - \nu''\right) \\ -3\nu' \nu {d_3}^2 + d_3\left(2\nu^2 {d_1}' - 3\nu^2 {d_3}'\right) + \nu {d_2}'' + 2\nu' {d_2}' \end{array} \right).$$

Hence, the Frenet invariants κ and τ of β_2 can be expressed as

$$\kappa_{\beta_2} = \frac{2\sqrt{2}\sqrt{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}}{\nu^3\left(2{d_2}^2 + \left(d_1 - d_3\right)^2\right)^{\frac{3}{2}}}, \qquad \tau_{\beta_2} = \frac{\chi_1 \omega_1 + \chi_2 \omega_2 + \chi_3 \omega_3}{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}.$$

By using again (6), the Flc apparatus of β_2 can be given by means of the Flc components of β as

$$T_{\beta_2} = \frac{\sqrt{2}}{\vartheta}\Big(- d_1 T + d_1 D_2 + (d_2 + d_3)\, D_1\Big),$$

$$D_{2\beta_2} = \frac{1}{\vartheta\chi}\left(\begin{array}{c} (\nu cos\theta_2 \left(\chi_2 d_2 - \chi_3 \left(d_1 - d_3\right)\right) + \vartheta sin\theta_2 \chi_1)\, T \\ - (\nu cos\theta_2 \left(d_2 \left(\chi_1 + \chi_3\right)\right) - \vartheta sin\theta_2 \chi_2)\, D_2 \\ + (\nu cos\theta_2 \left(\chi_1 \left(d_1 - d_3\right) + \chi_2 d_2\right) + \vartheta sin\theta_2 \chi_3)\, D_1 \end{array}\right),$$

$$D_{1\beta_2} = \frac{1}{\vartheta\chi}\left(\begin{array}{c} - (\nu sin\theta_2 \left(\chi_2 d_2 - \chi_3 \left(d_1 - d_3\right)\right) - \vartheta cos\theta_2 \chi_1)\, T \\ + (\nu sin\theta_2 \left(d_2 \left(\chi_1 + \chi_3\right)\right) + \vartheta cos\theta_2 \chi_2)\, D_2 \\ - (\nu sin\theta_2 \left(d_2 \left(\chi_1 + \chi_3\right)\right) - \vartheta cos\theta_2 \chi_3)\, D_1 \end{array}\right),$$

where $\vartheta = \sqrt{4{d_2}^2 + 2\left(d_1 - d_3\right)^2}$, $\chi = \sqrt{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}$, $\theta_2 = \sphericalangle\left(N_{\beta_2}, D_{2\beta_2}\right)$ and

$$d_{1\beta_2} = \left(\frac{2\sqrt{2}\sqrt{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}}{\nu^3 \left(2{d_2}^2 + \left(d_1 - d_3\right)^2\right)^{\frac{3}{2}}} \right) cos\theta_2,$$

$$d_{2\beta_2} = -\left(\frac{2\sqrt{2}\sqrt{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}}{\nu^3 \left(2{d_2}^2 + \left(d_1 - d_3\right)^2\right)^{\frac{3}{2}}} \right) sin\theta_2,$$

$$d_{3\beta_2} = -\frac{d_{2\beta_2}' d_{1\beta_2} - d_{2\beta_2} d_{1\beta_2}'}{\nu\left(d_{1\beta_2}^2 + d_{2\beta_2}^2\right)} + \frac{\chi_1\omega_1 + \chi_2\omega_2 + \chi_3\omega_3}{{\chi_1}^2 + {\chi_2}^2 + {\chi_3}^2}.$$

3.3. D_2D_1 Smarandache Curve

Definition 3.3 *The curve β_3 defined by the linear combination of two vectors D_2 and D_1 of Flc frame is called the D_2D_1 Smarandache curve and is defined as;*

$$\beta_3(s) = \frac{1}{\sqrt{2}}\left(D_2 + D_1\right). \tag{11}$$

We examine the Flc frame invariants of the D_2D_1 Smarandache curve β_3 by means of the main curve β. By differentiating (11) with respect to s, first and referring the relations given at (8) to get

$$\beta_3' = \frac{\nu}{\sqrt{2}}\left(-(d_1 + d_2)T - d_3 D_2 + d_3 D_1\right).$$

By taking the norm of above and considering the equations (1), we obtain the tangent vector T_{β_2} as;

$$T_{\beta_3} = \frac{-(d_1 + d_2)T + d_3 \mathrm{D}_2 + d_3 \mathrm{D}_1}{\sqrt{2{d_3}^2 + \left(d_1 + d_2\right)^2}}.$$

On the other hand, by recalling (8) again, the second order derivative of (11) with respect to s is given as

$$\beta_3'' = \phi_1 T + \phi_2 D_2 + \phi_3 D_1,$$

where

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu^2 d_3 (d_1 - d_2) - (\nu d_1)' - (\nu d_2)' \\ -\nu^2 \left({d_1}^2 + d_2 + {d_3}^2\right) - (\nu d_3)' \\ -\nu^2 \left(d_1 d_2 + {d_2}^2 + {d_3}^2\right) + (\nu d_3)' \end{bmatrix}.$$

Next, the cross product of first and second order derivatives is given

$$\beta_3' \wedge \beta_3'' = \upsilon_1 T + \upsilon_2 D_2 + \upsilon_3 D_1,$$

where

$$\begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \end{bmatrix} = \frac{\nu}{\sqrt{2}} \begin{bmatrix} -d_3 (\phi_2 + \phi_3) \\ \phi_3 (d_1 + d_2) + \phi_1 d_3 \\ -\phi_2 (d_1 + d_2) + \phi_1 d_3 \end{bmatrix}.$$

Hence, we express the principal normal and the binormal vector field of β_3 as in the following;

$$N_{\beta_3} = \frac{\nu}{\sqrt{2}} \frac{(d_3 (\upsilon_2 + \upsilon_3)) T - (\upsilon_1 d_3 + \upsilon_3 (d_1 + d_2)) D_2 - (\upsilon_1 d_3 - \upsilon_2 (d_1 + d_2)) D_1}{\left(\sqrt{2{d_3}^2 + (d_1 + d_2)^2}\right) \left(\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}\right)},$$

$$B_{\beta_3} = \frac{\upsilon_1 T + \upsilon_2 D_2 + \upsilon_3 D_1}{\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}}.$$

Moreover, the third derivative of β_3 Smarandache curve is

$$\beta_3''' = \epsilon_1 T + \epsilon_2 D_2 + \epsilon_3 D_1,$$

where

$$\epsilon_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{l} {d_1}^2 \nu^3 (d_1 + d_2) + d_1 \left(\nu^3 \left({d_2}^2 + {d_3}^2\right) + 3\nu d_3 \nu' + 2\nu^2 {d_3}' - \nu''\right) + \nu^2 d_3 \left({d_1}' - {d_2}'\right) \\ +{d_2}^3 \nu^3 + d_2 \left(\nu d_3 \left(\nu^2 d_3 - 3\nu'\right) - 2\nu^2 {d_3}' - \nu''\right) - 2\nu' \left({d_1}' + {d_2}'\right) - \nu \left({d_1}'' + {d_2}''\right) \end{array} \right),$$

$$\epsilon_2 = \frac{1}{\sqrt{2}} \left(\begin{array}{l} \nu {d_1}^2 \left(\nu^2 d_3 - 3\nu'\right) - \nu d_1 \left(3\nu' d_2 + \nu \left(3{d_1}' + 2{d_2}'\right)\right) + {d_2}^2 \nu^3 d_3 \\ -\nu \left({d_1}' d_2 \nu - \nu^2 {d_3}^3 + 3{d_3}^2 \nu' + {d_3}''\right) - d_3 \left(-3\nu^2 {d_3}' + \nu''\right) - 2\nu' {d_3}' \end{array} \right),$$

$$\epsilon_3 = \frac{1}{\sqrt{2}} \left(\begin{array}{l} -{d_1}^2 \nu^3 d_3 - \nu d_1 \left(3\nu' d_2 + {d_2}' \nu\right) - \nu {d_2}^2 \left(\nu^2 d_3 + 3\nu'\right) \\ -\nu^2 d_2 \left(2{d_1}' + 3{d_2}'\right) - \nu {d_3}^2 \left(\nu^2 d_3 + 3\nu'\right) - d_3 \left(3\nu^2 {d_3}' - \nu''\right) + 2\nu' {d_3}' + \nu {d_3}'' \end{array} \right).$$

Therefore, the Frenet curvatures κ and τ of β_3 can be expressed as

$$\kappa_{\beta_3} = \frac{2\sqrt{2}\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}}{\nu^3 \left(2{d_3}^2 + (d_1 + d_2)^2\right)^{\frac{3}{2}}}, \qquad \tau_{\beta_3} = \frac{\upsilon_1 \epsilon_1 + \upsilon_2 \epsilon_2 + \upsilon_3 \epsilon_3}{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}.$$

By using again (6), the Flc apparatus of β_3 can be given by means of the Flc components of β as

$$T_{\beta_3} = \frac{\sqrt{2}}{\delta}\Big(-(d_1+d_2)T + d_3 D_2 + d_3 D_1\Big),$$

$$D_{2\beta_3} = \frac{1}{\delta\upsilon}\left(\begin{array}{r} (\nu cos\theta_3\left(d_3\left(\upsilon_2+\upsilon_3\right)\right) + \delta sin\theta_3\upsilon_1)\,T \\ -\left(\nu cos\theta_3\left(\upsilon_1 d_3 + \upsilon_3(d_1+d_2)\right) - \delta sin\theta_3\upsilon_2\right)D_2 \\ +\left(\nu cos\theta_3\left(\upsilon_1 d_3 - \upsilon_2(d_1+d_2)\right) + \delta sin\theta_3\upsilon_3\right)D_1 \end{array}\right),$$

$$D_{1\beta_3} = \frac{1}{\delta\upsilon}\left(\begin{array}{r} -\left(\nu sin\theta_3\left(d_3\left(\upsilon_2+\upsilon_3\right)\right) - \delta cos\theta_3\upsilon_1\right)T \\ +\left(\nu sin\theta_3\left(\upsilon_1 d_3 + \upsilon_3(d_1+d_2)\right) + \delta cos\theta_3\upsilon_2\right)D_2 \\ +\left(\nu sin\theta_3\left(\upsilon_1 d_3 - \upsilon_2(d_1+d_2)\right) + \delta cos\theta_3\upsilon_3\right)D_1 \end{array}\right),$$

where $\delta = \sqrt{4{d_3}^2 + 2\left(d_1+d_2\right)^2}$, $\upsilon = \sqrt{{\upsilon_1}^2+{\upsilon_2}^2+{\upsilon_3}^2}$, $\theta_3 = \sphericalangle\left(N_{\beta_3}, D_{2\beta_3}\right)$ and

$$d_{1\beta_3} = \left(\frac{2\sqrt{2}\sqrt{{\upsilon_1}^2+{\upsilon_2}^2+{\upsilon_3}^2}}{\nu^3\left(2{d_3}^2+(d_1+d_2)^2\right)^{\frac{3}{2}}}\right)cos\theta_3,$$

$$d_{2\beta_3} = -\left(\frac{2\sqrt{2}\sqrt{{\upsilon_1}^2+{\upsilon_2}^2+{\upsilon_3}^2}}{\nu^3\left(2{d_3}^2+(d_1+d_2)^2\right)^{\frac{3}{2}}}\right)sin\theta_3,$$

$$d_{3\beta_3} = -\frac{d_2{}'_{\beta_3} d_{1\beta_3} - d_{2\beta_3} d_1{}'_{\beta_3}}{\nu\left(d_{1\beta_3}^2 + d_{2\beta_3}^2\right)} + \frac{\upsilon_1\epsilon_1+\upsilon_2\epsilon_2+\upsilon_3\epsilon_3}{{\upsilon_1}^2+{\upsilon_2}^2+{\upsilon_3}^2}.$$

3.4. TD_2D_1 Smarandache Curve

Definition 3.4 *The curve β_4 defined by the linear combination of the vectors T, D_2 and D_1 of Flc frame is called the TD_2D_1 Smarandache curve and is defined as;*

$$\beta_4(s) = \frac{1}{\sqrt{3}}\left(T + D_2 + D_1\right). \tag{12}$$

We examine the Flc frame invariants of the D_2D_1 Smarandache curve β_4 by means of the main curve β. By differentiating (12) with respect to s, first and referring the relations given at (8) to get

$$\beta_4' = \frac{\nu}{\sqrt{3}}\left(-(d_1+d_2)T + (d_1-d_3)D_2 + (d_2+d_3)D_1\right).$$

By taking the norm of above and considering the equations (1), we obtain the tangent vector T_{β_2} as;

$$T_{\beta_3} = \frac{-(d_1+d_2)T + (d_1-d_3)D_2 + (d_2+d_3)D_1}{\sqrt{(d_1+d_2)^2 + (d_1-d_3)^2 + (d_2+d_3)^2}}.$$

On the other hand, by recalling (8) again, the second order derivative of (12) with respect to s is given as

$$\beta_4'' = \gamma_1 T + \gamma_2 D_2 + \gamma_3 D_1,$$

where

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\nu^2\left({d_1}^2 - d_1 d_3 + {d_2}^2 + d_2 d_3\right) - \nu'\left(d_1 + d_2\right) - \nu\left({d_1}' + {d_2}'\right) \\ -\nu^2\left({d_1}^2 + d_1 d_2 + d_2 d_3 + {d_3}^2\right) + \nu\left({d_1}' - {d_3}'\right) + \nu'\left(d_1 - d_3\right) \\ -\nu^2\left({d_2}^2 + d_1 d_2 - d_1 d_3 + {d_3}^2\right) + \nu\left({d_2}' + {d_3}'\right) + \nu'\left(d_2 + d_3\right) \end{bmatrix}.$$

The cross product of first and second order derivatives is given

$$\beta_4' \wedge \beta_4'' = \psi_1 T + \psi_2 D_2 + \psi_3 D_1,$$

where

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \frac{\nu}{\sqrt{3}} \begin{bmatrix} \gamma_3\left(d_1 - d_3\right) - \gamma_2\left(d_2 + d_3\right) \\ \gamma_3\left(d_1 + d_2\right) + \gamma_1\left(d_2 + d_3\right) \\ \gamma_1\left(d_3 - d_1\right) - \gamma_2\left(d_1 + d_2\right) \end{bmatrix}.$$

Hence, we express the principal normal and the binormal vector field of β_4 as in the following;

$$N_{\beta_4} = \frac{\nu}{\sqrt{3}} \frac{\begin{array}{c}\left(\psi_2\left(d_2 + d_3\right) - \psi_3\left(d_1 - d_3\right)\right)T - \left(\psi_1\left(d_2 + d_3\right) + \psi_3\left(d_1 + d_2\right)\right)D_2 \\ +\left(\psi_1\left(d_1 - d_3\right) + \psi_2\left(d_1 + d_2\right)\right)D_1\end{array}}{\left(\sqrt{(d_1 + d_2)^2 + (d_1 - d_3)^2 + (d_2 + d_3)^2}\right)\left(\sqrt{{\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2}\right)},$$

$$B_{\beta_4} = \frac{\psi_1 T + \psi_2 D_2 + \psi_3 D_1}{\sqrt{{\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2}}.$$

Moreover, the third derivative of β_4 Smarandache curve is

$$\beta_4''' = \iota_1 T + \iota_2 D_2 + \iota_3 D_1,$$

where

$$\iota_1 = \frac{1}{\sqrt{3}} \left(\begin{array}{l} {d_1}^3\nu^3 + \nu {d_1}^2\left(\nu^2 d_2 - 3\nu'\right) + d_1\left(\nu^3\left({d_2}^2 + {d_3}^2\right) + 3\nu'\nu d_3 - \nu'' - \nu^2\left(3{d_1}' - 2{d_3}'\right)\right) \\ \nu {d_2}^2\left(\nu^2 d_2 - 3\nu'\right) + d_2\left(\nu d_3\left(\nu^2 d_3 - 3\nu'\right) - \nu'' - \nu^2\left(3{d_2}' + 2{d_3}'\right)\right) \\ \nu^2 d_3\left({d_1}' - {d_2}'\right) - 2\nu'\left({d_1}' + {d_2}'\right) - \nu\left({d_1}'' + {d_2}''\right) \end{array} \right),$$

$$\iota_2 = \frac{1}{\sqrt{3}} \left(\begin{array}{l} -{d_1}^3\nu^3 + \nu {d_1}^2\left(\nu^2 d_3 - 3\nu'\right) - d_1\left(\nu^3\left({d_2}^2 + {d_3}^2\right) + 3d_2\nu'\nu - \nu'' + \nu^2\left(3{d_1}' + 2{d_2}'\right)\right) \\ +\nu^3 {d_2}^2 d_3 - \nu d_2\left(3\nu' d_3 + \nu\left({d_1}' + {d_3}'\right)\right) + \nu {d_3}^2\left(\nu^2 d_3 - 3\nu'\right) \\ -d_3\left(\nu'' + \nu^2\left(3{d_3}' + 2{d_2}'\right)\right) + 2\nu'\left({d_1}' - {d_3}'\right) + \nu\left({d_1}'' - {d_3}''\right) \end{array} \right),$$

$$\iota_3 = \frac{1}{\sqrt{3}} \left(\begin{array}{l} -\nu^3 {d_1}^2\left(d_3 + d_2\right) - d_1\left(3\nu\nu'\left(d_2 + d_3\right) - \nu^2\left({d_3}' - {d_2}'\right)\right) - \nu^3 {d_2}^3 \\ +d_2\left(\nu'' - \nu^3 {d_3}^2 - \nu^2\left(2{d_1}' + 3{d_2}'\right)\right) - \left(\nu {d_2}^2 + \nu {d_3}^2\right)\left(\nu^2 d_3 + 3\nu'\right) \\ +\left(\nu'' - \nu^2\left(3{d_3}' - 2{d_1}'\right)\right)d_3 + 2\nu'\left({d_3}' + {d_2}'\right) + \left({d_2}'' + {d_3}''\right)\nu \end{array} \right).$$

Therefore, the Frenet curvatures κ and τ of β_4 can be expressed as

$$\kappa_{\beta_4} = \frac{3\sqrt{3}\sqrt{{\psi_1}^2+{\psi_2}^2+{\psi_3}^2}}{\nu^3((d_1+d_2)^2+(d_1-d_3)^2+(d_2+d_3)^2)^{\frac{3}{2}}}, \qquad \tau_{\beta_4} = \frac{\psi_1\iota_1+\psi_2\iota_2+\psi_3\iota_3}{{\psi_1}^2+{\psi_2}^2+{\psi_3}^2}.$$

By using again (6), the Flc apparatus of β_4 can be given by means of the Flc components of β as

$$T_{\beta_4} = \frac{\sqrt{3}}{\Delta}\Big(-(d_1+d_2)T+d_3D_2+d_3D_1\Big),$$

$$D_{2\beta_4} = \frac{1}{\Delta\psi}\left(\begin{array}{r} (\nu cos\theta_4\left(\psi_2\left(d_2+d_3\right)-\psi_3\left(d_1-d_3\right)\right)+\Delta sin\theta_4\psi_1)\,T \\ -\left(\nu cos\theta_4\left(\psi_1\left(d_2+d_3\right)+\psi_3\left(d_1+d_2\right)\right)-\Delta sin\theta_4\psi_2\right)D_2 \\ +\left(\nu cos\theta_4\left(\psi_1\left(d_1-d_3\right)+\psi_2\left(d_1+d_2\right)\right)+\Delta sin\theta_4\psi_3\right)D_1 \end{array}\right),$$

$$D_{1\beta_4} = \frac{1}{\Delta\psi}\left(\begin{array}{r} -\left(\nu sin\theta_4\left(\psi_2\left(d_2+d_3\right)-\psi_3\left(d_1-d_3\right)\right)-\Delta cos\theta_4\psi_1\right)T \\ +\left(\nu sin\theta_4\left(\psi_1\left(d_2+d_3\right)+\psi_3\left(d_1+d_2\right)\right)+\Delta cos\theta_4\psi_2\right)D_2 \\ -\left(\nu sin\theta_4\left(\psi_1\left(d_1-d_3\right)+\psi_2\left(d_1+d_2\right)\right)-\Delta cos\theta_4\psi_3\right)D_1 \end{array}\right),$$

where $\Delta = \sqrt{3\left((d_1+d_2)^2+(d_1-d_3)^2+(d_2+d_3)^2\right)}$, $\psi = \sqrt{{\psi_1}^2+{\psi_2}^2+{\psi_3}^2}$, $\theta_4 = \sphericalangle\left(N_{\beta_4}, D_{2\beta_4}\right)$ and

$$d_{1\beta_4} = \left(\frac{3\sqrt{3}\sqrt{{\psi_1}^2+{\psi_2}^2+{\psi_3}^2}}{\nu^3((d_1+d_2)^2+(d_1-d_3)^2+(d_2+d_3)^2)^{\frac{3}{2}}}\right)cos\theta_4,$$

$$d_{2\beta_4} = -\left(\frac{3\sqrt{3}\sqrt{{\psi_1}^2+{\psi_2}^2+{\psi_3}^2}}{\nu^3((d_1+d_2)^2+(d_1-d_3)^2+(d_2+d_3)^2)^{\frac{3}{2}}}\right)sin\theta_4,$$

$$d_{3\beta_4} = -\frac{{d_2}'_{\beta_4}d_{1\beta_4}-d_{2\beta_4}{d_1}'_{\beta_4}}{\nu\left(d_{1\beta_4}^2+d_{2\beta_4}^2\right)}+\frac{\psi_1\iota_1+\psi_2\iota_2+\psi_3\iota_3}{{\psi_1}^2+{\psi_2}^2+{\psi_3}^2}.$$

Example 3.5 *Let us consider $\alpha=\alpha(t)$ be a 4^{th} order polynomial curve parametrized by*

$$\alpha(t) = \left(t^2, \frac{2t^3}{3}, \frac{t^4}{4}\right).$$

The corresponding Frenet apparatus of this curve are given as

$$T(t) = \left(\tfrac{2t}{|t|(t^2+2)}, \tfrac{2|t|}{t^2+2}, \tfrac{|t|t}{t^2+2}\right), \quad N(t) = \left(-\tfrac{2|t|}{t^2+2}, -\tfrac{|t|t-2sign(t)}{t^2+2}, \tfrac{2|t|}{t^2+2}\right),$$

$$B(t) = \left(\tfrac{t^2}{t^2+2}, -\tfrac{2t}{t^2+2}, \tfrac{2t(|t|t+sign(t))}{|t|(t^2+2)}\right), \quad \kappa = \tfrac{2}{|t|(t^2+2)^2}, \quad \tau = \tfrac{2}{t(t^2+2)^2}.$$

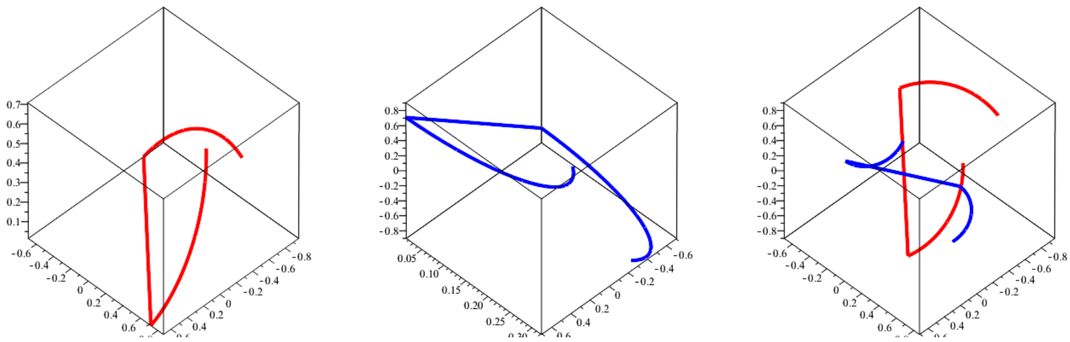
On the other hand, as $\|\alpha'\| = |t|\,(t^2+2)$, *the corresponding FLC apparatus of* α *are*

$$
\begin{aligned}
T(t) &= \left(\tfrac{2t}{|t|(t^2+2)}, \tfrac{2|t|}{t^2+2}, \tfrac{|t|t}{t^2+2}\right),\\
D_2(t) &= \left(-\tfrac{t|t|}{(t^2+2)\sqrt{t^2+1}}, -\tfrac{|t|^3}{(t^2+2)\sqrt{t^2+1}}, \tfrac{2(t|t|+sign(t))}{(t^2+2)\sqrt{t^2+1}}\right),\\
D_1(t) &= \left(\tfrac{t}{\sqrt{t^2+1}}, -\tfrac{1}{\sqrt{t^2+1}}, 0\right),\ \ d_1 = \tfrac{t}{\sqrt{t^2+1}},\ \ d_2 = -\tfrac{sign(t)}{\sqrt{t^2+1}},\ \ d_3 = \tfrac{t|t|}{2(t^2+1)}.
\end{aligned}
$$

To compare the two frames namely the Frenet frame and the Frenet like frame, let us denote α_1 *and* β_1 *as the* TN*-Smarandache curve and* TD_2*-Smarandache curve, and define these as*

$$
\alpha_1(t) = \frac{T(t)+N(t)}{\sqrt{2}}, \qquad \beta_1(t) = \frac{T(t)+D_2(t)}{\sqrt{2}}.
$$

The graph of these curves are given in Figure 1.



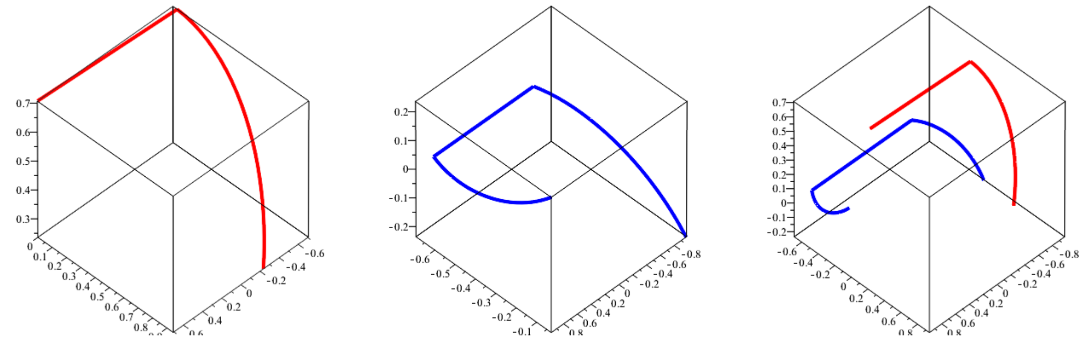
(a) α_1 alone (b) β_1 alone (c) α_1 and β_1 together

Figure 1: $TN-$ vs TD_2- Smarandache curves for $t \in (-1, 1)$

Next let us denote this time α_2 and β_2 as the TB and TD_1-Smarandache curves, respectively, and define these as

$$
\alpha_2(t) = \frac{T(t)+B(t)}{\sqrt{2}}, \qquad \beta_2(t) = \frac{T(t)+D_1(t)}{\sqrt{2}}.
$$

The corresponding pictures for these curves are provided in Figure 2.



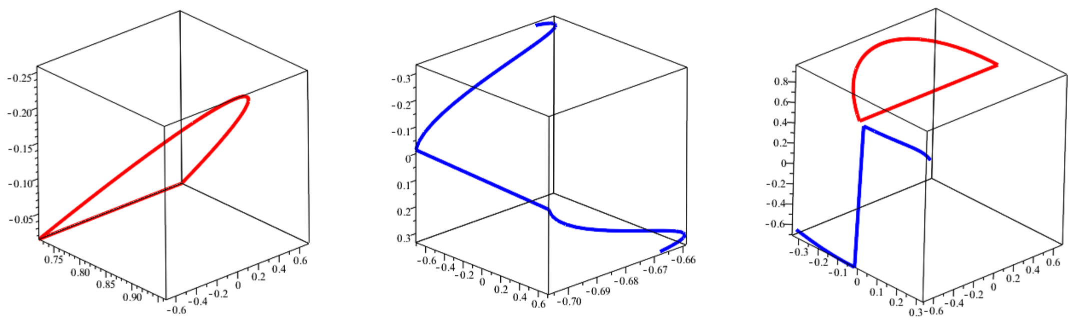
(a) α_2 alone (b) β_2 alone (c) α_2 and β_2 together

Figure 2: $TB-$ vs TD_1- Smarandache curves for $t \in (-1, 1)$

If we define α_3 and β_3 as the $NB-$ and D_2D_1- Smarandache curves, respectively, then we have

$$\alpha_3(t) = \frac{N(t) + B(t)}{\sqrt{2}}, \qquad \beta_3(t) = \frac{D_2(t) + D_1(t)}{\sqrt{2}},$$

where these curves are presented in Figure 3.



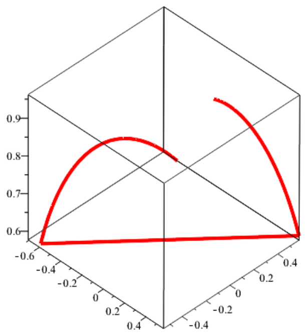
(a) α_3 alone (b) β_3 alone (c) α_3 and β_3 together

Figure 3: $NB-$ vs D_2D_1- Smarandache curves for $t \in (-1, 1)$

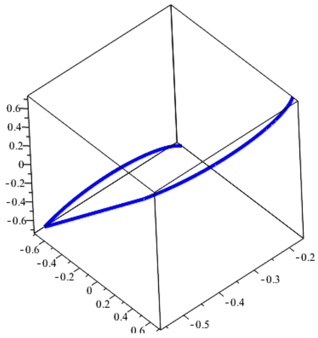
Finally, if we take α_4 and β_4 as the $TNB-$ and TD_2D_1- Smarandache curves, respectively, then we write

$$\alpha_4(t) = \frac{T(t) + N(t) + B(t)}{\sqrt{3}}, \qquad \beta_4(t) = \frac{T(t) + D_2(t) + D_1(t)}{\sqrt{3}}.$$

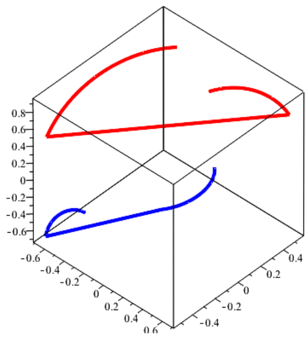
The Figure 4 shows these curves.



(a) α_4 alone



(b) β_4 alone



(c) α_4 and β_4 together

Figure 4: $TNB-$ vs TD_2D_1- Smarandache curves for $t \in (-1, 1)$

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Süleyman Şenyurt]: Thought and designed the research/problem, contributed to research method (%40).

Author [Kebire Hilal Ayvacı]: Evaluation of data, wrote the manuscript, Contributed to completing the research and solving the problem (%30).

Author [Davut Canlı]: Contributed to completing the research and solving the problem, visualization (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Smarandache Curves of Involute-Evolute Curve According to Frenet Frame

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Abstract: In this paper, the invariants of the Smarandache curves, which consist of Frenet vectors of the involute curve, are calculated in terms of the evolute curve.

Keywords: Curvature, evolute curve, Frenet frame, involute curve, Smarandache curves, torsion.

1. Introduction and Preliminaries

A regular curve in Minkowski space-time, whose position vector is composed of Frenet frame vectors on another regular curve, is called a Smarandache curve [17]. Special Smarandache curves have been studied by some authors. Turgut and Yılmaz's article deals with interesting knowledge of special Smarandache curves in the space $\mathbb{E}_1^4$. For example, they obtained another orthonormal frame [17]. In the light of the reference [17], Ali adapted Smarandache curve to regular curves in the $\mathbb{E}^3$ [2]. Ergüt et al. defined the isotropic types of Smarandache curves. Then they examined these kinds of isotropic Smarandache curves according to the Cartan frame in the complex 4-space [6]. By using the Darboux frame, Bektaş and Yüce obtained the results about Smarandache curves [4]. In another study, they studied the spatial quaternionic curve and the relationship between Frenet frames of the involute curve of the spatial quaternionic curve which are expressed by using the angle between the Darboux vector and binormal vector [15]. Şenyurt et al. used special curves as a base to create Smarandache curves, and then studied their geometric properties [12–14]. Al-Dayal and Solouma study some properties of spacelike Smarandache curves regarding Bishop frame of a spacelike curve in Minkowski 3-space [1]. There are many studies about Smarandache curves [9, 11, 16]

Huygens discovered an involute-evolute curve while trying to build a more accurate clock. The involute of a curve is a well-known concept in the Euclidean space [7, 8, 10]. The involute-

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evolute curve has attracted mathematicians' attention. In [5], authors found the relationships between the Frenet frames of the timelike curve and the spacelike involute curve. In another study, Bishop curvatures of the involute-evolute curve were examined and some important results were found [3].

In this paper, the invariants of the Smarandache curves, which consist of Frenet vectors of the involute curve, are calculated in terms of the evolute curve.

The inner product can be given by

$$\langle\ ,\rangle = x_1^2 + x_2^3 + x_3^2,$$

where $(x_1, x_2, x_3) \in \mathbb{E}^3$. Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with the moving Frenet frame $\{V_1(s), V_2(s), V_3(s)\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature k_1 and k_2, respectively, the Frenet formulas are given by [7]

$$\begin{cases} V_1{}'(s) = k_1(s)V_2(s), \\ V_2{}'(s) = -k_1(s)V_1(s) + k_2(s)V_3(s), \\ V_3{}'(s) = -k_2(s)V_2(s). \end{cases}$$

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = k_2 V_1 + k_1 V_3.$$

If we consider the normalization of the Darboux vector $C = \frac{1}{\|W\|}W$, we have

$$\sin\varnothing = \frac{k_2}{\|W\|}, \quad \cos\varnothing = \frac{k_1}{\|W\|}$$

and

$$C = \sin\varnothing V_1 + \cos\varnothing V_3,$$

where $\langle W, V_3\rangle = \varnothing$.

Theorem 1.1 *[10] Let the Frenet frames of α and α^* be $\{V_1(s), V_2(s), V_3(s)\}$ and $\{V_1{}^*(s), V_2{}^*(s), V_3{}^*(s)\}$ respectively. The relations between the Frenet frames are as follows;*

$$\begin{cases} V_1{}^*(s) = V_2(s), \\ V_2{}^*(s) = -\cos\varnothing V_1(s) + \sin\varnothing V_3(s), \\ V_3{}^*(s) = \sin\varnothing V_1(s) + \cos\varnothing V_3(s). \end{cases} \tag{1}$$

Definition 1.2 *[7] Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. For all $s \in I$, the curve α^* is called the involute of the curve α if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^* and $\langle V_1(s), V_1{}^*(s)\rangle = 0$. The curve α is called the evolute curve.*

Theorem 1.3 *[7] The distance between corresponding points of the evolute-involute curve in* $\mathbb{E}^3$ *is, for all* $s \in I$

$$d\big(\alpha(s), \alpha^*(s)\big) = |c - s|,$$

where c *is a constant.*

Theorem 1.4 *[10] Let* (α, α^*) *be a evolute-involute curves in* $\mathbb{E}^3$*. For the curvatures and the torsions of the evolute-involute curve* (α, α^*) *we have*

$$\begin{cases} {k_1}^* = \dfrac{\sqrt{{k_1}^2 + {k_2}^2}}{(c-s)k_1}, \lambda = c - s, \\ \\ {k_2}^* = \dfrac{k_1 {k_2}' - {k_1}' k_2}{(c-s)k_1({k_1}^2 + {k_2}^2)}. \end{cases}$$

2. Smarandache Curves of Involute-Evolute Curve Couple According to Frenet Frame

In this subsection, special Smarandache curves belonging to involute curves such as ${V_1}^*{V_2}^*$, ${V_2}^*{V_3}^*$, ${V_1}^*{V_3}^*$ and ${V_1}^*{V_2}^*{V_3}^*$ drawn by Frenet frame are defined. Curvatures and torsions of involute curves are expressed depending upon the evolute curve and some related results are given.

Definition 2.1 *Let* (α, α^*) *be a evolute-involute curves in* $\mathbb{E}^3$*.* ${V_1}^*{V_2}^*$ *- Smarandache curve can be defined by*

$$\beta_1(s) = \frac{1}{\sqrt{2}}({V_1}^* + {V_2}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_1(s) = \frac{-\cos\varnothing V_1 + V_2 + \sin\varnothing V_3}{\sqrt{2}}. \tag{2}$$

Theorem 2.2 *Frenet vectors of Smarandache curve* β_1 *are given as follows;*

$$T_{\beta_1} = \frac{(\varnothing' \sin\varnothing - k_1)V_1 - \|W\|V_2 + (\varnothing' \cos\varnothing + k_2)V_3}{\sqrt{\varnothing'^2 + 2\|W\|^2}}, \tag{3}$$

$$N_{\beta_1} = \frac{\bar{\omega_1} V_1 + \bar{\omega_2} V_2 + \bar{\omega_3} V_3}{\sqrt{\bar{\omega_1}^2 + \bar{\omega_2}^2 + \bar{\omega_3}^2}},$$

$$B_{\beta_1} = \frac{(-\|W\|\bar{\omega_3} - (\varnothing' \cos\varnothing + k_2)\bar{\omega_2})V_1 + (\bar{\omega_1}(\varnothing' \cos\varnothing + k_2) - \bar{\omega_3}(\varnothing' \sin\varnothing - k_1))V_2}{\sqrt{(\varnothing'^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}$$

$$+ \frac{(\bar{\omega_2}(\varnothing' \sin\varnothing - k_1) + \bar{\omega_1}\|W\|)V_3}{\sqrt{(\varnothing'^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}.$$

Here, the coefficients are

$$\begin{cases} \bar{\omega_1} = (\varnothing'' \sin\varnothing + {\varnothing'}^2 \cos\varnothing - {k_1}' + k_1\|W\|)\sqrt{{\varnothing'}^2 + 2\|W\|^2} \\ \qquad -(\varnothing' \sin\varnothing - k_1)\left(\sqrt{{\varnothing'}^2 + 2\|W\|^2}\right)', \\ \bar{\omega_2} = (-\|W\|^2 - \|W\|')\sqrt{{\varnothing'}^2 + 2\|W\|^2} + \|W\|\left(\sqrt{{\varnothing'}^2 + 2\|W\|^2}\right)', \\ \bar{\omega_3} = (\varnothing'' \cos\varnothing - {\varnothing'}^2 \sin\varnothing + {k_2}' - k_2\|W\|)\sqrt{{\varnothing'}^2 + 2\|W\|^2} \\ \qquad -(\varnothing' \cos\varnothing + k_2)\left(\sqrt{{\varnothing'}^2 + 2\|W\|^2}\right)'. \end{cases}$$

Proof The derivative of the equation (2) is

$$\beta_1' = T_{\beta_1}\frac{ds_{\beta_1}}{ds} = \frac{(\varnothing' \sin\varnothing - k_1)V_1 - \|W\|V_2 + (\varnothing' \cos\varnothing + k_2)V_3}{\sqrt{2}}.$$

By taking the norm of the above equation, we can write

$$\frac{ds_{\beta_1}}{ds} = \sqrt{\frac{{\varnothing'}^2 + 2\|W\|^2}{2}}.$$

If necessary operations are taken, the tangent vector is

$$T_{\beta_1}(s) = \frac{(\varnothing' \sin\varnothing - k_1)V_1 - \|W\|V_2 + (\varnothing' \cos\varnothing + k_2)V_3}{\sqrt{{\varnothing'}^2 + 2\|W\|^2}}. \tag{4}$$

In the light of the pieces of information, the principal normal and the binormal vectors are respectively given by

$$N_{\beta_1} = \frac{\bar{\omega_1}V_1 + \bar{\omega_2}V_2 + \bar{\omega_3}V_3}{\sqrt{{\bar{\omega_1}}^2 + {\bar{\omega_2}}^2 + {\bar{\omega_3}}^2}},$$

$$B_{\beta_1} = \frac{(-\|W\|\bar{\omega_3} - (\varnothing' \cos\varnothing + k_2)\bar{\omega_2})V_1 + (\bar{\omega_1}(\varnothing' \cos\varnothing + k_2) - \bar{\omega_3}(\varnothing' \sin\varnothing - k_1))V_2}{\sqrt{({\varnothing'}^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}$$

$$+\frac{(\bar{\omega_2}(\varnothing' \sin\varnothing - k_1) + \bar{\omega_1}\|W\|)V_3}{\sqrt{({\varnothing'}^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}.$$

□

Theorem 2.3 *Curvature and torsion belonging to Smarandache curve* β_1 *are, respectively*

$$\begin{aligned} k_{1\beta_1} &= \frac{\sqrt{2}}{(\varnothing'^2+2\|W\|^2)^{\frac{3}{2}}}\Bigg(\Big((\varnothing''\sin\varnothing+\varnothing'^2\cos\varnothing-k_1'+k_1\|W\|)\sqrt{\varnothing'^2+2\|W\|^2} \\ &\quad -(\varnothing'\sin\varnothing-k_1)\Big(\sqrt{\varnothing'^2+2\|W\|^2}\Big)'\Big)^2+\Big((-\|W\|^2-\|W\|')\sqrt{\varnothing'^2+2\|W\|^2} \\ &\quad +\|W\|\Big(\sqrt{\varnothing'^2+2\|W\|^2}\Big)'\Big)^2+\Big((\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing+k_2'-k_2\|W\|) \\ &\quad \sqrt{\varnothing'^2+2\|W\|^2}-(\varnothing'\cos\varnothing+k_2).\Big(\sqrt{\varnothing'^2+2\|W\|^2}\Big)'\Big)^2\Bigg)^{\frac{1}{2}}, \end{aligned}$$

$$k_{2\beta_1} = \frac{\begin{array}{l}\sqrt{2}\Big[\bar{\nu_1}(\varnothing'''\sin\varnothing+3\varnothing'\varnothing''\cos\varnothing-\varnothing'^3\sin\varnothing-k_1''+k_1'\|W\|+2k_1\|W\|' \\ +k_1\|W\|^2)+\bar{\nu_2}(\varnothing'^2\|W\|-k_1k_1'-k_2k_2'+\|W\|^3-2\|W\|\|W\|'+\|W\|'') \\ +\bar{\nu_3}(\varnothing'''\cos\varnothing-3\varnothing'\varnothing''\sin\varnothing-\varnothing'^3\cos\varnothing+k_2''-k_2'\|W\|-2k_2\|W\|'-k_2\|W\|^2)\Big]\end{array}}{\bar{\nu_1}^2+\bar{\nu_2}^2+\bar{\nu_3}^2},$$

where

$$\begin{cases} \bar{\nu_1} = -\|W\|(\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing+k_2'-k_2\|W\|)+(\|W\|^2+\|W\|')(\varnothing'\cos\varnothing+k_2), \\ \bar{\nu_2} = (\varnothing'\cos\varnothing+k_2)(\varnothing''\sin\varnothing+\varnothing'^2\cos\varnothing-k_1'+k_1\|W\|)-(\varnothing'\sin\varnothing-k_1)(\varnothing''\cos\varnothing \\ \qquad -\varnothing'^2\sin\varnothing+k_2'-k_2\|W\|), \\ \bar{\nu_3} = (\varnothing'\sin\varnothing-k_1)(-\|W\|^2-\|W\|')+\|W\|(\varnothing''\sin\varnothing+\varnothing'^2\cos\varnothing-k_1'+k_1\|W\|). \end{cases}$$

Proof The first curvature is

$$k_{1\beta_1} = \|T'_{\beta_1}\|. \tag{5}$$

Taking the derivative of the equation (4), we obtain

$$T'_{\beta_1}(s) = \sqrt{2}\frac{\bar{\omega_1}V_1+\bar{\omega_2}V_2+\bar{\omega_3}V_3}{(\varnothing'^2+2\|W\|^2)^{\frac{3}{2}}}. \tag{6}$$

If the expression (6) is written in (5), the first curvature is

$$k_{1\beta_1} = \|T'_{\beta_1}\| = \sqrt{2}\frac{\sqrt{\bar{\omega_1}^2+\bar{\omega_2}^2+\bar{\omega_3}^2}}{(\varnothing'^2+2\|W\|^2)^{\frac{3}{2}}}.$$

If the coefficients are written instead, the desired result is obtained.

To calculate the torsion of the curve β_1, we differentiate

$$\beta_1'' = \left(-\cos\varnothing\left(\left(\frac{\|W\|}{(c-s)k_1} \right)' - \frac{\|W\|^2}{(c-s)^2{k_1}^2} - \frac{(k_1{k_2}' - {k_1}'k_2)^2}{(c-s)^2{k_1}^2\|W\|^4} \right)\right.$$

$$+\sin\varnothing\Big(\frac{\|W\|(k_1{k_2}' - {k_1}'k_2)}{(c-s)^2{k_1}^2\|W\|^4} + \big(\frac{(k_1{k_2}' - {k_1}'k_2)}{(c-s)k_1\|W\|^2} \big)' \Big)\Bigg) V_1 + \Bigg(\frac{\|W\|^2}{(c-s)^2{k_1}^2}$$

$$+\left(\frac{\|W\|}{(c-s)k_1} \right)' \Bigg) V_2 + \Bigg(\sin\varnothing\left(\left(\frac{\|W\|}{(c-s)k_1} \right)' - \frac{\|W\|^2}{(c-s)^2{k_1}^2} - \frac{(k_1{k_2}' - {k_1}'k_2)^2}{(c-s)^2{k_1}^2\|W\|^4} \right)$$

$$\left.+\cos\varnothing\left(\frac{\|W\|(k_1{k_2}' - {k_1}'k_2)}{(c-s)^2{k_1}^2\|W\|^4} + \left(\frac{(k_1{k_2}' - {k_1}'k_2)}{(c-s)k_1\|W\|^2} \right)' \right)\right) V_3$$

and thus

$$\beta_1''' = \frac{(-\bar{\eta}_2\cos\varnothing + \bar{\eta}_3\sin\varnothing)V_1 + \bar{\eta}_1 V_2 + (\bar{\eta}_2\sin\varnothing + \bar{\eta}_3\cos\varnothing)V_3}{\sqrt{2}},$$

where

$$\begin{cases} \bar{\eta}_1 = \varnothing'''\sin\varnothing + 3\varnothing'\varnothing''\cos\varnothing - {\varnothing'}^3\sin\varnothing - {k_1}'' + {k_1}'\|W\| + 2k_1\|W\|' + k_1\|W\|^2, \\ \bar{\eta}_2 = {\varnothing'}^2\|W\| - k_1{k_1}' - k_2{k_2}' + \|W\|^3 - 2\|W\|\|W\|' + \|W\|'', \\ \bar{\eta}_3 = \varnothing'''\cos\varnothing - 3\varnothing'\varnothing''\sin\varnothing - {\varnothing'}^3\cos\varnothing + {k_2}'' - {k_2}'\|W\| - 2k_2\|W\|' - k_2\|W\|^2. \end{cases}$$

The torsion is then given by

$$k_{2_{\beta_1}} = \frac{det(\beta_1', \beta_1'', \beta_1''')}{\|\beta_1' \wedge \beta_1''\|^2} \ ,$$

$$k_{2_{\beta_1}} = \frac{\begin{array}{l} \sqrt{2}\Big[\bar{\nu}_1(\varnothing'''\sin\varnothing + 3\varnothing'\varnothing''\cos\varnothing - {\varnothing'}^3\sin\varnothing - {k_1}'' + {k_1}'\|W\| + 2k_1\|W\|' \\ +k_1\|W\|^2) + \bar{\nu}_2({\varnothing'}^2\|W\| - k_1{k_1}' - k_2{k_2}' + \|W\|^3 - 2\|W\|\|W\|' \\ +\|W\|'') + \bar{\nu}_3(\varnothing'''\cos\varnothing - 3\varnothing'\varnothing''\sin\varnothing - {\varnothing'}^3\cos\varnothing + {k_2}'' - {k_2}'\|W\| \\ -2k_2\|W\|' - k_2\|W\|^2)\Big] \end{array}}{{\bar{\nu}_1}^2 + {\bar{\nu}_2}^2 + {\bar{\nu}_3}^2}.$$

□

Definition 2.4 *Let* (α, α^*) *be a evolute-involute curves in* $\mathbb{E}^3$. ${V_2}^*{V_3}^*$ *- Smarandache curve can be defined by*

$$\beta_2(s) = \frac{1}{\sqrt{2}}({V_2}^* + {V_3}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_2(s) = \frac{(\sin\varnothing - \cos\varnothing)V_1 + (\sin\varnothing + \cos\varnothing)V_3}{\sqrt{2}}.$$

Theorem 2.5 *The Frenet invariants of the* β_2 *curve are given as follows;*

$$T_{\beta_2} = \frac{(\varnothing'\cos\varnothing + \varnothing'\sin\varnothing)V_1 - \|W\|V_2 + (\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)V_3}{\sqrt{2\varnothing'^2 + \|W\|^2}},$$

$$N_{\beta_2} = \frac{\bar{\varsigma}_1 V_1 + \bar{\varsigma}_2 V_2 + \bar{\varsigma}_3 V_3}{\sqrt{\bar{\varsigma}_1{}^2 + \bar{\varsigma}_2{}^2 + \bar{\varsigma}_3{}^2}},$$

$$\begin{aligned} B_{\beta_2} = {} & \frac{-\|W\|\bar{\varsigma}_3 - (\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)\bar{\varsigma}_2}{\sqrt{2\varnothing'^2 + \|W\|^2(\bar{\varsigma}_1{}^2 + \bar{\varsigma}_2{}^2 + \bar{\varsigma}_3{}^2)}}V_1 \\ & + \frac{(\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)\bar{\varsigma}_1 - (\varnothing'\cos\varnothing + \varnothing'\sin\varnothing)\bar{\varsigma}_3}{\sqrt{2\varnothing'^2 + \|W\|^2(\bar{\varsigma}_1{}^2 + \bar{\varsigma}_2{}^2 + \bar{\varsigma}_3{}^2)}}V_2 \\ & + \frac{\|W\|\bar{\varsigma}_1 + (\varnothing'\cos\varnothing + \varnothing'\sin\varnothing)\bar{\varsigma}_2}{\sqrt{2\varnothing'^2 + \|W\|^2(\bar{\varsigma}_1{}^2 + \bar{\varsigma}_2{}^2 + \bar{\varsigma}_3{}^2)}}V_3, \end{aligned}$$

$$\begin{aligned} k_{1\beta_2} = {} & \frac{\sqrt{2}}{(2\varnothing'^2 + \|W\|^2)^{\frac{3}{2}}}\Bigg(\Big(\big((\varnothing''\cos\varnothing - \varnothing'^2\sin\varnothing + \varnothing''\sin\varnothing + \varnothing'^2\cos\varnothing \\ & + k_1\|W\|)\sqrt{2\varnothing'^2 + \|W\|^2} - \left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'\Big)^2(\varnothing'\cos\varnothing \\ & + \varnothing'\sin\varnothing) + \Bigg((\|W\|\varnothing' - \|W\|')\sqrt{2\varnothing'^2 + \|W\|^2} + \|W\| \\ & .\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'\Bigg)^2 + \Big((\cos\varnothing(\varnothing'' - \varnothing'^2) - \sin\varnothing(\varnothing'' + \varnothing'^2) - k_2\|W\|) \\ & \sqrt{2\varnothing'^2 + \|W\|^2} - (\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'\Big)^2\Bigg)^{\frac{1}{2}}, \end{aligned}$$

$$k_{2\beta_2} = \frac{\begin{array}{l}\sqrt{2}\Big[h_1\big(\varnothing''' \cos\varnothing - 3\varnothing'\varnothing'' \sin\varnothing - \varnothing'^3 \cos\varnothing + \varnothing''' \sin\varnothing + 3\varnothing'\varnothing'' \cos\varnothing \\ -\varnothing'^3 \sin\varnothing + {k_1}'\|W\| + 2k_1\|W\|' - k_1\varnothing'\|W\|\big) + h_2\big(2\varnothing''\|W\| + \varnothing'^2\|W\| \\ +\|W\|^3 + \|W\|'\varnothing' - \|W\|''\big) + h_3\big(\varnothing''' \cos\varnothing - 3\varnothing'\varnothing'' \sin\varnothing - \varnothing'^3 \cos\varnothing \\ -\varnothing''' \sin\varnothing - 3\varnothing'\varnothing'' \cos\varnothing + \varnothing'^3 \sin\varnothing - {k_2}'\|W\| - 2k_2\|W\|' + k_2\varnothing'\|W\|\big)\end{array}}{{h_1}^2 + {h_2}^2 + {h_3}^2},$$

where

$$\begin{cases} \bar{\varsigma}_1 = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing + k_1\|W\|)\sqrt{2\varnothing'^2 + \|W\|^2} \\ \qquad -(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing)\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)', \\ \bar{\varsigma}_2 = (\|W\|\varnothing' - \|W\|')\sqrt{2\varnothing'^2 + \|W\|^2} + \|W\|\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)', \\ \bar{\varsigma}_3 = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing - k_2\|W\|)\sqrt{2\varnothing'^2 + \|W\|^2} \\ \qquad -(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing)\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'. \end{cases}$$

$$\begin{cases} h_1 = -\|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing - k_2\|W\|) \\ \qquad -(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing)(\|W\|\varnothing' - \|W\|'), \\ h_2 = (\varnothing' \cos\varnothing - \varnothing' \sin\varnothing)(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing + k_1\|W\|), \\ \qquad -(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing)(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing - k_2\|W\|) \\ h_3 = \|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing + k_1\|W\|) \\ \qquad +(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing)(\|W\|\varnothing' - \|W\|')\cdot \end{cases}$$

Proof The theorem is similar to Theorem 2.2 and Theorem 2.3, therefore we omit its proof. □

Definition 2.6 *Let* (α, α^*) *be a evolute-involute curves in* $\mathbb{E}^3$. ${V_1}^*{V_3}^*$ *- Smarandache curve can be defined by*

$$\beta_3(s) = \frac{1}{\sqrt{2}}({V_1}^* + {V_3}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_3(s) = \frac{\sin\varnothing V_1 + V_2 + \cos\varnothing V_3}{\sqrt{2}}\cdot$$

Theorem 2.7 *The Frenet invariants of the β_3 curve are given as follows;*

$$T_{\beta_3} = \frac{(\varnothing'\cos\varnothing - k_1)V_1 + (-\varnothing'\sin\varnothing + k_2)V_3}{\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}},$$

$$N_{\beta_3} = \frac{\bar{o_1}V_1 + \bar{o_2}V_2 + \bar{o_3}V_3}{\sqrt{\bar{o_1}^2 + \bar{o_1}^2 + \bar{o_3}^2}},$$

$$B_{\beta_3} = \frac{[(\varnothing'\sin\varnothing - k_2)\bar{o_2}]V_1 + [(\varnothing'\sin\varnothing - k_2)\bar{o_1} - (\varnothing'\cos\varnothing - k_1)\bar{o_3}]V_2}{\sqrt{(\bar{o_1}^2 + \bar{o_1}^2 + \bar{o_3}^2)(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)}}$$

$$+\frac{[(\varnothing'\cos\varnothing - k_1)\bar{o_2}]V_3}{\sqrt{(\bar{o_1}^2 + \bar{o_1}^2 + \bar{o_3}^2)(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)}},$$

$$\begin{aligned} k_{1\beta_3} = {} & \frac{\sqrt{2}}{(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)^{\frac{3}{2}}}\Bigg(\Big((\varnothing''\cos\varnothing - \varnothing'^2\sin\varnothing - k_1')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2} \\ & -(\varnothing'\cos\varnothing - k_1)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)'\Big)^2 + (\varnothing'\|W\| - \|W\|^2)^2 \\ & .(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2) + \Big((-\varnothing''\sin\varnothing - \varnothing'^2\cos\varnothing + k_2')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2} \\ & -(-\varnothing'\sin\varnothing + k_2)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)'\Big)^2\Bigg)^{\frac{1}{2}}, \end{aligned}$$

$$k_{2\beta_3} = \frac{\begin{array}{c}\sqrt{2}\Big[\big(\varnothing'''\cos\varnothing - 3\varnothing'\varnothing''\sin\varnothing - \varnothing'^3\cos\varnothing - k_1'' - k_1\varnothing'\|W\| + k_1\|W\|^2\big)\bar{p_1} \\ +\big(\varnothing''\|W\| - k_1k_1' - k_2k_2' + \varnothing''\|W\| + \varnothing'\|W\|' - 2\|W\|\|W\|'\big)\bar{p_2} \\ +\big(-\varnothing'''\sin\varnothing - 3\varnothing'\varnothing'\cos\varnothing + \varnothing'^3\sin\varnothing + k_2'' + k_2\varnothing'\|W\| - k_2\|W\|^2\big)\bar{p_3}\end{array}}{\bar{p_1}^2 + \bar{p_2}^2 + \bar{p_3}^2},$$

where

$$\begin{cases} \bar{o_1} = (\varnothing''\cos\varnothing - \varnothing'^2\sin\varnothing - k_1')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2} \\ \qquad -(\varnothing'\cos\varnothing - k_1)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)', \\ \bar{o_2} = (\varnothing'\|W\| - \|W\|^2)\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}, \\ \bar{o_3} = (-\varnothing''\sin\varnothing - \varnothing'^2\cos\varnothing + k_2')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2} \\ \qquad -(-\varnothing'\sin\varnothing + k_2)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)'; \end{cases}$$

$$\begin{cases} \bar{p_1} = (\varnothing' \sin\varnothing - k_2)(\varnothing'\|W\| - \|W\|^2), \\ \bar{p_2} = (\varnothing' \sin\varnothing - k_2)(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - k_1') - (\varnothing' \cos\varnothing - k_1) \\ \qquad .(-\varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + k_2'), \\ \bar{p_3} = (\varnothing' \cos\varnothing - k_1)(\varnothing'\|W\| - \|W\|^2). \end{cases}$$

Proof The theorem is similar to Theorem 2.2 and Theorem 2.3, therefore we omit its proof. □

Definition 2.8 *Let (α, α^*) be a evolute-involute curves in $\mathbb{E}^3$. ${V_1}^*{V_2}^*{V_3}^*$ - Smarandache curve can be defined by*

$$\beta_4(s) = \frac{1}{\sqrt{3}}({V_1}^* + {V_2}^* + {V_3}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_4(s) = \frac{(\sin\varnothing - \cos\varnothing)V_1 + V_2 + (\cos\varnothing + \sin\varnothing)V_3}{\sqrt{3}}.$$

Theorem 2.9 *The Frenet invariants of the β_4 curve are given as follows;*

$$T_{\beta_4} = \frac{\big(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1\big)V_1 - \|W\|V_2 + \big(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2\big)V_3}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)}},$$

$$N_{\beta_4} = \frac{\bar{g_1}V_1 + \bar{g_2}V_2 + \bar{g_3}V_3}{\sqrt{\bar{g_1}^2 + \bar{g_1}^2 + \bar{g_3}^2}},$$

$$\begin{aligned} B_{\beta_4} = {} & \left(\frac{-\|W\|\bar{g_3} - \big(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2\big)\bar{g_2}}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)(\bar{g_1}^2 + \bar{g_2}^2 + \bar{g_3}^2)}}\right)V_1 \\ & + \left(\frac{\big(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2\big)\bar{g_1} - \big(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1\big)\bar{g_3}}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)(\bar{g_1}^2 + \bar{g_2}^2 + \bar{g_3}^2)}}\right)V_2 \\ & + \left(\frac{\|W\|\bar{g_1} + \big(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1\big)\bar{g_2}}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)(\bar{g_1}^2 + \bar{g_2}^2 + \bar{g_3}^2)}}\right)V_3, \end{aligned}$$

$$
\begin{aligned}
k_{1_{\beta_4}} = {} & \frac{\sqrt{3}}{2((\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)^{\frac{3}{2}})}\Bigg(\Big((\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \cos\varnothing + \varnothing'^2 \cos\varnothing - {k_1}' \\
& + k_1\|W\|)\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2} - \Big(\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2}\Big)'\Big)^2 (\varnothing' \cos\varnothing \\
& + \varnothing' \sin\varnothing - k_1) + \Big((\|W\|\varnothing' - \|W\|^2 - \|W\|')\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2} \\
& + \|W\|\Big(\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2}\Big)'\Big)^2 + \Big((\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing \\
& + {k_2}' - k_2\|W\|)\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2} - \Big(\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2}\Big)'\Big)^2 \\
& .(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2)\Bigg)^{\frac{1}{2}},
\end{aligned}
$$

$$
k_{2_{\beta_4}} = \frac{\begin{array}{l}
\sqrt{3}\Big[\big(\varnothing''' \cos\varnothing - 3\varnothing'\varnothing'' \sin\varnothing - \varnothing'^3 \cos\varnothing + \varnothing''' \sin\varnothing + 3\varnothing'\varnothing'' \cos\varnothing - \varnothing''' \sin\varnothing - {k_1}'' \\
+ {k_1}'\|W\| + 2k_1\|W\|' - k_1\varnothing'\|W\| + k_1\|W\|^2\big)\bar{f}_1 + \big(\varnothing''\|W\| + \varnothing'^2\|W\|\|W\|^3 \\
- k_1{k_1}' - k_2{k_2}' + \varnothing'\|W\|' + \varnothing''\|W\| - 2\|W\|\|W\|' - \|W\|''\big)\bar{f}_2 + \big(\varnothing''' \cos\varnothing \\
- 3\varnothing'\varnothing'' \sin\varnothing - \varnothing'^3 \cos\varnothing - \varnothing''' \sin\varnothing - 3\varnothing'\varnothing'' \cos\varnothing - \varnothing'^3 \sin\varnothing + {k_2}'' - {k_2}'\|W\| \\
- 2k_2\|W\|' + k_2\varnothing'\|W\| - k_2\|W\|^2\big)\bar{f}_3\Big]
\end{array}}{\bar{f_1}^2 + \bar{f_2}^2 + \bar{f_3}^2},
$$

where

$$
\begin{cases}
\bar{g_1} = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \cos\varnothing + \varnothing'^2 \cos\varnothing - {k_1}' + k_1\|W\|) \\
\quad .\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2} - (\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1)\Big(\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2}\Big)', \\
\bar{g_2} = (\|W\|\varnothing' - \|W\|^2 - \|W\|')\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2} \\
\quad + \|W\|\Big(\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2}\Big)', \\
\bar{g_3} = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + {k_2}' - k_2\|W\|) \\
\quad .\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2} - (\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2)\Big(\sqrt{\varnothing'^2 - \varnothing'\|W\| + \|W\|^2}\Big)';
\end{cases}
$$

$$\begin{cases} \bar{f}_1 = -\|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + k_2{}' - k_2\|W\|) - (\varnothing'' \cos\varnothing \\ \quad -\varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + k_2{}' - k_2\|W\|)(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2), \\ \bar{f}_2 = (\varnothing'(\cos\varnothing - \sin\varnothing) + k_2)(2\varnothing'' \cos\varnothing - \varnothing'^2(\sin\varnothing - \cos\varnothing) - k_1{}' + k_1\|W\|), \\ \quad -(\varnothing'(\cos\varnothing + \sin\varnothing) - k_1)(\cos\varnothing(\varnothing'' - \varnothing'^2) - \sin\varnothing(\varnothing'' + \varnothing'^2) + k_2{}' - k_2\|W\|) \\ \bar{f}_3 = (\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1)(\|W\|\varnothing' - \|W\|^2 - \|W\|') \\ \quad +\|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \cos\varnothing + \varnothing'^2 \cos\varnothing - k_1{}' + k_1\|W\|). \end{cases}$$

Proof The theorem is similar to Theorem 2.2 and Theorem 2.3, therefore we omit its proof. □

3. Examples

Example 3.1 *Let us consider the unit speed helix curve and involute curve:*

$$\alpha(s) = \left(3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\right),$$

$$\alpha^*(s) = \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right)c + \frac{3}{5}\sin\left(\frac{s}{5}\right)s + 3\cos\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right)c - \frac{3}{5}\cos\left(\frac{s}{5}\right)s + 3\sin\left(\frac{s}{5}\right), \frac{4c}{5}\right).$$

The Smarandache curves, which consist of Frenet vectors of the involute curve, are, respectively, given as follows;

$$\begin{cases} \beta_1(s) = \left(\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), \frac{24\sqrt{2}}{25}\right), \\ \beta_2(s) = \left(\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), \frac{109\sqrt{2}}{50}\right), \\ \beta_3(s) = \left(-\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), \frac{61\sqrt{2}}{50}\right), \\ \beta_4(s) = \left(\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), \frac{109\sqrt{2}}{50}\right). \end{cases}$$

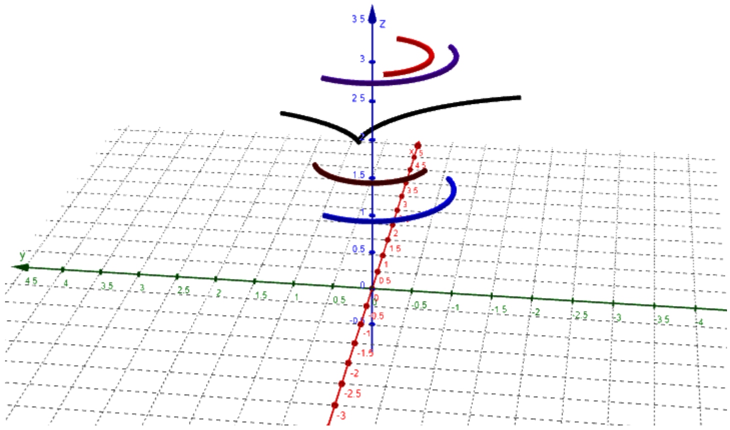

Figure 1: The black curve is the involute curve of the curve α (c=1). The blue, red, brown and purple curves are Smarandache curves, which consist of Frenet vectors of the involute curve, respectively

Example 3.2 *Let us consider the unit speed curve and involute curve:*

$$\alpha(s) = \left(\frac{(1+s)^{\frac{3}{2}}}{3}, \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{s\sqrt{2}}{2}\right),$$

$$\alpha^*(s) = \left(\frac{c\sqrt{1+s}}{2} - \frac{s\sqrt{1+s}}{2} + \frac{(1+s)^{\frac{3}{2}}}{3}, -\frac{c\sqrt{1-s}}{2} + \frac{s\sqrt{1-s}}{2} + \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{c\sqrt{2}}{2}\right).$$

The Smarandache curves, which consist of Frenet vectors of the involute curve, are, respectively, given as follows;

$$\begin{cases} \beta_1(s) = \left(\frac{-\sqrt{1+s}}{2} + \frac{\sqrt{1-s}}{2}, \frac{\sqrt{1-s}}{2} + \frac{\sqrt{1+s}}{2}, 0\right), \\ \beta_2(s) = \left(\frac{-\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{\sqrt{2}}{2}\right), \\ \beta_3(s) = \left(\frac{\sqrt{1-s}}{2}, \frac{\sqrt{1+s}}{2}, \frac{\sqrt{2}}{2}\right), \\ \beta_4(s) = \left(\frac{-\sqrt{1+s}}{2} + \frac{\sqrt{1-s}}{2}, \frac{\sqrt{1-s}}{2} + \frac{\sqrt{1+s}}{2}, \frac{\sqrt{2}}{2}\right). \end{cases}$$

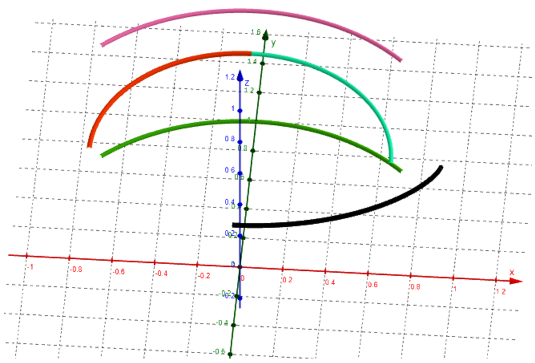

Figure 2: The black curve is the involute curve of the curve α (c=1). The green, orange, turquoise and purple curves are Smarandache curves, which consist of Frenet vectors of the involute curve, respectively

4. Conclusion

We examined the Smarandache curves formed by the Frenet vectors of the involute curve. Then curvatures and torsions of Smarandache curves are calculated. These invariants (Frenet vectors and curvatures) which depend on the evolute curve are explained. Besides, we illustrate the Smarandache curves formed by taking the helix curve.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Selin Sivas]: Thought and designed the research/problem, contributed to research method or evaluation of data (%55).

Author [Süleyman Şenyurt]: Contributed to research method or evaluation of data (%30).

Author [Abdussamet Çalışkan]: Collected the data, contributed to research examples and figures, wrote the manuscript (%15).

Conflicts of Interest

The authors declare no conflict of interest.

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Research Article

Some Important Classes of the Continuous and Complex Interval-Valued Functions

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Abstract: This paper presents some important classes of the continuous functions defined from the set of real numbers to the space of complex intervals. These function spaces have an algebraic structure named as a quasilinear space which is suggested by Aseev in 1986. In this work, we analysis the quasilinear structure on the classes of the continuous and complex interval-valued functions. Further, we show that these spaces are the normed Ω-spaces. Finally, we examine the dimension of these function spaces.

Keywords: Normed quasilinear space, complex interval, continuous function, dimension of a quasilinear space.

1. Introduction

As is known the Fourier transform is the main building block of many application areas, especially in the electrical engineering. This transform that is used for analyzing the signals in the frekans domain has a wide range of applications in the digital signal processing.

Many real world problems may contain uncertainties due to environmental factors, especially in signal processing [7–9, 14]. Such problems are modelled with intervals. For this reason there has been increasing interest in interval-valued functions [1, 2, 4]. We need the space of the continuous functions defined from $\mathbb{R}$ to the set of complex intervals to analyzes the signals with inexact data.

An interval x is the compact-convex subset of real numbers and x is denoted by $x = [\underline{x}, \overline{x}]$ where $\underline{x}$ and $\overline{x}$ are the left and right endpoints of x, respectively [13]. Further, if $\underline{x} = \overline{x}$, then we say that x is a degenerate interval and it can be shown by $\{x\}$ or $[x, x]$. The set of all real intervals is denoted by $\mathbb{I}_{\mathbb{R}}$.

To get a comprehensive and healthy interval-valued signal processing we need the notion of the complex interval. Therefore, we defined the space $\mathbb{I}_{\mathbb{C}}$ which is the set of all complex intervals

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in [11]. A complex interval is defined by

$$u = \left[\underline{u_r}, \overline{u_r}\right] + i\left[\underline{u_s}, \overline{u_s}\right],$$

where $\left[\underline{u_r}, \overline{u_r}\right]$ and $\left[\underline{u_s}, \overline{u_s}\right]$ are real intervals and $i = \sqrt{-1}$ is the complex unit. $\left[\underline{u_r}, \overline{u_r}\right]$ and $\left[\underline{u_s}, \overline{u_s}\right]$ are called real and imaginary part of u, respectively. Unfortunately, both $\mathbb{I}_{\mathbb{R}}$ and $\mathbb{I}_{\mathbb{C}}$ have an algebraic structure which is not linear space which is called as a "quasilinear space"by Aseev in 1986 (for details, see [3]). The most popular examples are $\Omega(E)$ and $\Omega_C(E)$ which are defined as the sets of all nonempty closed bounded and nonempty convex closed bounded subsets of any normed linear space E, respectively. Both are a quasilinear space with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}},$$

where the closure is taken on the norm topology of E. The real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

Especially, $\mathbb{I}_{\mathbb{R}}$ is a quasilinear space with the Minkowski sum and scalar multiplication operations are defined by

$$x + y = [\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

and

$$\lambda x = \begin{cases} [\lambda\underline{x}, \lambda\overline{x}] & , \quad \lambda \geq 0 \\ [\lambda\overline{x}, \lambda\underline{x}] & , \quad \lambda < 0, \end{cases}$$

$x, y \in \mathbb{I}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$, respectively.

The Minkowski sum and scalar multiplication on $\mathbb{I}_{\mathbb{C}}$ are defined by

$$\begin{aligned} u + v &= \left[\underline{u_r}, \overline{u_r}\right] + i\left[\underline{u_s}, \overline{u_s}\right] + \left[\underline{v_r}, \overline{v_r}\right] + i\left[\underline{v_s}, \overline{v_s}\right] \\ &= \left[\underline{u_r} + \underline{v_r}, \overline{u_r} + \overline{v_r}\right] + i\left[\underline{u_s} + \underline{v_s}, \overline{u_s} + \overline{v_s}\right] \\ &= \left\{a + ib : a \in \left[\underline{u_r} + \underline{v_r}, \overline{u_r} + \overline{v_r}\right],\ b \in \left[\underline{u_s} + \underline{v_s}, \overline{u_s} + \overline{v_s}\right]\right\} \end{aligned}$$

and

$$\begin{aligned} \lambda u &= \lambda\left[\underline{u_r}, \overline{u_r}\right] + i\left(\lambda\left[\underline{u_s}, \overline{u_s}\right]\right) \\ &= \left\{\lambda a + i\lambda b : a \in \left[\underline{u_r}, \overline{u_r}\right],\ b \in \left[\underline{u_s}, \overline{u_s}\right]\right\} \end{aligned}$$

on $\mathbb{I}_{\mathbb{C}}$, where $i = \sqrt{-1}$ and $\lambda \in \mathbb{C}$. Further, the relation

$$u \preceq v \text{ iff } \left[\underline{u_r}, \overline{u_r}\right] \subseteq \left[\underline{v_r}, \overline{v_r}\right] \text{ and } \left[\underline{u_s}, \overline{u_s}\right] \subseteq \left[\underline{v_s}, \overline{v_s}\right]$$

is a partial order relation on $\mathbb{I}_{\mathbb{C}}$. Thus, $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space.

This article is organized as follows: In Section 2, we present some definitions and theorems with respect to the normed quasilinear spaces. In Section 3, we introduce some the classes of the continuous complex interval-valued functions defined from $\mathbb{R}$ to $\mathbb{I}_{\mathbb{C}}$. Further, we prove that these function spaces are the consolidate spaces and we investigate the dimensions of these spaces.

2. Preliminaries

We will start by giving some main definitions and notions.

Suppose that X is a quasilinear space and $Y \subseteq X$. Then Y is called a *subspace of* X whenever Y is a quasilinear space with the same partial order and the restriction to Y of the operations on X. Y is subspace of a quasilinear space X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{K}$, $\alpha x + \beta y \in Y$. Proof of this theorem is quite similar to its classical linear space analogue. Let Y be a subspace of a quasilinear space X and suppose each element x in Y has an inverse in Y. Then the partial order on Y is determined by the equality. In this case, Y is a linear subspace of X [16].

An element x in a quasilinear space X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. Also, X_r stands for the set of all regular elements of X while X_s stands for the sets of all singular elements and zero in X. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular, symmetric* and *singular subspaces* of X, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of X is a linear space while the singular one is nonlinear at all. Further, $\mathbb{I}_{\mathbb{C}}$ is a closed subspace of $\Omega(\mathbb{C})$ [6].

A real-valued function $\|.\|$ on the quasilinear space X is called a *norm* if the following conditions hold;

$$\|x\| > 0 \text{ if } x \neq 0, \tag{1}$$

$$\|x + y\| \leq \|x\| + \|y\|, \tag{2}$$

$$\|\alpha x\| = |\alpha| \, \|x\|, \tag{3}$$

$$\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \tag{4}$$

$$\begin{gathered}\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that}\\ x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon, \text{ then } x \preceq y,\end{gathered} \tag{5}$$

here x, y, x_ε are arbitrary element in X and α is any scalar. A quasilinear space X with a norm defined on it, is called *normed quasilinear space* [3].

For a normed linear space E, a norm on $\Omega(E)$ is defined by

$$\|A\|_{\Omega} = \sup_{a\in A} \|a\|_{E} \ .$$

Hence $\Omega_C(E)$ and $\Omega(E)$ are normed quasilinear spaces. A norm on $\mathbb{I}_{\mathbb{R}}$ is defined by

$$\|x\| = \|[\underline{x}, \overline{x}]\| = \sup_{t\in[\underline{x},\overline{x}]} |t| \,.$$

Moreover, $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space with the norm

$$\begin{aligned}\|X\|_{\mathbb{I}_{\mathbb{C}}} &= \sup\{|z| : z \in X\} \\ &= \sup\{|a+ib| : a \in \left[\underline{x_r}, \overline{x_r}\right], b \in \left[\underline{x_s}, \overline{x_s}\right]\}\end{aligned}$$

for $X = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]$ [15].

Now we will give the notion of consolidate quasilinear space defined in [15]. Thanks to this definition, we were able to give a representation to every element in a quasilinear space and we were able to define an inner-product quasilinear space.

Definition 2.1 *[15] Let X be a quasilinear space and $y \in X$. The floor of y is the set of all regular elements y of X such that $x \preceq y$. It is denoted by F_y^X and $F_y^X \subset X$. Hence $F_y^X = \{x \in X_r : x \preceq y\}$.*

For example, $[3,7]$ is an element of $(\mathbb{I}_{\mathbb{R}})_s$ and hence of $\mathbb{I}_{\mathbb{R}}$ since $(\mathbb{I}_{\mathbb{R}})_s \subset \mathbb{I}_{\mathbb{R}}$. The floor of $[3,7]$ in $(\mathbb{I}_{\mathbb{R}})_s$ is empty set, that is,

$$F_{[3,7]}^{(\mathbb{I}_{\mathbb{R}})_s} = \{x \in ((\mathbb{I}_{\mathbb{R}})_s)_r : x \subseteq [3,7]\} = \{x \in \{0\} : x \subseteq [3,7]\} = \varnothing$$

since $((\mathbb{I}_{\mathbb{R}})_s)_r = \{0\}$. But, the floor of $[3,7]$ in $\mathbb{I}_{\mathbb{R}}$ is

$$F_{[3,7]}^{\mathbb{I}_{\mathbb{R}}} = \{x \in (\mathbb{I}_{\mathbb{R}})_r : x \subseteq [3,7]\} \equiv [3,7]$$

since $(\mathbb{I}_{\mathbb{R}})_r \equiv \mathbb{R}$.

Definition 2.2 *[15] A quasilinear space X is called consolidate or Solid-Floored whenever*

$$\sup_{\preceq}\{x \in X_r : x \preceq y\} = \sup_{\preceq} F_y^X$$

exists and

$$y = \sup_{\preceq}\{x \in X_r : x \preceq y\}$$

for each $y \in X$. Otherwise, X is called a nonconsolidate QLS, or briefly, a nc-QLS.

From above example immediately we can see that $\mathbb{I}_{\mathbb{R}}$ is consolidate while $(\mathbb{I}_{\mathbb{R}})_s$ is not. Analogous results are also true for the spaces $\mathbb{I}_{\mathbb{C}}$ and $(\mathbb{I}_{\mathbb{C}})_s$.

3. Main Results

In this section, we present some important class of the continuous functions defined from $\mathbb{R}$ to $\mathbb{I}_{\mathbb{C}}$ and we show that these sets are the normed quasilinear spaces.

Definition 3.1 *The support of the set-valued function $F:\mathbb{R}\to\mathbb{I}_{\mathbb{C}}$ is the smallest closed set outside which the function is equal to zero:*

$$suppF=\overline{\{x\in\mathbb{R}:F(x)\neq\{0\}\}}.$$

If $suppF$ is a bounded set, then we say that F has compact support.

Definition 3.2 *(Classes of Continuous Set-Valued Functions) Consider a set-valued function $F:\mathbb{R}\to\mathbb{I}_{\mathbb{C}}$.*

(i) *The set $C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ consists of all continuous set-valued functions having compact support:*

$$C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})=\{F:\mathbb{R}\to\mathbb{I}_{\mathbb{C}}\mid F \text{ is continuous and has compact support }\}.$$

(ii) *The set $C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ consists of all continuous set-valued functions that $F(x)\to\{0\}$ with respect to Hausdorff metric on $\mathbb{I}_{\mathbb{C}}$ as $x\to\pm\infty$:*

$$C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})=\{F:\mathbb{R}\to\mathbb{I}_{\mathbb{C}}\mid F \text{ is continuous and } F(x)\to\{0\} \text{ as } x\to\pm\infty\}.$$

Example 3.3 *Consider the complex interval-valued functions $F,G:\mathbb{R}\to\mathbb{I}_{\mathbb{C}}$ given by*

$$F(t)=\begin{cases}\{i\} & , \quad \text{for } t\in[0,1];\\ \{0\} & , \quad \text{otherwise}\end{cases}$$

and

$$G(t)=\begin{cases}[0,1] & , \quad \text{for } t\in[-1,1);\\ \{0\} & , \quad \text{otherwise,}\end{cases}$$

respectively. Since F and G are continuous and $suppF=[0,1]$, $suppG=[-1,1]$ we say that $F,G\in C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$. In fact, F is a regular element of $C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ while G is a singular element of $C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$.

Theorem 3.4 *$C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ is a quasilinear space with the operations of algebraic sum, multiplication by complex numbers and partial order relation are defined as follows;*

$$(F_1+F_2)(x)=F_1(x)+F_2(x),$$

$$(\alpha F) = \alpha F(x)$$

and

$$F_1 \preccurlyeq F_2 \Leftrightarrow F_1(x) \subseteq F_2(x) \text{ for any } x \in \mathbb{R}.$$

Proof Verification of first five axioms to be a quasilinear space is to straighforward. Further, the function $F = \{0\}$ is the identity element of the addition. Obviously, $1.F = F$ and $0.F = 0$, for $1, 0 \in \mathbb{C}$ and $F \in C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$, easily see that $\alpha(\beta F) = (\alpha\beta)F$ and $\alpha(F + G) = \alpha F + \alpha G$ for $\alpha, \beta \in \mathbb{C}$ and $F, G \in C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. For any $x \in \mathbb{R}$,

$$((\alpha + \beta)F)(x) = (\alpha + \beta)F(x) \subseteq \alpha F(x) + \beta F(x) = (\alpha F)(x) + (\beta F)(x)$$

and so $(\alpha + \beta)F \preccurlyeq \alpha F + \beta F$. If $F_1 \preccurlyeq F_2$ and $F_3 \preccurlyeq F_4$, then $F_1(x) \preceq F_2(x)$ and $F_3(x) \preceq F_4(x)$ for any $x \in \mathbb{R}$. Since $F_1(x), F_2(x), F_3(x), F_4(x) \in \mathbb{I}_{\mathbb{C}}$, we write $F_1(x) + F_3(x) \preceq F_2(x) + F_4(x)$. This means $F_1 + F_3 \preccurlyeq F_2 + F_4$. Suppose that $F_1 \preccurlyeq F_2$. Then $\alpha F_1(x) \preceq \alpha F_2(x)$ for any $x \in \mathbb{R}$, $\alpha \in \mathbb{C}$ since $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space. Thus, we have $\alpha F_1 \preccurlyeq \alpha F_2$. □

Lemma 3.5 $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ *is a subspace of the quasilinear space* $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$.

Proof It is not hard to see that $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) \subset C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Suppose that $\lambda_1, \lambda_2 \in \mathbb{C}$ and $F, G \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Let us take an arbitrary $y \in A = \{x \in \mathbb{R} : \lambda_1 F(x) + \lambda_2 G(x) \neq 0\}$. Then we say that $\lambda_1 F(y) + \lambda_2 G(y) \neq 0$. In this case it is either $\lambda_1 F(y) \neq 0$ or $\lambda_2 G(y) \neq 0$. If $\lambda_1 F(y) \neq 0$, then $y \in B = \{x \in \mathbb{R} : F(x) \neq 0\}$. This means $A \subseteq B$. Thus,

$$\bar{A} = \mathrm{supp}(\lambda_1 F + \lambda_2 G) \subseteq \bar{B} = \mathrm{supp}F.$$

Further, there exists at least an interval $[a, b]$ such that $\mathrm{supp}F \subseteq [a, b]$ since $F \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Consequently, we say that $\mathrm{supp}(\lambda_1 F + \lambda_2 G) \subseteq [a, b]$ and so $\lambda_1 F + \lambda_2 G \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. If $\lambda_2 G(y) \neq 0$, then the proof is similar. Now suppose that both $\lambda_1 F(y) \neq 0$ and $\lambda_2 G(y) \neq 0$ are satisfied. Then we have that

$$\{x \in \mathbb{R} : \lambda_1 F(x) + \lambda_2 G(x) \neq 0\} \subseteq \{x \in \mathbb{R} : F(x) \neq 0\} \cap \{x \in \mathbb{R} : G(x) \neq 0\}$$

since $y \in \{x \in \mathbb{R} : \lambda_1 F(x) + \lambda_2 G(x) \neq 0\}$. This implies $A \subseteq B$. Because of the fact that $\bar{A} \subset \bar{B}$ we write $\bar{A} = \mathrm{supp}(\lambda_1 F + \lambda_2 G) \subseteq \bar{B} = \mathrm{supp}F$. Thus, $\mathrm{supp}(\lambda_1 F + \lambda_2 G)$ is bounded and $\lambda_1 F + \lambda_2 G \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. □

Theorem 3.6 *The expression*

$$\|F\|_\infty = \max_{x \in \mathbb{R}} \|F(x)\|_{\mathbb{I}_{\mathbb{C}}}$$

defines a norm on $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ *and this space is a normed quasilinear space.*

Proof It is obvious that the above equality is well-defined. It can be shown similarly to the classical analysis that the first three conditions of norm are satisfied. Let us only verify the last two conditions. Let F_1 and F_2 be arbitrary elements of $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. If $F_1 \preccurlyeq F_2$, then $F_1(x) \preceq F_2(x)$ for every $x \in \mathbb{R}$. This implies $\|F_1(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \|F_2(x)\|_{\mathbb{I}_{\mathbb{C}}}$ and so $\|F_1\|_\infty = \max_{x\in\mathbb{R}} \|F_1(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \max_{x\in\mathbb{R}} \|F_2(x)\|_{\mathbb{I}_{\mathbb{C}}} = \|F_2\|_\infty$. For the last condition of the norm, let $\varepsilon > 0$ be arbitrary and there exists an element $F_\varepsilon \in C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ such that $F \preccurlyeq G + F_\varepsilon$ and $\|F_\varepsilon\|_\infty = \max_{x\in\mathbb{R}} \|F_\varepsilon(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \varepsilon$. By the assumption, we write that $F(x) \preceq G(x) + F_\varepsilon(x)$ and $\|F_\varepsilon(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \varepsilon$. By the last condition of norm on $\mathbb{I}_{\mathbb{C}}$ we say that $F(x) \preceq G(x)$ for every $x \in \mathbb{R}$. Thus, we obtain that $F \preccurlyeq G$. □

Now we will show that $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ are consolidate spaces. Thus, we can give a representation to every element in these spaces.

Lemma 3.7 *$C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ are the consolidate quasilinear spaces.*

Proof We will give only the proof for the space $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ since a similar proof can be made for $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Let us take an arbitrary $g \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Because of the fact that $\mathbb{I}_{\mathbb{C}}$ is consolidate, we write for $t \in \mathbb{R}$

$$\sup_{\preceq}\{x \in (\mathbb{I}_{\mathbb{C}})_r : x \preceq G(t)\} = \sup_{\preceq} F^{\mathbb{I}_{\mathbb{C}}}_{G(t)} = G(t) = [\underline{G_r(t)}, \overline{G_r(t)}] + i[\underline{G_s(t)}, \overline{G_s(t)}].$$

Now let us choose an element $\{x_{G_r(t)}\} + i\{x_{G_s(t)}\} \in \mathbb{I}_{\mathbb{C}}$ for each $t \in \mathbb{R}$ such that let be

$$\{x_{G_r(t)}\} + i\{x_{G_s(t)}\} \preceq [\underline{G_r(t)}, \overline{G_r(t)}] + i[\underline{G_s(t)}, \overline{G_s(t)}]. \tag{6}$$

Consider the function $h : \mathbb{R} \to (\mathbb{I}_{\mathbb{C}})_r$ given by

$$h(t) = \{x_{G_r(t)}\} + i\{x_{G_s(t)}\}, \tag{7}$$

where $\{x_{G_r(t)}\} + i\{x_{G_s(t)}\}$ is the regular element of $\mathbb{I}_{\mathbb{C}}$ that satisfies the condition (6). Now we will prove that $\sup F_G^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})} = G$, i.e.,

$$\sup_{\preceq}\{h \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})_r : h(t) \preceq G(t), \forall t \in \mathbb{R}\}.$$

First we have $h \preccurlyeq G$ since $h(t) = \{x_{G_r(t)}\} + i\{x_{G_s(t)}\} \preceq G(t)$. This means $F_G^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})} \neq \varnothing$. Further, the set $F_G^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}$ is the upper bounded since $h \preccurlyeq G$ for $h \in F_G^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}$. Suppose that the function F is another upper bound of the set $F_G^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}$. Now let us assume that $G \not\preccurlyeq F$. Then there exists an element $t_0 \in \mathbb{R}$ such that $G(t_0) \not\preceq F(t_0)$. This implies that it is either

$[\underline{G_r(t_0)}, \overline{G_r(t_0)}] \nsubseteq [\underline{F_r(t_0)}, \overline{F_r(t_0)}]$ or $[\underline{G_s(t_0)}, \overline{G_s(t_0)}] \nsubseteq [\underline{F_s(t_0)}, \overline{F_s(t_0)}]$. If $[\underline{G_r(t_0)}, \overline{G_r(t_0)}] \nsubseteq [\underline{F_r(t_0)}, \overline{F_r(t_0)}]$, then there exists the singleton $\{x_{G_r(t_0)}\}$ such that $\{x_{G_r(t_0)}\} \subseteq [\underline{G_r(t_0)}, \overline{G_r(t_0)}]$ while $\{x_{G_r(t_0)}\} \nsubseteq [\underline{F_r(t_0)}, \overline{F_r(t_0)}]$. Further, we have that $\{x_{G_r(t_0)}\} + i\{x_{G_s(t_0)}\} \preceq G(t_0)$ and $\{x_{G_r(t_0)}\} + i\{x_{G_s(t_0)}\} \npreceq F(t_0)$. Thus, we write that $h(t_0) \preceq G(t_0)$ while $h(t_0) \npreceq F(t_0)$ for the function h defined in (7). Therefore, $h \npreceq F$. This is a contradiction. If $[\underline{G_s(t_0)}, \overline{G_s(t_0)}] \nsubseteq [\underline{F_s(t_0)}, \overline{F_s(t_0)}]$, then the proof is given in a similar way. Consequently, the proof is complete. □

Now we will examine the dimension of the quasilinear spaces $C_c(\mathbb{R}, \mathbb{I}_\mathbb{C})$ and $C_0(\mathbb{R}, \mathbb{I}_\mathbb{C})$. For this purpose, firstly let us give some algebraic definitions in a quasilinear space (for details, see [5]). Let X be a quasilinear space and $\{x_k\}_{k=1}^n$ be a subset of X, where n is a positive integer. A (linear) combination of the set $\{x_k\}_{k=1}^n$ is an element z of X in the form

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = z,$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ are real scalars. On the other hand, a quasilinear combination of the set $\{x_k\}_{k=1}^n$ is an element $z \in X$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \preceq z$$

for some real scalars $\alpha_1, \alpha_2, ..., \alpha_n$. Hence,the quasilinear combination, briefly ql-combination, is defined by the partial order relation on X. Further, for any nonempty subset A of a quasilinear space X, span of A is given by following known definition

$$SpA = \{\sum_{k=1}^{n} \alpha_k x_k : x_1, x_2, ..., x_n \in A, \ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}, \ n \in \mathbb{N}\}.$$

However, $QspA$, the *quasispan (q-span,* for short) *of* A, is defined by the set of all possible quasilinear combinations of A, that is,

$$QspA = \{x \in X : \sum_{k=1}^{n} \alpha_k x_k \preceq x, \ x_1, x_2, ..., x_n \in A, \ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}, \ n \in \mathbb{N}\}.$$

A given set $A = \{x_1, x_2, ..., x_n\}$ in a quasilinear space X is called *quasilinear independent (ql-independent,* briefly) whenever the inequality

$$\theta \preceq \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \tag{8}$$

holds if and only if $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. Otherwise, A is called *quasilinear dependent (ql-dependent,* briefly*)*. A ql-independent subset A of a quasilinear space X which q-spans X is called *a basis (or Hamel basis)* for X.

Let S be a ql-independent subset of a quasilinear space X. S is called *maximal ql-independent subset* of X whenever S is ql-independent, but any superset of S is ql-dependent.

Definition 3.8 *[5] Regular (Singular) dimension of any quasilinear space X is the cardinality of any maximal ql-independent subsets of $X_r(X_s)$. If this number is finite then X is said to be finite regular (singular)-dimensional, otherwise; is said to be infinite regular (singular)-dimensional. Regular dimension is denoted by $r\text{-}\dim X$ and singular dimension is denoted by $s\text{-}\dim X$. If $r\text{-}\dim X = a$ and $s\text{-}\dim X = b$, then we say that X is an (a_r, b_s)-dimensional quasilinear space.*

Using these information we can give the following theorem.

Theorem 3.9 *The quasilinear spaces $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ are the (∞_r, ∞_s)-dimensional spaces.*

Proof Consider the functions $x_n : \mathbb{R} \to \mathbb{I}_{\mathbb{C}}$ given by

$$x_n(t) = \begin{cases} \{t^n\} & , \quad \text{for } t \in [-1, 1]; \\ \{0\} & , \quad \text{otherwise} \end{cases}$$

for $n = 0, 1, \ldots$ and the set $M = \{x_0, x_1, \ldots\}$. It is obvious that M is a subset of the regular subspace of $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Now we will prove that M is ql-independent. Let us take an arbitrary and finite subset $\{x_{k_1}, x_{k_2}, \ldots, x_{k_n}\}$ of M. Suppose that

$$c_{k_1} x_{k_1} + c_{k_2} x_{k_2} + \ldots + c_{k_n} x_{k_n} = 0$$

for $c_{k_1}, c_{k_2}, \ldots, c_{k_n} \in \mathbb{C}$. Then we write

$$c_{k_1}\{t^{k_1}\} + c_{k_2}\{t^{k_2}\} + \ldots + c_{k_n}\{t^{k_n}\} = \{0\}$$

and so $c_{k_1} t^{k_1} + c_{k_2} t^{k_2} + \ldots + c_{k_n} t^{k_n} = 0$. This implies that $c_{k_1} = c_{k_2} = \ldots = c_{k_n} = 0$. Thus, we say that $r\text{-}\dim C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = \infty$. Further, $s\text{-}\dim C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = \infty$ since $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ is a consolidate quasilinear space. Furthermore, we can say that $r\text{-}\dim C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = s - \dim C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = 0$ since $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ is a subspace of $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. □

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Halise Levent]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%70).

Author [Yılmaz Yılmaz]: Collected the data, contributed to research method or evaluation of data (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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