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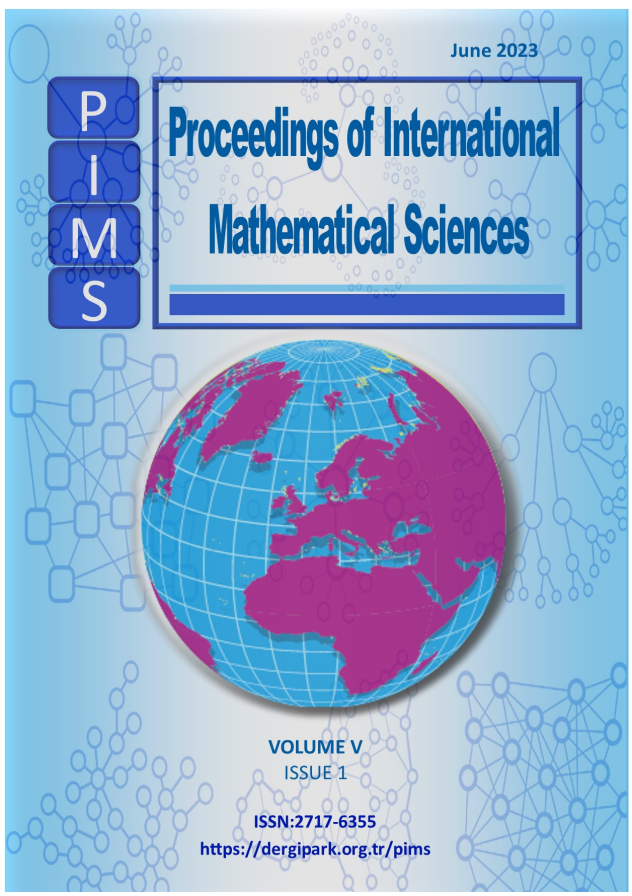

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SUZUKI TYPE P-CONTRACTIVE MAPPINGS

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Abstract. We introduce Suzuki type P-contractive mappings by taking into account the concepts of contractive, P-contractive, and Suzuki type contractive mappings. Then, for such mappings on compact metric spaces, we present a fixed point theorem that is more general than the well-known Edelstein fixed point theorem.

1. Introduction

Metric fixed point theory, as it is known, investigates the conditions that guarantee the existence and even uniqueness of fixed point of a self mapping on a metric space. These conditions are typically comprised of completeness of space and some type of contraction inequality. It is difficult to obtain a new result when the completeness of space is ignored. As a result, studies are conducted to ensure the existence of the fixed point by weakening the contraction inequalities. However, in complete metric space generalizations, the sum of the coefficients of the terms on the right side of the linear contraction inequalities is less than 1. Nonlinear contraction inequalities are subject to a similar constraint. Edelstein [4] introduced the concept of contractivity to overcome the coefficient problem and obtained a fixed point theorem. Although Edelstein extended the relevant class of mappings, he had to consider compactness of the space, which is a more strong condition than completeness. Many studies, covering Edelstein's fixed theorem, have been obtained by generalizing the concept of contractivity in the literature (for example see [2, 3, 5]). For the sake of completeness we recall the following:

Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then, T is said to be contractive if

$$d(Tx, Ty) < d(x, y) \tag{C}$$

for all $x, y \in X$ with $x \neq y$. Hence, Edelstein presented the following theorem:

Theorem 1.1 ([4]). *Let (X, d) be a compact metric space and $T : X \to X$ be a contractive mapping. Then, T has a unique fixed point.*

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Suzuki obtained a new fixed point theorem by weakening the concept of contractivity in 2009.

Theorem 1.2 ([5]). *Let (X,d) be a compact metric space and $T: X \to X$ be a mapping such that*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) < d(x,y) \tag{SC}$$

for all $x, y \in X$. Then, T has a unique fixed point.

For the sake of simplicity, we will refer to the mappings that provide the (SC) inequality as Suzuki type contractive mappings. In 2018, Altun et al. [2] defined the concept of P-contractivity. A self mapping T on X is said to be P-contractive if

$$d(Tx,Ty) < d(x,y) + |d(x,Tx) - d(y,Ty)| \tag{PC}$$

for all $x, y \in X$ with $x \neq y$. Then, the following theorem has been presented.

Theorem 1.3 ([2]). *Let (X,d) be a compact metric space and $T: X \to X$ be a continuous P-contractive mapping. Then, T has a unique fixed point.*

It is clear that every contractive (C) mapping is Suzuki type contractive (SC), also every contractive (C) mapping is P-contractive (PC). The following examples demonstrate that the converse of both propositions are not true.

Example 1.1 ([5]). *Let $X = [-11,-10] \cup \{0\} \cup [10,11]$ with the usual metric d and $T: X \to X$, defined by*

$$Tx = \begin{cases} \frac{11x+100}{x+9} & , \quad x \in [-11,-10) \\ 0 & \quad x \in \{-10, 0, 10\} \\ -\frac{11x-100}{x-9} & , \quad x \in (10,11] \end{cases} .$$

Then, T is Suzuki type contractive, but it is not contractive.

Example 1.2 ([3]). *Let $X = [0,1]$ with the usual metric d and $T: X \to X$, defined by*

$$Tx = \begin{cases} \frac{1}{2} & , \quad x = 0 \\ \frac{x}{2} & , \quad x \neq 0 \end{cases} .$$

Then, T is P-contractive, but it is not contractive.

The classes of Suzuki type contractive (SC) mappings and P-contractive (PC) mappings, on the other hand, are distinct. The following examples demonstrate this fact.

Example 1.3 ([2]). *Let $X = [0,2]$ with the usual metric d and $T: X \to X$, defined by*

$$Tx = \begin{cases} 1 & , \quad x \leq 1 \\ 0 & , \quad x > 1 \end{cases} .$$

Then, T is P-contractive, but it is not Suzuki type contractive.

Example 1.4 ([2]). *Let $X = \{(0,0),(4,0),(0,4),(4,5),(5,4)\} \subset \mathbb{R}^2$ with the metric*

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1, x_2), y = (y_1, y_2) \in X$*. Define a mapping* $T : X \to X$ *by*

$$T = \begin{pmatrix} (0,0) & (4,0) & (0,4) & (4,5) & (5,4) \\ \\ (0,0) & (0,0) & (0,0) & (4,0) & (0,4) \end{pmatrix}.$$

Then, T *is Suzuki type contractive, but it is not* P*-contractive.*

Remark. *Although contractive mappings are continuous, neither Suzuki type contractive nor* P*-contractive mappings are continuous. Note that Suzuki did not need the continuity in Theorem 1.2. However, in Theorem 1.3 the continuity of the mapping has been assumed. Example 1.2 above shows that the condition of continuity can not be removed in Theorem 1.3.*

In this paper, we introduce Suzuki type P-contractive mappings, which are inspired by the concepts of contractive, P-contractive, and Suzuki type contractive mappings. Then, we present a fixed point theorem that is more general than Theorem 1.1 and Theorem 1.3.

The following lemma will be used in our second theorem.

Lemma 1.4 ([1])**.** *Let* X *be a compact topological space and* $f : X \to \mathbb{R}$ *be a lower semicontinuous function. Then, there exists an element* $x_0 \in X$ *such that* $f(x_0) = \inf\{f(x) : x \in X\}$*.*

2. Main Result

First, we introduce a new concept for self mapping T on a metric space (X, d).

Definition 2.1. *Let* (X, d) *be a metric space and* $T : X \to X$ *be a mapping. Then* T *is said to be Suzuki type* P*-contractive if*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)| \quad \text{(SPC)}$$

for all $x, y \in X$*.*

Remark. *For the aforementioned contractivity concepts, we can draw the diagram below:*

$$\begin{array}{ccc} C & \Longrightarrow & PC \\ \Downarrow & & \Downarrow \\ SC & \Longrightarrow & SPC \end{array}.$$

Examples 1.1, 1.2, 1.3, 1.4 show that the converse of all implications are not true.

Now, we are ready to state our main result.

Theorem 2.1. *Let* (X, d) *be a compact metric space and* $T : X \to X$ *be a continuous Suzuki type* P*-contractive mapping. Then,* T *has a unique fixed point in* X*.*

Proof. Since X is compact and T is continuous, then there exists $u \in X$ such that

$$d(u, Tu) = \inf\{d(x, Tx) : x \in X\}.$$

We claim that $d(u, Tu) = 0$. Assume the contrary. In this case, since $0 < \frac{1}{2}d(u, Tu) < d(u, Tu)$, we have

$$\begin{aligned} d(Tu, T^2u) &< d(u, Tu) + \left|d(u, Tu) - d(Tu, T^2u)\right| \\ &= d(u, Tu) + d(Tu, T^2u) - d(u, Tu) \\ &= d(Tu, T^2u), \end{aligned}$$

which is a contradiction. Therefore, $d(u,Tu)=0$ and so u is a fixed point of T. Now, assume v is another fixed point of T. In this case, since $0=\frac{1}{2}d(u,Tu)<d(u,v)$, we have

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &< d(u,v)+|d(u,Tu)-d(v,Tv)| \\ &= d(u,v), \end{aligned}$$

which is a contradiction. Hence, the fixed point of T is unique. □

To see that the continuity condition in this theorem cannot be removed, one can refer to Example 1.2 again. However, a result can be obtained by assuming the lower semicontinuity of the function f defined by $f(x)=d(x,Tx)$ instead of the continuity of T. It is well known that if T is continuous, then f is also continuous (and so it is lower semicontinuous). However, if f is lower semicontinuous, then T may not be continuous (see Remark 2.8 in [2]).

Hence, by Lemma 1.4, we can state the following result:

Theorem 2.2. *Let (X,d) be a compact metric space and $T:X\to X$ be a Suzuki type P-contractive mapping. Then T has a unique fixed point in X provided that the function f defined by $f(x)=d(x,Tx)$ is lower semicontinuous.*

Proof. Since X is compact and $f:X\to\mathbb{R}$ is lower semicontinuous, then by Lemma 1.4, there exists $u\in X$ such that $f(u)=\inf f(X)$, that is, we have

$$d(u,Tu)=\inf\{d(x,Tx):x\in X\}.$$

Therefore, the proof can be completed as in the proof of Theorem 2.1. □

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FIXED POINTS OF ENRICHED CONTRACTION AND ALMOST ENRICHED CRR CONTRACTION MAPS WITH RATIONAL EXPRESSIONS AND CONVERGENCE OF FIXED POINTS

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Abstract. We define enriched Jaggi contraction map, enriched Dass and Gupta contraction map and almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$ in Banach spaces and prove the existence and uniqueness of fixed points of these maps. Further, we show that the sequence of fixed points of the corresponding enriched contraction maps converges to the fixed point of the uniform limit operator of these enriched contraction maps.

1. Introduction

Generalization of contraction conditions and finding the existence of fixed points play an important role in the development of fixed point theory. There are many works where the notion of fixed point play some role, apparently, in different context. For instance, we refer Mustafa, Hakan and Turkoglu [5], Mustafa, Hakan and Sadullah [6] and the references cited in these papers. Further, there are several generalizations of Banach contraction maps, one among them is contraction conditions involving rational expressions. Dass and Gupta [3] initiated and introduced contraction condition with rational expression as follows:

Let (X, d) be a metric space and $T : X \to X$. There exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and T satifies

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y) \tag{1.1}$$

for all $x, y \in X$. Dass and Gupta [3] proved that if $T : X \to X$, X complete metric space, satisfies the inequality (1.1) and if T is continuous then T has a unique fixed point in X.

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In 1977, Jaggi [4] introduced a different rational type contraction condition independent that of contraction condition (1.1), i.e., there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ and

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta d(x, y) \tag{1.2}$$

for all $x, y \in X, x \neq y$, and proved that every map $T : X \to X, X$ complete metric space, that satifies (1.2) has a unique fixed point in X, provided T is continuous.

A map T that satisfies (1.2) is said to be a Jaggi contraction map.

On the other hand, Berinde and Păcurar [1], introduced a larger class of mappings, namely, enriched contraction mappings in normed linear spaces which are more general than contraction maps.

Definition 1.1. (*Berinde and Păcurar* [2]) *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $k \in [0, +\infty)$ and $a \in [0, k+1)$ such that*

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\|, \tag{1.3}$$

for all $x, y \in X$, then we say that T is a $(k, a)-$enriched contraction.

Theorem 1.1. (*Berinde and Păcurar* [2]) *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a $(k, a)-$enriched contraction. Let $x_0 \in X$ and $\lambda \in (0, 1]$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, n \geq 0, \tag{1.4}$$

converges to p (say) in X and p is the unique fixed point of T.

On further extensions of $(k, a)-$enriched contractions, we refer (Berinde and Păcurar [2]).

Definition 1.2. (*Berinde and Păcurar* [2]) *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $k \in [0, +\infty)$ and $a, b \geq 0$, satisfying $a + 2b < 1$ such that*

$$\|k(x - y) + Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|), \tag{1.5}$$

for all $x, y \in X$, then we say that T is a $(k, a, b)-$enriched Ciric-Reich-Rus contraction map .

Here onwards, we call these maps by $(k, a, b)-$enriched CRR contraction maps. If $a = 0$ in (1.5) then T is said to a $(k, b)-$enriched Kannan mapping [2].

Theorem 1.2. (*Berinde and Păcurar* [2]) *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a $(k, a, b)-$enriched CRR contraction map. Let $x_0 \in X$ and $\lambda \in (0, 1]$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, n \geq 0, \tag{1.6}$$

converges to u (say) in X and u is the unique fixed point of T.

In Section 2 of this paper, we define enriched Jaggi contraction map, enriched Dass and Gupta contraction map in Banach spaces and prove the existence and uniqueness of fixed points.

In Section 3, we define almost $(k, a, b, \lambda)-$enriched CRR contraction maps with $\lambda = \frac{1}{k+1}$ in Banach spaces and prove the existence and uniqueness of fixed points.

In Section 4, we prove that, if the sequence of enriched contraction maps converges uniformly to an operator with a unique fixed point then the corresponding sequence of fixed points of sequence of enriched contraction maps also converges to the fixed point of the limit operator in Banach spaces.

2. Fixed point results on enriched contraction maps with rational expressions

Let $(X, \|\cdot\|)$ be a normed linear space and $T : X \to X$. For any $\lambda \in [0, 1)$, we denote

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx,\ x \in X. \tag{2.1}$$

Definition 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $k \in [0, +\infty)$ such that for $\lambda = \frac{1}{k+1}$, T satisfies the inequality*

$$\|k(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Tx\|\|y-T_\lambda y\|}{\|x-y\|} \tag{2.2}$$

for all $x, y \in X$ and $x \neq y$, then we say that T is an enriched Jaggi contraction map.

Here we note that every Jaggi contraction is a special case of enriched Jaggi contraction when $k = 0$. But, every enriched Jaggi contraction need not be a Jaggi contraction. The following example illustrates this fact.

Example 2.1. *Let $X = \mathbb{R}$ with the usual norm. We define $T : X \to X$ by $Tx = 1 - \frac{3}{2}x, x \in \mathbb{R}$. We choose $k = 2$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. We now consider*

$$\begin{aligned}|k(x-y) + Tx - Ty| &= |2(x-y) + 1 - \tfrac{3}{2}x - 1 + \tfrac{3}{2}y|\\ &= \tfrac{1}{2}|x-y|\\ &\leq \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|\frac{5}{2}x-1||\frac{5}{6}y-\frac{1}{3}|}{|x-y|}\\ &= \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|x+\frac{3}{2}x-1||y-\frac{1}{6}y-\frac{1}{3}|}{|x-y|}\\ &= \alpha|x-y| + \beta\tfrac{|x-Tx||y-T_\lambda y|}{|x-y|},\end{aligned}$$

so that T satisfies the inequality (2.2) with $\alpha + \beta < 1$.
Hence T is an enriched Jaggi contraction map.
Now, by choosing $x = 0, y = \frac{2}{5}$, we have

$$\begin{aligned}|Tx - Ty| = |T0 - T(\tfrac{2}{5})| = \tfrac{3}{5} \nleq \alpha\cdot\tfrac{2}{5} + \beta\cdot 0 &= \alpha|0-\tfrac{2}{5}| + \beta\tfrac{|0-T0|\cdot|\frac{2}{5}-T(\frac{2}{5})|}{|0-\frac{2}{5}|}\\ &= \alpha|x-y| + \beta\tfrac{|x-Tx||y-Ty|}{|x-y|},\end{aligned}$$

for any $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta < 1$.
Hence T is not a Jaggi contraction map.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$ be continuous. Assume that T is an enriched Jaggi contraction map. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n, n = 0, 1, 2, ...,$ converges to s (say) in X, and s is the unique fixed point of T_λ. Further, s is the unique fixed point of T.*

Proof. Let $x_0 \in X$. We consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, ...$.
For $\lambda = \frac{1}{k+1} < 1$, we have $k = \frac{1}{\lambda} - 1$ and thus the condition (2.2) becomes
$\|(\frac{1}{\lambda} - 1)(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Tx\|\|y-T_\lambda y\|}{\|x-y\|}$ for all $x, y \in X$ for $x \neq y$.
i.e.,
$\|(1-\lambda)(x-y) + \lambda Tx - \lambda Ty\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|\lambda x-\lambda Tx\|\|y-T_\lambda y\|}{\|x-y\|}$, $x \neq y$ and hence

$$\|T_\lambda x - T_\lambda y\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|x-T_\lambda x\|\|y-T_\lambda y\|}{\|x-y\|} \text{ for all } x, y \in X \text{ and } x \neq y. \tag{2.3}$$

By taking $x = x_{n-1}$ and $y = x_n$ in (2.3), we get
$\|T_\lambda x_{n-1} - T_\lambda x_n\| \leq \alpha\lambda\|x_{n-1} - x_n\| + \beta\frac{\|x_{n-1}-T_\lambda x_{n-1}\|\|x_n - T_\lambda x_n\|}{\|x_{n-1}-x_n\|}$, *i.e.*,
$\|x_n - x_{n+1}\| \leq \alpha\lambda\|x_{n-1} - x_n\| + \beta\frac{\|x_{n-1}-x_n\|\|x_n-x_{n+1}\|}{\|x_{n-1}-x_n\|}$. This implies that
$\|x_n - x_{n+1}\| \leq \alpha\lambda\|x_{n-1} - x_n\| + \beta\|x_n - x_{n+1}\|$, so that
$\|x_n - x_{n+1}\| \leq \eta\|x_{n-1} - x_n\|$ for $n = 1, 2, ...$, where $\eta = \frac{\alpha\lambda}{1-\beta} < 1$.
Hence, inductively, it follows that
$\|x_n - x_{n+1}\| \leq \eta^n\|x_0 - x_1\|$ for $n = 1, 2, ...$.
Therefore it is easy to see that the sequence $\{x_n\}$ is Cauchy.
Since X is complete, we have $\lim\limits_{n\to\infty} x_n = s$ (say), $s \in X$.
Since T is continuous on X, we have T_λ is so and hence
$s = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} T_\lambda x_n = T_\lambda \lim\limits_{n\to\infty} x_n = T_\lambda s$.
Therefore s is a fixed point of T_λ.
Let t be another fixed point of T_λ and $s \neq t$. Now, from the inequality (2.3), we have

$$0 < \|s - t\| = \|T_\lambda s - T_\lambda t\| \leq \alpha\lambda\|s - t\| + \beta\frac{\|s-T_\lambda s\|\|t-T_\lambda t\|}{\|s-t\|},$$

which implies that
$0 < \|s - t\| \leq \alpha\lambda\|s - t\|$,
a contradiction.
Therefore $t = s$, and T_λ has a unique fixed point s.
Thus, it follows that T has a unique fixed point s in X. □

Remark. *If $k = 0$ and $\beta = 0$ in the inequality (2.2), then T is a contraction and in this case, contraction principle follows as a corollary to Theorem 2.1.*

Example 2.2. *Let $X = \mathbb{R}$ with the usual norm and we define $T : X \to X$ by $Tx = -2x - 3, x \in \mathbb{R}$. We choose $k = \frac{3}{2}$, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. We now consider*

$$\begin{aligned}|k(x - y) + Tx - Ty| &= |\tfrac{3}{2}(x - y) - 2x - 3 - (-2y - 3)| \\ &= \tfrac{1}{2}|x - y| \\ &\leq \tfrac{1}{2}|x - y| + \tfrac{1}{3}\tfrac{|x-(-2x-3)||y-(-\frac{1}{5}y-\frac{6}{5})|}{|x-y|} \\ &= \alpha|x - y| + \beta\tfrac{|x-Tx||y-T_\lambda y|}{|x-y|}.\end{aligned}$$

Therefore T satisfies the inequality (2.2) of Theorem 2.1 with $\alpha + \beta < 1$ and '$-\frac{1}{3}$' is the unique fixed point of T.

Here we observe that T is not a contraction. So contraction mapping principle is not applicable.

For any positive integer p, we denote T^p, the composition of p number of selfmaps T. Here we note that $T^1 = T$. Also we denote $T^0 = I$, I the identity map of X. In this case, $T^0_\lambda = I$ for every $\lambda \in [0, 1]$.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$. Assume that T is an enriched Jaggi contraction map. Let $x_0 \in X$. If T^p is continuous for some positive integer p, then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ by $x_{n+1} = T^p_\lambda x_n, n = 0, 1, 2, ...$.
Then by applying Theorem 2.1 to T^p_λ, we get that the sequence $\{x_n\}$ converges to s, and $T^p_\lambda(s) = s$, and this s is unique.
We now show that $T_\lambda(s) = s$.
Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Then $\{x_{n_k}\}$ also converges to s. Now

$T_\lambda^p(s) = T_\lambda^p(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} T_\lambda^p x_{n_k} = \lim_{k\to\infty} x_{n_k+1} = s.$
Let r be the smallest positive integer such that $T_\lambda^r(s) = s$. Then $T_\lambda^i(s) \neq s$ for all $i = 1, 2, ..., r-1$.
For $i \in \{1, 2, ..., r-1, r\}$, we have

$$\begin{aligned}\|T_\lambda^i(s) - T_\lambda^{i-1}(s)\| &= \|T_\lambda(T_\lambda^{i-1}(s)) - T_\lambda(T_\lambda^{i-2}(s))\| \\ &\leq \alpha\lambda\|T_\lambda^{i-1}(s) - T_\lambda^{i-2}(s)\| + \beta\frac{\|T_\lambda^{i-1}(s)-T_\lambda^i(s)\|\|T_\lambda^{i-2}(s)-T_\lambda^{i-1}(s)\|}{\|T_\lambda^{i-1}(s)-T_\lambda^{i-2}(s)\|} \\ &= \alpha\lambda\|T_\lambda^{i-1}(s) - T_\lambda^{i-2}(s)\| + \beta\|T_\lambda^{i-1}(s) - T_\lambda^i(s)\|\end{aligned}$$

$$\|T_\lambda^i(s) - T_\lambda^{i-1}(s)\| \leq (\frac{\alpha\lambda}{1-\beta})\|T_\lambda^{i-1}(s) - T_\lambda^{i-2}(s)\|. \tag{2.4}$$

If $r > 1$, then

$$\begin{aligned}\|T_\lambda(s) - s\| &= \|T_\lambda s - T_\lambda^r(s)\| \\ &= \|T_\lambda s - T_\lambda(T_\lambda^{r-1}(s))\| \\ &\leq \alpha\lambda\|s - T_\lambda^{r-1}(s)\| + \beta\frac{\|s-T_\lambda(s)\|\|T_\lambda^{r-1}(s)-T_\lambda^r(s)\|}{\|s-T_\lambda^{r-1}(s)\|} \\ &= \alpha\lambda\|s - T_\lambda^{r-1}(s)\| + \beta\frac{\|s-T_\lambda(s)\|\|T_\lambda^{r-1}(s)-s\|}{\|s-T_\lambda^{r-1}(s)\|} \\ &= \alpha\lambda\|s - T_\lambda^{r-1}(s)\| + \beta\|s - T_\lambda(s)\| \text{ which implies that}\end{aligned}$$

$(1-\beta)\|s - T_\lambda(s)\| \leq \alpha\lambda\|s - T_\lambda^{r-1}(s)\|$. Therefore

$$\|s - T_\lambda(s)\| \leq (\frac{\alpha\lambda}{1-\beta})\|s - T_\lambda^{r-1}(s)\|. \tag{2.5}$$

Also, by (2.4) with $i = r$, we have

$$\begin{aligned}\|s - T_\lambda^{r-1}(s)\| &= \|T_\lambda^r(s) - T_\lambda^{r-1}(s)\| \\ &\leq (\tfrac{\alpha\lambda}{1-\beta})\|T_\lambda^{r-1}(s) - T_\lambda^{r-2}(s)\|.\end{aligned}$$

On repeated application of the inequality (2.4), we get

$$\begin{aligned}\|s - T_\lambda^{r-1}(s)\| = \|T_\lambda^r(s) - T_\lambda^{r-1}(s)\| &\leq (\tfrac{\alpha\lambda}{1-\beta})\|T_\lambda^{r-1}(s) - T_\lambda^{r-2}(s)\| \\ &\vdots \\ &\leq (\tfrac{\alpha\lambda}{1-\beta})^{r-1}\|T_\lambda(s) - T_\lambda^0(s)\|, \text{ and hence}\end{aligned}$$

$$\|s - T_\lambda^{r-1}(s)\| \leq (\frac{\alpha\lambda}{1-\beta})^{r-1}\|T_\lambda(s) - s\|, \text{ since } T_\lambda^0 \text{ is the identity map.} \tag{2.6}$$

From (2.5) and (2.6), we have
$\|s - T_\lambda(s)\| \leq (\frac{\alpha\lambda}{1-\beta})^r\|s - T_\lambda(s)\|$,
a contradiction, since $\frac{\alpha\lambda}{1-\beta} < 1$.
Therefore $T_\lambda s = s$.
Uniqueness of fixed point of T_λ follows as in the proof of Theorem 2.1.
Thus s is the unique fixed point of T. □

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$. Assume that there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $k \in [0, \infty)$ such that for $\lambda = \frac{1}{k+1}$, and for some positive integer q, T satisfies*

$$\|k(x-y) + T^q x - T^q y\| \leq \alpha\|x - y\| + \beta\frac{\|x - T^q x\|\|y - T_\lambda^q y\|}{\|x - y\|} \tag{2.7}$$

for all $x, y \in X$ and $x \neq y$; where $T_\lambda^q(x) = (1-\lambda)x + \lambda T^q x$.

If T^q is continuous then T has a unique fixed point in X.

Proof. By Theorem 2.1, T_λ^q has a unique fixed point s (say) in X. Then $T_\lambda(s) = T_\lambda(T_\lambda^q(s)) = T_\lambda^q(T_\lambda(s))$. Hence $T_\lambda(s)$ is also a fixed point of T_λ^q.
Now, by the uniqueness of fixed point of T_λ^q, we have $T_\lambda(s) = s$.
Since T_λ^q has a unique fixed point s, it follows that s is a unique fixed point of T_λ.
Hence it follows that s is the unique fixed point of T. □

The following example shows that Theorem 2.3 is more general than Theorem 2.1.

Example 2.3. *Let $X = \mathbb{R}$ with the usual norm. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{1}{3} & \text{if } x \in [0, \infty) \\ -x & \text{if } x \in (-\infty, 0) \end{cases}.$$

Then $T^2x = \frac{1}{3}$ for all $x \in \mathbb{R}$ so that T^2 is continuous on X. Indeed, inequality (2.7) of Theorem 2.3 holds with $q = 2$, $k = \frac{1}{2}, \alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. For, for any $x \in [0, \infty)$, $y \in (-\infty, 0)$, we have

$$\begin{aligned}
|k(x-y) + T^2x - T^2y| &= |\tfrac{1}{2}(x-y) + \tfrac{1}{3} - \tfrac{1}{3}| \\
&= \tfrac{1}{2}|x-y| \\
&\leq \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|x-\frac{1}{3}||y-\frac{1}{3}|}{|x-y|} \\
&= \tfrac{1}{2}|x-y| + \tfrac{1}{4}\tfrac{|x-T^2x||y-T_\lambda^2y|}{|x-y|} \\
&= \alpha|x-y| + \beta\tfrac{|x-T^2x||y-T_\lambda^2y|}{|x-y|}.
\end{aligned}$$

Thus T^2 satisfies the hypotheses of Theorem 2.3 and '$\frac{1}{3}$' is the unique fixed point of T. Here we observe that T is not continuous and so Theorem 2.1 is not applicable.

Definition 2.2. *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $k \in [0, +\infty)$ such that for $\lambda = \frac{1}{k+1}$, T satisfy the inequality*

$$\|k(x-y) + Tx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|y-Ty\|(1 + \|x-T_\lambda x\|)}{1 + \|x-y\|} \tag{2.8}$$

for all $x, y \in X$, then we say that T is an enriched Dass and Gupta contraction map.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$ be continuous. Assume that T is an enriched Dass and Gupta contraction map. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n, n = 0, 1, 2, ...$ converges to q (say) in X, and q is the unique fixed point of T.*

Proof. The proof of this theorem is similar to that of Theorem 2.1.
□

Definition 2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Let $S, T : X \to X$. If there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $k \in [0, +\infty)$ such that for $\lambda = \frac{1}{k+1}$, S and T satisfy the inequality*

$$\|k(x-y) + Sx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Sx\|\|y-T_\lambda y\|}{\|x-y\|} \tag{2.9}$$

for all $x, y \in X$ and $x \neq y$ then we say that the pair (S, T) is an enriched Jaggi contraction pair of maps. Here we note that if $S = T$ in the inequality (2.9), then T is an enriched Jaggi contraction map.

In the following, we extend Theorem 2.1 to a pair of selfmaps.

Theorem 2.5. *Let $(X, \|\cdot\|)$ be a Banach space. Let $S, T : X \to X$. Suppose that the pair (S, T) is an enriched Jaggi contraction pair of maps. Let $x_0 \in X$. We define the sequence $\{x_n\}_{n=0}^{\infty}$ by*

$$x_n = \begin{cases} S_\lambda x_{2m-1}, & if\ n = 2m,\ m = 1, 2, ... \\ T_\lambda x_{2m}, & if\ n = 2m+1,\ m = 0, 1, 2... . \end{cases}$$

Then $\{x_n\}$ converges to u (say) in X, and u is the unique common fixed point of S and T, provided S and T are continuous.

Proof. Let $\lambda = \frac{1}{k+1} < 1$. In this case, we have $k = \frac{1}{\lambda} - 1$ and thus the condition (2.9) becomes
$\|(\frac{1}{\lambda} - 1)(x-y) + Sx - Ty\| \leq \alpha\|x-y\| + \beta\frac{\|x-Sx\|\|y-T_\lambda y\|}{\|x-y\|}$ for all $x, y \in X, x \neq y$. *i.e.,*
$\|(1-\lambda)(x-y) + Sx - Ty\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|\lambda x-\lambda Sx\|\|y-T_\lambda y\|}{\|x-y\|}$. *i.e.,*
$\|S_\lambda x - T_\lambda y\| \leq \alpha\lambda\|x-y\| + \beta\frac{\|x-S_\lambda x\|\|y-T_\lambda y\|}{\|x-y\|}$ for any $x, y \in X$ and $x \neq y$.
Case (i) $n = 2m$. In this case, we consider

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_{2m+1} - x_{2m}\| \\ &= \|T_\lambda x_{2m} - S_\lambda x_{2m-1}\| \\ &= \|S_\lambda x_{2m-1} - T_\lambda x_{2m}\| \\ &\leq \alpha\lambda\|x_{2m-1} - x_{2m}\| + \beta\frac{\|x_{2m-1}-S_\lambda x_{2m-1}\|\|x_{2m}-T_\lambda x_{2m}\|}{\|x_{2m-1}-x_{2m}\|} \\ &= \alpha\lambda\|x_{2m-1} - x_{2m}\| + \beta\frac{\|x_{2m-1}-x_{2m}\|\|x_{2m}-x_{2m+1}\|}{\|x_{2m-1}-x_{2m}\|}\end{aligned}$$

$(1-\beta)\|x_{2m} - x_{2m+1}\| \leq \alpha\lambda\|x_{2m-1} - x_{2m}\|$. Thus, we have
$\|x_{2m+1} - x_{2m}\| \leq \eta\|x_{2m} - x_{2m-1}\|$ where $\eta = \frac{\alpha\lambda}{1-\beta} < 1$.
Case (ii) $n = 2m+1$. In this case, we consider

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|x_{2m+2} - x_{2m+1}\| \\ &= \|S_\lambda x_{2m+1} - T_\lambda x_{2m}\| \\ &\leq \alpha\lambda\|x_{2m+1} - x_{2m}\| + \beta\frac{\|x_{2m+1}-S_\lambda x_{2m+1}\|\|x_{2m}-T_\lambda x_{2m}\|}{\|x_{2m+1}-x_{2m}\|} \\ &= \alpha\lambda\|x_{2m+1} - x_{2m}\| + \beta\frac{\|x_{2m+1}-x_{2m+2}\|\|x_{2m}-x_{2m+1}\|}{\|x_{2m+1}-x_{2m}\|}\end{aligned}$$

$(1-\beta)\|x_{2m+2} - x_{2m+1}\| \leq \alpha\lambda\|x_{2m+1} - x_{2m}\|$
That is
$\|x_{2m+2} - x_{2m+1}\| \leq \eta\|x_{2m+1} - x_{2m}\|$ where $\eta = \frac{\alpha\lambda}{1-\beta} < 1$.
Thus from Case (i) and Case (ii), it follows that
$\|x_{n+1} - x_n\| \leq \eta\|x_n - x_{n-1}\|$ for all $n = 1, 2, 3, ...$.
Now, inductively, it follows that
$\|x_{n+1} - x_n\| \leq \eta^n\|x_1 - x_0\|$ for all $n = 1, 2, ...$.
Thus the sequence $\{x_n\}$ is Cauchy.
Since X is complete, we have $\lim_{n\to\infty} x_n = u$ (say), $u \in X$.
Suppose that S is continuous. So S_λ is continuous on X.
$u = \lim_{m\to\infty} x_{2m} = \lim_{m\to\infty} S_\lambda x_{2m-1} = S_\lambda \lim_{m\to\infty} x_{2m-1} = S_\lambda u$.
Therefore u is a fixed point of S_λ.
Suppose that T is continuous. So T_λ is continuous on X.
$u = \lim_{m\to\infty} x_{2m+1} = \lim_{m\to\infty} T_\lambda x_{2m} = T_\lambda \lim_{m\to\infty} x_{2m} = T_\lambda u$.
Therefore u is a common fixed point of T_λ and S_λ, and hence u is a common fixed point of S and T.

Uniqueness of this common fixed point follows trivially from the inequality (2.9).
□

Remark. *Theorem 2.1 follows by choosing $S = T$ in Theorem 2.5.*

3. Fixed points of almost $(k, a, b, \lambda)-$enriched CRR contraction maps

Definition 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \to X$. If there exist $k \in (0, +\infty), L \geq 0$ and $a, b \geq 0$ satisfying $a + 2b < 1$ such that*

$$\|k(x-y)+Tx-Ty\| \leq a\|x-y\| + b(\|x-Tx\| + \|y-Ty\|) + L\min\{\|y - T_\lambda x\|, \frac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\} \tag{3.1}$$

for all $x, y \in X$ with $\lambda = \frac{1}{k+1}$, then we say that T is an almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let $T : X \to X$ be an almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$. Let $x_0 \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n, n = 0, 1, 2, \ldots$ converges to p (say) in X, and p is the unique fixed point of T.*

Proof. Let $x_0 \in X$. We consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, \ldots$.
For $\lambda = \frac{1}{k+1} < 1$, we have $k = \frac{1}{\lambda} - 1$ and thus the condition (3.1) becomes
$$\|(\tfrac{1}{\lambda}-1)(x-y)+Tx-Ty\| \leq a\|x-y\| + b(\|x-Tx\| + \|y-Ty\|) + L\min\{\|y-T_\lambda x\|, \frac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\}$$
for all $x, y \in X$. Therefore
$$\|(1-\lambda)(x-y)+\lambda Tx-\lambda Ty\| \leq \lambda a\|x-y\| + b(\|\lambda x-\lambda Tx\| + \|\lambda y-\lambda Ty\|) + \lambda L\min\{\|y-T_\lambda x\|, \frac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\}.$$
That is
$$\|T_\lambda x - T_\lambda y\| \leq \lambda a\|x-y\| + b(\|x-T_\lambda x\| + \|y-T_\lambda y\|) + \lambda L\min\{\|y-T_\lambda x\|, \frac{\|x-T_\lambda x\|[1+\|x-T_\lambda y\|]}{1+\|x-y\|}\}. \tag{3.2}$$
By taking $x = x_{n-1}$ and $y = x_n$ in (3.2), we get
$$\|T_\lambda x_{n-1} - T_\lambda x_n\| \leq \lambda a\|x_{n-1}-x_n\| + b(\|x_{n-1}-T_\lambda x_{n-1}\| + \|x_n - T_\lambda x_n\|) + \lambda L\min\{\|x_n - T_\lambda x_{n-1}\|, \frac{\|x_{n-1}-T_\lambda x_{n-1}\|[1+\|x_{n-1}-T_\lambda x_n\|]}{1+\|x_{n-1}-x_n\|}\}$$
which implies that
$$\begin{aligned}\|x_n - x_{n+1}\| &\leq \lambda a\|x_{n-1}-x_n\| + b(\|x_{n-1}-x_n\| + \|x_n-x_{n+1}\|) + \lambda L\min\{\|x_n-x_n\|, \frac{\|x_n-x_{n+1}\|[1+\|x_{n-1}-x_{n+1}\|]}{1+\|x_{n-1}-x_n\|}\} \\ &\leq a\|x_{n-1}-x_n\| + b(\|x_{n-1}-x_n\| + \|x_n-x_{n+1}\|) + \lambda L\min\{0, \frac{\|x_n-x_{n+1}\|[1+\|x_{n-1}-x_{n+1}\|]}{1+\|x_{n-1}-x_n\|}\}\end{aligned}$$
so that
$(1-b)\|x_n - x_{n+1}\| \leq (a+b)\|x_{n-1} - x_n\|$
$\|x_n - x_{n+1}\| \leq \delta\|x_{n-1} - x_n\|$ where $\delta = \frac{a+b}{1-b} < 1$.
Inductively, it follows that
$\|x_n - x_{n+1}\| \leq \delta^n\|x_0 - x_1\|$ for $n = 1, 2, \ldots$.
Therefore $\{x_n\}$ is Cauchy. Since X is complete, we have $\lim_{n\to\infty} x_n = p$ (say), $p \in X$.
Now we show that p is the fixed point of T_λ.
We consider
$$\begin{aligned}\|p - T_\lambda p\| &\leq \|p - x_{n+1}\| + \|x_{n+1} - T_\lambda p\| \\ &= \|p - T_\lambda x_n\| + \|T_\lambda x_n - T_\lambda p\| \\ &\leq \|p - T_\lambda x_n\| + \lambda a\|x_n - p\| + b(\|x_n - T_\lambda x_n\| + \|p - T_\lambda p\|) + \lambda L\min\{\|p - T_\lambda x_n\|, \frac{\|x_n-T_\lambda x_n\|[1+\|x_n-T_\lambda p\|]}{1+\|x_n-p\|}\}.\end{aligned}$$

On letting $n \to \infty$, we get
$\|p - T_\lambda p\| \leq \|p-p\| + a\|p-p\| + b(\|p-p\| + \|p-T_\lambda p\|) + L\min\{\|p-p\|, \frac{\|p-p\|[1+\|p-T_\lambda p\|]}{1+\|p-p\|}\}$
$\leq b\|p - T_\lambda p\|$ so that
$(1-b)\|p - T_\lambda p\| \leq 0$. Since $(1-b) > 0$, it follows that
$\|p - T_\lambda p\| = 0$ and hence $T_\lambda p = p$.
Therefore p is a fixed point of T_λ.
Let q be another fixed point of T_λ and $q \neq p$. Then
$0 < \|p-q\| = \|T_\lambda p - T_\lambda q\| \leq a\|p-q\| + b(\|p - T_\lambda p\| + \|q - T_\lambda q\|) + \lambda L \min\{\|q - T_\lambda p\|, \frac{\|p-T_\lambda p\|[1+\|p-T_\lambda q\|]}{1+\|p-q\|}\}$
so that
$\|p-q\| \leq a\|p-q\|$,
a contradiction.
Therefore $p = q$.
Therefore T_λ has a unique fixed point. Thus, it follows that T has a unique fixed point in X. □

Remark. *Theorem 3.1 extends Theorem 1.2 to the case of almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$.*

Example 3.1. *Let $X = \mathbb{R}$ with the usual norm. We define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{8}, & 0 \leq x < 2 \\ 0, & (-\infty, 0) \cup [2, \infty) \end{cases}.$$

We choose $k = \frac{1}{2}, a = \frac{1}{2}$ and $b = \frac{1}{5}$ with $a + 2b < 1$.
Let $x \in [0, 2), y \in [2, \infty)$ We now consider

$$\begin{aligned} |k(x-y) + Tx - Ty| &= |\tfrac{1}{2}(x-y) + \tfrac{x}{8} - 0| \\ &= |\tfrac{1}{2}(x-y) + \tfrac{x}{8}| \\ &\leq \tfrac{1}{2}|x-y| + \tfrac{7}{40}x \\ &\leq \tfrac{1}{2}|x-y| + \tfrac{7}{40}x + \tfrac{1}{5}|y| + L\min\{|y - \tfrac{5}{12}x|, \tfrac{|x-\frac{1}{3}y|}{1+|x-y|}\} \\ &= \tfrac{1}{2}|x-y| + \tfrac{1}{5}(|x - \tfrac{x}{8}| + |y-0|) + L\min\{|y - \tfrac{5}{12}x|, \tfrac{|x-\frac{1}{3}y|}{1+|x-y|}\} \\ &= a|x-y| + b(|x-Tx| + |y-Ty|) + L\min\{|y - T_\lambda x|, \tfrac{|x-T_\lambda y|}{1+|x-y|}\}. \end{aligned}$$

Therefore inequality (3.1) holds for any $L \geq 0$. Hence T is an almost $(\frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{2}{3})-$enriched CRR contraction map on $\mathbb{R}$. So T satisfies the hypotheses of Theorem 3.1 and '0' is the unique fixed point of T.

4. Convergence of sequence of fixed points of enriched contraction maps

In the following, $\mathbb{Z}^+$ denotes the set of all natural numbers.

Theorem 4.1. *Let $\{T_n\}$ be a sequence of $(k, a)-$enriched contraction maps defined on a Banach space $(X, \|\cdot\|)$ and u_n, the fixed point of T_n for each $n = 1, 2, 3, ...,$ which exists by Theorem 1.1. If $\{T_n\}$ converges uniformly to T, then $u_n \to u$ implies that u is a fixed point of T. Conversely if u is a fixed point of T, then u_n converges to u provided $k < 1 - a$.*

Proof. First suppose that $u_n \to u$ as $n \to \infty$. Assume that $Tu \neq u$.
Let $\epsilon = \|Tu - u\| > 0$. Then there exists $N_1 \in \mathbb{Z}^+$ such that $\|u_n - u\| < \frac{\epsilon}{2(1+a+k)}$ for all $n \geq N_1$.

Since $T_n \to T$ uniformly, we have, there exists $N_2 \in \mathbb{Z}^+$ such that $\|T_n u - Tu\| < \frac{\epsilon}{2}$ for all $n \geq N_2$ and for all $u \in X$.
Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have

$$\begin{aligned}
0 < \epsilon = \|u - Tu\| &\leq \|u - u_n\| + \|u_n - T_n u\| + \|T_n u - Tu\| \\
&= \|u_n - u\| + \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\
&= \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\
&\leq \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u\| + k\|u_n - u\| + \|T_n u - Tu\| \\
&\leq \|u_n - u\| + a\|u_n - u\| + k\|u_n - u\| + \|T_n u - Tu\| \\
&= (1 + a + k)\|u_n - u\| + \|T_n u - Tu\| \\
&< (1 + a + k)\tfrac{\epsilon}{2(1+a+k)} + \tfrac{\epsilon}{2} \\
&= \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon,
\end{aligned}$$

a contradiction.
Therefore $Tu = u$.
Conversely, assume that $Tu = u$. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $\|T_n u - Tu\| < \frac{\epsilon}{c}$ for all $n \geq N$ and for all $u \in X$, where $c = \frac{1}{1-a-k} > 0$.
Let $n \geq N$. Then

$$\begin{aligned}
\|u_n - u\| &= \|T_n u_n - Tu\| \\
&\leq \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\
&= \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\
&\leq a\|u_n - u\| + k\|u_n - u\| + \|T_n u - Tu\| \\
&= (a + k)\|u_n - u\| + \|T_n u - Tu\|
\end{aligned}$$

$$(1 - a - k)\|u_n - u\| \leq \|T_n u - Tu\|$$
$$\|u_n - u\| \leq c\|T_n u - Tu\| < c.\tfrac{\epsilon}{c} = \epsilon.$$

Therefore $u_n \to u$ as $n \to \infty$.

Hence the theorem follows. □

Theorem 4.2. *Let $\{T_n\}$ be a sequence of enriched Jaggi contraction maps defined on a Banach space $(X, \|\cdot\|)$ and u_n, the fixed point of T_n for each $n = 1, 2, 3, ...,$ which exists by Theorem 2.1. If $\{T_n\}$ converges uniformly to T, then $u_n \to u$ implies that u is a fixed point of T. Conversely if u is a fixed point of T, then u_n converges to u provided $k < 1 - \alpha$.*

Proof. Follows as that of Theorem 4.1.

□

Theorem 4.3. *Let $\{T_n\}$ be a sequence of enriched Dass and Gupta contraction maps defined on a Banach space $(X, \|\cdot\|)$ and u_n, the fixed point of T_n for each $n = 1, 2, 3, ...,$ which exists by Theorem 2.4. If $\{T_n\}$ converges uniformly to T, then $u_n \to u$ implies that u is a fixed point of T. Conversely if u is a fixed point of T, then u_n converges to u provided $k < 1 - \alpha$.*

Proof. Suppose that $u_n \to u$ as $n \to \infty$.
Now, we consider

$$\begin{aligned}
\|u - Tu\| &\leq \|u - u_n\| + \|u_n - T_n u\| + \|T_n u - Tu\| \\
&= \|u_n - u\| + \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\
&= \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\
&\leq \|u_n - u\| + \|k(u_n - u) + T_n u_n - T_n u\| + k\|u_n - u\| + \|T_n u - Tu\| \\
&\leq \|u_n - u\| + \alpha\|u_n - u\| + \beta\tfrac{\|u - T_n u\|(1 + \|u_n - (T_n)_\lambda u_n\|)}{1 + \|u - u_n\|} + k\|u_n - u\| \\
&\qquad + \|T_n u - Tu\| \\
&= \|u_n - u\| + \alpha\|u_n - u\| + \beta\tfrac{\|u - T_n u\|}{1 + \|u_n - u\|} + k\|u_n - u\| + \|T_n u - Tu\|, \text{ since}
\end{aligned}$$

$$\|u_n - (T_n)_\lambda u_n\| = 0$$
$$\leq (1+\alpha+k)\|u_n - u\| + \beta\|u - T_n u\| + \|T_n u - Tu\|$$
$$\leq (1+\alpha+k)\|u_n - u\| + \beta[\|u - Tu\| + \|Tu - T_n u\|] + \|T_n u - Tu\|.$$

Therefore
$(1-\beta)\|u - Tu\| \leq (1+\alpha+k)\|u_n - u\| + (1+\beta)\|T_n u - Tu\|$, and hence
$\|u - Tu\| \leq \frac{1+\alpha+k}{1-\beta}\|u_n - u\| + \frac{1+\beta}{1-\beta}\|T_n u - Tu\| \to 0$ as $n \to \infty$, since $\{T_n\}$ converges to T uniformly.
Therefore $Tu = u$.
Conversely, we assume that $Tu = u$. We consider

$$\begin{aligned}
\|u_n - u\| &= \|T_n u_n - Tu\| \\
&\leq \|T_n u_n - T_n u\| + \|T_n u - Tu\| \\
&= \|k(u_n - u) + T_n u_n - T_n u - k(u_n - u)\| + \|T_n u - Tu\| \\
&\leq \alpha\|u_n - u\| + \beta\frac{\|u - T_n u\|(1+\|u_n - (T_n)_\lambda u_n\|)}{1+\|u - u_n\|} + k\|u_n - u\| + \|T_n u - Tu\| \\
&\leq \alpha\|u_n - u\| + \beta\|u - T_n u\| + k\|u_n - u\| + \|T_n u - Tu\| \\
&\leq \alpha\|u_n - u\| + \beta[\|u - Tu\| + \|Tu - T_n u\|] + k\|u_n - u\| + \|T_n u - Tu\| \\
&= (\alpha+k)\|u_n - u\| + (1+\beta)\|T_n u - Tu\|, \text{ and hence}
\end{aligned}$$

$(1-\alpha-k)\|u_n - u\| \leq (1+\beta)\|T_n u - Tu\|$. Therefore
$\|u_n - u\| \leq c\|T_n u - Tu\| \to 0$ as $n \to \infty$, where $c = \frac{1+\beta}{1-\alpha-k}$ is a positive constant.
Therefore $u_n \to u$ as $n \to \infty$.

Hence the theorem follows. □

5. Conclusion

In this paper, we defined enriched Jaggi contraction map, enriched Dass and Gupta contraction map and almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$ in Banach spaces. It is noted that every Jaggi contraction is an enriched Jaggi contraction but its converse is not true (Example 2.1) so that enriched Jaggi contraction maps are more general than Jaggi contraction maps. We proved the existence and uniqueness of fixed points of enriched Jaggi contraction map (Theorem 2.1). We provided an example in support of Theorem 2.1 and we observed that T is not a contraction and contraction mapping principle is not applicable. Hence Theorem 2.1 generalizes contraction mapping principle. Further, we extended Theorem 2.1 in which T^p is continuous for some positive integer p (Theorem 2.2). Also, we extended Theorem 2.1 for the map T^q for some positive integer q (Thereoem 2.3). An example (Example 2.3) is provided where T^q is satisfies the inequality (2.7), but T is not continuous. Since T is not continuous, Theorem 2.1 is not applicable. Also, it is easy to see that we can extend Theorem 2.1 to enriched Dass and Gupta contraction map. Further, enriched Jaggi contraction is extended to a pair of selfmaps and proved the existence and uniqueness of common fixed points. Also, we proved the existence and uniqueness of fixed points of almost $(k, a, b, \lambda)-$enriched CRR contraction map with $\lambda = \frac{1}{k+1}$.

Also, we proved that the sequence of fixed points $\{u_n\}$ of the corresponding enriched contraction maps $\{T_n\}$ converges to the fixed point u of the uniform limit operator T of these enriched contraction maps $\{T_n\}$. Conversely, if u is a fixed point of T then $\{u_n\}$ converges to u under certain assumption. Further, we extended this technique to a sequence of enriched Jaggi contraction maps and enriched Dass and Gupta contraction maps.

In the direction of future research, we would like to suggest the following:
1) Some new fixed point results can be investigated by introducing more general

enriched contraction conditions.
2) Some new fixed point results for multi-valued contractions can be investigated.

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GENERALIZED TOPOLOGICAL OPERATOR THEORY IN GENERALIZED TOPOLOGICAL SPACES

PART I. GENERALIZED INTERIOR AND GENERALIZED CLOSURE

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Abstract. In a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$-*space*), various ordinary topological operators ($\mathfrak{T}_{\mathfrak{g}}$-*operators*), namely, $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}$, $\mathrm{ext}_{\mathfrak{g}}$, $\mathrm{fr}_{\mathfrak{g}}$, $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{T}_{\mathfrak{g}}$-*interior*, $\mathfrak{T}_{\mathfrak{g}}$-*closure*, $\mathfrak{T}_{\mathfrak{g}}$-*exterior*, $\mathfrak{T}_{\mathfrak{g}}$-*frontier*, $\mathfrak{T}_{\mathfrak{g}}$-*derived*, $\mathfrak{T}_{\mathfrak{g}}$-*coderived operators*), are defined in terms of ordinary sets ($\mathfrak{T}_{\mathfrak{g}}$-*sets*). Accordingly, generalized $\mathfrak{T}_{\mathfrak{g}}$-operators ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators*), namely, $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Fr}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*interior*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closure*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*frontier*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operators*) may be defined in terms of generalized $\mathfrak{T}_{\mathfrak{g}}$-sets ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*), thereby making $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators theory in $\mathscr{T}_{\mathfrak{g}}$-spaces an interesting subject of inquiry. In this paper, we introduce the definitions and study the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in terms of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets which we studied earlier. The major findings to which the study has led to are: Firstly, $(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is $(\Omega, \emptyset)$-*grounded*, (*expansive, non-expansive*), (*idempotent, idempotent*) and $(\cap, \cup)$-*additive*. Secondly, $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. The elements supporting these facts are reported therein as sources of inspiration for more generalized operations.

1. Introduction

Just as the concepts of $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-interior operators in $\mathscr{T}$-spaces (ordinary and generalized interior operators in ordinary topological spaces) and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators in $\mathscr{T}$-spaces (ordinary and generalized closure operators in ordinary topological spaces) are essential operators in the study of $\mathfrak{T}$-sets in $\mathscr{T}$-spaces (arbitrary sets in ordinary topological spaces) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], so are the concepts

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of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized interior operators in generalized topological spaces) and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized closure operators in generalized topological spaces) essential operators in the study of $\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces (arbitrary sets in generalized topological spaces) [13, 14, 15, 16, 17, 18, 19].

Intuitively, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-interior operators, respectively, in a $\mathscr{T}$-space can be characterized as one-valued maps int, $\mathfrak{g}\text{-Int} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ from the power set $\mathscr{P}(\Omega)$ of Ω into itself, assigning to each $\mathfrak{T}$-set in the $\mathscr{T}$-space the $\cup$-operation (union operation) of all $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open subsets of the $\mathfrak{T}$-set [20, 21, 22, 23]. When the role of $\cup$-operation and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open subsets, respectively, are given to $\cap$-operation (intersection operation) and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closed supersets of the $\mathfrak{T}$-set, the dual notions, called $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space follow [21, 23, 24, 25, 26], which can likewise be characterized as one-valued maps cl, $\mathfrak{g}\text{-Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Finally, when $(\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathfrak{T}) \longmapsto (\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}})$, the notions of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space follow [15, 16, 27, 28, 29, 30, 31], which can in a similar manner be characterized as one-valued maps of the types $\text{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\text{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively.

Thus, in a $\mathscr{T}$-space, int, $\mathfrak{g}\text{-Int} : \mathscr{S} \longmapsto \text{int}(\mathscr{S})$, $\mathfrak{g}\text{-Int}(\mathscr{S})$ describe two types of collections of points interior in $\mathscr{S}$ and, cl, $\mathfrak{g}\text{-Cl} : \mathscr{S} \longmapsto \text{cl}(\mathscr{S})$, $\mathfrak{g}\text{-Cl}(\mathscr{S})$ describe another two types of collections of points but close to $\mathscr{S}$. Similarly, in a $\mathscr{T}_{\mathfrak{g}}$-space, $\text{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ describe two types of collections of points interior in $\mathscr{S}_{\mathfrak{g}}$ and, $\text{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ describe another two types of collections of points but close to $\mathscr{S}_{\mathfrak{g}}$. Of all such operators int, cl, $\mathfrak{g}$-Int, $\mathfrak{g}\text{-Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}$-spaces and $\text{int}_{\mathfrak{g}}$, $\text{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{g}}$-spaces, int, $\text{cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the oldest and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the newest. Hence, the studies of operators of these kinds have evolved from the studies of ordinary operators in ordinary topological spaces to the studies of generalized operators in generalized topological spaces.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, some new types of one-valued maps $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ have been defined and investigated by Mathematicians.

Based on θ-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, Min, W. K. [32, 33, 28] has introduced the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators i_{θ}, $c_{\theta} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and used them to study some properties of $\theta(g, g')$-continuity in $\mathscr{T}_{\mathfrak{g}}$-spaces. Cao, Yan, Wang and Wang [34] have introduced and then used the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators i_{λ}, $c_{\lambda} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (λ-*interior* and λ-*closure operators*), respectively, where $\lambda \in \{\alpha, \beta, \sigma, \pi\}$ in $\mathscr{T}$-spaces. Saravanakumar, Kalaivani and Krishnan [30] have studied the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $i_{\tilde{\mu}}$, $c_{\tilde{\mu}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\tilde{\mu}$-*interior* and $\tilde{\mu}$-*closure operators*), respectively, in terms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets ($\tilde{\mu}$-*open sets*) in $\mathscr{T}_{\mathfrak{g}}$-spaces. Srija and Jayanthi [35] have introduced the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators si_{g}, $\text{sc}_{\text{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (g-*semi interior* and g-*semi closure operators*), respectively. Boonpok, C. [36] has introduced the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $i_{\delta(\mu)}$, $c_{\delta(\mu)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\delta(\mu)$-*interior* and $\delta(\mu)$-*closure operators*), respectively, and utilized them to study the properties of $\zeta_{\delta(\mu)}$, $(\zeta, \delta(\mu))$-closed sets in strong $\mathscr{T}_{\mathfrak{g}}$-spaces. Later on, in extending the notion of μ-$\hat{\beta}g$-closed set introduced by Kannan and Nagaveni [37] in $\mathscr{T}$-spaces to $\mathscr{T}_{\mathfrak{g}}$-spaces and then studying their properties, Camargo, J. F. Z. [27] has also investigated the related $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\hat{\beta}gi_{\mu}$,

$\hat{\beta}gc_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (μ-$\hat{\beta}g$-*interior* and μ-$\hat{\beta}g$-*closure operators*), respectively. Relative to the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators introduced by Császár, A. [3, 38], the author found that the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under $\hat{\beta}gi_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a superset of that under $i_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and, the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under $\hat{\beta}gc_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a subset of its image under $c_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In this paper, the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented.

The rest of the paper is structured as: In Section 2, necessary and sufficient preliminary notions are described and the main results are reported in Section 3. In Section 4, various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed and an application of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented. Finally, the work is concluded in Section 5.

2. Theory

2.1. **Necessary Preliminaries.** The standard reference for notations and concepts is the Ph.D. Thesis of Khodabocus M. I. [16].

Throughout, $\mathfrak{U}$ is the *universe* of discourse, fixed within the framework of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator theory in $\mathscr{T}_{\mathfrak{g}}$-spaces; I_n^0, $I_n^* \subset \mathbb{Z}_+^0$ and I_∞^0, $I_\infty^* \subset \mathbb{Z}_+^0$ are index sets including and excluding 0 [15, 16]. To abstract definitions of concepts, let $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$.

Definition 2.1 ($\mathscr{T}_{\mathfrak{a}}$-Space [15, 16])**.** *A topological structure* $\mathfrak{T}_{\mathfrak{a}} \stackrel{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{a}})$*, consisting of an underlying set* $\Omega \subset \mathfrak{U}$ *and an* $\mathfrak{a}$*-topology* $\mathscr{T}_{\mathfrak{a}} : \begin{array}{ccc} \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ \mathscr{O}_{\mathfrak{a}} & \longmapsto & \mathscr{T}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}}) \end{array}$ *satisfying the compound* $\mathscr{T}_{\mathfrak{a}}$*-axiom:*

$$\operatorname{Ax}(\mathscr{T}_{\mathfrak{a}}) \stackrel{\text{def}}{\longleftrightarrow} \begin{cases} \big(\mathscr{T}_{\mathfrak{o}}(\emptyset) = \emptyset\big) \wedge \big(\mathscr{T}_{\mathfrak{o}}(\mathscr{O}_{\mathfrak{o},\nu}) \subseteq \mathscr{O}_{\mathfrak{o},\nu}\big) \\ \qquad \wedge \big(\mathscr{T}_{\mathfrak{o}}\big(\bigcap_{\nu \in I_n^*} \mathscr{O}_{\mathfrak{o},\nu}\big) = \bigcap_{\nu \in I_n^*} \mathscr{T}_{\mathfrak{o}}(\mathscr{O}_{\mathfrak{o},\nu})\big) \\ \qquad \wedge \big(\mathscr{T}_{\mathfrak{o}}\big(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{o},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{o}}(\mathscr{O}_{\mathfrak{o},\nu})\big) \quad (\mathfrak{a} = \mathfrak{o}), \\ \\ \big(\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset\big) \wedge \big(\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}\big) \\ \qquad \wedge \big(\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\big) \quad (\mathfrak{a} = \mathfrak{g}), \end{cases}$$

is called a $\mathscr{T}_{\mathfrak{a}}$*-space.*

On $\mathscr{T}_{\mathfrak{a}}$-spaces, neither ordinary nor generalized separation axioms are assumed unless otherwise stated. If $\mathfrak{a} = \mathfrak{o}$ (*ordinary*), then $\operatorname{Ax}(\mathscr{T}_{\mathfrak{o}})$ stands for an ordinary topology and if $\mathfrak{a} = \mathfrak{g}$ (*generalized*), then $\operatorname{Ax}(\mathscr{T}_{\mathfrak{g}})$ stands for a generalized topology. Accordingly, $\mathfrak{T} = (\Omega, \mathscr{T}) = (\Omega, \mathscr{T}_{\mathfrak{o}}) = \mathfrak{T}_{\mathfrak{o}} \neq \mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\Omega \in \mathscr{T}_{\mathfrak{g}}$, then $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space [3, 39] and if $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_n^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_n^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ for any $I_n^* \subset I_\infty^*$, then $\mathfrak{T}_{\mathfrak{g}}$ is a quasi $\mathscr{T}_{\mathfrak{g}}$-space [40].

Typically, $\big(\Gamma, \{\mathscr{O}_{\mathfrak{a}}\}, \mathscr{S}_{\mathfrak{a}}\big) \subset \Omega \times \mathscr{T}_{\mathfrak{a}} \times \mathfrak{T}_{\mathfrak{a}}$ denotes a triple of a Ω-subset, a unit set containing a $\mathscr{T}_{\mathfrak{a}}$-open set and a $\mathfrak{T}_{\mathfrak{a}}$-set. By $\complement_{\Omega}(\mathscr{O}_{\mathfrak{a}}) = \mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} \stackrel{\text{def}}{=} \big\{\mathscr{K}_{\mathfrak{a}} : \complement(\mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}_{\mathfrak{a}}\big\}$ is meant a $\mathscr{T}_{\mathfrak{a}}$-closed set. On the other hand, the operators $\operatorname{int}_{\mathfrak{a}}, \operatorname{cl}_{\mathfrak{a}} : \begin{array}{cl} \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} & \longmapsto \operatorname{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \operatorname{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ are called $\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Accordingly,

$$\operatorname{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \stackrel{\text{def}}{=} \bigcup_{\mathscr{O}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}}, \quad \operatorname{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \stackrel{\text{def}}{=} \bigcap_{\mathscr{K}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}}, \tag{2.1}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathscr{T}_{\mathfrak{a}} : \ \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} : \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$. In general, $(\mathrm{int}_{\mathfrak{g}}, \mathrm{cl}_{\mathfrak{g}}) \neq (\mathrm{int}_{\mathfrak{o}}, \mathrm{cl}_{\mathfrak{o}})$ [41]. Set $\mathscr{P}^{*}(\Omega) = \mathscr{P}(\Omega) \setminus \{\emptyset\}$, $\mathscr{T}^{*}_{\mathfrak{a}} = \mathscr{T}_{\mathfrak{a}} \setminus \{\emptyset\}$, and $\neg\mathscr{T}^{*}_{\mathfrak{a}} = \neg\mathscr{T}_{\mathfrak{a}} \setminus \{\emptyset\}$.

Definition 2.2 ($\mathfrak{g}$-Operation [15, 16]). *A mapping* $\begin{array}{rccl} \mathrm{op}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ *is called a generalized operation ($\mathfrak{g}$-operation) if and only if the following statements hold:*

$$\begin{aligned}\big(\forall \mathscr{S}_{\mathfrak{a}} \in \mathscr{P}^{*}(\Omega)\big)\big(\exists (\mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}^{*}_{\mathfrak{a}} \times \neg\mathscr{T}^{*}_{\mathfrak{a}}\big)\big[\big(\mathrm{op}_{\mathfrak{a}}(\emptyset) = \emptyset\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{a}}(\emptyset) = \emptyset\big) \\ \vee \big(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\big) \vee \big(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\big)\big],\end{aligned} \tag{2.2}$$

where $\begin{array}{rccl} \neg\,\mathrm{op}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ *is called its complementary $\mathfrak{g}$-operation, and for all $\mathfrak{T}_{\mathfrak{a}}$-sets $\mathscr{S}_{\mathfrak{a}}$, $\mathscr{S}_{\mathfrak{a},\nu}$, $\mathscr{S}_{\mathfrak{a},\mu} \in \mathscr{P}^{*}(\Omega)$, the following axioms are satisfied:*

- AX. I. $\big(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\big) \vee \big(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\big)$,
- AX. II. $\big(\mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathrm{op}_{\mathfrak{a}} \circ \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{a}} \circ \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\big)$,
- AX. III. $\big(\mathscr{S}_{\mathfrak{a},\nu} \subseteq \mathscr{S}_{\mathfrak{a},\mu} \ \longrightarrow\ \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a},\nu}) \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a},\mu})\big)$
  $\vee\big(\mathscr{S}_{\mathfrak{a},\mu} \subseteq \mathscr{S}_{\mathfrak{a},\nu} \ \longleftarrow\ \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a},\nu})\big)$,
- AX. IV. $\big(\mathrm{op}_{\mathfrak{a}}\big(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{a},\sigma}\big) \subseteq \bigcup_{\sigma=\nu,\mu} \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a},\sigma})\big)$
  $\vee\big(\neg\,\mathrm{op}_{\mathfrak{a}}\big(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{a},\sigma}\big) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a},\sigma})\big)$,

for some $\mathscr{T}_{\mathfrak{a}}$-sets $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{O}_{\mathfrak{a},\nu}$, $\mathscr{O}_{\mathfrak{a},\mu} \in \mathscr{T}^{*}_{\mathfrak{a}}$ and $\mathscr{K}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a},\nu}$, $\mathscr{K}_{\mathfrak{a},\mu} \in \neg\mathscr{T}^{*}_{\mathfrak{a}}$.

The formulation of DEF. 2.2 is based on the Čech closure operator axioms [42] and the axioms used by other mathematicians to define closure operators [43]. The class $\mathscr{L}_{\mathfrak{a}}[\Omega] \stackrel{\mathrm{def}}{=} \{\mathbf{op}_{\mathfrak{a},\nu} = (\mathrm{op}_{\mathfrak{a},\nu}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}) : \ \nu \in I^{0}_{3}\} \subseteq \mathscr{L}^{\omega}_{\mathfrak{a}}[\Omega] \times \mathscr{L}^{\kappa}_{\mathfrak{a}}[\Omega]$, where

$$\begin{aligned}\mathrm{op}_{\mathfrak{a}} \in \mathscr{L}^{\omega}_{\mathfrak{a}}[\Omega] &\stackrel{\mathrm{def}}{=} \{\mathrm{op}_{\mathfrak{a},0},\ \mathrm{op}_{\mathfrak{a},1},\ \mathrm{op}_{\mathfrak{a},2},\ \mathrm{op}_{\mathfrak{a},3}\} \\ &= \{\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}}\},\end{aligned} \tag{2.3}$$

$$\begin{aligned}\neg\,\mathrm{op}_{\mathfrak{a}} \in \mathscr{L}^{\kappa}_{\mathfrak{a}}[\Omega] &\stackrel{\mathrm{def}}{=} \{\neg\,\mathrm{op}_{\mathfrak{a},0},\ \neg\,\mathrm{op}_{\mathfrak{a},1},\ \neg\,\mathrm{op}_{\mathfrak{a},2},\ \neg\,\mathrm{op}_{\mathfrak{a},3}\} \\ &= \{\mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}}\},\end{aligned} \tag{2.4}$$

stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$. In general, $\mathscr{L}_{\mathfrak{g}}[\Omega] \ni \mathbf{op}_{\mathfrak{g}} = (\mathrm{op}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g}}) \neq (\mathrm{op}_{\mathfrak{o}}, \neg\,\mathrm{op}_{\mathfrak{o}}) = \mathbf{op}_{\mathfrak{o}} \in \mathscr{L}_{\mathfrak{o}}[\Omega]$.

Definition 2.3 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Sets [15, 16]). *Let $(\mathscr{S}_{\mathfrak{a}}, \{\mathscr{O}_{\mathfrak{a}}\}, \{\mathscr{K}_{\mathfrak{a}}\}) \subset \mathfrak{T}_{\mathfrak{a}} \times \mathscr{T}_{\mathfrak{a}} \times \neg\mathscr{T}_{\mathfrak{a}}$ and let $\mathbf{op}_{\mathfrak{a},\nu} \in \mathscr{L}_{\mathfrak{a}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Suppose the predicates*

$$\begin{aligned}\mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq, \supseteq\big) &\stackrel{\mathrm{def}}{=} \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq\big) \vee \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \supseteq\big), \\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq\big) &\stackrel{\mathrm{def}}{=} \big(\exists (\mathscr{O}_{\mathfrak{a}}, \mathrm{op}_{\mathfrak{a},\nu}) \in \mathscr{T}_{\mathfrak{a}} \times \mathscr{L}^{\omega}_{\mathfrak{a}}[\Omega]\big) \\ &\qquad \big[\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a},\nu}(\mathscr{O}_{\mathfrak{a}})\big], \\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \supseteq\big) &\stackrel{\mathrm{def}}{=} \big(\exists (\mathscr{K}_{\mathfrak{a}}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}) \in \neg\mathscr{T}_{\mathfrak{a}} \times \mathscr{L}^{\kappa}_{\mathfrak{a}}[\Omega]\big) \\ &\qquad \big[\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a},\nu}(\mathscr{K}_{\mathfrak{a}})\big]\end{aligned} \tag{2.5}$$

be Boolean-valued on $\mathfrak{T}_{\mathfrak{a}} \times (\mathscr{T}_{\mathfrak{a}} \cup \neg \mathscr{T}_{\mathfrak{a}}) \times \mathscr{L}_{\mathfrak{a}}[\Omega] \times \{\subseteq, \supseteq\}$*, then*

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] &\overset{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq, \supseteq\big)\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] &\overset{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq\big)\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] &\overset{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \supseteq\big)\big\}, \end{aligned} \tag{2.6}$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-sets,* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-open sets and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-closed sets of category* ν *in* $\mathfrak{T}_{\mathfrak{a}}$*.*

In particular, $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] \overset{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},0}; \subseteq\big)\big\}$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] \overset{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},0}; \supseteq\big)\big\}$ denote the classes of all $\mathfrak{T}_{\mathfrak{a}}$-open and $\mathfrak{T}_{\mathfrak{a}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{a}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E} \in \{\mathrm{O},\mathrm{K}\}} \mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ [15, 16]. Clearly,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] &\overset{\mathrm{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] \\ &= \bigcup_{(\nu,\mathrm{E}) \in I_3^0 \times \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E} \in \{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]. \end{aligned}$$

By virtue of the foregoing descriptions, $\mathscr{S}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-open or $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closed of category ν ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-open* or $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$*-closed*) if and only if there exist $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg \mathscr{T}_{\mathfrak{g}}$ such that

$$\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big), \tag{2.7}$$

where

$$\mathbf{op}_{\mathfrak{g},\nu} = \big(\mathrm{op}_{\mathfrak{g},\nu}, \neg\, \mathrm{op}_{\mathfrak{g},\nu}\big) \overset{\mathrm{def}}{=} \begin{cases} \big(\mathrm{int}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}}\big) & (\nu = 0), \\ \big(\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big) & (\nu = 1), \\ \big(\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big) & (\nu = 2), \\ \big(\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big) & (\nu = 3). \end{cases}$$

Thus, $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}$, $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ are of categories 0, 1, 2, 3, respectively, if and only if

$$\begin{aligned} &\big(\mathscr{R}_{\mathfrak{g}} \subseteq \mathrm{int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{R}_{\mathfrak{g}} \supseteq \mathrm{cl}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \\ &\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \\ &\big(\mathscr{U}_{\mathfrak{g}} \subseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{U}_{\mathfrak{g}} \supseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \\ &\big(\mathscr{V}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{V}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \end{aligned} \tag{2.8}$$

for some $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg \mathscr{T}_{\mathfrak{g}}$. The notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separateness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connectedness of category $\nu \in I_3^0$ are based on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of the same category ν.

Definition 2.4 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Connected [16])**.** *A* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-separation of category* ν *of two nonempty* $\mathfrak{T}_{\mathfrak{a}}$*-sets* $\mathscr{R}_{\mathfrak{a}}$*,* $\mathscr{S}_{\mathfrak{a}} \subseteq \mathfrak{T}_{\mathfrak{a}}$ *of a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is realised if and only if there exists either* $(\mathscr{O}_{\mathfrak{a},\xi}, \mathscr{O}_{\mathfrak{a},\zeta}) \in \times_{\alpha \in I_2^*} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{a}}\big]$ *or* $(\mathscr{K}_{\mathfrak{a},\xi}, \mathscr{K}_{\mathfrak{a},\zeta}) \in \times_{\alpha \in I_2^*} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{a}}\big]$ *such that:*

$$\Big(\bigsqcup_{\lambda = \xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{R}_{\mathfrak{a}} \sqcup \mathscr{S}_{\mathfrak{a}}\Big) \bigvee \Big(\bigsqcup_{\lambda = \xi,\zeta} \mathscr{K}_{\mathfrak{a},\lambda} = \mathscr{R}_{\mathfrak{a}} \sqcup \mathscr{S}_{\mathfrak{a}}\Big). \tag{2.9}$$

Otherwise, they are said to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-connected of category* ν*.*

Thus, $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connected if and only if $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separated if and only if $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{a}}]$ where,

$$\begin{aligned}
\mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \Big\{ \mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \big(\forall \big(\mathscr{O}_{\mathfrak{a},\lambda}, \mathscr{K}_{\mathfrak{a},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-O}\big[\mathfrak{T}_{\mathfrak{a}}\big] \times \mathfrak{g}\text{-}\nu\text{-K}\big[\mathfrak{T}_{\mathfrak{a}}\big]\big) \\
&\qquad \Big[\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big)\Big]\Big\}; \qquad (2.10)\\
\mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \Big\{ \mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \big(\exists \big(\mathscr{O}_{\mathfrak{a},\lambda}, \mathscr{K}_{\mathfrak{a},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-O}\big[\mathfrak{T}_{\mathfrak{a}}\big] \times \mathfrak{g}\text{-}\nu\text{-K}\big[\mathfrak{T}_{\mathfrak{a}}\big]\big) \\
&\qquad \Big[\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big)\Big]\Big\}. \qquad (2.11)
\end{aligned}$$

Evidently, by $\Omega \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{a}}]$ or $\Omega \in \mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{a}}]$ is meant a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connection of category ν or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separation of category ν of the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ is realised.

2.2. **Sufficient Preliminaries.** The dual concepts called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators of category ν in $\mathscr{T}_{\mathfrak{a}}$-spaces are presented from set-theoretic and vectorial viewpoints herein.

Definition 2.5 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Interior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Closure Operators)**.** *Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space, let $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{a}}] : \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-open subsets of $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{a}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-open sets, and let $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{a}}] : \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$ be the family of all $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closed supersets of $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{a}}]$ of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closed sets. Then, the one-valued maps of the types*

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \qquad (2.12)\\
\mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}},\\
\mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \qquad (2.13)\\
\mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}}
\end{aligned}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators of category ν. The classes $\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu} : \nu \in I_3^0\}$ and $\mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} : \nu \in I_3^0\}$, respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators.

Remark. *Note that* $\mathfrak{g}\text{-Int}_{\mathfrak{a}},\ \mathfrak{g}\text{-Cl}_{\mathfrak{a}} : \begin{array}{rl} \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-Cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ *are dual $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators because, the first is based on $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{a},1}$, $\mathscr{O}_{\mathfrak{a},2}$, $\ldots$ while the second on $\cap$, $\supseteq$, $\mathscr{K}_{\mathfrak{a},1}$, $\mathscr{K}_{\mathfrak{a},2}$, $\ldots$.*

Definition 2.6 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Vector Operator)**.** *Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, an operator of the type*

$$\begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned} \tag{2.14}$$

on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator of category ν. Then, $\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}) : \ \nu \in I_3^0\}$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark. *Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu})$ are based on $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$, respectively, it follows that $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = \mathbf{ic}_{\mathfrak{a}} \stackrel{\text{def}}{=} (\mathrm{int}_{\mathfrak{a}}, \mathrm{cl}_{\mathfrak{a}})$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. In this way,* $\begin{aligned} \mathbf{ic}_{\mathfrak{a}} : \ \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\mathrm{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned}$ *is called a $\mathfrak{T}_{\mathfrak{g}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.*

3. Main Results

The essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented below.

Lemma 3.1. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \ \nu \in I_\sigma^*\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcap_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$,
- II. $\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcup_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$.

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \ \nu \in I_\sigma^*\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then by virtue of $\mathfrak{T}_{\mathfrak{g}}$-set-theoretic $(\cap, \cup)$-operation, it results that

$$\begin{aligned} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\textstyle\bigcap_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\big] &= \big\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \textstyle\bigcap_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\big\} \\ &= \big\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \ \textstyle\bigwedge_{\nu \in I_\sigma^*} (\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g},\nu})\big\} \\ &= \bigcap_{\nu \in I_\sigma^*} \big\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g},\nu}\big\} = \bigcap_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]; \\ \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\textstyle\bigcup_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\big] &= \big\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{K}_{\mathfrak{g}} \supseteq \textstyle\bigcup_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\big\} \\ &= \big\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \ \textstyle\bigvee_{\nu \in I_\sigma^*} (\mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g},\nu})\big\} \\ &= \bigcup_{\nu \in I_\sigma^*} \big\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g},\nu}\big\} = \bigcup_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]. \end{aligned}$$

The proof of the lemma is complete. □

For any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ and $\mathscr{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$ hold, or alternatively, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently,

$$\big(\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\big) \wedge \big(\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\big).$$

As a consequence of the above lemma, the corollary follows.

Corollary 3.2. *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcap_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\nu}\right]$,
- II. $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcup_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\nu}\right]$.

Remark. *Clearly,* $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} = \emptyset\big] = \{\emptyset\}$ *and* $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \{\Omega\}$ *hold. Moreover,* $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g}} = \Omega\big] = \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *and* $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g}} = \emptyset\big] = \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

Proposition 3.3. *Let* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set and, let* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-interior and a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closure operators in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then, the necessary and sufficient conditions for* $(\xi, \zeta) \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \times \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *to hold in* $\mathfrak{T}_{\mathfrak{g}}$ *are:*

- I. $\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]$,
- II. $\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]$.

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and, let $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose

$$(\xi, \zeta) \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \times \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}\Bigg) \times \Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}\Bigg).$$

Then, since the relations

$$\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \longleftrightarrow \big\{\xi : \big(\exists \mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\xi \in \mathscr{O}_{\mathfrak{g}}\big]\big\},$$

$$\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \longleftrightarrow \big\{\zeta : \big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\zeta \in \mathscr{K}_{\mathfrak{g}}\big]\big\}$$

hold and $\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \supseteq \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]$, and, on the other hand, the relation $\xi \in \mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g},\xi}$ also holds for any $(\xi, \mathscr{O}_{\mathfrak{g},\xi}, \mathscr{K}_{\mathfrak{g},\xi}) \in \mathscr{S}_{\mathfrak{g}} \times \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]$, it follows that

$$\begin{aligned}
\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \big(\exists \mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\xi \in \mathscr{O}_{\mathfrak{g}}\big] \\
&\longleftrightarrow \big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]; \\
\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\zeta \in \mathscr{K}_{\mathfrak{g}}\big] \\
&\longleftrightarrow \big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big].
\end{aligned}$$

Hence, $\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]$ and $\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]$. The proof of the proposition is complete. □

Theorem 3.4. *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then:*

– I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \bigcap_{\nu\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu} \longmapsto \bigcap_{\nu\in I_\sigma^*} \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\nu}\big) \quad \forall\, \mathfrak{g}\text{-Int}_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\,[\mathfrak{T}_{\mathfrak{g}}]$,

– II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \bigcup_{\nu\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu} \longmapsto \bigcup_{\nu\in I_\sigma^*} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\nu}\big) \quad \forall\, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\,[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_\sigma^*\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then for any $\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-I}\,[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-C}\,[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \bigcap_{\nu\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu} \quad &\longmapsto \quad \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcap_{\nu\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right]} \mathscr{O}_{\mathfrak{g}} \\
&= \quad \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \bigcap_{\nu\in I_\sigma^*} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{O}_{\mathfrak{g}} \\
&= \quad \bigcap_{\nu\in I_\sigma^*} \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{O}_{\mathfrak{g}} \Bigg) = \bigcap_{\nu\in I_\sigma^*} \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\nu}\right);\\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \bigcup_{\nu\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu} \quad &\longmapsto \quad \bigcup_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\nu\in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right]} \mathscr{K}_{\mathfrak{g}} \\
&= \quad \bigcup_{\mathscr{K}_{\mathfrak{g}}\in \bigcup_{\nu\in I_\sigma^*} \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{K}_{\mathfrak{g}} \\
&= \quad \bigcup_{\nu\in I_\sigma^*} \Bigg(\bigcup_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]} \mathscr{K}_{\mathfrak{g}} \Bigg) = \bigcup_{\nu\in I_\sigma^*} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\nu}\right).
\end{aligned}$$

The proof of the theorem is complete. □

Theorem 3.5. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

$$\big(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\,[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathscr{S}_{\mathfrak{g}}\big) \wedge \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathscr{S}_{\mathfrak{g}}\big)\big]. \tag{3.1}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\,[\mathfrak{T}_{\mathfrak{g}}]$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, by virtue of the definition of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}\left(\Omega\right) \longrightarrow \mathscr{P}\left(\Omega\right)$, it results that,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \quad \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \quad \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},
\end{aligned}$$

respectively. But, for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]$, the relation $(\mathscr{O}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ holds. Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathscr{S}_{\mathfrak{g}}$. This completes the proof of the theorem. □

A consequence of the above theorem is the following corollary.

Corollary 3.6. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

$$\big(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\,[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big]. \tag{3.2}$$

Remark. *Employing the terminology of Levine, N.* [10], *any* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *which satisfies the relation* $\mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ *for some* $\mathfrak{g}$*-*$\mathfrak{T}_{\mathfrak{g}}$*-open set* $\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ *may well be termed a* $\mathfrak{g}$*-*$\mathfrak{T}_{\mathfrak{g}}$*-semi-open set.*

Proposition 3.7. *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *be a strong* $\mathscr{T}_{\mathfrak{g}}$*-space, then:*

$$\big(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \longmapsto (\Omega, \emptyset)\big]. \tag{3.3}$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, $(\Omega, \emptyset) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and, therefore, Ω is the biggest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subset contained in itself and, $\emptyset$ is the smallest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed superset containing itself. Consequently,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \quad & \longmapsto \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\Omega]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\emptyset]} \mathscr{K}_{\mathfrak{g}}\Bigg) \\
& = \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \{\Omega\} \cup \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\Omega]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \{\emptyset\} \cup \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\emptyset]} \mathscr{K}_{\mathfrak{g}}\Bigg) = (\Omega, \emptyset).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \longmapsto (\Omega, \emptyset)$. The proof of the proposition is complete. □

Proposition 3.8. *If* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\, \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and let $\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{g}}]$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} :\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad & \longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}; \\
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} :\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad & \longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}.
\end{aligned}$$

But, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently,

$$\begin{aligned}
\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} & = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}; \\
\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} & = \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}.
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. This completes the proof of the proposition. □

Proposition 3.9. *If* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, the first and second components of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ operated on $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ gives

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \mathscr{O}_{\mathfrak{g}} \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \left(\mathscr{K}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{K}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},
\end{aligned}$$

respectively. Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

Theorem 3.10. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then, for every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *such that* $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$*:*

$$\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}). \tag{3.4}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be given and $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ be an arbitrary pair of $\mathfrak{T}_{\mathfrak{g}}$-sets. Then, since for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}_{\mathfrak{g}}(\Omega)$, $(\mathscr{O}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$, it follows by virtue of the relation $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ that $(\mathscr{O}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ for any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$. Consequently, it results on the one hand that

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{R}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}} = \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]}\left(\mathscr{K}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\right)\\ &\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\left(\mathscr{K}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right).\end{aligned}$$

These show that the images of $\mathscr{R}_{\mathfrak{g}}$ under $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are subsets of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. ☐

Theorem 3.11. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

$$\left(\forall\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}\right)\left[\left(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\subseteq\left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\right]. \tag{3.5}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then,

$$\begin{aligned}\mathrm{int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\subseteq\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}});\\ \mathrm{cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\supseteq\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Therefore, it follows that the images of $\mathscr{S}_{\mathfrak{g}}$ under $\mathrm{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are subsets of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\left(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\subseteq\left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)$. The proof of the theorem is complete. ☐

Proposition 3.12. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *then, for any* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$,

$$\left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\longrightarrow\left(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right). \tag{3.6}$$

Proof. If $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and, let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But since $\left(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\subseteq\left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)$ it follows that

$$\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. ☐

Remark. *If* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\gtrsim\mathrm{int}_{\mathfrak{g}}$ *stands for* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ *and* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\precsim\mathrm{cl}_{\mathfrak{g}}$*, for* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$*, then the outstanding facts are:* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* finer *(or,* larger*,* stronger*) than* $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *or,* $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* coarser *(or,* smaller*,* weaker*) than* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$*;* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* coarser *(or,* smaller*,* weaker*) than* $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *or,* $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* finer *(or,* larger*,* stronger*) than* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$.

Proposition 3.13. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}} \in \mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *and* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *be any pair of* $\mathscr{T}_{\mathfrak{g}}$*-sets in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

$$(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \;\longrightarrow\; \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathbf{ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}). \tag{3.7}$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}} \in \mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and, let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathbf{ic}_{\mathfrak{g}} : (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \;&\longmapsto\; \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}},\; \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}},\; \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}},\; \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathbf{ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

Proposition 3.14. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

$$\begin{aligned}
\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big) \big[& \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
& \wedge \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big].
\end{aligned} \tag{3.8}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \;&\longmapsto\; \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \\
&\supseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}); \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \;&\longmapsto\; \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, the image of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ under $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a superset of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and that of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ under $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a subset of $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

Theorem 3.15. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\left(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right)\left[\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right]. \tag{3.9}$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then, by virtue of the definition of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}$, it results that,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}\Bigg); \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \\
&\supseteq \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) = \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}\Bigg).
\end{aligned}$$

But since

$$\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},\Bigg) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}},$$

it follows, consequently, that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. This proves the theorem. ☐

Corollary 3.16. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\big[\Omega\big]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then there exists* $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ *such that:*

$$\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right] \wedge \left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right]. \tag{3.10}$$

In view of THMS 3.2, 3.4 and PROPS 3.7, 3.8, it follows immediately that the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively possess similar properties analogous to the *Kuratowski closure Axioms* which can be grouped and stated in the form of a corollary.

Corollary 3.17. *Let* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-interior and a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closure operators in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega) = \Omega$,
  - II. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}$,
  - III. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - IV. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.
- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - V. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset) = \emptyset$,
  - VI. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}$,
  - VII. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - VIII. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Some nice Mathematical vocabulary follow. In Cor. 3.17, Items I., II., III. and IV. state that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω-*grounded*, *non-expansive*, *idempotent* and $\cap$-*additive*, respectively. Items V., VI., VII. and VIII. state that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$-*grounded*, *expansive*, *idempotent* and $\cup$-*additive*, respectively.

The axiomatic definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces follow.

Definition 3.1 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior Operator)**.** *A one-valued map of the type* $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is called a "*$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior operator" on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *if and only if, for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$*, it satisfies the following axioms:*

- Ax. I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}$,
- Ax. II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Thus, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a non-expansive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\text{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}\right] \wedge \left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right] \tag{3.11}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$.

Definition 3.2 (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operator)**.** *A one-valued map of the type* $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is called a "*$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closure operator" on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *if and only if, for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$*, it satisfies the following axioms:*

- Ax. I. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}$,
- Ax. II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Hence, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is an expansive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}\right] \wedge \left[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right] \tag{3.12}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$.

4. Discussion

4.1. **Categorical Classifications.** The notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators of category ν have been defined in terms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of the same category ν. Having adopted such a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, the twofold purposes here are, firstly, to establish the various relationships amongst the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and secondly, to illustrate them through diagrams.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$, $\text{op}_{\mathfrak{a},0}(\mathscr{O}_{\mathfrak{a}}) \subseteq \text{op}_{\mathfrak{a},1}(\mathscr{O}_{\mathfrak{a}}) \subseteq \text{op}_{\mathfrak{a},3}(\mathscr{O}_{\mathfrak{a}}) \supseteq \text{op}_{\mathfrak{a},2}(\mathscr{O}_{\mathfrak{a}})$ for every $\mathscr{O}_{\mathfrak{a}} \in \text{O}[\mathfrak{T}_{\mathfrak{a}}]$. Consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{a},0}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{a},1}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{a},3}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{a},2}(\mathscr{S}_{\mathfrak{a}})$ for any $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{T}_{\mathfrak{a}}$. But, $\mathscr{O}_{\mathfrak{a}} \subseteq \text{op}_{\mathfrak{o},\nu}(\mathscr{O}_{\mathfrak{a}}) \subseteq \text{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{a}})$ for every $\nu \in I_3^0$, implying $\mathfrak{g}\text{-Int}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}})$ for any $(\nu,\mathscr{S}_{\mathfrak{a}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$. Thus, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$\begin{array}{ccccccc} \mathscr{O}_{\mathfrak{a}} & = & \mathscr{O}_{\mathfrak{a}} & = & \mathscr{O}_{\mathfrak{a}} & = & \mathscr{O}_{\mathfrak{a}} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathrm{op}_{\mathfrak{o},0}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \mathrm{op}_{\mathfrak{o},1}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \mathrm{op}_{\mathfrak{o},3}(\mathscr{O}_{\mathfrak{a}}) & \supseteq & \mathrm{op}_{\mathfrak{o},2}(\mathscr{O}_{\mathfrak{a}}) \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathrm{op}_{\mathfrak{g},0}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \mathrm{op}_{\mathfrak{g},1}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \mathrm{op}_{\mathfrak{g},3}(\mathscr{O}_{\mathfrak{a}}) & \supseteq & \mathrm{op}_{\mathfrak{g},2}(\mathscr{O}_{\mathfrak{a}}). \end{array} \tag{4.1}$$

In FIG. 1, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Int}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}}$ and $\{\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; FIG. 1 may well be called a $(\mathfrak{g}\text{-Int}_{\mathfrak{o}}, \mathfrak{g}\text{-Int}_{\mathfrak{g}})$*-valued diagram.*

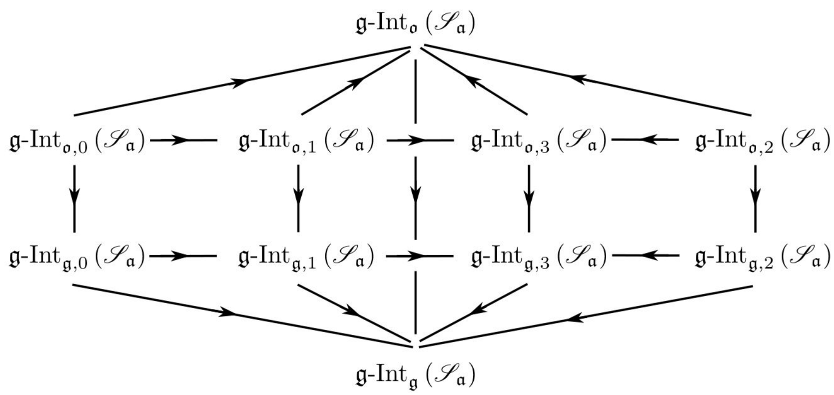

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-interior operators in $\mathscr{T}_{\mathfrak{o}}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$, $\neg\,\mathrm{op}_{\mathfrak{a},0}(\mathscr{K}_{\mathfrak{a}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{a},1}(\mathscr{K}_{\mathfrak{a}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{a},3}(\mathscr{K}_{\mathfrak{a}}) \subseteq \neg\,\mathrm{op}_{\mathfrak{a},2}(\mathscr{K}_{\mathfrak{a}})$ for every $\mathscr{K}_{\mathfrak{a}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. Consequently, $\mathfrak{g}\text{-Cl}_{\mathfrak{a},0}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{a},1}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{a},3}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{a},2}(\mathscr{S}_{\mathfrak{a}})$ for any $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{T}_{\mathfrak{a}}$. But, $\mathscr{K}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{o},\nu}(\mathscr{K}_{\mathfrak{a}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{a}})$ for every $\nu \in I_3^0$, implying, $\mathfrak{g}\text{-Cl}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}})$ for any $(\nu,\mathscr{S}_{\mathfrak{a}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$. Hence, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$\begin{array}{ccccccc} \mathscr{K}_{\mathfrak{a}} & = & \mathscr{K}_{\mathfrak{a}} & = & \mathscr{K}_{\mathfrak{a}} & = & \mathscr{K}_{\mathfrak{a}} \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \neg\,\mathrm{op}_{\mathfrak{o},0}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\mathrm{op}_{\mathfrak{o},1}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\mathrm{op}_{\mathfrak{o},3}(\mathscr{K}_{\mathfrak{a}}) & \subseteq & \neg\,\mathrm{op}_{\mathfrak{o},2}(\mathscr{K}_{\mathfrak{a}}) \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \neg\,\mathrm{op}_{\mathfrak{g},0}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathscr{K}_{\mathfrak{a}}) & \subseteq & \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathscr{K}_{\mathfrak{a}}). \end{array} \tag{4.2}$$

In FIG. 2, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Cl}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}}$ and $\{\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; FIG. 2 may well be called a $(\mathfrak{g}\text{-Cl}_{\mathfrak{o}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}})$*-valued diagram.*

As in the works of other authors [44, 45, 46, 47], the manner we have positioned the arrows in the $(\mathfrak{g}\text{-Int}_{\mathfrak{o}}, \mathfrak{g}\text{-Int}_{\mathfrak{g}})$, $(\mathfrak{g}\text{-Cl}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}})$*-valued diagrams* (FIGS 1, 2) is solely to stress that, in general, the implications in FIGS 1, 2 are irreversible.

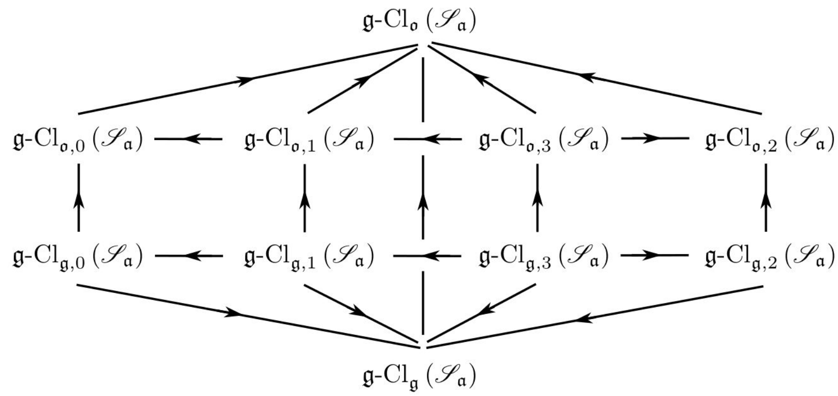

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-closure operators in $\mathscr{T}_{\mathfrak{o}}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

4.2. **A Nice Application.** The focus is on essential concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in an attempt to shed lights on the essential properties established in the earlier sections. Let $\Omega = \{\xi_\nu : \nu \in I_5^*\}$ denotes the underlying set and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where Ω is topologized by the choice:

$$\begin{aligned} \mathscr{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset,\ \{\xi_1\},\ \{\xi_1, \xi_3, \xi_5\},\ \Omega\} \qquad (4.3)\\ &= \{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\};\\ \neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \{\Omega,\ \{\xi_2, \xi_3, \xi_4, \xi_5\},\ \{\xi_2, \xi_4\}, \emptyset\} \qquad (4.4)\\ &= \{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\}. \end{aligned}$$

Evidently, $\mathscr{T}_{\mathfrak{g}}, \neg\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_\nu : \nu \in I_5^*\})$ establish the classes of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively. Since conditions $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, $\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_4^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_4^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ are satisfied, $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_\nu : \nu \in I_5^*\})$ is a strong $\mathfrak{g}$-topology and hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space. Because $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_4^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_4^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ is satisfied, $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_\nu : \nu \in I_5^*\})$ is also an $\mathfrak{o}$-topology and thus, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$. Moreover, $\mathscr{O}_{\mathfrak{g},\mu} \in \mathfrak{g}$-$\nu$-$\mathrm{O}[\mathfrak{T}_{\mathfrak{o}}]$ for every $(\nu, \mu) \in I_3^0 \times I_4^*$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\{\xi_\nu : \nu \in I_5^*\})$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-open sets relative to the $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$.

For convenience of notation, express $\mathscr{P}(\Omega)$ in set-builder notation as a collection indexed by the Cartesian product $I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}$:

$$\mathscr{P}(\Omega) = \{\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega) : (\nu, \mu) \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}\}, \qquad (4.5)$$

where $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega)$ denotes a $\mathfrak{T}_{\mathfrak{g}}$-set labeled $\nu \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}$ and containing $\mu \in I^0_{\mathrm{card}(\Omega)}$ elements. Below is established the indexing by the Cartesian product $I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}$ by the choice: $\mathscr{S}_{\mathfrak{g},(1,0)} = \emptyset$, ..., $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} = \{\xi_1, \xi_2, \ldots, \xi_\mu\}$, ..., $\mathscr{S}_{\mathfrak{g},(32,5)} = \Omega$.

For $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}}) \in \{0, 5\}$, let $\mathscr{S}_{\mathfrak{g},(1,0)} = \emptyset$ and $\mathscr{S}_{\mathfrak{g},(32,5)} = \Omega$. For $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}}) \in \{1, 4\}$, let $\mathscr{S}_{\mathfrak{g},(2,1)} = \{\xi_1\}$, $\mathscr{S}_{\mathfrak{g},(3,1)} = \{\xi_2\}$,

$\mathscr{S}_{\mathfrak{g},(4,1)} = \{\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(5,1)} = \{\xi_4\}$, and $\mathscr{S}_{\mathfrak{g},(6,1)} = \{\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(27,4)} = \{\xi_1,\xi_2,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(28,4)} = \{\xi_2,\xi_3,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(29,4)} = \{\xi_1,\xi_3,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(30,4)} = \{\xi_1,\xi_2,\xi_3,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(31,4)} = \{\xi_1,\xi_2,\xi_4,\xi_5\}$. For $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\operatorname{card}(\mathscr{S}_{\mathfrak{g}}) \in \{2,3\}$, let $\mathscr{S}_{\mathfrak{g},(7,2)} = \{\xi_1,\xi_2\}$, $\mathscr{S}_{\mathfrak{g},(8,2)} = \{\xi_1,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(9,2)} = \{\xi_1,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(10,2)} = \{\xi_1,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(11,2)} = \{\xi_2,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(12,2)} = \{\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(13,2)} = \{\xi_2,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(14,2)} = \{\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(15,2)} = \{\xi_3,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(16,2)} = \{\xi_4,\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(17,3)} = \{\xi_1,\xi_2,\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(18,3)} = \{\xi_1,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(19,3)} = \{\xi_1,\xi_4,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(20,3)} = \{\xi_1,\xi_2,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(21,3)} = \{\xi_1,\xi_2,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(22,3)} = \{\xi_1,\xi_3,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(23,3)} = \{\xi_2,\xi_3,\xi_4\}$, $\mathscr{S}_{\mathfrak{g},(24,3)} = \{\xi_2,\xi_3,\xi_5\}$, $\mathscr{S}_{\mathfrak{g},(25,3)} = \{\xi_3,\xi_4,\xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(26,3)} = \{\xi_2,\xi_4,\xi_5\}$.

Then, from a series of calculations it results that

$$\begin{aligned} \operatorname{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) &= \mathscr{S}_{\mathfrak{g},(\nu,\mu)} \\ &= \mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \end{aligned} \tag{4.6}$$

for every $(\nu,\mu) \in I^{*}_{\operatorname{card}(\mathscr{P}(\Omega))} \times I^{0}_{\operatorname{card}(\Omega)}$. On inspecting Eq. (4.6), some interesting features can be remarked and thus, some interesting conclusions can be drawn.

Having ordered the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators $\operatorname{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, by setting $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} \gtrsim \operatorname{int}_{\mathfrak{g}}$ if and only if $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\operatorname{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, by setting $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} \lesssim \operatorname{cl}_{\mathfrak{g}}$ if and only if $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, where $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ is arbitrary, Eq. (4.6), then, is but a result validating the following outstanding facts: $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

If the discussions of this nice application be explored a step further, other interesting conclusions can be drawn.

5. Conclusion

In this paper, the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were presented in as general and unified a manner as possible and, their essential properties were discussed in such a way as to show that much of the fundamental structure of $\mathscr{T}_{\mathfrak{g}}$-spaces is better considered for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ than for the $\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$-closure operators $\operatorname{int}_{\mathfrak{g}}$, $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. If $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} \gtrsim \operatorname{int}_{\mathfrak{g}}$ stands for $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} \lesssim \operatorname{cl}_{\mathfrak{g}}$, for $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then the outstanding facts are: $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

Moreover, the paper offers very nice features for the passage from $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(interior, closure) to $\mathfrak{T}_{\mathfrak{g}}$-(interior, closure) operators, respectively. Hence, several concepts and proven results it contained hold equally well when $(\Omega,\mathscr{T}_{\mathfrak{g}}) = (\Omega,\mathscr{T}_{\mathfrak{o}})$, while adapting other set-theoretic and topological features accordingly. For instance, the theoretical framework categorises $\big(\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}}), \mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big)$ as a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closed sets of categories ν, where $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ and $(\nu,\mathfrak{a}) \in I^{0}_{3} \times \{\mathfrak{o},\mathfrak{g}\}$, and theorises the concepts in a unified way.

The study of the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces will be presented in a subsequent paper, and the discussion of this paper ends here.

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GENERALIZED TOPOLOGICAL OPERATOR THEORY IN GENERALIZED TOPOLOGICAL SPACES

PART II. GENERALIZED INTERIOR AND GENERALIZED CLOSURE

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Abstract. In a recent paper (Cf. [19]), we have presented the definitions and the essential properties of the generalized topological operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*interior* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closure operators*) in a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$-*space*). Principally, we have shown that $(\mathfrak{g}\text{-Int}_{\mathfrak{g}},\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}) : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ is $(\Omega,\emptyset)$-*grounded*, (*expansive, non-expansive*), (*idempotent, idempotent*) and $(\cap,\cup)$-*additive*. We have also shown that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. In this paper, we study the commutativity of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}$-sets having some $(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}})$-based properties ($\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-*properties*) in $\mathscr{T}_{\mathfrak{g}}$-spaces. The main results of the study are: The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *duals* and $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under their $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operations*. A $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-*property* is *equivalent* to the $\mathfrak{T}_{\mathfrak{g}}$-set or its complement having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-*property*. The $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\cup$-*operation* and $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\{\cup,\cap,\complement\}$-*operations*. Finally, a $\mathfrak{T}_{\mathfrak{g}}$-set having $\{\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}\}$-*property* also has $\{\mathfrak{P}_{\mathfrak{g}}, \mathfrak{Q}_{\mathfrak{g}}\}$-*property*.

1. Introduction

Many mathematicians have studied several kinds of ordinary and generalized topological operators ($\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*operators*) in ordinary ($\mathfrak{a} = \mathfrak{o}$) and generalized ($\mathfrak{a} = \mathfrak{g}$) topological spaces ($\mathscr{T}_{\mathfrak{a}}$-*spaces*) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Jung and Nam [3] have used the $\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{T}_{\mathfrak{o}}$-closure operators $(\cdot)^{\circ}$, $\overline{(\cdot)}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to establish several necessary and sufficient conditions related

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to openness and closeness properties of sets in a $\mathscr{T}_{\mathfrak{o}}$-space. Lei and Zhang [4] have considered the $\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{T}_{\mathfrak{o}}$-closure operators $\mathbf{Int}$, $\mathbf{Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in studying some topological characterizations axiomatically in $\mathscr{T}_{\mathfrak{o}}$-spaces. Gupta and Sarma [5] have established a variety of generalized sets ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*) under the possible compositions of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators i_{γ}, $c_{\gamma} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (γ-*interior* and γ-*closure operators*), respectively, where $\gamma \in \{\alpha, \beta, \pi, \sigma\}$, in $\mathscr{T}_{\mathfrak{g}}$-spaces. Rajendiran and Thamilselvan [6] have studied the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-closure operators $\mathrm{g}^{*}\mathrm{s}^{*}\mathrm{Int}$, $\mathrm{g}^{*}\mathrm{s}^{*}\mathrm{Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($g^{*}s^{*}$-*interior* and $g^{*}s^{*}$-*closure operators*), respectively, in $\mathscr{T}_{\mathfrak{o}}$-spaces. In $\mathscr{T}_{\mathfrak{g}}$-spaces, Tyagi and Choudhary [7] have study stronger forms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $I_{(\cdot)}$, $C_{(\cdot)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ while Pankajam, V. [9] has presented some properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators i_{δ}, $\mathrm{c}_{\delta} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (δ-*interior* and δ-*closure operators*), respectively, to mention but a few references.

Despite these references, in regard to the study of the commutativity of $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$), the literature is, to our knowledge, almost void of studies in this direction [17, 16]. Levine, N. [17] has studied the commutativity of the $\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{T}_{\mathfrak{o}}$-closure operators $\mathrm{int}_{\mathfrak{o}}$, $\mathrm{cl}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{a}}$-space. Staley, D. H. [16] has presented some characterizations of ordinary sets ($\mathfrak{T}_{\mathfrak{o}}$-*sets*) for which the $\mathfrak{T}_{\mathfrak{o}}$-interior operator $\mathrm{int}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ commutes with the $\mathfrak{T}_{\mathfrak{o}}$-boundary operator $\mathrm{bd}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{o}}$-space. In general, since $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}}) \neq (\Omega, \mathscr{T}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}}$ by virtue of $\mathscr{T}_{\mathfrak{o}} \neq \mathscr{T}_{\mathfrak{g}}$ and, $\left(\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\right) \neq \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a}}\right)$ for each $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, so it seems reasonable to expect the existence of nice and interesting results in a $\mathscr{T}_{\mathfrak{g}}$-space with respect to those established by Levine, N. [17] and Staley, D. H. [16] in a $\mathscr{T}_{\mathfrak{o}}$-space.

Having made the study of the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces one subject of inquiry (Cf. [19]), the study of the commutativity properties of these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces may be made another subject of inquiry. In this paper, we endeavor to undertake such inquiry.

The rest of the paper is structured as thus: In Sect. 2, necessary and sufficient preliminary notions are described in Subsects 2.1, 2.2 and the main results are reported in Sect. 3. In Sect. 4, the establishment of the various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed in Sects 4.1. To support the work, a nice application of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, the work is concluded in Sect. 5.

2. Theory

2.1. Necessary Preliminaries.

As in Part I. (Cf. [19]), the standard reference for notations and concepts is the Ph.D. Thesis of Khodabocus, M. I. [2].

Herein, $\mathfrak{U}$ symbolizes the *universe* of discourse, fixed within the framework of $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator theory in $\mathscr{T}_{\mathfrak{a}}$-spaces, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, and containing *underlying sets*, *underlying subsets*, and so forth. By convention, the relation $(\alpha_1, \alpha_2, \ldots)\,\mathrm{R}\,\mathscr{A}_1 \times \mathscr{A}_2 \times \cdots$ means $\alpha_1\,\mathrm{R}\,\mathscr{A}_1$, $\alpha_2\,\mathrm{R}\,\mathscr{A}_2$, … where $\mathrm{R} = \in, \subset, \supset, \ldots$. The pairs $\left(I_n^0, I_n^*\right) \subset \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ and $\left(I_\infty^0, I_\infty^*\right) \sim \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ are pairs of *finite* and *infinite index sets* [1, 2].

Definition 2.1 ($\mathscr{T}_{\mathfrak{a}}$-Space [1, 2])**.** *A $\mathscr{T}_{\mathfrak{a}}$-space is a topological structure* $\mathfrak{T}_{\mathfrak{a}} \overset{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{a}})$ *in which $\Omega \subset \mathfrak{U}$ is an underlying set and* $\begin{array}{rccc} \mathscr{T}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{O}_{\mathfrak{a}} & \longmapsto & \mathscr{T}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}}) \end{array}$ *is* *an $\mathfrak{a}$-topology satisfying the compound $\mathscr{T}_{\mathfrak{a}}$-axiom:*

$$\mathrm{Ax}\left(\mathscr{T}_{\mathfrak{a}}\right) \overset{\text{def}}{\longleftrightarrow} \begin{cases} \left(\mathscr{T}_{\mathfrak{o}}(\emptyset)=\emptyset\right) \wedge \left(\mathscr{T}_{\mathfrak{o}}\left(\mathscr{O}_{\mathfrak{o},\nu}\right) \subseteq \mathscr{O}_{\mathfrak{o},\nu}\right) \\ \qquad \wedge\left(\mathscr{T}_{\mathfrak{o}}\left(\bigcap_{\nu\in I_n^*}\mathscr{O}_{\mathfrak{o},\nu}\right) = \bigcap_{\nu\in I_n^*}\mathscr{T}_{\mathfrak{o}}\left(\mathscr{O}_{\mathfrak{o},\nu}\right)\right) \\ \qquad \wedge\left(\mathscr{T}_{\mathfrak{o}}\left(\bigcup_{\nu\in I_\infty^*}\mathscr{O}_{\mathfrak{o},\nu}\right) = \bigcup_{\nu\in I_\infty^*}\mathscr{T}_{\mathfrak{o}}\left(\mathscr{O}_{\mathfrak{o},\nu}\right)\right) \quad (\mathfrak{a}=\mathfrak{o}), \\ \\ \left(\mathscr{T}_{\mathfrak{g}}(\emptyset)=\emptyset\right) \wedge \left(\mathscr{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right) \subseteq \mathscr{O}_{\mathfrak{g},\nu}\right) \\ \qquad \wedge\left(\mathscr{T}_{\mathfrak{g}}\left(\bigcup_{\nu\in I_\infty^*}\mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu\in I_\infty^*}\mathscr{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)\right) \quad (\mathfrak{a}=\mathfrak{g}). \end{cases}$$

By assumption, the $\mathscr{T}_{\mathfrak{a}}$-space is void of any $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separation axioms (*ordinary* and *generalized separation axioms*) unless otherwise stated [1, 2, 20]. If $\mathfrak{a}=\mathfrak{o}$ (*ordinary*), then $\mathrm{Ax}\left(\mathscr{T}_{\mathfrak{o}}\right)$ stands for an $\mathfrak{o}$-topology (*ordinary topology*) and $\mathfrak{T}_{\mathfrak{o}} = (\Omega,\mathscr{T}_{\mathfrak{o}}) = (\Omega,\mathscr{T}) = \mathfrak{T}$ is called a $\mathscr{T}_{\mathfrak{o}}$-space (*ordinary topological space*) and if $\mathfrak{a}=\mathfrak{g}$ (*generalized*), then $\mathrm{Ax}\left(\mathscr{T}_{\mathfrak{g}}\right)$ stands for a $\mathfrak{g}$-topology (*generalized topology*) and $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ is called a $\mathscr{T}_{\mathfrak{g}}$-space (*generalized topological space*). If $\Omega\in\mathscr{T}_{\mathfrak{g}}$, then $\mathfrak{T}_{\mathfrak{a}}$ is a *strong* $\mathscr{T}_{\mathfrak{a}}$-space [2, 21, 22] and if $\mathscr{T}_{\mathfrak{g}}\left(\bigcap_{\nu\in I_n^*}\mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcap_{\nu\in I_n^*}\mathscr{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)$ for any $I_n^*\subset I_\infty^*$, then $\mathfrak{T}_{\mathfrak{g}}$ is a *quasi* $\mathscr{T}_{\mathfrak{g}}$-space [2, 23]. The notations $\Gamma\subset\Omega$, $\mathscr{O}_{\mathfrak{a}}\in\mathscr{T}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}}\in\neg\mathscr{T}_{\mathfrak{a}} \overset{\text{def}}{=} \left\{\mathscr{K}_{\mathfrak{a}}:\ \complement_{\Omega}\left(\mathscr{K}_{\mathfrak{a}}\right)\in\mathscr{T}_{\mathfrak{a}}\right\}$ and $\mathscr{S}_{\mathfrak{a}}\subset\mathfrak{T}_{\mathfrak{a}}$ state that Γ, $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}}$ and $\mathscr{S}_{\mathfrak{a}}$ are a Ω-*subset*, $\mathscr{T}_{\mathfrak{a}}$-*open set*, $\mathscr{T}_{\mathfrak{a}}$-*closed set* and $\mathfrak{T}_{\mathfrak{a}}$-*set*, respectively [1, 2]. The operators $\begin{array}{rrl}\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathrm{int}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right),\ \mathrm{cl}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ are the $\mathfrak{T}_{\mathfrak{a}}$-*interior* and $\mathfrak{T}_{\mathfrak{a}}$-*closure operators*, respectively [1, 2]. For convenience of notation, let $\left(\mathscr{P}^*,\mathscr{T}_{\mathfrak{a}}^*,\neg\mathscr{T}_{\mathfrak{a}}^*\right)(\Omega) = \left(\mathscr{P}\setminus\{\emptyset\},\mathscr{T}_{\mathfrak{a}}\setminus\{\emptyset\},\neg\mathscr{T}_{\mathfrak{a}}\setminus\{\emptyset\}\right)(\Omega)$.

Definition 2.2 ($\mathfrak{g}$-Operation [1, 2])**.** *A mapping* $\begin{array}{rrl}\mathrm{op}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ *is called a $\mathfrak{g}$-operation if and only if the following statements hold:*

$$\begin{aligned}&\left(\forall\mathscr{S}_{\mathfrak{a}}\in\mathscr{P}^*(\Omega)\right)\left(\exists\left(\mathscr{O}_{\mathfrak{a}},\mathscr{K}_{\mathfrak{a}}\right)\in\mathscr{T}_{\mathfrak{a}}^*\times\neg\mathscr{T}_{\mathfrak{a}}^*\right)\left[\left(\mathrm{op}_{\mathfrak{a}}(\emptyset)=\emptyset\right)\vee\left(\neg\,\mathrm{op}_{\mathfrak{a}}(\emptyset)=\emptyset\right)\right.\\ &\qquad\left.\vee\left(\mathscr{S}_{\mathfrak{a}}\subseteq\mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right)\vee\left(\mathscr{S}_{\mathfrak{a}}\supseteq\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)\right], \quad (2.1)\end{aligned}$$

where $\begin{array}{rrl}\neg\,\mathrm{op}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\end{array}$ *is called its complementary $\mathfrak{g}$-operation, and for all $\mathfrak{T}_{\mathfrak{a}}$-sets $\mathscr{S}_{\mathfrak{a}}$, $\mathscr{S}_{\mathfrak{a},\nu}$, $\mathscr{S}_{\mathfrak{a},\mu}\in\mathscr{P}^*(\Omega)$, the following axioms are satisfied:*

- Ax. I. $\left(\mathscr{S}_{\mathfrak{a}}\subseteq\mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right)\vee\left(\mathscr{S}_{\mathfrak{a}}\supseteq\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)$,
- Ax. II. $\left(\mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\subseteq\mathrm{op}_{\mathfrak{a}}\circ\mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right)\vee\left(\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{a}}\circ\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)$,
- Ax. III. $\left(\mathscr{S}_{\mathfrak{a},\nu}\subseteq\mathscr{S}_{\mathfrak{a},\mu}\ \longrightarrow\ \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a},\nu}\right)\subseteq\mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a},\mu}\right)\right)$
  $\vee\left(\mathscr{S}_{\mathfrak{a},\mu}\subseteq\mathscr{S}_{\mathfrak{a},\nu}\ \longleftarrow\ \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a},\mu}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a},\nu}\right)\right)$,
- Ax. IV. $\left(\mathrm{op}_{\mathfrak{a}}\left(\bigcup_{\sigma=\nu,\mu}\mathscr{S}_{\mathfrak{a},\sigma}\right)\subseteq\bigcup_{\sigma=\nu,\mu}\mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a},\sigma}\right)\right)$
  $\vee\left(\neg\,\mathrm{op}_{\mathfrak{a}}\left(\bigcup_{\sigma=\nu,\mu}\mathscr{S}_{\mathfrak{a},\sigma}\right)\supseteq\bigcup_{\sigma=\nu,\mu}\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a},\sigma}\right)\right)$,

for some $\mathscr{T}_{\mathfrak{a}}$-sets $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{O}_{\mathfrak{a},\nu}$, $\mathscr{O}_{\mathfrak{a},\mu}\in\mathscr{T}_{\mathfrak{a}}^*$ and $\mathscr{K}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a},\nu}$, $\mathscr{K}_{\mathfrak{a},\mu}\in\neg\mathscr{T}_{\mathfrak{a}}^*$.

The class $\mathscr{L}_{\mathfrak{a}}[\Omega] \overset{\text{def}}{=} \left\{\mathbf{op}_{\mathfrak{a},\nu} = \left(\mathrm{op}_{\mathfrak{a},\nu},\neg\,\mathrm{op}_{\mathfrak{a},\nu}\right):\ \nu\in I_3^0\right\} \subseteq \mathscr{L}_{\mathfrak{a}}^{\omega}[\Omega]\times\mathscr{L}_{\mathfrak{a}}^{\kappa}[\Omega] = \left\{\mathrm{op}_{\mathfrak{a},\nu}:\ \nu\in I_3^0\right\}\times\left\{\neg\,\mathrm{op}_{\mathfrak{a},\nu}:\ \nu\in I_3^0\right\}$, where

$$\begin{aligned}\left\langle\mathrm{op}_{\mathfrak{a},\nu}:\ \nu\in I_3^0\right\rangle &= \left\langle\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}}\right\rangle,\\ \left\langle\neg\,\mathrm{op}_{\mathfrak{a},\nu}:\ \nu\in I_3^0\right\rangle &= \left\langle\mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}}\right\rangle,\end{aligned}$$

is the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$.

Definition 2.3 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Sets [1, 2]). *Let* $\left(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}, \mathbf{op}_{\mathfrak{a},\nu}\right) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{a}} \times \neg\mathscr{T}_{\mathfrak{a}} \times \mathscr{L}_{\mathfrak{a}}[\Omega]$ *and let the predicates*

$$\begin{aligned} \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathrm{op}_{\mathfrak{a},\nu}; \subseteq\big) &\stackrel{\text{def}}{=} \big(\exists\left(\mathscr{O}_{\mathfrak{a}}, \mathrm{op}_{\mathfrak{a},\nu}\right) \in \mathscr{T}_{\mathfrak{a}} \times \mathscr{L}_{\mathfrak{a}}^{\omega}[\Omega]\big)\big[\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a},\nu}\left(\mathscr{O}_{\mathfrak{a}}\right)\big], \\ \mathrm{Q}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \neg\,\mathrm{op}_{\mathfrak{a},\nu}; \supseteq\big) &\stackrel{\text{def}}{=} \big(\exists\left(\mathscr{K}_{\mathfrak{a}}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}\right) \in \neg\mathscr{T}_{\mathfrak{a}} \times \mathscr{L}_{\mathfrak{a}}^{\kappa}[\Omega]\big) \\ &\qquad \big[\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a},\nu}\left(\mathscr{K}_{\mathfrak{a}}\right)\big] \end{aligned} \tag{2.2}$$

be Boolean-valued functions on $\mathscr{P}(\Omega) \times (\mathscr{T}_{\mathfrak{a}} \cup \neg\mathscr{T}_{\mathfrak{a}}) \times \left(\mathscr{L}_{\mathfrak{a}}^{\omega} \cup \mathscr{L}_{\mathfrak{a}}^{\kappa}\right)[\Omega] \times \{\subseteq, \supseteq\}$, *then* $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] \,\&\stackrel{\text{def}}{=}\&\, \mathfrak{g}$-$\nu$-$\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] \cup \mathfrak{g}$-$\nu$-$\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$ *is the class of all* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-sets and,*

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathrm{op}_{\mathfrak{a},\nu}; \subseteq\big)\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} :\ \mathrm{Q}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \neg\,\mathrm{op}_{\mathfrak{a},\nu}; \supseteq\big)\big\}, \end{aligned} \tag{2.3}$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-open and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-closed sets of category* ν *in* $\mathfrak{T}_{\mathfrak{a}}$.

Then, $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \big\{\mathscr{S}_{\mathfrak{a}} :\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}; \mathrm{op}_{\mathfrak{a},0}; \subseteq\big)\big\} \cup \big\{\mathscr{S}_{\mathfrak{a}} :\ \mathrm{Q}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}; \neg\,\mathrm{op}_{\mathfrak{a},0}; \supseteq\big)\big\} = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-*open* and $\mathfrak{T}_{\mathfrak{a}}$-*closed sets* in $\mathfrak{T}_{\mathfrak{a}}$ [1, 2]. Further,

$$\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$$

Definition 2.4 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Connected [2]). *A* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-separation of two* $\mathfrak{T}_{\mathfrak{a}}$*-sets* $\emptyset \neq \mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}} \subseteq \mathfrak{T}_{\mathfrak{a}}$ *of a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is realised if and only if there exists either* $(\mathscr{O}_{\mathfrak{a},\xi}, \mathscr{O}_{\mathfrak{a},\zeta}) \in \times_{\alpha\in I_2^*} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{a}}\big]$ *or* $(\mathscr{K}_{\mathfrak{a},\xi}, \mathscr{K}_{\mathfrak{a},\zeta}) \in \times_{\alpha\in I_2^*} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{a}}\big]$ *such that:*

$$\bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{R}_{\mathfrak{a}} \sqcup \mathscr{S}_{\mathfrak{a}}\bigg) \bigvee \bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{a},\lambda} = \mathscr{R}_{\mathfrak{a}} \sqcup \mathscr{S}_{\mathfrak{a}}\bigg). \tag{2.4}$$

Otherwise, $\mathscr{R}_{\mathfrak{a}}$, $\mathscr{S}_{\mathfrak{a}}$ *are said to be* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-connected.*

Thus, $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connected if and only if $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separated if and only if $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{a}}]$ where,

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \bigg\{ &\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \big(\forall\big(\mathscr{O}_{\mathfrak{a},\lambda}, \mathscr{K}_{\mathfrak{a},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{a}}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{a}}\big]\big) \\ &\bigg[\neg\bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\bigg) \bigwedge \neg\bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\bigg)\bigg]\bigg\}; \end{aligned} \tag{2.5}$$

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \bigg\{ &\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \big(\exists\big(\mathscr{O}_{\mathfrak{a},\lambda}, \mathscr{K}_{\mathfrak{a},\lambda}\big)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{a}}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{a}}\big]\big) \\ &\bigg[\bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\bigg) \bigvee \bigg(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\bigg)\bigg]\bigg\}. \end{aligned} \tag{2.6}$$

Definition 2.5 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Interior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Closure Operators [19]). *In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the one-valued maps*

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}} \end{aligned} \tag{2.8}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] : \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ *and* $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] : \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$ *are called $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Then,* $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \nu \in I_3^0\}$ *and* $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \nu \in I_3^0\}$ *are the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively.*

Definition 2.6 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Vector Operator [19]). *In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the two-valued map*

$$\begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big) \end{aligned} \tag{2.9}$$

is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-vector operator. Then, $\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}) : \nu \in I_3^0\}$ *is the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.*

Remark. *For every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = \mathbf{ic}_{\mathfrak{a}} \stackrel{\mathrm{def}}{=} (\mathrm{int}_{\mathfrak{a}}, \mathrm{cl}_{\mathfrak{a}})$ if based on* $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. *Then,* $\begin{aligned} \mathbf{ic}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto (\mathrm{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})) \end{aligned}$ *is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.*

2.2. **Sufficient Preliminaries.** The notions of $\mathfrak{T}_{\mathfrak{a}}$-sets having $\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$-properties and $\mathfrak{Q}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-properties in $\mathscr{T}_{\mathfrak{a}}$-spaces are now presented.

Definition 2.7 (Complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Operator). *Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map*

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}), \end{aligned} \tag{2.10}$$

where $\complement_{\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$, is called a natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

For clarity, $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a}}$ whenever $\mathscr{R}_{\mathfrak{a}} = \Omega$ and $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} = \mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}$ (*natural complement $\mathfrak{T}_{\mathfrak{a}}$-operator*) whenever $\mathscr{R}_{\mathfrak{a}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$.

Definition 2.8 (Symmetric Difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Operator). *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map*

$$\mathfrak{g}\text{-}\mathrm{Sd}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \tag{2.11}$$

$$(\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}})\,\&\longmapsto\&\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}) \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{a},\mathscr{S}_{\mathfrak{a}}}(\mathscr{R}_{\mathfrak{a}})$$

is called the symmetric difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

If $\mathfrak{g}\text{-}\mathrm{Sd}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is based on $\mathrm{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the concept of *symmetric difference* $\mathfrak{T}_{\mathfrak{a}}$*-operator* $\mathrm{Sd}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ presents itself.

Definition 2.9 ($\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{a}}$-Property)**.** *A* $\mathfrak{T}_{\mathfrak{a}}$*-set* $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ *in a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is said to have* $\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$ *if and only if it belongs to:*

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} :\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big\}, \quad (2.12)$$

called the class of all $\mathfrak{T}_{\mathfrak{a}}$*-sets having* $\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$*.*

Then, $\mathrm{P}[\mathfrak{T}_{\mathfrak{a}}] \,\&\stackrel{\mathrm{def}}{=}\&\big\{\mathscr{S}_{\mathfrak{a}} :\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \longleftrightarrow \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big\}$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-sets having $\mathfrak{P}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$. By $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{a}}]$ is meant a $\mathfrak{T}_{\mathfrak{a}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$.

Definition 2.10 ($\mathfrak{g}$-ν-$\mathfrak{Q}_{\mathfrak{a}}$-Property)**.** *A* $\mathfrak{T}_{\mathfrak{a}}$*-set* $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ *in a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is said to have* $\mathfrak{g}$-ν-$\mathfrak{Q}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$ *if and only if it belongs to:*

$$\mathfrak{g}\text{-}\nu\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} :\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{S}_{\mathfrak{a}} \longmapsto \emptyset\big\}, \quad (2.13)$$

called the class of all $\mathfrak{T}_{\mathfrak{a}}$*-set having* $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$*.*

Then, $\mathrm{Nd}[\mathfrak{T}_{\mathfrak{a}}] \,\&\stackrel{\mathrm{def}}{=}\&\big\{\mathscr{S}_{\mathfrak{a}} :\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} : \mathscr{S}_{\mathfrak{a}} \longmapsto \emptyset\big\}$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-sets having $\mathfrak{Q}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$. By $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{a}}]$ is meant a $\mathfrak{T}_{\mathfrak{a}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$.

3. Main Results

The main results relative to the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators, and $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-properties in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented.

Lemma 3.1. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the natural complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator of its components in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{aligned}\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[&\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\ &\wedge\big(\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big].\end{aligned} \quad (3.1)$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given and, let $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator of its components in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for a $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}\Bigg); \\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}\Bigg).\end{aligned}$$

Let $\left\{\mathscr{O}_{\mathfrak{g},\nu}:\ (\forall\nu\in I_{\infty}^{*})\left[\mathscr{O}_{\mathfrak{g},\nu}\subseteq\mathscr{S}_{\mathfrak{g}}\right]\right\}$ and $\left\{\mathscr{K}_{\mathfrak{g},\nu}:\ (\forall\nu\in I_{\infty}^{*})\left[\mathscr{K}_{\mathfrak{g},\nu}\supseteq\mathscr{S}_{\mathfrak{g}}\right]\right\}$ stand for $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\subseteq\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\subseteq\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]}\mathscr{O}_{\mathfrak{g}}\Bigg) &= \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu\in I_{\infty}^{*}}\left(\mathscr{O}_{\mathfrak{g},\nu}\subseteq\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\Bigg)\\
&= \complement_{\Omega}\Bigg(\bigcup_{\nu\in I_{\infty}^{*}}\left(\mathscr{O}_{\mathfrak{g},\nu}\subseteq\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\Bigg)\\
&= \bigcap_{\nu\in I_{\infty}^{*}}\left(\complement_{\Omega}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)\supseteq\complement_{\Omega}\left(\complement_{\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\right)\\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]}\mathscr{K}_{\mathfrak{g}};\\
\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]}\mathscr{K}_{\mathfrak{g}}\Bigg) &= \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu\in I_{\infty}^{*}}\left(\mathscr{O}_{\mathfrak{g},\nu}\subseteq\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\Bigg)\\
&= \complement_{\Omega}\Bigg(\bigcap_{\nu\in I_{\infty}^{*}}\left(\mathscr{K}_{\mathfrak{g},\nu}\supseteq\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\Bigg)\\
&= \bigcup_{\nu\in I_{\infty}^{*}}\left(\complement_{\Omega}\left(\mathscr{K}_{\mathfrak{g},\nu}\right)\subseteq\complement_{\Omega}\left(\complement_{\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\right)\\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]}\mathscr{O}_{\mathfrak{g}}.
\end{aligned}$$

Since $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$ is arbitrary, it follows that, for every $\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$, the relations

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right),\\
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)
\end{aligned}$$

hold. The proof of the lemma is complete. □

Theorem 3.2. *A* $\mathfrak{T}_{\mathfrak{g}}$*-sets* $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *is said to have* $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$*-property in* $\mathfrak{T}_{\mathfrak{g}}$ *if and only if:*

$$\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\in\mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]. \tag{3.2}$$

Proof. Necessity. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \quad & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
= \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
= \quad & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Thus, it follows that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big),$$

and hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$. The condition of the theorem is, therefore, necessary.

Sufficiency. Conversely, suppose $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Set $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then,

$$\mathscr{S}_{\mathfrak{g}} \ \longleftrightarrow\ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ \longleftrightarrow\ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

But $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$ and it in turn implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$. Hence, it follows that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$. The condition of the theorem is, therefore, sufficient. □

Proposition 3.3. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. I. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned}
&\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\quad \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)
\end{aligned}$$

Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]$. The proof of ITEM I. of the proposition is complete.

II. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)
\end{aligned}$$

Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of ITEM II. of the proposition is complete. □

Theorem 3.4. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$, then $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.*

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then:

CASE I. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it follows that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. But $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows, furthermore, that $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and, hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

CASE II. Suppose $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then, by virtue of the above case, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and by virtue of the fact that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is equivalent to $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it results that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of the theorem is complete. □

Theorem 3.5. *Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, where $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$. Then:*

– I. $\Gamma \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *implies* $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$,
– II. $\Gamma \in \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *implies* $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$.

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and let $\big(\mathfrak{g}\text{-Int}_{\mathfrak{g},\Lambda}, \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Lambda}\big) \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right] \times \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Lambda} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$, respectively, where $\Lambda \in \{\Omega, \Gamma\}$. Then:

I. Suppose $\Gamma \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\ &= \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\Gamma\cap\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\ &\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\Gamma]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\Gamma) = \Gamma.\end{aligned}$$

Thus, $\Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\ &\longleftrightarrow \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{O}_{\mathfrak{g}}\cap\Gamma) \\ &\longleftrightarrow \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{O}_{\mathfrak{g}}\cap\Gamma) \\ &\longleftrightarrow \Gamma\cap\left(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}\right) = \Gamma\cap\mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

But $\Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$.

II. Suppose $\Gamma \in \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then,

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \\ &\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\Gamma]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\Gamma) = \Gamma.\end{aligned}$$

Consequently, $\Gamma\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\\ &\longleftrightarrow \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\Gamma)\\ &\longleftrightarrow \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\Gamma)\\ &\longleftrightarrow \Gamma\cap\left(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\right)=\Gamma\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

But $\Gamma\cap\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. ☐

Theorem 3.6. *Let $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set and let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$, then:*

$$\big(\forall\,\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{g}}]\big)\left[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big)=\displaystyle\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\sigma})\right]. \tag{3.3}$$

Proof. Let $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set, let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, suppose $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$. Then, for every $\mathscr{S}_{\mathfrak{g}}\in\{\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\}$,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\\ &\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\alpha}\cup\mathscr{S}_{\mathfrak{g},\beta}]}\mathscr{O}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big)\supseteq\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\sigma})$. Set $\hat{\mathscr{S}}_{\mathfrak{g},\alpha}=\mathscr{S}_{\mathfrak{g},\alpha}\cap\mathscr{Q}_{\mathfrak{g}}$ and $\hat{\mathscr{S}}_{\mathfrak{g},\beta}=\mathscr{S}_{\mathfrak{g},\beta}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$. Then, since $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned}
\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\right] &= \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\sigma=\alpha,\beta}\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right] \\
&= \left\{\mathscr{O}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]:\ \mathscr{O}_{\mathfrak{g}}\subseteq\bigcup_{\sigma=\alpha,\beta}\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right\} \\
&= \left\{\mathscr{O}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]:\ \bigvee_{\sigma=\alpha,\beta}\left(\mathscr{O}_{\mathfrak{g}}\subseteq\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right)\right\} \\
&= \bigcup_{\sigma=\alpha,\beta}\left\{\mathscr{O}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]:\ \mathscr{O}_{\mathfrak{g}}\subseteq\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right\} \\
&= \bigcup_{\sigma=\alpha,\beta}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\right] = \bigcup_{\sigma=\alpha,\beta}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\sigma}\right].
\end{aligned}$$

Therefore, $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\right] = \bigcup_{\sigma=\alpha,\beta}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\sigma}\right]$, as a consequence of the condition $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\left(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\right)$. Taking this fact into account, it follows, moreover, that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\alpha}\cup\mathscr{S}_{\mathfrak{g},\beta}]}\mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\bigcup_{\sigma=\alpha,\beta}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\sigma}]}\mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\sigma=\alpha,\beta}\left(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\sigma}]}\mathscr{O}_{\mathfrak{g}}\right) = \bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\sigma}\right).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\right)\subseteq\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\sigma}\right)$. The proof of the theorem is complete. □

Theorem 3.7. *Let* $\mathfrak{T}_{\mathfrak{g},\Gamma}=(\Gamma,\mathscr{T}_{\mathfrak{g},\Gamma})$ *be a* $\mathscr{T}_{\mathfrak{g}}$*-subspace of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$*, where* $\mathscr{T}_{\mathfrak{g},\Gamma}:\mathscr{P}(\Gamma)\longmapsto\mathscr{T}_{\mathfrak{g},\Gamma}=\left\{\mathscr{O}_{\mathfrak{g}}\cap\Gamma:\ \mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g},\Omega}\right\}$*. If* $\Gamma\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\cap\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *and* $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$*, then* $\mathscr{S}_{\mathfrak{g}}\cap\Gamma\in\mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]$*.*

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Gamma}=(\Gamma,\mathscr{T}_{\mathfrak{g},\Gamma})$ be a $\mathscr{T}_{\mathfrak{g}}$-subspace of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathscr{T}_{\mathfrak{g},\Omega})$ and, suppose $\Gamma\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\cap\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ and $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Then, since $\Gamma\in\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]\cap\mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ implies $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right)=\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right)=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}:\mathscr{S}_{\mathfrak{g}}\cap\Gamma &\longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\cap\Gamma\right) \\
&\subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right).
\end{aligned}$$

Since $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, it follows, moreover, that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}:\mathscr{S}_{\mathfrak{g}}\longmapsto\emptyset$. Consequently, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma}\circ\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}:\mathscr{S}_{\mathfrak{g}}\cap\Gamma\longmapsto\emptyset$ and hence, $\mathscr{S}_{\mathfrak{g}}\cap\Gamma\in\mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]$. The proof of the theorem is complete. □

Theorem 3.8. *In order that a* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ *in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *satisfies the condition* $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$*, it is necessary and sufficient that there exist a*

$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property such that it be expressible as:

$$\mathscr{S}_{\mathfrak{g}}=(\mathscr{Q}_{\mathfrak{g}}-\mathscr{R}_{\mathfrak{g}})\cup(\mathscr{R}_{\mathfrak{g}}-\mathscr{Q}_{\mathfrak{g}}). \tag{3.4}$$

Proof. Sufficiency. Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and let there exist $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ such that the relation $\mathscr{S}_{\mathfrak{g}}=(\mathscr{Q}_{\mathfrak{g}}-\mathscr{R}_{\mathfrak{g}})\cup(\mathscr{R}_{\mathfrak{g}}-\mathscr{Q}_{\mathfrak{g}})$ holds. Clearly, $(\mathscr{Q}_{\mathfrak{g}}-\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}-\mathscr{Q}_{\mathfrak{g}})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$, implying

$$\begin{aligned}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\big[(\mathscr{Q}_{\mathfrak{g}}-\mathscr{R}_{\mathfrak{g}})\cup(\mathscr{R}_{\mathfrak{g}}-\mathscr{Q}_{\mathfrak{g}})\big] &= \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{Q}_{\mathfrak{g}}-\mathscr{R}_{\mathfrak{g}}\big]\\ &\cup\ \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{R}_{\mathfrak{g}}-\mathscr{Q}_{\mathfrak{g}}\big].\end{aligned}$$

Set $\mathscr{S}_{\mathfrak{g},(q,r)}=\mathscr{Q}_{\mathfrak{g}}-\mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(r,q)}=\mathscr{R}_{\mathfrak{g}}-\mathscr{Q}_{\mathfrak{g}}$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\cup\mathscr{S}_{\mathfrak{g},(r,q)}\big)=\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\cup\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)$. Since $\big(\mathscr{S}_{\mathfrak{g},(q,r)},\mathscr{S}_{\mathfrak{g},(r,q)}\big)\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$ and $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big),\\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big),\\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big),\\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]}\mathscr{O}_{\mathfrak{g}}\\ &= \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)})\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})]}\mathscr{O}_{\mathfrak{g}}\\ &= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)})]}\mathscr{O}_{\mathfrak{g}}\Bigg)\\ &\cup \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})]}\mathscr{O}_{\mathfrak{g}}\Bigg)\\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)\big)\\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\\ &\cup\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)\\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\\ &\cup\ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big).\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\\ &\cup\ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big).\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\right]} \mathscr{K}_{\mathfrak{g}} \\
&= \left(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right)\right]} \mathscr{K}_{\mathfrak{g}}\right) \\
&\cup \left(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\right]} \mathscr{K}_{\mathfrak{g}}\right) \\
&= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right)\right) \\
&= \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\
&\cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) \\
&= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\
&\cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right).
\end{aligned}$$

Hence, it results that

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\
&\cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right).
\end{aligned}$$

By virtue of the relation $\left(\mathscr{S}_{\mathfrak{g},(q,r)}, \mathscr{S}_{\mathfrak{g},(r,q)}\right) \subseteq \left(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\right)$, it is plain that $\mathscr{S}_{\mathfrak{g},(q,r)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(r,q)} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$. But $\mathscr{S}_{\mathfrak{g},(q,r)} = \complement_{\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathscr{R}_{\mathfrak{g}}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$ and hence,

$$\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right).$$

On the other hand, the statement that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$ implies that $\mathscr{S}_{\mathfrak{g},(r,q)}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$ and therefore,

$$\begin{aligned}
&\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) \\
&\qquad = \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right).
\end{aligned}$$

When all the foregoing set-theoretic expressions are taken into account, it results that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\ &\cup \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) \\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{S}_{\mathfrak{g},(q,r)}\right) \\ &\cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\left(\mathscr{S}_{\mathfrak{g},(r,q)}\right) \\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The condition of the theorem is, therefore, sufficient.

Necessity. Conversely, suppose that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Set $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then, $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, meaning that $\mathscr{Q}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Set $\mathscr{S}_{\mathfrak{g},(s,q)} = \mathscr{S}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(q,s)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{S}_{\mathfrak{g}}$. Then,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(s,q)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}}; \\ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(s,q)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}).\end{aligned}$$

But $\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$ and consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g},(s,q)} \longmapsto \emptyset$, meaning that $\mathscr{Q}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{S}_{\mathfrak{g}}$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}}; \\ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}).\end{aligned}$$

Since $\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$ it follows, consequently, that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g},(q,s)} \longmapsto \emptyset$, meaning that $\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$. Set $\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g},(q,s)} \cup \mathscr{S}_{\mathfrak{g},(s,q)}$. Then,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)} \cup \mathscr{S}_{\mathfrak{g},(s,q)}\big) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\big) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(s,q)}\big) \\ &= \emptyset \cup \emptyset = \emptyset,\end{aligned}$$

implying that $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Having evidenced the existence of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property, it only remains to show that $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is expressible as $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$.

Observe that

$$\begin{aligned}
&\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} \\
&\quad = \big\{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}}\right)\big\} \cup \big\{\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big\} \\
&\quad = \big\{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big[\big(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \cup \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big)\big]\big\} \\
&\quad \cup \big\{\big[\big(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \cup \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big)\big] \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big\} \\
&\quad = \big\{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big)\big\} \\
&\quad \cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big\} \\
&\quad = \big\{\mathscr{Q}_{\mathfrak{g}} \cap \big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right) \cup \mathscr{S}_{\mathfrak{g}}\big) \cap \big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \cup \mathscr{Q}_{\mathfrak{g}}\big)\big\} \\
&\quad \cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big\} \\
&\quad = \big\{\big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big) \cap \big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \cup \mathscr{Q}_{\mathfrak{g}}\big)\big\} \cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big\} \\
&\quad = \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big) \cup \big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big).
\end{aligned}$$

But since $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathscr{Q}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and the latter in turn implies $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big)$, it follows that $\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right) = \emptyset$. Consequently, $\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} = \mathscr{S}_{\mathfrak{g}}$. But, $\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$ and hence, $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$. The condition of the theorem is, therefore, necessary. ☐

Observe that $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\left(\mathscr{R}_{\mathfrak{g}}\right) \cup \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}\left(\mathscr{Q}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big)$. Thus, an immediate consequence of the above theorem is the following corollary.

Corollary 3.9. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ if and only if:*

$$\big(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big(\exists \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big)\big]. \tag{3.5}$$

Proposition 3.10. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property, then $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \neq \Omega$:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \;\longrightarrow\; \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \neq \Omega. \tag{3.6}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows that $\Omega \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Consequently, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\Omega\right) = \Omega$. But, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) = \emptyset$. Thus, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) = \emptyset \neq \Omega = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\Omega\right)$, implying $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \neq \Omega$. The proof of the proposition is complete. ☐

Proposition 3.11. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and $\mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected, then:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longleftrightarrow \big(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big) \vee \big(\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big). \tag{3.7}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and $\mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-connected. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Then, there exist a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property such that $\mathscr{S}_{\mathfrak{g}}$ be expressible as $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$. Since the strong $\mathscr{T}_{\mathfrak{g}}$-space

$\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, the only $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set are the improper $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset$, $\Omega \subset \mathfrak{T}_{\mathfrak{g}}$. Consequently,

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longleftrightarrow \big(\mathscr{Q}_{\mathfrak{g}} \in \{\emptyset, \Omega\}\big)\big[\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})\big].$$

Case i. Suppose $\mathscr{Q}_{\mathfrak{g}} = \emptyset$. Then $\mathscr{S}_{\mathfrak{g}} = (\emptyset - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \emptyset)$. But $\emptyset - \mathscr{R}_{\mathfrak{g}} = \emptyset$ and $\mathscr{R}_{\mathfrak{g}} - \emptyset = \mathscr{R}_{\mathfrak{g}}$. Therefore, $\mathscr{S}_{\mathfrak{g}} = \emptyset \cup \mathscr{R}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}$. Thus, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

Case ii. Suppose $\mathscr{Q}_{\mathfrak{g}} = \Omega$. Then $\mathscr{S}_{\mathfrak{g}} = (\Omega - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \Omega)$. But $\Omega - \mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}} - \Omega = \emptyset$. Consequently, $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \emptyset = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and therefore, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of the proposition is complete. ☐

Lemma 3.12. *If* $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ *be a triple of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets and* $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the symmetric difference* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- i. $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\mathscr{Q}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,
- ii. $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big), \mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big)\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,
- iii. $\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big)$.

Proof. Let $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and, let $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the symmetric difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. The proof that $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\mathscr{Q}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ holds for any $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is first supplied. It is evident that

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \\ &= \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big) \subseteq \mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}, \end{aligned}$$

implying $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) \subseteq \mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it follows that $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Items i., ii. and iii. are now proved.

i. Since the order of the operands under the $\cup$-operation does not change, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}\big). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

ii. For any $(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it is plain that $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore,

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big), \mathscr{S}_{\mathfrak{g}}\big) &= \big\{\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\ &\cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big)\big)\big\} \\ &= \big\{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\ &\cup \big\{\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\ &\cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big\} \\ &\cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}\}. \end{aligned}$$

If $\text{P}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \overset{\text{def}}{=} \mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big), \mathscr{S}_{\mathfrak{g}}\big) &= \text{P}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \cup \text{P}(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \\ &\cup \text{P}(\mathscr{S}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \cup \big(\mathscr{S}_{\mathfrak{g}} \cap \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}\big). \end{aligned}$$

Since $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\big) &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}=\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}})\big)\\ &= \mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})\cup\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})\\ &\cup\ \mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\cup\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big).\end{aligned}$$

But by virtue of the associativity and distributive properties of the $\cap$, $\cup$-operations, the relations $\mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})=\mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})$, $\mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})=\mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})$, $\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})=\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})$, and $\mathscr{S}_{\mathfrak{g}}\cap\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}$ hold. Thus, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\big)\in\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

III. Since the relation $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})=\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds for any $(\mathscr{S}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it results that

$$\begin{aligned}\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) &= \mathscr{Q}_{\mathfrak{g}}\cap\big(\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &= \big(\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\big)\cup\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &= \big(\mathscr{Q}_{\mathfrak{g}}\cap\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big)\cup\big(\mathscr{Q}_{\mathfrak{g}}\cap\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\big)\\ &= \big((\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cup\big((\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big).\end{aligned}$$

Hence, $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big)\in\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The proof of the lemma is complete. □

Theorem 3.13. *If $\mathscr{S}_{\mathfrak{g},1},\ \mathscr{S}_{\mathfrak{g},2},\ \ldots,\ \mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then $\bigcap_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.*

Proof. Let $\mathscr{S}_{\mathfrak{g},1},\ \mathscr{S}_{\mathfrak{g},2},\ \ldots,\ \mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g},1},\ \mathscr{S}_{\mathfrak{g},2},\ \ldots,\ \mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exist $\sigma\geq 1$ $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed sets $\mathscr{Q}_{\mathfrak{g},1},\ \mathscr{Q}_{\mathfrak{g},2},\ \ldots,\ \mathscr{Q}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g},1},\ \mathscr{R}_{\mathfrak{g},2},\ \ldots,\ \mathscr{R}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property such that

$$\begin{aligned}\mathscr{S}_{\mathfrak{g},1} &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},1},\mathscr{R}_{\mathfrak{g},1}\big),\\ \mathscr{S}_{\mathfrak{g},2} &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},2},\mathscr{R}_{\mathfrak{g},2}\big),\ \ldots,\ \mathscr{S}_{\mathfrak{g},\sigma}=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\sigma},\mathscr{R}_{\mathfrak{g},\sigma}\big).\end{aligned}$$

For an arbitrary pair $(\nu,\mu)\in I_{\sigma}^{*}\times I_{\sigma}^{*}$, set $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)}=\mathscr{Q}_{\mathfrak{g},\nu}\cap\mathscr{Q}_{\mathfrak{g},\mu}$, $\mathscr{W}_{\mathfrak{g},(\nu,\mu)}=\mathscr{Q}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}$, and $\mathscr{R}_{\mathfrak{g},(\nu,\mu)}=\mathscr{R}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}$. Then,

$$\begin{aligned}\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{S}_{\mathfrak{g},\mu} &= \mathscr{S}_{\mathfrak{g},\nu}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\mu},\mathscr{R}_{\mathfrak{g},\mu}\big)\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{Q}_{\mathfrak{g},\mu},\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}\big)\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\nu}\big)\cap\mathscr{Q}_{\mathfrak{g},\mu},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\nu}\big)\cap\mathscr{R}_{\mathfrak{g},\mu}\big]\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\mathscr{W}_{\mathfrak{g},(\mu,\nu)}\big),\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\big]\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big\{\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathscr{W}_{\mathfrak{g},(\mu,\nu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\big]\big\}.\end{aligned}$$

But, $\mathscr{R}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, $(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\mu})\in\big(\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\big)\times\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{W}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and, $\mathscr{Q}_{\mathfrak{g},\nu},\ \mathscr{Q}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Thus, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, implying $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathscr{W}_{\mathfrak{g},(\mu,\nu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\big]=\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Therefore, $\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{S}_{\mathfrak{g},\mu}=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)}\big)$, where $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap$

$\mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}]$ and $\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}$-Nd$[\mathfrak{T}_{\mathfrak{g}}]$, and consequently, $\mathscr{S}_{\mathfrak{g},\nu} \cap \mathscr{S}_{\mathfrak{g},\mu} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ for any $(\nu,\mu) \in I_{\sigma}^{*} \times I_{\sigma}^{*}$. Hence, $\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}$-P$\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The proof of the theorem is complete. □

Proposition 3.14. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets each of which having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}$ has also $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\bigwedge\nolimits_{\nu\in I_{\sigma}^{*}} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \longrightarrow \bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.8}$$

Proof. Let $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ be $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu}) = \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}\big(\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu})\big)$ for any arbitrary pair $(\nu,\mu) \in I_{\sigma}^{*} \times I_{\sigma}^{*}$. But, $\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu})$, $\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu}) \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ and therefore, $\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu}) \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$. Set $\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) = \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu})$. Then, since $\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ is equivalent to $\mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ and, the relation $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big)$ holds, it follows that $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is complete. □

Theorem 3.15. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, then it has also $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\big(\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}\big)\big[\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathrm{P}\,[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.9}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}$-P$[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, it satisfies the relation $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Since $\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \big(\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned} &\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\\ &\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &= \mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathrm{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \end{aligned}$$

implying $\mathrm{cl}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathrm{cl}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently, it results that $\mathrm{int}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ which, in turn, implies $\mathrm{cl}_{\mathfrak{g}} \circ \mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, meaning that $\mathscr{S}_{\mathfrak{g}}$ has also $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. □

Proposition 3.16. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}$ has also $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\bigwedge\nolimits_{\nu\in I_{\sigma}^{*}} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \longrightarrow \bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.10}$$

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] : \ \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\bigwedge_{\nu \in I_{\sigma}^{*}}(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])$ implies $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is an untrue logical statement. Then, $\bigwedge_{\nu \in I_{\sigma}^{*}}(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])$ is true and $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \longmapsto \emptyset$ is untrue. Thus, to prove the proposition, it suffices to prove that $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is a contradiction. For arbitrary $(\nu, \mu(\nu)) \in I_{\sigma}^{*} \times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*} = I_{\sigma}^{*} \setminus \{\nu\}$, set $\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))} = \mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu(\nu)}$, where $\{\mathscr{S}_{\mathfrak{g},\nu}, \mathscr{S}_{\mathfrak{g},\mu(\nu)}\} \subset \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) &\cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) \\ &\subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) \\ &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}).\end{aligned}$$

Thus, for arbitrary $(\nu, \mu(\nu)) \in I_{\sigma}^{*} \times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*} = I_{\sigma}^{*} \setminus \{\nu\}$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})\big]& \\ \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) = \emptyset.&\end{aligned}$$

Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it results that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) = \emptyset,$$

and therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})$. On the other hand, since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)}) = \emptyset,$$

Thus, $\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ holds for arbitrary $(\nu, \mu(\nu)) \in I_{\sigma}^{*} \times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*} = I_{\sigma}^{*} \setminus \{\nu\}$ and hence, $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. The relation $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is therefore a contradiction. The proof of the proposition is complete. □

Theorem 3.17. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, then it has also $\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$(\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}})\big[\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}] \ \longleftarrow \ \mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.11}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is an untrue logical statement. Then, $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$ is untrue. Thus, to prove the theorem, it suffices to prove that $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is a contradiction. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently,

$$\text{int}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big] \subseteq \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Since $\mathscr{S}_{\mathfrak{g}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows that $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$ and therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset,$$

implying $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. The relation $\mathscr{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is therefore a contradiction. The proof of the theorem is complete. □

The important remark given below ends the present section.

Remark. *In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, the converse of the following statements with respect to some $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ are in general untrue:*

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$,
- III. $\big(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]\big) \longrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$.

Because, in the event that $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}}) = \big(\mathbb{R}, \mathscr{T}_{\mathfrak{g},\mathbb{R}}\big) = \mathfrak{T}_{\mathfrak{g},\mathbb{R}}$ and $\mathscr{S}_{\mathfrak{g}} = \mathbb{Q}$ ($\mathbb{Q}$ and $\mathbb{R}$, respectively, denote the sets of rational and real numbers, where $\mathbb{R} \supset \mathbb{Q}$), the converse of ITEMS I.,II. *and* III.*, reading*

- IV. $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \longleftarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,
- V. $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \longleftarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,
- VI. $\big(\mathbb{Q} \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \longleftarrow \mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,

respectively, are all untrue. In fact, every $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathbb{R}}$ contains both points $\xi \in \mathbb{Q}$ and $\zeta \in \mathbb{R} \setminus \mathbb{Q}$. Consequently, there are no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior points of $\mathbb{Q}$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) = \emptyset$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) = \mathbb{R}$ and thus, $\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \ni \mathbb{R} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{R}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) \neq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset) = \emptyset \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$; $\big(\mathbb{Q}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q})\big) \notin \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \times \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$. In ITEMS IV., V. *and* VI.*, the consequents $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$, $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ and $\big(\mathbb{Q} \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big)$ are all untrue and on the other hand, their antecedents $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ and $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ are all true. Consequently,* ITEMS IV., V. *and* VI. *are all untrue statements and hence, the converse of* ITEMS I., II. *and* III. *are untrue statements. In addition, since $\big(\mathbb{Q}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q})\big) \notin \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \times \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ it follows that, for some $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the condition $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ can be satisfied without the condition $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ being satisfied, though $\mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \emptyset$ for every $\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ is a consequence of $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$.*

4. DISCUSSION

4.1. **Categorical Classifications.** Having adopted a categorical approach in the classifications of $\mathfrak{T}_{\mathfrak{a}}$-sets with $\{\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}\}$-property, the twofold purposes here are, firstly, to establish the various relationships amongst the classes of $\mathfrak{T}_{\mathfrak{a}}$-sets with $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}$-properties, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and secondly, to illustrate them through diagrams.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{g}}$, since $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{a}}]$ implies $\bigvee_{\nu \in I_3^0}\big(\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}_{\mathfrak{a}}]\big)$, it follows that, $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{a}}$ for each $\nu \in I_3^0$. Therefore, $\mathfrak{g}\text{-0-}\mathfrak{P}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-1-}\mathfrak{P}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-3-}\mathfrak{P}_{\mathfrak{a}} \longleftarrow \mathfrak{g}\text{-2-}\mathfrak{P}_{\mathfrak{a}}$. But, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{g}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{o}}$ for each $\nu \in I_3^0$. Hence, EQ. (4.1) present itself which may well be called $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$*-property diagram.*

$$
\begin{array}{ccccccc}
\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{P}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{P}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{P}_{\mathfrak{o}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{P}_{\mathfrak{o}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{P}_{\mathfrak{g}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{P}_{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}
\end{array}
\tag{4.1}
$$

In terms of the class $\{\mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}_{\mathfrak{a}}] : \nu \in I_3^*\}$, FIG. 1 present itself which may well be called $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$-*class diagram*.

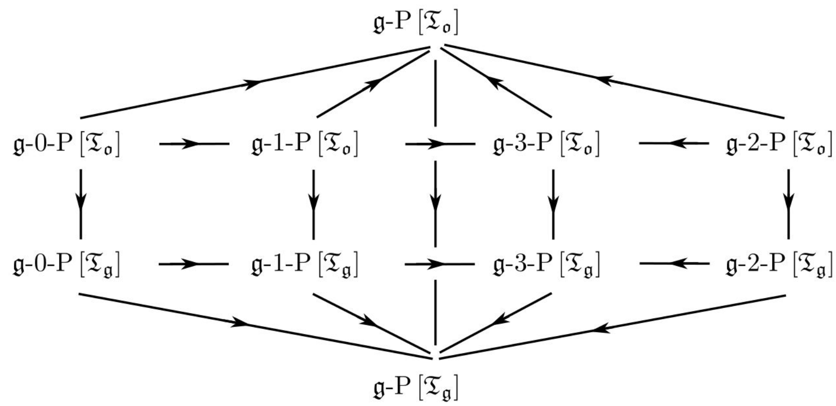

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$-*class diagram* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

In $\mathfrak{T}_{\mathfrak{a}}$, since $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{a}}]$ implies $\bigvee_{\nu\in I_3^0}(\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{a}}])$, it follows that, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{a}}$ for every $\nu \in I_3^0$. Therefore, $\mathfrak{g}\text{-0-}\mathfrak{Q}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-1-}\mathfrak{Q}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-3-}\mathfrak{Q}_{\mathfrak{a}} \longleftarrow \mathfrak{g}\text{-2-}\mathfrak{Q}_{\mathfrak{a}}$. But, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{o}} \longrightarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{g}}$ for each $\nu \in I_3^0$. Thus, EQ. (4.2) present itself which may well be called $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-*property diagram*.

$$\begin{array}{ccccccc}
\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{Q}_{\mathfrak{o}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{Q}_{\mathfrak{o}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{Q}_{\mathfrak{g}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{Q}_{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}
\end{array} \tag{4.2}$$

In terms of the class $\{\mathfrak{g}\text{-}\nu\text{-Nd}[\mathfrak{T}_{\mathfrak{a}}] : \nu \in I_3^*\}$, FIG. 2 present itself which may well be called $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-*class diagram*.

In $\mathfrak{T}_{\mathfrak{a}}$, since $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{a}}]$, $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathscr{S}_{\mathfrak{a}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{a}}]$ imply $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{a}}]$, $\mathscr{S}_{\mathfrak{a}} \in \text{P}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{a}}]$, respectively, it follows that $\mathfrak{Q}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}} \longrightarrow \mathfrak{P}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$. Finally, $\mathscr{S}_{\mathfrak{a}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{o}}]$ and $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{o}}]$ imply $\mathscr{S}_{\mathfrak{a}} \in \text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, and, $\mathscr{S}_{\mathfrak{a}} \in \text{P}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ imply $\mathscr{S}_{\mathfrak{a}} \in \text{P}[\mathfrak{T}_{\mathfrak{o}}]$ and $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{o}}]$, respectively. Altogether, EQ. (4.3) present itself which may well be called $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*properties diagram*.

$$\begin{array}{ccccccc}
\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longrightarrow & \mathfrak{P}_{\mathfrak{o}} \\
\downarrow & & \downarrow & & \uparrow & & \uparrow \\
\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{P}_{\mathfrak{g}}
\end{array} \tag{4.3}$$

In terms of the class $\{\text{Nd}[\mathfrak{T}_{\mathfrak{a}}], \text{P}[\mathfrak{T}_{\mathfrak{a}}], \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{a}}], \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{a}}]\}$, FIG. 3 present itself which may well be called $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*classes diagram*.

As in our previous works [1, 2, 19, 20], the manner we have positioned the arrows in the $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$, $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*properties diagrams* (EQS (4.1),

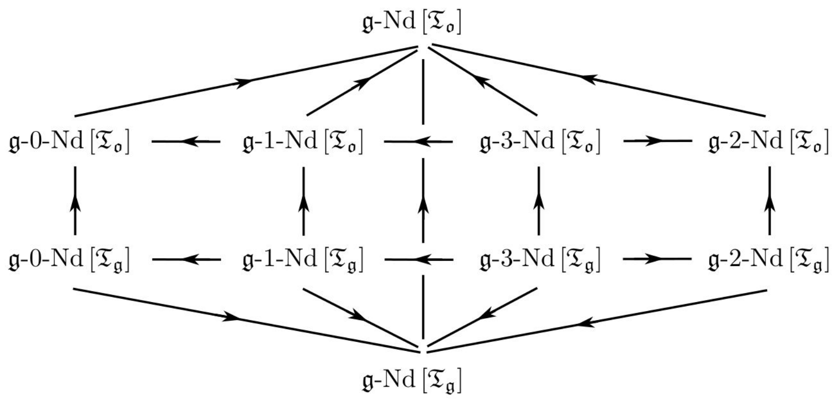

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-*property diagram* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

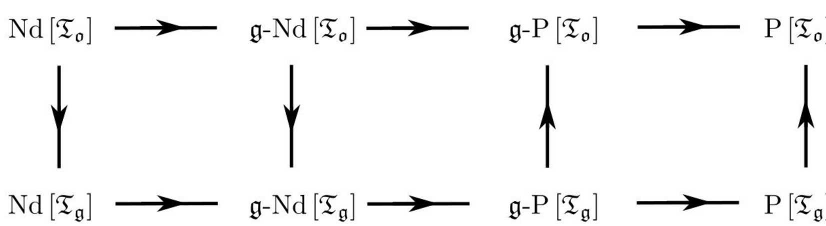

FIGURE 3. Relationships: $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*classes diagram* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

(4.2), (4.3)) and the $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$, $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*classes diagrams* (FIGS 1, 2, 3) is solely to stress that, in general, the implications in EQS (4.1)–(4.3) and FIGS 1–3 are irreversible.

4.2. **A Nice Application.** It is the purpose of this section to reveal through a nice application some characterizations on the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, and to give some other characterizations associated with $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-properties in a $\mathscr{T}_{\mathfrak{g}}$-space. Consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where $\Omega = \{\zeta_\nu : \ \nu \in I_5^*\}$ and is topologized by the choice:

$$\mathscr{T}_{\mathfrak{g}}(\Omega) = \{\emptyset, \{\zeta_1\}, \{\zeta_1, \zeta_3, \zeta_5\}, \Omega\} = \{\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}, \mathscr{O}_{\mathfrak{g},3}, \mathscr{O}_{\mathfrak{g},4}\}; \tag{4.4}$$

$$\neg\mathscr{T}_{\mathfrak{g}}(\Omega) = \{\Omega, \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}, \{\zeta_2, \zeta_4\}, \emptyset\} = \{\mathscr{K}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},2}, \mathscr{K}_{\mathfrak{g},3}, \mathscr{K}_{\mathfrak{g},4}\}. \tag{4.5}$$

For convenience of notation, let

$$\mathscr{P}(\Omega) = \{\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega) : \ (\nu, \mu) \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}\}, \tag{4.6}$$

where $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega)$ denotes a $\mathfrak{T}_{\mathfrak{g}}$-set labeled $\nu \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}$ and containing $\mu \in I^0_{\mathrm{card}(\Omega)}$ elements. Then, $\mathscr{R}_{\mathfrak{g},(1,0)} = \emptyset$, …, $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} = \{\zeta_1, \zeta_2, \dots, \zeta_\mu\}$, …, $\mathscr{R}_{\mathfrak{g},(32,5)} = \Omega$.

For $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) \in \{0, 5\}$, let $\mathscr{R}_{\mathfrak{g},(1,0)} = \emptyset$ and $\mathscr{R}_{\mathfrak{g},(32,5)} = \Omega$. For $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) \in \{1, 4\}$, let $\mathscr{R}_{\mathfrak{g},(2,1)} = \{\zeta_1\}$, $\mathscr{R}_{\mathfrak{g},(3,1)} = \{\zeta_2\}$, $\mathscr{R}_{\mathfrak{g},(4,1)} = \{\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(5,1)} = \{\zeta_4\}$, and $\mathscr{R}_{\mathfrak{g},(6,1)} = \{\zeta_5\}$; $\mathscr{R}_{\mathfrak{g},(27,4)} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(28,4)} = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(29,4)} = \{\zeta_1, \zeta_3, \zeta_4, \zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(30,4)} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$, and $\mathscr{R}_{\mathfrak{g},(31,4)} = \{\zeta_1, \zeta_2, \zeta_4, \zeta_5\}$. For $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) \in \{2, 3\}$, let

$\mathscr{R}_{\mathfrak{g},(7,2)} = \{\zeta_1,\zeta_2\}$, $\mathscr{R}_{\mathfrak{g},(8,2)} = \{\zeta_1,\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(9,2)} = \{\zeta_1,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(10,2)} = \{\zeta_1,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(11,2)} = \{\zeta_2,\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(12,2)} = \{\zeta_2,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(13,2)} = \{\zeta_2,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(14,2)} = \{\zeta_3,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(15,2)} = \{\zeta_3,\zeta_5\}$, and $\mathscr{R}_{\mathfrak{g},(16,2)} = \{\zeta_4,\zeta_5\}$; $\mathscr{R}_{\mathfrak{g},(17,3)} = \{\zeta_1,\zeta_2,\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(18,3)} = \{\zeta_1,\zeta_3,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(19,3)} = \{\zeta_1,\zeta_4,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(20,3)} = \{\zeta_1,\zeta_2,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(21,3)} = \{\zeta_1,\zeta_2,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(22,3)} = \{\zeta_1,\zeta_3,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(23,3)} = \{\zeta_2,\zeta_3,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(24,3)} = \{\zeta_2,\zeta_3,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(25,3)} = \{\zeta_3,\zeta_4,\zeta_5\}$, and $\mathscr{R}_{\mathfrak{g},(26,3)} = \{\zeta_2,\zeta_4,\zeta_5\}$. Then,

$$\begin{aligned} \operatorname{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) &= \mathscr{R}_{\mathfrak{g},(\nu,\mu)} \\ &= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \end{aligned} \tag{4.7}$$

for every $(\nu,\mu) \in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^{0}_{\mathrm{card}(\Omega)}$. Consequently,

$$\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) = \mathscr{R}_{\mathfrak{g},(\nu,\mu)} = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \tag{4.8}$$

for every $(\nu,\mu) \in I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^{0}_{\mathrm{card}(\Omega)}$. Introduce $J^{*}_{28} = I^{*}_{1} \cup (I^{*}_{7} \setminus I^{*}_{2}) \cup (I^{*}_{16} \setminus I^{*}_{10}) \cup (I^{*}_{26} \setminus I^{*}_{22}) \cup (I^{*}_{28} \setminus I^{*}_{27})$. Then,

$$\begin{aligned} \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) = \emptyset = \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big), \\ \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) = \Omega = \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) \end{aligned} \tag{4.9}$$

From Eq. (4.8), it follows that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, do *commute*. Thus, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is both *coarser and finer* (or, *smaller and larger*, *weaker and stronger*) than $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Consequently, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Furthermore, it is easily checked from Eq. (4.8) that, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue if and only if $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue.

From Eq. (4.9), both $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\nu,\mu) \in J^{*}_{28} \times I^{0}_{4}$ and $\mathscr{R}_{\mathfrak{g},(\delta,\eta)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\delta,\eta) \in \big(I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))} \setminus J^{*}_{28}\big) \times I^{0}_{\mathrm{card}(\Omega)}$ are easily checked. Moreover, it results from Eqs (4.8), (4.9) that, $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue for every $(\nu,\mu) \in \big(J^{*}_{28} \setminus I^{*}_{1}\big) \times I^{0}_{4}$. This confirms the statement that, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}] \longleftarrow \mathscr{R}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue if and only if $\mathscr{R}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue. Observing that, for every $(\nu,\mu) \in J^{*}_{28} \times I^{0}_{4}$ and every $(\delta,\eta) \in \big(I^{*}_{\mathrm{card}(\mathscr{P}(\Omega))} \setminus J^{*}_{28}\big) \times I^{0}_{\mathrm{card}(\Omega)}$, the relations

$$\begin{aligned} \emptyset = \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) &\subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \\ &= \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \supseteq \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) = \emptyset, \\ \operatorname{int}_{\mathfrak{g}} \circ \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) = \Omega &\supseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) \\ &= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) \subseteq \Omega = \operatorname{cl}_{\mathfrak{g}} \circ \operatorname{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big), \end{aligned}$$

respectively, hold, of which the first relation is the dual of the second, and conversely, it follows that the logical statement $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{R}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is satisfied for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

5. Conclusion

In a recent paper (Cf. [19]), we defined and studied the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces. We showed in a $\mathscr{T}_{\mathfrak{g}}$-space that $\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is $(\Omega,\emptyset)$-*grounded*, (*expansive, non-expansive*), (*idempotent, idempotent*) and $(\cap,\cup)$-*additive*. We also showed in a $\mathscr{T}_{\mathfrak{g}}$-space that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In this paper, we have studied in $\mathscr{T}_{\mathfrak{g}}$-spaces the commutativity of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}$-sets having some $(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}})$-based properties called $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$*-properties*. We have shown that the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *duals* and $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under their $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-*operations*. We have also shown that a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$*-property* is *equivalent* to the $\mathfrak{T}_{\mathfrak{g}}$-set or its complement having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$*-property*. The $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\cup$-*operation* and $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\{\cup, \cap, \complement\}$-*operations*. Finally, a $\mathfrak{T}_{\mathfrak{g}}$-set having $\{\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}\}$*-property* also has $\{\mathfrak{P}_{\mathfrak{g}}, \mathfrak{Q}_{\mathfrak{g}}\}$*-property*.

An interestingly promising avenue for future research arises if the theorization of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators of mixed categories in $\mathscr{T}_{\mathfrak{g}}$-spaces be made a new subject of inquiry. For instance, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to study the $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$-*interior* and $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$-*closure operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu\mu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces, where $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu\mu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points interior in $\mathscr{S}_{\mathfrak{g}}$ and interiorness are characterized by $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$-*open sets* belonging to the class $\{\mathscr{O}_{\mathfrak{g}} = \mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu} : (\mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\}$; $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu\mu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points close to $\mathscr{S}_{\mathfrak{g}}$ and closeness are characterized by $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$-*closed sets* belonging to the class $\{\mathscr{K}_{\mathfrak{g}} = \mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu} : (\mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\}$. Such a study is what we thought would be worth considering, and the discussion of this paper ends here.

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