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Categorification of Algebras: 2-Algebras

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2-Categories,
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Homotopy

Abstract — This paper introduces a categorification of k-algebras called 2-algebras, where k is a commutative ring. We define the 2-algebras as a 2-category with single object in which collections of all 1-morphisms and all 2-morphisms are k-algebras. It is shown that the category of 2-algebras is equivalent to the category of crossed modules in commutative k-algebras.

Subject Classification (2020): 18F99, 18G30, 18G55, 13C60.

1. Introduction

The term "categorification" coined by Louis Crane refers to the process of replacing set theoretic concepts by category-theoretic analogues in mathematics. A categorified version of a group is a 2-group. Internal categories in the category of groups are exactly the same as 2-groups. The Brown-Spencer theorem [3] thus constructs the associated 2-group of a crossed module given by Whitehead [11] to define an algebraic model for a "(connected) homotopy 2-type". The fact that the composition in the internal category must be a group homomorphism implies that the "interchange law" must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity.

We will concerne in this paper exclusively with categorification of algebras. We will obtain analogous results in (commutative) algebras with regard to Porter's work [9]. He states that there is an equivalence of categories between the category of internal categories in the category of k-algebras and the category of crossed modules of commutative k-algebras. Since the internal category in the category of k-algebras is a categorification of k-algebras, this internal category will be called as "strict 2-algebra" in this work. We define the strict 2-algebra by means of 2-module being a category in the category of modules as a 2-category with single object in which collections of 1-morphisms and 2-morphisms are k-algebras and we denote the category of strict 2-algebras by **2Alg** . Given a group G, it is known that automorphisms of G yield a 2-group. Analogous result in commutative algebras can be given that multiplications of C yield a strict 2-algebra where C is a commutative R-algebra and R is a commutative k-algebra.

A crossed module $\mathscr{A} = (\partial : C \longrightarrow R)$ of commutative algebras is given by an algebra morphism $\partial : C \longrightarrow R$

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together with an action $\cdot$ of R on C such that the relations below hold for each $r \in R$ and each $c, c' \in C$,

$$\begin{aligned} \partial(r \cdot c) &= r\partial(c) \\ \partial(c) \cdot c' &= cc'. \end{aligned}$$

In this paper we show that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras.

2. Internal Categories and 2-categories

We begin by recalling internal categories as well as 2-categories. Ehresmann defined internal categories in [5], and by now they are an important part of category theory [4].

2.1. Internal categories

Definition 2.1. Let **C** be any category. An internal category in **C**, say **A**, consists of:

• an object of objects $A_0 \in \mathbf{C}$

• an object of morphisms $A_1 \in \mathbf{C}$,

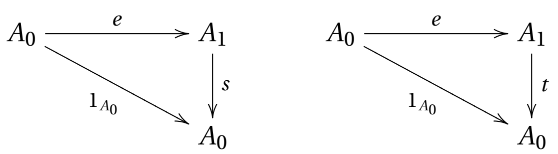
together with

• source and target morphisms $s, t : A_1 \longrightarrow A_0$,

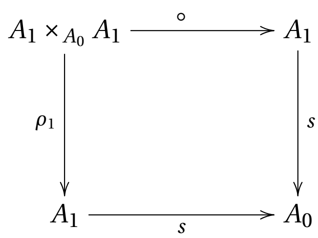
• an identity-assigning morphism $e : A_0 \longrightarrow A_1$,

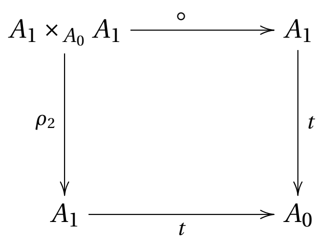
• a composition morphism $\circ : A_1 \times_{A_0} A_1 \longrightarrow A_1$ such that the following diagrams commute, expressing the usual category laws:

• laws specifying the source and target of identity morphisms:



• laws specifying the source and target of composite morphisms:





• the associative law for composition of morphisms:

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 \times_{A_0} A_1 & \xrightarrow{\ \circ\ } & A_1 \times_{A_0} A_1 \\ \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle t} \\ A_1 \times_{A_0} A_1 & \xrightarrow[t]{} & A_0 \end{array}$$

• the left and right unit laws for composition of morphisms:

$$\begin{array}{ccccc} A_0 \times_{A_0} A_1 & \xrightarrow{e\times_{A_0} 1_{A_1}} & A_1 \times_{A_0} A_1 & \xleftarrow{1_{A_1}\times_{A_0} e} & A_1 \times_{A_0} A_0 \\ & {\scriptstyle \rho_2}\searrow & \downarrow{\scriptstyle \circ} & \swarrow{\scriptstyle \rho_1} & \\ & & A_1 & & \end{array}$$

Here, the pullback $A_1 \times_{A_0} A_1$ is defined via the square:

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{\rho_2} & A_1 \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle s} \\ A_1 & \xrightarrow[t]{} & A_0. \end{array}$$

We denote this internal category with $A = (A_0, A_1, s, t, e, \circ)$.

Definition 2.2. Let **C** be a category. Given internal categories A and A' in **C**, an **internal functor** between them, say $F : A \longrightarrow A'$, consists of

• a morphism $F_0 : A_0 \longrightarrow A'_0$,

• a morphism $F_1 : A_1 \longrightarrow A'_1$

such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:

• preservation of source and target:

$$\begin{array}{ccc} A_1 & \xrightarrow{s} & A_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ A'_1 & \xrightarrow[s']{} & A'_0 \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{t} & A_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ A'_1 & \xrightarrow[t']{} & A'_0 \end{array}$$

• preservation of identity morphisms:

$$\begin{array}{ccc} A_0 & \xrightarrow{e} & A_1 \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\ A'_0 & \xrightarrow[e']{} & A'_1 \end{array}$$

• preservation of composite morphisms:

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{F_1 \times_{A_0} F_1} & A'_1 \times_{A'_0} A'_1 \\ \Big\downarrow{\circ} & & \Big\downarrow{\circ'} \\ A_1 & \xrightarrow[F_1]{} & A'_1 \end{array}$$

Given two internal functors $F: A \longrightarrow A'$ and $G: A' \longrightarrow A''$ in some category **C**, we define their composite $FG: A \longrightarrow A''$ by taking $(FG)_0 = F_0G_0$ and $(FG)_1 = F_1G_1$. Similarly, we define the identity internal functor in **C**, $1_A: A \longrightarrow A$ by taking $(1_A)_0 = 1_{A_0}$ and $(1_A)_1 = 1_{A_1}$.

Definition 2.3. Let **C** be a category. Given two internal functors $F, G: A \longrightarrow A'$ in **C**, an **internal natural transformation** in **C** between them, say $\theta: F \Longrightarrow G$, is a morphism $\theta: A_0 \longrightarrow A'_1$ for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:

• laws specifying the source and target of a natural transformation:

$$\begin{array}{ccc} A_0 & \xrightarrow{\theta} & A'_1 \\ & {\scriptstyle F_0}\searrow & \Big\downarrow{s'} \\ & & A'_0 \end{array} \qquad \begin{array}{ccc} A'_0 & \xrightarrow{\theta} & A'_1 \\ & {\scriptstyle G_0}\searrow & \Big\downarrow{t'} \\ & & A_0 \end{array}$$

• the commutative square law:

$$\begin{array}{ccc} A_1 & \xrightarrow{\triangle(s\theta \times G)} & A'_1 \times_{A'_0} A'_1 \\ \Big\downarrow{\triangle(F \times t\theta)} & & \Big\downarrow{\circ'} \\ A'_1 \times_{A'_0} A'_1 & \xrightarrow[\circ']{} & A'_1 \end{array}$$

Given an internal functor $F: A \longrightarrow A'$ in **C**, the identity internal natural transformation $1_F: F \Longrightarrow F$ in C is given by $1_F = F_0e$.

2.2. 2-categories

Definition 2.4. A 2-category $\mathscr{G}$ consists of a class of objects G_0 and for any pair of objects (A, B) a small category of morphisms $\mathscr{G}(A, B)$-with objects $G_1(A, B)$ and morphisms $G_2(A, B)$-, along with composition functors

$$\bullet \quad : \quad \mathscr{G}(A, B) \times \mathscr{G}(B, C) \quad \longrightarrow \quad \mathscr{G}(A, C)$$

for every triple (A, B, C) of objects and identity functors from the terminal category to $\mathscr{G}(A, A)$

$$iA: 1 \longrightarrow \mathscr{G}(A, A)$$

for all objects A such that $\bullet$ is associative and

$$F \bullet i_B = F = i_A \bullet F \quad \text{as well as} \quad \vartheta \bullet I_{i_B} = \vartheta = I_{i_A} \bullet \vartheta$$

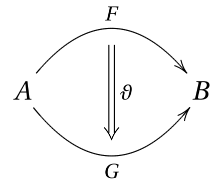
hold for all $F \in G_1(A,B)$ and $\vartheta \in G_2(A,B)$ where source and target morphisms are defined by

$$A \xrightarrow{F} B$$

$$\begin{array}{ccc} s : G_1(A,B) & \longrightarrow & G_0 \\ F & \longmapsto & s(F) = A \end{array}$$

$$\begin{array}{ccc} t : G_1(A,B) & \longrightarrow & G_0 \\ F & \longmapsto & t(F) = B \end{array}$$

for $F \in G_1(A,B)$ and

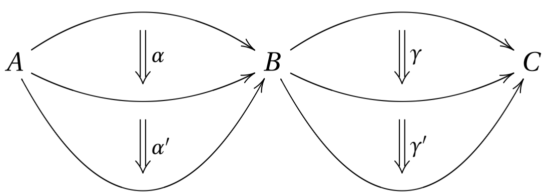


$$\begin{array}{ccc} s : G_2(A,B) & \longrightarrow & G_1 \\ \vartheta & \longmapsto & s(\vartheta) = F \end{array}$$

$$\begin{array}{ccc} t : G_2(A,B) & \longrightarrow & G_0 \\ \vartheta & \longmapsto & t(\vartheta) = G \end{array}$$

for $\vartheta : F \longrightarrow G \in G_2(A,B)$. For all pairs of objects (A,B) elements of $G_1(A,B)$ are called 1-morphisms or 1-cells of $\mathscr{G}$ and elements of $G_2(A,B)$ are called 2-morphisms or 2-cells of $\mathscr{G}$. We write G_1 and G_2 for the classes of all 1-morphisms and 2-morphisms respectively.

There are two ways of composing 2-morphisms: using the composition $\circ$ inside the categories $\mathscr{G}(A,B)$, called vertical composition, and using the morphism level of the functor $\bullet$, called horizontal composition. These compositions must be satisfy the following equation: for $\alpha, \alpha' \in G_2(A,B)$ with $t(\alpha) = s(\alpha')$ and $\gamma, \gamma' \in G_2(B,C)$ with $t(\gamma) = s(\gamma')$



$$(\alpha \circ \alpha') \bullet (\gamma \circ \gamma') = (\alpha \bullet \gamma) \circ (\alpha' \bullet \gamma')$$

which is called "interchange law".

3. Constructions of Two-Algebras

In this section we will construct 2-algebras by categorification. We can categorify the notion of an algebra by replacing the equational laws by isomorphisms satisfying extra structure and properties we expect. In [2]

Baez and Crans introduce the Lie 2-algebra by means of the concept of 2-vector space defined as an internal category in the category of vector spaces by them. Obviously we get a new notion of “2-module”which can be considered as an internal category in the category of modules and we categorify the notion of an algebra.

3.1. 2-Modules

A categorified module or “2-module”should be a category with structure analogous to that of a $k-$module, with functors replacing the usual $k-$module operations. Here we instead define a 2-module to be an internal category in a category of $k-$modules **Mod** . Since the main component part of a $k-$algebra is a $k-$module, a 2-algebra will have an underlying 2-module of this sort. In this section we thus first define a category of these 2-modules.

In the rest of this paper, the terms a module and an algebra will always refer to a $k-$module and a $k-$algebra.

Definition 3.1. A 2-module is an internal category in **Mod** .

Thus, a 2-module M is a category with a module of objects M_0 and a module of morphisms M_1, such that the source and target maps $s, t : M_1 \longrightarrow M_0$, the identity assigning map $e : M_0 \longrightarrow M_1$, and the composition map $\circ : M_1 \times_{M_0} M_1 \longrightarrow M_1$ are all module morphisms. We write a morphism as $a : x \longrightarrow y$ when $s(a) = x$ and $t(a) = y$, and sometimes we write $e(x)$ as 1_x.

The following proposition is given for the **Vect** vector space category in [2]. But we rewrite this proposition for **Mod** .

Proposition 3.2. It is defined a 2-module by specifying the modules M_0 and M_1 along with the source, target and identity module morphisms and the composition morphism $\circ$, satisfying the conditions of Definition 2.1. The composition map is uniquely determined by

$$\begin{array}{rccl} \circ : & M_1 \times_{M_0} M_1 & \longrightarrow & M_1 \\ & (a,b) & \longmapsto & \circ(a,b) = a \circ b = a + b - (es)(b). \end{array}$$

Proof.

First given modules M_0, M_1 and module morphisms $s, t : M_1 \longrightarrow M_0$ and $e : M_0 \longrightarrow M_1$, we will define a composition operation that satisfies the laws in the definition of internal category, obtaining a 2-module.

Given a, $b \in M_1$ such that $t(a) = s(b)$, i.e.

$$a : x \longrightarrow y \text{ and } b : y \longrightarrow z$$

we define their composite $\circ$ by

$$\begin{array}{rccl} \circ : & M_1 \times_{M_0} M_1 & \longrightarrow & M_1 \\ & (a,b) & \longmapsto & \circ(a,b) = a \circ b = a + b - (es)(b). \end{array}$$

We will show that with this composition $\circ$ the diagrams of the definition of internal category commute. The triangles specifying the source and target of the identity-assigning morphism do not involve composition.

The second pair of diagrams commute since

$$\begin{aligned} s(a\circ b) &= s(a+b-(es)(b)) \\ &= s(a)+s(b)-(se)(s(b)) \\ &= s(a)+s(b)-s(b) \\ &= s(a)=x \end{aligned}$$

and since $t(a)=s(b)$,

$$\begin{aligned} t(a\circ b) &= t(a+b-(es)(b)) \\ &= t(a)+t(b)-(te)(s(b)) \\ &= t(a)+t(b)-s(b) \\ &= t(b)=z. \end{aligned}$$

The associative law holds for composition because module addition is associative. Finally the left and right unit laws are satisfied since given $a: x\longrightarrow y$,

$$\begin{aligned} e(x)\circ a &= e(x)+a-(es)(a) \\ &= e(x)+a-e(x) \\ &= a \end{aligned}$$

and

$$\begin{aligned} a\circ e(y) &= a+e(y)-(es)(e(y)) \\ &= a+e(y)-e(y) \\ &= a. \end{aligned}$$

We thus have a 2-module.

Given a 2-module M, we shall show that its composition must be defined by the formula given above. Suppose that (a,g) and (a',g') are composable pairs of morphisms in M_1, i.e.

$$a: x\longrightarrow y \text{ and } b: y\longrightarrow z$$

and

$$a': x'\longrightarrow y' \text{ and } b': y'\longrightarrow z'.$$

Since the source and target maps are module morphisms, $(a+a', b+b')$ also forms a composable pair, and since that the composition is module morphism

$$(a+a')\circ(b+b') = a\circ b + a'\circ b'.$$

Then if (a,b) is a composable pair, i.e, $t(a)=s(b)$, we have

$$\begin{aligned} a\circ b &= (a+1_{M_1})\circ(1_{M_1}+b)\\ &= (a+e(s(b)-s(b)))\circ(e(s(b)-s(b))+b)\\ &= (a-e(s(b))+e(s(b)))\circ(e(s(b))-e(s(b))+b)\\ &= (a\circ e(s(b)))+(-e(s(b))+e(s(b)))\circ(-e(s(b))+b)\\ &= a\circ e(s(b))+(-e(s(b))\circ(-e(s(b))))+(e(s(b))\circ b)\\ &= a-e(s(b))+b\\ &= a+b-e(s(b)). \end{aligned}$$

This show that we can define $\circ$ by

$$\begin{aligned} \circ:\ M_1\times_{M_0}M_1 &\longrightarrow M_1\\ (a,b) &\longmapsto \circ(a,b)=a\circ b=a+b-e(s(b)). \end{aligned}$$

Corollary 3.3. For $b\in\ker s$, we have

$$\begin{aligned} a\circ b &= a+b-(es)(b)\\ &= a+b. \end{aligned}$$

Definition 3.4. Let M and N be 2-modules, a 2-module functor $F:M\longrightarrow N$ is an internal functor in **Mod** from M to N. 2-modules and 2-module functors between them is called the category of 2-modules denoted by **2Mod**.

After we get the definition of a 2-module, we define the definition of a categorified algebra which is main concept of this paper.

3.2. Two-algebras

Definition 3.5. A weak 2-algebra consists of

· a 2-module A equipped with a functor $\bullet: A\times A\longrightarrow A$, which is defined by $(x,y)\mapsto x\bullet y$ and bilinear on objects and defined by $(f,g)\mapsto f\bullet g$ on morphisms satisfying interchange law, i.e.,

$$(f_1\bullet g_1)\circ(f_2\bullet g_2)=(f_1\circ f_2)\bullet(g_1\circ g_2)$$

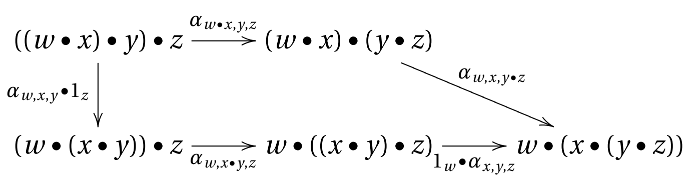
· $k-$bilinear natural isomorphisms

$$\alpha_{x,y,z}:(x\bullet y)\bullet z\longrightarrow x\bullet(y\bullet z)$$

$$l_x:1\bullet x\longrightarrow x$$

$$r_x:x\bullet 1\longrightarrow x$$

such that the following diagrams commute for all objects $w,x,y,z\in A_0$.



$$\begin{array}{ccc} (x \bullet 1) \bullet y & \xrightarrow{\alpha_{x,1,y}} & x \bullet (1 \bullet y) \\ & {\scriptstyle r_x \bullet 1_y} \searrow & \downarrow {\scriptstyle 1_x \bullet l_y} \\ & & x \bullet y \end{array}$$

A strict 2-algebra is the special case where $\alpha_{x,y,z}$, l_x, r_x are all identity morphisms. In this case we have

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

$$1 \bullet x = x, x \bullet 1 = x$$

Strict 2-algebra is called commutative strict 2-algebra if $x \bullet y = y \bullet x$ for all objects $x, y \in A_0$ and $f \bullet g = g \bullet f$ for all morphisms $f, g \in A_1$.

In the rest of this paper, the term 2-algebra will always refer to a commutative strict 2-algebra. A homomorphism between 2-algebras should preserve both the 2-module structure and the $\bullet$ functor.

Definition 3.6. Given 2-algebras A and A', a homomorphism

$$F : A \longrightarrow A'$$

consists of

· a linear functor F from the underlying 2-module of A to that of A', and

· a bilinear natural transformation

$$F_2(x, y) : F_0(x) \bullet F_0(y) \longrightarrow F_0(x \bullet y)$$

· an isomorphism $F : 1' \longrightarrow F_0(1)$ where 1 is the identity object of A and $1'$ is the identity object of A',

such that the following diagrams commute for $x, y, z \in A_0$,

$$\begin{array}{ccccc} (F(x) \bullet F(y)) \bullet F(z) & \xrightarrow{F_2 \bullet 1} & F(x \bullet y) \bullet F(z) & \xrightarrow{F_2} & F((x \bullet y) \bullet z) \\ {\scriptstyle \alpha_{F(x),F(y),F(z)}} \downarrow & & & & \downarrow {\scriptstyle F(\alpha_{x,y,z})} \\ F(x) \bullet (F(y) \bullet F(z)) & \xrightarrow[1 \bullet F_2]{} & F(x) \bullet F(y \bullet z) & \xrightarrow[F_2]{} & F(x \bullet (y \bullet z)). \end{array}$$

$$\begin{array}{ccc} 1' \bullet F(x) & \xrightarrow{l'_{F(x)}} & F(x) \\ {\scriptstyle F_0 \bullet 1} \downarrow & & \uparrow {\scriptstyle F(l_x)} \\ F(1) \bullet F(x) & \xrightarrow[F_2]{} & F(1 \bullet x). \end{array}$$

$$\begin{array}{ccc} F(x) \bullet 1' & \xrightarrow{r'_{F(x)}} & F(x) \\ {\scriptstyle 1 \bullet F_0} \downarrow & & \uparrow {\scriptstyle F(r_x)} \\ F(x) \bullet F(1) & \xrightarrow[F_2]{} & F(x \bullet 1). \end{array}$$

Definition 3.7. 2-algebras and homomorphisms between them give the category of 2-algebras denoted by

2Alg .

Therefore if $A = (A_0, A_1, s, t, e, \circ, \bullet)$ is a 2-algebra, A_0 and A_1 are algebras with this $\bullet$ bilinear functor. Thus we can take that 2-algebra is a 2-category with a single object say $*$, and A_0 collections of its 1-morphisms and A_1 collections of its 2-morphisms are algebras with identity.

3.3. Multiplication Algebras yield a 2-algebra

In [8] Norrie developed Lue's work [6] and introduced the notion of an actor of crossed modules of groups where it is shown to be the analogue of the automorphism group of a group. In the category of commutative algebras the appropriate replacement for automorphism groups is the multiplication algebra $\mathscr{M}(C)$ of an algebra C which is defined by MacLane [7].

Let C be an associative (not necessarily unitary or commutative) R-algebra. We recall Mac Lane's construction of the R-algebra $\mathrm{Bim}(C)$ of bimultipliers of C [7].

An element of $\mathrm{Bim}(C)$ is a pair (γ, δ) of R-linear mappings from C to C such that

$$\gamma(cc') = \gamma(c)c'$$

$$\delta(cc') = c\delta\left(c'\right)$$

and

$$c\gamma\left(c'\right) = \delta(c)c'.$$

$\mathrm{Bim}(C)$ has an obvious R-module structure and a product

$$(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta'\delta),$$

the value of which is still in $\mathrm{Bim}(C)$.

Suppose that $\mathrm{Ann}(C) = 0$ or $C^2 = C$. Then $\mathrm{Bim}(C)$ acts on C by

$$\begin{array}{lcll} \mathrm{Bim}(C) \times C & \to & C; & ((\gamma, \delta), c) \mapsto \gamma(c), \\ C \times \mathrm{Bim}(C) & \to & C; & (c, (\gamma, \delta)) \mapsto \delta(c) \end{array}$$

and there is a

$$\begin{array}{rccl} \mu : & C & \longrightarrow & \mathrm{Bim}(C) \\ & c & \longmapsto & (\gamma_c, \delta_c) \end{array}$$

with

$$\gamma_c(x) = cx \qquad \text{and} \qquad \delta_c(x) = xc.$$

Commutative case: we still assume $\mathrm{Ann}(C) = 0$ or $C^2 = C$. If C is a commutative R-algebra and $(\gamma, \delta) \in \mathrm{Bim}(C)$, then $\gamma = \delta$. This is because for every $x \in C$:

$$\begin{aligned} x\delta(c) &= \delta(c)x = c\gamma(x) = \gamma(x)c \\ &= \gamma(xc) = \gamma(cx) = \gamma(c)x = x\gamma(c). \end{aligned}$$

Thus $\mathrm{Bim}(C)$ may be identified with the R-algebra $\mathscr{M}(C)$ of multipliers of C. Recall that a multiplier of C is a linear mapping $\lambda : C \longrightarrow C$ such that for all $c, c' \in C$

$$\lambda(cc') = \lambda(c)c'.$$

Also $\mathscr{M}(C)$ is commutative as

$$\lambda'\lambda(xc) = \lambda'(\lambda(x)c) = \lambda(x)\lambda'(c) = \lambda'(c)\lambda(x) = \lambda\lambda'(cx) = \lambda\lambda'(xc)$$

for any $x \in C$. Thus $\mathscr{M}(C)$ is the set of all multipliers λ such that $\lambda\gamma = \gamma\lambda$ for every multiplier γ.

In [10] Porter states that automorphisms of a group G yield a 2-group. The appropriate analogue of this result in algebra case can be given. We claim that multiplications of an R-algebra C give a 2-algebra which is called a multiplication 2-algebra.

Let k be a commutative ring, R be a k-algebra with identity and C be a commutative R-algebra with $Ann(C) = 0$ or $C^2 = C$. Take $A_0 = \mathscr{M}(C)$ and say 1-morphisms to the elements of A_0. We define the action of $\mathscr{M}(C)$ on C as follows:

$$\begin{array}{ccc} \mathscr{M}(C) \times C & \longrightarrow & C \\ (f, x) & \longmapsto & f \blacktriangleright x = f(x). \end{array}$$

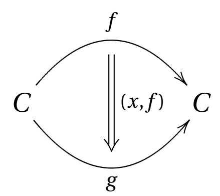
Using the action of $\mathscr{M}(C)$ on C, we can form the semidirect product

$$C \rtimes \mathscr{M}(C) = \{(x, f) | x \in C,\ f \in \mathscr{M}(C)\}$$

with multiplication

$$(x, f)(x', f') = (f \blacktriangleright x' + f' \blacktriangleright x + x'x, f'f).$$

Take $A_1 = C \rtimes \mathscr{M}(C)$ and say 2-morphisms to the elements of A_1. Therefore we get the following diagram for $(x, f) \in C \rtimes \mathscr{M}(C)$,



and we define the source, target and identity assigning maps as follows;

$$\begin{array}{cccc} s: & C \rtimes \mathscr{M}(C) & \longrightarrow & \mathscr{M}(C) \\ & (x, f) & \longmapsto & s(x, f) = f \end{array} \qquad \begin{array}{cccc} t: & C \rtimes \mathscr{M}(C) & \longrightarrow & \mathscr{M}(C) \\ & (x, f) & \longmapsto & t(x, f) = M_x \cdot f \end{array}$$

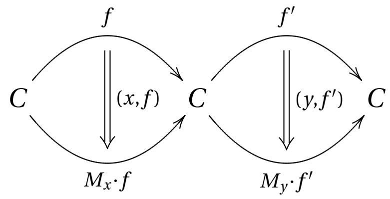
and

$$\begin{array}{cccc} e: & \mathscr{M}(C) & \longrightarrow & C \rtimes \mathscr{M}(C) \\ & f & \longmapsto & e(f) = (0, f) \end{array}$$

where $M_x \cdot f$ is defined by $(M_x \cdot f)(u) = xu + f(u)$ for $u \in C$.

There are two ways of composing 2-morphisms: vertical and horizontal composition. Now we define this compositions.

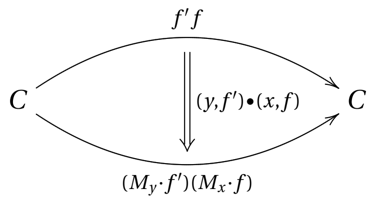
For $(x,f),(y,f')\in C\rtimes\mathscr{M}(C)$



the horizontal composition is defined by

$$(x,f)\bullet(y,f')=(f'(x)+f(y)+xy,f'f),$$

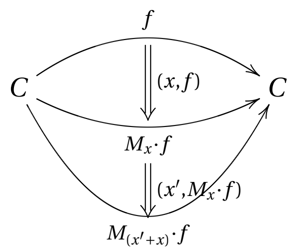
thus we have



and

$$\begin{aligned} t(f'(x)+f(y)+xy,f'f) &= M_{f'(x)+f(y)+xy}\cdot f'f \\ &= (M_y\cdot f')(M_x\cdot f) \end{aligned}$$

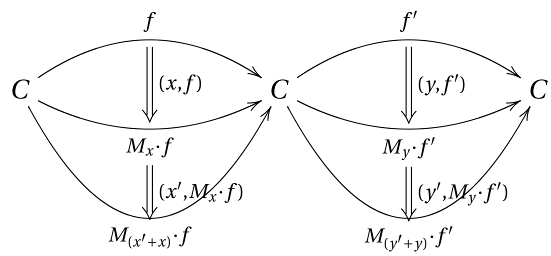
The vertical composition is defined by



$$(x,f)\circ(x',M_x\cdot f)=(x'+x,f)$$

for $(x,f),(x',M_x\cdot f)\in C\rtimes\mathscr{M}(C)$ with $t(x,f)=s(x',M_x\cdot f)=M_x\cdot f$.

It remains to satisfy the interchange law, i.e.



$$\begin{aligned} [(x,f)\circ(x',M_x\cdot f)]\bullet[(y,f')\circ(y',M_y\cdot f')] &= [(x,f)\bullet(y,f')] \\ &\quad \circ[(x',M_x\cdot f)\bullet(y',M_y\cdot f')]. \end{aligned}$$

Evaluating the two sides separately, we get

$$\begin{aligned}\textbf{LHS} &= (x'+x,f)\bullet(y'+y,f')\\ &= (f'(x'+x)+f(y'+y)+(x'+x)(y'+y),f'f)\\ &= (f'(x')+f'(x)+f(y')+f(y)+x'y'+x'y+xy'+xy,f'f)\end{aligned}$$

and

$$\begin{aligned}\textbf{RHS} &= (f'(x)+f(y)+xy,f'f)\circ((M_y\cdot f')(x')\\ &\quad +(M_x\cdot f)(y')+x'y',(M_y\cdot f')(M_x\cdot f))\\ &= (f'(x)+f(y)+xy+(M_y\cdot f')(x')+(M_x\cdot f)(y')+x'y',f'f)\\ &= (f'(x)+f(y)+xy+yx'+f'(x')+xy'+f(y')+x'y',f'f)\end{aligned}$$

LHS and RHS are equal, thus interchange law is satisfied. Therefore we get a 2-algebra consists of the R-algebra C as single object and the R-algebra A_0 of 1-morphisms and the R-algebra A_1 of 2-morphisms.

4. Crossed modules and 2-algebras

Crossed modules have been used widely and in various contexts since their definition by Whitehead [11] in his investigations of the algebraic structure of relative homotopy groups. We recalled the definition of crossed modules of commutative algebras given by Porter [10].

Let R be a k-algebra with identity. A pre-crossed module of commutative algebras is an R-algebra C together with a commutative action of R on C and a morphism

$$\partial : C \longrightarrow R$$

such that for all $c \in C$, $r \in R$

$$\text{CM1) } \partial(r \blacktriangleright c) = r\partial c.$$

This is a crossed R-module if in addition for all $c, c' \in C$

$$\text{CM2) } \partial c \blacktriangleright c' = cc'.$$

The last condition is called the Peiffer identity. We denote such a crossed module by (C,R,∂).

A morphism of crossed modules from (C,R,∂) to (C',R',∂') is a pair of k-algebra morphisms $\phi : C \longrightarrow C', \psi : R \longrightarrow R'$ such that

$$\partial'\phi = \psi\partial \qquad \text{and} \qquad \phi(r \blacktriangleright c) = \psi(r) \blacktriangleright \phi(c).$$

Thus we get a category $\mathbf{XMod}_k$ of crossed modules (for fixed k).

Examples of Crossed Modules

1. Any ideal I in R gives an inclusion map, $inc : I \longrightarrow R$ which is a crossed module. Conversely given an arbitrary R-module $\partial : C \longrightarrow R$ one easily sees that the Peiffer identity implies that ∂C is an ideal in R.

2. Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ sending everything in M to the zero element of R is a crossed module. Conversely: If (C,R,∂) is a crossed module, $\partial(C)$ acts trivially on $\ker\partial$, hence $\ker\partial$ has a natural $R/\partial(C)$-module structure.

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

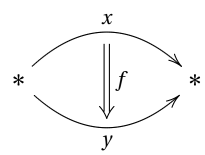
3. Let be $\mathscr{M}(C)$ multiplication algebra. Then $\left(C,\mathscr{M}(C),\mu\right)$ is multiplication crossed module. $\mu : C \to \mathscr{M}(C)$ is defined by $\mu(r) = \delta_r$ with $\delta_r\left(r'\right) = rr'$ for all $r,r' \in C$, where δ is multiplier $\delta : C \to C$ such that for all $r,r' \in C$, $\delta\left(rr'\right) = \delta(r)\,r'$. Also $\mathscr{M}(C)$ acts on C by $\delta \blacktriangleright r = \delta(r)$.(See [1] for details).

In [10] Porter states that there is an equivalence of categories between the category of internal categories in the category of k-algebras and the category of crossed modules of commutative k-algebras. In the following theorem, we will give a categorical presentation of this equivalence.

Theorem 4.1. The category of crossed modules $\mathbf{XMod}_k$ is equivalent to that of 2-algebras, **2Alg**.

Proof.

Let $A = (A_0, A_1, s, t, e, \circ, \bullet)$ be a 2-algebra consisting of a single object say $*$ and an algebra A_0 of 1-morphisms and an algebra A_1 of 2-morphisms. For $x, y \in A_0$ and $f : x \to y \in A_1$, we get the following diagram



We define s,t morphisms $s : A_1 \longrightarrow A_0, s(f) = x, t : A_1 \longrightarrow A_0, t(f) = y$ and e morphism $e : A_0 \longrightarrow A_1$ for $x \in A_0$, $e(x) : x \longrightarrow x \in A_1$.

The s,t and e morphisms are algebra morphisms and we have

$$\begin{aligned} se(x) &= s(e(x)) = x = Id_{A_0}(x) \\ te(x) &= t(e(x)) = x = Id_{A_0}(x) \end{aligned}$$

We define

$$\text{Ker } s = H = \{f \in A_1 \mid s(f) = Id_{A_0}\} \subseteq A_1$$

and $\partial = t\,|_H$ algebra homomorphism by $\partial : H \longrightarrow A_0, \partial(h) = t(h)$. We have semidirect product Ker $s \rtimes A_0 = \{(h,x) \mid h \in \text{Ker}s, x \in A_0\}$ with multiplication $(h,x)\bullet(h',x') = (x \blacktriangleright h' + x' \blacktriangleright h + h'\bullet h, x\bullet x')$ where action of A_0 on Ker$s$ is defined by $x \blacktriangleright h = e(x)\bullet h$. For each $f \in A_1$, we can write $f = n + e(x)$ where $n = f - es(f) \in$Ker$s$ and $x = s(f)$. Suppose $f' = n' + e(x')$. Then

$$\begin{aligned} f\bullet f' &= (n+e(x))\bullet(n'+e(x')) \\ &= n\bullet n' + n\bullet e(x') + e(x)\bullet n' + e(x)\bullet e(x') \\ &= e(x')\bullet n + e(x)\bullet n' + n\bullet n' + e(x\bullet x') \\ &= x' \blacktriangleright n + x \blacktriangleright n' + n\bullet n' + e(x\bullet x'). \end{aligned}$$

There is a map

$$\begin{aligned} \phi : \quad A_1 &\longrightarrow \text{Ker}s \rtimes A_0 \\ n + e(x) &\longmapsto \phi(n+e(x)) = (n,x). \end{aligned}$$

Now

$$\begin{aligned}\phi(f \bullet f') &= \phi(x' \blacktriangleright n + x \blacktriangleright n' + n \bullet n' + e(x \bullet x')) \\ &= (x' \blacktriangleright n + x \blacktriangleright n' + n \bullet n', x \bullet x') \\ &= (n,x) \bullet (n',x') \\ &= \phi(f) \bullet \phi(f')\end{aligned}$$

so ϕ is a homomorphism. Also, there is an obvious inverse

$$\begin{aligned}\phi^{-1}: \quad \mathrm{Ker}s \rtimes A_0 &\longrightarrow A_1 \\ (n,x) &\longmapsto \phi^{-1}(n,x) = n + e(x)\end{aligned}$$

which is also a homomorphism. Hence ϕ is an isomorphism and we have established that Ker $s \rtimes A_0 \simeq A_1$. Since A is a 2-algebra and Ker $s \rtimes A_0 \simeq A_1$, we can define algebra morphisms

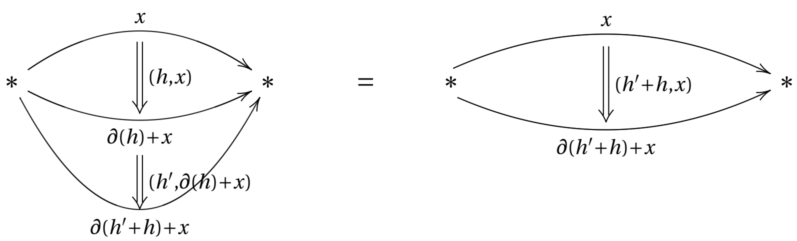
$$\begin{aligned}s: \quad \mathrm{Ker}s \rtimes A_0 &\longrightarrow A_0 \\ (h,x) &\longmapsto s(h,x) = x\end{aligned} \qquad \begin{aligned}t: \quad \mathrm{Ker}s \rtimes A_0 &\longrightarrow A_0 \\ (h,x) &\longmapsto t(h,x) = \partial(h) + x\end{aligned}$$

and

$$\begin{aligned}e: \quad A_0 &\longrightarrow \mathrm{Ker}s \rtimes A_0 \\ x &\longmapsto e(x) = (0,x)\end{aligned}$$

and for $t(h,x) = s(h', \partial(h) + x) = \partial(h) + x$ we define

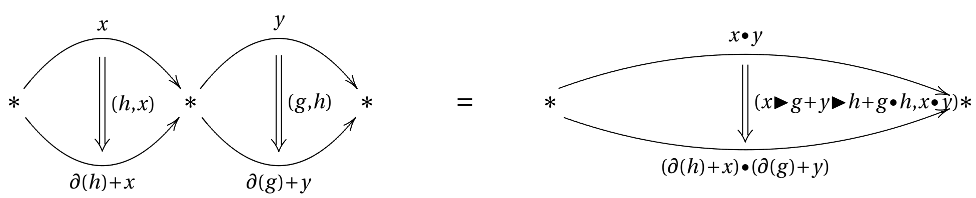
$$\begin{aligned}\circ: \quad \mathrm{Ker}s \rtimes A_0 \; {}_t\!\times_s \mathrm{Ker}s \rtimes A_0 &\longrightarrow \mathrm{Ker}s \rtimes A_0 \\ \big((h,x),(h',\partial(h)+x)\big) &\longmapsto (h'+h,x)\end{aligned}$$



which is vertical composition;

$$(h,x) \circ (h', \partial(h) + x) = (h' + h, x).$$

For $(h,x),(g,y) \in \mathrm{Ker}s \rtimes A_0$, horizontal composition is defined by



$$\begin{aligned}(h,x) \bullet (g,y) &= (x \blacktriangleright g + y \blacktriangleright h + g \bullet h, x \bullet y) \\ &= (e(x) \bullet g + e(y) \bullet h + g \bullet h, x \bullet y).\end{aligned}$$

Thus we have

CM1)

$$\begin{aligned}\partial(x \blacktriangleright h) &= \partial(e(x) \bullet h)\\ &= \partial(e(x)) \bullet \partial(h)\\ &= (te)(x) \bullet \partial(h)\\ &= x \bullet \partial(h).\end{aligned}$$

Also by interchange law we have

$$\begin{aligned}\left[(h,x)\bullet(g,y)\right]\circ\left[(h',\partial(h)+x)\bullet(g',\partial(g)+y)\right] &= \left[(h,x)\circ(h',\partial(h)+x)\right]\\ &\quad \bullet\left[(g,y)\circ(g',\partial(g)+y)\right].\end{aligned}$$

Therefore, evaluating the two sides of this equation gives:

$$\begin{aligned}LHS &= (x \blacktriangleright g + y \blacktriangleright h + g \bullet h, x \bullet y)\\ &\quad \circ((\partial(h)+x) \blacktriangleright g' + \left(\partial(g)+y\right) \blacktriangleright h' + g' \bullet h', (\partial(h)+x) \bullet \left(\partial(g)+y\right))\\ &= ((\partial(h)+x) \blacktriangleright g' + \left(\partial(g)+y\right) \blacktriangleright h' + g' \bullet h' + x \blacktriangleright g + y \blacktriangleright h + g \bullet h, x \bullet y)\\ &= (\partial(h) \blacktriangleright g' + e(x) \bullet g' + \partial(g) \blacktriangleright h'\\ &\quad + e(y) \bullet h' + g' \bullet h' + e(x) \bullet g + e(y) \bullet h + g \bullet h, x \bullet y)\\ RHS &= (h'+h, x) \bullet (g'+g, y)\\ &= (x \blacktriangleright \left(g'+g\right) + y \blacktriangleright (h'+h) + \left(g'+g\right) \bullet (h'+h), x \bullet y)\\ &= \left(e(x) \bullet g' + e(x) \bullet g + e(y) \bullet h' + e(y) \bullet h + g' \bullet h' + g' \bullet h + g \bullet h' + g \bullet h, x \bullet y\right).\end{aligned}$$

Since the two sides are equal, we know that their first components must be equal, so we have

$$\partial(h) \blacktriangleright g' + \partial(g) \blacktriangleright h' = h \bullet g' + g \bullet h'$$

and

$$\begin{aligned}h \bullet g' + g \bullet h' &= \partial(h) \blacktriangleright g' + \partial(g) \blacktriangleright h'\\ &= \partial(h+g) \blacktriangleright (g'+h') - \partial(h) \blacktriangleright h' - \partial(g) \blacktriangleright g'\\ &= \partial(h+g) \blacktriangleright (g'+h') - \left(h \bullet h' + g \bullet g'\right),\end{aligned}$$

thus

$$\begin{aligned}\partial(h+g) \blacktriangleright (g'+h') &= h \bullet g' + g \bullet h' + \left(h \bullet h' + g \bullet g'\right)\\ &= \left(h+g\right) \bullet \left(h'+g'\right)\end{aligned}$$

and writing $(h+g) = l, \left(h'+g'\right) = l' \in Kers$, we get

$$\partial(l) \blacktriangleright l' = l \bullet l'$$

which is the Peiffer identity as required. Hence $(Kers, A_0, \partial)$ is a crossed module.

Let $\mathscr{A} = (A_0, A_1, s, t, e, \circ, \bullet)$ and $\mathscr{A}' = (A_0', A_1', s', t', e', \circ', \bullet')$ be 2-algebras and $F = (F_0, F_1) : \mathscr{A} \longrightarrow \mathscr{A}'$ be a 2-algebra morphism. Then $F_0 : A_0 \longrightarrow A_0'$ and $F_1 : A_1 \longrightarrow A_1'$ are the k-algebra morphisms. We define $f_1 =$

$F_1|_{Kers} : Kers \longrightarrow Kers^{'}$ and $f_0 = F_0 : A_0 \longrightarrow A_0^{'}$. For all $a \in Kers$ and $x \in A_0$,

$$\begin{aligned} f_0\partial(a) &= F_0 t(a) \\ &= t^{'} F_1(a) \\ &= \partial^{'} f_1(a) \end{aligned}$$

and

$$\begin{aligned} f_1(x \blacktriangleright a) &= F_1(e(x)a) \\ &= F_1(e(x))F_1(a) \\ &= e^{'} F_0(x)F_1(a) \\ &= e^{'} f_0(x) f_1(a) \\ &= f_0(x) \blacktriangleright f_1(a). \end{aligned}$$

Thus (f_1, f_0) map is a crossed module morphism $(Kers, A_0, \partial) \longrightarrow (Kers^{'}, A_0^{'}, \partial^{'})$. So we have a functor

$$\Gamma : \mathbf{2Alg} \longrightarrow \mathbf{XMod}_k.$$

Conversely, let (G, C, ∂) be a crossed module of algebras. Therefore there is an algebra morphism $\partial : G \to C$ and an action of C on G such that

CM1) $\partial(x \blacktriangleright g) = x\partial(g)$,

CM2) $\partial(g) \blacktriangleright g' = gg'$.

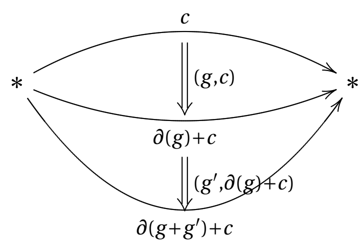
Since C acts on G, we can form the semidirect product $G \rtimes C$ as defined by

$$G \rtimes C = \{(g, c) \mid g \in G, c \in C\}$$

with multiplication

$$(g, c)(g', c') = (c \blacktriangleright g' + c' \blacktriangleright g + g'g, cc')$$

and define maps $s, t : G \rtimes C \to C$ and $e : C \to G \rtimes C$ by $s(g, c) = c$, $t(g, c) = \partial(g) + c$ and $e(c) = (0, c)$. These maps are clearly algebra morphisms.



For $t(g, c) = s(g', \partial(g) + c) = \partial(g) + c$, we define composition

$$\begin{aligned} \circ : \quad (G \rtimes C)_t \times_s (G \rtimes C) &\longrightarrow (G \rtimes C) \\ ((g, c), (g', \partial(g) + c)) &\longmapsto (g + g', c), \end{aligned}$$

for $(g, c), (h, d) \in G \rtimes C$ and $(g, c), (g', \partial(g) + c) \in G \rtimes C$, following equations give horizontal and vertical composition respectively.

$$(g, c) \bullet (h, d) = (c \blacktriangleright h + d \blacktriangleright g + gh, cd)$$

$$(g,c)\circ(g',\partial(g)+c)=(g+g',c)$$

Finally, since it must be that $\circ$ is an algebra morphism and by the crossed module conditions, interchange law is satisfied. Therefore we have constructed a 2-algebra $\mathscr{A}=(C,G\rtimes C,s,t,e,\circ,\bullet)$ consists of the single object say $*$ and the k-algebra C of 1-morphisms and the k-algebra $G\rtimes C$ of 2-morphisms. Let (G,C,∂) and $(G^{'},C^{'},\partial^{'})$ be crossed modules and $f=(f_1,f_0):(G,C,\partial)\longrightarrow(G^{'},C^{'},\partial^{'})$ be a crossed module morphism. We define

$$\begin{array}{rccc} F_1: & G\rtimes C & \longrightarrow & G^{'}\rtimes C^{'} \\ & (g,c) & \longmapsto & F_1(g,c)=(f_1(g),f_0(c)) \end{array}$$

and

$$\begin{array}{rccc} F_0: & C & \longrightarrow & C^{'} \\ & c & \longmapsto & F_0(c)=f_0(c). \end{array}$$

Then

$$\begin{aligned} s^{'}F_1(g,c) &= s^{'}(f_1(g),f_0(c)) \\ &= f_0(c) \\ &= F_0(c) \\ &= F_0s(g,c), \end{aligned}$$

$$\begin{aligned} t^{'}F_1(g,c) &= t^{'}(f_1(g),f_0(c)) \\ &= \partial^{'}f_1(g)+f_0(c) \\ &= f_0\partial(g)+f_0(c) \\ &= F_0(\partial(g)+c) \\ &= F_0t(g,c), \end{aligned}$$

$$\begin{aligned} e^{'}F_0(c) &= (0,f_0(c)) \\ &= F_1(0,c) \\ &= F_1e(c), \end{aligned}$$

$$\begin{aligned} F_1((g,c)\circ(g^{'},c^{'})) &= F_1(g+g^{'},c) \\ &= (f_1(g+g^{'}),f_0(c)) \\ &= (f_1(g)+f_1(g^{'}),f_0(c)) \\ &= (f_1(g),f_0(c))\circ(f_1(g^{'}),f_0(c^{'})) \\ &= F_1(g,c)\circ F_1(g^{'},c^{'}), \end{aligned}$$

$$\begin{aligned} F_1((g,c)\bullet(h,d)) &= F_1(c\blacktriangleright h+d\blacktriangleright g+gh,cd) \\ &= (f_1(c\blacktriangleright h)+f_1(d\blacktriangleright g)+f_1(gh),f_0(cd)) \\ &= (f_0(c)\blacktriangleright f_1(h)+f_0(d)\blacktriangleright f_1(g)+f_1(g)f_1(h),f_0(c)f_0(d)) \\ &= (f_1(g),f_0(c))\bullet(f_1(h),f_0(d)) \\ &= F_1(g,c)\bullet F_1(h,d) \end{aligned}$$

for all $(g,c)\in G\rtimes C$ and $c\in C$. Therefore $F=(F_1,F_0)$ is a 2-algebra morphism from $(C,G\rtimes C,s,t,e,\circ,\bullet)$ to $(C^{'},G^{'}\rtimes C^{'},s^{'},t^{'},e^{'},\circ^{'},\bullet^{'})$. Thus we get a functor

$$\Psi:\mathbf{XMod}_k\longrightarrow\mathbf{2Alg}.$$

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Constructing The Ellipse and Its Application in Analytical Fuzzy Plane Geometry

Seçil Özekinci[1], Cansel Aycan[2]

Keywords:
Fuzzy Ellipse,
Fuzzy Points,
Membership Function,
Same Fuzzy Points,
α - cuts

Abstract — In this paper, we studied about a detailed analysis of fuzzy ellipse. In the previously studies, some methods for fuzzy parabola are discussed (Ghosh and Chakraborty, 2019). To define the fuzzy ellipse, it is necessary to modify the method applied for the fuzzy parabola. First, need to get five same points with the same membership grade to create crisp ellipse and the union of crisp ellipses passing through these points will form the fuzzy ellipse. Although it is difficult to determine the points with this property, it is important for constructing the fuzzy ellipse equation. In this study, we determine the points that satisfy this condition and prove the properties required to obtain the fuzzy ellipse to be formed by using these points. We have drawn a graph of a fuzzy ellipse and depicted the geometric location of fuzzy points with different membership grades on graph. We have also shown some geometric application on examples. In the third part of this study, it has been shown that the determinants defined in the calculation of the coefficients of the fuzzy ellipse can be calculated for different points and angles with the examples given, thus different fuzzy ellipses can be obtained.

Subject Classification (2020): 51H25, 51M05, 51M15, 53C80, 58A40, 58A05.

1. Introduction

In the case of crisp sets, a given object x may belong to a set A or not belong to this set and these two options are denoted by $x \in A$ or $x \notin A$, A classic set may be described by the characteristic function (χ_A) that takes two values:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Fuzzy sets are introduced and described using membership functions by Zadeh in 1965 [10]. As opposed to crisp set, if $\bar{A}$ is a fuzzy set, we write its membership function as $\mu(x|\bar{A})$, $\mu(x|\bar{A})$ is in $[0,1]$ for all x.

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Many studies are available to understand fuzzy logic [10 − 12]. Likewise, certain ideas in fuzzy plane geometry have been introduced and studied by Buckley and Eslami in the study [1] may be the first to analyze fuzzy sets. Fuzzy points and the fuzzy distance between fuzzy points was defined by Buckley an Eslami in [8]. And they showed it is a (weak) fuzzy metric and fuzzy point, fuzzy line segment, fuzzy distance and the angle between two fuzzy segments and same and inverse points are defined by Ghosh and Chakraborty [3]. Buckley and Eslami studied fuzzy points and fuzzy lines and gave the theorems about them in [1] and studied fuzzy circles, fuzzy rectangles, fuzzy triangles and fuzzy polygons and showed that the area and perimeter of a fuzzy circle and a fuzzy polygon are a fuzzy number in [2]. A fuzzy line passing through several fuzzy points whose cores are collinear and introduced four different forms of fuzzy lines were introduced by Ghosh and Chakraborty in the study [4]. Ghosh and Chakraborty constructed a fuzzy circle in a fuzzy geometrical plane and showed that the center of a fuzzy circle may not be a fuzzy point in [5]. Rosenfeld presented fuzzy geometry and fuzzy topology of image subsets [9]. The fuzzy triangle as the intersection of three fuzzy half-planes and computed area and perimeter of the fuzzy triangle were discussed by Rosenfeld in the study [8]. Zimmermann dealt with types of fuzzy sets, fuzzy measures, fuzzy functions, applications of fuzzy set theory and gave basic definitions and theorems about fuzzy sets [12]. A fuzzy parabola that passes through five fuzzy points are constructed by Ghosh and Chakraborty in the study [6]. Then Özekinci and Aycan introduced a method to construct a fuzzy hyperbola and made applications about fuzzy hyperbola [7].

Fuzzy set theory provides a convenient method that is easy to implement in real-time applications, and also enables designers and operators to transfer their knowledge to the dynamic control systems. Fuzzy logic is also used in different fields such as artificial intelligence, computers, face recognition systems, cybernetic internet technologies, space vehicles, robot and war technologies, the formation of the universe, etc. Fuzzy logic has been the subject of many studies since it is an approach that is not only theoretical but also practical. When all these studies are examined geometrically, it is seen that only fuzzy circle, fuzzy parabola and fuzzy hyperbola curves are studied from the conics. No study has got been to construct a fuzzy ellipse. Fuzzy systems are used in the planning of technological structures developing in the field of engineering nowadays. Then fuzzy ellipse can be use kidney stones crushing machines, billiard games, aerospace engineering and lazer technology etc. Therefore, in this study, we studied how to construct a fuzzy ellipse and to obtain the equation for the geometric location of a fuzzy ellipse by using the properties of conics. While we aimed to analyze how the fuzzy ellipse could be defined, calculated and graphed mathematically, we thought that it would be useful to work on combining the applications mentioned above. We examine and prove these calculations with the evaluation of previous studies.

2. Preliminaries

In this section, we will mention the basic fuzzy definitions that will be used in this paper.

We will draw "a bar" over capital letters to denote a fuzzy subset of R^n, $i.e\ \bar{A}, \bar{B}, \bar{X}, \bar{Y}, \ldots$ and we will write membership of fuzzy set $\bar{A}$ as $\mu(x|\ \bar{A}\ ), x \in R^n$ and $\mu(R^n)$ is in [0,1].

Definition 2.1: (Fuzzy Set) The set of ordered pairs $\bar{A} = \{(x, \mu(x|\bar{A}\ )): x \in X\}$, where $\mu: X \to [0,1]$ is called a fuzzy set in X. The function $\mu: X \to [0,1]$ evaluates membership degree of x in the fuzzy set $\bar{A}$ [2].

Definition 2.2: For a fuzzy set $\bar{A}$ of R^n, its $\alpha - cut$ is denoted by $\bar{A}(\alpha)$ and it is defined by:

$$\bar{A}(\alpha) = \begin{cases} \{x, \mu(x|\bar{A}\ ) \geq \alpha\ \}\ if\ 0 < \alpha \leq 1 \\ \text{Clouse}\ \{x, \mu(x|\bar{A}\ ) > 0\}\ if\ \alpha = 0 \end{cases}$$

The set $\{x, \mu(x|\bar{A}\ ) > 0\ \}$ is called as support of the fuzzy set $\bar{A}$. The set $\bar{A}(0)$ ise often said as base of $\bar{A}$ and the set $\bar{A}(1) = \{x, \mu(x|\bar{A}\ ) = 1\}$ is said to be core of the fuzzy set $\bar{A}$. If the core is non-empty, the fuzzy set is called as a normal fuzzy set. A fuzzy set is said to be convex if all of its α-cuts are convex [3].

Definition 2.3 (Fuzzy Points): A fuzzy point at (a, b) in R^2, written as $\bar{P}(a, b)$ is defined by its membership function:

(i) $\mu((x, y)|\ \bar{P}(a, b))$ is upper semi-continuous,

(ii) $\mu((x, y)|\ \bar{P}(a, b)) = 1$ if and only if $(x, y) = (a, b)$,

(iii) $\bar{P}(a, b)(\alpha)$ is a compact, convex subset of R^2 of all α in [0,1].

The notations $\bar{P}_1(a, b), \bar{P}_2(a, b), \bar{P}_3(a, b), \ldots$ or $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots$ are used to represent fuzzy points [2].

Definition 2.4 (Same points with respect to fuzzy points): Let take two points (x_1, y_1) and (x_2, y_2). Such that (x_1, y_1) is support of fuzzy point $\bar{P}(a, b)$ and similarly (x_2, y_2) is support of fuzzy point $\bar{P}(c, d)$. Let L_1 is a line joining (x_1, y_1) and (a, b). As $\bar{P}(a, b)$ is a fuzzy point, along L_1, a fuzzy number, $\bar{r}_1$ say, is situated on the support of $\bar{P}(a, b)$. The membership function of this fuzzy number $\bar{r}_1$ can be written as $\mu((x, y)|\bar{r}_1) = \mu((x, y)|\ \bar{P}(a, b))$ for (x, y) in L_1 , and 0 otherwise. Similarly, along a line, L_2 say, joining (x_2, y_2) and (c, d), the exists a fuzzy number, $\bar{r}_2$ say, on the support of $\bar{P}(c, d)$. The points (x_1, y_1) and (x_2, y_2) are said to be same points with respect to $\bar{P}(a, b)$ and $\bar{P}(c, d)$ if :

(i) (x_1, y_1) and (x_2, y_2) are same -points with respect to $\bar{r}_1$ and $\bar{r}_2$,

(ii) L_1, L_2 have equal angle with line joining (a, b) and (c, d) [3].

3. Fuzzy Ellipse

In this section we will develop a method for obtaining a fuzzy ellipse. As its known the general conic equation has the form:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

If we divide both sides of this equation by a $(a \neq 0)$, this equation takes the form:

$$x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0$$

Thus, the number of unknown coefficients, in the equation containing six terms, becomes five. Then the common solution of the five equations, will be obtained by substituting five different pairs for x and y, will be sufficient to find these unknowns. So, five points in the plane is enough to write a conic equation. Namely five points on the plane will denote a single conic.

In this study, since we will define an ellipse in fuzzy plane geometry, first of all, these five points must be points in the fuzzy space that ensure the necessary properties. It will also be seen that the ellipse in fuzzy space is formed by different crisp elliptic curves. Their combination will form the fuzzy ellipse. The curve of ellipse passing through the core of five fuzzy points will be called a crisp ellipse and be denoted by CE . However, since these points are fuzzy points, their membership degrees may change. Differences in membership degrees affect the drawing of the resulting ellipse curves. Therefore, calculating five different coefficients for five same-points in the conic equation. Calculation of these coefficients is possible with five by determinants. For this reason, five different curves emerge for the fuzzy ellipse that we want to reach in our study. Therefore, in terms of the importance of the fuzzy membership degree, the definite ellipse CE with membership degree one is taken. The other four curves are ellipse and the combination of all of them gives the fuzzy ellipse and is denoted by FE. The system formed by these curves can also be considered as a curvilinear system or distribution in mathematical applications.

Now, we will denote a method to create a fuzzy ellipse in a fuzzy plane by taking five fuzzy points. These points will be the same-points which we gave in Definition 2.4 in preliminaries section.

Necessary explanations and proofs are presented below.

Let $\bar{E}_i(a_i, b_i), i = 1,2, \dots, 5$ be given five fuzzy points whose cores lie on a crisp ellipse CE . We will construct a fuzzy ellipse that passes through these five fuzzy points $\bar{E}_1, \bar{E}_2, \dots, \bar{E}_5$. We will denote fuzzy ellipse as $\overline{FE}$, briefly. Below are the steps of the method we used to create the fuzzy ellipse.

3.1. Construction of Fuzzy Ellipse $\overline{FE}$

In this section we construct the segment $\overline{FE}_{\;1\ldots5}$ for the fuzzy ellipse. This segment defined as,

$$\overline{FE} = \bigvee_{\alpha\in[0,1]} \left\{ \begin{matrix} FE_\alpha : Where\ FE_\alpha\ is\ a\ crisp\ ellipse\ that\ passes \\ through\ five\ same\ points\ an\ \bar{E}_i(a_i, b_i), \\ i = 1,2, \ldots ,5\ with\ membership\ value\ \alpha \end{matrix} \right\}$$

The ellipse $\overline{FE}$ can be defined by membership function below:

$$\mu\big((x,y)|\overline{FE}\big) = sup \left\{ \begin{matrix} \alpha : Where\ (x,y)\ lies\ on\ FE_\alpha\ that \\ passes\ through\ five\ same\ points\ on \\ \bar{E}_i, i = 1,2, \ldots ,5\ with\ membership\ value\ \alpha \end{matrix} \right\}$$

As this definition show that the fuzzy elliptic-segment $\overline{FE}_{\;1\ldots5}$ is a collection of crisp points with various membership degrees. However, the definition of membership function $\mu\left((x,y)|\overline{FE}_{\;1\ldots5}\right)$ shows that a fuzzy ellipse is the union of all crisp ellipses that pass through five same-points on the supports of $\bar{E}_i, i = 1,2, \ldots ,5$.

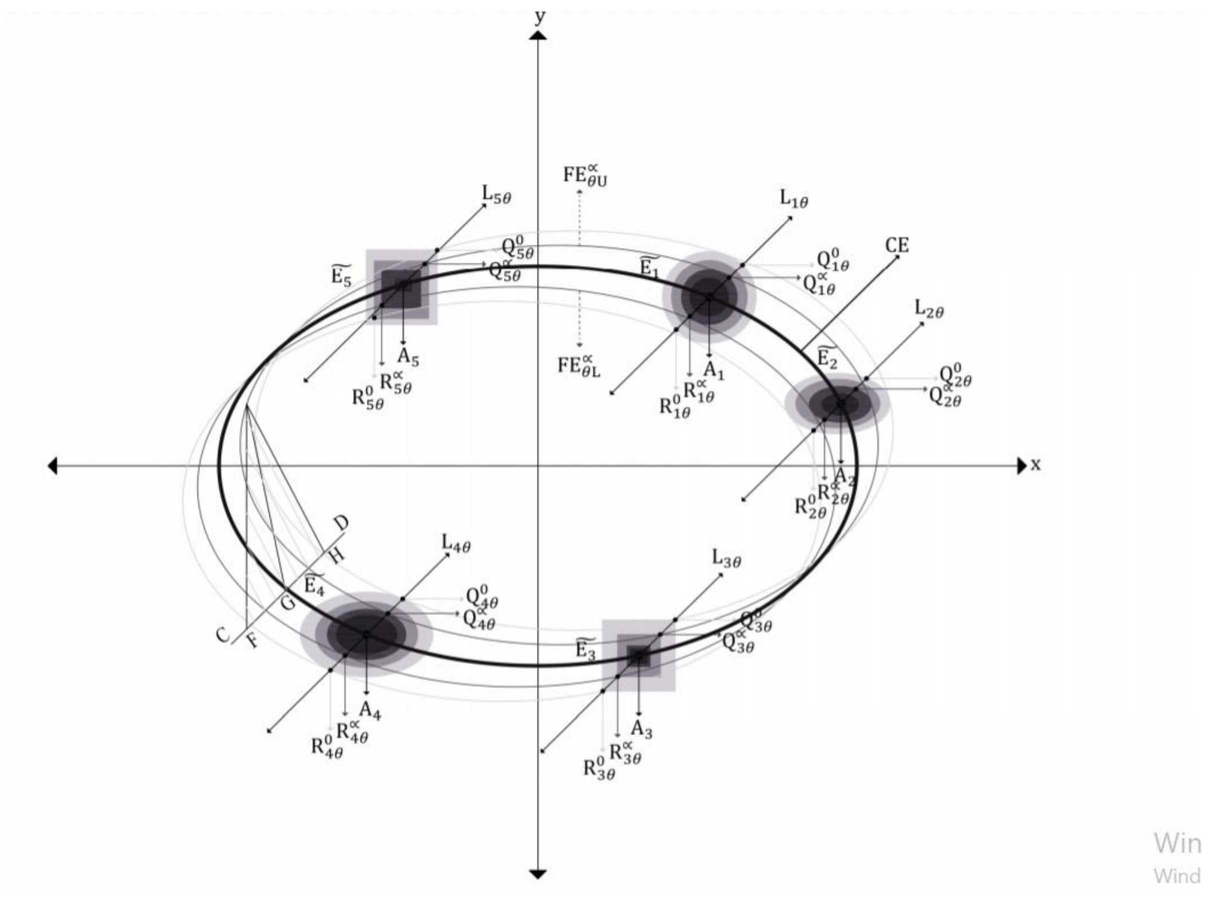

Figure.3.1. Construction of Fuzzy Ellipse in the Method

In Figure 3.1, we depict the fuzzy ellipse with membership degrees of given fuzzy points in detail. $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4$ and $\bar{E}_5$ are five fuzzy points. The regions under the circle centered at A_1, ellipse centered at A_2, square centered A_3, ellipse centered at A_4 and square centered at A_5 are the supports of the points $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4$ and $\bar{E}_5$, respectively. The grey -shaded regions inside the supports of the fuzzy points represent different α-cuts. The variation of the membership grades for fuzzy points is indicated by the intensity of the grey levels. The regions that, depicted in dark grey in the graph are formed by points with a high membership grade. Light grey regions on the graph are obtained as the membership grades approach 0. So, the membership grades of the centers of circle, squares and ellipses are one and it decreases gradually to zero on the periphery of the support of $\bar{E}_i$ for each $i = 1,2,3,4,5$.

In the Figure 3.1, $L_{i\theta}$'s are five lines that passes through A_i, for each $i = 1,2,3,4,5$. These five lines have an angle θ with the positive x-axis. Because $\bar{E}_i(A_i)(\alpha)$, being α −cut of a fuzzy point, is convex and A_i is an interior point of $\bar{E}_i(A_i)(\alpha)$, the line $L_{i\theta}$ must intersect with the boundary of $\bar{E}_i(A_i)(\alpha)$ at exactly two points. Let these two intersecting points be $Q_{i\theta}^{\alpha}$ and $R_{i\theta}^{\alpha}$. Thus, $Q_{1\theta}^{\alpha}, Q_{2\theta}^{\alpha}, Q_{3\theta}^{\alpha}, Q_{4\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$ constitute a set of five same-points with membership degree α. And similarly, the collection of $R_{i\theta}^{\alpha}$'s are also represent the set of five same-points with membership degree α.

Let $FE_{\theta U}^{\alpha}$ is the ellipse that passes through the points $Q_{i\theta}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ is the ellipse that passes through the points $R_{i\theta}^{\alpha}$'s in Figure 3.1 Since membership degree of all the points $Q_{i\theta}^{\alpha}$ and $R_{i\theta}^{\alpha}$ is α, we put a membership degree of α to the ellipse $FE_{\theta U}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ on the fuzzy ellipse $\overline{FE}, i = 1,2,3,4,5$.
Trough varying θ in $[0,2\pi]$ and α in $[0,1]$, several ellipses such as $FE_{\theta U}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ will be obtained. According to the definition, the fuzzy ellipse $\overline{FE}$ is the collection of all the ellipses $FE_{\theta U}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ with membership degree α.

Namely, we say

$$\overline{FE} = \bigvee_{\substack{\theta\in[0,2\pi] \\ \alpha\in[0,1]}} \{FE_{\theta U}^{\alpha}, FE_{\theta L}^{\alpha}\}$$

Let FE be any ellipse in the support of the fuzzy ellipse $\overline{FE}$. We define the membership degree of on ellipse FE in $\overline{FE}$ by

$$\mu(FE|\,\overline{FE}) = \min_{(x,y)\in FE} \mu((x,y)|\overline{FE}).$$

The underlying theorem shows how to obtain the membership degree ellipse FE in $\overline{FE}$ using the same -points in $\bar{E}_i$ $'s$, $i = 1,2,3,4,5$.

Theorem 3.1. Suppose that FE is an ellipse in $\overline{FE}$ and same -points $(x_i, y_i) \in \bar{E}_i(0)$ with $\mu\big((x_i, y_i)|\ \overline{FE}\big) = \alpha$ for all $i = 1,2,3,4,5$ such that FE is the ellipse that passes through the five (x_i, y_i) 's and $\mu(FE|\ \overline{FE}) = \alpha$.

Proof .

We examine the proof in two different cases that (i) $\mu(FE|\overline{FE}) \nless \alpha$ and (ii) $\mu(FE|\overline{FE}) \ngtr \alpha$.

(i) By contrast, let assume that $\mu(FE|\overline{FE}) < \alpha$. In that case, by the definition of $\mu(FE|\overline{FE})$, there exist (x_0, y_0) in $\overline{FE}$ such that $(x_0, y_0) \in FE$ and

$\mu((x_0, y_0)|\overline{FE}) < \alpha$. Let say $\mu((x_0, y_0)|\overline{FE}) = \beta$. Since $(x_0, y_0) \in FE$ and FE is an ellipse that joins the five same-points with membership degree α,

$$\mu((x_0, y_0)|\overline{FE}) = sup\begin{Bmatrix} \psi: where\ (x,y)\ lies\ on\ the\ ellipse \\ that\ joins\ the\ five\ same\ points \\ with\ membership\ degree\ \psi \end{Bmatrix} \geq \alpha.$$

But this contradicts our acceptance $\beta < \alpha$. So, $\mu(FE|\overline{FE}) \nless \alpha$.

(ii) It is clear that $\mu(FE|\overline{FE}) \ngtr \alpha$. Since $\mu(FE|\overline{FE}) = \min\begin{Bmatrix} \alpha\text{: where } (x,y) \text{ lies on} \\ FE \text{ and } \mu(FE|\overline{FE}) = \alpha \end{Bmatrix}$, and all the points $(x_i, y_i), i = 1,2,3,4,5$ lie on FE.

Therefore $\mu(FE|\overline{FE}) = \alpha$ is obtained.

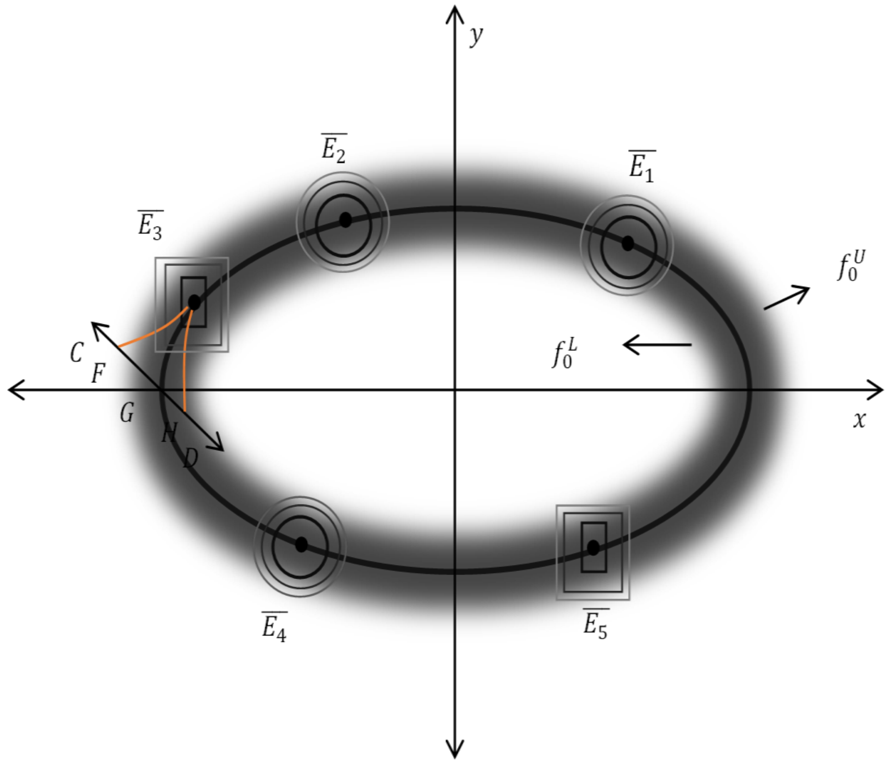

Figure 3.2 Fuzzy Ellipse (towards completing the fuzzy ellipse in the Figure 3.1)

The complete fuzzy ellipse $\overline{FE}$ is depicted in Figure 3.2. The region between the curves f_0L and f_0U is the support of the $\overline{FE}$. The core ellipse is the curve CE on which the five core points A_i of the fuzzy points $\overline{E_i}$ lies.

Let mention the line perpendicular to $CE \equiv \overline{FE}(1)$ that we take as the CD line in Figure 3.2. Along the CD, there exist a LR type fuzzy number that we denoted by $(F/G/H)_{LR}$. If we explain LR type fuzzy number like this, L and R are reference functions $L \; and \; R: [0, +\infty) \rightarrow [0,1]$ that doesn't decrease and satisfies two conditions $L(x) = L(-x)$ and $L(0) = 1$. Where α and β are positive and $\bar{A}$ is a fuzzy number, $\mu(x|\bar{A})$ can be written as:

$$\mu(x|\bar{A}) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & if \; x \leq m \\ R\left(\dfrac{x-m}{\beta}\right) & if \; x \geq m \end{cases}$$

The notation $(m-\alpha/\,m/m+\beta)_{LR}$ is used to represent an LR-type fuzzy number. That is, all fuzzy ellipses can be visualized as a three-dimensional figure. (a subset of $(x,y) \times [0,1]$) whose cross-section across $\overline{FE}$ is a fuzzy number such as $(F/G/H)_{LR}$.

Let $(F/G/H)_{LR}$ is a fuzzy number on fuzzy ellipse $\overline{FE}$ and take a convex region on $\overline{FE}(0)$ such that, except F and H, all points on the line segment $[FH]$ are inner of convex region. When we take a fuzzy point $\bar{E}$ such that the membership function is $\mu\big((x,y)|\,\bar{E}\big) = \mu\big((x,y)|(F/G/H)_{LR}\big)$, if $(x,y) \in [FH]$, $\mu\big((x,y)|\,\bar{E}\big) \leq \mu\big((x,y)|\,\overline{FE}\big)$. Only at G, $\mu\big((x,y)|\,\bar{E}\big) = 1$. Membership degree decreases gradually to '0' that approach F or H.

3.2. Construction of Membership Function

The membership degree $\mu\big((x,y)|\,\overline{FE}\,\big)$ might not always be simple to evaluate. Furthermore, it is really a difficult task to obtain the closed form of the membership function of $\overline{FE}$. Because, the membership degree at a particular point is the supremum of a set of real numbers that is obtained by solving a set of nonlinear equations. First, we get the closed form of membership function of $\overline{FE}$.

We note that the definition of fuzzy ellipse implies

$$\mu\big((x,y)|\,\overline{FE}\,\big) = sup \left\{ \begin{matrix} \alpha: where \; (x,y) \; lies \; in \; an \; ellipse \; that \\ passes \; through \; five \; same \; points \; in \; \bar{E}_i, \\ i = 1,2,3,4,5 \; with \; membership \; \text{degree} \; \alpha \end{matrix} \right\}.$$

For obtaining $\mu\big((x,y)|\,\overline{FE}\big)$, first we must find five same-points with membership degree $\alpha \in [0,1]$. Then, all possible values of α are identified for which (x,y) lies on the ellipse that joins five same-points with membership degrees. The evaluation of α may require to solving a nonlinear equation. From the solution of the equation, there may be real values between 0 and 1. The supremum of all these real α values is the membership degree of $\mu\big((x,y)|\,\overline{FE}\,\big)$. We refer the ellipse for which the supremum is attained as the adjoining ellipse of the points (x,y).

Now, we obtain a systematic procedure to identify the membership degree of a point (x_0,y_0) in a fuzzy ellipse $\overline{FE}$ which passes through five fuzzy points $\bar{E}_i$, $i = 1,2,3,4,5$ We show the expansion of the same-points on $\bar{E}_i$'s as $(x_{i\theta}^{\alpha}, y_{i\theta}^{\alpha})$, $i = 1,2,3,4,5$ $(0 \le \theta \le 2\pi\ , \alpha \in [0,1])$

As a result, we will have to examine the existence of solution of non-linear equations by giving various values to θ and determining the α membership degrees according to the angle θ.

Let the angle $\theta = \theta_0$ $(0 \le \theta \le 2\pi)$ and S_{θ_0}'s are the sets of membership degrees that can be compatible with respect to the various angle θ_0.

We assume that the supremum of the set S_{θ_0} as s_{θ_0}. It can be seen from the given examples that non-linear equation systems may not have a solution for some θ_0. Fuzzy ellipse $\overline{FE}$ are obtained by determining and giving appropriate values. Then the membership degree of (x_0,y_0) in the $\overline{FE}$ fuzzy ellipse is given by

$$\mu\big((x_0,y_0)|\overline{FE}\big) = \sup_{\theta} s_{\theta_0}.$$

The explanation of this part is given also in this section where the membership function is explained. Let give the application of the procedure with following examples.

The following examples illustrate the procedure numerically.

Example 3.1: Let $\overline{E_1}(0,1), \overline{E_2}\left(\frac{1}{5},\frac{4\sqrt{6}}{5}\right), \overline{E_3}\left(-\frac{1}{4},\frac{\sqrt{15}}{2}\right), \overline{E_4}\left(-\frac{1}{2},-\sqrt{3}\right)$ and $\overline{E_5}\left(\frac{1}{3},\frac{4\sqrt{2}}{3}\right)$ be five fuzzy points. Let's get the fuzzy ellipse that passes through these points. The equation of the core ellipse through the points is

$$\left\{(x,y): x^2 + \frac{y^2}{4} = 1\right\}$$

In this example we take core of that the points are in different regions on the curve.

The membership function of these five fuzzy points are circular and elliptical cones with bases, respectively.

$$\{(x,y): (x-1)^2 + y^2 \le 1\} \text{ (circular)}$$

$$\left\{(x,y): \left(x-\frac{1}{5}\right)^2 + 4\left(y-\frac{4\sqrt{6}}{5}\right)^2 \le 1\right\} \text{ (eliptical)}$$

$$\left\{(x,y):\left(x+\frac{1}{4}\right)^2+\left(y-\frac{\sqrt{15}}{2}\right)^2\leq 1\right\}\text{ (circular)}$$

$$\left\{(x,y):\left(x+\frac{1}{2}\right)^2+\left(y+\sqrt{3}\right)^2\leq 1\right\}\text{ (eliptical)}$$

$$\left\{(x,y):\left(x-\frac{1}{3}\right)^2+\left(y-\frac{4\sqrt{2}}{3}\right)^2\leq 1\right\}\text{(circular)}$$

The vertices of the membership functions are $(1,0),\left(\frac{1}{5},\frac{4\sqrt{6}}{5}\right),\left(-\frac{1}{4},\frac{\sqrt{15}}{2}\right),$ $\left(-\frac{1}{2},-\sqrt{3}\right)$ and $\left(\frac{1}{3},\frac{4\sqrt{2}}{3}\right)$ respectively.

Now, for $\alpha\in[0,1]$, we may find the same-points with membership degree α on $\bar{E}_1,\bar{E}_2,\bar{E}_3,\bar{E}_4$ and $\bar{E}_5$ as below;

$$Q_{1\theta}^{\alpha}:(x_{1\theta}^{\alpha},y_{1\theta}^{\alpha})=(1+(1-\alpha)\cos\theta\,,(1-\alpha)\sin\theta)$$

$$Q_{2\theta}^{\alpha}:(x_{2\theta}^{\alpha},y_{2\theta}^{\alpha})=\left(\frac{1}{5}+(1-\alpha)\frac{\cos\theta}{\sqrt{1+3\sin^2\theta}},\frac{4\sqrt{6}}{5}+(1-\alpha)\frac{\sin\theta}{\sqrt{1+3\sin^2\theta}}\right)$$

$$Q_{3\theta}^{\alpha}:(x_{3\theta}^{\alpha},y_{3\theta}^{\alpha})=\left(-\frac{1}{4}+(1-\alpha)\cos\theta\,,\frac{\sqrt{15}}{2}+(1-\alpha)\sin\theta\right)$$

$$Q_{4\theta}^{\alpha}:(x_{4\theta}^{\alpha},y_{4\theta}^{\alpha})=\left(-\frac{1}{2}+(1-\alpha)\frac{\cos\theta}{\sqrt{1+3\sin^2\theta}},-\sqrt{3}+(1-\alpha)\frac{\sin\theta}{\sqrt{1+3\sin^2\theta}}\right)$$

$$Q_{5\theta}^{\alpha}:(x_{5\theta}^{\alpha},y_{5\theta}^{\alpha})=\left(\frac{1}{3}+(1-\alpha)\cos\theta\,,\frac{4\sqrt{2}}{3}+(1-\alpha)\sin\theta\right)\tag{1}$$

The ellipse E_θ^α that passes through $Q_{1\theta}^{\alpha},Q_{2\theta}^{\alpha},Q_{3\theta}^{\alpha},Q_{4\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$ can be determinant by the equation

$$a_\theta^\alpha x^2+\ 2h_\theta^\alpha xy+\ b_\theta^\alpha y^2+2g_\theta^\alpha x+2f_\theta^\alpha y+c_\theta^\alpha=0\tag{2}$$

with ${h_\theta^\alpha}^2<a_\theta^\alpha.b_\theta^\alpha$ where

$$a_\theta^\alpha=\frac{2h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix}-x_{1\theta}^\alpha y_{1\theta}^\alpha & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & 1\\ -x_{2\theta}^\alpha y_{2\theta}^\alpha & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & 1\\ -x_{3\theta}^\alpha y_{3\theta}^\alpha & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & 1\\ -x_{4\theta}^\alpha y_{4\theta}^\alpha & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & 1\\ -x_{5\theta}^\alpha y_{1\theta}^\alpha & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & 1\end{vmatrix}$$

$$b_\theta^\alpha=\frac{2h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix}{x_{1\theta}^\alpha}^2 & -x_{1\theta}^\alpha y_{1\theta}^\alpha & x_{1\theta}^\alpha & y_{1\theta}^\alpha & 1\\ {x_{2\theta}^\alpha}^2 & -x_{2\theta}^\alpha y_{2\theta}^\alpha & x_{2\theta}^\alpha & y_{2\theta}^\alpha & 1\\ {x_{3\theta}^\alpha}^2 & -x_{3\theta}^\alpha y_{3\theta}^\alpha & x_{3\theta}^\alpha & y_{3\theta}^\alpha & 1\\ {x_{4\theta}^\alpha}^2 & -x_{4\theta}^\alpha y_{4\theta}^\alpha & x_{4\theta}^\alpha & y_{4\theta}^\alpha & 1\\ {x_{5\theta}^\alpha}^2 & -x_{5\theta}^\alpha y_{1\theta}^\alpha & x_{5\theta}^\alpha & y_{5\theta}^\alpha & 1\end{vmatrix}$$

$$g_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & -x_{1\theta}^\alpha y_{1\theta}^\alpha & y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & -x_{2\theta}^\alpha y_{2\theta}^\alpha & y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & -x_{3\theta}^\alpha y_{3\theta}^\alpha & y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & -x_{4\theta}^\alpha y_{4\theta}^\alpha & y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & -x_{5\theta}^\alpha y_{5\theta}^\alpha & y_{5\theta}^\alpha & 1 \end{vmatrix}$$

$$f_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & -x_{1\theta}^\alpha y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & -x_{2\theta}^\alpha y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & -x_{3\theta}^\alpha y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & -x_{4\theta}^\alpha y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & -x_{5\theta}^\alpha y_{5\theta}^\alpha & 1 \end{vmatrix}$$

$$c_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & -x_{1\theta}^\alpha y_{1\theta}^\alpha \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & -x_{2\theta}^\alpha y_{2\theta}^\alpha \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & -x_{3\theta}^\alpha y_{3\theta}^\alpha \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & -x_{4\theta}^\alpha y_{4\theta}^\alpha \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & -x_{5\theta}^\alpha y_{5\theta}^\alpha \end{vmatrix}$$

and

$$k_\theta^\alpha = \begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & 1 \end{vmatrix}.$$

These determinants are composed by writing column

$$\begin{bmatrix} -x_{1\theta}^\alpha & y_{1\theta}^\alpha \\ -x_{2\theta}^\alpha & y_{2\theta}^\alpha \\ -x_{3\theta}^\alpha & y_{3\theta}^\alpha \\ -x_{4\theta}^\alpha & y_{4\theta}^\alpha \\ -x_{5\theta}^\alpha & y_{5\theta}^\alpha \end{bmatrix}$$

instead of columns in determinant

$$\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & 1 \end{vmatrix}$$

Let A, B, C, F, G and K be the determinant values used to find the values of $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$ and k_θ^α respectively. So,

$$a_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}.A$$

$$b_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}.B$$

$$c_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}.C$$

$$f_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}.F$$

$$g_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}.G$$

$$k_\theta^\alpha = K \tag{3}$$

are obtained.

The fuzzy ellipse $\overline{FE}$ that passes through $\bar{E}_i$'s, $i = 1,2,3,4,5$ is the union of all possible ellipse E_θ^α 's that lies between $Q_{1\theta}^\alpha$ and $Q_{5\theta}^\alpha$'s.

That

$$\overline{FE} = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \left\{ \begin{matrix} (x,y) = a_\theta^\alpha x^2 + 2h_\theta^\alpha xy + b_\theta^\alpha y^2 \\ +2g_\theta^\alpha x + 2f_\theta^\alpha y + c_\theta^\alpha = 0 \end{matrix} \right\}$$

Now we find the membership degree of the point (1,0.5) on the fuzzy ellipse $\overline{FE}$. First, we adjust the set of ellipses E_θ^α 's which the point (1,0.5) lies.

Let replace point (1,0.5) in Equation (2). We need to identify the passible values of α. Then, we get the equation below;

$$a_\theta^\alpha.(1)^2 + 2.h_\theta^\alpha.(1).(0.5) + b_\theta^\alpha.(0.5)^2 + 2.g_\theta^\alpha.(1) + 2.f_\theta^\alpha.(0.5) + c_\theta^\alpha = 0$$

which simplifies to

$$a_\theta^\alpha + h_\theta^\alpha + 0.25\, b_\theta^\alpha + 2g_\theta^\alpha + f_\theta^\alpha + c_\theta^\alpha = 0 \tag{4}$$

Now let's examine the angular values that L lines make with the x-axis.

First, we admit that $\theta_0 = 45°$. We calculate above determinant values for this angle and will find k_θ^α using the Maple program.

$$k_\theta^\alpha = K$$

$$K=\begin{vmatrix} [1+(1-\alpha)\cos 45°]^2 & [(1-\alpha)\sin 45°]^2 & 1+(1-\alpha)\cos 45° & (1-\alpha)\sin 45° & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & \left[\frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}} & \frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}} & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 45°\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 45°\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 45° & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 45° & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}} & -\sqrt{3}+\frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}} & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 45°\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 45°\right]^2 & \left[\frac{1}{3}+(1-\alpha)\cos 45°\right] & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 45° & 1 \end{vmatrix}=0$$

But this angle is not suitable for calculating a_θ^α, $b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha$ and g_θ^α. Because, it makes equations (3) undefined.

Then we put $\theta_0 = 30°$ in (3.3) and we calculate a_θ^α, $b_\theta^\alpha, g_\theta^\alpha, f_\theta^\alpha, c_\theta^\alpha$ and k_θ^α.

$$k_\theta^\alpha = K$$

$$K=\begin{vmatrix} [1+(1-\alpha)\cos 30°]^2 & [(1-\alpha)\sin 30°]^2 & 1+(1-\alpha)\cos 30° & (1-\alpha)\sin 30° & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \left[\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & \frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 30°\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30°\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30° & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30° & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 30°\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30°\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30° & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30° & 1 \end{vmatrix}$$

and

$$K = -0{,}02\alpha^3 + 0{,}37\alpha^2 + 0{,}69\alpha - 1{,}02$$

We continue to find the other determinant values.

For the value of A,

$$\begin{vmatrix} -[1+(1-\alpha)\cos 30°][(1-\alpha)\cos 30°] & [(1-\alpha)\sin 30°]^2 & 1+(1-\alpha)\cos 30° & (1-\alpha)\sin 30° & 1 \\ -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]\left[\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right] & \left[\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & \frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ -\left[-\frac{1}{4}+(1-\alpha)\cos 30°\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30°\right] & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30°\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30° & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30° & 1 \\ -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right] & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ -\left[\frac{1}{3}+(1-\alpha)\cos 30°\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30°\right] & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30°\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30° & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30° & 1 \end{vmatrix}$$

$$=0.011\alpha^3 - 0.477\alpha^2 + 3.221\alpha - 1.696$$

For the value of B,

$$\begin{vmatrix} [1+(1-\alpha)\cos 30^\circ]^2 & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] & 1+(1-\alpha)\cos 30^\circ & (1-\alpha)\sin 30^\circ & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & \frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & \frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] & -\frac{1}{4}+(1-\alpha)\cos 30^\circ & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & -\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] & \frac{1}{3}+(1-\alpha)\cos 30^\circ & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ & 1 \end{vmatrix}$$

$$=0.033\alpha^3-0.255\alpha^2+0.230\alpha+0.717$$

For the value of G,

$$\begin{vmatrix} [1+(1-\alpha)\cos 30^\circ]^2 & [(1-\alpha)\sin 30^\circ]^2 & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] & (1-\alpha)\sin 30^\circ & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & \frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right]^2 & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & -\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]^2 & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ & 1 \end{vmatrix}$$

$$=-0.555\alpha^3+5.777\alpha^2-10.915\alpha+5.693$$

For the value of F,

$$\begin{vmatrix} [1+(1-\alpha)\cos 30^\circ]^2 & [(1-\alpha)\sin 30^\circ]^2 & 1+(1-\alpha)\cos 30^\circ & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30^\circ & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30^\circ & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] & 1 \end{vmatrix}$$

$$=0.057\alpha^3+0.141\alpha^2+0.649\alpha-0.848$$

For the value of C,

$$\begin{vmatrix} [1+(1-\alpha)\cos 30^\circ]^2 & [(1-\alpha)\sin 30^\circ]^2 & 1+(1-\alpha)\cos 30^\circ & (1-\alpha)\sin 30^\circ & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & \frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] \\ \left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30^\circ & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] \\ \left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30^\circ & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] \end{vmatrix}$$

$$=-0.194\alpha^4+2.264\alpha^3-5.842\alpha^2+6.232\alpha-3.518$$

Let substitute these values in the equation (3) and we obtain the coefficients $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$ and c_θ^α. If we substitude these coefficients in the conic equation (4) and simplify the equation with h_θ^α $(h_\theta^\alpha \neq 0)$, we obtain the following equation;

$$-0{,}389\alpha^4+3{,}475\alpha^3-0{,}692\alpha^2-1{,}706\alpha-0{,}555=0 \qquad (5)$$

By solving the equation (5), the real values of α;

$$0.98, 0.85$$

As alpha represents the grade of membership, it must be [0,1].

So the appropriate alpha real number is $\theta = 0.98$ as it is supremum. Thus for $\theta_0 = 30°$, the set S_{θ_0} of all passible value of α is the set $\{0.98\}$.

We vary θ_0 across $[0,2\pi]$ and keep an identifying the value of S_{θ_0}. Finally, we get supremum of all S_{θ_0}'s. One can easily verify that the supremum value for the considered point is 0,98 which is attained for the value of $\theta_0 = 30°$. Eventually, we note that the conic passing through to the same points for $\theta_0 = 30°$ and $\alpha = 0{,}98$. As a result, we find that the conic which passing through to the same-points are;

$(1.01,0.01) \in \overline{E_1}, (0.21,1.96) \in \overline{E_2}, (-0.23,1.94) \in \overline{E_3}, (-0.48,-1.72) \in \overline{E_4},$ and $(0.35,1.89) \in \overline{E_5}$

Then we obtain the conic equation (6) which passing through to the same -points:

$$0.96x^2 - 0.04xy + 0.27y^2 + 0.05x - 0.03y - 1.03 = 0 \quad (6)$$

This conic equation contains the point $(1, 0.5)$. And, we have

$$\mu\big((1, 0.5)|\overline{FE}\big) = 0.98$$

alpha cut of fuzzy ellipse.

Now we give an example of a fuzzy ellipse whose core ellipse is decentralized.

Example 3.2. Let $\overline{E_1} = (1,4), \overline{E_2} = \left(0, \frac{2\sqrt{35}}{3}\right), \overline{E_3} = (-2,2\sqrt{3}), \overline{E_4} = \left(-3, -\frac{4\sqrt{5}}{3}\right)$ and $\overline{E_5} = (4,-2\sqrt{3})$ are fuzzy poinys. We get the fuzzy ellipse that passes through these points.

The core ellipse equation is that passes through these points;

$$\left\{(x,y): \frac{(x-1)^2}{36} + \frac{y^2}{16} = 1\right\}$$

Now let take the membership functions of these five ellipses as circle, ellipse, circle, circle and ellipse respectively;

$$\{(x,y): (x-1)^2 + (y-4)^2 \leq 1\}$$

$$\left\{(x,y): x^2 + 9\left(y - \frac{2\sqrt{35}}{3}\right)^2 \leq 1\right\}$$

$$\left\{(x,y): (x+2)^2 + \left(y + 2\sqrt{3}\right)^2 \leq 1\right\}$$

$$\left\{(x,y): (x+3)^2 + \left(y + \frac{4\sqrt{5}}{3}\right)^2 \leq 1\right\}$$

$$\left\{(x,y): 9(x-4)^2 + \left(y+2\sqrt{3}\right)^2 \leq 1\right\}$$

The vertices of the membership function are $(1,4)$, $\left(0,\frac{2\sqrt{35}}{3}\right)$, $\left(-2,2\sqrt{3}\right)$, $\left(-3,-\frac{4\sqrt{5}}{3}\right)$ and $\left(4,-2\sqrt{3}\right)$ respectively. Now let write five same-points on $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4$ and $\bar{E}_5$ whose membership degrees are alpha

$$Q_{1\theta}^{\alpha}: (x_{1\theta}^{\alpha}, y_{1\theta}^{\alpha}) = (1 + (1-\alpha).\cos\theta\, , 4 + (1-\alpha).\sin\theta)$$

$$Q_{2\theta}^{\alpha}: (x_{2\theta}^{\alpha}, y_{2\theta}^{\alpha}) = \left((1-\alpha).\frac{\cos\theta}{\sqrt{1+8\sin^2\theta}}, \frac{2\sqrt{35}}{3} + (1-\alpha).\frac{\sin\theta}{\sqrt{1+8\sin^2\theta}}\right)$$

$$Q_{3\theta}^{\alpha}: (x_{3\theta}^{\alpha}, y_{3\theta}^{\alpha}) = \left(-2 + (1-\alpha).\cos\theta\, , 2\sqrt{3} + (1-\alpha).\sin\theta\right)$$

$$Q_{4\theta}^{\alpha}: (x_{4\theta}^{\alpha}, y_{4\theta}^{\alpha}) = \left(-3 + (1-\alpha).\cos\theta\, , -\frac{4\sqrt{5}}{3} + (1-\alpha).\sin\theta\right)$$

$$Q_{5\theta}^{\alpha}: (x_{5\theta}^{\alpha}, y_{5\theta}^{\alpha}) = \left(4 + (1-\alpha).\frac{\cos\theta}{\sqrt{1+8\cos^2\theta}}, -2\sqrt{3} + (1-\alpha).\frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}\right)$$

The ellipse E_{θ}^{α} that passes through $Q_{1\theta}^{\alpha}, Q_{2\theta}^{\alpha}, Q_{3\theta}^{\alpha}, Q_{4\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$ can be determinant by the equation below again as in the previous example;

$$a_{\theta}^{\alpha}x^2 + 2h_{\theta}^{\alpha}xy + b_{\theta}^{\alpha}y^2 + 2g_{\theta}^{\alpha}x + 2\, f_{\theta}^{\alpha}y + c_{\theta}^{\alpha} = 0$$

with ${h_{\theta}^{\alpha}}^2 < a_{\theta}^{\alpha}.b_{\theta}^{\alpha}$.

We gave on the previous example how to find a_{θ}^{α}, b_{θ}^{α}, g_{θ}^{α}, f_{θ}^{α} and c_{θ}^{α}. In this example we apply the same. The fuzzy ellipse $\overline{FE}_{1\ldots5}$ that passes through $\bar{E}_i$'s $i = 1,2,3,4,5$ is the union of all possible ellipse E_{θ}^{α} 's that lies between $Q_{1\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$'s. That is,

$$\overline{FE}_{1\ldots5} = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \left\{ \begin{array}{c} (x,y) = a_{\theta}^{\alpha}x^2 + 2h_{\theta}^{\alpha}xy + b_{\theta}^{\alpha}y^2 \\ +2g_{\theta}^{\alpha}x + 2f_{\theta}^{\alpha}y + c_{\theta}^{\alpha} = 0 \end{array} \right\}$$

Now we find the membership degree of the point (1,4.1) on the fuzzy ellipse $\overline{FE}$. We adjust the set of ellipses E_{θ}^{α} 's on which the point (1,4.1) lies.

Let replace point (1, 4.1) in equation (2) we need to identify the possible values of α. Then, we get the equation below:

$$a_{\theta}^{\alpha}(1)^2 + 2h_{\theta}^{\alpha}(1).(4.1) + b_{\theta}^{\alpha}(4.1)^2 + 2g_{\theta}^{\alpha}(1) + 2f_{\theta}^{\alpha}(0.5) + c_{\theta}^{\alpha} = 0$$

which simplifies to

$$a_{\theta}^{\alpha} + 8.2h_{\theta}^{\alpha} + 16.81b_{\theta}^{\alpha} + 2g_{\theta}^{\alpha} + 8.2f_{\theta}^{\alpha} + c_{\theta}^{\alpha} = 0 \quad (7)$$

Then we put $\theta_0 = 45°$ in (7) and we calculate $a_\theta^\alpha, b_\theta^\alpha, g_\theta^\alpha,\ f_\theta^\alpha$ and c_θ^α.

We find them and replace in (7) then we obtain the following non-linear equation which determinants are found:

$$-31.18\alpha^4 - 1303.24\alpha^3 - 5634.93\alpha^2 - 6568.61\alpha + 10404.88 = 0$$

By solving this equation, real alpha values are found;

$$\alpha = 0.85, -2.78$$

But we get 0.85 from 0 to 1 from these real two values.

Thus, for $\theta_0 = 45°$, the set S_{θ_0} of all passible value of α is the set $\{0.85\}$.

We vary θ_0 across $[0,2\pi]$ and keep an identifying the value of s_{θ_0}. Finaly, we get supremum of all s_{θ_0}'s. One can easily verify that the supremum value fort he considered point is 0,85 which is attained for the value of $\theta_0 = 45°$. Eventually, we note that the conic passing through to the same points, for $\theta_0 = 45°$ and $\alpha = 0{,}85$.

As a result, we find that the conic which passing through to the same-points are;

$$(1.31, 4.31) \in \bar{E}_1, (0.13, 4.08) \in \bar{E}_2, (-1.68, 3.77) \in \bar{E}_3,$$

$$(-2.68, -2.67) \in \bar{E}_4, (4.13, -3.32) \in \bar{E}_5$$

Then we obtain the following conic equation (8) which passing through to the same-points:

$$-524.84x^2 + 74.07xy - 2027.74y^2 + 1902.47x + 907.46y + 28574.06 = 0 \tag{8}$$

This conic equation contains the point (1,4.1). And, we have

$$\mu\big((1,4.1)\big|\overline{FE}\big) = 0.85$$

alpha cut of fuzzy ellipse and membership degree of a core ellipse in a fuzzy ellipse.

4. Conclusion

The concept of fuzzy ellipse has been initiated and basic properties of fuzzy ellipse have been explained in this study in details. The membership degrees of fuzzy points have a specific role for the graph of ellipse. The needed explanations based on these roles were made on the drawn graphics. Equations of conics such as hyperbola and ellipse can be obtained by determining five points. Starting from these, we developed a method for obtaining the fuzzy ellipse equation in the study. But we can't do this with five random points. We used the points which called the same-points. In the study, we have presented a method by determining the necessary properties for selecting points. As seen in the figures, the fuzzy ellipses can be depicted with different curves. Depending on the degree of membership Fuzzy ellipse can be use kidney stones crushing machines, billiard games, aerospace engineering and laser technology etc. We have shown the applicability of the method in the examples. When we create the fuzzy ellipse equations in the 3rd section, it is seen that the necessary coefficients for the calculation of these equations will be made with high-dimensional determinants. In the examples given, the maple program was used to calculate the membership of the coefficients using the selected points and angles. Thus, it will be possible to find fuzzy ellipses at different point and angle selections. The determinants we presented in section 3 can be easily calculated with mathematical programs.

Fuzzy ellipse can be use kidney stones crushing machines, billiard games, aerospace engineering and laser technology etc. The method and applications we have shown in this study will be a guide for studies in these areas.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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On Dynamics and Solutions Expressions of Higher-Order Rational Difference Equations

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Abstract — The principle goal of this paper is to look at some of the qualitative behavior of the critical point of the rational difference equation

$$\Psi_{n+1} = \alpha \Psi_{n-2} + \frac{\beta \Psi_{n-2} \Psi_{n-3}}{\gamma \Psi_{n-3} + \delta \Psi_{n-6}}, \quad n = 0, 1, 2, ...,$$

where α, β, γ and δ are arbitrary positive real numbers. We also used the proposed equation to get the general solution for particular cases and provided numerical examples to demonstrate our results.

Subject Classification (2020): 39A10.

1. Introduction

One of the most important scientific topics is difference equations, often known as discrete dynamical systems. The study of the qualitative properties of rational difference equations has sparked a lot of attention recently.

Many researchers have opted to utilize difference equations in mathematical models to explain the problems in various sciences, including allowing scientists to introduce their study's predictions and producing more precise results.

It is particularly fascinating to look into the behavior of the solutions to a system of nonlinear differential equations and examine the local asymptotic stability of their equilibrium points. Numerous studies have been conducted on the technique of identifying the general form of the solution for some special cases of the problem. The systems and behavior of rational difference equations have been the subject of numerous works (can be obtained in the references).

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Alayachi et al. [3] studied the qualitative properties of:

$$y_{n+1} = Ay_{n-1} + \frac{By_{n-1}y_{n-3}}{Cy_{n-3} + Dy_{n-5}}.$$

Almatrafi et al. [6] studied the global behavior of:

$$\chi_{n+1} = \alpha\chi_n + \frac{\beta\chi_n^2 + \gamma\chi_n\chi_{n-1} + \delta\chi_{n-1}^2}{\lambda\chi_n^2 + \mu\chi_n\chi_{n-1} + \sigma\chi_{n-1}^2}.$$

Alzubaidi and Elsayed [8] examined the dynamics behavior and gave the general form of:

$$\varphi_{n+1} = \alpha\varphi_{n-2} \pm \frac{\beta\varphi_{n-1}\varphi_{n-2}}{\gamma\varphi_{n-2} \pm \delta\varphi_{n-4}}.$$

Ibrahim et al. [26] investigated the global stability and boundedness of solutions for:

$$\Upsilon_{n+1} = \alpha + \sum_{i=0}^{k} a_i\Upsilon_{n-i} + \frac{\Upsilon_n\Upsilon_{n-k}}{\beta + \sum_{j=0}^{k} b_j\Upsilon_{n-j}}.$$

Kara and Yazlik [27] found the exact formulas for the solutions of the system:

$$x_n = \frac{x_{n-2}z_{n-3}}{z_{n-1}(a_n + b_n x_{n-2}z_{n-3})},$$
$$y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}(\alpha_n + \beta_n y_{n-2}x_{n-3})},$$
$$z_n = \frac{z_{n-2}y_{n-3}}{y_{n-1}(A_n + B_n z_{n-2}y_{n-3})}.$$

Karatas et al. [28] investigated the solutions of:

$$U_{n+1} = \frac{U_{n-5}}{1 + bU_{n-2}U_{n-5}}.$$

Abdul Khaliq et al. [30] investigated the asymptotic behavior of the solutions of:

$$\omega_{n+1} = \omega_{n-p}\left(\alpha + \frac{\beta\omega_n}{\gamma\omega_n + \delta\omega_{n-r}}\right).$$

In [35] Muna and Mohammad deal with:

$$V_{n+1} = \frac{(\alpha + \beta V_n)}{(A + BV_n + CV_{n-k})}.$$

The goal of this paper is to find a general solution to some special cases of the fractional recursive equation

$$\Psi_{n+1} = \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3} + \delta\Psi_{n-6}}, \quad n = 0, 1, 2, ..., \tag{1}$$

where α, β, γ and δ are arbitrary positive real numbers.

2. Local Stability of the Critical Point

The critical point of Eq.(1), is given by

$$\bar{\Psi} = \alpha\bar{\Psi} + \frac{\beta\bar{\Psi}^2}{\gamma\bar{\Psi} + \delta\bar{\Psi}},$$

$$(1-\alpha)\bar{\Psi} = \frac{\beta\bar{\Psi}^2}{(\gamma+\delta)\bar{\Psi}} \Rightarrow (1-\alpha)(\gamma+\delta)\bar{\Psi}^2 = \beta\bar{\Psi}^2.$$

Thus,

$$\left[(1-\alpha)(\gamma+\delta) - \beta\right]\bar{\Psi}^2 = 0.$$

If $(1-\alpha)(\gamma+\delta) \neq \beta$ then the unique critical point is $\bar{\Psi} = 0$.

Assume $\Phi : (0,\infty)^3 \to (0,\infty)$ be a C^1 function defined by

$$\Phi(w_1, w_2, w_3) = \alpha w_1 + \frac{\beta w_1 w_2}{\gamma w_2 + \delta w_3}. \tag{2}$$

In consequence,

$$\frac{\partial\Phi}{\partial w_1} = \alpha + \frac{\beta w_2}{\gamma w_2 + \delta w_3}, \quad \frac{\partial\Phi}{\partial w_2} = \frac{\beta\delta w_1 w_3}{(\gamma w_2 + \delta w_3)^2}, \quad \frac{\partial\Phi}{\partial w_3} = \frac{-\beta\delta w_1 w_2}{(\gamma w_2 + \delta w_3)^2}. \tag{3}$$

At $\bar{\Psi} = 0$, we see that

$$\begin{aligned} \frac{\partial\Phi}{\partial w_1}(\bar{\Psi},\bar{\Psi},\bar{\Psi}) &= \alpha + \frac{\beta}{\gamma+\delta} = \gamma_1, \\ \frac{\partial\Phi}{\partial w_2}(\bar{\Psi},\bar{\Psi},\bar{\Psi}) &= \frac{\beta\delta}{(\gamma+\delta)^2} = \gamma_2, \\ \frac{\partial\Phi}{\partial w_3}(\bar{\Psi},\bar{\Psi},\bar{\Psi}) &= \frac{-\beta\delta}{(\gamma+\delta)^2} = \gamma_3. \end{aligned} \tag{4}$$

Hence,

$$Z_{n+1} - \left(\alpha + \frac{\beta}{\gamma+\delta}\right)Z_{n-2} - \left(\frac{\beta\delta}{(\gamma+\delta)^2}\right)Z_{n-3} + \left(\frac{\beta\delta}{(\gamma+\delta)^2}\right)Z_{n-6} = 0.$$

Theorem 2.1. The critical point $\bar{\Psi} = 0$ is locally asymptotically stable if

$$\beta(\gamma+3\delta) < (1-\alpha)(\gamma+\delta)^2.$$

Proof.

By using the values in the Eq.(4) and by Lemma 1 in [30], ensures that Eq.(1) is asymptotically stable if

$$|\gamma_1| + |\gamma_2| + |\gamma_3| < 1,$$

$$\left|\alpha + \frac{\beta}{\gamma+\delta}\right| + \left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| + \left|\frac{-\beta\delta}{(\gamma+\delta)^2}\right| < 1,$$

or

$$\alpha + \frac{\beta(\gamma+\delta)}{(\gamma+\delta)^2} + \frac{\beta\delta}{(\gamma+\delta)^2} + \frac{\beta\delta}{(\gamma+\delta)^2} < 1,$$

$$\frac{\beta\gamma+3\beta\delta}{(\gamma+\delta)^2}<(1-\alpha),$$

therefore,

$$\beta(\gamma+3\delta)<(1-\alpha)(\gamma+\delta)^2.$$

3. Global Attractive of the Critical Point

In this section, we aim to investigate the global asymptotic stability of the positive solutions of Eq.(1).

Theorem 3.1. The critical point $\bar{\Psi}=0$ of Eq.(1) is a global attracting if

$$\gamma(1-\alpha)\neq\beta.$$

Proof.
From Eq.(3), we note that, the function $\Phi(w_1,w_2,w_3)$ is increasing in w_1 and w_2 and is decreasing in w_3. Assume that whenever (H,h) is a solution of the system

$$H=\Phi(H,H,h),$$
$$h=\Phi(h,h,H),$$

then, we have

$$H=\alpha H+\frac{\beta H^2}{\gamma H+\delta h}, \ \Rightarrow \ (1-\alpha)H=\frac{\beta H^2}{\gamma H+\delta h},$$

$$\gamma(1-\alpha)H^2+\delta(1-\alpha)hH=\beta H^2. \tag{5}$$

$$h=\alpha h+\frac{\beta h^2}{\gamma h+\delta H}, \ \Longrightarrow \ (1-\alpha)h=\frac{\beta h^2}{\gamma h+\delta H}.$$

$$\gamma(1-\alpha)h^2+\delta(1-\alpha)hH=\beta h^2. \tag{6}$$

By substrate Eq.(5) from Eq.(6) we obtain

$$\left[\gamma(1-\alpha)-\beta\right]\left(H^2-h^2\right)=0.$$

In consequence, $H=h$ if $\gamma(1-\alpha)\neq\beta$. It follows by Theorem 1 in [30] the equilibrium point $\bar{\Psi}=0$ of Eq.(1) is a global attractor.

4. Boundedness of solutions

Here, we demonstrate how the positive solutions to Eq.(1) have boundedness.

Theorem 4.1. Every solution of Eq.(1) is bounded if

$$\left(\alpha+\frac{\beta}{\gamma}\right)<1.$$

Proof.
Assume that $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(1), then

$$\begin{aligned}\Psi_{n+1} &= \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3}+\delta\Psi_{n-6}} \\ &\leq \alpha\Psi_{n-2} + \frac{\beta\Psi_{n-2}\Psi_{n-3}}{\gamma\Psi_{n-3}} \\ &= \left(\alpha+\frac{\beta}{\gamma}\right)\Psi_{n-2}.\end{aligned}$$

Hence,

$$\Psi_{n+1} \leq \Psi_{n-2}, \quad for\ all\ n \geq 0.$$

This implies that the subsequences are bounded from above by

$$\Psi_{\max} = \max\{\Psi_{-6},\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1},\Psi_0\}.$$

5. General Solution for Special Cases

In this section, we will find expressions of solution for some special cases of Eq.(1)

5.1. First Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha=\beta=\delta=\gamma=1,$ so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}, \quad n=0,1,2,..., \tag{7}$$

where the initial conditions $\Psi_{-6},\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.1. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(7). Thus for n=0,1,2,...,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+7}\sigma+\mathscr{F}_{6i+6}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i+6}\sigma+\mathscr{F}_{6i+5}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-1} = \lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+7}\lambda+\mathscr{F}_{6i+6}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i+6}\lambda+\mathscr{F}_{6i+5}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n} = \eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+7}\eta+\mathscr{F}_{6i+6}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i+6}\eta+\mathscr{F}_{6i+5}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)},$$

$$\Psi_{12n+1} = \sigma\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i-3}\eta+\mathscr{F}_{6i-4}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i-4}\eta+\mathscr{F}_{6i-5}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n+2} = \lambda\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i-1}\zeta+\mathscr{F}_{6i-2}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i-2}\zeta+\mathscr{F}_{6i-3}\kappa)},$$

$$\Psi_{12n+3} = \eta\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n+4}=\sigma\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n+5}=\lambda\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n+6}=\eta\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n+7}=\sigma\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n+8}=\lambda\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n+9}=\eta\prod_{i=0}^{n}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)},$$

where $\Psi_{-6}=\kappa,\Psi_{-5}=\tau,\Psi_{-4}=\mu,\Psi_{-3}=\zeta,\Psi_{-2}=\sigma,\Psi_{-1}=\lambda,\Psi_{0}=\eta$ and $\{\mathscr{F}_i\}_{i=-5}^{\infty}=\{1,1,1,1,1,1,1,1,2,3,5,8,13,21,...,\}$.

Proof.

For $n=0$ the result holds. Now suppose that $n>0$ and our assumption holds for $n-1$, that is

$$\Psi_{12n-14}=\sigma\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+7}\sigma+\mathscr{F}_{6i+6}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i+6}\sigma+\mathscr{F}_{6i+5}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-13}=\lambda\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{6i+5}\eta+\mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+7}\lambda+\mathscr{F}_{6i+6}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta+\mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i+6}\lambda+\mathscr{F}_{6i+5}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n-12}=\eta\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{6i+7}\eta+\mathscr{F}_{6i+6}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta+\mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i+6}\eta+\mathscr{F}_{6i+5}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta+\mathscr{F}_{6i+5}\kappa)},$$

$$\Psi_{12n-11}=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-3}\eta+\mathscr{F}_{6i-4}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i-4}\eta+\mathscr{F}_{6i-5}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-10}=\lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i-1}\zeta+\mathscr{F}_{6i-2}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i-2}\zeta+\mathscr{F}_{6i-3}\kappa)},$$

$$\Psi_{12n-9}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n-8}=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i-1}\lambda+\mathscr{F}_{6i-2}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i-2}\lambda+\mathscr{F}_{6i-3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-7}=\lambda\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i-1}\eta+\mathscr{F}_{6i-2}\zeta)(\mathscr{F}_{6i+1}\lambda+\mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma+\mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta+\mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i-2}\eta+\mathscr{F}_{6i-3}\zeta)(\mathscr{F}_{6i}\lambda+\mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma+\mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta+\mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n-6}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)},$$

$$\Psi_{12n-5}=\sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{6i+3}\eta+\mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda+\mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma+\mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta+\mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta+\mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda+\mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma+\mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta+\mathscr{F}_{6i+1}\kappa)},$$

$$\Psi_{12n-4} = \lambda \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+5}\eta + \mathscr{F}_{6i+4}\zeta)(\mathscr{F}_{6i+1}\lambda + \mathscr{F}_{6i}\mu)(\mathscr{F}_{6i+3}\sigma + \mathscr{F}_{6i+2}\tau)(\mathscr{F}_{6i+5}\zeta + \mathscr{F}_{6i+4}\kappa)}{(\mathscr{F}_{6i+4}\eta + \mathscr{F}_{6i+3}\zeta)(\mathscr{F}_{6i}\lambda + \mathscr{F}_{6i-1}\mu)(\mathscr{F}_{6i+2}\sigma + \mathscr{F}_{6i+1}\tau)(\mathscr{F}_{6i+4}\zeta + \mathscr{F}_{6i+3}\kappa)},$$

$$\Psi_{12n-3} = \eta \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+1}\eta + \mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda + \mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma + \mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+7}\zeta + \mathscr{F}_{6i+6}\kappa)}{(\mathscr{F}_{6i}\eta + \mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda + \mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma + \mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i+6}\zeta + \mathscr{F}_{6i+5}\kappa)}.$$

Now, we prove that the results are holds for n. From Eq.(7), it follows that

$$\begin{aligned}
\Psi_{12n-2} &= \Psi_{12n-5} + \frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6} + \Psi_{12n-9}} \\
&= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}
\end{aligned}$$

$$\left[1 + \frac{\eta \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)}}{\eta \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)} + \eta \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+1}\eta+\mathscr{F}_{6i}\zeta)(\mathscr{F}_{6i+3}\lambda+\mathscr{F}_{6i+2}\mu)(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)(\mathscr{F}_{6i+1}\zeta+\mathscr{F}_{6i}\kappa)}{(\mathscr{F}_{6i}\eta+\mathscr{F}_{6i-1}\zeta)(\mathscr{F}_{6i+2}\lambda+\mathscr{F}_{6i+1}\mu)(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)(\mathscr{F}_{6i}\zeta+\mathscr{F}_{6i-1}\kappa)}}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}$$

$$\left[1 + \frac{\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)}{(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)}}{\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+5}\sigma+\mathscr{F}_{6i+4}\tau)}{(\mathscr{F}_{6i+4}\sigma+\mathscr{F}_{6i+3}\tau)} + \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i-1}\sigma+\mathscr{F}_{6i-2}\tau)}{(\mathscr{F}_{6i-2}\sigma+\mathscr{F}_{6i-3}\tau)}}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}$$

$$\left[1 + \frac{\frac{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n-2}\sigma+\mathscr{F}_{6n-3}\tau)}}{\frac{(\mathscr{F}_{6n-1}\sigma+\mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n-2}\sigma+\mathscr{F}_{6n-3}\tau)} + 1}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}$$

$$\left[1 + \frac{(\mathscr{F}_{6n-1}\sigma + \mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n-1}\sigma + \mathscr{F}_{6n-2}\tau) + (\mathscr{F}_{6n-2}\sigma + \mathscr{F}_{6n-3}\tau)}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}$$

$$\left[1 + \frac{(\mathscr{F}_{6n-1}\sigma + \mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau)}\right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)} \left[\frac{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau) + (\mathscr{F}_{6n-1}\sigma + \mathscr{F}_{6n-2}\tau)}{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau)} \right]$$

$$= \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+1}\sigma + \mathscr{F}_{6i}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i}\sigma + \mathscr{F}_{6i-1}\tau)(F_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)} \left[\frac{(\mathscr{F}_{6n+1}\sigma + \mathscr{F}_{6n}\tau))}{(\mathscr{F}_{6n}\sigma + \mathscr{F}_{6n-1}\tau)} \right].$$

Hence, we get

$$\Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{6i+3}\eta + \mathscr{F}_{6i+2}\zeta)(\mathscr{F}_{6i+5}\lambda + \mathscr{F}_{6i+4}\mu)(\mathscr{F}_{6i+7}\sigma + \mathscr{F}_{6i+6}\tau)(\mathscr{F}_{6i+3}\zeta + \mathscr{F}_{6i+2}\kappa)}{(\mathscr{F}_{6i+2}\eta + \mathscr{F}_{6i+1}\zeta)(\mathscr{F}_{6i+4}\lambda + \mathscr{F}_{6i+3}\mu)(\mathscr{F}_{6i+6}\sigma + \mathscr{F}_{6i+5}\tau)(\mathscr{F}_{6i+2}\zeta + \mathscr{F}_{6i+1}\kappa)}.$$

Other expressions can be investigated in the same way. The proof has been completed.

5.2. Second Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha = \gamma = \beta = 1$ and $\delta = -1$, so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3} - \Psi_{n-6}}, \quad n = 0, 1, 2, ..., \tag{8}$$

where the initial conditions $\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.2. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(8). Thus for n=0,1,2,...,

$$\Psi_{12n-2} = \sigma \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta - \mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta - \mathscr{F}_{3i-1}\kappa)},$$

$$\Psi_{12n-1} = \lambda \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta - \mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma - \mathscr{F}_{3i-1}\tau)(\mathscr{F}_{3i+2}\zeta - \mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n} = \eta \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda - \mathscr{F}_{3-1}\mu)(\mathscr{F}_{3i+2}\sigma - \mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n+1} = \frac{\sigma(2\zeta - \kappa)}{(\zeta - \kappa)} \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta - \mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n+2} = \frac{\lambda(2\sigma - \tau)}{(\sigma - \tau)} \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+6}\sigma - \mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta - \mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda - \mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta - \mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n+3} = \frac{\eta(2\lambda - \mu)}{(\lambda - \mu)} \prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+5}\eta - \mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda - \mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma - \mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta - \mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta - \mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma - \mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta - \mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n+4} = \frac{\sigma(2\eta - \zeta)(2\zeta - \kappa)}{(\eta - \zeta)(\zeta - \kappa)} \prod_{i=0}^{n-1} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda - \mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda - \mathscr{F}_{3i}\mu)\\(\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)\end{array}},$$

$$\Psi_{12n+5}=\frac{\lambda(2\sigma-\tau)(3\zeta-\kappa)}{\zeta(\sigma-\tau)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta-\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)\\(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}},$$

$$\Psi_{12n+6}=\frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)\end{array}},$$

$$\Psi_{12n+7}=\frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}},$$

$$\Psi_{12n+8}=\frac{\lambda(3\eta-\zeta)(2\sigma-\tau)(3\zeta-\kappa)}{\eta\zeta(\sigma-\tau)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+7}\eta-\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta-\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)\\(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}},$$

$$\Psi_{12n+9}=\frac{\eta(2\lambda-\mu)(3\sigma-\tau)(5\zeta-2\kappa)}{\sigma(\lambda-\mu)(2\zeta-\kappa)}\prod_{i=0}^{n-1}\frac{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+8}\zeta-\mathscr{F}_{3i+6}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}},$$

where $\Psi_{-6}=\kappa,\Psi_{-5}=\tau,\Psi_{-4}=\mu,\Psi_{-3}=\zeta,\Psi_{-2}=\sigma,\Psi_{-1}=\lambda,\Psi_{0}=\eta$ and $\{\mathscr{F}_i\}_{i=-1}^{\infty}=\{1,0,1,1,2,3,5,8,13,21,\ldots\}$.

Proof.

For $n=0$ the result holds. Now suppose that $n>0$ and our assumption holds for $n-1$, that is

$$\Psi_{12n-14}=\sigma\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta-\mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta-\mathscr{F}_{3i-1}\kappa)},$$

$$\Psi_{12n-13}=\lambda\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma-\mathscr{F}_{3i-1}\tau)(\mathscr{F}_{3i+2}\zeta-\mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n-12}=\eta\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda-\mathscr{F}_{3-1}\mu)(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n-11}=\frac{\sigma(2\zeta-\kappa)}{(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta-\mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-10}=\frac{\lambda(2\sigma-\tau)}{(\sigma-\tau)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta-\mathscr{F}_{3i}\kappa)},$$

$$\Psi_{12n-9}=\frac{\eta(2\lambda-\mu)}{(\lambda-\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)},$$

$$\Psi_{12n-8}=\frac{\sigma(2\eta-\zeta)(2\zeta-\kappa)}{(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}},$$

$$\Psi_{12n-7}=\frac{\lambda(2\sigma-\tau)(3\zeta-\kappa)}{\zeta(\sigma-\tau)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta-\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+2}\eta-\mathscr{F}_{3i}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)\\(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}},$$

$$\Psi_{12n-6}=\frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)\end{array}},$$

$$\Psi_{12n-5}=\frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}},$$

$$\Psi_{12n-4}=\frac{\lambda(3\eta-\zeta)(2\sigma-\tau)(3\zeta-\kappa)}{\eta\zeta(\sigma-\tau)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+7}\eta-\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+6}\sigma-\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+7}\zeta-\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda-\mathscr{F}_{3i+1}\mu)\\(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)\end{array}},$$

$$\Psi_{12n-3}=\frac{\eta(2\lambda-\mu)(3\sigma-\tau)(5\zeta-2\kappa)}{\sigma(\lambda-\mu)(2\zeta-\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+8}\zeta-\mathscr{F}_{3i+6}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}.$$

Now, we prove that the results are holds for n. From Eq.(8), it follows that

$$\Psi_{12n-2} = \Psi_{12n-5} + \frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6} - \Psi_{12n-9}}$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda - \mathscr{F}_{3i+5}\mu) \\ (\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu) \\ (\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)\end{array}}$$

$$\left[1 + \frac{\frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}}{\begin{array}{c}\frac{\eta(2\lambda-\mu)(3\sigma-\tau)}{\sigma(\lambda-\mu)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)} - \\ \frac{\eta(2\lambda-\mu)}{(\lambda-\mu)} \prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+5}\eta-\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+6}\lambda-\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+5}\zeta-\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}\end{array}}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda - \mathscr{F}_{3i+5}\mu) \\ (\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu) \\ (\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)\end{array}}$$

$$\left[1 + \frac{\prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)}}{\prod_{i=0}^{n-2} \frac{(\mathscr{F}_{3i+7}\sigma-\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)} - \prod_{i=1}^{n-2} \frac{(\mathscr{F}_{3i+4}\sigma-\mathscr{F}_{3i+2}\tau)}{(\mathscr{F}_{3i+2}\sigma-\mathscr{F}_{3i}\tau)}}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda - \mathscr{F}_{3i+5}\mu) \\ (\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu) \\ (\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)\end{array}}$$

$$\left[1 + \frac{\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma-\mathscr{F}_{3n-3}\tau)}}{\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma-\mathscr{F}_{3n-3}\tau)} - 1}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta - \mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda - \mathscr{F}_{3i+5}\mu) \\ (\mathscr{F}_{3i+5}\sigma - \mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta - \mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta - \mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda - \mathscr{F}_{3i+3}\mu) \\ (\mathscr{F}_{3i+3}\sigma - \mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta - \mathscr{F}_{3i+2}\kappa)\end{array}}$$

$$\left[1 + \frac{(\mathscr{F}_{3n+1}\sigma - \mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n+1}\sigma - \mathscr{F}_{3n-1}\tau) - (\mathscr{F}_{3n-1}\sigma - \mathscr{F}_{3n-3}\tau)}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}} \left[1+\frac{(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}} \left[\frac{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)+(\mathscr{F}_{3n+1}\sigma-\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)}\right]$$

$$= \frac{\sigma(2\eta-\zeta)(3\lambda-\mu)(2\zeta-\kappa)}{\lambda(\eta-\zeta)(\zeta-\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+6}\eta-\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+7}\lambda-\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+6}\zeta-\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta-\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+5}\lambda-\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+4}\zeta-\mathscr{F}_{3i+2}\kappa)\end{array}} \left[\frac{(\mathscr{F}_{3n+2}\sigma-\mathscr{F}_{3n}\tau)}{(\mathscr{F}_{3n}\sigma-\mathscr{F}_{3n-2}\tau)}\right].$$

Therefore,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+3}\eta-\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+4}\lambda-\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+5}\sigma-\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta-\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta-\mathscr{F}_{3i-1}\zeta)(\mathscr{F}_{3i+2}\lambda-\mathscr{F}_{3i}\mu)(\mathscr{F}_{3i+3}\sigma-\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta-\mathscr{F}_{3i-1}\kappa)}.$$

The following cases can be proved using a similar technique.

5.3. Third Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha=\gamma=\delta=1$ and $\beta=-1,$ so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}, \quad n=0,1,2,..., \tag{9}$$

where the initial conditions $\Psi_{-6},\Psi_{-5},\Psi_{-4},\Psi_{-3},\Psi_{-2},\Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.3. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(9). Thus for n=0,1,2,...,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i}\zeta+\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-1} = \lambda\prod_{i=0}^{n-1} \frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i}\sigma+\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n}=\eta\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i}\lambda+\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n+1}=\frac{\sigma\kappa}{(\zeta+\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)},$$

$$\Psi_{12n+2}=\frac{\lambda\tau}{(\sigma+\tau)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n+3}=\frac{\eta\mu}{(\lambda+\mu)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n+4}=\frac{\sigma\zeta\kappa}{(\eta+\zeta)(\zeta+\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)},$$

$$\Psi_{12n+5}=\frac{\lambda\tau(\zeta+\kappa)}{(\sigma+\tau)(\zeta+2\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)},$$

$$\Psi_{12n+6}=\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n+7}=\frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)},$$

$$\Psi_{12n+8}=\frac{\lambda\tau(\eta+\zeta)(\zeta+\kappa)}{(\eta+2\zeta)(\sigma+\tau)(\zeta+2\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)}{(\mathscr{F}_{3i+5}\eta+\mathscr{F}_{3i+6}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)},$$

$$\Psi_{12n+9}=\frac{\eta\mu(\sigma+\tau)(\zeta+2\kappa)}{(\lambda+\mu)(\sigma+2\tau)(2\zeta+3\kappa)}\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+6}\zeta+\mathscr{F}_{3i+7}\kappa)},$$

where $\Psi_{-6}=\kappa, \Psi_{-5}=\tau, \Psi_{-4}=\mu, \Psi_{-3}=\zeta, \Psi_{-2}=\sigma, \Psi_{-1}=\lambda, \Psi_0=\eta$ and $\{\mathscr{F}_i\}_{i=0}^{\infty}=\{0,1,1,2,3,5,8,13,21,...,\}$.

Proof.

For $n=0$ the result holds. Now suppose that $n>0$ and our assumption holds for $n-1$, that is

$$\Psi_{12n-14}=\sigma\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i}\zeta+\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)},$$

$$\Psi_{12n-13}=\lambda\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i}\sigma+\mathscr{F}_{3i+1}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n-12}=\eta\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i}\lambda+\mathscr{F}_{3i+1}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n-11}=\frac{\sigma\kappa}{(\zeta+\kappa)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)},$$

$$\Psi_{12n-10}=\frac{\lambda\tau}{(\sigma+\tau)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)},$$

$$\Psi_{12n-9}=\frac{\eta\mu}{(\lambda+\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)},$$

$$\Psi_{12n-8}=\frac{\sigma\zeta\kappa}{(\eta+\zeta)(\zeta+\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}},$$

$$\Psi_{12n-7}=\frac{\lambda\tau(\zeta+\kappa)}{(\sigma+\tau)(\zeta+2\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}},$$

$$\Psi_{12n-6}=\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}},$$

$$\Psi_{12n-5}=\frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)}\prod_{i=0}^{n-2}\frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}},$$

$$\Psi_{12n-4} = \frac{\lambda\tau(\eta+\zeta)(\zeta+\kappa)}{(\eta+2\zeta)(\sigma+\tau)(\zeta+2\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+5}\eta+\mathscr{F}_{3i+6}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}},$$

$$\Psi_{12n-3} = \frac{\eta\mu(\sigma+\tau)(\zeta+2\kappa)}{(\lambda+\mu)(\sigma+2\tau)(2\zeta+3\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)\\(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+5}\zeta+\mathscr{F}_{3i+6}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+6}\zeta+\mathscr{F}_{3i+7}\kappa)\end{array}}.$$

Now, we prove that the results are holds for n. From Eq.(9), it follows that

$$\begin{aligned}\Psi_{12n-2} &= \Psi_{12n-5} - \frac{\Psi_{12n-5}\Psi_{12n-6}}{\Psi_{12n-6}+\Psi_{12n-9}}\\
&= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}\\
&\left[1-\frac{\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}}{\begin{array}{c}\frac{\eta\mu(\sigma+\tau)}{(\lambda+\mu)(\sigma+2\tau)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}+\\ \frac{\eta\mu}{(\lambda+\mu)}\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+2}\eta+\mathscr{F}_{3i+3}\zeta)(\mathscr{F}_{3i+3}\lambda+\mathscr{F}_{3i+4}\mu)(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)(\mathscr{F}_{3i+2}\zeta+\mathscr{F}_{3i+3}\kappa)}{(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)}\end{array}}\right]\end{aligned}$$

$$\begin{aligned}&= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}\\
&\left[1-\frac{\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)}}{\prod_{i=0}^{n-2}\frac{(\mathscr{F}_{3i+4}\sigma+\mathscr{F}_{3i+5}\tau)}{(\mathscr{F}_{3i+5}\sigma+\mathscr{F}_{3i+6}\tau)}+\prod_{i=1}^{n-2}\frac{(\mathscr{F}_{3i+1}\sigma+\mathscr{F}_{3i+2}\tau)}{(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)}}\right]\end{aligned}$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$

$$\left[1-\frac{\frac{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}}{\frac{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}+1}\right]$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$

$$\left[1-\frac{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)+(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}\right]$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$

$$\left[\frac{(\mathscr{F}_{3n}\sigma+\mathscr{F}_{3n+1}\tau)-(\mathscr{F}_{3n-2}\sigma+\mathscr{F}_{3n-1}\tau)}{(\mathscr{F}_{3n}\sigma+\mathscr{F}_{3n+1}\tau)}\right]$$

$$= \frac{\sigma\zeta\kappa(\lambda+\mu)}{(\eta+\zeta)(\lambda+2\mu)(\zeta+\kappa)} \prod_{i=0}^{n-2} \frac{\begin{array}{c}(\mathscr{F}_{3i+3}\eta+\mathscr{F}_{3i+4}\zeta)(\mathscr{F}_{3i+4}\lambda+\mathscr{F}_{3i+5}\mu)\\(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i+3}\zeta+\mathscr{F}_{3i+4}\kappa)\end{array}}{\begin{array}{c}(\mathscr{F}_{3i+4}\eta+\mathscr{F}_{3i+5}\zeta)(\mathscr{F}_{3i+5}\lambda+\mathscr{F}_{3i+6}\mu)\\(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+4}\zeta+\mathscr{F}_{3i+5}\kappa)\end{array}}$$

$$\left[\frac{(\mathscr{F}_{3n-1}\sigma+\mathscr{F}_{3n}\tau)}{(\mathscr{F}_{3n}\sigma+\mathscr{F}_{3n+1}\tau)}\right].$$

Thus,

$$\Psi_{12n-2} = \sigma\prod_{i=0}^{n-1}\frac{(\mathscr{F}_{3i}\eta+\mathscr{F}_{3i+1}\zeta)(\mathscr{F}_{3i+1}\lambda+\mathscr{F}_{3i+2}\mu)(\mathscr{F}_{3i+2}\sigma+\mathscr{F}_{3i+3}\tau)(\mathscr{F}_{3i}\zeta+\mathscr{F}_{3i+1}\kappa)}{(\mathscr{F}_{3i+1}\eta+\mathscr{F}_{3i+2}\zeta)(\mathscr{F}_{3i+2}\lambda+\mathscr{F}_{3i+3}\mu)(\mathscr{F}_{3i+3}\sigma+\mathscr{F}_{3i+4}\tau)(\mathscr{F}_{3i+1}\zeta+\mathscr{F}_{3i+2}\kappa)}.$$

Other relations can be proved in the same way.

5.4. Fourth Equation

In this subsection, we will find the solution of Eq.(1) when $\alpha = \gamma = 1$, and $\beta = \delta = -1$, so the Eq.(1) become as

$$\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3} - \Psi_{n-6}}, \quad n = 0,1,2,..., \tag{10}$$

where the initial conditions $\Psi_{-6}, \Psi_{-5}, \Psi_{-4}, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are arbitrary positive real numbers.

Theorem 5.4. Assume $\{\Psi_n\}_{n=-6}^{\infty}$ be a solution of Eq.(10). Thus for n=0,1,2,...,

$$\begin{aligned}
\Psi_{12n-6} &= \frac{(-1)^n \eta^n \mu^n (\sigma-\tau)^n}{\sigma^n \kappa^{n-1} (\lambda-\mu)^n}, \\
\Psi_{12n-5} &= \frac{\sigma^n \zeta^n \kappa^n (\lambda-\mu)^n}{\lambda^n \tau^{n-1} (\eta-\zeta)^n (\zeta-\kappa)^n}, \\
\Psi_{12n-4} &= \frac{(-1)^n \lambda^n \tau^n (\eta-\zeta)^n (\zeta-\kappa)^n}{\eta^n \zeta^n \mu^{n-1} (\sigma-\tau)^n}, \\
\Psi_{12n-3} &= \frac{(-1)^n \eta^n \zeta \mu^n (\sigma-\tau)^n}{\sigma^n \kappa^n (\lambda-\mu)^n}, \\
\Psi_{12n-2} &= \frac{\sigma^{n+1} \zeta^n \kappa^n (\lambda-\mu)^n}{\lambda^n \tau^n (\eta-\zeta)^n (\zeta-\kappa)^n}, \\
\Psi_{12n-1} &= \frac{(-1)^n \lambda^{n+1} \tau^n (\eta-\zeta)^n (\zeta-\kappa)^n}{\eta^n \zeta^n \mu^n (\sigma-\tau)^n}, \\
\Psi_{12n} &= \frac{(-1)^n \eta^{n+1} \mu^n (\sigma-\tau)^n}{\sigma^n \kappa^n (\lambda-\mu)^n}, \\
\Psi_{12n+1} &= -\frac{\sigma^{n+1} \zeta^n \kappa^{n+1} (\lambda-\mu)^n}{\lambda^n \tau^n (\eta-\zeta)^n (\zeta-\kappa)^{n+1}}, \\
\Psi_{12n+2} &= \frac{(-1)^{n+1} \lambda^{n+1} \tau^{n+1} (\eta-\zeta)^n (\zeta-\kappa)^n}{\eta^n \zeta^n \mu^n (\sigma-\tau)^{n+1}}, \\
\Psi_{12n+3} &= \frac{(-1)^{n+1} \eta^{n+1} \mu^{n+1} (\sigma-\tau)^n}{\sigma^n \kappa^n (\lambda-\mu)^{n+1}}, \\
\Psi_{12n+4} &= \frac{\sigma^{n+1} \zeta^{n+1} \kappa^{n+1} (\lambda-\mu)^n}{\lambda^n \tau^n (\eta-\zeta)^{n+1} (\zeta-\kappa)^{n+1}}, \\
\Psi_{12n+5} &= \frac{(-1)^{n+1} \lambda^{n+1} \tau^{n+1} (\eta-\zeta)^n (\zeta-\kappa)^{n+1}}{\eta^n \zeta^{n+1} \mu^n (\sigma-\tau)^{n+1}},
\end{aligned}$$

where $\Psi_{-6} = \kappa, \Psi_{-5} = \tau, \Psi_{-4} = \mu, \Psi_{-3} = \zeta, \Psi_{-2} = \sigma, \Psi_{-1} = \lambda, \Psi_0 = \eta$.

Proof.

For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is

$$\begin{aligned}
\Psi_{12n-18} &= \frac{(-1)^{n-1}\eta^{n-1}\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-2}(\lambda-\mu)^{n-1}},\\
\Psi_{12n-17} &= \frac{\sigma^{n-1}\zeta^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-2}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}},\\
\Psi_{12n-16} &= \frac{(-1)^{n-1}\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-2}(\sigma-\tau)^{n-1}},\\
\Psi_{12n-15} &= \frac{(-1)^{n-1}\eta^{n-1}\zeta\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}},\\
\Psi_{12n-14} &= \frac{\sigma^{n}\zeta^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}},\\
\Psi_{12n-13} &= \frac{(-1)^{n-1}\lambda^{n}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n-1}},\\
\Psi_{12n-12} &= \frac{(-1)^{n-1}\eta^{n}\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}},\\
\Psi_{12n-11} &= -\frac{\sigma^{n}\zeta^{n-1}\kappa^{n}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n}},\\
\Psi_{12n-10} &= \frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}},\\
\Psi_{12n-9} &= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}},\\
\Psi_{12n-8} &= \frac{\sigma^{n}\zeta^{n}\kappa^{n}(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^{n}(\zeta-\kappa)^{n}},\\
\Psi_{12n-7} &= \frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n}}{\eta^{n-1}\zeta^{n}\mu^{n-1}(\sigma-\tau)^{n}}.
\end{aligned}$$

Now, we prove that the results are holds for n. From Eq.(10), it follows that

$$\begin{aligned}
\Psi_{12n-6} &= \Psi_{12n-9} - \frac{\Psi_{12n-9}\Psi_{12n-10}}{\Psi_{12n-10}-\Psi_{12n-13}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}} - \frac{\frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}}\frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}}}{\frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}} - \frac{(-1)^{n-1}\lambda^{n}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n-1}}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}} - \frac{\frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}}\frac{(-1)^{n}\lambda^{n}\tau^{n}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}}}{\frac{(-1)^{n-1}\lambda^{n}\tau^{n-1}(\eta-\zeta)^{n-1}(\zeta-\kappa)^{n-1}[-\tau-\sigma+\tau]}{\eta^{n-1}\zeta^{n-1}\mu^{n-1}(\sigma-\tau)^{n}}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n}} - \frac{(-1)^{n}\eta^{n}\mu^{n}\tau(\sigma-\tau)^{n-1}}{\sigma^{n}\kappa^{n-1}(\lambda-\mu)^{n}}\\
&= \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n-1}[\sigma-\tau]}{\sigma^{n}\kappa^{n-1}(\lambda-\mu)^{n}}.
\end{aligned}$$

So we have

$$\Psi_{12n-6} = \frac{(-1)^{n}\eta^{n}\mu^{n}(\sigma-\tau)^{n}}{\sigma^{n}\kappa^{n-1}(\lambda-\mu)^{n}}.$$

Similarly,

$$\begin{aligned}
\Psi_{12n-5} &= \Psi_{12n-8} - \frac{\Psi_{12n-8}\Psi_{12n-9}}{\Psi_{12n-9} - \Psi_{12n-12}} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} - \frac{\frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}\frac{(-1)^n\eta^n\mu^n(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n}}{\frac{(-1)^n\eta^n\mu^n(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n} - \frac{(-1)^{n-1}\eta^n\mu^{n-1}(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^{n-1}}} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} - \frac{\frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}\frac{(-1)^n\eta^n\mu^n(\sigma-\tau)^{n-1}}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n}}{\frac{(-1)^{n-1}\eta^n\mu^{n-1}(\sigma-\tau)^{n-1}[-\mu-\lambda+\mu]}{\sigma^{n-1}\kappa^{n-1}(\lambda-\mu)^n}} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}}{\lambda^{n-1}\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} - \frac{\sigma^n\zeta^n\mu\kappa^n(\lambda-\mu)^{n-1}}{\lambda^n\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n} \\
&= \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^{n-1}[\lambda-\mu]}{\lambda^n\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}.
\end{aligned}$$

Hence, we obtain

$$\Psi_{12n-5} = \frac{\sigma^n\zeta^n\kappa^n(\lambda-\mu)^n}{\lambda^n\tau^{n-1}(\eta-\zeta)^n(\zeta-\kappa)^n}.$$

Similarly, by using the same method, we can investigate other relations.

6. Numerical Examples

For our prior results, we present some numerical examples to explain the solution behavior of Eq.(1).

Example 1. In numerical simulation they assumed that for Eq.(7) the initial value are $\Psi_{-6} = 0.3, \Psi_{-5} = 0.6, \Psi_{-4} = 0.9, \Psi_{-3} = 1.2, \Psi_{-2} = 1.5, \Psi_{-1} = 1.8$ and $\Psi_0 = 2.1$. Then the solution appear in Figure 1.

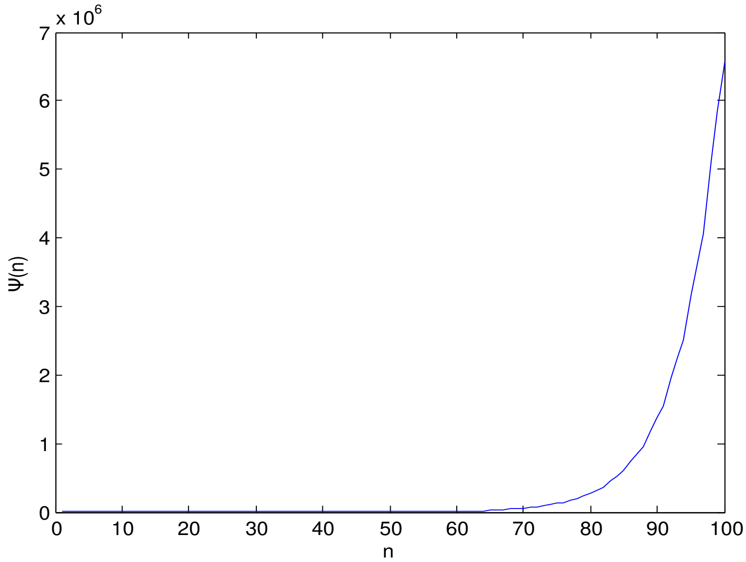

Figure 1. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}$.

Example 2. Numerically when the initial value are $\Psi_{-6} = 4.6, \Psi_{-5} = 2.5, \Psi_{-4} = 1.4, \Psi_{-3} = 3, \Psi_{-2} = 4.5, \Psi_{-1} = 6.3$ and $\Psi_0 = 3.5$. Figure 2 shows the results of Eq.(8).

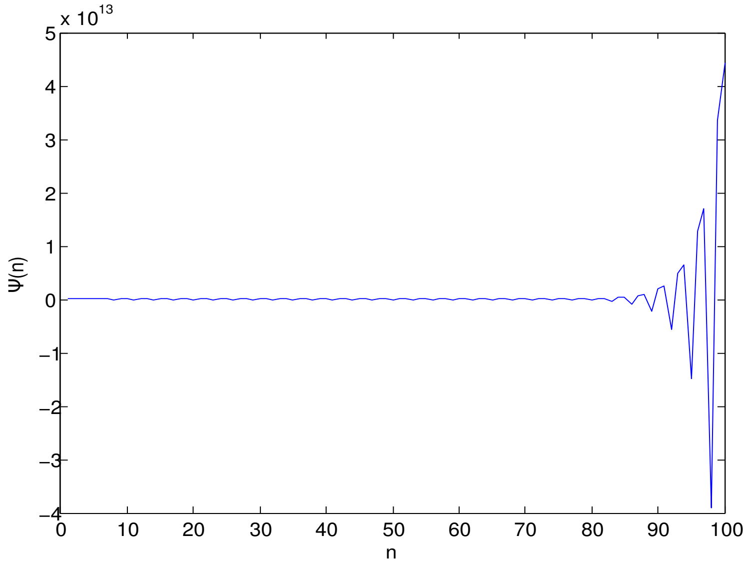

Figure 2. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} + \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}-\Psi_{n-6}}$.

Example 3. Figures 3 depict the behavior of Eq.(9), with initial conditions are $\Psi_{-6} = 2.8, \Psi_{-5} = 5.9, \Psi_{-4} = 8.5, \Psi_{-3} = 4.2, \Psi_{-2} = 7.4, \Psi_{-1} = 3.2$ and $\Psi_0 = 6.7$.

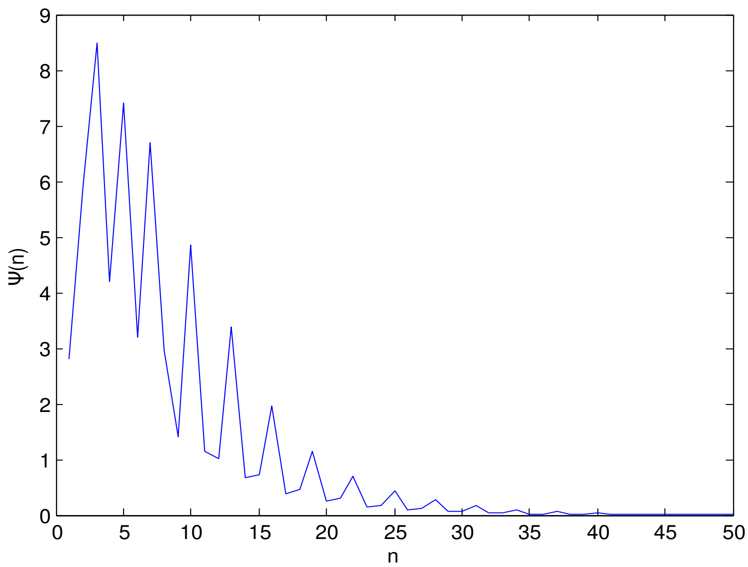

Figure 3. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}+\Psi_{n-6}}$.

Example 4. For Eq.(10) the initial conditions are set as follows: $\Psi_{-6} = 2.2, \Psi_{-5} = 3.9, \Psi_{-4} = 7.5, \Psi_{-3} = 4.2, \Psi_{-2} = 4.8, \Psi_{-1} = 3.2$ and $\Psi_0 = 6.7$, results shows in Figure 4.

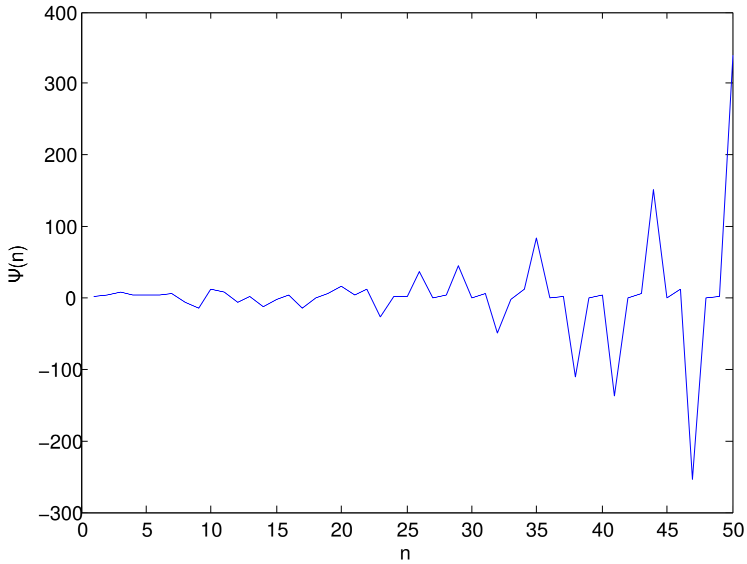

Figure 4. Plotting the solution of $\Psi_{n+1} = \Psi_{n-2} - \frac{\Psi_{n-2}\Psi_{n-3}}{\Psi_{n-3}-\Psi_{n-6}}$.

7. Conclusions

Studying the dynamics of such equations is a very significant mathematical topic since these equations are strongly related to models in population dynamics and biological sciences. The basic goal of equations dynamics is to predict the global behavior of a equation based on the information of its current state. In this article, we have found general form of the solutions of rational difference equations and we investigated the dynamics of equilibrium point. In sections 2 and 3, we have investigated the existence and uniqueness of equilibrium point and the solutions qualitative behavior is explored, such as local and global stability. Also, we have proven that the solution is bounded in section 4. In section 5, we have obtained expressions of solutions of four special cases of the studied equations 7,8,9 and 10, as applications of Eq.(1). Finally, to support our theoretical discussion some illustrative examples are provided in section 6.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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Lie Algebra Contributions to Instantaneous Plane Kinematics

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Keywords
Geometric Kinematics, Instantaneous Invariants, Lie Algebra, Time Independent Planar Motion

Abstract — To investigate the instantaneous properties of a planar motion, Roth and Bottema [1] obtain the instantaneous invariants of a planar motion using Veldkamp's canonical frame [7]. We investigate the derivatives of time-independent planar motions with respect to the fixed frame up to thrid order and their instantaneous invariants are obtained by using the Lie algebra to the planar motion group near identity element.

Subject Classification (2020): 53Z30, 70E17, 70G65.

1. Introduction

Lie theory connects almost every branch of mathematics. It has a wide range of applications from harmonic analysis to quantum groups. In this work, our interest is Lie algebra in plane kinematics [4]. The group of rigid body motions are all related to Lie groups. Planar motion group, Spherical motion group and Spatial motion group are represented by $SE(2)$, $SO(3)$ and $SE(3)$ respectively.

Rigid body motions in $\mathbb{R}^2$ has a 3×3 homogeneous matrix representation. Any element of planar motion group is given by,

$$G = \begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix} \tag{1.1}$$

where R is the (2×2) rotation matrix and the vector $\vec{t}$ is a (2×1) translation vector. Well known Mozzi-Chasles's theorem says that each spatial motion is a screw motion. That is, any spatial displacement can be seen as a rotation about a line, the screw axis, and followed by a translation parallel to that line. In the planar case, there is a fixed point instead of a line. Any planar motion is a rotation about this fixed point.

To study planar motion, we attach a coordinate frame, M, to the moving body and a coordinate frame, F to the ground (reference frame). In plane kinematics, except pure translations, there is a single point whose coordinates are the same both in the fixed frame and in the moving frame before and after the displacement.

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This point is called the pole point of the planar displacement. The rotation angle θ and the pole points are the geometric invariants of planar kinematics.

Any rigid transformation is the combination of a rotation followed by a translation, given by,

$$\vec{X} = R\vec{x} + \vec{t} \tag{1.2}$$

where $\vec{x}$ is the coordinates of a point in the moving frame M and $\vec{X}$ is the coordinates of the point in the fixed frame F. In 1.2 the rotation matrix R and the translation vector $\vec{t}$ are given by,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \vec{t} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1.3}$$

Then the fundamental equations of the plane kinematics can be written as;

$$X = x\cos\theta - y\sin\theta + a, \tag{1.4}$$

$$Y = x\sin\theta + y\cos\theta + b.$$

If θ, a and b are functions of a time parameter μ, one can determine a continuous motion of a point with its positions, velocity, accelaration etc. The parameters depending on μ are the concerns of *time-dependent kinematics* of the motion. But some other properties are independent of time; such as curves, tangents, poles etc. which are called the *geometric kinematics* of the motion. In this case

$$X = x\cos\theta - y\sin\theta + a(\theta), \tag{1.5}$$

$$Y = x\sin\theta + y\cos\theta + b(\theta)$$

where θ is the only parameter for the planar motion, and a, b are the functions of θ [1, 3]. In the equation 1.5 the planar motion is completely defined by the functions $a(\theta)$ and $b(\theta)$. The equation 1.5 is called *time-independent motion.*

2. Veldkamp's Canonical Frame

To discuss the instantaneous geometric invariants of a planar motion, we introduce Veldkamp's canonical frame. In his dissertation [7], for a given time-independent motion $a(\theta)$ and $b(\theta)$ are the power series of θ;

$$a(\theta) = \sum_{n=0}^{\infty} a_n(\frac{\theta^n}{n!}), \quad b(\theta) = \sum_{n=0}^{\infty} b_n(\frac{\theta^n}{n!}) \tag{2.1}$$

and the following are satisfield:

i) The moving and fixed frame are chosen such that they coincide in the "zero-position", so we have from 2.1 and 1.5

$$a_0 = b_0 = 0. \tag{2.2}$$

ii) At the moment $\theta = 0$ we place common origin of the frames at the pole, which implies

$$a_1 = b_1 = 0. \tag{2.3}$$

iii) The axes O_X and O_x are chosen along the common tangent of the pole curves (at the pole), which yields

$$a_2 = 0. \tag{2.4}$$

iv) Assuming $b_2 \neq 0$, we can take the possitive direction of the X axis as to set $b_2 > 0$. The coinciding frames O_{XY} and O_{xy} defined by i)-iv) are called canonical. Here O_{XY} and O_{xy} denote the fixed frame and the moving frame respectively. These canonical frames can be used to study instantaneous kinematics, for details see [1, 7].

Roth and Bottema [1] obtained the geometry of the planar motion by differentiating the coordinate axes given in equation 1.5.

$$\begin{array}{llll} X = x & \dot{X} = -y & \ddot{X} = -x & \dddot{X} = y + a_3 \quad \dots \\ Y = y & \dot{Y} = x & \ddot{Y} = -y + b_2 & \dddot{Y} = -x + b_3 \quad \dots \end{array} \tag{2.5}$$

at $\theta = 0$, where dot over an alphabet denotes the derivative with respect to θ. Hence instantaneous properties of the motion depend on the constants $a_3, a_4, \dots, a_n, \dots$ and $b_2, b_3, b_4, \dots, b_n, \dots$. which are called the instantaneous invariants of the kinematics.

2.1. The Group Planar Motions and Its Lie Algebra

The (3×3) matrix G in equation 1.1 represents the planar motion group, it is the element of the Lie group $SE(2)$. The time independent motion in equation 1.5 defines a one parameter subgroup of $SE(2)$. Let $D(\theta)$ denotes the planar displacement defined in equation 1.5 in the homogeneous matrix representation,

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & a(\theta) \\ \sin\theta & \cos\theta & b(\theta) \\ 0 & 0 & 1 \end{pmatrix}.$$

If we consider an initial point $P(0)$, then the transfomed point $P(\theta)$ is written by,

$$\begin{pmatrix} P(\theta) \\ 1 \end{pmatrix} = D(\theta) \begin{pmatrix} P(0) \\ 1 \end{pmatrix}. \tag{2.6}$$

Differentiating the equation 2.6 gives,

$$\begin{pmatrix} \dot{P}(\theta) \\ 0 \end{pmatrix} = D_F(\theta) \begin{pmatrix} P(\theta) \\ 1 \end{pmatrix}, \tag{2.7}$$

where $D_F(\theta)$ is the derivative with respect to the fixed frame. The geometric velocity matrix $D_F(\theta)$ of the group element $D(\theta)$ can be found as,

$$D_F(\theta) = \dot{D}(\theta) D(\theta)^{-1} = \begin{pmatrix} 0 & -1 & \dot{a}(\theta) + b(\theta) \\ 1 & 0 & -a(\theta) + \dot{b}(\theta) \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

Let $(X,Y,1)^t$ and $(x,y,1)^t$ be the homogeneous coordinates of $P(\theta)$ in the fixed frame and the moving frame respectively. Then the equation 2.7 at $\theta=0$ gives,

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ 0\end{pmatrix} = D_F(0)\begin{pmatrix}X\\ Y\\ 1\end{pmatrix} = \begin{pmatrix}0 & -1 & a_1\\ 1 & 0 & b_1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\\ 1\end{pmatrix}. \tag{2.9}$$

Since the moving frame and the fixed frame are coincident at $\theta=0$, we get

$$\dot{X} = -y + a_1 \quad \dot{Y} = x + b_1.$$

In equation 2.3 a_1 and b_1 are equal to zero. Hence,

$$\dot{X} = -y \quad \dot{Y} = x.$$

The second derivatives can be obtained by differentiating the equation 2.7 ,

$$\begin{pmatrix}\ddot{P}(\theta)\\ 0\end{pmatrix} = D_{F_2}(\theta)\begin{pmatrix}P(\theta)\\ 1\end{pmatrix}, \tag{2.10}$$

where $D_{F_2}(\theta)$ denotes the second derivative matrix with respect to the fixed frame. The matrix $D_{F_2}(\theta)$ can be written as follows,

$$D_{F_2}(\theta) = \dot{D}_F(\theta) + D_F^2(\theta) = \begin{pmatrix}-1 & 0 & a(\theta)+\ddot{a}(\theta)\\ 0 & -1 & b(\theta)+\ddot{b}(\theta)\\ 0 & 0 & 0\end{pmatrix}. \tag{2.11}$$

Then the equation 2.10 at $\theta=0$ is,

$$\begin{pmatrix}\ddot{X}\\ \ddot{Y}\\ 0\end{pmatrix} = D_{F_2}(0)\begin{pmatrix}X\\ Y\\ 1\end{pmatrix} = \begin{pmatrix}-1 & 0 & a_2\\ 0 & -1 & b_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\\ 1\end{pmatrix}. \tag{2.12}$$

Since $a_2=0$ in 2.4 and the frames are coincident at $\theta=0$,

$$\ddot{X} = -x \quad \ddot{Y} = -y + b_2.$$

Finally, the third derivative of the equation 2.6 can be found as,

$$\begin{pmatrix}\dddot{P}(\theta)\\ 0\end{pmatrix} = D_{F_3}(\theta)\begin{pmatrix}P(\theta)\\ 1\end{pmatrix}, \tag{2.13}$$

where $D_{F_3}(\theta)$ denotes the third derivative with respect to the fixed frame. The third order derivative matrix $D_{F_3}(\theta)$ can be found as follows,

$$\begin{aligned} D_{F_3}(\theta) &= \ddot{D}_F(\theta) + 2\dot{D}_F(\theta)D_F(\theta) + D_F(\theta)\dot{D}_F(\theta) + D_F^3(\theta) \\ &= \begin{pmatrix} 0 & 1 & \ddot{a}(\theta) - b(\theta) \\ -1 & 0 & a(\theta) + \dddot{b}(\theta) \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.14}$$

Then the equation 2.13 at $\theta = 0$ is,

$$\begin{pmatrix} \dddot{X} \\ \dddot{Y} \\ 0 \end{pmatrix} = D_{F_3}(0) \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & a_3 \\ -1 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}. \tag{2.15}$$

That is,

$$\dddot{X} = y + a_3 \quad \dddot{Y} = -x + b_3.$$

Hence the kinematic invariants of time-independent planar motion are obtained by using the elements of Lie algebra, $se(2)$ to the planar motion group, $SE(2)$. Similarly, higher order terms in the equation 2.5 can be obtained by using the higher order derivatives of equation 2.6.

3. Conclusion

Instantaneous properties of a planar motion are obtained by Bottema and Roth [1] using canonical frame which was introduced by Veldkamp [7]. In this study, the derivatives of time-independent planar motions with respect to the fixed frame are given and the instantaneous invariants of planar motions are obtained by using the Lie algebra to $SE(2)$.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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